THE UNIVERSITY *of* EDINBURGH

Extreme Black Holes and Near-Horizon Geometries

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is original and my own, unless referenced to the contrary in the text. No part of this thesis has been submitted here or elsewhere for any other degree or qualification. Chapters 2 and 3 of this thesis are based on the published papers [95] and [96] authored by myself and my supervisor James Lucietti, with the exception of section 2.3 which was not published by the time this thesis was first submitted. [97] appeared after the initial submission and is based on the work presented in Chapter 4 of this thesis. The main result of chapter 5 was stated without derivation in the review paper [88].

(*Carmen Ka Ki Li*)

Tuesday 17th November, 2015

Edinburgh

To my parents.

Abstract

In this thesis we study near-horizon geometries of extreme black holes. We first consider stationary extreme black hole solutions to the Einstein-Yang-Mills theory with a compact semi-simple gauge group in four dimensions, allowing for a negative cosmological constant. We prove that any axisymmetric black hole of this kind possesses a near-horizon AdS_2 symmetry and deduce its near-horizon geometry must be that of the abelian embedded extreme Kerr-Newman (AdS) black hole. We show that the near-horizon geometry of any static black hole is a direct product of AdS_2 and a constant curvature space. We then consider near-horizon geometry in Einstein gravity coupled to a Maxwell field and a massive complex scalar field, with a cosmological constant. We prove that assuming non-zero coupling between the Maxwell and the scalar fields, there exists no solution with a compact horizon in any dimensions where the massive scalar is non-trivial. This result generalises to any scalar potential which is a monotonically increasing function of the modulus of the complex scalar.

Next we determine the most general three-dimensional vacuum spacetime with a negative cosmological constant containing a non-singular Killing horizon. We show that the general solution with a spatially compact horizon possesses a second commuting Killing field and deduce that it must be related to the BTZ black hole (or its near-horizon geometry) by a diffeomorphism. We show there is a general class of asymptotically AdS_3 extreme black holes with arbitrary charges with respect to one of the asymptotic-symmetry Virasoro algebras and vanishing charges with respect to the other. We interpret these as descendants of the extreme BTZ black hole. However descendants of the non-extreme BTZ black hole are absent from our general solution with a non-degenerate horizon.

We then show that the first order deformation along transverse null geodesics about any near-horizon geometry with compact cross-sections always admits a finite-parameter family of solutions as the most general solution. As an application, we consider the first order expansion from the near-horizon geometry of the extreme Kerr black hole. We uncover a local uniqueness theorem by demonstrating that the only possible black hole solutions which admit a $U(1)$ symmetry are gauge equivalent to the first order expansion of the extreme Kerr solution itself. We then investigate the first order expansion from the near-horizon geometry of the extreme self-dual Myers-Perry black hole in $5D$. The only solutions which inherit the enhanced $SU(2) \times U(1)$ symmetry and are compatible with black holes correspond to the first order expansion of the extreme self-dual Myers-Perry black hole itself and the extreme $J = 0$ Kaluza-Klein black hole. These are the only known black holes to possess this near-horizon geometry. If only $U(1) \times U(1)$ symmetry is assumed in first order, we find that the most general solution is a three-parameter family which is more general than the two known black hole solutions. This hints the possibility of the existence of new black holes.

Lay summary

The key to formulate a quantum theory of gravity is to understand the microscopic, quantum mechanical origin of the entropy of the most gravitational object - the black hole. Extreme black holes are black holes with zero surface gravity, and they are expected to have simpler quantum descriptions because they do not emit Hawking radiation. All known extreme black holes possess a special geometric structure in their near-horizon geometries, which has played an important role in developing their quantum descriptions.

One of the most intriguing results in the mathematical theories of black holes is the uniqueness theorem, which states that any stationary black hole is uniquely determined by a finite set of charges. For instance, a stationary, asymptotically flat, electromagnetically charged rotating black hole in $4D$ must be a Kerr-Newman black hole, which is completely characterised by its mass, angular momentum, and electromagnetic charges. However, the uniqueness theorem is well known to be violated in gravitational theories coupled to more complex matter fields, as well as in higher dimensions, even in vacuum. This therefore opens up the question of black hole classification.

In this thesis we study the near-horizon geometries of extreme black holes. We prove several near-horizon theorems under various assumptions regarding the matter content, dimensionality and rotational symmetry. We then investigate the inverse problem of determining the corresponding full black hole solution given a near-horizon geometry, which is particularly important in black hole classification, by linearising the Einstein equation to first order transverse to the near-horizon geometry. We show that there always exists a finite dimensional space of solutions to the linearised Einstein equation on a compact horizon. We also uncover a local uniqueness theorem for the extreme Kerr black hole in 4-dimensions, and show that there could be new black hole solutions in 5-dimensions. In 3-dimensions the first order analysis in fact determines the full exact spacetime, and our general solution is the first explicit example of a special family of black hole spacetimes called descendant black holes.

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Chapter 1

Introduction

Black holes arise naturally as solutions to the Einstein equation. However the existence of black holes is perhaps the most interesting prediction from general relativity. In every black hole spacetime there is an event horizon, a null hypersufrace of no return beyond which the gravitational effect is so strong that nothing, not even light can escape. Black holes are the ultimate fate of sufficiently massive stars. Numerous pieces of indirect observational evidence have been found pointing to the existence of such massive *black* objects.

1.1 Black hole uniqueness

One of the most intriguing results in the mathematical theories of black holes is the *uniqueness theorem*, also known as the "no-hair theorem". Originally appeared in the late 1960's under the slogan "a black hole has no hair" [105], the *(generalised) no-hair conjecture* states that given a sensible matter model, any suitably regular stationary black hole solution is *uniquely* determined by a finite set of global charges defined from asymptotic Gaussian flux integrals. In this view, black holes are very simple objects even though they may have come from the gravitational collapse of trillions of particles. The uniqueness theorem in $4D$ Einstein-Maxwell theory was finally proved four decades on [24, 25, 31]. In this section we outline and review briefly the main steps towards the uniqueness theorem, the precise statement of the theorem will be presented at the end of section 1.1.5 (theorem 1.1). We follow largely the review paper [21] here.

1.1.1 Preliminaries

Let us begin with some definitions. Consider an asymptotically flat, stationary spacetime (M, g) with $\mathscr{I}^{\pm}$ denoting the future / past null infinity. Let $J^-(\mathscr{I}^+)$ be the causal past of the future null infinity $\mathscr{I}^+$, the *black hole region* is then $B = [M - J^-(\mathscr{I}^+)]$ and the *black hole (future) event horizon* is $\mathcal{N}^+ = \partial B$ (see figure 1.1) . Similarly the *white hole region* is $W = [M - J^+(\mathscr{I}^-)]$, where $J^+(p)$ denotes the causal future of a point $p \in M$, and the *white hole (past) event horizon* is $\mathcal{N}^- = \partial W$. The *domain of* *outer communications* (DOC) is defined to be $J^+(\mathscr{I}^-) \cap J^-(\mathscr{I}^+)$.

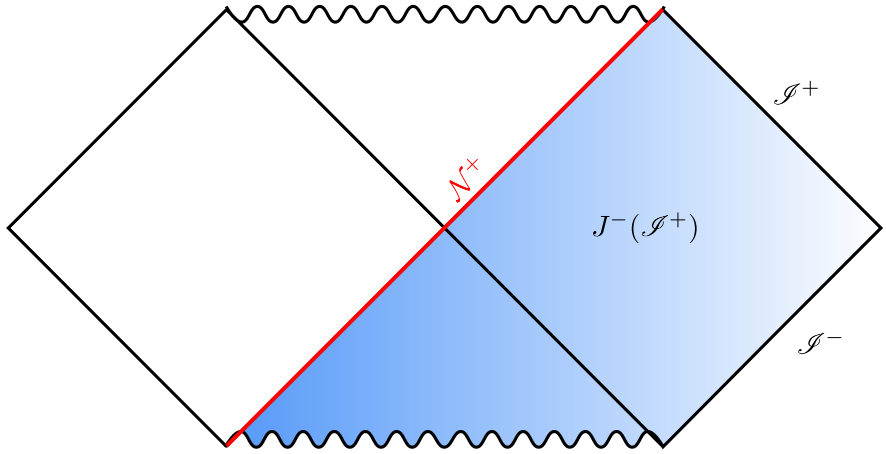


Figure 1.1: A black hole spacetime. The shaded region is the causal past of the future null infinity and the red line is the black hole horizon.

In addition to asymptotic flatness and being stationary, the uniqueness proof also relies on the spacetime (M, g) in question being "reasonably regular" and satisfying some extra global regularity conditions. We briefly discuss them here for completeness. Firstly, the stationary Killing field must be *complete* so it describes an action of $\mathbb{R}$ on the whole M. Secondly, it requires that the DOC is globally hyperbolic and possesses a connected, acausal spacelike hypersurface Σ which is "adequately well-behaving[1]" to enable elliptic partial differential equation analysis.

Another notion of regularity to be considered is the differentiability of the event horizon. The horizon of a black hole can be very "rough" and nowhere differentiable [23], after all causality theory only implies the event horizon is a Lipschitz topological submanifold [69, 109]. It was proved in [22, 24] that, assuming the null energy condition, as a corollary to the (non-decreasing horizon) area theorem [22, 69] an event horizon in a sufficiently well-behaving stationary spacetime is as smooth as the spacetime metric, so it is analytic if the metric is analytic. In fact *analyticity* of the spacetime is required to establish the uniqueness proof (in the rigidity theorem). This however is not entirely satisfactory, because it implies the whole spacetime can be determined by the behaviour of a single point and its neighbourhood. Removing the analyticity assumption remains a major challenge.

[1] In particular, it requires that its closure $\bar{\Sigma}$ is a topological manifold with boundary consists of the union of a compact set and a finite number of asymptotic ends, such that its boundary $\partial\bar{\Sigma} = \bar{\Sigma}/\Sigma$ satisfies $\partial\bar{\Sigma} = \partial\,(DOC) \cap I^+(M_{ext})$ where M_{ext} is the asymptotic exterior region. See [21] for further details. These condition are sometimes collectively referred as "I^+-regularity" in the literature.

1.1.2 Horizon topology theorem

The first step towards the uniqueness theorem is the horizon topology theorem, which asserts that each connected component of the event horizon has $\mathbb{R}\times S^2$ topology [28,67]. By "black hole" compactness of horizon cross-sections is already implied. In Hawking's original proof [67] he argued that if a spatial cross-section H of the horizon of a stationary asymptotically flat black hole satisfying the dominant energy condition has higher genus, then it can be deformed along a null hypersurface to an outer trapped surface *outside* the horizon which is not allowed. The T^2 cross-section is actually a borderline case and is not completely excluded by this argument.

The more recent proof [28] on the other hand draws heavily on the *topological censorship*. Topological censorship states that in an asymptotically flat globally hyperbolic spacetime satisfying the null energy condition, every causal curve from $\mathscr{I}^-$ to $\mathscr{I}^+$ is homotopic to a topologically trivial curve from $\mathscr{I}^-$ to $\mathscr{I}^+$ [49]. The key ingredient in the proof [28] is simple connectedness of the DOC, which can be shown to be a direct consequence of the topological censorship when applied to a black hole spacetime satisfying the same asymptotic, regularity and energy conditions. Then consider an achronal *asymptotically flat* slice S of the DOC such that its closure $\bar{S}$ in M is a compact manifold with boundary with a simply connected interior, whose boundary is homeomorphic to the disjoint union of H and the S^2 at the asymptotically flat end. It follows that each connected component of H must be homeomorphic to S^2 by standard results in (compact) 3-manifolds, see lemma 4.9 in [72].

1.1.3 Static solutions

A stationary spacetime is also static if the stationary Killing field ξ is hypersurface orthogonal i.e. $\xi \wedge \mathrm{d}\xi = 0$. As a result of this restrictive definition, the uniqueness theory for static electrovacuum black holes is relatively straightforward compared to the general stationary ones. For instance analyticity is not required in the proof, see [21] and the references therein. The static uniqueness theorem states that the DOC of any $4D$ static asymptotically flat electrovacuum black hole which is sufficiently regular must be isometric to the DOC of a Reissner-Nordström or Majumdar-Papapetrou spacetime. The static uniqueness theorem thus covers also *multi-black hole* solutions which, as we shall see, is not the case in the general stationary uniqueness theorem.

1.1.4 Rigidity Theorem

For the general stationary case, more input is required to draw any meaningful conclusion. The (strong) rigidity theorem $[20, 51, 69, 107]$ asserts that *assuming analyticity*, the event horizon of any stationary black hole is a Killing horizon. This powerful statement relates the event horizon, which is a globally defined object, to the locally, independently defined Killing horizon. If the Killing horizon is generated by the stationary Killing field ξ so ξ is normal to it, the horizon is called "non-rotating". Then by the staticity theorem [27, 131, 132], the spacetime is necessarily static and the static uniqueness theorem follows. However this only applies to *non-degenerate*[2] static horizons, because in order to apply the staticity theorem, the intermediate step of showing that the DOC contains a maximal Cauchy surface is only available for non-degenerate horizons.

If on the other hand the stationary Killing field ξ is *not* normal to the horizon, the horizon is said to be "rotating". The rigidity theorem then states that, again by assuming analyticity, there must exist a *second* Killing field K normal to the horizon, whose orbits are complete [29]. Furthermore if the spacetime satisfies some reasonable regularity conditions, the positive energy theorem [9] then implies there exists a linear combination m of ξ and K which has periodic orbits and an axis of rotation (i.e. a $2D$ totally geodesic submanifold in M on which m vanishes), and that m commutes with ξ (and obviously K too). Thus m generates a $U(1)$ isometry and the black hole is axisymmetric as well as stationary. In particular, the Killing field K that generates the Killing horizon is the combination

$$K = \xi + \Omega m \tag{1.1}$$

where Ω is the angular velocity of the horizon. Thus the horizon rotates *rigidly* with respect to infinity, hence the name "rigidity theorem".

1.1.5 The uniqueness theorem

The "stationary implies axisymmetry" result then allows one to apply the Carter-Robinson-Mazur-Bunting theorems [17,19,103,112] to complete the proof of the uniqueness theorem [24, 25, 31]

Theorem 1.1 (Uniqueness theorem). *Let (M, g) be a 4-dimensional stationary, asymptotically flat, reasonably regular electrovacuum black hole spacetime. If the event horizon is connected and rotating, then the domain of outer communications is isometric to the domain of outer communications of a Kerr-Newman black hole.*

In Boyer-Lindquist coordinates, the Kerr-Newman metric is

$$\mathrm{d}s^2 = -\frac{\Delta}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)\mathrm{d}\phi - a\mathrm{d}t\right)^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma \mathrm{d}\theta^2 \tag{1.2}$$

with

$$\Delta = r^2 - 2Mr + a^2 + e^2 \qquad \Sigma = r^2 + a^2\cos^2\theta \qquad a = \frac{J}{M} \qquad e^2 = Q^2 + P^2\,,$$

[2] A Killing horizon is degenerate if the surface gravity κ vanishes and non-degenerate otherwise; the definition of surface gravity can be found in section 1.3.

where M, J, Q and P are the mass, angular momentum, electric and magnetic charges respectively.

1.1.6 Black hole non-uniqueness beyond Einstein-Maxwell

The no-hair conjecture is however violated when Einstein gravity is coupled to non-linear matter. The most notable example is the Einstein-Yang-Mills theory. Under the conjecture, any stationary black hole solution should be uniquely specified by its mass, angular momentum and Yang-Mills charges. However [13,83,118,120,134] demonstrated the existence of an infinite number of static black hole solutions indexed by a winding number, all of which admit a regular horizon and are asymptotically Schwarzschild with vanishing Yang-Mills charges.

1.2 Black hole mechanics

Stationary black holes bear striking resemblance to bodies in thermal equilibrium - the laws of black hole mechanics [7] are analogous to the laws of thermodynamics with the few parameters that characterise a *classical*[3] black hole playing the role of the state parameters in thermodynamics. In particular the mass M, surface gravity κ (see section 1.3 for definition and interpretation) and the area A of the spatial cross section of the horizon of a black hole are found to behave exactly like the total energy E, temperature T and entropy S in a thermodynamic system, see table 1.1. The fact that M and E are physically equivalent seems to suggest that this is not just an accidental analogy but there exists some deeper underlying principles relating the other quantities too. This nevertheless immediately falls to two obvious contradictions: a *classical* black hole being a perfect absorber cannot acquire a non-zero temperature, besides under the uniqueness theorem there is only one microstate and therefore no entropy for any given mass, charge and angular momentum.

Law	Thermodynamics	Black hole mechanics
0th	T is constant throughout body in thermal equilibrium	κ is constant over stationary horizon
1st	$\mathrm{d}E = T\mathrm{d}S + \mathrm{d}(\text{work})$	$\mathrm{d}M = \frac{\kappa}{8\pi}\mathrm{d}A + \Omega\mathrm{d}J + \Phi\mathrm{d}Q$
2nd	$\delta S \geq 0$	$\delta A \geq 0$
3rd	$T = 0$ unachievable	$\kappa = 0$ unachievable

Table 1.1: Laws of thermodynamics versus laws of black hole mechanics

This picture however is purely *classical* without taking *quantum mechanics* into account. Hawking demonstrated that a black hole is in fact a perfect black body by considering the particle creation effect that arises from quantum field theory near the

[3] By classical we mean in the context of general relativity.

event horizon [68]. An observer at infinity sees a thermal flux of particles with black body spectrum at temperature

$$T = \frac{\hbar\kappa}{2\pi} \tag{1.3}$$

appearing to be emitted from the black hole, so the surface gravity really *is* the (Hawking) temperature of the black hole. Following this result one can then deduce from the first law of black hole mechanics the Bekenstein-Hawking entropy

$$S_{BH} = \frac{A}{4\hbar} \ . \tag{1.4}$$

But this appears to violate the second law which states that $\delta S_{BH} \geq 0$, because now A can decrease as the black hole evaporates from Hawking radiation. However the *total* entropy $S = S_{radiation} + S_{BH}$ never decreases, so the *generalised second law* $\delta S' = \delta(S_{out} + S_{BH}) \geq 0$, where S_{out} is the entropy of matter outside the black hole, is never violated [10].

Although first formulated in $4D$, the laws of black hole mechanics hold in *any* dimensions in general. The only difference in higher dimensions ($D > 4$) is that the first law may acquire new "work" terms which arise from the new features exhibited in the novel black hole solutions in $D \geq 5$ (see e.g. [91, 108]).

1.3 Extreme black holes

Associated to every Killing horizon $\mathcal{N}$ of Killing field K, there exists a notion of *surface gravity* κ. It is interpreted as the force required to exert on a unit test mass at infinity in order to hold it at rest and follow a orbit of the stationary Killing field near the horizon. Obviously the force exerted *locally* on the unit test mass diverges on the horizon, nor can a test mass be held stationary with respect to infinity if the black hole is rotating, but κ is still referred as the surface gravity in line with Newtonian surface gravity. The surface gravity κ on $\mathcal{N}$ is defined by

$$\mathrm{d}K^2 = -2\kappa K \ . \tag{1.5}$$

Form this definition and the Einstein equation, it follows that under the dominant energy condition, κ must be constant on a Killing horizon, which is the zeroth law of black hole mechanics.

Extreme (degenerate) black holes are black hole spacetimes with $\kappa = 0$, and non-extreme (non-degenerate) otherwise. Extreme black holes therefore do not radiate Hawking radiation, so they are inert objects even semi-classically[4], hence they are ex-

[4]Classically, a *stationary* black hole is necessary in equilibrium so they are inert objects regardless of extremality.

pected to have a simpler quantum description.

Supersymmetric black holes are necessarily extreme. Killing spinors are the parameters of preserved supersymmetry of a solution, so a supersymmetric solution to any supergravity theory necessarily admits a Killing spinor ϵ. The bilinear $K^\mu = \bar{\epsilon}\Gamma^\mu\epsilon$ is a non-spacelike Killing vector field i.e. $K^2 \leq 0$. Suppose a supersymmetric Killing horizon $\mathcal{N}$ is generated by K^μ, then K^μ must be null and must attain a maximum on $\mathcal{N}$, therefore $\mathrm{d}K^2 = 0$ on $\mathcal{N}$ and $\mathcal{N}$ must be degenerate.

1.4 Black holes in quantum gravity

1.4.1 Black holes in string theory

The fact that the Bekenstein-Hawking entropy (1.4) relates a statistical quantity to a purely geometric one implies it has a much deeper origin. A quantum theory of gravity should explain the microscopic state counting of the "quantum degrees of freedom" of a black hole, which is given macroscopically by the Bekenstein-Hawking entropy. The entropy of a black hole S_{BH} is related to the number of its microstates N by the standard $S_{BH} = \log N$.

String theory is the most promising candidate to date for a quantum theory of gravity. The entropy-area relation was first derived microscopically in the context of string theory for a certain class of 5-dimensional asymptotically flat supersymmetric black holes by counting the degeneracy of BPS states [130]. The derivation itself also marks one of the most notable successes of string theory. One caveat in this type of calculations is the assumption that a black hole is *uniquely determined* by its conserved charges. This is not always true as we will discuss in more details in section 1.5.1, and it remains unclear what effect classical black hole non-uniqueness has on the microscopic degeneracy of black holes in string theory.

Supersymmetric black holes are charged and they are typically supported by scalar fields. The attractor mechanism is the phenomenon that, although these scalar fields are in general not constant in the radial direction, they are *attracted* to certain values on the horizon depending only on the charges of the black hole. So the black hole entropy is completely independent of the scalar fields (this in fact also holds for any other moduli of the theory) and is totally determined by the charges [44,45,127]. The attractor mechanism was later generalised to extreme but non-supersymmetric black holes assuming that their *near-horizon geometries* admit $SO(2,1)$ symmetry [59,82,114]. All known extreme black holes possess $SO(2,1)$ symmetry in their near-horizon geometries; the presence of $SO(2,1)$ symmetry in near-horizon geometries has been proved in a wide class of gravitational theories in various dimensions through *near-horizon symmetry enhancement theorems* (a more detailed discussion can be found in section 2.1, or see e.g. [93] or the review paper [88]). This implies that rather than supersymmetry, it is extremality that holds the key to the success of the microstates counting calculations in string theory [2,32]. In particular, this shows that black hole entropy is independent of the string coupling, so it can be computed at weak coupling which is easier.

1.4.2 Black Holes in AdS/CFT

The latest break through in quantum gravity is the AdS/CFT duality [60,100,139,140], the statement of a fully non-perturbative equivalence between strongly coupled quantum field theories in D-dimensions and classical theories of gravity in $D+1$-dimensions. In particular, it states that classical gravitational theory in asymptotically AdS spacetime is dual to the strongly coupled regime in the CFT living on its boundary. The original correspondence [100] was established between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ maximally supersymmetric $SU(N)$ Yang-Mills theory on the $R \times S^3$ boundary of the AdS_5. Since then it has been generalised to other dimensions and to theories which are not AdS or CFT, and with numerous phenomenological applications in the context of condensed matter physics where the systems are typically strongly interacting, see e.g. the reviews [1, 66, 104].

Since then, asymptotically AdS black holes are studied extensively due to their applications in AdS/CFT. This is because AdS/CFT typically involves a string theory on spacetime asymptotic to $\mathrm{AdS}_d \times X$, which is then dimensionally reduced on X to a d-dimensional gauged supergravity theory with a negative cosmological constant, where one solves for asymptotically AdS black hole solutions (although there could be black hole solutions in the original theory that cannot be dimensionally reduced). The black hole horizon serves as an important "in going boundary condition" to the holographic calculation in the asymptotically AdS bulk because black holes are dual to *thermal states* in the CFT on the boundary.

Applying the duality the other way round, the AdS/CFT dictionary then provides a precise framework to derive black hole entropy microscopically from the CFT; precise agreement has been found for BTZ black holes [128]. Because all known extreme black holes have an AdS_2 factor in their near-horizon geometries, it is then natural to speculate that $\mathrm{AdS}_2/\mathrm{CFT}_1$ [115, 129] would account for their entropy from the degeneracy of the ground states of the one-dimensional dual CFT. Progress has been made in establishing a precise relation [101, 115]. Furthermore, the Kerr/CFT correspondence has been proposed by adapting AdS/CFT to the extreme Kerr black hole, which conjectures that quantum gravity in the near-horizon geometry of extreme Kerr black hole is dual to a 2-dimensional chiral CFT [63]. This has led to the successful microscopic entropy counting for many black holes, see e.g. the reviews [15, 30] and the references therein.

Black holes (and branes) are increasing used to study phase transitions in superfluids and superconductors following the AdS/CFT correspondence. It has been demonstrated that superconductivity occurs when a charged scalar condensate forms just outside the horizon and break the $U(1)$ gauge symmetry in the Einstein-Maxwell-Higgs theory [61]. A classical non-extreme black hole describes a thermal state in the dual CFT, while an extreme one represents the zero-temperature phase where quantum critical behaviour occurs. The fact that an extreme black hole has finite entropy suggests that the ground state of the corresponding CFT is highly degenerate. The classification of NHG in Einstein gravity coupled to abelian vectors and *charged* scalars therefore corresponds to mapping out the zero-temperature phases of the dual CFT system, which is a largely open problem.

1.5 Black holes in higher dimensions

1.5.1 Black hole non-uniqueness

Mathematical theories of *classical* black holes in higher dimensions have gained increasing interest in recent years. This is mainly due to the fact that string theory only works in higher dimensions. Applications in the context of AdS/CFT duality are also typically set in higher dimensions, for instance one needs to consider a 5-dimensional dual spacetime in order to study a real life $(3 + 1)$-dimensional field theory system. Form the mathematical point of view, in light of the profound success of the black hole uniqueness theorem in $4D$, it would be interesting to see how it generalises to higher dimensions. A comprehensive review on black holes in higher dimensions can be found in [42].

Black hole uniqueness is in fact violated even in the absence of matter field in higher dimensions. In $5D$ Einstein vacuum there exists more than one stationary asymptotically flat black hole solutions with the same mass and angular momenta: there is the Myers-Perry black hole which has S^3 horizon topology [108] and the black ring which has $S^1 \times S^2$ horizon topology [41], the two different horizon topologies cannot be distinguished asymptotically by a Gaussian flux integral. There is however uniqueness theorem for *static* black holes in higher dimensions, which states that any static asymptotically flat black hole in Einstein-Maxwell-dilaton theory is described by the higher dimensional analogue of the Reissner-Nordström black hole and therefore uniquely determined by its mass and charges [57,58]. Interestingly, the Schwarzschild(-Tangherlini) black hole in $5D$ is not unique in the sense that a black Saturn, which is a $5D$ vacuum stationary asymptotically flat multi-black hole spacetime with an S^3 black hole inside a black ring, can have *vanishing total angular momentum* by counter-rotation [38]. This however does not violate the uniqueness theorem for static black holes because the black Saturn is not static.

Recall that topological censorship asserts that the domain of outer communications of any asymptotically flat, globally hyperbolic spacetime which obeys the null-energy condition is simply connected. In $4D$ this turns out to be a very restrictive constraint: any spacelike hypersurface Σ in the DOC must be trivially $\mathbb{R}^3 - B$ since $H_2 = 0$ as well by the Poincaré duality. This in turn implies that the spatial cross-section H of the event horizon must have S^2 topology by the existence of a cobordism between H and the S^2 at the asymptotically flat null infinity, as we discussed in section 1.1.2.

This argument however does not generalise to impose such strong constraints in higher dimensions. For instance, Σ can have non-trivial higher homology groups $H_p(\Sigma)$ for $p \geq 2$, so there can be non-trivial 2-cycles outside the black hole horizon. These incontractible S^2 are called "bubbles"; soliton spacetimes with bubbles are well established in supergravity (see [11] for review). Stationary asymptotically flat vacuum soliton solutions are forbidden by the Lichnerowicz theorem, so there can be no such solutions with bubbles either. However the possibility remains wide open for bubbly vacuum *black hole solutions.* Although much less is known about black hole spacetimes with bubbles, asymptotically flat bubbly black hole solutions have recently been found in $5D$ minimal supergravity [89].

The horizon topology theorem is also much less stringent in higher dimensions. The existence of black rings alone is enough to show that horizon cross-sections are not necessarily homeomorphic to S^{D-2}. However there are still some restrictions on the allowed topologies. Cobordism theory still imposes some restrictions [71] as in the $4D$ case. Further in [55] and [53], it was shown that the horizon cross-sections of a stationary asymptotically flat black hole which obeys the dominant energy condition must by of positive Yamabe type, i.e. they must admit metrics of positive Ricci scalar.

Nevertheless, the rigidity theorem still holds in higher dimensions. It was proved in [78] for a non-degenerate (non-extreme) Killing horizon, and partially proved for the degenerate (extreme) case in [76] under certain assumptions regarding the angular momenta. However these results only guarantee the existence of *one* rotational Killing field as in the $4D$ case, whereas all known black hole solutions and classification theorems in higher dimensions admit more than one rotational symmetries. It remains unclear how "rigidly" horizons must rotate relative to infinity, although some evidence pointing towards the existence of stationary vacuum black holes with only $\mathbb{R} \times U(1)$ symmetry does exists [35, 39].

1.5.2 Black Hole Classification

Even though higher dimensional stationary black holes cannot be uniquely specified by a finite set of global charges, it may still be possible to *classify* them (see [77] for a review on uniqueness theorems in higher dimensions). This has been done in $5D$ for stationary asymptotically flat vacuum black holes with two commuting axisymmetries: these black holes are completely determined by their mass, angular momenta and a quantity called "rod structure" [79]. The rod structure describes the relative positions of the horizon and the rotation axes, thus encodes information about the horizon topology. It was also shown that the topology of the horizon cross section must be a sphere S^3, a ring $S^1 \times S^2$ or a lens space $L(p,q)$ with $p,q \in \mathbb{Z}$. Whereas the Myers-Perry solution and the black ring belong to the first two classes, the "black lens" is yet to be found. The first asymptotically flat black hole spacetime with the lens space topology has recently been constructed in $5D$ supergravity [90].

The picture beyond $D = 5$ is far less understood, their classification remains largely unexplored. The only known exact vacuum stationary asymptotically flat black hole solutions are those in the Myers-Perry family. Progress has been made in generalising the Weyl-Papapetrou solutions in $4D$ to $D > 4$, but for $D \geq 6$ the *generalised Weyl solutions* are not compatible with asymptotic flatness. It was demonstrated in [40] and [65] that if a spacetime in Einstein vacuum possesses $\mathbb{R} \times U(1)^{D-3}$ isometry generated by one stationary and $D-3$ rotational Killing fields, then the metric can be factored into the form

$$\mathrm{d}s^2 = e^{2\nu(\rho,z)}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + g_{ab}(\rho,z)\mathrm{d}x^a\mathrm{d}x^b \tag{1.6}$$

where ∂_a with $a = 0,...,D-3$ are the Killing fields, and the Einstein equations can be shown to be integrable. This class of solutions are called the generalised Weyl solutions. In particular if all Killing fields are orthogonal and the spacetime is static, the problem reduces to solving a Laplace equation. Recall that if a spacetime is globally asymptotically flat then it is diffeomorphic to $\mathbb{R} \times S^{D-2}$ asymptotically. The spatial factor S^{D-2} has isometry group $O(D-2)$ which has Cartan subgroup $U(1)^N$, where the rank $N = \lfloor (D-1)/2 \rfloor$. In $D = 4, 5$, the number of commuting axisymmetries $D-3$ one needs to assume in order for the spacetime to take the (generalised) Weyl form (1.6) matches exactly with N the number of commuting axisymmetries at the asymptotic. But in $D \geq 6$, the number of axisymmetries $D-3$ one assumes always exceeds N and therefore (1.6) is incompatible with asymptotic flatness.

1.6 Near-horizon geometry

The Einstein equation is a complicated system of second order coupled non-linear partial differential equations, so analytic solutions are very difficult to find in general without enough symmetries. The rigidity theorem assets that the event horizon of any stationary black hole is a Killing horizon. Any *extreme* Killing horizon has a well-defined *near-horizon geometry* (the precise definition will be given in the next section), so one may instead turn to the simpler problem of solving near-horizon geometries

(NHG). Classifying NHG is equivalent to classifying the possible horizon topologies and geometries of the extreme black hole spacetimes, see [88] for a review. It also provides a natural testing ground for the $SO(2,1)$ near-horizon symmetry enhancement phenomenon.

1.6.1 Gaussian Null Coordinates and Near-Horizon Limit

In order to study near-horizon geometries we need to introduce a coordinate system which is *regular* on the horizon. We will work in an adapted coordinate system called *Gaussian null coordinates* (GNC) [106]. The coordinate chart is defined as follows.

Let (M,g) be a D dimensional spacetime and $\mathcal{N}$ be a smooth codimension-1 null hypersurface in M. Let N be a vector field normal to $\mathcal{N}$ such that the integral curves of N are future directed null geodesic generators of $\mathcal{N}$. Let H be a smooth spacelike cross-section of $\mathcal{N}$ such that each integral curve of N crosses H exactly once, we can assign local coordinates (x^a) with $(a = 1, ..., D-2)$ to H. Starting from a point $p \in H$, we assign a point $q \in \mathcal{N}$ lying a parameter (need not be affine) value v away along the integral curve of N the coordinates (v, x^a), keeping the functions x^a constant along the curve. Thus (v, x^a) describe a coordinate system on $\mathcal{N}$ in the neighbourhood of the integral curves of N through H with $N = \frac{\partial}{\partial v}$. Recall that N is normal to $\mathcal{N}$ and is null on $\mathcal{N}$, so we have the metric functions $g_{vv} = N \cdot N = 0$ and $g_{va} = N \cdot \frac{\partial}{\partial x^a} = 0$ on $\mathcal{N}$.

Now at every point $q \in \mathcal{N}$, let L be the unique past directed null vector satisfying the (energy) normalization $N \cdot L = 1$ and orthogonality $L \cdot \frac{\partial}{\partial x^a} = 0$. Starting from q, the point $s \in M$ lying an affine parameter value r along the null geodesic with tangent L is assigned coordinates (v, r, x^a), with the functions v and x^a kept constant along the geodesic as they are extended into M. Therefore (v, r, x^a) describe a coordinate system in the neighbourhood of $\mathcal{N}$ in M, where the null hypersurface $\mathcal{N}$ is $\{r = 0\}$.

Using these coordinates, we can also extend the definitions of the vector fields $N = \frac{\partial}{\partial v}$, $L = \frac{\partial}{\partial r}$ and $\frac{\partial}{\partial x^a}$ into M, and since these are coordinate vector fields, they all commute. By construction L is null and $\nabla_L L = 0$ as the integral curves of L are null geodesics, so we have $g_{rr} = L \cdot L = 0$ everywhere. Consider the directional derivatives

$$\nabla_L (L \cdot N) = L \cdot \nabla_L N = L \cdot ([L, N] + \nabla_N L) = \frac{1}{2} \nabla_N (L \cdot L) = 0 \tag{1.7}$$

$$\nabla_L \left(L \cdot \frac{\partial}{\partial x^a} \right) = L \cdot \nabla_L \partial_a = L \cdot ([L, \partial_a] + \nabla_a L) = \frac{1}{2} \nabla_a (L \cdot L) = 0 \; , \tag{1.8}$$

therefore we also have $g_{rv} = L \cdot N = 1$ and $g_{ra} = L \cdot \partial_a = 1$ for all r in the open set where the coordinates are defined, not only on $\mathcal{N}$. Nevertheless, $g_{vv} = N \cdot N$ and $g_{va} = N \cdot \partial_a$ need not vanish outside $\mathcal{N}$. Hence the metric in Gaussian Null coordinates takes the form

$$g = 2\mathrm{d}v\left(\frac{1}{2}rf\mathrm{d}v + \mathrm{d}r + rh_a\mathrm{d}x^a\right) + \gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{1.9}$$

where γ is the metric on H and f, h_a and γ_{ab} are *smooth* (or at least C^2) functions of all coordinates (v, r, x^a) so that the spacetime is *smooth* (or C^2). See figure 1.2 for a diagrammatic way of constructing the coordinate system.

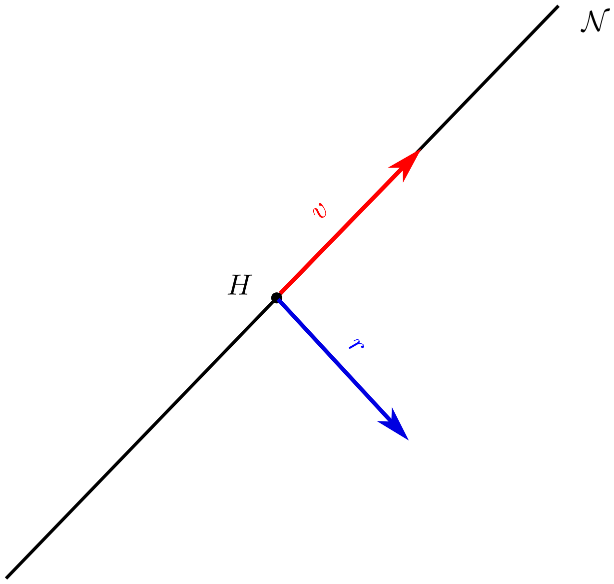


Figure 1.2: Gaussian Null Coordinates.

A Killing horizon of Killing field K is a null hypersurface $\mathcal{N}$ along which K is null and non-vanishing[5]. Since K is necessarily normal to $\mathcal{N}$, we can set $N = K$ and define Gaussian Null coordinates (v, r, x^a) in the neighbourhood of the Killing horizon. Because K is Killing, all metric functions are independent of v. It follows that in the neighbourhood of a Killing Horizon $\mathcal{N}$, the spacetime metric can be written in GNC as (1.9), where all metric functions depend only on (r, x^a) and the horizon is located at $r = 0$.

The surface gravity κ of a Killing horizon $\mathcal{N}$ is defined via the Killing field K on $\mathcal{N}$ by $\mathrm{d}(K \cdot K) = -2\kappa K$ or equivalently, $\nabla_K K = \kappa K$ i.e. it measures the extend to which the Killing parameter v fails to be affine. When κ vanishes, the Killing horizon is degenerate or extreme, and non-degenerate or non-extreme otherwise.

Let us now focus on the degenerate case where v is affine, and $\mathrm{d}(K.K)|_{\mathcal{N}} = 0$. Expanding this 1-form in components and using the linear independence of GNC basis, the $\mathrm{d}r$ component then gives $(\partial_r g_{vv})|_{r=0} = 0$, so $g_{vv} = r^2F(r, x)$ where $F(r, x)$ is some

[5]Thus we exclude any *bifurcate Killing horizons* since K vanishes on the intersection.

smooth function. Hence the spacetime metric in the neighbourhood of a *degenerate* Killing horizon in GNC is given by

$$g = 2\mathrm{d}v\left(\frac{1}{2}r^2F(r,x)\mathrm{d}v + \mathrm{d}r + rh_a(r,x)\mathrm{d}x^a\right) + \gamma_{ab}(r,x)\mathrm{d}x^a\mathrm{d}x^b \tag{1.10}$$

The double zero which arises from extremality turns out to be a crucial point in defining *near horizon geometry*, as we demonstrate below.

Let $\varepsilon > 0$, consider the diffeomorphism

$$r \to \varepsilon r\ , \qquad v \to \frac{v}{\varepsilon} \tag{1.11}$$

so the metric (1.10) transforms as

$$g \to g_\varepsilon = 2\mathrm{d}v\left(\frac{1}{2}r^2F(\varepsilon r,x)\mathrm{d}v + \mathrm{d}r + rh_a(\varepsilon r,x)\mathrm{d}x^a\right) + \gamma_{ab}(\varepsilon r,x)\mathrm{d}x^a\mathrm{d}x^b\ . \tag{1.12}$$

The *near-horizon limit* is defined as the limit $\varepsilon \to 0$ [111]. Since the metric functions $(F,\ h_a,\ \gamma_{ab})$ are assumed to be smooth at $r = 0$ and all ε factors cancel outside the metric functions, such limit always exists. However for a *non-degenerate* Killing horizon, $g_{vv} = \mathcal{O}(r)$ so $g_{\varepsilon vv} = \frac{1}{\varepsilon}rf(\varepsilon r,x)$, therefore the near horizon limit does *not* exist. Thus the *near horizon geometry* is only defined for a spacetime containing a *degenerate* Killing horizon, which is given by the metric

$$g_{NH} = 2\mathrm{d}v\left(\frac{1}{2}r^2F(x)\mathrm{d}v + \mathrm{d}r + rh_a(x)\mathrm{d}x^a\right) + \gamma_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b \tag{1.13}$$

where $F(x) = F(0,x)$ and similarly for the other functions. Note that the r dependence of the metric is completely fixed, and the near horizon geometry is completely specified by a set of functions $\{F(x),\ h_a(x),\ \gamma_{ab}(x)\}$ on the codimension-2 spacelike hypersurface H which corresponds to $r = 0$ and $v = const$. This set of functions are collectively called the *near horizon data*.

Regardless of the forms of the near-horizon data, the metric (1.13) is always invariant under the translation $v \to v + a$ generated by ∂_v and "dilation" $v \to v/\Omega$ and $r \to \Omega r$ generated by $v\partial_v - r\partial_r$, together they form a 2-dimensional non-abelian symmetry group $\mathcal{G}_2$. Whereas the translational symmetry is guaranteed in the first place by the presence of the Killing field K that generates the Killing horizon, the dilation symmetry is an *enhanced* symmetry which arises from the definition of NHG. *Near-horizon symmetry enhancement theorems* state that in all those cases considered, this $\mathcal{G}_2$ is *further enhanced* to an even bigger group $SO(2,1)$.

An important remark is that near-horizon geometries are generically *exact* solutions to the Einstein equation. This is because if a spacetime containing a extreme Killing horizon written in the form (1.10) is an exact solution, the one-parameter family of diffeomorphic metrics g_ε as well as the $\varepsilon \to 0$ limit g_{NH} are also exact solutions.

1.6.2 Curvature of Near-Horizon Geometry

It is often more convenient to work in *null orthonormal basis* (e^A) so that the metric takes the form $g_{NH} = \eta_{AB} e^A e^B = 2e^+e^- + \delta_{ab}\hat{e}^a\hat{e}^b$, with $\hat{e}^a$ denoting the vielbeins for the metric γ on H such that $\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b = \delta_{ab}\hat{e}^a\hat{e}^b$. Comparing with (1.13), it is straightforward to deduce that

$$e^+ = \mathrm{d}v \qquad e^- = \mathrm{d}r + rh_a\hat{e}^a + \frac{1}{2}r^2F\mathrm{d}v \qquad e^a = \hat{e}^a \; , \tag{1.14}$$

and their dual basis vectors are

$$e_+ = \partial_v - \frac{1}{2}Fr^2\partial_r \qquad e_- = \partial_r \qquad e_a = \hat{\partial}_a - rh_a\partial_r \; , \tag{1.15}$$

where $\hat{\partial}_a$ are the dual vectors of $\hat{e}^a$. The connection 1-forms are given by $\mathrm{d}e^A = -\omega^A{}_B \wedge e^B$, so we have

$$\begin{array}{ll} \omega_{+-} = rFe^+ + \frac{1}{2}h_a e^a & \omega_{+a} = \frac{1}{2}r^2(\hat{\partial}_aF - Fh_a)e^+ - \frac{1}{2}h_a e^- + r\hat{\nabla}_{[a}h_{b]}e^b \\ \omega_{-a} = -\frac{1}{2}h_a e^+ & \omega_{ab} = \hat{\omega}_{ab} - r\hat{\nabla}_{[a}h_{b]}e^+ \; , \end{array} \tag{1.16}$$

where $\hat{\nabla}_a$ and $\hat{\omega}_{ab}$ are the covariant derivative and the connection 1-forms of the metric γ_{ab} respectively. In terms of the connection 1-forms, the curvature 2-forms are given by $\Omega_{AB} = \mathrm{d}\omega_{AB} + \omega_{AC} \wedge \omega^C{}_B$

$$\begin{aligned}
\Omega_{ab} \;&=\; \hat{\Omega}_{ab} + \hat{\nabla}_{[a}h_{b]}\; e^+ \wedge e^- \\
&\quad + r\left(-h_d\hat{\nabla}_{[a}h_{b]} + \hat{\nabla}_d\hat{\nabla}_{[a}h_{b]} + \frac{1}{2}h_a\hat{\nabla}_{[b}h_{d]} - \frac{1}{2}h_b\hat{\nabla}_{[a}h_{d]}\right) e^+ \wedge e^d \\
\Omega_{+-} \;&=\; \left(\frac{1}{4}h_ah_a - F\right) e^+ \wedge e^- + r\left(\hat{\partial}_bF - Fh_b + \frac{1}{2}h_a\hat{\nabla}_{[a}h_{b]}\right) e^b \wedge e^+ \\
&\quad + \frac{1}{2}\hat{\nabla}_{[a}h_{b]}\; e^a \wedge e^b \\
\Omega_{+a} \;&=\; r^2\left[\left(-\frac{1}{2}\hat{\nabla}_d + h_d\right)(\hat{\partial}_aF - Fh_a) + \frac{1}{2}F\hat{\nabla}_{[a}h_{d]} + \hat{\nabla}_{[c}h_{a]}\hat{\nabla}_{[c}h_{d]} + \frac{1}{2}h_{[a}\hat{\nabla}_{d]}F\right] e^+ \wedge e^d \\
&\quad + r\left(h_aF - \hat{\partial}_aF - \frac{1}{2}h_b\hat{\nabla}_{[b}h_{a]}\right) e^+ \wedge e^- + \frac{1}{2}\left(\hat{\nabla}_ah_b - \frac{1}{2}h_ah_b\right) e^- \wedge e^b \\
&\quad + r\left(-\hat{\nabla}_d\hat{\nabla}_{[a}h_{b]} + \frac{1}{2}h_a\hat{\nabla}_{[d}h_{b]} - \frac{1}{2}h_b\hat{\nabla}_{[a}h_{d]}\right) e^b \wedge e^d \\
\Omega_{-a} \;&=\; \frac{1}{2}\left(\hat{\nabla}_bh_a - \frac{1}{2}h_ah_b\right) e^+ \wedge e^b \; ,
\end{aligned} \tag{1.17}$$

from which the Riemann tensor can be computed with the relation $\Omega_{AB} = \frac{1}{2} R_{ABCD}\, e^C \wedge e^D$. The components of the Ricci tensor are then

$$
\begin{aligned}
R_{--} &= R_{-a} = 0 \\
R_{+-} &= F - \frac{1}{2} h^a h_a + \frac{1}{2} \hat{\nabla}^a h_a \\
R_{ab} &= \hat{R}_{ab} + \hat{\nabla}_{(a} h_{b)} - \frac{1}{2} h_a h_b \\
R_{++} &= r^2 \left[-\frac{1}{2} \hat{\nabla}^2 F + \frac{3}{2} h^a \hat{\nabla}_a F + \frac{1}{2} F \hat{\nabla}^a h_a - F h^a h_a + \hat{\nabla}^{[c} h^{a]} \hat{\nabla}_{[c} h_{a]} \right] \\
R_{+a} &= r \left[\hat{\nabla}_a F - F h_a - 2 h_b \hat{\nabla}_{[a} h_{b]} + \hat{\nabla}_b \hat{\nabla}_{[a} h_{b]} \right] . \qquad (1.18)
\end{aligned}
$$

In fact R_{++} can be written in terms of R_{+a}, which in turn can be expressed in terms of $R{+-}$ and R_{ab} as follows

$$
\begin{aligned}
R_{++} &= -\frac{1}{2} r (\hat{\nabla}^a - 2h^a) R_{+a} \qquad (1.19) \\
R_{+a} &= r \left\{ -\hat{\nabla}^b \left[R_{ba} - \frac{1}{2} \gamma_{ba} (R^c{}_c + 2R_{+-}) \right] + h^b R_{ba} - h_a R_{+-} \right\} . \qquad (1.20)
\end{aligned}
$$

Thus not all components of the Ricci tensor are independent; this is due to the contracted Bianchi identity.

1.6.3 The Einstein equation

We are interested in solutions to the Einstein equation. For convenience we write it in the form

$$
R_{\mu\nu} = \Lambda g_{\mu\nu} + T_{\mu\nu} - \frac{T}{n} g_{\mu\nu} \, , \qquad (1.21)
$$

where $T_{\mu\nu}$ is the near-horizon limit of the stress energy tensor and $T = g^{\mu\nu} T_{\mu\nu}$ denotes its trace, and $n = D - 2$ is the dimension of H. The $--$ and $-a$ components of $T_{\mu\nu}$ must vanish in order for it to admit a near-horizon limit, thus the $--$ and $-a$ components of the Einstein equation (1.21) are always trivially satisfied. The $+-$ and ab equations are

$$
\begin{aligned}
F &= \frac{1}{2} h^2 - \frac{1}{2} \hat{\nabla}_a h^a + \Lambda + \left(\frac{n-2}{n} \right) T_{+-} - \frac{\gamma^{ab} T_{ab}}{n} \qquad (1.22) \\
R_{ab} &= \frac{1}{2} h_a h_b - \hat{\nabla}_{(a} h_{b)} + \Lambda \gamma_{ab} + T_{ab} - \frac{1}{n} (\gamma^{cd} T_{cd} + 2T_{+-}) \gamma_{ab} \, , \qquad (1.23)
\end{aligned}
$$

which are equations defined purely on H. In fact (1.22) and (1.23) are *the near-horizon geometry equations*. By the virtue of the contracted Bianchi identity the $++$ and $+a$ components are redundant; this can also be verified explicitly through a tedious calculation.

In summary, near-horizon geometry dimensionally reduces the problem of solving the full spacetime (M, g) to the horizon cross-section (H, γ). NHG is also easier to solve than (1.10) because of the enhanced $\mathcal{G}_2$ symmetry. Nevertheless, while NHG preserves all the important information about the topology and geometry of the horizon, we emphasise that given a NHG, there is no guarantee that it is the near-horizon limit of some full *black hole* solution. Two different black hole solutions can also have the same near-horizon limit, for example in $5D$ the extreme self-dual Myers-Perry black hole and the extreme $J = 0$ Kaluza-Klein black hole both share the same NHG [84]. Even a non-static black hole may admit a static NHG, as in the case of supersymmetric black rings [37].

1.7 This Thesis

In chapter 2 we investigate near-horizon geometries beyond $4D$ Einstein-Maxwell theory. We prove that in $4D$ Einstein Yang-Mills theory, the near-horizon geometry of any stationary axisymmetric extreme black hole solution possesses enhanced $SO(2, 1)$ symmetry, and the solution must be that of the abelian embedded extreme Kerr-Newman (AdS) black hole. We also prove that in Einstein-Maxwell-charged scalar theory in any dimensions, there exists no near-horizon geometry with a compact horizon in which the massive complex scaler field takes non-zero value.

In chapter 3 we determine the most general solution containing a non-singular Killing horizon with a compact spatial cross-section in $3D$ Einstein vacuum with a negative cosmological constant. We show that they must be related to the BTZ black hole by *some* diffeomorphism: a large diffeomorphism for the extreme case and a small diffeomorphism for the non-extreme case.

In chapter 4 we address the inverse problem of "given a near-horizon geometry, find the full black hole spacetime that admits this near-horizon limit" by examining the first order expansion from near-horizon geometry. We apply the analysis to the extreme Kerr NHG and uncover a local uniqueness theorem. In $5D$ we show that there exists more general solutions to the known black hole solutions which possess the NHG of the extreme Myers-Perry black hole.

In chapter 5 we write down the near-horizon geometries of the BPS extremal M2, M5 and D3 black branes in coordinates which are regular on the horizon and verify their famous near-horizon geometries, written in the form of a warped product of AdS_2 with a hyperbolic space plus a sphere.

In chapter 6 we summarise and conclude with a discussion on future work.

Chapter 2

Extreme Horizons Beyond Einstein-Maxwell

2.1 Introduction

As we discussed in chapter 1, extreme black holes are important in the studies of quantum gravity since they possess zero temperature. A key geometric structure which exists in all known examples is a near-horizon AdS_2 symmetry. This symmetry has played a fundamental role in developing various quantum descriptions of extreme black holes, see e.g. [115, 129]. It has even lead to the proposal that the extreme Kerr black hole is described by a two-dimensional CFT [63].

The AdS_2 near-horizon symmetry has been established in a wider context via near-horizon symmetry enhancement theorems for $D = 4, 5$ extreme black holes [93] and also for $D > 5$ [46, 98], under various assumptions regarding the rotational symmetry. In $D = 4, 5$ the theorem is valid in a general class of theories of Einstein gravity coupled to an arbitrary number of Maxwell fields and uncharged scalars (with a non-positive potential) [93]. This includes a number of consistent truncations of higher dimensional supergravity theories, such as $D = 4, 5$ minimal (gauged) supergravity coupled to vector multiplets. Typically, these are special cases of more general consistent truncations such as $D = 4, 5$ maximal (gauged) supergravity, which contain more general types of matter such as charged scalar fields and non-abelian gauge fields. It is therefore of interest to investigate whether the near-horizon symmetry enhancement phenomenon persists in the presence of such fields. In this chapter we first focus on four dimensional extreme black holes with *non-abelian gauge fields*, then we comment briefly in the case where a charged scalar is coupled to a Maxwell field.

It has been known for sometime that the four dimensional black hole uniqueness theorems fail in the presence of non-abelian gauge fields, see [133] for a review. Most strikingly Einstein-Yang-Mills theory admits an infinite number of asymptotically flat, static and spherically symmetric solutions, including both smooth solitons [8] and regular black holes [13, 83, 134]. These were first found numerically and subsequently their existence was established rigourously [118, 120]. In fact many of the components of the black hole uniqueness theorems do not work when coupled to a non-abelian gauge field [21, 73]. Non-rotating black holes need not be static, static ones need not be spherically symmetric, and as already mentioned even spherically symmetric ones are not unique. On the other hand, the rigidity theorem still applies, which guarantees that a rotating black hole must be axisymmetric. However, the Einstein equations for stationary and axisymmetric spacetimes do not guarantee orthogonal transitivity of the isometry group (as in Einstein-Maxwell theory). Hence the Weyl-Papapetrou form for the metric is overly restrictive, and furthermore, even assuming this does not lead to an integrable 2d theory (as in Einstein-Maxwell).

Interestingly, if the gauge group is $SU(2)$, four-dimensional Einstein-Yang-Mills theory with a negative cosmological constant is a consistent truncation of 11d supergravity on a squashed S^7 [110]. It is worth noting that in this context non-abelian Anti de Sitter black hole solutions also exist [137, 138].[1] Such solutions are of interest in the context of the AdS/CFT dualities and in the case of planar horizons have been used to model phase transitions analogous to superfluidity/superconductivity [62].

Most of the investigations of non-abelian black holes have focused on non-extreme black holes. Given the importance of non-abelian equilibrium states in quantum theory it is natural to ask: does the gross violation of uniqueness persist for *extreme* black holes? It appears this question has not been fully investigated even in four dimensions. A result in this direction suggesting this is not the case is that static and spherically symmetric extreme black holes to $SU(2)$-Einstein-Yang-Mills theory are uniquely given by the abelian embedding of the Reissner-Norström black hole [14, 56, 119]. A natural method for investigating the question more generally is to attempt to classify *near-horizon geometries* of non-abelian extreme black holes. In fact for *static* black holes this has been already considered under certain restrictive assumptions [12, 64]. Thus, our main focus will be stationary (non-static) black holes. The analogous problem in Einstein-Maxwell theory, including a cosmological constant, has been previously solved [84, 85, 94].

The first hurdle is that the AdS_2 near-horizon symmetry theorems mentioned above, do not immediately apply in the presence of non-abelian gauge fields. In fact recently it was shown that the enhancement of symmetry of the near-horizon geometry follows from orthogonal transitivity of stationary and axisymmetric solutions [98]. As mentioned above, the Einstein-Yang-Mills equations do not imply orthogonal transitivity (unlike in Einstein-Maxwell theory), thus raising the question: are there non-abelian

[1] Of course, in the presence of a cosmological constant even the (electro-)vacuum black hole uniqueness theorems are not valid.

near-horizon geometries without an AdS_2 symmetry?

In the first part of this chapter we will solve this problem within the simplest set up: four dimensional Einstein-Yang-Mills with a compact semi-simple gauge group and a cosmological constant Λ (mainly focusing on $\Lambda \leq 0$). We show that in fact the AdS_2 symmetry theorem can be generalised to axisymmetric near-horizon geometries with cross-sections of the horizon of spherical topology. This requires an extra global argument as compared to the Einstein-Maxwell case [85]. Given this, the system of equations is then essentially equivalent to the Einstein-Maxwell case, allowing us to show that the most general solution of this kind is the near-horizon geometry of the abelian embedded extreme Kerr-Newman black hole (with cosmological constant).

We also show that there are no *non-static* axisymmetric near-horizon geometries with toroidal cross-sections of the horizon. This is also the case in Einstein-Maxwell theory, a fact that does not seem to have been shown before for $\Lambda < 0$.[2] For completeness, by following the method used for Einstein-Maxwell theory we completely classify static near-horizon geometries, revealing that the only solutions with a compact horizon section are direct products of AdS_2 and a constant curvature space.

We then comment briefly on near-horizon geometries in Einstein-Maxwell theory with the presence of a charged scalar. We prove a no-hair theorem for compact horizons, valid in any dimensions with any cosmological constant. Static near-horizon geometries were previously investigated in [43], providing a lower bound for the effective mass of the scalar hair. We however do not assume staticity and our result holds also for static horizons. Thus our analysis is more general in this sense, although [43] covers also non-compact horizons. NHGs in Einstein gravity coupled to an arbitrary number of ableian vectors and *uncharged* scalars were also considered in [93].

It was found that in the Einstein-Maxwell-Higgs theory with a negative cosmological constant, a charged scalar condensate can appear *slightly* outside an non-extreme horizon therefore spontaneously breaking the $U(1)$ gauge invariance [61]. Following the AdS/CFT correspondence, this corresponds to a phase transition to the superconducting phase in the dual CFT with the scalar field playing the role of the order parameter.[3] Since an extreme black hole is dual to the ground state of the dual CFT, it is of interest to investigate the analogous problem for extreme horizons. A natural starting point is then to look at their near-horizon geometries, as we discuss towards the end of this chapter.

[2] For $\Lambda \geq 0$ this fact immediately follows by integrating the horizon scalar curvature [85] .

[3] Note that this is conceptually different from the Meissner effect exhibited in extreme black holes which arises geometrically [135].

2.2 Uniqueness of extreme horizons in Einstein-Yang-Mills Theory

2.2.1 Non-abelian gauge fields near an extreme horizon

Let $(M, g_{\mu\nu})$ be a four-dimensional spacetime satisfying the Einstein-Yang-Mills equations with a cosmological constant Λ. We will assume the gauge group is a compact Lie group whose Lie algebra $\mathfrak{g}$ is semisimple. Hence $\mathfrak{g}$ admits a positive definite invariant metric $(\cdot,\cdot)$ which we will denote by $\mathrm{Tr}(AB) \equiv (A, B)$ where $A, B \in \mathfrak{g}$ (i.e. the Killing form).

We denote the $\mathfrak{g}$-valued Yang-Mills gauge field by $\mathcal{A}_\mu$ and the the gauge-covariant derivative of any $\mathfrak{g}$-valued differential form is $\mathcal{D}X = \mathrm{d}X + [\mathcal{A}, X]$. The Yang-Mills field strength is $\mathcal{F} = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}]$ and the Bianchi identity is $\mathcal{D}\mathcal{F} = 0$. Gauge transformations act as $X \mapsto UXU^{-1}$ and $\mathcal{A} \mapsto U\mathcal{A}U^{-1} - \mathrm{d}UU^{-1}$ where U is a group-valued function. The Einstein-Yang-Mills equations are then

$$R_{\mu\nu} = 2\,\mathrm{Tr}\left(\mathcal{F}_\mu{}^\delta \mathcal{F}_{\nu\delta} - \tfrac{1}{4} g_{\mu\nu} \mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}\right) + \Lambda g_{\mu\nu} \tag{2.1}$$

$$\mathcal{D}\star\mathcal{F} = 0 \tag{2.2}$$

where $\star$ denotes the Hodge dual with respect to $g_{\mu\nu}$.

We will consider smooth solutions $(g_{\mu\nu}, \mathcal{A}_\mu)$ which are invariant – up to gauge transformations – under some symmetry group. Let us recall various well known general facts for such solutions [48,73]. Explicitly, if ξ^μ is a Killing vector field of $g_{\mu\nu}$ then $\mathcal{L}_\xi\mathcal{A} = \mathcal{D}\mathcal{V}_\xi$ where $\mathcal{V}_\xi$ is a $\mathfrak{g}$-valued function. This condition is gauge covariant provided gauge transformations act as $\mathcal{V}_\xi \mapsto U\mathcal{V}_\xi U^{-1} - (\mathcal{L}_\xi U)U^{-1}$. It follows that $\mathcal{L}_\xi\mathcal{F} = [\mathcal{F}, \mathcal{V}_\xi]$; hence for non-abelian fields there is no gauge-invariant notion of an invariant field strength. It is convenient to introduce the "electric" 1-form $\mathcal{E} = -i_\xi\mathcal{F}$; it is then easy to show that there exists a $\mathfrak{g}$-valued potential $\mathcal{W} = i_\xi\mathcal{A} - \mathcal{V}_\xi$ such that $\mathcal{E} = \mathcal{D}\mathcal{W}$. Observe that $\mathcal{D}\mathcal{E} = [\mathcal{F}, \mathcal{W}]$. One can also introduce a "magnetic" 1-form $\mathcal{B} = i_\xi\star\mathcal{F}$; for a non-abelian field there is no associated potential, although by contracting the Yang-Mills equation with ξ one can show $\mathcal{D}\mathcal{B} = [\mathcal{W}, \star\mathcal{F}]$.

We are now ready to introduce our setup. Suppose $(M, g_{\mu\nu})$ contains a smooth *degenerate* Killing horizon $\mathcal{N}$ of a complete Killing vector field K^μ, with a compact cross-section H (i.e. a 2-dimensional submanifold of $\mathcal{N}$ intersected by each orbit of K exactly once). Let L^μ be tangent to the null geodesics which are orthogonal to H and satisfy $K \cdot L = 1$. In the neighbourhood of such a horizon one can define Gaussian null coordinates (v, r, x^1, x^2) introduced in chapter 1.6, so that $K = \partial/\partial v$, $L = \partial/\partial r$, where $r = 0$ is the horizon $\mathcal{N}$, and (x^1, x^2) are arbitrary coordinates on H (which we recall corresponds to $r = 0$ and $v = \mathrm{const}$). The metric in these coordinates takes the form [106]

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 2\,\mathrm{d}v\left(\mathrm{d}r + rh_a(r,x)\mathrm{d}x^a + \tfrac{1}{2}r^2F(r,x)\mathrm{d}v\right) + \gamma_{ab}(r,x)\mathrm{d}x^a\mathrm{d}x^b \tag{2.3}$$

where F, h_a, γ_{ab} are all smooth functions. Degeneracy of the horizon corresponds to $g_{vv} = \mathcal{O}(r^2)$.

We will assume that the Killing field K leaves the gauge field invariant up to gauge transformations, i.e. $\mathcal{L}_K\mathcal{A} = \mathcal{D}\mathcal{V}_K$, and denote the associated potential defined above by $\mathcal{W} = i_K\mathcal{A} - \mathcal{V}_K$. Now, for any Killing horizon one must have $R_{\mu\nu}K^\mu K^\nu|_\mathcal{N} = 0$. On the other hand the Einstein equations (2.1) imply $R_{\mu\nu}K^\mu K^\nu|_\mathcal{N} = \mathrm{Tr}(\mathcal{E}_\mu\mathcal{E}^\mu + \mathcal{B}_\mu\mathcal{B}^\mu)|_\mathcal{N}$. It follows that in Gaussian null coordinates $\mathcal{E}_a|_{r=0} = \mathcal{B}_a|_{r=0} = 0$. We will now recast these as equations on H.

Let us denote the restriction of any quantity to H by a "hat". In particular we write the gauge field on H as $\hat{\mathcal{A}} = \hat{\mathcal{A}}_a\mathrm{d}x^a$ and the corresponding Yang-Mills field strength on H is $\hat{\mathcal{F}} \equiv \hat{\mathrm{d}}\hat{\mathcal{A}} + \frac{1}{2}[\hat{\mathcal{A}}, \hat{\mathcal{A}}]$. Also let $\hat{\mathcal{D}} \equiv \hat{\mathrm{d}}\,\cdot + [\hat{\mathcal{A}}, \cdot]$ be the gauge-covariant derivative on H. The condition $\mathcal{E}_a|_{r=0} = 0$ then implies the equation on H

$$\hat{\mathcal{D}}_a\hat{\mathcal{W}} = 0\ . \tag{2.4}$$

We deduce that $\hat{\mathcal{D}}^2\hat{\mathcal{W}} = [\hat{\mathcal{F}}, \hat{\mathcal{W}}] = 0$. On the other hand, by contracting $\mathcal{D}\mathcal{B} = [\mathcal{W}, \star\mathcal{F}]$ with vector fields tangent to H, the condition $\mathcal{B}_a|_{r=0} = 0$ implies that $[\hat{\mathcal{W}}, \hat{\mathcal{F}}_{vr}] = 0$ on H.

By a gauge transformation we may set $\mathcal{V}_K = 0$, so that $\mathcal{L}_K\mathcal{A} = 0$; it follows that $\mathcal{L}_K\mathcal{F} = 0$ and the electric potential $\mathcal{W} = i_K\mathcal{A}$. In Gaussian null coordinates the components of the gauge field and field strength are now both v-independent: $\partial_v\mathcal{A}_\mu = \partial_v\mathcal{F}_{\mu\nu} = 0$. There is a residual gauge freedom which includes any gauge transformation satisfying $\partial_v U = 0$: using this we can further fix the gauge $\mathcal{A}_r = 0$. In this gauge, the most general gauge field is thus

$$\mathcal{A} = \mathcal{W}(r,x)\mathrm{d}v + \mathcal{A}_a(r,x)\mathrm{d}x^a\ . \tag{2.5}$$

Residual gauge transformations now satisfy $\partial_v U = \partial_r U = 0$.

The remaining gauge field data on H is therefore explicitly given by $\hat{\mathcal{A}}_a = \mathcal{A}_a|_{r=0}$ and $\hat{\mathcal{W}} = \mathcal{W}|_{r=0}$. For convenience we also define $\hat{E} \equiv \partial_r\mathcal{W}|_{r=0}$ and $\hat{G} \equiv \hat{\star}_2\hat{\mathcal{F}}$, which are $\mathfrak{g}$-valued functions on H. From above it follows that

$$[\hat{\mathcal{W}}, \hat{E}] = 0 \qquad [\hat{\mathcal{W}}, \hat{G}] = 0\ , \tag{2.6}$$

where we have used $\mathcal{F}_{vr} = -\partial_r \mathcal{W}$.

We wish to investigate the constraints imposed by the Einstein-Yang-Mills equations on the horizon geometry. Usually, a convenient way to do this is to consider the field equations for the *near-horizon geometry*. This is defined by taking the near-horizon limit, which consists of first performing the diffeomorphism $(v, r) \to (v/\epsilon, \epsilon r)$ and then taking the limit $\epsilon \to 0$ [93]. Due to the degeneracy of the horizon this limit always exists for the metric. As we will show below, in our gauge, the limit also always exists for the Yang-Mills field strength $\mathcal{F}$. However, if $\hat{\mathcal{W}} \neq 0$ this limit does not exist for the gauge field $\mathcal{A}$. Since $\mathcal{A}$ appears explicitly in the Yang-Mills equation (i.e not just through $\mathcal{F}$), it is therefore not clear if one can take the near-horizon limit of this equation (for the Einstein equation this is not an issue). Therefore we will not immediately take the near-horizon limit, but instead expand the Einstein-Yang-Mills equations for the full spacetime fields for small values of the affine parameter r, i.e. near the horizon $\mathcal{N}$.

First we consider the Einstein equation near $\mathcal{N}$. Restricted to $r = 0$ this implies the following set of geometrical equations on H (see (1.22) and (1.23) in chapter 1.6):

$$\hat{R}_{ab} = \tfrac{1}{2}\hat{h}_a\hat{h}_b - \hat{\nabla}_{(a}\hat{h}_{b)} + \Lambda\hat{\gamma}_{ab} + \mathrm{Tr}\left(\hat{E}^2 + \hat{G}^2\right)\hat{\gamma}_{ab} \tag{2.7}$$

$$\hat{F} = \tfrac{1}{2}\hat{h}_a\hat{h}^a - \tfrac{1}{2}\hat{\nabla}_a\hat{h}^a + \Lambda - \mathrm{Tr}\left(\hat{E}^2 + \hat{G}^2\right) \tag{2.8}$$

where $\hat{F} \equiv F|_{r=0}, \hat{h}_a \equiv h_a|_{r=0}$ are the components of a smooth function and 1-form on H, whereas $\hat{\gamma}_{ab} \equiv \gamma_{ab}|_{r=0}$ is Riemannian metric on H with Ricci curvature $\hat{R}_{ab}$ and metric connection $\hat{\nabla}_a$. Observe, that as is typical of degenerate horizons, the equations (2.7, 2.8) only contain quantities which are *intrinsic* to the horizon.

We now consider the Yang-Mills equation (2.2) near $\mathcal{N}$. The Yang-Mills equation is a 3-form and there are three independent components vra, vab and rab, by restricting to $r = 0$ they become equations on H. It turns out the vab component is trivial at $r = 0$, and the rab equation is

$$\hat{\epsilon}_{ab}\hat{\mathcal{W}}_{,rr} + \hat{\epsilon}^c{}_b(\hat{\mathcal{D}}_a\hat{\mathcal{A}}_{c,r} - \hat{h}_a\hat{\mathcal{A}}_{c,r}) = 0 \tag{2.9}$$

where $\hat{\epsilon}_{ab}$ is the volume form of $\hat{\gamma}_{ab}$, $\hat{\mathcal{W}}_{,rr} \equiv (\partial_r^2\mathcal{W})_{r=0}$ and $\hat{\mathcal{A}}_{a,r} \equiv (\partial_r\mathcal{A}_a)_{r=0}$. Thus the rab equation only determines the higher order quantity $\hat{\mathcal{W}}_{,rr}$ in terms of $\hat{\mathcal{A}}_{a,r}$, which is not relevant here. The horizon Yang-Mills equation is given by the vra component

$$\hat{\mathcal{D}}_a\hat{G} - \hat{h}_a\hat{G} - \hat{\epsilon}_a{}^b(\hat{\mathcal{D}}_b\hat{E} - \hat{h}_b\hat{E}) + 2[\hat{\mathcal{W}}, \hat{\epsilon}_a{}^b\hat{\mathcal{A}}_{b,r}] = 0\ , \tag{2.10}$$

it is easy to check that by making the gauge group abelian it reduces to the previously obtained horizon Maxwell equation, see e.g. [85]. However, (2.10) is not just a gauge-covariant version of the Maxwell equation [85]; it includes a new type of term $[\hat{\mathcal{W}}, \hat{\mathcal{A}}_{,r}]$ which encodes information *extrinsic* to the horizon. Hence, if $\hat{\mathcal{W}} \neq 0$, a priori it is unclear if the Yang-Mills field on the horizon is constrained as in the abelian case. If fact, for the class of Lie algebras we are considering we may argue this extra term always vanishes.

If $\hat{\mathcal{W}} \neq 0$ at some point on H, then (2.4) implies there is a gauge such that $\partial_a \hat{\mathcal{W}} = 0$ and $[\hat{\mathcal{A}}_a, \hat{\mathcal{W}}] = 0$. Thus $w \equiv \hat{\mathcal{W}}$ is a fixed element of $\mathfrak{g}$ and by conjugation (i.e constant gauge transformation), we may always assume that $w \in \mathfrak{h}$ where $\mathfrak{h}$ is a Cartan subalgebra. From (2.6) we deduce that the fields $\hat{\mathcal{A}}_a, \hat{G}, \hat{E}$ are all in the centralizer of w which we denote by Z_w. The horizon Yang-Mills equation (2.10) now implies that $[\hat{\mathcal{W}}, \hat{\mathcal{A}}_{a,r}] \in Z_w$. Semi-simplicity of ad_w then implies that $\hat{\mathcal{A}}_{a,r} \in Z_w$ and hence $[\hat{\mathcal{W}}, \hat{\mathcal{A}}_{a,r}] = 0$ after all. Therefore, even if $\hat{\mathcal{W}} \neq 0$, the horizon Yang-Mills equation now simplifies to

$$\hat{\mathcal{D}}\hat{G} - \hat{h}\hat{G} = \hat{\star}_2(\hat{\mathcal{D}}\hat{E} - \hat{h}\hat{E}) \ . \tag{2.11}$$

Note that if w is a regular element of $\mathfrak{h}$, then it lies *inside* one of the Weyl chambers that partition $\mathfrak{h}$ and so $Z_w = \mathfrak{h}$, hence the Yang-Mills field is equivalent to $\mathrm{rank}(\mathfrak{g})$ Maxwell fields. However w could also lie *on a boundary* between two Weyl chambers, in which case it belongs to non-abelian centralizers. We shall focus on the latter.

As in the Einstein-Maxwell case these horizon equations can now be thought of as the full Einstein-Yang-Mills equations for the near-horizon geometry defined above. The limit of the metric is:

$$g_{NH} = 2\,\mathrm{d}v \left(\mathrm{d}r + r\hat{h}_a(x)\mathrm{d}x^a + \tfrac{1}{2}r^2\hat{F}(x)\mathrm{d}v\right) + \hat{\gamma}_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b \ . \tag{2.12}$$

The field strength also always[4] admits a near-horizon limit due to (2.4):

$$\mathcal{F}_{NH} \quad = \quad \hat{E}(x)\mathrm{d}r \wedge \mathrm{d}v - r\hat{\mathcal{D}}_a\hat{E}\,\mathrm{d}v \wedge \mathrm{d}x^a + \tfrac{1}{2}\hat{G}(x)\,\hat{\epsilon}_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b \ . \tag{2.13}$$

However, as mentioned above, the gauge field only admits a near-horizon limit if $\hat{\mathcal{W}} \equiv 0$; in this case it given by

$$\mathcal{A}_{NH} = \hat{E}(x)r\mathrm{d}v + \hat{\mathcal{A}}_a(x)\mathrm{d}x^a. \tag{2.14}$$

Then, the near-horizon metric and near-horizon gauge field $(g_{NH}, \mathcal{A}_{NH})$ must satisfy the Einstein-Yang-Mills equations (2.1), (2.2). Indeed one can check directly that the Einstein-Yang-Mills equations for $(g_{NH}, \mathcal{A}_{NH})$ are equivalent to (2.7) and (2.8) and (2.11). Furthermore, since we have argued that $\hat{\mathcal{W}}$ does not actually appear in the horizon equations even when $\hat{\mathcal{W}} \neq 0$, if we simply *define* the near-horizon gauge field by (2.14) we can still think of the near-horizon geometry as a solution to the Einstein-

[4]Recall we are working in a gauge where $\mathcal{F}$ is independent of v. If one does not pick this gauge the near-horizon limit of $\mathcal{F}$ cannot be even defined.

Yang-Mills equations.

To summarise, the Einstein-Yang-Mills equations for a near-horizon geometry are equivalent to the set of equations (2.7), (2.8) and (2.11) for the near-horizon data $(\hat{\gamma}_{ab}, \hat{h}_a, \hat{F}, \hat{E}, \hat{G})$ which are all purely defined on H. These equations have inherited the original gauge invariance restricted to H; this acts as $(\hat{E}, \hat{G}) \mapsto \hat{U}(\hat{E}, \hat{G})\hat{U}^{-1}$ and $\hat{\mathcal{A}} \mapsto \hat{U}\hat{\mathcal{A}}\hat{U}^{-1} - \mathrm{d}\hat{U}\hat{U}^{-1}$, where $\hat{U}$ is a group-valued function on H. We will consider the classification of solutions to this system of equations on a compact manifold H. We will focus on $\Lambda \leq 0$, although all of our local results remain valid for $\Lambda > 0$.

Before moving on, we note that the contracted Bianchi identity for the horizon metric $\hat{\gamma}_{ab}$ can be written in the useful form

$$\hat{\nabla}_a \hat{F} = \hat{F}\hat{h}_a + 2\hat{h}^b \hat{\nabla}_{[a}\hat{h}_{b]} - \hat{\nabla}^b \hat{\nabla}_{[a}\hat{h}_{b]} - 2\mathrm{Tr}\left[(\hat{G}\hat{\epsilon}_{ab} + \hat{E}\hat{\gamma}_{ab})(\hat{\mathcal{D}}^b \hat{E} - \hat{h}^b \hat{E})\right] \quad , \qquad (2.15)$$

where we have used (2.7), (2.8) and (2.11). Henceforth, we will deal with quantities purely defined on H and will drop the "hats".

2.2.2 Static near-horizon geometries

Any static black hole must have a static near-horizon geometry[5], that is $K \wedge \mathrm{d}K = 0$ everywhere (not just on H). This is equivalent to $\mathrm{d}h = 0$ and $\mathrm{d}F = hF$ on H. These conditions are solved by $h = \mathrm{d}\lambda$ and $F = F_0 e^{\lambda}$ for some constant F_0 and are sufficient to show that the near-horizon geometry is locally warped product of AdS_2 and H [93]. To solve explicitly for the geometry, we may use the same method as in the Einstein-Maxwell case [85]. We note that the results of this section generalise those found in [12, 64].

Staticity implies Ricci staticity $K \wedge R(K) = 0$, where $R(K)_\mu = R_{\mu\nu}K^\nu$. Using Einstein's equation for a near-horizon geometry Ricci staticity implies that $\mathcal{D}_a E = h_a E$ on H. Hence $\mathcal{D}(e^{-\lambda}E) = 0$ and thus $q^2 \equiv e^{-2\lambda}\mathrm{Tr}\, E^2$ is a constant on H. The horizon Yang-Mills equation (2.11) reduces to $\mathcal{D}_a G = h_a G$ and hence $\mathcal{D}(e^{-\lambda}G) = 0$, so we learn that $p^2 \equiv e^{-2\lambda}\mathrm{Tr}\, G^2$ is also a constant on H.

So far the analysis has been local; it applies to any coordinate patch U_i such that $h = \mathrm{d}\lambda_i$. Since $\mathrm{Tr}\, E^2$ and $\mathrm{Tr}\, G^2$ are invariants of the solution from the above we deduce that on any overlap $U_i \cap U_j$ we must have $q_i^2 e^{2\lambda_i} = q_j^2 e^{2\lambda_j}$ and $p_i^2 e^{2\lambda_i} = p_j^2 e^{2\lambda_j}$; for a non-trivial Yang-Mills field we see that either all the q_i are non-zero or all the p_i are non-zero. Since the λ_i in each U_i are defined only up to an additive constant we may always arrange $q_i = q_j$ or $p_i = p_j$ and hence $\lambda_i = \lambda_j$. Therefore there exists a globally defined function λ such that $h = \mathrm{d}\lambda$ irrespective of the topology of H (of

[5] The converse is not always true.

course if $H = S^2$ this is automatic).

Observe that the source terms in the horizon Einstein equations $\text{Tr}(E^2 + G^2) = (q^2+p^2)e^{2\lambda}$ are of the same form as in the Einstein-Maxwell case. If λ non-constant, one can use the same method as in [85] to explictly solve for the horizon metric γ_{ab} and show that it can not be extended smoothly onto a compact H (at least for $\Lambda \leq 0$). Hence compactness requires $h = \mathrm{d}\lambda \equiv 0$, which implies E and G are covariantly constant: thus $\text{Tr}\, E^2, \text{Tr}\, G^2$ must be constants. The horizon equations now reduce to

$$R_{ab} = (\Lambda + \text{Tr}\,(E^2 + G^2))\gamma_{ab}\ , \qquad\qquad F = \Lambda - \text{Tr}\,(E^2 + G^2)\ , \tag{2.16}$$

so that H is a constant curvature space and $F = F_0$ is a constant. The near-horizon geometry is simply the direct product of a 2d Lorenzian maximally symmetric space and a 2d constant curvature space. For $F < 0$, as must be the case if $\Lambda \leq 0$, it is $\text{AdS}_2 \times H$ with $H = S^2, T^2, \Sigma_g$ depending on the sign of curvature, where Σ_g is a Riemann surface of genus g (only S^2 is allowed for $\Lambda = 0$).

If $E \equiv 0$ the problem reduces to solving $\mathcal{D}_a G = 0$ on H, which is equivalent classifying Yang-Mills connections on S^2 and more generally on a Riemann surface of higher genus, a problem which has been solved [3]. In particular, for S^2 the moduli space Yang-Mills connections is in one-to-one correspondence with conjugacy classes of closed geodesics on the gauge group [3, 52]. We deduce that for $SU(2)$ gauge group all solutions on S^2 must be abelian. For more general gauge group it may be interesting to construct explicit non-abelian solutions, although we will not pursue this here.

Finally consider the case where $E \neq 0$ at least at a point. We have that $0 = \mathcal{D}^2 E = [\mathcal{F}, E]$ and hence $[E, G] = 0$. Furthermore, since E is covariantly constant we may choose a gauge such that it is a constant on H; then $[\mathcal{A}_a, E] = 0$. It follows that $\mathcal{A}_a, G \in Z_E$. By a constant conjugation we may assume $E \in \mathfrak{h}$ is in a Cartan subalgebra. If E is a regular element of $\mathfrak{g}$ then all fields are in $\mathfrak{h}$ and hence the system is equivalent to rank$(\mathfrak{g})$ Maxwell fields. For $SU(2)$ gauge group this is the only possibility so in this case there are no non-abelian solutions. For more general gauge group, if E is a singular element then the centralizer Z_E is non-abelian and the problem reduces again to the 2d Yang-Mills equations on a Riemann surface with a gauge group broken to the centralizer of E.

2.2.3 Axisymmetric near-horizon geometries

Motivated by the rigidity theorem for rotating black holes, we will now assume the spacetime and extreme horizon are axisymmetric. That is, we assume there exists a $U(1)$ isometry which commutes with the $\mathbb{R}$ isometry generated by K (and hence leaves the horizon $\mathcal{N}$ invariant). We denote the corresponding Killing field by m and we assume the spacetime gauge field is also invariant up to gauge transformations, so $\mathcal{L}_m \mathcal{A}_\mu = \mathcal{D}_\mu \mathcal{V}_m$ for a group-valued function $\mathcal{V}_m$.

The vector field m must be tangent to H and hence generates a $U(1)$ action on H; it restricts to a Killing field of the metric γ_{ab} on H which also leaves the rest of the near-horizon data F, h_a invariant. The near-horizon gauge field data inherits the following invariance properties $\mathcal{L}_m \mathcal{A}_a = \mathcal{D}_a \mathcal{V}_m$, $\mathcal{L}_m G = [G, \mathcal{V}_m]$, $\mathcal{L}_m E = [E, \mathcal{V}_m]$, where $\mathcal{V}_m$ is now a function on H.

The existence of a $U(1)$-action on H constraints its topology: if the action is free it must be T^2, otherwise it must be S^2 in which case there are exactly two fixed points (the poles). We will first consider the S^2 case. Now consider the closed 1-form on H defined by $i_m \epsilon$. It follows there exists a function x such that $\mathrm{d}x = i_m \epsilon$. Compactness implies that there exists a global maximum and minimum for x, so $x_1 \leq x \leq x_2$. Since $(\mathrm{d}x)^2 = |m|^2$ we see that x can be used as a coordinate at any point where $m \neq 0$. We deduce that the fixed points of m correspond to the endpoints $x = x_1, x_2$. Therefore we can introduce coordinates (x, ϕ) for $x_1 < x < x_2$ such that $m = \partial/\partial\phi$, in which the near-horizon metric can be parameterised as

$$\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{\mathrm{d}x^2}{B(x)} + B(x)\mathrm{d}\phi^2, \qquad h_a\mathrm{d}x^a = \Gamma(x)^{-1}(Bk(x)\mathrm{d}\phi - \Gamma'(x)\mathrm{d}x)\ , \tag{2.17}$$

where $B(x) > 0$ and $B(x_1) = B(x_2) = 0$ and $\Gamma(x) > 0$ everywhere. Smoothness requires the absence of conical singularities at the end points $x = x_1, x_2$: this is equivalent to $B'(x_1) = -B'(x_2) = 2$ and $\phi \sim \phi + 2\pi$.

Now consider the gauge field. We may choose a gauge such that $\mathcal{V}_m = 0$ and hence $\partial_\phi \mathcal{A}_a = \partial_\phi E = \partial_\phi G = 0$, i.e. $\mathcal{A}_a, E, G$ are only functions of x. Furthermore by a residual axisymmetric gauge transformation we can also set $\mathcal{A}_x = 0$. In this gauge the horizon gauge field is simply

$$\mathcal{A}_a\mathrm{d}x^a = a(x)\mathrm{d}\phi \tag{2.18}$$

where $a \equiv i_m \mathcal{A}$ is a $\mathfrak{g}$-valued function on H. It follows that

$$G(x) = a'(x)\ . \tag{2.19}$$

The horizon Yang-Mills equations (2.11) now reduce to the coupled ODE system

$$B(\Gamma G)' + BkE = \Gamma[a, E] \tag{2.20}$$

$$B(\Gamma E)' - BkG = -\Gamma[a, G]\ . \tag{2.21}$$

Now consider the $x\phi$ component of (2.7). The terms from the Yang-Mills fields do not contribute and one finds as in the vacuum case $k' = 0$, so k must be a constant. If

$k = 0$ the near-horizon geometry is in fact static; as shown in the previous section static near-horizon geometries can be treated more generally without the assumption of axisymmetry.

The x component of (2.15) can be simplified using $k' = 0$ and (2.21), resulting in the expression

$$BA' = 4\Gamma\,\mathrm{Tr}(E[a, G])\ , \tag{2.22}$$

where we have defined the function

$$A \equiv \Gamma F - k^2\Gamma^{-1}B\ . \tag{2.23}$$

The significance of this quantity is revealed by changing $r \to \Gamma(x)r$ in the full near horizon geometry, which results in

$$g_{NH} = \Gamma(x)[Ar^2\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r] + \frac{\mathrm{d}x^2}{B(x)} + B(x)(\mathrm{d}\phi + kr\mathrm{d}v)^2\ . \tag{2.24}$$

In an abelian theory, such as Einstein-Maxwell theory, the righthand side of (2.22) must vanish. In that case A is a constant which can be shown to be negative for $\Lambda \leq 0$; then the metric in the square brackets is AdS_2 and the near-horizon geometry inherits all its isometries (since k is constant). The non-abelian structure of Einstein-Yang-Mills theory thus appears to obstruct this symmetry enhancement phenomena.

Let us now study the obstruction term

$$T \equiv \mathrm{Tr}(\Gamma E[a, \Gamma G]) \tag{2.25}$$

where the extra factors of Γ appear for convenience. Let us also define

$$S \equiv \Gamma^2\mathrm{Tr}(E^2 + G^2)\ . \tag{2.26}$$

These quantities can be constrained using the Yang-Mills equations. Indeed equations (2.20), (2.21) allow one to establish the crucial identities

$$BS' = -4T \tag{2.27}$$
$$BT' = -\Gamma^2\mathrm{Tr}([a, G]^2 + [a, E]^2)\ . \tag{2.28}$$

We may use these identities together with a global argument on H as follows.

First note that the vector field $X \equiv B\partial/\partial x$ is globally defined on S^2 and vanishes at $x = x_1, x_2$. Hence given any smooth function f on H, the function $X(f)$ must also be smooth everywhere on H and also vanishes at $x = x_1, x_2$. It is clear that S is invariantly defined on H; hence (2.27) implies T is smooth on H and vanishes at the endpoints

$$T(x_1) = T(x_2) = 0. \tag{2.29}$$

It then follows from (2.28) that $X(T) \leq 0$ and $X(T)|_{x=x_1,x_2} = 0$. Assume there is a single point in the open interval $x_1 < x < x_2$ such that $X(T) < 0$. At this point $T' < 0$ and therefore

$$T(x_2) - T(x_1) = \int_{x_1}^{x_2} \mathrm{d}x\, T' < 0\,. \tag{2.30}$$

This clearly contradicts (2.29) and therefore we deduce that $T \equiv 0$ for all $x_1 < x < x_2$. Hence we have shown that the obstruction term in (2.22) vanishes and deduce that

$$A(x) = A_0 \tag{2.31}$$

where A_0 is a constant. The sign of A_0 can be determined using (2.8) which gives

$$A_0 = \frac{1}{2}\nabla^2\Gamma - \frac{k^2 B}{2\Gamma} + \Gamma\Lambda - \frac{S}{\Gamma}\,. \tag{2.32}$$

By integrating this equation over H we deduce that for $\Lambda \leq 0$ a non-trivial solution (i.e. either $k \neq 0$ or $S \neq 0$) must have $A_0 < 0$. By the above remarks this shows the near-horizon geometry possesses the AdS_2 symmetry enhancement as in the abelian theory.

Observe that from (2.27) and (2.28) the condition $T = 0$ allows us to deduce that $S = S_0$ is a constant and $[a, G] = [a, E] \equiv 0$. Commuting (2.21) with a then shows that $[a, E'] = 0$, which can be used to deduce $[G, E] = 0$. Then commuting (2.21) with E and G, shows $[E, E'] = 0$ and $[G, E'] = 0$ respectively. This shows that all the components of near-horizon gauge field and field strength (2.13) commute.

The classification problem now essentially reduces to that in Einstein-Maxwell theory which has been previously solved, so we will be brief (although the present argument is more efficient than in [85]). Take the xx component of (2.7) and subtract B^{-2} times the $\phi\phi$ component of (2.7) to get

$$\Gamma'' - \frac{\Gamma'^2}{2\Gamma} - \frac{k^2}{2\Gamma} = 0\,. \tag{2.33}$$

If Γ is a constant then $k = 0$ and hence we recover the static case $h_a \equiv 0$. If Γ is non-constant then this equation for Γ can be used to rewrite (2.32) as

$$\left(\frac{B\Gamma}{\Gamma'}\right)' = \frac{2(A_0\Gamma - \Lambda\Gamma^2 + S_0)}{\Gamma'^2}\,. \tag{2.34}$$

The general solution to (2.33) is

$$\Gamma = \frac{k^2}{\beta} + \frac{\beta x^2}{4} \tag{2.35}$$

where $\beta > 0$ is an integration constant and we have used the shift freedom in the definition of x to fix the other constant. This can then used to integrate for B using (2.34)

$$B(x) = \frac{P(x)}{\Gamma} \tag{2.36}$$

where P is a polynomial given by

$$P(x) = -\frac{\beta \Lambda x^4}{12} + (A_0 - 2\Lambda k^2 \beta^{-1})x^2 + c_1 x - \frac{4k^2}{\beta^2}(A_0 - \Lambda k^2 \beta^{-1}) - \frac{4S_0}{\beta} \tag{2.37}$$

and c_1 is an integration constant. We have thus completely solved for the metric and a global analysis reveals that the horizon metric extends to smoothly onto S^2 if and only if $c_1 = 0$ (at least for $\Lambda \leq 0$).

We now turn to the Yang-Mills equations (2.20) and (2.21) which reduce to:

$$(\Gamma G)' + kE = 0 \tag{2.38}$$
$$(\Gamma E)' - kG = 0\,. \tag{2.39}$$

By expanding in any Lie algebra basis the components of (E, G) each satisfy the same equations as in the Maxwell case. Assuming $k \neq 0$ one finds

$$E = \frac{xq - \left(\frac{k^2}{\beta} - \frac{\beta x^2}{4}\right)p}{\Gamma^2}, \qquad a = a_0 + \frac{xq - \left(\frac{k^2}{\beta} - \frac{\beta x^2}{4}\right)p}{k\Gamma} \tag{2.40}$$

where a_0, q, p are fixed elements in $\mathfrak{g}$ and recall $G = a'$. Since these fields and their first derivatives must commute for all x, we deduce that a_0, q, p all commute.

The above shows that the most general axisymmetric near-horizon geometry and gauge field with $H = S^2$ is isometric to that of the abelian embedded extreme Kerr-Newman black with a cosmological constant (see [85] to deduce the explicit coordinate and parameter change).

We close by considering toroidal horizon topology $H = T^2$. In this case one can again introduce coordinates (x, ϕ), this time both periodic, such that the horizon metric take the same form with $B(x) > 0$ everywhere, but instead h may have an extra term of the form $cB(x)^{-1}\mathrm{d}x$ for some constant c, see e.g. [75]. The $x\phi$ component of (2.7) now implies $k' = c\,k/B$ which integrates to $k(x) = k_0 \exp(c \int_{x_0}^{x} B(x)^{-1}\mathrm{d}x)$. Since the integrand in the exponent is positive this means that $k(x)$ is a monotonic function which is in contradiction to the fact that k must be a periodic function of x. Hence for $k \neq 0$ we must have $c = 0$ after all. Therefore the horizon equations in the toroidal case are identical to the S^2 case. From above we saw that if $k \neq 0$ then $\Gamma(x)$ is given by (2.35); but since Γ is a globally defined function on H it must be periodic in x and hence we have a contradiction. This shows there are no axisymmetric near-horizon geometries which are non-static (i.e. $k \neq 0$) and have $H = T^2$. Note that this proof is equally valid in pure Einstein-Maxwell theory or even pure gravity; for $\Lambda < 0$ this fact does not seem to have been shown before (for $\Lambda \geq 0$ it is easily follows by integrating the trace of the general horizon equation (2.7)).

The above results thus completely classify all axisymmetric near-horizon geometries with a compact horizon section in Einstein-Yang-Mills theory with a cosmological constant. We conclude that the near-horizon uniqueness present in Einstein-Maxwell theory persists in the non-abelian Einstein-Yang-Mills theory.

2.3 A no-hair theorem for near-horizon geometry in Einstein-Maxwell-charged scalar theory

We now move onto the near-horizon geometry in $4D$ Einstein-Maxwell-charged scalar theory. We begin with the Lagrangian

$$\mathcal{L} = \sqrt{-g}\left[\tfrac{1}{2}\left(R - 2\Lambda\right) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - g^{\mu\nu}(\nabla_\mu - ieA_\mu)\phi^*(\nabla_\nu + ieA_\nu)\phi - V(|\phi|^2)\right] \tag{2.41}$$

where ϕ is a complex scalar, $V(|\phi|^2)$ is some arbitrary potential, A_μ is the Maxwell gauge field, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\mu A_\nu$ is the Maxwell field strength and e is the coupling constant. We can rewrite the field contents to extract the physical degrees of freedom in the broken symmetry phase by setting

$$\phi = X(x^\mu)\exp\left[i\,\xi(x^\mu)\right] \tag{2.42}$$

$$A_\mu = \tilde{A}_\mu - \nabla_\mu \xi \tag{2.43}$$

where $X(x^\mu) = |\phi| \geq 0$ and $\xi(x^\mu)$ are both real scalar field. Because A_μ and $\tilde{A}_\mu$ differ by a gauge transformation, the field strength remains unchanged i.e. $F_{\mu\nu} = \tilde{F}_{\mu\nu}$. This results in

$$\mathcal{L} = \sqrt{-g}\left[\tfrac{1}{2}\left(R - 2\Lambda\right) - \tfrac{1}{4}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} - g^{\mu\nu}(\nabla_\mu X \nabla_\nu X + e^2 X^2 \tilde{A}_\mu \tilde{A}_\nu) - V(X)\right] \tag{2.44}$$

The vector field becomes massive by absorbing the Goldstone mode ξ via the Higgs mechanism. We will work with this Lagrangian and drop the tildes from now on. The energy-momentum tensor for the matter fields is given by

$$\begin{aligned} T_{\mu\nu} &= F_\mu{}^\rho F_{\nu\rho} + 2(\nabla_\mu X \nabla_\nu X + e^2 X^2 A_\mu A_\nu) \\ &\quad + g_{\mu\nu}\left[-\tfrac{1}{4}F_{\rho\sigma}F^{\rho\sigma} - \nabla_\rho X \nabla^\rho X + e^2 X^2 A_\rho A^\rho - V(X)\right] , \end{aligned} \tag{2.45}$$

and the equations of motion are

$$0 = \nabla_\mu F^{\mu\nu} - 2e^2 X^2 A^\nu \tag{2.46}$$

$$0 = \nabla^\mu \nabla_\mu X - e^2 X A_\mu A^\mu - \tfrac{1}{2}V'(X) \tag{2.47}$$

$$\begin{aligned} R_{\mu\nu} &= F_\mu{}^\rho F_{\nu\rho} - \tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + 2(\nabla_\mu X \nabla_\nu X + e^2 X^2 A_\mu A_\nu) \\ &\quad + g_{\mu\nu}V(X) + g_{\mu\nu}\Lambda . \end{aligned} \tag{2.48}$$

Our goal is to find the most general *near-horizon geometry* solution $(g_{\mu\nu}, A, X)$ to these equations of motion.

We shall work in Gaussian null coordinates as in the previous section. First of all since the vector A appears explicitly in the equations of motion, it is a physical quantity so it should admit a well-defined near horizon limit. This can be verified by considering the Raychaudhuri's equation $R_{\mu\nu}K^\mu K^\nu|_{\mathcal{N}} = 0$. According to the Einstein equation (2.48), this is equivalent to

$$\left(F_{va}F_{vb}\gamma^{ab} + 2e^2 X^2 A_v^2\right)\Big|_{r=0} = 0 . \tag{2.49}$$

Clearly each of the two terms must vanish individually. Since we are interested in cases where neither e nor $\hat{X}(x) = X(r,x)|_{r=0}$ is identically zero (otherwise we just go back to Einstein-Maxwell with an uncharged scalar or Einstein-Maxwell), assuming that $A_\mu(r,x)$ are regular functions, we must have $A_v = r\Delta(x) + \mathcal{O}(r^2)$. Thus in the near-horizon limit

$$\hat{A} = A|_{r=0} = r\Delta(x)\mathrm{d}v + \hat{\mathcal{A}}_a(x)\mathrm{d}x^a \tag{2.50}$$

$$\hat{F} = F|_{r=0} = -\Delta(x)\mathrm{d}v \wedge \mathrm{d}r + r\hat{\nabla}_a\Delta(x)\mathrm{d}x^a \wedge \mathrm{d}v + \tfrac{1}{2}\epsilon_{ab}\hat{\mathcal{F}}(x)\mathrm{d}x^a \wedge \mathrm{d}x^b \tag{2.51}$$

where $\hat{\nabla}_a$ the is Levi-Civita connection of the metric $\gamma_{ab}(x)$ on H and $\frac{1}{2}\epsilon_{ab}\hat{\mathcal{F}} = \hat{\nabla}_{[a}\hat{\mathcal{A}}_{b]}$.

We are now ready to write down the near-horizon equations. Since all equations are defined on H we shall drop the "hats". The scalar and massive Maxwell field equations (2.46) and (2.47) then give

$$0 = \nabla^a \nabla_a X - h^a \nabla_a X - e^2 X \mathcal{A}^a \mathcal{A}_a - \tfrac{1}{2}V' \tag{2.52}$$

$$0 = \epsilon_{ab}(h^b \mathcal{F} - \nabla^b \mathcal{F}) + (h_a \Delta - \nabla_a \Delta) - 2e^2 X^2 \mathcal{A}_a , \tag{2.53}$$

and the Einstein equation (2.48) gives, in orthonormal basis,

$$\begin{aligned}
&\hat{R}_{ab} + \nabla_{(a}h_{b)} - \tfrac{1}{2}h_a h_b = \\
&\qquad \gamma_{ab}\left[\tfrac{1}{2}(\mathcal{F}^2 + \Delta^2) + V + \Lambda\right] + 2(\nabla_a X \nabla_b X + e^2 X^2 \mathcal{A}_a \mathcal{A}_b) &(2.54)\\
&F - \tfrac{1}{2}h_a h^a + \tfrac{1}{2}\nabla_a h^a = -\tfrac{1}{2}(\mathcal{F}^2 + \Delta^2) + V + \Lambda &(2.55)\\
&R_{++} = r^2\left[\gamma^{ab}(\nabla_a\Delta - h_a\Delta)(\nabla_b\Delta - h_b\Delta) + 2e^2X^2\Delta^2\right] &(2.56)\\
&R_{+a} = r\left[\mathcal{F}\varepsilon_{ab}(h^b\Delta - \nabla^b\Delta) + \Delta(h_a\Delta - \nabla_a\Delta) + 2e^2X^2\Delta\mathcal{A}_a\right] . &(2.57)
\end{aligned}$$

(2.54) is the ab component of (2.48), we have reinstated the "hat" on $\hat{R}_{ab}$ to emphasise that it is the Ricci tensor of the horizon metric $\gamma_{ab}(x)$. (2.55) is the $+-$ equation and $F = F(x)$ here is the near-horizon datum which is a 0-form, not to be confused with the Maxwell field strength in (2.41) which is a 2-form. (2.56) and (2.57) are clearly the $++$ and $+a$ components. We have skipped writing down explicitly the full expressions for R_{++} and R_{+a}, they can be found in (1.18).

Recall that $X(x) \geq 0$, so if the potential $V(X)$ satisfies $V' \geq 0$ e.g. a mass term $V(X) = \frac{1}{2}m^2X^2$, then (2.52) becomes

$$\nabla^2 X - h\cdot\nabla X = e^2 X \mathcal{A}^2 + \frac{1}{2}V' \geq 0 , \qquad (2.58)$$

which according to the maximum principle on a closed manifold forces $X = const.$ This results holds no matter what value $\mathcal{A}$ takes, including zero, but let us assume $\mathcal{A}$ is not everywhere vanishing for now so that the solution would not be purely electric. If $X = 0$ we just return to Einstein-Maxwell, so let us focus on $X \neq 0$. Feeding this result back into (2.52) then implies $V' = 0$ therefore V is constant, and $\mathcal{A}_a = 0$ so (2.53) reduces to $h_a\Delta - \nabla_a\Delta = 0$. Plugging these into (2.57) gives $R_{+a} = 0$, (1.19) then states that $R_{++} = 0$ too so the right hand side of (2.56) must vanish. This is only possible if $\Delta = 0$ and the system reduces to Einstein-Λ', where the new cosmological constant $\Lambda' = V + \Lambda$. Hence we are led to the following proposition:

Proposition 2.1. *Consider the near-horizon geometry with compact cross sections in Einstein gravity coupled to a Maxwell field A and a massive complex scalar ϕ, allowing for a cosmological constant. Assuming non-zero minimal coupling between A and ϕ, there exists no solution where ϕ is non-trivial in any dimensions. This result also holds for any scalar potential $V(|\phi|)$ which is a monotonically increasing function of $|\phi|$.*

2.4 Summary

We considered near-horizon geometries beyond Einstein-Maxwell theory. We first investigated stationary and axisymmetric solutions in Einstein-Yang-Mills theory with a non-positive cosmological constant in $4D$. We proved that any solution of this kind possesses a near-horizon AdS_2 symmetry and deduced that its near-horizon geometry must be that of the abelian embedded extreme Kerr-Newman(-AdS) black hole. We also showed that the near-horizon geometry of any static black hole is a direct product of AdS2 and a constant curvature space. We then considered the Einstein-Maxwell-charged scalar theory with a cosmological constant in general dimensions. Assuming non-zero coupling between the Maxwell field A and the massive complex scalar ϕ, there exists no near-horizon geometry with a non-trivial ϕ if the horizon is compact. This result generalises to any scalar potential $V(|\phi|)$ which satisfies $V'(|\phi|) \geq 0$.

Chapter 3

Three Dimensional Black Holes and Descendants

3.1 Introduction

A central result in the theory of equilibrium black holes in four and higher dimensions is the rigidity theorem [67, 78, 107]. This states that the event horizon of a stationary, rotating, black hole is a Killing horizon with respect to the Killing field

$$K = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \, , \tag{3.1}$$

where $\partial/\partial t$ is the stationary Killing field, $\partial/\partial\phi$ is a Killing field generating the rotational symmetry, and Ω the angular velocity of the black hole with respect to the static asymptotic frame.[1] For asymptotically flat space times, it is clear that if the angular velocity is non-zero, the Killing field K must become spacelike outside a large enough ball. Therefore, there is no possibility of having matter in equilibrium and co-rotating with the black hole (since it would have to exceed the speed of light).

Black holes in anti de Sitter (AdS) spacetimes are of central importance in the context of the AdS/CFT duality [140]. For such black holes the situation is quite different. In particular, for $D \geq 4$ asymptotically globally AdS black holes, such as the Kerr-AdS black hole and its higher dimensional generalisation, the Killing field K is timelike everywhere outside the horizon if $|\ell\Omega| \leq 1$, where ℓ is the radius of AdS. In this case K defines a frame in which matter can co-rotate in equilibrium with the black hole. This raises the interesting possibility of having black holes which are invariant under a *single* Killing field K. Matter here refers also to gravitons, hence this argument suggests the possibility of new vacuum solutions. Note that such solutions would not violate the rigidity theorem i.e "stationary implies axisymmetry" means there there are at least

[1] In greater than four spacetime dimensions a black hole may have multiple rotational Killing fields with corresponding angular velocities which must be included in (3.1).

two Killing fields, because the single stationary Killing field K would necessarily be *normal* to the horizon. This however does not imply the spacetime is static since to apply the staticity theorem [132], it requires the existence of an asymptotically flat Killing initial data.

Although Kerr-AdS black holes are thought to be stable if $|\ell\Omega| \leq 1$ [70], it has been proposed that new solutions invariant under just the co-rotating Killing field may arise as the endpoint of a superradiant instability which occurs for rapidly rotating Kerr-AdS black holes (i.e. $|\ell\Omega| > 1$) [18, 92]: consider a wave e.g. a scalar field with mode $e^{i(\omega t - m\phi)}$, it can scatter off a fast rotating ($m\Omega > \omega$) black hole, amplify and slow down the black hole via the Penrose process. In an asymptotically AdS spacetime the conformal boundary acts at a mirror[2], so the scalar field is reflected back to the black hole, amplifies again and slows the black hole further, until the equilibrium $m\Omega_E = \omega_E$ is reached. The resulting stationary spacetime is therefore a black hole with a lump of scalar hair co-rotating with it. Neither $\partial/\partial t$ nor $\partial/\partial\phi$ is Killing but the combination K is. This process can be generalised to gravitons, the stationary spacetime is then a vacuum black hole solution with a single Killing field, which has higher entropy than the original black hole.

Finding new vacuum black hole solutions invariant under a single Killing field is a daunting task. Remarkably after this work was published, asymptotically AdS vacuum black hole solutions with a single Killing field were constructed numerically in $4D$ [34]. These solutions prove non-uniqueness of Kerr-AdS. However, although they have higher entropy than Kerr-AdS, they are unstable and therefore cannot be the end point of superradiant instability. One may also avoid the complication of dealing metrics with a single Killing field by coupling matter fields which are invariant under just the co-rotating Killing field. Indeed, examples with a complex scalar field have been found numerically [36] (see also [124, 126]). In this chapter we follow a different strategy by examining this problem in pure gravity in lower dimensions.

Although there are no local degrees of freedom, three-dimensional Einstein gravity with a negative cosmological constant provides a valuable toy model for examining certain higher dimensional questions [33]. Brown and Henneaux demonstrated that there exist boundary conditions such that the asymptotic symmetry algebra is the infinite dimensional conformal symmetry of a cylinder [16]. In particular, they showed that in any asymptotically AdS_3 spacetime there exists a coordinate system (t, r, ϕ) such that

[2]Since AdS space is not globally hyperbolic, a boundary condition needs to be chosen at conformal infinity. The reflecting boundary condition, such that energy and angular momentum are conserved, is usually chosen to ensure well posedness of the initial value problem [50].

at $r \to \infty$, the metric functions take the form

$$\begin{aligned} g_{tt} &= -\frac{r^2}{\ell^2} + \mathcal{O}(1) \qquad & g_{rr} &= \tfrac{\ell^2}{r^2} + \mathcal{O}(r^{-4}) \qquad & g_{tr} &= \mathcal{O}(r^{-3}) \\ g_{t\phi} &= \mathcal{O}(1) \qquad & g_{r\phi} &= \mathcal{O}(r^{-3}) \qquad & g_{\phi\phi} &= r^2 + \mathcal{O}(1) \end{aligned} \tag{3.2}$$

where $\phi \sim \phi + 2\pi$, and the conserved charges associated to the asymptotic symmetry group can be computed from the subleading terms. Furthermore, Banados-Teitelboim-Zanelli (BTZ) found explicit black hole solutions to the $D = 3$ Einstein equations [5,6]. Although locally AdS$_3$, globally this is a family of stationary and axisymmetric black holes which are asymptotically AdS$_3$ with a cylinder conformal boundary and possess mass M and angular momentum J. See appendix 3.A for further discussions on the BTZ solutions.

The BTZ black holes always satisfy $|\ell\Omega| \leq 1$. For the non-extreme black hole ($|\ell\Omega| < 1$) the Killing field K is everywhere timelike outside the horizon, whereas for the extreme black hole ($|\ell\Omega| = 1$) the Killing field K is everywhere null. By the above arguments, this raises the possibility of black hole solutions invariant under a single Killing field. However, the BTZ black hole does not suffer from a superradiant instability (since it never rotates faster than the speed of light, the stability argument used in higher dimensions can be applied [70]). Therefore, such putative solutions may not arise from the evolution of some perturbation of the BTZ black hole. Indeed, stationary and axisymmetric black holes which are coupled to a complex scalar field invariant under a co-rotating Killing field, have been argued not to exist [123, 125].

In this chapter we show that in fact black holes with a single Killing field do *not* exist in three dimensional Einstein gravity, by explicitly determining the most general Einstein metric with a (non-singular) Killing horizon. It turns out the general solution with a spatially compact horizon always possesses a second commuting Killing field and hence must be related to the BTZ black hole (or its near-horizon geometry) by a diffeomorphism. Interestingly, in the case of a degenerate horizon the general solution is related to the extreme BTZ black hole by a *large* diffeomorphism. Our results establish a new type of uniqueness theorem for three-dimensional AdS black holes.[3]

In fact, the general solution may have an interesting interpretation in the dual CFT. One expects that by acting on the BTZ black hole with a general element of the asymptotic-symmetry diffeomorphism group, one would obtain AdS$_3$ solutions with arbitrary Virasoro charges. We refer to these as *descendants* of the BTZ black hole.[4] These new solutions should also have two commuting Killing fields, corresponding to

[3]See e.g. [47, 113] for other types of uniqueness results.
[4]Note that these are not descendants of pure states in the dual CFT.

the push-forward of the Killing fields of the BTZ black hole.[5] If these geometries still contain a Killing horizon, they must be within our general class of Einstein metrics. Indeed, we identify a general class of extreme black holes that are asymptotically AdS_3 with cylinder boundary, which carry arbitrary charges with respect to one of the Virasoro algebras and vanishing charges with respect to the other. Hence these geometries are descendants (in the above sense) of the extreme BTZ black hole.

Before moving on, we mention a technical motivation which led us to investigating this problem in the extreme case. As we mentioned in chapter 1, an important inverse problem is to understand how, given a near-horizon geometry, one determines the possible corresponding extreme black holes. As we will show, three dimensional gravity provides a simple setup which allows one to examine this question explicitly.

3.2 General solution

3.2.1 Derivation

Consider a general $2+1$ dimensional spacetime containing a smooth[6] Killing horizon $\mathcal{N}$ of a future-pointing, complete, Killing field K with a one-dimensional spacelike cross-section H. In the neighbourhood of $\mathcal{N}$ the metric in Gaussian null coordinates reads, see e.g. [88],

$$\mathrm{d}s^2 = 2\mathrm{d}v\left(\mathrm{d}\lambda + \lambda h(\lambda,x)\mathrm{d}x + \tfrac{1}{2}\lambda f(\lambda,x)\mathrm{d}v\right) + \gamma(\lambda,x)^2\mathrm{d}x^2 \ , \tag{3.3}$$

where $K = \partial/\partial v$ is the Killing field which is null on $\mathcal{N}$ and $\partial/\partial\lambda$ is tangent to null geodesics which are transverse to the horizon $\mathcal{N}$ such that $\lambda > 0$ is the exterior region and $\mathcal{N} = \{\lambda = 0\}$. The coordinate (x) is on the one-dimensional spacelike cross-section H, which by assumption has a non-degenerate induced metric so $\gamma > 0$ in the neighbourhood of $\mathcal{N}$.

We wish to find the general vacuum solution of this form with a cosmological constant $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Of course any Einstein metric in three dimensions is *locally* isometric to one of the maximally symmetric spaces: we are concerned with spacetimes with a *global* Killing horizon as above.

To compute the Ricci tensor it is convenient to use the null-orthonormal basis (e^+, e^-, e^x) defined by

$$e^+ = \mathrm{d}v \ , \qquad e^- = \mathrm{d}\lambda + \lambda h\mathrm{d}x + \tfrac{1}{2}\lambda f\mathrm{d}v \ , \qquad e^x = \gamma\mathrm{d}x \ , \tag{3.4}$$

[5]We thank Harvey Reall for this observation.

[6]In fact, rather than smooth, we will only need to assume the functions f, h are C^1 and γ is C^2.

so that the metric reads $\mathrm{d}s^2 = 2e^+e^- + e^xe^x$. It turns out that the function defined by

$$b \equiv \partial_x f + \lambda f \partial_\lambda h - \lambda h \partial_\lambda f \;, \tag{3.5}$$

appears naturally in the curvature calculations. The dual basis vectors are

$$E_+ = \partial_v - \frac{1}{2}\lambda f \partial_\lambda \;, \qquad E_- = \partial_\lambda \;, \qquad E_x = \frac{1}{\gamma}(\partial_x - \lambda h \partial_\lambda) \;. \tag{3.6}$$

We find that with respect to the above basis the connection 1-forms defined by $\mathrm{d}e^A = -\omega^A_{\ B} \wedge e^B$ are given by

$$\begin{aligned} \omega_{+-} &= \frac{1}{2}\partial_\lambda(\lambda f)e^+ + \frac{1}{2\gamma}\partial_\lambda(\lambda h)e^x \\ \omega_{+a} &= \frac{1}{2\gamma}\left(\lambda b e^+ - \partial_\lambda(\lambda h)e^- + \lambda f \partial_\lambda \gamma e^x\right) \\ \omega_{-a} &= -\frac{1}{2\gamma}\left(2\partial_\lambda \gamma e^x + \partial_\lambda(\lambda h)e^+\right) \;, \end{aligned} \tag{3.7}$$

and the Ricci tensor is

$$\begin{aligned} R_{++} &= \frac{1}{2\gamma}\left[\lambda h \partial_\lambda \left(\frac{1}{\gamma}\lambda b\right) - \partial_x\left(\frac{1}{\gamma}\lambda b\right) - \tfrac{1}{2}\lambda^2 f^2 \partial_\lambda^2 \gamma\right] \;, \\ R_{+-} &= \tfrac{1}{2}\partial_\lambda^2(\lambda f) + \frac{1}{2\gamma}\left[\partial_\lambda(\lambda f \partial_\lambda \gamma) - \frac{1}{\gamma}(\partial_\lambda(\lambda h))^2 + \partial_x\left(\frac{1}{\gamma}\partial_\lambda(\lambda h)\right) - \lambda h \partial_\lambda\left(\frac{1}{\gamma}\partial_\lambda(\lambda h)\right)\right] \;, \\ R_{+x} &= \tfrac{1}{2}\partial_\lambda\left(\frac{1}{\gamma}\lambda b\right) - \tfrac{1}{4}\lambda f \partial_\lambda\left(\frac{1}{\gamma}\partial_\lambda(\lambda h)\right) \;, \\ R_{--} &= -\frac{1}{\gamma}\partial_\lambda^2\gamma \;, \qquad R_{-x} = \tfrac{1}{2}\partial_\lambda\left(\frac{1}{\gamma}\partial_\lambda(\lambda h)\right) \;, \\ R_{xx} &= \frac{1}{\gamma}\left[\partial_\lambda(\lambda f \partial_\lambda \gamma) - \frac{1}{2\gamma}(\partial_\lambda(\lambda h))^2 + \partial_x\left(\frac{1}{\gamma}\partial_\lambda(\lambda h)\right) - \lambda h \partial_\lambda\left(\frac{1}{\gamma}\partial_\lambda(\lambda h)\right)\right] \;. \end{aligned}$$

The $--$ component of the Einstein equations immediately implies $\gamma = \gamma_0(x) + \lambda\gamma_1(x)$, where $\gamma_0(x), \gamma_1(x)$ are arbitrary functions. We may use the coordinate freedom on H to set $\gamma_0 = 1$, which we will assume henceforth. The $-x$ component can be easily integrated for h and the most general solution which is regular at $\lambda = 0$ is

$$h = h_0(x)\left(1 + \tfrac{1}{2}\lambda\gamma_1(x)\right) , \tag{3.8}$$

where h_0 is an arbitrary function. Now consider the $+-$ and xx components. This is facilitated by noting that the Einstein equation implies $\frac{1}{2}R_{xx} - R_{+-} = -\frac{1}{2}\Lambda$, which explicitly reads

$$\partial_\lambda^2(\lambda f) - \tfrac{1}{2}\left(\frac{\partial_\lambda(\lambda h)}{\gamma}\right)^2 = \Lambda \;. \tag{3.9}$$

This can now be integrated for f and the most general solution regular at $\lambda = 0$ is

$$f = f_0(x) + \tfrac{1}{2}\left(\Lambda + \tfrac{1}{2}h_0(x)^2\right)\lambda \tag{3.10}$$

where f_0 is an arbitrary function. Now, the xx equation is satisfied iff

$$\partial_x h_0 - \tfrac{1}{2}h_0^2 + f_0\gamma_1 = \Lambda \,. \tag{3.11}$$

It remains to consider the $++$ and $+x$ components. It is easy to see these are satisfied if and only if $\lambda b/\gamma$ is a constant. Hence by regularity at $\lambda = 0$ we deduce $b = 0$. Finally, by substituting into (3.5) one finds $b = \partial_x f_0$ and so we deduce that $f_0(x) = -2\kappa$, where κ is a constant. We have now satisfied all components of the Einstein equation.

To summarise, we have found that the most general solution with a non-singular Killing horizon is given by:

$$\begin{aligned} \gamma(\lambda,x) &= 1+\lambda\gamma_1(x) \\ h(\lambda,x) &= h_0(x)\left(1+\tfrac{1}{2}\lambda\gamma_1(x)\right) \\ f(\lambda,x) &= -2\kappa+\tfrac{1}{2}\lambda\left(\Lambda+\tfrac{1}{2}h_0(x)^2\right) \,, \end{aligned} \tag{3.12}$$

where κ is a constant and h_0, γ_1 are arbitrary functions subject to the constraint

$$\partial_x h_0 - \tfrac{1}{2}h_0^2 - 2\kappa\gamma_1 = \Lambda \,. \tag{3.13}$$

The various quantities which appear in the solution all have a direct geometrical meaning. The 1-form $h_0 \mathrm{d}x$ is the connection of the normal bundle on H, viewed as a submanifold of the spacetime. The function $\gamma_1 = \theta|_{\lambda=0}$ where θ is the expansion of the null geodesic congruence tangent to $\partial/\partial\lambda$, i.e. $\theta = \gamma^{-1}\partial_\lambda\gamma$. The constant κ is the surface gravity on the Killing horizon, i.e. $\mathrm{d}K^2|_{\lambda=0} = -2\kappa K|_{\lambda=0}$.

In the non-degenerate case, $\kappa \neq 0$, the constraint equation (3.13) can be solved to determine the extrinsic data γ_1 in terms of the intrinsic data h_0, so the solution depends on the constant κ and one freely specifiable function on the horizon $h_0(x)$. On the other hand, in the degenerate case, $\kappa = 0$, we deduce that the constraint equation (3.13) reduces to the Einstein equation for the near-horizon geometry [88] and $\gamma_1(x)$ is an *arbitrary* function on H. Hence, once the near-horizon solution has been fixed, the degenerate solution depends only on one freely specifiable function $\gamma_1(x)$. This explicitly shows that decoupling of intrinsic and extrinsic data occur if and only if the horizon is degenerate.

In general, Gaussian null coordinates are only defined in a neighbourhood on the horizon, in particular, as long as the transverse null geodesic congruence $\partial/\partial\lambda$ does not caustic. For our solution, observe that if $\gamma_1(x_0) < 0$ for some x_0 the transverse null geodesics converge initially, i.e. $\theta(\lambda, x_0) < 0$ for small λ, and furthermore $\theta \to -\infty$ as $\lambda \to 1/|\gamma_1(x_0)|$. On the other hand, if $\gamma_1(x) \geq 0$ it is clear the coordinate system can be extended to all positive values of λ.

We emphasise that our general solution is valid for any cosmological constant. Motivated by the discussion in the introduction, in this chapter we will focus on AdS solutions with compact cross-sections of the horizon. Therefore, henceforth we set $\Lambda = -2/\ell^2$ and $H \cong S^1$. We thus identify $x \sim x + 2\pi R$, where $R > 0$ is the radius of the horizon, and assume the functions $h_0(x), \gamma_1(x)$ are $2\pi R$-periodic. Thus, if $\gamma_1(x) > 0$, then $\lambda \to \infty$ is a conformal boundary with boundary metric

$$\lambda^{-2}\mathrm{d}s^2 \to -\frac{\mathrm{d}v^2}{\ell^2} + (\gamma_1 \mathrm{d}x + \tfrac{1}{2}h_0 \mathrm{d}v)^2 \ . \tag{3.14}$$

If h_0 is constant we may define coordinates $t = v$ and $\mathrm{d}\phi = \gamma_1(x)\mathrm{d}x + \frac{1}{2}h_0\mathrm{d}v$ which explicitly show the boundary is a flat cylinder. This will be relevant below.

3.2.2 Extra Killing field

We will now show that under the assumptions $H \cong S^1$ and $\Lambda < 0$, our general solution in fact always possesses a second Killing field which commutes with K and is globally defined (i.e. it is compatible with the periodic identification $x \sim x + 2\pi R$).

A tedious calculation (see appendix 3.B) shows that for the general non-degenerate case $\kappa \neq 0$, the most general Killing field which commutes with K is (a multiple of)

$$X = \left(c + \frac{h_0}{2\kappa}\right)\partial_v + \frac{\lambda^2 h_0 h_0'}{4\kappa\gamma}\partial_\lambda + \left(1 - \frac{\lambda h_0'}{2\kappa\gamma}\right)\partial_x \ , \tag{3.15}$$

where c is a constant and we have used (3.13). Observe that this Killing field is globally defined, tangent to the horizon $\mathcal{N}$ and has closed orbits.

For the general degenerate case $\kappa = 0$, equation (3.13) shows h_0 is determined by the near-horizon equation

$$\partial_x h_0 - \tfrac{1}{2}h_0^2 = -\frac{2}{\ell^2} \ . \tag{3.16}$$

It has been shown that the most general solution on $H \cong S^1$ is $h_0 = 2/\ell$ (choosing a sign), which corresponds to the near-horizon geometry of the extreme BTZ black hole [88]. In this case, it can be shown that (see appendix 3.B) the most general globally defined Killing field which commutes with K is (a multiple of)

$$X = (c + y)\partial_v + \frac{\lambda^2 y'}{\ell\gamma}\partial_\lambda + \left(1 - \frac{\lambda y'}{\gamma}\right)\partial_x \ , \tag{3.17}$$

where c is a constant and $y(x)$ is the unique periodic solution to

$$y' - \frac{2}{\ell}y = \gamma_1(x) \ . \tag{3.18}$$

Again, note that this Killing field has closed orbits and is also tangent to the horizon.

Thus in either case we see that a general spacetime containing a Killing horizon with compact cross-sections always possesses a second Killing field X with closed orbits which commutes with K, i.e. it is *axisymmetric*.[7] Since X is tangent to the horizon, we could always choose a different cross-section $\tilde{H} \cong S^1$ such that X is tangent to $\tilde{H}$ for some constant c. In this case, the solution written in Gaussian null coordinates $(\tilde{v}, \tilde{\lambda}, \tilde{x})$ adapted to this new cross-section $\tilde{H}$, must take our general form (3.12) but with $\tilde{h}_0, \tilde{\gamma}_1$ constant functions. It is then easy to see the solution is given by the BTZ black hole or its near-horizon geometry, as we show next.

Thus suppose that $\partial/\partial x$ is a Killing field with closed orbits so h_0 and γ_1 are constant functions. Using the discrete transformations $x \to -x$ and $(v, \lambda, \kappa) \to -(v, \lambda, \kappa)$ we may always arrange $h_0 \geq 0$ and $\gamma_1 \geq 0$, respectively.

If $\gamma_1 > 0$ define two positive parameters (r_+, r_-) by $\gamma_1 = 1/r_+$ and $h_0 = 2r_-/(\ell r_+)$. Solving the constraint (3.13) implies $\kappa = (r_+^2 - r_-^2)/(\ell^2 r_+)$. Now, performing the coordinate change

$$\begin{aligned} \lambda &= r - r_+ \, , \\ \mathrm{d}v &= \mathrm{d}t + \frac{\mathrm{d}r}{N^2} \\ \mathrm{d}x &= r_+ \mathrm{d}\phi - \frac{r_-}{\ell}\mathrm{d}t + r_+\left(N^\phi - \frac{r_-}{r_+\ell}\right)\frac{\mathrm{d}r}{N^2} \, , \end{aligned} \tag{3.19}$$

where we have defined the functions

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} \, , \qquad N^\phi = \frac{r_- r_+}{\ell r^2} \, , \tag{3.20}$$

gives

$$\mathrm{d}s^2_{\mathrm{BTZ}} = -N^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{N^2} + r^2\left(\mathrm{d}\phi + N^\phi \mathrm{d}t\right)^2 \, , \tag{3.21}$$

which is the BTZ black hole solution (see (3.56) in appendix 3.A). If $\kappa \geq 0$ the horizon $\lambda = 0$ corresponds to the outer horizon, whereas if $\kappa < 0$ the horizon $\lambda = 0$ corresponds to the inner horizon.

If $\gamma_1 = 0$, the constraint (3.13) can be immediately solved to get $h_0 = 2/\ell$ and hence the solution in this case simply reads

$$\mathrm{d}s^2 = \left(-2\kappa\lambda - \frac{\lambda^2}{\ell^2}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}\lambda + \left(\mathrm{d}x + \frac{\lambda \mathrm{d}v}{\ell}\right)^2 \, . \tag{3.22}$$

[7] We emphasise that, although related, this does not follow from the usual rigidity theorem for stationary rotating black holes.

If $\kappa = 0$ this is the near-horizon limit of the extreme BTZ black hole. If $\kappa \neq 0$ this is the decoupling limit of the near-extreme BTZ black hole.

3.3 General solution with a degenerate horizon

In this section we will study the general solution containing a degenerate horizon ($\kappa = 0$) with compact cross-sections $H \cong S^1$. A shown above, the general spacetime in this case is given by

$$\mathrm{d}s^2 = 2\mathrm{d}v\left[\mathrm{d}\lambda + \frac{2}{\ell}\lambda(1 + \tfrac{1}{2}\lambda\gamma_1(x))\mathrm{d}x\right] + (1 + \lambda\gamma_1(x))^2\mathrm{d}x^2 \ , \tag{3.23}$$

where $\gamma_1(x)$ is an arbitrary periodic function $\gamma_1(x + 2\pi R) = \gamma_1(x)$.

3.3.1 Large diffeomorphism

We now explicitly show that this solution is globally isometric to the BTZ black hole, or its near-horizon geometry, by introducing coordinates adapted to the two commuting Killing fields $K = \partial/\partial v$ and X given by (3.17).

The inner products of the Killing fields are thus:

$$K^2 = 0, \qquad K \cdot X = \frac{2\lambda}{\ell}\left(1 - \frac{\lambda y}{\ell}\right), \qquad X^2 = 1 + \frac{4c\lambda}{\ell}\left(1 - \frac{\lambda y}{\ell}\right) \ . \tag{3.24}$$

Define a third vector field U by: $U^2 = 0, U \cdot X = 0, U \cdot K = 1$. It is easy to show that

$$U = -\frac{1}{2}C^2\partial_v + \partial_\lambda + Ce_x \tag{3.25}$$

where $e_x = \frac{1}{\gamma}(\partial_x - \lambda h\partial_\lambda)$ is a dual vector to the basis (3.4) and the function C satisfies

$$y + c + \left(1 - \frac{2\lambda y}{\ell}\right)C - \frac{\lambda}{\ell}\left(1 - \frac{\lambda y}{\ell}\right)C^2 = 0 \ . \tag{3.26}$$

The discriminant of this quadratic is simply X^2. Hence the unique solution which is regular on the horizon is

$$C = \frac{1 - \frac{2\lambda y}{\ell} - \sqrt{1 + \frac{4c\lambda}{\ell}\left(1 - \frac{\lambda y}{\ell}\right)}}{\frac{2}{\ell}\lambda(1 - \frac{\lambda y}{\ell})} \ , \tag{3.27}$$

where we must have $X^2 > 0$. Now a tedious calculation shows that $[X, U] = 0$ if and only if

$$\frac{\lambda^2 y'}{\ell}\partial_\lambda C + \left(1 - \frac{2\lambda y}{\ell}\right)\partial_x C + y' = 0 \ . \tag{3.28}$$

Remarkably, it can be shown that (3.27) automatically satisfies (3.28). This allows us to deduce that a new coordinate system $(\tilde{v}, \tilde{\lambda}, \tilde{x})$ exists such that

$$K = \frac{\partial}{\partial \tilde{v}}, \qquad U = \frac{\partial}{\partial \tilde{\lambda}}, \qquad X = \frac{\partial}{\partial \tilde{x}} \ . \tag{3.29}$$

From (3.25) we may read off $\frac{\partial \lambda}{\partial \tilde{\lambda}} = 1 - \frac{\lambda h C}{\gamma}$ and $\frac{\partial x}{\partial \tilde{\lambda}} = \frac{C}{\gamma}$ which imply

$$\partial_{\tilde{\lambda}} \sqrt{X^2} \;=\; \frac{2c}{\ell} \frac{\left[1 - \frac{2\lambda y}{\ell} - \frac{2\lambda}{\ell}\left(1 - \frac{\lambda y}{\ell}\right) C\right]}{\sqrt{1 + \frac{4c\lambda}{\ell}\left(1 - \frac{\lambda y}{\ell}\right)}} = \frac{2c}{\ell} \ , \tag{3.30}$$

where in the second equality we used (3.27). Hence, integrating and fixing the horizon to be at $\tilde{\lambda} = 0$ we get

$$\sqrt{X^2} = 1 + \frac{2c\tilde{\lambda}}{\ell} \ , \qquad K \cdot X = \frac{2\tilde{\lambda}}{\ell}\left(1 + \frac{c\tilde{\lambda}}{\ell}\right) \ . \tag{3.31}$$

Therefore, the metric in the new coordinates is

$$\mathrm{d}s^2 = 2\mathrm{d}\tilde{v}\left[\mathrm{d}\tilde{\lambda} + \frac{2}{\ell}\tilde{\lambda}\left(1 + \frac{c\tilde{\lambda}}{\ell}\right)\mathrm{d}\tilde{x}\right] + \left(1 + \frac{2c\tilde{\lambda}}{\ell}\right)^2 \mathrm{d}\tilde{x}^2 \ . \tag{3.32}$$

This expresses the solution in Gaussian null coordinates adapted to a cross-section $\tilde{H} \cong S^1$ which is tangent to X. It is thus takes our general form (3.23) with $\tilde{\gamma}_1 = 2c/\ell$ a constant. As we showed above this is the extreme BTZ black hole ($c \neq 0$) or its near-horizon geometry ($c = 0$).

The results of the next section will show that the diffeomorphism constructed above must be a large diffeomorphism.

3.3.2 Asymptotic charges

We now consider the extreme solution in a chart adapted to a general cross-section, i.e. the spacetime (3.23). We will assume that the transverse null geodesics $\partial/\partial\lambda$ are strictly expanding, i.e. $\gamma_1(x) > 0$. This ensures it is asymptotically AdS_3 with a cylinder conformal boundary. In fact since h_0 is constant and x is periodically identified, this immediately follows from (3.14).

To see this in more detail, consider the coordinate change defined by

$$\begin{aligned}
r &= \lambda + \frac{1}{\gamma_1(x)} \\
t &= v + \frac{\ell^2}{r}\left(1 + \frac{\beta(x)}{3r^2}\right) \\
\phi &= \int \gamma_1(x)\mathrm{d}x + \frac{v}{\ell} - \frac{\beta(x)}{3r^3}
\end{aligned} \tag{3.33}$$

where the function

$$\beta \equiv \frac{1}{\gamma_1^2}\left(1 - \frac{\ell\gamma_1'}{\gamma_1}\right) . \tag{3.34}$$

To derive this coordinate change, we expanded the one for extreme BTZ (3.19) for large r and then allowed the subleading terms to depend on x. Observe that the coordinate change (3.33) forces ϕ to be a periodic coordinate with period $\int_0^{2\pi R} \gamma_1(x)\mathrm{d}x$. By scaling $(v, \lambda, \gamma_1) \to (cv, c^{-1}\lambda, c\gamma_1)$, we may always fix the period of ϕ to be 2π. In these coordinates our general metric (3.23) has the following asymptotics

$$\begin{aligned}
g_{tt} &= -\frac{r^2}{\ell^2} + \frac{2\beta(x)}{\ell^2} + \mathcal{O}(r^{-1}) , \qquad g_{t\phi} = -\frac{\beta(x)}{\ell} + \mathcal{O}(r^{-1}) , \qquad g_{\phi\phi} = r^2 + \mathcal{O}(r^{-1}) , \\
g_{tr} &= -\frac{2\ell\beta'(x)}{3\gamma_1(x)r^3} + \mathcal{O}(r^{-4}) , \qquad g_{r\phi} = \frac{\ell^2\beta'(x)}{3\gamma_1(x)r^3} + \mathcal{O}(r^{-4}) , \\
g_{rr} &= \frac{\ell^2}{r^2} + \frac{2\ell^2\beta(x)}{r^4} + \mathcal{O}(r^{-5}) ,
\end{aligned} \tag{3.35}$$

for $r \to \infty$. By comparing with (3.2), this explicitly shows that our spacetime is asymptotically AdS_3 in the sense of Brown and Henneaux [16]. Observe that for large r

$$\phi - \frac{t}{\ell} = \int \gamma_1(x)\mathrm{d}x + \mathcal{O}(r^{-1}) \tag{3.36}$$

and hence asymptotically x is purely a function of $\phi - \frac{t}{\ell}$ (note the coordinate change is invertible due to our assumption $\gamma_1 > 0$).

From the subleading terms in (3.35) we may compute the asymptotic charges of this solution. The asymptotic-symmetry generators are [16]

$$L_n^{\pm} = \frac{1}{2}e^{in(\frac{t}{\ell}\mp\phi)}\left(\ell\frac{\partial}{\partial t} \mp \frac{\partial}{\partial\phi}\right) + \dots \tag{3.37}$$

where $\dots$ denotes subleading terms and also terms proportional to ∂_r which will not be needed. The conserved charge $Q[\xi]$ associated to an asymptotic symmetry generated by a vector field ξ is an integral at fixed time t over the boundary circle at spacelike infinity $r \to \infty$. We find that, relative to the zero mass BTZ solution, the Virasoro charges are

$$Q[L_n^+] = \frac{1}{\ell\pi}\int_0^{2\pi} d\phi\, e^{in(\frac{t}{\ell}-\phi)}\beta(x) \quad (3.38)$$

$$Q[L_n^-] = 0\,, \quad (3.39)$$

where as noted above asymptotically x is only a function of $\phi - \frac{t}{\ell}$. Thus our general solution generically carries non-zero charges only in one of the Virasoro algebras. In particular, the mass $\ell M = Q[L_0^+] + Q[L_0^-]$ and angular momentum $J = Q[L_0^+] - Q[L_0^-]$ are given by

$$\ell M = J = \frac{1}{\ell\pi}\int_0^{2\pi} d\phi\,\beta(x) = \frac{1}{\ell\pi}\int_0^{2\pi R}\frac{dx}{\gamma_1(x)}\,, \quad (3.40)$$

where in the final equality we converted to the x coordinate (at constant t) and used the explicit form of β together with periodicity.

Thus we see that the mass/angular-momentum relation satisfied by the extreme BTZ black hole persists for this class of spacetimes. However, unlike the BTZ black hole, these carry arbitrary non-zero charges with respect to all the Virasoro generators L_n^+ and vanishing ones with respect to L_n^-. In particular, the general solution is characterised by the Virasoro charges $Q[L_n^+]$ with $n \neq 0$. It is worth noting that if $Q[L_n^+] = 0$ for all $n \neq 0$, then the function β must be a constant and hence (3.34) implies γ_1 must be a constant (using periodicity) and we recover the BTZ black hole. Therefore, these geometries may be interpreted as descendants of the extreme BTZ black hole.

3.4 Non-degenerate horizon

In this section we study the general solution containing a non-degenerate horizon ($\kappa \neq 0$) with compact cross-sections $H \cong S^1$. As shown above, the general solution is given by (3.3), (3.12) and is determined by the constant κ and an arbitrary function $h_0(x)$, with $\gamma_1(x)$ then determined by (3.13). These functions must be periodic with $x \sim x + 2\pi R$.

We will also assume that $\gamma_1(x) > 0$ so the transverse null geodesics $\partial/\partial\lambda$ are strictly expanding and that $\kappa > 0$ to ensure the null generators are future complete. Under these conditions, it can be shown that (3.13) implies

$$h_0(x)^2 < \frac{4}{\ell^2}\,, \quad (3.41)$$

for *all* x (otherwise h_0 is monotonic, contradicting periodicity). In fact, this condition implies the Killing field K is timelike *everywhere* outside the horizon.

We will first analyse the conformal boundary of this Einstein spacetime. The main complication arises due to the fact that the conformal boundary metric in the frame defined by equation (3.14) is not flat for non-constant $h_0(x)$. Remarkably, we find there is a simple Weyl transformation on the boundary which makes (3.14) a flat cylinder and is consistent with the global identifications we already have (i.e. x periodic and v not). It may be verified that

$$\mathrm{d}s_b^2 = \frac{-\frac{\mathrm{d}v^2}{\ell^2} + (\gamma_1 \mathrm{d}x + \frac{1}{2}h_0 \mathrm{d}v)^2}{\Omega(x)^2} \ , \tag{3.42}$$

where

$$\Omega(x) \equiv \sqrt{1 - \frac{\ell^2 h_0^2}{4}} \ , \tag{3.43}$$

is a flat metric for any $h_0(x), \gamma_1(x)$. Indeed, this can be seen by performing the coordinate change

$$\mathrm{d}t = c_\beta \mathrm{d}v + \frac{\ell\gamma_1}{\Omega^2}\left(s_\beta - \frac{c_\beta \ell h_0}{2}\right)\mathrm{d}x \ , \qquad \mathrm{d}\phi = s_\beta \frac{\mathrm{d}v}{\ell} + \frac{\gamma_1}{\Omega^2}\left(c_\beta - \frac{s_\beta \ell h_0}{2}\right)\mathrm{d}x \ , \tag{3.44}$$

where $c_\beta = \cosh\beta, s_\beta = \sinh\beta$ and β is a constant "boost" parameter, which gives

$$\mathrm{d}s_b^2 = -\frac{\mathrm{d}t^2}{\ell^2} + \mathrm{d}\phi^2 \ . \tag{3.45}$$

Observe that we have to include the boost β (which is a large diffeomorphism) in order to ensure that the coordinate t is not periodically identified. The condition to avoid identifications of t corresponds to the following specific choice of boost:

$$\tanh\beta = \frac{\int_0^{2\pi R} \frac{\ell h_0 \gamma_1}{2\Omega^2}\mathrm{d}x}{\int_0^{2\pi R} \frac{\gamma_1}{\Omega^2}\mathrm{d}x} \ , \tag{3.46}$$

which we assume henceforth. Observe that due to $\gamma_1 > 0$ and (3.41),

$$\left| \frac{\int_0^{2\pi R} \frac{\ell h_0 \gamma_1}{2\Omega^2}\mathrm{d}x}{\int_0^{2\pi R} \frac{\gamma_1}{\Omega^2}\mathrm{d}x} \right| \le \frac{\int_0^{2\pi R} \frac{\ell |h_0| \gamma_1}{2\Omega^2}\mathrm{d}x}{\int_0^{2\pi R} \frac{\gamma_1}{\Omega^2}\mathrm{d}x} < 1, \tag{3.47}$$

so a unique value for this special boost always exists. Furthermore, by a discrete transformation $x \to -x$ we may always arrange $\beta \ge 0$, which we will assume below. Also note that $\partial\phi/\partial x > 0$ so the coordinate ϕ inherits a periodicity from x. By scaling $(v, \lambda, \gamma_1) \to (cv, c^{-1}\lambda, c\gamma_1)$, we may always fix the period of ϕ to be 2π. Hence, in the above conformal frame the boundary is indeed globally a flat cylinder, as claimed.

We will now compute the asymptotic charges by working in the cylinder conformal frame. Consider the coordinate change defined by

$$
\begin{aligned}
r &= \Omega\left(\lambda + \frac{\ell^2\kappa}{\Omega^2}\right) \ , \\
t &= c_\beta v + \int \frac{\ell\gamma_1}{\Omega^2}\left(s_\beta - \frac{c_\beta \ell h_0}{2}\right) \mathrm{d}x + \frac{\ell^2}{\Omega r}\left[c_\beta - \frac{s_\beta \ell h_0}{2} + \frac{\kappa^2\ell^4}{3\Omega^2 r^2}\left(c_\beta - \frac{s_\beta \ell^3 h_0^3}{8}\right)\right] \ , \\
\phi &= s_\beta \frac{v}{\ell} + \int \frac{\gamma_1}{\Omega^2}\left(c_\beta - \frac{s_\beta \ell h_0}{2}\right) \mathrm{d}x \\
&\quad + \frac{\ell}{\Omega r}\left[s_\beta - \frac{c_\beta \ell h_0}{2} + \frac{\kappa^2\ell^4}{3\Omega^2 r^2}\left(s_\beta - \frac{c_\beta \ell^3 h_0^3}{8}\right)\right] \ .
\end{aligned} \tag{3.48}
$$

In these coordinates we find that the metric for $r \to \infty$ has the following behaviour:

$$
\begin{aligned}
g_{tt} &= -\frac{r^2}{\ell^2} + \frac{\kappa^2\ell^2}{\Omega^2}\left(c_\beta^2 - \frac{s_\beta^2 \ell^2 h_0^2}{4}\right) + \mathcal{O}(r^{-3}) \ , \qquad g_{t\phi} = -s_\beta c_\beta \kappa^2 \ell^3 + \mathcal{O}(r^{-3}) \ , \\
g_{\phi\phi} &= r^2 + \frac{\kappa^2\ell^4}{\Omega^2}\left(s_\beta^2 - \frac{c_\beta^2 \ell^2 h_0^2}{4}\right) + \mathcal{O}(r^{-3}) \ , \qquad g_{tr} = \mathcal{O}(r^{-3}) \ , \\
g_{\phi r} &= \mathcal{O}(r^{-3}) \ , \qquad g_{rr} = \frac{\ell^2}{r^2} + \frac{\kappa^2\ell^6}{\Omega^2 r^4}\left(1 + \frac{\ell^2 h_0^2}{4}\right) + \mathcal{O}(r^{-5}) \ ,
\end{aligned} \tag{3.49}
$$

which is of Brown-Henneaux form (3.2) and hence suitable for reading off the conserved charges. Observe that asymptotically x is a function of *both* $\phi \pm t/\ell$. Hence a priori, one would expect the Virasoro charges $Q[L_n^\pm] \neq 0$ for all integers n. In fact, performing the computation, we actually find that

$$
Q[L_n^\pm] = \ell^3\kappa^2\left(\frac{1}{2} + s_\beta^2 \pm s_\beta c_\beta\right)\delta_{n,0} \ . \tag{3.50}
$$

Therefore, we see that all higher Virasoro charges vanish, whereas the zero mode charges give the following mass and angular momentum

$$
M = \ell^2\kappa^2(c_\beta^2 + s_\beta^2), \qquad J = 2\ell^3\kappa^2 s_\beta c_\beta \ . \tag{3.51}
$$

Observe that

$$
\ell M - J = \ell^3\kappa^2(c_\beta - s_\beta)^2 > 0. \tag{3.52}
$$

Therefore, our spacetime has precisely the same Virasoro charges as the non-extreme BTZ black hole. It follows that it must be diffeomorphic to the non-extreme BTZ black hole.[8] Hence, in contrast to the extreme case, we do not obtain descendants of the non-extreme BTZ black hole (which would possess arbitrary charges with respect to all $L_n^\pm$ and thus be related by a large diffeomorphism).

[8]This follows from the fact that the Fefferman-Graham expansion (3.53) terminates in three-dimensions and that $Q[L_n^\pm] \sim \int_0^{2\pi} \mathrm{d}\phi \; e^{inx^\pm} T^\pm(x^\pm)$.

3.5 Discussion

Another way of understanding our results may be as follows. The Fefferman-Graham expansion for three-dimensional Einstein spacetimes terminates and hence the conformal boundary metric and stress tensor determine the full spacetime [116]. For asymptotically globally AdS_3 spacetimes, so a cylinder conformal boundary metric $-\frac{\mathrm{d}t^2}{\ell^2}+\mathrm{d}\phi^2$ where $\phi \sim \phi + 2\pi$, it is easy to determine the general Einstein spacetime [4], which is

$$\mathrm{d}s^2 = \frac{1}{z^2}\left[\ell^2 \mathrm{d}z^2 + 2(\mathrm{d}x^+ + \tfrac{1}{2}z^2 T^-(x^-)\mathrm{d}x^-)(\mathrm{d}x^- + \tfrac{1}{2}z^2 T^+(x^+)\mathrm{d}x^+)\right] , \qquad (3.53)$$

where $z = 0$ is the conformal boundary, $\sqrt{2}x^\pm = \phi \pm \frac{t}{\ell}$ are lightcone coordinates on the cylinder, and the two arbitrary functions $T^\pm(x^\pm)$ are the components of the boundary stress tensor.

Now, suppose (3.53) describes a black hole with a horizon invariant under a Killing field of the form (3.1). If $|\ell\Omega| \neq 1$, then it is straightforward to show that *both* $T^\pm(x^\pm)$ must be constant functions and hence the spacetime is stationary and axisymmetric (if $T^\pm > 0$ this is the BTZ black hole). On the other hand, if $|\ell\Omega| = 1$, then only one of ∂_+ or ∂_- is a Killing field; without loss of generality suppose $\Omega\ell = 1$ so $K \propto \partial_+$. Then $T^+(x^+)$ is again a constant, although now $T^-(x^-)$ can be an arbitrary function. It then follows that $|K|^2 = T^+$ is a constant and therefore such a Killing horizon exists if and only if $T^+ = 0$. In this case, K is a globally null Killing field and since by assumption it is tangent to the black hole horizon, the horizon must be *degenerate*.

This simple argument allows for extreme black holes more general than extreme BTZ, which are related by a large diffeomorphism to the extreme BTZ. Furthermore, it also does not appear to allow for more general non-extreme black holes with a *Killing* horizon. This picture is consistent with the results derived in this paper, which were obtained by determining the most general three-dimensional Einstein metric containing a Killing horizon.

The asymptotic Virasoro charges of our general extreme black hole show that these geometries can be interpreted as descendants of the extreme BTZ (as defined in the introduction). It would be interesting to better understand their CFT interpretation. On the other hand, we did not find descendants of the non-extreme BTZ black hole within our general solution with a non-extreme horizon (these would be related by a large diffeomorphism). It would be interesting to understand this by directly analysing under what conditions the general Einstein metric (3.53) contains a non-singular horizon.

The only assumption in our analysis was that the transverse null geodesics are strictly expanding i.e. $\gamma_1(x) > 0$ for all x, so that the whole spacetime can be covered by a single coordinate patch. A natural extension is therefore to remove this assumption. In fact $3D$ wormhole spacetimes with multiple asymptotic regions have been constructed [117]. In this class of solutions, each region is isometric to the BTZ black hole separated by horizons. It is possible that descendants black holes of the non-extreme BTZ black hole may be found in these wormhole spacetimes.

3.6 Summary

We determined the most general three-dimensional vacuum spacetime with a negative cosmological constant containing a non-singular Killing horizon. We showed that the general solution with a spatially compact horizon possesses a second commuting Killing field and deduced that it must be related to the BTZ black hole (or its near-horizon geometry) by a diffeomorphism. We showed there is a general class of asymptotically AdS_3 extreme black holes with arbitrary charges with respect to one of the asymptotic-symmetry Virasoro algebras and vanishing charges with respect to the other. We interpret these as descendants of the extreme BTZ black hole and they are related to the extreme BTZ black hole by a large diffeomorphism. We did not find descendants of the non-extreme BTZ black hole within our general solution admitting a non-degenerate horizon, our general non-extreme solution is therefore necessarily diffeomorphic to the BTZ solution by a small diffeomorphism.

3.A Banados-Teitelboim-Zanelli black holes

In this section we discuss some basic properties of the Banados-Teitelboim-Zanelli (BTZ) black hole. Consider the Einstein field equation in 3 dimensions with a negative cosmological constant $\Lambda = -\frac{1}{\ell^2}$

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} \ . \tag{3.54}$$

The *maximally symmetric* solution to (3.54) is the anti-de-Sitter space AdS_3, in global coordinates it is given by

$$-\left(1+\frac{r^2}{\ell^2}\right)\mathrm{d}t^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \tag{3.55}$$

with $t \in \mathbb{R}$, $r \in \mathbb{R}^+$ and $\phi \sim \phi + 2\pi$. In (2+1) dimensional gravity the Ricci and Riemann tensors have the same number of degrees of freedom (six), so the Weyl tensor which encodes information about "shape distortion" is identically zero, therefore any solutions to (3.54) is *locally* AdS_3.

The BTZ solution is a *black hole* solution to (3.54), which is given by

$$\mathrm{d}s^2 = -N^2(r)\mathrm{d}t^2 + N^{-2}(r)\mathrm{d}r^2 + r^2(N^\phi(r)\mathrm{d}t + \mathrm{d}\phi)^2 \tag{3.56}$$

where the lapse and shift functions are

$$N^2(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} \qquad\qquad N^\phi(r) = -\frac{J}{2r^2} \ , \tag{3.57}$$

M and J are the mass and angular momentum of the black hole given by Gaussian flux integrals at infinity. The event horizon is located at r_+, which is the larger of the two roots of the lapse function N

$$r_\pm = \ell\sqrt{\frac{M}{2}\left(1 \pm \sqrt{1 - \left(\frac{J}{M\ell}\right)^2}\right)} \ . \tag{3.58}$$

The extreme solution is attained when $M = J\ell$ and the two roots coincide. Clearly (3.58) only makes sense for $M > 0$ and $|J| \leq M\ell$, as $M \to 0$ the horizon shrinks to zero size. From (3.54) it is also clear that the singularity at $r = 0$ is cannot be a curvature singularity; rather it is a singularity in the causal structure because continuing past $r = 0$ leads to closed timelike curves.

It is easy to see that when $M = -1$ and $J = 0$ the solution (3.56) reduces to AdS_3 (3.55), thus AdS_3 is separated from the continuous black hole spectrum by a mass gap. Within the mass gap i.e. $-1 < M < 0$ the solution (3.56) develops a naked conical singularity therefore it is excluded from the configuration space.

3.B The extra Killing fields

By construction $K = \partial/\partial v$ is a Killing field in our general solution. Let $X = A(\lambda, x)e_+ + B(\lambda, x)e_- + C(\lambda, x)e_x$ be another Killing field which commutes with K, we seek the most A, B and C satisfying the Killing's equation

$$\nabla_{(A}X_{B)} = 0 \ . \tag{3.59}$$

Non-degenerate horizon

For the non-degenerate case we have $\partial_\lambda(\lambda h) = \gamma h_0$ and let $f_1 \equiv \partial_\lambda(\lambda f) = -2\kappa + \lambda(\frac{1}{2}h_0^2 - \frac{2}{\ell^2})$. Using the connection 1-forms given in (3.7), it is straightforward to compute the

$--$, $+-$, $++$, $-x$, $+x$ and xx components of the Killing's equation (3.59), which are

$$\partial_\lambda A = 0 \tag{3.60}$$

$$\partial_\lambda B - h_0 C - \frac{1}{2} f_1 A - \frac{1}{2}\lambda f \partial_\lambda A = 0 \tag{3.61}$$

$$-\frac{1}{2}\lambda f \partial_\lambda B + \frac{1}{2} f_1 B = 0 \tag{3.62}$$

$$\partial_\lambda C - \frac{\partial_\lambda \gamma}{\gamma} C + \frac{1}{\gamma}\partial_x A - \frac{\lambda h}{\gamma}\partial_\lambda A = 0 \tag{3.63}$$

$$\frac{1}{\gamma}\partial_x B - \frac{\lambda h}{\gamma}\partial_\lambda B + h_0 B - \frac{\lambda f}{2}\left(\partial_\lambda C - \frac{\partial_\lambda \gamma}{\gamma} C\right) = 0 \tag{3.64}$$

$$\frac{1}{\gamma}\partial_x C - \frac{\lambda h}{\gamma}\partial_\lambda C + \frac{\partial_\lambda \gamma}{\gamma} B - \frac{\lambda f \partial_\lambda \gamma}{2\gamma} A = 0\ . \tag{3.65}$$

First of all the $--$ equation (3.60) states $A = A(x)$, so the $-x$ equation (3.63) can be written as

$$\partial_\lambda \left(\frac{C}{\gamma}\right) = -\frac{A'}{\gamma^2}\ , \tag{3.66}$$

which has solution

$$C = \frac{A'}{\gamma_1} + \alpha(x)(1 + \lambda\gamma_1) \tag{3.67}$$

where $\alpha(x)$ is an integration function. This allows us to integrate the $+-$ equation (3.61) and obtain

$$B = \frac{\lambda^2}{2}\left[\alpha h_0 \gamma_1 + \frac{A}{4}\left(h_0^2 - \frac{4}{\ell^2}\right)\right] + \lambda\left[h_0\left(\frac{A'}{\gamma_1} + \alpha\right) - \kappa A\right] + \beta(x) \tag{3.68}$$

where $\beta(x)$ is an integration function. In fact evaluating (3.62) at $\lambda = 0$ gives $\beta(x) = 0$. Substituting our the above expressions for C and B into (3.65) and evaluate at $\lambda = 0$ implies $\partial_x C|_{\lambda=0} = 0$, so

$$\frac{A'}{\gamma_1} + \alpha = k_1 = const\ . \tag{3.69}$$

We can eliminate α and obtain

$$C = k_1\gamma - \lambda A' \tag{3.70}$$

$$B = \frac{\lambda^2}{2}\left[h_0(k_1\gamma_1 + A') + \frac{A}{4}\left(h_0^2 - \frac{4}{\ell^2}\right)\right] + \lambda\left(k_1 h_0 - \kappa A\right)\ . \tag{3.71}$$

The higher order terms in (3.62), (3.64) and (3.65) along with the constraint (3.13) then imply

$$-k_1 h_0 + 2\kappa A = k_2 = const\ . \tag{3.72}$$

Therefore the general solution is

$$
\begin{aligned}
A &= \frac{k_1 h_0 + k_2}{2\kappa} && (3.73)\\
B &= \frac{\lambda}{2}(k_1 h_0 - k_2)\left[1 + \frac{\lambda}{2}\left(\gamma_1 - \frac{h_0'}{2\kappa}\right)\right] && (3.74)\\
C &= k_1\left(1 + \lambda\gamma_1 - \frac{\lambda h_0'}{2\kappa}\right) . && (3.75)
\end{aligned}
$$

By converting back to coordinate basis and with suitable normalisation for the constants, it follows that

$$X = \left(c + \frac{h_0}{2\kappa}\right)\partial_v + \frac{\lambda^2 h_0 h_0'}{4\kappa\gamma}\partial_\lambda + \left(1 - \frac{\lambda h_0'}{2\kappa\gamma}\right)\partial_x , \qquad (3.76)$$

where c is a constant. Clearly X commutes with $K = \partial_v$.

Degenerate horizon

For the degenerate case the calculation is simpler. The components of the Killing's equation (3.59), in the same order as previous case, minus the $++$ component which is now trivial, are

$$
\begin{aligned}
&\partial_\lambda A = 0 && (3.77)\\
&\partial_\lambda B - \frac{2}{\ell}C = 0 && (3.78)\\
&\partial_\lambda C - \frac{\partial_\lambda\gamma}{\gamma}C + \frac{1}{\gamma}\partial_x A - \frac{\lambda h}{\gamma}\partial_\lambda A = 0 && (3.79)\\
&\frac{1}{\gamma}\partial_x B - \frac{\lambda h}{\gamma}\partial_\lambda B + \frac{2}{\ell}B = 0 && (3.80)\\
&\frac{1}{\gamma}\partial_x C - \frac{\lambda h}{\gamma}\partial_\lambda C + \frac{\partial_\lambda\gamma}{\gamma}B = 0 . && (3.81)
\end{aligned}
$$

It can be checked that in fact the $+-$ and $+x$ equations (3.78) and (3.80) imply the xx equation (3.81). So we only have four equations to consider. As in the previous case the $--$ equation (3.77) gives $A = A(x)$ and we can integrate the $-x$ equation (3.79) to obtain

$$C = \frac{A'}{\gamma_1} + \alpha(x)(1 + \lambda\gamma_1) . \qquad (3.82)$$

Substituting C with $C = \frac{\ell}{2}\partial_\lambda B$ from the $+-$ equation (3.78) allows us to integrate with respect to λ and get

$$\frac{\ell}{2}B = \frac{\lambda^2}{2}\alpha(x)\gamma_1 + \lambda\left(\alpha(x) + \frac{A'}{\gamma_1}\right) + \beta(x) . \qquad (3.83)$$

We can then use this expression for B in (3.80), which gives a polynomial equation in λ

$$0 = \lambda^2 \left[\ell(\alpha\gamma_1)' + 2A'\right] + 2\lambda \left[\ell \left(\frac{A'}{\gamma_1} + \alpha\right)' + 2\gamma_1\beta\right] + 2\ell\beta' + 4\beta \; . \tag{3.84}$$

Since the coefficients must vanish for each order in λ, it leads to the following ODE's

$$0 = (\alpha\gamma_1)' + \frac{2A'}{\ell} \tag{3.85}$$

$$0 = \left(\frac{A'}{\gamma_1} + \alpha\right)' + \frac{2\gamma_1\beta}{\ell} \tag{3.86}$$

$$0 = \beta' + \frac{2\beta}{\ell} \; . \tag{3.87}$$

(3.85) can easily be integrated to give $\alpha\gamma_1 + \frac{2A}{\ell} = \frac{2k_1}{\ell}$ where k_1 is the constant of integration. Since x is periodic the only solution to (3.87) is $\beta = 0$, therefore (3.86) integrates to $A' + \alpha\gamma_1 = k_2\gamma_1$ where k_2 is another integration constant. We can thus eliminate α to obtain the ODE

$$A' - \frac{2A}{\ell} = -\frac{2k_1}{\ell} + k_2\gamma_1 \; . \tag{3.88}$$

Since now everything is determined by A, the problem reduces to solving this ODE for a periodic A. By linearity we can consider the cases for $k_1 = 0$ and $k_2 = 0$ separately. If $k_2 = 0$ then the only periodic solution for A is $A = k_1$, which implies $B = C = 0$ and we are back to the original Killing field K. On the other hand if $k_1 = 0$, then we may write $A = k_2 y(x)$ so (3.88) becomes

$$y' - \frac{2}{\ell}y = \gamma_1(x) \; . \tag{3.89}$$

Hence by taking a linear combination of the two solutions and converting back to coordinate basis, it follows that, with suitable normalisation for the constants k_1 and k_2,

$$X = (c + y)\partial_v + \frac{\lambda^2 y'}{\ell\gamma}\partial_\lambda + \left(1 - \frac{\lambda y'}{\gamma}\right)\partial_x \tag{3.90}$$

where c is a constant and X clearly commutes with K as in the non-degenerate case.

Chapter 4

First Order Expansion From Near-Horizon Geometry

4.1 Introduction

In this chapter we address the following question: given a near-horizon geometry, what is the *full black hole solution* with this near-horizon limit? There is no guarantee that such full solution actually exists; even if a full solution does exit it needs not be unique. Furthermore, the existence of a Killing horizon is necessary but not sufficient for a solution to describe a stationary black hole, for example a Poincaré horizon in AdS space is a Killing horizon but there is no black hole. In this chapter we address this important *inverse problem* by considering the first order transverse expansion from some well-known near-horizon geometries and investigate the possibility of the existence of *new black hole* solutions. We follow a slightly different treatment here than in our paper [97].

4.2 Near-horizon geometry

We proceed by revisiting the definition of near-horizon geometry; this allows us to illustrate the problem explicitly and give a precise definition of "first order transverse expansion". Let us consider a D-dimensional spacetime (M, g) containing a *degenerate* Killing Horizon $\mathcal{N}$ of Killing field K with spatial cross-section H. In the neighbourhood of $\mathcal{N}$, the spacetime metric can be written in Gaussian Null Coordinates (GNC) as

$$g = 2\mathrm{d}v\left(\frac{1}{2}r^2F(r,x)\mathrm{d}v + \mathrm{d}r + rh_a(r,x)\mathrm{d}x^a\right) + \gamma_{ab}(r,x)\mathrm{d}x^a\mathrm{d}x^b \tag{4.1}$$

such that $K = \partial/\partial v$, $\mathcal{N} = \{r = 0\}$ and x^a are coordinates on H with $a = 1, ..., D-2$. For any $\varepsilon > 0$, we can take the diffeomorphism $v \to v/\varepsilon$ and $r \to \varepsilon r$ generated by "dilation" $v\,\partial/\partial v - r\,\partial/\partial r$ to define a 1-parameter family of metrics

$$g(\varepsilon) = 2\mathrm{d}v\left(\frac{1}{2}r^2F(\varepsilon r, x)\mathrm{d}v + \mathrm{d}r + rh_a(\varepsilon r, x)\mathrm{d}x^a\right) + \gamma_{ab}(\varepsilon r, x)\mathrm{d}x^a\mathrm{d}x^b\ . \tag{4.2}$$

The *near-horizon limit* is defined as the limit $\varepsilon \to 0$ and $g(0)$ is the *near-horizon geometry*, which we also denote as g_{NH}. Clearly if g is an exact solution to the Einstein equation $\mathscr{E}(g) = 0$, its near-horizon geometry $g(0)$ is also an exact solution

$$\mathscr{E}(g(\varepsilon)) = 0 \Rightarrow \mathscr{E}(g(0)) = 0\ . \tag{4.3}$$

Note that all dependence on r is completely fixed in *near-horizon geometry*

$$g_{NH} = 2\mathrm{d}v\left(\frac{1}{2}r^2F(x)\mathrm{d}v + \mathrm{d}r + rh_a(x)\mathrm{d}x^a\right) + \gamma_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b \tag{4.4}$$

where $F(x) = F(0, x)$, $h_a(x) = h_a(0, x)$ and $\gamma_{ab}(x) = \gamma_{ab}(0, x)$; g_{NH} is completely specified by these three geometric quantities on the spatial cross-section H which are collectively referred to as *near-horizon data.* This has motivated the study of near-horizon geometries - exact solutions of the form (4.4) to the Einstein equation $\mathscr{E}(g_{NH}) = 0$, which is much easier to solve than the general $\mathscr{E}(g) = 0$ while keeping all the essential information about the horizon. Nevertheless, a solution g_{NH} which satisfies $\mathscr{E}(g_{NH}) = 0$ need not be the near-horizon limit of some g which solves $\mathscr{E}(g) = 0$. Even if it is the corresponding g may not be a *black hole*, as we demonstrate in section 4.4 with the plane wave solution[1]; afterall near-horizon geometry only ensures that the spacetime contains a degenerate *Killing horizon.* There is also no guarantee for uniqueness of the corresponding g given a g_{NH}. For instance, the near-horizon limit of the $5D$ extreme self-dual Myers-Perry black hole turns out to be the same as the near-horizon limit of the $J = 0$ extreme Kaluza-Klein black hole in $5D$. We will investigate the first order expansion to the $5D$ vacuum near-horizon geometry with a homogeneous S^3 horizon in detail in section 4.7.

Our strategy to tackle this problem is to extend our solution from the near-horizon geometry to *just* outside the horizon and see what we can learn from the *first order expansion*: since g_{NH} is an exact solution to the Einstein equation, we consider a small transverse expansion (deformation) from the NHG to first order and solve the linearised Einstein equation.

[1]In fact the $4D$ flat T^2 NHG alone can be ruled out as a black hole candidate according to horizon topology theorems.

4.3 First order expansion

4.3.1 The linearised Einstein equation

Suppose $g(\varepsilon)$ is a full solution to the vacuum Einstein equation with zero cosmological constant $R_{\mu\nu}(g(\varepsilon)) = 0$. It is clear that the parameter ε in $g(\varepsilon)$ can be treated as a perturbation parameter from the exact solution g_{NH}. Hence the linearised Einstein equation is

$$R^{(1)}_{\mu\nu} = \left.\frac{\mathrm{d}R_{\mu\nu}(g(\varepsilon))}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0\ , \tag{4.5}$$

which is a linear equation for the first order expansion of $g(\varepsilon)$ from g_{NH}, defined as

$$g^{(1)} = \left.\frac{\mathrm{d}g(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}. \tag{4.6}$$

Recall that the change in the Ricci tensor $R_{\mu\nu}(g(\varepsilon))$ with respect to ε is given by, to lowest order in ε,

$$\frac{\mathrm{d}R_{\mu\nu}}{\mathrm{d}\varepsilon} = \Delta_L g^{(1)}_{\mu\nu} + \nabla_{(\mu}v_{\nu)} + \mathcal{O}(\varepsilon) \tag{4.7}$$

where $v_\mu = \nabla_\nu g^{(1)}{}_\mu{}^\nu - \frac{1}{2}\partial_\mu(g^{(1)}_{\rho\sigma}g^{\rho\sigma})$ and Δ_L is the Lichnerowicz operator

$$\Delta_L g^{(1)}_{\mu\nu} = -\tfrac{1}{2}\nabla^2 g^{(1)}_{\mu\nu} - R_\mu{}^\kappa{}_\nu{}^\lambda g^{(1)}_{\kappa\lambda} + R_{(\mu}{}^\kappa g^{(1)}_{\nu)\kappa}\ . \tag{4.8}$$

Therefore the linearised Einstein equation (4.5) is just

$$\Delta^{(0)}_L g^{(1)}_{\mu\nu} + \nabla^{(0)}_{(\mu}v^{(0)}_{\nu)} = 0 \tag{4.9}$$

where the (0) superscripts on the curvature operators mean they are those of the near-horizon geometry g_{NH} and all indices are lowered and raised with the metric $g_{NH\mu\nu}$ and its inverse.

Before we can write the linearised Einstein equation (4.9) in components we need to first write $g^{(1)}_{\mu\nu}$ in components. Since the metric functions F, h_a and γ_{ab} must be smooth on $\mathcal{N}$ we can Taylor expand them around $\varepsilon r = 0$:

$$F(\varepsilon r, x) = F^{(0)}(x) + \varepsilon r F^{(1)}(x) + \mathcal{O}(\varepsilon^2 r^2) \qquad h_a(\varepsilon r, x) = h^{(0)}_a(x) + \varepsilon r h^{(1)}_a(x) + \mathcal{O}(\varepsilon^2 r^2)$$
$$\gamma_{ab}(\varepsilon r, x) = \gamma^{(0)}_{ab}(x) + \varepsilon r \gamma^{(1)}_{ab}(x) + \mathcal{O}(\varepsilon^2 r^2) \tag{4.10}$$

so the Taylor expansion of expansion of the metric $g(\varepsilon)$ to first order in εr is

$$g(\varepsilon) = g_{NH} + \varepsilon\left[2\mathrm{d}v\left(\frac{1}{2}r^3F^{(1)}(x)\mathrm{d}v + r^2h^{(1)}_a(x)\mathrm{d}x^a\right) + r\gamma^{(1)}_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b\right] + \mathcal{O}(\varepsilon^2) \tag{4.11}$$

and $g^{(1)}$ is the quantity in the square bracket.

It is however more convenient to work in *null orthonormal basis* defined in (1.14), such that $g_{NH} = 2e^+e^- + \delta_{ab}\hat{e}^a\hat{e}^b$ which makes lowering and raising indices simpler. In this basis, the Taylor expanded metric (4.11) becomes

$$g(\varepsilon) = 2e^+e^- + \delta_{ab}\hat{e}^a\hat{e}^b + \varepsilon\left[r^3F^{(1)}(x)e^+e^+ + 2r^2\hat{h}^{(1)}_a(x)e^+\hat{e}^a + r\hat{\gamma}^{(1)}_{ab}(x)\hat{e}^a\hat{e}^b\right] + \mathcal{O}(\varepsilon^2)\,. \tag{4.12}$$

Since we will be working this basis for the rest of the section, we will drop the "hats". We also use capital Latin letters instead of Greek letters to denote spacetime indices in null orthonormal basis, so there should be no confusion. The triplet of "first order data" $F^{(1)}(x), h^{(1)}_a(x)$ and $\gamma^{(1)}_{ab}(x)$, which are geometric quantities defined purely on H (analogous to near-horizon data) are thus related to $g^{(1)}_{MN}$ by

$$g^{(1)}_{++} = r^3F^{(1)}\,, \qquad g^{(1)}_{+a} = r^2h^{(1)}_a\,, \qquad g^{(1)}_{ab} = r\gamma^{(1)}_{ab}\,, \tag{4.13}$$

and they completely characterise the first order expansion $g^{(1)}$.

Expanding the Lichnerowicz operator and the vector v, the linearised Einstein equation (4.9) reads

$$\begin{aligned} 0 = R^{(1)}_{AB} &= \frac{1}{2}\left(-\nabla^2 g^{(1)}_{AB} + \nabla_A\nabla_C g^{(1)C}{}_B + \nabla_B\nabla_C g^{(1)C}{}_A - \nabla_{(A}\nabla_{B)}\, g^{(0)AB}g^{(1)}_{AB}\right) \\ &\quad -R_A{}^C{}_B{}^D g^{(1)}_{CD} + \frac{1}{2}\left(R_A{}^C g^{(1)}_{BC} + R_B{}^C g^{(1)}_{AC}\right) \end{aligned} \tag{4.14}$$

where all curvature operators are those of g_{NH} and the $^{(0)}$ superscripts are omitted. Notice that the trace $g^{(0)AB}g^{(1)}_{AB}$ actually simplifies to $r\gamma^{(1)} = r\,\gamma^{(0)ab}\gamma^{(1)}_{ab}$.

We are now ready to write down all its components. Using the dual basis and the connection 1-forms given in section 1.6.2, its components can be written explicitly as equations defined entirely on H by separating out the null coordinates $+$ and $-$. We expect 6 equations linear in the first order data: 3 scalar equations from the $++$, $--$ and $+-$ components, 2 vector equations $+a$ and $-a$, and one tensor equation ab; each of them must vanish in order to solve the linearised Einstein equation. Note they are tensorial on H so they hold in coordinate basis too, although here we have explicitly used orthonormal basis to compute them. These 6 components of $R^{(1)}_{AB}$ are:

$$R^{(1)}_{--} = 0 \tag{4.15}$$

$$R^{(1)}_{-a} = h^{(1)}_a - \frac{1}{2}h^{(0)b}\gamma^{(1)}_{ab} + \frac{1}{2}\nabla^b\gamma^{(1)}_{ab} - \frac{1}{2}\nabla_a\gamma^{(1)} + \frac{1}{4}h^{(0)}_a\gamma^{(1)} \tag{4.16}$$

$$\begin{aligned} R^{(1)}_{+-} &= r\Big[3F^{(1)} - 3h^{(0)a}h^{(1)}_a + \nabla^a h^{(1)}_a + h^{(0)a}h^{(0)b}\gamma^{(1)}_{ab} - \frac{1}{2}h^{(0)(a}\nabla^{b)}\gamma^{(1)}_{ab} \\ &\quad -\frac{1}{2}\left(\nabla^{(a}h^{(0)b)}\right)\gamma^{(1)}_{ab} + \frac{1}{2}F^{(0)}\gamma^{(1)} + \frac{1}{4}h^{(0)a}\left(\nabla_a\gamma^{(1)} - h^{(0)}_a\gamma^{(1)}\right)\Big] \end{aligned} \tag{4.17}$$

$$
\begin{aligned}
R^{(1)}_{ab} &= r\Big[-4h^{(0)}_{(a}h^{(1)}_{b)} + 2\nabla_{(a}h^{(1)}_{b)} + F^{(0)}\gamma^{(1)}_{ab} - h^{(0)^2}\gamma^{(1)}_{ab} + 2h^{(0)}_{(a}h^{(0)c}\gamma^{(1)}_{b)c} \\
&+\frac{1}{2}\nabla^c h^{(0)}_c\gamma^{(1)}_{ab} - \left(\nabla^c h^{(0)}_{(a}\right)\gamma^{(1)}_{b)c} + \frac{3}{2}h^{(0)c}\nabla_c\gamma^{(1)}_{ab} - 2h^{(0)c}\nabla_{(a}\gamma^{(1)}_{b)c} \\
&-h^{(0)}_{(a}\nabla^c\gamma^{(1)}_{b)c} - \frac{1}{2}\nabla^2\gamma^{(1)}_{ab} + \nabla_{(a}\nabla^c\gamma^{(1)}_{b)c} - \frac{1}{2}h^{(0)}_a h^{(0)}_b\gamma^{(1)} + \frac{1}{2}\left(\nabla_{(a}h^{(0)}_{b)}\right)\gamma^{(1)} \\
&+h^{(0)}_{(a}\nabla_{b)}\gamma^{(1)} - \frac{1}{2}\nabla_{(a}\nabla_{b)}\gamma^{(1)} + R_{(a}{}^c\gamma^{(1)}_{b)c} - R_a{}^c{}_b{}^d\gamma^{(1)}_{cd}\Big] \qquad (4.18)
\end{aligned}
$$

$$
\begin{aligned}
R^{(1)}_{+a} &= r^2\Big[-3h^{(0)}_a F^{(1)} + \frac{3}{2}\nabla_a F^{(1)} - 2h^{(0)^2}h^{(1)}_a + \frac{1}{2}F^{(0)}h^{(1)}_a + \nabla^b h^{(0)}_b h^{(1)}_a \\
&+\left(\nabla^b h^{(0)}_a\right)h^{(1)}_b + \frac{3}{2}h^{(0)}_a h^{(0)b}h^{(1)}_b + \frac{1}{2}\hat{R}_a{}^b h^{(1)}_b + 2h^{(0)b}\nabla_b h^{(1)}_a \\
&-\frac{3}{2}\nabla_a\left(h^{(0)b}h^{(1)}_b\right) - \frac{1}{2}h^{(0)}_a\nabla^b h^{(1)}_b + \frac{1}{2}\nabla_a\nabla^b h^{(1)}_b - \frac{1}{2}\nabla^2 h^{(1)}_a \\
&-\frac{1}{2}\nabla^b F^{(0)}\gamma^{(1)}_{ab} + \frac{3}{4}F^{(0)}h^{(0)b}\gamma^{(1)}_{ab} + h^{(0)}_c\nabla^{[b}h^{(0)c]}\gamma^{(1)}_{ab} + 3h^{(0)}_b\nabla_{[a}h^{(0)}_{c]}\gamma^{(1)bc} \\
&-\nabla_b\nabla_{[a}h^{(0)}_{c]}\gamma^{(1)bc} - \frac{1}{4}F^{(0)}\nabla^b\gamma^{(1)}_{ab} + \nabla^{[b}h^{(0)c]}\nabla_b\gamma^{(1)}_{ac} + \nabla_{[b}h^{(0)}_{a]}\nabla_c\gamma^{(1)bc} \\
&+\frac{1}{4}\left(\nabla_a F^{(0)}\right)\gamma^{(1)} - \frac{3}{8}F^{(0)}h^{(0)}_a\gamma^{(1)} + \frac{1}{2}h^{(0)b}\nabla_{[b}h^{(0)}_{a]}\gamma^{(1)} \\
&+\frac{1}{4}F^{(0)}\nabla_a\gamma^{(1)} - \frac{1}{2}\nabla_{[b}h^{(0)}_{a]}\nabla^b\gamma^{(1)}\Big] \qquad (4.19)
\end{aligned}
$$

$$
\begin{aligned}
R^{(1)}_{++} &= r^3\Big[F^{(0)}F^{(1)} - \frac{7}{2}h^{(0)^2}F^{(1)} + \frac{3}{2}\nabla^a h^{(0)}_a F^{(1)} + \frac{5}{2}h^{(0)a}\nabla_a F^{(1)} - \frac{1}{2}\nabla^2 F^{(1)} \\
&+\nabla^a F^{(0)}h^{(1)}_a - h^{(0)a}F^{(0)}h^{(1)}_a - 2h^{(0)}_a\nabla^{[a}h^{(0)b]}h^{(1)}_b + 2\nabla^{[a}h^{(0)b]}\nabla_{[a}h^{(1)}_{b]} \\
&+\frac{3}{2}F^{(0)}h^{(0)a}h^{(0)b}\gamma^{(1)}_{ab} - 2h^{(0)(a}\left(\nabla^{b)}F^{(0)}\right)\gamma^{(1)}_{ab} - 2\nabla^{[a}h^{(0)b]}\nabla_{[a}h^{(0)c]}\gamma^{(1)}_{bc} \\
&+\frac{1}{2}\nabla^{(a}F^{(0)}\nabla^{b)}\gamma^{(1)}_{ab} - \frac{1}{2}h^{(0)(a}F^{(0)}\nabla^{b)}\gamma^{(1)}_{ab} + \frac{1}{2}\nabla^a\nabla^b F^{(0)}\gamma^{(1)}_{ab} \\
&-\frac{1}{2}F^{(0)}\nabla^{(a}h^{(0)b)}\gamma^{(1)}_{ab} + \frac{1}{4}\Big(-\nabla^a F^{(0)}\nabla_a\gamma^{(1)} + \nabla^a F^{(0)}h^{(0)}_a\gamma^{(1)} \\
&+h^{(0)a}F^{(0)}\nabla_a\gamma^{(1)} - h^{(0)^2}F^{(0)}\gamma^{(1)}\Big)\Big] \qquad (4.20)
\end{aligned}
$$

Here all indices are lowered, and all curvature operators are defined, with respect to the horizon metric $\gamma^{(0)}_{ab}$. Note that the combination $-\frac{1}{2}\nabla^2\gamma^{(1)}_{ab} + R_{(a}{}^c\gamma^{(1)}_{b)c} - R_a{}^c{}_b{}^d\gamma^{(1)}_{cd} = \Delta^H_L\gamma^{(1)}_{ab}$ in the ab equation (4.18) is the Lichnerowicz operator on H acting on $\gamma^{(1)}_{ab}$ as one would expect: by fixing the coordinates v and r (up to positive scaling) we are dimensionally reducing the problem from M to H.

The $-a$ equation (4.16) gives $h^{(1)}_a$ in terms of $\gamma^{(1)}_{ab}$ and its derivatives, whereas the ab equation (4.18) relate $h^{(1)}_a$ and $\gamma^{(1)}_{ab}$ and their derivatives. Thus we can eliminate $h^{(1)}_a$

to write down an equation purely for $\gamma^{(1)}_{ab}$

$$\begin{aligned} 0 &= \Delta^H_L \gamma^{(1)}_{ab} + \frac{1}{2}\nabla_{(a}\nabla_{b)}\gamma^{(1)} + \frac{3}{2}h^{(0)c}\nabla_c\gamma^{(1)}_{ab} - h^{(0)c}\nabla_{(a}\gamma^{(1)}_{b)c} + h^{(0)}_{(a}\nabla^c\gamma^{(1)}_{b)c} \\ &\quad -\frac{3}{2}h^{(0)}_{(a}\nabla_{b)}\gamma^{(1)} + F^{(0)}\gamma^{(1)}_{ab} - h^{(0)^2}\gamma^{(1)}_{ab} + \frac{1}{2}\nabla^c h^{(0)}_c\gamma^{(1)}_{ab} \\ &\quad + \left(\nabla_{(a}h^{(0)^c}\right)\gamma^{(1)}_{b)c} - \left(\nabla^c h^{(0)}_{(a}\right)\gamma^{(1)}_{b)c} + \frac{1}{2}h^{(0)}_a h^{(0)}_b\gamma^{(1)} \, . \end{aligned} \tag{4.21}$$

This can be simplified further by eliminating $F^{(0)}$ using the near-horizon geometry, which imposes

$$F^{(0)} = \frac{1}{2}h^{(0)a}h^{(0)}_a - \frac{1}{2}\nabla^a h^{(0)}_a + \Lambda \, , \tag{4.22}$$

so (4.21) becomes

$$\boxed{\begin{aligned} 0 = \Delta^H_L \gamma^{(1)}_{ab} &+ \frac{1}{2}\nabla_{(a}\nabla_{b)}\gamma^{(1)} + \frac{3}{2}h^{(0)c}\nabla_c\gamma^{(1)}_{ab} - h^{(0)c}\nabla_{(a}\gamma^{(1)}_{b)c} + h^{(0)}_{(a}\nabla^c\gamma^{(1)}_{b)c} \\ -\frac{3}{2}h^{(0)}_{(a}\nabla_{b)}\gamma^{(1)} &- \frac{1}{2}h^{(0)2}\gamma^{(1)}_{ab} + \frac{1}{2}h^{(0)}_a h^{(0)}_b\gamma^{(1)} + \left(\nabla_{(a}h^{(0)^c}\right)\gamma^{(1)}_{b)c} - \left(\nabla^c h^{(0)}_{(a}\right)\gamma^{(1)}_{b)c} \, . \end{aligned}} \tag{4.23}$$

It is easy to see that by taking the trace of (4.23), the right hand side yields 0 and so the equation is trivially satisfied. This means not all equations given by (4.23) are independent and one is always redundant.

Once we solve for $\gamma^{(1)}_{ab}$ it is straight forward for find the other two first order data: $h^{(1)}_a$ can be computed using the $-a$ equation (4.16)

$$h^{(1)}_a = \frac{1}{2}h^{(0)b}\gamma^{(1)}_{ab} - \frac{1}{2}\hat{\nabla}^b\gamma^{(1)}_{ab} + \frac{1}{2}\hat{\nabla}_a\gamma^{(1)} - \frac{1}{4}h^{(0)}_a\gamma^{(1)} \tag{4.24}$$

and the $+-$ equation (4.17) gives $F^{(1)}$ in terms of $h^{(1)}_a$ and $\gamma^{(1)}_{ab}$

$$\begin{aligned} F^{(1)} &= h^{(0)a}h^{(1)}_a - \tfrac{1}{3}\nabla^a h^{(1)}_a - \tfrac{1}{3}h^{(0)a}h^{(0)b}\gamma^{(1)}_{ab} + \frac{1}{6}h^{(0)(a}\nabla^{b)}\gamma^{(1)}_{ab} \\ &\quad + \frac{1}{6}\left(\nabla^{(a}h^{(0)b)}\right)\gamma^{(1)}_{ab} - \frac{1}{6}F^{(0)}\gamma^{(1)} - \frac{1}{12}h^{(0)a}\left(\nabla_a\gamma^{(1)} - h^{(0)}_a\gamma^{(1)}\right) \, . \end{aligned} \tag{4.25}$$

Hence (4.23) is *the* linearised Einstein equation to solve. Note that because of linearity, if $\gamma^{(1)}_{ab}$ is a solution then any constant multiple $\Omega\gamma^{(1)}_{ab}$ is also a solution, and $h^{(1)}_a \to \Omega h^{(1)}_a$ and $F^{(1)} \to \Omega F^{(1)}$ accordingly from (4.24) and (4.25). However scaling by Ω should not lead to a physically different solution; in fact from the form of the metric (4.11), this scaling freedom is equivalent to the residual gauge freedom in our coordinate system: g_{NH} is invariant under the scaling $r \to \Omega r$ and $v \to v/\Omega$. Thus scaling the solution to the Einstein equation $\gamma^{(1)}_{ab} \to \Omega\gamma^{(1)}_{ab}$ is equivalent to scaling the coordinates $r \to \Omega r$ and

$v \to v/\Omega$. This also means that only scaling by $\Omega > 0$ is allowed; negative scaling is forbidden as it effectively changes the sign of r and v.

The $+a$ and $++$ components (4.19) and (4.20) of the linearised Einstein equation are in fact redundant due to the linearised contracted Bianchi identity (see appendix 4.A for derivation)

$$0 = \nabla^{(0)\mu} R^{(1)}_{\mu\nu} - \frac{1}{2}\nabla^{(0)}_{\nu}(g^{\alpha\beta}_{NH} R^{(1)}_{\alpha\beta}) \,, \tag{4.26}$$

as we demonstrate below. By writing the non-zero components of $R^{(1)}_{\mu\nu}$ as

$$R^{(1)}_{++} = r^3 S_{++} \qquad R^{(1)}_{+a} = r^2 S_{+a} \qquad R^{(1)}_{ab} = rS_{ab} \qquad R^{(1)}_{+-} = rS_{+-} \qquad R^{(1)}_{-a} = S_{-a} \tag{4.27}$$

where all $S_{\mu\nu}$ are independent of r, (4.26) implies, in the order of $\nu = +, -$ and a,

$$\begin{aligned} 3S_{++} &= 3h^{(0)a} S_{+a} - \nabla^a S_{+a} - \frac{1}{2}(\nabla^a F^{(0)} + h^{(0)a} F^{(0)}) S_{-a} - \frac{1}{4} F^{(0)} \gamma^{ab} S_{ab} &(4.28)\\ S^a{}_a &= 2\nabla^a S_{-a} - 4h^{(0)a} S_{-a} &(4.29)\\ S_{+a} &= \frac{1}{2} F^{(0)} S_{-a} + \nabla_{[a} h^{(0)}_{b]} S_-{}^b - \frac{1}{2} h^{(0)}_a S_{+-} + \frac{1}{2}\nabla_a S_{+-} - \frac{1}{2} h^{(0)}_a S_{+-} \\ &\quad + h^{(0)b} S_{ab} - \frac{1}{2}\nabla^b S_{ab} + \frac{1}{4}\nabla_a(\gamma^{bc} S_{bc}) - \frac{1}{4} h^{(0)}_a \gamma^{bc} S_{bc} \,. &(4.30) \end{aligned}$$

We have again dropped the $^{(0)}$ superscript on ∇ and all indices are raised and lowed with γ_{NH}. (4.28) and (4.30) thus state that the $R^{(1)}_{+a}$ and $R^{(1)}_{++}$ equations can be expressed entirely in terms of the other three non-trivial components of the linearised Einstein equation; this can also be verified directly using (4.16)-(4.20), so we need not consider them. (4.29) on the other hand explains the origin of the redundancy we pointed out earlier in the linearised Einstein equation (4.23).

Our task is to examine the solution space of $g^{(1)}_{\mu\nu}$ (or in practice $\gamma^{(1)}_{ab}$) for some given near-horizon geometry under certain symmetry assumptions (e.g. axisymmetry), subjected to smoothness and other restrictions from global arguments on H. In particular, we want to investigate whether there are solutions which do not correspond to any known *black hole* solutions that possess the given near-horizon limit.

4.3.2 Gauge freedom

Nevertheless, any first order expansion $g^{(1)}_{\mu\nu}$ could only be determined up to a gauge: any two spacetimes which are related by diffeomorphism are physically the same. Since we may treat $g^{(1)}$ as a perturbation about g_{NH}, we can follow the same procedure as described in [136] (Chapter 7.5) to work out this gauge freedom. Consider our one-parameter family of spacetimes $(M, g_{\mu\nu}(\varepsilon))$ and let ϕ_ε be a one-parameter group of diffeomorphisms generated by an arbitrary vector field V. Denoting the push-forward map by ϕ^*_ε, then $(M, g_{\mu\nu}(\varepsilon))$ and $(M, \phi^*_\varepsilon g_{\mu\nu}(\varepsilon))$ are physically the same one-parameter family. Therefore when we take the first order expansion,

$$\tilde{g}^{(1)}_{\mu\nu} = \left.\frac{\mathrm{d}\phi^*_\varepsilon g_{\mu\nu}(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \tag{4.31}$$

is physically equivalent to $g^{(1)}_{\mu\nu}$, hence

$$g^{(1)}_{\mu\nu} \to \tilde{g}^{(1)}_{\mu\nu} = g^{(1)}_{\mu\nu} - \mathcal{L}_V g^{(0)}_{\mu\nu} = g^{(1)}_{\mu\nu} - \nabla^{(0)}_\mu V_\nu - \nabla^{(0)}_\nu V_\mu \tag{4.32}$$

where $\nabla^{(0)}$ is the covariant derivative of g_{NH}, is a gauge transformation that leaves the physical spacetime invariant. We can now compute explicitly this gauge transformation for the first order data. Since the coordinates v and r are fixed, up to the residual scaling $r \to \Omega r$ and $v \to v/\Omega$ for any $\Omega > 0$, the form of the metric (4.11) should be preserved under the gauge transformation i.e. $\nabla^{(0)}_{(+}V_{-)} = \nabla^{(0)}_{-}V_{-} = \nabla^{(0)}_{(-}V_{a)} = 0$, with $V_A = V_A(r,x)$ because ∂_v should remain Killing.

Observe that $\nabla^{(0)}_{-}V_{-} = 0$ simply gives $V_- = V_-(x)$, plugging this into the other two constraints yields

$$V_a = r\tilde{V}_a + \mathcal{O}(r^2) \qquad V_+ = \frac{1}{2}r^2\left(F^{(0)}V_- + h^{(0)a}\tilde{V}_a\right) + \mathcal{O}(r^3)\,, \quad \text{with} \quad \tilde{V}_a = -\nabla_a V_- \tag{4.33}$$

where ∇_a denotes the covariant derivative of $\gamma^{(0)}_{ab}$, therefore the gauge transformation can be expressed in terms of a *single* function $f(x) = 2V_-$ defined on H. A straight forward calculation then reveals

$$\gamma^{(1)}_{ab} \;\to\; \gamma^{(1)}_{ab} + \nabla_a\nabla_b f - h^{(0)}_{(a}\nabla_{b)}f \tag{4.34}$$

$$h^{(1)}_a \;\to\; h^{(1)}_a - \frac{1}{2}F^{(0)}\nabla_a f - \frac{1}{4}\nabla_a h^{(0)}_b\nabla^b f - \frac{1}{4}h^{(0)}_a h^{(0)}_b \nabla^b f + \frac{1}{2}\nabla_b h^{(0)}_a \nabla^b f + \frac{1}{4}h^{(0)}_b\nabla_a\nabla^b f \tag{4.35}$$

$$F^{(1)} \;\to\; F^{(1)} + \frac{1}{2}\nabla^a f\left(\nabla_a F^{(0)} - h^{(0)}_a F^{(0)}\right) \tag{4.36}$$

are the transformation rules for the first order data. Because the new $g^{(1)}_{\mu\nu}$ after the gauge transformation are still metric functions, they must remain regular and so the function $f(x)$ must also be regular. It can be checked explicitly that under the above transformations the linearised Einstein equations (4.23) to (4.25) remain invariant.

One may naively expect $\gamma^{(1)}_{ab}$ to transform as the general diffeomorphism on H, which would be represented by $\gamma^{(1)}_{ab} \to \gamma^{(1)}_{ab} + \nabla_{(a}W_{b)}$ for any arbitrary vector field W on H. However as shown in the derivation above, (4.34) turns out to differ from expectation as a consequence of preserving the form (4.11) of the spacetime coordinate system[2]. The

[2] Of course on H we still have the general diffeomorphism $\gamma^{(0)}_{ab} \to \gamma^{(0)}_{ab} + \nabla_{(a}W_{b)}$.

gauge transformation described by (4.34) - (4.36) is also independent of the freedom to scale the first order data by Ω, which as we argued above is effectively the residual gauge freedom in scaling v and r and is generated by the vector field $X = v\partial_v - r\partial_r$, although both transformations preserve the form of the metric (4.11) and leave g_{NH} invariant. In order to distinguish these two "gauge freedoms", we will use the term "gauge freedom" to refer to the coordinate freedom generated by $f(x)$ i.e. (4.34) - (4.36)), and "scaling freedom" for scaling by Ω.

Nevertheless this gauge freedom represented by $f(x)$ is nothing but an artefact of the first order expansion. Once the cross-section H and the coordinates on H are fixed, Gaussian Null coordinates for the full spacetime metric is uniquely defined. Demanding the spacetime metric to be in GNC only to first order thus leaves a huge redundancy in the coordinate chart, which is manifested by $f(x)$. Since we simply cannot know which $f(x)$ actually leads to the full GNC, we should solve (4.23) in terms of gauge invariant variables - quantities which are independent of $f(x)$. Gauge invariant quantities can be constructed using (4.34) as linear combinations of components of $\gamma^{(1)}_{ab}$ and their derivatives. On the other hand, it makes no sense to demand invariance with respect to scaling by Ω because one can always scale the whole (4.23) by any constant without changing anything, because (4.23) is linear. Obviously the choice of gauge invariant variables is not unique, but the resulting equations in terms of the gauge invariant variables remain linear. However there is no easy way to write them down in general, so we will construct them separately case by case.

Let us close this section by counting the degrees of freedom in the linearised Einstein equation (4.23). H is co-dimension 2 so $\dim(H) = d = D - 2$ and $\gamma^{(1)}_{ab}$ has $d(d+1)/2$ independent components. However, due to gauge freedom there are really only $d(d+1)/2 - 1$ independent gauge invariant variables to determine $\gamma^{(1)}_{ab}$. On the other hand (4.23) has $d(d+1)/2$ component equations but due to traceless-ness, there are only $d(d+1)/2 - 1$ independent equations, thus we have the same number of equations as variables to solve. Upon gauge fixing the trace $\gamma^{(1)} = 0$ using (4.34), the linearised Einstein equation (4.23) becomes *elliptic*. Since now it is an elliptic equation on H which is compact, we can apply the Fredholm theorem to arrive at the following proposition:

Proposition 4.1. *Consider the first order expansion from a near-horizon geometry with a compact horizon. The moduli space of solutions to the linearised Einstein equation* (4.23) *is finite dimensional up to the gauge transformation* (4.34).

In other words, (4.23) admits a family of solutions with a finite number of parameters, although in general the equations we need to solved are still a complicated system of coupled second order linear partial differential equations. However if we assume enough symmetries (e.g. rotational symmetries) as we do in the cases we consider below, they reduce to ordinary differential equations which are easier to solve.

4.4 Stable marginally outer trapped surfaces

So far in our discussion we have only focussed on Killing horizons - the rigidity theorem asserts that the event horizon of a stationary black hole is a Killing horizon. Another important feature of the event horizon which plays a crucial role in black hole topology theorem is that cross-sections of the event horizon of a stationary black hole are (outermost) *marginally outer trapped surfaces* (MOTSs). In this section we investigate the constraints this imposes on the first order expansions.

We proceed by giving a brief review on the definition of MOTSs; we follow closely the construction given in [54]. Starting with a spacetime (M, g) (with $\dim(M) \geq 4$), let S be a spacelike hypersurface in M with unit future directed timelike normal u. Define Σ to be be a *compact* hypersurface in S which naturally separates S into "inside" and "outside" and let v be the outward unit normal on Σ. We can therefore define a pair of *null* vector fields $\ell_\pm = u \pm v$ to be future directed outward (ℓ_+) and inward (ℓ_-) pointing normal vector fields on Σ; note that $\ell_\pm$ are only unique up to boost (positive rescaling). Thus $\ell_\pm$ are tangent to a pair of orthogonal, future directed null geodesics emerging from Σ on Σ. The null expansion scalars $\theta_\pm$ of the surface Σ are then defined as the divergence of the out($+$) and in($-$) going light rays coming from Σ

$$\theta_\pm = \nabla_\Sigma \cdot \ell_\pm \; . \tag{4.37}$$

The sign of $\theta_\pm$ is invariant under boost of $\ell_\pm$.

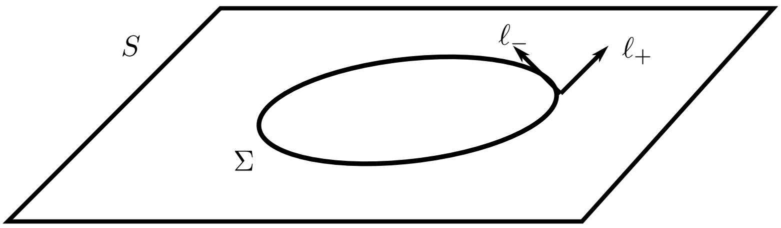


Figure 4.1: A marginally outer trapped surface.

Intuitively on an spatial S^2 slice of Minkowski spacetime, one has $\theta_- < 0$ and $\theta_+ > 0$ based on the definition of in and out on S^2. However inside a *non-extreme* black hole, one finds both $\theta_- < 0$ and $\theta_+ < 0$ because there is no way "out", so Σ is called an *outer trapped surface*; the boundary case where $\theta_+ = 0$ identically is called a *marginally outer trapped surface (MOTS).*

Since we want to investigate *black hole* solutions, we demand H to be a MOTS. For reasons which will become clear later, in this section we shall work in GNC in *gen-*

eral gauge[3]. This slightly modified GNC is related to (4.1) by the coordinate change $r \to \Gamma(x)r$, where $\Gamma(x)$ is a positive function on H. By changing also $F \to F/\Gamma^2$ and $h_a \to (h_a - \nabla_a\Gamma)/\Gamma$, the metric (4.1) becomes

$$\mathrm{d}s^2 = 2\mathrm{d}v\left(\frac{1}{2}r^2F(r,x)\mathrm{d}v + \Gamma(x)\mathrm{d}r + rh_a(r,x)\mathrm{d}x^a\right) + \gamma_{ab}(r,x)\mathrm{d}x^a\mathrm{d}x^b\,. \tag{4.38}$$

All indices in this section are lowered and raised by this metric and its inverse, which can be written explicitly in matrix form as

$$g_{\mu\nu} = \begin{pmatrix} r^2F & \Gamma & rh_a \\ \Gamma & 0 & 0 \\ rh_a & 0 & \gamma_{ab} \end{pmatrix}_{\mu\nu} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{\Gamma} & 0 \\ \frac{1}{\Gamma} & \frac{r^2}{\Gamma^2}(-F+h^2) & -\frac{rh^a}{\Gamma} \\ 0 & -\frac{rh^a}{\Gamma} & \gamma^{ab} \end{pmatrix}^{\mu\nu} . \tag{4.39}$$

It is straightforward to see that on H the null vector fields $\ell_+ = \partial_v|_{r=0}$ and $\ell_- = -\partial_r|_{r=0}$. Denoting surfaces of constant v and r by $\mathscr{S}_{v,r}$ (so $H = \mathscr{S}_{0,0}$), let k be the unique null future directed outward pointing vector field normal to $\mathscr{S}_{v,r}$ which coincides with ℓ_+ on the horizon (i.e. $k|_{r=0} = \ell_+$). By construction $k^v = 1$ and $\ell = \partial_r$ is a past directed outward pointing null vector field orthogonal to both k and $\mathscr{S}_{v,r}$. Let $X = X^a\partial_a$ be a vector field tangent to $\mathscr{S}_{v,r}$, we thus have $k \cdot X = 0$ which implies $k^a = -rh^a$. Since k is null we also have $k \cdot k = 0$, which gives $k^r = \frac{r^2}{2\Gamma}\left(h^2 - F\right)$ with $h^2 = h_ah_b\gamma^{ab}$. So in summary,

$$k = \partial_v + \frac{r^2}{2\Gamma}\left(h^2 - F\right)\partial_r - rh^a\partial_a\,. \tag{4.40}$$

With the explicit form of k and ℓ we can find the induced metric (projection tensor) $q_{\mu\nu} = g_{\mu\nu} - A_\mu A_\nu/A\cdot A - B_\mu B_\nu/B\cdot B$ on $\mathscr{S}_{v,r}$ via the orthogonal normals $A = (k+\ell)/\sqrt{2}$ and $B = (k-\ell)/\sqrt{2}$. The fact that $A \cdot A = \Gamma$, $B \cdot B = -\Gamma$ and $A \cdot B = 0$ means it simplifies to

$$q_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu\ell_\nu + k_\nu\ell_\mu}{\Gamma}\,, \tag{4.41}$$

where as one-forms $k_\mu = \begin{pmatrix}\frac{r^2}{2}(F-h^2) & \Gamma & 0\end{pmatrix}_\mu$ and $\ell_\mu = (1 \quad 0 \quad 0)_\mu$. The resulting induced metric and inverse can be expressed in matrix form as

$$q_{\mu\nu} = \begin{pmatrix} r^2h^2 & 0 & rh_a \\ 0 & 0 & 0 \\ rh_a & 0 & \gamma_{ab} \end{pmatrix}_{\mu\nu} \qquad q^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma^{ab} \end{pmatrix}^{\mu\nu} . \tag{4.42}$$

[3] This diffeomorphism corresponding to the freedom in rescaling the affine parameter $r \to \Gamma(x)r$ *changes* the form of the metric (4.11) and thus should not be confused with the diffeomorphism (associated with another function $f(x)$) in (4.34) - (4.36) in section 4.3.2 which preserves the form of (4.11). For any choice of $\Gamma(x)$ there is still a residual freedom in scaling $r \to \Omega r$ and $v \to v/\Omega$ for any $\Omega > 0$, which has also been discussed in section 4.3.2. Thus in this section there are *three* different notions of "gauge freedom" associated with $\Gamma(x)$, $f(x)$ and Ω.

The null geodesic expansion of ℓ_i on $\mathscr{S}_{v,r}$ is then defined by $\mathcal{L}_i \epsilon_q = \theta_i \epsilon_q$ where ϵ_q is the volume element of the induced metric $q_{\mu\nu}$ on $\mathscr{S}_{v,r}$. Hence by denoting q as the determinant of $q_{\mu\nu}$,

$$\theta_i = \frac{\mathcal{L}_{\ell_i}\sqrt{q}}{\sqrt{q}} \,, \tag{4.43}$$

which gives

$$\theta_\ell = \frac{1}{2}\gamma^{ab}\partial_r\gamma_{ab} \tag{4.44}$$

$$\theta_k = \frac{r^2}{4\Gamma}(h^2 - F)\gamma^{ab}\partial_r\gamma_{ab} - r\nabla_a h^a \tag{4.45}$$

where ∇_a is the covariant derivative with respect to γ_{ab}. Clearly on H, the "marginal" criterion $\theta_+ = \theta_k|_{r=0} = 0$ for MOTS is always satisfied; however from (4.44) is not obvious if the other MOTS condition $\theta_- < 0$, which ensures that the ingoing light rays are indeed going "in" and converging, is also satisfied. Recall that whereas ℓ_- is defined to be future directed and inward pointing, ℓ is past directed and outward pointing. Thus $\theta_- < 0$ is equivalent to $\theta_\ell|_{r=0} > 0$, so that *all* light rays with tangent ℓ diverges on H. However we shall impose the weaker condition

$$\int_H \theta_\ell|_{r=0} \; \epsilon_H > 0 \,, \tag{4.46}$$

such that at least *some* light diverges and escapes on H; obviously, the stronger condition $\theta_\ell|_{r=0} > 0$ implies (4.46).

In fact (4.46) is the more natural notion of "in" in terms of decreasing cross-sectional area. This is especially true when the horizon contains a concave part. The change of the area $\mathcal{A}_{v,r}$ of $\mathscr{S}_{v,r}$ along the null vector fields k and ℓ are related to the expansion scalars θ_i by

$$\mathcal{L}_\ell \mathcal{A}_{v,r} = \int_{\mathscr{S}_{v,r}} \theta_\ell \; \epsilon_q \qquad\qquad \mathcal{L}_k \mathcal{A}_{v,r} = \int_{\mathscr{S}_{v,r}} \theta_k \; \epsilon_q \,. \tag{4.47}$$

Hence (4.46) is equivalent to saying that the area of the cross section $\mathscr{S}_{v,r}$ is *decreasing* along the future directed null vector field $-\ell$ on H, which means that that null geodesics with tangent $-\ell$ are indeed going *in* and overall converging on H. Note that the latter in (4.47) is always trivial with the area remaining constant along k on H as k is Killing on H. As $\theta_\ell|_{r=0} = \frac{1}{2}\gamma^{(0)ab}\gamma^{(1)}_{ab} = \frac{1}{2}\gamma^{(1)}$, demanding H to be a MOTS thus gives rise to the following constraint on the first order quantity $\gamma^{(1)}$:

$$\boxed{\int_H \gamma^{(1)} \; \epsilon_H > 0 \,.} \tag{4.48}$$

This is also the appropriate constraint to impose on our first order expansion because it is *gauge invariant* with respect to $f(x)$. Of course the sign of $\theta_\ell|_{r=0}$, and hence the existence of MOTS are invariant under general diffeomorphism, but recall that we can only determine $\gamma^{(1)}$ up to a function $f(x)$ since our metric is in GNC only to first order. To see that (4.48) is gauge invariant, we take advantage of the fact that in the new coordinate system (4.38), there is still a freedom in choosing the positive scaling function $\Gamma(x)$. In fact there always exists a choice for Γ where $\int_H \gamma^{(1)}\epsilon_q$ is a gauge invariant quantity. A little calculation shows that in the new GNC (4.38) with any Γ, the analogue of (4.34) is

$$\gamma_{ab}^{(1)} \to \gamma_{ab}^{(1)} + \Gamma\nabla^{(0)}_{(a}\nabla^{(0)}_{b)} f - h^{(0)}_{(a}\nabla^{(0)}_{b)} f + \nabla^{(0)}_{(a}\Gamma\nabla^{(0)}_{b)} f \tag{4.49}$$

where $\nabla^{(0)}_a$ is the covariant derivative of $\gamma^{(0)}_{ab} = \gamma_{ab}|_{r=0}$, which can also be written as

$$\gamma_{ab}^{(1)} \to \gamma_{ab}^{(1)} + \nabla^{(0)}_{(a}\left(\Gamma\nabla^{(0)}_{b)} f - h^{(0)}_{b)} f\right) + f\nabla^{(0)}_{(a} h^{(0)}_{b)} \ , \tag{4.50}$$

so that the trace transforms as

$$\gamma^{(1)} \to \gamma^{(1)} + \nabla^{(0)}\cdot\left(\Gamma\nabla^{(0)} f - h^{(0)} f\right) + f\nabla^{(0)}\cdot h^{(0)} \ . \tag{4.51}$$

It was shown in [99] (lemma 0) that there always exists a choice for Γ such that the last term vanishes with $\nabla_a h^a|_{r=0} = \nabla^{(0)}\cdot h^{(0)} = 0$. Hence with this choice of Γ, $\gamma^{(1)}$ only transforms by a total divergence which vanishes when we integrate over the closed hypersurface H, and we prove our claim.

Nevertheless even after fixing a Γ, the quantity $\int_H \gamma^{(1)}\epsilon_H$ is not invariant under the residual scaling of the coordinates v and r by Ω: while this scaling leaves g_{NH} unchanged, in first order it can also be interpreted as scaling the solution by an overall factor $\gamma^{(1)}_{ab} \to \Omega\gamma^{(1)}_{ab}$ because of linearity of (4.23), so $\int_H \gamma^{(1)}\epsilon_H \to \Omega\int_H \gamma^{(1)}\epsilon_H$. This is not a problem, in fact this means $\int_H \gamma^{(1)}\epsilon_H$ provides also a measure of the scale of r.

H is a *strictly stable outermost* MOTS if $\mathcal{L}_-\theta_+ \leq 0$ with $\mathcal{L}_-\theta_+ < 0$ somewhere on H (or up to some Ω rescaling of the null fields $\ell_\pm$), and *marginally* stable if $\mathcal{L}_-\theta_+ = 0$. Applying the same argument as above, in terms of the vector field ℓ the stability condition is then

$$\int_H \mathcal{L}_\ell\theta_k|_{r=0}\,\epsilon_H \geq 0 \ , \tag{4.52}$$

where we integrate over H to examine the *overall* rate of change of θ_k on H. This is a weaker condition than the conventional stability bound, which in terms of ℓ reads $\mathcal{L}_\ell\theta_k|_{r=0} \geq 0$. In the Gaussian null coordinates with the choice of Γ where $\int_H \gamma^{(1)}\epsilon_H$ is a gauge invariant quantity, the first derivative on the horizon

$$\mathcal{L}_\ell\theta_k|_{r=0} = -\,\nabla_a h^a|_{r=0} \tag{4.53}$$

simply vanishes, so in fact the *strong* stability bound is always saturated and H is always *marginally* stable if it is a MOTS. This is in agreement with the result found in [102], where it was shown that extremality implies marginal stability and vice versa.

But logically one would expect θ_k to be increasing so that the light rays with tangent k is diverging *in some way* along the ℓ direction *just* outside the horizon, so let us consider the second derivative, which gives

$$\mathcal{L}_\ell \mathcal{L}_\ell \theta_k|_{r=0} = -\frac{1}{2} A\gamma^{(1)} - 2\nabla_a^{(0)} (h^{(1)a} + h^{(0)a}\gamma^{(1)}) \tag{4.54}$$

where $A = (F^{(0)} - h^{(0)2})/\Gamma$ is a quantity that depends only on the near-horizon geometry. The second term in (4.54) is a total divergence so it vanishes upon integrating. Thus according to this argument, one would expect by integrating over H

$$\int_H \mathcal{L}_\ell \mathcal{L}_\ell \theta_k \epsilon_H = -\frac{1}{2} \int_H A\gamma^{(1)} \epsilon_H \geq 0 \,. \tag{4.55}$$

Equality in (4.55) would imply that higher derivatives need to be taken in order to see that θ_k is *somehow* increasing along ℓ.

In terms of the quantity A, the near-horizon limit of the spacetime metric (4.38) in general Γ gauge can be written as

$$\mathrm{d}s^2_{NH} = \Gamma(x)(A(x)r^2 \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r) + \gamma_{ab}(x)\,(\mathrm{d}x^a + rh^a(x)\mathrm{d}v)(\,\mathrm{d}x^b + rh^b(x)\mathrm{d}v)\;. \tag{4.56}$$

As we already mentioned in chapter 2, the near-horizon geometries for all known extreme black holes contain an AdS_2 factor. This $SO(2,1)$ near-horizon symmetry enhancement has been *proved* in different dimensions under various assumptions of matter fields and rotational symmetries, as well as the strong energy condition. Their NHGs all take the form (4.56) with $A = A_0$ being a negative constant and h^a Killing. So for any NHG that falls under the $SO(2,1)$ symmetry enhancement, the condition (4.55) simplifies to

$$\frac{|A|}{2} \int_H \gamma^{(1)} \epsilon_H \geq 0 \,. \tag{4.57}$$

Therefore if H is a MOTS so that (4.48) holds, we are guaranteed to have

$$\frac{|A|}{2} \int_H \gamma^{(1)} \epsilon_H > 0. \tag{4.58}$$

Since the first derivative vanishes and the second derivative is positive with respect to ℓ, the quantity $\int_{\mathscr{S}_{v,r}} \theta_k\ \epsilon_q = \mathcal{L}_k \mathcal{A}_{v,r}$ which measures the overall expansion of the null geodesics tangent to k, has a *local minimum* on H and so light rays are overall *expanding* as we go along *both* $\pm\ell$ (in and out) directions from the horizon. This shows that there are *no* trapped surfaces inside or outside an extreme horizon, and light rays only ever become *marginally* trapped on the event horizon. This is another peculiar property of extreme black holes not exhibited in their non-extreme counterparts. In fact trapped surfaces only exists in the region between the event horizon and the inner Cauchy horizon of a non-extreme black hole, where the two horizons are themselves foliated by MOTSs; for extreme black holes where the two horizons coincide, only MOTSs are present without any trapped surfaces. This also Israel's *definition* of extremality in [81].

4.5 First order expansion to flat T^2 near-horizon geometry

We begin our first order analysis with a very simple case: consider the first order expansion from the near-horizon geometry in $4D$ which is just the direct product of the $2D$ Minkowski space and a flat square T^2

$$\mathrm{d}s^2_{NH} = 2\mathrm{d}v\mathrm{d}r + \mathrm{d}x^2 + \mathrm{d}y^2 \tag{4.59}$$

so that x and y are periodic. It was shown in [26] this is the unique static NHG in $4D$ Einstein vacuum with $\Lambda = 0$. However it does not qualify as a *black hole horizon* according to horizon topology theorem [28]; this is another example where a NHG does not correspond to any black hole even though it contains a Killing horizon by construction. Nevertheless, this exercise could help us clarify a few concepts.

In this case we have $h^{(0)}_a = F^{(0)} = R^{(0)}_{ab} = 0$, so the $\gamma^{(1)}_{ab}$ equation (4.23) is simply

$$0 = -\frac{1}{2}\nabla^2\gamma_{ab} + \frac{1}{2}\nabla_{(a}\nabla_{b)}\gamma \ . \tag{4.60}$$

It should be clear that here the notation $\gamma_{ab} = \gamma^{(1)}_{ab}$ because $\gamma^{(0)}_{ab}$ is just the flat metric in $2D$ in Cartesian coordinates, and $\nabla_a = \partial_a$ is the covariant derivative associated to the flat metric of T^2. γ_{ab} must also be periodic in both x and y in respect of the T^2 topology. The gauge transformation rule (4.34) states that

$$\begin{aligned} \gamma_{xx} &\to \gamma_{xx} + \partial_x^2 f \\ \gamma_{xy} &\to \gamma_{xy} + \partial_x\partial_y f \\ \gamma_{yy} &\to \gamma_{yy} + \partial_y^2 f \end{aligned} \tag{4.61}$$

leave (4.23) invariant. Let us define the gauge invariant quantities

$$X = \partial_y\gamma_{xx} - \partial_x\gamma_{xy} \qquad Y = \partial_x\gamma_{yy} - \partial_y\gamma_{xy} \ , \tag{4.62}$$

so the xx (or yy, they are the same) and the xy components of (4.23) give

$$\partial_y X = \partial_x Y \qquad \partial_x X = -\partial_y Y \ , \tag{4.63}$$

which we recognise are just the Cauchy-Riemann equations, so on the complex plane with $z = x + iy$ the function $F(z) = Y(x, y) + iX(x, y)$ is holomorphic. Differentiating again of course yields both X and Y are harmonic functions $(\partial_x^2 + \partial_y^2)X = (\partial_x^2 + \partial_y^2)Y = 0$. Recalling that on T^2 all harmonic functions are constant, X and Y are therefore constants. Using the definition of X and Y, we get

$$\gamma_{xx} = \int (\partial_x \gamma_{xy} + X)\, \mathrm{d}y + f_1(x) \tag{4.64}$$

$$\gamma_{yy} = \int (\partial_y \gamma_{xy} + Y)\, \mathrm{d}x + f_2(y) \tag{4.65}$$

where f_i are some arbitrary periodic functions. Hence if X and Y are non-zero, there will be terms linear in y and x in γ_{xx} and γ_{yy} respectively. This contradicts with the global constraint that γ_{ab} must be periodic functions of both x and y, so we conclude that $X = Y = 0$ is the only solution to the first order expansion.

Thus the general solution in terms of the metric functions γ_{ab} is

$$\gamma_{xx} = \int \partial_x \gamma_{xy}(x, y)\ \mathrm{d}y + f_1(x) \tag{4.66}$$

$$\gamma_{yy} = \int \partial_y \gamma_{xy}(x, y)\ \mathrm{d}x + f_2(y) \tag{4.67}$$

with $\gamma_{xy}(x, y)$ being an arbitrary periodic function due to the gauge freedom. In fact the answer can be simplified by picking some particular gauge; individual components of γ_{ab} are not gauge invariant quantities themselves. Let us set the gauge function

$$f(x, y) = -\int \tilde{\gamma}_{xy}\ \mathrm{d}x\mathrm{d}y - \int \left(\int \tilde{f}_1(x)\ \mathrm{d}x \right) \mathrm{d}x - \int \left(\int \tilde{f}_2(y)\ \mathrm{d}y \right) \mathrm{d}y\ , \tag{4.68}$$

where the tilde in $\tilde{X}$ denotes the higher Fourier modes of the function X, so that we can write in Fourier series $X(x, y) = X_0 + \tilde{X}(x, y)$ with $\int_{T^2} \tilde{X} \mathrm{d}x\mathrm{d}y = 0$ and X_0 is a constant (the 0th Fourier mode). The 0th modes do not enter the integrals because f being globally defined on H, must respect its periodicity, but integrating over any constant would spoil it. Then according to (4.61) it is obvious that in this gauge all oscillation parts in all three γ_{ab} components drop out, therefore the solution can be characterised by just three constants $\gamma_{xx} = a$, $\gamma_{xy} = b$ and $\gamma_{yy} = c$.

The *plane wave solution* [122] can be written in GNC as

$$\mathrm{d}s^2 = 2\mathrm{d}v\mathrm{d}r + g_{ab}(r)\mathrm{d}x^a\mathrm{d}x^b \tag{4.69}$$

where Latin indices take the value of either x or y, with $g_{ab}(r)$ satisfying

$$2g^{ab}\ddot{g}_{ab} + \dot{g}^{ab}\dot{g}_{ab} = 0\ . \tag{4.70}$$

By compactifying and scaling x and y setting $r = 0$, it clearly admits the flat T^2 NHG we considered at the beginning. Because $g_{ab}(r)$ is independent of x or y, its Taylor expansion about $r = 0$ is simply $g_{ab}(r) = \sum_{n=0}^{\infty} g_{ab}^{(n)} r^n$ where $g_{ab}^{(n)}$ are constant $\forall n$ subjected to the constraint (4.70). Thus our general solution clearly includes the first order expansion of plane wave. But since the equation (4.70) is second order (and *very underdetermined*), there always exists a power series solution of the form $g_{ab}(r) = \sum_{n=0}^{\infty} g_{ab}^{(n)} r^n$ with any choice of the constants $g_{ab}^{(0)}$ and $g_{ab}^{(1)}$, therefore our general solution *is* the plane wave solution.

The MOTS condition (4.48) states that for the general solution to describe a *black hole* with H being a MOTS, it must satisfies

$$\int_H \gamma^{(1)} \epsilon_H = 4\pi^2 (a + c) > 0 \quad \Rightarrow \quad a + c > 0 \, . \tag{4.71}$$

However we know that the plane waves are not black hole solutions, so the MOTS condition is only *necessary* but still *not sufficient* for a *black hole* solution. After all the event horizon of a black hole is a *globally* defined notion, therefore expansion to higher orders (or perhaps *all orders*) is probably needed to sufficiently describe a black hole. Anyway the MOTS condition (4.48) is consistent with the fact that the linearly polarised plane gravitational wave [122]

$$\mathrm{d}s^2 = 2\mathrm{d}v\mathrm{d}r + (1 - \sin\omega r)\mathrm{d}x^2 + (1 + \sin\omega r)\mathrm{d}y^2 \, , \tag{4.72}$$

which is a special case of (4.69), is *not* a black hole since $\int_H \gamma^{(1)} \epsilon_H = 0$ in this case.

4.6 First order expansion to extreme Kerr near-horizon geometry

It has been shown that any non-static near-horizon geometry invariant under an $\mathbb{R} \times U(1)$ isometry in $4D$ Einstein vacuum with $\Lambda = 0$ and a compact cross section must be the near-horizon limit of the extreme Kerr black hole [80, 94]. This result has again made use of topological censorship [28] to exclude T^2 horizons from black hole solutions. In this section we investigate whether we recover *only* the extreme Kerr black hole itself in first order expansion. Note that this is not guaranteed because the proof of Kerr uniqueness involves the *full* spacetime which we have no access to. In particular, it assumes asymptotic flatness as well as a number of global regularity conditions, including completeness of the stationary Killing filed and global hyperbolicity of the domain of outer communications.

The extreme Kerr NHG can be parametrised by a single constant a (the mass / angular momentum parameter). In GNC, it can be written as

$$\begin{aligned} ds^2_{NH} &= \frac{3-6x^2-x^4}{a^2(1+x^2)^3}r^2 dv^2 + 2dvdr + 2rdv\left(-\frac{2x}{1+x^2}dx + \frac{4(1-x^2)}{(1+x^2)^2}d\phi\right) \\ &\quad + a^2\frac{1+x^2}{1-x^2}dx^2 + 4a^2\frac{1-x^2}{1+x^2}d\phi^2 \end{aligned} \tag{4.73}$$

with x and ϕ being coordinates on the S^2, $\phi \sim \phi + 2\pi$ is periodic and $x = (-1, 1)$. The fixed points of the rotational Killing field ∂_ϕ coincide with the poles of the S^2 at $x = \pm 1$ where x is not a well defined coordinate, but we can explicitly change to polar coordinates in the neighbourhood of the poles to show that everything is regular there and we do have a smooth topologically S^2 horizon. Thus the range of x can be extended to $-1 \le x \le 1$ in the following analysis.

We focus on the the case where axisymmetry is preserved outside the horizon, thus ∂_ϕ is still a Killing field and the first order data depends only on x. We need to solve (4.23) which we may recall actually only gives two independent equations

$$\begin{aligned} 0 &= -4x^2(1+x^2)(x^2-3)^2\gamma_{\phi\phi} + (1-x^2)\left\{16x^3(x^2-1)\gamma_{x\phi} + 6x(x^6-3x^2-2)\gamma'_{\phi\phi}\right. \\ &\quad \left.(x^2-1)\left[8(x^4-1)\gamma'_{x\phi} - 8x(x^2-1)\gamma'_{xx} + (1+x^2)^3\gamma''_{\phi\phi}\right]\right\} \end{aligned} \tag{4.74}$$

$$\begin{aligned} 0 &= 8x(x^4-2x^2-3)\gamma_{\phi\phi} + 8(-x^6+3x^4+x^2+1)\gamma_{x\phi} + (x^2-1)\left\{5(1+x^2)^2\gamma'_{\phi\phi}\right. \\ &\quad \left.+2x(-6x^4-4x^2+2)\gamma'_{x\phi} - 2(x^2-1)\left[2(1-x^2)\gamma'_{xx} + (1+x^2)^2\gamma''_{x\phi}\right]\right\} . \end{aligned} \tag{4.75}$$

where the prime denotes derivative with respect to x and we have dropped the superscript $^{(1)}$ on $\gamma^{(1)}_{ab}$. For the rest of this section we shall use the notation $\gamma_{ab} = \gamma^{(1)}_{ab}$, although in some places the superscript $^{(1)}$ will be put back for clarity. Next we want to write them in terms of two gauge invariant variables. Under the gauge transformation rule (4.34), the three components of $\gamma^{(1)}_{ab}$ change as

$$\gamma_{xx} \to \gamma_{xx} - \frac{2x^3}{1-x^4}f' + f'' \tag{4.76}$$

$$\gamma_{x\phi} \to \gamma_{x\phi} - \frac{2(1-x^2)}{(1+x^2)^2}f' \tag{4.77}$$

$$\gamma_{\phi\phi} \to \gamma_{\phi\phi} - \frac{8x(1-x^2)}{(1+x^2)^3}f' . \tag{4.78}$$

It is straightforward to see that the quantity

$$X = 4x\gamma_{x\phi} - (1+x^2)\gamma_{\phi\phi} \tag{4.79}$$

is gauge invariant and well defined on the entire S^2. If we let

$$\begin{aligned} Y &= x^2\left(x^2+1\right)^2\left(x^4-1\right)\left(x^2-1\right)\gamma''_{\phi\phi} + 2x\left(4x^8-11x^4-6x^2+1\right)\left(x^2-1\right)\gamma'_{\phi\phi} \\ &\quad +2\left(5x^{10}-15x^8+6x^6+22x^4-3x^2+1\right)\gamma_{\phi\phi} - 8x^3\left(x^2-1\right)^3\gamma'_{xx} , \end{aligned} \tag{4.80}$$

which can be check easily is also gauge invariant and well defined, then (4.74) and (4.75) become

$$\begin{aligned} 0 &= 2x\left(x^2+1\right)\left(x^2-1\right)^3 X' + 2\left(x^4+1\right)\left(x^2-1\right)^2 X + Y \qquad (4.81)\\ 0 &= x^2\left(x^4-1\right)^2 X'' + 2x\left(2x^8-x^6-x^4+x^2-1\right)X' \\ &\quad -2\left(5x^6+x^4+3x^2-1\right)X + Y\ . \qquad (4.82) \end{aligned}$$

Subtracting them gives simply a second order ODE for X

$$0 = 2(x^2+1)X - (x^2-1)\left[2xX' + (x^2-1)X''\right] \qquad (4.83)$$

which has solution

$$X = \frac{Ax(x^2-3)+B}{x^2-1}\ . \qquad (4.84)$$

At the endpoints $x = \pm 1$ we must have $\gamma_{\phi\phi} = |\partial_\phi|^2$ vanishing as they are the fixed points of rotation. The metric function $\gamma_{x\phi}$ on the other hand must also vanish as $(1-x^2)$ at the poles for the metric to be smooth at the poles, hence X vanishes at $x = \pm 1$ according to its definition (4.79). This is true iff $A = B = 0$, which implies $Y = 0$ too according to (4.81) and (4.82). Thus there is a *unique* solution to the first order expansion of the extreme Kerr NHG in terms of the gauge invariant quantities X and Y. However since in general the gauge invariant quantities one can write down conveniently contain derivatives, finding solutions to the individual metric components γ_{ab} often requires solving differential equations. Here $Y = 0$ in (4.80) takes the form of a second order inhomogeneous ODE for $\gamma_{\phi\phi}$, so it is far from clear whether there is indeed a *unique* solution to the first order expansion, which must then correspond to the extreme Kerr black hole itself.

In order to make a comparison, we need to write down the Kerr metric in GNC. The conversion has to be done order by order in r and we find to first order, the Kerr metric is (see appendix 4.B for derivation)

$$\begin{aligned} \mathrm{d}s^2 &= \left(\frac{3-6x^2-x^4}{a^2(1+x^2)^3} - \frac{2(x^6+7x^4-21x^2+5)\lambda}{a^3(1+x^2)^5}\right)\lambda^2\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r \\ &\quad +2\lambda\mathrm{d}v\left[\left(-\frac{2x}{1+x^2} + \frac{4x(2-x^2)\lambda}{a(1+x^2)^3}\right)\mathrm{d}x\right. \\ &\quad \left.+\left(\frac{4(1-x^2)}{(1+x^2)^2} + \frac{2(1-x^2)(x^4+10x^2-3)\lambda}{a(1+x^2)^4}\right)\mathrm{d}\chi\right] \\ &\quad +\mathrm{d}x^2\left(a^2\frac{1+x^2}{1-x^2} + \frac{4a\lambda}{1-x^4}\right) + 2\mathrm{d}x\mathrm{d}\chi\frac{4ax(1-x^2)\lambda}{(1+x^2)^2} \\ &\quad +\mathrm{d}\chi^2\left(4a^2\frac{1-x^2}{1+x^2} + \frac{16ax^2(1-x^2)\lambda}{(1+x^2)^3}\right)\ . \qquad (4.85) \end{aligned}$$

A straightforward calculation using (4.79) and (4.80) shows that the Kerr black hole (4.85) indeed gives $X = Y = 0$, so our general first order solution definitely includes extreme Kerr itself - as it should. Now we want to investigate whether extreme Kerr is the *only* solution.

By inverting (4.79) and (4.80) with $X = Y = 0$, we can write everything in terms of $\gamma_{x\phi}(x)$ which now plays the role of the gauge function $f(x)$:

$$\gamma_{xx} = C + \frac{(1+x^2)\left[2x(2x^2-3)\gamma_{x\phi} - (1-x^4)\gamma'_{x\phi}\right]}{2(1-x^2)^2} \qquad \gamma_{\phi\phi} = \frac{4x\gamma_{x\phi}}{1+x^2} \tag{4.86}$$

where C arises from the fact that only γ'_{xx} appears in the linearised Einstein equation but not γ_{xx}. Using the gauge freedom we can fix

$$\gamma_{x\phi} = \frac{4ax(1-x^2)}{(1+x^2)^2} \ ; \tag{4.87}$$

this is always possible as we show in the following. Let $\gamma_{x\phi}(x)$ be a solution; smoothness at the poles imposes that it has to vanish at the poles as $\gamma_{x\phi} \sim 1 - x^2$, so without loss of generality, we may write $\gamma_{x\phi} = g(x)(1-x^2)/(1+x^2)^2$ where $g(x)$ is some regular function. Thus according to (4.77), we want to show that there always exists a regular gauge function $f(x)$ such that

$$\frac{g(x)(1-x^2)}{(1+x^2)^2} - \frac{2(1-x^2)}{(1+x^2)^2} f'(x) = \frac{4ax(1-x^2)}{(1+x^2)^2} \ , \tag{4.88}$$

which is in fact the true as it reduces to integrating a regular function

$$f'(x) = \frac{1}{2} g(x) - 2ax \ . \tag{4.89}$$

In this gauge (4.87) we have

$$\gamma_{xx} = \frac{2a(3x^4-1) + C(1-x^4)}{1-x^4} \qquad \gamma_{\phi\phi} = \frac{16ax^2(1-x^2)}{(1+x^2)^3} \ , \tag{4.90}$$

therefore by *choosing* $C = 6a$ we recover exactly the extreme Kerr metric (4.85). This seems to suggest that by expanding the extreme Kerr NHG to first order we are allowed to have something other than extreme Kerr itself.

However it can be shown explicitly that for any *positive* C we can always gauge transform it to the extreme Kerr solution (4.85). Recall also that we are free to scale $\gamma_{ab} \to \Omega\gamma_{ab}$, so proving equivalence becomes a matter of showing there always exists a

Ω and f for any choice of C such that

$$\gamma_{xx} \to \quad \Omega\left(\tfrac{2a(3x^4-1)+C(1-x^4)}{1-x^4}\right) - \tfrac{2x^3}{1-x^4}f' + f'' \quad = \frac{4a}{1-x^4} \tag{4.91}$$

$$\gamma_{x\phi} \to \quad \Omega\left(\tfrac{4ax(1-x^2)}{(1+x^2)^2}\right) - \tfrac{2(1-x^2)}{(1+x^2)^2}f' \quad = \frac{4ax(1-x^2)}{(1+x^2)^2} \tag{4.92}$$

$$\gamma_{\phi\phi} \to \quad \Omega\left(\tfrac{16ax^2(1-x^2)}{(1+x^2)^3}\right) - \tfrac{8x(1-x^2)}{(1+x^2)^3}f' \quad = \frac{16ax^2(1-x^2)}{(1+x^2)^3}\ . \tag{4.93}$$

It is easy to see that both (4.92) and (4.93) give $f' = 2ax(\Omega - 1)$ so $f'' = 2a(\Omega - 1)$; plugging these into (4.91) then gives $\Omega = 6a/C$. Hence there always exists such $f(x)$ and Ω to do the job. Note that since Ω is positive C must also be positive. But how about $C \leq 0$ which is also included in the general solution?

Let us now look at the MOTS condition (4.48). This requires us to change to a new GNC with $\Gamma(x) = (1 + x^2)$. For the general solution (4.90) we find [4]

$$\int_H \gamma^{(1)}\epsilon_H = \frac{8\pi C}{3a^2}\ , \tag{4.95}$$

hence the MOTS condition is satisfied for any $C > 0$. This shows that only $C > 0$ could give rise to a black hole solution, in particular the extreme Kerr black hole. In fact for the extreme Kerr metric (4.85)

$$\int_H \gamma^{(1)}_{\text{eK}}\epsilon_H = \frac{16\pi}{a}\ , \tag{4.96}$$

therefore if we want to compare directly our general solution with (4.85) such that $\Omega = 1$, $C = 6a$ is the correct scale to choose. This is consistent with the interpretation that $\int_H \gamma^{(1)}\epsilon_H$ serves as a measure of the scale of r.

In summary, we prove:

Theorem 4.2 (Local uniqueness theorem for extreme Kerr)**.** *Consider the first order expansion from the near-horizon geometry of extreme Kerr. The most general axisymmetric solution which is compatible with black hole is gauge equivalent to the first order expansion of the Extreme Kerr black hole itself.*

This provides a *local* uniqueness theorem for the extreme Kerr black hole without asymptotic flatness or any other global assumptions which we discussed in chapter 1.

[4]Because $\int_H \gamma^{(1)}\epsilon_H$ is a gauge invariant quantity the pre-gauge-fixed solution (4.86) should lead the the same result with any (allowed) $\gamma_{x\phi}(x)$. It can be checked explicitly that in this case

$$\int_H \gamma^{(1)}\epsilon_H = \frac{8\pi C}{3a^2} + \left[(1+x^2)^2\gamma_{x\phi}(x)\right]_{-1}^{1}\ , \tag{4.94}$$

but since smoothness requires $\gamma_{x\phi}(x)$ to vanish at the poles as $\gamma_{x\phi}(x) \sim 1 - x^2$, the result is the same as (4.95).

Our result also suggests that there could be non-black hole solutions in $4D$ which also happen to admit the extreme Kerr NHG (when $C \leq 0$).

4.7 First order expansion to $5D$ extreme self-dual Myers-Perry near-horizon geometry

The Myers-Perry black holes are the higher dimensional analogue of the Kerr black hole. They are stationary, asymptotically flat and possess a spherical horizon, but unlike Kerr they can rotate in more than one plane. In general the $5D$ Myers-Perry black hole admits a stationary and two independent rotational Killing fields which all commute, forming a $\mathbb{R} \times U(1) \times U(1)$ isometry group. In the self-dual case where the two angular momenta are equal, the isometry group is enhanced to $\mathbb{R} \times SU(2) \times U(1)$.

In this section we investigate the first order expansion to the NHG of the $5D$ extreme self-dual Myers-Perry black hole. Near-horizon geometry theorem asserts that in $5D$ vacuum Einstein gravity with $\Lambda = 0$, any *homogeneous* and non-static NHG is *locally* isometric to the near-horizon limit of the extreme self-dual Myers-Perry black hole [87], which is given by the one-parameter family (*extreme* and *self-dual* reduce the three parameters M, J_1 and J_2 of the general non-extreme black hole to just one $M = J_1 = J_2$)

$$\begin{aligned} \mathrm{d}s^2_{NH} &= \frac{1}{2}k^2 r^2 \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + 4r\mathrm{d}v(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi) + \frac{4}{k^2}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi)^2 \\ &\quad + \frac{2}{k^2}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) , \end{aligned} \tag{4.97}$$

where k is the one-parameter and $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and $\psi \in [0, 4\pi]$ are the Euler angles on the topologically S^3 horizon. γ_{NH} being a homogeneously squashed metric on S^3 implies that the entire NHG (4.97) is a homogeneous spacetime; the overall isometry group of (4.97) is $SO(2,1) \times SU(2) \times U(1)$. Extreme self-dual Myers-Perry NHG also arises naturally as a special symmetry enhanced solution in one of the three families of NHGs ($S^1 \times S^2$, S^3, Lens space horizon topologies) in non-static $5D$ Einstein vacuum with $U(1) \times U(1)$ rotational symmetry and compact cross sections [84].

Although $\partial/\partial\phi$ and $\partial/\partial\psi$ are clearly Killing in (4.97) they are not orthogonal. Denoting the orthogonal generators of the two rotational symmetries on the S^3 by $\partial/\partial\phi_1$ and $\partial/\partial\phi_2$ with $\phi_{1,2} \sim \phi_{1,2} + 2\pi$, it is easy to see that the two sets of angles are related by $\phi = \phi_1 - \phi_2$ and $\psi = \phi_1 + \phi_2$, and γ_{NH} can be written as

$$\gamma_{NH} = \frac{2}{k^2}\left(\frac{\mathrm{d}x^2}{1-x^2} + (1+x)(3+x)\mathrm{d}\phi_1^2 - 2(1-x^2)\mathrm{d}\phi_1\mathrm{d}\phi_2 + (1-x)(3-x)\mathrm{d}\phi_2^2\right) \tag{4.98}$$

where we have defined $x = \cos\theta$. Note that unlike in the Kerr case where *the* rotational Killing field vanishes at both the north and south poles, here $\partial/\partial\phi_1$ vanishes at $x = -1$ and $\partial/\partial\phi_2$ vanishes at $x = 1$ due to the S^3 topology. For our analysis it is more convenient to use x instead of θ, so we rewrite (4.97) and examine the first order expansion about

$$\begin{aligned} \mathrm{d}s^2_{NH} &= \frac{1}{2}k^2r^2\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + 4r\mathrm{d}v(x\mathrm{d}\phi + \mathrm{d}\psi) \\ &+ \frac{2}{k^2(1-x^2)}\mathrm{d}x^2 + \frac{2(1+x^2)}{k^2}\mathrm{d}\phi^2 + \frac{8x}{k^2}\mathrm{d}\phi\mathrm{d}\psi + \frac{4}{k^2}\mathrm{d}\psi^2 \,. \end{aligned} \tag{4.99}$$

4.7.1 First order expansion with $SU(2) \times U(1)$ symmetry

We first explore the more symmetric case where the full enhanced $SU(2) \times U(1)$ rotational symmetry on the horizon is preserved in first order, so $\gamma^{(1)}_{ab}$ takes the form

$$\gamma^{(1)}_{ab}\mathrm{d}x^a\mathrm{d}x^b = a(\mathrm{d}\psi + x\mathrm{d}\phi)^2 + b\left(\frac{\mathrm{d}x^2}{1-x^2} + (1-x^2)\mathrm{d}\phi^2\right) \tag{4.100}$$

where a and b are constants. Plugging this directly into the linearised Einstein equation (4.23) shows that a must vanish whereas b does not appear in any of the six equations at all, thus b can be any arbitrary real constant. However since the trace $\gamma^{(1)} = bk^2$, the MOTS condition (4.48) then restricts $b > 0$ for the solution to admit MOTSs and describe a black hole candidate. Using (4.24) and (4.25), the remaining first order data $h^{(1)}_a$ and $F^{(1)}$ can be expressed entirely in terms of the NHG parameter k and the first order parameter b

$$F^{(1)} = -\frac{k^4b}{4} \qquad h^{(1)}_\phi = -\frac{k^2b}{2}x \qquad h^{(1)}_x = 0 \qquad h^{(1)}_\psi = -\frac{k^3b}{4} \,. \tag{4.101}$$

Hence the first order expansion is a *one*-parameter family of solutions for any k fixed by the NHG.

Let us compare this result with the first order expansion of the $5D$ extreme self-dual Myers-Perry black hole itself. Since it is parametrised by a single mass and angular momenta parameter, b must be related to k in some way. The full black hole metric is often written as

$$\mathrm{d}s^2 = -f^2\mathrm{d}t^2 + g^2\mathrm{d}r^2 + \frac{h^2}{4}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi - \Omega\mathrm{d}t)^2 + \frac{r^2}{4}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{4.102}$$

where

$$g^{-2} = 1 - \frac{4a^2}{r^2} + \frac{4a^4}{r^4} \qquad h^2 = r^2\left(1 + \frac{4a^4}{r^4}\right) \qquad f^2 = \frac{r^2}{h^2g^2} \qquad \Omega = \frac{8a^3}{r^2h^2}$$

are all functions of r only and the horizon is located at $r = r_+ = \sqrt{2}a$. In order to compare with our result we need to write the metric in GNC $(v, \lambda, \Theta, \Phi, \Psi)$, which can be done as follows. First we define $\Omega_H = \Omega|_{r=r_+} = 1/a$ and substitute $\psi' = \psi - \Omega_H t$, so the $\mathrm{d}t$ term in the first bracket becomes $(\Omega_H - \Omega)\mathrm{d}t$ and the cross terms $g_{t\phi}$ and $g_{t\psi'}$ vanish on the horizon. Next we set $\mathrm{d}v = \mathrm{d}t + X(r)\mathrm{d}r$ for some function $X(r)$ to be determined; demanding that the g_{rr} terms cancel fixes $X^2(r) = g^2/f^2$. Then we define another new coordinate ψ'' such that its total derivative is $\mathrm{d}\psi'' = \mathrm{d}\psi' - (g/f)(\Omega_H - \Omega)\mathrm{d}r$ to remove unwanted cross terms with $\mathrm{d}r$, where g/f is the positive root of X^2 as ∂_v is future and outward directed. Finally we define a new coordinate $\lambda(r)$ such that the new cross-term $2f^2 X \mathrm{d}v\mathrm{d}r$ becomes $2\mathrm{d}v\mathrm{d}\lambda$, so

$$\mathrm{d}\lambda = fg\ \mathrm{d}r = \frac{\lambda^2}{\sqrt{r^4 + 4a^4}}\ \mathrm{d}r\ . \tag{4.103}$$

The positive square root is chosen since in GNC the vector field ∂_λ points outwards. The horizon is located at $\lambda = 0$ in GNC, so we impose the the boundary condition $\lambda(r = r_+) = 0$. By Taylor expanding around r_+ and integrate with respect to $\tilde{r} = r - r_+$, we find

$$\lambda = \frac{1}{\sqrt{2}}(r - r_+) + \frac{1}{4a}(r - r_+)^2 + \mathcal{O}\left((r - r_+)^3\right)\ . \tag{4.104}$$

Keeping only the first two terms, we can rearrange and write $r^2 = 2a^2 + 4a\lambda$. Substituting this into the metric (4.102) and expand around $\lambda = 0$, we obtain, to first two leading orders in λ,

$$\begin{aligned} \mathrm{d}s^2 &= \left(-\frac{2\lambda^2}{a^2} + \frac{4\lambda^3}{a^3}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + (a^2 + 2\lambda^2)\left(\mathrm{d}\Psi + \cos\Theta\ \mathrm{d}\Phi + \left(\frac{2\lambda}{a^2} - \frac{2\lambda^2}{a^3}\right)\mathrm{d}v\right)^2 \\ &\quad + \left(\frac{a^2}{2} + a\lambda\right)(\mathrm{d}\Theta^2 + \sin^2\Theta\ \mathrm{d}\Phi^2)\ . \end{aligned} \tag{4.105}$$

where we have relabelled the angles $\psi'' = \Psi$, $\theta = \Theta$ and $\phi = \Phi$. Thus we see that our near-horizon parameter k in (4.99) is related to the parameter a above by $k = 2/a$, and in first order expansion $b = a = 2/k$ corresponds to the extreme self-dual Myers-Perry Black hole itself.

Recall that the general solution to the first order expansion is a *one-parameter* family, so how about $b \neq 2/k$? In fact (4.97) is also the near-horizon limit of the $J = 0$ extreme Kaluza-Klein black hole in $5D$. Let us examine if it covers the rest of the solution space.

The $J = 0$ extreme Kaluza-Klein black hole is a *two*-parameter family

$$\mathrm{d}s^2 = -\frac{r^2}{H_q}\ \mathrm{d}t^2 + \frac{H_q}{H_p}\ (\mathrm{d}\psi - 2P\cos\theta\ \mathrm{d}\phi - \Omega\mathrm{d}t)^2 + \frac{H_p}{r^2}\ \mathrm{d}r^2 + H_p(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2) \tag{4.106}$$

with

$$
\begin{aligned}
&H_p = r^2 + rp + \frac{p^2 q}{2(p+q)} \qquad H_q = r^2 + rq + \frac{pq^2}{2(p+q)} \qquad \Omega = \frac{Q(2r+p)}{H_q} \\
&P^2 = \frac{p^3}{4(p+q)} \qquad Q^2 = \frac{q^3}{4(p+q)}
\end{aligned} \tag{4.107}
$$

where the two parameters p and q are positive constants and they are related to the magnetic and electric charges. The horizon is located at $r = 0$. We follow the same method as above to write it in GNC $(v, \lambda, \Theta, \Phi, \Psi)$. After some calculations we find

$$
\lambda = \int \sqrt{\tfrac{H_p}{H_q}}\, \mathrm{d}r = \sqrt{\tfrac{q}{p}} r + \mathcal{O}(r^3) \,, \tag{4.108}
$$

where we have imposed the boundary condition $\lambda(r = 0) = 0$ and chosen the positive root so λ points outwards. Keeping only the first order term (note that there is no second order term), we can simply substitute $r = \sqrt{q/p}\lambda$, so the metric in GNC reads, to leading orders,

$$
\begin{aligned}
\mathrm{d}s^2 \;=\;& \left(-\frac{2(p+q)}{p^2 q}\, \lambda^2 + \frac{4(p+q)^2}{\sqrt{p^7 q^3}}\, \lambda^3 \right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}\lambda + \left(1 + \frac{2(p^2 - q^2)}{p^3 q}\, \lambda^2 \right) \\
& \left[\sqrt{\frac{q}{p}} \mathrm{d}\Psi - \frac{p\sqrt{q}}{\sqrt{p+q}} \cos\Theta \mathrm{d}\Phi + \left(\frac{2\sqrt{p+q}}{p\sqrt{q}}\, \lambda - \frac{2\sqrt{(p+q)^3}}{\sqrt{p^5 q}}\, \lambda^2 \right) \mathrm{d}v \right]^2 \\
& + \left(\frac{p^2 q}{2(p+q)} + \sqrt{pq}\lambda \right) (\mathrm{d}\Theta^2 + \sin^2\Theta\, \mathrm{d}\Phi^2) \,.
\end{aligned} \tag{4.109}
$$

By scaling $\Phi \to -\Phi$ and $\Psi \to \frac{\sqrt{p^3}}{\sqrt{p+q}}\Psi$, we recover (4.97) in leading order with NHG parameter $k = \frac{2\sqrt{p+q}}{p\sqrt{q}}$; this implies $b = \sqrt{pq}$ in first order. Since the full black hole solution is a two-parameter family, We can interchange the charges p and q with the NHG and first order parameters k and b by

$$
p = 2b\sqrt{\frac{1}{k^2 b^2 - 4}} \qquad\qquad q = b\sqrt{\frac{k^2 b^2}{4} - 1} \,. \tag{4.110}
$$

Note that we must have $b^2 > 4/k^2$ for p and q to be real, finite and non-zero.

Let us return to the MOTS condition (4.48), for our general solution with $SU(2)\times U(1)$ symmetry,

$$
\int_H \gamma^{(1)} \epsilon_H = 16\pi^2 k^2 b \,, \tag{4.111}
$$

so it requires $b > 0$ for a solution to describe a black hole. Since the two known black hole solutions with the extreme self-dual Myers-Perry NHG have $b \geq 4/k^2$, does it mean there is room for new black hole solutions? Recall that we are free to scale the solution $\gamma^{(1)}_{ab}$ by any positive constant Ω due to linearity. Therefore we can always scale the positive parameter b in any black hole candidate solution by some positive Ω to take $b \to k$ or $b \to \tilde{b} > 4/k^2$ and recover the extreme self-dual Myers-Perry or the extreme non-rotating Kaluza-Klein black holes to first order. Hence we deduce that any first order solution compatible with black hole with $SU(2) \times U(1)$ symmetry is gauge equivalent to the known solutions.

4.7.2 First order expansion with $U(1) \times U(1)$ symmetry

What happens if we relax the $SU(2)\times U(1)$ symmetry and assume only two independent rotational symmetries in first order? In this section, we investigate the general solution to the first order expansion with only $U(1)\times U(1)$ symmetry from the extreme self-dual Myers-Perry NHG. Obviously, the general solution must include the more symmetric, one-parameter family of solutions we found in the previous section. But we want to see if there exists more general solutions with "less symmetries" outside the horizon, which are also compatible with black holes and happen to admit the extreme self-dual Myers-Perry NHG. Such solutions would exhibit a rotational symmetry enhancement on the horizon.

We start with the same NHG (4.97) as before. However since now we only demand $U(1)^2$ rotational symmetry in first order, $\gamma^{(1)}_{ab}$ takes the general form

$$\gamma^{(1)}_{ab} = \begin{pmatrix} \gamma_{xx}(x) & \gamma_{x\phi}(x) & \gamma_{x\psi}(x) \\ \gamma_{x\phi}(x) & \gamma_{\phi\phi}(x) & \gamma_{\phi\psi}(x) \\ \gamma_{x\psi}(x) & \gamma_{\phi\psi}(x) & \gamma_{\psi\psi}(x) \end{pmatrix}_{ab} \tag{4.112}$$

where we have dropped the superscript $^{(1)}$ on the right hand side. So we have 6 metric functions to determine, all of which are functions of x as ∂_ϕ and ∂_ψ are the rotational Killing fields . The gauge freedom of these 6 metric functions are summarised below:

$$\begin{array}{lll} \gamma_{xx} \to \frac{-xf'}{1-x^2} + f'' & \gamma_{x\phi} \to -xf' & \gamma_{x\psi} \to -f' \\ \gamma_{\phi\phi} \to x(1-x^2)f' & \gamma_{\phi\psi} \to (1-x^2)f' & \gamma_{\psi\psi} \to \gamma_{\psi\psi} \end{array}$$

where $f(x)$ is an arbitrary function. Note that $\gamma_{\psi\psi}$ is itself a gauge invariant quantity.

Using the linearised Einstein equation (4.23), we can write down, along with $\gamma_{\psi\psi}$, five gauge invariant quantities which are regular on H, including the poles at $x = \pm 1$ (using the same argument as in the Kerr case):

$$\begin{aligned} W(x) &= (1-x^2)^3\gamma''_{x\psi} - x(1-x^2)^2\gamma'_{x\psi} - (1-x^4)\gamma_{x\psi} + (1-x^2)^3\gamma'_{xx} \\ X(x) &= \gamma_{\phi\phi} + x(1-x^2)\gamma_{x\psi} \\ Y(x) &= x(1-x^2)\gamma_{x\psi} - (1-x^2)\gamma_{x\phi} \\ Z(x) &= \gamma_{\phi\psi} + (1-x^2)\gamma_{x\psi} \ , \end{aligned} \tag{4.113}$$

so the six equations can be written as, in order of xx, $x\phi$, $x\psi$, $\phi\phi$, $\phi\psi$ and $\psi\psi$:

$$\begin{aligned}
0 &= -\left(1+x^2\right)\left(1-x^2\right)^2\gamma''_{\psi\psi} - x\left(x^4-8x^2+7\right)\gamma'_{\psi\psi} + 2\left(x^4-4x^2-1\right)\gamma_{\psi\psi} \\
&\quad -2xW - 2\left(1-x^2\right)^2 X'' - 6x\left(1-x^2\right)X' - 4\left(1+x^2\right)X \\
&\quad +4x\left(1-x^2\right)^2 Z'' - 4\left(x^4+x^2-2\right)Z' + 16xZ \qquad (4.114)\\
0 &= -3x\left(1-x^4\right)\gamma'_{\psi\psi} + 2\left(x^4-6x^2+1\right)\gamma_{\psi\psi} - 2xW - 6x\left(1-x^2\right)X' - 8x^2X \\
&\quad +2\left(1-x^2\right)^2 Y'' + 4\left(1-x^2\right)Y - 2\left(7x^4-8x^2+1\right)Z' + 8x\left(x^2+1\right)Z \qquad (4.115)\\
0 &= \left(x^4+4x^2-5\right)\gamma'_{\psi\psi} + 4x\left(x^2-3\right)\gamma_{\psi\psi} - 2W - 6\left(1-x^2\right)X' - 8xX \\
&\quad -4\left(1-x^2\right)Y' + 12x\left(1-x^2\right)Z' + 4\left(x^2+3\right)Z \qquad (4.116)\\
0 &= -\left(1-x^4\right)x\gamma'_{\psi\psi} - 2\left(x^4+4x^2-1\right)\gamma_{\psi\psi} - 2xW + 2\left(1-x^2\right)^2 X'' - 6x\left(1-x^2\right)X' \\
&\quad +4\left(1-3x^2\right)X - 8x\left(x^2-1\right)Y' - 4\left(2x^4-3x^2+1\right)Z' + 16x^3Z \qquad (4.117)\\
0 &= -\left(x^4-4x^2+3\right)\gamma'_{\psi\psi} - 8x\gamma_{\psi\psi} - 2W - 8xX - 6\left(1-x^2\right)X' + 4\left(1-x^2\right)Y' \\
&\quad +2\left(1-x^2\right)^2 Z'' + 4x\left(1-x^2\right)Z' + 8\left(1+x^2\right)Z \qquad (4.118)\\
0 &= -\left(1-x^2\right)\gamma''_{\psi\psi} - 2x\gamma'_{\psi\psi} + 4Z' \,. \qquad (4.119)
\end{aligned}$$

Recall that only five of the six equations are independent, we may discard the xx equation which is the longest. In fact, it is easy to check that it can be written in terms of the $\phi\phi$, $\phi\psi$ and $\psi\psi$ equations. Observe that none of the equations contains any derivatives of W, so we can use the $x\psi$ equation (since it is first order) to eliminate W. This allows us to dramatically simplify the $x\phi$, $\phi\phi$ and $\phi\psi$ equations into

$$\begin{aligned}
0 &= x\left(1-x^2\right)\gamma'_{\psi\psi} + \left(x^2+1\right)\gamma_{\psi\psi} + \left(1-x^2\right)Y'' + 2xY' + 2Y \\
&\quad -\left(1-x^2\right)Z' - 2xZ \qquad (4.120)\\
0 &= 2x\gamma'_{\psi\psi} + \left(3x^2+1\right)\gamma_{\psi\psi} + \left(1-x^2\right)X'' + 2X(x) + 6xY' \\
&\quad -2\left(1+x^2\right)Z' - 6xZ \qquad (4.121)\\
0 &= \left(x^2+1\right)\gamma'_{\psi\psi} + 2x\gamma_{\psi\psi} + 4Y' + \left(1-x^2\right)Z'' - 4xZ' - 2Z \,. \qquad (4.122)
\end{aligned}$$

These, along with the $\psi\psi$ equation (4.119), are the only four equations left to consider. Note that (4.122) is in fact a total derivative, which can be integrated to give $\gamma_{\psi\psi}$ in terms of Y and Z and a constant of integration C

$$\gamma_{\psi\psi} = \frac{\left(x^2-1\right)Z'(x) + 2xZ(x) - 4Y(x) + C}{\left(x^2+1\right)} \,. \qquad (4.123)$$

Substituting this into the $x\phi$ equation (4.121) leads to a second order ODE involving just Y and Z

$$\begin{aligned}
0 &= -C\left(3x^4+1\right) + \left(x^2+1\right)^2\left(x^2-1\right)Y'' - 2x\left(3x^4+2x^2-1\right)Y' \\
&\quad +2\left(5x^4-2x^2+1\right)Y + x\left(x^2+1\right)\left(1-x^2\right)^2 Z'' + 2\left(1-x^2\right)^2 Z' \\
&\quad -2x\left(1-x^2\right)^2 Z \,; \qquad (4.124)
\end{aligned}$$

similarly the $\psi\psi$ equation (4.119) becomes

$$\begin{aligned} 0 \;=\; & \left(1-x^4\right)^2 Z^{(3)}(x) + 8x\left(x^4-1\right)Z''(x) - 4\left(x^6+3x^4-5x^2+1\right)Z'(x) \\ & +8x\left(1-x^2\right)^2 Z(x) - 4\left(x^2-1\right)\left(x^2+1\right)^2 Y''(x) + 8x\left(3x^4+2x^2-1\right)Y'(x) \\ & -8\left(5x^4-2x^2+1\right)Y(x) + 2C(5x^4-2x^2+1)\ . \end{aligned} \tag{4.125}$$

Thus once we solve for Y and Z we can use (4.123) to compute $g_{\psi\psi}$ algebraically, and then solve the ODE for X using the $\phi\phi$ equation (4.121). Finally W is determined algebraically using the $x\psi$ equation (4.116).

Remarkably, adding 4×(4.124) and (4.125) gives simply a *second order* ODE for Z'

$$0 = \left(1-x^2\right)^2 Z^{(3)} - 4(1-x^2)(xZ''-Z') - 2C \tag{4.126}$$

which has the general solution

$$Z = \frac{1}{4}x(4Ax - C - 3D) + B + \frac{1}{8}\left(Cx^2 + 3Dx^2 + C - D\right)\,\ln\left(\frac{1+x}{1-x}\right)\ , \tag{4.127}$$

where A, B, D are constants. Z is regular on H iff $C = D = 0$, hence $Z = Ax^2 + B$. Plugging this into (4.124) results in a second order ODE for Y

$$\begin{aligned} 0 \;=\; & \left(x^2+1\right)^2\left(x^2-1\right)Y'' + \left(-6x^5-4x^3+2x\right)Y' + 2\left(5x^4-2x^2+1\right)Y \\ & +2x\left(x^2-1\right)^2(3A-B)\ , \end{aligned} \tag{4.128}$$

which has general solution regular on H

$$Y = -\frac{3}{2}Ax + \frac{1}{2}Bx + M\left(x^2+1\right) + Nx\left(x^2-3\right)\left(x^2+1\right) \tag{4.129}$$

with M and N constant. (4.123) then gives

$$\gamma_{\psi\psi} = 4\left(Ax - M + Nx(3-x^2)\right)\,, \tag{4.130}$$

and so (4.121) implies the following second order ODE for X

$$0 = \left(1-x^2\right)X'' + 2X + Ax(2x^2-1) - 3Bx - 4M + 2Nx(9x^4-14x^2+9)\ . \tag{4.131}$$

The general solution for X is

$$\begin{aligned} X \;=\; & -\frac{1}{2}Ax(1-x^2) + \frac{3}{4}Bx + 2Mx^2 + \frac{1}{2}Nx(2x^4-4x^2-3) + P(1-x^2) + \frac{1}{4}Qx \\ & +\frac{1}{8}(2A-3B+6N+Q)\left(1-x^2\right)\log\left(\frac{x+1}{1-x}\right)\ , \end{aligned} \tag{4.132}$$

so for X to be regular on H, we must have $Q = -2A + 3B - 6N$. Thus it simplifies to

$$X = \frac{1}{2}Ax(x^2-2) + \frac{3}{2}Bx + 2Mx^2 + Nx(x^4 - 2x^2 - 3) + P(1-x^2)\,. \qquad (4.133)$$

Finally from (4.116),

$$W = \frac{1}{2}A\left(5x^4 - 9x^2 - 8\right) + \frac{1}{2}B\left(3x^2+1\right) + 8Mx + N\left(7x^6 - 17x^4 + 9x^2 - 15\right) + 2Px\left(1-x^2\right)\,. \qquad (4.134)$$

Therefore the general solution to the first order expansion is a five-parameter family of solutions.

However these solutions are given in terms of the gauge invariant quantities; individual components of γ_{ab} must also be smooth on H. By inverting (4.113) we can write everything in terms of a free function (due to gauge freedom) - which we may choose to be $\gamma_{x\psi}(x)$. In orthogonal angles ϕ_1 and ϕ_2, the metric components are given by

$$\begin{aligned}
\gamma_{xx} &= S - A\left(\frac{5x}{2(1-x^2)} + \frac{3x}{2(1-x^2)^2}\right) + \frac{Bx}{2(1-x^2)^2} + \frac{2M}{(1-x^2)^2} \\
&\quad - N\left(\frac{4x}{1-x^2} + \frac{4x}{(1-x^2)^2} + 7x\right) + \frac{P}{1-x^2} - \gamma'_{x\psi} + \frac{x\gamma_{x\psi}}{1-x^2} \qquad (4.135)\\
\gamma_{x1} &= \frac{3Ax}{2(1-x^2)} - \frac{Bx}{2(1-x^2)} - \frac{M(1+x^2)}{1-x^2} - \frac{Nx(x^4-2x^2-3)}{1-x^2} \\
&\quad + (1+x)\gamma_{x\psi} \qquad (4.136)\\
\gamma_{x2} &= -\frac{3Ax}{2(1-x^2)} + \frac{Bx}{2(1-x^2)} + \frac{M(1+x^2)}{1-x^2} + \frac{Nx(x^4-2x^2-3)}{1-x^2} \\
&\quad + (1-x)\gamma_{x\psi} \qquad (4.137)\\
\gamma_{11} &= Ax\left(\frac{x^2}{2} + 2x + 3\right) + B\left(\frac{3x}{2} + 2\right) + 2M\left(x^2-2\right) + N\left(x^5 - 6x^3 + 9x\right) \\
&\quad + P\left(1-x^2\right) + \left(x^3 + 2x^2 - x - 2\right)\gamma_{x\psi} \qquad (4.138)\\
\gamma_{12} &= Ax\left(5 - \frac{x^2}{2}\right) - \frac{3Bx}{2} - 2M\left(x^2+2\right) + N\left(-x^5 - 2x^3 + 15x\right) - P\left(1-x^2\right) \\
&\quad + x\left(1-x^2\right)\gamma_{x\psi} \qquad (4.139)\\
\gamma_{22} &= Ax\left(\frac{x^2}{2} - 2x + 3\right) + B\left(\frac{3x}{2} - 2\right) + 2M\left(x^2-2\right) + N\left(x^5 - 6x^3 + 9x\right) \\
&\quad + P\left(1-x^2\right) + \left(x^3 - 2x^2 - x + 2\right)\gamma_{x\psi}\,, \qquad (4.140)
\end{aligned}$$

where S is a constant that comes from integrating γ_{xx}. Therefore in terms of the individual metric functions γ_{ab} (which are what matter ultimately), it appears that the general solution is a *six* parameter family.

However in order for the metric to be smooth, *especially at the poles*, the first order quantity γ_{ab} in orthogonal coordinates should look like

$$\begin{aligned}\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b &= \mathcal{A}(x)\frac{\mathrm{d}x^2}{1-x^2}+\mathcal{B}(x)(1+x)\mathrm{d}\phi_1^2+\mathcal{C}(x)(1-x)\mathrm{d}\phi_2^2+\mathcal{D}(x)(1-x^2)\mathrm{d}\phi_1\mathrm{d}\phi_2\\ &\quad +\mathcal{E}(x)(1+x)\mathrm{d}x\mathrm{d}\phi_1+\mathcal{F}(x)(1-x)\mathrm{d}x\mathrm{d}\phi_2 \qquad (4.141)\end{aligned}$$

where the functions $\mathcal{A}(x),\mathcal{B}(x)$... are regular functions on the S^3. Note this also implies the gauge function $\gamma_{x\psi}=\left[\mathcal{E}(x)(1+x)+\mathcal{F}(x)(1-x)\right]/2$ so it is regular everywhere. At the pole $x=1$, (4.141) imposes $3A-B-4M+8N=0$. Similarly at $x=-1$, it imposes $3A-B+4M+8N=0$. Therefore $M=0$ and $N=-3A/8+B/8$, and we are left with a *four*-parameter family of solutions characterised by the constants A,B,P and S.

Hence in terms of the gauge invariant quantities defined in (4.113), the general solution to the first order expansion is

$$\begin{aligned}W &= \frac{1}{8}A\left(21x^4-50x^2+13\right)\left(1-x^2\right)+\frac{1}{8}B\left(-7x^4+10x^2-11\right)\left(1-x^2\right)\\ &\quad +2Px\left(1-x^2\right) \qquad (4.142)\\ X &= \frac{1}{8}Ax\left(-3x^4+10x^2+1\right)+\frac{1}{8}Bx\left(x^4-2x^2+9\right)+P\left(1-x^2\right) \qquad (4.143)\\ Y &= \frac{3}{8}Ax\left(1-x^2\right)^2-\frac{1}{8}Bx\left(1-x^2\right)^2 \qquad (4.144)\\ Z &= Ax^2+B \qquad (4.145)\\ \gamma_{\psi\psi} &= -\frac{1}{2}Ax\left(1-3x^2\right)-\frac{1}{2}Bx\left(x^2-3\right)\ , \qquad (4.146)\end{aligned}$$

along with a constant S which is absent here (because only γ'_{xx} appears in (4.113) and S arises from integrating it to find γ_{xx}). It can be checked explicitly that the general one-parameter solution with enhanced $SU(2)\times U(1)$ symmetry found in the previous section is included in the general solution above: from (4.100) it is easy to see that the $SU(2)\times U(1)$ case gives

$$\gamma_{\psi\psi}=0 \qquad W=2bx(1-x^2) \quad X=b(1-x^2) \quad Y=0 \quad Z=0 \qquad (4.147)$$

i.e. $A=B=0$ and $P=b$, along with $S=0$.

However because of the scaling freedom, the most general solution to the first order expansion is in fact a *three*-parameter family, as we argue as follows. Let us consider the MOTS condition (4.48). By inverting (4.142) to (4.146) and keeping $\gamma_{x\psi}$ arbitrary as before, so that $\gamma_{x\psi}$ serves effectively as the gauge function $f(x)$, the integral evaluates to

$$\int_H \gamma^{(1)}\epsilon_H = 16\pi^2k^2\left(P+\frac{S}{3}\right)-\frac{1}{2}k^2\left[(1-x^2)\gamma_{x\psi}(x)\right]_{-1}^{1}\ , \qquad (4.148)$$

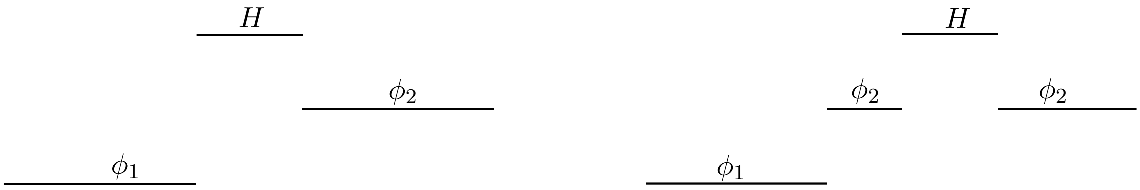
Thus by fixing a scale for $\int_H \gamma^{(1)}\epsilon_H$ and a gauge for $\gamma_{x\psi}$, for any given P the constant S is fixed. Therefore S is not really a free parameter and the general solution is characterised by only *three* parameters.

To summarise, we show that when the enhanced $SU(2) \times U(1)$ symmetry is preserved in first order, the most general solutions which are compatible with black holes are gauge equivalent to the first order expansion of the extreme self-dual Myers-Perry solution itself and the extreme $J = 0$ Kaluza Klein black hole. These are the only known black holes to possess this the extreme self-dual Myers-Perry NHG. If only $U(1) \times U(1)$ symmetry is assumed, then the most general solution compatible with black holes is a three-parameter family. This vast solution space includes much more than the two known black hole solutions. Could these extra solutions really arise from some unknown black holes?

4.7.3 New black hole solutions?

In contrary to the extreme Kerr case, we see that in $5D$ there exists a much larger solution space for black hole candidates in first order which do not correspond to any known black hole solutions. This is perhaps not so surprising because black hole geometries are well known to be much richer in higher dimensions without the stringent restrictions which apply only in $4D$. Let us also point out that our general first order solution *cannot* include the general extreme Myers-Perry black hole with non-equal angular momenta $J_1 \neq J_2$, since this would then involve perturbation *tangent* to the horizon, whereas our formalism only captures perturbation *transverse* to the horizon. This can be verified explicitly by comparing our general solution with the general extreme Myers-Perry metric, which can be found written in GNC in appendix 4.C.

One possible origin of these more general solutions to the first order expansion is spacetimes which contain a black hole as well as some non-trivial exterior 2-cycles. Such "bubbly" solutions may admit $\mathbb{R} \times U(1)^2$ symmetry, like the Myers-Perry black hole and the black ring, without violating any known theorems. Recall that a stationary asymptotically flat vacuum black hole in $5D$ with two commuting axisymmetries is uniquely specified by its mass, angular momenta and "rod structure" [79]. The rod structure is a diffeomorphism invariant datum that describes the relative positions of the horizon and the rotation axes. Each rod segment either represents the orbit space of a connect component of the horizon or an axis where a linear combination of the two rotational Killing fields vanishes. The horizon topology can therefore be read directly from the rod structure and the length of the rods encodes information about M and J_i. However it is not clear what rod structure may lead to black holes. All known solutions possess the simplest rod structure compatible with their horizon topology. For example, the Myers-Perry and black ring are represented by figures 4.2a and 4.2b respectively



(a) Myers-Perry black hole (b) Black ring

Figure 4.2: Rod structures of the Myers-Perry black hole and the black ring.

where H labels the horizon rod and ϕ_i labels which rotational Killing vector $\partial/\partial\phi_i$ vanishes, both ends of the diagram run to infinity. On the other hand, a spacetime admitting a doubly-spinning S^3 horizon and a "bubble" in the exterior would be described by

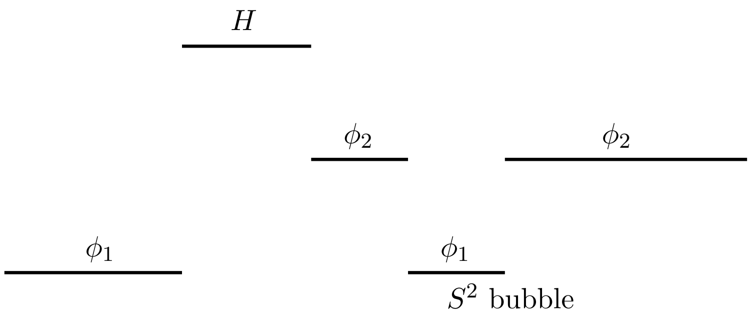


Figure 4.3: A black hole spacetime with an S^2 bubble outside the horizon.

To date there is no evidence for the existence of such bubbly vacuum black holes, but if they do exist they must be contained in our general three-parameter family of solutions.

Of course the first order solutions which are more general than the Myers-Perry and the Kaluza Klein black holes may not correspond to any black holes at all. After all the conditions imposed in our analysis are only necessarily but not sufficient for black hole solutions. It is also possible that novel black hole solutions with near-horizon geometry of the extreme self-dual Myers-Perry black hole may have trivial first order expansion, then one may consider expansion to higher orders to detect them.

4.8 Summary

We examined the first order expansion from near-horizon geometry. We demonstrated that for any compact horizon, the general solution to the first order expansion is a finite-parameter family of solutions. In particular, we considered the first order expansion from the near-horizon geometry of the extreme Kerr black hole. This has led to the following local uniqueness theorem: the only possible black hole solutions which admit a $U(1)$ symmetry are gauge equivalent to the first order expansion of the extreme Kerr solution itself. We then investigated the first order expansion from the near-horizon geometry of the extreme self-dual Myers-Perry black hole in $5D$. The only solutions which can describe black holes and inherit the enhanced $SU(2) \times U(1)$ symmetry are gauge equivalent to the first order expansion of the extreme self-dual Myers-Perry black hole itself and the extreme $J = 0$ Kaluza Klein black hole. When only the $U(1) \times U(1)$ symmetry is preserved, the most general solution compatible with black holes is a 3-parameter family of solutions. This 3-parameter family is more general than the above-mentioned solutions, which are the only known black holes to possess the extreme self-dual Myers-Perry near-horizon geometry. This may hint the existence of new black hole solutions.

4.A Linearised contracted Bianchi identity

Here we give the derivation of the linearised contracted Bianchi identity (4.26) in section 4.3. The contracted Bianchi identity for the full spacetime $(M, g(\varepsilon))$ is given by

$$0 = \nabla^\nu G_{\nu\rho} = g^{\mu\nu}\nabla_\mu \left(R_{\nu\rho} - \frac{1}{2} g_{\nu\rho} g^{\alpha\beta} R_{\alpha\beta} \right) \tag{4.149}$$

where ∇_μ is the covariant derivative of $g_{\mu\nu}(\varepsilon)$. The *linearised* contracted Bianchi identity is then

$$0 = \left. \frac{\mathrm{d}(\nabla^\mu G_{\mu\rho}(\varepsilon))}{\mathrm{d}\varepsilon} \right|_{r=0} . \tag{4.150}$$

Recall that $\nabla_\mu = \nabla_\mu(\varepsilon)$ is related to $\nabla^{(0)}_\mu = \nabla_\mu(0)$, the covariant derivative of $g_{\mu\nu}(0)$, by a tensor $C^\rho_{\mu\nu}(\varepsilon)$ such that for a covector ω_ρ

$$\nabla_\mu(\varepsilon)\, \omega_\nu = \nabla^{(0)}_\mu \omega_\nu - C^\rho_{\mu\nu}(\varepsilon)\omega_\rho . \tag{4.151}$$

Obviously, $C^\rho_{\mu\nu}(0) = 0$ and $\nabla^{(0)}_\mu$ is independent of ε. (4.150) thus expands into

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left[g^{\mu\nu}(\varepsilon) \left(\nabla^{(0)}_\mu G_{\nu\rho}(\varepsilon) - C^\sigma_{\mu\nu}(\varepsilon) G_{\rho\sigma}(\varepsilon) - C^\sigma_{\mu\rho}(\varepsilon) G_{\sigma\nu}(\varepsilon) \right) \right] \Bigg|_{r=0} , \tag{4.152}$$

which becomes

$$0 = g^{(1)\mu\nu} \nabla^{(0)}_\mu G_{\nu\rho}(0) + g^{\mu\nu}(0) \left(\nabla^{(0)}_\mu \left. \frac{\mathrm{d}G_{\nu\rho}}{\mathrm{d}\varepsilon} \right|_{r=0} - \left. \frac{\mathrm{d}C^\sigma_{\mu\nu}}{\mathrm{d}\varepsilon} \right|_{r=0} G_{\rho\sigma}(0) - \left. \frac{\mathrm{d}C^\sigma_{\mu\rho}}{\mathrm{d}\varepsilon} \right|_{r=0} G_{\sigma\nu}(0) . \right) \tag{4.153}$$

But since $g(0) = g_{NH}$ is itself an exact solution to the vacuum Einstein equation, $G_{\mu\nu}(0) = 0$ and $\nabla^\mu G_{\mu\nu}(0) = 0$, so only the second term survives. Expressing $G_{\mu\nu}$ in terms of the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R, and noting that

$$\left. \frac{\mathrm{d}R}{\mathrm{d}\varepsilon} \right|_{r=0} = g^{(1)\rho\sigma} R_{\rho\sigma}(0) + g^{\rho\sigma}_{NH} R^{(1)}_{\rho\sigma} = g^{\rho\sigma}_{NH} R^{(1)}_{\rho\sigma} \tag{4.154}$$

where the last equality is due to $R_{\mu\nu}(0) = 0$ as $g(0)$ solves the Einstein equation, the linearised contracted Bianchi identity is therefore

$$0 = \nabla^{(0)\mu} R^{(1)}_{\mu\nu} - \frac{1}{2} \nabla^{(0)}_{\nu} (g^{\rho\sigma}_{NH} R^{(1)}_{\rho\sigma}) \; . \tag{4.155}$$

4.B Kerr metric in GNC

In Boyer-Lindquist coordinates, the full extreme Kerr is given by

$$\mathrm{d}s^2 = -\frac{\Delta}{\rho^2}(\mathrm{d}t^2 - a\sin^2\theta \mathrm{d}\phi)^2 + \frac{\sin^2\theta}{\rho^2}(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 \tag{4.156}$$

where $\Delta = (r-a)^2$ and $\rho^2 = r^2 + a^2\cos^2\theta$. For our purpose of converting Boyer-Lindquist coordinates (t, r, ϕ, θ) into GNC (v, λ, χ, ψ) it is more convenient to expand the brackets and write it as

$$\mathrm{d}s^2 = \tfrac{1}{\rho^2}\left(A\mathrm{d}t^2 + 2B\mathrm{d}t\mathrm{d}\phi + C\mathrm{d}\phi^2\right) + \tfrac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 \; , \tag{4.157}$$

with

$$A = -r^2 - a^2\cos^2\theta + 2ra \qquad B = -2a^2 r\sin^2\theta \qquad C = \sin^2\theta\left((r^2+a^2)^2 - \Delta a^2\sin^2\theta\right) \; . \tag{4.158}$$

However the conversion can only be done perturbatively order by order form the horizon, which is associated to the Killing field $K = \frac{\partial}{\partial t} + \frac{1}{2a}\frac{\partial}{\partial\phi}$ and is located at $r = a$. Recall that we need to "shoot out" affinely parametrised null geodesics from the horizon to write the metric in GNC, we shall adopt the notation $\dot{f} \equiv \partial f/\partial\lambda$ where λ is the affine parameter of past-directed radial null geodesics defined in section 1.6 (there we used r). In this notation, the Lagrangian is

$$\mathscr{L} = \tfrac{1}{\rho^2}\left(A\dot{t}^2 + 2B\dot{t}\dot{\phi} + C\dot{\phi}^2\right) + \tfrac{\rho^2}{\Delta}\dot{r}^2 + \rho^2\dot{\theta}^2 \; . \tag{4.159}$$

Using the Euler-Lagrange equation

$$\frac{\partial\mathscr{L}}{\partial x^\mu} = \frac{\partial}{\partial\lambda}\frac{\partial\mathscr{L}}{\partial\dot{x}^\mu} \tag{4.160}$$

the t and ψ equations give, upon rearranging,

$$\dot{t} = \frac{\rho^2(Bf_\phi - Cf_t)}{B^2 - AC} \qquad \dot{\phi} = \frac{\rho^2(-Af_\phi + Bf_t)}{B^2 - AC} \; , \tag{4.161}$$

where the functions f_t and f_ϕ arise from integrating the Euler-Lagrange equations with respect to λ, thus they are functions of v, χ and ψ only and independent of λ. The energy condition $K \cdot L = 1$ in GNC then implies $f_\phi = 2a(1 - f_t)$. By imposing that only ϕ depends on χ and recalling that in GNC $L \cdot \frac{\partial}{\partial\chi} = 0$, we obtain $B\dot{t} + C\dot{\phi} = 0$ and so $f_t = 1$. Therefore

$$\dot{t} = \frac{-(a^2+r^2)^2 + a^2(r-a)^2\sin^2\theta}{(r-a)^2(r^2+a^2\cos^2\theta)} \qquad \dot{\phi} = \frac{-2a^2 r}{(r-a)^2(r^2+a^2\cos^2\theta)} \ , \tag{4.162}$$

where $r = r(\lambda,\psi)$ and $\theta = \theta(\lambda,\psi)$ are some functions of λ and ψ yet to be determined. Since we are interested in first order expansion in small λ, we can write r and θ as power series expansions in λ of the forms

$$r = a + \sum_{n=1}^{N} r_n(\psi)\lambda^n \qquad \theta = \psi + \sum_{n=1}^{N} T_n(\psi)\lambda^n \tag{4.163}$$

for some large enough N, and we have exploited the reparametrisation freedom to set $\psi = \theta(\lambda = 0)$. The functions $r_n(\psi)$ and $T_n(\psi)$ are then determined order by order using the null constraint $\mathscr{L} = 0$ and the Euler-Lagrange equation for θ (the Euler-Lagrange equation for r is now redundant). In fact the Euler-Lagrange equation for θ can be simplified significantly by substituting in the expression of $\dot{\theta}^2$ from the null constraint. We observe that all $r_n(\psi)$ and $T_n(\psi)$ for $n \geq 2$ depend on one undetermined function $T_1(\psi)$; since we only need *a* coordinate transformation to GNC, we may choose $T_1(\psi) = 0$. Following a change of coordinate $x = \cos(\psi)$, we find to third order in λ

$$r = a + \frac{2\lambda}{1+x^2} + \frac{2(x^2-1)\lambda^2}{a(1+x^2)^3} + \frac{(x^2+5)(x^2-1)^2\lambda^3}{a(1+x^2)^5} \tag{4.164}$$

$$\theta = \cos^{-1}(x) - \frac{x\sqrt{1-x^2}\lambda^2}{2a^2(1+x^2)^2} + \frac{2x\sqrt{1-x^2}\lambda^3}{a^3(1+x^2)^4} \ . \tag{4.165}$$

Substituting these into the expressions for $\dot{t}$ and $\dot{\phi}$ above and power series expanding around $\lambda = 0$ allows us to write $\mathrm{d}t$ and $\mathrm{d}\phi$ in terms of only v, λ, χ and x near the horizon

$$\mathrm{d}t = \mathrm{d}v + \dot{t}\mathrm{d}\lambda + \left(\int \tfrac{\partial \dot{t}}{\partial x}\,\mathrm{d}\lambda + f_t'(x)\right)\mathrm{d}x \tag{4.166}$$

$$\mathrm{d}\phi = \tfrac{1}{2a}\mathrm{d}v + \dot{\phi}\mathrm{d}\lambda + \left(\int \tfrac{\partial \dot{\phi}}{\partial x}\,\mathrm{d}\lambda + f_\phi'(x)\right)\mathrm{d}x + \mathrm{d}\chi \ , \tag{4.167}$$

where to leading orders

$$\begin{aligned}\dot{t} \;=\; & -\frac{a^2\left(x^2+1\right)}{\lambda^2} - \frac{2a}{\lambda} + \frac{-x^6-5x^4-x^2-1}{\left(x^2+1\right)^3} + \frac{\left(-3x^8-4x^6+34x^4-28x^2+1\right)\lambda}{2a\left(x^2+1\right)^5} \\ & + \frac{\left(-21x^{10}+169x^8+442x^6-1338x^4+779x^2-31\right)\lambda^2}{10a^2\left(x^2+1\right)^7}\end{aligned} \tag{4.168}$$

$$\begin{aligned}\dot{\phi} &= -\frac{a\left(x^2+1\right)}{2\lambda^2}+\frac{x^4+3x^2}{a\left(x^2+1\right)^3}+\frac{\left(-x^8+4x^6-2x^4-68x^2+3\right)\lambda}{4a^2\left(x^2+1\right)^5} \\ &+\frac{\left(12x^{10}+77x^8-124x^6-474x^4+872x^2-43\right)\lambda^2}{10a^3\left(x^2+1\right)^7} \\ &+\frac{\left(-x^{14}+9x^{12}-13x^{10}-451x^8-35x^6+2155x^4-1743x^2+79\right)\lambda^3}{4a^4\left(x^2+1\right)^9}\end{aligned} \tag{4.169}$$

The property $L \cdot \partial_x = 0$ of GNC fixes

$$f_t'(x) = \frac{4ax^3}{(1+x^2)^2} \ ; \tag{4.170}$$

using the coordinate freedom on H we may choose $\partial_x \cdot \partial_\chi|_{\lambda=0} = 0$, which fixes

$$f_\phi'(x) = \frac{-2x}{(1+x^2)^2} \ . \tag{4.171}$$

Therefore to first order, the extreme Kerr metric in GNC is

$$\begin{aligned}\mathrm{d}s^2 &= \left(\frac{3-6x^2-x^4}{a^2(1+x^2)^3} - \frac{2(x^6+7x^4-21x^2+5)\lambda}{a^3(1+x^2)^5}\right)\lambda^2\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r \\ &+2\lambda\mathrm{d}v\left[\left(-\frac{2x}{1+x^2}+\frac{4x(2-x^2)\lambda}{a(1+x^2)^3}\right)\mathrm{d}x\right. \\ &\left.+\left(\frac{4(1-x^2)}{(1+x^2)^2}+\frac{2(1-x^2)(x^4+10x^2-3)\lambda}{a(1+x^2)^4}\right)\mathrm{d}\chi\right] \\ &+\mathrm{d}x^2\left(a^2\frac{1+x^2}{1-x^2}+\frac{4a\lambda}{1-x^4}\right)+2\mathrm{d}x\mathrm{d}\chi\frac{4ax(1-x^2)\lambda}{(1+x^2)^2} \\ &+\mathrm{d}\chi^2\left(4a^2\frac{1-x^2}{1+x^2}+\frac{16ax^2(1-x^2)\lambda}{(1+x^2)^3}\right) \ .\end{aligned} \tag{4.172}$$

4.C General angular momenta Myers-Perry black hole in GNC

Let the angular momenta be parametrised by two constants a and b. In Boyer-Linquist coordinates (t, r, x, ϕ, ψ), the metric is

$$\begin{aligned}\mathrm{d}s^2 &= -\mathrm{d}t^2 + \left(r^2+a^2+\frac{x^2(r^2+b^2)}{1-x^2}\right)\mathrm{d}x^2 + \frac{\Pi F}{\Pi-\mu r^2}\mathrm{d}r^2 \\ &+\frac{\mu r^2}{\Pi F}\left(\mathrm{d}t - ax^2\mathrm{d}\phi - b(1-x^2)\mathrm{d}\psi\right)^2 + x^2(r^2+a^2)\mathrm{d}\phi^2 + (1-x^2)(r^2+b^2)\mathrm{d}\psi^2\end{aligned} \tag{4.173}$$

where μ is the mass parameter and

$$F = 1 - \frac{x^2a^2}{r^2+a^2} - \frac{(1-x^2)b^2}{r^2+b^2} \ , \qquad \Pi = (r^2+a^2)(r^2+b^2) \ .$$

The horizon is located at $r = r_+$ which is the largest root of $\Pi(r_+) = \mu r_+^2$ and extremality requires $\Pi'(r_+) = 2\mu r_+$. Using these two conditions we can express r_+ and μ in terms of a and b only

$$r_+^2 = ab \, , \qquad \mu = (a+b)^2 \, . \tag{4.174}$$

It is convenient to write the metric in a form such that ∂_t is the Killing field that generates the horizon. This can be achieved by shifting the coordinates ϕ and ψ by some constant multiples of t. In this case, the shift required is

$$\phi \to \phi + \frac{t}{a+b} \, , \qquad \psi \to \psi + \frac{t}{a+b} \tag{4.175}$$

and so the metric becomes

$$\begin{aligned} \mathrm{d}s^2 &= \frac{(r^2-ab)^2}{\sigma^2(a+b)^2}\,\mathrm{d}t^2 + \frac{r^2\sigma^2}{(r^2-ab)^2}\,\mathrm{d}r^2 + \frac{\sigma^2}{1-x^2}\,\mathrm{d}x^2 \\ &+ 2\mathrm{d}t\left(\frac{(r^2-ab)x^2(\sigma^2+a^2+ab)}{\sigma^2(a+b)}\,\mathrm{d}\phi + \frac{(r^2-ab)(1-x^2)(\sigma^2+b^2+ab)}{\sigma^2(a+b)}\,\mathrm{d}\psi\right) \\ &+ \frac{x^2\left(r^2\sigma^2+a^2(\sigma^2+x^2(a+b)^2)\right)}{\sigma^2}\,\mathrm{d}\phi^2 + \frac{2ab(a+b)^2x^2(1-x^2)}{\sigma^2}\,\mathrm{d}\phi\,\mathrm{d}\psi \\ &+ \frac{(1-x^2)\left(r^2\sigma^2+b^2(\sigma^2+(1-x^2)(a+b)^2)\right)}{\sigma^2}\,\mathrm{d}\psi^2 \, . \end{aligned} \tag{4.176}$$

Next we shall consider null geodesics. In GNC $\left(v, R, y, \tilde{\phi}, \tilde{\psi}\right)$, R is the affine parameter for the radial past directed null geodesics, and we denote the derivative of any quantity X with respect to the affine parameter by $\dot{X} = \partial X/\partial R$. Thus we have the Lagrangian

$$\begin{aligned} \mathcal{L}(r,x,\dot{t},\dot{r},\dot{x},\dot{\phi},\dot{\psi}) &= \frac{(r^2-ab)^2}{\sigma^2(a+b)^2}\,\dot{t}^2 + \frac{r^2\sigma^2}{(r^2-ab)^2}\,\dot{r}^2 + \frac{\sigma^2}{1-x^2}\,\dot{x}^2 \\ &+2\dot{t}\left(\frac{(r^2-ab)x^2(\sigma^2+a^2+ab)}{\sigma^2(a+b)}\,\dot{\phi} + \frac{(r^2-ab)(1-x^2)(\sigma^2+b^2+ab)}{\sigma^2(a+b)}\,\dot{\psi}\right) \\ &+\frac{x^2\left(r^2\sigma^2+a^2(\sigma^2+x^2(a+b)^2)\right)}{\sigma^2}\,\dot{\phi}^2 + \frac{2ab(a+b)^2x^2(1-x^2)}{\sigma^2}\,\dot{\phi}\,\dot{\psi} \\ &+\frac{(1-x^2)\left(r^2\sigma^2+b^2(\sigma^2+(1-x^2)(a+b)^2)\right)}{\sigma^2}\,\dot{\psi}^2 \end{aligned} \tag{4.177}$$

where the spacetime coordinates (t, r, x, ϕ, ψ) and their R derivatives are functions of the five GNC $\left(v, R, y, \tilde{\phi}, \tilde{\psi}\right)$. The null geodesics satisfy the Euler-Lagrange equation

$$\frac{\partial}{\partial R}\frac{\partial \mathcal{L}}{\partial \dot{X}} = \frac{\partial \mathcal{L}}{\partial X} \, . \tag{4.178}$$

Note that $\mathcal{L}$ is independent of t, ϕ and ψ, so let us consider their Euler-Lagrange equations first. These simplify to

$$\frac{\partial \mathcal{L}}{\partial \dot{X}_i} = 2f_i\left(v, y, \tilde{\phi}, \tilde{\psi}\right) \qquad i = t, \phi, \psi \tag{4.179}$$

which explicitly, can be written in matrix form

$$\begin{pmatrix} g_{tt} & g_{t\phi} & g_{t\psi} \\ g_{t\phi} & g_{\phi\phi} & g_{\phi\psi} \\ g_{t\psi} & g_{\phi\psi} & g_{\psi\psi} \end{pmatrix} \begin{pmatrix} \dot{t} \\ \dot{\phi} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} f_t \\ f_\phi \\ f_\psi \end{pmatrix} . \tag{4.180}$$

We can fix the functions f_i using the properties $L \cdot K = 0$, $L \cdot \partial/\partial\tilde{\phi} = 0$ and $L \cdot \partial/\partial\tilde{\psi} = 0$ of GNC, where the vectors fields

$$\begin{aligned} K = \frac{\partial}{\partial v} = \frac{\partial}{\partial t} \qquad \frac{\partial}{\partial \tilde{\phi}} = \frac{\partial}{\partial \phi} \qquad \frac{\partial}{\partial \tilde{\psi}} = \frac{\partial}{\partial \psi} \\ L = \frac{\partial}{\partial R} = \dot{t}\frac{\partial}{\partial t} + \dot{r}\frac{\partial}{\partial r} + \dot{y}\frac{\partial}{\partial y} + \dot{\phi}\frac{\partial}{\partial \phi} + \dot{\psi}\frac{\partial}{\partial \psi} \; . \end{aligned} \tag{4.181}$$

Whereas K has to be ∂_v in GNC and ∂_t in BL coordinates, $\partial_{\tilde{\phi}}$ and $\partial_{\tilde{\psi}}$ in general can be linear independent combinations of ∂_ϕ and ∂_ψ, but it is convenient to make the choice we made above. This also fixes the following relation between BL coordinates and GNC as functions: $t = t(v, R, y)$, $r = r(R, y)$, $x = x(R, y)$, $\phi = \phi(R, y, \tilde{\phi})$ and $\psi = \phi(R, y, \tilde{\psi})$.

Thus in matrix form the three GNC conditions given above can be written compactly as

$$\begin{pmatrix} g_{tt} & g_{t\phi} & g_{t\psi} \\ g_{t\phi} & g_{\phi\phi} & g_{\phi\psi} \\ g_{t\psi} & g_{\phi\psi} & g_{\psi\psi} \end{pmatrix} \begin{pmatrix} \dot{t} \\ \dot{\phi} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.182}$$

therefore the functions f_i are simply the constants $f_t = 1$ and $f_\phi = f_\psi = 0$ and the derivatives $\dot{t}$, $\dot{\phi}$ and $\dot{\psi}$ are given by

$$\begin{pmatrix} \dot{t} \\ \dot{\phi} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} g_{tt} & g_{t\phi} & g_{t\psi} \\ g_{t\phi} & g_{\phi\phi} & g_{\phi\psi} \\ g_{t\psi} & g_{\phi\psi} & g_{\psi\psi} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.183}$$

where $^{-1}$ denotes the matrix inverse. All metric functions here are explicitly functions of r and x, so we have the explicit expressions for $\dot{t}(r, x)$, $\dot{\phi}(r, x)$ and $\dot{\psi}(r, x)$, which are needed later.

Since we can only compute the coordinate transformation from BL coordinates to GNC perturbatively near the horizon i.e. around $r = r_+ = \sqrt{ab}$ in BL coordinates and in small R in GNC, we may write r and x as power series expansions in R

$$r = r_+ + \sum_{n=1}^{N} r_n(y) R^n \qquad x = y + \sum_{n=1}^{N} x_n(y) R^n \tag{4.184}$$

for some large enough N. We also used reparametrisation freedom in the coordinate y to fix x to be y in 0th order in R. We still have the Euler-Lagrange equations for

r and x to consider, and since we are considering *null* geodesics we also have the null constraint $\mathcal{L} = 0$. But since not all three are independent, we shall discard the r Euler-Lagrange equation which is the longest. We use the expressions for $\dot{t}$, $\dot{\phi}$ and $\dot{\psi}$ we found above and substitute r and x with the power series expansions in the Euler-Lagrange equation for x and the null constraint. We demand these two equations to be satisfied simultaneously order by order in R to find the coefficients $r_n(y)$ and $x_n(y)$. This leads to

$$\begin{aligned} x &= y + \frac{y\left(1-y^2\right)\left(b^2-a^2\right)R^2}{2\sigma_+^4} - \frac{y\left(1-y^2\right)\sqrt{ab}(a+b)^2\left(b^2-a^2\right)R^3}{\sigma_+^8} \\ &\quad +\mathcal{O}(R^4) \qquad (4.185) \\ r &= \sqrt{ab} + \frac{(a+b)^2 R}{\sigma_+^2} - \frac{\sqrt{ab}(a+b)^4 R^2}{\sigma_+^6} \\ &\quad -\frac{(a+b)^5}{3\sigma_+^{10}}\left[a^3\left(3y^6-10y^4+10y^2-3\right) - a^2 b\left(9y^6-19y^4+9y^2+6\right)\right. \\ &\quad \left. +ab^2\left(9y^6-8y^4-2y^2-5\right) - b^3y^2\left(3y^4+y^2-1\right)\right] R^3 + \mathcal{O}(R^4) \qquad (4.186) \end{aligned}$$

where $\sigma_+^2 = \sigma^2(R=0) = ab + a^2(1-y^2) + b^2y^2$. We then substitute the expressions for x and r into the metric functions $g_{\mu\nu}$ and expand in power series of R. We also need to expand $\mathrm{d}x^\mu$ in power series, which are explicitly

$$\mathrm{d}t = \mathrm{d}v + \dot{t}\,\mathrm{d}R + \left(\int \frac{\partial \dot{t}}{\partial y}\,\mathrm{d}R + h_t'(y)\right)\mathrm{d}y \qquad (4.187)$$

$$\mathrm{d}\phi = \mathrm{d}\tilde{\phi} + \dot{\phi}\,\mathrm{d}R + \left(\int \frac{\partial \dot{\phi}}{\partial y}\,\mathrm{d}R + h_\phi'(y)\right)\mathrm{d}y \qquad (4.188)$$

$$\mathrm{d}\psi = \mathrm{d}v + \dot{\psi}\,\mathrm{d}R + \left(\int \frac{\partial \dot{\psi}}{\partial y}\,\mathrm{d}R + h_\psi'(y)\right)\mathrm{d}y \qquad (4.189)$$

$$\mathrm{d}x = \frac{\partial x}{\partial R}\,\mathrm{d}R + \frac{\partial x}{\partial y}\,\mathrm{d}y \qquad (4.190)$$

$$\mathrm{d}r = \frac{\partial r}{\partial R}\,\mathrm{d}R + \frac{\partial r}{\partial y}\,\mathrm{d}y\ . \qquad (4.191)$$

where the functions $h_i'(y)$ are determined by demanding $L\cdot\partial/\partial y = 0$, $\partial/\partial y\cdot\partial/\partial\tilde{\phi}|_{R=0} = 0$ and $\partial/\partial y\cdot\partial/\partial\tilde{\psi}|_{R=0} = 0$ in GNC:

$$h_t' = \frac{y\left(b^2-a^2\right)(a+b)^2(\sigma_+^2-ab)}{2\sigma_+^2\sqrt{ab}} \qquad (4.192)$$

$$h_{\tilde{\phi}}' = -\frac{y\sqrt{ab}\left(b^2-a^2\right)}{a\sigma_+^2} \qquad (4.193)$$

$$h_{\tilde{\psi}}' = -\frac{y\sqrt{ab}\left(b^2-a^2\right)}{b\sigma_+^2}\ . \qquad (4.194)$$

We have now everything we need to write the metric (4.173) in GNC near the horizon located at $R = 0$. To first order in R, the metric reads

$$
\begin{aligned}
\mathrm{d}s^2 \;=\;& \left(\frac{4ab(a+b)^2R^2}{\sigma_+^6} + \frac{4\sqrt{ab}(a+b)^4(\sigma_+^2 - 4ab)R^3}{\sigma_+^{10}}\right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R \\
&+ 2\mathrm{d}v\mathrm{d}y \left(\frac{2\left(a^2-b^2\right)yR}{\sigma_+^2} + \frac{6\sqrt{ab}\left(b^2-a^2\right)(a+b)^2yR^2}{\sigma_+^6}\right) \\
&+ 2R\mathrm{d}v \left(\frac{2\sqrt{ab}(a+b)^2}{\sigma_+^4}\right)\left(y^2(a(2-y^2)+by^2)\,\mathrm{d}\phi + (1-y^2)(a(1-y^2)+b(1+y^2))\,\mathrm{d}\psi\right) \\
&+ 2R^2\mathrm{d}v \left(\frac{a+b}{\sigma_+^2}\right)^4 \left(y^2(2a^2(a-3b) - a(a-b)(3a+b)y^2 + (a-b)^2(a+b)y^4)\mathrm{d}\phi\right. \\
&\left.+ (1-y^2)\left(a^3 - 5ab^2 + \left(-2a^3 + a^2b + b^3\right)y^2 + (a-b)^2(a+b)y^4\right)\mathrm{d}\psi\right) \\
&+ \left(\frac{\sigma_+^2}{1-y^2} + \frac{2\sqrt{ab}(a+b)^2R}{\sigma_+^2\,(1-y^2)}\right)\mathrm{d}y^2 \\
&+ 4R\mathrm{d}y \left(\frac{\left(b^2-a^2\right)(a+b)^2y}{\sigma_+^2}\right)\left(ay^2\mathrm{d}\phi + b\left(1-y^2\right)\mathrm{d}\psi\right) \\
&+ \left(\frac{a(a+b)^2y^2\left(a+by^2\right)}{\sigma_+^2} + \frac{2\sqrt{ab}(a+b)^4y^2\left(a^2 + (2ab-3a^2)y^2 + (a-b)^2y^4\right)R}{\sigma_+^6}\right)\mathrm{d}\phi^2 \\
&+ \left(\frac{b(a+b)^2\left(1-y^2\right)\left(a(1-y^2)+b\right)}{\sigma_+^2}\right. \\
&\left.+\frac{2\sqrt{ab}(a+b)^4\left(1-y^2\right)\left(b^2y^2 + 2\left(1-y^2\right)\left(ab-b^2\right) + \left(1-y^2\right)^2(a-b)^2\right)R}{\sigma_+^6}\right)\mathrm{d}\psi^2 \\
&+ 2\mathrm{d}\phi\mathrm{d}\psi\left(\frac{ab(a+b)^2y^2\left(1-y^2\right)}{\sigma_+^2} - \frac{2ab\sqrt{ab}(a+b)^4y^2\left(1-y^2\right)R}{\sigma_+^6}\right) . \qquad (4.195)
\end{aligned}
$$

It can be checked that by setting $a = b$ it reduces to the self-dual case (4.105).

Chapter 5

Near-Horizon Geometries of M2, M5 and D3 Black Branes

5.1 Introduction

In the previous chapters we have focussed only on solutions with a *compact* horizon cross-section H, because we are mainly interested in *black hole* spacetimes in this thesis. However as illustrated in chapter 1.6, near-horizon geometry is defined for *any* spacetime containing an extreme Killing horizon regardless of the topology of H. Thus we can extend our analyses to extreme *black branes* which have non-compact H. As mentioned also in the introduction, black brane spacetimes often appear in the context of AdS/CMT and the extremal limit corresponds to the 0 temperature phase on the CMT side. However the classification of extreme black branes is a major open problem, even for their near-horizon geometries. Many of the near-horizon geometry symmetry enhancement and classification theorems that apply to black *holes* cannot be generalised directly to black *branes*, because they often involve *global arguments* which rely on *compactness*, thereby allowing one to bypass solving for the general local metric.

It was proved in [26] that any static vacuum $\Lambda \leq 0$ near-horizon geometry with a *compact* H must be given by $h_a = 0$ and $F = \Lambda$, with the metric on H satisfying $R_{ab} = \Lambda\gamma_{ab}$ (recall these quantities are the near-horizon data (1.13)). In $4D$ this result is also valid for $\Lambda > 0$. This is not true for non-compact H as we demonstrate with Poincaré-AdS as a counterexample in the next section. Although the local analysis in [26] applies regardless of compactness, not much more is known about the general solutions to the near-horizon geometries of non-compact horizons, even for the static case. Progress has been made on the AdS/CFT interpretation of the general static solutions which are dual to vacuum states of some CFT on a cone [74].

As a starting point, it would therefore be useful to revisit some well known extreme black brane solutions and write them in Gaussian null coordinates to get a taste of how near-horizon geometries with non-compact cross-sections typically look like.

A p-brane is a p-spatial-dimensional object in string theory which sweeps out a $(p+1)$-dimensional hyperplane called "world-volume" in the ambient D-dimensional spacetime. The special cases of $p = 0$ and $p = 1$ refer to point particle and string respectively. According to the string ansatz, the simplest class of classical p-brane solutions possess $(\text{Poincaré})_d \times SO(D-d)$ symmetry. A D-brane is a black p-brane where open strings can end on with the Dirichlet boundary condition. (a review on p-branes can be found in [121].)

In this chapter we write the BPS extremal M2, M5 and D3 black brane solutions of 11 and 10-dimensional supergravity in Gaussian null coordinates and take the near-horizon limit to recover their well-known near-horizon geometries: $\text{AdS}_4 \times S^7$, $\text{AdS}_7 \times S^4$ and $\text{AdS}_5 \times S^5$. Their horizons are the Poincaré horizons in their AdS factors, but they are usually given in Poincaré coordinates which are singular on the Poincaré horizon. We want to derive their near-horizon geometries in a more precise manner.

5.2 AdS$_D$ in Gaussian Null Coordinates

First of all we need to write the AdS$_D$ metric Gaussian null coordinates. In Poincaré coordinates it is given by

$$\mathrm{d}s^2 = \lambda^2(-\mathrm{d}t + \delta_{ab}\mathrm{d}x^a\mathrm{d}x^b) + \frac{\mathrm{d}\lambda^2}{\lambda^2} \tag{5.1}$$

where the Latin indices $a, b = 1, ..., D-2$. The Poincaré horizon is the surface $\lambda = 0$ and it is an extreme Killing horizon of the Killing field $\partial/\partial t$. However since λ is singular on the horizon, the Poincaré coordinates are not suitable for describing near-horizon geometry or extending the geometry beyond the horizon.

It can be shown that by the coordinate transformation (see section 7.5 in [88])

$$\lambda = r\cosh\eta \qquad t = v + \frac{1}{r} \qquad x^a = \frac{\mu^a \tanh\eta}{r} \tag{5.2}$$

with $\delta_{ab}\mu^a\mu^b = 1$, the metric (5.1) becomes

$$\mathrm{d}s^2 = \cosh^2\eta\,(-r^2\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r) + \mathrm{d}\eta^2 + \sinh^2\eta\,\mathrm{d}\Omega^2_{D-3}\,, \tag{5.3}$$

which is in the desired GNC. The Poincaré horizon is now located at $r = 0$ which is a manifestly a smooth extreme Killing horizon of the Killing field $\partial/\partial v$. One may also cross and go beyond the horizon by taking r to negative values. The metric (5.3) is manifestly a warped product of AdS_2 with a hyperbolic space $\mathbb{H}^{D-2}$. This is reminiscent to static near-horizon geometry even though we have not taken any near-horizon limit:

it was proved that any static NHG with a simply connected H is a warped product of AdS_2, dS_2 or $\mathbb{R}^{1,1}$ with H [93]. Thus the AdS space can be regarded as a static vacuum ($\Lambda < 0$) near-horizon geometry with a non-compact H; furthermore, it is its own NHG. Clearly by changing $r\cosh^2\eta \to r$ in (5.3), the near-horizon data $F = \cosh^2\eta \neq const$ and $h_\eta = 2\cosh\eta\sinh\eta \neq 0$, therefore the static vacuum NHG theorem for compact H mentioned in the previous section cannot generalise to non-compact H.

5.3 M2 Brane

The M2 brane solution in the 11-dimensional supergravity theory has $(\text{Poincaré})_3 \times SO(8)$ symmetry. In isotropic coordinates it takes the form

$$\begin{aligned} \mathrm{d}s^2 &= H^{-\frac{2}{3}}\left(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2\right) + H^{\frac{1}{3}}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_7^2\right) \\ H &= 1 + \frac{Q}{r^6} \end{aligned} \tag{5.4}$$

where $Q > 0$ is a constant which sets the mass scale of the solution. A horizon is located at $r = 0$.

Since $\lim_{r\to 0} H \sim Q/r^6$, one (simplest) way to argue that the near-horizon geometry is $\mathrm{AdS}_4 \times S^7$ is by considering the the coordinate transformation $r^2 = R$. Let $Q = 1$ for simplicity, the metric in the $r \to 0$ limit becomes

$$\mathrm{d}s^2_{r\to 0} = R^2\left(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2\right) + \frac{1}{4R^2}\mathrm{d}R^2 + \mathrm{d}\Omega_7^2$$

which is just $\mathrm{AdS}_4 \times S^7$ with AdS_4 written in Poincaré coordinates. The Poincaré horizon $\{R = 0\}$ is a degenerate Killing horizon of the Killing vector field ∂_t; however, the metric is singular at $R = 0$ and therefore it is unsuitable for describing the near-horizon geometry.

Our goal is to find a coordinate transformation $(t, r, x_i, \Omega_7) \to (v, \lambda, y^a, \Omega_7)$ to get a regular metric on the horizon with the S^7 remaining unchanged. The near-horizon geometry should take the form

$$\mathrm{d}s^2_{NH} = \lambda^2 F(y)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}\lambda + 2\lambda h_a(y)\mathrm{d}v\mathrm{d}y^a + \gamma_{\alpha\beta}(y)z^\alpha z^\beta \tag{5.5}$$

where

$$z^\alpha = (y^a, \Omega_7) \tag{5.6}$$

are the coordinates on the spatial cross-section H of the horizon. Since λ is an affine parameter of past directed null geodesic, it is natural to proceed from the Lagrangian along the geodesic $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, where we use the notation $\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$, (recall that

(v, y_i) are unchanged along the geodesic)

$$\mathcal{L} = H^{-\frac{2}{3}}(-\dot{t}^2 + \dot{x}_i^2) + H^{\frac{1}{3}}\dot{r}^2 \ .$$

The "equations of motion" are given by the Euler Lagrange equation $\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$, which can be integrated to

$$-H^{-\frac{2}{3}}\dot{t} = k_0(v, y) \tag{5.7}$$

$$H^{-\frac{2}{3}}\dot{x}_i = k_i(v, y) \ . \tag{5.8}$$

Applying these two equations into the null geodesic constraint $\mathcal{L} = 0$, we obtain an equation for r

$$\dot{r} = H^{\frac{1}{6}}\sqrt{-k^2} \qquad\qquad k^2 \equiv \eta^{\mu\nu}k^\mu k^\nu = -k_0^2 + k_1^2 + k_2^2 \tag{5.9}$$

and we must have $k^2 < 0$.

The future directed vector field N which defines the GNC is Killing and null on the horizon so it can be chosen to be $N = \frac{\partial}{\partial t}$. By a change of basis, we can expand $L = \frac{\partial}{\partial \lambda} = \frac{\partial t}{\partial \lambda}\frac{\partial}{\partial t} + \frac{\partial r}{\partial \lambda}\frac{\partial}{\partial r} + \frac{\partial x_i}{\partial \lambda}\frac{\partial}{\partial x_i}$. By the construction of GNC $N \cdot L = 1$ everywhere, meaning that $\frac{\partial t}{\partial \lambda} = 1$ and $k_0 = 1$, and we can integrate (5.7) to get

$$t = -\int \mathrm{d}\lambda \ H^{\frac{2}{3}} + f_0(v, y) \ .$$

Expanding N in the same way and equating $N = \frac{\partial}{\partial t}$ then leads to $\frac{\partial t}{\partial v} = 1$ and $\frac{\partial r}{\partial v} = \frac{\partial x_i}{\partial v} = 0$. Hence

$$t = -\int \mathrm{d}\lambda \ H^{\frac{2}{3}} + v + f_0(y) \ , \tag{5.10}$$

and the equations (5.8) and (5.9) for x_i and r integrate to give

$$x_i = k_i(y)\int \mathrm{d}\lambda \ H^{\frac{2}{3}} + f_i(y) \tag{5.11}$$

$$r = \sqrt{-k^2(y)}\int \mathrm{d}\lambda \ H^{\frac{1}{6}} + g(y) \ . \tag{5.12}$$

Setting the horizons $\{r = 0\}$ and $\{\lambda = 0\}$ to coincide imposes the boundary condition $r(\lambda = 0) = 0$.

In order to write the near-horizon geometry in GNC we write $\mathrm{d}t$, $\mathrm{d}x_i$ and $\mathrm{d}r$ in terms of $\mathrm{d}v$, $\mathrm{d}y^a$ and $\mathrm{d}\lambda$. Differentiating the above gives

$$
\begin{aligned}
\mathrm{d}t &= -H^{\frac{2}{3}}\mathrm{d}\lambda + \mathrm{d}v + \partial_a(f_0 - I)\mathrm{d}y^a \\
\mathrm{d}x_i &= k_i H^{\frac{2}{3}}\mathrm{d}\lambda + \partial_a(f_i + k_i I)\mathrm{d}y^a \\
\mathrm{d}r &= H^{\frac{1}{6}}\sqrt{-k^2}\mathrm{d}\lambda + \partial_a\left(\sqrt{-k^2}\int \mathrm{d}\lambda\ H^{\frac{1}{6}} + g\right)\mathrm{d}y^a \ ,
\end{aligned}
$$

with $I = \int \mathrm{d}\lambda\ H^{\frac{2}{3}}$ and $\partial_a I = \frac{\partial I}{\partial y^a}$. From here it is easy to see that $g_{v\lambda} = 1$ and $g_{\lambda\lambda} = 0$ as in (5.5). The property $L.\frac{\partial}{\partial y^a} = 0$ of GNC sets the constraints on $f_\mu(y)$ and $g(y)$ and guarantees $g_{\lambda a} = 0$ as demanded.

Let us now focus on the near-horizon geometry; first let us consider the equation (5.9) for r along the null geodesic. Expanding $H^{-\frac{1}{6}}$ in power series of r and integrating with respect to $\mathrm{d}r$ allows us to express λ in terms of r:

$$
\begin{aligned}
\frac{\partial r}{\partial \lambda} &= H^{\frac{1}{6}}\sqrt{-k^2(y)} \\
H^{-\frac{1}{6}} &= \frac{r}{Q} - \frac{r^7}{6Q^7} + \mathcal{O}(r^8) \\
\therefore\ \lambda\sqrt{-k^2} &= \int H^{-\frac{1}{6}}\ \mathrm{d}r + C(y) = \frac{r^2}{2Q} - \frac{r^8}{48Q^7} + \mathcal{O}(r^9) + C(y)\ .
\end{aligned}
$$

Because of the boundary condition $r(\lambda = 0) = 0$, the integration constant $C(y)$ in the last line necessarily vanishes; in the near-horizon limit of $r \to 0$ we need only keep the leading term, thus for the purpose of studying the near-horizon geometry

$$
r^2 = 2Q\lambda\sqrt{-k^2}\ . \tag{5.13}
$$

This allows us to write in the near-horizon limit the metric function $H = H(r)$ as a function of λ and y^a, and we have

$$
\begin{aligned}
H^{-\frac{2}{3}} &= \frac{4(-k^2)\lambda^2}{Q^2} - \frac{64(-k^2)^{\frac{5}{2}}\lambda^5}{3Q^5} + \mathcal{O}(\lambda^6) \\
H^{\frac{1}{3}} &= \frac{Q}{2\sqrt{-k^2}\lambda} + \frac{4(-k^2)\lambda^2}{3Q^2} + \mathcal{O}(\lambda^3) \\
\partial_a I &= \frac{Q^2}{4k^4\lambda}\frac{\mathrm{d}(-k^2)}{\mathrm{d}y^a} + \frac{\lambda^2}{3\sqrt{-k^2}Q}\frac{\mathrm{d}(-k^2)}{\mathrm{d}y^a} + \mathcal{O}(\lambda^3)\ .
\end{aligned}
$$

By writing $H^{\frac{1}{6}} = \frac{Q}{r} + \frac{r^5}{6Q^5} + \mathcal{O}(r^7)$ in power series of λ using (5.13) and integrate (5.9) along the geodesic, we deduce that $g(y) = 0$ from the boundary condition $r(\lambda = 0) = 0$. Hence in the near-horizon limit

$$
r\mathrm{d}r = Q\sqrt{-k^2}\mathrm{d}\lambda + Q\lambda\partial_a\sqrt{-k^2}\mathrm{d}y^a
$$

The near-horizon geometry is obtained by substituting the above expressions for $\mathrm{d}t$, $\mathrm{d}x_i$ and $\mathrm{d}r$ into (5.4). Hence in power series of λ

$$\begin{aligned} g_{vv} &= -H^{-\frac{2}{3}} = \frac{4k^2\lambda^2}{Q^2} + \mathcal{O}(\lambda^5) \\ g_{va} &= -H^{-\frac{2}{3}}\partial_a(f_0 - I) = \frac{\lambda}{-k^2}\frac{\mathrm{d}(-k^2)}{\mathrm{d}y^a} + \frac{4k^2}{Q^2}\frac{\mathrm{d}f_0}{\mathrm{d}y^a}\lambda^2 + \mathcal{O}(\lambda^4) \\ g_{ab} &= \frac{Q^2}{4k^4}(\delta^{ij}\delta^{mn}k_ik_m\partial_ak_j\partial_bk_n - k^2\delta^{ij}\partial_ak_i\partial_bk_j) + \mathcal{O}(\lambda) \end{aligned}$$

where we have used the fact that $k_0 = 1$ and therefore $\frac{\mathrm{d}k^2}{\mathrm{d}y^a} = 2\delta^{ij}k_i\frac{\mathrm{d}k_j}{\mathrm{d}y^a}$ in the last line. Since to leading order all dependence on y enters from $k^2(y)$, the functions k_1 and k_2 can be used to define the GNC instead of y_1 and y_2. k^2 can be conveniently split into $k^2 = -1 + |\underline{k}^2|$ with $|\underline{k}^2| < 1$ as $k^2 < 0$, such that $|\underline{k}^2| = k_1^2 + k_2^2$, thus it is natural to convert k_i into polar coordinates with $k_1 = \rho\cos\theta$, $k_2 = \rho\sin\theta$ and $\rho^2 = |\underline{k}^2|$, which yields

$$g_{ab}\mathrm{d}y^a\mathrm{d}y^b = \frac{Q^2}{4(1-\rho^2)}\left(\rho^2\mathrm{d}\theta^2 + \frac{\mathrm{d}\rho^2}{1-\rho^2}\right) .$$

With the change of coordinates $R \equiv -k^2\lambda$, the near-horizon geometry in Gaussian Null Coordinates is simply

$$\mathrm{d}s^2_{NH} = \frac{1}{1-\rho^2}\left[\frac{-4R^2}{Q^2}\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R + \frac{Q^2}{4}\left(\rho^2\mathrm{d}\theta^2 + \frac{\mathrm{d}\rho^2}{1-\rho^2}\right)\right] + Q^2\mathrm{d}\Omega_7^2 .$$

In order to make the $\mathrm{AdS}_4 \times S^7$ structure manifest, let us consider a further coordinate transformation $\cosh^2\eta = 1/(1-\rho^2)$. The metric then becomes

$$\mathrm{d}s^2_{NH} = \cosh^2\eta\left(\frac{-4R^2}{Q^2}\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R\right) + \frac{Q^2}{4}\left(\sinh^2\eta\ \mathrm{d}\theta^2 + \mathrm{d}\eta^2\right) + Q^2\mathrm{d}\Omega_7^2 ;$$

comparing with the AdS metric in (5.3) confirms the near-horizon geometry is indeed $\mathrm{AdS}_4 \times S^7$, written as a warped product of AdS_2 with $\mathbb{H}^2$ plus an S^7.

5.4 M5 Brane

The M5 brane solution of 11-dimensional supergravity in isotropic coordinates is given by

$$\begin{aligned} \mathrm{d}s^2 &= H^{-\frac{1}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}x_i^2\right) + H^{\frac{1}{2}}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_4^2\right) \qquad i = 1, 2, ..., 5 \\ H &= 1 + \frac{Q^3}{r^3} . \end{aligned}$$

Although the metric is very similar to the M2 brane (5.4), unlike the M2 brane which contains a genuine timelike singularity, the M5 brane has no singularity at all. Nevertheless, there is still a horizon at $r=0$. Following an analogous calculation as in the M2 case, it is straightforward to verify that the near-horizon geometry is $\mathrm{AdS}_7 \times S^4$

$$\begin{aligned}
\mathrm{d}s^2_{NH} &= \frac{1}{1-\rho^2}\left[\frac{-R^2}{4Q^2}\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R + 4Q^2\left(\rho^2\mathrm{d}\Omega_4^2 + \frac{\mathrm{d}\rho^2}{1-\rho^2}\right)\right] + Q^2\mathrm{d}\Omega_4^2 \\
&= \cosh^2\eta\left(\frac{-R^2}{4Q^2}\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R\right) + 4Q^2\left(\sinh^2\eta\ \mathrm{d}\Omega_4^2 + \mathrm{d}\eta^2\right) + Q^2\mathrm{d}\Omega_4^2
\end{aligned}$$

expressed as a warped product of AdS_2 with $\mathbb{H}^5$ plus an S^4.

5.5 D3 Brane

The D3 brane solution in the 10-dimensional type IIB supergravity has $(\text{Poincaré})_4 \times SO(6)$ symmetry. In isotropic coordinates it takes the form

$$\begin{aligned}
\mathrm{d}s^2 &= H^{-\frac{1}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}x_i^2\right) + H^{\frac{1}{2}}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\right) \qquad i = 1,2,3 \\
H &= 1 + \frac{Q^4}{r^4}
\end{aligned} \tag{5.14}$$

with again Q being a constant which sets the mass scale to the theory and a horizon is located at $r=0$. By taking $H \sim \frac{Q^4}{r^4}$ in the of limit $r \to 0$, one finds (naively) that its near-horizon geometry is $\mathrm{AdS}_5 \times S^5$

We follow the same strategy as for the M2 brane. However, for the D3 brane the equivalent of equation (5.9) is greatly simplified to

$$\dot{r} = \sqrt{-k^2(y)}$$

due to the powers in H cancelling. The coordinate transformation rules are

$$\begin{aligned}
t &= -\int \mathrm{d}\lambda\ H^{\frac{1}{2}} + v + f_0(y) \\
x_i &= k_i(y)\int \mathrm{d}\lambda\ H^{\frac{1}{2}} + f_i(y) \\
r &= \sqrt{-k^2(y)}\lambda + g(y)
\end{aligned} \tag{5.15}$$

and $r(\lambda = 0) = 0$ means that the function $g(y) = 0$. By differentiating, we obtain

$$\begin{aligned}
\mathrm{d}t &= -H^{\frac{1}{2}}\mathrm{d}\lambda + \mathrm{d}v + \partial_a\left(f_0 - \int \mathrm{d}\lambda\ H^{\frac{1}{2}}\right)\mathrm{d}y^a \\
\mathrm{d}x_i &= k_i H^{\frac{1}{2}}\mathrm{d}\lambda + \partial_a\left(f_i + k_i\int \mathrm{d}\lambda\ H^{\frac{1}{2}}\right)\mathrm{d}y^a \\
\mathrm{d}r &= \sqrt{-k^2}\mathrm{d}\lambda + \lambda\frac{\mathrm{d}\sqrt{-k^2(y)}}{\mathrm{d}y^a}\mathrm{d}y^a\ .
\end{aligned}$$

To find the near-horizon geometry we again write H in terms of λ, which is simply $H = 1 + \frac{Q^4}{k^4\lambda^4}$ by applying (5.15). Expanding in power series of λ,

$$\begin{aligned} H^{\frac{1}{2}} &= \frac{Q^2}{-k^2\lambda^2} + \frac{-k^2\lambda^2}{2Q^2} + \mathcal{O}(\lambda^6) \\ H^{-\frac{1}{2}} &= \frac{-k^2\lambda^2}{Q^2} - \frac{k^4(-k^2)\lambda^6}{Q^6} + \mathcal{O}(\lambda^7) \\ \partial_a \int \mathrm{d}\lambda\ H^{\frac{1}{2}} &= \frac{Q^2}{k^4}\frac{\mathrm{d}(-k^2)}{\mathrm{d}y^a}\frac{1}{\lambda} + \frac{\lambda^3}{6Q^2}\frac{\mathrm{d}(-k^2)}{\mathrm{d}y^a} + \mathcal{O}(\lambda^7)\ . \end{aligned}$$

Using k_i as coordinates instead of y^a as before, the sought-for metric functions in near-horizon limit are

$$\begin{aligned} g_{vv} &= -H^{-\frac{1}{2}} = \frac{k^2}{Q^2}\lambda^2 + \mathcal{O}(\lambda^6) \\ g_{vi} &= \frac{2}{k^2}\delta_i^j k_j \lambda + \mathcal{O}(\lambda^2) \\ g_{ij} &= \frac{Q^2}{k^4}(\delta_i^m \delta_j^n k_m k_n - k^2\delta_{ij}) + \mathcal{O}(\lambda)\ . \end{aligned}$$

By changing the coordinates (k_1, k_2, k_3) into polar coordinates (ρ, θ, ϕ) such that $k_1 = \rho\cos\theta\cos\phi$, $k_2 = \rho\cos\theta\sin\phi$ and $k_3 = \rho\sin\theta$, the near-horizon geometry takes the form

$$\begin{aligned} \mathrm{d}s^2_{NH} &= \frac{1}{1-\rho^2}\left[\frac{-R^2}{Q^2}\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R + Q^2\left(\rho^2\mathrm{d}\Omega_2^2 + \frac{\mathrm{d}\rho^2}{1-\rho^2}\right)\right] + Q^2\mathrm{d}\Omega_5^2 \\ &= \cosh^2\eta\left(\frac{-R^2}{Q^2}\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R\right) + Q^2\left(\sinh^2\eta\mathrm{d}\Omega_2^2 + \mathrm{d}\eta^2\right) + Q^2\mathrm{d}\Omega_5^2 \end{aligned}$$

which is indeed $\mathrm{AdS}_5 \times S^5$ written as a warped product of AdS_2 with $\mathbb{H}^3$ plus an S^5.

5.6 Summary

We wrote the M2, M5 and D3 brane solutions in Gaussian null coordinates. We verified that their near-horizon geometries are $\mathrm{AdS}_4 \times S^7$, $\mathrm{AdS}_7 \times S^4$ and $\mathrm{AdS}_5 \times S^5$. We showed that the cross-sections of their horizons are some hyperbolic space plus some sphere.

Chapter 6

Conclusion

In chapter 2 we considered near-horizon geometries beyond Einstein-Maxwell theory. We showed that in $4D$ any stationary and axisymmetric near-horizon geometry in Einstein-Yang-Mills theory with a compact semi-simple gauge group and a non-negative cosmological constant must be that of abelian embedded extreme Kerr-Newman(-AdS) black hole. This near-horizon uniqueness theorem is somewhat unexpected given that $4D$ Einstein-Yang-Mills black holes are well known to grossly violate the "no-hair" theorem. Further, any such solution necessarily admits a near-horizon AdS_2 symmetry, thus our result generalises the AdS_2 near-horizon symmetry enhancement theorem to include *non-abelian* gauge fields. We also showed that any static solution must be a direct product of AdS and a constant curvature space. We then showed that in Einstein gravity with a cosmological constant coupled to a Maxwell field A and a massive complex scalar field ϕ, assuming non-zero coupling between A and ϕ, there exists no near-horizon geometry with a compact horizon in which ϕ is non-trivial. This result is valid in any dimensions and can be extended to any scalar potential $V(|\phi|)$ that satisfies $V' \geq 0$.

In chapter 3 we constructed the most general vacuum spacetime containing a smooth Killing horizon with compact spatial cross-sections in $3D$ Einstein gravity with a negative cosmological constant. We found that the general solution always admits a second commuting Killing field hence black hole solutions with a single Killing field do not exist, and deduced that the general solution must be related to the BTZ black hole (or its near-horizon geometry) by a diffeomorphism. We showed there is a general class of asymptotically AdS_3 extreme black holes with arbitrary charges with respect to one of the asymptotic-symmetry Virasoro algebras and vanishing charges with respect to the other. We interpret these as descendants of the extreme BTZ black hole and they are related to the extreme BTZ black hole by a large diffeomorphism. These are the first examples of descendant black holes. However, we did not find descendants of the non-extreme BTZ black hole within our general solution admitting a non-degenerate horizon, the general solution in this case is necessarily diffeomorphic to the BTZ solution by a small diffeomorphism. This suggests that descendants of the non-extreme BTZ black hole do not possess a Killing horizon.

In chapter 4 we considered the first order expansion from near-horizon geometry. We showed that for a compact horizon, the general solution to the first order expansion is a finite-parameter family, so this is not an intractable problem. As a concrete application, we determined the first order expansion from the near-horizon geometry of the extreme Kerr black hole. This has led to the following local uniqueness theorem: the only solution which admit a $U(1)$ symmetry compatible with a black hole is first order expansion of the extreme Kerr solution itself. Unlike the $4D$ electrovacuum black hole uniqueness theorem we reviewed in chapter 1, this local uniqueness theorem does not require asymptotic flatness or global hyperbolicity or any other global regularity assumptions. We then investigated the first order expansion from the near-horizon geometry of the extreme self-dual Myers-Perry black hole in $5D$. The only solutions which inherit the full $SU(2) \times U(1)$ symmetry and can describe black holes are the first order expansion of the extreme self-dual Myers-Perry black hole itself and the extreme $J = 0$ Kaluza Klein black hole, which are the only known solutions to admit this near-horizon geometry. If the symmetry assumption is relaxed to $U(1) \times U(1)$ in first order, then the most general solution is a 3-parameter family. This 3-parameter family is much more general than the two known solutions, thus it hints the possible existence of new black holes. In particular, this leaves the possibility of the existence asymptotically flat $5D$ vacuum black hole spacetimes containing bubbles wide open.

In chapter 5 we discussed near-horizon geometries with non-compact cross-sections. We revisited the near-horizon geometries of the BPS extremal M2, M5 and D3 black brane solutions of 11 and 10-dimensional supergravity by expressing them in Gaussian coordinates and taking the near-horizon limit. We verified their famous near-horizon geometries and showed that they can be expressed as a warped product of AdS_2 with a hyperbolic space plus a suitable sphere. Since the classification of near-horizon geometries of extreme black branes is largely unexplored, this exercise may help to shed some light on and initiate it.

In fact due to their applications in AdS/CMT correspondence, it would be interesting to generalise our work on Einstein-Maxwell-Higgs near-horizon geometry in chapter 2 to non-compact horizons. Such solutions would correspond to the ground states of the dual strongly coupled condensed matter system which is charged under the electromagnetic field. Then with the near-horizon geometry in hand, the next step would be to explore its first order expansion. In fact the first order analysis can already be done for compact horizons using the result found in section 2.3, to investigate if the scalar field turns on and breaks the gauge symmetry outside the horizon as in the non-extreme case [61]. This spontaneous symmetry breaking is not only limited to scalars, it can also be achieved with a non-abelian $SU(2)$ gauge field [62]. Therefore it will also be of interest to find the general solution to the near-horizon geometry in the Einstein-Yang-Mills(-Higgs) theory and explore its first order expansion.

As we have seen in chapter 4, the first order analysis provides an effective method to find possible new black hole solutions or to establish local uniqueness theorems. The near-horizon geometries we considered in this thesis are all near-horizon limits of some known vacuum black holes. However even in vacuum, there exists also a zoo of exotic near-horizon geometries in various dimensions which are not yet linked to any black holes (these solutions all contain a Killing horizon by construction, but this is not enough to qualify as a black hole). For example in Einstein vacuum with a non-positive cosmological constant, it was shown that there exists near-horizon geometries with spatial cross-sections homeomorphic to $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ in $6D$ [86]. In odd $D \geq 7$, there exists an infinite class of near-horizon geometries whose cross-sections are inhomogeneous Sasakian metrics [87]. It would therefore be very useful to apply the first order analysis on these near-horizon geometries with exotic horizon topologies, and see whether they can possibly arise from any new black hole solutions by checking against the MOTS condition (4.48). This will also help towards developing the classification of higher dimensional black holes.

Recall that in our analysis in chapter 3, we have assumed $\gamma^{(1)}(x) > 0 \ \forall x$ so the coordinate system does not break down anywhere and the whole spacetime is covered by a single coordinate patch. This however may be too restrictive from the point of view of the MOTS condition (4.48). Recall that the first order in fact extends to *all orders* in this case: according to (3.12) and (3.13), only the first order quantity $\gamma^{(1)}$ is required to determine the full general solution. It would therefore be interesting to revise this analysis under the weaker MOTS condition (4.48) and investigate descendant *black hole* solutions directly.

The first order formalism also serves as a powerful technique to probe black holes with a single Killing field. This is done by retaining only the $\mathbb{R}$ symmetry generated by the horizon generator K in first order. As we discussed in chapter 3, asymptotically AdS stationary black holes with a single Killing field are speculated to exist as the endpoint of superradiant instability, but such solutions are extremely difficult to fine due to the lack of symmetry. We have already shown that black holes with a single Killing field do not exist in $3D$ Einstein vacuum with a negative cosmological constant, the next logical step is therefore to examine the analogous problem in $4D$. The near-horizon geometry of the extreme Kerr-AdS black hole is the unique near-horizon geometry solution which satisfies the Einstein equation with a negative cosmological constant [85]. From (4.23) it is clear that its first order expansion with only the symmetry generated by K yields a system of two coupled second order PDE's of two variables, which does not seem too complicated a problem to solve.

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