A STUDY OF ARCH DAM ANALYSIS ON THE

BASIS OF THIN SHELL THEORY.

by

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INTRODUCTION.

An arch dam is a concrete arched structure which is constructed across a valley in order to retain water. The analysis of this curved surface of variable thickness is complex due to the redundant nature of the structure and the number of variables involved so that various assumptions and approximations must be made in order to simplify the problem before a solution can be found. The existing methods of analysis can be divided into two classes viz: a) formulating exact equations and solving by approximate methods, or b) simplifying the equations so that an exact solution of these equations can be found. It cannot be stated a priori that one of these will yield results which are more correct than the results obtained by the other, and in order to test the validity of any new method of analysis the results obtained must be compared with the values actually measured in the prototype.

The analysis of an arch dam by both the above methods is laborious and in this thesis a simpler method of analysis is proposed and developed.

The maximum ratio of thickness to radius of many of the arch dams that have been constructed in recent years has been of such small magnitude that a thin cylindrical shell theory could be applied for their analysis and this method has been suggested by previous authors. However, in order to obtain a formal solution the shell equations were simplified by the use of additional assumptions which increased the error. In this thesis use is made of a thin shell theory, but the shell equations are solved by relaxation so that some of these
additional error-producing assumptions are no longer necessary. In order to apply the relaxation technique it is necessary to assume that the developed surface of the dam is covered with a grid of imaginary horizontal and vertical lines and the solution at each of the intersection points is then obtained by a controlled trial method. The accuracy of the solution depends on the mesh length of the grid system selected, but a reduced mesh length, as well as increasing accuracy, increases the number of points to be 'relaxed' and, therefore, the amount of labour involved in obtaining the solution. This is one of the disadvantages of the relaxational technique, for the amount of labour involved in obtaining a reasonable solution might prove to be excessive due to the number of intersection points that have to be considered. It is shown in this work that the results obtained by means of a graded mesh, i.e., a fine mesh at the crest and a coarser mesh throughout the rest of the dam is almost equivalent in accuracy to the results obtained when using a fine mesh over the complete surface. The accuracy is thus maintained with a reduction in the amount of computation involved.

After a brief description of existing design methods a comparison is made of some of the thin cylindrical shell theories that have been developed and one of these is selected and expressed in a suitable form for relaxation. The resulting equations are then applied to a particular case for which a solution, using exact elastic equations, had already been published and comparison of the displacements obtained by both methods indicates that, at the crest, there is a large
discrepancy in the results. In order to discover the cause of this error a relaxational solution is obtained for the problem of the complete cylinder, fixed at the base and subjected to external water pressure, for which the formal solution was known. By comparing the results of the analytical solution of the complete cylinder with a number of relaxational solutions using different mesh lengths, it is shown that the error can be reduced by reducing the mesh length at the crest and in order to enable the fine and coarse mesh to be relaxed simultaneously the graded mesh technique is developed.

Finally, the graded mesh relaxational solution is applied to the solution of a constant angle arch dam, situated in a rectangular valley and having a linear variation of thickness with depth. Since for this hypothetical dam there is no prototype, the results obtained by relaxation are compared with the experimental results obtained from a reduced scale model. The model material selected was a mixture of plaster and diatomite, the load being applied by means of mercury, and the strains were measured with wire resistance strain gauges. Previous authors have already shown that results obtained by means of plaster-diatomite models compare favourably with the results measured directly in the prototype, and thus the agreement obtained between the relaxational solution and the model results indicates that the theoretical solution is reasonably accurate.
CHAPTER 1.
ARCH DAMS.

INTRODUCTION. (1,1)

An arch dam is a dam which is curved in plan so that the water load applied to the upstream surface is transmitted by arch action to the abutments. As far as records show the first dam of this type was built in 1611 in Austria (1), but it was not until the beginning of this century that any serious attempt was made to analyse the arch action.

In 1923 the trial-load system of analysis was developed in the U.S.A., and a large number of dams were designed by this method, whilst in Europe in the last twenty years designers have turned their attentions to design by model investigations.

Although, until recently, arch dams were considered suitable for V-shaped valleys only, their use has now been extended to U-shaped valleys and since they are more economic and have a greater factor of safety than other types of dam, the practicability of an arch dam is usually considered in preference to other possible solutions (2).

In a recent paper by Leliavsky (3) the profiles of two arch dams – Boulder and Vajont – were contrasted in order to illustrate the wide variation that must exist in design methods when they produce dams of such different shape. (Fig. 1,1)

The problem of arch dam design must rank amongst the

* A bibliography is given on Page /28
most complex of design problems when one considers the number of variables and unknowns that exist.

Classification of Arch Dams. (1.2)

Arch dams are usually classified as follows:

(a) **Constant angle.**

(b) **Constant radius.**

(c) **Variable radius and angle.**

(a) **Constant angle** dams are those in which the angle subtended at the centre is constant for all arches throughout the depth of the dam. It has been shown by Jorgensen (4) that for the volume of the dam to be a minimum, the constant angle should be $133^\circ 34'$ or

$$ R = 0.544 \ell $$

where $\ell$ is the arch span and $R$ the corresponding radius. This condition is only true for dams designed by the cylinder theory.

Since the arch span increases from base to crest, the radius increases and the upper arches will overhang downstream.

(b) **Constant Radius.** The constant radius maybe to the upstream face, downstream face or arch axis. The angle subtended by the arch varies with the depth being a maximum at the crest. At some elevation the angle subtended will be the economic angle of $133^\circ 34'$ but at all other elevations the angle will be greater or less than this value. For this reason the constant radius dam is not as economic as the constant angle dam and also is best suited to U-shaped valleys where the deviation from the best angle is least.
(c) **Variable Radius and Angle.** If the thicknesses of the lower arches of a constant angle dam are increased it is possible to eliminate the overhang. This, however, would not be economical and since the lower arches are now thicker than required the radius can be increased. By adjusting both the thickness and the radius the overhang can be eliminated and the volume of the dam kept within economical limits. This leads to a dam of variable radius and angle.

**Loads on Arch Dams.** (1.3)

The stresses in arch dams are usually computed due to the effects of hydraulic pressure, temperature variation, weight of concrete and yielding of abutments.

Other effects which may be allowed for include ice pressure, earthquakes, uplift pressure, concrete shrinkage, and foundation settlement.

**Methods of Analysis.** (1.4)

The analysis of arch dams may be reduced to the solution of the problem of a curved plate of varying thickness, fixed along part of its edge, which is subjected to a particular loading. No exact solution has been found for this problem and various approximate solutions have been proposed. This thesis is concerned with the development of one of these approximate methods of solution.

The earlier arch dams were of the constant radius type and these were designed by use of the approximation that the dam was part of a complete cylinder. The upper portion of this cylinder was designed by using the thin-cylinder theory and
the lower and thicker sections at the base by the use of the thick cylinder theory. Since arch dams are neither thin nor complete cylinders, the results obtained by this method are inaccurate.

A more reasonable method is to consider the dam divided into individual arches and this method, according to Jaeger (2), was first proposed by Moersch (5) in 1908. The dam was assumed to be divided into independent horizontal arches fixed at the abutments and the entire load of the dam assumed to be carried by these arches. Using formulae developed by W. Cain (6) the stresses at the intrados and extrados of each arch could then be calculated. This method is still used for preliminary design and charts have been produced by Fowler (7) and Cravitz (8) for this purpose.

In 1913 H. Ritter (9) suggested that the dam should be considered as being made up of a system of horizontal arches and another system of vertical cantilevers. The deflection of the arches and cantilevers were first determined by assuming that unit load was applied at each intersection. The unknown loads acting at the points of intersection of the grid could then be calculated using the fact that the deflection at any intersection point must be the same for both arch and cantilever. Once the loads were obtained the stresses could be evaluated.

In the method of active arches proposed in 1922 by Resal (28) and developed by Materre (10) the tension zones occurring at the extrados of the springings and the intrados of the crown of an actual arch were assumed to be fissured so that a fictitious arch was established which satisfied the
condition of no tension (Fig. 1.2). This fictitious arch, called the active arch, was assumed to be circular and of uniform thickness. The material in the tension zones was neglected and only the material in the active arch resisted the stresses to which the arch was subjected.

The trial load method (ll) of analysis is similar to the method of H. Ritter in-so-far as the dam is divided into horizontal and vertical slices, but for this method the load is distributed between the two systems, by trial, on the basis of equal deflections at the intersection points. The total static load is first divided so that part is assumed carried by the cantilevers and the rest by the arches. Deflections, due to these loads, are calculated and the load division re-adjusted to obtain equal deflections at intersection points. Adjustments are made for radial displacements, tangential displacements and rotations for both arch and cantilever elements.

A full analysis of an arch dam by the trial-load method is a major undertaking requiring a staff of trained designers and takes a considerable time to complete. A simplified trial load analysis has been developed in which radial adjustments of deflections are made to horizontal arches and one cantilever only. The arches are assumed to be uniformly loaded so that the Cain formulae can be used to determine the deflections. Only the central cantilever is considered and the problem reduces to determining the law of loading distribution on this cantilever.
In a method attributed to M. Ritter (12) and developed by Jaeger (2), calculation is again confined to a series of horizontal arches and one central cantilever. Radial deflections are adjusted and allowance can be made for yielding of the abutments and foundations.

The total radial deflection $\delta_{tr}$ of the crown of any arch $r$ is determined for uniform load $P_r = 1$. The total radial displacement $\delta_{sr}$ of a point of the crown cantilever loaded at $r$ with a load $P_r = 1$ is also determined (Fig. 13).

Equating deflections at the various arch levels of the crown cantilever the following equations can be obtained (2):

\[
\begin{align*}
P_1 \delta_{10} + P_2 \delta_{11} + P_3 \delta_{12} &= (Z_1 - P_1) \delta_{1r} \\
P_1 \delta_{20} + P_2 \delta_{21} + P_3 \delta_{22} &= (Z_2 - P_2) \delta_{2r} \\
&\vdots \\
P_1 \delta_{n0} + P_2 \delta_{n1} + P_3 \delta_{n2} &= (Z_n - P_n) \delta_{nr}
\end{align*}
\]

where $Z_1, Z_2, Z_3, \ldots, Z_n$ are the total water loads acting over the arch slices at different levels and $P_1, P_2, \ldots, P_n$ the part of the total water load taken by the cantilevers.

There are, therefore, $n$ equations for calculating the unknowns $P_1, P_2, P_3, \ldots, P_n$. 
Instead of assuming that the arch dam is divided into a series of horizontal and vertical elements it is possible to consider the dam as a portion of a cylindrical shell and analyse the surface on the basis of thin cylindrical shell theory. This approach has been followed by Tolke, Lombardi, Hertzog, Prieu and others. The development of this method of analysis will be considered in Chapter 3.

In 1956 Pippard, Allen, Chitty and Severn (13) published the results of an analysis of an arch dam using exact elastic equations, formulated in terms of the displacements, and a three dimensional relaxation solution was obtained, using Southwell's relaxation technique. In order to calculate an approximate solution the authors first considered the dam as a section of a complete cylinder and the results obtained from this 'tumbler' analogy were used as a first approximation in the three-dimensional solution. Experimental investigations were also conducted using scale models cast in rubber and loaded with water.
Before considering the application of a thin cylindrical shell theory to the analysis of an arch dam, it becomes necessary to select one of the various sets of shell equations that have been proposed. The most important of the shell equations are those developed by Love (14), Flügge (15), Timoshenko (16), and Donnell (17), and in this chapter a brief comparison is made between the theories of these authors. Such a comparison is complicated by the fact that the different authors make their approximations and assumptions at different stages of the development of the theory.

Comparison of the Various Shell Theories. (2.2)

In order to develop a shell theory it is necessary to obtain the relationships which connect the following:

a) Middle surface strains and changes of curvature with the displacements of the middle surface.

b) Parallel surface strains with middle surface strains and changes of curvature.

c) Normal and shear stresses with parallel surface strains.

d) Forces and Moments with normal and shear stresses.

e) The forces and moments in the form of the equilibrium equations.

By consecutive substitution of each stage of these relationships into the next, one obtains three partial differential equations connecting the displacements of the
middle surface and external forces. (See App. II). Further manipulation of these three equations produces an 8th order equation in one variable and the solution of this equation which satisfies the boundary conditions of the particular problem would furnish a complete solution. However, it is not generally possible to obtain an exact solution of this equation and it is usually more convenient to attempt a solution of the three displacement equations.

(a) Strain - Displacement Relationships.

Comparison of the work of Love, Flügge, Timoshenko and Donnell shows that the only differences in the strain-displacement relationships occur in the expressions for change of curvature $x_2$ and twist $\tau$.*

<table>
<thead>
<tr>
<th></th>
<th>Love</th>
<th>Flügge</th>
<th>Timoshenko</th>
<th>Donnell</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>$W'' + \frac{V''}{R}$</td>
<td>$W'' + \frac{V''}{R}$</td>
<td>$W'' + \frac{V''}{R}$</td>
<td>$W''$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>$W'' + \frac{V''}{R}$</td>
<td>$W'' - \frac{V''}{R}$</td>
<td>$W'' + \frac{V''}{R}$</td>
<td>$W''$</td>
</tr>
</tbody>
</table>

Where $u$, $v$ and $w$ refer to the displacements in the vertical $(x)$, tangential $(y)$ and radial $(z)$ directions respectively and

$$(x)' = \frac{\partial}{\partial x} ; \quad (y)' = \frac{\partial}{\partial y}.$$  

It is interesting to note that Timoshenko first establishes the same formula as Flügge for $x_2$ and then substitutes $\frac{V'}{R}$ for $\frac{W}{R}$ which is true for the case of the inextensional deformation of a cylindrical shell. The difference for $\tau$ can be accounted for similarly since for inextensional deformation $u = -v$ at the middle surface. It would appear, therefore, that Flügge's

*A list of symbols is given on Page 126.
(b) **Parallel Surface Strains to Middle Surface Strains.**

The strains occurring in a parallel surface distance $z$ from the middle surface (Fig. II.1) can be expressed in terms of a power series in $z$. Love's equation for a cylindrical shell reduce to

$$
\varepsilon_x = \varepsilon_l - z x_l,
$$
$$
\varepsilon_y = \varepsilon_2 - z x_2 - z^2 \frac{x_2}{R} - \frac{\mu}{2(1-\mu)} \left( \frac{x_1 + x_2}{R} \right)
$$
$$
\gamma_{xy} = \omega - 2 z \gamma - z^2 \frac{\gamma}{R}
$$

Timoshenko and Donnell simplify these equations by neglecting all terms containing $z^2$. In order to compare Flügge's equations with the above it is necessary to recall that $x_l$ and $\gamma$ are only approximate values of the changes of curvature since the effect of middle surface strain is neglected. Allowing for middle surface strain

$$
x'_l = x_l + \frac{\sigma_2}{R}
$$
$$
\gamma' = \gamma + \frac{\sigma_2}{2R}
$$

where $x'_l$ and $\gamma'$ denote the true change of curvature.

Substituting (2.2.2) in (2.2.1) leads to

$$
\varepsilon_x = \varepsilon_l - z x'_l
$$
$$
\varepsilon_y = \varepsilon_2 - z x_2' + z \frac{x_2}{R} - \frac{\mu}{2(1-\mu)} \left( \frac{x_1 + x_2'}{R} \right)
$$
$$
\gamma_{xy} = \omega - 2 z \gamma' + \frac{\omega_2}{R} - \frac{\gamma^2}{R - \frac{\omega_2}{2R^2}}
$$

which are identical with the Flügge equations if the last term of $\gamma_y$ is neglected, i.e., for $\mu = 0$.

(c) **Stress - Strain Equations.**

The only difference in the stress-strain equations occurs in Love's expressions, where the effect of the direct radial stress $\sigma_z$ is included. Timoshenko, Flügge and also Donnell take $\sigma_z = 0$. 

expressions are the more exact.
(d) **Forces and Moments in terms of Stresses.**

These relationships can be typified by the expressions

\[
N_r = \int_{\frac{1}{2}}^{1} \sigma_r (1 + \frac{Z}{R}) dz \quad \text{and} \quad M_r = \int_{\frac{1}{2}}^{1} \sigma_r z (1 + \frac{Z}{R}) dz \quad \cdots \quad (2.2.4)
\]

where \( \frac{Z}{R} \) is included to allow for the difference in the length of the parallel surface compared with the middle surface. If the trapezoidal shape of the element perpendicular to the axis of the cylinder (Fig. II.2) is not allowed for, i.e., the outer edges are assumed to be of the same length as the middle surface, then \( \frac{Z}{R} \) is neglected in the above expressions.

Love and Flügge both retain the \( \frac{Z}{R} \) term whilst Timoshenko and Donnell neglect it.

Since \( \tau_{xy} = \tau_{yx} \) it follows that the neglect of the \( \frac{Z}{R} \) term implies that \( M_{xy} = M_{yx} \) and \( N_{xy} = N_{yx} \). Both Timoshenko and Donnell, therefore, assume these equalities.

(e) **Equilibrium Equations.**

The equations of equilibrium are expressed by Love and Flügge in the following form:

1. \( N_x + N_y' + X = 0 \)
2. \( N_y + N_y' + Y - \frac{Q_y}{R} = 0 \)
3. \( Q_x + Q_y' + \frac{N_y}{R} + Z = 0 \) \( \cdots \quad (2.2.5) \)
4. \( M_{xx} + M_{xy} = Q_x = 0 \)
5. \( M_{yy} + M_{xy} = Q_y = 0 \)
6. \( M_{yx} + (N_{xy} - N_{yx}) R = 0 \)

Where the forces and moments have the direction indicated in Fig II.3 and Fig. II.4. Timoshenko and Donnell use only
the first five of these equations; since, from (d) 
\( N \) is neglected and \( N_{xy} = N_{yx} \), equation six cannot be 
satisfied. It can be shown that equation six is 
automatically satisfied by Love and Flügge. Donnell 
also omits the term \( Q_{xy} N \) in the second equation.

**SUMMARY.** (2.3)

It is apparent from the above that the theories 
of thin shells as developed by Love, Flügge, Timoshenko 
and Donnell cannot be easily compared since the assump-
tions and approximations are made at different stages in 
the development and it is difficult to assess what effect 
these approximations would have if introduced at some 
other stage.

A comparison of the relative accuracy of the relations-
ships that are necessary in any shell theory is given in 
the following table for the theories of Love, Flügge, 
Timoshenko and Donnell.

<table>
<thead>
<tr>
<th>Relationship</th>
<th>Love</th>
<th>Flügge</th>
<th>Timoshenko</th>
<th>Donnell</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>III</td>
</tr>
<tr>
<td>b</td>
<td>I</td>
<td>II</td>
<td>III</td>
<td>III</td>
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<tr>
<td>c</td>
<td>I</td>
<td>II</td>
<td>II</td>
<td>II</td>
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<td>d</td>
<td>I</td>
<td>I</td>
<td>II</td>
<td>II</td>
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<tr>
<td>e</td>
<td>I</td>
<td>I</td>
<td>II</td>
<td>II</td>
</tr>
</tbody>
</table>

In this table the roman numerals indicate the relative 
accuracy of the particular relationship, e.g. for relation-
ship (c) (Stress - Strain) the table shows that Flügge, 
Timoshenko and Donnell use the same equations II which are 
not as accurate as the equations I used by Love.
Simplifications other than those indicated above have been suggested by various authors from time to time, the most important being the neglect of the vertical displacement u. Some of these theories are considered in Chapter 3.

Since the application of the cylindrical shell theory to arch dam analysis is further complicated by the fact that both the thickness of the dam and the external load vary with the depth, it was decided to use the simplest of the above theories, i.e., Donnell's, and to modify this to allow for the variable thickness.

**APPENDIX II.**

**Development of Theory of Thin Cylindrical Shells**

**as used by Author.**

**Assumptions.**

(a) The thickness of the shell is small compared with the radius. Various authors give different values for the limiting magnitude of \( \frac{t}{R} \) below which they consider that a thin shell theory is applicable. Novozhilov (29) states that \( \frac{t}{R} \) should be < \( \frac{1}{20} \) but Lombardi (22) when considering arch dams assumes a limiting value of \( \frac{1}{5} \). Others, such as Priscu (24) quote a figure as high as \( \frac{1}{3} \).

(b) Normal sections to the middle surface remain normal after deformation.

(c) Normal stresses on planes parallel to the middle surface are neglected in comparison with other stresses.

(d) The ratio \( \frac{Z}{R} \) can be neglected compared with unity.
Strain - Displacement Equations.

Figure II.1 shows a normal fibre as extended by direct stress to bb and rotated due to bending to cc. Figure II.2 may be considered similar since assumption (d) is equivalent to neglecting the trapezoidal shape of the section.

The displacements of a parallel surface can be expressed:

\[ \begin{align*}
    u^b &= u - z \frac{\partial \nu^b}{\partial x} \\
    v^b &= v - z \frac{\partial \nu^b}{\partial y} \\
    w^b &= w
\end{align*} \] -- (II.1)

where \( u^b, v^b, \) and \( w^b \) are the displacements of a surface parallel to the mid-surface.

Since

\[ \varepsilon_{xx} = \frac{\partial u^b}{\partial x} \quad \varepsilon_{yy} = \frac{\partial v^b}{\partial y} - \frac{w^b}{R} \quad \gamma_{xy} = \left( \frac{\partial v^b}{\partial x} + \frac{\partial u^b}{\partial y} \right) \] -- -- (II.2)

Therefore, from (II.2) and (II.1)

\[ \begin{align*}
    \varepsilon_{xx} &= u' - z \varepsilon_{w} \\
    \varepsilon_{yy} &= v' - z \varepsilon_{w} - \frac{w'}{R} \\
    \gamma_{xy} &= v' - 2z \varepsilon_{w} + u'
\end{align*} \] -- -- (II.3)

From Hooke's Law neglecting the stress \( \sigma_z \) (assumption (d)) the stress-strain relationships are:

\[ \begin{align*}
    \sigma_x &= \frac{E}{1 - \mu^2} (\varepsilon_{xx} + \mu \varepsilon_{yy}) \\
    \sigma_y &= \frac{E}{1 - \mu^2} (\varepsilon_{yy} + \mu \varepsilon_{xx}) \\
    \tau_{xy} &= \tau_{yx} = G \gamma_{xy}
\end{align*} \] -- -- (II.4)

The stresses \( \tau_{xy} \) and \( \tau_{yx} \) (Fig. II.5) are not required in order to derive the displacement equations since the forces \( Q_x \) and \( Q_y \), which are related to these stresses, can be eliminated from the equations of equilibrium.
\[ U^p = U - Z \frac{\partial w}{\partial x} \]

**FIG II.1**

\[ V^p = V - Z \frac{\partial w}{\partial y} \]

**FIG II.2**

**STRESSES**

**FIG II.5**
Since $\frac{Z}{R}$ is considered small compared with unity, the forces (Fig. II.3) and moments (Fig. II.4) can be defined by the following equations:

$$N_x = \int_{Z/2}^{Z} \sigma_x \, dz; \quad N_y = \int_{Z/2}^{Z} \sigma_y \, dz$$

$$N_{xy} = N_{yx} = \int_{Z/2}^{Z} \tau_{xy} \, dz$$

$$M_{xx} = \int_{Z/2}^{Z} \sigma_x \, z \, dz; \quad M_{yy} = \int_{Z/2}^{Z} \sigma_y \, z \, dz$$

$$M_{xy} = M_{yx} = \int_{Z/2}^{Z} \tau_{xy} \, z \, dz$$

Using (II.3), (II.4) and (II.5) the forces and moments can be expressed in terms of the middle surface displacements.

$$N_x = D(u'' + \mu u'' - \mu \frac{Z}{R})$$

$$N_y = D(v'' + \mu u'' - \mu \frac{Z}{R})$$

$$N_{xy} = N_{yx} = \frac{D}{2} (1 + \mu) (u'' + v'')$$

$$M_{xy} = M_{yx} = -K (W'' + \mu W'')$$

$$M_{yy} = -K (W'' + \mu W'')$$

$$M_{xy} = M_{yx} = -K (1 - \mu) W'$$

where $D = \frac{E I}{1 - \mu^2}$ and $K = \frac{E I^3}{12 (1 - \mu^2)}$

The equations of equilibrium can be written down by inspection from Fig. II.3 and Fig. II.4 thus:

**Equilibrium of Forces.**

- In the x direction $N_x + N_{xy} + X = 0$  
- In the y direction $N_y' + N_{yx} + Y = 0$  
- In the z direction $Q_x' + Q_y' + \frac{N_y}{R} + Z = 0$

**Equilibrium of Moments.**

- About the x axis $M_{yy} + M_{xy} - Q_y = 0$  
- About the y axis $M_{xx} + M_{yx} - Q_x = 0$
FORCES ON AN ELEMENT

FIG II.3

MOMENTS ON AN ELEMENT

FIG II.4
As stated previously the term \( \frac{QV}{R} \) is neglected in the second equation of (II.7). Neglect of this term would appear to be of the same order as assumption (d).

\( Q_x \) and \( Q_y \) are eliminated from (II.7) and (II.8) and the three equations of equilibrium obtained are:

\[
\begin{align*}
N_x' + N_yx' + X &= 0 \\
N_y' + N_xy' + Y &= 0 \quad \ldots \quad (II.9) \\
M_{xx}'' + 2M_{xy}' + M_{yy}'' + \frac{N_y + Z}{R} &= 0
\end{align*}
\]

Substitution of (II.6) into (II.9) yields the three displacement equations:

\[
\begin{align*}
(\mu u')' + (\frac{\mu}{2})(u')'' + \mu (v')' + (\frac{\mu}{2})(v')'' - \frac{\mu}{R} (w')' + X &= 0 \\
\mu (u')' + (\frac{\mu}{2})(\mu u')' + (v')'' + (\frac{\mu}{2})(v')'' - \frac{\mu}{R} (w')' + Y &= 0 \\
\mu \frac{\mu u''}{R} + \frac{v''}{R} - \frac{w''}{R^2} - (k\omega'')'' - (k\omega'')' - \mu (k\omega'')''' - \mu (k\omega'')'' | - \mu (k\omega'')'' | - 2(1-\mu)(k\omega''')' + Z &= 0
\end{align*}
\]
CHAPTER 3

APPLICATION OF THIN SHELL THEORY TO ARCH DAM ANALYSIS.

INTRODUCTION. (3.1)

In this chapter, after modification of the three displacement equations obtained in Appendix II, to allow for the thickness variation of an arch dam, the boundary conditions of the problem are considered and the various methods that have been suggested for the solution of these equations outlined. The application of Southwell's relaxation technique is then briefly described.

Modification of Displacement Equations. (3.2)

In this treatment allowance is made for variation of thickness in the vertical direction only. Thickness variations in horizontal sections, although they occur in practice, are not considered but the methods outlined in this thesis would be applicable to a constant thickness arch within the actual arch - similar to the active arches of Reesal and Materre.

Since \( D = \frac{E t}{1-\mu^2} \) and \( K = \frac{E t^3}{12(1-\mu^2)} \)...

\[ D' = \frac{E t}{1-\mu^2} \frac{t'}{t} \quad \text{and} \quad K' = \frac{E t^2}{4(1-\mu^2)} \frac{t''}{t} \]...

Also \( K'' = \frac{E t}{2(1-\mu^2)} \left( \frac{(t')^2}{t^2} + \frac{t t''}{2} \right) \)...

where \( (\ )' = \frac{\partial}{\partial x} \); and \( (\ )'' = \frac{\partial^2}{\partial x^2} \).

Substitution of equations (3.2.1), (3.2.2) and (3.2.3) into the equations (II,10) lead to the following displacement equations:
\[
\mu V''_n - \frac{\mu N''_n}{R} + u'' + \frac{\xi}{\ell} \{ \mu V''_n - \mu \omega'' + \omega'' \} + \frac{(1-\mu)}{2} \{ u'' + v'' \} + \frac{x(1-\mu^2)}{E} = 0
\]
\[
(1-\mu) \{ u'' + v'' \} + (1-\mu) \ell \{ u'' + v'' \} + \mu \{ u'' + v'' - \frac{\omega''}{R} \} + \frac{\ell}{E} = 0
\]
\[
-\frac{\xi}{\ell} \nabla^2 w + \frac{\xi}{\ell} \{ w'''' + w'''' \} - \{ \ell \xi (1-\frac{1}{2}) \} \{ \mu \omega'' + w'' \} + \frac{(1-\mu^2)}{E} = 0
\]

In these equations \( R \) is the radius of curvature of the middle surface and in practice would vary with \( x \), but since the dam is assumed to be thin, this variation is neglected. The error involved due to the neglect of the variation of \( R \) with \( x \) is considered in Chapter 6.

**Boundary Conditions.** (3.3)

![Figure 3.1](image)

Although, in the past the tendency was to consider the foundations to be perfectly rigid, later developments included the effects of elastic movement in the foundation rock. In 1925 Vogt (18) proposed that the dam should be considered rigid at a depth below the actual foundation boundary. Along this artificial boundary line the displacements and rotations are assumed to be zero:

\[
\begin{align*}
    u &= 0, & v &= 0, & w &= 0, \\
    w' &= 0, & w' &= 0
\end{align*}
\]

... (3.3.1)
At the crest of the arch dam the moments \( M_{xx}, M_{xy} \) and the forces \( N_x, N_{xy} \) and \( Q_x \) are all zero. These five conditions are not independent, however, and it can be shown that \( M_{xy} \) must be coupled with \( Q_x \) to give \( Q_x + M_{xy}' = 0 \), and with \( N_{xy} \) to give \( N_{xy} - \frac{M_{xy}}{R} = 0 \) (Appendix III.2). This first condition is due to Kirchhoff (19) and the second to Bassett (20).

The four conditions to be satisfied at the free boundary are therefore:

\[
\begin{align*}
N_x &= 0 \\
M_{xx} &= 0 \\
N_{xy} - \frac{M_{xy}}{R} &= 0 \\
Q_x + M_{xy}' &= 0
\end{align*}
\] ... (3.3.2)

The contribution of \( \frac{M_{xy}}{R} \) to \( N_{xy} \) in the third equation of (3.3.2) is small and this term is neglected in this analysis; if this equation is retained in its present form it cannot be used for obtaining the 'fictitious' points (See Fig. 3.2) required in the relaxation process without some additional assumption (Appendix II.2). Neglecting, therefore, this term and using the force-displacement and moment-displacement equations (II.6), the conditions at the free boundary can be expressed in the form:

\[
\begin{align*}
u' + \mu v'' - \frac{\mu w'}{R} &= 0 \\
v'' + \mu w''' &= 0 \\
\mu' + q' &= 0 \\
w'' + (2-\mu) w' &= 0
\end{align*}
\] ... (3.3.3)

Other Solutions Based on Shell Theory. (3.4)

The first important application of a shell theory to arch dam analysis was made by Tolke (21) in 1938. The dis-
placement equations were reduced to two by neglecting the vertical displacement \( u \), and the equations were further simplified by neglecting certain other terms. The simplified equations were then reduced to one fourth order partial differential equation in \( w \). In order to solve this equation Tolke assumed that the solution was made up of two parts, the first part being the solution for a complete cylinder of the same profile as the dam and the second part a solution introduced to correct for the difference between the cylinder and the actual arch.

Lombardi (22) in 1955 obtained a simplified set of displacement equations by first considering the relative magnitudes of some of the terms in Flugge's force-displacement relationships, neglecting terms considered to be small and then substituting the remaining terms into the equations of equilibrium. As in the Tolke equations the vertical displacements were neglected so that there were only two displacement equations. The method of solution of these equations suggested by Lombardi was based on the selection of suitable relationships to replace the displacement equations. The coefficients of these relationships were found by means of a least squares error method.

Instead of using three displacement equations of the type (II.10) Hertzog (23) in 1956 expressed the equations of the dam as two fourth order partial differential equations in \( w \) and \( \phi \) where \( \phi \) was a stress function which was defined so as to automatically satisfy the first two equations of equilibrium; \( \frac{\partial^2 \phi}{\partial y^2} \) was first neglected.
To obtain a solution Hertzog suggested that these equations could be written out for each intersection point of a grid system and the resulting equations (two for each intersection point) solved by relaxation or by means of a computer. Hertzog also indicated a method of solution of a simplified set of these fourth order equations by assuming that the equations can be separated into two parts, one part being due to membrane action and the other plate action.

In 1959 Prisco (24), starting with Lombardi’s simplified displacement equations, substituted into them a set of finite difference approximations and the resulting equations were applied to the nodes of a grid system. This lead to a system of n equations with n unknowns, the unknowns being the radial and tangential displacements at the intersection points. In the example given in the paper there were 27 intersection points and the number of equations and unknowns were reduced to 49.

**Relaxation (3.5)**

In order to apply the relaxation technique to the solution of equations (3.2.4) the developed surface of the dam was first partitioned using a horizontal and vertical grid system of convenient grid length. The finite difference approximations at any node, i.e., expressions for the derivatives of the displacements at any node in terms of the values of the displacements at surrounding nodes, were then obtained (Appendix III.1).

The approximation made in deriving these expressions is that certain terms which are a product of the grid length
raised to some power and a derivative of the displacement
are neglected, e.g. in the finite difference expression for
\( w^n \) the term \( \frac{2a^4}{4!} W^{iii} \) and terms containing higher powers of
the grid length \( a \) are neglected. The expressions are thus
more accurate for smaller grid dimensions.

Substitution of these expressions into equations
\( (3.2.4) \) will, therefore, give a set of equations which are
functions of the displacements occurring at the inter-
section points of the grid. The equations for all nodes
occurring in a horizontal line will be identical since
the thickness is constant but the equations will be different
for each vertical node.

At some points (near the boundaries) the equations will
contain terms which refer to the values of the displacements
at nodal points outside the developed surface of the dam.
This is illustrated in Fig. 3.2.

![Diagram showing grid points and fictitious points](https://example.com/diagram)

The equation for the radial displacement at \( O \) would be a
function of the displacements at the surrounding nodes
including nodes 9 and 10 which are outside the boundary.
Points outside the boundary are known as fictitious points
and in order to eliminate these from the equations the
finite difference equivalents of the boundary conditions
\( (3.3.1) \) and \( (3.3.3) \) are used. (Appendix III.2).
If an axis of symmetry can be drawn within the region considered then the number of nodal points can be halved for each such axis by considering only the surface to one side of the axis and assuming that the axis itself is a boundary.

The solution of the equations is obtained by assuming values of the displacements at each node, substituting these values into the equations and calculating the 'out of balance' or residual amount. The residuals should all be zero if a correct set of displacements had been selected. The trial values are now corrected by means of a systematic process, based on the finite difference equivalents of equations (3.2.4), so as to reduce the magnitude of the residuals.

It is convenient, in application, to consider the effect on the surrounding residuals of unit change of displacement at a node. These are drawn in the form of patterns as shown in Fig. 3.3 where the numbers in the circles represent the change occurring in the residuals due to unit positive change at the central node.

Since point relaxation by means of patterns such as Fig. 3.3 would prove to be tedious it is more usual to employ block relaxation, i.e. the application of unit displacement to a number of intersection points simultaneously.
In applying these principles to arch dams it was found that due to the variation of thickness the usual symmetry associated with point and block patterns did not exist. The same patterns cannot be used at different depths and the coefficients within any one pattern were not symmetrical.

Although the three equations (3, 2, 4) contain u, v and w terms, it is convenient to refer to the first as the \( u \)-equation, the second as the \( v \)-equation and the third as the \( w \)-equation. The relaxation process is commenced by first assuming trial values for the three displacements at each intersection point. Assume that the trial values at any node \( i \) are denoted by \( w_{i1} \), \( v_{i1} \), and \( u_{i1} \). Using the \( w \) equation and keeping \( u_{i1} \) and \( v_{i1} \) constant, by the relaxation process outlined above, a value of the radial displacements \( w_{i2} \) can be found which, when used in the \( w \)-equation, produces smaller residuals. Treating \( w_{i2} \) and \( u_{i1} \) as constants, new values of the tangential displacements \( v_{i2} \) can be obtained. Similarly, with \( w_{i2} \) and \( v_{i2} \) treated as constants, the vertical displacement \( u_{i2} \) can be calculated. Referring to these operations, of obtaining new values \( w_{i2} \), \( v_{i2} \) and \( u_{i2} \) from the trial values of \( w_{i1} \), \( v_{i1} \) and \( u_{i1} \) as a complete cycle, the cycle is repeated until the values of the residuals are small for each stage of the cycle. The practicability of employing a relaxational solution depends on the number of cycles required to obtain a solution.

**APPENDIX (III.1)**

In the finite difference expressions listed below the partial differential equations are expressed in terms of the values of the displacements occurring at the nodes of a
rectangular mesh. The derivation of these expressions is given by Southwell (25), Allen (26) and Shaw (27).

For any quantity \( q \), the finite difference expressions for the point \( o \) in terms of the values of \( q \) at the surrounding nodes are:

\[
\begin{align*}
q'_o &= \frac{q_2 - q_4}{2a} & \quad q''_o &= \frac{q_1 - q_3}{2b} \\
q''_o &= \frac{q_2 + q_4 - 2q_o}{a^2} & \quad q'''_o &= \frac{q_1 + q_3 - 2q_o}{b^2} \\
q_{10} &= \frac{q_5 + q_7 - q_8 + q_9}{4a} \\
q_{10}'' &= \frac{q_{10} - 2q_1 + 2q_4 - q_{13}}{2a^3} \\
q_{10}''' &= \frac{q_5 + q_7 - 2q_8 - q_9 + 2q_4}{2a b^2} \\
\nabla^4 q_o &= \frac{(q_{10} - 4q_2 + 6q_o - 4q_4 + q_{12}) + (q_9 - 4q_1 + 6q_0 - 4q_3 + q_{11})}{a^4} \\
&\quad + 2 \left( q_5 + q_6 + q_7 + q_8 - 2q_2 - 2q_3 - 2q_9 + 2q_0 \right) \frac{1}{a^2 b^2}
\end{align*}
\]
APPENDIX (III.2)

THE BOUNDARY CONDITIONS AT THE FREE SURFACE OF THE DAM.

The forces and moments per unit length acting on an elemental length of a horizontal plane section, \( x = \text{constant} \), are shown in Fig. (III.2.1) and Fig. (III.2.2)

\[ Q_x \delta y; N_{xy} \delta y; N_x \delta y; M_{xx} \delta y; M_{xy} \delta y \]

The twisting moment \( M_{xy} \delta y \) can be replaced by a pair of equal and opposite forces acting at the ends of the element in a direction parallel to the normal at the mid-point of the segment. The magnitudes of these forces will be \( M_{xy} \).

On an adjacent segment the total twisting moment will have increased to \( M_{xy} \delta y + (M_{xy} \delta y) \delta y \) and the pair of forces required to replace this moment will be of magnitude \( M_{xy} + M_{xy} \delta y \).

At the common normal of these two segments the forces acting in the plane are, therefore, as shown in Fig. (III.2.3).
Resolving these forces and taking $\cos \frac{\delta \theta}{2} = 1$

and $\sin \frac{\delta \theta}{2} = \frac{\delta \theta}{2} = \frac{\delta y}{2R}$

leads to the following:

Normal force/unit length

$$Q_x + M_{xy} \sin \frac{\delta \theta}{2} = Q_x + M_{xy}^*$$  \hspace{1cm} (III.21)

Tangential force/unit length

$$N_{xy} = N_{xy}^* - M_{xy}^* \frac{\delta y}{R}$$  \hspace{1cm} (III.22)

The term $M_{xy}^* \frac{\delta y}{R}$ is obviously small compared with $\frac{M_{xy}}{R}$ and is neglected. At the free boundary the forces and moments shown in Fig. (III.2.1) and Fig. (III.2.2) can, therefore, be combined to give the four boundary conditions (3,3.2).

Expressing the third of these equations in finite difference form it can be shown that:

$$(u_1 - u_3) + (v_2 - v_4) + \frac{\epsilon^2}{12R^3} (w_5 + w_7 - w_6 - w_8) = 0 \hspace{1cm} (III.2.3)$$

In this expression $(v_2 - v_4)$ and $(w_5 + w_7 - w_6 - w_8)$ are of similar magnitude and $(u_1 - u_3)$ is small. Thus, since the thickness at the crest is small and $R$ is large, the term $\frac{\epsilon^2}{12R^3} (w_5 + w_7 - w_6 - w_8)$ will be small compared with $(v_2 - v_4)$.

Equation (III.2.3) is used for eliminating the fictitious points $v_5$ and $v_6$ occurring in the $u$-equation.
at the crest. In order to do this it is necessary first to obtain the expression for \( v_2 \) from equation (III,2.3) and then rewrite this equation for the adjacent points \( v_5 \) and \( v_6 \). However, since the above expression for \( v_2 \) contains the values \( w_5 \) and \( w_8 \), the value of \( v_5 \) obtained from this expression will contain an extra fictitious point and an extra intersection point - points \( f_1 \) and \( n_1 \) respectively in Fig. III.2.4.

Similarly the expression for \( v_6 \) will contain \( f_2 \) and \( n_2 \). Thus, in attempting to eliminate the fictitious points \( v_5 \) and \( v_6 \) two other fictitious points are introduced.

The term \( \frac{M_{xy}}{R} \) of equation (III,2.3) cannot, therefore, be used to eliminate the fictitious points without some extra simplifying assumption. Since the term is small compared with \( M_{xy} \) it was decided to neglect it and the finite difference approximations obtained for eliminating the fictitious points thus reduce to:

\[
\begin{align*}
(u_2 - u_4) + \mu (u - u_3) - \frac{\mu_2 \phi_6}{R} &= 0 \quad \text{III.2.4} \\
(w_2 + w_4) + \mu (w_1 + w_3) - 2(\mu + 1)w_0 &= 0 \quad \text{III.2.5} \\
(u_1 - u_3) + (v_2 - v_4) &= 0 \quad \text{III.2.6}
\end{align*}
\]
Using equations (3.2.4) and the finite difference approximations (Appendix III.1) the finite difference expressions of the displacement equations could be obtained. Since the only external forces considered were the horizontal hydrostatic pressures $Z$, the vertical and tangential forces $X$ and $Y$ appearing in equation 3.2.4 were equal to zero.

The finite difference approximations were also simplified by making $a = b$ so that the following displacement equations were only applicable to the nodes of a square mesh. After substitution and simplification the following equations were obtained:

**U- Equation**

\[-U_0\left\{12t - 4\mu t\right\} + (U_1 + U_3)\left\{2(1-\mu)t\right\} + U_2\left\{4t + 2at^2\right\}
+ U_4\left\{2t - 2at^2\right\} + (W_1 - W_3)\left\{\frac{(1+\mu)t}{2}\right\} + (W_1 - W_3)\left\{2a\mu t^2\right\}
- (W_2 - W_4)\left\{2a\mu t^2\right\} - W_0\left\{4\frac{a\mu t^2}{R}\right\} = 0\]

**V- Equation**

\[-V_0\left\{12t - 4\mu t\right\} + (V_1 + V_3)\left\{4t\right\} + V_2\left\{2t + 2at^2\right\} + V_4\left\{2\frac{a\mu t^2}{2}\right\}
+ (U_5 - U_7 + U_7)\left\{\frac{(1+\mu)t}{2}\right\} + (U_1 - U_3)\left\{1-\mu\right\}at^2\right\} - (W_7 - W_3)\left\{2at^2\right\} = 0\]

**W- Equation**

\[-W_0\left\{\frac{2t^2}{3} - 8a^2\mu T + 4a^4 - 8a^2 T^2\right\} + (W_1 + W_3)\left\{\frac{a^2 t^2}{3} - 4a^2 \mu T\right\}
+ W_2\left\{3t^2 + 4at t' - 4a^2 T^2\right\} + W_4\left\{\frac{a^2 t^2}{3} - 4a^2 T^2\right\} - (W_5 + W_7)\left\{\frac{2t^2}{3} + att'\right\}
- (W_7 + W_9)\left\{\frac{2t^2}{3} - att'\right\} - W_0\left\{\frac{2t^2}{3} + att'\right\} - W_2\left\{\frac{2t^2}{3} - att'\right\}
+ (U_2 - U_4)\left\{\frac{2a^2 \mu}{R}\right\} + (V_1 - V_3)\left\{\frac{2a^2 \mu}{R}\right\} + Z(1-\mu)^2 4a^2 = 0\]

where $T_1 = \left\{\frac{t^2}{2} + \frac{1}{2}(T')^2\right\}$
These equations apply to all the horizontal grid lines if the appropriate values of \( t, t', \) and \( t'' \) for the grid line considered are substituted. However, when the equations are applied to Grid 0 and Grid 2 (Fig. III.3.1) they contain the following fictitious points:

**Grid 0:**
- \( u \) equation: \( u_2, v_5 \) and \( v_6 \)
- \( v \) equation: \( v_2, u_5 \) and \( u_6 \)
- \( w \) equation: \( w_2, w_5, w_6, w_{10} \) and \( u_2 \)

**Grid 2:**
- \( w \) equation: \( w_{10} \)

In order to eliminate these fictitious values, equations III.2.4 to III.2.7 must be used.

For Grid 0 the values of \( u_2 \) and \( w_2 \) and \( v_2 \) can be obtained directly from (III.2.4); (III.2.5) and (III.2.6) respectively and the values of \( u_5, u_6, w_5, w_6, v_5 \) and \( v_6 \) written down by inspection, e.g. Since from (III.2.6) \( v_2 = v_6 - (u_1 - u_3) \) it follows from inspection of Fig. III.3.1 that \( v_5 = v_8 - (u_9 - u_0) \).

To obtain an expression for the remaining fictitious point \( W_{10} \) the displacements \( w_2, w_5 \) and \( w_6 \) must first be
eliminated from equation (III.2.7). Using expressions for \( w_2 \), \( w_5 \) and \( w_7 \) from equation (III.2.5) and substituting in (III.2.7) leads to:

\[
W_{10} = W_0\left[12+12\mu-6\mu^2\right] - (W_1+W_3)\left[4+8\mu-4\mu^2\right]-W_0\left[12-4\mu\right] + (W_7+W_8)\left[4-2\mu\right] + (W_9+W_11)\left[2\mu-\mu^2\right] + W_12 \quad \text{--- (III.3.2)}
\]

For Grid 2 the fictitious point \( w_{10} \) can be obtained by first writing down the value of \( w_2 \), for Grid O, from equation (III.2.5) and then applying this to Grid 2:

\[
W_2 = 2(\mu+1)W_0 - \mu(N_1+N_3) - W_4 \quad \text{--- for Grid O}
\]

becomes \( W_{10} = 2(\mu+1)W_2 - \mu(N_5+N_6) - W_0 \quad \text{--- for Grid 2} \quad \text{--- (III.3.5)}
\]

The fictitious points indicated above are not the only fictitious points that occur, for, in applying the \( w \) equation to nodes near the fixed boundary and all three equations to nodes near an axis of symmetry, certain values become fictitious. This is illustrated in Fig. III.3.2.

To calculate the residual of the \( w \) equation at the node marked \( O \) in Fig. III.3.2 the values of \( w_{11} \), \( w_7 \) and \( w_{12} \) are required. These can be obtained from the boundary condition \( w^* = 0 \) which leads to:
\[ w_{11} = w_0; \quad w_7 = w_8 \text{ and } w_{12} = w_{13} \quad \ldots \quad (III, 3.4) \]

However, it is convenient not to eliminate these fictitious points from the equations but to use the equalities \((III, 3.4)\) during the process of calculation of residuals. Similarly, in calculating the residuals at \(A\) near the axis of symmetry, the axis is considered to be a line of reflection so that \(A' = A\). If unit displacement is applied to the nodes \(A\) or \(B\), the residuals at the surrounding nodes must be changed in accordance with unit displacement also being applied to \(A'\) and \(B'\).

If the fictitious points are eliminated from equations \((III, 3.1)\) in accordance with the above the three equations which must be applied to Grid 0 and Grid 2 are obtained.

**GRID 0.**

**U-Equation.**
\[ -\mu_0 \{ \mu - 5\mu \} + (n + \mu_3) \{ 2(1 - \mu) \} + \mu_4 \{ \frac{2\pi \mu^2}{R^2} \} - \left( \nu_v \nu_3 \right) \{ 4\mu \} = 0 \]

**v-Equation.**
\[ -\nu_0 \{ \mu - 5\mu \} + \nu_v \{ \frac{4\pi}{R^2} \} + \nu_4 \{ (1 - \mu) \} - \left( \nu_v \nu_3 \right) \{ \frac{4\pi \nu^2}{R^2} \} = 0 \]

**w-Equation.**
\[ -\nu_0 \left\{ \mu^3 \frac{12 - 5\mu - \mu' \nu}{R^2} + \frac{4\pi \mu' \nu}{R^2} + \mu \nu' \frac{2\pi \nu^2}{R^2} - \frac{4\pi \nu' \nu^2}{R^2} - \frac{2\pi \nu^2}{R^2} - \frac{6\mu^2 \nu^2}{R^2} \right\} \]
\[ + \left\{ \nu_4 \frac{4\pi \mu^2}{R^2} + 2\mu \nu' \frac{2\pi \nu^2}{R^2} - \frac{4\pi \mu^2 \nu^2}{R^2} + \mu \nu' \frac{2\pi \nu^2}{R^2} \right\} + \nu_4 \left\{ 4\pi \mu^2 \nu' \frac{2\pi \nu^2}{R^2} - 4\mu \nu' \frac{2\pi \nu^2}{R^2} - \frac{4\pi \nu^2}{R^2} \right\} \]
\[ - \left( \nu_4 + \nu_3 \right) \left\{ \frac{4\pi \mu^2}{R^2} + 2\mu \nu' \frac{2\pi \nu^2}{R^2} - 2\mu \nu' \frac{2\pi \nu^2}{R^2} \right\} \]
\[ + \left( \nu_4 - \nu_3 \right) \left\{ \frac{2\pi \nu^2}{R^2} - \frac{2\pi \nu^2}{R^2} \right\} + \left( 1 - \mu \right) \frac{4\pi \nu^2}{R^2} - \frac{2\pi \nu^2}{R^2} = 0 \]
Equations (III.3.1), (III.3.5) and (III.3.6) are the finite difference equations to be used for obtaining the displacements of the nodes of a square mesh assumed to be drawn on the developed middle surface of an arch dam.
CHAPTER 4.

NUMERICAL APPLICATION.

INTRODUCTION (4.1)

In order to test the validity of the assumptions used and methods proposed in the previous chapters it was necessary to select a dam profile for which the solution was already known so that this solution could be compared with the results obtained by relaxation of the shell equations. Since the results of an exact analysis of Dokan Dam had been published (Ref. 13) it was decided to consider a dam of the same profile as Dokan.

To reduce the volume of the calculation required the following simplifications were made:

(a) It was assumed that there was a vertical plane of elastic symmetry so that the number of nodes to be considered was halved. This assumption was also made by Pippard et al.

(b) The profile was modified at the base so that there were no sudden changes of thickness (Fig. 4.1). Since a thin shell theory was being applied and the base of the dam considered was thick, it was assumed that slight changes of profile at the base would not affect the results to any great extent.

(c) The position of the fixed boundary was altered from that considered for the Dokan solution so that the boundary passed through the intersection points of the
SECTION OF DOKAN DAM

SECTION AT CENTRE LINE

FIG 4.1
EXCAVATION LINE FOR DOKAN DAM

BOUNDARY OF GRID FOR RELAXATION.

DEVELOPED VERTICAL WATER FACE

FIG 4.2
grid system; this avoided the use of fractional grid lengths near the boundary (Fig. 4.2). Since, using the method suggested by Vogt, the boundary is considered fixed at a depth below the actual boundary, the magnitude of this depth can always be modified slightly so as to allow the fictitious boundary to pass through the nearest node.

(d) Although the radius R in the shell equations is the radius of curvature of the mid-surface, the actual radius used in the calculations was the constant radius of the upstream face. At the crest $R = 120\text{m}$ was used instead of $R = 116.9\text{m}$ and the change of radius of curvature of the mid-surface with depth was neglected. The effect of the neglect of this change in radius is considered in Chapter 6.

The Displacement Equation, \((4.2)\)

The cross-section at the centre line and the developed vertical water face are shown in Fig. 4.1 and Fig. 4.2 respectively. The constants used in the calculations were:

\[ \mu = 0.15 \]
\[ E = 141,000 \text{ kN/cm}^2 \]
\[ a = 12.4 \text{ metres} \]
\[ R = 120 \text{ metres} \]

and with these values the \(w\), \(v\) and \(u\) equations can be expressed in the form:

\[ w - \text{Equation,} \]
\[ -R_i w_0 + B_i (w_i + w_3) + C_i w_2 + D_i w_4 - E_i (w_5 + w_6) - F_i (w_i + w_8) - G_i (w_i + w_1) - H_i w_0 - J_i w_2 + K_i (u_2 - u_4) + L_i (u_1 - u_3) = 0 \]

Where \(i = \text{horizontal grid 0, 2, 4} \) --- 18
\( \textbf{v - Equation.} \)

\begin{align*}
\text{Grid 0:} & \quad -A_{20} v_0 + B_{20} (v_1 + v_2) + D_{20} v_4 - E_{20} (u_1 - u_3) \\
& \quad -F_{20} (v_4 + v_1) - G_{20} (w_1 - w_3) = 0
\end{align*}

\( \text{Grid 2} \rightarrow 18. \)

\begin{align*}
-A_{2i} + B_{2i} (v_1 + v_2) + C_{2i} v_2 + D_{2i} v_4 \\
- E_{2i} (u_1 - u_3) + F_{2i} (u_5 + u_6 + u_7 - u_8) - G_{2i} (w_1 - w_3) = 0
\end{align*}

Where \( i = \) horizontal grid 2, 4 ..., 18

\( \textbf{u - Equation.} \)

\begin{align*}
\text{Grid 0:} & \quad -A_{30} u_0 + B_{30} (u_1 + u_3) + D_{30} u_4 + E_{30} (w_1 + w_3) \\
& \quad -F_{30} (v_1 - v_3) - G_{30} (u_1 + u_4) + H_{30} w_0 + J_{30} w_4 = 0
\end{align*}

\( \text{Grid 2} \rightarrow 18. \)

\begin{align*}
-A_{3i} u_0 + B_{3i} (u_1 + u_3) + C_{3i} u_2 + D_{3i} u_4 - E_{3i} (w_2 - w_4) \\
- F_{3i} (v_1 - v_3) + G_{3i} (v_5 - v_6 + v_7 - v_8) + H_{3i} w_0 = 0
\end{align*}

Where \( i = \) horizontal grid 2, 4 ..., 18

The coefficients

\[ A_{3i}, B_{3i}, C_{3i}, \ldots \]

\[ A_{2i}, B_{2i}, C_{2i}, \ldots \]

\[ A_{3i}, B_{3i}, C_{3i}, \ldots \]

have the values shown in tables 4.1, 4.2 and 4.3.

\textbf{The Relaxation (4.3)}

Starting from assumed values of \( u, v \) and \( w \) the residuals at the nodes were calculated and reduced by relaxation. The relaxation was completed in two cycles, but this gave no indication of the rate of convergence to the correct solution since the relaxation was commenced from trial values, and if these values had been of similar magnitude to the correct solution, then the amount of relaxation required would have been small and vice versa. In the example considered in Chapter 6, the relaxation was commenced from zero values and an indication of the rate
of convergence was obtained from that example. It was found that the major part of the work was involved in the relaxation of the fourth order w-Equation, and that the effect of changes in u and v on the w residuals was small.

Calculation of the residuals was made with the aid of a desk calculator and the reduction of the residuals speeded by frequent use of block patterns.

The final results of the relaxation are shown in Table 4.4 where the displacement u, v and w of the middle surface are given. In order to compare these displacements with those obtained in Ref.13, it is necessary to calculate the displacements at the air and water faces of the dam by means of equations (II,1). The result of this calculation is given in Tables 4.5 and 4.6 where the vertical and tangential displacements of both faces are shown. The radial displacement is assumed constant throughout the dam thickness. Table 4.7 gives the results obtained by Pippard et al. of the displacements occurring at the water face. The displacements of the air face are only given at vertical grids 0, 4, 8, 12 and 16 in Ref.13 and these are shown in Figs. 4.3, 4.4, 4.5 and 4.6, where the displacements obtained by both methods are compared graphically. All displacements are given in cm x 10^2 units.

Calculation of Stresses. (4.4)

Using equations II,4, II,3 and the finite difference approximation (Appendix III,1) applied to a square mesh the following expressions for the stresses occurring at the faces of the dam can be obtained:
Fig. 4.3

```plaintext
Radial Displacement [cms]

Vertical Grid 0

Shell Theory

Pippard Analysis

Vertical Grid 12

Vertical Depth: Grid Units

0 1 2

3.5 3.0 2.5 2.0 1.5 1.0 0.5

0 2 4 6

8 10 12

14 16 18 20
```
Radial Displacement [ems]

Vertical Grid 4

Vertical Grid 16

Shell Theory
Pippard Analysis

Fig. 4.4
Vertical Displacement [cms]

Vertical Grid 0

Vertical Grid 4

Vertical Grid 8

Vertical Grid 16

Water Face
Shell Theory
Pippard Analysis

Air Face
Shell Theory
Pippard Analysis

Fig 4.6
The stresses calculated using these equations are shown in Table 4.8 and 4.9 for both the upstream and downstream faces. Table 4.10 shows the stresses calculated by Pippard et al for the upstream face and in Figs. 4.7, 4.8, 4.9, 4.10 and 4.11 the stresses calculated by both methods are compared graphically. All stresses are given in lb/ins.$^2$.

It is interesting to consider the equivalent equations to 4.4.1 which would be obtained if Lombardi's Force - displacement and Moment - displacement relationships were used in the derivation. After substitution of the finite difference approximations the Lombardi Stress-Displacement equations obtained would be:

\[ \sigma_x = \frac{E}{1-\mu^2} \left\{ \frac{1}{2a} \left[ (\nu_x - \nu_y) - \mu (\nu_x - \nu_y) - \mu (\nu_x + \nu_y - 2\nu) \right] \right\} + \frac{6E}{1-\mu^2} \left[ \frac{W_2 + W_4}{R} \right]. \]

\[ \sigma_y = \frac{E}{1-\mu^2} \left\{ \frac{1}{2a} \left[ (\nu_y - \nu_x) - \mu (\nu_x - \nu_y) - \mu (\nu_x + \nu_y - 2\nu) \right] \right\} + \frac{6E}{1-\mu^2} \left[ \frac{W_1 + W_3}{R} \right]. \]

\[ \tau_{xy} = G \left\{ \frac{1}{2a} \left[ (\nu_x - \nu_y) + \frac{E}{2a} (\nu_x - \nu_y) \right] \right\} + \frac{6E}{1-\mu^2} \left[ \frac{W_5 + W_7}{R} \right]. \]

These equations were used for calculating the stresses shown in Tables 4.11 and 4.12, using the displacements...
Stresses at Vertical Grid 4

[Units: lbs/ins², Tension +]
Stresses at Vertical Grid 8
[Units 16/ins², Tension +]

Vertical Stress [fs]

Tangential Stress [tg]

Shear Stress [ty]

WATER FACE
Shell Theory

Pippard Analysis

AIR FACE

Shell Theory

Fig 49
Stresses at Vertical Grid 12

[Units lbs/in², Tension t]

FIG. 4.10

Water Face
Shell Theory
Pippard Analysis
Air Face
Shell Theory
Stresses at Vertical Grid 16

[Units lbs/ins. Tension +]

Fig 4:11

Water Face
Shell Theory
Pippard Analysis
Air Face
Shell Theory
calculated above (Table 4.4). A graphical comparison between the vertical and tangential stresses obtained by means of equation 4.4.1 and 4.4.2 is shown in Figs. 4.12, 4.13 and 4.14 for vertical grids 0.3 and 16 respectively.

Conclusions. (4.5)

On comparing the results obtained by application of a thin shell theory with those obtained by means of a more exact analysis it can be seen from the tables and graphs that:

(a) The errors in the radial and vertical displacements are greatest at the crest of the dam. In the case of the radial displacements the error decreases from Grid 0 to Grid 4 and then increases again with depth whilst the error in the vertical displacements decrease with depth. The tangential displacements are in close agreement at all depths and sections.

Since the thickness of the dam increases with depth a thin shell theory becomes less applicable, and the error can be expected to increase with depth. The large error at the crest must, therefore, be due mainly to the method of solution. In the next chapter slight modifications to the method of solution are indicated in order to correct for this error.

(b) The predominant term in the first of equations 4.4.1 is \( w'' \) and the magnitude of the vertical stress thus depends on the slope of the radial displacement-depth curve. The error in the radial displacements from
Stresses at Vertical Grid 0

[Units: 161/m², Tension +]

**Fig 4.12**
STRESSES AT VERTICAL GRID 8

[Units: 10^5 in² Tension]

Fig 4.13
Stresses at Vertical Grid 16

[Units 16/ins², Tension +]

Water Face
Shell Theory
Lombardi Analysis

Air Face
Shell Theory
Lombardi Analysis

Vertical Stress [lb]

Tangential Stress [lb]

Fig 4.14
Grid 0 to Grid 4 is thus reflected in the results for vertical stress where the maximum deviation occurs at Grid 4.

The predominant term in calculating the tangential stresses is the actual value of the radial displacement \( w_0 \) and the error in these stresses for the water face is greatest at the centre sections and increases with depth.

The shear stresses are small and similar values are obtained by both methods.

(a) The stresses obtained using equations 4.4.2 differ from those obtained using equations 4.4.1. The changes decrease the error for the stresses occurring at some sections and increase it at others. The stresses shown in Tables 4.11 and 4.12 are not necessarily identical with the stresses that would be obtained by full application of the equations proposed by Lombardi since the displacements used in the evaluation of the stresses from equations 4.4.2 are the displacements calculated by means of the Donnell type shell equations.

In order to compare equations 4.4.1 and 4.4.2 the vertical and tangential stresses have been calculated for the central vertical section for the following conditions:

(a) Equations 4.4.1 with \( u = 0 \) and \( \mu = 0 \)

(b) Equations 4.4.2

(c) Equations 4.4.1

(d) Equations 4.4.1 with \( u = 0 \)

(e) Equations 4.4.1 with \( \mu = 0 \)

The results of these computations are shown in Table 4.13.
The change from equations 4.4.1 to 4.4.2 can be considered in three stages:— (a) make \( \nu = 0 \), (b) make \( u = 0 \) and (c) add terms which depend on the radial displacement and the square of the thickness. The additional terms in 4.4.2 are small compared to the predominant terms, especially in the thin shell upper portion of the dam and the stresses obtained using 4.4.1 with \( \nu = 0 \) and \( u = 0 \) are similar to those obtained using 4.4.2, as shown by comparison of columns A and B of Table 4.13.

If, however, columns A and C of Table 4.13 are compared it can be seen that for calculating the stresses the neglect of both \( \nu \) and \( u \) introduces a large error. Comparison of columns D and E with C indicates that the neglect of either \( \nu \) or \( u \) would also introduce a large error. It can thus be concluded that in order to calculate the stresses in an arch dam from known values of the displacement, the vertical displacement \( u \) and Poisson's ratio \( \nu \) should not be assumed equal to zero.
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**TABLE 4.1**

**COEFFICIENTS OF w-EQUATIONS.**
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**TABLE 4.2**

Coefficients of $v$-equations.
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**TABLE 4.3**

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**TABLE 4.4**

Middle surface displacements calculated using shell theory.

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**TABLE 4.5**

Water face displacements calculated using shell theory.

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TABLE 4.7

Water face displacements
Pippard et al. Ref.13.

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**Table 4.8**

*Water face stresses calculated using shell theory.*

*(Units: lb/in²)*
## Table 4.9

Air face stresses
Calculated using shell theory.
(Units - lb/ins²)
### Table 4.10

Water face stresses, Pippard et al Ref.13

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**Table 4.11**

Water face stresses. Calculated using Lombardi's equations

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**Table 4.12**

Air face stresses

Calculated using Lombardi's equations.

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**TABLE 4.13**

**COMPARISON OF STRESSES AT CENTRAL VERTICAL SECTION.**

(a - waterface; b - air face) (Units = lb/ins²)

A = Author's equations with \( u = 0 \) and \( \mu = 0 \)
B = Lombardti's equations,
C = Author's equations,
D = Author's equations with \( u = 0 \),
E = Author's equations with \( \mu = 0 \),
CHAPTER 5.

THE 'TUMBLER' SOLUTION.

INTRODUCTION (5.1)

If the thin cylindrical shell theory is applied to the problem of a complete cylinder, or tumbler, subjected to water pressure and having either constant or linearly varying thickness with depth then a formal solution of the resulting equations can be obtained. Using such a solution as a standard, it was decided to compare the solution of the same problem by means of the relaxation process, in an attempt to discover the cause of the large discrepancy in the displacements, which was found to occur at the crest of the dam. The effect on the value of the displacements of neglecting Poisson's ratio was also determined.

Although Pippard, Allan, Chitty and Severn in Ref.13 obtained an approximate relaxational solution by means of the 'tumbler' analogy, this was for the case of a tumbler having the same profile as that of Dokan at the greatest height (See Fig.4.1), i.e., non-linear variation of thickness, and with the use of exact elastic equations. In the correspondence to Ref.13, Zienkiewicz showed that the results obtained by application of a thin shell theory to the tumbler, having the Dokan profile, were in good agreement with those obtained by Pippard et al.

Analytical Solution (5.2)

To obtain the displacement equations for a tumbler,
equation 11.6 and the third of equation 11.10 may be modified by making \( N_{xx} = 0 \) and \( v = 0 \). With this substitution it can be shown that:

\[
\frac{\mu v}{k} \quad \text{(5.2.1)}
\]

\[
\left( \frac{E''}{R^2} \right) + \frac{12(1-\mu^2)H}{R^2} = \frac{Z}{E} - (5.2.2)
\]

which are identical with the equations given by Timoshenko (16).

For the case of hydrostatic pressure and linear variation of wall thickness, as shown in Fig. 5.1, the solution of equation 5.2.2 is given in Ref. 16 in the form:

\[
W = K \left[ G \psi'( \rho \sqrt{x} ) + C_2 \psi_2' ( \rho \sqrt{x} ) + C_3 \psi_3' ( \rho \sqrt{x} ) + C_4 \psi_4' ( \rho \sqrt{x} ) \right], \quad (5.2.3)
\]

where \( i = \frac{12(1-\mu^2)}{\alpha^4 R^2} \) and \( \alpha \) and \( x \) are measured as shown in Fig. 5.1.

The values of the functions \( \psi', \psi_1', \psi_2', \psi_3' \) and \( \psi_4' \) can be obtained from tables or the use of approximate formulae and the constant \( C_1, C_2, C_3 \) and \( C_4 \) can be evaluated from the boundary conditions.

It was decided to select the dimensions of the cylinder so that \( \rho \sqrt{x} \) would be less than 10 in order that the value of the \( \psi' \) functions could be obtained from the tables published in Ref. (30). For \( \rho \sqrt{x} > 10 \) the values of the functions can be obtained by means of the approximate expressions given by Timoshenko, but preliminary calculations indicated that these expressions were not accurate enough.

With this restriction the selected dimensions of the cylinder were:
\[ R = 120 \text{ metres} \]
\[ t_1 = 6.824 \text{ metres} \]
\[ t_2 = 26.086 \text{ metres} \]
\[ d_1 = 106.6 \text{ metres} \]
\[ d = 111.6 \text{ metres} \]

Timoshenko has stated that the constants \( C_3 \) and \( C_4 \) appearing in 5.2.3 could be neglected if the cylinder is long but in the example considered in this chapter it was found, by trial, that these constants could not be neglected without introducing a large error.

Equation 5.2.2 was solved for a tumbler of the above dimensions for values of \( \mu = 0, 0.15 \) and 0.5 and the radial displacements obtained are shown in Table 5.1 and compared graphically in Fig. 5.2.

**TABLE 5.1**

Radial Displacements (cm x \( 10^2 \))

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<tr>
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<th>( \mu = 0 )</th>
<th>( \mu = 0.15 )</th>
<th>( \mu = 0.5 )</th>
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<td>114</td>
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<td>167</td>
<td>184</td>
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<td>210</td>
<td>210</td>
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<td>10</td>
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<tr>
<td>14</td>
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<td>72</td>
<td>58</td>
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<tr>
<td>16</td>
<td>30</td>
<td>20</td>
<td>17</td>
</tr>
<tr>
<td>18</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Relaxational Solution (5.3)

Since all vertical sections of the tumbler were identical it was convenient to re-number the grid points surrounding node 0 as shown:

```
 4 2 0 2 3
```

With this numeration, the appropriate finite difference expressions (App. III.1) can be written in the form:

\[
W_{iii} = \frac{W_0 - 4W_1 + 6W_0 - 4W_2 + W_4}{\alpha^4}
\]

\[
W_{iii} = \frac{W_0 - 2W_1 + 2W_2 - W_4}{\alpha^2}
\]

\[
W_{ii} = \frac{W_1 + W_2 - 2W_0}{\alpha^2}
\]

Substitution of 5.3.1 into equation 5.2.2 leads to the equation:

\[
-w_o\left\{24a^2t\left(t''\right)^2 + 12a^2t^3 + 6a^2t^2t' - 12at^2 + 8t^3\right\} + w_1\left\{12a^2t\left(t''\right)^2 + 6a^2t^2t' - 12at^2 + 8t^3\right\} + w_2\left\{12a^2t\left(t''\right)^2 + 6a^2t^2t' + 12at^2 + 8t^3\right\} + w_3\left\{6at^2 + 2t^3\right\}
\]

\[
-w_4\left\{6at^2 + 2t^3\right\} = \frac{24a^4(1-\mu^2)Z}{R^2} - \frac{24a^4t(1-\mu^2)w_0}{R^2}
\]

Equation 5.3.2 is the general equation which must be applied to each vertical grid for a constant mesh length \(\alpha\). The fictitious points at the free end can be eliminated using the boundary conditions \(w''' = w'' = 0\) so that for Grid 0 and Grid 2 (See Fig. 5.1) the equation reduces to:

**Grid 0.**

\[
-w_o\left\{24a^2t\left(t''\right)^2 + 12a^2t^3 + 6a^2t^2t' + 12at^2 + 8t^3\right\} + w_1\left\{12a^2t\left(t''\right)^2 + 6a^2t^2t' - 12at^2 + 8t^3\right\} + w_2\left\{12a^2t\left(t''\right)^2 + 6a^2t^2t' + 12at^2 + 8t^3\right\} - w_4\left\{6at^2 + 2t^3\right\}
\]

\[
= \frac{24a^4(1-\mu^2)Z}{R^2} - \frac{24a^4t(1-\mu^2)w_0}{R^2}
\]

---

**Grid 2.**

\[
-w_o\left\{24a^2t\left(t''\right)^2 + 12a^2t^3 + 6a^2t^2t' - 12at^2 + 8t^3\right\} + w_1\left\{12a^2t\left(t''\right)^2 + 6a^2t^2t' - 12at^2 + 8t^3\right\} + w_2\left\{12a^2t\left(t''\right)^2 + 6a^2t^2t' + 12at^2 + 8t^3\right\} - w_4\left\{6at^2 + 2t^3\right\}
\]

\[
= \frac{24a^4(1-\mu^2)Z}{R^2} - \frac{24a^4t(1-\mu^2)w_0}{R^2}
\]

---
RADIAL DISPLACEMENTS [cms x 10^3]

Fig 52

Fig 53

Fig 54
If the general equation 5.3.2 is expressed in the form
\[ A_i \psi_i + B_i \psi_i + C_i \psi_i + D_i \psi_i + E_i \psi_i = F_i Z + G_i W_i \]
then for a tumbler having the dimensions given in (5.2) the coefficients \( A_1, B_1, C_1 \ldots \) at the various grid levels for \( \mu = 0.15 \) and \( \mu = 0.5 \) have the values shown in Table 5.2.

The radial displacements obtained as a result of the relaxation of these equations are given in Table 5.4 for both \( \mu = 0.15 \) and \( \mu = 0.5 \).

Since the accuracy of the relaxational solution is dependent on the magnitude of the grid length selected, the solution of equation 5.3.2 was also determined for a reduced grid length of 6.2 metres. The values of the coefficients of the general equation, for \( \mu = 0.15 \) and \( \mu = 0.5 \) are shown in Table 5.3. The radial displacements obtained for the reduced grid length are given in Table 5.4 and a graphical comparison between the results given in Table 5.4 is shown in Figs. 5.3 and 5.4.

The Graded Mesh. (5.4)

From Fig. 5.3 and Fig. 5.4 it could be seen that:

1. The radial displacements over the upper portion of the tumbler obtained by means of a relaxational solution using a mesh length of 12.4 metres differed from the analytical solution and that the error was of the same form as the discrepancy obtained in Chapter 4, i.e. maximum error at the crest and error reducing rapidly from Grid 0 to Grid 4.

2. The use of a finer mesh of length 6.2 metres eliminated the discrepancy for both the values of \( \mu \) considered.
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**TABLE 5.2**
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**TABLE 5.3**
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**Table 5.4**

RADIAL DISPLACEMENTS - (cm x 10²)
Since the use of a finer mesh over the complete depth of the tumbler doubles the number of grid points (and would quadruple the number to be considered if the finer mesh was applied to an arch dam), it was decided to attempt to reduce the volume of computation by the use of a graded mesh. The finer mesh was used from Grid 0 to Grid 2 only and the coarser mesh for the remainder of the depth. (Fig. 5.5).

Equation 5.3.2 was still applicable to the grid points of the coarse mesh (2, 4, 6, ..., 18) and the equation could also be used for Grid 0 if \( \frac{a}{2} \) was substituted for \( a \). However, for Grid 1, the finite difference expressions 5.3.1 were no longer true since the mesh length of the surrounding nodes were of different length. For the graded mesh surrounding Grid 1 the finite difference expressions to be used were (App.V.1):

\[
\begin{align*}
W^{iii} &= \frac{8(W_3 + W_4) - 72(W_1 + W_2) + 128W_0}{3a^2} \\
W^{iii} &= \frac{W_3 - W_4 - 3W_1 + 3W_2}{a^3} \\
W^{ii} &= \frac{4(W_1 + W_2) - 8W_0}{a^2}
\end{align*}
\]

Substitution of equations 5.4.1 into 5.2.2 leads to the equivalent expressions to 5.3.2 which were to be applied to Grid 1:

\[
\begin{align*}
-W_0\left\{ 12a^2\left( \frac{l}{2} \right)^2 t + 6a^2 t^2 \left( \frac{32}{9} t^2 \right) + W_1\left\{ 6a^2 \left( \frac{l}{2} \right)^2 t + 3a^2 t^2 \left( \frac{9}{2} a t^2 + 6t^2 \right) \right\} + W_2\left\{ 6a^2 \left( \frac{l}{2} \right)^2 t + 3a^2 t^2 \left( \frac{9}{2} a t^2 + 6t^2 \right) \right\} + W_3\left\{ \frac{3}{2} a t^2 + \frac{2}{3} t^3 \right\} \right\} \\
-W_4\left\{ \frac{3}{2} a t^2 - \frac{2}{3} t^3 \right\} = \frac{3a^4 z (1 - \mu^2)}{R^2} - \frac{3a^4 t (1 - \mu^2) W_0}{R^2}
\end{align*}
\]
In this equation $w_3$ is a fictitious point and must be eliminated by using the boundary conditions $w''' = w'' = 0$. From these conditions it can be shown that for Grid 1

$$w_3 = w_2 - 4w_0 + 4w_1 \quad \cdots \quad (5.4.3)$$

and thus $w_3$ can be eliminated from 5.4.2.

The coefficients of the equations of the graded mesh for the tumbler considered above (with $\mu = 0.15$) are given in Table 5.5 for Grid 0 and Grid 1, whilst the coefficients of the other grid levels are identical with the values given in Table 5.2.

**Table 5.5**

Coefficients of Graded Mesh ($\mu = 0.15$)

<table>
<thead>
<tr>
<th>Grid</th>
<th>$A_i$</th>
<th>$B_i$</th>
<th>$C_i$</th>
<th>$D_i$</th>
<th>$E_i$</th>
<th>$F_i$</th>
<th>$G_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>-</td>
<td>-20</td>
<td>-</td>
<td>10</td>
<td>-</td>
<td>-0.129</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>-3.772</td>
<td>-7.455</td>
<td>-</td>
<td>1.227</td>
<td>1.372</td>
<td>-0.088</td>
</tr>
</tbody>
</table>

The radial displacements obtained as a result of the relaxation of these equations are given in Table 5.6 where they are compared with the results obtained from the relaxation of the equations of both the coarse and fine mesh.

**CONCLUSIONS (5.5)**

1. Comparison of the analytical solutions for $\mu = 0$, 0.15 and 0.50 ([Fig 5.2] and Table 5.1) indicates that:

   a) The values of the radial displacements for $\mu = 0$ and 0.15 are in close agreement, so that, in calculating the displacements of a tumbler made of concrete for which $\mu = 0.15$, Poisson's ratio can be neglected without
affecting the results greatly. In Chapter 6 it is assumed that this conclusion can be applied to an arch dam. Lombardi, Priscu and Hertzog also make this assumption.

b) There is a large discrepancy between the values of the radial displacements obtained for $\mu = 0.15$ and $\mu = 0.5$ so that the values of the displacements obtained by means of a rubber model ($\mu = 0.5$) could not be expected to yield the results for a concrete prototype ($\mu = 0.15$) without the use of a correction factor. However, it is shown in Chapter 6 that the discrepancy between the results for $\mu = 0$ and $\mu = 0.5$ is not so large for an arch dam as it is for the complete cylinder.

TABLE 5.6

<table>
<thead>
<tr>
<th>GRID</th>
<th>Coarse Mesh</th>
<th>Fine Mesh</th>
<th>Graded Mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.092</td>
<td>0.113</td>
<td>0.110</td>
</tr>
<tr>
<td>1</td>
<td>0.141</td>
<td>0.140</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.160</td>
<td>0.168</td>
<td>0.168</td>
</tr>
<tr>
<td>4</td>
<td>0.209</td>
<td>0.211</td>
<td>0.212</td>
</tr>
<tr>
<td>6</td>
<td>0.229</td>
<td>0.230</td>
<td>0.229</td>
</tr>
<tr>
<td>8</td>
<td>0.219</td>
<td>0.221</td>
<td>0.219</td>
</tr>
<tr>
<td>10</td>
<td>0.183</td>
<td>0.187</td>
<td>0.183</td>
</tr>
<tr>
<td>12</td>
<td>0.130</td>
<td>0.133</td>
<td>0.130</td>
</tr>
<tr>
<td>14</td>
<td>0.072</td>
<td>0.072</td>
<td>0.072</td>
</tr>
<tr>
<td>16</td>
<td>0.023</td>
<td>0.021</td>
<td>0.023</td>
</tr>
<tr>
<td>18</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
2. Comparison of the analytical and relaxational solutions for both $\mu = 0.15$ and $\mu = 0.5$ (Figs. 5.3 and 5.4 and Table 5.4) shows that the radial displacements obtained by means of the relaxational solution of the equations of the coarse mesh are in error at the crest of the dam for both values of $\mu$ and that the use of finer mesh entirely eliminates this error.

3. The application of the graded mesh leads to results which are almost identical with those obtained for the finer mesh (Table 5.6). In the case of the tumbler, therefore, use of the graded mesh maintains the accuracy of the finer mesh solution with a considerable reduction in the amount of computation involved. In chapter 6 the graded mesh technique is applied to an arch dam.

Appendix V.1

**The finite difference expressions for the Graded mesh.**

The Taylor series expansion around a point $x_0$ can be expressed in the form

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \ldots$$

and this expression can be applied to the points 1, 2, 3 and 4 of the graded net shown in Fig. V.1 by making $(x-x_0)$ equal to $\frac{a}{2}$; $-\frac{a}{2}$; $\frac{3a}{2}$; and $-\frac{3a}{2}$ respectively.

![Fig. V.1](image)

The result of this substitution leads to the following equations:

...
\[ W_1 = W_0 + \frac{a}{2} (w')_0 + \frac{a^2}{2^2 2!} (w'')_0 + \frac{a^3}{2^3 3!} (w''')_0 + \quad (V.1.1) \]

\[ W_2 = W_0 - \frac{a}{2} (w')_0 + \frac{a^2}{2^2 2!} (w'')_0 - \frac{a^3}{2^3 3!} (w''')_0 + \quad (V.1.2) \]

\[ W_3 = W_0 + \frac{3}{2} a (w')_0 + \frac{3^2 a^2}{2^2 2!} (w'')_0 + \frac{3^3 a^3}{2^3 3!} (w''')_0 + \quad (V.1.3) \]

\[ W_4 = W_0 - \frac{3}{2} a (w')_0 + \frac{3^2 a^2}{2^2 2!} (w'')_0 - \frac{3^3 a^3}{2^3 3!} (w''')_0 + \quad (V.1.4) \]

and from these equations can be obtained:

\[ W_1 + W_2 = 2W_0 + \frac{a^2}{2^2 2!} (w'')_0 + \frac{a^4}{2^4 4!} (w''')_0 + \frac{a^6}{2^6 6!} (w''')_0 - \quad (V.1.5) \]

\[ W_3 + W_4 = 2W_0 + \frac{3^2 a^2}{2^2 2!} (w'')_0 + \frac{3^4 a^4}{2^4 4!} (w''')_0 + \frac{3^6 a^6}{2^6 6!} (w''')_0 - \quad (V.1.6) \]

Multiplying equation (V.1.5) by 3^2 and subtracting the resulting expression from (V.1.6) yields,

\[ W_3 + W_4 - 9(W_1 + W_2) = -16W_0 + \frac{(3^2 + 3^2) a^4 (w''')_0}{3^2 4!} + \text{terms of small order} \]

\[ \therefore (w''')_0 = \frac{8(W_3 + W_4) - 72(W_1 + W_2) + 128W_0}{3a^4} \quad (V.1.7) \]

If, from the first four equations, the expression for \((w_3 - w_4) - 3(w_1 - w_2)\) is found, then in a similar manner to the above the finite difference approximation for \(w''\) can be obtained.

The finite difference expression for \(w''\) is unchanged for the graded mesh shown in Fig. V.1 since the nodes 1 and 2 are equidistant from node 0.
CHAPTER 6.

APPLICATION OF THE GRADED MESH TECHNIQUE TO AN ARCH DAM SITUATED IN A RECTANGULAR VALLEY.

INTRODUCTION. (6.1) For the complete cylinder considered in Chapter 5, it was found that the application of a graded mesh to the upper portion improved the accuracy of the solution. In this chapter it is shown that the application of a graded mesh to the case of a dam in a rectangular valley affects the results in a similar manner to the change observed in the complete cylinder results.

In applying any simplified method of stress analysis, it is not only necessary to compare the accuracy of the results with those obtained by more exact methods, but also to compare the time required to attain this accuracy and thus it was decided to commence the relaxation from zero trial values so that the rate of convergence to the final solution could be ascertained. The time required to obtain the final solution from zero values would represent the worst possible case since in practice the discrepancy between the assumed non-zero trial values and the final results would be smaller so that the actual amount of relaxation required would be reduced.

In chapter 4, to obtain the displacements of Doken Dam by application of a thin shell theory, the radius of the dam was taken to the upstream face and not the mid-surface and the radius was also assumed to be constant throughout the depth. The effect of these assumptions on the results
are also considered in this chapter.

**SELECTION OF DIMENSIONS. (6.2)** A comparison was to be made between the results obtained by the application to an arch dam of the graded mesh relaxational technique and the values deduced from measurements obtained from a scale model made of plaster of Paris and diatomite. To simplify the work of building the model it was decided to select a dam of constant angle, having a linear variation of thickness with depth and situated in a rectangular valley. Since it was known that research work in progress at the Imperial College was being conducted on a rubber model of an arch dam situated in a rectangular valley, it was decided to select the dimensions of the dam so that the results obtained by relaxation, and the use of a plaster-diatomite model, could be compared with the results obtained at Imperial College by means of the rubber model tests.

With these considerations the dimensions of the dam selected (See Fig. 6.1) were:

- Central Angle ... ... 2 radians
- Radius of upstream face ... 300 ft.
- Height ... 300 ft.
- Thickness at crest ... 15 ft.
- Thickness at Base ... 90 ft.

The ratio \( \frac{R}{H} \) changes from 1/20 at the crest to 3/10 at the base and the limit of \( \frac{1}{5} \), given by Lombardi, for which a thin shell theory is applicable is reached at a depth of 0.6H. However, the limit of \( \frac{1}{3} \), given by Priscu, is not exceeded and it was thus assumed that the example would provide some evidence as to the correct limit that should
ARCH DAM IN RECTANGULAR VALLEY

FIG 61
be imposed.

The abutments and foundations were assumed to be perfectly rigid.

NON-DIMENSIONAL EQUATION. (6.3) The relaxational grid drawn on the developed surface of the upstream face was selected so that the horizontal grid length $b$ was twice the vertical grid length $a$ before application of the graded mesh. The equations were further simplified by making $\mu = 0$ since it was shown in Chapter 5 that the values of the displacements found for $\mu = 0$ and $\mu = 0.15$ were almost identical. Using equations (3.2,4) and the finite difference approximations (App. III,1) the displacement equations obtained for a rectangular mesh with $b = 2a$ were:

$u$ - Equation

\[
\begin{align*}
\text{GRID 0:} & \quad -8\frac{1}{2} u_0 + \frac{1}{2} (u_1 + u_2) + 8 u_4 - \frac{(u_9 + u_{10})}{8} = 0 \\
\text{GRIDS 2, 4, 6 \ldots 18:} & \quad -9u_6 + \frac{1}{2} (u_1 + u_3) + 1 \{4 - 2 \frac{a}{b} \ell'\} + u_4 \{4 - 2 \frac{a}{b} \ell'\} \\
& \quad + \left(\frac{v_5 - v_6 - v_8 + v_{10}}{4}\right) = 0 \quad (6.3.1)
\end{align*}
\]

$v$ - Equation

\[
\begin{align*}
\text{GRID 0:} & \quad -6V_0 + (u_1 + u_3) + 4V_4 - (u_1 - u_3) - (u_1 - u_3)a = 0 \\
\text{GRIDS 2, 4, 6 \ldots 18:} & \quad -6V_6 + (u_1 + u_3) + 1 \left\{2 + \frac{a}{b} \ell'\right\} + V_4 \left\{2 - \frac{a}{b} \ell'\right\} \\
& \quad + \frac{(u_1 - u_3) a \ell'}{2} + \frac{(u_5 - u_6 - u_8 + u_{10}) - (u_1 - u_3)a}{4} = 0 \quad (6.3.2)
\end{align*}
\]
**GRID 0**

\[ -w_0 \left\{ \frac{h_t}{2} E^2 + a t l^2 \right\} + (N_1 + W_3) \left\{ \frac{s}{12} E^2 + a t l^2 \right\} + W_4 \left\{ 2E^2 + a t l^2 \right\} \\
- (N_1 + W_3) \left\{ \frac{e^2}{2} + a t l^2 \right\} - (W_9 + W_1) \left\{ \frac{E^2}{4} \right\} - W_2 \left\{ \frac{L^2}{3} \right\} \\
+ \frac{a^3}{R^2} (V_1 - V_3) + 4a^4 Z^2 - 4a^4 W_0 = 0 \]

... (6.3.3)

**GRID 2**

\[ -w_0 \left\{ \frac{2h_t}{2} E^2 - 2a^2 [T] - a t l^2 \right\} + (W_1 + W_3) \left\{ \frac{s}{12} E^2 \right\} \\
+ W_2 \left\{ e^2 - a^2 [T] + a t l^2 \right\} + W_4 \left\{ s^2 E^2 - a^2 [T] - \frac{a}{2} a t l^2 \right\} - (W_9 + W_1) \left\{ \frac{L^2}{6} + a t l^2 \right\} \\
- (W_9 + W_1) \left\{ \frac{E^2}{4} \right\} - W_2 \left\{ \frac{L^2}{3} - a t l^2 \right\} \\
+ \frac{a^3}{R^2} (V_1 - V_3) + 4a^4 Z^2 - 4a^4 W_0 = 0 \]

... (6.3.4)

**GRIDS 4, 6, 8 ... 18.**

\[ -w_0 \left\{ \frac{2h_t}{2} e^2 - 2a^2 [T] \right\} + (N_1 + W_3) \left\{ \frac{s}{12} E^2 \right\} + W_2 \left\{ e^2 - a^2 [T] + \frac{a}{2} a t l^2 \right\} \\
+ W_4 \left\{ s^2 E^2 - a^2 [T] - \frac{a}{2} a t l^2 \right\} - (W_9 + W_1) \left\{ \frac{L^2}{6} + a t l^2 \right\} \\
- (W_9 + W_1) \left\{ \frac{E^2}{4} \right\} - W_2 \left\{ \frac{L^2}{3} + a t l^2 \right\} - W_2 \left\{ \frac{L^2}{3} - a t l^2 \right\} \\
+ \frac{a^3}{R^2} (V_1 - V_3) + 4a^4 Z^2 - 4a^4 W_0 = 0 \]

... (6.3.5)

where \[ [T] = [2(t^t)^2 + t t''] \]

It is convenient to rewrite the above equations in terms of non-dimensional quantities \( \tilde{u}, \tilde{v} \) and \( \tilde{w} \) so that the calculation of the coefficients of the equations for the different mesh lengths and the comparison between the experimental and analytical results are simplified.

The non-dimensional quantities \( \tilde{u}, \tilde{v} \) and \( \tilde{w} \) are defined by:

\[ \tilde{u}_n = \frac{U_n}{\gamma H^2} \quad (n = 0, 1, 2, \ldots, 11) \]

\[ \tilde{v}_n = \frac{V_n}{\gamma H^2} \quad (n = 0, 1, 2, \ldots, 8) \]

\[ \tilde{w}_n = \frac{W_n}{\gamma H^2} \quad (n = 0, 1, 2, \ldots, 12) \]
where $\gamma =$ density of water

$H =$ height of dam.

If $\bar{u}$, $\bar{v}$ and $\bar{w}$ are substituted into the $u$, $v$ and $w$ equations then the non-dimensional expressions can be obtained in the case of the $u$ and $v$ equations by dividing each term by $\frac{\gamma H^2}{E}$ and for the $w$ equation by dividing by $\frac{\gamma H^4}{E^2}$. The $u$ and $v$ equations (6.3.1) and (6.3.2) remain the same, except that $u_n$, $v_n$ and $w_n$ are replaced by $\bar{u}_n$, $\bar{v}_n$ and $\bar{w}_n$, but the $w$-equations become after substitution:

Non-dimensional $w$ - Equations.

**GRID 0.**

$$\begin{align*}
-\bar{w}_1 \{1 & \bar{w}_4 \left[ \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right] + (\bar{w}_1 + \bar{w}_3) \left[ \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right] + \bar{w}_4 \left[ \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right] \\
- (\bar{w}_1 + \bar{w}_3) & \left[ \frac{1}{3} \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] + \bar{w}_4 \left[ \frac{1}{4} \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] \\
+ (\bar{v}_1 - \bar{v}_3) & \left[ \frac{1}{3} \left( \frac{\bar{v}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{v}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] + 4 \left( \frac{\bar{v}_1 (\bar{H})^2}{(\bar{H})^2} \right) - 4 \left( \frac{\bar{v}_3 (\bar{H})^2}{(\bar{H})^2} \right) \bar{w}_0 = 0
\end{align*}$$

*** (6.3.6)

**GRID 2.**

$$\begin{align*}
-\bar{w}_0 \left\{ \frac{1}{2} \bar{w}_4 \left[ \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right] + 2 \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) \right] & + (\bar{w}_1 + \bar{w}_3) \left[ \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right] + \bar{w}_4 \left[ \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right] \\
+ \bar{w}_1 & \left[ \frac{1}{3} \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] + \bar{w}_4 \left[ \frac{1}{4} \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] \\
- (\bar{w}_1 + \bar{w}_3) & \left[ \frac{1}{3} \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] - (\bar{w}_1 + \bar{w}_3) \left[ \frac{1}{4} \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] \\
+ (\bar{v}_1 - \bar{v}_3) & \left[ \frac{1}{3} \left( \frac{\bar{v}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{v}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] + 4 \left( \frac{\bar{v}_1 (\bar{H})^2}{(\bar{H})^2} \right) - 4 \left( \frac{\bar{v}_3 (\bar{H})^2}{(\bar{H})^2} \right) \bar{w}_0 = 0
\end{align*}$$

*** (6.3.7)

**GRIDS 4, 6, 18.**

$$\begin{align*}
-\bar{w}_0 \left\{ \frac{1}{2} \bar{w}_4 \left[ \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right] + 2 \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) \right] & + (\bar{w}_1 + \bar{w}_3) \left[ \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right] + \bar{w}_4 \left[ \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right] \\
+ \bar{w}_1 & \left[ \frac{1}{3} \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] + \bar{w}_4 \left[ \frac{1}{4} \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] \\
- (\bar{w}_1 + \bar{w}_3) & \left[ \frac{1}{3} \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] - (\bar{w}_1 + \bar{w}_3) \left[ \frac{1}{4} \left( \frac{\bar{w}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{w}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] \\
+ (\bar{v}_1 - \bar{v}_3) & \left[ \frac{1}{3} \left( \frac{\bar{v}_1 (\bar{H})^2}{(\bar{H})^2} \right) + \frac{1}{2} \left( \frac{\bar{v}_3 (\bar{H})^2}{(\bar{H})^2} \right) \right] + 4 \left( \frac{\bar{v}_1 (\bar{H})^2}{(\bar{H})^2} \right) - 4 \left( \frac{\bar{v}_3 (\bar{H})^2}{(\bar{H})^2} \right) \bar{w}_0 = 0
\end{align*}$$

*** (6.3.8)
If the units of \( u, v \) and \( w \) are changed, the coefficients in the above equations remain unaltered except for the term \( 4(\varepsilon)^4 \frac{h}{H} t \) which does not contain \( u, v \) or \( w \); this term would be changed by a factor equivalent to the change of unit.

For convenience the displacement equations were multiplied by \( 10^2 \) so that in order to convert the values of \( \bar{w}, \bar{v} \) and \( \bar{u} \) given in this chapter into any length dimension they must be multiplied by \( \frac{\gamma h^2}{E x 10^3} \) expressed in that dimension.

e.g. For the prototype \( \gamma = 62.5 \text{ lb/ft}^3; \quad H = 300 \text{ ft} \); and \( E = 2 \times 10^6 \text{ lb/in}^2 \), and thus to convert the non-dimensional units into cms, multiply by \( \frac{62.5 \times 300^2 \times 12^2 \times 2.54}{12^2 \times 2 \times 10^6 \times 10^2} \) (i.e., .006).

THE RELAXATION (6.4)

Since the relaxation was to be commenced from zero values of \( \bar{u}, \bar{v} \) and \( \bar{w} \) it was decided to select a very coarse mesh, having only 10 intersection points, in order to arrive at suitable trial values for the finer mesh. (Fig. 6.2)
The dimensions of the rectangular grid of the coarse mesh were selected so that \( b = 2a \) and equations (6.3.1), (6.3.2), (6.3.6), (6.3.7) and (6.3.8) were still applicable. The coefficients of the non-dimensional equations, applied to the coarse mesh, are given in Tables 6.1, 6.2 and 6.3 and the results of the relaxation are shown in Table 6.4. These results were obtained after two cycles of relaxation and, as in the solution for Dokan, the major part of the computation was involved in the relaxation of the \( w \)-equation during the first cycle, and it appears that one cycle would have been sufficiently accurate during the coarse stage.

For the fine mesh there were 50 nodal points (Fig. 6.2) and the coefficients of \( \tilde{u} \), \( \tilde{v} \) and \( \tilde{w} \) equations for this case are given in Tables 6.5, 6.6 and 6.7. Starting with the values obtained from the coarse mesh the relaxation was completed in two cycles, but the amount of relaxation required after the first cycle was very small since the values of \( \tilde{u} \) and \( \tilde{v} \) did not change greatly from one cycle to the next. The second and third cycles are concerned almost entirely with the relaxation of the \( \tilde{w} \)-equation. The results of the relaxation are given in Table 6.8 where the values of \( \tilde{u} \), \( \tilde{v} \) and \( \tilde{w} \) obtained at the end of the third cycle are shown.

**THE GRADED MESH FOR \( \mu = 0 \), (6.5)**

If in addition to the horizontal grid lines shown in Fig. 6.2 an extra grid line is inserted between 0 and 2, then the number of intersection points is increased
from 50 to 55. The equations given in sections 6.3 would still apply to the nodes occurring along grid line 2 and along the grid lines below this depth, but the equations must be changed for the nodes along Grid 0 since $b$ would now be equal to $4a$.

The intersection points occurring along the inserted grid line 1 must be treated in a similar manner to the graded mesh considered for the complete cylinder in Chapter 5 since the mesh length of the surrounding nodes in the vertical direction are not equal.

With the arrangement of nodes shown in Fig. 6.3 the finite difference expressions for the graded mesh derived in App. (V,1) still apply, but must be re-written due to the change of numeration. The expressions now become:

$$w'''' = \frac{8(W_0 + W_2) - 12(W_2 + W_4) + 12W_0}{3a^4}$$

$$W''' = \frac{W_0 - W_2 - 3W_2 + 3W_4}{a^3}$$

$$W'' = \frac{4(W_2 + W_4) - 8W_0}{a^2}$$

All the other finite difference expressions required can be obtained from Appendix III.1 if $\frac{a}{2}$ is substituted for $a$.

Substitution of these expressions into the general equations applicable to Grid 1 leads to the $\bar{W}$, $\bar{V}$ and $\bar{U}$ equations. The $\bar{W}$ equation will contain a fictitious point $\bar{W}_{10}$ and this must be eliminated using the boundary conditions (3.3.3). The final equations after elimination of $\bar{W}_{10}$ are:
FIG 6.5

(a) - PROFILE OF PLASTER-DIATOMITE MODEL
(b) - PROFILE ANALYSED BY THIN SHELL THEORY
GRID 1. \( \bar{w} \)-Equation:
\[
- \bar{W}_0 \left\{ \frac{323}{150} \left( \frac{a}{H} \right)^2 + \frac{1}{18} \left( \frac{a}{H} \right)^4 \right\} + \left( \bar{W}_1 + \bar{W}_3 \right) \left\{ \frac{17}{192} \left( \frac{a}{H} \right)^2 + \frac{1}{12} \left( \frac{a}{H} \right)^4 \right\} \\
+ \bar{W}_4 \left\{ \frac{3}{2} \left( \frac{a}{H} \right)^2 + \frac{1}{4} \left( \frac{a}{H} \right)^4 \right\} - \left( \bar{W}_1 + \bar{W}_3 \right) \left\{ \frac{1}{12} \left( \frac{a}{H} \right)^2 + \frac{1}{12} \left( \frac{a}{H} \right)^4 \right\} \\
- \bar{W}_2 \left\{ \frac{9}{5} \left( \frac{a}{H} \right)^2 - \frac{2}{9} \left( \frac{a}{H} \right)^4 \right\} - \left( \bar{W}_7 + \bar{W}_8 \right) \left\{ \frac{1}{2} \left( \frac{a}{H} \right)^2 - \frac{1}{2} \left( \frac{a}{H} \right)^4 \right\} \\
- \bar{W}_9 + \bar{W}_11 \left\{ \frac{1}{45} \left( \frac{a}{H} \right)^2 \right\} - \bar{W}_{12} \left\{ \frac{8}{9} \left( \frac{a}{H} \right)^2 - 2 \left( \frac{a}{H} \right)^4 \right\} \\
+ \left( \bar{V} - \bar{V}_3 \right) \left\{ \frac{1}{2} \left( \frac{a}{H} \right)^2 \right\} + 4 \left( \frac{a}{H} \right)^2 \left( \frac{a}{H} \right)^2 \bar{W}_0 = 0 \quad \ldots \quad (6.5.1)
\]

\( \bar{v} \)-Equation:
\[
- \bar{V}_0 \left\{ \frac{1}{2} \right\} + (\bar{V} + \bar{V}_3) \left\{ \frac{1}{2} \right\} + \bar{V}_2 \left\{ 2 + \frac{1}{2} \left( \frac{a}{H} \right)^2 \right\} + \bar{V}_4 \left\{ 2 - \frac{3}{2} \left( \frac{a}{H} \right)^2 \right\} \\
+ \frac{1}{6} \left\{ \bar{V}_5 - \bar{V}_4 - \bar{V}_8 + \bar{U}_7 \right\} + \frac{1}{6} \left( \frac{a}{H} \right)^2 \left\{ \bar{V}_1 - \bar{V}_3 \right\} - \frac{1}{2} \left( \frac{a}{H} \right)^2 \left( \bar{W}_1 - \bar{W}_3 \right) = 0 \quad \ldots \quad (6.5.2)
\]

\( \bar{u} \)-Equation:
\[
- \bar{U}_0 \left\{ \frac{8}{9} \right\} + \frac{1}{6} (\bar{U} + \bar{U}_3) + \bar{U}_4 \left\{ 4 + \frac{1}{2} \left( \frac{a}{H} \right)^2 \right\} + \bar{U}_4 \left\{ 4 - \frac{3}{2} \left( \frac{a}{H} \right)^2 \right\} \\
+ \frac{1}{6} \left( \bar{V} - \bar{V}_5 - \bar{V}_4 + \bar{V}_7 \right) = 0 \quad \ldots \quad (6.5.3)
\]

For Grid 0 the equations must be changed, since \( b = 4a \) for this case, and the fictitious points eliminated using the boundary conditions. After elimination the appropriate equations become:

GRID 0.

\( \bar{w} \)-Equation:
\[
- \bar{W}_0 \left\{ \frac{323}{150} \left( \frac{a}{H} \right)^2 + \frac{1}{18} \left( \frac{a}{H} \right)^4 \right\} + \left( \bar{W}_1 + \bar{W}_3 \right) \left\{ \frac{17}{192} \left( \frac{a}{H} \right)^2 + \frac{1}{12} \left( \frac{a}{H} \right)^4 \right\} \\
+ \bar{W}_4 \left\{ \frac{3}{2} \left( \frac{a}{H} \right)^2 + \frac{1}{4} \left( \frac{a}{H} \right)^4 \right\} - \left( \bar{W}_1 + \bar{W}_3 \right) \left\{ \frac{1}{12} \left( \frac{a}{H} \right)^2 + \frac{1}{12} \left( \frac{a}{H} \right)^4 \right\} \\
- \bar{W}_2 \left\{ \frac{9}{5} \left( \frac{a}{H} \right)^2 - \frac{2}{9} \left( \frac{a}{H} \right)^4 \right\} - \left( \bar{W}_7 + \bar{W}_8 \right) \left\{ \frac{1}{2} \left( \frac{a}{H} \right)^2 - \frac{1}{2} \left( \frac{a}{H} \right)^4 \right\} \\
+ \left( \bar{W}_9 + \bar{W}_11 \right) \left\{ \frac{1}{45} \left( \frac{a}{H} \right)^2 \right\} - \left( \bar{W}_12 \right) \left\{ \frac{8}{9} \left( \frac{a}{H} \right)^2 - 2 \left( \frac{a}{H} \right)^4 \right\} \\
+ 4 \left( \frac{a}{H} \right)^2 \left( \frac{a}{H} \right)^2 \bar{W}_0 = 0 \quad \ldots \quad (6.5.4)
\]

\( \bar{v} \)-Equation:
\[
- \bar{V}_0 \left\{ \frac{1}{2} \right\} + (\bar{V} + \bar{V}_3) \left\{ \frac{1}{2} \right\} + \bar{V}_4 \left\{ 2 + \frac{1}{2} \left( \frac{a}{H} \right)^2 \right\} - (\bar{V}_1 - \bar{V}_3) \left\{ \frac{1}{6} \right\} \\
- \left( \bar{W}_1 - \bar{W}_3 \right) \left\{ 00625 \right\} = 0 \quad \ldots \quad (6.5.5)
\]

\( \bar{u} \)-Equation:
\[
- \bar{U}_0 \left\{ \frac{3}{8} \right\} + (\bar{U} + \bar{U}_3) \left\{ \frac{1}{12} \right\} + \bar{U}_4 \left\{ 4 + \frac{1}{2} \left( \frac{a}{H} \right)^2 \right\} + \bar{U}_4 \left\{ 4 - \frac{3}{2} \left( \frac{a}{H} \right)^2 \right\} \\
- \left( \bar{U}_1 + \bar{U}_3 \right) \left\{ \frac{1}{12} \right\} - \bar{U}_4 \left\{ \frac{1}{12} \right\} \left( \bar{W}_11 + \bar{W}_9 \right) = 0 \quad \ldots \quad (6.5.6)
\]
For an arch dam of the dimensions given in (6.2) the coefficients of the above equations have the values shown in Tables 6.9, 6.10 and 6.11 where the coefficients for grids 0, 1 and 2 are indicated. The coefficients for all other grids are identical with the values given in Tables 6.5, 6.6 and 6.7. The relaxation of these equations was completed in one cycle starting with trial values obtained from the fine mesh relaxation solution shown in Table 6.8. The displacements obtained are shown in Table 6.12 and comparison of these results with those shown in Table 6.8 indicates that only minor changes occur below Grid 4 so that the extra labour involved in application of the graded mesh is confined to the upper portion of the dam.

The Graded Mesh With $\mu = 0.5$ (6.6).

In order to compare the results obtained by application of a thin shell theory with those obtained by means of a rubber model loaded with water, it was decided to calculate the radial displacements using a value of Poisson's ratio of 0.5. Following a similar procedure to that outlined above the non-dimensional equations obtained for this case are:

$\bar{w}$-Equation.

\[ \bar{w}_0 \left\{ \frac{\bar{w}_0}{h_{36}} \left( \frac{1}{H} \right)^2 + \frac{\bar{w}_1}{h_{12}} \left( \frac{a}{H} \right) \bar{w}_1 \right\} + \left( \bar{w}_1 + \bar{w}_3 \right) \frac{35}{48} \left( \frac{1}{H} \right)^2 + \frac{\bar{w}_2}{25} \left( \frac{a}{H} \right) \bar{w}_2 \right\}
\]
\[ + \bar{w}_3 \left\{ \frac{35}{48} \left( \frac{1}{H} \right)^2 + \frac{\bar{w}_2}{12} \left( \frac{a}{H} \right) \bar{w}_2 \right\} - \left( \bar{w}_7 + \bar{w}_8 \right) \frac{1}{\bar{w}_7} \left( \frac{1}{H} \right)^2 + \frac{1}{12} \left( \frac{a}{H} \right) \bar{w}_7 \right\}
\]
\[ - (\bar{w}_9 + \bar{w}_11) \left\{ \frac{1}{1024} \left( \frac{1}{H} \right)^2 + \frac{1}{1024} \left( \frac{a}{H} \right) \bar{w}_11 \right\} - \bar{w}_{12} \left\{ \frac{1}{3} \left( \frac{1}{H} \right)^2 \right\}
\]
\[ + \frac{3}{8} \bar{w}_1 \left( \frac{1}{H} \right)^2 - 3 \bar{w}_0 \left( \frac{a}{H} \right)^2 + 3 \left( \frac{a}{H} \right)^2 \left( \frac{1}{H} \right) = 0 \quad \ldots \quad (6.6.1) \]
GRID I.
\[
- \bar{\omega}_0 \left\{ \frac{\rho_0^2}{\lambda^2} \left( \frac{\sigma}{\lambda} \right)^2 - \frac{\rho_0^2}{\lambda^2} \left[ \frac{1}{2} \left( \frac{\sigma}{\lambda} \right)^2 \right] - \frac{\rho_0^2}{\lambda^2} \left( \frac{\sigma}{\lambda} \right)^2 \right\} + \bar{\omega}_3 \left\{ \frac{\rho_0^2}{\lambda^2} \left( \frac{\sigma}{\lambda} \right)^2 - \frac{\rho_0^2}{\lambda^2} \left( \frac{\sigma}{\lambda} \right)^2 \right\} + \frac{1}{2} \left( \frac{\sigma}{\lambda} \right)^2 \left( \frac{\sigma}{\lambda} \right)^2 \right\}
\]
\[
+ \bar{\omega}_1 \left\{ \frac{10}{12} \left( \frac{\sigma}{\lambda} \right)^2 - \frac{5}{12} \left( \frac{\sigma}{\lambda} \right)^2 \right\} - \frac{1}{2} \left( \frac{\sigma}{\lambda} \right)^2 \left( \frac{\sigma}{\lambda} \right)^2 \right\} - \bar{\omega}_4 \left\{ \frac{8}{12} \left( \frac{\sigma}{\lambda} \right)^2 - \frac{4}{12} \left( \frac{\sigma}{\lambda} \right)^2 \right\} - \frac{1}{2} \left( \frac{\sigma}{\lambda} \right)^2 \left( \frac{\sigma}{\lambda} \right)^2 \right\}
\]
\[
- \left( \bar{\omega}_5 + \bar{\omega}_4 \right) \left\{ \frac{125}{256} \left( \frac{\sigma}{\lambda} \right)^2 - \frac{3}{12} \left( \frac{\sigma}{\lambda} \right)^2 \right\} - \left( \bar{\omega}_7 - \bar{\omega}_6 \right) \left\{ \frac{3}{12} \left( \frac{\sigma}{\lambda} \right)^2 - \frac{1}{2} \left( \frac{\sigma}{\lambda} \right)^2 \right\}
\]
\[
- \left( \bar{\omega}_9 + \bar{\omega}_5 \right) \left\{ \frac{1}{49} \left( \frac{\sigma}{\lambda} \right)^2 \right\} - \left( \bar{\omega}_3 + \bar{\omega}_4 \right) \left\{ \frac{125}{256} \left( \frac{\sigma}{\lambda} \right)^2 + \frac{3}{12} \left( \frac{\sigma}{\lambda} \right)^2 \right\}
\]
\[
- \bar{\omega}_2 \left\{ \frac{9}{12} \left( \frac{\sigma}{\lambda} \right)^2 - 2 \left( \frac{\sigma}{\lambda} \right)^2 \right\} + \left( \bar{\omega}_5 - \bar{\omega}_6 \right) \left\{ \frac{9}{12} \left( \frac{\sigma}{\lambda} \right)^2 + 3 \left( \frac{\sigma}{\lambda} \right)^2 \right\}
\]
\[
- \frac{3}{2} \left( \frac{\sigma}{\lambda} \right)^2 \bar{\omega}_0 + \frac{1}{2} \left( \frac{\sigma}{\lambda} \right)^2 \left( \bar{\omega}_0 - \bar{\omega}_5 \right) = 0
\]
--- (6.6.2)

**General w-Equation.**

As before plus the following extra terms,
\[
- \frac{1}{2} \left( \frac{\sigma}{\lambda} \right)^2 \left( \bar{\omega}_0 + \bar{\omega}_5 \right) \left\{ \frac{3}{12} \left( \frac{\sigma}{\lambda} \right)^2 + \left( \bar{\omega}_0 - \bar{\omega}_5 \right) \left\{ \frac{3}{12} \left( \frac{\sigma}{\lambda} \right)^2 \right\}
\]
--- (6.6.3)

**V-Equation.**

GRID 0.
\[
- \bar{\nu}_0 \left\{ \frac{15}{12} \right\} + \left( \bar{\nu}_6 + \bar{\nu}_5 \right) \left\{ \frac{1}{2} \right\} + \bar{\nu}_7 \left\{ \frac{1}{2} \right\} - \left( \bar{\nu}_6 + \bar{\nu}_5 \right) \left\{ \frac{1}{2} \right\} - \left( \bar{\nu}_7 - \bar{\nu}_6 \right) \left\{ \frac{1}{2} \right\} = 0
\]
--- (6.6.4)

GRID 1.
\[
- \bar{\nu}_0 \left\{ \frac{1}{2} \right\} + \left( \bar{\nu}_6 + \bar{\nu}_5 \right) \left\{ \frac{1}{2} \right\} + \bar{\nu}_7 \left\{ \frac{1}{2} \right\} - \left( \bar{\nu}_6 + \bar{\nu}_5 \right) \left\{ \frac{1}{2} \right\} - \left( \bar{\nu}_7 - \bar{\nu}_6 \right) \left\{ \frac{1}{2} \right\} = 0
\]
--- (6.6.5)

**GENERAL.**

As before plus the following terms:
\[
- \left( \bar{\nu}_2 + \bar{\nu}_4 \right) \left( \frac{\nu_5 - \nu_6 - \nu_8 + \nu_7}{\nu_5 - \nu_6} \right) - \frac{1}{2} \left\{ \frac{\nu_5 - \nu_6 + \nu_2 - \nu_4}{\nu_5} \right\}
\]
--- (6.6.6)

**u-Equation.**

GRID 0.
\[
- \bar{u}_0 \left\{ \frac{1}{2} \right\} + \left( \bar{u}_1 + \bar{u}_3 \right) \left\{ \frac{1}{2} \right\} + \bar{u}_4 \left\{ \frac{1}{2} \right\} - \left( \bar{u}_1 + \bar{u}_3 \right) \left\{ \frac{1}{2} \right\} - \left( \bar{u}_4 - \bar{u}_2 \right) \left\{ \frac{1}{2} \right\}
\]
\[
+ \frac{1}{2} \left( \frac{\sigma}{\nu_5} \right) \bar{\omega}_0 + \frac{1}{2} \left( \frac{\sigma}{\nu_5} \right) \bar{\omega}_4 + \frac{1}{2} \left( \frac{\sigma}{\nu_5} \right) \left( \bar{\omega}_0 + \bar{\omega}_3 \right) = 0
\]
--- (6.6.7)

GRID 1.
\[
- \bar{u}_0 \left\{ \frac{1}{2} \right\} + \left( \bar{u}_1 + \bar{u}_3 \right) \left\{ \frac{1}{2} \right\} + \bar{u}_4 \left\{ \frac{1}{2} \right\} - \left( \bar{u}_1 + \bar{u}_3 \right) \left\{ \frac{1}{2} \right\} - \left( \bar{u}_4 - \bar{u}_2 \right) \left\{ \frac{1}{2} \right\}
\]
\[
+ \frac{1}{2} \left( \frac{\sigma}{\nu_5} \right) \left( \bar{\nu}_5 - \bar{\nu}_3 \right) + \frac{1}{2} \left( \frac{\sigma}{\nu_5} \right) \left( \bar{\nu}_5 - \bar{\nu}_3 \right) - \frac{1}{2} \left( \frac{\sigma}{\nu_5} \right) \left( \bar{\omega}_0 + \bar{\omega}_3 \right) = 0
\]
--- (6.6.8)
GENERAL.

As before plus the following extra terms:

\[
\frac{(\tilde{V}_5 - \tilde{V}_6 - \tilde{V}_8 + \tilde{V}_7)}{8} - \frac{a}{E} (\tilde{W}_2 - \tilde{W}_4) + \frac{1}{2} \left[ \frac{\tilde{u}'}{E} (\tilde{V}_1' - \tilde{V}_3') - \tilde{W}_6 (\tilde{u} + \tilde{u}_0) + \frac{\tilde{u}_0}{\tilde{x}} \right] = 0
\]

The coefficients of these equations for an arch dam having the dimensions given in section 6.2 have the values shown in Table 6.13, 6.14 and 6.15 and the values of \( \tilde{w} \) obtained after one cycle of relaxation are shown in Table 6.16 where they are compared with the values obtained at the Imperial College by means of a rubber model loaded with water. The experimental values differed slightly at the air and water faces and the values shown in Table 6.16 represent the average of both faces converted into non-dimensional units.

The radial displacements \( \tilde{w} \) at three vertical sections are compared graphically in Fig. 6.4. Although the results are in fairly close agreement the curves connecting the experimental values do not approach the foundation level tangentially as would be expected for a fixed base. This lack of fixity at the foundations is also present at the abutments and the proximity of vertical Grid 8 to the abutments is probably the cause of the discrepancy at the crest which is evident at this section.

From table 6.16 it can be seen that an increase in Poisson's ratio from 0 to 0.5 causes only minor changes in the radial displacements and the results obtained for these values of \( \mu \) differ by a very much smaller amount than the changes that were observed in Chapter 5 for the complete cylinder.
Radial Displacements [non-dimensional $\frac{WE}{8H^2} \times 10^2$]

Vertical Grid 0

Vertical Grid 4

Vertical Grid 8

Shell Theory $(n = 0.5)$
Rubber Model

Constants for Rubber Model
- $E = 110 \text{ lb/ins}^2$
- $H = 15 \text{ ins}$
- $\gamma = 62.5 \text{ lb/ft}^3$

Fig. 6.4
The variation of the Radius with depth. (6.7)

The radius used in the above calculations was the radius of the upstream face and no allowance had been made for the variation of radius of the middle surface with depth. The solution obtained theoretically was, in fact, for a dam of a slightly different profile from that considered practically (Fig. 6.5)

The shell equations developed in Appendix II are unchanged if the radius of the middle surface is assumed to vary with \( x \), for, in obtaining these equations the radius of the elemental length would still be considered as a constant and no term containing \( R \) is differentiated with respect to \( x \) in the development of these equations. If, therefore, the radius of each elemental length \( \delta x \) of a dam of constant radius was decreased by a value \( \frac{t}{2} \), where \( t \) is the thickness of the element, then the same equations apply (using the new values of \( R \)) and a dam, with a profile similar to that shown in Fig. (6.5a), would be obtained.

The non-dimensional displacement equations obtained in section 6.3 were applied to an arch dam having the dimensions given in section 6.2 and in order to calculate the coefficients at each horizontal grid the radius of the mid-surface at the grid level was used. Changes occurred in the coefficients of the \( \bar{w} \) and \( \bar{v} \) equations only and since the changes in the coefficient of the \( \bar{v} \) equations were small, it was decided to neglect them and consider only the alterations to the coefficients of the \( \bar{w} \) equations. The
new coefficients of the $\bar{w}$ equations are shown in Table 6.17 where the corrected values of the coefficients of the $(\bar{v}_1 - \bar{v}_2)$ and the $(\bar{w}_0)$ terms are shown; all other coefficients are identical with the values given in Tables 6.9 and 6.5. Using these coefficients, the equations were solved by relaxation to obtain the values of the non-dimensional displacements $\bar{w}$. The calculated values are shown in Table 6.16 and comparison of these results with those obtained using a constant radius show that the values of $\bar{w}$ near the base of the dam are almost unchanged, even though the change in the value of the radius (compared with the constant radius) is largest in this region. The greatest deviation between the results of the two cases occurs in the upper portion of the dam near the centre line. If, in the original calculations, instead of using the constant radius of the upstream face, the radius of the mid-surface at the crest had been used throughout the depth of the dam, then the differences between the results for constant and variable radius would be reduced to almost negligible proportions since the error in the value of the radius of the upper portion would be considerably reduced. By this means, therefore, a thin dam of variable radius could be considered as a dam of constant radius without introducing a large error.

Since the radius $R$ enters also into the equations connecting the stresses and displacements, it follows that there will be a change in the value of the stresses due to two factors:-
1. The alteration in the displacements previously calculated and

2. The change in the radius.

The calculation of the stresses is considered in the next section where a comparison between the results obtained using a constant and variable radius is included.

Calculations of the Stresses. (6.8)

Using equations II.3, II.4 and the finite difference approximations (Appendix III.1) applied to a rectangular mesh with \( b = 2a \), the following expressions in terms of the non-dimensional displacement, for the stresses occurring at the faces of the dam can be obtained:

\[
\sigma_x = \frac{3H}{1-\mu^2} \left[ \left( \frac{\bar{w}_2 - \bar{w}_4}{2a} \right) + \mu \left( \frac{\bar{v}_1 - \bar{v}_3}{4a} \right) - \frac{\bar{w}_0}{R} \right] + \frac{t}{2a^2} \left\{ \left( \bar{w}_1 + \bar{w}_4 - 2\bar{w}_0 \right) + \mu \left( \bar{w}_1 + \bar{w}_3 - 2\bar{w}_0 \right) \right\} \tag{6.8.1}
\]

\[
\sigma_y = \frac{3H}{1-\mu^2} \left[ \left( \frac{\bar{w}_1 - \bar{w}_3}{2a} \right) - \frac{\bar{w}_0}{R} + \mu \left( \frac{\bar{u}_2 - \bar{u}_4}{4a} \right) \right] + \frac{t}{2a^2} \left\{ \left( \bar{w}_1 + \bar{w}_3 - 2\bar{w}_0 \right) + \mu \left( \bar{w}_2 + \bar{w}_4 - 2\bar{w}_0 \right) \right\}
\]

The shear stresses \( \tau_{xy} \) were not calculated since, as shown in Chapter 4, these stresses are small and, therefore, comparison of the results obtained by various methods would not yield reliable information.

The vertical and tangential stresses were calculated using equation 6.8.1 for the following cases:

(a) Using values of \( \bar{u}, \bar{v}, \) and \( \bar{w} \) obtained in Section 6.5 for \( \mu = 0. \)

(b) Using values of \( \bar{u}, \bar{v}, \) and \( \bar{w} \) obtained in section 6.5 but making \( \mu = 0.15 \) for calculating the stresses.

(c) Using values of \( \bar{u}, \bar{v}, \) and \( \bar{w} \) obtained from section 6.7 with \( \mu = 0. \) (allowing for the variation in the radius).

The results of these calculations, in lb/ins², for the arch dam having the dimensions given in Section 6.2, are
shown in Table 6.18 and 6.19 for both the air and water faces.

A graphical comparison between the calculated stresses using $\mu = 0$ and $\mu = 0.15$ (results (a) and (b) of Tables 6.18 and 6.19), starting with displacements calculated from $\mu = 0$ in both cases, is shown in Figs. 6.6 and 6.7. The graphs indicate that there is a large discrepancy in the vertical stresses over the upper portion of the dam and this result confirms the results obtained in Chapter 4 where the effect of using $\mu = 0.15$ for calculating the stresses was considered.

From Table 6.18 and 6.19 it can be seen that the stresses obtained for the constant radius (results a) and the variable radius (result c) are in close agreement.

CONCLUSIONS. (6.9)

1. Comparison of the non-dimensional displacements obtained for the coarse, fine and graded mesh (Table 6.4, 6.8 and 6.12 show that the maximum change occurs in the radial displacement at the crest of the dam. The change is of the same form as that obtained in Chapter 5 for the complete cylinder and it is reasonable to assume that the reduction of the mesh length increases the accuracy of the solution.

2. In order to determine the displacements the number of cycles of relaxation for the coarse, fine and graded mesh were 2, 3 and 1 respectively, but as stated in section 6.4 this could probably be reduced to 1, 3 and 1 since 1 cycle would have been sufficient at the coarse stage. The major part of the relaxation was concerned with reducing the $\bar{w}$-equation residuals, and it was found convenient to prepare a number
VERTICAL STRESSES [lb/ins²]
TANGENTIAL STRESSES [lb/ins²]

AIR FACE
- $\mu = 0.15$
- $\mu = 0$

WATER FACE
- $\mu = 0.15$
- $\mu = 0$

Fig 6.7
of different types of block patterns for the relaxation of this equation.

It is difficult to estimate the amount of time spent in the actual relaxation since, of necessity, the work was performed at short intervals over a long period, but for 50 points, starting from non-zero trial values, the relaxation could probably be completed in two months.

3. In Chapter 5, it was shown that for a complete cylinder a change in the value of \( \mu \) from 0 to 0.5 caused a large discrepancy in the calculated displacements, but comparison of results (b) and (c) of Table 6.16 indicates that the discrepancy is negligible for an arch dam having the dimensions given in Section 6.2. The error is reduced if only part of the cylinder is considered instead of the complete cylinder. The assumption that \( \mu = 0 \), for calculating the displacement would, therefore, appear to be justified.

The conclusion reached in Chapter 4, that \( \mu \) must be given its correct value when calculating the stresses from the displacements, is confirmed in Section 6.8.

4. The values of the displacements and stresses calculated assuming the radius is constant and the values obtained allowing for the variation in radius were found to be in close agreement so that an arch dam of variable radius could be analysed as a dam of constant radius without introducing a large error. The constant radius used should be the radius of the mid-surface at the crest of the dam.
### Table 6.1

**Coefficients of \( \bar{w} \)-Equation (Coarse Stage).**

<table>
<thead>
<tr>
<th>GRID</th>
<th>(-w_0)</th>
<th>(+(w_1+w_3))</th>
<th>(+w_2)</th>
<th>(+w_4)</th>
<th>(- (w_5+w_6))</th>
<th>(- (w_7+w_8))</th>
<th>(- (w_9+w_{11}))</th>
<th>(-w_{10})</th>
<th>(-w_{12})</th>
<th>(+ (v_1+v_3))</th>
<th>(+K)</th>
<th>(-w_0)</th>
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<tr>
<td>0</td>
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<td>+0.25</td>
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<td>-</td>
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<tr>
<td>8</td>
<td>6</td>
<td>1</td>
<td>1.666</td>
<td>2.333</td>
<td>+1.66'</td>
<td>+1.66'</td>
<td></td>
<td>-</td>
<td>5.680</td>
<td>3.777</td>
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<td>3.777</td>
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<tr>
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**Coefficients of \( \bar{v} \)-Equation (Coarse Stage).**

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<th>(+v_2)</th>
<th>(+v_4)</th>
<th>(- (u_1-u_3))</th>
<th>(+ (u_5-u_6-u_8)</th>
<th>(+u_7)</th>
<th>(- (w_1-w_3))</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
<td>-</td>
<td>4</td>
<td>+1</td>
<td>-</td>
<td></td>
<td>-</td>
</tr>
<tr>
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<td>6</td>
<td>1</td>
<td>1.5</td>
<td>2.5</td>
<td>+0.25</td>
<td>+0.25</td>
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<td>2</td>
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<td>1</td>
<td>1.666</td>
<td>2.333</td>
<td>+1.66'</td>
<td>+1.66'</td>
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<td>2.25</td>
<td>+0.125</td>
<td>+0.125</td>
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<td>6</td>
<td>1</td>
<td>1.80</td>
<td>2.20</td>
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### Table 6.3

Coefficients of $\bar{u}$-Equation (Coarse Stage).

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<th>$-u_0$</th>
<th>$(u_1+u_3)$</th>
<th>$u_2$</th>
<th>$u_4$</th>
<th>$(v_5-v_6-v_8+v_7)$</th>
<th>$-(u_9+u_{11})$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td></td>
<td>16</td>
<td>-</td>
<td>2.25</td>
</tr>
<tr>
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<td>18.0</td>
<td>1</td>
<td>6</td>
<td>10</td>
<td>+.5</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>18.0</td>
<td>1</td>
<td>6.666</td>
<td>9.333</td>
<td>+.5</td>
<td>-</td>
</tr>
<tr>
<td>12</td>
<td>18.0</td>
<td>1</td>
<td>7</td>
<td>9.0</td>
<td>+.5</td>
<td>-</td>
</tr>
<tr>
<td>16</td>
<td>18.0</td>
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<td>-</td>
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### Table 6.4

VALUES OF $\bar{u}$, $\bar{v}$ and $\bar{w}$ OBTAINED FROM COARSE MESH RELAXATION.
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<th>(+w_4)</th>
<th>(-w_5-w_6)</th>
<th>(-w_7-w_8)</th>
<th>(-w_9-w_{11})</th>
<th>(-w_{10})</th>
<th>(-w_{12})</th>
<th>((v_1-v_3))</th>
<th>(+K)</th>
<th>(-w_0)</th>
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<td>5.680</td>
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<td>0.606</td>
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<td>8.258</td>
<td>15.858</td>
<td>0.887</td>
<td>1.647</td>
<td>0.158</td>
<td>1.013</td>
<td>4.053</td>
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<td>3.731</td>
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<td>0.126</td>
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**TABLE 6.5**

Coefficients of \(\bar{w}\)-Equation (Fine Mesh).
<table>
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<th>GRID</th>
<th>$-v_0$</th>
<th>$(v_1+v_3)$</th>
<th>$+v_2$</th>
<th>$+v_4$</th>
<th>$-(u_1-u_3)$</th>
<th>$(u_5-u_6-u_8+u_7)$</th>
<th>$-(w_1-w_3)$</th>
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</thead>
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<td>1</td>
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<td>1</td>
<td>1,667</td>
<td>2,333</td>
<td>.167</td>
<td>.25</td>
<td>.1</td>
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**TABLE 6.6**

Coefficients of $\bar{v}$-Equation (Fine Mesh).
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<th>$+U_4$</th>
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<th>$-(U_9 + U_{11})$</th>
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<tbody>
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**TABLE 6.7**

*Coefficients of u Equation (Fine Mesh).*
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</table>

### TABLE 6.8

VALUES OF $\bar{u}$, $\bar{v}$ and $\bar{w}$ OBTAINED FROM FINE MESH RELAXATION.
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<th>$w_0$</th>
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<th>$-(w_7 + w_8)$</th>
<th>$-(w_9 + w_{11})$</th>
<th>$w_{10}$</th>
<th>$w_{12}$</th>
<th>$(v_1 - v_3)$</th>
<th>+k</th>
<th>$w_0$</th>
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<td>-</td>
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<td>.146</td>
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<td>20</td>
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<td>5.207</td>
<td>19.408</td>
<td>.110</td>
<td>2.130</td>
<td>-5.680</td>
<td>1.515</td>
<td>.808</td>
<td>.606</td>
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</table>

**TABLE 6.9**

Coefficient of $\bar{w}$-Equations (Graded Mesh).

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<th>$u_1 + u_3$</th>
<th>$u_2$</th>
<th>$u_4$</th>
<th>$(v_5 - v_6 - v_8 + v_7)$</th>
<th>$-(v_9 + v_{11})$</th>
</tr>
</thead>
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<td>17.588</td>
<td>-</td>
<td>.069</td>
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<td>9.600</td>
<td>.273</td>
<td>-</td>
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<tr>
<td>2</td>
<td>18</td>
<td>1</td>
<td>6.667</td>
<td>9.333</td>
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<td>-</td>
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**TABLE 6.10**

Coefficients of $\bar{u}$-Equations (Graded Mesh).
### Table 6.11

Coefficients of \( \bar{v} \)-Equations (Graded Mesh).

<table>
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<tr>
<th>Grid</th>
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<th>(v_2)</th>
<th>(v_4)</th>
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<th>((u_5-u_6-u_8+u_7))</th>
<th>(-(w_1-w_3))</th>
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Values for Grids 10 - 18 are identical with values given in Table 6.8.

### Table 6.12

Values of \( \bar{u}, \bar{v} \) and \( \bar{w} \) obtained from Graded Mesh Relaxation.
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<th>$+w_4$</th>
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<th>$-(w_7+w_8)$</th>
<th>$-(w_9+w_11)$</th>
<th>$-w_{10}$</th>
<th>$-w_{12}$</th>
<th>$+(v_1-v_3)$</th>
<th>$+(u_2-u_4)$</th>
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**Table 6.13**

Coefficients of $\bar{w}$ Equations. (Graded Mesh with $\kappa = 0.5$)
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<th>$(u_5-u_7-u_8)$</th>
<th>$-(w_1-w_3)$</th>
<th>$-(v_9+v_{11})$</th>
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**Table 6.14**

Coefficients of $\bar{v}$-Equations (Graded Mesh with $\mu = 0.5$)
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<th>$-(u_9+u_{11})$</th>
<th>$+w_0$</th>
<th>$+w_4$</th>
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**TABLE 6.15**

Coefficients of $\overline{u}$-Equation (Graded Mesh with $\mu = 0.5$)
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**Table 6.16**

Radial Displacements (Non Dimensional)
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**TABLE 6.17**

Altered Coefficients of $\bar{w}$-Equation.
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**TABLE 6.18**

Tangential and Vertical Stresses
(AIR FACE) lb/in$^2$.

(a) $\mu = 0$, Constant R.
(b) $\mu = 0.5$, Constant R.
(c) $\mu = 0$, Variable R.
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**TABLE 6.19**

Tangential and Vertical Stresses Water Face lb/sq.in.

(a) $\mu = 0$ Constant R
(b) $\mu = 0.5$ Constant R
(c) $\mu = 0$ Variable R.
CHAPTER 7.
MODEL EXPERIMENTS.

INTRODUCTION (7.1)

Although, ideally, the results obtained by application of a thin shell theory to the analysis of an arch dam should be compared with the results actually measured in the prototype, this procedure could not be adopted for the hypothetical case considered in Chapter 6 and it was thus decided to compare the results with those obtained by means of a scale model.

In order to design a scale model of an arch dam which will provide similar and measurable strains or displacements, a careful selection must be made of:

(a) the model material
(b) the method of applying the load
(c) the method of measuring the strains or displacements
(d) the scale.

The design and construction of a model, similar to the prototype considered in Chapter 6, is described in this chapter and the experimental results obtained from this model are compared with those obtained by means of a thin shell theory.

DESIGN OF MODEL. (7.2)

Since the purpose of the experimental work described in this thesis was to provide a standard by which the theoretical results could be compared, it was necessary that the design and construction of the model should follow methods which were
known to provide fairly accurate experimental results compared with those actually measured in a prototype.

Experimental investigations conducted by Rocha and Serafin in Portugal (32) indicate that the experimental results, measured by means of wire-resistance strain gauges on a model made of a plaster of Paris and diatomite mixture and loaded with either jacks or mercury, were in good agreement with those obtained by direct measurement from the actual dam which the models represented (31). It was thus decided to follow this technique and construct a model of plaster of Paris-diatomite and to use mercury for applying the load. The scale of the model had to be a minimum consistent with the condition that the strains induced would be measurable and thus it was necessary to select a dense loading medium such as mercury.

To predict the stresses and strains in a prototype from values measured in a model, it is necessary that certain similarity conditions are satisfied. Apart from geometrical similarity of model and prototype the model material must be perfectly elastic, homogeneous and isotropic if the material of the prototype is assumed to have these properties. Plaster-diatomite mixes can be moulded easily, do not shrink and are elastic for low stresses and strains. Rocha and Serafin have shown that for a plaster-diatomite mixture there is a variation of the modulus of elasticity with time of application of load and also an elastic after effect so that the strains are not removed immediately on removal of load. However,
from their results (32) it can be shown that the variation of the modulus is approximately 5% for a strain of $300 \times 10^{-6}$ and it was decided to design the model so that the strain did not exceed this figure and thus the variation could be neglected. To allow for the elastic after effect a time (approximately 20 minutes) must be allowed to elapse before successive load applications. The applied load and scale of the model must be such that the yield stresses of the model material are not exceeded. It is assumed by some authors that Poisson's ratio must be the same for both model and prototype, but this appears to be controversial and experiments have been made using models of rubber and plastics for which $\mu \geq 0.5$. In Ref. 13, Pippard et al show that the displacements measured by means of a rubber model compare favourably with the calculated values if a suitable correction factor is employed. The value of Poisson's ratio for plaster of Paris - diatomite ($\mu = 0.2$) is similar to the value assumed for the concrete ($\mu = 0.15$).

It can be shown that, if the above similarity conditions are fulfilled, the stress and strain ratios at corresponding points of the prototype and model are given by:

$$\frac{\sigma_p}{\sigma_m} = \frac{E_p}{E_m} \left(\frac{\rho_E}{\rho_m}\right)^n$$

\[ \text{(7.2.1)} \]

$$\frac{\epsilon_p}{\epsilon_m} = \frac{E_p}{E_m} \left(\frac{\rho_E}{\rho_m}\right)^n$$

\[ \text{(7.2.2)} \]

where $\frac{\rho_m}{\rho_p}$ is the ratio of the density of the loading medium on the model to that on prototype, $\frac{E_m}{E_p}$ the ratio of the moduli of elasticity and $\sqrt{n}$ the scale of the model.

The factors which must be considered in order to select suitable dimensions for the model are:
(a) the maximum tensile stress in the model must be less than the ultimate tensile stress of the model material. Approximate values of the ultimate tensile stress of a 2/1 mixture (by weight) of plaster of Paris and diatomite for different water contents are given in Ref. (32).

(b) the maximum strain in the model should be below $300 \times 10^{-6}$ so that the modulus of elasticity can be considered constant with respect to time of application of load. This also ensures that the strain limit of the gauges ($1,500 \times 10^{-6}$) is not exceeded. The maximum strain, however, should not fall much below $300 \times 10^{-6}$ or the lower strain occurring in the model would be too small to measure.

(c) the water/plaster ratio must be large enough to produce an easily workable mixture since it was intended to cast both the dam and foundations in one pouring.

(d) the scale is limited so that the model conforms to conditions (a) and (b) and also by practical limitations such as space available.

These factors are best considered by plotting graphs which enable all the variables to be considered simultaneously and in order to obtain convenient graphs for this purpose equations (7.2.1) and (7.2.2) must be re-written in the form:

$$\frac{C_p}{C_m} = \frac{C_p H}{\rho}$$

and

$$\frac{E_h}{E_m} = \frac{C_p H E_m}{\rho E_p}$$

where $\rho =$ pressure at the base of the model and $H =$ height of the prototype.

For the arch dam considered in Chapter 6, $H = 300$ ft;

$E_p = 2 \times 10^6 \text{ lb/ft}^2$ and $\rho_p = 62.5 \text{ lb/ft}^3$ so that equations (7.2.3) become
In Fig. 7.1 equations (7.2.4) are plotted for values of $E_m$ within the range 57,000 - 568,000 lb/in$^2$ which is the range given by Roche and Serafin for water ratios varying from 3.0 to 0.8. Since the ultimate tensile stress can also be obtained from Ref. (32) it is possible to calculate the limiting value of the ratio $\lambda$ for each value of the modulus of elasticity.

where $\lambda = \frac{\text{maximum tensile stress expected in the prototype}}{\text{Ultimate tensile stress of model material}}$.

It is necessary to assume a value of the maximum tensile stress expected in the prototype but for the example considered in Chapter 6 a value of +275 lb/sq.in. was deduced from the experimental results obtained at Imperial College was available. The limiting values of $\lambda$ are indicated in Fig. 7.1 and using these values it is possible to obtain a 'design region' within which the model parameters must lie in order that the model material does not crack.

Since, for the experiments conducted by the author, the loading medium was to be mercury, selection of a suitable pressure at the base would fix the depth of the model and, therefore, the scale.

If the maximum stress (compressive) in the prototype is assumed to be 4.00 lb/sq.in., then since $E_p = 2 \times 10^6$ lb/sq.in., the maximum strain $\varepsilon_p$ will be approximately $200 \times 10^{-6}$ and thus in order that $E_m < 300 \times 10^{-6}$ it is
necessary that \( \frac{E_m}{E_p} < \frac{3}{2} \). The maximum prototype stress must, in general, be estimated from previous experience with similar dams and for the case considered above the stress values obtained by Richmond (33) by means of the rubber model were used for obtaining the maximum value of 400 lb/sq.in.

Trials with various water/plaster ratios demonstrated that a 2/1 ratio produced a fairly easily workable mixture and with this water content the value of \( E_m \) given in Ref. (32) is 112,000 lb/sq.in. This value of \( E_m \) was accepted for the preliminary design although tests conducted at a later stage on the actual material used for casting the dam showed that the value of \( E_m \) was slightly larger than this figure. (Table 7.2)

From Fig. 7.1 the value of the pressure required at the base of the model to conform with the above was found to be 11 lb/sq.in. and since this lies within the design region is also safe from maximum tensile stress consideration.

The actual height of the model decided upon was 20" so that the pressure at the base was to be 9.85 lb/sq.in. and the scale of the model 1/180.

PRELIMINARY INVESTIGATIONS. 7.3

The casting of the dam and foundation was to be undertaken in one pouring. Since the plaster-diatomite mixture takes up its initial set after only 8 minutes, it was decided that it would be convenient if the setting time could be retarded and some preliminary investigations were made in order to test the effect of various retarding agents. A number of beams were cast using different quantities of
**Design Graph for Model**

\[ \lambda = \frac{\text{Maximum Tensile Stress in Prototype (}\sigma_{\text{p}}\text{)} \text{Ult. Tensile Stress of Model Material (}f_{\text{m}}\text{)}}{1} \]

---

**Constants**
- \( H = 300 \text{ ft.} \)
- \( \sigma_{\text{p}} = 62.5 \text{ lb/ins}^2 \)
- \( E_p = 2 \times 10^6 \text{ lb/ins}^2 \)
- \( f_{\text{m}} = +275 \text{ lb/ins}^2 \)

---

**Design Region**

**Fig 7.1**
household size and although the setting was retarded by its addition, it produced a large number of bubbles through the beams and there was a tendency for their surfaces to adhere to the moulds. The setting time was increased to 20 minutes with a 1.3% mixture (by weight) of size and plaster of Paris and to 100 minutes with a 2.5% mixture. Cream of tartar was also used as a retarding medium, but it was found that although the setting time was only slightly increased the strength of the mixture was considerably reduced. The retarders were considered unsatisfactory and it was decided to organise the available labour so that the casting of the complete dam could be performed in one short operation before the mixture began to set.

To obtain the value of the modulus of elasticity of the material, beams 14" x 1½" x 1½" were cast and tested in pure bending. (Fig. 7.2). Strains were measured by means of wire-resistance strain gauges placed at the centre of the beams and tests were made with the gauge in both the compressive and tensile positions.

CONSTRUCTION OF THE MODEL, 7.4

The three views of the model arch dam and the Isometric Projection of the concrete supporting tank is shown in Fig. 7.3 and in this figure it can be seen that the foundations and abutments of the dam were to be large blocks of plaster-diatomite monolithic with the actual arch. It was assumed that the use of the large block would ensure that the edge line of the model
Profile as cast

Profile obtained after removal of surface layer

Section AA

Section BB

Details of Model and Surround

Isometric projection of concrete surround with part of left abutment removed

Fig 75
would be perfectly rigid and, therefore, conform with the assumptions made for the theoretical calculations in Chapter 6. However, comparison of the stresses deduced from the model with the theoretical results obtained by application of a thin shell theory indicated that this method of fixing the abutments and foundations was not adequate enough and another method of fixing the foundations was employed (See section 7.6) with a consequent improvement in the results.

It is convenient to consider the construction of the model divided into five stages:

(a) **Casting of the Concrete Surround.** The timber and hardboard formwork constructed for the shuttering of the concrete is shown in Fig. 7.4 and the concrete surround after removal of the shuttering is shown in Fig. 7.3 and Fig. 7.5.

(b) **Construction of Mould for Arch Dam.** Using the bolts shown in Fig. 7.5 wooden brackets were fixed to support the hardboard shuttering which was designed so that the cast dam would be slightly larger than the required model (Fig. 7.3). The faces of the hardboard which were to be in contact with the plaster-diatomite were shellaced to prevent any sticking.

(c) **Casting of the Dam.** The correct quantities of plaster of paris and diatomite were thoroughly dry mixed and separated into two equal parts and the required amount of water for each part was placed into two large containers. The plaster-diatomite mixes were then added to the water, which was stirred briskly for four minutes and then poured
into each abutment until the mould was completely full. During pouring, which took approximately two minutes, the material in the mould was stirred to remove any entrapped air and the mixture was then left undisturbed until set. During the casting, samples of the material from each container were taken and cast into six beams which were to be used for the determination of Young's modulus.

(d) Final Shaping of the Dam. The shuttering was removed after seven days and the dam was then left to dry for a period of three months in a thermostatically controlled room temperature of 68°F ± 2°F. Steel templates were cut to fit the required dam profile and the curve of the upstream face, and the dam was carved by hand until the required shape as gauged by the templates was obtained. (Fig. 7.6 and 7.7). For dams with more complicated sections it would probably be more convenient to use specially designed mechanical carving instruments but for the simple linear profile used by the author it was felt that carving by means of hand tools would provide the required shape in a short time. After carving the material was moisture-proofed with the mixture of Durofix and Acetone.

(e) The procedure adopted by Rocha and Serafim for waterproofing the material was to first coat the surface with shellac and then apply a waterproofing wax. The author was of the opinion, however, that the application of shellac plus the two coats of adhesive necessary for sticking the gauges to the material would result in a relatively thick separating layer between gauges and the surface of the material and after some initial tests it was decided to apply a coat of Durofix
which had been thinned down with acetone. The Durofix-Acetone solution was sprayed on to the surface with a paint sprayer and after two thin coats had been applied tests showed that the material was waterproof. The strain gauges could now be stuck to the surface using only one coat of strain gauge glue so that the resulting separating layer between material and gauge was very thin.

METHODS OF LOADING AND STRAIN MEASUREMENT (7.5)

For applying the load to the dam a rubber bag, which was to be filled with mercury, was placed between the surface of the dam and a suitably curved steel sheet. The rubber bag was open at the top and connected from the base by rubber tube to a container. (Fig. 7.8 and 7.9). The level of mercury could be controlled by raising or lowering the container and thus the load could be removed entirely by placing the container on the ground.

The distance between the steel plate and the surface of the dam was kept to a minimum for economic reasons since extra 1" between plate and dam would require an extra 100 cu.in. of mercury which at the present time would cost approximately £100.

The wire resistance strain gauges were attached to the prepared surface of the model in horizontal and vertical positions as indicated in Fig. 7.10. Fifty eight gauges were placed on the downstream face, but only twelve gauges were attached to the upstream face since the connecting leads from the gauges to the measuring instrument might have interfered with the loading system. Due to the symmetry of the model most of the gauges were placed to one side of the centre
Gauges on water face are
on Q only and at similar depths

Position of gauges on air face

Fig 7.10
line, but four gauges were attached on the other side so as to check that the dam was, in fact, behaving as a symmetrical structure. The twelve gauges placed on the upstream face were placed on the centre line at the same vertical depth as the corresponding gauges on the centre line of the downstream face. The gauges are shown in Figs. 7, 11 and 7.12.

With the strain measuring equipment available, only four gauges could be balanced at a time and it was necessary to load, record and unload for each set of four gauges. After the results had been obtained from a set of gauges, three were disconnected and the fourth left to act as a control gauge for the next set. A different control gauge was used for each group and the measuring procedure was repeated three times unless the results for any particular gauge differed by more than $10 \times 10^{-6}$ in which case the measurements were again repeated. All readings were taken when the temperature of the room was between $66^\circ$ and $68^\circ$ F. The readings were taken at approximately the same time after loading and at least 20 minutes was allowed to elapse between successive loadings. During preliminary tests the load was applied continuously over a period of 6 hours, but no change was recorded in the strain measurements taken during this period so that there was no measurable change in the modulus of elasticity with time.

**Modification to the Foundation.** (7.6).

Comparison of the stresses deduced from the model with the values obtained by application of a thin shell
theory indicated that the foundation of the model was not perfectly rigid, as assumed in the theory, and it was decided to modify the foundations in an attempt to increase the rigidity.

The plaster of Paris – diatomite foundations were first cut away both upstream and downstream of the dam so that the only part that remained was the portion directly under the dam. This remaining part of the foundation was then drilled at four points and \( \frac{3}{4} \) dia. bolts inserted horizontally as shown in Fig. 7.13. The part of the foundation that had been removed was replaced with concrete leaving access to one end of each bolt to enable the nuts to be tightened, and any shrinkage gaps between concrete and plaster to be closed. In order to ensure that the strains induced in the lower portion of the dam were not excessive, four of the lowest gauges were connected to the measuring instruments and observed whilst the nuts were being tightened. The tightening was stopped as soon as strains were induced and it was found that quite a large force had to be exerted before this condition was produced.

The strain measurements are shown in Table 7.1 where A refers to the measurements obtained with the original model and B the measurements taken after modification of the base. The measured strains are also compared graphically in Figs. 7.14 and 7.17. The most significant effect of the modification occurs near the base of the dam where there is an increase in the vertical strains and a decrease in the horizontal strains.
MEASURED VERTICAL STRAINS x 10^6

Results A
Results B

FIG 7.14
Measured Vertical Strains \times 10^6

Results A
Results B

FIG 7.15
MEASURED HORIZONTAL STRAINS × 10^6

RESULTS A
RESULTS B

VERTICAL GRID 0
AIR FACE

VERTICAL GRID 2
AIR FACE

VERTICAL GRID 4
AIR FACE

FIG 7:0
Measured Horizontal Strains $\times 10^6$

Results A
Results B

Vertical Grid 6
Air Face

Vertical Grid 8
Air Face

Vertical Grid 0
Water Face

Fig 7:17
Of the six beams that were cast simultaneously with the casting of the dam only three were suitable for the determination of Young's Modulus of Elasticity. The surfaces of two of the beams were rough and pitted due to the sticking of these surfaces to the mould and another of these beams broke during handling. The value of E found experimentally for the remaining three beams were in agreement to within 7% and an average value of these three readings was used for determining the stresses. The values of Young's Modulus obtained for each beam is shown in Table 7.2.

**CALCULATIONS OF THE STRESSES. (7.3)**

In order to calculate the vertical and horizontal stresses occurring at any point in the model, it is necessary to know the values of vertical and horizontal strains at that point. Using graphs 7.14 and 7.17 the values of the strains occurring at depths of 2, 6, 10, 14 and 18 ins. from the crest were interpolated and these values were used for determining the stresses at these points. The equations required for these calculations were:

\[
\sigma_x = \frac{E_m}{1-\mu^2} \left( \varepsilon_{xx} + \mu \varepsilon_{yy} \right) \\
\sigma_y = \frac{E_m}{1-\mu^2} \left( \varepsilon_{yy} + \mu \varepsilon_{xx} \right)
\]

In chapter 6 the stresses were calculated using a thin shell theory for a hypothetical prototype 300 ft. high, and in order to compare these stresses with the prototype stresses deduced from the model the values obtained using equations 7.3.1 must be substituted into equation (7.2.1).
With this substitution the prototype stresses become:

\[
\sigma_x = n \frac{C_p}{C_m} \left( \frac{E_m}{1 - \mu^2} \right) (G_{xx} + \mu G_{yy}) \\
\sigma_y = n \frac{C_p}{C_m} \left( \frac{E_m}{1 - \mu^2} \right) (G_{yy} + \mu G_{xx})
\]

Since \( \frac{C_p}{C_m} = \frac{1}{13.6} \); \( n = 180 \); \( E_m = 131,000 \text{ lb/in}^2 \) and \( \mu = 0.2 \),

equation 7.8.2 becomes

\[
\sigma_x = 1.8 \left\{ G_{xx} + 0.2 G_{yy} \right\} \\
\sigma_y = 1.8 \left\{ G_{yy} + 0.2 G_{xx} \right\}
\]

The prototype stresses obtained using (7.8.3) are given in Table 7.3 and these values are compared graphically in Fig. 7.18 - 7.22 with the values obtained in Chapter 6 (Table 6.18(b)).

The stresses deduced for a similar prototype using the displacements measured in a rubber model are given in Table 7.4. These values obtained by Richmond are compared with the values obtained from the plaster-diatomite model and shell theory in Figs. 7.23 - 7.24 for vertical Grid 0, 4 & 8.

In obtaining these stresses from the rubber model no allowance or correction factor was used for the difference in the values of Poisson's ratio in the model and prototype.

CONCLUSIONS. 7.9

1. The stresses deduced from the strain measurements obtained from the plaster-diatomite model are in close agreement with the strains calculated using a thin shell theory. The largest discrepancy between the results occurs in the vertical stresses at the base of the dam, but since the dam is thick at the base a thin shell theory becomes less applicable.
VERTICAL STRESSES DEDUCED FROM MODEL

[UNITS 10^6 lbf/ft^2  TENSION +]

RESULTS A  --.--
RESULTS B  X---
SHELL THEORY  +---+

VERTICAL GRID 0  AIR FACE
VERTICAL GRID 2  AIR FACE

Fig 7/8
VERTICAL STRESSES DEDUCED FROM MODEL
[UNITS 1b/in³ TENSION +]

RESULTS A
RESULTS B
SHELL THEORY

VERTICAL GRID A
AIR FACE

VERTICAL GRID B
AIR FACE

FIG 7:19
VERTICAL STRESSES DEDUCED FROM MODEL

[UNITS lb/ins², TENSION +]

RESULTS A
RESULTS B
SHELL THEORY

VERTICAL GRID B
AIR FACE

VERTICAL GRID C
WATER FACE

FIG 7.20
Tangential Stresses Deduced From Model

[Units 10/ins² Tension ±]

Results A
Results B
Shell Theory

VERTICAL DEPTH - GRID UNITS

VERTICAL DEPTH - GRID UNITS

VERTICAL DEPTH - GRID UNITS

VERTICAL GRID 0
AIR FACE

VERTICAL GRID 2
AIR FACE

VERTICAL GRID 4
AIR FACE

FIG 7:21.
Tangential Stresses Deduced from Model
[Units 10/ins² Tension +]

Results A
Results B
Shell Theory

Fig 7.22
and such a discrepancy was expected. It is doubtful whether any existing method of analysis would yield the 'correct' solution at the base of a thick dam.

At a depth of 0.6 \( H \) from the crest, \( \frac{t}{R} \) reaches the limiting value of 1/5th and it can be seen from the graphs (Fig. 7.18 to Fig. 7.22) that the discrepancy over this depth is negligible. As \( \frac{t}{R} \) becomes \( > 1/5 \) the error increases and thus for the example considered in this thesis the Lombardi limit of \( \frac{t}{R} < 1/5 \), for the applicability of a thin shell theory to an arch dam would appear to be justified.

2. The vertical stresses deduced from the rubber model are in close agreement with those obtained by means of the thin shell theory for the central portion of the dam, but nearer the abutments the discrepancy increases. (Fig. 7.23).

Comparison of the tangential stresses (Fig. 7.24) deduced from the rubber model with those obtained using the thin shell theory indicate that there is a large discrepancy which is again largest near the abutments and the foundations, and of such a magnitude as to suggest a lack of fixity in the rubber model.
VERTICAL STRESSES [lbf/ins²]
[TENSION +]

Plaster Model
Rubber Model
Shell Theory

VERTICAL GRID 0
AIR FACE

VERTICAL GRID 4
AIR FACE

VERTICAL GRID 8
AIR FACE

FIG 7.23
TANGENTIAL STRESSES [lb/ins²] (TENSION)

Vertical Grids

- Air Face

Plaster Model
Rubber Model
Shell Theory

Fig 7.24
<table>
<thead>
<tr>
<th>GRID Depth (ins.)</th>
<th>AIR FACE 8 A</th>
<th>AIR FACE 6 A</th>
<th>AIR FACE 4 A</th>
<th>AIR FACE 2 A</th>
<th>AIR FACE 4 B</th>
<th>WATER FACE 4 A</th>
<th>WATER FACE 4 B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.50</td>
<td>0.031</td>
<td>0.032</td>
<td>0.037</td>
<td>0.047</td>
<td>0.045</td>
<td>0.052</td>
<td>0.046</td>
</tr>
<tr>
<td>2.25</td>
<td>-0.154</td>
<td>-0.146</td>
<td>-0.132</td>
<td>-0.130</td>
<td>-0.142</td>
<td>-0.136</td>
<td>-0.149</td>
</tr>
<tr>
<td>4.50</td>
<td>0.051</td>
<td>0.051</td>
<td>0.067</td>
<td>0.078</td>
<td>0.068</td>
<td>0.081</td>
<td>0.076</td>
</tr>
<tr>
<td>5.25</td>
<td>-0.163</td>
<td>-0.145</td>
<td>-0.138</td>
<td>-0.122</td>
<td>-0.146</td>
<td>-0.133</td>
<td>-0.145</td>
</tr>
<tr>
<td>8.00</td>
<td>-0.050</td>
<td>-0.051</td>
<td>0.069</td>
<td>0.067</td>
<td>0.077</td>
<td>0.079</td>
<td>0.084</td>
</tr>
<tr>
<td>8.75</td>
<td>-0.146</td>
<td>-0.118</td>
<td>-0.110</td>
<td>-0.085</td>
<td>-0.141</td>
<td>-0.117</td>
<td>-0.131</td>
</tr>
<tr>
<td>11.50</td>
<td>0.031</td>
<td>0.022</td>
<td>0.041</td>
<td>0.032</td>
<td>0.053</td>
<td>0.038</td>
<td>0.050</td>
</tr>
<tr>
<td>12.25</td>
<td>-0.108</td>
<td>-0.078</td>
<td>-0.072</td>
<td>-0.045</td>
<td>-0.108</td>
<td>-0.078</td>
<td>-0.098</td>
</tr>
<tr>
<td>15.00</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-0.004</td>
<td>-0.042</td>
<td>-</td>
</tr>
<tr>
<td>15.75</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-0.061</td>
<td>-0.025</td>
<td>-</td>
</tr>
<tr>
<td>18.50</td>
<td>-0.067</td>
<td>-1.111</td>
<td>-0.081</td>
<td>-1.111</td>
<td>-0.091</td>
<td>-1.144</td>
<td>-0.094</td>
</tr>
<tr>
<td>19.25</td>
<td>-0.032</td>
<td>0.011</td>
<td>-0.030</td>
<td>0.021</td>
<td>-0.005</td>
<td>0.038</td>
<td>-0.013</td>
</tr>
</tbody>
</table>

**Table 7.1**

Vertical and Horizontal Strains \(x \times 10^6\).

A. - Plaster-diatomite base.

B. - Concrete Base.
<table>
<thead>
<tr>
<th>Beam</th>
<th>Test</th>
<th>Compression Loading</th>
<th>Compression Unloading</th>
<th>Tension Loading</th>
<th>Tension Unloading</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>130,000</td>
<td>128,000</td>
<td>135,000</td>
<td>125,000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>135,000</td>
<td>135,000</td>
<td>131,000</td>
<td>130,000</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>127,000</td>
<td>130,000</td>
<td>125,000</td>
<td>127,000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>125,000</td>
<td>128,000</td>
<td>125,000</td>
<td>127,000</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>137,000</td>
<td>138,000</td>
<td>135,000</td>
<td>136,000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>136,000</td>
<td>138,000</td>
<td>135,000</td>
<td>137,000</td>
</tr>
</tbody>
</table>

Average Value = 131,000 lb/ins².

**Table 7.2**

*Young's Modulus of Elasticity of Model Material.*
<table>
<thead>
<tr>
<th>GRID</th>
<th>AIR FACE</th>
<th></th>
<th></th>
<th>2</th>
<th>WATER FACE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>A B</td>
<td>A B</td>
<td>ε</td>
<td>A B</td>
</tr>
<tr>
<td>8</td>
<td>7 11</td>
<td>32 52</td>
<td>36 54</td>
<td>43</td>
<td>47 72</td>
</tr>
<tr>
<td>6</td>
<td>-257 -241</td>
<td>-226 -214</td>
<td>-221 -214</td>
<td>-234</td>
<td>-244 -230</td>
</tr>
<tr>
<td>6</td>
<td>45 54</td>
<td>85 103</td>
<td>83 104</td>
<td>103</td>
<td>117 99 123</td>
</tr>
<tr>
<td>6</td>
<td>-274 -234</td>
<td>-216 -182</td>
<td>-242 -212</td>
<td>-226</td>
<td>-240 -216</td>
</tr>
<tr>
<td>6</td>
<td>-232 -178</td>
<td>-160 -113</td>
<td>-212 -171</td>
<td>-190</td>
<td>-200 -155</td>
</tr>
<tr>
<td>6</td>
<td>-159 -110</td>
<td>-99 -52</td>
<td>-151 -103</td>
<td>-137</td>
<td>-126 -94</td>
</tr>
</tbody>
</table>

**TABLE 7.3**

STRESSES DEDUCED FROM MEASURED STRAINS (UNITS lb/sq.in.) (TENSION +)
<table>
<thead>
<tr>
<th>GRID</th>
<th>( \sigma_x )</th>
<th>( \sigma_y )</th>
<th>( \sigma_z )</th>
<th>( \sigma_{xy} )</th>
<th>( \sigma_{xz} )</th>
<th>( \sigma_{yz} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-204 -279</td>
<td>-194 -259</td>
<td>-194 -259</td>
<td>-000 000</td>
<td>-197 -157</td>
<td>-240 -133</td>
</tr>
<tr>
<td>2</td>
<td>-166 -345</td>
<td>-166 -332</td>
<td>-175 -312</td>
<td>-175 -312</td>
<td>-175 -312</td>
<td>-175 -312</td>
</tr>
<tr>
<td>6</td>
<td>-140 -385</td>
<td>-134 -389</td>
<td>-156 -360</td>
<td>-156 -360</td>
<td>-156 -360</td>
<td>-156 -360</td>
</tr>
<tr>
<td>12</td>
<td>-150 -279</td>
<td>-144 -258</td>
<td>-141 -249</td>
<td>-141 -249</td>
<td>-141 -249</td>
<td>-141 -249</td>
</tr>
<tr>
<td>14</td>
<td>-168 -201</td>
<td>-152 -182</td>
<td>-144 -198</td>
<td>-144 -198</td>
<td>-144 -198</td>
<td>-144 -198</td>
</tr>
<tr>
<td>16</td>
<td>-183 -127</td>
<td>-181 -130</td>
<td>-150 -069</td>
<td>-150 -069</td>
<td>-150 -069</td>
<td>-150 -069</td>
</tr>
<tr>
<td>18</td>
<td>-342 275</td>
<td>-312 -229</td>
<td>-280 165</td>
<td>-280 165</td>
<td>-280 165</td>
<td>-280 165</td>
</tr>
</tbody>
</table>

**TABLE 7.4**

Stresses deduced from Rubber Model by Richmond (Ref. 33)

(Units lb/sq. in. Tension +)

а. = Air Face

b. = Water face.
CHAPTER 8.
CONCLUSIONS.

From the results obtained and the methods of analysis and solution used in the previous chapters, for arch dams subjected to hydrostatic pressure only and having a constant thickness over horizontal planes, the following conclusions may be drawn.

(1) The equations obtained from the application of a thin shell theory to an arch dam when solved by Southwell's relaxation technique and using the methods described in this thesis lead to a reasonably accurate solution in a comparatively short time.

This conclusion is reached after comparing the results obtained from the application of a thin shell theory with:-

(a) The results obtained by means of more exact analysis by Pippard et al.

(b) The results obtained by the author from measurements with a plaster of Paris-diatomite model loaded with mercury.

Comparison (a) indicated that there were discrepancies in the results over the upper portion of the dam, but this was shown to be due to the size of the mesh used in the relaxation process. It would appear that for a constant mesh length throughout the depth of the dam, the error near the crest will be larger than the error over the remainder of the dam.

Comparison (b) indicated that there was good agreement
between the results and since the experimental technique employed for obtaining the strains from a model has been shown by previous authors to yield accurate results it is reasonable to assume that the agreement implies that both are accurate.

(2) The use of a graded mesh near the crest of the dam increases the accuracy of the solution without much increase in the amount of labour involved in obtaining this solution. This result was first established for the case of a complete cylinder subjected to external water pressure for which an analytical solution was known, and then applied to the arch dam in Chapter 6 with a consequent improvement in the results.

(3) Assuming that the thin shell equations applied to arch dams are expressed in terms of the three displacements $u$, $v$ and $w$ then, for calculating these displacements the value of Poisson's ratio can be neglected, but in calculating the stresses from these displacements the correct value of Poisson's ratio should be used. Similarly the vertical displacement $u$ should not be neglected when calculating the stresses.

These conclusions are drawn from the results of the comparisons, made in Chapter 4, between the stresses calculated for the different conditions, and are applied in Chapter 6 to the calculation of the stresses for the dam in the rectangular valley.

(4) For the purpose of calculating the displacements and stresses by the methods outlined, the radius of the dam may be assumed to be constant throughout the depth of the dam at
a value equal to the radius of the mid-surface at the crest without introducing a large error.

(5) The vertical stresses deduced from a rubber model by Richmond (33) are in close agreement over the central portion of the dam with those obtained from a plaster of Paris-diatomite model, but there is a large discrepancy between the tangential stresses obtained from these models.

(6) The limit \( \frac{t}{R} < \frac{1}{5} \), given by Lombardi, below which an arch dam can be considered as thin is verified in this thesis, for, over the upper portion of the dam where \( \frac{t}{R} < \frac{1}{5} \) the vertical and tangential stresses calculated from the application of a shell theory are in close agreement with the values deduced from the model and it is only after this depth that a discrepancy appears in the results.

Since the above calculations have been derived for particular cases, it may be objected that generalisations should not be made, but the author is of the opinion that the sections of the dams considered are typical of actual practical dams and with the modern tendency of constructing arch dams with even slenderer sections the above conclusions become even more applicable.

It is convenient to summarise the condition for which the above conclusions have been derived.

(a) The only loads considered were the water load and no calculations were made to allow for the effect of temperature variation. This might form the subject for future research, but it was considered to be too lengthy for inclusion in the present study.
(b) The thickness of the dam must be constant over horizontal planes, for, if the thickness varies with $y$, the displacement equations would change for each node of the horizontal as well as the vertical grid lines and this would involve extra difficulties and labour which have not been considered. However, since the variation of the thickness with $y$ in practice would be small it may be possible to ignore the variation without introducing large errors.

(c) Both the arch dams considered in this thesis were symmetrical about a vertical centre line but for the practical cases for which such symmetry cannot be assumed the methods outlined in thesis would still apply but the dam would take longer to analyse because the displacements would have to be calculated at a greater number of nodes.
\[ K_i(p) = P_1 \times \frac{2}{3} \Delta x + P_2 \times \frac{1}{3} + P_3 \times \frac{1}{2} = \frac{4}{3} + \frac{1}{2} (p_2 + p_3) \]

\[ \frac{1}{2} \left( \Delta x \times p_{i-1} \right) \]

\[ p_2 = \frac{1}{2} \times p_{i-1} \times \Delta x \]

\[ p_1 = \frac{1}{2} \times p_i \times \Delta x \]

\[ p_3 = \frac{1}{2} \times p_{i+1} \times \Delta x \]

\[ = \frac{4}{3} \times \frac{1}{2} p_i \Delta x \]

\[ + \frac{1}{2} \left( \frac{1}{2} \times p_{i-1} \Delta x + \frac{1}{2} \times p_{i+1} \Delta x \right) \]

\[ = \frac{\Delta x}{6} \left( p_{i-1} + 4p_i + p_{i+1} \right) \]

\[ \rightarrow \frac{\Delta x}{12} \left( 2p_{i-1} + 8p_i + 2p_{i+1} \right) \]

\[ \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} \times \Delta x + \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} \times \Delta x \]

\[ = \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} \times \Delta x \left( s_1 + s_2 \right) \]

\[ = \frac{\Delta x}{12} \left( p_i - p_{i-1} \frac{3}{2} + p_1 - p_{i+1} \right) \]

\[ = \frac{\Delta x}{12} \left( -p_{i-1} + 2p_i - p_{i+1} \right) \]

\[ \therefore K_i(p) = \tilde{K}_i(p) + \Delta K_i(p) = \frac{\Delta x}{12} \left( p_{i-1} + 10p_i + p_{i+1} \right) - (2g) \]
LIST OF SYMBOLS.

\( u, v, w \)  
Component displacements of a point in the middle surface in the direction \( x, y \) and \( z \) respectively.

\( x, y, z \)  
Right handed system of co-ordinates - axial, circumferential and radial direction respectively.

\( X, Y, Z \)  
External loads in direction \( x, y, z \).

\( \varepsilon, \varepsilon_2 \)  
Middle surface direct strains in \( x \) and \( y \) directions.

\( \omega \)  
Middle surface shear strain.

\( \chi, \chi_1 \)  
Change of curvature of middle surface in directions \( x \) and \( y \).

\( \tau \)  
Twist of middle surface.

\( \varepsilon_x, \varepsilon_y \)  
Direct strains.

\( \gamma_{xy} \)  
Shear strain.

\( \sigma_x, \sigma_y, \sigma_3 \)  
Direct stresses

\( \tau_{xy}, \tau_{yx} \)  
Shear stresses

\( N_x, N_y \)  
Direct forces.

\( N_{xy}, N_{yx}, Q_x, Q_y \)  
Shear forces

\( M_{xx}, M_{yy}, M_{xy}, M_{yx} \)  
Bending moments.

\( Q_x, Q_y \)  
Twisting moments.

\( \gamma \)  
Specific weight of water.

\( H \)  
Maximum height of dam.

\( e_m/e_p \)  
Ratio of density of the loading medium on the model to that on the prototype.

\( \lambda \)  
Scale of model

\( \lambda \)  
Max. tensile stress expected in the prototype U.T.S. of model material.

\( a, b \)  
Mesh lengths in \( x \) and \( y \) directions.

\( \mu \)  
Poisson's ratio.
R  
Radius of curvature.

T  
Thickness

E, G.  
Moduli of Elasticity and Rigidity.

( )'  
Differentiation with respect to x.

( )''  
Differentiation with respect to y.

D  
\[ \frac{E t}{1-\mu^2} \]

K  
\[ \frac{E t^3}{12(1-\mu^2)} \]

T  
\[ 2(t')^2 + t t'' \]

\[ \frac{1}{t} \]

u, v, w.  
Non dimensional displacement expressions,

\[ \left( \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial w}{\partial x} \right) \] respectively
BIBLIOGRAPHY.


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