Restriction & Isoperimetric Inequalities in Harmonic Analysis

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. The work has not been submitted for any other degree or professional qualification.

(*Stephen Harris*)

To my girls, Helen and Lucy

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Lay Summary

This thesis looks at two related inequalities that arise in Harmonic Analysis: restriction type inequalities and isoperimetric inequalities.

Restriction type inequalities have been the subject of much interest since they were first conceived in the 1960s. In this thesis we look at the restriction estimate for curves where the curvature vanishes. In particular we look at homogeneous polynomial surfaces and build upon an existing result.

Of the two types, isoperimetric inequalities are the more readily understood. The classical problem is, given a string of fixed length, which shape would enclose the largest area, from which we derive the term *isoperimetric* (literally, “having the same perimeter”). The answer, as was known to the Ancient Greeks, is a circle. A variation on this problem is the *affine isoperimetric inequality*, where the ‘length’ of the perimeter is measured not by the usual arclength but by the affine arclength. The affine arclength is related to the curvature of the curve and, like arclength, is preserved by translations and rotations, but unlike arclength is also preserved by stretching where area is preserved. As an example, all ellipses of the same volume have equal affine arclength. The problem is now to *maximise* the affine arclength of the perimeter used to enclose a fixed volume. In two dimensions this problem is well understood, and the solution is any ellipse with the given volume. We consider the problem of curves in higher dimensions, where the enclosed volume is taken to be the convex hull of the curve.

A related problem is one inspired by the tale of Dido of Carthage. Fleeing her tyrant brother, she arrived in North Africa and asked for a small plot of land: that which can be encompassed by an oxhide. When this was agreed she cut the oxhide into strips and made a long perimeter. The problem she then faced was how to encompass the largest area. This is known as the *relative isoperimetric inequality* and differs subtly from the classical problem as we can now make use of the coast in forming the shape. In this thesis we consider the *affine relative isoperimetric inequality*: this is a similar problem, but we don’t measure the arclength of the perimeter, but the affine arclength.

Abstract

We study two related inequalities that arise in Harmonic Analysis: restriction type inequalities and isoperimetric inequalities.

The (L^p, L^q) Restriction type inequalities have been the subject of much interest since they were first conceived in the 1960s. The classical restriction type inequality involving surfaces of non-vanishing curvature is only fully resolved in two dimensions and there have been a lot of recent developments to establish the conjectured (p,q) range in higher dimensions. However, it also interesting to consider what can be said for curves where the curvature does vanish. In particular we build upon a restriction result for homogeneous polynomial surfaces, using what is considered the natural weight - the one induced by the affine curvature of the surface. This is known to hold with a non-universal constant which depends in some way on the coefficients of the polynomial. In this dissertation we shall quantify that relationship.

Restriction estimates (for curves or surfaces) using the affine curvature weight can be shown to lead to an affine isoperimetric inequality for such curves or surfaces. We first prove, directly, this inequality for polynomial curves, where the constant depends only on the degree of the underlying polynomials. We then adapt this method, to prove an isoperimetric inequality for a wide class of curves, which includes curves for which a restriction estimate is not yet known.

Next we state and prove an analogous result of the relative affine isoperimetric inequality, which applies to unbounded convex sets. Lastly we demonstrate that this relative affine isoperimetric inequality for unbounded sets is in fact equivalent to the classical affine isoperimetric inequality.

Contents

Chapter 1

Introduction

It is well-known that the Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is continuous, and as such is well-defined on any set in $\mathbb{R}^n$. On the other hand, the Fourier transform maps L^2 functions onto L^2 functions, which are only defined almost everywhere and so are arbitary on sets of measure zero. At first glance therefore it seems there is no meaningful way to restrict the Fourier transform of a function to sets of measure zero. Indeed, the well-known example

$$f(x_1, \ldots, x_n) = \frac{\psi(x_2, \ldots, x_n)}{|x_1| + 1},$$

where ψ is a smooth bump function, has an infinite Fourier transform on the hyperlane $\{\xi \in \mathbb{R}^n : \xi_1 = 0\}$.

However in the 1960s Stein made the remarkable observation that it was possible to restrict the Fourier transform to the sphere S^{n-1}. This observation lead to the Stein-Tomas restriction inequality, which we now state.

Let Γ be a smooth hypersurface in $\mathbb{R}^n$, and $d\sigma$ a measure supported on Γ. Then a $L(p, q)$ Fourier restriction inequality for Γ is one of the form

$$||\hat{f}||_{L^q(d\sigma)} \leq C_{p,q}||f||_p \tag{1.0.1}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, which can then be extended to $f \in L^p(\mathbb{R}^n)$ by continuity.

The archetypical restriction inequality, where Γ is the unit sphere in $\mathbb{R}^n$, was originally established by Tomas for the range $1 \leq p < \frac{2(n+1)}{n+3}$, $q = 2$ (see [40]) and Stein who obtained the endpoint case $\frac{2(n+1)}{n+3}$ (see [35]). Zygmund later established, for two dimensions, the full range of p and q for which the above inequality holds ([42]).

More generally if Γ is a hypersurface with non-vanishing curvature then (1.0.1) is known to hold for $1 \leq p \leq \frac{2(n+1)}{n+3}$ and $q \leq \frac{n-1}{n+1}p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$ (see [35]). A well-known homogeneity argument establishes the necessity of the condition $q \leq \frac{n-1}{n+1}p'$. Additionally, by considering the dual "extension" inequality, it can be shown that a necessary condition for (1.0.1) to hold is $\hat{\sigma} \in L^{p'}$. Taking this observation together with the following bound on $\hat{\sigma}$ (see again, for example, [35]) for sufficiently large $|\xi|$, with ξ belonging to a sufficiently small cone

$$|\hat{\sigma}(\xi)| \geq C_n |\xi|^{\frac{-(n-1)}{2}},$$

we can see that (1.0.1) fails for $p \geq \frac{2n}{n+1}$.

It is conjectured that these necessary conditions are in fact sufficient. That is, that the full range for which (1.0.1) holds should be

$$q \leq \frac{n-1}{n+1}p' \qquad\qquad 1 \leq p < \frac{2n}{n+1}.$$

At present this is known to be true only in two dimensions. In higher dimensions there has been a great deal of attention in extending the known range of p (See [8], [41], [23], [39], [37], and for an extensive summary of known results see [38]).

Additionally there are similar results where Γ is a hypersurface whose Gaussian curvature may vanish, but satisfies an additional property such as being of finite type (see again [35]). It is natural therefore to ask what can be said about flat hypersurfaces, which may not be of finite type. To set the context let Γ be a hypersurface, with the following parametrisation:

$$\Gamma(t) = (t, \gamma(t)).$$

where $t \in \mathbb{R}^{n-1}, \gamma : \mathbb{R}^{n-1} \to \mathbb{R}$.

In [19] the authors demonstrate that we cannot hope to the use the induced Lebesgue measure to achieve (1.0.1) for such hypersurfaces. Therefore we must introduce a mitigating factor to compensate for the possible vanishing Gaussian curvature:

$$d\sigma(t) = |K(t)|^{\frac{1}{n+1}} dt.$$

Although other measures have been chosen (see for example [24]), typically K is chosen to be the Gaussian curvature of Γ (see [1], [10], [32],[33], [25], [11]):

$$K(t) = \det \operatorname{Hess}(\gamma(t)).$$

The question then becomes under what conditions on Γ (and for what range of p and q) do we have an inequality of the form

$$\left(\int_{\mathbb{R}^{n-1}} |\hat{f}(\Gamma(t))|^q |K(t)|^{\frac{1}{n+1}} dt\right)^{\frac{1}{q}} \leq C_{p,q}(n)||f||_{L^p} \tag{1.0.2}$$

where $p' = \frac{(n+1)}{n-1}q$.

This choice of weight is largely considered to be the "natural" one because with the induced measure the restriction inequality is both affine invariant and invariant under reparametrisation. It also the largest weight for which the Fourier restriction inequality can hold for full range of (p, q) (see [19]).

To see this we first remark that by Gauss' Theorema Egregium ([18]), Gaussian curvature is an intrinsic property of the hypersurface, and invariant under isometries. If $\Gamma : I \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a parametrisation of a surface and $\Phi : I \to I$ is a diffeomorphism then

$$K_{\Gamma\circ\Phi}(t)^{\frac{1}{n+1}} dt = K_\Gamma(\Phi(t))^{\frac{1}{n+1}} d\Phi(t).$$

Now let $\Gamma : U \to \mathbb{R}^n$ be a parametrisation of a surface which satisfies (1.0.2), and $\tilde{\Gamma} = \Gamma \circ \Phi$ it follows

$$\begin{aligned}\left(\int_{\mathbb{R}^{n-1}} |\hat{f}(\tilde{\Gamma}(t))|^q |K_{\tilde{\Gamma}}(t)|^{\frac{1}{n+1}} dt\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{R}^{n-1}} |\hat{f}(\Gamma(\Phi(t)))|^q |K_{\Gamma\circ\Phi}(t)|^{\frac{1}{n+1}} dt\right)^{\frac{1}{q}} \\ &= \left(\int_{\mathbb{R}^{n-1}} |\hat{f}(\Gamma(\omega))|^q |K_\Gamma(\omega)|^{\frac{1}{n+1}} d\omega\right)^{\frac{1}{q}} \\ &\leq C_{p,q}(n)||f||_{L^p}.\end{aligned}$$

We remark that (1.0.2) is invariant under Euclidean motions for any exponent of K. As we shall see our choice of exponent is critical for invariance under more general affine transformations. We now demonstrate this fact.

As before let $\Gamma : I \to \mathbb{R}^n$ be a parametrisation of a surface, satisfying (1.0.2). We first consider translations, so let $\tilde{\Gamma} = \Gamma + v$. Then the left hand side of (1.0.2) becomes

$$\begin{aligned}\left(\int_{\mathbb{R}^{n-1}} |\hat{f}(\tilde{\Gamma}(t))|^q |K_{\tilde{\Gamma}}(t)|^{\frac{1}{n+1}} dt\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{R}^{n-1}} |\hat{f}(\Gamma(t)+v)|^q |K_{\Gamma+v}(t)|^{\frac{1}{n+1}} dt\right)^{\frac{1}{q}} \\ &= \left(\int_{\mathbb{R}^{n-1}} |\widehat{e^{2\pi i v\cdot} f}(\Gamma(t))|^q |K_\Gamma(t)|^{\frac{1}{n+1}} dt\right)^{\frac{1}{q}} \\ &\leq C_{p,q}(n)||e^{2\pi i v\cdot} f||_{L^p}\end{aligned}$$

$$= C_{p,q}(n)||f||_{L^p}.$$

where we have used the fact that Gaussian curvature is invariant under translation of the surface.

It remains to deal with general affine transformations, $T = Ax + v$. Given a hypersurface, by the above we are free to choose a parametrisation for it, $\Gamma = (t, \gamma(t))$. Consider the image of our hypersurface under T. Again, by the above we may parametrise this as $\tilde{\Gamma} = (t, \tilde{\gamma}(t))$. Furthermore $\tilde{\Gamma} = A\Gamma$, and

$$K_{\tilde{\Gamma}}(t) = \det(A)^{n-1} K_{\Gamma}(t).$$

For any $f \in L^p(\mathbb{R}^n)$, let $f_A(x) = f(A^{*-1}x)$, then it is straightforward to show

$$\widehat{f}(\tilde{\Gamma}(t)) = \frac{1}{\det(A)} \widehat{f}_A(\Gamma(t))$$
$$||f||_{L^p} = (\det(A))^{-\frac{1}{p}} ||f_A||_{L^p}.$$

Then,

$$\begin{aligned}
\left(\int_{\mathbb{R}^{n-1}} |\hat{f}(\tilde{\Gamma}(t))|^q |K_{\tilde{\Gamma}}(t)|^{\frac{1}{n+1}} dt\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{R}^{n-1}} |\tfrac{1}{\det(A)} \widehat{f_A}(\Gamma(t))|^q |(\det(A))^{n-1} K(t)|^{\frac{1}{n+1}} dt\right)^{\frac{1}{q}} \\
&= \det(A)^{\frac{(n-1)}{(n+1)q} - 1} \left(\int_{\mathbb{R}^{n-1}} |\widehat{f_A}(\Gamma(t))|^q |K(t)|^{\frac{1}{n+1}} dt\right)^{\frac{1}{q}} \\
&\leq \det(A)^{\frac{(n-1)}{(n+1)q} - 1} C ||f_A||_{L^p} \\
&= \det(A)^{\frac{(n-1)}{(n+1)q} - 1 + \frac{1}{p}} C ||f||_{L^p} \\
&= C ||f||_{L^p}
\end{aligned}$$

where equality in the last line follows from $p' = \frac{(n+1)}{(n-1)} q$.

In two dimensions Sjölin ([34]) proved the following result.

Theorem 1 ([34])**.** *Let $n = 2$, and Γ be described as above, where $\gamma \in C^2(0, 1)$ is such that $K(t)$ is single signed. Then (1.0.2) holds for $p < \frac{4}{3}$, $q' \leq \frac{p'}{3}$, with a constant C independent of Γ when $q' = \frac{p'}{3}$.*

In [34] Sjölin also provided an example which demonstrates the necessity of the condition on $K(t)$. In two dimensions, for Γ of the form above, this is equivalent to convexity (or concavity) of γ.

In three dimensions the situation becomes more complicated. In [13] Carbery and Ziesler prove that there is no such universal result for hypersurfaces where

$K(t) \geq 0$. However, as the authors observe the standard homogenity argument for finding counterexamples to the restriction inequality leads to the affine isoperimetric inequality (although the constant obtained by doing so is not sharp).

The affine isoperimetric inequality relates the affine surface area of a convex body $\mathcal{K}$ (which in our context is the convex hull of our hypersurface Γ) and its volume. It states

$$\operatorname{AffineSurf}(\partial\mathcal{K}) \leq C(n) \operatorname{Vol}(\mathcal{K})^{\frac{n-1}{n+1}}$$

where

$$\operatorname{AffineSurf}(\partial\mathcal{K}) := \int_{\partial\mathcal{K}} |k(x)|^{\frac{1}{n+1}} d\mu(x)$$

and μ is the surface measure on $\partial\mathcal{K}$ and $k(x)$ the Gauss-Kronecker curvature. In our context, this can be equivalently defined as

$$\operatorname{AffineSurf}(\partial\mathcal{K}) := \int_I |K(t)|^{\frac{1}{n+1}} dt.$$

We refer the reader to [20] for further details.

To see that the restriction estimate implies the isoperimetric inequality consider the dual problem, the extension inequality:

$$\left|\left|\int e^{2\pi i x\cdot\Gamma(t)} g(\Gamma(t))|K(t)|^{\frac{1}{(n+1)}} dt\right|\right|_{L^{p'}(dx)} \leq C_{p,q}(n)||g||_{L^{q'}(d\mu)}. \tag{1.0.3}$$

Let $\mathcal{K} = \operatorname{ConvexHull}(\Gamma)$, and define the *polar body* of $\mathcal{K}$,

$$\mathcal{K}^* = \{\xi \in \mathbb{R}^n : |\xi \cdot x| \leq 1, \forall x \in \mathcal{K}\}.$$

The Blaschke-Santaló inequality (originally proved in [7] and [31], see also [22], [6]) states that

$$\operatorname{Vol_n}(\mathcal{K}) \operatorname{Vol_n}(\mathcal{K}^*) \leq \omega_n^2$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Additionally for $x \in \mathcal{K}^*$ we have $|x \cdot \Gamma(t)| \leq 1$. Therefore restricting the $L^{p'}(dx)$ integral on the left hand side of (1.0.3) to $\mathcal{K}^*$ and taking $g \equiv 1$ we can see that it is bounded below by $\operatorname{AffineSurf}(\Gamma)(\operatorname{Vol}(\mathcal{K}^*))^{\frac{1}{p'}}$. The right hand side is essentially $\operatorname{AffineSurf}(\Gamma)^{\frac{1}{q'}}$, and after rearranging and employing the Blaschke-Santaló inequality, we get

$$\operatorname{AffineSurf}(\Gamma) \leq C_n \operatorname{Vol}(\mathcal{K})^{\frac{q}{p'}}, \tag{1.0.4}$$

from which the affine isoperimetric inequality follows as $p' = \frac{(n+1)}{n-1}q$.

Since this inequality holds for convex hypersurfaces, this leaves open the possiblity that a universal restriction theorem for convex hypersurfaces may be obtainable. In [1] Abi-Khuzam and Shayya prove a result for radial convex hypersurfaces satisfying an additional condition, a result which is improved upon by Carbery and Ziesler in their paper.

In the absence of a universal result for convex γ we can instead look at a different class of hypersurfaces which are piecewise convex/concave but over which we can bring to bear a finer analysis. In view of the results in two dimensions, we would then aim for a constant in (1.0.2) which depends on the number of these pieces.

In [11] the authors looked at the restriction inequality for surfaces in $\mathbb{R}^3$ of the form,

$$\Gamma(t_1, t_2) = (t_1, t_2, \gamma(t_1, t_2)),$$

where $\gamma : \mathbb{R}^2 \to \mathbb{R}$ is a homogeneous polynomial of degree d. They established (1.0.2) for the full Stein-Tomas range in three dimensions, but the constant in [11] depended, in a not particularly transparent way, on γ and not just its degree. In Chapter 2 we are able to quantify the relationship between the constant and γ. It is still of interest to determine if a universal result can be achieved.

If we ignore the endpoint of the Stein-Tomas range, then Oberlin in [25] achieves a stronger and more general result. It is stronger in the sense that it is a universal result, and more general in the sense it applies to any dimension $n \geq 2$, and a wider class of surfaces. However, [25] misses the endpoint $L^{\frac{4}{3}} \to L^2$, establishing instead a weak-type inequality $L^{\frac{4}{3}} \to L^{2,\infty}$.

We can also ask a similar question for restriction to curves in higher dimensions. If we now consider $\Gamma : I \to \mathbb{R}^n$ to be a parametrisation of a curve in $\mathbb{R}^n$.

$$\Gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)).$$

Then we can define the quantity

$$L_\Gamma(t) = \det(\Gamma'(t), \ldots, \Gamma^{(n)}(t));$$

which is the determinant of the $n \times n$ matrix whose ith column is the ith derivative of Γ. The natural choice measure is now the *affine arclength measure*,

$$d\mu(t) = |L_\Gamma(t)|^{\frac{2}{n(n+1)}}.$$

As with surfaces the question is now for which class of curves Γ, and for what range of (p, q), do we have an inequality of the form

$$\left(\int_I |\hat{f}(\Gamma(t))|^q |L_\Gamma(t)|^{\frac{2}{n(n+1)}} dt\right)^{\frac{1}{q}} \leq C_{p,q}(n)||f||_p. \tag{1.0.5}$$

The earlier argument which deduced the affine isoperimetric inequality for surfaces from (1.0.2) also deduces the affine isoperimetric inequality for curves from (1.0.5). The affine isoperimetric inequality bounds the affine arclength of a curve $\Gamma : I \to \mathbb{R}^n$, which is defined as

$$\text{AffineArclength}(\Gamma) := \int_I |L_\Gamma(t)|^{\frac{2}{n(n+1)}} dt,$$

by the volume of the convex hull formed by that curve. We now summarise the argument of deducing such an inequality from (1.0.5), but we remark that as was the case with surfaces this argument does not give us the optimal constant.

Again we consider the extension inequality:

$$\left|\left|\int e^{2\pi i x\cdot\Gamma(t)} g(\Gamma(t))|L_\Gamma(t)|^{\frac{2}{n(n+1)}} dt\right|\right|_{L^{p'}(dx)} \leq C_{p,q}(n)||g||_{L^{q'}(d\mu)}. \tag{1.0.6}$$

Let $\mathcal{K} = \text{ConvexHull}(\Gamma)$, then for $x \in \mathcal{K}^*$ we have $|x \cdot \Gamma(t)| \leq 1$. Restricting the $L^{p'}(dx)$ integral on the left hand side of (1.0.6) to $\mathcal{K}^*$ and taking $g \equiv 1$ we can see that it is bounded below by $\text{AffineArclength}(\Gamma)(\text{Vol}(\mathcal{K}^*))^{\frac{1}{p'}}$. For $g \equiv 1$ the right hand side is essentially $\text{AffineArclength}(\Gamma)^{\frac{1}{q'}}$. Rearranging and applying the Blaschke-Santaló inequality, we get

$$\text{AffineArclength}(\Gamma) \leq C_n \,\text{Vol}(\mathcal{K})^{\frac{q}{p'}}. \tag{1.0.7}$$

If we consider the *moment curve*,

$$\Gamma(t) = (t, t^2, \ldots, t^n), \tag{1.0.8}$$

which has $L_\Gamma(t) \equiv n!$, then we find

$$n!|I| \leq C_n |I|^{\frac{qn(n+1)}{2p'}}.$$

Since we would like $C_{p,q}(n)$ to be independent of $|I|$, then we must have $p' = \frac{n(n+1)}{2}q$, leading to the affine isoperimetric inequality (for curves). Additionally with such a choice of (p, q), (1.0.5) is affine invariant.

Furthermore by considering the moment curve on $[0,1]$ and taking $g \equiv 1$ in the extension inequality (1.0.6) we see immediately that we have the necessary condition that the quantity

$$\left|\left|\int e^{2\pi i x\cdot\Gamma(t)}dt\right|\right|_{L^{p'}(dx)}$$

is finite. Thus determining for which p' the above is finite also determines a necessary p range. When considering surfaces of non-vanishing curvature, one can use lower bounds on the decay of the Fourier transform of the surface measure to obtain this range, but here the correct range is not so clear. However it was known by number theorists (see [2]) that this quantity is finite precisely when $p' > \frac{n(n+1)}{2} + 1$, or equivalently $p < \frac{n^2+n+2}{n(n+1)}$.

The restriction inequality (1.0.5) in higher dimensions was originally considered by Drury and Marshall ([15]), in which they establish the inequality for a range of (p,q) for *simple curves*. In [3] the authors established (1.0.5) for the "monomial" curves of the form,

$$(|t|^{a_1},\ldots,|t|^{a_n})$$

with constant independent of the a_i. More recently Dendrinos and Wright (see [14]) established (1.0.5) for the range $p' = \frac{n(n+1)}{2}q$, $1 \le p < \frac{n(n+2)}{n^2+2n-2}$ for polynomial curves,

$$(p_1(t),\ldots,p_n(t))$$

with the constant depending only on n, p and the degrees of the polynomials $\{p_i\}_{i=1}^n$. It is these results, together with the earlier observation, which motivates the problem investigated in Chapter 3. That is, for what class of curves do we have the affine isoperimetric inequality,

$$\text{AffineArclength}(\Gamma) \le C_n \,\text{Vol}(\mathcal{K})^{\frac{2}{n(n+1)}}? \tag{1.0.9}$$

We remark that we shall be not be concerned in obtaining a sharp constant.

1.1 d-crossing curves

By considering the curve

$$\Gamma(t) = \frac{1}{t+1}(\sin(2\pi N t),\cos(2\pi N t)),$$

as $N \to \infty$, we see that we must require some sort of convexity condition on Γ if (1.0.9) is to hold, even in $\mathbb{R}^2$.

In view of this, we follow [4] and define a curve Γ to be k-crossing if it intersects every hyperplane at most k times. Explicitly, for every hyperplane h in $\mathbb{R}^n$

$$\#\{t : \Gamma(t) \in h\} \leq k,$$

where $\#\{\cdot\}$ is the counting measure.

As we shall see this notion generalises both polynomial curves (which are either embedded in a plane or are d-crossing, where d is the degree of the polynomials) and simple curves (which are n-crossing). Furthermore this definition reasonably captures the intuitive notion of convexity. Indeed, in two dimensions 2-crossing curves form a connected subset of the boundary of a (strictly) convex body. By drawing a picture it is easy to convince oneself of this fact, and an elementary proof is given in Chapter 3. Motivated by this fact we shall call a curve in $\mathbb{R}^n$ *convex* if it is n-crossing.

Finally we remark that if Γ is k-crossing, then so is the image of Γ under any affine transformation.

In Chapter 3 we provide a direct proof of (1.0.9) for polynomial curves. We then provide an abstract proof for (1.0.9) and discuss its relevance to n-crossing curves. In the course of that proof we shall make repeated use of the main result of [4]:

Theorem 2 ([4])**.** *For every integer $n \geq 2$ there exists $M = M(n)$ such that for every $(n+1)$-crossing curve $\Gamma : I \to \mathbb{R}^n$ in $\mathbb{R}^n$ there exists a partition $I = \cup_{i=1}^{M(n)} I_i$ such that $\Gamma|_{I_i}$ is convex.*

This theorem allows to reduce the n-dimensional problem to an $(n-1)$-dimensional one, and so by employing an inductive argument, to the two dimensional base case which is well understood.

1.2 Notation

We conclude the introduction with a few remarks on notation.

Throughout we shall use $|\cdot|$ to denote the usual Euclidean norm on $\mathbb{R}^n$. For $n = 1$, this coincides with the modulus, so we use this notation to also denote the modulus of $x \in \mathbb{R}$. Where I is an interval in $\mathbb{R}$, $|I|$ shall denote its length.

We shall denote the L^p norm of a function $f : \mathbb{R}^n \to \mathbb{R}$ using the standard notation, $||f||_{L^p}$. To avoid confusion we shall sometimes use the notation $||f||_{L^p(d\mu)}$ and $||f||_{L^p(\mathbb{R}^n)}$, explicitly stating the measure of the integral and the domain of the function respectively. In the case of the latter it should be assumed that the Lebesgue measure is being used.

For any body $K \subset \mathbb{R}^n$, K^o shall denote the interior of K, and K^c the complement of K.

We shall use the notation $A \sim B$ to mean that there exist absolute constants $c_2 \geq c_1 > 0$ such that

$$c_2 A \leq B \leq c_1 A.$$

Addititonally we shall use $A \sim_d B$ if the above holds, but if the constants may depend in some way on d.

Chapter 2

A restriction theorem for homogeneous polynomial surfaces in $\mathbb{R}^3$

A polynomial function $\gamma : \mathbb{R}^k \to \mathbb{R}$ is a *homogeneous polynomial* of degree d if it satisfies the homogeneity condition:

$$\gamma(\lambda t_1, \ldots, \lambda t_k) = \lambda^d \gamma(t_1, \ldots, t_k) \text{ for all } \lambda \in \mathbb{R}$$

When $k = 1$, homogeneous polynomials are precisely the monomials. In this chapter $k = 2$, and with an abuse of notation we can write,

$$\gamma(t, ts) = t^d \gamma(s),$$

where formally $\gamma(s) := \gamma(1, s)$.

In [11] the authors proved the following restriction theorem for homogeneous polynomial surfaces, $\Gamma : \mathbb{R}^2 \to \mathbb{R}^3$

$$\Gamma(t) = (t_1, t_2, \gamma(t_1, t_2)),$$

where $t = (t_1, t_2) \in \mathbb{R}^2$ and $\gamma : \mathbb{R}^2 \to \mathbb{R}$ is a homogeneous polynomial of degree d.

Theorem 3 ([11]). *Let Γ and γ be as defined above and let*

$$K_\gamma(t_1, t_2) = \det \operatorname{Hess} \gamma.$$

Then there exists a constant C depending on γ such that

$$\left(\int_{\mathbb{R}^2} |\hat{f}(\Gamma(t_1,t_2))|^2 |K_\gamma(t_1,t_2)|^{\frac{1}{4}} dt_1 dt_2\right)^{\frac{1}{2}} \leq C||f||_{L^{\frac{4}{3}}} \qquad (2.0.1)$$

for all $f \in L^{\frac{4}{3}}$.

In this chapter we shall quantify the relationship between γ and C. It remains of interest to determine whether a "universal" result, where the constant depends only on the degree of the polynomial, holds.

First we state the following lemma, the proof of which is given in [11].

Lemma 1. *Given a γ of the above form there exists a partition of $\mathbb{R}^2 = \cup_{j=1}^{R(d)} Z_j$ into two types of regions: type I and type II. For each Z_j of type I there exists $0 < \epsilon < 1$, such that*

$$|K_\gamma(1, \frac{t_2}{t_1})| \geq \epsilon |\gamma'(1, \frac{t_2}{t_1})|^2, \forall (t_1, t_2) \in Z_j \qquad (2.0.2)$$

and for each Z_j of type II there exists an affine transformation $T_j : \mathbb{R}^2 \to \mathbb{R}^2$ and $0 < \epsilon < 1$, such that with $\gamma_j = \gamma \circ T_j^{-1}$

$$|K_{\gamma_j}(1, \frac{t_2}{t_1})| \geq \epsilon |\gamma_j'(1, \frac{t_2}{t_1})|^2, \forall (t_1, t_2) \in Z_j$$

Furthermore $T_j(Z_j)$ is contained within a set of type I for γ_j. The number of regions $R(d)$ making up the partition is bounded by $2(d-1)+1$.

Remark. *This lemma summarises Lemmas 2.1, 2.2, 2.3 and 2.4 in [11].*

Theorem 4. *Let Γ, γ and ϵ be as defined above and let*

$$K_\gamma(t_1, t_2) = \det \operatorname{Hess} \gamma.$$

Then there exists a constant $C = C(d, \epsilon)$ such that

$$\left(\int_{\mathbb{R}^2} |\hat{f}(\Gamma(t_1,t_2))|^2 |K_\gamma(t_1,t_2)|^{\frac{1}{4}} dt_1 dt_2\right)^{\frac{1}{2}} \leq C||f||_{L^{\frac{4}{3}}} \qquad (2.0.3)$$

for all $f \in L^{\frac{4}{3}}$. For a fixed d,

$$C(d, \epsilon) = O(\epsilon^{-36}).$$

Proof. This proof is based on the one given in [11], and for some details we shall refer the reader to that proof. The proof itself is analogous to the one given by the same authors in [10]. First Littlewood-Paley theory is used to reduce the region of integration. Then the fact that $||f||_{L^4}^2 = ||f\overline{f}||_{L^2}$ is exploited, together with a change of variables, Plancherel's theorem and the Cauchy-Schwarz inequality to reduce the result to a 'simpler' inequality on which analysis of the specific nature of the curve (in this case a homogeneous polynomial) can be brought to bear.

Lemma 1 allows us to restrict our attention to regions of type I. If, for Z_j of type I we can prove

$$\left(\int_{(t_1,t_2)\in Z_j} |\hat{f}(\Gamma(t_1,t_2))|^2 |K_\gamma(t_1,t_2)|^{\frac{1}{4}} dt_1 dt_2\right)^{\frac{1}{2}} \leq C||f||_{L^{\frac{4}{3}}} \tag{2.0.4}$$

with a constant $C = C(d,\epsilon)$ independent of j, then we can extend this to regions of type II by the following argument. Let Z_j now be a region of type II. Then $T_j(Z_j)$ is contained a region of type I for γ_j, we can apply the above to γ_j and $T_j(Z_j)$:

$$\left(\int_{(t_1,t_2)\in T(Z_j)} |\hat{f}(\Gamma_j(t_1,t_2))|^2 |K_{\gamma_j}(t_1,t_2)|^{\frac{1}{4}} dt_1 dt_2\right)^{\frac{1}{2}} \leq C||f||_{L^{\frac{4}{3}}}$$

from which the desired result for Z_j follows from the parametrisation invariance property of the restriction inequality. Since there are most $2(d-1)+1$ regions, (2.0.4) can be extended to (2.0.3) independently of ϵ.

For the remainder of the proof we consider Z_j to be a fixed, but arbitrary region of type I. We now consider the dual problem and denote the extension operator:

$$\mathcal{E}_j g(z,x,y) = \int_{Z_j} e^{iz\gamma(t_1,t_2)} e^{i(xt_1+yt_2)} \hat{g}(t_1,t_2) |K(t_1,t_2)|^{\frac{1}{4}} dt_1 dt_2$$

Then by duality (2.0.4) holds for all $f \in L^{\frac{4}{3}}(\mathbb{R}^3)$, if and only if

$$||\mathcal{E}_j g||_{L^4(\mathbb{R}^3)} \leq C||g||_{L^2(\mathbb{R}^2)}, \tag{2.0.5}$$

for $g \in L^2(\mathbb{R}^2)$. Furthermore if we define

$$\widehat{Ug}(t_1,t_2) = \chi_{\{|t_2|<|t_1|\}}(t_1,t_2)\hat{g}(t_1,t_2)$$
$$\widehat{\tilde{U}g}(t_1,t_2) = \chi_{\{|t_2|>|t_1|\}}(t_1,t_2)\hat{g}(t_1,t_2)$$

then we will prove

$$||\mathcal{E}_j U g||_{L^4(\mathbb{R}^3)} \leq C||g||_{L^2(\mathbb{R}^2)}. \tag{2.0.6}$$

A bound for $||\mathcal{E}_j \tilde{U} g||_{L^4(\mathbb{R}^3)}$ follows a similar argument and the remaining set $\{(t_1, t_2) : |t_1| = |t_2|\}$ is a set of measure zero in $\mathbb{R}^2$.

We now proceed as in [11] and define Littlewood-Paley operators. To aid this we make the following definitions:

$$\begin{aligned} c_d &= (d-1)2^{2+4d}\left(2^{16+2d} + 9d2^{13+4d}\right) \\ \delta &= \min\{\frac{\epsilon(d-1)}{2^8 c_d}, \frac{d-1}{3d}\} \\ \lambda &= (1+\delta)^{\frac{1}{3d}} \\ \alpha &= (1+\delta)^{\frac{1}{3}} \end{aligned}$$

The choice of values will become clear later in the proof.

We define, for fixed $\omega > 1$, a smooth bump function $\psi^\omega : \mathbb{R} \to \mathbb{R}$,

$$\psi^\omega(x) = \begin{cases} 1 & 1 \leq |x| \leq \omega \\ 0 & |x| \leq \frac{1}{\omega} \text{ or } |x| \geq \omega^2, \end{cases} \tag{2.0.7}$$

and $\psi_k^\omega(x) = \psi^\omega(\omega^k x)$. We can choose ψ^ω such that $\sum_k |\psi_k^\omega(x)|^2 = 2$ for $x \neq 0$. We shall omit the ω, if $\omega = 2$.

We define the Littlewood-Paley operators as in [11],

$$\begin{aligned} \widehat{S_k^\lambda g} &= \psi_{2k}^{\lambda^2}(t_1^2 + t_2^2)\hat{g}(t_1, t_2) \\ \widehat{P_k g} &= \psi_k(K(1, \tfrac{t_2}{t_1}))\hat{g}(t_1, t_2) \\ \widehat{Q_k^\alpha g} &= \psi_k^\alpha(\gamma'(\tfrac{t_2}{t_1}))\hat{g}(t_1, t_2) \end{aligned}$$

and the additonal operator

$$\widehat{R_k^\lambda g} = \psi_k^\lambda(t_1)\hat{g}(t_1, t_2).$$

Proposition 1. *If, for Z_j of type I we have*

$$||\mathcal{E}_j U S_0^\lambda P_k Q_l^\alpha R_m^\lambda g||_{L^4(\mathbb{R}^3)} \leq C(d)||g||_{L^2(\mathbb{R}^2)}, \forall g \in L^2(\mathbb{R}^2) \tag{2.0.8}$$

with a constant $C(d)$ depending only on d, but independent of k,l and m, then

$$||\mathcal{E}_j U g||_{L^4(\mathbb{R}^3)} \leq C(d, \epsilon)||g||_{L^2(\mathbb{R}^2)}, \forall g \in L^2(\mathbb{R}^2).$$

Furthermore, for a fixed d,

$$C(d,\epsilon) = O(\epsilon^{-36}).$$

Proof. For the purposes of this section we shall use the notation $A \sim_C B$ to mean that there exists a constant C such that

$$\frac{1}{C}A \leq B \leq CA.$$

By annular Littlewood-Paley theory there exists a constant $c_1 = c_1(d,\lambda)$, depending on d and λ, such that for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^2)$

$$\left|\left|\left(\sum_k |S_k^\lambda f|^2\right)^{\frac{1}{2}}\right|\right|_{L^p} \sim_{c_1} ||f||_{L^p}. \tag{2.0.9}$$

By Proposition 1.2(b) in [11], there exists constant $c_2 = c_2(d)$, depending only on d, and constant $c_3 = c_3(d,\alpha)$, depending only on d and α, such that for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^2)$

$$\left|\left|\left(\sum_k |P_k f|^2\right)^{\frac{1}{2}}\right|\right|_{L^p} \sim_{c_2} ||f||_{L^p}, \tag{2.0.10}$$

$$\left|\left|\left(\sum_k |Q_k^\alpha f|^2\right)^{\frac{1}{2}}\right|\right|_{L^p} \sim_{c_3} ||f||_{L^p}. \tag{2.0.11}$$

Finally by standard Littlewood-Paley theory there exists a constant $c_4 = c_4(\lambda)$, depending only on λ, such that for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^2)$

$$\left|\left|\left(\sum_k |R_k^\lambda f|^2\right)^{\frac{1}{2}}\right|\right|_{L^p} \sim_{c_4} ||f||_{L^p}. \tag{2.0.12}$$

We remark that we have suppressed the dependence on p of the constants $\{c_i\}_{i=1}^4$, and that when $p = 2$, these constants are independent of α and λ. Results concerning the behaviour of these constants with respect to p can be found in [27].

Furthermore, by a standard argument, there exist constants $\tilde{c}_1 = \tilde{c}_1(d,\lambda)$,

$\tilde{c}_2 = \tilde{c}_2(d)$, $\tilde{c}_3 = \tilde{c}_3(d,\alpha)$ and $\tilde{c}_4 = \tilde{c}_4(\lambda)$ such that for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^2)$

$$\left\| \left(\sum_k |S_k^\lambda S_k^\lambda f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \sim_{\tilde{c}_1} ||f||_{L^p}, \tag{2.0.13}$$

$$\left\| \left(\sum_k |P_k P_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \sim_{\tilde{c}_2} ||f||_{L^p}, \tag{2.0.14}$$

$$\left\| \left(\sum_k |Q_k^\alpha Q_k^\alpha f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \sim_{\tilde{c}_3} ||f||_{L^p}, \tag{2.0.15}$$

$$\left\| \left(\sum_k |R_k^\lambda R_k^\lambda f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \sim_{\tilde{c}_4} ||f||_{L^p}. \tag{2.0.16}$$

Following the proof of [11, Proposition 2.5] if (2.0.8) holds then

$$||\mathcal{E}_j U g||_{L^4(\mathbb{R}^3)} \leq c_1\tilde{c}_1 c_2\tilde{c}_2 c_3\tilde{c}_3 c_4\tilde{c}_4 ||g||_{L^2(\mathbb{R}^2)}.$$

It remains to prove, that for a fixed d,

$$c_1\tilde{c}_1 c_2\tilde{c}_2 c_3\tilde{c}_3 c_4\tilde{c}_4 = O(\epsilon^{-36}).$$

Since the terms c_2 and $\tilde{c}_2$ depend only on d we may ignore them.

For c_4 we follow the proof of the Littlewood-Paley theorem (in one dimension), keeping track of the effect our choice of ω in (2.0.7) has on the constants involved.

Let $(\widehat{\psi^\lambda})_k$ be the kernel of the operator R_k. In particular, $(\widehat{\psi^\lambda})_k(x) = \lambda^k \widehat{\psi^\lambda}(\lambda^k x)$. For $p = 2$, ω has no affect, and we simply apply Plancherel's theorem

$$\left\| \left(\sum_k |R_k^\lambda f|^2 \right)^{\frac{1}{2}} \right\|_{L^2}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_k |\psi_k^\lambda(t_1)|^2 |\hat{f}(t_1,t_2)|^2 dt_1 dt_2 = 2||f||_{L^2}^2. \tag{2.0.17}$$

For $1 < p \neq 2 < \infty$ it suffices to show the operator satisfies the Hörmander condition. By [16, Proposition 5.2] it suffices to show that

$$||(\widehat{\psi^\lambda})_k'(t_1)||_{l^2} \leq c_4 |t_1|^{-2}$$

In fact we shall show that $c_4 = C(\lambda - 1)^{-4}$. First fix i such that $\lambda^{-i} \leq |t_1| <$

λ^{-i+1}, and note that since $\widehat{\psi^\lambda} \in \mathcal{S}$, $|(\widehat{\psi^\lambda})'(t_1)| \leq C\min(1, |(\lambda-1)t_1|^{-3})$. Then

$$\begin{aligned}\left(\sum_k |(\widehat{\psi^\lambda})_k'(t_1)|^2\right)^{\frac{1}{2}} &\leq \sum_k |(\widehat{\psi^\lambda})_k'(t_1)| \\ &= \sum_k \lambda^{2k}|(\widehat{\psi^\lambda})'(\lambda^k t_1))| \\ &\leq C\sum_{k\leq i}\lambda^{2k} + C|(\lambda-1)t_1|^{-3}\sum_{k>i}\lambda^{-k} \\ &\leq C\frac{\lambda^{2i}}{1-\lambda^{-2}} + C|(\lambda-1)t_1|^{-3}\frac{\lambda^{-i+1}}{1-\lambda^{-1}} \\ &\leq C|t_1|^{-2}(\lambda-1)^{-4},\end{aligned}$$

where in the last inequality we have used the fact that $(1-\lambda^{-2}) \geq (1-\lambda^{-1})$ as $\lambda > 1$. We remark that the value of C may change from line to line but is independent of λ.

By using the polarisation identity and (2.0.17), we have that

$$\int_{\mathbb{R}} \sum_k R_k^\lambda f \overline{R_k^\lambda g} = \int_{\mathbb{R}} fg.$$

A standard argument establishes the opposite inequality in the Littlewood-Paley theorem.

$$\begin{aligned}||f||_{L^p} &= \sup\left\{\left|\int_{\mathbb{R}} fg\right| : ||g||_{L^{p'}} \leq 1\right\} \\ &= \sup\left\{\left|\int_{\mathbb{R}} \sum_k R_k^\lambda f\overline{R_k^\lambda g}\right| : ||g||_{L^{p'}} \leq 1\right\} \\ &\leq \sup\left\{\left|\left|\left(\sum_k |R_k^\lambda f|^2\right)^{\frac{1}{2}}\right|\right|_{L^p}\left|\left|\left(\sum_k |R_k^\lambda g|^2\right)^{\frac{1}{2}}\right|\right|_{L^{p'}} : ||g||_{L^{p'}} \leq 1\right\} \\ &\leq c_4\left|\left|\left(\sum_k |R_k^\lambda f|^2\right)^{\frac{1}{2}}\right|\right|_{L^p}\end{aligned}$$

where $c_4 = C(\lambda-1)^{-4}$. Additionally $\tilde{c}_4 = C(\lambda-1)^{-8}$. Then since $\lambda - 1 \sim_d \delta \sim_d \epsilon$ we have,

$$c_4\tilde{c}_4(d,\lambda) = O(\epsilon^{-12}).$$

A similar argument establishes the same estimate for $c_1\tilde{c}_1$ and $c_3\tilde{c}_3$. $\square$

If we define

$$\Psi_{0klm}(t_1,t_2) = \psi_0^{\lambda^2}(t_1^2+t_2^2)\psi_k(K(1,\tfrac{t_2}{t_1}))\psi_l^{\alpha}(\gamma'(1,\tfrac{t_2}{t_1}))\psi_m^{\lambda}(t_1)$$

then the additional operator R_k^λ ensures that for $(t_1,t_2)\in \mathrm{supp}(\Psi_{0klm})$, $\lambda^{-m+1}\le t_1\le \lambda^{-m+2}$. In particular, for $(t_1,t_2),(s_1,s_2)\in \mathrm{supp}(\Psi_{0klm})$,

$$\lambda^{-3}\le \frac{t_1}{s_1}\le \lambda^3$$

The remaining analysis of the support of Ψ_{0klm} remains the same, and we note that the support of Ψ_{0klm} may be written as union of at most $N(d)$ regions, on each of which (letting $t=\frac{t_2}{t_1}$)

$$\left.\begin{aligned} 2^{-k-1}<|K(1,t)|<2^{-k+2}\\ K(1,t)\text{ is single signed}\\ \alpha^{-l-1}<|\gamma'(t)|<\alpha^{-l+2}\\ \gamma'(t)\text{ is single signed}\\ \lambda^{-m-1}<|t_1|<\lambda^{-m+2}\\ t_1\text{ is single signed}\end{aligned}\right\} \tag{2.0.18}$$

We denote the intersection of the region in $\mathbb{R}^2$ defined by (2.0.18) with Z_j by $R_{jn}(klm)$, where $n\in 1,\dots,N(d)$. Then define for $a>0$,

$$A_{jklmn}(a)=\{(t_1,t_2,s_1,s_2):(t_1,t_1s_1),(t_2,t_2s_2)\in R_{jn}(klm),s_2>s_1+a\}$$

Then we have the following lemma (an analogue of Lemma 2.6 in [11]).

Lemma 2. *Suppose that for fixed j (of Type I), n and $a>0$,*

$$\Bigg|\Bigg|\int_{A_{jklmn}(a)} e^{i(t_1^d t\gamma(s_1)+t_2^d t\gamma(s_2))}e^{i(x(t_1+t_2)+y(s_1t_1+s_2t_2))}|t_1t_2|^{\frac{d-2}{4}}|K(1,s_1)K(1,s_2)|^{\frac{1}{8}}$$
$$\psi_{mlk}(t_1,t_1s_1)\psi_{mlk}(t_2,t_2s_2)\hat{g}(t_1,t_1s_1)\hat{g}(t_2,t_2s_2)t_1t_2dt_1dt_2ds_1ds_2\Bigg|\Bigg|_{L^2_{\mathbb{R}^3}}$$
$$\le C||g||^2_{L^2_{\mathbb{R}^2}}, \tag{2.0.19}$$

with C depending only on d, and independent of j,k,l,m and a. Then

$$||\mathcal{E}_jUS_0^\lambda P_kQ_l^\alpha R_m^\lambda g||_{L^4(\mathbb{R}^3)}\le C||g||_{L^2(\mathbb{R}^2)},\forall g\in L^2(\mathbb{R}^2)$$

with C depending only on d and independent of j,k, l and m.

Next, we define a change of variables. On each of the $A_{jklm}(a)$, $K(1,s)$ is single signed. When $K(1,s)$ is positive, we define:

$$\begin{aligned} V(t_1,t_2,s_1,s_2) &= (u,v,w,z) \\ u &= t_1+t_2 \\ v &= t_1s_1+t_2s_2 \\ w &= t_1^d\gamma(s_1)+t_2^d\gamma(s_2) \\ z &= t_1^d\gamma(s_1)-t_2^d\gamma(s_2) \end{aligned}$$

For $K(s)$ negative we use instead:

$$z = t_1^d\gamma(s_1)-t_2^d\gamma(s_2).$$

(We note that the proof follows in a similar way for K negative). The argument proceeds as in [11] so we omit the details. It is possible to show that on $A_{jklm}(a)$ the Jacobian of V satisfies

$$\frac{1}{d-1}2^{-6-4d}2^{-k}|s_2-s_1| < |J_V(t_1,t_2,s_1,s_2)| < \frac{2^{9+4d}}{d-1}(d)2^{-k}|s_2-s_1|. \quad (2.0.20)$$

An application of the Inverse Function Theorem gives V is locally one-to-one. Additionally, by an argument given in [11] it can be shown that V is $N(d)$-to-one on $A_{jklm}(a)$, where $N(d)$ is a constant that depends only on d.

For fixed u_0,v_0,w_0 such that $(u_0,v_0,w_0,\overline{z})$ (for some $\overline{z}$) is in the image of $A_{jklm}(a)$ under V, define the set

$$\tilde{A}(u_0,v_0,w_0) = \{(t_1,t_2,s_1,s_2)\in A_{jklm}(a) : V(t_1,t_2,s_1,s_2) = (u_0,v_0,w_0,z) \text{ for some } z\}$$

where we have suppressed the references to j,k,l,m and a. It can be shown that $\tilde{A}(u_0,v_0,w_0)$ is the union of at most $N(d)$ connected components,

$$\tilde{A}(u_0,v_0,w_0) = \cup_r^{N(d)}\tilde{A}_r(u_0,v_0,w_0)$$

and the image of these components, $V_4(\tilde{A}_r(u_0,v_0,w_0)) = \mathcal{J}_r$, is an interval. Additionally $V_4 : \tilde{A}_r(u_0,v_0,w_0) \to \mathcal{J}_r$ is bijective.

We now claim, for $A_{jklm}(a)$ (recall Z_j is of type I) and fixed u_0,v_0,w_0, that the following inequality holds

$$\int_{V_4(\tilde{A}(u_0,v_0,w_0))} \sum_{V^{-1}(u_0,v_0,w_0,z)\cap\tilde{A}} \frac{|K(t_1,t_1s_1)||K(t_2,t_2s_2)|^{\frac{1}{4}}}{|J_V(t_1,t_2,s_1,s_2)|}$$

$$\Psi_{0klm}(t_1,t_1s_1)^2\Psi_{0klm}(t_2,t_2s_2)^2|t_1t_2|dz \leq C(d) \qquad (2.0.21)$$

where C is independent of k,l,m,a,u_0,v_0,w_0 and depends only on d. Before proceeding to prove this claim we state the following lemma, referring the proof to [11, Corollary 2.12a].

Lemma 3. *If, for fixed Z_j (of type I), k,l,m,a,u_0,v_0,w_0 (2.0.21) holds with constant C independent of k,l,m,a,u_0,v_0,w_0 and depending only on d, then*

$$||\mathcal{E}_j U S_0^{\lambda} P_k Q_l^{\alpha} R_m^{\lambda} g||_{L^4(\mathbb{R}^3)} \leq C||g||_{L^2(\mathbb{R}^2)}, \forall g \in L^2(\mathbb{R}^2)$$

with constant C depending on d, but independent of k,l and m.

We now proceed to prove our claim. From the above analysis, we know $V_4(\tilde{A}(u_0,v_0,w_0))$ is the union of at most $N(d)$ (not necessarily disjoint) intervals, $\{\mathcal{J}_r\}_{r=1}^{N(d)}$. We can replace the integral with a summation of integrals on each of the $\mathcal{J}_r$ and it suffices to prove

$$\int_{\mathcal{J}_r} \sum_{V^{-1}(u_0,v_0,w_0,z)\cap\tilde{A}} \frac{|K(t_1,t_1s_1)||K(t_2,t_2s_2)|^{\frac{1}{4}}}{|J_V(t_1,t_2,s_1,s_2)|}$$

$$\Psi_{0klm}(t_1,t_1s_1)^2\Psi_{0klm}(t_2,t_2s_2)^2|t_1t_2|dz \leq C(d).$$

A similar argument reduces matters to

$$\int_{\mathcal{J}_r} \sum_{V^{-1}(u_0,v_0,w_0,z)\cap\tilde{A}_r} \frac{|K(t_1,t_1s_1)||K(t_2,t_2s_2)|^{\frac{1}{4}}}{|J_V(t_1,t_2,s_1,s_2)|}$$

$$\Psi_{0klm}(t_1,t_1s_1)^2\Psi_{0klm}(t_2,t_2s_2)^2|t_1t_2|dz \leq C(d).$$

Since $V_4 : A_r(u_0,v_0,w_0) \to \mathcal{J}_r$ is bijective, the summation is in fact over just one term. Thus it suffices to bound, for each r:

$$\int_{\mathcal{J}_r} \frac{|K(t_1,t_1s_1)||K(t_2,t_2s_2)|^{\frac{1}{4}}}{|J_V(t_1,t_2,s_1,s_2)|}\Psi_{0klm}(t_1,t_1s_1)^2\Psi_{0klm}(t_2,t_2s_2)^2|t_1t_2|dz \leq C(d).$$

where (t_1,t_2,s_1,s_2) are uniquely determined by z. Recall that $(t_1,t_2,s_1,s_2) \in A_{jklm}(a)$, so using the estimates (2.0.18) and (2.0.20), along with the fact that

$\lambda < 2$, it suffices to prove

$$\int_{\mathcal{J}_r} \frac{2^{\frac{k}{2}}}{|s_2 - s_1|} dz \leq C(d)$$

where C depends only on d.

Since we are considering $A_{jklm}(a)$ where Z_j is of type j, we also have the estimate (2.0.2). Applying the mean value theorem we have, for some $s_3 \in (s_1, s_2)$,

$$\begin{aligned} \int_{\mathcal{J}_r} \frac{2^{\frac{k}{2}}}{|s_2 - s_1|} dz &= 2^{\frac{k}{2}} \int_a^b t_2^d \gamma'(s_3) \tfrac{1}{t_2^d(\gamma(s_2) - \gamma(s_1))} dz \\ &\leq 2^{2+2d} C \epsilon^{-\frac{1}{2}} \int_a^b \tfrac{1}{t_2^d(\gamma(s_2) - \gamma(s_1))} dz. \end{aligned}$$

Thus it is enough to show that

$$\int_{\mathcal{J}_r} \frac{1}{|t_2^d(\gamma(s_2) - \gamma(s_1))|} dz \leq C(d) \epsilon^{\frac{1}{2}}$$

Since $V_4 : A_r(u_0, v_0, w_0) \to V_4(A_r(u_0, v_0, w_0))$ is bijective, its inverse is well-defined, which we denote by $L = (L_1, L_2, L_3, L_4)$. Furthermore $V(L(z)) = (u_0, v_0, w_0, z)$, and the chain rule gives $DV(L(z))L'(z) = (0, 0, 0, 1)$. Routine calculations give:

$$\begin{aligned} L_3'(z) &= \tfrac{1}{|J_V(L(z))|} & \big[dL_2(z)[L_2(z)^{d-1}\gamma(L_4(z) - L_1(z)^{d-1}\gamma(L_3(z)))] \\ & & -L_2(z)^d \gamma'(L_4(z))(L_4(z) - L_3(z)) \big] \\ L_4'(z) &= \tfrac{1}{|J_V(L(z))|} & \big[-dL_1(z)[L_2(z)^{d-1}\gamma(L_4(z)) - L_1(z)^{d-1}\gamma(L_3(z))] \\ & & +L_1(z)^d \gamma'(L_3(z))(L_4(z) - L_3(z)) \big] \end{aligned}$$

Letting $G(z) = (\gamma(L_4(z)) - \gamma(L_3(z))$, then

$$\begin{aligned} G'(z) &= \gamma'(L_4(z))L_4'(z) - \gamma'(L_3(z))L_3'(z) \\ &= \frac{1}{|J_V(L(z))|} \big[(L_4(z) - L_3(z))\gamma'(L_3(z))\gamma'(L_4(z))(L_1^d(z) + L_2^d(z)) \\ &\quad -d(Ł_1 \gamma'(L_4(z)) + Ł_2 \gamma'(L_3(z))(Ł_2^{d-1}\gamma'(L_4(z)) - Ł_1^{d-1}\gamma'(L_3(z))) \big] . \end{aligned}$$

Supressing the dependence on z we can write this as

$$\begin{aligned} G'(z) &= \frac{1}{|J_V|} \big[(s_2 - s_1)\gamma'(s_1)\gamma'(s_2)(t_1 + t_2^d) \\ &\quad -d(t_1 \gamma'(s_2) + t_2 \gamma'(s_1)(t_2^{d-1}\gamma(s_2) - t_1^{d-1}\gamma(s_1)) \big] \end{aligned}$$

$$=: \quad \frac{1}{|J_V(t_1,t_2,s_1,s_2)|} R(t_1,t_2,s_1,s_2)$$

We note that,

$$|z| = |(t_2^d(\gamma(s_2)-\gamma(s_1)) - (t_2^d - t_2^d)\gamma'(s_1)| \tag{2.0.22}$$

and consider two cases:

(a) $A = \{z : |z| \le 2^{-(i_0+1)} \max\{|(t_2^d - t_1^d)\gamma(s_1)|, |(t_2^d(\gamma(s_2)-\gamma(s_1))|\}\}$

(b) $B = \{z : |z| \ge 2^{-(i_0+1)} \max\{|(t_2^d - t_1^d)\gamma(s_1)|, |(t_2^d(\gamma(s_2)-\gamma(s_1))|\}\}$,

where $i_0 = i_0(\epsilon, d) \ge 0$ is to be chosen later. We note that to be in case (a) it is necessary that the two terms on the right-hand side of (2.0.22) have opposite sign. We deal with these two cases in Lemmas 4 and 5 respectively.

Lemma 4.

$$\int_{\mathcal{J}_r \cap A} \frac{1}{|t_2^d(\gamma(s_2)-\gamma(s_1))|} dz \le C(d)\epsilon^{\frac{1}{2}}$$

Proof. Since A is defined by polynomial inequalities (of degree d), $\mathcal{J}_r \cap A$ is a union of at most d intervals. Therefore it suffices to consider just one of these intervals, which we label $[a,b]$.

$$\begin{aligned} I_i &:= \left\{(t_1,t_2,s_1,s_2) : 1+2^{-(i+1)} \le \left|\frac{t_2^d(\gamma(s_2)-\gamma(s_1))}{(t_2^d-t_1^d)\gamma(s_1)}\right| \le 1+2^{-i}\right\} \cap A_r(u_0,v_0,w_0) \\ \tilde{I}_i &:= \left\{(t_1,t_2,s_1,s_2) : 1+2^{-(i+1)} \le \left|\frac{(t_2^d-t_1^d)\gamma(s_1)}{t_2^d(\gamma(s_2)-\gamma(s_1))}\right| \le 1+2^{-i}\right\} \cap A_r(u_0,v_0,w_0) \end{aligned}$$

We remark it follows from our assumption on $|z|$ that

$$\begin{aligned} 1+2^{-(i_0+1)} &\le \left|\frac{t_2^d(\gamma(s_2)-\gamma(s_1))}{(t_2^d-t_1^d)\gamma(s_1)}\right| \le 1+2^{-i_0} \\ 1+2^{-(i_0+1)} &\le \left|\frac{(t_2^d-t_1^d)\gamma(s_1)}{t_2^d(\gamma(s_2)-\gamma(s_1))}\right| \le 1+2^{-i_0} \end{aligned}$$

so we only need to consider $i \ge i_0$. On $I_i \cup \tilde{I}_i$ we have

$$2^{-i-2}|t_2^d(\gamma(s_2)-\gamma(s_1))| \quad \le \quad 2^{-i-1}\min\{|(t_2^d-t_1^d)\gamma(s_1)|, |(t_2^d(\gamma(s_2)-\gamma(s_1))|\}$$

$$
\begin{aligned}
&\leq \ |z| \\
&\leq \ 2^{-i}\min\{|(t_2^d - t_1^d)\gamma(s_1)|, |(t_2^d(\gamma(s_2)-\gamma(s_1))|\} \\
&\leq \ 2^{-i}|t_2^d(\gamma(s_2)-\gamma(s_1))|
\end{aligned}
$$

Additionally we shall show that $|\frac{z}{z'}| \leq C(d)$ on each $I_i \cup \tilde{I}_i$ $(i \geq i_0)$, but assuming it in the meantime,

$$
\begin{aligned}
\int_a^b \frac{1}{|t_2^d(\gamma(s_2)-\gamma(s_1))|}dz &= \sum_{i=i_0}^{\infty}\int_{I_i\cup\tilde{I}_i}\frac{1}{|t_2^d(\gamma(s_2)-\gamma(s_1))|}dz \\
&\leq \sum_{i=i_0}^{\infty} 2^{-i}\int_{I_i\cup\tilde{I}_i}\frac{1}{|z|}dz \\
&\leq \sum_{i=i_0}^{\infty} 2^{-i}\frac{1}{|z'|}\int_{I_i\cup\tilde{I}_i} C(d)dz \\
&\leq 2\sum_{i=i_0}^{\infty} 2^{-i}C(d) \\
&= 2^{-i_0+2}C(d) \qquad (2.0.23)
\end{aligned}
$$

Next we prove that $|\frac{z}{z'}| \leq C(d)$ on $I_i \cup \tilde{I}_i$. Since

$$
\left|\frac{z}{z'}\right| \ \leq \ 4\lambda^{4d}\frac{(\gamma(s_2)-\gamma(s_1))(z)}{(\gamma(s_2)-\gamma(s_1))(z')} = 4\lambda^{4d}\left|\frac{G(z)}{G(z')}\right|
$$

it suffices to bound $|\frac{G(z)}{G(z')}|$ by constant depending only on d, (we can assume $|G(z')| \leq |G(z)|$ as otherwise the bound is trivial)

$$
\begin{aligned}
\left|\frac{G(z)}{G(z')}\right| &\leq |1 + \frac{G(z)-G(z')}{G(z')}| \\
&\leq |1 + \frac{G'(\theta)(z-z')}{G(z')}|
\end{aligned}
$$

Using the inequality $|z - z'| \leq |z| + |z'| \leq 2\lambda^{2d}|G(z)|2^{-i}$

$$
\leq \left|1 + 2\lambda^{2d}\left|\frac{G'(\theta)}{2^i}\frac{G(z)}{G(z')}\right|\right|
$$

Thus we are done if we can show $\frac{\lambda^{2d}G'(\theta)}{2^{i-1}} \leq \frac{1}{2}$ for $i \geq i_0$. A trivial estimate gives

$$
|G'(z)| \ \leq \ \frac{1}{|J_V|}\left\{|(s_2-s_1)\gamma'(s_1)\gamma'(s_2)(|t_1^d| + |t_2^d|)|\right.
$$

$$
\begin{aligned}
&+ d|(t_2^{d-1} - t_1^{d-1})\gamma(s_1)| \cdot |t_1\gamma'(s_2) + t_2\gamma'(s_1)| \\
&+ d|t_2^{d-1}| \cdot |(\gamma(s_2) - \gamma(s_1)| \cdot |t_1\gamma'(s_2) + t_2\gamma'(s_1)|\} \\
= \;& I + II + III
\end{aligned}
$$

Then using the estimates (2.0.18) and (2.0.20) gives

$$
\begin{aligned}
I &\leq (d-1)2^{6+2d}\alpha^{2(-l+2)}2^{-k} \\
&\leq \frac{(d-1)2^{16+2d}}{\epsilon}.
\end{aligned}
$$

For II we observe

$$
\begin{aligned}
|J_V|II &= d(t_2^{d-1} - t_1^{d-1})\gamma(s_1) \cdot |t_1\gamma'(s_2) + t_2\gamma'(s_1)| \\
&= d(t_2^d - t_1^d)\gamma(s_1)\frac{(t_2^{d-1} - t_1^{d-1})}{(t_2^d - t_1^d)} \cdot |t_1\gamma'(s_2) + t_2\gamma'(s_1)| \\
&\leq d(t_2^d - t_1^d)\gamma(s_1)\frac{\lambda^3(d-1)}{d} \cdot |t_1\gamma'(s_2) + t_2\gamma'(s_1)| \\
&\leq 2d|t_2^d| \cdot |(\gamma(s_2) - \gamma(s_1)|\frac{\lambda^3(d-1)}{d} \cdot |t_1\gamma'(s_2) + t_2\gamma'(s_1)| \\
&\leq \frac{\sqrt{2}\lambda^2(d-1)}{d} d|t_2^{d-1}| \cdot |(\gamma(s_2) - \gamma(s_1)| \cdot |t_1\gamma'(s_2) + t_2\gamma'(s_1)| \\
&\leq \frac{\sqrt{2}\lambda^2(d-1)}{d}|J_V||III| \\
&\leq 8|J_V||III|
\end{aligned}
$$

For III, using the estimates (2.0.18) and (2.0.20) again and the mean value theorem,

$$
\begin{aligned}
III &\leq d(d-1)2^{6+2d}2^k|t_2^{d-1}| \cdot \alpha^{-l+2} \cdot |t_1\gamma'(s_2) + t_2\gamma'(s_1)| \\
&\leq d(d-1)2^{5+4d}2^k\alpha^{2(-l+2)} \\
&\leq \frac{d(d-1)2^{13+4d}}{\epsilon}
\end{aligned}
$$

So,

$$
|G'(\theta)| \leq \frac{(d-1)}{\epsilon}\left(2^{16+2d} + 9d2^{13+4d}\right) = \frac{c_d}{2^{2+4d}\epsilon}
$$

Recall that the choice of i_0 is at our disposal. Since we want $|G'(\theta)| \leq \frac{2^i}{4\lambda^{2d}}$

for $i \geq i_0$, then it suffices to take i_0 be the integer which satisfies

$$\frac{c_d}{\epsilon} \leq 2^{i_0} < 2\frac{c_d}{\epsilon}$$

Furthermore substituting this into (2.0.23) gives,

$$\int_a^b \frac{1}{|t_2^d(\gamma(s_2) - \gamma(s_1))|} dz \leq C(d)\epsilon$$
$$\leq C(d)\epsilon^{\frac{1}{2}},$$

where we have used the fact $\epsilon < 1$. □

Lemma 5.

$$\int_{\mathcal{J}_r \cap B} \frac{1}{|t_2^d(\gamma(s_2) - \gamma(s_1))|} dz \leq C(d)\epsilon^{\frac{1}{2}}$$

Proof. As in Lemma 4, since B is defined by polynomial inequalities (of degree d), $\mathcal{J}_r \cap B$ is a union of at most d intervals. It again suffices to consider just one of these intervals, which we label $[a, b]$. By dividing the interval further if required we may assume that either $b \geq a \geq 0$ or $a \leq b \leq 0$.

Recall $G(z) = \gamma(L_4(z)) - \gamma(L_3(z))$, which we shall more succinctly write as $\gamma(s_2) - \gamma(s_1)$. We will show that

$$-C_1(d)\frac{1}{\epsilon} \leq \frac{G(z)G'(z)}{z} \leq -C_1(d)\frac{1}{\epsilon}. \tag{2.0.24}$$

Recall $G'(z) = \frac{R(z)}{J_V(L(z))}$ where

$$\begin{aligned} R(z) = \big\{ &(s_2 - s_1)\gamma'(s_1)\gamma'(s_2)(t_1^d + t_2^d) \\ &-d(t_2^{(d-1)}\gamma(s_2) - t_1^{(d-1)}\gamma(s_1))(t_1\gamma'(s_2) + t_2\gamma'(s_1)) \big\} \end{aligned} \tag{2.0.25}$$

Now since we are considering the region $s_2 > s_1$, it follows that $\text{sign}(G) = \text{sign}(\gamma')$. Additionally since $J_V(L(z)) > 0$, from (2.0.20) we have

$$0 \leq \frac{C_1(d)}{(d-1)}\gamma'(s_3)^2 2^k \leq \frac{G(z)\gamma'(s_3)}{J_V(L(z))} \leq \frac{C_2(d)}{(d-1)}\gamma'(s_3)^2 2^k. \tag{2.0.26}$$

We will obtain (2.0.24) once we have shown

$$-3(d-1) \leq \frac{R}{\gamma'(s_3)z} \leq -(d-1). \tag{2.0.27}$$

We have:

$$
\begin{aligned}
\frac{R}{\gamma'(s_3)z} =& \Big\{(s_2-s_1)\frac{\gamma'(s_1)\gamma'(s_2)}{\gamma'(s_3)}(t_1^d+t_2^d) \\
& -d(t_2^{(d-1)}\gamma(s_2)-t_1^{(d-1)}\gamma(s_1))\frac{(t_1\gamma'(s_2)+t_2\gamma'(s_1))}{\gamma'(s_3)}\Big\} \\
=& \Big\{(s_2-s_1)\frac{\gamma'(s_1)\gamma'(s_2)}{\gamma'(s_3)}(t_1^d+t_2^d) \\
& -d(t_2-t_1)t_1^{(d-1)}\gamma(s_1)\frac{t_1\gamma'(s_2)+t_2\gamma'(s_1)}{t_2\gamma'(s_3)} \\
& -d\frac{t_1\gamma'(s_2)+t_2\gamma'(s_1)}{t_2\gamma'(s_3)}\Big\}.
\end{aligned}
$$

Thus it suffices to show $|\frac{R}{z\gamma'(s_3)}+2(d-1)|\leq(d-1)$. So,

$$
\begin{aligned}
\frac{R}{z\gamma'(s_3)}+2(d-1) &= \frac{(s_2-s_1)\frac{\gamma'(s_1)\gamma'(s_2)}{\gamma'(s_3)}(t_1^d+t_2^d)-2t_2^d(\gamma(s_2)-\gamma(s_1))}{z} \\
& \quad +\left(2(t_2^d-t_1^d)-d\tfrac{t_1\gamma'(s_2)+t_1\gamma'(s_2))}{t_2\gamma'(s_3)}t_1^{d-1}(t_2-t_1)\right)\frac{\gamma(s_1)}{z} \\
& \quad -d(2-\frac{t_1\gamma'(s_2)+t_2\gamma'(s_1)}{t_2\gamma'(s_3)}) \\
&= I+II+III
\end{aligned}
$$

For I, we note that $|z|\geq\frac{1}{2^{i_0+1}}t_2^d(\gamma(s_2)-\gamma(s_1))$, so

$$
\begin{aligned}
I &\leq 2^{i_0+1}\left|\frac{\gamma'(s_1)\gamma'(s_2)}{(\gamma'(s_3))^2}\frac{(t_1^d+t_2^d)}{t_2^d}-2\right| \\
&\leq 2^{i_0+1}2\left|\frac{\gamma'(s_1)\gamma'(s_2)}{(\gamma'(s_3))^2}-1\right| \\
& \quad +2^{i_0+1}\left|\left(1-(\tfrac{t_1}{t_2})^d\right)\frac{\gamma'(s_1)\gamma'(s_2)}{\gamma'(s_3)^2}\right| \\
&\leq 2^{i_0+2}\left(|\alpha^3-1|+2^2|\lambda^{3d}-1|\right) \\
&\leq 5\delta 2^{i_0+2} \\
&\leq \frac{1}{3}(d-1)
\end{aligned}
$$

For II, we use that $|z|\geq\frac{1}{2^{i_0+1}}|\gamma(s_1)(t_2^d-t_1^d)|$. By the mean value theorem, for some t_3 between t_1 and t_2 we have

$$
II \leq 2^{i_0+1}\left|2-d[\tfrac{t_1\gamma'(s_2)+t_2\gamma'(s_1))}{t_2\gamma'(s_3)}]t_1^{d-1}\frac{(t_2-t_1)}{(t_2^d-t_1^d)}\right|
$$

$$
\begin{aligned}
&\leq 2^{i_0+1}\left|2-[\tfrac{t_1\gamma'(s_2)}{t_2\gamma'(s_3)}+\tfrac{\gamma'(s_1))}{\gamma'(s_3)}]t_1^{d-1}\frac{1}{t_3^{d-1}}\right| \\
&\leq 2^{i_0+1}\left|1-\tfrac{\gamma'(s_1)}{\gamma'(s_3)}\right|+2^{i_0+1}\left|\tfrac{\gamma'(s_1)}{\gamma'(s_3)}\right|\left|1-\left(\tfrac{t_1}{t_3}\right)^{d-1}\right| \\
&\quad +2^{i_0+1}\left|1-\tfrac{\gamma'(s_2)}{\gamma'(s_3)}\right|+2^{i_0+1}\left|\tfrac{\gamma'(s_2)}{\gamma'(s_3)}\right|\left|1-\left(\tfrac{t_1^d}{t_3^{d-1}t_2}\right)\right| \\
&\leq 2^{i_0+2}\left(|\alpha^3-1|+|2^3(\lambda^{3d}-1)|\right) \\
&\leq 9\delta 2^{i_0+2} \\
&\leq \frac{1}{3}(d-1)
\end{aligned}
$$

Finally for III we have the easy estimate

$$
III \leq d\left|(\frac{t_1}{t_2}-1)\right| \leq d(\lambda^3-1) \leq d\delta \leq \frac{1}{3}(d-1)
$$

and so,

$$
I+II+II \leq (d-1)
$$

We now assume that $z>0$ on our interval of integration. We have the following bound:

$$
-2C_2(d)\frac{1}{\epsilon}z \leq \frac{d}{dz}(G(z)^2) \leq -2C_1(d)\frac{1}{\epsilon}z, \tag{2.0.28}
$$

Integrating and rearranging we get:

$$
G(z)^2 \geq \frac{c(d)}{\epsilon}(b^2-z^2)+G(b)^2.
$$

Together with the fact that $G(z)$ is singled-signed on $[a,b]$,

$$
\begin{aligned}
\int_a^b \frac{1}{|t_2^d(\gamma(s_2)-\gamma(s_1))|}dz &\leq \left|\int_a^b \frac{1}{G(z)}dz\right| \\
&\leq \int_a^b \left(\frac{\epsilon}{C(d)\sqrt{b^2-z^2}}\right)^{\frac{1}{2}} dz \\
&\leq C(d)\epsilon^{\frac{1}{2}}\int_a^b \frac{1}{\sqrt{b^2-z^2}}dz \\
&\leq C(d)\epsilon^{\frac{1}{2}}
\end{aligned}
$$

The case $a \leq z \leq b \leq 0$ follows a similar argument, and since in case (b) we have a lower bound on $|z|$, $z \neq 0$. □

This, together with the remarks made at the beginning completes the proof.

□

Chapter 3

An affine isoperimetric inequality for polynomial and convex curves

In this chapter we first prove an affine isoperimetric type inequality for polynomial curves in $\mathbb{R}^n$ (Section 3.3). In Section 3.4 we establish an abstract theorem and in Section 3.5 discuss the application of that theorem to $(n+1)$-crossing curves.

First we make a few definitions. A *polynomial curve*, $\Gamma(t) : I \to \mathbb{R}^n$, is one which we can be expressed as $\Gamma(t) = (p_1(t), ..., p_n(t))$ where $\{p_i\}_{i=1}^n$ are polynomials. The degree of a polynomial curve Γ is defined to be the maximum of the degrees of its components:

$$\deg(\Gamma) = \max_{i=1,\ldots,n} \deg(p_i).$$

For general curves $\Gamma = (\gamma_1, \ldots, \gamma_n) : I \to \mathbb{R}^n$, the curve is defined to be *single signed* if each γ_i is single signed. If $I = \cup_{i=1}^M I_i$ is a partition of I into intervals such that $\{\Gamma|_{I_i}\}_{i=1}^M$ are all single signed, then Γ is said to change sign no more than M times. Finally a curve is *regular* if its derivative never vanishes, $\Gamma'(t) \neq (0, \ldots, 0)$ $\forall t \in I$.

Now we can state the first main result of this chapter.

Theorem 5. *There exists a constant, $C = C(n,d)$, such that for any polynomial curve, $\Gamma : I \to \mathbb{R}^n$ of degree at most d, the following inequality holds:*

$$\text{AffineArclength}(\Gamma) \leq C(n,d)\,\text{Vol}(\text{ConvexHull}(\Gamma))^{\frac{2}{n(n+1)}}. \tag{3.0.1}$$

Although we provide a direct proof here, Theorem 5 is also an immediate consequence of the restriction inequality to polynomial curves (established by Dendrinos and Wright in [14]) via the argument presented in the introduction.

We start by establishling a claim mentioned in the Introduction.

Proposition 2. *A polynomial curve of degree d that is not embedded in any hyperplane is d-crossing.*

Proof. Let $\Gamma(t) = (p_1(t), \dots, p_n(t))$ be a polynomial curve of degree d in $\mathbb{R}^n$. Given a hyperplane, h, with normal $\eta = (\eta_1, \dots, \eta_n)$ and containing the point $a = (a_1, \dots, a_n)$, let $A = a \cdot \eta$. Then $\Gamma(t)$ intersects h whenever

$$\Gamma(t) \cdot \eta = A. \tag{3.0.2}$$

Equivalently, $\Gamma(t)$ intersects h at the zeroes of the polynomial

$$q(t) = p_1(t)\eta_1 + \dots p_n(t)\eta_n - A,$$

which has degree at most d. If $q(t) \equiv 0$ then $\{A, p_1, \dots, p_n\}$ are linearly dependent and Γ is embedded in h. Otherwise (3.0.2) has at most d solutions, and $\Gamma(t)$ is d-crossing. □

3.1 Convex curves in the plane

In the plane, convex (in the sense of 2-crossing) curves form a connected (proper) subset of a boundary of a strictly convex subset. We cannot replace strict convexity with convexity here, because by our definition k-crossing curves in $\mathbb{R}^2$ cannot contain straight line segments.

Lemma 6. *Let $\Gamma : I \to \mathbb{R}^2$ be a convex curve, then its image forms a connected (proper) subset of the boundary of a strictly convex set.*

Proof. Let $\Gamma(t) : [0, 1] \to \mathbb{R}^2$ be a 2-crossing curve. We first note that it cannot self-intersect, because if $t_0 \neq t_1$ were such that $\Gamma(t_0) = \Gamma(t_1)$, then a line through the point of self-intersection and $\Gamma(t_3)$, for any $t_3 \neq t_1, t_2$ would meet the curve at three points: t_1, t_2 and t_3.

We claim that we can extend this curve (formally, an extension of the mapping Γ to $\tilde{\Gamma} : I \supset [0, 1] \to \mathbb{R}^2$), so that the extended curve is closed, and in such a way so that it is still 2-crossing. The details of this construction, and the proof of the claim we leave for the appendix (Appendix A).

We note that since $\tilde{\Gamma}$ is 2-crossing, $\tilde{\Gamma}$ is a Jordan curve, so the notion of an interior and exterior are well defined. Next we show the interior of $\tilde{\Gamma}$, K^0, is convex. By way of contradiction suppose there exists $x_1, x_2 \in K$ such that there exists $z \in K^c \cap \{x_1 t + (1-t)x_2 : t \in (0,1)\}$. Then there exists $y_i \in \partial K \cap \{x_i t + (1-t)z : t \in (0,1)\}$, $i = 1, 2$. Furthermore, since K is bounded, there exists $y_3 \in \partial K \cap \{x_1 - (x_2 - x_1)t : t > 0\}$. Consequently we have found three co-linear points in ∂K, which contradicts the fact that the curve is 2-crossing.

Similarly, the fact that there cannot be three co-linear points in ∂K, proves that K is a strictly convex set, of which Γ forms a connected subset of its boundary. $\square$

Remark. *Unfortunately, as demonstrated by this lemma, the set of n-crossing curves in $\mathbb{R}^n$ excludes all closed curves. In $\mathbb{R}^2$, for example, this means curves we might reasonably describe as convex are formally not 2-crossing, for example:*

$$(\cos(2\pi t), \sin(2\pi t)) : [0,1] \to \mathbb{R}^2.$$

Such a family of curves could be included if we modified the definition of k-crossing to count the endpoints of the curve as one-crossing if they both lie on the given hyperplane. This however, complicates the definition, and we simply remark that such curves can be written as the union of two n-crossing curves.

3.2 Preliminary results

In order to prove Theorem 5 we shall need to make use of the following lemma.

Lemma 7. *Given a polynomial curve, $\Gamma : I \to \mathbb{R}^2$ of degree d, which is not embedded in a line, there exists a disjoint partition $\{I_i\}_{i=1}^N$ of I, such that $\Gamma|_{I_i}$ is 2-crossing. N is bounded by a constant $M = M(d)$, which depends only on d.*

Proof. We first consider the polynomial

$$h(t) = p''(t)q'(t) - q''(t)p'(t).$$

We note that $h \not\equiv 0$ as otherwise $(\frac{p'}{q'})(t)$ (and $(\frac{q'}{p'})(t)$) are constant, and p and q are linearly dependent and Γ is embedded in a hyperplane. We first partition I into intervals, $\{J_j\}_{j=1}^N$ with the roots of h for endpoints. As $\max\{\deg(p), \deg(q)\} \leq d$, it follows $N \leq (d-1)(d-2)$.

We further break up each of these intervals into intervals on which p' is single signed. This leaves us with a final decomposition consisting of at most $(d-1)^2(d-2)$ intervals which form our $\{I_i\}_{i=1}^N$.

We now show that $\Gamma|_{I_i}$ is 2-crossing for an arbitrary I_i. Since p is monotone on I_i and strictly monotone on the interior of I_i it a one-to-one function. So $p^{-1}(t)$ is well defined. Reparametrising Γ we get

$$\tilde{\Gamma}(s) = (s, \gamma(s)) = (s, q \circ p^{-1}(s))$$

Since the definition of 2-crossing is invariant under reparametrisation, it suffices to show that $\tilde{\Gamma}$ is 2-crossing, i.e. $\gamma''(s)$ is strictly single-signed on the interior of $p(I_i)$.

By employing the chain rule (for two derivatives) and the quotient rule we can calculate:

$$\begin{aligned}(q \circ p^{-1})''(s) &= \frac{q''(p^{-1}(s))}{[p'(p^{-1}(s))]^2} - \frac{q'(p^{-1}(s))p''(p^{-1}(s))}{[p'(p^{-1}(s))]^3} \\ &= \frac{(h \circ p^{-1})(s)}{[(p' \circ p^{-1})(s)]^3}.\end{aligned}$$

Since $h \circ p^{-1}$ and $p' \circ p^{-1}$ are both strictly single-signed on the interior of $p(I_i)$, the result follows. □

3.3 Proof of Theorem 5

Proof of Theorem 5. We prove this theorem by induction on n, noting that the case $n = 2$ has been dealt with by Lemma 7, Lemma 6, and the classical affine isoperimetric inequality. Let $\Gamma(t)$ be a polynomial curve of degree at most d and let $K = \text{ConvexHull}(\{\Gamma(t) : t \in I\})$. We note that if it is embedded in a hyperplane its affine arclength vanishes and there is nothing to prove.

Because (3.0.1) is invariant under affine transformations, by applying an affine transformation we need only consider the case where the volume of this convex hull, $\text{Vol}_\text{n}(K)$, is 1. Thus it is sufficient to show

$$\text{AffineArclength}(\Gamma) \leq C(n, d)$$

where the constant $C(n, d)$ depends only on n and the degree of Γ.

Let E be K's Löwner-John ellipsoid. Again by applying a volume preserving affine transformation, we can further reduce matters to where E is a ball centred at the origin (with radius $r \leq C(n)$). Consequently,

$$B_r \subset K \subset B_{nr} \tag{3.3.1}$$

We observe

$$L_\Gamma(t) := \det \begin{vmatrix} p_1'(t) & \cdots & p_1^{(n)}(t) \\ \vdots & \vdots & \vdots \\ p_n'(t) & \cdots & p_n^{(n)}(t) \end{vmatrix}.$$

Expanding about the last column, we have

$$\begin{aligned} |L_\Gamma(t)| &= \left| \begin{pmatrix} L_{\Gamma_1} \\ \vdots \\ L_{\Gamma_n} \end{pmatrix} \cdot \begin{pmatrix} (-1)^0 p_1^{(n)}(t) \\ \vdots \\ (-1)^{n-1} p_n^{(n)}(t) \end{pmatrix} \right| \\ &\le \left| \begin{pmatrix} L_{\Gamma_1} \\ \vdots \\ L_{\Gamma_n} \end{pmatrix} \right| \left| \begin{pmatrix} p_1^{(n)}(t) \\ \vdots \\ p_n^{(n)}(t) \end{pmatrix} \right| \end{aligned} \tag{3.3.2}$$

where $\Gamma_i = (p_1(t), ..., p_{i-1}(t), p_{i+1}(t), p_n(t))$ is Γ with the i-th component removed. Now

$$\text{AffineArclength}(\Gamma) = \int_I |L_\Gamma(t)|^{\frac{2}{n(n+1)}}\, dt.$$

By inequality (3.3.2) and Hölder's inequality,

$$\begin{aligned} \text{AffineArclength}(\Gamma) &\le \int_I \left|\Gamma^{(n)}(t)\right|^{\frac{2}{n(n+1)}} \left| \begin{pmatrix} L_{\Gamma_1}(t) \\ \vdots \\ L_{\Gamma_n}(t) \end{pmatrix} \right|^{\frac{2}{n(n+1)}} dt \\ &\le \underbrace{\left[\int_I |\Gamma^{(n)}(t)|^{\frac{1}{n}} dt\right]^{\frac{2}{n+1}}}_{=A} \underbrace{\left[\int_I \left| \begin{pmatrix} L_{\Gamma_1}(t) \\ \vdots \\ L_{\Gamma_n}(t) \end{pmatrix} \right|^{\frac{2}{(n-1)n}} dt \right]^{\frac{n-1}{n+1}}}_{=B}. \end{aligned} \tag{3.3.3}$$

First, focusing on the B term,

$$\begin{aligned} B &= \left[\int_I \left(\sum_{i=1}^n |L_{\Gamma_i}(t)|^2 \right)^{\frac{1}{(n-1)n}} dt \right]^{\frac{n-1}{n+1}} \\ &\le \left[\sum_{i=1}^n \int_I |L_{\Gamma_i}(t)|^{\frac{2}{(n-1)n}} dt \right]^{\frac{n-1}{n+1}} \end{aligned} \tag{3.3.4}$$

$$\leq \sum_{i=1}^{n} \left[\int_I |L_{\Gamma_i}(t)|^{\frac{2}{(n-1)n}} dt \right]^{\frac{n-1}{n+1}}$$

$$= \sum_{i=1}^{n} (\text{AffineArclength}(\Gamma_i))]^{\frac{n-1}{n+1}}.$$

Note that $\Gamma_i : I \to \mathbb{R}^{n-1}$ is a polynomial curve in $\mathbb{R}^{n-1}$ of degree at most d. Furthermore, by (3.3.1), $\sup_{t\in I} |\Gamma_i(t)| \leq \sup_{t\in I} |\Gamma(t)| \leq C(n)$ for each i, so it follows that

$$\text{Vol}_{n-1}(\text{ConvexHull}(\Gamma_i)) \leq C(n)$$

where the constant $C(n)$ has changed.

By employing the inductive hypothesis we get,

$$\text{AffineArclength}(\Gamma_i) \leq C(n-1, d).$$

So $B \leq C(n, d)$, where the constant has changed but still depends only on n and d.

Now we turn to the A term. By Bernstein's inequality ([5, p.39]) $|\Gamma^{(n)}(t)| \leq \frac{n^{\frac{1}{2}} d!}{(d-n)!} \frac{1}{|I|^n} \sup_{t\in I} |\Gamma(t)|$. Hence,

$$A = \left[\int_I |\Gamma^{(n)}(t)|^{\frac{1}{n}} dt \right]^{\frac{2}{n+1}} \tag{3.3.5}$$

$$\leq \left[\int_I \left(\tfrac{d!}{(d-n)!} \right)^{\frac{1}{n}} \frac{1}{|I|} | \sup_{t\in I} |\Gamma(t)| dt \right]^{\frac{2}{n+1}}$$

$$\leq \left[\int_I \left(\tfrac{d!}{(d-n)!} \right)^{\frac{1}{n}} \frac{C(n)}{|I|} dt \right]^{\frac{2}{n+1}}$$

$$\leq C(n, d).$$

Consequently in (3.3.3) we get

$$\text{AffineArclength}(\Gamma) \leq C(n, d).$$

□

3.4 An abstract proof of the affine isoperimetric inequality

In the previous section we focused solely on polynomial curves, in this section we aim to establish the result for an abstract family of curves.

We shall consider the family of curves $\{\mathcal{F}_m\}_{m\geq 2}$ where $\mathcal{F}_m$ represents a set of curves in $\mathbb{R}^m$ statisyfing the following:

A1 The family $\mathcal{F}_m$ is affine-invariant; that is, for every curve $\Gamma \in \mathcal{F}_m$ and any affine transformation $A : \mathbb{R}^m \to \mathbb{R}^m$, the curve $A\Gamma$ lies in $\mathcal{F}_m$.

A2 For $\Gamma = (\gamma_1, ..., \gamma_n) \in \mathcal{F}_m$, $k \leq m$ and any $0 < \alpha_1 < \ldots < \alpha_k$ where $\alpha_j \in \{1, \ldots, n\}$ then the projection $P_{\alpha_1,...,\alpha_k}\Gamma = (\gamma_{\alpha_1}, ..., \gamma_{\alpha_k})$ lies in $\mathcal{F}_k$.

A3 The affine isoperimetric inequality holds with some uniform constant C_2 for all $\Gamma \in \mathcal{F}_2$:

$$\text{AffineSurf}(\Gamma) \leq C_2 \,\text{Vol}(\text{ConvexHull}(\{\Gamma(t) : t \in I\}))^{\frac{1}{3}}.$$

A4 For each $m \geq 2$ there exists $k = k(m)$, $0 \leq k \leq m$ such that the following Bernstein-type inequality holds:

$$\int_I |\Gamma^{(m)}(t)|^{\frac{1}{m}} dt \leq B_{m,k} \left(\sup_{t\in I} |\Gamma^{(m-k)}(t)||I|^{(m-k)} \right)^{\frac{1}{m}} \qquad (3.4.1)$$

A5 For each $m \geq 2$ any $R > 0$, there is a constant $D = D(m, R)$ such that $\sup_{t\in I} |\Gamma^{(m-k(m))}(t)||I|^{m-k(m)} \leq D$ for every $\Gamma : I \to \mathbb{R}^m$ lying in $\mathcal{F}_m$ with the normalisation $\Gamma \subset B_R$. Here $k(m)$ is given by (A4).

Theorem 6. *Suppose $\mathcal{F} = \cup_{n\geq 2}\mathcal{F}_n$ is a family of curves in the Euclidean spaces which satisfies the properties (A1)-(A5) above. Then For every $n \geq 2$ there exists a constant C_n which depends on n such that for $\Gamma \in \mathcal{F}_n$*

$$\text{AffineArclength}(\Gamma) \leq C_n \,\text{Vol}(\text{ConvexHull}(\Gamma))^{\frac{2}{n(n+1)}}. \qquad (3.4.2)$$

Proof of Theorem 6. We start by normalising the curve Γ. Let $K = \text{ConvexHull}(\{\Gamma(t) : t \in I\})$. We note that if it is embedded in a hyperplane its affine arclength vanishes and there is nothing to prove, therefore we may assume it has positive volume. By (A1) and the affine invariant nature of the affine isoperimetric inequality we need only consider the case where the volume of this convex hull, $\mathrm{Vol_n}(K)$, is 1. Thus it is sufficient to show

$$\text{AffineArclength}(\Gamma) \leq C(n)$$

where the constant $C(n)$ depends only on n.

We prove this theorem by induction in n, noting that the case $n = 2$ is dealt with by assumption (A3). The inductive argument reduces the affine isoperimetric inequality for $\Gamma \in \mathcal{F}_n$ to an estimate of the product of two pieces.

To estimate the first piece (labelled B below) we invoke the inductive hypothesis on each of the n co-ordinate projections of Γ into $\mathbb{R}^{n-1}$; that is, the n projections of the form: $(\gamma_1, \dots, \gamma_{k-1}, \gamma_{k+1}, \dots \gamma_n)$ where $1 \leq k \leq n$. For this inductive argument to work it is critical that we can control the $(n-1)$-dimension volume of the convex hull of each of these projections. Thus we must first 'square' the curve Γ, so that the convex hull of each of these projections have comparable $(n-1)$-volume, and thus have volume comparable to the convex hull of Γ.

We achieve this by a second, volume preserving, affine transformation. Let E be K's Löwner-John ellipsoid. Let A be the affine transformation which maps the ellipsoid E to ball centred at the origin (with radius $r \leq C(n)$) of the same volume. Relabelling $A\Gamma \in \mathcal{F}_n$ as Γ and defining K as above, we have:

$$B_r \subset K \subset B_{nr} \tag{3.4.3}$$

To estimate the second piece (labelled A below) we use the Bernstein-type inequality (3.4.1), and the estimate in (A5) with $D = D(n, R)$ where $R = nC(r)$ (here again we make use the fact we have 'squared' the curve Γ),

In a similar vein to (3.3.2):

$$|L_\Gamma(t)| \leq \left| \begin{pmatrix} \gamma_1^{(n)}(t) \\ \vdots \\ \gamma_n^{(n)}(t) \end{pmatrix} \right| \left| \begin{pmatrix} L_{\Gamma_1} \\ \vdots \\ L_{\Gamma_n} \end{pmatrix} \right|$$

where $\Gamma_i = (\gamma_1(t), ..., \gamma_{i-1}(t), \gamma_{i+1}(t), \gamma_n(t))$ is Γ with the i-th component removed. Thus we have

$$\text{AffineArclength}(\Gamma) = \int_I |L_\Gamma(t)|^{\frac{2}{n(n+1)}} \, dt$$

$$\leq \int_I \left|\Gamma^{(n)}(t)\right|^{\frac{2}{n(n+1)}} \left|\begin{pmatrix} L_{\Gamma_1}(t) \\ \vdots \\ L_{\Gamma_n}(t) \end{pmatrix}\right|^{\frac{2}{n(n+1)}} dt$$

$$\leq \underbrace{\left[\int_I |\Gamma^{(n)}(t)|^{\frac{1}{n}} dt\right]^{\frac{2}{n+1}}}_{=A} \underbrace{\left[\int_I \left|\begin{pmatrix} L_{\Gamma_1}(t) \\ \vdots \\ L_{\Gamma_n}(t) \end{pmatrix}\right|^{\frac{2}{(n-1)n}} dt\right]^{\frac{n-1}{n+1}}}_{=B}. \tag{3.4.4}$$

First we bound the B term,

$$\begin{aligned} B &= \left[\int_I \left(\sum_{i=1}^n |L_{\Gamma_i}(t)|^2\right)^{\frac{1}{(n-1)n}} dt\right]^{\frac{n-1}{n+1}} \qquad (3.4.5) \\ &\leq \left[\sum_{i=1}^n \int_I |L_{\Gamma_i}(t)|^{\frac{2}{(n-1)n}} dt\right]^{\frac{n-1}{n+1}} \\ &\leq \sum_{i=1}^n \left[\int_I |L_{\Gamma_i}(t)|^{\frac{2}{(n-1)n}} dt\right]^{\frac{n-1}{n+1}} \\ &= \sum_{i=1}^n (\text{AffineArclength}(\Gamma_i))]^{\frac{n-1}{n+1}}. \end{aligned}$$

By assumpion (A2) $\Gamma_i \in \mathcal{F}_{n-1}$ for each $1 \leq i \leq n$. Additionally, we have the trivial bound $K_i = \text{ConvexHull}(\{\Gamma_i\}) \subset nB_r$ where here B_r is a ball in $\mathbb{R}^{n-1}$ of radius r. Recall $r < C(n)$.

Then by the induction hypothesis,

$$\begin{aligned} \text{AffineArclength}(\Gamma_i) &\leq C(n)\,\text{Vol}_{n-1}(K_i)^{\frac{2}{(n-1)n}} \\ &\leq C(n)\,\text{Vol}_{n-1}(B_1)^{\frac{2}{(n-1)n}} C(n)^{\frac{2}{n}} \\ &\leq C(n), \end{aligned}$$

where the constant $C = C(n)$ changes, but only depends on n. It follows that B is also bounded by a constant depending only on n.

For the A term we use (3.4.1) in (A2) and (A5):

$$
\begin{aligned}
A &= \left[\int_I |\Gamma^{(n)}(t)|^{\frac{1}{n}}dt\right]^{\frac{2}{n+1}} \\
&\leq B_{n,k}^{\frac{2}{n+1}}\left(\sup_{t\in I}|\Gamma(t)||I|\right)^{\frac{2}{n(n+1)}} \\
&\leq B_{n,k}^{\frac{2}{n+1}}(D)^{\frac{2}{n(n+1)}}
\end{aligned}
$$

where $D = C(n,R)$ and R is bounded by a constant depending on n. □

3.5 $(n+1)$-crossing curves in $\mathbb{R}^n$

In view of the abstract theorem of the previous section we shall now discuss what family of curves may satisfy the above conditions.

First we remark that (A1)-(A5) hold for polynomial curves, which we discussed in Section 3.3. (A1) and (A2) are trivial, and we established (A3) by Lemma 7 and Lemma 6. (A4), with $k = n$, is an immediate corollary of the Bernstein inequality, and for $k = n$, (A5) is a triviality.

Nevertheless, while polynomial curves are affine invariant, they are not geometric; that is, the family of polynomial curves is not closed under reparametrisation. Since the affine isoperimetric inequality is both affine and parametrisation invariant, it is reasonable to look for a class of curves which satisfy those properties. Additionally, such a class of curves should extend the notion of convexity to higher dimensions.

As remarked in the Introduction, n-crossing curves in $\mathbb{R}^n$ satisfy such properties. We now establish a slightly weaker variant of (A2) for $(n+1)$-crossing curves.

Lemma 8. *Let $\Gamma : I \to (\gamma_1, ..., \gamma_n)$ be an $(n+1)$-crossing curve, and for $n \geq k \geq 2$, let $P_{\alpha_1,...,\alpha_k}\Gamma = (\gamma_{\alpha_1}, ..., \gamma_{\alpha_k})$, where $1 \leq \alpha_1 < ... < \alpha_k \leq n$, be the projection on to the $(\alpha_1, ..., \alpha_k)$ plane. Then there exists a partition of $I = \uplus_{j=1}^{C(n,k)} I_j$ such that $\Gamma_{\alpha_1,...,\alpha_k}|_{I_j}$ is convex (k-crossing).*

Remark. *While in general, k-crossing curves ($k > n+1$) in $\mathbb{R}^n$ cannot be partitioned into n-crossing segments (see [4]), this lemma shows that given such a curve $\Gamma : I \to \mathbb{R}^n$, if you can 'extend' that curve to $\mathbb{R}^k$: $(\Gamma, \gamma_{n+1}, ..., \gamma_k) : I \to \mathbb{R}^k$ in such a way that its k-crossing property is preserved, then Γ can be partitioned into n-crossing segments.*

Proof. We prove this by induction on n. First consider $n = 2$. In this case $k = 2$. Given a 3-crossing curve $\Gamma : I \to (\gamma_1, \gamma_2)$, by Theorem 2, there exists a partition of $I = \uplus_{i=1}^{4} I_j$, such that $\Gamma|_{I_j}$ is 2-crossing.

For general n, we note that by Theorem 2 there exists a partition of $I = \uplus_{i=1}^{C(n)} I_j$, such that $\Gamma|_{I_j}$ is n-crossing. Consider the part of this curve restricted to an arbitrary interval, I_j. Since $\Gamma|_j$ is n-crossing in $\mathbb{R}^n$, it follows that $P_{1,\dots,\beta-1,\beta+1,\dots,n}(\Gamma|_{I_j}) : I_j \to \mathbb{R}^{n-1}$ is n-crossing in $\mathbb{R}^{n-1}$, for any β, $1 \le \beta \le n$. Choose $\beta \notin \{\alpha_1, ..., \alpha_k\}$. By the inductive hypothesis there exists a partition of $I_j = \{I_{j,i}\}_{i=1}^{C(n-1,k)}$ such that $P_{\alpha_1,\dots,\alpha_k}(P_{1,\dots,\beta-1,\beta+1,\dots,n}(\Gamma|_{I_j}))|_{I_{j,i}} = P_{\alpha_1,\dots,\alpha_k}(\Gamma)|_{I_{j,i}}$ is k-crossing. The desired partition is formed by enumerating over the $\{I_{j,i}\}$ of which there are at most $C(n)C(n-1,k)$ pieces. □

Although this doesn't quite establish (A2), for the purposes of the proof of Theorem 6 this weaker variant suffices.

We remark that, as demonstrated below, Lemma 8 allows us to extend Lemma 7 to n dimensions. Morally this demonstrates that n-crossing curves are in some sense a wider family of curves that polynomial curves of degree n.

Lemma 9. *Given a polynomial curve, $\Gamma : I \to \mathbb{R}^n$ of degree d, which is not embedded in a hyperplane, there exists a disjoint partition $\{I_i\}_{i=1}^{M}$, such that $\Gamma|_{I_i}$ is n-crossing and $M = M(n, d)$ depends only on n and d.*

Proof. We observe that by the hypothesis, $d \ge n$. To see this first note that $\{p_i\}_{i=1}^{n}$ cannot contain a constant term, and indeed any constant term is linearly independent of $\{p_i\}_{i=1}^{n}$ as otherwise Γ is contained with a hyperplane. Then $\{1, p_1, \dots, p_n\}$ for are $n+1$ polynomials of degree no greater than $n-1$, and so linearly dependent. It follows Γ is embedded inside a hyperplane. For $d = n$ we are done by Proposition 2. For $d = n+1$ we appeal to Theorem 2, and Proposition 2. For $d > n+1$ we iteratively choose

$$p_i \notin \text{Span}(1, p_1, ..., p_{i-1}) \text{ and degree}(p_i) \le d$$

for $i = n+1, ..., d$. The curve $\tilde{\Gamma} = (p_1, ..., p_d) : I \to \mathbb{R}^d$ is therefore not embedded in a hyperplane and has degree at most d. Consequently, it is d-crossing and so by Lemma 8 there exists a partition $I = \cup_{j=1}^{M(n,d)} I_j$ such that

$$\Gamma|_{I_j} = P_{1,\dots,n}(\tilde{\Gamma})|_{I_j}$$

is n-crossing. □

For $(n+1)$-crossing curves, (A3) follows from Lemma 8, Lemma 6 and the classical affine isoperimetric inequality.

We now establish that (A4) holds for each $1 \leq k \leq n-1$, under the additional property of bounded changes of sign of the nth derivative of $\Gamma \in \mathcal{F}_n$.

Proposition 3. *Let $\Gamma = (\gamma_1, ..., \gamma_n) : I \to \mathbb{R}^n$ be a curve such that $\Gamma^{(n)}$ changes sign at most M times. Then for each $1 \leq l \leq n-1$ there exists $C = C(n, l, M)$ such that*

$$\int_I |\Gamma^{(n)}(t)|^{\frac{1}{n}} dt \leq C_{n,l,M} \left(\sup_{t \in I} |\Gamma^{(n-l)}(t)||I|^{(n-l)} \right)^{\frac{1}{n}}$$

Proof. By assumption, for a given $i \in \{1, ..., n\}$, there exists a partition of I into M closed intervals with disjoint interiors such that $\gamma_i^{(n)}$ is single-signed. Additionally there also exists a partition of I into $C(n, M)$ intervals such that γ_i' is single-signed. Consequently there exists a partition $I = \cup_k I_k = \cup_k [a_k, a_{k+1}]$, where the $\{I_k\}_k$ are closed intervals with disjoint interiors, and $\gamma_i^{(n)}|_{I_k}$ and $\gamma_i'|_{I_k}$ are both single-signed for $i = 1, ..., n$. Furthermore this partition contains no more than $n^2 M C(n, M)$ intervals. We denote this constant by N.

By the trivial inequality

$$\int_I |\Gamma^{(n)}(t)|^{\frac{1}{n}} dt \leq \sum_{i=1}^{n} \int_I |\gamma_i^{(n)}(t)|^{\frac{1}{n}} dt,$$

it suffices to consider an arbitrary term in that summation, as the same argument will apply to the other $n-1$ terms. In the following γ represents a chosen, (but arbitrary) γ_i.

Let $\zeta_k = \text{sign}(\gamma^{(n)}|_{I_k})$, and define

$$h(t) = h_k(t) = \frac{(t - a_k)^{(n-2)} (a_{k+1} - t)^{(n-2)}}{(a_{k+1} - a_k)^{(n-3)}} \text{ for } t \in I_k.$$

Then $h_k(t) \leq (a_{k+1} - a_k)^{n-1}$. An application of Bernstein's inequality ([5, p.39]) gives us the following for any $0 \leq m \leq 2(n-2)$,

$$h_k^{(m)}(t) \leq \frac{(2(n-2))!}{(2(n-2)-m)!} (a_{k+1} - a_k)^{n-m-1}. \tag{3.5.1}$$

Now,

$$\int_I |\gamma^{(n)}(t)|^{\frac{1}{n}} dt = \sum_{k=1}^{N} \int_{I_k} (\gamma^{(n)}(t)\zeta_k)^{\frac{1}{n}} dt$$

$$= \sum_{k=1}^{N} \int_{I_k} (\gamma^{(n)}(t)\zeta_k h_k(t))^{\frac{1}{n}} (h_k(t))^{-\frac{1}{n}} dt$$

$$\leq \sum_{k=1}^{N} \Bigg(\underbrace{\int_{I_k} \gamma^{(n)}(t)\zeta_k h_k(t) dt}_{A_k} \Bigg)^{\frac{1}{n}} \Bigg(\underbrace{\int_{I_k} (h_k(t))^{-\frac{1}{n-1}} dt}_{B_k} \Bigg)^{\frac{n-1}{n}} .$$

The first terms can be bounded by iteratively applying integration by parts on each of the I_k using the fact that all the boundary terms vanish:

$$\begin{aligned} A_k &= \int_{I_k} \gamma^{(n)}(t)\zeta_k h_k(t) dt \\ &= \cancel{\left[\gamma^{(n-1)}(t)\zeta_k h_k(t)\right]_{a_k}^{a_{k+1}}} - \int_{I_k} \gamma^{(n-1)}(t)\zeta_k h_k'(t) dt \end{aligned}$$

$\ldots$ step l,

$$= \cancel{(-1)^{l+1} \left[\gamma^{(n-l)}(t)\zeta_k h_k^{(l-1)}(t)\right]_{a_k}^{a_{k+1}}} + (-1)^l \int_{I_k} \gamma^{(n-l)}(t)\zeta_k h_k^{(l)}(t) dt.$$

If we stop at step l, $1 \leq l \leq n-2$, we have the trivial inequality

$$\begin{aligned} |A_k| &\leq \left| \int_{I_k} \gamma^{(n-l)}(t)\zeta_k h_k^{(l)}(t) dt \right| \\ &\leq \sup_{t\in I} |\Gamma^{(n-l)}(t)| \left| \int_{I_k} h_k^{(l)}(t) dt \right|, \end{aligned}$$

using (3.5.1),

$$\begin{aligned} &\leq C(n,l) \sup_{t\in I} |\Gamma^{(n-l)}(t)| \left| \int_{I_k} (a_{k+1} - a_k)^{n-l-1} \, dt \right| \\ &\leq C(n,l) \sup_{t\in I} |\Gamma^{(n-l)}(t)| |I_k|^{(n-l)}. \end{aligned}$$

So for $1 \leq l \leq n-2$,

$$|A_k| \leq C(n,l) \sup_{t\in I} |\Gamma^{(n-l)}(t)| |I_k|^{(n-l)}.$$

If we instead continue up to step $n-2$:

$$\begin{aligned} A_k &= \cancel{(-1)^{n-1} \left[\gamma''(t)\zeta_k h_k^{(n-3)}(t)\right]_{a_k}^{a_{k+1}}} + (-1)^{n-2} \int_{I_k} \gamma''(t)\zeta_k h_k^{(n-2)}(t) dt \\ &\leq \left| \left[\gamma'(t)\zeta_k h_k^{(n-2)}(t)\right]_{a_k}^{a_{k+1}} - \int_{I_k} \gamma'(t)\zeta_k h_k^{(n-1)}(t) dt \right|. \end{aligned}$$

Expanding out the left hand term and using (3.5.1) with $m = n-1$ on the right

$$\leq \left|\left(\gamma'(a_{k+1})\zeta_k h_k^{(n-2)}(a_{k+1}) - \gamma'(a_k)\zeta_k h_k^{(n-2)}(a_k)\right)\right| + C(n)\left|\int_{I_k} \gamma'(t)dt\right|.$$

Now using (3.5.1) with $m = n-2$

$$\leq 3C(n)\sup_{t\in I}|\Gamma'(t)||I_k|$$

where $C(n)$ may change from line to line, but depends only n.

Returning now to the B terms,

$$B_k = \int_{a_k}^{a_{k+1}} (h_k(t))^{-\frac{1}{n-1}} dt$$

(letting $c_k = \frac{a_k+a_{k+1}}{2}$)

$$\begin{aligned}
&\leq \int_{a_k}^{c_k} (a_{k+1}-a_k)^{\frac{-1}{n-1}} \left(\frac{2}{(t-a_k)}\right)^{\frac{n-2}{n-1}} dt \\
&+ \int_{c_k}^{a_{k+1}} (a_{k+1}-a_k)^{\frac{-1}{n-1}} \left(\frac{2}{(a_{k+1}-t)}\right)^{\frac{n-2}{n-1}} dt \\
&\leq 2\left[\left(\frac{(t-a_k)}{(a_{k+1}-a_k)}\right)^{\frac{1}{n-1}}\right]_{a_k}^{c_k} - 2\left[\left(\frac{(a_{k+1}-t)}{(a_{k+1}-a_k)}\right)^{\frac{1}{n-1}}\right]_{c_k}^{a_{k+1}} \\
&\leq 2\left(\frac{(c_k-a_k)}{(a_{k+1}-a_k)}\right)^{\frac{1}{n-1}} + 2\left(\frac{(a_{k+1}-c_k)}{(a_{k+1}-a_k)}\right)^{\frac{1}{n-1}} \\
&\leq C(n).
\end{aligned}$$

It follows that, for each $1 \leq l \leq n-1$,

$$\begin{aligned}
\sum_{k=1}^{N}(A_k)^{\frac{1}{n}} B_k^{\frac{n-1}{n}} &\leq \sum_{k=1}^{N}\left(C(n)\sup_{t\in I}|\Gamma^{(n-l)}(t)||I|^{(n-l)}\right)^{\frac{1}{n}} \\
&\leq C(n,M)\left(\sup_{t\in I}|\Gamma^{(n-l)}(t)||I|^{(n-l)}\right)^{\frac{1}{n}}.
\end{aligned}$$

□

Lastly we would need to establish (A5). Unfortunately this is simply not true in general. However, if our family of curves had the additional property that it was closed under arclength reparametrisation, then it suffices for (A5) to hold for such curves with arclength parametrisation. For such curves we have $|\Gamma'(t)| = 1$ and by the following lemma $|I| \leq C(n)$.

Lemma 10. *Let* $\Gamma(t) : I \to \mathbb{R}^n$ *be a convex curve satisfying,*

$$\{\Gamma(t) : t \in I\} \subset \overline{B}_r,$$

where $\overline{B}_r$ *is a closed ball of radius* r*, then the arclength of* Γ *is bounded by a constant* $C(n,r)$ *depending only on* n *and* r*.*

Proof. We begin with the $n = 2$ case. Let $K \subset \overline{B}_r$ be the convex hull of Γ. Then the *nearest point projection* (Appendix B), $P : \overline{B}_r \to K$ is a contraction, which maps $\partial\overline{B}_r$ onto ∂K. It follows Arclength$(\Gamma) \leq 2\pi r$.

For $n > 2$ we reduce to the two dimensional case as follows. First write $\Gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$. By Lemma 8 there exists a partiton of $I = \cup_{i=1}^{M(n)} I_i$ such that each member of following family of two dimensional curves:

$$\{(\gamma_j, \gamma_{j+1}) : j = 1, .., n-1\},$$

is convex on each I_i. By assumption, $(\gamma_j, \gamma_{j+1}) \subset \overline{B}_r$ where $\overline{B}_r$ now denotes a ball in $\mathbb{R}^2$. The result for $n > 2$ then follows from the two dimensional case with the following inequality.

$$\begin{aligned}
\text{Arclength}(\Gamma) &= \int_I \left(\gamma_1^2(t) + \ldots + \gamma_n^2(t)\right)^{\frac{1}{2}} dt \\
&= \sum_{i=1}^{M(n)} \int_{I_i} \left(\gamma_1^2(t) + \ldots + \gamma_n^2(t)\right)^{\frac{1}{2}} dt \\
&\leq \sum_{i=1}^{M(n)} \int_{I_i} \sum_{j=1}^{n-1} \left(\gamma_j^2(t) + \gamma_{j+1}^2(t)\right)^{\frac{1}{2}} dt \\
&= \sum_{i=1}^{M(n)} \sum_{j=1}^{n-1} \text{Arclength}((\gamma_j, \gamma_{j+1})|_{I_i}).
\end{aligned}$$

□

Of course, because of the parametrisation invariant nature of the problem, it's reasonable to expect that this propery be reflected in the family of curves of Theorem 6. The condition that we would hope to get rid of is that of the bound on the changes of sign of $\Gamma^{(n)}$ because that property is intrinsically tied to our choice of parametrisation. With this in mind we make the following conjecture.

Conjecture 1. *There exists a constant, $C = C(n)$, such that for any $(n+1)$-crossing curve, $\Gamma : I \to \mathbb{R}^n$, the following inequality holds:*

$$\text{AffineArclength}(\Gamma) \leq C(n)\,\text{Vol}(\text{ConvexHull}(\Gamma))^{\frac{2}{n(n+1)}}. \tag{3.5.2}$$

where $C(n)$ depends only on n.

Whilst Conjecture 1, when compared to Theorem 5, would establish the inequality for a far wider class of curves and is parametrisation invariant, we note that this comes at the expense of restricting ourselves to $(n+1)$-crossing curves (as opposed to d-crossing in Theorem 5). This limitation is due to the fact that Lemma 7 above, cannot be extended to general d-crossing curves. A counter example demonstrating this fact is given in [4].

It is known that Conjecture 1 holds for $n = 2$. As proved earlier in this chapter, 2-crossing curves in $\mathbb{R}^2$ form a connected subset of the boundary of a strictly convex body. The result then follows from the classical two dimensional affine isoperimetric inequality once we observe that 3-crossing curves in $\mathbb{R}^2$ can be subdivided into at most 4 2-crossing curves.

Chapter 4

A relative affine isoperimetric inequality

4.1 A relative affine isoperimetric inequality for unbounded convex sets

As is well-known the classical isoperimetric inequality places a lower bound on the surface area required to enclose a fixed volume. A related inequality is the *relative isoperimetric inequality*. This concerns the problem of minimising a perimeter inside a convex body which encloses a given volume (see [30], [29]). Given a fixed, bounded and sufficiently smooth (see for example [36]) convex body B in $\mathbb{R}^n$, the relative isoperimetric inequality states there exists a constant in $C_{n,B}$ such that for any convex body $\mathcal{K}$,

$$\mathrm{Vol}(\mathcal{K} \cap B) \leq C_{n,B} \left(\mathrm{Arclength}(\partial\mathcal{K} \cap B^0)\right)^{\frac{n}{n-1}} . \tag{4.1.1}$$

The relative isoperimetric inequality lends itself to a variety of different fields in analysis and geometry. In "Geometry of Numbers" the inequality leads to a lower bound for the number of points of an integer lattice inside a convex body ([26]). In Analysis the relative inequality of a domain B leads to Sobolev embedding result on that domain [21].

In harmonic analysis the inequality plays a role in the proof of the Multilinear Kakeya Theorem (see [12]). As part of this proof it is shown that, given a finite subset of a unit-cube lattice, there exists a polynomial of low degree whose zero set bisects each cube in that subset. The relative isoperimetric inequality ensures that the part of the hypersurface intersecting each cube has surface area bounded away from zero.

Another related inequality is the *affine isoperimetric inequality*, which relates the affine surface area required to enclose a fixed volume:

Theorem 7 (Affine Isoperimetric Inequality). *[20][Theorem 5.1.3.1] Let $\mathcal{K}$ be a bounded convex body, then*

$$\text{AffineSurf}(\partial\mathcal{K}) \leq C_n \text{Vol}(\mathcal{K})^{\frac{n-1}{n+1}}$$

with equality if and only if $\mathcal{K}$ is an ellipsoid, where

$$C_n = n\omega_n^{\frac{2}{n+1}}$$

and ω_n is the n-dimensional volume of the unit ball in $\mathbb{R}^n$

Therefore it is natural to ask whether an affine variant of the relative isoperimetric inequality exists. If we let B be a bounded convex body, and $\mathcal{K} \cap B$ represent the region enclosed by $\partial\mathcal{K}$ inside B, can we bound AffineSurf$(\partial\mathcal{K} \cap B)$, by an appropriate power of the enclosed volume Vol$(\mathcal{K} \cap B)$.

If we let $\tilde{\mathcal{K}} = \mathcal{K} \cap B$ then clearly by the affine isoperimetric inequality,

$$\begin{aligned}
\text{AffineSurf}(\partial\mathcal{K} \cap B) &\leq \text{AffineSurf}(\partial\tilde{\mathcal{K}}) \\
&\leq n\omega_n^{\frac{2}{n+1}} \text{Vol}(\tilde{\mathcal{K}})^{\frac{n-1}{n+1}} \\
&= n\omega_n^{\frac{2}{n+1}} \text{Vol}(\mathcal{K} \cap B)^{\frac{n-1}{n+1}},
\end{aligned}$$

where ω_n is the n-dimensional volume of the unit ball in $\mathbb{R}^n$.

In this chapter we prove the following result.

Theorem 8. *Let $\mathcal{K}$ and B be two closed convex bodies with C^2 boundary in $\mathbb{R}^n$, where B is bounded and $\mathcal{K}$ is unbounded. Then the following inequality holds*

$$\text{AffineSurf}(\partial\mathcal{K} \cap B) \leq n(\tfrac{\omega_n}{2})^{\frac{2}{n+1}} \text{Vol}(\mathcal{K} \cap B)^{\frac{n-1}{n+1}}. \tag{4.1.2}$$

Furthermore we have equality in (4.1.2) if and only if $\mathcal{K} \cap B$ is an affine image of a semiball and $\partial\mathcal{K} \cap B$ is the affine image of a hemisphere.

Remark. *The constant in (4.1.2) is smaller by a factor of $2^{\frac{-2}{n+1}}$, than the constant that appears in the classical affine isoperimetric inequality. It is an open question* *as to whether this inequality which such an inequality can be extended to all convex $\mathcal{K}$.*

In the case of the the classical relative isoperimetric inequality, while such a constant, $C_{n,B}$, is known to exist subject to the domain B being sufficiently smooth (e.g. Lipschitz domains), the constant does depend on B. Additionally the correct constant is only known in a limited number of cases (for the ball see [21], for the cone see [17]) and little is known about arbitrary domains.

Conversely the constant in the affine relative isoperimetric inequality is idendepent of B. As we shall see we can 'deform' B and $\mathcal{K}$, without decreasing the affine arclength of $\partial\mathcal{K} \cap B$ to produce a flat edge, and this then allows us to reflect along that edge and appeal to the affine isoperimetric inequality. We remark that this type of argument does not work for general domans in the classical case because we wish to *minimise* the arclength, but the deformation will also increase the arclength.

4.2 Proof of the relative affine isoperimetric inequality for unbounded convex sets

The moral of the proof is to reduce to the case where $\mathcal{K}^o \cap \partial B$ is a hyperplane, and then by reflecting $\mathcal{K} \cap B$ in this hyperplane we can appeal to the classical affine isoperimetric equality.

We start with two simple reductions. If $\partial\mathcal{K} = \emptyset$ then there is nothing to prove in (4.1.2), so we may assume $\mathcal{K}$ is contained within some half-space, by the following propositions:

Proposition 4 (Separating hyperplane theorem). *[9, 2.5.1] Let $\mathcal{K}$ be a closed convex body, with non-empty complement and $z \in \mathcal{K}^c$. Then there exists a hyperplane, H with corresponding half-spaces, $H^{\pm}$, for which $\mathcal{K} \subset H^+$ and $z \in H^-$.*

Remark. *The separating hyperplane theorem is a simple application of the Hahn-Banach Theorem.*

Proposition 5 (Supporting hyperplane theorem). *[9, 2.5.2] Let $\mathcal{K}$ be a closed convex body, with non-empty boundary and $y \in \partial\mathcal{K}$. Then there exists a hyperplane containing y such that $\mathcal{K}$ is contained in one of the corresponding half-spaces.*

Proof of separating hyperplane theorem. By choice of co-ordinate axis we may assume that z is the origin. Let $\delta = \inf\{|x| : x \in \mathcal{K}\}$, then $\delta > 0$ since $\mathcal{K}^c$ is open and non-empty. This minimum is attained at some (in fact unique) point $\tilde{x} \in \mathcal{K}$.

We define our hyperplane:

$$H = \{x : x \cdot \tilde{x} = \tfrac{1}{2}\delta^2\}$$

Clearly $z = 0 \in \{x : x \cdot \tilde{x} < \frac{1}{2}\delta^2\}$. Now let $x \in \mathcal{K}$, then $y(t) = \tilde{x} + t(x - \tilde{x}) \in \mathcal{K}$ for $t \in (0, 1)$. It follows from the definition of δ that

$$\begin{aligned} \delta^2 &\leq |y(t)|^2 \\ &= |\tilde{x}|^2 + 2t\tilde{x} \cdot (x - \tilde{x}) + t^2(x - \tilde{x}) \cdot (x - \tilde{x}) \end{aligned}$$

which gives the inequality

$$t|(x - \tilde{x})|^2 + 2\tilde{x} \cdot (x - \tilde{x}) \geq 0.$$

Taking the limit as $t \to 0$ gives $\tilde{x} \cdot x \geq |\tilde{x}|^2 > \frac{1}{2}\delta^2$. So $\mathcal{K} \subset \{x : x \cdot \tilde{x} > \frac{1}{2}\delta^2\}$

□

Proof of supporting hyperplane theorem. Since $\partial\mathcal{K}$ is non-empty, the complement of $\mathcal{K}$, $\mathcal{K}^c$, is non-empty. Let $\{y_i\}_{i=0}^{\infty}$ be a sequence in $\mathcal{K}^c$ such that $y_i \to y \in \partial\mathcal{K}$. Via the separating hyperplane theorem, for each y_i we can construct a hyperplane separating y_i and $\mathcal{K}$. Denote its unit normal by n_i, then (by choosing the orientation of n_i appropriately)

$$x \cdot n_i \geq y_i \cdot n_i \text{ for every } x \in \mathcal{K}.$$

The sequence $\{n_i\}_{i=1}^{\infty} \subset S^{n-1}$ is bounded, so by extracting a subsequence if necessary, $n_i \to n \in S^{n-1}$ and

$$x \cdot n = \lim_{i\to\infty} x \cdot n_i \geq \lim_{i\to\infty} y_i \cdot n_i = y \cdot n \text{ for every } x \in K.$$

So the desired supporting hyperplane is $\{x : x \cdot n = y \cdot n\}$ □

The second reduction is to observe that we can assume B is contained within

$\mathcal{K}$, indeed we have the trivial equalities:

$$\begin{aligned} \partial\mathcal{K} \cap B &= \partial\mathcal{K} \cap (B \cap \mathcal{K}), \\ \mathcal{K} \cap B &= \mathcal{K} \cap (B \cap \mathcal{K}), \end{aligned}$$

allowing us to replace B with $B \cap \mathcal{K}$ in (4.1.2).

With these two simple observations, we can now proceed to reduce matters further still to the case where $\mathcal{K}$ is an 'infinite test tube' with its base given by $\partial\mathcal{K} \cap B$. We shall see that it follows from the claim below that $\mathcal{K}$ must contain such a test tube.

Claim 1. *There exists an $\omega \in S^{n-1}$ and a Euclidean motion T such that*

$$l = \{\lambda\omega : \lambda \geq 0\} \subset \mathcal{K}$$

and $\mathcal{K} \cap l = T($positive x_n-axis$)$.

Proof. By translating if necessary we may assume the origin lies in $\mathcal{K}$. It suffices to show the existence of l as a further Euclidean motion translating the origin to $\partial\mathcal{K}$ (in the direction $-\omega$) and rotating completes the proof.

Since $\mathcal{K}$ is unbounded there exists a sequence of points in $\mathcal{K}$, $\{x_n\}_{n=1}^{\infty}$, such that $|x_n| > n$. Let

$$\omega_n = \frac{x_n}{|x_n|}$$

then $\{\omega_n\}_{n=1}^{\infty}$ is a sequence in S^{n-1}. By extracting a subsequence if necessary we have that ω_n converges to some point $\omega \in S^{n-1}$. Since $\mathcal{K}$ is closed, and $\omega_n \in \mathcal{K}$ by convexity, it follows that $\omega \in \mathcal{K}$. In fact we have,

$$\lambda\omega \in \mathcal{K} \text{ for all non-negative } \lambda.$$

To see this, for any given $\lambda \geq 0$ let $y_n = \lambda\omega_n$, then for $n \geq \lambda$, y_n lies on the line between the origin and x_n. So by by convexity, $y_n \in \mathcal{K}$ for all $n \geq \lambda$. Since $y_n \to \lambda\omega$ it follows that $\lambda\omega \in \mathcal{K}$. □

Using the above claim we can choose a co-ordinate system in which l coincides with the positive x_n axis. Now let $\Omega = \Pi(B)$ be the projection of B onto the hyperplane $x_n = 0$. Define the graph of the convex cap of B, $f_+ : \Omega \to \mathbb{R}$ as

$$f_+(x') = \sup\{x_n : (x', x_n) \in B\}.$$

and the graph of the convex bowl of B, $f_- : \Omega \to \mathbb{R}$ as

$$f_-(x') = \inf\{x_n : (x', x_n) \in B\}.$$

It is easy to see that f_+ is concave, and f_- is convex, and by assumption they both are C^2.
We now construct a $\tilde{\mathcal{K}} \subset \mathcal{K}$ which satisfies the following conditions

$$\mathrm{Vol}(\mathcal{K} \cap B) = \mathrm{Vol}(\tilde{\mathcal{K}} \cap B), \tag{4.2.1}$$

$$\mathrm{AffineSurf}(\partial\mathcal{K} \cap B) \leq \mathrm{AffineSurf}(\partial\tilde{\mathcal{K}} \cap B). \tag{4.2.2}$$

Then it suffices to show

$$\mathrm{AffineSurf}(\partial\tilde{\mathcal{K}} \cap B) \leq C_n \,\mathrm{Vol}(\tilde{\mathcal{K}} \cap B)^{\frac{n-1}{n+1}}. \tag{4.2.3}$$

We shall construct $\tilde{\mathcal{K}}$ to be the infinite tube above Ω with its bowl given by f_-. Specifically:

$$\tilde{\mathcal{K}} := \{(x', f_-(x') + \lambda) \in \Omega \times \mathbb{R} : \lambda \geq 0\}.$$

Clearly $\tilde{\mathcal{K}}$ is unbounded.

Claim 2. *$\tilde{\mathcal{K}}$ is convex*

Proof. Let $x = (x', \lambda_1 + f_-(x'))$, $y = (y', \lambda_2 + f_-(y')) \in \tilde{\mathcal{K}}$, with $\lambda_i \geq 0$. For a given $t \in (0,1)$, let $z = tx + (1-t)y$.

Now $z' = tx' + (1-t)y' \in \Omega$ by convexity of Ω. Furthermore,

$$\begin{aligned} z_n &= tf_-(x') + (1-t)f_-(y') + t\lambda_1 + (1-t)\lambda_2 \\ &\geq tf_-(x') + (1-t)f_-(y') \end{aligned}$$

By the convexity of f_-,

$$\begin{aligned} &\geq f_-(tx' + (1-t)y') \\ &= f_-(z') \end{aligned}$$

Thus $z = (z', f_-(z') + \lambda) \in \tilde{\mathcal{K}}$, for some $\lambda \geq 0$. □

Claim 3. *$\tilde{K} \subset \mathcal{K}$.*

Proof. Suppose by way of contradiction, that there exists $x \in \tilde{\mathcal{K}}$ such that $x \notin \mathcal{K}$. By definition $x = (x', f_-(x') + \lambda)$, for some $\lambda \geq 0$ and $x' \in \Omega$. Since $(x', f_-(x')) \in$

$\partial B \subset \mathcal{K}$, it follows that $\lambda > 0$. Because $\mathcal{K}$ is closed, for some ϵ sufficiently small (and $\epsilon < 1$) $B_\epsilon(x) \subset \mathcal{K}^c$. Let, R be such that $|x'| < R$ for all $x' \in \Omega$, and define:

$$\begin{aligned} x_\epsilon &= (x'(1-\tfrac{\epsilon}{R}), f_-(x'(1-\tfrac{\epsilon}{R})) + \lambda_\epsilon) \\ \lambda_\epsilon &= f_-(x') - f_-(x'(1-\tfrac{\epsilon}{R})) + \lambda. \end{aligned}$$

Then $|x - x_\epsilon| = |(\frac{\epsilon}{R}x', 0)| < \epsilon$, so $x_\epsilon \in B_\epsilon(x)$. By continuity of f_-, and taking ϵ to be smaller if required we have $\lambda_\epsilon > 0$. Now define the points which lie on the line passing through $(x', f_-(x'))$ and x_ϵ

$$y(t) = [(x', f_-(x')) - x_\epsilon]\, t + (x', f_-(x')).$$

Then $y(0) = (x', f_-(x')) \in \partial B \subset \mathcal{K}$, and $y(\frac{-1}{\epsilon}) = (0, f_-(x') + \frac{\lambda}{\epsilon}) \in \mathcal{K}$. Since $0 > -1 > \frac{-1}{\epsilon}$, by convexity it follows that $y(-1) = x_\epsilon \in \mathcal{K}$, which contradicts $x_\epsilon \in B_\epsilon(x) \subset \mathcal{K}^c$. $\square$

By construction, $B \subset \tilde{\mathcal{K}} \subset \mathcal{K}$, so

$$\mathcal{K} \cap B = \tilde{\mathcal{K}} \cap B = B$$

and we immediately have (4.2.1). Furthermore since $B \subset \mathcal{K}$, $\partial\mathcal{K} \cap B = \partial\mathcal{K} \cap \partial B$ and $\partial\mathcal{K} \cap \partial B$ is contained in the set

$$\{(x', f_-(x')) : x' \in \Omega\} \cup \{(x', x_n) : x' \in \partial\Omega, x_n \geq f_-(x')\} = \partial\tilde{\mathcal{K}} \cap B.$$

Thus $\partial\mathcal{K} \cap B \subset \partial\tilde{\mathcal{K}} \cap B$, from which (4.2.2) follows.

We now 'deform' $\tilde{\mathcal{K}}$ and B by flattening and translating the concave cap of B so that the convex top of the deformed B now lies within the hyperplane $x_n = 0$. Set

$$\begin{aligned} g_+(x') &= f_+(x') - f_+(x') \ (= 0), \\ g_-(x') &= f_-(x') - f_+(x'). \end{aligned}$$

We note that g_- is convex and C^2. Let $\mathcal{K}_g$ and B_g be $\tilde{\mathcal{K}}$ and B after this deformation, i.e.

$$\begin{aligned} B_g &= \{x : (x', x_n + f_+(x')) \in B\}, \\ \mathcal{K}_g &= \{x : (x', x_n + f_+(x')) \in \mathcal{K}\}. \end{aligned}$$

By the concavity of f_+, B_g and $\mathcal{K}_g$ are convex. The concave cap of B_g is given by $g_+(x') \equiv 0$ - which coincides with the $x_n = 0$ hyperplane, and the convex bowl of B_g is given by g_-.

So we now have an unbounded and convex $\mathcal{K}_g$ - which is an infinite test tube perpendicular to the $x_n = 0$ hyperplane, with a convex bowl given by $\partial\mathcal{K}_g \cap B_g$. B_g is a bounded convex body that sits between $\partial\mathcal{K}_g \cap B_g$ and the $x_n = 0$ hyperplane, with volume Vol(B). Furthermore

$$\begin{aligned} \text{AffineSurf}(\partial\mathcal{K}_g \cap B_g) &= \int_\Omega |\det \text{Hess}\, g_-(x')|^{\frac{1}{n+1}}\, dx' \\ &= \int_\Omega |\det \text{Hess}(f_-(x') - f_+(x'))|^{\frac{1}{n+1}}\, dx' \\ &\geq \int_\Omega |\det \text{Hess}(f_-(x'))|^{\frac{1}{n+1}}\, dx' \\ &= \text{AffineSurf}(\partial\tilde{\mathcal{K}} \cap B), \end{aligned} \tag{4.2.4}$$

where the third line comes from the following lemma.

Lemma 11. *Let T,Q be two positive definite matrices.*

$$\det(T + Q) \geq \det(T)$$

with equality if and only if $Q = 0$.

Proof. As T,Q are both positive definite symmetric matrices there exists a non-singular E such that simultaneously

$$T = E \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} E^T$$

$$Q = E \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix} E^T.$$

A proof of the existance of such an E can be found in [28, Chapter IV, p.99]. We remark that E is not necessarily orthogonal, and indeed is so if and only if T and Q commute.

Then it suffices to note

$$\prod_{i=1}^{n}(\lambda_i + \mu_i) \geq \prod_{i=1}^{n}(\lambda_i)$$

as $\lambda_i, \mu_i \geq 0$, with equality if and only if $\mu_i = 0$. □

Reflecting $\mathcal{K}_g \cap B_g$ in the $x_n = 0$ hyperplane gives us a convex body L such that

$$\begin{aligned} \mathrm{Vol}(L) &= 2\,\mathrm{Vol}(\mathcal{K}_g \cap B_g) \\ \mathrm{AffineSurf}(L) &= 2\,\mathrm{AffineSurf}(\partial\mathcal{K}_g \cap B_g). \end{aligned}$$

By applying the classical affine isoperimetric inequality to L we get

$$\mathrm{AffineSurf}(\partial\mathcal{K}_g \cap B_g) \leq C_n 2^{\frac{-2}{n+1}}\,\mathrm{Vol}(\mathcal{K}_g \cap B_g)$$

where $C_n = n\omega_n^{\frac{2}{n+1}}$ is the constant in the classical affine isoperimetric inequality.

Using (4.2.4) and the equality $\mathrm{Vol}(\mathcal{K}_g \cap B_g) = \mathrm{Vol}(\tilde{\mathcal{K}} \cap B)$ we obtain (4.2.3) and hence the first part of the Theorem 8.

By the classical affine isoperimetric inequality we have equality if and only if L is an ellipsoid. Following the argument of the above proof, this holds if and only if $\mathcal{K}_g \cap B_g$ and $\partial\mathcal{K}_g \cap B_g$ are the affine images of a semiball and hemisphere, respectively. By lemma 11 and (4.2.4) we can have equality in (4.1.2) only when f_+ is of the form

$$f_+(x') = \sum_{i=1}^{n-1} a_i x_i + b. \tag{4.2.5}$$

In particular deformation by f_+ is an affine transformation. It follows (modulo an affine transformation) that $\tilde{\mathcal{K}} \cap B$ is a semiball and $\partial\tilde{\mathcal{K}} \cap B$ a hemisphere. By the construction of $\tilde{\mathcal{K}}$, we have that $B = \mathcal{K} \cap B = \tilde{\mathcal{K}} \cap B$ is the affine image of semiball. We also have $\partial\mathcal{K} \cap B \subset \partial\tilde{\mathcal{K}} \cap B$ and so equality in (4.2.2) requires

$$\mathrm{AffSurf}((\partial\tilde{\mathcal{K}} \cap B)\backslash(\partial\mathcal{K} \cap B)) = 0. \tag{4.2.6}$$

But since $\partial\tilde{\mathcal{K}} \cap B$, as the affine image of a hemisphere, has non-zero affine curvature everywhere, it follows $\partial\mathcal{K} \cap B = \partial\tilde{\mathcal{K}} \cap B$ is also the affine image of a hemisphere.

Thus equality in (4.1.2) implies, up to an affine transformation, that $\partial\mathcal{K} \cap B$ is a hemisphere and $\mathcal{K} \cap B$ semiball. The opposite implication is easily verified.

$\square$

4.3 Affine isoperimetric inequality for unbounded sets implies the classical affine isoperimetric inequality for convex sets

The moral of the proof given below is that any bounded convex body, which is symmetric about a hyperplane can be written as the intersection of two unbounded convex sets. The idea is then to apply (4.1.2) to each of these unbounded sets where B is taken to be the finite convex body enclosed by our hyperplane and unbounded set. From the resulting inequalities we can deduce the classical affine isoperimetric inequality.

Lemma 12. *A bounded convex body, which is symmetric about a hyperplane, can be written as the intersection of two unbounded convex sets*

Proof. Choose a co-ordinate system such that our convex body B is symmetric about $x_n = 0$ and let B^+, and B^- be the parts of B living in the positive, and negative, half-space respectively. We shall extend B^+ to an unbounded convex set, $\mathcal{K}^+$, such that $B \subset \mathcal{K}^+$.

Let $\mathcal{K}^+$ to be the infinite test tube with cap B^+. Specifically define

$$\mathcal{K}^+ = \{(x', x_n - \lambda) : \lambda \geq 0, (x', x_n) \in (\partial B) \cap \{x_n > 0\}\}.$$

It is clear $\mathcal{K}^+$ is unbounded; it also convex by the argument used in the previous section.

Clearly $B^+ \subset \mathcal{K}^+$. That $B \subset \mathcal{K}^+$ follows from symmetry: suppose $y = (y', -y_n) \in B^-$, then $(y', y_n) \in B^+ \subset \mathcal{K}^+$, so $(y', y_n) = (x', x_n - \lambda)$ for some $\lambda \geq 0$, and thus $y = (x', x_n - \lambda - 2y_n) \in \mathcal{K}^+$.

Using the same argument we can define the unbounded convex set $\mathcal{K}^-$

$$\mathcal{K}^- = \{(y', y_n + \lambda) : \lambda \geq 0 (y', y_n) \in (\partial B) \cap \{x_n < 0\}\}.$$

We claim that $B = \mathcal{K}^+ \cap \mathcal{K}^-$. We have already seen that $B \subset \mathcal{K}^+ \cap \mathcal{K}^-$ so it only remains to prove the opposite inclusion. Let $z \in \mathcal{K}^+ \cap \mathcal{K}^-$, then for some $w_n \geq 0, y_n \geq 0$, $(z', y_n) \in (\partial B) \cap \{x_n < 0\}$ and $(z', w_n) \in (\partial B) \cap \{x_n > 0\}$. Then we can write $z = (z', y_n + \lambda_1) = (z', w_n - \lambda_2)$.

Let $t = \frac{\lambda_2}{\lambda_1+\lambda_2} \in [0,1]$, then

$$\begin{aligned} tw_n + (1-t)y_n &= tw_n + (1-t)(w_n - (\lambda_1 + \lambda_2)) \\ &= w_n - \frac{\lambda_2}{\lambda_1 + \lambda_2}(\lambda_1 + \lambda_2)) \\ &= z_n \end{aligned}$$

Thus $z = (z', z_n) = t(z', w_n) + (1-t)(z', y_n) \in B$, by convexity.

□

Remark. *Symmetry is not a necessary requirement in Lemma 12. The argument of the proof can be adapted to require only that $(z', 0) \in B$ whenever $(z', z_n) \in \partial B$. This follows, for example if the body formed by reflecting the body B^+ in the $x_n = 0$ and the body formed by reflecting the body B^- in the $x_n = 0$ hyperplane are both convex.*

Remark. *By construction $\partial\mathcal{K}^\pm \cap B^\pm = \partial B \cap B^\pm$.*

Theorem 9. *The isoperimetric inequality for unbounded sets (4.1.2) implies the classical affine isoperimetric inequality for convex sets.*

Proof. Let B be our bounded convex set, by applying Steiner symmetrisation (see [20][§5.1.1]) if necessary, we may assume that B is symmetric about some hyperplane. For ease of notation, we choose a coordinate system such that B is symmetric about the hyperplane $x_n = 0$, and let B^+ and B^- be defined as above.

Let $\mathcal{K}^+$ and $\mathcal{K}^-$ be the convex unbounded sets from lemma 12, then $B^\pm$ and $\mathcal{K}^\pm$ satisfy the conditions of (4.1.2). Furthermore $B^\pm \subset \mathcal{K}^\pm$, so we have the following inequalities:

$$\begin{aligned} \text{AffineSurf}(\partial\mathcal{K}^+ \cap B^+) \leq C_n \text{Vol}(B^+)^{\frac{n-1}{n+1}}, \\ \text{AffineSurf}(\partial\mathcal{K}^- \cap B^-) \leq C_n \text{Vol}(B^-)^{\frac{n-1}{n+1}}. \end{aligned}$$

By construction $\partial\mathcal{K}^\pm \cap B^\pm = \partial B \cap B^\pm$ so,

$$\begin{aligned} \text{AffineSurf}(\partial B) &= \text{AffineSurf}(\partial B \cap B^+) + \text{AffineSurf}(\partial B \cap B^-) \\ &= \text{AffineSurf}(\partial\mathcal{K}^+ \cap B^+) + \text{AffineSurf}(\partial\mathcal{K}^- \cap B^-) \end{aligned}$$

$$
\begin{aligned}
&\leq C_n \left(\mathrm{Vol}(B^+)^{\frac{n-1}{n+1}} + \mathrm{Vol}(B^-)^{\frac{n-1}{n+1}} \right) \\
&= C_n 2^{1-\frac{n-1}{n+1}} \left(\mathrm{Vol}(B)^{\frac{n-1}{n+1}} \right).
\end{aligned}
$$

where we have used $\mathrm{Vol}(B^{\pm}) = \frac{1}{2}\,\mathrm{Vol}(B)$

$\square$

Appendix A

2-crossing closed curves

Below we prove that given a 2-crossing curve $\Gamma : I \to \mathbb{R}^2$, one can extend that curve in such a way that the extended curve is also 2-crossing. Formally, there exists $\tilde{\Gamma}(t) : \tilde{I} = [a,b] \to \mathbb{R}^2$ where $I \subset \tilde{I}$, such that $\tilde{\Gamma}(t) = \Gamma(t)$ for $t \in I$, $\tilde{\Gamma}(a) = \tilde{\Gamma}(b)$ and $\tilde{\Gamma}$ is 2-crossing.

First we note that if Γ is closed, there is nothing to do, so we need only to consider the case where it is not. For notational convenience we shall assume that $\Gamma(t) : [0,1] \to \mathbb{R}^2$ and that $\Gamma(0) = (\alpha, 0), \Gamma(1) = (\beta, 0)$ lie on the x-axis with $\alpha < \beta$ and Γ lying in the $y \leq 0$ plane.

Let $t \mapsto (\sin(2\pi t), \cos(2\pi t))$ be a parametrisation of a circle. Denote by s_0 and s_1 the points on the circle with unit tangent vectors $\widehat{\nabla\Gamma(0)}$ and $\widehat{\nabla\Gamma(1)}$ (where, for a vector v, $\widehat{v}$ denotes the unit vector of v).

Let t_0 and t_1 be such that $s_0 = (\sin(2\pi t_0), \cos(2\pi t_0))$ and $s_1 = (\sin(2\pi t_1), \cos(2\pi t_1))$, and $\theta = 2\pi(t_0 - t_1)$. Let $R = \frac{|\beta - \alpha|}{2\sin(\frac{\theta}{2})}$, and let $r(t) = 2\pi((2-t)t_1 + (t-1)t_0)$. Finally we define our extended curve as follows:

$$\widetilde{\Gamma}(t) = \begin{cases} \Gamma(t) & t \in [0,1] \\ R(\sin(r(t)), \cos(r(t)) & t \in [1,2] \end{cases}$$

It now remains to prove our claim that the interior of this curve is strictly convex. We first note that $\widetilde{\Gamma}$ is a Jordan curve, so the notion of an interior is well defined. Secondly its interior is the union of two convex bodies: the convex hull of Γ (B_1) and the convex hull of the subset of a circle (B_2).

Take the lines formed by extending the tangents at s_0 and s_1. By convexity of B_1 and B_2 these lines either form a cone or "slice", S, of the plane, in which $B_1 \cup B_2$ is contained. Furthemore S is divided into two parts by the line segment between s_0 and s_1, with $B_1 \subset H_- \cap S$ and $B_2 \subset H_+ \cap S$.

We note then, by convexity of S, that any line from $a \in B_2$ to $b \in B_1$ must pass through $S \cap \{x\text{-axis}\}$.

By way of contradiction, suppose that $\widehat{\Gamma}(t)$ is not 2-crossing. That is, there exist $t_1 < t_2 < t_3$ such that $\{\widehat{\Gamma}(t_i)\}_{i=1}^{3}$ are co-linear. Because $\widehat{\Gamma}|_{[0,1]}$ and $\widehat{\Gamma}|_{[1,2]}$ are 2-crossing curves it follows that $t_1 \in [0,1]$ and $t_3 \in [1,2]$. Furthermore it is clear that $t_2 \in [0,1]$ (by strict convexity of circles).

By the above observation there exists $x \in S \cap \{x\text{-axis}\} \cap \overrightarrow{\Gamma(t_1)\Gamma(t_3)}$. But then, $\Gamma(t_2) \in \overrightarrow{\Gamma(t_1)x}$, which contradicts the fact $\widehat{\Gamma}|_{[0,1]}$ is 2-crossing.

Appendix B

Nearest Point Projection

First, we observe that for a fixed $y \in \mathbb{R}^n$, the function

$$g_y(x) = |x - y|^2$$

is strictly convex. This follows from the following identity

$$\begin{aligned}
g_y(\lambda x + (1-\lambda)z) =& |\lambda x + (1-\lambda)z - y|^2 \\
=& |\lambda(x-y) + (1-\lambda)(z-y)|^2 \\
=& \lambda^2|(x-y)|^2 + (1-\lambda)^2|(z-y)|^2 + 2\lambda(1-\lambda)\langle x-y, z-y\rangle \\
=& \lambda g_y(x) + (1-\lambda)g_y(z) + \lambda(\lambda-1)\left(|(x-y)|^2\right. \\
& \left. -2\langle x-y, z-y\rangle + |(z-y)|^2\right) \\
=& \lambda g_y(x) + (1-\lambda)g_y(z) + \lambda(\lambda-1)\left(|(x-z)|^2\right).
\end{aligned}$$

For $x \neq z$ and $\lambda \in (0,1)$, therefore, it is clear $g_y(\lambda x + (1-\lambda)z) < \lambda g_y(x) + (1-\lambda)g_y(z)$.

Now, let $C_1 \subset C_2$ be closed convex (bounded) sets. We can define the *nearest point projection* as a map $P : C_2 \to C_1$ that satisfies:

$$|P(y) - y| = \inf_{z \in C_1} \{|z - y|\}.$$

Since $|\cdot - y|$ is continuous, and C_1 is a closed bounded set, there exists such a minimiser. That such a minimiser is unique follows from the fact that g_y is strictly convex.

Lemma 13. *The map P is a contraction. Specifically, for all $x, y \in C_2$*

$$|P(x) - P(y)| \leq |x - y|$$

Proof. Consider the map

$$h(t) = |t(x - y) + (1 - t)(P(x) - P(y)|^2$$

so that $h(0) = |P(x) - P(y)|^2$ and $h(1) = |x - y|^2$. We will show that $h'(t) \geq 0$, from which the desired inequality follows. By convexity of h, it suffices to show that $h'(0) \geq 0$.

First we note that by definition of P, and the convexity of C_1, for any $x \in C_2$.

$$|(1 - t)P(x) + tP(y) - x| \geq |P(x) - x|.$$

Differentiating with respect to t and evaluating at $t = 0$

$$2\langle P(y) - P(x), P(x) - x\rangle \geq 0.$$

Similarly we find

$$2\langle P(x) - P(y), P(y) - y\rangle \geq 0.$$

Differentiating h at $t = 0$, we get

$$\begin{aligned} h'(0) &= 2\langle (x - y) - (P(x) - P(y), t(x - y) + (1 - t)(P(x) - P(y))\rangle|_{t=0} \\ &= 2\langle (x - y) - (P(x) - P(y), P(x) - P(y)\rangle \\ &= \underbrace{2\langle x - P(x), P(x) - P(y)\rangle}_{\geq 0} + \underbrace{2\langle P(y) - y, P(x) - P(y)\rangle}_{\geq 0}. \end{aligned}$$

□

Lemma 14. $P(\partial C_2) = \partial C_1$.

Proof. Let $y \in \partial C_2$ and suppose $P(y) \notin \partial C_1$. Therefore $P(y)$ is in the interior of C_1. The line

$$\{ty + (1 - t)P(y) : t \in [0, 1]\}$$

has one endpoint ($t = 0$) in the interior of C_1 and the other ($t = 1$) in C_2. Consequently there exists $t_0 \in (0, 1)$ such that $t_0 y + (1 - t_0)P(y) \in \partial C_1 \subset C_1$.

But then

$$|t_0 y + (1 - t_0)P(y) - y| = (1 - t_0)|P(y) - y| < |P(y) - y|,$$

which contradicts the fact that $P(y)$ minimises $\{|z - y| : z \in C_1\}$.

That P is onto follows from the convexity of C_1 and the supporting hyperplane theorem. $\square$

Bibliography

[1] Faruk Abi-Khuzam and Bassam Shayya. Fourier restriction to convex surfaces of revolution in $\mathbb{R}^3$. *Publicacions Matemàtiques*, 50(1):71–85, 2006.

[2] G. I. Arkipov, V. N. Chubarikov, and A. A. Karatsuba. Exponent of convergence of the singular integral in the Tarry problem. *Dokl. Akad. Nauk SSSR*, 248(2):268272, 1979. In Russian.

[3] Jong-Guk Bak, Daniel M. Oberlin, and Andreas Seeger. Restriction of Fourier transforms to curves and related oscillatory integrals. *Amer. J. Math.*, 131(2):277–311, 2009.

[4] Imre Bárány, Jiří Matoušek, and Attila Pór. Curves in $\mathbb{R}^d$ intersecting every hyperplane at most $d+1$ times. In *Proceedings of the Thirtieth Annual Symposium on Computational Geometry*, SOCG'14, pages 565–571, New York, NY, USA, 2014. ACM.

[5] Sergei Bernstein. *Leçons sur les propriétés extremales et meilleure approximation des fonetions analytiques d'une variable réelle professées a la Sorbonne.* Collection de Monographies sur la Théorie des Fonctions sous la direction de Emile Borel. Gauthier-Villars et Cie., 1926.

[6] Gabriele Bianchi and Michael Kelly. The Blaschke-Santaló Inequality. *ArXiv e-prints*, December 2013. arXiv:1312.0244.

[7] Wilhelm Blaschke. Über affine Geometrie VII: Neue Extremeingenschaften von Ellipse und Ellipsoid. *Ber. Verh. Sächs. Akad. Wiss.*, 69:412–420, 1917.

[8] Jean Bourgain. Some New Estimates on Oscillatory Integrals. In *Essays in Fourier Analysis in honor of E. M. Stein*, pages 83–112. Princeton University Press, 1995.

[9] Stephen Boyd and Lieven Vandenberghe. *Convex optimization.* Cambridge University Press, Cambridge, UK New York, 2004.

[10] Anthony Carbery, Carlos Kenig, and Sarah Ziesler. Restriction for flat surfaces of revolution in $\mathbb{R}^3$. *Proc. Amer. Math. Soc.*, 135(6):1905–1914 (electronic), 2007.

[11] Anthony Carbery, Carlos Kenig, and Sarah Ziesler. Restriction for homogeneous polynomial surfaces in $\mathbb{R}^3$. *Transactions of the American Mathematical Society*, 365(5):2367–2407, 2013.

[12] Anthony Carbery and Stefán Ingi Valdimarsson. The endpoint multilinear Kakeya theorem via the Borsuk-Ulam theorem. *J. Funct. Anal.*, 264(7):1643–1663, 2013.

[13] Anthony Carbery and Sarah Ziesler. Restriction and decay for flat hypersurfaces. *Publ. Mat.*, 46(2):405–434, 2002.

[14] Spyridon Dendrinos and James Wright. Fourier restriction to polynomial curves I: a geometric inequality. *Amer. J. Math.*, 132(4):1031–1076, 2010.

[15] Stephen W. Drury and Bernard P. Marshall. Fourier restriction theorems for curves with affine and Euclidean arclengths. *Math. Proc. Camb. Phil. Soc.*, 97(01):111, Jan 1985.

[16] Javier Duoandikoetxea. *Fourier analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[17] A. Figalli and E. Indrei. A sharp stability result for the relative isoperimetric inequality inside convex cones. *J. Geom. Anal.*, 23(2):938–969, 2013.

[18] Alfred Gray, Elsa Abbena, and Simon Salamon. *Modern Differential Geometry of Curves and Surfaces with Mathematica*. CRC Press, Inc., Boca Raton, FL, USA, 3rd edition, 2006.

[19] Alex Iosevich and Guozhen Lu. Sharpness results and Knapp's homogeneity argument. *Canad. Math. Bull.*, 43(1):63–68, 2000.

[20] An-Min Li, Udo Simon, and Guosong Zhao. *Global affine differential geometry of hypersurfaces*. De Gruyter expositions in mathematics. De Gruyter, 1993.

[21] Vladimir Maz'ya. *Sobolev Spaces: With Applications to Elliptic Partial Differential Equations*. Die Grundlehren der mathematischen Wissenschaften

in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Springer Berlin Heidelberg, 2011.

[22] Mathieu Meyer and Alain Pajor. On the Blaschke-Santaló inequality. *Arch. Math. (Basel)*, 55(1):82–93, 1990.

[23] Adela Moyua, Ana Vargas, and Luis Vega. Schrödinger maximal function and restriction properties of the Fourier transform. *Internat. Math. Res. Notices*, (16):793–815, 1996.

[24] Daniel M. Oberlin. A uniform Fourier restriction theorem for surfaces in $\mathbb{R}^3$. *Proc. Amer. Math. Soc.*, 132(4):1195–1199 (electronic), 2004.

[25] Daniel M. Oberlin. A uniform Fourier restriction theorem for surfaces in $\mathbb{R}^d$. *Proc. Amer. Math. Soc.*, 140(1):263–265, 2012.

[26] Carla Peri. On relative isoperimetric inequalities. In *Conferenze del Seminario di Matematica dell'Universit di Bari*, volume 279, 2001.

[27] Stylianos K. Pichorides. A remark on the constants of the Littlewood-Paley inequality. *Proc. Amer. Math. Soc.*, 114(3):787–789, 1992.

[28] Viktor V. Prasolov. *Problems and Theorems in Linear Algebra*, volume 134 of *Translations of mathematical monographs.* American Mathematical Society, 1994.

[29] Manuel Ritoré and Antonio Ros. Some updates on isoperimetric problems. *Math. Intelligencer*, 24(3):9–14, 2002.

[30] Manuel Ritoré and Efstratios Vernadakis. Isoperimetric inequalities in Euclidean convex bodies. *Transactions Amer. Math. Soc.* to appear, arXiv:1302.4588.

[31] Luis A. Santaló. An affine invariant for convex bodies of n-dimensional space. *Portugaliae Math.*, 8:155–161, 1949.

[32] Bassam Shayya. An affine restriction estimate in $\mathbb{R}^d$. *Proceedings of the American Mathematical Society*, pages 1107–1113, 2007.

[33] Bassam Shayya. Affine restriction for radial surfaces. *Mathematische Zeitschrift*, 262(1):41–55, 2009.

[34] Per Sjölin. Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in $\mathbb{R}^2$. *Studia Math.*, 51:169–182, 1974.

[35] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Princeton Univ Pr, 1993.

[36] Peter Sternberg and Kevin Zumbrun. On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint. *Comm. Anal. Geom.*, 7(1):199–220, 1999.

[37] Terence Tao. A sharp bilinear restriction estimate for paraboloids. *Geometric & Functional Analysis GAFA*, 13(6):1359–1384, 2003.

[38] Terence Tao. Some recent progress on the restriction conjecture. In *Fourier analysis and convexity*, Appl. Numer. Harmon. Anal., pages 217–243. Birkhäuser Boston, Boston, MA, 2003.

[39] Terence Tao, Ana Vargas, and Luis Vega. A bilinear approach to the restriction and Kakeya conjectures. *J. Amer. Math. Soc.*, 11(4):967–1000, 1998.

[40] Peter A. Tomas. A restriction theorem for the Fourier transform. *Bull. Amer. Math. Soc.*, 81:477–478, 1975.

[41] Thomas Wolff. An improved bound for Kakeya type maximal functions. *Revista Matemtica Iberoamericana*, 11(3):651–674, 1995.

[42] Antoni Zygmund. On Fourier coefficients and transforms of functions of two variables. *Studia Math.*, 50(2):189–201, 1974.