

# ALGEBRAIC AND COMBINATORIAL CODIMENSION 1 TRANSVERSALITY

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ABSTRACT. The Waldhausen construction of Mayer-Vietoris splittings for chain complexes over an injective generalized free product is extended to Seifert-van Kampen splittings for  $CW$  complexes with fundamental group an injective generalized free product.

*Dedicated to Andrew Casson*

## INTRODUCTION

The close relationship between the topological properties of codimension 1 submanifolds and the algebraic properties of groups with a generalized free product structure first became apparent with the Seifert-van Kampen Theorem on the fundamental group of a union, the work of Kneser on 3-dimensional manifolds with fundamental group a free product, and the topological proof of Grushko's theorem by Stallings.

This paper describes two abstractions of the geometric codimension 1 transversality properties of manifolds (in all dimensions) :

- (1) the algebraic transversality construction of Mayer-Vietoris splittings of chain complexes of free modules over the group ring of an injective generalized free product,
- (2) the combinatorial transversality construction of Seifert-van Kampen splittings for  $CW$  complexes with fundamental group an injective generalized free product.

By definition, a group  $G$  is a *generalized free product* if it has one of the following structures :

- (A)  $G = G_1 *_H G_2$  is the amalgamated free product determined by group morphisms  $i_1 : H \rightarrow G_1$ ,  $i_2 : H \rightarrow G_2$ , so that there is defined a pushout square of groups

$$\begin{array}{ccc} H & \xrightarrow{i_1} & G_1 \\ i_2 \downarrow & & \downarrow j_1 \\ G_2 & \xrightarrow{j_2} & G \end{array}$$

The amalgamated free product is *injective* if  $i_1, i_2$  are injective, in which case so are  $j_1, j_2$ , with

$$G_1 \cap G_2 = H \subseteq G .$$

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The amalgamated free product is *finitely presented* if the groups  $G_1, G_2, H$  are finitely presented, in which case so is  $G$ . (If  $G$  is finitely presented, it does not follow that  $G_1, G_2, H$  need be finitely presented).

- (B)  $G = G_1 *_H \{t\}$  is the *HNN* extension determined by group morphisms  $i_1, i_2 : H \rightarrow G_1$

$$H \begin{array}{c} \xrightarrow{i_1} \\ \xrightarrow{i_2} \end{array} G_1 \xrightarrow{j_1} G$$

with  $t \in G$  such that

$$j_1 i_1(h)t = t j_1 i_2(h) \in G \quad (h \in H).$$

The *HNN* extension is *injective* if  $i_1, i_2$  are injective, in which case so is  $j_1$ , with

$$G_1 \cap t G_1 t^{-1} = i_1(H) = t i_2(H) t^{-1} \subseteq G.$$

The *HNN* extension is *finitely presented* if the groups  $G_1, H$  are finitely presented, in which case so is  $G$ . (If  $G$  is finitely presented, it does not follow that  $G_1, H$  need be finitely presented).

A subgroup  $H \subseteq G$  is *2-sided* if  $G$  is either an injective amalgamated free product  $G = G_1 *_H G_2$  or an injective *HNN* extension  $G = G_1 *_H \{t\}$ . (See Stallings [13] and Hausmann [5] for the characterization of 2-sided subgroups in terms of bipolar structures.) If  $G \neq \{1\}$  is an injective generalized free product then the subgroups

$$\begin{cases} G_1, G_2, H \\ G_1 \end{cases} \text{ are of infinite index in } G = \begin{cases} G_1 *_H G_2 \\ G *_H \{t\} \end{cases}.$$

A *CW* pair  $(X, Y \subset X)$  is *2-sided* if  $Y$  has an open neighbourhood  $Y \times \mathbb{R} \subset X$ . The pair is *connected* if  $X$  and  $Y$  are connected. By the Seifert-van Kampen Theorem  $\pi_1(X)$  is a generalized free product :

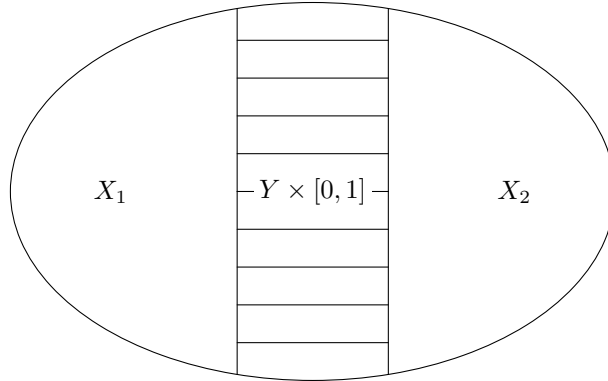
- (A) if  $Y$  separates  $X$  then  $X - Y$  has two components, and

$$X = X_1 \cup_Y X_2$$

for connected  $X_1, X_2 \subset X$  with

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$$

the amalgamated free product determined by the morphisms  $i_1 : \pi_1(Y) \rightarrow \pi_1(X_1)$ ,  $i_2 : \pi_1(Y) \rightarrow \pi_1(X_2)$  induced by the inclusions  $i_1 : Y \rightarrow X_1$ ,  $i_2 : Y \rightarrow X_2$ .



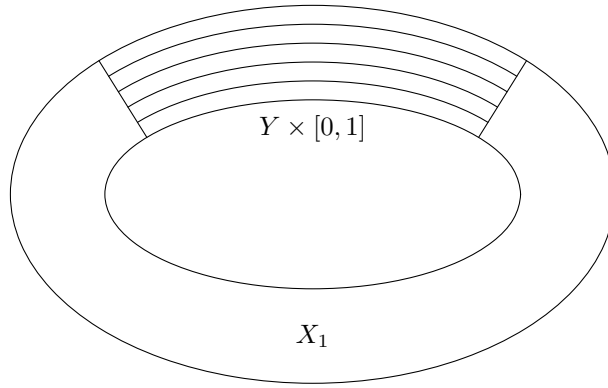
(B) if  $Y$  does not separate  $X$  then  $X - Y$  is connected and

$$X = X_1 \cup_{Y \times \{0,1\}} Y \times [0, 1]$$

for connected  $X_1 \subset X$ , with

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \{t\}$$

the *HNN* extension determined by the morphisms  $i_1, i_2 : \pi_1(Y) \rightarrow \pi_1(X_1)$  induced by the inclusions  $i_1, i_2 : Y \rightarrow X_1$ .



The generalized free product is injective if and only if the morphism  $\pi_1(Y) \rightarrow \pi_1(X)$  is injective, in which case  $\pi_1(Y)$  is a 2-sided subgroup of  $\pi_1(X)$ .

A codimension 1 submanifold  $N^{n-1} \subset M^n$  is 2-sided if the normal bundle is trivial, in which case  $(M, N)$  is a 2-sided *CW* pair.

For a 2-sided *CW* pair  $(X, Y)$  every map  $f : M \rightarrow X$  from an  $n$ -dimensional manifold  $M$  is homotopic to a map (also denoted by  $f$ ) which is transverse at  $Y \subset X$ , with

$$N^{n-1} = f^{-1}(Y) \subset M^n$$

a 2-sided codimension 1 submanifold, by the Sard-Thom theorem.

By definition, a *Seifert-van Kampen splitting* of a connected *CW* complex  $W$  with

$$\pi_1(W) = G = \begin{cases} G_1 *_{H} G_2 \\ G_1 *_{H} \{t\} \end{cases} \quad \text{an injective generalized free product is a connected}$$

2-sided *CW* pair  $(X, Y)$  with a homotopy equivalence  $X \rightarrow W$  such that

$$\text{im}(\pi_1(Y) \rightarrow \pi_1(X)) = H \subseteq \pi_1(X) = \pi_1(W) = G.$$

The splitting is *injective* if  $\pi_1(Y) \rightarrow \pi_1(X)$  is injective, in which case

$$X = \begin{cases} X_1 \cup_Y X_2 \\ X_1 \cup_{Y \times \{0,1\}} Y \times [0, 1] \end{cases}$$

with

$$\begin{cases} \pi_1(X_1) = G_1, \pi_1(X_2) = G_2 \\ \pi_1(X_1) = G_1 \end{cases}, \pi_1(Y) = H.$$

The splitting is *finite* if the complexes  $W, X, Y$  are finite, and *infinite* otherwise.

A connected  $CW$  complex  $W$  with  $\pi_1(W) = G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$  an injective generalized free product is a homotopy pushout

$$\begin{cases} \begin{array}{ccc} \widetilde{W}/H & \xrightarrow{i_1} & \widetilde{W}/G_1 \\ i_2 \downarrow & & \downarrow j_1 \\ \widetilde{W}/G_2 & \xrightarrow{j_2} & W \end{array} \\ \begin{array}{ccc} \widetilde{W}/H \times \{0, 1\} & \xrightarrow{i_1 \cup i_2} & \widetilde{W}/G_1 \\ \downarrow & & \downarrow j_1 \\ \widetilde{W}/H \times [0, 1] & \longrightarrow & W \end{array} \end{cases}$$

with  $\widetilde{W}$  the universal cover of  $W$  and  $\begin{cases} i_1, i_2, j_1, j_2 \\ i_1, i_2, j_1 \end{cases}$  the covering projections. Thus  $W$  has a canonical infinite injective Seifert-van Kampen splitting  $(X(\infty), Y(\infty))$  with

$$\begin{cases} Y(\infty) = \widetilde{W}/H \times \{1/2\} \subset X(\infty) = \widetilde{W}/G_1 \cup_{i_1} \widetilde{W}/H \times [0, 1] \cup_{i_2} \widetilde{W}/G_2 \\ Y(\infty) = \widetilde{W}/H \times \{1/2\} \subset X(\infty) = \widetilde{W}/G_1 \cup_{i_1 \cup i_2} \widetilde{W}/H \times [0, 1] . \end{cases}$$

For finite  $W$  it is easy to obtain finite injective Seifert-van Kampen splittings by codimension 1 manifold transversality. In fact, there are two ways of doing so :

- (i) Consider a regular neighbourhood  $(M, \partial M)$  of  $W \subset S^N$  ( $N$  large), apply codimension 1 manifold transversality to a map

$$\begin{cases} f : M \rightarrow BG = BG_1 \cup_{BH} BG_2 \\ f : M \rightarrow BG = BG_1 \cup_{BH \times \{0, 1\}} BH \times [0, 1] \end{cases}$$

inducing the identification  $\pi_1(M) = G$  to obtain a finite Seifert-van Kampen splitting  $(M, f^{-1}(BH))$ , and then make the splitting injective by low-dimensional handle exchanges.

- (ii) Assume inductively that the  $n$ -skeleton  $W^{(n)}$  already has a Seifert-van Kampen splitting  $(X, Y)$ . For each  $(n+1)$ -cell  $D^{n+1} \subset W^{(n+1)}$  make the attaching map  $S^n \rightarrow W^{(n)} \simeq X$  transverse at  $Y \subset X$ , and make the composite  $f : D^{n+1} \rightarrow W^{(n+1)} \rightarrow BG$  transverse at  $BH \subset BG$ . The transversality gives  $D^{n+1}$  a  $CW$  structure in which  $f^{-1}(BH) \subseteq D^{n+1}$  is a subcomplex, and

$$(X', Y') = \left( X \cup \bigcup_{D^{n+1} \subset W^{(n+1)}} D^{n+1}, Y \cup \bigcup_{D^{n+1} \subset W^{(n+1)}} f^{-1}(BH) \right)$$

is an extension of the Seifert-van Kampen splitting to the  $(n+1)$ -skeleton  $W^{(n+1)}$ . Again, the finite splitting can be made injective by low-dimensional handle exchanges.

However, the geometric nature of manifold transversality does not give any insight into the  $CW$  structures of the splittings  $(X, Y)$  of  $W$ , let alone into the algebraic analogue of transversality for  $\mathbb{Z}[G]$ -module chain complexes. Here, we

obtain Seifert-van Kampen splittings combinatorially, in the following converse of the Seifert-van Kampen Theorem.

**Combinatorial Transversality Theorem** *Let  $W$  be a finite connected CW complex with  $\pi_1(W) = G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$  an injective generalized free product.*

(i) *The canonical infinite Seifert-van Kampen splitting  $(X(\infty), Y(\infty))$  of  $W$  is a union of finite Seifert-van Kampen splittings  $(X, Y) \subset (X(\infty), Y(\infty))$*

$$(X(\infty), Y(\infty)) = \bigcup (X, Y) .$$

(ii) *If  $\pi_1(W)$  is a finitely presented generalized free product then for any finite Seifert-van Kampen splitting  $(X, Y)$  of  $W$  it is possible to attach a finite number of 2- and 3-cells to obtain a finite injective Seifert-van Kampen splitting  $(X', Y')$  such that the inclusion  $X \rightarrow X'$  is a homotopy equivalence and the inclusion  $Y \rightarrow Y'$  is a  $\mathbb{Z}[H]$ -coefficient homology equivalence.  $\square$*

The Theorem is proved in §2. The main ingredient of the proof is the construction of a finite Seifert-van Kampen splittings of  $W$  from a finite domain of the universal cover  $\widetilde{W}$ , as given by finite subcomplexes  $\begin{cases} W_1, W_2 \subseteq \widetilde{W} \\ W_1 \subseteq \widetilde{W} \end{cases}$  such that

$$\begin{cases} G_1 W_1 \cup G_2 W_2 = \widetilde{W} \\ G_1 W_1 = \widetilde{W} . \end{cases}$$

Algebraic transversality makes much use of the induction and restriction functors associated to a ring morphism  $i : A \rightarrow B$

$$\begin{aligned} i_! & : \{A\text{-modules}\} \rightarrow \{B\text{-modules}\} ; M \mapsto i_! M = B \otimes_A M , \\ i^! & : \{B\text{-modules}\} \rightarrow \{A\text{-modules}\} ; N \mapsto i^! N = N . \end{aligned}$$

Let  $G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$  be a generalized free product. By definition, a Mayer-Vietoris splitting (or presentation)  $\mathcal{E}$  of a  $\mathbb{Z}[G]$ -module chain complex  $C$  is :

(A) an exact sequence of  $\mathbb{Z}[G]$ -module chain complexes

$$\mathcal{E} : 0 \rightarrow k_1 D \xrightarrow{\begin{pmatrix} 1 \otimes e_1 \\ 1 \otimes e_2 \end{pmatrix}} (j_1)_! C_1 \oplus (j_2)_! C_2 \rightarrow C \rightarrow 0$$

with  $C_1$  a  $\mathbb{Z}[G_1]$ -module chain complex,  $C_2$  a  $\mathbb{Z}[G_2]$ -module chain complex,  $E$  a  $\mathbb{Z}[H]$ -module chain complex,  $e_1 : (i_1)_! D \rightarrow C_1$  a  $\mathbb{Z}[G_1]$ -module chain map and  $e_2 : (i_2)_! D \rightarrow C_2$  a  $\mathbb{Z}[G_2]$ -module chain map,

(B) an exact sequence of  $\mathbb{Z}[G]$ -module chain complexes

$$\mathcal{E} : 0 \rightarrow (j_1 i_1)_! D \xrightarrow{1 \otimes e_1 - t \otimes e_2} (j_1)_! C_1 \rightarrow C \rightarrow 0$$

with  $C_1$  a  $\mathbb{Z}[G_1]$ -module chain complex,  $D$  a  $\mathbb{Z}[H]$ -module chain complex, and  $e_1 : (i_1)_! D \rightarrow C_1$ ,  $e_2 : (i_2)_! D \rightarrow C_1$   $\mathbb{Z}[G_1]$ -module chain maps.

A Mayer-Vietoris splitting  $\mathcal{E}$  is *finite* if every chain complex in  $\mathcal{E}$  is finite f.g. free, and *infinite* otherwise.

Let  $X$  be a connected CW complex with fundamental group  $\pi_1(X) = G$  and universal covering projection  $p : \widetilde{X} \rightarrow X$ , with  $G$  acting on the left of  $\widetilde{X}$ . Let  $C(\widetilde{X})$

be the cellular free (left)  $\mathbb{Z}[G]$ -module chain complex. For any subgroup  $H \subseteq G$  the covering  $Z = \tilde{X}/H$  of  $X$  has universal cover  $\tilde{Z} = \tilde{X}$  with cellular  $\mathbb{Z}[H]$ -module chain complex

$$C(\tilde{Z}) = k^!C(\tilde{X})$$

with  $k : \mathbb{Z}[H] \rightarrow \mathbb{Z}[G]$  the inclusion. If  $H = \pi_1(Y)$  for a connected subcomplex  $Y \subseteq X$  then  $p^{-1}(Y) \subseteq \tilde{X}$  is a disjoint union of copies of the universal cover  $\tilde{Y}$  of  $Y$

$$p^{-1}(Y) = \bigcup_{g \in [G:H]} g\tilde{Y} \subset \tilde{X}$$

with  $[G : H]$  the set of right  $H$ -cosets  $g = xH \subseteq G$  ( $x \in G$ ). The cellular  $\mathbb{Z}[G]$ -module chain complex of  $p^{-1}(Y)$  is induced from the cellular  $\mathbb{Z}[H]$ -module chain complex of  $\tilde{Y}$

$$C(p^{-1}(Y)) = k_!C(\tilde{Y}) = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C(\tilde{Y}) \subseteq C(\tilde{X}).$$

Furthermore, if  $(X, Y)$  is a connected 2-sided  $CW$  pair then  $C(\tilde{X})$  has a Mayer-Vietoris splitting with respect to the injective generalized free product structure on  $\pi_1(X)$  :

(A) By the Mayer-Vietoris Theorem applied to

$$\begin{aligned} \tilde{X} &= p^{-1}(X_1) \cup_{p^{-1}(Y)} p^{-1}(X_2) \\ &= \bigcup_{g_1 \in [G:G_1]} g_1\tilde{X}_1 \cup \bigcup_{h \in [G:H]} h\tilde{Y} \cup \bigcup_{g_2 \in [G:G_2]} g_2\tilde{X}_2 \end{aligned}$$

$C(\tilde{X})$  has a Mayer-Vietoris splitting

$$\begin{aligned} 0 \rightarrow k_!C(\tilde{Y}) &\xrightarrow{\begin{pmatrix} 1 \otimes e_1 \\ 1 \otimes e_2 \end{pmatrix}} (j_1)_!C(\tilde{X}_1) \oplus (j_2)_!C(\tilde{X}_2) \\ &\xrightarrow{(f_1 - f_2)} C(\tilde{X}) \rightarrow 0 \end{aligned}$$

with  $\tilde{X}_1, \tilde{X}_2$  the universal covers of  $X_1, X_2$  and  $e_1 : Y \rightarrow X_1, e_2 : Y \rightarrow X_2, f_1 : X_1 \rightarrow X, f_2 : X_2 \rightarrow X$  the inclusions.

(B) By the Mayer-Vietoris Theorem applied to

$$\begin{aligned} \tilde{X} &= p^{-1}(X_1) \cup_{p^{-1}(Y \times \{0,1\})} p^{-1}(Y \times [0,1]) \\ &= \bigcup_{g_1 \in [G:G_1]} g_1\tilde{X}_1 \cup \bigcup_{h \in [G:H]} (h\tilde{Y}, 0) \cup (th\tilde{Y}, 1) \cup \bigcup_{h \in [G:H]} h\tilde{Y} \times [0,1] \end{aligned}$$

(with  $H = i_1(H) \subseteq G$ ) there is defined a short exact sequence of  $\mathbb{Z}[G]$ -module chain complexes

$$\begin{aligned} 0 \rightarrow k_!C(\tilde{Y}) \oplus k_!C(\tilde{Y}) &\xrightarrow{\begin{pmatrix} 1 \otimes e_1 & t \otimes e_2 \\ 1 & 1 \end{pmatrix}} (j_1)_!C(\tilde{X}_1) \oplus k_!C(\tilde{Y}) \\ &\xrightarrow{(f_1 - f_1(1 \otimes e_1))} C(\tilde{X}) \rightarrow 0 \end{aligned}$$

with  $\tilde{X}_1$  the universal cover of  $X_1$ , and  $e_1, e_2 : Y \rightarrow X_1$ ,  $f_1 : X_1 \rightarrow X$  the inclusions, so that  $C(\tilde{X})$  has a Mayer-Vietoris splitting

$$\mathcal{E} : 0 \rightarrow k_! C(\tilde{Y}) \xrightarrow{1 \otimes e_1 - t \otimes e_2} (j_1)_! C(\tilde{X}_1) \xrightarrow{f_1} C(\tilde{X}) \rightarrow 0 .$$

If  $(X, Y)$  is a finite  $CW$  pair the above Mayer-Vietoris splittings are finite.

For any injective generalized free product  $G = \begin{cases} G_1 *_H G_2 \\ G *_H \{t\} \end{cases}$  every free  $\mathbb{Z}[G]$ -module chain complex  $C$  has a canonical infinite Mayer-Vietoris splitting

$$\begin{aligned} \text{(A)} \quad \mathcal{E}(\infty) &: 0 \rightarrow k_! k^! C \rightarrow (j_1)_! j_1^! C \oplus (j_2)_! j_2^! C \rightarrow C \rightarrow 0 \\ \text{(B)} \quad \mathcal{E}(\infty) &: 0 \rightarrow k_! k^! C \rightarrow (j_1)_! j_1^! C \rightarrow C \rightarrow 0 . \end{aligned}$$

For finite  $C$  we shall obtain finite Mayer-Vietoris splittings in the following converse of the Mayer-Vietoris Theorem.

**Algebraic Transversality Theorem** *Let  $G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$  be an injective generalized free product. For a finite f.g. free  $\mathbb{Z}[G]$ -module chain complex  $C$  the canonical infinite Mayer-Vietoris splitting  $\mathcal{E}(\infty)$  of  $C$  is a union of finite Mayer-Vietoris splittings  $\mathcal{E} \subset \mathcal{E}(\infty)$*

$$\mathcal{E}(\infty) = \bigcup \mathcal{E} . \quad \square$$

The existence of finite Mayer-Vietoris splittings was first proved by Waldhausen [14], [15]. The proof of the Theorem in §1 is a simplification of the original argument, using chain complex analogues of the  $CW$  domains.

Suppose now that  $(X, Y)$  is the finite 2-sided  $CW$  pair defined by a (compact) connected  $n$ -dimensional manifold  $X^n$  together with a connected codimension 1 submanifold  $Y^{n-1} \subset X$  with trivial normal bundle. By definition, a homotopy equivalence  $f : M^n \rightarrow X$  from an  $n$ -dimensional manifold *splits* at  $Y \subset X$  if  $f$  is homotopic to a map (also denoted by  $f$ ) which is transverse at  $Y$ , such that the restriction  $f| : N^{n-1} = f^{-1}(Y) \rightarrow Y$  is also a homotopy equivalence. In general, homotopy equivalences do not split. For  $(X, Y)$  with injective  $\pi_1(Y) \rightarrow \pi_1(X)$  there are algebraic  $K$ - and  $L$ -theory obstructions to splitting, involving the Nil-groups of Waldhausen [14], [15] and the UNil-groups of Cappell [2], and for  $n \geq 6$  these are the complete obstructions to splitting. As outlined in Ranicki [8, §8], algebraic transversality for chain complexes is an essential ingredient for a systematic treatment of both the algebraic  $K$ - and  $L$ -theory obstructions. Although the algebraic  $K$ - and  $L$ -theory of generalized free products will not actually be considered here, it is worth noting that the early results of Higman [6], Bass, Heller and Swan [1] and Stallings [12] on the Whitehead groups of polynomial extensions and free products were followed by the work of the dedicatee on the Whitehead group of amalgamated free products (Casson [4]) prior to the general results of Waldhausen [14], [15] on the algebraic  $K$ -theory of generalized free products.

## 1. ALGEBRAIC TRANSVERSALITY

We shall deal separately with the amalgamated free and  $HNN$  cases.

**1.1. Algebraic transversality for amalgamated free products.** Given an injective amalgamated free product  $G = G_1 *_H G_2$  with  $G \neq \{1\}$  let  $T$  be the infinite tree defined by

$$T^{(0)} = [G : G_1] \cup [G : G_2], \quad T^{(1)} = [G : H]$$

(Serre [11], Waldhausen [15]). The edge  $h \in [G : H]$  joins the unique vertices  $g_1 \in [G : G_1]$ ,  $g_2 \in [G : G_2]$  with

$$g_1 \cap g_2 = h \subset G.$$

The group  $G$  acts on  $T$  by

$$G \times T \rightarrow T; (g, x) \mapsto gx$$

with  $T/G = [0, 1]$ ,  $G_i \subseteq G$  the isotropy subgroup of  $G_i \in T^{(0)}$  and  $H \subseteq G$  the isotropy subgroup of  $H \in T^{(1)}$ . Conversely, if a group  $G$  acts on a tree  $T$  with  $T/G = [0, 1]$  then  $G = G_1 *_H G_2$  is an injective amalgamated free product with tree  $T$ .

As before write the injections as

$$\begin{aligned} i_1 &: H \rightarrow G_1, \quad i_2 : H \rightarrow G_2, \quad j_1 : G_1 \rightarrow G, \quad j_2 : G_2 \rightarrow G, \\ k &= j_1 i_1 = j_2 i_2 : H \rightarrow G. \end{aligned}$$

**Definition 1.1.** (i) A *domain*  $(C_1, C_2)$  of a  $\mathbb{Z}[G]$ -module chain complex  $C$  is a pair of subcomplexes  $(C_1 \subseteq j_1^! C, C_2 \subseteq j_2^! C)$  such that the chain maps

$$\begin{aligned} e_1 &: (i_1)_!(C_1 \cap C_2) \rightarrow C_1; \quad b_1 \otimes y_1 \mapsto b_1 y_1, \\ e_2 &: (i_2)_!(C_1 \cap C_2) \rightarrow C_2; \quad b_2 \otimes y_2 \mapsto b_2 y_2, \\ f_1 &: (j_1)_! C_1 \rightarrow C; \quad a_1 \otimes x_1 \mapsto a_1 x_1, \\ f_2 &: (j_2)_! C_2 \rightarrow C; \quad a_2 \otimes x_2 \mapsto a_2 x_2 \end{aligned}$$

fit into a Mayer-Vietoris splitting of  $C$

$$\mathcal{E}(C_1, C_2) : 0 \rightarrow k_!(C_1 \cap C_2) \xrightarrow{e} (j_1)_! C_1 \oplus (j_2)_! C_2 \xrightarrow{f} C \rightarrow 0$$

with  $e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ ,  $f = (f_1 \quad -f_2)$ .

(ii) A domain  $(C_1, C_2)$  is *finite* if  $C_i$  ( $i = 1, 2$ ) is a finite f.g. free  $\mathbb{Z}[G_i]$ -module chain complex,  $C_1 \cap C_2$  is a finite f.g. free  $\mathbb{Z}[H]$ -module chain complex, and *infinite* otherwise.  $\square$

**Proposition 1.2.** *Every free  $\mathbb{Z}[G]$ -module chain complex  $C$  has a canonical infinite domain  $(C_1, C_2) = (j_1^! C, j_2^! C)$  with*

$$C_1 \cap C_2 = k^! C,$$

so that  $C$  has a canonical infinite Mayer-Vietoris splitting

$$\mathcal{E}(\infty) = \mathcal{E}(j_1^! C, j_2^! C) : 0 \rightarrow k_! k^! C \rightarrow (j_1)_! j_1^! C \oplus (j_2)_! j_2^! C \rightarrow C \rightarrow 0.$$

*Proof.* It is enough to consider the special case  $C = \mathbb{Z}[G]$ , concentrated in degree 0. The pair

$$(C_1, C_2) = (j_1^! \mathbb{Z}[G], j_2^! \mathbb{Z}[G]) = \left( \bigoplus_{[G:G_1]} \mathbb{Z}[G_1], \bigoplus_{[G:G_2]} \mathbb{Z}[G_2] \right)$$



is a canonical infinite domain for  $C$ , with

$$\mathcal{E}(\infty) = \mathcal{E}(C_1, C_2) : 0 \rightarrow k_1 k^1 \mathbb{Z}[G] \rightarrow (j_1)! j_1^1 \mathbb{Z}[G] \oplus (j_2)! j_2^1 \mathbb{Z}[G] \rightarrow \mathbb{Z}[G] \rightarrow 0$$

the simplicial  $\mathbb{Z}[G]$ -module chain complex of  $T \times G$  with the diagonal  $G$  action, along with its augmentation to  $H_0(T \times G) = \mathbb{Z}[G]$ .  $\square$

**Definition 1.3.** (i) For a based f.g. free  $\mathbb{Z}[G]$ -module  $B = \mathbb{Z}[G]^b$  and a subtree  $U \subseteq T$  define a domain for  $B$  (regarded as a chain complex concentrated in degree 0)

$$(B(U)_1, B(U)_2) = \left( \sum_{U_1^{(0)}} \mathbb{Z}[G_1]^b, \sum_{U_2^{(0)}} \mathbb{Z}[G_2]^b \right)$$

with

$$\begin{aligned} U_1^{(0)} &= U^{(0)} \cap [G : G_1], \quad U_2^{(0)} = U^{(0)} \cap [G : G_2], \\ B(U)_1 \cap B(U)_2 &= \sum_{U^{(1)}} \mathbb{Z}[H]^b. \end{aligned}$$

The associated Mayer-Vietoris splitting of  $B$  is the subobject  $\mathcal{E}(U) \subseteq \mathcal{E}(\infty)$  with

$$\mathcal{E}(U) : 0 \rightarrow k_1 \sum_{U^{(1)}} \mathbb{Z}[H]^b \rightarrow (j_1)! \sum_{U_1^{(0)}} \mathbb{Z}[G_1]^b \oplus (j_2)! \sum_{U_2^{(0)}} \mathbb{Z}[G_2]^b \rightarrow B \rightarrow 0.$$

If  $U \subset T$  is finite then  $(B(U)_1, B(U)_2)$  is a finite domain.

(ii) Let  $C$  be an  $n$ -dimensional based f.g. free  $\mathbb{Z}[G]$ -module chain complex, with  $C_r = \mathbb{Z}[G]^{c_r}$ . A sequence  $U = \{U_n, U_{n-1}, \dots, U_1, U_0\}$  of subtrees  $U_r \subseteq T$  is *realized* by  $C$  if the differentials  $d : C_r \rightarrow C_{r-1}$  are such that

$$d(C_r(U_r)_i) \subseteq C_{r-1}(U_{r-1})_i \quad (1 \leq r \leq n, i = 1, 2),$$

so that there is defined a Mayer-Vietoris splitting of  $C$

$$\mathcal{E}(U) : 0 \rightarrow k_1 \sum_{U^{(1)}} C(U)_1 \cap C(U)_2 \rightarrow (j_1)! \sum_{U_1^{(0)}} C(U)_1 \oplus (j_2)! \sum_{U_2^{(0)}} C(U)_2 \rightarrow C \rightarrow 0.$$

The sequence  $U$  is *finite* if each subtree  $U_r \subseteq T$  is finite, in which case  $\mathcal{E}(U)$  is finite.  $\square$

**Proposition 1.4.** For a finite based f.g. free  $\mathbb{Z}[G]$ -module chain complex  $C$  the canonical infinite domain is a union of finite domains

$$(j_1^1 C, j_2^1 C) = \bigcup_U (C(U)_1, C(U)_2),$$

with  $U$  running over all the finite sequences which are realized by  $C$ . The canonical infinite Mayer-Vietoris splitting of  $C$  is thus a union of finite Mayer-Vietoris splittings

$$\mathcal{E}(\infty) = \bigcup_U \mathcal{E}(U).$$

*Proof.* The proof is based on the following observations :

(a) for any subtrees  $V \subseteq U \subseteq T$

$$\mathcal{E}(V) \subseteq \mathcal{E}(U) \subseteq \mathcal{E}(T) = \mathcal{E}(\infty)$$

(b) the infinite tree  $T$  is a union

$$T = \bigcup U$$

of the finite subtrees  $U \subset T$ ,

- (c) for any finite subtrees  $U, U' \subset T$  there exists a finite subtree  $U'' \subset T$  such that  $U \subseteq U''$  and  $U' \subseteq U''$ ,
- (d) for any finite subtree  $U \subset T$  and any element  $d \in \mathbb{Z}[G]$  there exists a finite subtree  $U' \subset T$  such that the  $\mathbb{Z}[G]$ -module morphism  $d : \mathbb{Z}[G] \rightarrow \mathbb{Z}[G]$  is resolved by a morphism  $d : \mathcal{E}(U) \rightarrow \mathcal{E}(U')$  of finite Mayer-Vietoris splittings.

Assume  $C$  is  $n$ -dimensional, with  $C_r = \mathbb{Z}[G]^{c_r}$ . Starting with any finite subtree  $U_n \subseteq T$  let

$$U = \{U_n, U_{n-1}, \dots, U_1, U_0\}$$

be a sequence of finite subtrees  $U_r \subset T$  such that the f.g. free submodules

$$\begin{aligned} C_r(U)_1 &= \sum_{U_{i,1}^{(0)}} \mathbb{Z}[G_1]^{c_r} \subset j_1^! C_r = \sum_{T_1^{(0)}} \mathbb{Z}[G_1]^{c_r}, \\ C_r(U)_2 &= \sum_{U_{r,2}^{(0)}} \mathbb{Z}[G_2]^{c_r} \subset j_2^! C_r = \sum_{T_2^{(0)}} \mathbb{Z}[G_2]^{c_r}, \\ D(U)_r &= \sum_{U_r^{(1)}} \mathbb{Z}[H]^{c_r} \subset k^! C_r = \sum_{T^{(1)}} \mathbb{Z}[H]^{c_r} \end{aligned}$$

define subcomplexes  $C(U)_1 \subset j_1^! C$ ,  $C(U)_2 \subset j_2^! C$ ,  $D(U) \subset k^! C$ . Then  $(C(U)_1, C(U)_2)$  is a domain of  $C$  with  $C(U)_1 \cap C(U)_2 = D(U)$ , and  $U$  is realized by  $C$ .  $\square$

**Remark 1.5.** (i) The existence of finite Mayer-Vietoris splittings was first proved by Waldhausen [14],[15], using essentially the same method. See Quinn [7] for a proof using controlled algebra. The construction of generalized free products by noncommutative localization (cf. Ranicki [10]) can be used to provide a different proof.

(ii) The construction of the finite Mayer-Vietoris splittings  $\mathcal{E}(U)$  in 1.4 as subobjects of the universal Mayer-Vietoris splitting  $\mathcal{E}(T) = \mathcal{E}(\infty)$  is taken from Remark 8.7 of Ranicki [8].  $\square$

This completes the proof of the Algebraic Transversality Theorem for amalgamated free products.

**1.2. Algebraic transversality for  $HNN$  extensions.** The proof of algebraic transversality for  $HNN$  extensions proceeds exactly as for amalgamated free products, so only the statements will be given.

Given an injective  $HNN$  extension  $G = G_1 *_H \{t\}$  with  $G \neq \{1\}$  let  $T$  be the infinite tree defined by

$$T^{(0)} = [G : G_1], \quad T^{(1)} = [G : H],$$

identifying  $H = i_1(H) \subseteq G$ . The edge  $h \in [G : H]$  joins the unique vertices  $g_1, g_2 \in [G : G_1]$  with

$$g_1 \cap g_2 t^{-1} = h \subset G.$$

The group  $G$  acts on  $T$  by

$$G \times T \rightarrow T; (g, x) \mapsto gx$$

with  $T/G = S^1$ ,  $G_1 \subseteq G$  the isotropy subgroup of  $G_1 \in T^{(0)}$  and  $H \subseteq G$  the isotropy subgroup of  $H \in T^{(1)}$ . Conversely, if a group  $G$  acts on a tree  $T$  with  $T/G = S^1$  then  $G = G_1 *_H \{t\}$  is an injective  $HNN$  extension with tree  $T$ . As before, write the injections as

$$i_1, i_2 : H \rightarrow G_1, \quad j : G_1 \rightarrow G, \quad k = j_1 i_1 = j_1 i_2 : G_1 \rightarrow G.$$

**Definition 1.6.** (i) A *domain*  $C_1$  of a  $\mathbb{Z}[G]$ -module chain complex  $C$  is a subcomplex  $C_1 \subseteq j_1^! C$  such that the chain maps

$$\begin{aligned} e_1 &: (i_1)!(C_1 \cap tC_1) \rightarrow C_1 ; b_1 \otimes y_1 \mapsto b_1 y_1 , \\ e_2 &: (i_2)!(C_1 \cap tC_1) \rightarrow C_1 ; b_2 \otimes y_2 \mapsto b_2 t^{-1} y_2 , \\ f &: (j_1) ! C_1 \rightarrow C ; a \otimes x \mapsto ax \end{aligned}$$

fit into a Mayer-Vietoris splitting of  $C$

$$\mathcal{E}(C_1) : 0 \rightarrow k_!(C_1 \cap tC_1) \xrightarrow{1 \otimes e_1 - t \otimes e_2} (j_1) ! C_1 \xrightarrow{f} C \rightarrow 0 .$$

(ii) A domain  $C_1$  is *finite* if  $C_1$  is a finite f.g. free  $\mathbb{Z}[G_1]$ -module chain complex and  $C_1 \cap tC_1$  is a finite f.g. free  $\mathbb{Z}[H]$ -module chain complex.  $\square$

**Proposition 1.7.** *Every free  $\mathbb{Z}[G]$ -module chain complex  $C$  has a canonical infinite domain  $C_1 = j_1^! C$  with*

$$C_1 \cap tC_1 = k^! C_1 ,$$

so that  $C$  has a canonical infinite Mayer-Vietoris splitting

$$\mathcal{E}(\infty) = \mathcal{E}(j_1^! C) : 0 \rightarrow k_! k^! C \rightarrow (j_1) ! j_1^! C \rightarrow C \rightarrow 0 .$$

$\square$

**Definition 1.8.** For any subtree  $U \subseteq T$  define a domain for  $\mathbb{Z}[G]$

$$C(U)_1 = \sum_{U^{(0)}} \mathbb{Z}[G_1]$$

with

$$C(U)_1 \cap tC(U)_1 = \sum_{U^{(1)}} \mathbb{Z}[H] .$$

The associated Mayer-Vietoris splitting of  $\mathbb{Z}[G]$  is the subobject  $\mathcal{E}(U) \subseteq \mathcal{E}(\infty)$  with

$$\mathcal{E}(U) : 0 \rightarrow k_! \sum_{U^{(1)}} \mathbb{Z}[H] \rightarrow (j_1) ! \sum_{U^{(0)}} \mathbb{Z}[G_1] \rightarrow \mathbb{Z}[G] \rightarrow 0 .$$

If  $U \subset T$  is finite then  $C(U)_1$  is finite.  $\square$

**Proposition 1.9.** *For a finite f.g. free  $\mathbb{Z}[G]$ -module chain complex  $C$  the canonical infinite domain is a union of finite domains*

$$j_1^! C = \bigcup C_1 .$$

The canonical infinite Mayer-Vietoris splitting of  $C$  is thus a union of finite Mayer-Vietoris splittings

$$\mathcal{E}(\infty) = \bigcup \mathcal{E}(C_1) .$$

$\square$

This completes the proof of the Algebraic Transversality Theorem for  $HNN$  extensions.

## 2. COMBINATORIAL TRANSVERSALITY

The Combinatorial Transversality Theorem stated in the Introduction will now be proved, treating the cases of an amalgamated free product and an  $HNN$  extension separately.

**2.1. Mapping cylinders.** We review some basic mapping cylinder constructions.

The *mapping cylinder* of a map  $e : V \rightarrow W$  is the identification space

$$\mathcal{M}(e) = (V \times [0, 1] \cup W) / \{(x, 0) \sim e(x) \mid x \in V\}$$

such that  $V = V \times \{1\} \subset \mathcal{M}(e)$ , and the projection

$$p : \mathcal{M}(e) \rightarrow W ; \begin{cases} (x, s) \mapsto e(x) & \text{for } x \in V, s \in [0, 1] \\ y \mapsto x & \text{for } y \in W \end{cases}$$

is a homotopy equivalence.

If  $e$  is a cellular map of  $CW$  complexes then  $\mathcal{M}(e)$  is a  $CW$  complex, such that cellular chain complex  $C(\mathcal{M}(e))$  is the algebraic mapping cylinder of  $e : C(V) \rightarrow C(W)$  with

$$\begin{aligned} d_{C(\mathcal{M}(e))} &= \begin{pmatrix} d_{C(W)} & (-1)^r e & 0 \\ 0 & d_{C(V)} & 0 \\ 0 & (-1)^{r-1} & d_{C(V)} \end{pmatrix} : \\ C(\mathcal{M}(e))_r &= C(W)_r \oplus C(V)_{r-1} \oplus C(V)_r \\ &\rightarrow C(\mathcal{M}(e))_{r-1} = C(W)_{r-1} \oplus C(V)_{r-2} \oplus C(V)_{r-1} . \end{aligned}$$

The chain equivalence  $p : C(\mathcal{M}(e)) \rightarrow C(W)$  is given by

$$p = (1 \ 0 \ e) : C(\mathcal{M}(e))_r = C(W)_r \oplus C(V)_{r-1} \oplus C(V)_r \rightarrow C(W)_r .$$

The *double mapping cylinder*  $\mathcal{M}(e_1, e_2)$  of maps  $e_1 : V \rightarrow W_1$ ,  $e_2 : V \rightarrow W_2$  is the identification space

$$\begin{aligned} \mathcal{M}(e_1, e_2) &= \mathcal{M}(e_1) \cup_V \mathcal{M}(e_2) \\ &= W_1 \cup_{e_1} V \times [0, 1] \cup_{e_2} W_2 \\ &= (W_1 \cup V \times [0, 1] \cup W_2) / \{(x, 0) \sim e_1(x), (x, 1) \sim e_2(x) \mid x \in V\} . \end{aligned}$$

Given a commutative square of spaces and maps

$$\begin{array}{ccc} V & \xrightarrow{e_1} & W_1 \\ e_2 \downarrow & & \downarrow f_1 \\ W_2 & \xrightarrow{f_2} & W \end{array}$$

define the map

$$f_1 \cup f_2 : \mathcal{M}(e_1, e_2) \rightarrow W ; \begin{cases} (x, s) \mapsto f_1 e_1(x) = f_2 e_2(x) & (x \in V, s \in [0, 1]) \\ y_i \mapsto f_i(y_i) & (y_i \in W_i, i = 1, 2) . \end{cases}$$

The square is a *homotopy pushout* if  $f_1 \cup f_2 : \mathcal{M}(e_1, e_2) \rightarrow W$  is a homotopy equivalence.

If  $e_1 : V \rightarrow W_1$ ,  $e_2 : V \rightarrow W_2$  are cellular maps of  $CW$  complexes then  $\mathcal{M}(e_1, e_2)$  is a  $CW$  complex, such that cellular chain complex  $C(\mathcal{M}(e_1, e_2))$  is the algebraic mapping cone of the chain map

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} : C(V) \rightarrow C(W_1) \oplus C(W_2)$$

with

$$d_{C(\mathcal{M}(e_1, e_2))} = \begin{pmatrix} d_{C(W_1)} & (-1)^r e_1 & 0 \\ 0 & d_{C(V)} & 0 \\ 0 & (-1)^r e_2 & d_{C(W_2)} \end{pmatrix} :$$

$$C(\mathcal{M}(e_1, e_2))_r = C(W_1)_r \oplus C(V)_{r-1} \oplus C(W_2)_r$$

$$\rightarrow C(\mathcal{M}(e_1, e_2))_{r-1} = C(W_1)_{r-1} \oplus C(V)_{r-2} \oplus C(W_2)_{r-1} .$$

**2.2. Combinatorial transversality for amalgamated free products.** In this section  $W$  is a connected  $CW$  complex with fundamental group an injective amalgamated free product

$$\pi_1(W) = G = G_1 *_H G_2$$

with tree  $T$ . Let  $\widetilde{W}$  be the universal cover of  $W$ , and let

$$\begin{array}{ccc} \widetilde{W}/H & \xrightarrow{i_1} & \widetilde{W}/G_1 \\ i_2 \downarrow & & \downarrow j_1 \\ \widetilde{W}/G_2 & \xrightarrow{j_2} & W \end{array}$$

be the commutative square of covering projections.

**Definition 2.1.** (i) Suppose given subcomplexes  $W_1, W_2 \subseteq \widetilde{W}$  such that

$$G_1 W_1 = W_1, G_2 W_2 = W_2$$

so that

$$H(W_1 \cap W_2) = W_1 \cap W_2 \subseteq \widetilde{W} .$$

Define a commutative square of  $CW$  complexes and cellular maps

$$\begin{array}{ccc} (W_1 \cap W_2)/H & \xrightarrow{e_1} & W_1/G_1 \\ e_2 \downarrow & \Phi & \downarrow f_1 \\ W_2/G_2 & \xrightarrow{f_2} & W \end{array}$$

with

$$(W_1 \cap W_2)/H \subseteq \widetilde{W}/H, W_1/G_1 \subseteq \widetilde{W}/G_1, W_2/G_2 \subseteq \widetilde{W}/G_2,$$

$$e_1 = i_1| : (W_1 \cap W_2)/H \rightarrow W_1/G_1, e_2 = i_2| : (W_1 \cap W_2)/H \rightarrow W_2/G_2,$$

$$f_1 = j_1| : W_1/G_1 \rightarrow W, f_2 = j_2| : W_2/G_2 \rightarrow W .$$

(ii) A *domain*  $(W_1, W_2)$  for the universal cover  $\widetilde{W}$  of  $W$  consists of connected subcomplexes  $W_1, W_2 \subseteq \widetilde{W}$  such that  $W_1 \cap W_2$  is connected, and such that for each cell  $D \subseteq \widetilde{W}$  the subgraph  $U(D) \subseteq T$  defined by

$$U(D)^{(0)} = \{g_1 \in [G : G_1] \mid g_1 D \subseteq W_1\} \cup \{g_2 \in [G : G_2] \mid g_2 D \subseteq W_2\}$$

$$U(D)^{(1)} = \{h \in [G : H] \mid h D \subseteq W_1 \cap W_2\}$$

is a tree.

(iii) A domain  $(W_1, W_2)$  for  $\widetilde{W}$  is *fundamental* if the subtrees  $U(D) \subseteq T$  are either single vertices or single edges, so that

$$g_1 W_1 \cap g_2 W_2 = \begin{cases} h(W_1 \cap W_2) & \text{if } g_1 \cap g_2 = h \in [G : H] \\ \emptyset & \text{if } g_1 \cap g_2 = \emptyset, \end{cases}$$

$$W = (W_1/G_1) \cup_{(W_1 \cap W_2)/H} (W_2/G_2).$$

□

**Proposition 2.2.** *For a domain  $(W_1, W_2)$  of  $\widetilde{W}$  the pair of cellular chain complexes  $(C(W_1), C(W_2))$  is a domain of the cellular chain complex  $C(\widetilde{W})$ .*

*Proof.* The union of  $GW_1, GW_2 \subseteq \widetilde{W}$  is

$$GW_1 \cup GW_2 = \widetilde{W}$$

since for any cell  $D \subseteq \widetilde{W}$  there either exists  $g_1 \in [G : G_1]$  such that  $g_1 D \subseteq W_1$  or  $g_2 \in [G : G_2]$  such that  $g_2 D \subseteq W_2$ . The intersection of  $GW_1, GW_2 \subseteq \widetilde{W}$  is

$$GW_1 \cap GW_2 = G(W_1 \cap W_2) \subseteq \widetilde{W}.$$

The Mayer-Vietoris exact sequence of cellular  $\mathbb{Z}[G]$ -module chain complexes

$$0 \rightarrow C(GW_1 \cap GW_2) \rightarrow C(GW_1) \oplus C(GW_2) \rightarrow C(\widetilde{W}) \rightarrow 0$$

is the Mayer-Vietoris splitting of  $C(\widetilde{W})$  associated to  $(C(W_1), C(W_2))$

$$0 \rightarrow k_1 C(W_1 \cap W_2) \rightarrow (j_1)_! C(W_1) \oplus (j_2)_! C(W_2) \rightarrow C(\widetilde{W}) \rightarrow 0$$

with  $C(W_1 \cap W_2) = C(W_1) \cap C(W_2)$ . □

**Example 2.3.**  $W$  has a canonical infinite domain  $(W_1, W_2) = (\widetilde{W}, \widetilde{W})$  with  $(W_1 \cap W_2)/H = \widetilde{W}/H$ , and  $U(D) = T$  for each cell  $D \subseteq \widetilde{W}$ . □

**Example 2.4.** (i) Suppose that  $W = X_1 \cup_Y X_2$ , with  $X_1, X_2, Y \subseteq W$  connected subcomplexes such that the isomorphism

$$\pi_1(W) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \xrightarrow{\cong} G = G_1 *_H G_2$$

preserves the amalgamated free structures. Thus  $(X, Y)$  is a Seifert-van Kampen splitting of  $W$ , and the morphisms

$$\pi_1(X_1) \rightarrow G_1, \pi_1(X_2) \rightarrow G_2, \pi_1(Y) \rightarrow H$$

are surjective. (If  $\pi_1(Y) \rightarrow \pi_1(X_1)$  and  $\pi_1(Y) \rightarrow \pi_1(X_2)$  are injective these morphisms are isomorphisms, and the splitting is injective). The universal cover of  $W$  is

$$\widetilde{W} = \bigcup_{g_1 \in [G : G_1]} g_1 \widetilde{X}_1 \cup \bigcup_{h \in [G : H]} h \widetilde{Y} \cup \bigcup_{g_2 \in [G : G_2]} g_2 \widetilde{X}_2$$

with  $\widetilde{X}_i$  the regular cover of  $X_i$  corresponding to  $\ker(\pi_1(X_i) \rightarrow G_i)$  ( $i = 1, 2$ ) and  $\widetilde{Y}$  the regular cover of  $Y$  corresponding to  $\ker(\pi_1(Y) \rightarrow H)$  (which are the universal covers of  $X_1, X_2, Y$  in the case  $\pi_1(X_1) = G_1, \pi_1(X_2) = G_2, \pi_1(Y) = H$ ). The pair

$$(W_1, W_2) = (\widetilde{X}_1, \widetilde{X}_2)$$

is a fundamental domain of  $\widetilde{W}$  such that

$$(W_1 \cap W_2)/H = Y ,$$

$$g_1 W_1 \cap g_2 W_2 = (g_1 \cap g_2) \widetilde{Y} \subseteq \widetilde{W} \quad (g_1 \in [G : G_1], g_2 \in [G : G_2]) .$$

For any cell  $D \subseteq \widetilde{W}$

$$U(D) = \begin{cases} \{g_1\} & \text{if } g_1 D \subseteq \widetilde{X}_1 - \bigcup_{h_1 \in [G_1 : H]} h_1 \widetilde{Y} \text{ for some } g_1 \in [G : G_1] \\ \{g_2\} & \text{if } g_2 D \subseteq \widetilde{X}_2 - \bigcup_{h_2 \in [G_2 : H]} h_2 \widetilde{Y} \text{ for some } g_2 \in [G : G_1] \\ \{g_1, g_2, h\} & \text{if } h D \subseteq \widetilde{Y} \text{ for some } h = g_1 \cap g_2 \in [G : H]. \end{cases}$$

(ii) If  $(W_1, W_2)$  is a fundamental domain for any connected CW complex  $W$  with  $\pi_1(W) = G = G_1 *_H G_2$  then  $W = X_1 \cup_Y X_2$  as in (i), with

$$X_1 = W_1/G_1, \quad X_2 = W_2/G_2, \quad Y = (W_1 \cap W_2)/H .$$

□

**Definition 2.5.** Suppose that  $W$  is  $n$ -dimensional. Lift each cell  $D^r \subseteq W$  to a cell  $\widetilde{D}^r \subseteq \widetilde{W}$ . A sequence  $U = \{U_n, U_{n-1}, \dots, U_1, U_0\}$  of subtrees  $U_r \subseteq T$  is *realized* by  $W$  if the subspaces

$$W(U)_1 = \bigcup_{r=0}^n \bigcup_{D^r \subset W} \bigcup_{g_1 \in U_{r,1}^{(0)}} g_1 \widetilde{D}^r, \quad W(U)_2 = \bigcup_{r=0}^n \bigcup_{D^r \subset W} \bigcup_{g_2 \in U_{r,2}^{(0)}} g_2 \widetilde{D}^r \subseteq \widetilde{W}$$

are connected subcomplexes, in which case  $(W(U)_1, W(U)_2)$  is a domain for  $\widetilde{W}$  with

$$W(U)_1 \cap W(U)_2 = \bigcup_{r=0}^n \bigcup_{D^r \subset W} \bigcup_{h \in U_r^{(1)}} h \widetilde{D}^r \subseteq \widetilde{W}$$

a connected subcomplex. Thus  $U$  is realized by  $C(\widetilde{W})$  and

$$(C(W(U)_1), C(W(U)_2)) = (C(\widetilde{W})(U)_1, C(\widetilde{W})(U)_2) \subseteq (C(\widetilde{W}), C(\widetilde{W}))$$

is the domain for  $C(\widetilde{W})$  given by  $(C_r(\widetilde{W})_1(U_r), C_r(\widetilde{W})(U_2))$  in degree  $r$ . □

If a sequence  $U = \{U_n, U_{n-1}, \dots, U_1, U_0\}$  realized by  $W$  is finite (i.e. if each  $U_r \subseteq T$  is a finite subtree) then  $(W(U)_1, W(U)_2)$  is a finite domain for  $\widetilde{W}$ .

**Proposition 2.6.** (i) For any domain  $(W_1, W_2)$  there is defined a homotopy pushout

$$\begin{array}{ccc} (W_1 \cap W_2)/H & \xrightarrow{e_1} & W_1/G_1 \\ \downarrow e_2 & \Phi & \downarrow f_1 \\ W_2/G_2 & \xrightarrow{f_2} & W \end{array}$$

with  $e_1 = i_1|$ ,  $e_2 = i_2|$ ,  $f_1 = j_1|$ ,  $f_2 = j_2|$ . The connected 2-sided CW pair

$$(X, Y) = (\mathcal{M}(e_1, e_2), (W_1 \cap W_2)/H \times \{1/2\})$$

is a Seifert-van Kampen splitting of  $W$ , with a homotopy equivalence

$$f = f_1 \cup f_2 : X = \mathcal{M}(e_1, e_2) \xrightarrow{\cong} W .$$

(ii) *The commutative square of covering projections*

$$\begin{array}{ccc} \widetilde{W}/H & \xrightarrow{i_1} & \widetilde{W}/G_1 \\ i_2 \downarrow & & \downarrow j_1 \\ \widetilde{W}/G_2 & \xrightarrow{j_2} & W \end{array}$$

is a homotopy pushout. The connected 2-sided CW pair

$$(X(\infty), Y(\infty)) = (\mathcal{M}(i_1, i_2), \widetilde{W}/H \times \{1/2\})$$

is a canonical injective infinite Seifert-van Kampen splitting of  $W$ , with a homotopy equivalence  $j = j_1 \cup j_2 : X(\infty) \rightarrow W$  such that

$$\pi_1(Y(\infty)) = H \subseteq \pi_1(X(\infty)) = G_1 *_H G_2 .$$

(iii) *For any (finite) sequence  $U = \{U_n, U_{n-1}, \dots, U_0\}$  of subtrees of  $T$  realized by  $W$  there is defined a homotopy pushout*

$$\begin{array}{ccc} Y(U) & \xrightarrow{e_1} & X(U)_1 \\ e_2 \downarrow & & \downarrow f_1 \\ X(U)_2 & \xrightarrow{f_2} & W \end{array}$$

with

$$\begin{aligned} X(U)_1 &= W(U)_1/G_1, \quad X(U)_2 = W(U)_2/G_2, \\ Y(U) &= (W(U)_1 \cap W(U)_2)/H, \\ e_1 &= i_1|, \quad e_2 = i_2|, \quad f_1 = j_1|, \quad f_2 = j_2|. \end{aligned}$$

Thus

$$(X(U), Y(U)) = (\mathcal{M}(e_1, e_2), Y(U) \times \{1/2\})$$

is a (finite) Seifert-van Kampen splitting of  $W$ .

(iv) *The canonical infinite domain of a finite CW complex  $W$  with  $\pi_1(W) = G_1 *_H G_2$  is a union of finite domains*

$$(\widetilde{W}, \widetilde{W}) = \bigcup_U (W(U)_1, W(U)_2)$$

with  $U$  running over all the finite sequences realized by  $W$ . The canonical infinite Seifert-van Kampen splitting of  $W$  is thus a union of finite Seifert-van Kampen splittings

$$(X(\infty), Y(\infty)) = \bigcup_U (X(U), Y(U)) .$$

*Proof.* (i) Given a cell  $D \subseteq W$  let  $\widetilde{D} \subseteq \widetilde{W}$  be a lift. The inverse image of the interior  $\text{int}(D) \subseteq W$

$$f^{-1}(\text{int}(D)) = U(\widetilde{D}) \times \text{int}(D) \subseteq \mathcal{M}(i_1, i_2) = T \times_G \widetilde{W}$$

is contractible. In particular, point inverses are contractible, so that  $f : X \rightarrow W$  is a homotopy equivalence. (Here is a more direct proof that  $f : X \rightarrow W$  is a  $\mathbb{Z}[G]$ -coefficient homology equivalence. The Mayer-Vietoris Theorem applied to



the union  $\widetilde{W} = GW_1 \cup GW_2$  expresses  $C(\widetilde{W})$  as the cokernel of the  $\mathbb{Z}[G]$ -module chain map

$$e = \begin{pmatrix} 1 \otimes e_1 \\ 1 \otimes e_2 \end{pmatrix} : \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C(W_1 \cap W_2) \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_1]} C(W_1) \oplus \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_2]} C(W_2)$$

with a Mayer-Vietoris splitting

$$\begin{aligned} 0 \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C(W_1 \cap W_2) &\xrightarrow{e} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_1]} C(W_1) \oplus \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_2]} C(W_2) \\ &\longrightarrow C(\widetilde{W}) \rightarrow 0. \end{aligned}$$

The decomposition  $X = \mathcal{M}(e_1, e_2) = X_1 \cup_Y X_2$  with

$$X_i = \mathcal{M}(e_i) \ (i = 1, 2), \ Y = X_1 \cap X_2 = (W_1 \cap W_2)/H \times \{1/2\}$$

lifts to a decomposition of the universal cover as

$$\widetilde{X} = \bigcup_{g_1 \in [G:G_1]} g_1 \widetilde{X}_1 \cup \bigcup_{h \in [G:H]} h \widetilde{Y} \cup \bigcup_{g_2 \in [G:G_2]} g_2 \widetilde{X}_2.$$

The Mayer-Vietoris splitting

$$0 \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C(\widetilde{Y}) \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_1]} C(\widetilde{X}_1) \oplus \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_2]} C(\widetilde{X}_2) \rightarrow C(\widetilde{X}) \rightarrow 0,$$

expresses  $C(\widetilde{X})$  as the algebraic mapping cone of the chain map  $e$

$$C(\widetilde{X}) = \mathcal{C}(e : \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C(W_1 \cap W_2) \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_1]} C(W_1) \oplus \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_2]} C(W_2)).$$

Since  $e$  is injective the  $\mathbb{Z}[G]$ -module chain map

$$\widetilde{f} = \text{projection} : C(\widetilde{X}) = \mathcal{C}(e) \rightarrow C(\widetilde{W}) = \text{coker}(e)$$

induces isomorphisms in homology.)

(ii) Apply (i) to  $(W_1, W_2) = (\widetilde{W}, \widetilde{W})$ .

(iii) Apply (i) to the domain  $(W(U)_1, W(U)_2)$ .

(iv) Assume that  $W$  is  $n$ -dimensional. Proceed as for the chain complex case in the proof of Proposition 1.4 for the existence of a domain for  $C(\widetilde{W})$ , but use only the sequences  $U = \{U_n, U_{n-1}, \dots, U_0\}$  of finite subtrees  $U_r \subset T$  realized by  $W$ . An arbitrary finite subtree  $U_n \subset T$  extends to a finite sequence  $U$  realized by  $W$  since for  $r \geq 2$  each  $r$ -cell  $\widetilde{D}^r \subset \widetilde{W}$  is attached to an  $(r-1)$ -dimensional finite connected subcomplex, and every 1-cell  $\widetilde{D}^1 \subset \widetilde{W}$  is contained in a 1-dimensional finite connected subcomplex. Thus finite sequences  $U$  realized by  $W$  exist, and can be chosen to contain arbitrary finite collections of cells of  $\widetilde{W}$ , with

$$(\widetilde{W}, \widetilde{W}) = \bigcup_U (W(U)_1, W(U)_2).$$

□

This completes the proof of part (i) of the Combinatorial Transversality Theorem, the existence of finite Seifert-van Kampen splittings. Part (ii) deals with existence of finite injective Seifert-van Kampen splittings: the adjustment of fundamental groups needed to replace  $(X(U), Y(U))$  by a homology-equivalent finite injective Seifert-van Kampen splitting will use the following rudimentary version of the Quillen plus-construction.

**Lemma 2.7.** *Let  $K$  be a connected CW complex with a finitely generated fundamental group  $\pi_1(K)$ . For any surjection  $\phi : \pi_1(K) \rightarrow \Pi$  onto a finitely presented group  $\Pi$  it is possible to attach a finite number  $n$  of 2- and 3-cells to  $K$  to obtain a connected CW complex*

$$K' = K \cup \bigcup_n D^2 \cup \bigcup_n D^3$$

such that the inclusion  $K \rightarrow K'$  is a  $\mathbb{Z}[\Pi]$ -coefficient homology equivalence inducing  $\phi : \pi_1(K) \rightarrow \pi_1(K') = \Pi$ .

*Proof.* The kernel of  $\phi : \pi_1(K) \rightarrow \Pi$  is the normal closure of a finitely generated subgroup  $N \subseteq \pi_1(K)$  by Lemma 1.4 of Cappell [3]. (Here is the proof. Choose finite generating sets

$$g = \{g_1, g_2, \dots, g_r\} \subseteq \pi_1(K), \quad h = \{h_1, h_2, \dots, h_s\} \subseteq \Pi$$

and let  $w_k(h_1, h_2, \dots, h_s)$  ( $1 \leq k \leq t$ ) be words in  $h$  which generate  $\Pi$ . As  $\phi$  is surjective, can choose  $h'_j \in \pi_1(K)$  with  $\phi(h'_j) = h_j$  ( $1 \leq j \leq s$ ). As  $h$  generates  $\Pi$   $\phi(g_i) = v_i(h_1, h_2, \dots, h_s)$  ( $1 \leq i \leq r$ ) for some words  $v_i$  in  $h$ . The kernel of  $\phi$  is the normal closure  $N = \langle N' \rangle \triangleleft \pi_1(K)$  of the subgroup  $N' \subseteq \pi_1(K)$  generated by the finite set  $\{v_i(h'_1, \dots, h'_s)g_i^{-1}, w_k(h'_1, \dots, h'_s)\}$ . Let  $x = \{x_1, x_2, \dots, x_n\} \subseteq \pi_1(K)$  be a finite set of generators of  $N$ , and set

$$L = K \cup_x \bigcup_{i=1}^n D^2.$$

The inclusion  $K \rightarrow L$  induces

$$\phi : \pi_1(K) \rightarrow \pi_1(L) = \pi_1(K) / \langle x_1, x_2, \dots, x_n \rangle = \pi_1(K) / \langle N \rangle = \Pi.$$

Let  $\tilde{L}$  be the universal cover of  $L$ , and let  $\tilde{K}$  be the pullback cover of  $K$ . Now

$$\pi_1(\tilde{K}) = \ker(\phi) = \langle x_1, x_2, \dots, x_n \rangle = \langle N \rangle$$

so that the attaching maps  $x_i : S^1 \rightarrow K$  of the 2-cells in  $L-K$  lift to null-homotopic maps  $\tilde{x}_i : S^1 \rightarrow \tilde{K}$ . The cellular chain complexes of  $\tilde{K}$  and  $\tilde{L}$  are related by

$$C(\tilde{L}) = C(\tilde{K}) \oplus \bigoplus_n (\mathbb{Z}[\Pi], 2)$$

where  $(\mathbb{Z}[\Pi], 2)$  is just  $\mathbb{Z}[\Pi]$  concentrated in degree 2. Define

$$x^* = \{x_1^*, x_2^*, \dots, x_n^*\} \subseteq \pi_2(L)$$

by

$$x_i^* = (0, (0, \dots, 0, 1, 0, \dots, 0)) \in \pi_2(L) = H_2(\tilde{L}) = H_2(\tilde{K}) \oplus \mathbb{Z}[\Pi]^n \quad (1 \leq i \leq n),$$

and set

$$K' = L \cup_{x^*} \bigcup_{i=1}^n D^3.$$

The inclusion  $K \rightarrow K'$  induces  $\phi : \pi_1(K) \rightarrow \pi_1(K') = \pi_1(L) = \Pi$ , and the relative cellular  $\mathbb{Z}[\Pi]$ -module chain complex is

$$C(\tilde{K}', \tilde{K}) : \dots \rightarrow 0 \rightarrow \mathbb{Z}[\Pi]^n \xrightarrow{1} \mathbb{Z}[\Pi]^n \rightarrow 0 \rightarrow \dots$$

concentrated in degrees 2,3. In particular,  $K \rightarrow K'$  is a  $\mathbb{Z}[\Pi]$ -coefficient homology equivalence.  $\square$

**Proposition 2.8.** *Let  $(X, Y)$  be a finite connected 2-sided CW pair with  $X = X_1 \cup_Y X_2$  for connected  $X_1, X_2, Y$ , together with an isomorphism*

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \xrightarrow{\cong} G = G_1 *_H G_2$$

*preserving amalgamated free product structures, with the structure on  $G$  injective. It is possible to attach a finite number of 2- and 3-cells to  $(X, Y)$  to obtain a finite injective Seifert-van Kampen splitting  $(X', Y')$  with  $X' = X'_1 \cup_{Y'} X'_2$  such that*

- (i)  $\pi_1(X') = G$ ,  $\pi_1(X'_i) = G_i$  ( $i = 1, 2$ ),  $\pi_1(Y') = H$ ,
- (ii) the inclusion  $X \rightarrow X'$  is a homotopy equivalence,
- (iii) the inclusion  $X_i \rightarrow X'_i$  ( $i = 1, 2$ ) is a  $\mathbb{Z}[G_i]$ -coefficient homology equivalence,
- (iv) the inclusion  $Y \rightarrow Y'$  is a  $\mathbb{Z}[H]$ -coefficient homology equivalence.

*Proof.* Apply the construction of Lemma 1.4 to the surjections  $\pi_1(X_1) \rightarrow G_1$ ,  $\pi_1(X_2) \rightarrow G_2$ ,  $\pi_1(Y) \rightarrow H$ , to obtain

$$\begin{aligned} X'_i &= (X_i \cup_{x_i} \bigcup_{m_i} D^2) \cup_{x_i^*} \bigcup_{m_i} D^3 \quad (i = 1, 2), \\ Y' &= (Y \cup_y \bigcup_n D^2) \cup_{y^*} \bigcup_n D^3 \end{aligned}$$

for any  $y = \{y_1, y_2, \dots, y_n\} \subseteq \pi_1(Y)$  such that  $\ker(\pi_1(Y) \rightarrow H)$  is the normal closure of the subgroup of  $\pi_1(Y)$  generated by  $y$ , and any

$$x_i = \{x_{i,1}, x_{i,2}, \dots, x_{i,m_i}\} \subseteq \pi_1(X_i)$$

such that  $\ker(\pi_1(X_i) \rightarrow G_i)$  is the normal closure of the subgroup of  $\pi_1(X_i)$  generated by  $x_i$  ( $i = 1, 2$ ). Choosing  $x_1, x_2$  to contain the images of  $y$ , we obtain the required 2-sided CW pair  $(X', Y')$  with  $X' = X'_1 \cup_{Y'} X'_2$ .  $\square$

This completes the proof of the Combinatorial Transversality Theorem for amalgamated free products.

**2.3. Combinatorial transversality for HNN extensions.** The proof of combinatorial transversality for HNN extensions proceeds exactly as for amalgamated free products, so only the statements will be given.

In this section  $W$  is a connected CW complex with fundamental group an injective HNN extension

$$\pi_1(W) = G = G_1 *_H \{t\}$$

with tree  $T$ . Let  $\widetilde{W}$  be the universal cover of  $W$ , and let

$$\begin{array}{ccc} \widetilde{W}/H & \xrightarrow{i_1} & \widetilde{W}/G_1 \\ & \xrightarrow{i_2} & \xrightarrow{j_1} W \end{array}$$

be the covering projections, and define a commutative square

$$\begin{array}{ccc} \widetilde{W}/H \times \{0, 1\} & \xrightarrow{i_1 \cup i_2} & \widetilde{W}/G_1 \\ i_3 \downarrow & & \downarrow j_1 \\ \widetilde{W}/H \times [0, 1] & \xrightarrow{j_2} & W \end{array}$$

where

$$\begin{aligned} i_3 &= \text{inclusion} : \widetilde{W}/H \times \{0, 1\} \rightarrow \widetilde{W}/H \times [0, 1], \\ j_2 &: \widetilde{W}/H \times [0, 1] \rightarrow W ; (x, s) \mapsto j_1 i_1(x) = j_1 i_2(x). \end{aligned}$$

**Definition 2.9.** (i) Suppose given a subcomplex  $W_1 \subseteq \widetilde{W}$  with

$$G_1 W_1 = W_1$$

so that

$$H(W_1 \cap tW_1) = W_1 \cap tW_1 \subseteq \widetilde{W}.$$

Define a commutative square of  $CW$  complexes and cellular maps

$$\begin{array}{ccc} (W_1 \cap tW_1)/H \times \{0, 1\} & \xrightarrow{e_1} & W_1/G_1 \\ \downarrow e_2 & \Phi & \downarrow f_1 \\ (W_1 \cap tW_1)/H \times [0, 1] & \xrightarrow{f_2} & W \end{array}$$

with

$$\begin{aligned} (W_1 \cap tW_1)/H &\subseteq \widetilde{W}/H, \quad W_1/G_1 \subseteq \widetilde{W}/G_1, \\ e_1 &= (i_1 \cup i_2)| : (W_1 \cap tW_1)/H \times \{0, 1\} \rightarrow W_1/G_1, \\ e_2 &= i_3| : (W_1 \cap tW_1)/H \times \{0, 1\} \rightarrow (W_1 \cap tW_1)/H \times [0, 1], \\ f_1 &= j_1| : W_1/G_1 \rightarrow W, \quad f_2 = j_2| : (W_1 \cap tW_1)/H \times [0, 1] \rightarrow W. \end{aligned}$$

(ii) A *domain*  $W_1$  for the universal cover  $\widetilde{W}$  of  $W$  is a connected subcomplex  $W_1 \subseteq \widetilde{W}$  such that  $W_1 \cap tW_1$  is connected, and such that for each cell  $D \subseteq \widetilde{W}$  the subgraph  $U(D) \subseteq T$  defined by

$$\begin{aligned} U(D)^{(0)} &= \{g_1 \in [G : G_1] \mid g_1 D \subseteq W_1\} \\ U(D)^{(1)} &= \{h \in [G_1 : H] \mid h D \subseteq W_1 \cap tW_1\} \end{aligned}$$

is a tree.

(iii) A domain  $W_1$  for  $\widetilde{W}$  is *fundamental* if the subtrees  $U(D) \subseteq T$  are either single vertices or single edges, so that

$$g_1 W_1 \cap g_2 W_1 = \begin{cases} h(W_1 \cap tW_1) & \text{if } g_1 \cap g_2 t^{-1} = h \in [G_1 : H] \\ g_1 W_1 & \text{if } g_1 = g_2 \\ \emptyset & \text{if } g_1 \cap g_2 t^{-1} = \emptyset, \end{cases}$$

$$W = (W_1/G_1) \cup_{(W_1 \cap tW_1)/H \times \{0, 1\}} (W_1 \cap tW_1)/H \times [0, 1].$$

□

**Proposition 2.10.** For a domain  $W_1$  of  $\widetilde{W}$  the cellular chain complex  $C(W_1)$  is a domain of the cellular chain complex  $C(\widetilde{W})$ . □

**Example 2.11.**  $W$  has a canonical infinite domain  $W_1 = \widetilde{W}$  with

$$(W_1 \cap tW_1)/H = \widetilde{W}/H$$

and  $U(D) = T$  for each cell  $D \subseteq \widetilde{W}$ . □

**Example 2.12.** (i) Suppose that  $W = X_1 \cup_{Y \times \{0, 1\}} Y \times [0, 1]$ , with  $X_1, Y \subseteq W$  connected subcomplexes such that the isomorphism

$$\pi_1(W) = \pi_1(X_1) *_{\pi_1(Y)} \{t\} \xrightarrow{\cong} G = G_1 *_{H} \{t\}$$

preserves the  $HNN$  extensions. The morphisms  $\pi_1(X_1) \rightarrow G_1$ ,  $\pi_1(Y) \rightarrow H$  are surjective. (If  $i_1, i_2 : \pi_1(Y) \rightarrow \pi_1(X_1)$  are injective these morphisms are also injective, allowing identifications  $\pi_1(X_1) = G_1$ ,  $\pi_1(Y) = H$ ). The universal cover of  $W$  is

$$\widetilde{W} = \bigcup_{g_1 \in [G:G_1]} g_1 \widetilde{X}_1 \cup \bigcup_{h \in [G_1:H]} (h\widetilde{Y} \cup ht\widetilde{Y}) \bigcup_{h \in [G_1:H]} h\widetilde{Y} \times [0, 1]$$

with  $\widetilde{X}_1$  the regular cover of  $X_1$  corresponding to  $\ker(\pi_1(X_1) \rightarrow G_1)$  and  $\widetilde{Y}$  the regular cover of  $Y$  corresponding to  $\ker(\pi_1(Y) \rightarrow H)$  (which are the universal covers of  $X_1, Y$  in the case  $\pi_1(X_1) = G_1$ ,  $\pi_1(Y) = H$ ). Then  $W_1 = \widetilde{X}_1$  is a fundamental domain of  $\widetilde{W}$  such that

$$\begin{aligned} (W_1 \cap tW_1)/H &= Y, \quad W_1 \cap tW_1 = \widetilde{Y}, \\ g_1 W_1 \cap g_2 W_1 &= (g_1 \cap g_2 t^{-1}) \widetilde{Y} \subseteq \widetilde{W} \quad (g_1 \neq g_2 \in [G:G_1]). \end{aligned}$$

For any cell  $D \subseteq \widetilde{W}$

$$U(D) = \begin{cases} \{g_1\} & \text{if } g_1 D \subseteq \widetilde{X}_1 - \bigcup_{h \in [G_1:H]} (h\widetilde{Y} \cup ht\widetilde{Y}) \text{ for some } g_1 \in [G:G_1] \\ \{g_1, g_2, h\} & \text{if } hD \subseteq \widetilde{Y} \times [0, 1] \text{ for some } h = g_1 \cap g_2 t^{-1} \in [G_1:H]. \end{cases}$$

(ii) If  $W_1$  is a fundamental domain for any connected  $CW$  complex  $W$  with  $\pi_1(W) = G = G_1 *_H \{t\}$  then  $W = X_1 \cup_{Y \times \{0,1\}} Y \times [0, 1]$  as in (i), with

$$X_1 = W_1/G_1, \quad Y = (W_1 \cap tW_1)/H.$$

□

**Definition 2.13.** Suppose that  $W$  is  $n$ -dimensional. Lift each cell  $D^r \subseteq W$  to a cell  $\widetilde{D}^r \subseteq \widetilde{W}$ . A sequence  $U = \{U_n, U_{n-1}, \dots, U_1, U_0\}$  of subtrees  $U_r \subseteq T$  is *realized* by  $W$  if the subspace

$$W(U)_1 = \bigcup_{r=0}^n \bigcup_{D^r \subset W} \bigcup_{g_1 \in U_r^{(0)}} g_1 \widetilde{D}^r \subseteq \widetilde{W}$$

is a connected subcomplex, in which case  $W(U)_1$  is a domain for  $\widetilde{W}$  with

$$W(U)_1 \cap tW(U)_1 = \bigcup_{r=0}^n \bigcup_{D^r \subset W} \bigcup_{h \in U_r^{(1)}} h \widetilde{D}^r \subseteq \widetilde{W}$$

a connected subcomplex. Thus  $U$  is realized by  $C(\widetilde{W})$  and

$$C(W(U)_1) = C(\widetilde{W})(U)_1 \subseteq j_1^! C(\widetilde{W})$$

is the domain for  $C(\widetilde{W})$  given by  $C_r(\widetilde{W})_1(U_r)$  in degree  $r$ . □

**Proposition 2.14.** (i) For any domain  $W_1$  there is defined a homotopy pushout

$$\begin{array}{ccc} (W_1 \cap tW_1)/H \times \{0, 1\} & \xrightarrow{e_1} & W_1/G_1 \\ \downarrow e_2 & \Phi & \downarrow f_1 \\ (W_1 \cap tW_1)/H \times [0, 1] & \xrightarrow{f_2} & W \end{array}$$

with  $e_1 = i_1 \cup i_2|$ ,  $e_2 = i_3|$ ,  $f_1 = j_1|$ ,  $f_2 = j_2|$ . The connected 2-sided CW pair

$$(X, Y) = (\mathcal{M}(e_1, e_2), (W_1 \cap tW_1)/H \times \{1/2\})$$

is a Seifert-van Kampen splitting of  $W$ , with a homotopy equivalence

$$f = f_1 \cup f_2 : X = \mathcal{M}(e_1, e_2) \xrightarrow{\cong} W .$$

(ii) The commutative square of covering projections

$$\begin{array}{ccc} \widetilde{W}/H \times \{0, 1\} & \xrightarrow{i_1 \cup i_2} & \widetilde{W}/G_1 \\ i_3 \downarrow & & \downarrow j_1 \\ \widetilde{W}/H \times [0, 1] & \xrightarrow{j_2} & W \end{array}$$

is a homotopy pushout. The connected 2-sided CW pair

$$(X(\infty), Y(\infty)) = (\mathcal{M}(i_1 \cup i_2, i_3), \widetilde{W}/H \times \{0\})$$

is a canonical injective infinite Seifert-van Kampen splitting of  $W$ , with a homotopy equivalence  $j = j_1 \cup j_2 : X(\infty) \rightarrow W$  such that

$$\pi_1(Y(\infty)) = H \subseteq \pi_1(X(\infty)) = G_1 *_H \{t\} .$$

(iii) For any (finite) sequence  $U = \{U_n, U_{n-1}, \dots, U_0\}$  of subtrees of  $T$  realized by  $W$  there is defined a homotopy pushout

$$\begin{array}{ccc} Y(U) \times \{0, 1\} & \xrightarrow{e_1} & X(U)_1 \\ e_2 \downarrow & & \downarrow f_1 \\ Y(U) \times [0, 1] & \xrightarrow{f_2} & W \end{array}$$

with

$$Y(U) = (W(U)_1 \cap tW(U)_1)/H , \quad X(U)_1 = W(U)_1/G_1 ,$$

$$e_1 = i_1 \cup i_2| , \quad e_2 = i_3| , \quad f_1 = j_1| , \quad f_2 = j_2| .$$

Thus

$$(X(U), Y(U)) = (\mathcal{M}(e_1, e_2), Y(U) \times \{1/2\})$$

is a (finite) Seifert-van Kampen splitting of  $W$ .

(iv) The canonical infinite domain of a finite CW complex  $W$  with  $\pi_1(W) = G_1 *_H \{t\}$  is a union of finite domains  $W(U)_1$

$$\widetilde{W} = \bigcup_U W(U)_1$$

with  $U$  running over all the finite sequences realized by  $W$ . The canonical infinite Seifert-van Kampen splitting is thus a union of finite Seifert-van Kampen splittings

$$(X(\infty), Y(\infty)) = \bigcup_U (X(U), Y(U)) .$$

□

**Proposition 2.15.** Let  $(X, Y)$  be a finite connected 2-sided CW pair with  $X = X_1 \cup_{Y \times \{0,1\}} Y \times [0, 1]$  for connected  $X_1, Y$ , together with an isomorphism

$$\pi_1(X) = \pi_1(X_1) *_H \pi_1(Y) \{t\} \xrightarrow{\cong} G = G_1 *_H \{t\}$$

preserving the  $HNN$  structures, with the structure on  $G$  injective. It is possible to attach a finite number of 2- and 3-cells to the finite Seifert-van Kampen splitting  $(X, Y)$  of  $X$  to obtain a finite injective Seifert-van Kampen splitting  $(X', Y')$  with  $X' = X'_1 \cup_{Y' \times \{0,1\}} Y' \times [0, 1]$  such that

- (i)  $\pi_1(X') = G$ ,  $\pi_1(X'_1) = G_1$ ,  $\pi_1(Y') = H$ ,
- (ii) the inclusion  $X \rightarrow X'$  is a homotopy equivalence,
- (iii) the inclusion  $X_1 \rightarrow X'_1$  is a  $\mathbb{Z}[G_1]$ -coefficient homology equivalence,
- (iv) the inclusion  $Y \rightarrow Y'$  is a  $\mathbb{Z}[H]$ -coefficient homology equivalence.

□

This completes the proof of the Combinatorial Transversality Theorem for  $HNN$  extensions.

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