GENERALIZED ARF INVARIANTS IN ALGEBRAIC L-THEORY

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Abstract. The difference between the quadratic L-groups $L_*(A)$ and the symmetric L-groups $L^*(A)$ of a ring with involution A is detected by generalized Arf invariants. The special case $A = \mathbb{Z}[x]$ gives a complete set of invariants for the Cappell UNil-groups $\mathrm{UNil}_*(\mathbb{Z};\mathbb{Z},\mathbb{Z})$ for the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, extending the results of Connolly and Ranicki [10], Connolly and Davis [8].

Contents

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Introduction

The invariant of Arf [1] is a basic ingredient in the isomorphism classification of quadratic forms over a field of characteristic 2. The algebraic L-groups of a ring with involution A are Witt groups of quadratic structures on A-modules and A-module chain complexes, or equivalently the cobordism groups of algebraic Poincaré complexes over A. The cobordism formulation of algebraic L-theory is used here to obtain generalized Arf invariants detecting the difference between the quadratic and symmetric L-groups of an arbitrary ring with involution A, with applications to the computation of the Cappell UNil-groups.

The (projective) *quadratic L-groups* of Wall [19] are 4-periodic groups

$$L_n(A) \ = \ L_{n+4}(A) \ .$$

The $2k$-dimensional L-group $L_{2k}(A)$ is the Witt group of nonsingular $(-1)^k$-quadratic forms (K, ψ) over A, where K is a f.g. projective A-module and ψ is an equivalence class of A-module morphisms

$$\psi \ : \ K \to K^* \ = \ \mathrm{Hom}_A(K, A)$$

such that $\psi + (-1)^k\psi^* : K \to K^*$ is an isomorphism, with $\psi \sim \psi + \chi + (-1)^{k+1}\chi^*$ for $\chi \in \mathrm{Hom}_A(K, K^*)$. A lagrangian L for (K, ψ) is a direct summand $L \subset K$ such that

$$L^\perp \ = \ L, \text{ where } L^\perp \ = \ \{x \in K \,|\, (\psi + (-1)^k\psi^*)(x)(y) = 0 \text{ for all } y \in L\} \ ,$$
$$\psi(x)(x) \in \{a + (-1)^{k+1}\overline{a} \,|\, a \in A\} \text{ for all } x \in L \ .$$

A form (K, ψ) admits a lagrangian L if and only if it is isomorphic to the hyperbolic form $H_{(-1)^k}(L) = (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$, in which case

$$(K, \psi) \ = \ H_{(-1)^k}(L) \ = \ 0 \in L_{2k}(A) \ .$$

The $(2k+1)$-dimensional L-group $L_{2k+1}(A)$ is the Witt group of $(-1)^k$-quadratic formations $(K, \psi; L, L')$ over A, with $L, L' \subset K$ lagrangians for (K, ψ).

The *symmetric L-groups* $L^n(A)$ of Mishchenko [13] are the cobordism groups of *n-dimensional symmetric Poincaré complexes* (C, ϕ) over A, with C an n-dimensional f.g. projective A-module chain complex

$$C \ : \ \cdots \to 0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0 \to \dots$$

and $\phi \in Q^n(C)$ an element of the n-dimensional symmetric Q-group of C (about which more in §1 below) such that $\phi_0 : C^{n-*} \to C$ is a chain equivalence. In particular, $L^0(A)$ is the Witt group of nonsingular symmetric forms (K, ϕ) over A, with

$$\phi \ = \ \phi^* \ : \ K \to K^*$$

an isomorphism, and $L^1(A)$ is the Witt group of symmetric formations $(K,\phi;L,L')$ over A. An analogous cobordism formulation of the quadratic L-groups was obtained in Ranicki [15], expressing $L_n(A)$ as the cobordism group of *n-dimensional quadratic Poincaré complexes* (C,ψ), with $\psi\in Q_n(C)$ an element of the n-dimensional quadratic Q-group of C such that $(1+T)\psi_0: C^{n-*}\to C$ is a chain equivalence. The *hyperquadratic L-groups* $\widehat{L}^n(A)$ of [15] are the cobordism groups of n-dimensional (symmetric, quadratic) Poincaré pairs $(f:C\to D,(\delta\phi,\psi))$ over A such that

$$(\delta\phi_0,(1+T)\psi_0)\ :\ D^{n-*}\to \mathcal{C}(f)$$

is a chain equivalence, with $\mathcal{C}(f)$ the algebraic mapping cone of f. The various L-groups are related by an exact sequence

$$\cdots\longrightarrow L_n(A)\xrightarrow{1+T}L^n(A)\longrightarrow \widehat{L}^n(A)\xrightarrow{\partial}L_{n-1}(A)\longrightarrow\cdots\ .$$

The symmetrization maps $1+T:L_*(A)\to L^*(A)$ are isomorphisms modulo 8-torsion, so that the hyperquadratic L-groups $\widehat{L}^*(A)$ are of exponent 8. The symmetric and hyperquadratic L-groups are not 4-periodic in general. However, there are defined natural maps

$$L^n(A)\to L^{n+4}(A)\ ,\ \widehat{L}^n(A)\to\widehat{L}^{n+4}(A)$$

(which are isomorphisms modulo 8-torsion), and there are 4-periodic versions of the L-groups

$$L^{n+4*}(A)\ =\ \lim_{k\to\infty}L^{n+4k}(A)\ ,\ \widehat{L}^{n+4*}(A)\ =\ \lim_{k\to\infty}\widehat{L}^{n+4k}(A)\ .$$

The 4-periodic symmetric L-group $L^{n+4*}(A)$ is the cobordism group of n-dimensional symmetric Poincaré complexes (C,ϕ) over A with C a finite (but not necessarily n-dimensional) f.g. projective A-module chain complex, and similarly for $\widehat{L}^{n+4*}(A)$.

The Tate $\mathbb{Z}_2$-cohomology groups of a ring with involution A

$$\widehat{H}^n(\mathbb{Z}_2;A)\ =\ \frac{\{x\in A\,|\,\overline{x}=(-1)^nx\}}{\{y+(-1)^n\overline{y}\,|\,y\in A\}}\ (n(\mathrm{mod}\,2))$$

are A-modules via

$$A\times\widehat{H}^n(\mathbb{Z}_2;A)\to\widehat{H}^n(\mathbb{Z}_2;A)\ ;\ (a,x)\mapsto ax\overline{a}\ .$$

The Tate $\mathbb{Z}_2$-cohomology A-modules give an indication of the difference between the quadratic and symmetric L-groups of A. If $\widehat{H}^*(\mathbb{Z}_2;A)=0$ (e.g. if $1/2\in A$) then the symmetrization maps $1+T:L_*(A)\to L^*(A)$ are isomorphisms and $\widehat{L}^*(A)=0$. If A is such that $\widehat{H}^0(\mathbb{Z}_2;A)$ and $\widehat{H}^1(\mathbb{Z}_2;A)$ have 1-dimensional f.g. projective A-module resolutions then the symmetric and hyperquadratic L-groups of A are 4-periodic (Proposition 1.30).

We shall say that a ring with the involution A is *r-even* for some $r\geqslant 1$ if

(i) A is commutative with the identity involution, so that $\widehat{H}^0(\mathbb{Z}_2;A)=A_2$ as an additive group with

$$A\times\widehat{H}^0(\mathbb{Z}_2;A)\to\widehat{H}^0(\mathbb{Z}_2;A)\ ;\ (a,x)\mapsto a^2x\ ,$$

and

$$\widehat{H}^1(\mathbb{Z}_2;A)\ =\ \{a\in A\,|\,2a=0\}\ ,$$

(ii) $2\in A$ is a non-zero divisor, so that $\widehat{H}^1(\mathbb{Z}_2;A)=0$,

(iii) $\widehat{H}^0(\mathbb{Z}_2;A)$ is a f.g. free A_2-module of rank r with a basis $\{x_1=1,x_2,\dots,x_r\}$.

If A is r-even then $\widehat{H}^0(\mathbb{Z}_2;A)$ has a 1-dimensional f.g. free A-module resolution

$$0 \to A^r \xrightarrow{2} A^r \xrightarrow{x} \widehat{H}^0(\mathbb{Z}_2;A) \to 0$$

so that the symmetric and hyperquadratic L-groups of A are 4-periodic (1.30).
A ring with involution A is 1-even if and only if it satisfies (i), (ii) and also

$$a - a^2 \in 2A \text{ for all } a \in A\ .$$

Theorem 0.1. *The hyperquadratic L-groups of a 1-even ring with involution A are given by :*

$$\widehat{L}^n(A) \ = \ \begin{cases} A_8 & \text{if } n \equiv 0 (\bmod 4) \\ A_2 & \text{if } n \equiv 1,3 (\bmod 4) \\ 0 & \text{if } n \equiv 2 (\bmod 4)\ . \end{cases}$$

The boundary maps $\partial : \widehat{L}^n(A) \to L_{n-1}(A)$ are given by :

$$\begin{aligned} \partial \ &: \ \widehat{L}^0(A) = A_8 \to L_{-1}(A)\ ;\ a \mapsto (A\oplus A, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; A, \mathrm{im}(\begin{pmatrix} 1-a \\ a \end{pmatrix} : A \to A\oplus A))\ , \\ \partial \ &: \ \widehat{L}^1(A) = A_2 \to L_0(A)\ ;\ a \mapsto (A\oplus A, \begin{pmatrix} (a-a^2)/2 & 1-2a \\ 0 & -2 \end{pmatrix})\ , \\ \partial \ &: \ \widehat{L}^3(A) = A_2 \to L_2(A)\ ;\ a \mapsto (A\oplus A, \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix})\ . \end{aligned}$$

The map

$$L^0(A) \to \widehat{L}^0(A) \ = \ A_8\ ;\ (K,\phi) \mapsto \phi(v,v)$$

is defined using any element $v \in K$ such that

$$\phi(u,u) \ = \ \phi(u,v) \in A_2\ (u \in K)\ .$$

□

Theorem 0.1 is proved in §2 (Corollary 2.31). In particular, $A = \mathbb{Z}$ is 1-even, and in this case Theorem 0.1 recovers the computation of $\widehat{L}^*(\mathbb{Z})$ obtained in [15] – the algebraic L-theory of $\mathbb{Z}$ is recalled further below in the Introduction.

Theorem 0.2. *If A is 1-even then the polynomial ring $A[x]$ is 2-even, with $A[x]_2$-module basis $\{1,x\}$ for $\widehat{H}^0(\mathbb{Z}_2;A[x])$. The hyperquadratic L-groups of $A[x]$ are given by :*

$$\widehat{L}^n(A[x]) \ = \ \begin{cases} A_8 \oplus A_4[x] \oplus A_2[x]^3 & \text{if } n \equiv 0 (\bmod 4) \\ A_2 & \text{if } n \equiv 1 (\bmod 4) \\ 0 & \text{if } n \equiv 2 (\bmod 4) \\ A_2[x] & \text{if } n \equiv 3 (\bmod 4)\ . \end{cases}$$

□

Theorems 0.1 and 0.2 are special cases of the following computation :

Theorem 0.3. *The hyperquadratic L-groups of an r-even ring with involution A are given by :*

$$\widehat{L}^n(A) =$$

$$\begin{cases} \dfrac{\{M \in \mathrm{Sym}_r(A) \,|\, M - MXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A) + \{2(N+N^t) - N^tXN \,|\, N \in M_r(A)\}} & \text{if } n = 0 \\ \dfrac{\{N \in M_r(A) \,|\, N + N^t \in 2\mathrm{Sym}_r(A), \frac{1}{2}(N+N^t) - N^tXN \in \mathrm{Quad}_r(A)\}}{2M_r(A)} & \text{if } n = 1 \\ 0 & \text{if } n = 2 \\ \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A) + \{L - LXL \,|\, L \in \mathrm{Sym}_r(A)\}} & \text{if } n = 3 \end{cases}$$

with $\mathrm{Sym}_r(A)$ *the additive group of symmetric $r \times r$ matrices $(a_{ij}) = (a_{ji})$ in A,* $\mathrm{Quad}_r(A) \subset \mathrm{Sym}_r(A)$ *the subgroup of the matrices such that $a_{ii} \in 2A$, and*

$$X = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & x_r \end{pmatrix} \in \mathrm{Sym}_r(A)$$

for an A_2-module basis $\{x_1 = 1, x_2, \dots, x_r\}$ of $\widehat{H}^0(\mathbb{Z}_2; A)$. The boundary maps $\partial : \widehat{L}^n(A) \to L_{n-1}(A)$ are given by :

$$\partial \;:\; \widehat{L}^0(A) \to L_{-1}(A) \;;\; M \mapsto (H_-(A^r); A^r, \mathrm{im}(\begin{pmatrix} 1 - XM \\ M \end{pmatrix} : A^r \to A^r \oplus (A^r)^*)) \;,$$

$$\partial \;:\; \widehat{L}^1(A) \to L_0(A) \;;\; N \mapsto (A^r \oplus A^r, \begin{pmatrix} \frac{1}{4}(N + N^t - 2N^tXN) & 1 - 2NX \\ 0 & -2X \end{pmatrix}) \;,$$

$$\partial \;:\; \widehat{L}^3(A) \to L_2(A) \;;\; M \mapsto (A^r \oplus (A^r)^*, \begin{pmatrix} M & 1 \\ 0 & X \end{pmatrix}) \;.$$

□

In §§1,2 we recall and extend the Q-groups and algebraic chain bundles of Ranicki [15], [18] and Weiss [20], including a proof of Theorem 0.3 (Theorem 2.30).

We shall be dealing with two types of generalized Arf invariant: for forms on f.g. projective modules, and for linking forms on homological dimension 1 torsion modules, which we shall be considering separately.

In §3 we define the *generalized Arf invariant* of a nonsingular $(-1)^k$-quadratic form (K, ψ) over an arbitrary ring with involution A with a lagrangian $L \subset K$ for $(K, \psi + (-)^k\psi^*)$ to be the element

$$(K, \psi; L) \in \widehat{L}^{4*+2k+1}(A)$$

with image

$$(K, \psi) \in \mathrm{im}(\partial : \widehat{L}^{4*+2k+1}(A) \to L_{2k}(A)) \;=\; \ker(1 + T : L_{2k}(A) \to L^{4*+2k}(A)) \;.$$

Theorem 3.8 gives an explicit formula for the generalized Arf invariant $(K,\psi;L) \in \widehat{L}^3(A)$ for an r-even A. Generalizations of the Arf invariants in L-theory have been previously studied by Clauwens [7] and Bak [2].

In §4 we consider a ring with involution A with a localization $S^{-1}A$ inverting a multiplicative subset $S \subset A$ of central non-zero divisors such that $\widehat{H}^*(\mathbb{Z}_2;S^{-1}A)=0$ (e.g. if $2 \in S$). The relative L-group $L_{2k}(A,S)$ in the localization exact sequence

$$\dots \to L_{2k}(A) \to L_{2k}(S^{-1}A) \to L_{2k}(A,S) \to L_{2k-1}(A) \to L_{2k-1}(S^{-1}A) \to \dots$$

is the Witt group of nonsingular $(-1)^k$-quadratic linking forms (T,λ,μ) over (A,S), with T a homological dimension 1 S-torsion A-module, λ an A-module isomorphism

$$\lambda \ = \ (-1)^k\lambda\widehat{\ } \ : \ T \to T\widehat{\ } \ = \ \mathrm{Ext}^1_A(T,A) \ = \ \mathrm{Hom}_A(T,S^{-1}A/A)$$

and

$$\mu \ : \ T \to Q_{(-1)^k}(A,S) \ = \ \frac{\{b \in S^{-1}A \,|\, \overline{b} = (-1)^k b\}}{\{a+(-1)^k\overline{a} \,|\, a \in A\}}$$

a $(-1)^k$-quadratic function for λ. The *linking Arf invariant* of a nonsingular $(-1)^k$-quadratic linking form (T,λ,μ) over (A,S) with a lagrangian $U \subset T$ for (T,λ) is defined to be an element

$$(T,\lambda,\mu;U) \in \widehat{L}^{4*+2k}(A) \ ,$$

with properties analogous to the generalized Arf invariant defined for forms in §3. Theorem 4.10 gives an explicit formula for the linking Arf invariant $(T,\lambda,\mu;U) \in \widehat{L}^{2k}(A)$ for an r-even A, using

$$S \ = \ (2)^\infty \ = \ \{2^i \,|\, i \geqslant 0\} \subset A \ , \ S^{-1}A \ = \ A[1/2] \ .$$

In §5 we apply the generalized and linking Arf invariants to the algebraic L-groups of a polynomial extension $A[x]$ ($\overline{x}=x$) of a ring with involution A, using the exact sequence

$$\cdots \longrightarrow L_n(A[x]) \xrightarrow{1+T} L^n(A[x]) \longrightarrow \widehat{L}^n(A[x]) \longrightarrow L_{n-1}(A[x]) \longrightarrow \cdots \ .$$

For a Dedekind ring A the quadratic L-groups of $A[x]$ are related to the UNil-groups $\mathrm{UNil}_*(A)$ of Cappell [4] by the splitting formula of Connolly and Ranicki [10]

$$L_n(A[x]) \ = \ L_n(A) \oplus \mathrm{UNil}_n(A) \ ,$$

and the symmetric and hyperquadratic L-groups of $A[x]$ are 4-periodic, and such that

$$L^n(A[x]) \ = \ L^n(A) \ , \ \widehat{L}^{n+1}(A[x]) \ = \ \widehat{L}^{n+1}(A) \oplus \mathrm{UNil}_n(A) \ .$$

Any computation of $\widehat{L}^*(A)$ and $\widehat{L}^*(A[x])$ thus gives a computation of $\mathrm{UNil}_*(A)$. Combining the splitting formula with Theorems 0.1, 0.2 gives :

Theorem 0.4. *If A is a 1-even Dedekind ring then*

$$\begin{aligned}\mathrm{UNil}_n(A) \ &= \ \widehat{L}^{n+1}(A[x])/\widehat{L}^{n+1}(A) \\ &= \ \begin{cases} 0 & \text{if } n \equiv 0,1 (\bmod 4) \\ xA_2[x] & \text{if } n \equiv 2 (\bmod 4) \\ A_4[x] \oplus A_2[x]^3 & \text{if } n \equiv 3 (\bmod 4) \ . \end{cases}\end{aligned}$$

□

In particular, Theorem 0.4 applies to $A = \mathbb{Z}$. The twisted quadratic Q-groups were first used in the partial computation of

$$\mathrm{UNil}_n(\mathbb{Z}) \;=\; \widehat{L}^{n+1}(\mathbb{Z}[x])/\widehat{L}^{n+1}(\mathbb{Z})$$

by Connolly and Ranicki [10]. The calculation in [10] was almost complete, except that $\mathrm{UNil}_3(\mathbb{Z})$ was only obtained up to extensions. The calculation was first completed by Connolly and Davis [8], using linking forms. We are grateful to them for sending us a preliminary version of their paper. The calculation of $\mathrm{UNil}_3(\mathbb{Z})$ in [8] uses the results of [10] and the classifications of quadratic and symmetric linking forms over $(\mathbb{Z}[x],(2)^\infty)$. The calculation of $\mathrm{UNil}_3(\mathbb{Z})$ here uses the linking Arf invariant measuring the difference between the Witt groups of quadratic and symmetric linking forms over $(\mathbb{Z}[x],(2)^\infty)$, developing further the Q-group strategy of [10].

The algebraic L-groups of $A = \mathbb{Z}_2$ are given by :

$$L^n(\mathbb{Z}_2) \;=\; \begin{cases} \mathbb{Z}_2 \text{ (rank (mod 2))} & \text{if } n \equiv 0 (\mathrm{mod}\, 2) \\ 0 & \text{if } n \equiv 1 (\mathrm{mod}\, 2)\ , \end{cases}$$

$$L_n(\mathbb{Z}_2) \;=\; \begin{cases} \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 0 (\mathrm{mod}\, 2) \\ 0 & \text{if } n \equiv 1 (\mathrm{mod}\, 2)\ , \end{cases}$$

$$\widehat{L}^n(\mathbb{Z}_2) \;=\; \mathbb{Z}_2$$

with $1 + T = 0 : L_n(\mathbb{Z}_2) \to L^n(\mathbb{Z}_2)$. The classical Arf invariant is defined for a nonsingular quadratic form (K,ψ) over $\mathbb{Z}_2$ and a lagrangian $L \subset K$ for the symmetric form $(K,\psi+\psi^*)$ to be

$$(K,\psi;L) \;=\; \sum_{i=1}^{\ell} \psi(e_i,e_i).\psi(e_i^*,e_i^*) \in \widehat{L}^1(\mathbb{Z}_2) \;=\; L_0(\mathbb{Z}_2) \;=\; \mathbb{Z}_2$$

with $e_1,e_2,\dots,e_\ell$ any basis for $L \subset K$, and $e_1^*,e_2^*,\dots,e_\ell^*$ a basis for a direct summand $L^* \subset K$ such that

$$(\psi+\psi^*)(e_i^*,e_j^*) \;=\; 0\ ,\ (\psi+\psi^*)(e_i^*,e_j) \;=\; \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\ . \end{cases}$$

The Arf invariant is independent of the choices of L and L^*.

The algebraic L-groups of $A = \mathbb{Z}$ are given by :

$$L^n(\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z} \text{ (signature)} & \text{if } n \equiv 0 (\mathrm{mod}\, 4) \\ \mathbb{Z}_2 \text{ (de Rham invariant)} & \text{if } n \equiv 1 (\mathrm{mod}\, 4) \\ 0 & \text{otherwise}\ , \end{cases}$$

$$L_n(\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z} \text{ (signature/8)} & \text{if } n \equiv 0 (\mathrm{mod}\, 4) \\ \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 2 (\mathrm{mod}\, 4) \\ 0 & \text{otherwise}\ , \end{cases}$$

$$\widehat{L}^n(\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z}_8 \text{ (signature (mod 8))} & \text{if } n \equiv 0 (\mathrm{mod}\, 4) \\ \mathbb{Z}_2 \text{ (de Rham invariant)} & \text{if } n \equiv 1 (\mathrm{mod}\, 4) \\ 0 & \text{if } n \equiv 2 (\mathrm{mod}\, 4) \\ \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 3 (\mathrm{mod}\, 4)\ . \end{cases}$$

Given a nonsingular symmetric form (K,ϕ) over $\mathbb{Z}$ there is a congruence (Hirzebruch, Neumann and Koh [12, Theorem 3.10])

$$\mathrm{signature}(K,\phi) \;\equiv\; \phi(v,v)\ (\mathrm{mod}\,8)$$

with $v\in K$ any element such that

$$\phi(u,v) \;\equiv\; \phi(u,u)\ (\mathrm{mod}\,2)\ (u\in K)\ ,$$

so that

$$\begin{aligned}(K,\phi) \;&=\; \mathrm{signature}(K,\phi) \;=\; \phi(v,v)\\ &\in \mathrm{coker}(1+T: L_0(\mathbb{Z})\to L^0(\mathbb{Z})) \;=\; \widehat{L}^0(\mathbb{Z}) \;=\; \mathrm{coker}(8:\mathbb{Z}\to\mathbb{Z}) \;=\; \mathbb{Z}_8\ .\end{aligned}$$

The projection $\mathbb{Z}\to\mathbb{Z}_2$ induces an isomorphism $L_2(\mathbb{Z})\cong L_2(\mathbb{Z}_2)$, so that the Witt class of a nonsingular (-1)-quadratic form (K,ψ) over $\mathbb{Z}$ is given by the Arf invariant of the mod 2 reduction

$$(K,\psi;L) \;=\; \mathbb{Z}_2\otimes_{\mathbb{Z}}(K,\psi;L)\in L_2(\mathbb{Z}) \;=\; L_2(\mathbb{Z}_2) \;=\; \mathbb{Z}_2$$

with $L\subset K$ a lagrangian for the (-1)-symmetric form $(K,\psi-\psi^*)$. Again, this is independent of the choice of L.

The Q-groups are defined for an A-module chain complex C to be $\mathbb{Z}_2$-hyperhomology invariants of the $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $C\otimes_A C$. The involution on A is used to define the tensor product over A of left A-module chain complexes C,D, the abelian group chain complex

$$C\otimes_A D \;=\; \frac{C\otimes_{\mathbb{Z}} D}{\{ax\otimes y - x\otimes \overline{a}y\,|\,a\in A, x\in C, y\in D\}}\ .$$

Let $C\otimes_A C$ denote the $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex defined by $C\otimes_A C$ via the transposition involution

$$T\;:\; C_p\otimes_A C_q\to C_q\otimes_A C_p\ ;\ x\otimes y\mapsto (-1)^{pq}y\otimes x\ .$$

The $\begin{cases}\textit{symmetric}\\ \textit{quadratic}\\ \textit{hyperquadratic}\end{cases}$ *Q-groups* of C are defined by

$$\begin{cases}Q^n(C) \;=\; H^n(\mathbb{Z}_2;C\otimes_A C)\\ Q_n(C) \;=\; H_n(\mathbb{Z}_2;C\otimes_A C)\\ \widehat{Q}^n(C) \;=\; \widehat{H}^n(\mathbb{Z}_2;C\otimes_A C)\ .\end{cases}$$

The Q-groups are covariant in C, and are chain homotopy invariant. The Q-groups are related by an exact sequence

$$\cdots\longrightarrow Q_n(C)\xrightarrow{1+T} Q^n(C)\xrightarrow{J}\widehat{Q}^n(C)\xrightarrow{H} Q_{n-1}(C)\longrightarrow\cdots\ .$$

A *chain bundle* (C,γ) over A is an A-module chain complex C together with an element $\gamma\in\widehat{Q}^0(C^{-*})$. The *twisted quadratic Q-groups* $Q_*(C,\gamma)$ were defined in Weiss [20] using simplicial abelian groups, to fit into an exact sequence

$$\cdots\longrightarrow Q_n(C,\gamma)\xrightarrow{N_\gamma} Q^n(C)\xrightarrow{J_\gamma}\widehat{Q}^n(C)\xrightarrow{H_\gamma} Q_{n-1}(C,\gamma)\longrightarrow\cdots$$

with

$$J_\gamma\;:\; Q^n(C)\to\widehat{Q}^n(C)\ ;\ \phi\mapsto J(\phi)-(\widehat{\phi}_0)^{\%}(\gamma)\ .$$

An *n-dimensional algebraic normal complex* (C,ϕ,γ,θ) over A is an n-dimensional symmetric complex (C,ϕ) together with a chain bundle $\gamma \in \widehat{Q}^0(C^{-*})$ and an element $(\phi,\theta) \in Q_n(C,\gamma)$ with image $\phi \in Q^n(C)$. Every n-dimensional symmetric Poincaré complex (C,ϕ) has the structure of an algebraic normal complex (C,ϕ,γ,θ) : the *Spivak normal chain bundle* (C,γ) is characterized by

$$(\widehat{\phi}_0)^{\%}(\gamma) \;=\; J(\phi) \in Q^n(C) \;,$$

with

$$(\widehat{\phi}_0)^{\%} \;:\; \widehat{Q}^0(C^{-*}) \;=\; \widehat{Q}^n(C^{n-*}) \to \widehat{Q}^n(C)$$

the isomorphism induced by the Poincaré duality chain equivalence $\phi_0 : C^{n-*} \to C$, and the *algebraic normal invariant* $(\phi,\theta) \in Q_n(C,\gamma)$ is such that

$$N_\gamma(\phi,\theta) \;=\; \phi \in Q^n(C) \;.$$

See Ranicki [18, §7] for the one-one correspondence between the homotopy equivalence classes of n-dimensional (symmetric, quadratic) Poincaré pairs and n-dimensional algebraic normal complexes. Specifically, an n-dimensional algebraic normal complex (C,ϕ,γ,θ) determines an n-dimensional (symmetric, quadratic) Poincaré pair $(\partial C \to C^{n-*},(\delta\phi,\psi))$ with

$$\partial C \;=\; \mathcal{C}(\phi_0 : C^{n-*} \to C)_{*+1} \;.$$

Conversely, an n-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D,(\delta\phi,\psi))$ determines an n-dimensional algebraic normal complex $(\mathcal{C}(f),\gamma,\phi,\theta)$, with $\gamma \in \widehat{Q}^0(\mathcal{C}(f)^{-*})$ the Spivak normal chain bundle and $\phi = \delta\phi/(1{+}T)\psi$; the class $(\phi,\theta) \in Q_n(\mathcal{C}(f),\gamma)$ is the algebraic normal invariant of $(f : C \to D,(\delta\phi,\psi))$. Thus $\widehat{L}^n(A)$ is the cobordism group of n-dimensional normal complexes over A.

Weiss [20] established that for any ring with involution A there exists a universal chain bundle (B^A,β^A) over A, such that every chain bundle (C,γ) is classified by a chain bundle map

$$(g,\chi) \;:\; (C,\gamma) \to (B^A,\beta^A) \;,$$

with

$$H_*(B^A) \;=\; \widehat{H}^*(\mathbb{Z}_2;A) \;.$$

The function

$$\widehat{L}^{n+4*}(A) \to Q_n(B^A,\beta^A) \;;\; (C,\phi,\gamma,\theta) \mapsto (g,\chi)_{\%}(\phi,\theta)$$

was shown in [20] to be an isomorphism. Since the Q-groups are homological in nature (rather than of the Witt type) they are in principle effectively computable. The algebraic normal invariant defines the isomorphism

$$\begin{array}{l} \ker(1+T : L_n(A) \to L^{n+4*}(A)) \xrightarrow{\;\cong\;} \operatorname{coker}(L^{n+4*+1}(A) \to Q_{n+1}(B^A,\beta^A)) \;; \\ \qquad\qquad\qquad\qquad (C,\psi) \mapsto (g,\chi)_{\%}(\phi,\theta) \end{array}$$

with $(\phi,\theta) \in Q_{n+1}(\mathcal{C}(f),\gamma)$ the algebraic normal invariant of any $(n+1)$-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D,(\delta\phi,\psi))$, with classifying chain bundle map $(g,\chi) : (\mathcal{C}(f),\gamma) \to (B^A,\beta^A)$. For $n = 2k$ such a pair with $H_i(C) = H_i(D) = 0$ for $i \neq k$ is just a nonsingular $(-1)^k$-quadratic form $(K = H^k(C),\psi)$ with a lagrangian

$$L \;=\; \operatorname{im}(f^* : H^k(D) \to H^k(C)) \subset K \;=\; H^k(C)$$

for $(K,\psi+(-1)^k\psi^*)$, such that the generalized Arf invariant is the image of the algebraic normal invariant

$$(K,\psi;L) \ = \ (g,\chi)_{\%}(\phi,\theta)\in \widehat{L}^{4*+2k+1}(A) \ = \ Q_{2k+1}(B^A,\beta^A)\ .$$

For $A=\mathbb{Z}_2$ and $n=0$ this is just the classical Arf invariant isomorphism

$$\begin{array}{rl} L_0(\mathbb{Z}_2) \ = & \ker(1+T=0:L_0(\mathbb{Z}_2)\to L^0(\mathbb{Z}_2)) \\ & \xrightarrow{\ \cong\ } \ \mathrm{coker}(L^1(\mathbb{Z}_2)=0\to Q_1(B^{\mathbb{Z}_2},\beta^{\mathbb{Z}_2})) \ = \ \mathbb{Z}_2\ ; \\ & (K,\psi)\mapsto (K,\psi;L) \end{array}$$

with $L\subset K$ an arbitrary lagrangian of $(K,\psi+\psi^*)$. The isomorphism

$$\mathrm{coker}(1+T:L_n(A)\to L^{n+4*}(A)) \ \xrightarrow{\ \cong\ }\ \ker(\partial:Q_n(B^A,\beta^A)\to L_{n-1}(A))$$

is a generalization from $A=\mathbb{Z}$, $n=0$ to arbitrary A, n of the identity signature$(K,\phi)\equiv \phi(v,v)\pmod 8$ described above.

[Here is some of the geometric background. Chain bundles are algebraic analogues of vector bundles and spherical fibrations, and the twisted Q-groups are the analogues of the homotopy groups of the Thom spaces. A $(k-1)$-spherical fibration $\nu:X\to BG(k)$ over a connected CW complex X determines a chain bundle $(C(\widetilde{X}),\gamma)$ over $\mathbb{Z}[\pi_1(X)]$, with $C(\widetilde{X})$ the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex of the universal cover $\widetilde{X}$, and there are defined Hurewicz-type morphisms

$$\pi_{n+k}(T(\nu))\to Q_n(C(\widetilde{X}),\gamma)$$

with $T(\nu)$ the Thom space. An n-dimensional normal space $(X,\nu:X\to BG(k),\rho:S^{n+k}\to T(\nu))$ (Quinn [14]) determines an n-dimensional algebraic normal complex $(C(\widetilde{X}),\phi,\gamma,\theta)$ over $\mathbb{Z}[\pi_1(X)]$. An n-dimensional geometric Poincaré complex X has a Spivak normal structure (ν,ρ) such that the composite of the Hurewicz map and the Thom isomorphism

$$\pi_{n+k}(T(\nu))\to \widetilde{H}_{n+k}(T(\nu)) \ \cong\ H_n(X)$$

sends ρ to the fundamental class $[X]\in H_n(X)$, and there is defined an n-dimensional symmetric Poincaré complex $(C(\widetilde{X}),\phi)$ over $\mathbb{Z}[\pi_1(X)]$, with

$$\phi_0 \ = \ [X]\cap - \ :\ C(\widetilde{X})^{n-*}\to C(\widetilde{X})\ .$$

The symmetric signature of X is the symmetric Poincaré cobordism class

$$\sigma^*(X) \ = \ (C(\widetilde{X}),\phi)\in L^n(\mathbb{Z}[\pi_1(X)])$$

which is both a homotopy and $K(\pi_1(X),1)$-bordism invariant. The algebraic normal invariant of a normal space (X,ν,ρ)

$$[\rho] \ = \ (\phi,\theta)\in Q_n(C(\widetilde{X}),\gamma)$$

is a homotopy invariant. The classifying chain bundle map

$$(g,\chi)\ :\ (C(\widetilde{X}),\gamma)\to (B^{\mathbb{Z}[\pi_1(X)]},\beta^{\mathbb{Z}[\pi_1(X)]})$$

sends $[\rho]$ to the hyperquadratic signature of X

$$\widehat{\sigma}^*(X) \ = \ [\phi,\theta]\in Q_n(B^{\mathbb{Z}[\pi_1(X)]},\beta^{\mathbb{Z}[\pi_1(X)]}) \ = \ \widehat{L}^{n+4*}(\mathbb{Z}[\pi_1(X)])\ ,$$

which is both a homotopy and $K(\pi_1(X),1)$-bordism invariant. The (simply-connected) symmetric signature of a $4k$-dimensional geometric Poincaré complex X is just the signature

$$\sigma^*(X) \ = \ \text{signature}(X) \in L^{4k}(\mathbb{Z}) \ = \ \mathbb{Z}$$

and the hyperquadratic signature is the mod 8 reduction of the signature

$$\widehat{\sigma}^*(X) \ = \ \text{signature}(X) \in \widehat{L}^{4k}(\mathbb{Z}) \ = \ \mathbb{Z}_8 \ .$$

See Ranicki [18] for a more extended discussion of the connections between chain bundles and their geometric models.]

1. The Q- and L-groups

1.1. **Duality.** Let $T \in \mathbb{Z}_2$ be the generator. The *Tate $\mathbb{Z}_2$-cohomology* groups of a $\mathbb{Z}[\mathbb{Z}_2]$-module M are given by

$$\widehat{H}^n(\mathbb{Z}_2;M) \ = \ \frac{\{x \in M \,|\, T(x) = (-1)^n x\}}{\{y + (-1)^n T(y) \,|\, y \in M\}} \ ,$$

and the $\begin{cases} \mathbb{Z}_2\text{-}\textit{cohomology} \\ \mathbb{Z}_2\text{-}\textit{homology} \end{cases}$ groups are given by

$$H^n(\mathbb{Z}_2;M) \ = \ \begin{cases} \{x \in M \,|\, T(x) = x\} & \text{if } n = 0 \\ \widehat{H}^n(\mathbb{Z}_2;M) & \text{if } n > 0 \\ 0 & \text{if } n < 0 \ , \end{cases}$$

$$H_n(\mathbb{Z}_2;M) \ = \ \begin{cases} M/\{y - T(y) \,|\, y \in M\} & \text{if } n = 0 \\ \widehat{H}^{n+1}(\mathbb{Z}_2;M) & \text{if } n > 0 \\ 0 & \text{if } n < 0 \ . \end{cases}$$

We recall some standard properties of $\mathbb{Z}_2$-(co)homology :

Proposition 1.1. *Let M be a $\mathbb{Z}[\mathbb{Z}_2]$-module.*
(i) *There is defined an exact sequence*

$$\dots \to H_n(\mathbb{Z}_2;M) \xrightarrow{\ N\ } H^{-n}(\mathbb{Z}_2;M) \to \widehat{H}^n(\mathbb{Z}_2;M) \to H_{n-1}(\mathbb{Z}_2;M) \to \dots$$

with

$$N \ = \ 1 + T \ : \ H_0(\mathbb{Z}_2;M) \to H^0(\mathbb{Z}_2;M) \ ; \ x \mapsto x + T(x) \ .$$

(ii) *The Tate $\mathbb{Z}_2$-cohomology groups are 2-periodic and of exponent 2*

$$\widehat{H}^*(\mathbb{Z}_2;M) \ = \ \widehat{H}^{*+2}(\mathbb{Z}_2;M) \ , \ 2\widehat{H}^*(\mathbb{Z}_2;M) \ = \ 0 \ .$$

(iii) $\widehat{H}^*(\mathbb{Z}_2;M) = 0$ *if M is a free $\mathbb{Z}[\mathbb{Z}_2]$-module.* □

Let A be an associative ring with 1, and with an involution

$$^- : \ A \to A \ ; \ a \mapsto \overline{a} \ ,$$

such that

$$\overline{a+b} \ = \ \overline{a} + \overline{b} \ , \ \overline{ab} \ = \ \overline{b}.\overline{a} \ , \ \overline{1} \ = \ 1 \ , \ \overline{\overline{a}} \ = \ a \ .$$

When a ring A is declared to be commutative it is given the identity involution.

Definition 1.2. For a ring with involution A and $\epsilon = \pm 1$ let (A, ϵ) denote the $\mathbb{Z}[\mathbb{Z}_2]$-module given by A with $T \in \mathbb{Z}_2$ acting by

$$T_\epsilon \ : \ A \to A \ ; \ a \mapsto \epsilon \overline{a} \ .$$

□

For $\epsilon = 1$ we shall write

$$\widehat{H}^*(\mathbb{Z}_2; A, 1) \ = \ \widehat{H}^*(\mathbb{Z}_2; A) \ , \ H^*(\mathbb{Z}_2; A, 1) \ = \ H^*(\mathbb{Z}_2; A) \ , \ H_*(\mathbb{Z}_2; A, 1) \ = \ H_*(\mathbb{Z}_2; A) \ .$$

The *dual* of a f.g. projective (left) A-module P is the f.g. projective A-module

$$P^* \ = \ \mathrm{Hom}_A(P, A) \ , \ A \times P^* \to P^* \ ; \ (a, f) \mapsto (x \mapsto f(x)\overline{a}) \ .$$

The natural A-module isomorphism

$$P \to P^{**} \ ; \ x \mapsto (f \mapsto \overline{f(x)})$$

is used to identify

$$P^{**} \ = \ P \ .$$

For any f.g. projective A-modules P, Q there is defined an isomorphism

$$P \otimes_A Q \to \mathrm{Hom}_A(P^*, Q) \ ; \ x \otimes y \mapsto (f \mapsto \overline{f(x)}y)$$

regarding Q as a right A-module by

$$Q \times A \to Q \ ; \ (y, a) \mapsto \overline{a}y \ .$$

There is also a duality isomorphism

$$T \ : \ \mathrm{Hom}_A(P, Q) \to \mathrm{Hom}_A(Q^*, P^*) \ ; \ f \mapsto f^*$$

with

$$f^* \ : \ Q^* \to P^* \ ; \ g \mapsto (x \mapsto g(f(x))) \ .$$

Definition 1.3. For any f.g. projective A-module P and $\epsilon = \pm 1$ let $(S(P), T_\epsilon)$ denote the $\mathbb{Z}[\mathbb{Z}_2]$-module given by the abelian group

$$S(P) \ = \ \mathrm{Hom}_A(P, P^*)$$

with $\mathbb{Z}_2$-action by the ϵ-duality involution

$$T_\epsilon \ : \ S(P) \to S(P) \ ; \ \phi \mapsto \epsilon\phi^* \ .$$

Furthermore, let

$$\begin{aligned} \mathrm{Sym}(P, \epsilon) \ &= \ H^0(\mathbb{Z}_2; S(P), T_\epsilon) \ = \ \{\phi \in S(P) \,|\, T_\epsilon(\phi) = \phi\} \ , \\ \mathrm{Quad}(P, \epsilon) \ &= \ H_0(\mathbb{Z}_2; S(P), T_\epsilon) \ = \ \frac{S(P)}{\{\theta \in S(P) \,|\, \theta - T_\epsilon(\theta)\}} \ . \end{aligned}$$

□

An element $\phi \in S(P)$ can be regarded as a sesquilinear form

$$\phi \ : \ P \times P \to A \ ; \ (x, y) \mapsto \langle x, y \rangle_\phi \ = \ \phi(x)(y)$$

such that

$$\langle ax, by \rangle_\phi \ = \ b\langle x, y \rangle_\phi \overline{a} \in A \ (x, y \in P, a, b \in A) \ ,$$

with

$$\langle x, y \rangle_{T_\epsilon(\phi)} \ = \ \epsilon\overline{\langle y, x \rangle}_\phi \in A \ .$$

An A-module morphism $f : P \to Q$ induces contravariantly a $\mathbb{Z}[\mathbb{Z}_2]$-module morphism

$$S(f) \ : \ (S(Q),T_\epsilon) \to (S(P),T_\epsilon) \ ; \ \theta \mapsto f^*\theta f \ .$$

For a f.g. free A-module $P = A^r$ we shall use the A-module isomorphism

$$A^r \to (A^r)^* \ ; \ (a_1,a_2,\dots,a_r) \mapsto ((b_1,b_2,\dots,b_r) \mapsto \sum_{i=1}^{r} b_i\overline{a}_i)$$

to identify

$$(A^r)^* \ = \ A^r \ , \ \mathrm{Hom}_A(A^r,(A^r)^*) \ = \ M_r(A)$$

noting that the duality involution T corresponds to the conjugate transposition of a matrix. We can thus identify

$$\begin{aligned}
&M_r(A) \ = \ S(A^r) \ = \ \text{additive group of } r\times r \text{ matrices } (a_{ij}) \text{ with } a_{ij}\in A \ , \\
&T \ : \ M_r(A) \to M_r(A) \ ; \ M \ = \ (a_{ij}) \mapsto M^t \ = \ (\overline{a}_{ji}) \ , \\
&\mathrm{Sym}_r(A,\epsilon) \ = \ \mathrm{Sym}(A^r,\epsilon) \ = \ \{(a_{ij})\in M_r(A)\,|\,a_{ij} = \epsilon\overline{a}_{ji}\} \ , \\
&\mathrm{Quad}_r(A,\epsilon) \ = \ \mathrm{Quad}(A^r,\epsilon) \ = \ \frac{M_r(A)}{\{(a_{ij}-\epsilon\overline{a}_{ji})\,|\,(a_{ij})\in M_r(A)\}} \ , \\
&1+T_\epsilon \ : \ \mathrm{Quad}_r(A,\epsilon) \to \mathrm{Sym}_r(A,\epsilon) \ ; \ M \mapsto M+\epsilon M^t \ .
\end{aligned}$$

The homology of the chain complex

$$\cdots \longrightarrow M_r(A) \xrightarrow{1-T} M_r(A) \xrightarrow{1+T} M_r(A) \xrightarrow{1-T} M_r(A) \longrightarrow \cdots$$

is given by

$$\frac{\ker(1-(-1)^nT : M_r(A)\to M_r(A))}{\mathrm{im}(1+(-1)^nT : M_r(A)\to M_r(A))} \ = \ \widehat{H}^n(\mathbb{Z}_2;M_r(A)) \ = \ \bigoplus_r \widehat{H}^n(\mathbb{Z}_2;A) \ .$$

The $(-1)^n$-symmetrization map $1+(-1)^nT : \mathrm{Sym}_r(A) \to \mathrm{Quad}_r(A)$ fits into an exact sequence

$$\begin{aligned}
0 \to \bigoplus_r \widehat{H}^{n+1}(\mathbb{Z}_2;A) \ &\to \mathrm{Quad}_r(A,(-1)^n) \\
&\xrightarrow{1+(-1)^nT} \ \mathrm{Sym}_r(A,(-1)^n) \to \bigoplus_r \widehat{H}^n(\mathbb{Z}_2;A) \to 0 \ .
\end{aligned}$$

For $\epsilon = 1$ we abbreviate

$$\begin{aligned}
&\mathrm{Sym}(P,1) \ = \ \mathrm{Sym}(P) \ , \ \mathrm{Quad}(P,1) \ = \ \mathrm{Quad}(P) \ , \\
&\mathrm{Sym}_r(A,1) \ = \ \mathrm{Sym}_r(A) \ , \ \mathrm{Quad}_r(A,1) \ = \ \mathrm{Quad}_r(A) \ .
\end{aligned}$$

Definition 1.4. An involution on a ring A is *even* if

$$\widehat{H}^1(\mathbb{Z}_2;A) \ = 0 \ ,$$

that is if

$$\{a\in A\,|\,a+\overline{a}=0\} \ = \ \{b-\overline{b}\,|\,b\in A\} \ .$$

□

Proposition 1.5. (i) *For any f.g. projective A-module P there is defined an exact sequence*

$$0 \to \widehat{H}^1(\mathbb{Z}_2;S(P),T) \to \mathrm{Quad}(P) \xrightarrow{1+T} \mathrm{Sym}(P) \ ,$$

with

$$1+T \ : \ \mathrm{Quad}(P) \to \mathrm{Sym}(P) \ ; \ \psi \mapsto \psi+\psi^* \ .$$

(ii) *If the involution on A is even the symmetrization* $1+T:\mathrm{Quad}(P)\to\mathrm{Sym}(P)$ *is injective, and*

$$\widehat{H}^n(\mathbb{Z}_2;S(P),T)~=~\begin{cases}\dfrac{\mathrm{Sym}(P)}{\mathrm{Quad}(P)} & \textit{if } n \textit{ is even}\\ 0 & \textit{if } n \textit{ is odd},\end{cases}$$

identifying $\mathrm{Quad}(P)$ *with* $\mathrm{im}(1+T)\subseteq\mathrm{Sym}(P)$.

Proof. (i) This is a special case of 1.1 (i).
(ii) If Q is a f.g. projective A-module such that $P\oplus Q=A^r$ is f.g. free then

$$\begin{aligned}\widehat{H}^1(\mathbb{Z}_2;S(P),T)\oplus\widehat{H}^1(\mathbb{Z}_2;S(Q),T)~&=~\widehat{H}^1(\mathbb{Z}_2;S(P\oplus Q),T)\\ &=~\bigoplus_r\widehat{H}^1(\mathbb{Z}_2;A,-T)~=~0\end{aligned}$$

and so $\widehat{H}^1(\mathbb{Z}_2;S(P),T)=0$. □

In particular, if the involution on A is even there is defined an exact sequence

$$0\to\mathrm{Quad}_r(A)\xrightarrow{1+T}\mathrm{Sym}_r(A)\to\bigoplus_r\widehat{H}^0(\mathbb{Z}_2;A)\to 0$$

with

$$\mathrm{Sym}_r(A)\to\bigoplus_r\widehat{H}^0(\mathbb{Z}_2;A)~;~(a_{ij})\mapsto(a_{ii})~.$$

For any involution on A, $\mathrm{Sym}_r(A)$ is the additive group of symmetric $r\times r$ matrices $(a_{ij})=(\overline{a}_{ji})$ with $a_{ij}\in A$. For an even involution $\mathrm{Quad}_r(A)\subseteq\mathrm{Sym}_r(A)$ is the subgroup of the matrices such that the diagonal terms are of the form $a_{ii}=b_i+\overline{b}_i$ for some $b_i\in A$, with

$$\frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}~=~\bigoplus_r\widehat{H}^0(\mathbb{Z}_2;A)~.$$

Definition 1.6. A ring A is *even* if $2\in A$ is a non-zero divisor, i.e. $2:A\to A$ is injective. □

Example 1.7. (i) An integral domain A is even if and only if it has characteristic $\neq 2$.
(ii) The identity involution on a commutative ring A is even (1.4) if and only if A is even, in which case

$$\widehat{H}^n(\mathbb{Z}_2;A)~=~\begin{cases}A_2 & \text{if } n\equiv 0(\bmod 2)\\ 0 & \text{if } n\equiv 1(\bmod 2)\end{cases}$$

and

$$\mathrm{Quad}_r(A)~=~\{(a_{ij})\in\mathrm{Sym}_r(A)\,|\,a_{ii}\in 2A\}~.$$

□

Example 1.8. For any group π there is defined an involution on the group ring $\mathbb{Z}[\pi]$

$$\bar{~}~:~\mathbb{Z}[\pi]\to\mathbb{Z}[\pi]~;~\sum_{g\in\pi}n_g g\mapsto\sum_{g\in\pi}n_g g^{-1}~.$$

If π has no 2-torsion this involution is even. □

1.2. **The Hyperquadratic Q-Groups.** Let C be a finite (left) f.g. projective A-module chain complex. The dual of the f.g. projective A-module C_p is written

$$C^p = (C_p)^* = \mathrm{Hom}_A(C_p, A) .$$

The dual A-module chain complex C^{-*} is defined by

$$d_{C^{-*}} = (d_C)^* : (C^{-*})_r = C^{-r} \to (C^{-*})_{r-1} = C^{-r+1} .$$

The n-dual A-module chain complex C^{n-*} is defined by

$$d_{C^{n-*}} = (-1)^r (d_C)^* : (C^{n-*})_r = C^{n-r} \to (C^{n-*})_{r-1} = C^{n-r+1} .$$

Identify

$$C \otimes_A C = \mathrm{Hom}_A(C^{-*}, C) ,$$

noting that a cycle $\phi \in (C \otimes_A C)_n$ is a chain map $\phi : C^{n-*} \to C$. For $\epsilon = \pm 1$ the ϵ-transposition involution T_ϵ on $C \otimes_A C$ corresponds to the ϵ-duality involution on $\mathrm{Hom}_A(C^{-*}, C)$

$$T_\epsilon : \mathrm{Hom}_A(C^p, C_q) \to \mathrm{Hom}_A(C^q, C_p) ; \ \phi \mapsto (-1)^{pq} \epsilon \phi^* .$$

Let $\widehat{W}$ be the complete resolution of the $\mathbb{Z}[\mathbb{Z}_2]$-module $\mathbb{Z}$

$$\widehat{W} : \ldots \longrightarrow \widehat{W}_1 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \widehat{W}_0 = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \widehat{W}_{-1} = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \widehat{W}_{-2} = \mathbb{Z}[\mathbb{Z}_2] \longrightarrow \ldots .$$

If we set

$$\widehat{W}^{\%} C = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, \mathrm{Hom}_A(C^{-*}, C)) ,$$

then an *n-dimensional ϵ-hyperquadratic structure on C* is a cycle $\theta \in (\widehat{W}^{\%} C)_n$, which is just a collection $\{\theta_s \in \mathrm{Hom}_A(C^r, C_{n-r+s}) \,|\, r, s \in \mathbb{Z}\}$ such that

$$d\theta_s + (-1)^r \theta_s d^* + (-1)^{n+s-1}(\theta_{s-1} + (-1)^s T_\epsilon \theta_{s-1}) = 0 : C^r \to C_{n-r+s-1} .$$

Definition 1.9. The *n-dimensional ϵ-hyperquadratic Q-group* $\widehat{Q}^n(C, \epsilon)$ is the abelian group of equivalence classes of n-dimensional ϵ-hyperquadratic structures on C, that is,

$$\widehat{Q}^n(C, \epsilon) = H_n(\widehat{W}^{\%} C) .$$

□

The ϵ-hyperquadratic Q-groups are 2-periodic and of exponent 2

$$\widehat{Q}^*(C, \epsilon) \cong \widehat{Q}^{*+2}(C, \epsilon) , \ 2\widehat{Q}^*(C, \epsilon) = 0 .$$

More precisely, there are defined isomorphisms

$$\widehat{Q}^n(C, \epsilon) \xrightarrow{\cong} \widehat{Q}^{n+2}(C, \epsilon) ; \ \{\theta_s\} \mapsto \{\theta_{s+2}\} ,$$

and for any n-dimensional ϵ-hyperquadratic structure $\{\theta_s\}$

$$2\theta_s = d\chi_s + (-1)^r \chi_s d^* + (-1)^{n+s}(\chi_{s-1} + (-1)^s T_\epsilon \chi_{s-1}) : C^r \to C_{n-r+s}$$

with $\chi_s = (-1)^{n+s-1}\theta_{s+1}$. There are also defined suspension isomorphisms

$$S : \widehat{Q}^n(C, \epsilon) \xrightarrow{\cong} \widehat{Q}^{n+1}(C_{*-1}, \epsilon) ; \ \{\theta_s\} \mapsto \{\theta_{s-1}\}$$

and skew-suspension isomorphisms

$$\overline{S} : \widehat{Q}^n(C, \epsilon) \xrightarrow{\cong} \widehat{Q}^{n+2}(C_{*-1}, -\epsilon) ; \ \{\theta_s\} \mapsto \{\theta_s\} .$$

Proposition 1.10. *Let C be a f.g. projective A-module chain complex which is concentrated in degree k*

$$C \ : \ \cdots \to 0 \to C_k \to 0 \to \dots \ .$$

The ϵ-hyperquadratic Q-groups of C are given by

$$\widehat{Q}^n(C,\epsilon) \ = \ \widehat{H}^{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_\epsilon)$$

(with $S(C^k) = \mathrm{Hom}_A(C^k, C_k)$).

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $V = \mathrm{Hom}_A(C^{-*}, C)$ is given by

$$V \ : \ \cdots \to V_{2k+1} \ = \ 0 \to V_{2k} \ = \ S(C^k) \to V_{2k-1} \ = \ 0 \to \dots$$

and

$$(\widehat{W}^{\%}C)_j \ = \ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_{2k-j}, V_{2k}) \ = \ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_{2k-j}, S(C^k)) \ .$$

Thus the chain complex $\widehat{W}^{\%}C$ is of the form

$$\begin{array}{lcl}
(\widehat{W}^{\%}C)_{2k+1} & = & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_{-1}, V_{2k}) \ = \ S(C^k) \\
\Big\downarrow {\scriptstyle d_{2k+1}=1+(-1)^k T_\epsilon} & & \\
(\widehat{W}^{\%}C)_{2k} & = & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_0, V_{2k}) \ = \ S(C^k) \\
\Big\downarrow {\scriptstyle d_{2k}=1+(-1)^{k+1} T_\epsilon} & & \\
(\widehat{W}^{\%}C)_{2k-1} & = & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_1, V_{2k}) \ = \ S(C^k) \\
\Big\downarrow {\scriptstyle d_{2k-1}=1+(-1)^k T_\epsilon} & & \\
(\widehat{W}^{\%}C)_{2k-2} & = & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}_2, V_{2k}) \ = \ S(C^k) \\
\Big\downarrow & & \\
\vdots & &
\end{array}$$

and

$$\widehat{Q}^n(C,\epsilon) \ = \ H_n(\widehat{W}^{\%}C) \ = \ \widehat{H}^{n-2k}(\mathbb{Z}_2; S(C^k), (-1)^k T_\epsilon) \ .$$

□

Example 1.11. The ϵ-hyperquadratic Q-groups of a 0-dimensional f.g. free A-module chain complex

$$C \ : \ \cdots \to 0 \to C_0 \ = \ A^r \to 0 \to \dots$$

are given by

$$\widehat{Q}^n(C,\epsilon) \ = \ \bigoplus_r \widehat{H}^n(\mathbb{Z}_2; A, \epsilon) \ .$$

□

The *algebraic mapping cone* $\mathcal{C}(f)$ of a chain map $f: C \to D$ is the chain complex defined as usual by

$$d_{\mathcal{C}(f)} \ = \ \begin{pmatrix} d_D & (-1)^{r-1} f \\ 0 & d_C \end{pmatrix} \ : \ \mathcal{C}(f)_r \ = \ D_r \oplus C_{r-1} \to \mathcal{C}(f)_{r-1} \ = \ D_{r-1} \oplus C_{r-2} \ .$$

The relative homology groups

$$H_n(f) \;=\; H_n(\mathcal{C}(f))$$

fit into an exact sequence

$$\dots \to H_n(C) \xrightarrow{f_*} H_n(D) \to H_n(f) \to H_{n-1}(C) \to \dots \;.$$

An A-module chain map $f : C \to D$ induces a chain map

$$\widehat{f}^{\%} \;=\; \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(1_{\widehat{W}}, \mathrm{Hom}_A(f^*, f)) \;:\; \widehat{W}^{\%}C \longrightarrow \widehat{W}^{\%}D$$

which induces

$$\widehat{f}^{\%} \;:\; \widehat{Q}^n(C, \epsilon) \longrightarrow \widehat{Q}^n(D, \epsilon)$$

on homology. The *relative ϵ-hyperquadratic Q-group*

$$\widehat{Q}^n(f, \epsilon) \;=\; H_n(\widehat{f}^{\%} : \widehat{W}^{\%}C \to \widehat{W}^{\%}D)$$

fits into a long exact sequence

$$\dots \longrightarrow \widehat{Q}^n(C, \epsilon) \xrightarrow{\widehat{f}^{\%}} \widehat{Q}^n(D, \epsilon) \longrightarrow \widehat{Q}^n(f, \epsilon) \longrightarrow \widehat{Q}^{n-1}(C, \epsilon) \longrightarrow \dots \;.$$

Proposition 1.12. (i) *The relative ϵ-hyperquadratic Q-groups of an A-module chain map $f : C \to D$ are isomorphic to the absolute ϵ-hyperquadratic Q-groups of the algebraic mapping cone $\mathcal{C}(f)$*

$$\widehat{Q}^*(f, \epsilon) \;\cong\; \widehat{Q}^*(\mathcal{C}(f), \epsilon) \;.$$

(ii) *If $f : C \to D$ is a chain equivalence the morphisms $\widehat{f}^{\%} : \widehat{Q}^*(C, \epsilon) \to \widehat{Q}^*(D, \epsilon)$ are isomorphisms, and*

$$\widehat{Q}^*(f, \epsilon) \;=\; 0 \;.$$

(iii) *The ϵ-hyperquadratic Q-groups are additive: for any collection $\{C(i) \,|\, i \in \mathbb{Z}\}$ of f.g. projective A-module chain complexes $C(i)$*

$$\widehat{Q}^n(\sum_i C(i), \epsilon) \;=\; \bigoplus_i \widehat{Q}^n(C(i), \epsilon) \;.$$

Proof. (i) See [15, §1,§3] for the definition of the $\mathbb{Z}_2$-isovariant chain map $t : \mathcal{C}(f \otimes f) \to \mathcal{C}(f) \otimes_A \mathcal{C}(f)$ inducing the algebraic Thom construction

$$t \;:\; \widehat{Q}^n(f, \epsilon) \to \widehat{Q}^n(\mathcal{C}(f), \epsilon) \;;\; (\theta, \partial\theta) \mapsto \theta/\partial\theta$$

with

$$\begin{aligned}(\theta/\partial\theta)_s \;&=\; \begin{pmatrix} \theta_s & 0 \\ \partial\theta_s f^* & T_\epsilon \partial\theta_{s-1} \end{pmatrix} \;: \\ \mathcal{C}(f)^{n-r+s} \;&=\; D^{n-r+s} \oplus C^{n-r+s-1} \to \mathcal{C}(f)_r \;=\; D_r \oplus C_{r-1} \;(r, s \in \mathbb{Z}) \;.\end{aligned}$$

Define a free $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex

$$E \;=\; (C_{*-1} \otimes_A \mathcal{C}(f)) \oplus (\mathcal{C}(f) \otimes_A C_{*-1})$$

with

$$T \;:\; E \to E \;;\; (a \otimes b, x \otimes y) \mapsto (y \otimes x, b \otimes a) \;,$$

such that

$$H_*(\widehat{W} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} E) \;=\; H_*(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, E)) \;=\; 0 \;.$$

Let $p : \mathcal{C}(f) \to C_{*-1}$ be the projection. It now follows from the chain homotopy cofibration

$$\mathcal{C}(f\otimes f) \xrightarrow{\;t\;} \mathcal{C}(f)\otimes_A \mathcal{C}(f) \xrightarrow{\begin{pmatrix} p\otimes 1 \\ 1\otimes p\end{pmatrix}} E$$

that t induces isomorphisms

$$\widehat{t} \;:\; \widehat{Q}^*(f,\epsilon) \;\cong\; \widehat{Q}^*(\mathcal{C}(f),\epsilon)\ .$$

(ii)+(iii) See [15, Propositions 1.1,1.4]. □

Proposition 1.13. *Let C be a f.g. projective A-module chain complex which is concentrated in degrees $k,k+1$*

$$C \;:\; \dots \to 0 \to C_{k+1} \xrightarrow{\;d\;} C_k \to 0 \to \dots\ .$$

(i) *The ϵ-hyperquadratic Q-groups of C are the relative Tate $\mathbb{Z}_2$-cohomology groups in the exact sequence*

$$\dots \to \widehat{H}^{n-2k}(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon) \xrightarrow{\;\widehat{d}^{\%}\;} \widehat{H}^{n-2k}(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon)$$
$$\to \widehat{Q}^n(C,\epsilon) \to \widehat{H}^{n-2k-1}(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon) \to \dots$$

that is

$$\widehat{Q}^n(C,\epsilon) \;=\; \frac{\{(\phi,\theta)\in S(C^{k+1})\oplus S(C^k)\,|\,\phi^*=(-1)^{n+k-1}\epsilon\phi,\, d\phi d^*=\theta+(-1)^{n+k-1}\epsilon\theta^*\}}{\{(\sigma+(-1)^{n+k-1}\epsilon\sigma^*, d\sigma d^*+\tau+(-1)^{n+k}\epsilon\tau^*)\,|\,(\sigma,\tau)\in S(C^{k+1})\oplus S(C^k)\}}\ ,$$

with (ϕ,θ) corresponding to the cycle $\beta\in \widehat{W}^{\%}C)_n$ given by

$$\beta_{n-2k-2} \;=\; \theta \;:\; C^{k+1}\to C_{k+1}\ ,\ \beta_{n-2k} \;=\; \phi \;:\; C^k\to C_k\ ,$$
$$\beta_{n-2k-1} \;=\; \begin{cases} d\phi \;:\; C^{k+1}\to C_k \\ 0 \;:\; C^k \to C_{k+1}\ .\end{cases}$$

(ii) *If the involution on A is even then*

$$\widehat{Q}^n(C) \;=\; \begin{cases} \operatorname{coker}(\widehat{d}^{\%} : \dfrac{\operatorname{Sym}(C^{k+1})}{\operatorname{Quad}(C^{k+1})} \to \dfrac{\operatorname{Sym}(C^k)}{\operatorname{Quad}(C^k)}) & \text{if } n-k \text{ is even}\ , \\[2ex] \ker(\widehat{d}^{\%} : \dfrac{\operatorname{Sym}(C^{k+1})}{\operatorname{Quad}(C^{k+1})} \to \dfrac{\operatorname{Sym}(C^k)}{\operatorname{Quad}(C^k)}) & \text{if } n-k \text{ is odd}\ .\end{cases}$$

Proof. (i) Immediate from Proposition 1.12.
(ii) Combine (i) and the vanishing $\widehat{H}^1(\mathbb{Z}_2;S(P),T)=0$ given by Proposition 1.5 (ii). □

For $\epsilon=1$ we write

$$T_\epsilon \;=\; T\ ,\ \widehat{Q}^n(C,\epsilon) \;=\; \widehat{Q}^n(C)\ ,\ \epsilon\text{-hyperquadratic} \;=\; \text{hyperquadratic}\ .$$

Example 1.14. Let A be a ring with an involution which is even (1.4).
(i) The hyperquadratic Q-groups of a 1-dimensional f.g. free A-module chain complex

$$C \;:\; \dots \to 0 \to C_1 = A^q \xrightarrow{\;d\;} C_0 = A^r \to 0 \to \dots$$

are given by

$$\widehat{Q}^n(C) \;=\; \frac{\{(\phi,\theta)\in M_q(A)\oplus M_r(A)\,|\,\phi^*=(-1)^{n-1}\phi, d\phi d^*=\theta+(-1)^{n-1}\theta^*\}}{\{(\sigma+(-1)^{n-1}\sigma^*, d\sigma d^*+\tau+(-1)^n\tau^*\,|\,(\sigma,\tau)\in M_q(A)\oplus M_r(A)\}}\ .$$

Example 1.11 and Proposition 1.13 give an exact sequence

$$\begin{aligned}
&\widehat{H}^1(\mathbb{Z}_2;S(C^1),T)=0\to\widehat{Q}^1(C)\\
&\qquad\longrightarrow \widehat{H}^0(\mathbb{Z}_2;S(C^1),T)=\bigoplus_q\widehat{H}^0(\mathbb{Z}_2;A)\\
&\qquad\xrightarrow{\widehat{d}^{\%}} \widehat{H}^0(\mathbb{Z}_2;S(C^0),T)=\bigoplus_r\widehat{H}^0(\mathbb{Z}_2;A)\\
&\qquad\qquad\longrightarrow \widehat{Q}^0(C)\to\widehat{H}^{-1}(\mathbb{Z}_2;S(C^1),T)=0\ .
\end{aligned}$$

(ii) If A is an even commutative ring and

$$d\;=\;2\;:\;C_1\;=\;A^r\to C_0\;=\;A^r$$

then $\widehat{d}^{\%}=0$ and there are defined isomorphisms

$$\begin{aligned}
\widehat{Q}^0(C)&\xrightarrow{\cong}\frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}\;=\;\bigoplus_r A_2\ ;\ (\phi,\theta)\mapsto\theta\;=\;(\theta_{ii})_{1\leqslant i\leqslant r}\\
\widehat{Q}^1(C)&\xrightarrow{\cong}\frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}\;=\;\bigoplus_r A_2\ ;\ (\phi,\theta)\mapsto\phi\;=\;(\phi_{ii})_{1\leqslant i\leqslant r}\ .
\end{aligned}$$

□

1.3. **The Symmetric Q-Groups.** Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$W\ :\ \dots\longrightarrow W_3=\mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1-T}W_2=\mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1+T}W_1=\mathbb{Z}[\mathbb{Z}_2]\xrightarrow{1-T}W_0=\mathbb{Z}[\mathbb{Z}_2]\longrightarrow 0.$$

Given a f.g. projective A-module chain complex C we set

$$W^{\%}C\;=\;\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Hom}_A(C^{-*},C))\ ,$$

with $T\in\mathbb{Z}_2$ acting on $C\otimes_A C=\mathrm{Hom}_A(C^{-*},C)$ by the ϵ-duality involution T_ϵ. An *n-dimensional ϵ-symmetric structure on C* is a cycle $\phi\in(W^{\%}C)_n$, which is just a collection $\{\phi_s\in\mathrm{Hom}_A(C^r,C_{n-r+s})\,|\,r\in\mathbb{Z},s\geqslant 0\}$ such that

$$\begin{aligned}
d\phi_s+(-1)^r\phi_s d^*+(-1)^{n+s-1}(\phi_{s-1}+(-1)^sT_\epsilon\phi_{s-1})\;=\;0\;:\;C^r\to C_{n-r+s-1}\\
(r\in\mathbb{Z},s\geqslant 0,\phi_{-1}=0)\ .
\end{aligned}$$

Definition 1.15. The *n-dimensional ϵ-symmetric Q-group* $Q^n(C,\epsilon)$ is the abelian group of equivalence classes of n-dimensional ϵ-symmetric structures on C, that is,

$$Q^n(C,\epsilon)\;=\;H_n(W^{\%}C)\ .$$

□

Note that there are defined skew-suspension isomorphisms

$$\overline{S}\ :\ Q^n(C,\epsilon)\xrightarrow{\cong}Q^{n+2}(C_{*-1},-\epsilon)\ ;\ \{\phi_s\}\mapsto\{\phi_s\}\ .$$

Proposition 1.16. *The ϵ-symmetric Q-groups of a f.g. projective A-module chain complex concentrated in degree k*

$$C \ :\ \dots \to 0 \to C_k \to 0 \to \dots$$

are given by

$$\begin{aligned} Q^n(C,\epsilon) &= H^{2k-n}(\mathbb{Z}_2; S(C^k), (-1)^k T_\epsilon) \\ &= \begin{cases} \widehat{H}^{2k-n}(\mathbb{Z}_2; S(C^k), (-1)^k T_\epsilon) & \text{if } n \leqslant 2k-1 \\ H^0(\mathbb{Z}_2; S(C^k), (-1)^k T_\epsilon) & \text{if } n = 2k \\ 0 & \text{if } n \geqslant 2k+1. \end{cases} \end{aligned}$$

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $V = \mathrm{Hom}_A(C^{-*}, C)$ is given by

$$V \ :\ \dots \to V_{2k+1} \ = \ 0 \to V_{2k} \ = \ S(C^k) \to V_{2k-1} \ = \ 0 \to \dots$$

and

$$(W^{\%}C)_j \ = \ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_{2k-j}, V_{2k}) \ = \ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_{2k-j}, S(C^k))$$

which vanishes for $j > 2k$. Thus the chain complex $W^{\%}C$ is of the form

$$\begin{array}{lcl} (W^{\%}C)_{2k+1} & = & 0 \\ \big\downarrow d_{2k+1} & & \\ (W^{\%}C)_{2k} & = & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_0, V_{2k}) \ = \ S(C^k) \\ \big\downarrow d_{2k}=1+(-1)^{k+1}T_\epsilon & & \\ (W^{\%}C)_{2k-1} & = & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_1, V_{2k}) \ = \ S(C^k) \\ \big\downarrow d_{2k-1}=1+(-1)^{k}T_\epsilon & & \\ (W^{\%}C)_{2k-2} & = & \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_2, V_{2k}) \ = \ S(C^k) \\ \big\downarrow & & \end{array}$$

and

$$Q^n(C,\epsilon) \ = \ H_n(W^{\%}C) \ = \ H^{2k-n}(\mathbb{Z}_2; S(C^k), (-1)^k T_\epsilon) \ .$$

□

For $\epsilon = 1$ we write

$$T_\epsilon \ = \ T \ ,\ Q^n(C,\epsilon) \ = \ Q^n(C) \ ,\ \epsilon\text{-symmetric} \ = \ \text{symmetric} \ .$$

Example 1.17. The symmetric Q-groups of a 0-dimensional f.g. free A-module chain complex

$$C \ :\ \dots \to 0 \to C_0 \ = \ A^r \to 0 \to \dots$$

are given by

$$Q^n(C) \;=\; \begin{cases} \bigoplus\limits_r \widehat{H}^n(\mathbb{Z}_2;A) & \text{if } n<0 \\ \mathrm{Sym}_r(A) & \text{if } n=0 \\ 0 & \text{otherwise.} \end{cases}$$

□

An A-module chain map $f: C \to D$ induces a chain map

$$\mathrm{Hom}_A(f^*,f) \;:\; \mathrm{Hom}_A(C^{-*},C) \to \mathrm{Hom}_A(D^{-*},D) \;;\; \phi \mapsto f\phi f^*$$

and thus a chain map

$$f^{\%} \;=\; \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(1_W, \mathrm{Hom}_A(f^*,f)) \;:\; W^{\%}C \longrightarrow W^{\%}D$$

which induces

$$f^{\%} \;:\; Q^n(C,\epsilon) \longrightarrow Q^n(D,\epsilon)$$

on homology. The *relative ϵ-symmetric Q-group*

$$Q^n(f,\epsilon) \;=\; H_n(f^{\%} : W^{\%}C \to W^{\%}D)$$

fits into a long exact sequence

$$\cdots \longrightarrow Q^n(C,\epsilon) \xrightarrow{f^{\%}} Q^n(D,\epsilon) \longrightarrow Q^n(f,\epsilon) \longrightarrow Q^{n-1}(C,\epsilon) \longrightarrow \cdots .$$

Proposition 1.18. (i) *The relative ϵ-symmetric Q-groups of an A-module chain map $f: C\to D$ are related to the absolute ϵ-symmetric Q-groups of the algebraic mapping cone $\mathcal{C}(f)$ by a long exact sequence*

$$\cdots \to H_n(\mathcal{C}(f)\otimes_A C) \xrightarrow{F} Q^n(f,\epsilon) \xrightarrow{t} Q^n(\mathcal{C}(f),\epsilon) \to H_{n-1}(\mathcal{C}(f)\otimes_A C) \to \dots$$

with

$$t \;:\; Q^n(f,\epsilon) \to Q^n(\mathcal{C}(f),\epsilon) \;;\; (\phi,\partial\phi) \mapsto \phi/\partial\phi$$

the algebraic Thom construction

$$(\phi/\partial\phi)_s \;=\; \begin{pmatrix} \phi_s & 0 \\ \partial\phi_s f^* & T_\epsilon \partial\phi_{s-1} \end{pmatrix} \;:$$

$$\mathcal{C}(f)^{n-r+s} \;=\; D^{n-r+s} \oplus C^{n-r+s-1} \to \mathcal{C}(f)_r \;=\; D_r \oplus C_{r-1} \;(r\in\mathbb{Z}, s\geqslant 0, \phi_{-1}=0) \;.$$

An element $(g,h) \in H_n(\mathcal{C}(f)\otimes_A C)$ is represented by a chain map $g: C^{n-1-*} \to C$ together with a chain homotopy $h: fg \simeq 0 : C^{n-1-*} \to D$, and

$$F \;:\; H_n(\mathcal{C}(f)\otimes_A C) \to Q^n(f,\epsilon) \;;\; (g,h) \mapsto (\phi,\partial\phi)$$

with

$$\partial\phi_s \;=\; \begin{cases} (1+T_\epsilon)g & \textit{if } s=0 \\ 0 & \textit{if } s\geqslant 1 \end{cases} \;,\; \phi_s \;=\; \begin{cases} (1+T_\epsilon)hf^* & \textit{if } s=0 \\ 0 & \textit{if } s\geqslant 1 \;. \end{cases}$$

The map

$$Q^n(\mathcal{C}(f),\epsilon) \to H_{n-1}(\mathcal{C}(f)\otimes_A C) \;;\; \phi \mapsto p\phi_0$$

is defined using $p =$ projection $: \mathcal{C}(f) \to C_{*-1}$.
(ii) *If $f: C\to D$ is a chain equivalence the morphisms $f^{\%}: Q^*(C,\epsilon) \to Q^*(D,\epsilon)$ are isomorphisms, and*

$$Q^*(\mathcal{C}(f),\epsilon) \;=\; Q^*(f,\epsilon) \;=\; 0 \;.$$

(iii) *For any collection* $\{C(i)\,|\,i\in\mathbb{Z}\}$ *of f.g. projective A-module chain complexes* $C(i)$

$$Q^n(\sum_i C(i),\epsilon) \;=\; \bigoplus_i Q^n(C(i),\epsilon)\oplus\bigoplus_{i<j}H_n(C(i)\otimes_A C(j))\ .$$

Proof. (i) The long exact sequence is induced by the chain homotopy cofibration of Proposition 1.12

$$\mathcal{C}(f\otimes f)\xrightarrow{\;t\;}\mathcal{C}(f)\otimes_A\mathcal{C}(f)\xrightarrow{\begin{pmatrix}p\otimes 1\\ 1\otimes p\end{pmatrix}}E$$

with

$$\begin{aligned} E &= (C_{*-1}\otimes_A\mathcal{C}(f))\oplus(\mathcal{C}(f)\otimes_A C_{*-1})\ ,\\ H_*(W^{\%}E) &= H_*(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,E)) \;=\; H_{*-1}(C\otimes_A\mathcal{C}(f))\ .\end{aligned}$$

(ii)+(iii) See [15, Propositions 1.1,1.4]. □

Proposition 1.19. *Let* C *be a f.g. projective A-module chain complex which is concentrated in degrees* $k,k+1$

$$C\ :\ \dots\to 0\to C_{k+1}\xrightarrow{\;d\;}C_k\to 0\to\dots\ .$$

The absolute ϵ*-symmetric Q-groups* $Q^*(C,\epsilon)$ *and the relative* ϵ*-symmetric Q-groups* $Q^*(d,\epsilon)$ *of* $d:C_{k+1}\to C_k$ *regarded as a morphism of chain complexes concentrated in degree* k *are given as follows.*
(i) *For* $n\neq 2k,2k+1,2k+2$

$$Q^n(C,\epsilon)\;=\;Q^n(d,\epsilon)\;=\;\begin{cases}\widehat{Q}^n(d,\epsilon)\;=\;\widehat{Q}^n(C,\epsilon) & \text{if } n\leqslant 2k-1\\ 0 & \text{if } n\geqslant 2k+3\end{cases}$$

with $\widehat{Q}^n(C,\epsilon)$ *as given by Proposition 1.13.*
(ii) *For* $n=2k,2k+1,2k+2$ *there are exact sequences*

$$\begin{aligned}0\to Q^{2k+1}(d,\epsilon)&\longrightarrow Q^{2k}(C_{k+1},\epsilon)=H^0(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon)\\ &\xrightarrow{d^{\%}} Q^{2k}(C_k,\epsilon)=H^0(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon)\longrightarrow Q^{2k}(d,\epsilon)\\ &\longrightarrow Q^{2k-1}(C_{k+1},\epsilon)=\widehat{H}^1(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon)\\ &\xrightarrow{d^{\%}} Q^{2k-1}(C_k,\epsilon)=\widehat{H}^1(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon)\ ,\end{aligned}$$

$$\begin{aligned}Q^{2k+2}(d,\epsilon)=0\to Q^{2k+2}(C,\epsilon)&\to C_{k+1}\otimes_A H_{k+1}(C)\xrightarrow{F}Q^{2k+1}(d,\epsilon)\\ &\xrightarrow{t}Q^{2k+1}(C,\epsilon)\to C_{k+1}\otimes_A H_k(C)\xrightarrow{F}Q^{2k}(d,\epsilon)\xrightarrow{t}Q^{2k}(C,\epsilon)\to 0\ .\end{aligned}$$

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $V=\mathrm{Hom}_A(C^{-*},C)$ is such that

$$V_n\;=\;\begin{cases}S(C^k) & \text{if } n=2k\\ \mathrm{Hom}_A(C^k,C_{k+1})\oplus\mathrm{Hom}_A(C^{k+1},C_k) & \text{if } n=2k+1\\ S(C^{k+1}) & \text{if } n=2k+2\\ 0 & \text{otherwise}\end{cases}$$

and

$$(W^{\%}C)_n \;=\; \sum_{s=0}^{\infty} \mathrm{Hom}_A(W_s, V_{n+s}) \;=\; 0 \text{ for } n \geqslant 2k+3 \;.$$

□

Example 1.20. Let C be a 1-dimensional f.g. free A-module chain complex

$$C \;:\; \dots \to 0 \to C_1 = A^q \xrightarrow{\;d\;} C_0 = A^r \to 0 \to \dots \;,$$

so that $C = \mathcal{C}(d)$ is the algebraic mapping cone of the chain map $d : C_1 \to C_0$ of 0-dimensional complexes, with

$$d^{\%} \;:\; \mathrm{Hom}_A(C^1, C_1) \;=\; M_q(A) \to \mathrm{Hom}_A(C^0, C_0) \;=\; M_r(A) \;;\; \phi \mapsto d\phi d^* \;.$$

Example 1.17 and Proposition 1.19 give exact sequences

$$Q^1(C_0) = 0 \to Q^1(d) \longrightarrow Q^0(C_1) = \mathrm{Sym}_q(A) \xrightarrow{\;d^{\%}\;} Q^0(C_0) = \mathrm{Sym}_r(A)$$

$$\longrightarrow Q^0(d) \longrightarrow Q^{-1}(C_1) = \bigoplus_q \widehat{H}^1(\mathbb{Z}_2; A) \xrightarrow{\;d^{\%}\;} Q^{-1}(C_0) = \bigoplus_r \widehat{H}^1(\mathbb{Z}_2; A)$$

$$H_1(C) \otimes_A C_1 \xrightarrow{\;F\;} Q^1(d) \xrightarrow{\;t\;} Q^1(C) \to H_0(C) \otimes_A C_1 \xrightarrow{\;F\;} Q^0(d) \xrightarrow{\;t\;} Q^0(C) \to 0 \;.$$

In particular, if A is an even commutative ring and

$$d \;=\; 2 \;:\; C_1 \;=\; A^r \to C_0 \;=\; A^r$$

then $d^{\%} = 4$ and

$$\begin{aligned} Q^0(d) \;&=\; \frac{\mathrm{Sym}_r(A)}{4\,\mathrm{Sym}_r(A)} \;,\; Q^1(d) \;=\; 0 \;, \\ Q^0(C) \;&=\; \mathrm{coker}(2(1+T) : M_r(A) \to \frac{\mathrm{Sym}_r(A)}{4\,\mathrm{Sym}_r(A)}) \;=\; \frac{\mathrm{Sym}_r(A)}{2\,\mathrm{Quad}_r(A)} \;, \\ Q^1(C) \;&=\; \ker(2(1+T) : \frac{M_r(A)}{2M_r(A)} \to \frac{\mathrm{Sym}_r(A)}{4\,\mathrm{Sym}_r(A)}) \\ &=\; \frac{\{(a_{ij}) \in M_r(A) \,|\, a_{ij} + a_{ji} \in 2A\}}{2M_r(A)} \;=\; \frac{\mathrm{Sym}_r(A)}{2\,\mathrm{Sym}_r(A)} \;. \end{aligned}$$

□

We refer to Ranicki [15] for the one-one correspondence between highly-connected algebraic Poincaré complexes/pairs and forms, lagrangians and formations.

1.4. **The Quadratic Q-Groups.** Given a f.g. projective A-module chain complex C we set

$$W_{\%}C \;=\; W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^{-*}, C) \;,$$

with $T \in \mathbb{Z}_2$ acting on $C \otimes_A C = \mathrm{Hom}_A(C^{-*}, C)$ by the ϵ-duality involution T_ϵ. An *n-dimensional ϵ-quadratic structure on* C is a cycle $\psi \in (W_{\%}C)_n$, a collection $\{\psi_s \in \mathrm{Hom}_A(C^r, C_{n-r-s}) \,|\, r \in \mathbb{Z}, s \geqslant 0\}$ such that

$$d\psi_s + (-1)^r \psi_s d^* + (-1)^{n-s-1}(\psi_{s+1} + (-1)^{s+1} T_\epsilon \psi_{s+1}) \;=\; 0 \;:\; C^r \to C_{n-r-s-1} \;.$$

Definition 1.21. The *n-dimensional ϵ-quadratic Q-group* $Q_n(C, \epsilon)$ is the abelian group of equivalence classes of n-dimensional ϵ-quadratic structures on C, that is,

$$Q_n(C, \epsilon) \;=\; H_n(W_{\%}C) \;.$$

□

Note that there are defined skew-suspension isomorphisms

$$\overline{S} \;:\; Q_n(C,\epsilon) \xrightarrow{\cong} Q_{n+2}(C_{*-1},-\epsilon)\;;\;\{\psi_s\}\mapsto\{\psi_s\}\ .$$

Proposition 1.22. *The ϵ-quadratic Q-groups of a f.g. projective A-module chain complex concentrated in degree k*

$$C \;:\; \dots \to 0 \to C_k \to 0 \to \dots$$

are given by

$$\begin{aligned} Q_n(C,\epsilon) &= H_{n-2k}(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon) \\ &= \begin{cases} \widehat{H}^{n-2k+1}(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon) & \text{if } n\geqslant 2k+1 \\ H_0(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon) & \text{if } n=2k \\ 0 & \text{if } n\leqslant 2k-1\ . \end{cases} \end{aligned}$$

Proof. The $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $V=\mathrm{Hom}_A(C^{-*},C)$ is given by

$$V \;:\; \dots\to V_{2k+1} \;=\; 0\to V_{2k} \;=\; \mathrm{Hom}_A(C^k,C_k)\to V_{2k-1} \;=\; 0\to\dots$$

and

$$(W_\%C)_j \;=\; W_{j-2k}\otimes_{\mathbb{Z}[\mathbb{Z}_2]}V_{2k} \;=\; \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W_{2k-j},S(C^k))$$

which vanishes for $j<2k$. Thus the chain complex $W_\%C$ is of the form

$$\begin{array}{lcl} (W_\%C)_{2k+2} &=& W_2\otimes_{\mathbb{Z}[\mathbb{Z}_2]}V_{2k} \;=\; S(C^k) \\ \big\downarrow{\scriptstyle d_{2k+2}=1+(-1)^kT_\epsilon} && \\ (W_\%C)_{2k+1} &=& W_1\otimes_{\mathbb{Z}[\mathbb{Z}_2]}V_{2k} \;=\; S(C^k) \\ \big\downarrow{\scriptstyle d_{2k+1}=1+(-1)^{k+1}T_\epsilon} && \\ (W_\%C)_{2k} &=& W_0\otimes_{\mathbb{Z}[\mathbb{Z}_2]}V_{2k} \;=\; S(C^k) \\ \big\downarrow && \\ (W_\%C)_{2k-1} &=& 0 \end{array}$$

and

$$Q_n(C,\epsilon) \;=\; H_n(W_\%C) \;=\; H_{n-2k}(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon)\ .$$

□

Example 1.23. The ϵ-quadratic Q-groups of the 0-dimensional f.g. free A-module chain complex

$$C \;:\; \dots\to 0\to C_0 \;=\; A^r\to 0\to\dots$$

are given by

$$Q_n(C) \;=\; \begin{cases} \bigoplus_r \widehat{H}^{n+1}(\mathbb{Z}_2;A) & \text{if } n>0 \\ \mathrm{Quad}_r(A) & \text{if } n=0 \\ 0 & \text{otherwise.} \end{cases}$$

□

An A-module chain map $f : C \to D$ induces a chain map

$$f_\% \ = \ 1_W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(f^*, f) \ : \ W_\% C \longrightarrow W_\% D$$

which induces

$$f_\% \ : \ Q_n(C, \epsilon) \longrightarrow Q_n(D, \epsilon)$$

on homology. The *relative ϵ-quadratic Q-group* $Q_n(f, \epsilon)$ is designed to fit into a long exact sequence

$$\ldots \longrightarrow Q_n(C,\epsilon) \xrightarrow{f_\%} Q_n(D,\epsilon) \longrightarrow Q_n(f,\epsilon) \longrightarrow Q_{n-1}(C,\epsilon) \longrightarrow \ldots \ ,$$

that is, $Q_n(f, \epsilon)$ is defined as the n-th homology group of the mapping cone of $f_\%$,

$$Q_n(f, \epsilon) \ = \ H_n(f_\% : W_\% C \longrightarrow W_\% D) \ .$$

Proposition 1.24. (i) *The relative ϵ-quadratic Q-groups of $f : C \to D$ are related to the absolute ϵ-quadratic Q-groups of the algebraic mapping cone $\mathcal{C}(f)$ by a long exact sequence*

$$\cdots \to H_n(\mathcal{C}(f) \otimes_A C) \xrightarrow{\ F\ } Q_n(f,\epsilon) \xrightarrow{\ t\ } Q_n(\mathcal{C}(f),\epsilon) \to H_{n-1}(\mathcal{C}(f) \otimes_A C) \to \ldots \ .$$

(ii) *If $f : C \to D$ is a chain equivalence the morphisms $f_\% : Q_*(C) \to Q_*(D)$ are isomorphisms, and*

$$Q_*(\mathcal{C}(f), \epsilon) \ = \ Q_*(f, \epsilon) \ = \ 0 \ .$$

(iii) *For any collection $\{C(i) \,|\, i \in \mathbb{Z}\}$ of f.g. projective A-module chain complexes $C(i)$*

$$Q_n(\sum_i C(i), \epsilon) \ = \ \bigoplus_i Q_n(C(i), \epsilon) \oplus \bigoplus_{i<j} H_n(C(i) \otimes_A C(j)) \ .$$

□

Proposition 1.25. *Let C be a f.g. projective A-module chain complex which is concentrated in degrees $k, k+1$*

$$C \ : \ \cdots \to 0 \to C_{k+1} \xrightarrow{\ d\ } C_k \to 0 \to \ldots \ .$$

The absolute ϵ-quadratic Q-groups $Q_*(C, \epsilon)$ and the relative ϵ-quadratic Q-groups $Q_*(d, \epsilon)$ of $d : C_{k+1} \to C_k$ regarded as a morphism of chain complexes concentrated in degree k are given as follows.
(i) *For $n \neq 2k, 2k+1, 2k+2$*

$$Q_n(C, \epsilon) \ = \ Q_n(d, \epsilon) \ = \ \begin{cases} \widehat{Q}^{n+1}(d, \epsilon) \ = \ \widehat{Q}^{n+1}(C, \epsilon) & \textit{if } n \geqslant 2k+3 \\ 0 & \textit{if } n \leqslant 2k-1 \end{cases}$$

with $\widehat{Q}^n(C,\epsilon)$ as given by Proposition 1.13.
(ii) *For $n=2k,2k+1,2k+2$ there are exact sequences*

$$\begin{aligned}
&Q_{2k+2}(C_{k+1},\epsilon)=\widehat{H}^1(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon) \xrightarrow{d_\%} Q_{2k+2}(C_k,\epsilon)=\widehat{H}^1(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon)\\
&\quad\longrightarrow Q_{2k+2}(d,\epsilon)=\widehat{Q}^{2k+3}(C,\epsilon) \xrightarrow{d_\%} Q_{2k+1}(C_{k+1},\epsilon)=\widehat{H}^0(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon)\\
&\quad\xrightarrow{d_\%} Q_{2k+1}(C_k,\epsilon)=\widehat{H}^0(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon) \longrightarrow Q_{2k+1}(d,\epsilon)\\
&\quad\longrightarrow Q_{2k}(C_{k+1},\epsilon)=H_0(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon)\\
&\quad\xrightarrow{d_\%} Q_{2k}(C_k,\epsilon)=H_0(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon) \longrightarrow Q_{2k}(d,\epsilon) \longrightarrow Q_{2k-1}(C_{k+1})=0\ ,
\end{aligned}$$

$$\begin{aligned}
&0\to Q_{2k+2}(d,\epsilon) \xrightarrow{t} Q_{2k+2}(C,\epsilon) \longrightarrow H_{k+1}(C)\otimes_A C_{k+1} \xrightarrow{F} Q_{2k+1}(d,\epsilon)\\
&\qquad\xrightarrow{t} Q_{2k+1}(C,\epsilon) \longrightarrow C_{k+1}\otimes_A H_k(C) \xrightarrow{F} Q_{2k}(d,\epsilon) \xrightarrow{t} Q_{2k}(C,\epsilon)\to 0\ .
\end{aligned}$$

□

For $\epsilon=1$ we write

$$T_\epsilon\ =\ T\ ,\ Q_n(C,\epsilon)\ =\ Q_n(C)\ ,\ \epsilon\text{-quadratic}\ =\ \text{quadratic}\ .$$

Example 1.26. Let C be a 1-dimensional f.g. free A-module chain complex

$$C\ :\ \dots\to 0\to C_1=A^q \xrightarrow{d} C_0=A^r\to 0\to\dots\ ,$$

so that $C=\mathcal{C}(d)$ is the algebraic mapping cone of the chain map $d:C_1\to C_0$ of 0-dimensional complexes, with

$$d^\%\ :\ \mathrm{Hom}_A(C^1,C_1)\ =\ M_q(A)\to \mathrm{Hom}_A(C^0,C_0)\ =\ M_r(A)\ ;\ \phi\mapsto d\phi d^*\ .$$

Example 1.23 and Proposition 1.25 give exact sequences

$$\begin{aligned}
&Q_1(C_1)=\bigoplus_q \widehat{H}^0(\mathbb{Z}_2;A) \xrightarrow{\widehat{d}^\%} Q_1(C_0)=\bigoplus_r \widehat{H}^0(\mathbb{Z}_2;A)\to Q_1(d)\\
&\longrightarrow Q_0(C_1)=\mathrm{Quad}_q(A) \xrightarrow{d_\%} Q_0(C_0)=\mathrm{Quad}_r(A) \longrightarrow Q_0(d)\to Q_{-1}(C_1)=0\ ,\\
&H_1(C)\otimes_A C_1\to Q_1(d)\to Q_1(C)\to H_0(C)\otimes_A C_1\to Q_0(d)\to Q_0(C)\to 0\ .
\end{aligned}$$

In particular, if A is an even commutative ring and

$$d\ =\ 2\ :\ C_1\ =\ A^r\to C_0\ =\ A^r$$

then $d_{\%} = 4$ and

$$\begin{aligned}
Q_0(d) &= \frac{\mathrm{Quad}_r(A)}{4\,\mathrm{Quad}_r(A)}\ ,\\
Q_1(d) &= \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)+4\,\mathrm{Sym}_r(A)}\ ,\\
Q_0(C) &= \mathrm{coker}(2(1+T):\frac{M_r(A)}{2M_r(A)} \to \frac{\mathrm{Quad}_r(A)}{4\,\mathrm{Quad}_r(A)}) \;=\; \frac{\mathrm{Quad}_r(A)}{2\,\mathrm{Quad}_r(A)}\ ,\\
Q_1(C) &= \frac{\{(\psi_0,\psi_1)\in M_r(A)\oplus M_r(A)\,|\,2\psi_0=\psi_1-\psi_1^*\}}{\{(2(\chi_0-\chi_0^*),4\chi_0+\chi_2+\chi_2^*)\,|\,(\chi_0,\chi_2)\in M_r(A)\oplus M_r(A)\}} \;=\; \bigoplus_{\frac{r(r+1)}{2}} A_2\ .
\end{aligned}$$

□

1.5. **L-groups.** An *n-dimensional* $\begin{cases} \epsilon\text{-}symmetric \\ \epsilon\text{-}quadratic \end{cases}$ *Poincaré complex* $\begin{cases} (C,\phi) \\ (C,\psi) \end{cases}$ over A is an n-dimensional f.g. projective A-module chain complex

$$C\ :\ \cdots \to 0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0 \to \dots$$

together with an element $\begin{cases} \phi \in Q^n(C,\epsilon) \\ \psi \in Q_n(C,\epsilon) \end{cases}$ such that the A-module chain map

$$\begin{cases} \phi_0 : C^{n-*} \to C \\ (1+T_\epsilon)\psi_0 : C^{n-*} \to C \end{cases}$$

is a chain equivalence. We refer to [18] for the detailed definition of the *n-dimensional* $\begin{cases} \epsilon\text{-}symmetric \\ \epsilon\text{-}quadratic \end{cases}$ *L-group* $\begin{cases} L^n(A,\epsilon) \\ L_n(A,\epsilon) \end{cases}$ as the cobordism group of n-dimensional $\begin{cases} \epsilon\text{-symmetric} \\ \epsilon\text{-quadratic} \end{cases}$ Poincaré complexes over A.

Definition 1.27. (i) The *relative* (ϵ*-symmetric,* ϵ*-quadratic*) *Q-group* $Q^n_n(f,\epsilon)$ of a chain map $f : C \to D$ of f.g. projective A-module chain complexes is the relative group in the exact sequence

$$\cdots \to Q_n(C,\epsilon) \xrightarrow{(1+T_\epsilon)f_\%} Q^n(D,\epsilon) \to Q^n_n(f,\epsilon) \to Q_{n-1}(C,\epsilon) \to \dots\ .$$

An element $(\delta\phi,\psi) \in Q^n_n(f,\epsilon)$ is an equivalence class of pairs

$$(\delta\phi,\psi) \in (W^\% D)_n \oplus (W_\% C)_{n-1}$$

such that

$$d(\psi)\ =\ 0 \in (W_\% C)_{n-2}\ ,\ (1+T_\epsilon)f_\%\psi\ =\ d(\delta\phi) \in (W^\% D)_{n-1}\ .$$

(ii) An *n-dimensional* (ϵ*-symmetric,* ϵ*-quadratic*) *pair over* A $(f : C \to D, (\delta\phi,\psi))$ is a chain map f together with a class $(\delta\phi,\psi) \in Q^n_n(f,\epsilon)$ such that the chain map

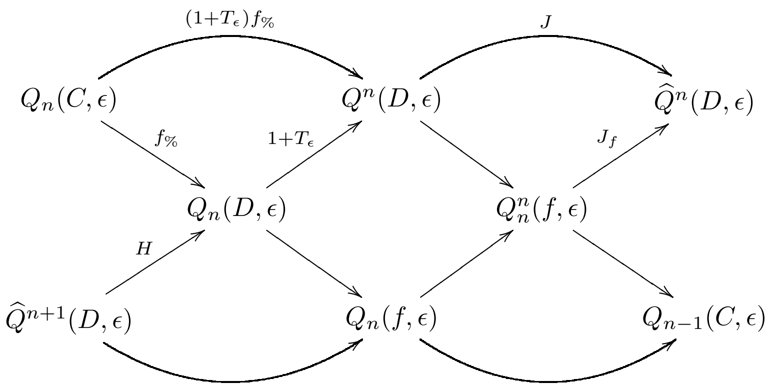
$$(\delta\phi,(1+T_\epsilon)\psi)_0\ :\ D^{n-*} \to \mathcal{C}(f)$$

defined by

$$(\delta\phi,(1+T_\epsilon)\psi)_0\ =\ \begin{pmatrix} \delta\phi_0 \\ (1+T_\epsilon)\psi_0 f^* \end{pmatrix}\ :\ D^{n-r} \to \mathcal{C}(f)_r\ =\ D_r \oplus C_{r-1}$$

is a chain equivalence. □

Proposition 1.28. *The relative (ϵ-symmetric, ϵ-quadratic) Q-groups $Q^n_n(f,\epsilon)$ of a chain map $f : C \to D$ fit into a commutative braid of exact sequences*



with

$$J_f \ : \ Q^n_n(f,\epsilon) \to \widehat{Q}^n(D,\epsilon) \ ; \ (\delta\phi,\psi) \mapsto \alpha \ ,$$
$$\alpha_s \ = \ \begin{cases} \delta\phi_s & \text{if } s \geqslant 0 \\ f\psi_{-s-1}f^* & \text{if } s \leqslant -1 \end{cases} \ : \ D^r \to D_{n-r+s} \ .$$

□

The *n-dimensional ϵ-hyperquadratic L-group* $\widehat{L}^n(A,\epsilon)$ is the cobordism group of n-dimensional (ϵ-symmetric, ϵ-quadratic) Poincaré pairs $(f : C \to D, (\phi,\psi))$ over A. As in [15], there is defined an exact sequence

$$\cdots \longrightarrow L_n(A,\epsilon) \xrightarrow{1+T_\epsilon} L^n(A,\epsilon) \longrightarrow \widehat{L}^n(A,\epsilon) \longrightarrow L_{n-1}(A,\epsilon) \longrightarrow \cdots \ .$$

The skew-suspension maps in the $\pm\epsilon$-quadratic L-groups are isomorphisms

$$\overline{S} \ : \ L_n(A,\epsilon) \xrightarrow{\ \cong\ } L_{n+2}(A,-\epsilon) \ ; \ (C,\{\psi_s\}) \mapsto (C_{*-1},\{\psi_s\}) \ ,$$

so the ϵ-quadratic L-groups are 4-periodic

$$L_n(A,\epsilon) \ = \ L_{n+2}(A,-\epsilon) \ = \ L_{n+4}(A,\epsilon) \ .$$

The skew-suspension maps in ϵ-symmetric and ϵ-hyperquadratic L-groups and $\pm\epsilon$-hyperquadratic L-groups

$$\overline{S} \ : \ L^n(A,\epsilon) \to L^{n+2}(A,-\epsilon) \ ; \ (C,\{\phi_s\}) \mapsto (C_{*-1},\{\phi_s\}) \ ,$$
$$\overline{S} \ : \ \widehat{L}^n(A,\epsilon) \to \widehat{L}^{n+2}(A,-\epsilon) \ ; \ (f : C \to D,\{\psi_s,\phi_s\}) \mapsto (f : C_{*-1} \to D_{*-1},\{(\psi_s,\phi_s)\})$$

are not isomorphisms in general, so the ϵ-symmetric and ϵ-hyperquadratic L-groups need not be 4-periodic. We shall write the 4-periodic versions of the ϵ-symmetric and ϵ-hyperquadratic L-groups of A as

$$L^{n+4*}(A,\epsilon) \ = \ \lim_{k\to\infty} L^{n+4k}(A,\epsilon) \ , \ \widehat{L}^{n+4*}(A,\epsilon) \ = \ \lim_{k\to\infty} \widehat{L}^{n+4k}(A,\epsilon) \ ,$$

noting that there is defined an exact sequence

$$\cdots \to L_n(A,\epsilon) \to L^{n+4*}(A,\epsilon) \to \widehat{L}^{n+4*}(A,\epsilon) \to L_{n-1}(A,\epsilon) \to \dots \ .$$

Definition 1.29. The *Wu classes* of an n-dimensional ϵ-symmetric complex (C,ϕ) over A are the A-module morphisms

$$\widehat{v}_k(\phi) \ : \ H^{n-k}(C) \to \widehat{H}^k(\mathbb{Z}_2;A,\epsilon) \ ; \ x \mapsto \phi_{n-2k}(x)(x) \ (k \in \mathbb{Z}) \ .$$

□

For an n-dimensional ϵ-symmetric Poincaré complex (C,ϕ) over A the evaluation of the Wu class $\widehat{v}_k(\phi)(x) \in \widehat{H}^k(\mathbb{Z}_2;A,\epsilon)$ is the obstruction to killing $x \in H^{n-k}(C) \cong H_k(C)$ by algebraic surgery ([15, §4]).

Proposition 1.30. (i) *If $\widehat{H}^0(\mathbb{Z}_2;A,\epsilon)$ has a 1-dimensional f.g. projective A-module resolution then the skew-suspension maps*

$$\overline{S} \ : \ L^{n-2}(A,-\epsilon) \to L^n(A,\epsilon) \ , \ \overline{S} \ : \ \widehat{L}^{n-2}(A,-\epsilon) \to \widehat{L}^n(A,\epsilon) \ (n \geqslant 2)$$

are isomorphisms. Thus if $\widehat{H}^1(\mathbb{Z}_2;A,\epsilon)$ also has a 1-dimensional f.g. projective A-module resolution the ϵ-symmetric and ϵ-hyperquadratic L-groups of A are 4-periodic

$$L^n(A,\epsilon) \ = \ L^{n+2}(A,-\epsilon) \ = \ L^{n+4}(A,\epsilon) \ ,$$
$$\widehat{L}^n(A,\epsilon) \ = \ \widehat{L}^{n+2}(A,-\epsilon) \ = \ \widehat{L}^{n+4}(A,\epsilon) \ .$$

(ii) *If A is a Dedekind ring then the ϵ-symmetric L-groups are 'homotopy invariant'*

$$L^n(A[x],\epsilon) \ = \ L^n(A,\epsilon)$$

and the ϵ-symmetric and ϵ-hyperquadratic L-groups of A and $A[x]$ are 4-periodic.

Proof. (i) Let D be a 1-dimensional f.g. projective A-module resolution of $\widehat{H}^0(\mathbb{Z}_2;A,\epsilon)$

$$0 \to D_1 \to D_0 \to \widehat{H}^0(\mathbb{Z}_2;A) \to 0 \ .$$

Given an n-dimensional ϵ-symmetric Poincaré complex (C,ϕ) over A resolve the A-module morphism

$$\widehat{v}_n(\phi)(\phi_0)^{-1} \ : \ H_0(C) \cong H^n(C) \to H_0(D) \ = \ \widehat{H}^0(\mathbb{Z}_2;A,\epsilon) \ ; \ u \mapsto (\phi_0)^{-1}(u)(u)$$

by an A-module chain map $f : C \to D$, defining an $(n+1)$-dimensional ϵ-symmetric pair $(f : C \to D, (\delta\phi,\phi))$. The effect of algebraic surgery on (C,ϕ) using $(f : C \to D, (\delta\phi,\phi))$ is a cobordant n-dimensional ϵ-symmetric Poincaré complex (C',ϕ') such that there are defined an exact sequence

$$0 \to H^n(C') \to H^n(C) \xrightarrow{\widehat{v}_n(\phi)} \widehat{H}^0(\mathbb{Z}_2;A,\epsilon) \to H^{n+1}(C') \to 0$$

and an $(n+1)$-dimensional ϵ-symmetric pair $(f' : C' \to D', (\delta\phi',\phi'))$ with f' the projection onto the quotient complex of C' defined by

$$D' \ : \ \cdots \to 0 \to D'_{n+1} \ = \ C'_{n+1} \to D'_n \ = \ C'_n \to 0 \to \dots \ .$$

The effect of algebraic surgery on (C',ϕ') using $(f' : C' \to D', (\delta\phi',\phi'))$ is a cobordant n-dimensional ϵ-symmetric Poincaré complex (C'',ϕ'') with $H_n(C'') = 0$, so that it is (homotopy equivalent to) the skew-suspension of an $(n-2)$-dimensional $(-\epsilon)$-symmetric Poincaré complex.

(ii) The 4-periodicity $L^*(A,\epsilon) = L^{*+4}(A,\epsilon)$ was proved in [15, §7]. The 'homotopy invariance' $L^*(A[x],\epsilon) = L^*(A,\epsilon)$ was proved in [17, 41.3] and [10, 2.1]. The 4-periodicity of the ϵ-symmetric and ϵ-hyperquadratic L-groups for A and $A[x]$ now follows from the 4-periodicity of the ϵ-quadratic L-groups $L_*(A,\epsilon) = L_{*+4}(A,\epsilon)$. □

2. Chain Bundle Theory

2.1. **Chain Bundles.**

Definition 2.1. (i) An *ϵ-bundle* over an A-module chain complex C is a 0-dimensional ϵ-hyperquadratic structure γ on C^{0-*}, that is, a cycle

$$\gamma \in (\widehat{W}^{\%} C^{0-*})_0$$

as given by a collection of A-module morphisms

$$\{\gamma_s \in \mathrm{Hom}_A(C_{r-s}, C^{-r}) \,|\, r, s \in \mathbb{Z}\}$$

such that

$$(-1)^{r+1} d^*\gamma_s + (-1)^s \gamma_s d + (-1)^{s-1}(\gamma_{s-1} + (-1)^s T_\epsilon \gamma_{s-1}) \;=\; 0 \;:\; C_{r-s+1} \to C^{-r} \;.$$

(ii) An *equivalence of ϵ-bundles* over C,

$$\chi \;:\; \gamma \longrightarrow \gamma'$$

is an equivalence of ϵ-hyperquadratic structures.
(iii) A *chain ϵ-bundle* (C,γ) over A is an A-module chain complex C together with an ϵ-bundle $\gamma \in (\widehat{W}^{\%} C^{0-*})_0$.

□

Let (D,δ) be a chain ϵ-bundle and $f : C \to D$ a chain map. The dual of f

$$f^* \;:\; D^{0-*} \longrightarrow C^{0-*}$$

induces a map

$$\widehat{(f^*)}{}^{\%}_0 \;:\; (\widehat{W}^{\%} D^{0-*})_0 \longrightarrow (\widehat{W}^{\%} C^{0-*})_0 \;.$$

Definition 2.2. (i) The *pullback chain ϵ-bundle* $(C, f^*\delta)$ is defined to be

$$f^*\delta \;=\; \widehat{(f^*)}{}^{\%}_0(\delta) \in (\widehat{W}^{\%} C^{0-*})_0.$$

(ii) A *map of chain ϵ-bundles*

$$(f,\chi) \;:\; (C,\gamma) \longrightarrow (D,\delta)$$

is a chain map $f : C \to D$ together with an equivalence of ϵ-bundles over C

$$\chi \;:\; \gamma \longrightarrow f^*\delta \;.$$

□

The ϵ-hyperquadratic Q-group $\widehat{Q}^0(C^{0-*},\epsilon)$ is thus the group of equivalence classes of chain ϵ-bundles on the chain complex C, the algebraic analogue of the topological K-group of a space. The Tate $\mathbb{Z}_2$-cohomology groups

$$\widehat{H}^n(\mathbb{Z}_2; A, \epsilon) \;=\; \frac{\{a \in A \,|\, \overline{a} = (-1)^n \epsilon a\}}{\{b + (-1)^n \epsilon \overline{b} \,|\, b \in A\}}$$

are A-modules via

$$A \times \widehat{H}^n(\mathbb{Z}_2; A, \epsilon) \to \widehat{H}^n(\mathbb{Z}_2; A, \epsilon) \;;\; (a, x) \mapsto a x \overline{a}.$$

Definition 2.3. The *Wu classes* of a chain ϵ-bundle (C,γ) are the A-module morphisms

$$\widehat{v}_k(\gamma)\ :\ H_k(C)\to \widehat{H}^k(\mathbb{Z}_2;A,\epsilon)\ ;\ x\mapsto \gamma_{-2k}(x)(x)\ (k\in\mathbb{Z})\ .$$

□

An n-dimensional ϵ-symmetric Poincaré complex (C,ϕ) with Wu classes (1.29)

$$\widehat{v}_k(\phi)\ :\ H^{n-k}(C)\to \widehat{H}^k(\mathbb{Z}_2;A,\epsilon)\ ;\ y\mapsto \phi_{n-2k}(y)(y)\ (k\in\mathbb{Z})$$

has a Spivak normal ϵ-bundle ([15])

$$\gamma\ =\ S^{-n}(\phi_0^{\%})^{-1}(J(\phi))\in \widehat{Q}^0(C^{0-*},\epsilon)$$

such that

$$\widehat{v}_k(\phi)\ =\ \widehat{v}_k(\gamma)\phi_0\ :\ H^{n-k}(C)\ \cong\ H_k(C)\to \widehat{H}^k(\mathbb{Z}_2;A,\epsilon)\ (k\in\mathbb{Z})\ ,$$

the abstract analogue of the formulae of Wu and Thom.

For any A-module chain map $f:C\to D$ Proposition 1.12 (i) gives an exact sequence

$$\cdots\to \widehat{Q}^1(C^{0-*},\epsilon)\to \widehat{Q}^0(\mathcal{C}(f)^{0-*},\epsilon)\to \widehat{Q}^0(D^{0-*},\epsilon)\xrightarrow{(\widehat{f}^*)^{\%}} \widehat{Q}^0(C^{0-*},\epsilon)\to\dots\ ,$$

motivating the following construction of chain ϵ-bundles :

Definition 2.4. The *cone* of a chain ϵ-bundle map $(f,\chi):(C,0)\to(D,\delta)$ is the chain ϵ-bundle

$$(B,\beta)\ =\ \mathcal{C}(f,\chi)$$

with $B=\mathcal{C}(f)$ the algebraic mapping cone of $f:C\to D$ and

$$\beta_s\ =\ \begin{pmatrix}\delta_s & 0\\ f^*\delta_{s+1} & \chi_{s+1}\end{pmatrix}\ :\ B_{r-s}\ =\ D_{r-s}\oplus C_{r-s-1}\to B^{-r}\ =\ D^{-r}\oplus C^{-r-1}\ .$$

Note that $(D,\delta)=g^*(B,\beta)$ is the pullback of (B,β) along the inclusion $g:D\to B$.

□

Proposition 2.5. *For a f.g. projective A-module chain complex concentrated in degree k*

$$C\ :\ \cdots\to 0\to C_k\to 0\to\dots$$

the kth Wu class defines an isomorphism

$$\widehat{v}_k\ :\ \widehat{Q}^0(C^{0-*},\epsilon)\xrightarrow{\cong}\mathrm{Hom}_A(C_k,\widehat{H}^k(\mathbb{Z}_2;A,\epsilon))\ ;\ \gamma\mapsto\widehat{v}_k(\gamma)\ .$$

Proof. By construction. □

Proposition 2.6. *For a f.g. projective A-module chain complex concentrated in degrees $k,k+1$*

$$C\ :\ \cdots\to 0\to C_{k+1}\xrightarrow{d} C_k\to 0\to\dots$$

there is defined an exact sequence

$$\mathrm{Hom}_A(C_k,\widehat{H}^{k+1}(\mathbb{Z}_2;A,\epsilon))\xrightarrow{d^*}\mathrm{Hom}_A(C_{k+1},\widehat{H}^{k+1}(\mathbb{Z}_2;A,\epsilon))$$

$$\longrightarrow \widehat{Q}^0(C^{0-*},\epsilon)\xrightarrow{p^*\widehat{v}_k}\mathrm{Hom}_A(C_k,\widehat{H}^k(\mathbb{Z}_2;A,\epsilon))\xrightarrow{d^*}\mathrm{Hom}_A(C_{k+1},\widehat{H}^k(\mathbb{Z}_2;A,\epsilon))$$

with $p : C_k \to H_k(C)$ the projection. Thus every chain ϵ-bundle (C,γ) is equivalent to the cone $\mathcal{C}(d,\chi)$ (2.4) of a chain ϵ-bundle map $(d,\chi) : (C_{k+1},0) \to (C_k,\delta)$, regarding $d : C_{k+1} \to C_k$ as a map of chain complexes concentrated in degree k, with

$$\begin{aligned}
&\delta^* ~=~ (-1)^k\delta ~:~ C_k \to C^k ~,~ d^*\delta d ~=~ \chi + (-1)^k\chi^* ~:~ C_{k+1} \to C^{k+1} ~,\\
&\gamma_{-2k} ~=~ \delta ~:~ C_k \to C^k ~,~ \gamma_{-2k-1} ~=~ \begin{cases} d^*\delta ~:~ C_k \to C^{k+1} \\ 0 ~:~ C_{k+1} \to C^k \end{cases} ~,\\
&\gamma_{-2k-2} ~=~ \chi ~:~ C_{k+1} \to C^{k+1} ~.
\end{aligned}$$

Proof. This follows from 2.6 and the algebraic Thom isomorphisms

$$\widehat{t} ~:~ \widehat{Q}^*(d,\epsilon) ~\cong~ \widehat{Q}^*(C,\epsilon)$$

of Proposition 1.12. □

2.2. **The Twisted Quadratic Q-Groups.** For any f.g. projective A-module chain complex C there is defined a short exact sequence of abelian group chain complexes

$$0 \to W_{\%}C \xrightarrow{1+T_\epsilon} W^{\%}C \xrightarrow{J} \widehat{W}^{\%}C \to 0$$

with $1+T_\epsilon$, J the chain maps

$$\begin{aligned}
&1+T_\epsilon ~:~ W_{\%}C \to W^{\%}C ~;~ \psi \mapsto (1+T_\epsilon)\psi ~,~ ((1+T_\epsilon)\psi)_s ~=~ \begin{cases} (1+T_\epsilon)(\psi_0) & \text{if } s=0 \\ 0 & \text{if } s \geqslant 1 ~, \end{cases}\\
&J ~:~ W^{\%}C \to \widehat{W}^{\%}C ~;~ \phi \mapsto J\phi ~,~ (J\phi)_s ~=~ \begin{cases} \phi_s & \text{if } s \geqslant 0 \\ 0 & \text{if } s \geqslant -1 ~. \end{cases}
\end{aligned}$$

The ϵ-symmetric, ϵ-quadratic and ϵ-hyperquadratic Q-groups of C are thus related by the exact sequence of Ranicki [15]

$$\dots \to \widehat{Q}^{n+1}(C,\epsilon) \xrightarrow{H} Q_n(C,\epsilon) \xrightarrow{1+T_\epsilon} Q^n(C,\epsilon) \xrightarrow{J} \widehat{Q}^n(C,\epsilon) \to \dots$$

with

$$H ~:~ \widehat{W}^{\%}C \to (W_{\%}C)_{*-1} ~;~ \theta \mapsto H\theta ~,~ (H\theta)_s ~=~ \theta_{-s-1} ~(s \geqslant 0) ~.$$

Weiss [20] used simplicial abelian groups to defined the twisted quadratic Q-groups $Q_*(C,\gamma,\epsilon)$ of a chain ϵ-bundle (C,γ), to fit into the exact sequence

$$\dots \to \widehat{Q}^{n+1}(C,\epsilon) \xrightarrow{H_\gamma} Q_n(C,\gamma,\epsilon) \xrightarrow{N_\gamma} Q^n(C,\epsilon) \xrightarrow{J_\gamma} \widehat{Q}^n(C,\epsilon) \to \dots ~.$$

The morphisms

$$J_\gamma ~:~ Q^n(C,\epsilon) \longrightarrow \widehat{Q}^n(C,\epsilon) ~;\phi \mapsto J_\gamma\phi ~,~ (J_\gamma\phi)_s ~=~ J(\phi) - (\phi_0)^{\%}(S^n\gamma)$$

are induced by a morphism of simplicial abelian groups, where

$$S^n ~:~ \widehat{Q}^0(C^{0-*},\epsilon) \xrightarrow{\cong} \widehat{Q}^n(C^{n-*},\epsilon) ~;~ \{\theta_s\} \mapsto \{(S^n\theta)_s = \theta_{s-n}\}$$

are the n-fold suspension isomorphisms.

The Kan-Dold theory associates to a chain complex C a simplicial abelian group $K(C)$ such that

$$\pi_*(K(C)) ~=~ H_*(C) ~.$$

For any chain complexes C, D a simplicial map $f : K(C) \to K(D)$ has a mapping fibre $K(f)$. The relative homology groups of f are defined by

$$H_*(f) ~=~ \pi_{*-1}(K(f))$$

and the fibration sequence of simplicial abelian groups

$$K(f) \longrightarrow K(C) \xrightarrow{f} K(D)$$

induces a long exact sequence in homology

$$\dots \to H_n(C) \to H_n(D) \to H_n(f) \to H_{n-1}(C) \to \dots .$$

For a chain map $f : C \to D$

$$K(f) \;=\; K(\mathcal{C}(f)) .$$

The applications involve simplicial maps which are not chain maps, and the *triad homology groups*: given a homotopy-commutative square of simplicial abelian groups

$$\Phi \;:\; \begin{array}{ccc} K(C) & \longrightarrow & K(D) \\ \big\downarrow & \rightsquigarrow & \big\downarrow \\ K(E) & \longrightarrow & K(F) \end{array}$$

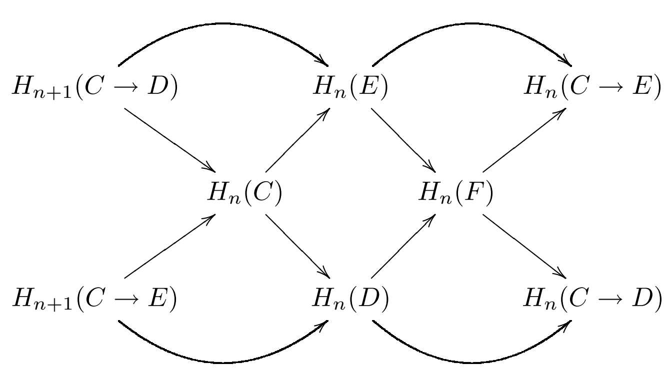
(with $\rightsquigarrow$ denoting an explicit homotopy) the triad homology groups of Φ are the homotopy groups of the mapping fibre of the map of mapping fibres

$$H_*(\Phi) \;=\; \pi_{*-1}(K(C \to D) \to K(E \to F))$$

which fit into a commutative diagram of exact sequences

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & H_{n+1}(D) & \longrightarrow & H_{n+1}(C \to D) & \longrightarrow & H_n(C) & \longrightarrow & H_n(D) & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & H_{n+1}(F) & \longrightarrow & H_{n+1}(E \to F) & \longrightarrow & H_n(E) & \longrightarrow & H_n(F) & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & H_{n+1}(D \to F) & \longrightarrow & H_{n+1}(\Phi) & \longrightarrow & H_n(C \to E) & \longrightarrow & H_n(D \to F) & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & H_n(D) & \longrightarrow & H_n(C \to D) & \longrightarrow & H_{n-1}(C) & \longrightarrow & H_{n-1}(D) & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

If $H_*(\Phi) = 0$ there is a commutative braid of exact sequences



The twisted ϵ-quadratic Q-groups were defined in [20] to be the relative homology groups of a simplicial map

$$J_\gamma \ : \ K(W^{\%}C) \to K(\widehat{W}^{\%}C) \ ,$$

with

$$Q_n(C,\gamma,\epsilon) \ = \ \pi_{n+1}(J_\gamma) \ .$$

A more explicit description of the twisted quadratic Q-groups was then obtained in Ranicki [18], as equivalence classes of ϵ-symmetric structures on the chain ϵ-bundle.

Definition 2.7. (i) An *ϵ-symmetric structure on a chain ϵ-bundle* (C,γ) is a pair (ϕ,θ) with $\phi \in (W^{\%}C)_n$ a cycle and $\theta \in (\widehat{W}^{\%}C)_{n+1}$ such that

$$d\theta \ = \ J_\gamma(\phi) \ ,$$

or equivalently

$$\begin{aligned} & d\phi_s + (-1)^r\phi_s d^* + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T_\epsilon\phi_{s-1}) \ = \ 0 \ : \ C^r \to C_{n-r+s-1} \ , \\ & \phi_s - \phi_0^*\gamma_{s-n}\phi_0 \ = \ d\theta_s + (-1)^r\theta_s d^* + (-1)^{n+s}(\theta_{s-1} + (-1)^s T_\epsilon\theta_{s-1}) \ : \ C^r \to C_{n-r+s} \\ & \qquad (r,s \in \mathbb{Z}, \phi_s = 0 \text{ for } s \leqslant -1) \ . \end{aligned}$$

(ii) Two structures (ϕ,θ) and (ϕ',θ') are *equivalent* if there exist $\xi \in (W^{\%}C)_{n+1}$, $\eta \in (\widehat{W}^{\%}C)_{n+2}$ such that

$$d\xi \ = \ \phi' - \phi, \ d\eta \ = \ \theta' - \theta + J(\xi) + (\xi_0,\phi_0,\phi'_0)^{\%}(S^n\gamma) \ ,$$

where $(\xi_0,\phi_0,\phi'_0)^{\%} : (\widehat{W}^{\%}C^{-*})_n \to (\widehat{W}^{\%}C)_{n+1}$ is the chain homotopy from $(\phi_0)^{\%}$ to $(\phi'_0)^{\%}$ induced by ξ_0. (See [15, 1.1] for the precise formula).
(iii) The *n-dimensional twisted ϵ-quadratic Q-group* $Q_n(C,\gamma,\epsilon)$ is the abelian group of equivalence classes of n-dimensional ϵ-symmetric structures on (C,γ) with addition by

$$(\phi,\theta) + (\phi',\theta') \ = \ (\phi+\phi', \theta+\theta'+\zeta) \ , \text{ where } \zeta_s \ = \ \phi_0\gamma_{s-n+1}\phi'_0 \ .$$

□

As for the $\pm\epsilon$-symmetric and $\pm\epsilon$-quadratic Q-groups, there are defined skew-suspension isomorphisms of twisted $\pm\epsilon$-quadratic Q-groups

$$\overline{S} \ : \ Q_n(C,\gamma,\epsilon) \xrightarrow{\cong} Q_{n+2}(C_{*-1},\gamma,-\epsilon) \ ; \ (\{\phi_s\},\{\theta_s\}) \mapsto (\{\phi_s\},\{\theta_s\}) \ .$$

Proposition 2.8. (i) *The twisted ϵ-quadratic Q-groups $Q_*(C,\gamma,\epsilon)$ are related to the ϵ-symmetric Q-groups $Q^*(C,\epsilon)$ and the ϵ-hyperquadratic Q-groups $\widehat{Q}^*(C,\epsilon)$ by the exact sequence*

$$\dots \to \widehat{Q}^{n+1}(C,\epsilon) \xrightarrow{H_\gamma} Q_n(C,\gamma,\epsilon) \xrightarrow{N_\gamma} Q^n(C,\epsilon) \xrightarrow{J_\gamma} \widehat{Q}^n(C,\epsilon) \to \dots$$

with

$$\begin{aligned} H_\gamma \ & : \ \widehat{Q}^{n+1}(C,\epsilon) \to Q_n(C,\gamma,\epsilon) \ ; \ \theta \mapsto (0,\theta) \ , \\ N_\gamma \ & : \ Q_n(C,\gamma,\epsilon) \to Q^n(C,\epsilon) \ ; \ (\phi,\theta) \mapsto \phi \ . \end{aligned}$$

(ii) *For a chain ϵ-bundle (C,γ) such that C splits as*

$$C \ = \ \sum_{i=-\infty}^{\infty} C(i) \ ,$$

the ϵ-hyperquadratic Q-groups split as

$$\widehat{Q}^n(C,\epsilon) \;=\; \sum_{i=-\infty}^{\infty} \widehat{Q}^n(C(i),\epsilon)$$

and

$$\gamma \;=\; \sum_{i=-\infty}^{\infty} \gamma(i) \in \widehat{Q}^0(C^{-*},\epsilon) \;=\; \sum_{i=-\infty}^{\infty} \widehat{Q}^0(C(i)^{-*},\epsilon)\ .$$

The twisted ϵ-quadratic Q-groups of (C,γ) fit into the exact sequence

$$\dots \to \sum_i Q_n(C(i),\gamma(i),\epsilon) \xrightarrow{\ q\ } Q_n(C,\gamma,\epsilon) \xrightarrow{\ p\ } \sum_{i<j} H_n(C(i)\otimes_A C(j)) \xrightarrow{\ \partial\ } \sum_i Q_{n-1}(C(i),\gamma(i),\epsilon) \to \dots$$

with

$$\begin{aligned}
p\ &:\ Q_n(C,\gamma,\epsilon) \to \sum_{i<j} H_n(C(i)\otimes_A C(j))\ ;\ (\phi,\theta) \mapsto \sum_{i<j} (p(i)\otimes p(j))(\phi_0) \\
&\qquad (p(i) = \text{projection} : C \to C(i))\ , \\
q\ &=\ \sum_i q(i)_{\%}\ :\ \sum_i Q_n(C(i),\gamma(i),\epsilon) \to Q_n(C,\gamma,\epsilon)\ , \\
&\qquad (q(i) = \text{inclusion} : C(i) \to C))\ , \\
\partial\ &:\ \sum_{i<j} H_n(C(i)\otimes_A C(j)) \to \sum_i Q_{n-1}(C(i),\gamma(i),\epsilon)\ ; \\
&\qquad \sum_{i<j} h(i,j) \mapsto \sum_i (0, \sum_{i<j} \widehat{h(i,j)}^{\%}(S^n\gamma(j)))\ \ (h(i,j) : C(j)^{n-*} \to C(i))\ .
\end{aligned}$$

Proof. (i) See Weiss [20].
(ii) See Ranicki [18, p.26]. □

Example 2.9. The twisted ϵ-quadratic Q-groups of the zero chain ϵ-bundle $(C,0)$ are just the ϵ-quadratic Q-groups of C, with isomorphisms

$$Q_n(C,\epsilon) \to Q_n(C,0,\epsilon)\ ;\ \psi \mapsto ((1+T)\psi,\theta)$$

defined by

$$\theta_s \;=\; \begin{cases} \psi_{-s-1} : C^{n-r+s+1} \to C_r & \text{if } s \leqslant -1 \\ 0 & \text{if } s \geqslant 0\ , \end{cases}$$

and with an exact sequence

$$\dots \to \widehat{Q}^{n+1}(C,\epsilon) \xrightarrow{H} Q_n(C,\epsilon) \xrightarrow{N} Q^n(C,\epsilon) \xrightarrow{J} \widehat{Q}^n(C,\epsilon) \to \dots\ .$$

□

For $\epsilon = 1$ we write

$$\text{chain 1-bundle} \;=\; \text{chain bundle}\ ,\ Q_n(C,\gamma,1) \;=\; Q_n(C,\gamma)\ .$$

2.3. **The Algebraic Normal Invariant.** Fix a chain ϵ-bundle (B,β) over A.

Definition 2.10. (i) A (B,β)*-structure* $(\gamma,\phi,\theta,g,\chi)$ on an n-dimensional (ϵ-symmetric, ϵ-quadratic) Poincaré pair $(f:C\to D,(\delta\phi,\psi))$ is a Spivak normal structure (γ,ϕ,θ) together with a chain ϵ-bundle map

$$(g,\chi)\ :\ (\mathcal{C}(f),\gamma)\to(B,\beta)\ .$$

(ii) The *n-dimensional (B,β)-structure ϵ-symmetric L-group* $L\langle B,\beta\rangle^n(A,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric Poincaré complexes $(D,\delta\phi)$ over A together with a (B,β)-structure $(\gamma,\phi,\theta,g,\chi)$ (so $(C,\psi)=(0,0)$).
(iii) The *n-dimensional (B,β)-structure ϵ-hyperquadratic L-group* $\widehat{L}\langle B,\beta\rangle^n(A,\epsilon)$ is the cobordism group of n-dimensional (ϵ-symmetric, ϵ-quadratic) Poincaré pairs $(f:C\to D,(\delta\phi,\psi))$ over A together with a (B,β)-structure $(\gamma,\phi,\theta,g,\chi)$. □

There are defined skew-suspension maps in the (B,β)-structure ϵ-symmetric and ϵ-hyperquadratic L-groups

$$\begin{aligned}\overline{S}\ &:\ L\langle B,\beta\rangle^n(A,\epsilon)\to L\langle B_{*-1},\beta_{*-1}\rangle^{n+2}(A,-\epsilon)\ ,\\ \overline{S}\ &:\ \widehat{L}\langle B,\beta\rangle^n(A,\epsilon)\to \widehat{L}\langle B_{*-1},\beta_{*-1}\rangle^{n+2}(A,-\epsilon)\end{aligned}$$

given by $C\mapsto C_{*-1}$ on the chain complexes, with (B_{*-1},β_{*-1}) a chain $(-\epsilon)$-bundle. We shall write the 4-periodic versions of the (B,β)-structure L-groups as

$$\begin{aligned}L\langle B,\beta\rangle^{n+4*}(A,\epsilon)\ &=\ \lim_{k\to\infty}L\langle B,\beta\rangle^{n+4k}(A,\epsilon)\ ,\\ \widehat{L}\langle B,\beta\rangle^{n+4*}(A,\epsilon)\ &=\ \lim_{k\to\infty}\widehat{L}\langle B,\beta\rangle^{n+4k}(A,\epsilon)\ .\end{aligned}$$

Example 2.11. An (ϵ-symmetric, ϵ-quadratic) Poincaré pair with a $(0,0)$-structure is essentially the same as an ϵ-quadratic Poincaré pair. In particular, an ϵ-symmetric Poincaré complex with a $(0,0)$-structure is essentially the same as an ϵ-quadratic Poincaré complex. The (0,0)-structure L-groups are given by

$$L\langle 0,0\rangle^n(A,\epsilon)\ =\ L_n(A,\epsilon)\ ,\ \widehat{L}\langle 0,0\rangle^n(A,\epsilon)\ =\ 0\ .$$

□

Proposition 2.12. (Ranicki [18, §7])
(i) *An n-dimensional ϵ-symmetric structure $(\phi,\theta)\in Q_n(B,\beta,\epsilon)$ on a chain ϵ-bundle (B,β) determines an n-dimensional (ϵ-symmetric, ϵ-quadratic) Poincaré pair $(f:C\to D,(\delta\phi,\psi))$ with*

$$f\ =\ \text{proj.}\ :\ C\ =\ \mathcal{C}(\phi_0:B^{n-*}\to B)_{*+1}\to D\ =\ B^{n-*}\ ,$$

$$\psi_0\ =\ \begin{pmatrix}\theta_0 & 0\\ 1+\beta_{-n}\phi_0^* & \beta^*_{-n-1}\end{pmatrix}\ :$$

$$C^r\ =\ B^{r+1}\oplus B_{n-r}\to C_{n-r-1}\ =\ B_{n-r}\oplus B^{r+1}\ ,$$

$$\psi_s\ =\ \begin{pmatrix}\theta_{-s} & 0\\ \beta_{-n-s}\phi_0^* & \beta^*_{-n-s-1}\end{pmatrix}\ :$$

$$C^r\ =\ B^{r+1}\oplus B_{n-r}\to C_{n-r-s-1}\ =\ B_{n-r-s}\oplus B^{r+s+1}\ (s\geqslant 1)\ ,$$

$$\delta\phi_s\ =\ \beta_{s-n}\ :\ D^r\ =\ B_{n-r}\to D_{n-r+s}\ =\ B^{r-s}\ (s\geqslant 0)$$

(up to signs) such that $(\mathcal{C}(f),\gamma)\simeq(B,\beta)$.
(ii) *An n-dimensional (ϵ-symmetric, ϵ-quadratic) Poincaré pair* $(f:C\to D,(\delta\phi,\psi)\in$

$Q_n^n(f,\epsilon)$) has a canonical equivalence class of 'algebraic Spivak normal structures' (γ,ϕ,θ) with γ a chain ϵ-bundle over $\mathcal{C}(f)$ and (ϕ,θ) an n-dimensional ϵ-symmetric structure on γ representing an element

$$(\phi,\theta)\in Q_n(\mathcal{C}(f),\gamma,\epsilon)\ .$$

The construction of (i) *applied to (ϕ,θ) gives an n-dimensional (ϵ-symmetric, ϵ-quadratic) Poincaré pair homotopy equivalent to $(f:C\to D,(\delta\phi,\psi)\in Q_n^n(f,\epsilon))$.*

Proof. (i) By construction.
(ii) The equivalence class $\phi=\delta\phi/(1+T_\epsilon)\psi\in Q^n(\mathcal{C}(f))$ is given by the algebraic Thom construction

$$\phi_s\ =\ \begin{cases}\begin{pmatrix}\delta\phi_0 & 0\\ (1+T_\epsilon)\psi_0 f^* & 0\end{pmatrix} & \text{if } s=0\\ \begin{pmatrix}\delta\phi_1 & 0\\ 0 & (1+T_\epsilon)\psi_0\end{pmatrix} & \text{if } s=1\\ \begin{pmatrix}\delta\phi_s & 0\\ 0 & 0\end{pmatrix} & \text{if } s\geqslant 2\end{cases}$$

$$:\ \mathcal{C}(f)^r\ =\ D^r\oplus C^{r-1}\to\mathcal{C}(f)_{n-r+s}\ =\ D_{n-r+s}\oplus C_{n-r+s-1}\ ,$$

such that

$$\phi_0\ :\ \mathcal{C}(f)^{n-*}\to D^{n-*}\xrightarrow[\simeq]{(\delta\phi,(1+T_\epsilon)\psi)_0}\mathcal{C}(f)\ .$$

The equivalence class $\gamma\in\widehat{Q}^0(\mathcal{C}(f)^{0-*},\epsilon)$ of the Spivak normal chain bundle is the image of $(\delta\phi,\psi)\in Q_n^n(f,\epsilon)$ under the composite

$$Q_n^n(f,\epsilon)\xrightarrow{J_f}\widehat{Q}^n(D,\epsilon)\xrightarrow[\cong]{((\delta\phi,(1+T_\epsilon)\psi)_0^{\%})^{-1}}\widehat{Q}^n(\mathcal{C}(f)^{n-*},\epsilon)\xrightarrow[\cong]{S^{-n}}\widehat{Q}^0(\mathcal{C}(f)^{0-*},\epsilon)\ .$$

□

Definition 2.13. (i) The *boundary* of an n-dimensional ϵ-symmetric structure $(\phi,\theta)\in Q_n(B,\beta,\epsilon)$ on a chain ϵ-bundle (B,β) over A is the ϵ-symmetric null-cobordant $(n-1)$-dimensional ϵ-quadratic Poincaré complex over A

$$\partial(\phi,\theta)\ =\ (C,\psi)$$

defined in Proposition 2.12 (i) above, with $C=\mathcal{C}(\phi_0:B^{n-*}\to B)_{*+1}$.
(ii) The *algebraic normal invariant* of an n-dimensional (ϵ-symmetric, ϵ-quadratic) Poincaré pair over A $(f:C\to D,(\delta\phi,\psi)\in Q_n^n(f,\epsilon))$ is the class

$$(\phi,\theta)\in Q_n(\mathcal{C}(f),\gamma,\epsilon)$$

defined in Proposition 2.12 (ii) above. □

Proposition 2.14. *Let (B,β) be a chain ϵ-bundle over A such that B is concentrated in degree k*

$$B\ :\ \cdots\to 0\to B_k\to 0\to\dots\ .$$

The boundary map $\partial:Q_{2k}(B,\beta,\epsilon)\to L_{2k-1}(A,\epsilon)$ sends an ϵ-symmetric structure $(\phi,\theta)\in Q_{2k}(B,\beta,\epsilon)$ to the Witt class of the $(-1)^{k-1}\epsilon$-quadratic formation

$$\partial(\phi,\theta)\ =\ (H_{(-1)^{k-1}\epsilon}(B^k);B^k,\mathrm{im}(\begin{pmatrix}1-\beta\phi\\ \phi\end{pmatrix}:B^k\to B^k\oplus B_k))$$

with

$$H_{(-1)^{k-1}\epsilon}(B^k) \;=\; (B^k \oplus B_k, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$$

the hyperbolic $(-1)^{k-1}\epsilon$-quadratic form.

Proof. The chain ϵ-bundle (equivalence class)

$$\beta \in \widehat{Q}^0(B^{0-*},\epsilon) \;=\; \widehat{H}^0(\mathbb{Z}_2;S(B_k),\epsilon)$$

is represented by an ϵ-symmetric form (B_k,β). An ϵ-symmetric structure $(\phi,\theta) \in Q_{2k}(B,\beta,\epsilon)$ is represented by an $(-1)^k\epsilon$-symmetric form (B^k,ϕ) together with $\theta \in S(B_k)$ such that

$$\phi - \phi\beta\phi \;=\; \theta + (-1)^k\epsilon\theta^* \in H^0(\mathbb{Z}_2;S(B^k),(-1)^k\epsilon) \ .$$

The boundary of (ϕ,θ) is the ϵ-symmetric null-cobordant $(2k-1)$-dimensional ϵ-quadratic Poincaré complex $\partial(\phi,\theta) = (C,\psi)$ concentrated in degrees $k-1,k$ corresponding to the formation in the statement. □

Proposition 2.15. *Let (B,β) be a chain ϵ-bundle over A such that B is concentrated in degrees $k,k+1$*

$$B \;:\; \dots \to 0 \to B_{k+1} \xrightarrow{\;d\;} B_k \to 0 \to \dots \ .$$

The boundary map $\partial : Q_{2k+1}(B,\beta,\epsilon) \to L_{2k}(A,\epsilon)$ sends an ϵ-symmetric structure $(\phi,\theta) \in Q_{2k+1}(B,\beta,\epsilon)$ to the Witt class of the nonsingular $(-1)^k\epsilon$-quadratic form over A

$$(\mathrm{coker}(\begin{pmatrix} -d^* \\ \phi_0^* \\ 1-\beta_{-2k}d\phi_0^* \end{pmatrix} : B^k \to B^{k+1}\oplus B_{k+1}\oplus B^k), \begin{pmatrix} \theta_0 & 0 & \phi_0 \\ 1 & \beta^*_{-2k-2} & d^* \\ 0 & 0 & 0 \end{pmatrix}) \ .$$

Proof. This is an application of the instant surgery obstruction of [15, 4.3], which identifies the cobordism class $(C,\psi) \in L_{2k}(A,\epsilon)$ of a $2k$-dimensional ϵ-quadratic Poincaré complex (C,ψ) with the Witt class of the nonsingular ϵ-quadratic form

$$I(C,\psi) \;=\; (\mathrm{coker}(\begin{pmatrix} d^* \\ (-1)^{k+1}(1+T_\epsilon)\psi_0 \end{pmatrix} : C^{k-1} \to C^k \oplus C_{k+1}), \begin{pmatrix} \psi_0 & d \\ 0 & 0 \end{pmatrix}) \ .$$

By Proposition 2.6 the chain ϵ-bundle β can be taken to be the cone of a chain ϵ-bundle map

$$(d,\beta_{-2k-2}) \;:\; (B_{k+1},0) \to (B_k,\beta_{-2k})$$

with

$$\begin{aligned} \beta^*_{-2k} \;&=\; (-1)^k\epsilon\beta_{-2k} \;:\; B_k \to B^k \ , \\ d^*\beta_{-2k}d \;&=\; \beta_{-2k-2} + (-1)^k\epsilon\beta^*_{-2k-2} \;:\; B_{k+1} \to B^{k+1} \ , \\ \beta_{-2k-1} \;&=\; \begin{cases} \beta_{-2k}d \;:\; B_{k+1} \to B^k \\ 0 \;:\; B_k \to B^{k+1} \ . \end{cases} \end{aligned}$$

An ϵ-symmetric structure $(\phi,\theta) \in Q_{2k+1}(B,\beta,\epsilon)$ is represented by A-module morphisms

$$\phi_0 \;:\; B^k \to B_{k+1} \ , \ \widetilde{\phi}_0 \;:\; B^{k+1} \to B_k \ , \ \phi_1 \;:\; B^{k+1} \to B_{k+1} \ ,$$

$$\theta_0 \;:\; B^{k+1} \to B_{k+1} \ , \ \theta_{-1} \;:\; B^k \to B_{k+1} \ , \ \widetilde{\theta}_{-1} \;:\; B^{k+1} \to B_k \ , \ \theta_{-2} \;:\; B^k \to B_k$$

such that

$$\begin{aligned}
&d\phi_0 + (-1)^k\widetilde{\phi}_0 d^* \;=\; 0 \;:\; B^k \to B_k \;,\\
&\phi_0 - \epsilon\widetilde{\phi}_0^* + (-1)^{k+1}\phi_1 d^* \;=\; 0 \;:\; B^k \to B_{k+1} \;,\\
&\phi_1 + (-1)^{k+1}\epsilon\phi_1^* \;=\; 0 \;:\; B^{k+1} \to B_{k+1} \;,\\
&\phi_0 - \phi_0\beta_{-2k}d\widetilde{\phi}_0^* \;=\; (-1)^k\theta_0 d^* - \theta_{-1} - \epsilon\widetilde{\theta}_{-1}^* \;:\; B^k \to B_{k+1} \;,\\
&\widetilde{\phi}_0 \;=\; d\theta_0 - \widetilde{\theta}_{-1} - \epsilon\theta_{-1}^* \;:\; B^{k+1} \to B_k \;,\\
&-\widetilde{\phi}_0\beta_{-2k-2}\widetilde{\phi}_0^* \;=\; \theta_{-2} + (-1)^{k+1}\epsilon\theta_{-2}^* \;:\; B^k \to B_k \;,\\
&\phi_1 - \phi_0\beta_{-2k}\phi_0^* \;=\; \theta_0 + (-1)^k\epsilon\theta_0^* \;:\; B^{k+1} \to B_{k+1} \;.
\end{aligned}$$

The boundary of (ϕ,θ) given by 2.13 (i) is an ϵ-symmetric null-cobordant $2k$-dimensional ϵ-quadratic Poincaré complex $\partial(\phi,\theta) = (C,\psi)$ concentrated in degrees $k-1, k, k+1$, with $I(C,\psi)$ the instant surgery obstruction form (2.15) in the statement. □

The ϵ-quadratic L-groups and the (B,β)-structure L-groups fit into an evident exact sequence

$$\cdots \to L_n(A,\epsilon) \to L\langle B,\beta\rangle^n(A,\epsilon) \to \widehat{L}\langle B,\beta\rangle^n(A,\epsilon) \xrightarrow{\;\partial\;} L_{n-1}(A,\epsilon) \to \dots \;,$$

and similarly for the 4-periodic versions

$$\cdots \to L_n(A,\epsilon) \to L\langle B,\beta\rangle^{n+4*}(A,\epsilon) \to \widehat{L}\langle B,\beta\rangle^{n+4*}(A,\epsilon) \xrightarrow{\;\partial\;} L_{n-1}(A,\epsilon) \to \dots \;.$$

Proposition 2.16. (Weiss [20])
(i) *The function*

$$Q_n(B,\beta,\epsilon) \to \widehat{L}\langle B,\beta\rangle^{n+4*}(A,\epsilon) \;;\; (\phi,\theta) \mapsto (f : C \to D, (\delta\phi,\psi)) \;(\mathit{2.12}\;\text{(ii)})$$

is an isomorphism, with inverse given by the algebraic normal invariant. The ϵ-quadratic L-groups of A, the 4-periodic (B,β)-structure ϵ-symmetric L-groups of A and the twisted ϵ-quadratic Q-groups of (B,β) are thus related by an exact sequence

$$\cdots \to L_n(A,\epsilon) \xrightarrow{\;1+T\;} L\langle B,\beta\rangle^{n+4*}(A,\epsilon) \to Q_n(B,\beta,\epsilon) \xrightarrow{\partial} L_{n-1}(A,\epsilon) \to \dots \;.$$

(ii) *The cobordism class of an n-dimensional (ϵ-symmetric, ϵ-quadratic) Poincaré pair $(f : C \to D, (\delta\phi,\psi))$ over A with a (B,β)-structure $(\gamma,\phi,\theta,g,\chi)$ is the image of the algebraic normal invariant $(\phi,\theta) \in Q_n(\mathcal{C}(f),\gamma,\epsilon)$*

$$(f : C \to D, (\delta\phi,\psi)) \;=\; (g,\chi)_\%(\phi,\theta) \in Q_n(B,\beta,\epsilon) \;.$$

Proof. The ϵ-symmetrization of an n-dimensional ϵ-quadratic Poincaré complex (C,ψ) is an n-dimensional ϵ-symmetric Poincaré complex $(C,(1+T_\epsilon)\psi)$ with (B,β)-structure $(0,(1+T)\psi,\theta,0,0)$ given by

$$\theta_s \;=\; \begin{cases} \psi_{-s-1} \in \mathrm{Hom}_A(C^{-*},C)_{n+s+1} & \text{if } s \leqslant -1 \\ 0 & \text{if } s \geqslant 0 \;. \end{cases}$$

The relative groups of the symmetrization map

$$1+T_\epsilon \;:\; L_n(A,\epsilon) \to L\langle B,\beta\rangle^n(A,\epsilon) \;;\; (C,\psi) \mapsto (C,(1+T_\epsilon)\psi)$$

are the cobordism groups of n-dimensional (ϵ-symmetric, ϵ-quadratic) Poincaré pairs $(f : C \to D, (\delta\phi,\psi))$ together with a (B,β)-structure $(\gamma,\phi,\theta,g,\chi)$. □

Proposition 2.17. *Let (B,β) be a chain ϵ-bundle over A with B concentrated in degree k*

$$B \ : \ \dots \to 0 \to B_k \to 0 \to \dots$$

so that $\beta \in \widehat{Q}^0(B^{0-*},\epsilon) = \widehat{H}^0(\mathbb{Z}_2;S(B^k),(-1)^kT_\epsilon)$ is represented by an element

$$\beta_{-2k} \ = \ (-1)^k\epsilon\beta^*_{-2k} \in S(B^k) \ .$$

The twisted ϵ-quadratic Q-groups $Q_n(B,\beta,\epsilon)$ are given as follows.
(i) *For $n \neq 2k-1, 2k$*

$$\begin{aligned} Q_n(B,\beta,\epsilon) \ &= \ Q_n(B,\epsilon) \\ &= \ \begin{cases} \widehat{Q}^{n+1}(B,\epsilon) = \widehat{H}^{n-2k+1}(\mathbb{Z}_2;S(B^k),(-1)^kT_\epsilon) & \text{if } n \geqslant 2k+1 \\ 0 & \text{if } n \leqslant 2k-2 \ . \end{cases} \end{aligned}$$

(ii) *For $n = 2k$*

$$Q_{2k}(B,\beta,\epsilon) \ = \ \frac{\{(\phi,\theta) \in S(B^k)\oplus S(B^k) \,|\, \phi = (-1)^k\epsilon\phi^*, \phi - \phi\beta_{-2k}\phi^* = \theta + (-1)^k\epsilon\theta^*\}}{\{(0,\eta + (-1)^{k+1}\epsilon\eta^*) \,|\, \eta \in S(B^k)\}}$$

with addition by

$$(\phi,\theta) + (\phi',\theta') \ = \ (\phi+\phi', \theta+\theta'+\phi'\beta_{-2k}\phi^*) \ .$$

The boundary of $(\phi,\theta) \in Q_{2k}(B,\beta,\epsilon)$ is the $(2k-1)$-dimensional ϵ-quadratic Poincaré complex over A concentrated in degrees $k-1, k$ corresponding to the $(-1)^{k+1}\epsilon$-quadratic formation over A

$$\partial(\phi,\theta) \ = \ (H_{(-1)^{k+1}\epsilon}(B^k); B^k, \mathrm{im}(\begin{pmatrix} 1-\beta_{-2k}\phi \\ \phi \end{pmatrix} : B^k \to B^k \oplus B_k)) \ .$$

(iii) *For $n = 2k-1$*

$$\begin{aligned} Q_{2k-1}(B,\beta,\epsilon) \ &= \ \mathrm{coker}(J_\beta : Q^{2k}(B,\epsilon) \to \widehat{Q}^{2k}(B,\epsilon)) \\ &= \ \frac{\{\sigma \in S(B^k) \,|\, \sigma = (-1)^k\epsilon\sigma^*\}}{\{\phi - \phi\beta_{-2k}\phi^* - (\theta + (-1)^k\epsilon\theta^*) \,|\, \phi = (-1)^k\epsilon\phi^*, \theta \in S(B^k)\}} \ . \end{aligned}$$

The boundary of $\sigma \in Q_{2k-1}(B,\beta,\epsilon)$ is the $(2k-2)$-dimensional ϵ-quadratic Poincaré complex over A concentrated in degree $k-1$ corresponding to the $(-1)^{k+1}\epsilon$-quadratic form over A

$$\partial(\sigma) \ = \ (B^k \oplus B_k, \begin{pmatrix} \sigma & 1 \\ 0 & \beta_{-2k} \end{pmatrix})$$

with

$$(1+T_{(-1)^{k+1}\epsilon})\partial(\sigma) \ = \ (B^k \oplus B_k, \begin{pmatrix} 0 & 1 \\ (-1)^{k+1}\epsilon & 0 \end{pmatrix}) \ .$$

(iv) *The maps in the exact sequence*

$$\begin{aligned} 0 \to \widehat{Q}^{2k+1}(B,\epsilon) \xrightarrow{H_\beta} Q_{2k}(B,\beta,\epsilon) \xrightarrow{N_\beta} Q^{2k}(B,\epsilon) \\ \xrightarrow{J_\beta} \widehat{Q}^{2k}(B,\epsilon) \xrightarrow{H_\beta} Q_{2k-1}(B,\beta,\epsilon) \to 0 \end{aligned}$$

are given by

$$\begin{aligned}
H_\beta &: \widehat{Q}^{2k+1}(B,\epsilon) = \widehat{H}^1(\mathbb{Z}_2;S(B^k),(-1)^kT_\epsilon) \to Q_{2k}(B,\beta,\epsilon)\ ;\ \theta \mapsto (0,\theta)\ ,\\
N_\beta &: Q_{2k}(B,\beta,\epsilon) \to Q^{2k}(B,\epsilon) = H^0(\mathbb{Z}_2;S(B^k),(-1)^kT_\epsilon)\ ;\ (\phi,\theta)\mapsto \phi\ ,\\
J_\beta &: Q^{2k}(B,\epsilon) = H^0(\mathbb{Z}_2;S(B^k),(-1)^kT_\epsilon) \to \widehat{Q}^{2k}(B,\epsilon) = \widehat{H}^0(\mathbb{Z}_2;S(B^k),(-1)^kT_\epsilon)\ ;\\
&\qquad\qquad \phi \mapsto \phi - \phi\beta_{-2k}\phi^*\ ,\\
H_\beta &: \widehat{Q}^{2k}(B,\epsilon) = \widehat{H}^0(\mathbb{Z}_2;S(B^k),(-1)^kT_\epsilon) \to Q_{2k-1}(B,\beta,\epsilon)\ ;\ \sigma\mapsto\sigma\ .
\end{aligned}$$

□

Example 2.18. Let (K,λ) be a nonsingular ϵ-symmetric form over A, which may be regarded as a 0-dimensional ϵ-symmetric Poincaré complex (D,ϕ) over A with

$$\phi_0 \;=\; \lambda \;:\; D^0 \;=\; K \to D_0 \;=\; K^*\ .$$

The composite

$$Q^0(D,\epsilon) \;=\; H^0(\mathbb{Z}_2;S(K),\epsilon) \xrightarrow{\;J\;} \widehat{Q}^0(D,\epsilon) \xrightarrow{(\phi_0)^{-1}} \widehat{Q}^0(D^{0-*},\epsilon)$$

sends $\phi\in Q^0(D,\epsilon)$ to the algebraic Spivak normal chain bundle

$$\gamma\in\widehat{Q}^0(D^{0-*},\epsilon) \;=\; \widehat{H}^0(\mathbb{Z}_2;S(K^*),\epsilon)$$

with

$$\gamma_0 \;=\; \epsilon\lambda^{-1} \;:\; D_0 \;=\; K^*\to D^0 \;=\; K\ .$$

By Proposition 2.17

$$Q_0(D,\gamma,\epsilon) \;=\; \frac{\{(\kappa,\theta)\in S(K)\oplus S(K)\,|\,\kappa=\epsilon\kappa^*,\kappa-\kappa\gamma_0\kappa^*=\theta+\epsilon\theta^*\}}{\{(0,\eta-\epsilon\eta^*)\,|\,\eta\in S(K)\}}$$

with addition by

$$(\kappa,\theta)+(\kappa',\theta') \;=\; (\kappa+\kappa',\theta+\theta'+\kappa'\gamma_0\kappa^*)\ .$$

The algebraic normal invariant of (D,ϕ) is given by

$$(\phi,0)\in Q_0(D,\gamma,\epsilon)\ .$$

□

Example 2.19. Let A be a ring with even involution (1.4), and let C be concentrated in degree k with $C_k=A^r$. For odd $k=2j+1$

$$\widehat{Q}^0(C^{0-*}) \;=\; 0$$

and there is only one chain ϵ-bundle $\gamma=0$ over C, with

$$Q_n(C,\gamma) \;=\; Q_n(C) \;=\; \begin{cases}\bigoplus_r \widehat{H}^0(\mathbb{Z}_2;A) & \text{if } n\geqslant 4j+2,\ n\equiv 0(\bmod 2)\\ 0 & \text{otherwise}\ .\end{cases}$$

For even $k=2j$

$$\widehat{Q}^0(C^{0-*}) \;=\; \bigoplus_r \widehat{H}^0(\mathbb{Z}_2;A)$$

a chain ϵ-bundle $\gamma \in \widehat{Q}^0(C^{0-*})$ is represented by a diagonal matrix

$$\gamma \ = \ X \ = \ \begin{pmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_r \end{pmatrix} \in \mathrm{Sym}_r(A)$$

with $\overline{x}_i = x_i \in A$, and there is defined an exact sequence

$$\widehat{Q}^{4j+1}(C) = 0 \to Q_{4j}(C,\gamma) \to Q^{4j}(C) \xrightarrow{J_\gamma} \widehat{Q}^{4j}(C) \to Q_{4j-1}(C,\gamma) \to Q^{4j-1}(C) = 0$$

with

$$J_\gamma \ : \ Q^{4j}(C) \ = \ \mathrm{Sym}_r(A) \to \widehat{Q}^{4j}(C) \ = \ \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \ ; \ M \mapsto M - MXM \ ,$$

so that

$$Q_n(C,\gamma) \ = \ \begin{cases} \bigoplus\limits_r \widehat{H}^0(\mathbb{Z}_2;A) & \text{if } n \geqslant 4j+1 \\ & \quad \text{and } n \equiv 1 (\mathrm{mod}\, 2) \\ \{M \in \mathrm{Sym}_r(A) \,|\, M - MXM \in \mathrm{Quad}_r(A)\} & \text{if } n = 4j \\ M_r(A)/\{M - MXM - (N+N^t) \,|\, M \in \mathrm{Sym}_r(A), N \in M_r(A)\} & \text{if } n = 4j-1 \\ 0 & \text{otherwise} \ . \end{cases}$$

Moreover, Proposition 2.8 (ii) gives an exact sequence

$$0 \to \bigoplus_{i=1}^r Q_{4j}(B,x_i) \to Q_{4j}(C,\gamma) \to \bigoplus_{r(r-1)/2} A \to \bigoplus_{i=1}^r Q_{4j-1}(B,x_i) \to Q_{4j-1}(C,\gamma) \to 0$$

with B concentrated in degree $2j$ with $B_{2j} = A$. □

2.4. **The Relative Twisted Quadratic Q-groups.** Let $(f,\chi) : (C,\gamma) \to (D,\delta)$ be a map of chain ϵ-bundles, and let (ϕ,θ) be an n-dimensional ϵ-symmetric structure on (C,γ), so that $\chi \in (\widehat{W}^{\%}C)_1$, $\phi \in (W^{\%}C)_n$ and $\theta \in (\widehat{W}^{\%}C)_{n+1}$. Composing the chain map $\phi_0 : C^{n-*} \to C$ with f, we get an induced map

$$(\widehat{f\phi_0})^{\%} \ : \ \widehat{W}^{\%}C^{n-*} \to \widehat{W}^{\%}D \ .$$

The morphisms of twisted quadratic Q-groups

$$(f,\chi)_{\%} \ : \ Q_n(C,\gamma,\epsilon) \to Q_n(D,\delta,\epsilon) \ ; \ (\phi,\theta) \mapsto (f^{\%}(\phi), \widehat{f}^{\%}(\theta) + (\widehat{f\phi_0})^{\%}(S^n\chi))$$

are induced by a simplicial map of simplicial abelian groups. The relative homotopy groups are the *relative twisted ϵ-quadratic Q-groups* $Q_n(f,\chi,\epsilon)$, designed to fit into a long exact sequence

$$\dots \longrightarrow Q_n(C,\gamma,\epsilon) \xrightarrow{(f,\chi)_{\%}} Q_n(D,\delta,\epsilon) \longrightarrow Q_n(f,\chi,\epsilon) \longrightarrow Q_{n-1}(C,\gamma,\epsilon) \longrightarrow \dots .$$

Proposition 2.20. *For any chain ϵ-bundle map $(f,\chi):(C,\gamma)\to(D,\delta)$ the various Q-groups fit into a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \widehat{Q}^{n+1}(C,\epsilon) & \xrightarrow{H_\gamma} & Q_n(C,\gamma,\epsilon) & \xrightarrow{N_\gamma} & Q^n(C,\epsilon) & \xrightarrow{J_\gamma} & \widehat{Q}^n(C,\epsilon) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle \widehat{f}^{\%}} & & \downarrow{\scriptstyle (f,\chi)_{\%}} & & \downarrow{\scriptstyle f^{\%}} & & \downarrow{\scriptstyle \widehat{f}^{\%}} & \\
\cdots \longrightarrow & \widehat{Q}^{n+1}(D,\epsilon) & \xrightarrow{H_\delta} & Q_n(D,\delta,\epsilon) & \xrightarrow{N_\gamma} & Q^n(D,\epsilon) & \xrightarrow{J_\delta} & \widehat{Q}^n(D,\epsilon) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \widehat{Q}^{n+1}(f) & \xrightarrow{H_\chi} & Q_n(f,\chi,\epsilon) & \xrightarrow{N_\chi} & Q^n(f,\epsilon) & \xrightarrow{J_\chi} & \widehat{Q}^n(f,\epsilon) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \widehat{Q}^{n}(C,\epsilon) & \xrightarrow{H_\gamma} & Q_{n-1}(C,\gamma,\epsilon) & \xrightarrow{N_\delta} & Q^{n-1}(C,\epsilon) & \xrightarrow{J_\gamma} & \widehat{Q}^{n-1}(C,\epsilon) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

Proof. These are the exact sequences of the homotopy groups of the simplicial abelian groups in the commutative diagram of fibration sequences

$$\begin{array}{ccccc}
K(J_\gamma) & \longrightarrow & K(W^{\%}C) & \xrightarrow{J_\gamma} & K(\widehat{W}^{\%}C) \\
\downarrow & & \downarrow{\scriptstyle f^{\%}} & & \downarrow{\scriptstyle \widehat{f}^{\%}} \\
K(J_\delta) & \longrightarrow & K(W^{\%}D) & \xrightarrow{J_\delta} & K(\widehat{W}^{\%}D) \\
\downarrow & & \downarrow & & \downarrow \\
K(J_\chi) & \longrightarrow & K(W^{\%}\mathcal{C}(f)) & \xrightarrow{J_\chi} & K(\widehat{W}^{\%}\mathcal{C}(f))
\end{array}$$

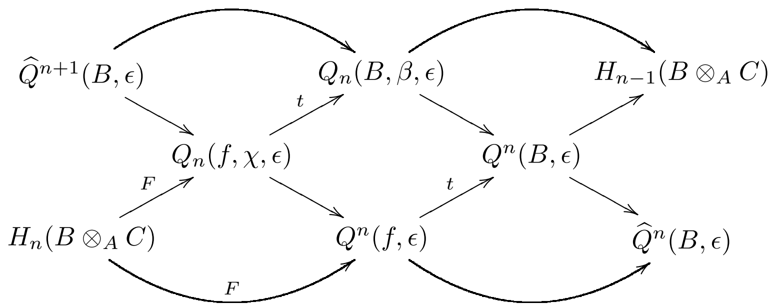
with

$$\pi_n(K(J_\chi)) \;=\; Q_n(f,\chi,\epsilon)\ .$$

□

There is also a twisted ϵ-quadratic Q-group version of the algebraic Thom constructions (1.12, 1.18, 1.24) :

Proposition 2.21. *Let $(f,\chi):(C,0)\to(D,\delta)$ be a chain ϵ-bundle map, and let $(B,\beta)=\mathcal{C}(f,\chi)$ be the cone chain ϵ-bundle (2.4). The relative twisted ϵ-quadratic Q-groups $Q_*(f,\chi,\epsilon)$ are related to the (absolute) twisted ϵ-quadratic Q-groups $Q_*(B,\beta,\epsilon)$ by a commutative braid of exact sequences*



involving the exact sequence of 1.18

$$\cdots \to H_n(B\otimes_A C) \xrightarrow{F} Q^n(f,\epsilon) \xrightarrow{t} Q^n(B,\epsilon) \to H_{n-1}(B\otimes_A C) \to \dots\ .$$

Proof. The identity

$$\widehat{f}^{*\%}(\delta) \ = \ d\chi \in (\widehat{W}C^{0-*})_0$$

determines a homotopy $\rightsquigarrow$ in the square

$$\begin{array}{ccc} K(W^{\%}C) & \xrightarrow{J} & K(\widehat{W}^{\%}C) \\ \Big\downarrow f^{\%} & \searrow & \Big\downarrow \widehat{f}^{\%} \\ K(W^{\%}D) & \xrightarrow{J_\delta} & K(\widehat{W}^{\%}D) \end{array}$$

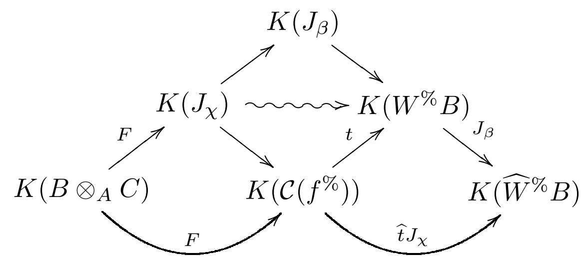
(with $J=J_0$) and hence maps of the mapping fibres

$$J_\chi \ : \ K(\mathcal{C}(f^{\%})) \to K(\mathcal{C}(\widehat{f}^{\%}))\ ,\ (f,\chi)_{\%} \ : \ K(J) \to K(J_\delta)\ .$$

The map J_χ is related to $J_\beta : K(W^{\%}B) \to K(\widehat{W}^{\%}B)$ by a homotopy commutative diagram

$$\begin{array}{ccc} K(\mathcal{C}(f^{\%})) & \xrightarrow{J_\chi} & K(\mathcal{C}(\widehat{f}^{\%})) \\ \Big\downarrow t & \searrow & \simeq \Big\downarrow \widehat{t} \\ K(W^{\%}B) & \xrightarrow{J_\beta} & K(\widehat{W}^{\%}B) \end{array}$$

with $\widehat{t} : K(\mathcal{C}(\widehat{f}^{\%})) \simeq K(\widehat{W}^{\%}B)$ a simplicial homotopy equivalence inducing the algebraic Thom isomorphisms $\widehat{t} : \widehat{Q}^*(f,\epsilon) \cong \widehat{Q}^*(B,\epsilon)$ of Proposition 1.12, and $t : K(\mathcal{C}(f^{\%})) \to K(W^{\%}B)$ a simplicial map inducing the algebraic Thom maps $t : Q^*(f,\epsilon) \to Q^*(B,\epsilon)$ of Proposition 1.18, with mapping fibre $K(t) \simeq K(B\otimes_A C)$. The braid in the statement is the commutative braid of homotopy groups induced by the homotopy commutative braid of fibrations



$\square$

Proposition 2.22. *Let (C, γ) be a chain ϵ-bundle over a f.g. projective A-module chain complex which is concentrated in degrees $k, k+1$*

$$C \ : \ \cdots \to 0 \to C_{k+1} \xrightarrow{\ d\ } C_k \to 0 \to \dots \ ,$$

so that (C, γ) can be taken (up to equivalence) to be the cone $\mathcal{C}(d, \chi)$ of a chain ϵ-bundle map $(d, \chi) : (C_{k+1}, 0) \to (C_k, \delta)$ (2.6), regarding C_k, C_{k+1} as chain complexes concentrated in degree k. The relative twisted ϵ-quadratic Q-groups $Q_*(d, \chi, \epsilon)$ and the absolute twisted ϵ-quadratic Q-groups $Q_*(C, \gamma, \epsilon)$ are given as follows.
(i) *For $n \neq 2k-1, 2k, 2k+1, 2k+2$*

$$Q_n(C, \gamma, \epsilon) \ = \ Q_n(d, \chi, \epsilon) \ = \ Q_n(C, \epsilon) \ = \ \begin{cases} \widehat{Q}^{n+1}(C, \epsilon) & \text{if } n \geqslant 2k+3 \\ 0 & \text{if } n \leqslant 2k-2 \end{cases}$$

with

$$\widehat{Q}^{n+1}(C, \epsilon) \ = \ \frac{\{(\phi, \theta) \in S(C^{k+1}) \oplus S(C^k) \,|\, \phi = (-1)^{n+k}\epsilon\phi^*, d\phi d^* = \theta + (-1)^{n+k}\epsilon\theta^*\}}{\{(\sigma + (-1)^{n+k}\epsilon\sigma^*, d\sigma d^* + \tau + (-1)^{n+k+1}\epsilon\tau^* \,|\, (\sigma, \tau) \in S(C^{k+1}) \oplus S(C^k)\}}$$

as given by Proposition 1.13.
(ii) *For $n = 2k-1, 2k, 2k+1, 2k+2$ the relative twisted ϵ-quadratic Q-groups are given by*

$$Q_n(d, \chi, \epsilon) \ =$$

$$\begin{cases} \dfrac{\{(\phi, \theta) \in S(C^{k+1}) \oplus S(C^k) \,|\, \phi = (-1)^k\epsilon\phi^*, d\phi d^* = \theta + (-1)^k\epsilon\theta^*\}}{\{(\sigma + (-1)^k\epsilon\sigma^*, d\sigma d^* + \tau + (-1)^{k+1}\epsilon\tau^* \,|\, (\sigma, \tau) \in S(C^{k+1}) \oplus S(C^k)\}} & \text{if } n = 2k+2 \\ \dfrac{\{(\psi, \eta) \in S(C^{k+1}) \oplus S(C^k) \,|\, (d, \chi)_\%(\psi) = (0, \eta + (-1)^{k+1}\epsilon\eta^*)\}}{\{(\sigma + (-1)^{k+1}\epsilon\sigma^*, d\sigma d^* + \tau + (-1)^k\epsilon\tau^*) \,|\, (\sigma, \tau) \in S(C^{k+1}) \oplus S(C^k)\}} & \text{if } n = 2k+1 \\ \operatorname{coker}((d, \chi)_\% : Q_{2k}(C_{k+1}) \to Q_{2k}(C_k, \delta)) & \text{if } n = 2k \\ Q_{2k-1}(C_k, \delta, \epsilon) & \text{if } n = 2k-1 \end{cases}$$

with

$$Q_{2k}(C_k,\delta,\epsilon) \;=\; \frac{\{(\phi,\theta)\in S(C^k)\oplus S(C^k)\,|\,\phi=(-1)^k\epsilon\phi^*,\phi-\phi\delta\phi^*=\theta+(-1)^k\epsilon\theta^*\}}{\{(0,\eta+(-1)^{k+1}\epsilon\eta^*)\,|\,\eta\in S(C^k)\}}\ ,$$

$$Q_{2k-1}(C_k,\delta,\epsilon) \;=\; \frac{\{\sigma\in S(C^k)\,|\,\sigma=(-1)^k\epsilon\sigma^*\}}{\{\phi-\phi\delta\phi^*-(\theta+(-1)^k\epsilon\theta^*)\,|\,\phi=(-1)^k\epsilon\phi^*,\theta\in S(C^k)\}}\ ,$$

$$(d,\chi)_\% \;:\; Q_{2k}(C_{k+1},\epsilon) \;=\; H_0(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon)\to Q_{2k}(C_k,\delta,\epsilon)\ ;$$
$$\psi\mapsto(d(\psi+(-1)^k\epsilon\psi^*)d^*,d\psi d^*-d(\psi+(-1)^k\epsilon\psi^*)\chi(\psi^*+(-1)^k\epsilon\psi)d^*)\ .$$

The absolute twisted quadratic Q-groups are such that

$$Q_{2k-1}(C,\gamma,\epsilon) \;=\; Q_{2k-1}(d,\chi,\epsilon) \;=\; Q_{2k-1}(C_k,\delta,\epsilon)$$

and there is defined an exact sequence

$$\begin{array}{l} 0\to Q_{2k+2}(d,\chi,\epsilon)\xrightarrow{\ t\ }Q_{2k+2}(C,\gamma,\epsilon)\\ \quad\to H_{k+1}(C)\otimes_A C_{k+1}\xrightarrow{\ F\ }Q_{2k+1}(d,\chi,\epsilon)\xrightarrow{\ t\ }Q_{2k+1}(C,\gamma,\epsilon)\\ \quad\to H_k(C)\otimes_A C_{k+1}\xrightarrow{\ F\ }Q_{2k}(d,\chi,\epsilon)\xrightarrow{\ t\ }Q_{2k}(C,\gamma,\epsilon)\to 0\end{array}$$

with

$$F \;:\; H_k(C)\otimes_A C_{k+1} \;=\; \mathrm{coker}(d^*:\mathrm{Hom}_A(C^{k+1},C_{k+1})\to\mathrm{Hom}_A(C^k,C_{k+1}))$$
$$\to Q_{2k}(d,\chi)\ ;\ \lambda\mapsto(\lambda d^*+(-1)^k\epsilon d\lambda^*-d\lambda^*\delta\lambda d^*,$$
$$\lambda d^*-\lambda\chi\lambda^*-d\lambda^*\delta\lambda\chi\lambda^*\delta\lambda d^*-d\lambda^*\delta(\lambda d^*+(-1)^k\epsilon d\lambda^*)-(\lambda d^*+(-1)^k\epsilon d\lambda^*)\delta d\lambda^*\delta\lambda d^*)\ .$$

Proof. The absolute and relative twisted ϵ-quadratic Q-groups are related by the exact sequence of 2.21

$$\cdots\to Q_n(d,\chi,\epsilon)\xrightarrow{\ t\ }Q_n(C,\gamma,\epsilon)\to H_{n-k-1}(C)\otimes_A C_{k+1}\xrightarrow{\ F\ }Q_{n-1}(d,\chi,\epsilon)\to\dots\ .$$

The twisted ϵ-quadratic Q-groups of $(C_{k+1},0)$ are given by Proposition 1.22

$$\begin{array}{rcl} Q_n(C_{k+1},0,\epsilon) \;=\; Q_n(C_{k+1},\epsilon) &=& H_{n-2k}(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon)\\ &=& \begin{cases}\widehat{H}^{n-2k+1}(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon) & \text{if } n\geqslant 2k+1\\ H_0(\mathbb{Z}_2;S(C^{k+1}),(-1)^kT_\epsilon) & \text{if } n=2k\\ 0 & \text{if } n\leqslant 2k-1\ .\end{cases}\end{array}$$

The twisted ϵ-quadratic Q-groups of (C_k,δ) are given by Proposition 2.17

$$Q_n(C_k,\delta,\epsilon)$$
$$=\begin{cases}\widehat{H}^{n-2k+1}(\mathbb{Z}_2;S(C^k),(-1)^kT_\epsilon) & \text{if } n\geqslant 2k+1\\ \dfrac{\{(\phi,\theta)\in S(C^k)\oplus S(C^k)\,|\,\phi=(-1)^k\epsilon\phi^*,\phi-\phi\delta\phi^*=\theta+(-1)^k\epsilon\theta^*\}}{\{(0,\eta+(-1)^{k+1}\epsilon\eta^*)\,|\,\eta\in S(C^k)\}} & \text{if } n=2k\\ \dfrac{\{\sigma\in S(C^k)\,|\,\sigma=(-1)^k\epsilon\sigma^*\}}{\{\phi-\phi\delta\phi^*-(\theta+(-1)^k\epsilon\theta^*)\,|\,\phi=(-1)^k\epsilon\phi^*,\theta\in S(C^k)\}} & \text{if } n=2k-1\\ 0 & \text{if } n\leqslant 2k-2\ .\end{cases}$$

The twisted ϵ-quadratic Q-groups of (d,χ) fit into the exact sequence

$$\dots\longrightarrow Q_n(C_{k+1},\epsilon)\xrightarrow{(d,\chi)_\%}Q_n(C_k,\delta,\epsilon)\longrightarrow Q_n(d,\chi,\epsilon)\longrightarrow Q_{n-1}(C_{k+1},\epsilon)\longrightarrow\dots$$

giving the expressions in the statements of (i) and (ii). □

2.5. **The Computation of** $Q_*(C(X),\gamma(X))$**.** In this we compute the twisted quadratic Q-groups $Q_*(C(X),\gamma(X))$ of the following chain bundles over an even commutative ring A.

Definition 2.23. For $X\in \mathrm{Sym}_r(A)$ let

$$(C(X),\gamma(X)) \;=\; \mathcal{C}(d,\chi)$$

be the cone of the chain bundle map over A

$$(d,\chi)\;:\;(C(X)_1,0)\to (C(X)_0,\delta)$$

defined by

$$\begin{aligned}
d \;&=\; 2 \;:\; C(X)_1 \;=\; A^r \to C(X)_0 \;=\; A^r\ ,\\
\delta \;&=\; X \;:\; C(X)_0 \;=\; A^r \to C(X)^0 \;=\; A^r\ ,\\
\chi \;&=\; 2X \;:\; C(X)_1 \;=\; A^r \to C(X)^1 \;=\; A^r\ .
\end{aligned}$$

□

By Proposition 2.6 every chain bundle (C,γ) with $C_1=A^r \xrightarrow{\;2\;} C_0=A^r$ is of the form $(C(X),\gamma(X))$ for some $X=(x_{ij})\in \mathrm{Sym}_r(A)$, with the equivalence class given by

$$\begin{aligned}
\gamma \;=\; \gamma(X) \;=\; X \;&=\; (x_{11},x_{22},\dots,x_{rr})\\
&\in \widehat{Q}^0(C(X)^{-*}) \;=\; \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \;=\; \bigoplus_r \widehat{H}^0(\mathbb{Z}_2;A)\ (1.14)\ .
\end{aligned}$$

The 0th Wu class of $(C(X),\gamma(X))$ is the A-module morphism

$$\begin{aligned}
\widehat{v}_0(\gamma(X))\;:\;\;& H_0(C(X)) \;=\; (A_2)^r \to \widehat{H}^0(\mathbb{Z}_2;A)\ ;\\
& a \;=\; (a_1,a_2,\dots,a_r)\mapsto aXa^t \;=\; \sum_{i=1}^{r} a_i x_{ij} a_j \;=\; \sum_{i=1}^{r}(a_i)^2 x_{ii}\ .
\end{aligned}$$

In Theorem 2.30 below the universal chain bundle (B^A,β^A) of a commutative even ring A with $\widehat{H}^0(\mathbb{Z}_2;A)$ a f.g. free A_2-module will be constructed from $(C(X),\gamma(X))$ for a diagonal $X\in \mathrm{Sym}_r(A)$ with $\widehat{v}_0(\gamma(X))$ an isomorphism, and the twisted quadratic Q-groups $Q_*(B^A,\beta^A)$ will be computed using the following computation of $Q_*(C(X),\gamma(X))$ (which holds for arbitrary X).

Theorem 2.24. *Let A be an even commutative ring, and let $X \in \mathrm{Sym}_r(A)$.*
(i) *The twisted quadratic Q-groups of $(C(X),\gamma(X))$ are given by*

$$Q_n(C(X),\gamma(X)) \;=\; \begin{cases} 0 & \text{if } n \leqslant -2 \\ \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)+\{M-MXM \,|\, M\in \mathrm{Sym}_r(A)\}} & \text{if } n=-1 \\ \dfrac{\{M\in \mathrm{Sym}_r(A) \,|\, M-MXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A)+\{2(N+N^t)-4N^tXN \,|\, N\in M_r(A)\}} & \text{if } n=0 \\ \dfrac{\{N\in M_r(A) \,|\, N+N^t\in 2\mathrm{Sym}_r(A),\, \frac{1}{2}(N+N^t)-N^tXN\in \mathrm{Quad}_r(A)\}}{2M_r(A)} & \text{if } n=1 \\ \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} & \text{if } n\geqslant 2\ . \end{cases}$$

(ii) *The boundary maps $\partial : Q_n(C(X),\gamma(X)) \to L_{n-1}(A)$ are given by*

$$\begin{aligned} \partial \;&:\; Q_{-1}(C(X),\gamma(X)) \to L_{-2}(A)\ ;\ M \mapsto (A^r\oplus (A^r)^*, \begin{pmatrix} M & 1 \\ 0 & X\end{pmatrix})\ , \\ \partial \;&:\; Q_0(C(X),\gamma(X)) \to L_{-1}(A)\ ;\ M \mapsto (H_-(A^r); A^r, \mathrm{im}(\begin{pmatrix} 1-XM \\ M\end{pmatrix} : A^r \to A^r\oplus (A^r)^*))\ , \\ \partial \;&:\; Q_1(C(X),\gamma(X)) \to L_0(A)\ ;\ N \mapsto (A^r\oplus A^r, \begin{pmatrix} \frac{1}{4}(N+N^t-2N^tXN) & 1-2NX \\ 0 & -2X\end{pmatrix})\ . \end{aligned}$$

(iii) *The twisted quadratic Q-groups of the chain bundles*

$$(B(i),\beta(i)) \;=\; (C(X),\gamma(X))_{*+2i}\ (i\in \mathbb{Z})$$

are just the twisted quadratic Q-groups of $(C(X),\gamma(X))$ with a dimension shift

$$Q_n(B(i),\beta(i)) \;=\; Q_{n-4i}(C(X),\gamma(X))\ .$$

Proof. (i) Proposition 2.22 (i) and Example 1.14 (ii) give

$$Q_n(C(X),\gamma(X)) \;=\; \begin{cases} 0 & \text{if } n\leqslant -2 \\ \widehat{Q}^{n+1}(C(X)) = \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} & \text{if } n\geqslant 3\ . \end{cases}$$

For $-1 \leqslant n \leqslant 2$ Examples 1.14, 1.20, 2.19 and Proposition 2.22 (ii) show that the commutative diagram with exact rows and columns

$$\begin{CD}
Q^1(C(X)_1) @>J>> \widehat{Q}^1(C(X)_1) @>H>> Q_0(C(X)_1) @>1+T>> Q^0(C(X)_1) @>J>> \widehat{Q}^0(C(X)_1) \\
@VV{d^{\%}}V @VV{\widehat{d}^{\%}}V @VV{(d,\chi)_{\%}}V @VV{d^{\%}}V @VV{\widehat{d}^{\%}}V \\
Q^1(C(X)_0) @>J_\delta>> \widehat{Q}^1(C(X)_0) @>H_\delta>> Q_0(C(X)_0,\delta) @>N_\delta>> Q^0(C(X)_0) @>J_\delta>> \widehat{Q}^0(C(X)_0) \\
@VVV @VVV @VVV @VVV @VVV \\
Q^1(d) @>J_\chi>> \widehat{Q}^1(d) @>H_\chi>> Q_0(d,\chi) @>N_\chi>> Q^0(d) @>J_\chi>> \widehat{Q}^0(d) \\
@VVV @VVV @VVV @VVV @VVV \\
Q^0(C(X)_1) @>J>> \widehat{Q}^0(C(X)_1) @>H>> Q_{-1}(C(X)_1) @>1+T>> Q^{-1}(C(X)_1) @>J>> \widehat{Q}^{-1}(C(X)_1)
\end{CD}$$

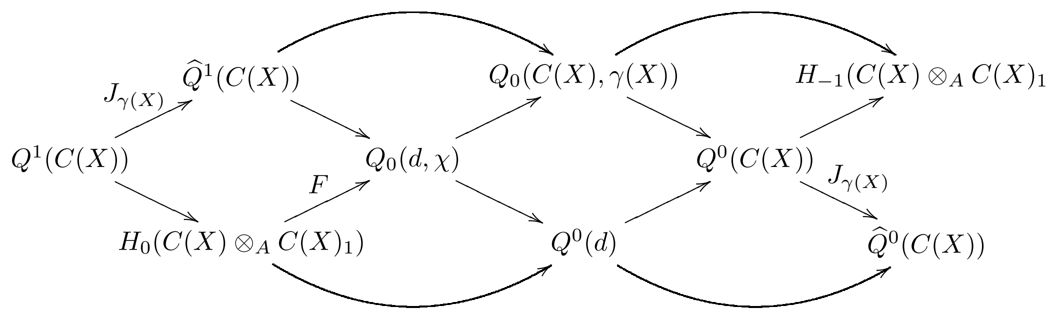
is given by

$$\begin{CD}
0 @>>> 0 @>>> \mathrm{Quad}_r(A) @>>> \mathrm{Sym}_r(A) @>>> \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \\
@VVV @VVV @VV{4}V @VV{4}V @VV{0}V \\
0 @>>> 0 @>>> Q_0(C(X)_0,\delta) @>>> \mathrm{Sym}_r(A) @>J_\delta>> \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \\
@VVV @VVV @VVV @VVV @VV{1}V \\
0 @>>> \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} @>4>> Q_0(d,\chi) @>>> \dfrac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)} @>>> \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \\
@VVV @VV{1}V @VVV @VVV @VVV \\
\mathrm{Sym}_r(A) @>>> \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} @>>> 0 @>>> 0 @>>> 0
\end{CD}$$

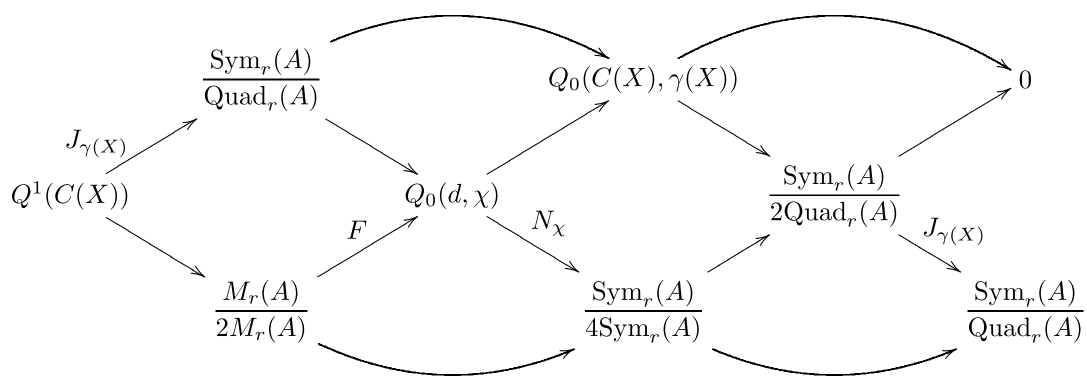
with

$$\begin{aligned}
J_\delta \ &: \ \mathrm{Sym}_r(A) \to \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \ ; \ M \mapsto M - MXM \ , \\
Q_0(C(X)_0,\delta) \ &= \ \ker(J_\delta) \ = \ \{M \in \mathrm{Sym}_r(A) \,|\, M - MXM \in \mathrm{Quad}_r(A)\} \ , \\
Q_0(d,\chi) \ &= \ \mathrm{coker}((d,\chi)_{\%} : Q_0(C(X)_1) \to Q_0(C(X)_0,\delta)) \\
&= \ \frac{\{M \in \mathrm{Sym}_r(A) \,|\, M - MXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A)} \ , \\
N_\chi \ &: \ Q_0(d,\chi) \to Q^0(d) \ = \ \frac{\mathrm{Sym}_r(A)}{4\mathrm{Sym}_r(A)} \ ; \ M \mapsto M \ .
\end{aligned}$$

Furthermore, the commutative braid of exact sequences



is given by



with

$$\frac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)} \;\cong\; Q^0(C(X))\ ;\ M \mapsto \phi \ (\text{where } \phi_0 = M : C^0 \to C(X)_0)\ ,$$

$$J_{\gamma(X)} \;:\; \frac{\mathrm{Sym}_r(A)}{2\mathrm{Quad}_r(A)} \to \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}\ ;\ M \mapsto M - MXM\ ,$$

$$F \;:\; H_0(C(X)\otimes_A C(X)_1) \;=\; \frac{M_r(A)}{2M_r(A)} \to Q_0(d,\chi)\ ;\ N \mapsto 2(N+N^t) - 4N^tXN\ ,$$

$$\begin{aligned} Q^1(C(X)) \;&=\; \ker(N_\chi F : H_0(C(X)\otimes_A C(X)_1) \to Q^0(d)) \\ &=\; \frac{\{N \in M_r(A)\,|\,N+N^t \in 2\mathrm{Sym}_r(A)\}}{2M_r(A)} \\ &\qquad (\text{where } \phi \in Q^1(C(X)) \text{ corresponds to } N = \phi_0 \in M_r(A))\ , \end{aligned}$$

$$J_{\gamma(X)} \;:\; Q^1(C(X)) \to \widehat{Q}^1(C(X)) \;=\; \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}\ ;\ N \mapsto \frac{1}{2}(N+N^t) - N^tXN\ .$$

It follows that

$$\begin{aligned} Q_0(C(X),\gamma(X)) &= \operatorname{coker}(F : \frac{M_r(A)}{2M_r(A)} \to Q_0(d,\chi)) \\ &= \frac{\{M\in \mathrm{Sym}_r(A)\,|\, M - MXM \in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A) + \{2(N+N^t) - 4N^tXN \,|\, N \in M_r(A)\}}\ , \\ Q_{-1}(C(X),\gamma(X)) &= Q_{-1}(d,\chi) \\ &= \operatorname{coker}(J_{\gamma(X)} : Q^0(C(X)) \to \widehat{Q}^0(C(X))) \\ &= \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A) + \{M - MXM \,|\, M \in \mathrm{Sym}_r(A)\}} \end{aligned}$$

with

$$\begin{aligned} \widehat{Q}^1(C(X)) &= \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \to Q_0(C(X),\gamma(X))\ ;\ M \mapsto 4M\ , \\ \widehat{Q}^0(C(X)) &= \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \to Q_{-1}(C(X),\gamma(X))\ ;\ M \mapsto M\ . \end{aligned}$$

Also

$$\begin{aligned} (d,\chi)_{\%} = 0 \ :\ Q_2(d,\chi) &= Q_1(C(X)_1) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)} \\ &\to Q_1(C(X)_0,\delta) = \widehat{Q}^2(C(X)_0) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}\ , \\ Q^1(C(X)) &= \ker(N_\chi F : H_0(C(X)\otimes_A C(X)_1) \to Q^0(d)) \\ &= \frac{\{N\in M_r(A)\,|\, N+N^t \in 2\mathrm{Sym}_r(A)\}}{2M_r(A)}\ , \\ J_{\gamma(X)}\ :\ Q^1(C(X)) \to \widehat{Q}^1(C(X)) &= \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}\ ;\ N \mapsto \frac{1}{2}(N+N^t) - N^tXN\ , \\ Q_1(C(X),\gamma(X)) &= \ker(J_{\gamma(X)} : Q^1(C(X)) \to \widehat{Q}^1(C(X))) \\ &= \frac{\{N\in M_r(A)\,|\, N+N^t \in 2\mathrm{Sym}_r(A), \frac{1}{2}(N+N^t) - N^tXN \in \mathrm{Quad}_r(A)\}}{2M_r(A)}\ , \\ Q_2(C(X),\gamma(X)) &= Q_2(d,\chi) = \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}\ . \end{aligned}$$

(ii) The expressions for $\partial : Q_n(C(X),\gamma(X)) \to L_{n-1}(A)$ are given by the boundary construction of Proposition 2.13 and its expression in terms of forms and formations (2.14, 2.15). The form in the case $n=-1$ (resp. the formation in the case $n=0$) is given by 2.15 (resp. 2.14) applied to the n-dimensional symmetric structure $(\phi,\theta)\in Q_n(C(X),\gamma(X))$ corresponding to $M\in \mathrm{Sym}_r(A)$. For $n=1$ the boundary of the 1-dimensional symmetric structure $(\phi,\theta)\in Q_1(C(X),\gamma(X))$ corresponding to $N\in M_r(A)$ with

$$N+N^t \in 2\mathrm{Sym}_r(A)\ ,\ \frac{1}{2}(N+N^t) - N^tXN \in \mathrm{Quad}_r(A)$$

is a 0-dimensional quadratic Poincaré complex (C,ψ) with

$$C = \mathcal{C}(N : C(X)^{1-*} \to C(X))_{*+1}\ .$$

The instant surgery obstruction (2.15) is the nonsingular quadratic form

$$I(C,\psi) \;=\; (\mathrm{coker}(\begin{pmatrix} -2 \\ N^t \\ 1+2XN^t \end{pmatrix} : A^r \to A^r \oplus A^r \oplus A^r), \begin{pmatrix} \frac{1}{4}(N+N^t-2NXN^t) & 1 & N \\ 0 & -2X & 2 \\ 0 & 0 & 0 \end{pmatrix})$$

such that there is defined an isomorphism

$$\begin{pmatrix} 1 & -4X & 2 \\ N^t & 1-2N^tX & N^t \end{pmatrix} \;:\; I(C,\psi) \to (A^r \oplus A^r, \begin{pmatrix} \frac{1}{4}(N+N^t-2N^tXN) & 1-2NX \\ 0 & -2X \end{pmatrix}) \;.$$

(iii) The even multiple skew-suspension isomorphisms of the symmetric Q-groups

$$\overline{S}^{2i} \;:\; Q^{n-4i}(C(X)_{*+2i}) \xrightarrow{\;\cong\;} Q^n(C(X)) \;;\; \{\phi_s \,|\, s \geqslant 0\} \mapsto \{\phi_s \,|\, s \geqslant 0\} \; (i \in \mathbb{Z})$$

are defined also for the hyperquadratic, quadratic and twisted quadratic Q-groups. □

2.6. **The Universal Chain Bundle.** For any A-module chain complexes B, C the additive group $H_0(\mathrm{Hom}_A(C,B))$ consists of the chain homotopy classes of A-module chain maps $f : C \to B$. For a chain ϵ-bundle (B,β) there is thus defined a morphism

$$H_0(\mathrm{Hom}_A(C,B)) \to \widehat{Q}^0(C^{0-*},\epsilon) \;;\; (f : C \to B) \mapsto \widehat{f}^*(\beta) \;.$$

Proposition 2.25. (Weiss [20]) (i) *For every ring with involution A and $\epsilon = \pm 1$ there exists a universal chain ϵ-bundle $(B^{A,\epsilon}, \beta^{A,\epsilon})$ over A such that for any finite f.g. projective A-module chain complex C the morphism*

$$H_0(\mathrm{Hom}_A(C, B^{A,\epsilon})) \to \widehat{Q}^0(C^{0-*},\epsilon) \;;\; (f : C \to B^{A,\epsilon}) \mapsto \widehat{f}^*(\beta^{A,\epsilon})$$

is an isomorphism. Thus every chain ϵ-bundle (C,γ) is classified by a chain ϵ-bundle map

$$(f,\chi) \;:\; (C,\gamma) \to (B^{A,\epsilon}, \beta^{A,\epsilon}) \;.$$

(ii) *The universal chain ϵ-bundle $(B^{A,\epsilon}, \beta^{A,\epsilon})$ is characterized (uniquely up to equivalence) by the property that its Wu classes are A-module isomorphisms*

$$\widehat{v}_k(\beta^{A,\epsilon}) \;:\; H_k(B^{A,\epsilon}) \xrightarrow{\;\cong\;} \widehat{H}^k(\mathbb{Z}_2; A, \epsilon) \; (k \in \mathbb{Z}) \;.$$

(iii) *An n-dimensional (ϵ-symmetric, ϵ-quadratic) Poincaré pair over A has a canonical universal ϵ-bundle $(B^{A,\epsilon}, \beta^{A,\epsilon})$-structure.*
(iv) *The 4-periodic $(B^{A,\epsilon}, \beta^{A,\epsilon})$-structure L-groups are the 4-periodic versions of the ϵ-symmetric and ϵ-hyperquadratic L-groups of A*

$$L\langle B^{A,\epsilon}, \beta^{A,\epsilon}\rangle^{n+4*}(A,\epsilon) \;=\; L^{n+4*}(A,\epsilon) \;,$$
$$\widehat{L}\langle B^{A,\epsilon}, \beta^{A,\epsilon}\rangle^{n+4*}(A,\epsilon) \;=\; \widehat{L}^{n+4*}(A,\epsilon) \;.$$

(v) *The twisted ϵ-quadratic Q-groups of $(B^{A,\epsilon}, \beta^{A,\epsilon})$ fit into an exact sequence*

$$\dots \to L_n(A,\epsilon) \xrightarrow{1+T_\epsilon} L^{n+4*}(A,\epsilon) \to Q_n(B^{A,\epsilon}, \beta^{A,\epsilon}, \epsilon) \xrightarrow{\partial} L_{n-1}(A,\epsilon) \to \dots$$

with

$$\partial \;:\; Q_n(B^{A,\epsilon}, \beta^{A,\epsilon}, \epsilon) \to L_{n-1}(A,\epsilon) \;;\; (\phi,\theta) \mapsto (C,\psi)$$

given by the construction of Proposition 2.12 (ii), with

$$C \;=\; \mathcal{C}(\phi_0 : (B^{A,\epsilon})^{n-*} \to B^{A,\epsilon})_{*+1} \; etc.$$

□

For $\epsilon = 1$ write

$$(B^{A,1}, \beta^{A,1}) \;=\; (B^A, \beta^A)$$

and note that

$$(B^{A,-1}, \beta^{A,-1}) \;=\; (B^A, \beta^A)_{*-1} \; .$$

In general, the chain A-modules $B^{A,\epsilon}$ are not finitely generated, although $B^{A,\epsilon}$ is a direct limit of f.g. free A-module chain complexes. In our applications the involution on A will satisfy the following conditions :

Proposition 2.26. (Connolly and Ranicki [10, Section 3.6])
Let A be a ring with an even involution such that $\widehat{H}^0(\mathbb{Z}_2; A)$ has a 1-dimensional f.g. projective A-module resolution

$$0 \to C_1 \xrightarrow{\;d\;} C_0 \xrightarrow{\;x\;} \widehat{H}^0(\mathbb{Z}_2; A) \to 0 \; .$$

Let $(C, \gamma) = \mathcal{C}(d, \chi)$ be the cone of a chain bundle map $(d, \chi) : (C_1, 0) \to (C_0, \delta)$ with

$$\widehat{v}_0(\delta) \;=\; x \;:\; C_0 \to \widehat{H}^0(\mathbb{Z}_2; A)$$

and set

$$(B^A(i), \beta^A(i)) \;=\; (C, \gamma)_{*+2i} \;\; (i \in \mathbb{Z}) \; .$$

(i) *The chain bundle over A*

$$(B^A, \beta^A) \;=\; \bigoplus_i (B^A(i), \beta^A(i))$$

is universal.
(ii) *The twisted quadratic Q-groups of (B^A, β^A) are given by*

$$Q_n(B^A, \beta^A) \;=\; \begin{cases} Q_0(C, \gamma) & \text{if } n \equiv 0 (\operatorname{mod} 4) \\ \ker(J_\gamma : Q^1(C) \to \widehat{Q}^1(C)) & \text{if } n \equiv 1 (\operatorname{mod} 4) \\ 0 & \text{if } n \equiv 2 (\operatorname{mod} 4) \\ Q_{-1}(C, \gamma) & \text{if } n \equiv 3 (\operatorname{mod} 4) \; . \end{cases}$$

The projection $(B^A, \beta^A) \to (B^A(2j), \beta^A(2j))$ induces isomorphisms

$$Q_n(B^A, \beta^A) \;\cong\; \begin{cases} Q_n(B^A(2j), \beta^A(2j)) & \text{if } n = 4j, 4j-1 \\ \ker(J_{\beta^A(2j)} : Q^n(B^A(2j)) \to \widehat{Q}^n(B^A(2j))) & \text{if } n = 4j+1 \; . \end{cases}$$

Proof. (i) The Wu classes of the chain bundle $(C, \gamma)_{*+2i}$ are isomorphisms

$$\widehat{v}_k(\gamma) \;:\; H_k(C_{*+2i}) \xrightarrow{\;\cong\;} \widehat{H}^k(\mathbb{Z}_2; A)$$

for $k = 2i, 2i+1$.
(ii) See [10] for the detailed analysis of the exact sequence of 2.8 (ii)

$$\cdots \to \sum_{i=-\infty}^{\infty} Q_n(B^A(i), \beta^A(i)) \to Q_n(B^A, \beta^A) \to \sum_{i<j} H_n(B^A(i) \otimes_A B^A(j)) \\ \to \sum_{i=-\infty}^{\infty} Q_{n-1}(B^A(i), \beta^A(i)) \to \dots \; .$$

□

As in the Introduction :

Definition 2.27. A ring with involution A is *r-even* for some $r \geqslant 1$ if

(i) A is commutative, with the identity involution,
(ii) $2 \in A$ is a non-zero divisor,
(iii) $\widehat{H}^0(\mathbb{Z}_2; A)$ is a f.g. free A_2-module of rank r with a basis $\{x_1 = 1, x_2, \dots, x_r\}$.

□

Example 2.28. $\mathbb{Z}$ is 1-even. □

Proposition 2.29. *If A is 1-even the polynomial extension $A[x]$ is 2-even, with $A[x]_2 = A_2[x]$ and $\{1, x\}$ an $A_2[x]$-module basis of $\widehat{H}^0(\mathbb{Z}_2; A[x])$.*

Proof. For any $a = \sum\limits_{i=0}^{\infty} a_i x^i \in A[x]$

$$\begin{aligned} a^2 &= \sum_{i=0}^{\infty} (a_i)^2 x^{2i} + 2 \sum_{0 \leqslant i < j < \infty} a_i a_j x^{i+j} \\ &= \sum_{i=0}^{\infty} a_i x^{2i} \in A_2[x] \ . \end{aligned}$$

The $A_2[x]$-module morphism

$$A_2[x] \oplus A_2[x] \to \widehat{H}^0(\mathbb{Z}_2; A[x]) \ ; \ (p, q) \mapsto p^2 + q^2 x$$

is thus an isomorphism, with inverse

$$\widehat{H}^0(\mathbb{Z}_2; A[x]) \xrightarrow{\cong} A_2[x] \oplus A_2[x] \ ; \ a \ = \ \sum_{i=0}^{\infty} a_i x^i \mapsto (\sum_{j=0}^{\infty} a_{2j} x^j, \sum_{j=0}^{\infty} a_{2j+1} x^j) \ .$$

□

Proposition 2.29 is the special case $k = 1$ of a general result: if A is 1-even and $t_1, t_2, \dots, t_k$ are commuting indeterminates over A then the polynomial ring $A[t_1, t_2, \dots, t_k]$ is 2^k-even with

$$\{x_1 = 1, x_2, x_3, \dots, x_{2^k}\} \ = \ \{(t_1)^{i_1} (t_2)^{i_2} \dots (t_k)^{i_k} \,|\, i_j = 0 \text{ or } 1, 1 \leqslant j \leqslant k\}$$

an $A_2[t_1, t_2, \dots, t_k]$-module basis of $\widehat{H}^0(\mathbb{Z}_2; A[t_1, t_2, \dots, t_k])$.

We can now prove Theorem 0.3 :

Theorem 2.30. *Let A be an r-even ring with involution.*
(i) *The A-module morphism*

$$x \ : \ A^r \to \widehat{H}^0(\mathbb{Z}_2; A) \ ; \ (a_1, a_2, \dots, a_r) \mapsto \sum_{i=1}^{r} (a_i)^2 x_i$$

fits into a 1-dimensional f.g. free A-module resolution of $\widehat{H}^0(\mathbb{Z}_2; A)$

$$0 \to C_1 \ = \ A^r \xrightarrow{\ 2\ } C_0 \ = \ A^r \xrightarrow{\ x\ } \widehat{H}^0(\mathbb{Z}_2; A) \to 0 \ .$$

The symmetric and hyperquadratic L-groups of A are 4-periodic

$$L^n(A) \ = \ L^{n+4}(A) \ , \ \widehat{L}^n(A) \ = \ \widehat{L}^{n+4}(A) \ .$$

(ii) *Let* $(C(X),\gamma(X))$ *be the chain bundle over* A *given by the construction of* (2.23) *for*

$$X \;=\; \begin{pmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_r \end{pmatrix} \in \mathrm{Sym}_r(A)\ ,$$

with $C(X) = \mathcal{C}(2 : A^r \to A^r)$. *The chain bundle over* A *defined by*

$$(B^A,\beta^A) \;=\; \bigoplus_i (C(X),\gamma(X))_{*+2i} \;=\; \bigoplus_i (B^A(i),\beta^A(i))$$

is universal. The hyperquadratic L*-groups of* A *are given by*

$$\widehat{L}^n(A) \;=\; Q_n(B^A,\beta^A) \;=\;$$

$$\begin{cases} Q_0(C(X),\gamma(X)) \;=\; \dfrac{\{M\in \mathrm{Sym}_r(A)\,|\,M-MXM\in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A)+\{2(N+N^t)-N^tXN\,|\,N\in M_r(A)\}} & \textit{if } n=0 \\ \mathrm{im}(N_{\gamma(X)}:Q_1(C(X),\gamma(X))\to Q^1(C(X))) \;=\; \ker(J_{\gamma(X)}:Q^1(C(X))\to \widehat{Q}^1(C(X))) & \\ = \;\dfrac{\{N\in M_r(A)\,|\,N+N^t\in 2\mathrm{Sym}_r(A),\,\frac{1}{2}(N+N^t)-N^tXN\in \mathrm{Quad}_r(A)\}}{2M_r(A)} & \textit{if } n=1 \\ 0 & \textit{if } n=2 \\ Q_{-1}(C(X),\gamma(X)) \;=\; \dfrac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)+\{L-LXL\,|\,L\in \mathrm{Sym}_r(A)\}} & \textit{if } n=3 \end{cases}$$

with

$$\partial \;:\; \widehat{L}^0(A)\to L_{-1}(A)\ ;\ M\mapsto (H_-(A^r);A^r,\mathrm{im}(\begin{pmatrix}1-XM\\ M\end{pmatrix}:A^r\to A^r\oplus(A^r)^*))\ ,$$

$$\partial \;:\; \widehat{L}^1(A)\to L_0(A)\ ;\ N\mapsto (A^r\oplus A^r,\begin{pmatrix}\frac{1}{4}(N+N^t-2N^tXN) & 1-2NX\\ 0 & -2X\end{pmatrix})\ ,$$

$$\partial \;:\; \widehat{L}^3(A)\to L_2(A)\ ;\ M\mapsto (A^r\oplus(A^r)^*,\begin{pmatrix}M & 1\\ 0 & X\end{pmatrix})\ .$$

Proof. Combine Proposition 1.30, Theorem 2.24 and Proposition 2.26. □

We can now prove Theorem 0.1 :

Corollary 2.31. *Let* A *be a 1-even ring.*
(i) *The universal chain bundle* (B^A,β^A) *over* A *is given by*

$$B^A \;:\; \dots \longrightarrow B^A_{2k+2}=A \xrightarrow{0} B^A_{2k+1}=A \xrightarrow{2} B^A_{2k}=A \xrightarrow{0} B^A_{2k-1}=A \longrightarrow \cdots\ ,$$
$$(\beta^A)_{-4k} \;=\; 1 \;:\; B^A_{2k} \;=\; A\to (B^A)^{2k} \;=\; A\ (k\in\mathbb{Z})\ .$$

(ii) *The hyperquadratic* L*-groups of* A *are given by*

$$\widehat{L}^n(A) \;=\; Q_n(B^A,\beta^A) \;=\; \begin{cases} A_8 & \textit{if } n\equiv 0(\mathrm{mod}\,4) \\ A_2 & \textit{if } n\equiv 1,3(\mathrm{mod}\,4) \\ 0 & \textit{if } n\equiv 2(\mathrm{mod}\,4) \end{cases}$$

with

$$\partial \;:\; \widehat{L}^0(A) = A_8 \to L_{-1}(A) \;;\; a \mapsto (H_-(A); A, \mathrm{im}(\begin{pmatrix} 1-a \\ a \end{pmatrix} : A \to A \oplus A)) \;,$$

$$\partial \;:\; \widehat{L}^1(A) = A_2 \to L_0(A) \;;\; a \mapsto (A \oplus A, \begin{pmatrix} a(1-a)/2 & 1-2a \\ 0 & -2 \end{pmatrix}) \;;$$

$$\partial \;:\; \widehat{L}^3(A) = A_2 \to L_2(A) \;;\; a \mapsto (A \oplus A, \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}) \;.$$

(iii) *The map* $L^0(A) \to \widehat{L}^0(A)$ *sends the Witt class* $(K,\lambda) \in L^0(A)$ *of a nonsingular symmetric form* (K,λ) *over* A *to*

$$[K,\lambda] \;=\; \lambda(v,v) \in \widehat{L}^0(A) \;=\; A_8$$

for any $v \in K$ *such that*

$$\lambda(x,x) \;=\; \lambda(x,v) \in A_2 \;(x \in K) \;.$$

Proof. (i)+(ii) The A-module morphism

$$\widehat{v}_0(\beta^A) \;:\; H_0(B^A) \;=\; A_2 \to \widehat{H}^0(\mathbb{Z}_2; A) \;;\; a \mapsto a^2$$

is an isomorphism. Apply Theorem 2.30 with $r = 1$, $x_1 = 1$.
(ii) The computation of $\widehat{L}^*(A) = Q_*(B^A, \beta^A)$ is given by Theorem 2.30, using the fact that $a - a^2 \in 2A$ $(a \in A)$ for a 1-even A. The explicit descriptions of ∂ are special cases of the formulae in Theorem 2.24 (ii).
(iii) As in Example 2.18 regard (K,λ) as a 0-dimensional symmetric Poincaré complex (D,ϕ) with

$$\phi_0 \;=\; \epsilon\lambda^{-1} \;:\; D^0 \;=\; K \to D^0 \;=\; K^* \;.$$

The Spivak normal chain bundle $\gamma = \lambda^{-1} \in \widehat{Q}^0(D^{0-*})$ is classified by the chain bundle map $(v, 0) : (D,\gamma) \to (B^A, \beta^A)$ with

$$g \;:\; D_0 \;=\; K^* \to \widehat{H}^0(\mathbb{Z}_2; A) \;;\; x \mapsto \lambda^{-1}(x,x) \;=\; x(v) \;.$$

The algebraic normal invariant $(\phi, 0) \in Q_0(D,\gamma)$ has image

$$g_\%(\phi, 0) \;=\; \lambda(v,v) \in Q_0(B^A, \beta^A) \;=\; A_8 \;.$$

□

Example 2.32. For $R = \mathbb{Z}$

$$\widehat{L}^n(\mathbb{Z}) \;=\; Q_n(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) \;=\; \begin{cases} \mathbb{Z}_8 & \text{if } n \equiv 0 (\bmod 4) \\ \mathbb{Z}_2 & \text{if } n \equiv 1,3 (\bmod 4) \\ 0 & \text{if } n \equiv 2 (\bmod 4) \end{cases}$$

as recalled (from [15]) in the Introduction. □

3. The Generalized Arf Invariant for Forms

A nonsingular ϵ-quadratic form (K,ψ) over A corresponds to a 0-dimensional ϵ-quadratic Poincaré complex over A. The 0-dimensional ϵ-quadratic L-group $L_0(A,\epsilon)$ is the Witt group of nonsingular ϵ-quadratic forms, and similarly for $L^0(A,\epsilon)$ and ϵ-symmetric forms. In this section we define the 'generalized Arf invariant'

$$(K,\psi;L) \in Q_1(B^{A,\epsilon}, \beta^{A,\epsilon}) \;=\; \widehat{L}^{4*+1}(A,\epsilon)$$

for a nonsingular ϵ-quadratic form (K,ψ) over A with a lagrangian L for the ϵ-symmetric form $(K,\psi+\epsilon\psi^*)$, so that

$$\begin{aligned}(K,\psi) \;=\; \partial(K,\psi;L) &\in \ker(1+T : L_0(A,\epsilon)\to L^{4*}(A,\epsilon))\\ &= \operatorname{im}(\partial : Q_1(B^{A,\epsilon},\beta^{A,\epsilon},\epsilon)\to L_0(A,\epsilon))\ .\end{aligned}$$

3.1. **Forms and Formations.** Given a f.g. projective A-module K and the inclusion $j : L\to K$ of a direct summand, let $f : C\to D$ be the chain map defined by

$$\begin{aligned} C &: \dots\to 0\to C_k \;=\; K^*\to 0\to\dots\ ,\\ D &: \dots\to 0\to D_k \;=\; L^*\to 0\to\dots\ ,\\ f &= j^* \;:\; C_k \;=\; K^*\to D_k \;=\; L^*\ .\end{aligned}$$

The symmetric Q-group

$$Q^{2k}(C) \;=\; H^0(\mathbb{Z}_2;S(K),(-1)^kT) \;=\; \{\phi\in S(K)\,|\,\phi^*=(-1)^k\phi\}$$

is the additive group of $(-1)^k$-symmetric pairings on K, and

$$f^{\%} \;=\; S(j) \;:\; Q^{2k}(C)\to Q^{2k}(D)\ ;\ \phi\mapsto f\phi f^* \;=\; j^*\phi j \;=\; \phi|_L$$

sends such a pairing to its restriction to L. A $2k$-dimensional symmetric (Poincaré) complex $(C,\phi\in Q^{2k}(C))$ is the same as a (nonsingular) $(-1)^k$-symmetric form (K,ϕ). The relative symmetric Q-group of f

$$\begin{aligned}Q^{2k+1}(f) &= \ker(f^{\%} : Q^{2k}(C)\to Q^{2k}(D))\\ &= \{\phi\in S(K)\,|\,\phi^*=(-1)^k\phi\in S(K),\phi|_L=0\in S(L)\}\ ,\end{aligned}$$

consists of the $(-1)^k$-symmetric pairings on K which restrict to 0 on L. The submodule $L\subset K$ is a lagrangian for (K,ϕ) if and only if ϕ restricts to 0 on L and

$$L^{\perp} \;=\; \{x\in K\,|\,\phi(x)(L)=\{0\}\subset A\} \;=\; L\ ,$$

if and only if $(f : C\to D,(0,\phi)\in Q^{2k+1}(f))$ defines a $(2k+1)$-dimensional symmetric Poincaré pair, with an exact sequence

$$0\longrightarrow D^k=L\xrightarrow{f^*=j} C^k=K\xrightarrow{f\phi=j^*\phi} D_k=L^*\longrightarrow 0\ .$$

Similarly for the quadratic case, with

$$\begin{aligned}Q_{2k}(C) &= H_0(\mathbb{Z}_2;S(K),(-1)^kT)\ ,\\ Q_{2k+1}(f) &= \frac{\{(\psi,\chi)\in S(K)\oplus S(L)\,|\,f^*\psi f=\chi+(-1)^{k+1}\chi^*\in S(L)\}}{\{(\theta+(-1)^{k+1}\theta^*,f\theta f^*+\nu+(-1)^k\nu^*)\,|\,\theta\in S(K),\nu\in S(L)\}}\end{aligned}$$

A quadratic structure $\psi\in Q_{2k}(C)$ determines and is determined by the pair (λ,μ) with $\lambda=\psi+(-1)^k\psi^*\in Q^{2k}(C)$ and

$$\mu \;:\; K\to H_0(\mathbb{Z}_2;A,(-1)^k)\ ;\ x\mapsto\psi(x)(x)\ .$$

A $(2k+1)$-dimensional (symmetric, quadratic) Poincaré pair $(f : C\to D,(\delta\phi,\psi))$ is a nonsingular $(-1)^k$-quadratic form (K,ψ) together with a lagrangian $L\subset K$ for the nonsingular $(-1)^k$-symmetric form $(K,\psi+(-1)^k\psi^*)$.

Lemma 3.1. *Let (K,ψ) be a nonsingular $(-1)^k$-quadratic form over A, and let $L\subset K$ be a lagrangian for $(K,\psi+(-1)^k\psi^*)$. There exists a direct complement for $L\subset K$ which is also a lagrangian for $(K,\psi+(-1)^k\psi^*)$.*

Proof. Choosing a direct complement $L' \subset K$ to $L \subset K$ write

$$\psi \ = \ \begin{pmatrix} \mu & \lambda \\ 0 & \nu' \end{pmatrix} \ : \ K \ = \ L \oplus L' \to K^* \ = \ L^* \oplus (L')^*$$

with $\lambda : L' \to L^*$ an isomorphism and

$$\mu + (-1)^k \mu^* \ = \ 0 \ : \ L \to L^* \ .$$

In general $\nu' + (-1)^k (\nu')^* \neq 0 : L^* \to L$, but if the direct complement L' is replaced by

$$L'' \ = \ \{(-(\lambda^{-1})^*(\nu')^*(x), x) \in L \oplus L' \,|\, x \in L'\} \subset K$$

and the isomorphism

$$\lambda'' \ : \ L'' \to L^* \ ; \ (-(\lambda^{-1})^*(\nu')^*(x), x) \mapsto \lambda(x)$$

is used as an identification then

$$\psi \ = \ \begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix} \ : \ K \ = \ L \oplus L^* \to K^* \ = \ L^* \oplus L$$

with $\nu = (\nu')^* \mu \nu' : L^* \to L$ such that

$$\nu + (-1)^k \nu^* \ = \ 0 \ : \ L^* \to L \ .$$

Thus $L'' = L^* \subset K$ is a direct complement for L which is a lagrangian for $(K, \psi + (-1)^k \psi^*)$, with

$$\psi + (-1)^k \psi^* \ = \ \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix} \ : \ K \ = \ L \oplus L^* \to K^* \ = \ L^* \oplus L \ .$$

□

A lagrangian L for the $(-1)^k$-symmetrization $(K, \psi + (-1)^k \psi^*)$ is a lagrangian for the $(-1)^k$-quadratic form (K, ψ) if and only if $\psi|_L = \mu$ is a $(-1)^{k+1}$-symmetrization, i.e.

$$\mu \ = \ \theta + (-)^{k+1} \theta^* \ : \ L \to L^*$$

for some $\theta \in S(L)$, in which case the inclusion $j : (L, 0) \to (K, \psi)$ extends to an isomorphism of $(-1)^k$-quadratic forms

$$\begin{pmatrix} 1 & -\nu^* \\ 0 & 1 \end{pmatrix} \ : \ H_{(-1)^k}(L) \ = \ (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \ \xrightarrow{\ \cong\ } \ (K, \psi)$$

with $\nu = \psi|_{L^*}$. The $2k$-dimensional quadratic L-group $L_{2k}(A)$ is the Witt group of stable isomorphism classes of nonsingular $(-1)^k$-quadratic forms over A, such that

$$(K, \psi) \ = \ (K', \psi') \in L_{2k}(A) \quad \text{if and only if there exists an isomorphism}$$
$$(K, \psi) \oplus H_{(-1)^k}(L) \ \cong \ (K', \psi') \oplus H_{(-1)^k}(L') \ .$$

Proposition 3.2. *Given a $(-1)^k$-quadratic form (L, μ) over A such that*

$$\mu + (-1)^k \mu^* \ = \ 0 \ : \ L \to L^*$$

let (B, β) be the chain bundle over A given by

$$B \ : \ \cdots \to 0 \to B_{k+1} \ = \ L \to 0 \to \dots \ ,$$
$$\beta \ = \ \mu \in \widehat{Q}^0(B^{0-*}) \ = \ \mathrm{Hom}_A(L, \widehat{H}^{k+1}(\mathbb{Z}_2; A)) \ = \ \widehat{H}^0(\mathbb{Z}_2; S(L), (-1)^{k+1}T) \ .$$

(i) *The* $(2k+1)$*-dimensional twisted quadratic* Q*-group of* (B,β)

$$\begin{aligned} Q_{2k+1}(B,\beta) &= \frac{\{\nu \in S(L^*) \,|\, \nu + (-1)^k \nu^* = 0\}}{\{\phi - \phi\mu\phi^* - (\theta + (-1)^{k+1}\theta^*) \,|\, \phi^* = (-1)^{k+1}\phi, \theta \in S(L^*)\}} \\ &= \text{coker}(J_\mu : H^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1}T) \to \widehat{H}^0(\mathbb{Z}_2; S(L^*), (-1)^{k+1}T)) \end{aligned}$$

classifies nonsingular $(-1)^k$*-quadratic forms* (K,ψ) *over* A *for which there exists a lagrangian* L *for* $(K, \psi + (-1)^k\psi^*)$ *such that*

$$\begin{aligned} \psi|_L \;=\; \mu \in\; & \text{im}(\widehat{H}^1(\mathbb{Z}_2; S(L), (-1)^kT) \to H_0(\mathbb{Z}_2; S(L), (-1)^kT)) \\ =\; & \ker(1 + (-1)^kT : H_0(\mathbb{Z}_2; S(L), (-1)^kT) \to H^0(\mathbb{Z}_2; S(L), (-1)^kT))\ . \end{aligned}$$

Specifically, for any $(-1)^k$*-quadratic form* (L^*, ν) *such that*

$$\nu + (-1)^k\nu^* \;=\; 0 \;:\; L^* \to L$$

the nonsingular $(-1)^k$*-quadratic form* (K,ψ) *defined by*

$$\psi \;=\; \begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix} \;:\; K \;=\; L \oplus L^* \to K^* \;=\; L^* \oplus L$$

is such that L *is a lagrangian of* $(K, \psi + (-1)^k\psi^*)$*, and*

$$\partial \;:\; Q_{2k+1}(B,\beta) \to L_{2k}(A) \;;\; \nu \mapsto (K,\psi)\ .$$

(ii) *The algebraic normal invariant of a* $(2k+1)$*-dimensional* (*symmetric, quadratic*) *Poincaré pair* $(f : C \to D, (\delta\phi, \psi))$ *concentrated in degree* k *with*

$$\begin{aligned} & C_k \;=\; K^*\ ,\ D_k \;=\; L^*\ , \\ & f\psi_0 f^* = \;\mu \in \ker(1 + (-1)^kT : H_0(\mathbb{Z}_2; S(L), (-1)^kT) \to H^0(\mathbb{Z}_2; S(L), (-1)^kT)) \end{aligned}$$

is given by

$$(\phi,\theta) \;=\; \nu \in Q_{2k+1}(\mathcal{C}(f),\gamma) \;=\; Q_{2k+1}(B,\beta)$$

with

$$\widehat{v}_{k+1}(\gamma) \;=\; \widehat{v}_{k+1}(\beta) \;:\; L \;=\; H_{k+1}(f) \;=\; H_{k+1}(B) \to \widehat{H}^{k+1}(\mathbb{Z}_2; A) \;;\; x \mapsto \mu(x)(x)$$

and $\nu = \psi|_{L^*}$ *the restriction of* ψ *to any lagrangian* $L^* \subset K$ *of* $(K, \psi + (-1)^k\psi^*)$ *complementary to* L*.*

Proof. (i) Given $(-1)^{k+1}$-symmetric forms (L^*, ν), (L^*, ϕ) and $\theta \in S(L^*)$ replacing ν by

$$\nu' \;=\; \nu + \phi - \phi\mu\phi^* - (\theta + (-1)^{k+1}\theta^*) \;:\; L^* \to L$$

results in a $(-1)^k$-quadratic form (K,ψ') such that there is defined an isomorphism

$$\begin{pmatrix} 1 & \phi^* \\ 0 & 1 \end{pmatrix} \;:\; (K,\psi') \to (K,\psi)$$

which is the identity on L.
(ii) This is the translation of Proposition 2.12 (iii) into the language of forms and lagrangians. □

More generally :

Proposition 3.3. *Given $(-1)^k$-quadratic forms (L,μ), (L^*,ν) over A such that*

$$\mu+(-1)^k\mu^* \;=\; 0 \;:\; L\to L^*\;,\;\; \nu+(-1)^k\nu^* \;=\; 0 \;:\; L^*\to L$$

define a nonsingular $(-1)^k$-quadratic form

$$(K,\psi) \;=\; (L\oplus L^*,\begin{pmatrix}\mu & 1\\ 0 & \nu\end{pmatrix})$$

such that L and L^* are complementary lagrangians of the nonsingular $(-1)^k$-symmetric form

$$(K,\psi+(-1)^k\psi^*) \;=\; (L\oplus L^*,\begin{pmatrix}0 & 1\\ (-1)^k & 0\end{pmatrix})\;,$$

and let $(f:C\to D,(\delta\phi,\psi))$ be the $(2k+1)$-dimensional (symmetric, quadratic) Poincaré pair concentrated in degree k defined by

$$f \;=\; \begin{pmatrix}1 & 0\end{pmatrix} \;:\; C_k \;=\; K^* \;=\; L^*\oplus L\to D_k \;=\; L^*\;,\;\; \delta\phi \;=\; 0\;,$$

with $\mathcal{C}(f)\simeq L_{*-k-1}$.

(i) *The Spivak normal bundle of $(f:C\to D,(\delta\phi,\psi))$ is given by*

$$\gamma \;=\; \mu\in \widehat{Q}^0(\mathcal{C}(f)^{0-*}) \;=\; \widehat{H}^0(\mathbb{Z}_2;S(L),(-1)^{k+1}T)\;,$$

and

$$\begin{aligned}Q_{2k+1}(\mathcal{C}(f),\gamma) \;&=\; \frac{\{\lambda\in S(L^*)\,|\,\lambda+(-1)^k\lambda^*=0\}}{\{\phi-\phi\mu\phi^*-(\theta+(-1)^{k+1}\theta^*)\,|\,\phi^*=(-1)^{k+1}\phi,\theta\in S(L^*)\}}\\ &=\; \mathrm{coker}(J_\mu:H^0(\mathbb{Z}_2;S(L^*),(-1)^{k+1}T)\to\widehat{H}^0(\mathbb{Z}_2;S(L^*),(-1)^{k+1}T))\;.\end{aligned}$$

The algebraic normal invariant of $(f:C\to D,(\delta\phi,\psi))$ is

$$(\phi,\theta) \;=\; \nu\in Q_{2k+1}(\mathcal{C}(f),\gamma)\;.$$

(ii) *Let (B,β) be a chain bundle concentrated in degree $k+1$*

$$\begin{aligned}&B \;:\; \cdots\to 0\to B_{k+1}\to 0\to\dots\;,\\ &\beta\in\widehat{Q}^0(B^{0-*}) \;=\; \mathrm{Hom}_A(B_{k+1},\widehat{H}^{k+1}(\mathbb{Z}_2;A)) \;=\; \widehat{H}^0(\mathbb{Z}_2;S(B_{k+1}),(-1)^{k+1}T)\;,\end{aligned}$$

so that

$$\begin{aligned}Q_{2k+1}(B,\beta) \;&=\; \frac{\{\lambda\in S(B^{k+1})\,|\,\lambda+(-1)^k\lambda^*=0\}}{\{\phi-\phi\beta\phi^*-(\theta+(-1)^{k+1}\theta^*)\,|\,\phi^*=(-1)^{k+1}\phi,\theta\in S(B^{k+1})\}}\\ &=\; \mathrm{coker}(J_\beta:H^0(\mathbb{Z}_2;S(B^{k+1}),(-1)^{k+1}T)\to\widehat{H}^0(\mathbb{Z}_2;S(B^{k+1}),(-1)^{k+1}T))\;.\end{aligned}$$

A (B,β)-structure on $(f:C\to D,(\delta\phi,\psi))$ is given by a chain bundle map $(g,\chi):(\mathcal{C}(f),\gamma)\to(B,\beta)$, corresponding to an A-module morphism $g:L\to B_{k+1}$ such that

$$g^*\beta g \;=\; \mu\in\widehat{H}^0(\mathbb{Z}_2;S(L),(-1)^{k+1}T)\;,$$

with

$$(g,\chi)_\% \;:\; Q_{2k+1}(\mathcal{C}(f),\gamma)\to Q_{2k+1}(B,\beta)\;;\;\lambda\mapsto g\lambda g^*\;.$$

The 4-periodic (B,β)-structure cobordism class is thus given by

$$\begin{aligned}(K,\psi;L) \;=\; (f:C\to D,(\delta\phi,\psi)) \;&=\; (g,\chi)_\%(\phi,\theta) \;=\; g\nu g^*\\ &\in \widehat{L}\langle B,\beta\rangle^{4*+2k+1}(A) \;=\; Q_{2k+1}(B,\beta)\;,\end{aligned}$$

with

$$\begin{aligned}(K,\psi) &= (B_{k+1}\oplus B^{k+1}, \begin{pmatrix}\beta & 1\\ 0 & g\nu g^*\end{pmatrix})\\ &\in \mathrm{im}(\partial : Q_{2k+1}(B,\beta)\to L_{2k}(A)) \;=\; \ker(L_{2k}(A)\to L\langle B,\beta\rangle^{4*+2k}(A))\ .\end{aligned}$$

□

3.2. **The Generalized Arf Invariant.**

Definition 3.4. The *generalized Arf invariant* of a nonsingular $(-1)^k$-quadratic form (K,ψ) over A together with a lagrangian $L\subset K$ for the $(-1)^k$-symmetric form $(K,\psi+(-1)^k\psi^*)$ is the image

$$(K,\psi;L) \;=\; (g,\chi)_{\%}(\phi,\theta)\in \widehat{L}^{4*+2k+1}(A) \;=\; Q_{2k+1}(B^A,\beta^A)$$

of the algebraic normal invariant $(\phi,\theta)\in Q_{2k+1}(\mathcal{C}(f),\gamma)$ (2.13) of the corresponding $(2k+1)$-dimensional (symmetric, quadratic) Poincaré pair $(f : C\to D,(\delta\phi,\psi)\in Q^{2k+1}_{2k+1}(f))$

$$\begin{aligned}(\phi,\theta) &= \nu\in Q_{2k+1}(\mathcal{C}(f),\gamma)\\ &= \mathrm{coker}(J_\mu : H^0(\mathbb{Z}_2;S(L^*),(-1)^{k+1}T)\to \widehat{H}^0(\mathbb{Z}_2;S(L^*),(-1)^{k+1}T))\end{aligned}$$

under the morphism $(g,\chi)_{\%}$ induced by the classifying chain bundle map $(g,\chi) : (\mathcal{C}(f),\gamma)\to(B^A,\beta^A)$. As in 3.2 $\nu=\psi|_{L^*}$ is the restriction of ψ to a lagrangian $L^*\subset K$ of $(K,\psi+(-1)^k\psi^*)$ complementary to L. □

A nonsingular $(-1)^k$-symmetric formation $(K,\phi;L,L')$ is a nonsingular $(-1)^k$-symmetric form (K,ϕ) together with two lagrangians L, L'. This type of formation is essentially the same as a $(2k+1)$-dimensional symmetric Poincaré complex concentrated in degrees $k,k+1$, and represents an element of $L^{4*+2k+1}(A)$.

Proposition 3.5. (i) *The generalized Arf invariant is such that*

$$(K,\psi;L) \;=\; 0\in Q_{2k+1}(B^A,\beta^A) \;=\; \widehat{L}^{4*+2k+1}(A)$$

if and only if there exists an isomorphism of $(-1)^k$-quadratic forms

$$(K,\psi)\oplus H_{(-1)^k}(L') \;\cong\; H_{(-1)^k}(L'')$$

such that

$$((K,\psi+(-1)^k\psi^*)\oplus(1+T)H_{(-1)^k}(L');L\oplus L',L'') \;=\; 0\in L^{4*+2k+1}(A)\ .$$

(ii) *If (K,ψ) is a nonsingular $(-1)^k$-quadratic form over A and $L,L'\subset K$ are lagrangians for $(K,\psi+(-1)^k\psi^*)$ then*

$$\begin{aligned}(K,\psi;L)-(K,\psi;L') &= (K,\psi+(-1)^k\psi^*;L,L')\\ &\in \mathrm{im}(L^{4*+2k+1}(A)\to\widehat{L}^{4*+2k+1}(A)) \;=\; \ker(\widehat{L}^{4*+2k+1}(A)\to L_{2k}(A))\ .\end{aligned}$$

Proof. This is the translation of the isomorphism $Q_{2k+1}(B^A,\beta^A)\cong\widehat{L}^{4*+2k+1}(A)$ given by 2.16 into the language of forms and formations. □

Example 3.6. Let A be a field, so that each $\widehat{H}^n(\mathbb{Z}_2;A)$ is a free A-module, and the universal chain bundle over A can be taken to be

$$B^A \;=\; \widehat{H}^*(\mathbb{Z}_2;A) \;:\; \dots\longrightarrow B^A_n \;=\; \widehat{H}^n(\mathbb{Z}_2;A)\xrightarrow{0} B^A_{n-1} \;=\; \widehat{H}^{n-1}(\mathbb{Z}_2;A)\xrightarrow{0}\cdots\ .$$

If A is a perfect field of characteristic 2 with the identity involution squaring defines an A-module isomorphism

$$A \xrightarrow{\cong} \widehat{H}^n(\mathbb{Z}_2;A)\ ;\ a \mapsto a^2\ .$$

Every nonsingular $(-1)^k$-quadratic form over A is isomorphic to one of the type

$$(K,\psi)\ =\ (L\oplus L^*, \begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix})$$

with $L = A^\ell$ f.g. free and

$$\mu\ =\ (-1)^{k+1}\mu^*\ :\ L \to L^*\ ,\ \nu\ =\ (-1)^{k+1}\nu^*\ :\ L^* \to L\ .$$

For $j = 1, 2, \ldots, \ell$ let

$$\begin{aligned} e_j\ &=\ (0,\ldots,0,1,0,\ldots,0) \in L\ ,\ g_j\ =\ \mu(e_j)(e_j) \in A\ ,\\ e_j^*\ &=\ (0,\ldots,0,1,0,\ldots,0) \in L^*\ ,\ h_j\ =\ \nu(e_j^*)(e_j^*) \in A\ . \end{aligned}$$

The generalized Arf invariant in this case was identified in Ranicki [18, §11] with the original invariant of Arf [1]

$$(K,\psi;L)\ =\ \sum_{j=1}^{\ell} g_j h_j \in Q_{2k+1}(B^A,\beta^A)\ =\ A/\{c+c^2\,|\,c\in A\}\ .$$

□

For $k = 0$ we have :

Proposition 3.7. *Suppose that the involution on A is even. If (K,ψ) is a nonsingular quadratic form over A and L is a lagrangian of $(K,\psi+\psi^*)$ then L is a lagrangian of (K,ψ), the Witt class is*

$$(K,\psi)\ =\ 0 \in L_0(A)\ ,$$

the algebraic normal invariant is

$$(\phi,\theta)\ =\ 0 \in Q_1(\mathcal{C}(f),\gamma)\ =\ 0$$

and the generalized Arf invariant is

$$(K,\psi;L)\ =\ (g,\chi)_{\%}(\phi,\theta)\ =\ 0 \in \widehat{L}^{4*+1}(A)\ =\ Q_1(B^A,\beta^A)\ .$$

Proof. By hypothesis $\widehat{H}^1(\mathbb{Z}_2;A) = 0$, and $L = A^\ell$, so that by Proposition 3.2 (i)

$$Q_1(\mathcal{C}(f),\gamma)\ =\ \widehat{H}^0(\mathbb{Z}_2;S(L^*),-T)\ =\ \bigoplus_{\ell}\widehat{H}^1(\mathbb{Z}_2;A)\ =\ 0\ .$$

□

For $k = 1$ we have :

Theorem 3.8. *Let A be an r-even ring with A_2-module basis $\{x_1 = 1, x_2, \ldots, x_r\} \subset \widehat{H}^0(\mathbb{Z}_2;A)$, and let*

$$X\ =\ \begin{pmatrix} x_1 & 0 & 0 & \ldots & 0 \\ 0 & x_2 & 0 & \ldots & 0 \\ 0 & 0 & x_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x_r \end{pmatrix} \in \mathrm{Sym}_r(A)$$

so that by Theorem 2.30

$$Q_3(B^A,\beta^A) \;=\; \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)+\{L-LXL\,|\,L\in \mathrm{Sym}_r(A)\}}\ .$$

(i) *Given* $M\in \mathrm{Sym}_r(A)$ *define the nonsingular* (-1)*-quadratic form over* A

$$(K_M,\psi_M) \;=\; (A^r\oplus (A^r)^*,\begin{pmatrix} X & 1 \\ 0 & M\end{pmatrix})$$

such that $L_M=A^r\subset K_M$ *is a lagrangian of* $(K_M,\psi_M-\psi_M^*)$*. The function*

$$Q_3(B^A,\beta^A)\to \widehat{L}^{4*+3}(A)\ ;\ M\mapsto (K_M,\psi_M;L_M)$$

is an isomorphism, with inverse given by the generalized Arf invariant.
(ii) *Let* (K,ψ) *be a nonsingular* (-1)*-quadratic form over* A *of the type*

$$(K,\psi) \;=\; (L\oplus L^*,\begin{pmatrix} \mu & 1 \\ 0 & \nu\end{pmatrix})$$

with

$$\mu-\mu^* \;=\; 0\ :\ L\to L^*\ ,\ \nu-\nu^* \;=\; 0\ :\ L^*\to L$$

and let $g:L\to A^r$, $h:L^*\to A^r$ *be* A*-module morphisms such that*

$$\mu \;=\; g^*Xg\in \widehat{H}^0(\mathbb{Z}_2;S(L),T)\ ,\ \nu \;=\; h^*Xh\in \widehat{H}^0(\mathbb{Z}_2;S(L^*),T)\ .$$

The generalized Arf invariant of $(K,\psi;L)$ *is*

$$(K,\psi;L) \;=\; g\nu g^* \;=\; gh^*Xhg^*\in Q_3(B^A,\beta^A)\ .$$

If $L=A^\ell$ *then*

$$g \;=\; (g_{ij})\ :\ L \;=\; A^\ell\to A^r\ ,\ h \;=\; (h_{ij})\ :\ L^* \;=\; A^\ell\to A^r$$

with the coefficients $g_{ij},h_{ij}\in A$ *such that*

$$\mu(e_j)(e_j) \;=\; \sum_{i=1}^r (g_{ij})^2x_i\ ,\ \nu(e_j^*)(e_j^*) \;=\; \sum_{i=1}^r (h_{ij})^2x_i\in \widehat{H}^0(\mathbb{Z}_2;A)$$

$$(e_j=(0,\dots,0,1,0,\dots,0)\in L=A^\ell\ ,\ e_j^*=(0,\dots,0,1,0,\dots,0)\in L^*=A^\ell)$$

and

$$(K,\psi;L) \;=\; gh^*Xhg^* \;=\; \begin{pmatrix} c_1 & 0 & 0 & \dots & 0 \\ 0 & c_2 & 0 & \dots & 0 \\ 0 & 0 & c_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_r\end{pmatrix}\in Q_3(B^A,\beta^A)$$

with

$$c_i \;=\; \sum_{k=1}^r\Big(\sum_{j=1}^\ell g_{ij}h_{kj}\Big)^2x_k\in \widehat{H}^0(\mathbb{Z}_2;A)\ .$$

(iii) *For any* $M=(m_{ij})\in \mathrm{Sym}_r(A)$ *let* $h=(h_{ij})\in M_r(A)$ *be such that*

$$m_{jj} \;=\; \sum_{i=1}^r (h_{ij})^2x_i\in \widehat{H}^0(\mathbb{Z}_2;A)\ (1\leqslant j\leqslant r)\ ,$$

so that

$$M \;=\; \begin{pmatrix} m_{11} & 0 & 0 & \dots & 0 \\ 0 & m_{22} & 0 & \dots & 0 \\ 0 & 0 & m_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & m_{rr} \end{pmatrix} \;=\; h^*Xh \in \widehat{H}^0(\mathbb{Z}_2; M_r(A), T) \;=\; \frac{\mathrm{Sym}_r(A)}{\mathrm{Quad}_r(A)}$$

and the generalized Arf invariant of the triple $(K_M, \psi_M; L_M)$ *in* (i) *is*

$$(K_M, \psi_M; L_M) \;=\; h^*Xh \;=\; M \in Q_3(B^A, \beta^A)$$

(*with* $g = (\delta_{ij})$ *here*).

Proof. (i) The isomorphism $Q_3(B^A, \beta^A) \to \widehat{L}^3(A); M \mapsto (K_M, \psi_M; L_M)$ is given by Proposition 2.16.
(ii) As in Definition 3.4 let $(\phi, \theta) \in Q_3(\mathcal{C}(f), \gamma)$ be the algebraic normal invariant of the 3-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi))$ concentrated in degree 1, with

$$f \;=\; \begin{pmatrix} 1 & 0 \end{pmatrix} \;:\; C_1 \;=\; K^* \;=\; L^* \oplus L \to D_1 \;=\; L^* \;,\; \delta\phi \;=\; 0 \;.$$

The A-module morphism

$$\widehat{v}_2(\gamma) \;:\; H_2(\mathcal{C}(f)) \;=\; H^1(D) \;=\; L \to \widehat{H}^0(\mathbb{Z}_2; A) \;;\; y \mapsto \mu(y)(y)$$

is induced by the A-module chain map

$$g \;:\; \mathcal{C}(f) \;\simeq\; L_{*-2} \to B^A(1)$$

and

$$(g, 0) \;:\; (\mathcal{C}(f), \gamma) \to (B^A(1), \beta^A(1)) \to (B^A, \beta^A)$$

is a classifying chain bundle map. The induced morphism

$$\begin{aligned} &(g,0)_{\%} \;:\; Q_3(\mathcal{C}(f), \gamma) \;=\; \mathrm{coker}(J_\mu : H^0(\mathbb{Z}_2; S(L^*), T) \to \widehat{H}^0(\mathbb{Z}_2; S(L^*), T)) \\ &\to Q_3(B^A, \beta^A) \;=\; \mathrm{coker}(J_X : H^0(\mathbb{Z}_2; M_r(A), T) \to \widehat{H}^0(\mathbb{Z}_2; M_r(A), T)) \;;\; \sigma \mapsto g\sigma g^* \end{aligned}$$

sends the algebraic normal invariant

$$(\phi, \theta) \;=\; \nu \;=\; h^*Xh \in Q_3(\mathcal{C}(f), \gamma)$$

to the generalized Arf invariant

$$(g, 0)_{\%}(\phi, \theta) \;=\; gh^*Xhg^* \in Q_3(B^A, \beta^A) \;.$$

(iii) By construction. □

In particular, the generalized Arf invariant for $A = \mathbb{Z}_2$ is just the classical Arf invariant.

4. The Generalized Arf Invariant for Linking Forms

An ϵ-quadratic formation $(Q,\psi;F,G)$ over A corresponds to a 1-dimensional ϵ-quadratic Poincaré complex. The 1-dimensional ϵ-quadratic L-group $L_1(A,\epsilon)$ is the Witt group of ϵ-quadratic formations, or equivalently the cobordism group of 1-dimensional ϵ-quadratic Poincaré complexes over A. We could define a generalized Arf invariant $\alpha \in Q_2(B^A,\beta^A,\epsilon)$ for any formation with a null-cobordism of the 1-dimensional ϵ-symmetric Poincaré complex, so that

$$\begin{aligned}(Q,\psi;F,G) \ &= \ \partial(\alpha) \in \ker(1+T_\epsilon : L_1(A,\epsilon) \to L^{4*+1}(A,\epsilon)) \\ &= \ \mathrm{im}(\partial : Q_2(B^{A,\epsilon},\beta^{A,\epsilon},\epsilon) \to L_1(A,\epsilon)) \ .\end{aligned}$$

However, we do not need quite such a generalized Arf invariant here. For our application to UNil, it suffices to work with a localization $S^{-1}A$ of A and to only consider a formation $(Q,\psi;F,G)$ such that

$$F \cap G \ = \ \{0\} \ , \ S^{-1}(Q/(F+G)) \ = \ 0$$

which corresponds to a $(-\epsilon)$-quadratic linking form (T,λ,μ) over (A,S) with

$$T \ = \ Q/(F+G) \ , \ \lambda \ : \ T \times T \to S^{-1}A/A \ .$$

Given a lagrangian $U \subset T$ for the $(-\epsilon)$-symmetric linking form (T,λ) we define in this section a 'linking Arf invariant'

$$(T,\lambda,\mu;U) \in Q_2(B^{A,\epsilon},\beta^{A,\epsilon},\epsilon) \ = \ \widehat{L}^{4*+2}(A,\epsilon)$$

such that

$$\begin{aligned}(Q,\psi;F,G) \ &= \ \partial(T,\lambda,\mu;U) \in \ker(1+T_\epsilon : L_1(A,\epsilon) \to L^{4*+1}(A,\epsilon)) \\ &= \ \mathrm{im}(\partial : Q_2(B^{A,\epsilon},\beta^{A,\epsilon}) \to L_1(A,\epsilon)) \ .\end{aligned}$$

4.1. **Linking Forms and Formations.** Given a ring with involution A and a multiplicative subset $S \subset A$ of central non-zero divisors such that $\overline{S} = S$ let $S^{-1}A$ be the localized ring with involution obtained from A by inverting S. We refer to [16] for the localization exact sequences in ϵ-symmetric and ϵ-quadratic algebraic L-theory

$$\begin{aligned}&\cdots \to L^n(A,\epsilon) \to L^n_I(S^{-1}A,\epsilon) \to L^n(A,S,\epsilon) \to L^{n-1}(A,\epsilon) \to \dots \ , \\ &\cdots \to L_n(A,\epsilon) \to L_n^I(S^{-1}A,\epsilon) \to L_n(A,S,\epsilon) \to L_{n-1}(A,\epsilon) \to \dots\end{aligned}$$

with $I = \mathrm{im}(\widetilde{K}_0(A) \to \widetilde{K}_0(S^{-1}A))$, $L^n(A,S,\epsilon)$ the cobordism group of $(n-1)$-dimensional ϵ-symmatric Poincaré complexes (C,ϕ) over A such that $H_*(S^{-1}C)=0$, and similarly for $L_n(A,S,\epsilon)$.

An (A,S)-*module* is an A-module T with a 1-dimensional f.g. projective A-module resolution

$$0 \longrightarrow P \xrightarrow{d} Q \longrightarrow T \longrightarrow 0$$

such that $S^{-1}d : S^{-1}P \to S^{-1}Q$ is an $S^{-1}A$-module isomorphism. In particular,

$$S^{-1}T \ = \ 0 \ .$$

The *dual* (A,S)-*module* is defined by

$$\begin{aligned}T\widehat{\ } \ &= \ \mathrm{Ext}^1_A(T,A) \ = \ \mathrm{Hom}_A(T,S^{-1}A/A) \\ &= \ \mathrm{coker}(d^* : Q^* \to P^*)\end{aligned}$$

with

$$A \times T\widehat{\ } \to T\widehat{\ }\ ;\ (a,f) \mapsto (x \mapsto f(x)\overline{a})\ .$$

For any (A,S)-modules T, U there is defined a duality isomorphism

$$\mathrm{Hom}_A(T,U) \to \mathrm{Hom}_A(U\widehat{\ },T\widehat{\ })\ ;\ f \mapsto f\widehat{\ }$$

with

$$f\widehat{\ }\ :\ U\widehat{\ } \to T\widehat{\ }\ ;\ g \mapsto (x \mapsto g(f(x)))\ .$$

An element $\lambda \in \mathrm{Hom}_A(T,T\widehat{\ })$ can be regarded as a sesquilinear linking pairing

$$\lambda\ :\ T \times T \to S^{-1}A/A\ ;\ (x,y) \mapsto \lambda(x,y)\ =\ \lambda(x)(y)$$

with

$$\begin{aligned}\lambda(x,ay+bz)\ &=\ a\lambda(x,y)+b\lambda(x,z)\ ,\\ \lambda(ay+bz,x)\ &=\ \lambda(y,x)\overline{a}+\lambda(z,x)\overline{b}\ ,\\ \lambda\widehat{\ }(x,y)\ &=\ \overline{\lambda(y,x)} \in S^{-1}A/A\ (a,b \in A,\ x,y,z \in T)\ .\end{aligned}$$

Definition 4.1. Let $\epsilon = \pm 1$.
(i) An *ϵ-symmetric linking form over* (A,S) (T,λ) is an (A,S)-module T together with $\lambda \in \mathrm{Hom}_A(T,T\widehat{\ })$ such that $\lambda\widehat{\ } = \epsilon\lambda$, so that

$$\overline{\lambda(x,y)}\ =\ \epsilon\lambda(y,x) \in S^{-1}A/A\ (x,y \in T)\ .$$

The linking form is *nonsingular* if $\lambda : T \to T\widehat{\ }$ is an isomorphism. A *lagrangian* for (T,λ) is an (A,S)-submodule $U \subset T$ such that the sequence

$$0 \longrightarrow U \xrightarrow{j} T \xrightarrow{j\widehat{\ }\lambda} U\widehat{\ } \longrightarrow 0$$

is exact with $j \in \mathrm{Hom}_A(U,T)$ the inclusion. Thus λ restricts to 0 on U and

$$U^{\perp}\ =\ \{x \in T\,|\,\lambda(x)(U) = \{0\} \subset S^{-1}A/A\}\ =\ U\ .$$

(ii) A (*nonsingular*) *ϵ-quadratic linking form over* (A,S) (T,λ,μ) is a (nonsingular) ϵ-symmetric linking form (T,λ) together with a function

$$\mu\ :\ T \to Q_\epsilon(A,S)\ =\ \frac{\{b \in S^{-1}A\,|\,\epsilon\overline{b} = b\}}{\{a+\epsilon\overline{a}\,|\,a \in A\}}$$

such that

$$\begin{aligned}\mu(ax)\ &=\ a\mu(x)\overline{a}\ ,\\ \mu(x+y)\ &=\ \mu(x)+\mu(y)+\lambda(x,y)+\lambda(y,x) \in Q_\epsilon(A,S)\ ,\\ \mu(x)\ &=\ \lambda(x,x) \in \mathrm{im}(Q_\epsilon(A,S) \to S^{-1}A/A)\ (x,y \in T, a \in A)\ .\end{aligned}$$

A *lagrangian* $U \subset T$ for (T,λ,μ) is a lagrangian for (T,λ) such that $\mu|_U = 0$. □

We refer to Ranicki [16, 3.5] for the development of the theory of ϵ-symmetric and ϵ-quadratic linking formations over (A,S).

From now on, we shall only be concerned with A,S which satisfy :

Hypothesis 4.2. A,S are such that

$$\widehat{H}^*(\mathbb{Z}_2;S^{-1}A)\ =\ 0\ .$$

□

Example 4.3. Hypothesis 4.2 is satisfied if $1/2 \in S^{-1}A$, e.g. if A is even and

$$S \;=\; (2)^{\infty} \;=\; \{2^i \,|\, i \geqslant 0\} \subset A \;,\; S^{-1}A \;=\; A[1/2] \;.$$

□

Proposition 4.4. (i) *For $n = 2$ (resp. 1) the relative group $L^n(A, S, \epsilon)$ in the ϵ-symmetric L-theory localization exact sequence*

$$\dots \to L^n(A, \epsilon) \to L^n_I(S^{-1}A, \epsilon) \to L^n(A, S, \epsilon) \to L^{n-1}(A, \epsilon) \to \dots$$

is the Witt group of nonsingular $(-\epsilon)$-symmetric linking forms (resp. ϵ-symmetric linking formations) over (A, S), with $I = \mathrm{im}(\widetilde{K}_0(A) \to \widetilde{K}_0(S^{-1}A))$. The skew-suspension maps

$$\overline{S} \;:\; L^n(A, S, \epsilon) \to L^{n+2}(A, S, -\epsilon) \;(n \geqslant 1)$$

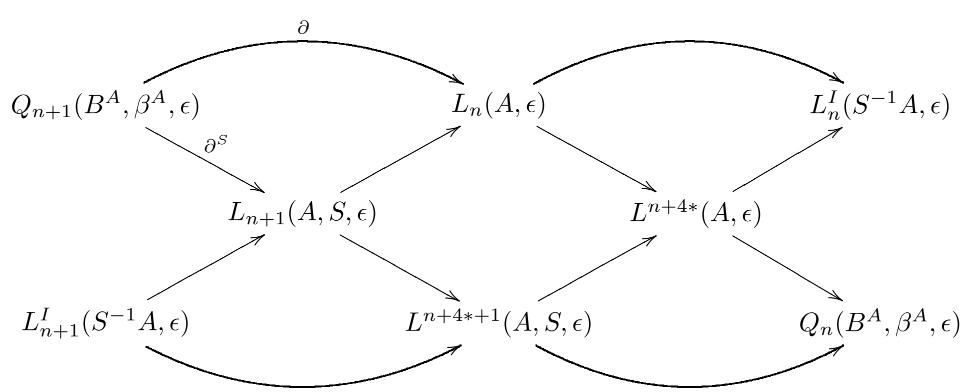
are isomorphisms if and only if the skew-suspension maps

$$\overline{S} \;:\; L^n(A, \epsilon) \to L^{n+2}(A, -\epsilon) \;(n \geqslant 0)$$

are isomorphisms.
(ii) *The relative group $L_n(A, S, \epsilon)$ for $n = 2k$ (resp. $2k+1$) in the ϵ-quadratic L-theory localization exact sequence*

$$\dots \to L_n(A, \epsilon) \to L^I_n(S^{-1}A, \epsilon) \to L_n(A, S, \epsilon) \to L_{n-1}(A, \epsilon) \to \dots$$

is the Witt group of nonsingular $(-1)^k\epsilon$-quadratic linking forms (resp. formations) over (A, S).
(iii) *The 4-periodic ϵ-symmetric and ϵ-quadratic localization exact sequences interleave in a commutative braid of exact sequences*



Proof. (i)+(ii) See [16, §3].
(iii) For A, S satisfying Hypothesis 4.2 the ϵ-symmetrization maps for the L-groups of $S^{-1}A$ are isomorphisms

$$1 + T_\epsilon \;:\; L^I_n(S^{-1}A, \epsilon) \xrightarrow{\;\cong\;} L^n_I(S^{-1}A, \epsilon) \;.$$

□

Definition 4.5. (i) An *ϵ-quadratic S-formation* $(Q,\psi;F,G)$ over A is an ϵ-quadratic formation such that

$$S^{-1}F \oplus S^{-1}G \;=\; S^{-1}Q \;,$$

or equivalently such that $Q/(F+G)$ is an (A,S)-module.
(iii) A *stable isomorphism* of ϵ-quadratic S-formations over A

$$[f] \;:\; (Q_1,\psi_1;F_1,G_1) \to (Q_2,\psi_2;F_2,G_2)$$

is an isomorphism of the type

$$f \;:\; (Q_1,\psi_1;F_1,G_1) \oplus (N_1,\nu_1;H_1,K_1) \to (Q_2,\psi_2;F_2,G_2) \oplus (N_2,\nu_2;H_2,K_2)$$

with $N_1 = H_1 \oplus K_1$, $N_2 = H_2 \oplus K_2$. □

Proposition 4.6. (i) *A $(-\epsilon)$-quadratic S-formation $(Q,\psi;F,G)$ over A determines a nonsingular ϵ-quadratic linking form (T,λ,μ) over (A,S), with*

$$\begin{aligned} &T \;=\; Q/(F+G)\;,\\ &\lambda \;:\; T\times T \to S^{-1}A/A\;;\; (x,y) \mapsto (\psi-\epsilon\psi^*)(x)(z)/s\\ &\mu \;:\; T \to Q_\epsilon(A,S)\;;\; y \mapsto (\psi-\epsilon\psi^*)(x)(z)/s - \psi(y)(y)\\ &(x,y\in Q\;,\; z\in G\;,\; s\in S\;,\; sy-z\in F)\;. \end{aligned}$$

(ii) *The isomorphism classes of nonsingular ϵ-quadratic linking forms over A are in one-one correspondence with the stable isomorphism classes of $(-\epsilon)$-quadratic S-formations over A.*

Proof. See Proposition 3.4.3 of [16]. □

For any $S^{-1}A$-contractible f.g. projective A-module chain complexes concentrated in degrees $k, k+1$

$$\begin{aligned} &C \;:\; \cdots \to 0 \to C_{k+1} \to C_k \to 0 \to \dots\;,\\ &D \;:\; \cdots \to 0 \to D_{k+1} \to D_k \to 0 \to \dots \end{aligned}$$

there are natural identifications

$$\begin{aligned} &H^{k+1}(C) \;=\; H_k(C)\widehat{\ }\;,\; H_k(C) \;=\; H^{k+1}(C)\widehat{\ }\;,\\ &H^{k+1}(D) \;=\; H_k(D)\widehat{\ }\;,\; H_k(D) \;=\; H^{k+1}(D)\widehat{\ }\;,\\ &H_0(\mathrm{Hom}_A(C,D)) \;=\; \mathrm{Hom}_A(H_k(C),H_k(D)) \;=\; \mathrm{Tor}_1^A(H^{k+1}(C),H_k(D))\;,\\ &H_1(\mathrm{Hom}_A(C,D)) \;=\; H^{k+1}(C)\otimes_A H_k(D) \;=\; \mathrm{Ext}_A^1(H_k(C),H_k(D))\;,\\ &H_{2k}(C\otimes_A D) \;=\; H_k(C)\otimes_A H_k(D) \;=\; \mathrm{Ext}_A^1(H^{k+1}(C),H_k(D))\;,\\ &H_{2k+1}(C\otimes_A D) \;=\; \mathrm{Hom}_A(H^{k+1}(C),H_k(D)) \;=\; \mathrm{Tor}_1^A(H_k(C),H_k(D))\;. \end{aligned}$$

In particular, an element $\lambda \in H_{2k+1}(C\otimes_A D)$ is a sesquilinear linking pairing

$$\lambda \;:\; H^{k+1}(C)\times H^{k+1}(D) \to S^{-1}A/A\;.$$

An element $\phi \in H_{2k}(C\otimes_A D)$ is a chain homotopy class of chain maps $\phi : C^{2k-*} \to D$, classifying the extension

$$0 \to H_k(D) \to H_k(\phi) \to H^{k+1}(C) \to 0\;.$$

Proposition 4.7. *Given an (A,S)-module T let*

$$B\ :\ \cdots \to 0 \to B_{k+1} \xrightarrow{\ d\ } B_k \to 0 \to \dots$$

be a f.g. projective A-module chain complex concentrated in degrees $k, k+1$ such that $H^{k+1}(B) = T$, $H^k(B) = 0$, so that $H_k(B) = T^\frown$, $H_{k+1}(B) = 0$. The Q-groups in the exact sequence

$$Q^{2k+2}(B) = 0 \longrightarrow \widehat{Q}^{2k+2}(B) \xrightarrow{\ H\ } Q_{2k+1}(B) \xrightarrow{\ 1+T\ } Q^{2k+1}(B) \xrightarrow{\ J\ } \widehat{Q}^{2k+1}(B)$$

have the following interpretation in terms of T.
(i) *The symmetric Q-group*

$$Q^{2k+1}(B)\ =\ H^0(\mathbb{Z}_2; \mathrm{Hom}_A(T, T^\frown), (-1)^{k+1})$$

is the additive group of $(-1)^{k+1}$-symmetric linking pairings λ on T, with $\phi \in Q^{2k+1}(B)$ corresponding to

$$\lambda\ :\ T \times T \to S^{-1}A/A\ ;\ (x,y) \mapsto \phi_0(d^*)^{-1}(x)(y)\ (x, y \in B^{k+1})\ .$$

(ii) *The quadratic Q-group*

$$Q_{2k+1}(B)\ = \frac{\{(\psi_0, \psi_1) \in \mathrm{Hom}_A(B^k, B_{k+1}) \oplus S(B^k) \,|\, d\psi_0 = \psi_1 + (-1)^{k+1}\psi_1^* \in S(B^k)\}}{\{((\chi_0 + (-1)^{k+1}\chi_0^*)d^*, d\chi_0 d^* + \chi_1 + (-1)^k \chi_1^*) \,|\, (\chi_0, \chi_1) \in S(B^{k+1}) \oplus S(B^k)\}}$$

is the additive group of $(-1)^{k+1}$-quadratic linking structures (λ, μ) on T. The element $\psi = (\psi_0, \psi_1) \in Q_{2k+1}(B)$ corresponds to

$$\begin{aligned}\lambda\ &:\ T \times T \to S^{-1}A/A\ ;\ (x,y) \mapsto \psi_0(d^*)^{-1}(x)(y)\ (x,y \in B^{k+1})\ ,\\ \mu\ &:\ T \to Q_{(-1)^{k+1}}(A,S)\ ;\ x \mapsto \psi_0(d^*)^{-1}(x)(x)\ .\end{aligned}$$

(iii) *The hyperquadratic Q-groups of B*

$$\widehat{Q}^n(B)\ =\ H_n(\widehat{d}^{\%} : \widehat{W}^{\%}B_{k+1} \to \widehat{W}^{\%}B_k)$$

are such that

$$\widehat{Q}^{2k}(B)\ =\ \frac{\{(\delta,\chi) \in S(B^{k+1}) \oplus S(B^k) \,|\, \delta^* = (-1)^{k+1}\delta, d\delta d^* = \chi + (-1)^{k+1}\chi^*\}}{\{(\mu + (-1)^{k+1}\mu^*, d\mu d^* + \nu + (-1)^k\nu^*) \,|\, (\mu,\nu) \in S(B^{k+1}) \oplus S(B^k)\}}\ ,$$

$$\widehat{Q}^{2k+1}(B)\ =\ \frac{\{(\delta,\chi) \in S(B^{k+1}) \oplus S(B^k) \,|\, \delta^* = (-1)^k\delta, d\delta d^* = \chi + (-1)^k\chi^*\}}{\{(\mu + (-1)^k\mu^*, d\mu d^* + \nu + (-1)^{k+1}\nu^*) \,|\, (\mu,\nu) \in S(B^{k+1}) \oplus S(B^k)\}}\ ,$$

with universal coefficient exact sequences

$$0 \to T^\frown \otimes_A \widehat{H}^k(\mathbb{Z}_2; A) \to \widehat{Q}^{2k}(B) \xrightarrow{\ \widehat{v}_{k+1}\ } \mathrm{Hom}_A(T, \widehat{H}^{k+1}(\mathbb{Z}_2; A)) \to 0\ ,$$

$$0 \to T^\frown \otimes_A \widehat{H}^{k+1}(\mathbb{Z}_2; A) \to \widehat{Q}^{2k+1}(B) \xrightarrow{\ \widehat{v}_k\ } \mathrm{Hom}_A(T, \widehat{H}^k(\mathbb{Z}_2; A)) \to 0\ .$$

□

Let $f : C \to D$ be a chain map of $S^{-1}A$-contractible A-module chain complexes concentrated in degrees $k, k+1$, inducing the A-module morphism

$$f^*\ =\ j\ :\ U\ =\ H^{k+1}(D) \to T\ =\ H^{k+1}(C)\ .$$

By Proposition 4.7 (i) a $(2k+1)$-dimensional symmetric Poincaré complex (C, ϕ) is essentially the same as a nonsingular $(-1)^{k+1}$-symmetric linking form (T, λ), and a

$(2k+2)$-dimensional symmetric Poincaré pair $(f : C \to D, (\delta\phi, \phi))$ is essentially the same as a lagrangian U for (T, λ), with $j = f^* : U \to T$ the inclusion. Similarly, a $(2k+1)$-dimensional quadratic Poincaré complex (C, ψ) is essentially the same as a nonsingular $(-1)^{k+1}$-quadratic linking form (T, λ, μ), and a $(2k+2)$-dimensional quadratic Poincaré pair $(f : C \to D, (\delta\psi, \psi))$ is essentially the same as a lagrangian $U \subset T$ for (T, λ, μ). A $(2k+2)$-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi))$ is a nonsingular $(-1)^{k+1}$-quadratic form (T, λ, μ) together with a lagrangian $U \subset T$ for the nonsingular $(-1)^{k+1}$-symmetric linking form (T, λ).

Proposition 4.8. *Let U be an (A, S)-module together with an A-module morphism $\mu_1 : U \to \widehat{H}^{k+1}(\mathbb{Z}_2; A)$, defining a $(-1)^{k+1}$-quadratic linking form (U, λ_1, μ_1) over (A, S) with $\lambda_1 = 0$.*
(i) *There exists a map of chain bundles $(d, \chi) : (B_{k+2}, 0) \to (B_{k+1}, \delta)$ concentrated in degree $k+1$ such that the cone chain bundle $(B, \beta) = \mathcal{C}(d, \chi)$ has*

$$H_{k+1}(B) = U \ , \ H^{k+2}(B) = U\widehat{\ } \ , \ H_{k+2}(B) = H^{k+1}(B) = 0 \ ,$$
$$\beta = [\delta] = \mu_1 \in \widehat{Q}^0(B^{0-*}) = \mathrm{Hom}_A(U, \widehat{H}^{k+1}(\mathbb{Z}_2; A)) \ .$$

(ii) *The $(2k+2)$-dimensional twisted quadratic Q-group of (B, β) as in* (i)

$$Q_{2k+2}(B, \beta) =$$
$$\frac{\{(\phi, \theta) \in S(B^{k+1}) \oplus S(B^{k+1}) \,|\, \phi^* = (-1)^{k+1}\phi, \phi - \phi\delta\phi^* = \theta + (-1)^{k+1}\theta^*\}}{\{(d, \chi)_{\%}(\nu) + (0, \eta + (-1)^k\eta^*) \,|\, \nu \in S(B^{k+2}), \eta \in S(B^{k+1})\}}$$
$$((d, \chi)_{\%}(\nu) = (d(\nu + (-1)^{k+1}\nu^*)d^*, d\nu d^* - d(\nu + (-1)^{k+1}\nu^*)\chi(\nu^* + (-1)^{k+1}\nu)d^*))$$

is the additive group of isomorphism classes of extensions of U to a nonsingular $(-1)^{k+1}$-quadratic linking form (T, λ, μ) over (A, S) such that $U \subset T$ is a lagrangian of the $(-1)^{k+1}$-symmetric linking form (T, λ) and

$$\beta = \mu|_U : H_{k+1}(B) = U \to \widehat{H}^{k+1}(\mathbb{Z}_2; A) = \ker(Q_{(-1)^{k+1}}(A, S) \to S^{-1}A/A) \ .$$

(iii) *An element $(\phi, \theta) \in Q_{2k+2}(B, \beta)$ is the algebraic normal invariant (2.13) of the $(2k+2)$-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi, \psi) \in Q^{2k+2}_{2k+2}(f))$ with*

$$d_C = \begin{pmatrix} d & \phi \\ 0 & d^* \end{pmatrix} : C_{k+1} = B_{k+2} \oplus B^{k+1} \to C_k = B_{k+1} \oplus B^{k+2} \ ,$$
$$f = \text{projection} : C \to D = B^{2k+2-*}$$

constructed as in Proposition 2.12 (ii), *corresponding to the quadruple $(T, \lambda, \mu; U)$ given by*

$$j = f^* : U = H^{k+1}(D) = H_{k+1}(B) \to T = H^{k+1}(C) \ .$$

The A-module extension

$$0 \to U \to T \to U\widehat{\ } \to 0$$

is classified by

$$[\phi] \in H_{2k+2}(B \otimes_A B) = U \otimes_A U = \mathrm{Ext}^1_A(U\widehat{\ }, U) \ .$$

(iv) *The $(-1)^{k+1}$-quadratic linking form (T,λ,μ) in* (iii) *corresponds to the $(-1)^k$-quadratic S-formation $(Q,\psi;F,G)$ with*

$$\begin{aligned}
(Q,\psi) &= H_{(-1)^k}(F)\ ,\ F\ =\ B_{k+2}\oplus B^{k+1}\ ,\\
G &= \mathrm{im}\Big(\begin{pmatrix} 1 & 0\\ -\delta d & 1-\delta\phi\\ 0 & (-1)^{k+1}d^*\\ d & \phi\end{pmatrix} : B_{k+2}\oplus B^{k+1}\to B_{k+2}\oplus B^{k+1}\oplus B^{k+2}\oplus B_{k+1}\Big)\subset F\oplus F^*
\end{aligned}$$

such that

$$F\cap G\ =\ \{0\}\ ,\ Q/(F+G)\ =\ H^{k+1}(C)\ =\ T\ .$$

The inclusion $U\to T$ is resolved by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_{k+2} & \xrightarrow{\ d\ } & B_{k+1} & \longrightarrow & U & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle\begin{pmatrix}1\\0\end{pmatrix}} & & \Big\downarrow{\scriptstyle\begin{pmatrix}0\\1\end{pmatrix}} & & \Big\downarrow & & \\
0 & \longrightarrow & B_{k+2}\oplus B^{k+1} & \xrightarrow{\begin{pmatrix}0 & (-)^{k+1}d^*\\ d & \phi\end{pmatrix}} & B^{k+2}\oplus B_{k+1} & \longrightarrow & T & \longrightarrow & 0
\end{array}$$

(v) *If the involution on A is even and $k=-1$ then*

$$Q_0(B,\beta)\ =\ \frac{\{\phi\in \mathrm{Sym}(B^0)\,|\,\phi-\phi\delta\phi\in \mathrm{Quad}(B^0)\}}{\{d\sigma d^*\,|\,\sigma\in \mathrm{Quad}(B^1)\}}\ .$$

An extension of $U=\mathrm{coker}(d:B_1\to B_0)$ to a nonsingular quadratic linking form (T,λ,μ) over (A,S) with $\mu|_U=\mu_1$ and $U\subset T$ a lagrangian of (T,λ) is classified by $\phi\in Q_0(B,\beta)$ such that $\lambda:T\to T\widehat{\ }$ is resolved by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_1\oplus B^0 & \xrightarrow{\begin{pmatrix}0 & d^*\\ d & \phi\end{pmatrix}} & B^1\oplus B_0 & \longrightarrow & T & \longrightarrow & 0\\
 & & {\scriptstyle\begin{pmatrix}1 & 0\\ -\delta d & 1-\delta\phi\end{pmatrix}}\Big\downarrow & & \Big\downarrow{\scriptstyle\begin{pmatrix}1 & -d^*\delta\\ 0 & 1-\phi\delta\end{pmatrix}} & & \Big\downarrow{\scriptstyle\lambda} & & \\
0 & \longrightarrow & B_1\oplus B^0 & \xrightarrow{\begin{pmatrix}0 & d^*\\ d & \phi\end{pmatrix}} & B^1\oplus B_0 & \longrightarrow & T\widehat{\ } & \longrightarrow & 0
\end{array}$$

and

$$\begin{aligned}
&T\ =\ \mathrm{coker}(\begin{pmatrix}0 & d^*\\ d & \phi\end{pmatrix}: B_1\oplus B^0\to B^1\oplus B_0)\ ,\\
&\lambda\ :\ T\times T\to S^{-1}A/A\ ;\\
&((x_1,x_0),(y_1,y_0))\mapsto -d^{-1}\phi(d^*)^{-1}(x_1)(y_1)+d^{-1}(x_1)(y_0)+(d^*)^{-1}(x_0)(y_1)\ ,\\
&\mu\ :\ T\to Q_{+1}(A,S)\ ;\\
&(x_1,x_0)\mapsto -d^{-1}\phi(d^*)^{-1}(x_1)(x_1)+d^{-1}(x_1)(x_0)+(d^*)^{-1}(x_0)(x_1)-\delta(x_0)(x_0)\ ,\\
&(x_0,y_0\in B_0\ ,\ x_1,y_1\in B^1)\ .
\end{aligned}$$

□

4.2. **The Linking Arf Invariant.**

Definition 4.9. The *linking Arf invariant* of a nonsingular $(-1)^{k+1}$-quadratic linking form (T,λ,μ) over (A,S) together with a lagrangian $U\subset T$ for (T,λ) is the image

$$(T,\lambda,\mu;U) \;=\; (g,\chi)_\%(\phi,\theta)\in \widehat{L}^{4*+2k+2}(A) \;=\; Q_{2k+2}(B^A,\beta^A)$$

of the algebraic normal invariant $(\phi,\theta)\in Q_{2k+2}(\mathcal{C}(f),\gamma)$ (2.13) of the corresponding $(2k+2)$-dimensional (symmetric, quadratic) Poincaré pair $(f:C\to D,(\delta\phi,\psi)\in Q^{2k+2}_{2k+2}(f))$ concentrated in degrees $k,k+1$ with

$$f^* \;=\; j \;:\; H^{k+1}(D) \;=\; U\to H^{k+1}(C) \;=\; T\ ,$$

and $(g,\chi)_\%$ induced by the classifying chain bundle map $(g,\chi):(\mathcal{C}(f),\gamma)\to(B^A,\beta^A)$. □

The chain bundle $(\mathcal{C}(f),\gamma)$ in 4.9 is (up to equivalence) of the type (B,β) considered in Proposition 4.8 (i): the algebraic normal invariant $(\phi,\theta)\in Q_{2k+2}(B,\beta)$ classifies the extension of (U,β) to a lagrangian of a $(-1)^{k+1}$-symmetric linking form (T,λ) with a $(-1)^{k+1}$-quadratic function μ on T such that $\mu|_U=\beta$. The linking Arf invariant $(T,\lambda,\mu;U)\in Q_{2k+2}(B^A,\beta^A)$ gives the Witt class of $(T,\lambda,\mu;U)$. The boundary map

$$\partial \;:\; Q_{2k+2}(B^A,\beta^A)\to L_{2k+1}(A)\ ;\ (T,\lambda,\mu;U)\mapsto (Q,\psi;F,G)$$

sends the linking Arf invariant to the Witt class of the $(-1)^k$-quadratic formation $(Q,\psi;F,G)$ constructed in 4.8 (iv).

Theorem 4.10. *Let A be an r-even ring with A_2-module basis $\{x_1=1,x_2,\dots,x_r\}\subset \widehat{H}^0(\mathbb{Z}_2;A)$, and let*

$$X \;=\; \begin{pmatrix} x_1 & 0 & 0 & \dots & 0\\ 0 & x_2 & 0 & \dots & 0\\ 0 & 0 & x_3 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & x_r\end{pmatrix}\in \mathrm{Sym}_r(A)\ ,$$

so that by Theorem 2.30

$$Q_{2k}(B^A,\beta^A) \;=\; \begin{cases}\dfrac{\{M\in \mathrm{Sym}_r(A)\,|\,M-MXM\in \mathrm{Quad}_r(A)\}}{4\mathrm{Quad}_r(A)+\{2(N+N^t)-N^tXN\,|\,N\in M_r(A)\}} & \text{if } k=0\\ 0 & \text{if } k=1\ .\end{cases}$$

(i) *Let*

$$S \;=\; (2)^\infty\subset A\ ,$$

so that

$$S^{-1}A \;=\; A[1/2]$$

and $\widehat{H}^0(\mathbb{Z}_2;A)$ is an (A,S)-module. The hyperquadratic L-group $\widehat{L}^0(A)$ fits into the exact sequence

$$\dots\to L^1(A,S)\to \widehat{L}^0(A)\to L_0(A,S)\to L^0(A,S)\to\dots\ .$$

The linking Arf invariant of a nonsingular quadratic linking form (T,λ,μ) over (A,S) with a lagrangian $U\subset T$ for (T,λ) is the Witt class

$$(T,\lambda,\mu;U)\in Q_0(B^A,\beta^A) \;=\; \widehat{L}^{4*}(A)\ .$$

(ii) *Given $M \in \mathrm{Sym}_r(A)$ such that $M - MXM \in \mathrm{Quad}_r(A)$ let (T_M, λ_M, μ_M) be the nonsingular quadratic linking form over (A,S) corresponding to the (-1)-quadratic S-formation over A (4.6)*

$$(Q_M, \psi_M; F_M, G_M) \;=\; (H_-(A^{2r}); A^{2r}, \mathrm{im}\Big(\begin{pmatrix} \begin{pmatrix} I & 0 \\ -2X & I - XM \end{pmatrix} \\ \begin{pmatrix} 0 & 2I \\ 2I & M \end{pmatrix} \end{pmatrix} : A^{2r} \to A^{2r} \oplus A^{2r}\Big))$$

and let

$$U_M \;=\; (A_2)^r \subset T_M \;=\; Q_M/(F_M + G_M) \;=\; \mathrm{coker}(G_M \to F_M^*)$$

be the lagrangian for the nonsingular symmetric linking form (T_M, λ_M) over (A,S) with the inclusion $U_M \to T_M$ resolved by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^r & \xrightarrow{\;2I\;} & A^r & \longrightarrow & U_M & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 0 \\ 1 \end{pmatrix}} & & \Big\downarrow & & \\ 0 & \longrightarrow & A^r \oplus A^r & \xrightarrow{\begin{pmatrix} 0 & 2I \\ 2I & M \end{pmatrix}} & A^r \oplus A^r & \longrightarrow & T_M & \longrightarrow & 0 \end{array}$$

The function

$$Q_0(B^A, \beta^A) \to \widehat{L}^{4*}(A) \;;\; M \mapsto (T_M, \lambda_M, \mu_M; U_M)$$

is an isomorphism, with inverse given by the linking Arf invariant.

(iii) *Let (T, λ, μ) be a nonsingular quadratic linking form over (A,S) together with a lagrangian $U \subset T$ for (T, λ). For any f.g. projective A-module resolution of U*

$$0 \to B_1 \xrightarrow{\;d\;} B_0 \to U \to 0$$

let

$$\delta \in \mathrm{Sym}(B_0) \;,\; \phi \in \mathrm{Sym}(B^0) \;,\; \beta \;=\; [\delta] \;=\; \mu|_U \in \widehat{Q}^0(B^{0-*}) \;=\; \mathrm{Hom}_A(U, \widehat{H}^0(\mathbb{Z}_2; A))$$

be as in Proposition 4.8 (i),(v)*, so that*

$$d^*\delta d \in \mathrm{Quad}(B_1) \;,\; \phi - \phi\delta\phi \in \mathrm{Quad}(B^0)$$

and

$$\phi \in Q_0(B, \beta) \;=\; \frac{\ker(J_\delta : \mathrm{Sym}(B^0) \to \mathrm{Sym}(B^0)/\mathrm{Quad}(B^0))}{\mathrm{im}((d^*)^{\%} : \mathrm{Quad}(B^1) \to \mathrm{Sym}(B^0))}$$

classifies $(T, \lambda, \mu; U)$. Lift $\beta : U \to \widehat{H}^0(\mathbb{Z}_2; A)$ to an A-module morphism $g : B_0 \to A^r$ such that

$$gd(B_1) \subseteq 2A^r \;,\; \delta \;=\; g^*Xg \in \widehat{H}^0(\mathbb{Z}_2; S(B^0), T) \;=\; \mathrm{Sym}(B^0)/\mathrm{Quad}(B^0) \;.$$

The linking Arf invariant is

$$(T, \lambda, \mu; U) \;=\; g\phi g^* \in Q_0(B^A, \beta^A) \;.$$

(iv) *For any $M = (m_{ij}) \in \mathrm{Sym}_r(A)$ with $m_{ij} \in 2A$*

$$M - MXM \;=\; 2(M/2 - 2(M/2)X(M/2)) \in \mathrm{Quad}_r(A)$$

and so M represents an element $M \in Q_0(B^A, \beta^A)$. The invertible matrix

$$\begin{pmatrix} -M/2 & I \\ I & 0 \end{pmatrix} \in M_{2r}(A)$$

is such that

$$\begin{pmatrix} -M/2 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & 2I \\ 2I & M \end{pmatrix} = \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix} ,$$

$$\begin{pmatrix} I & 0 \\ -2X & I-XM \end{pmatrix}\begin{pmatrix} -M/2 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} -M/2 & I \\ I & -2X \end{pmatrix}$$

so that $(Q_M,\psi_M;F_M,G_M)$ *is isomorphic to the* (-1)*-quadratic* S*-formation*

$$(Q'_M,\psi'_M;F'_M,G'_M) = (H_-(A^{2r});A^{2r},\mathrm{im}\Big(\begin{pmatrix} \begin{pmatrix} -M/2 & I \\ I & -2X \end{pmatrix} \\ \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix} \end{pmatrix} : A^{2r}\to A^{2r}\oplus A^{2r}\Big)) ,$$

corresponding to the nonsingular quadratic linking form over (A,S)

$$(T'_M,\lambda'_M,\mu'_M) = ((A_2)^r\oplus(A_2)^r,\begin{pmatrix} -M/4 & I/2 \\ I/2 & 0 \end{pmatrix},\begin{pmatrix} -M/4 \\ -X \end{pmatrix})$$

with $2T'_M = 0$*, and* $U'_M = 0\oplus(A_2)^r\subset T'_M$ *a lagrangian for the symmetric linking form* (T'_M,λ'_M)*. The linking Arf invariant of* $(T'_M,\lambda'_M,\mu'_M;U'_M)$ *is*

$$(T'_M,\lambda'_M,\mu'_M;U'_M) = M\in Q_0(B^A,\beta^A) .$$

Proof. (i) $\widehat{H}^0(\mathbb{Z}_2;A)$ has an $S^{-1}A$-contractible f.g. free A-module resolution

$$0 \longrightarrow A^r \xrightarrow{\;2\;} A^r \xrightarrow{\;x\;} \widehat{H}^0(\mathbb{Z}_2;A) \longrightarrow 0 .$$

The exact sequence for $\widehat{L}^0(A)$ is given by the exact sequence of Proposition 4.4 (iii)

$$\dots\to L^{4*+1}(A,S)\to Q_0(B^A,\beta^A)\to L_0(A,S)\to L^{4*}(A,S)\to\dots$$

and the isomorphism $Q_0(B^A,\beta^A)\cong\widehat{L}^{4*}(A)$.
(ii) The isomorphism

$$Q_0(B^A,\beta^A)\to\widehat{L}^{4*}(A) ;\ M\mapsto(T_M,\lambda_M,\mu_M;U_M)$$

is given by Proposition 2.16.
(iii) Combine (ii) and Proposition 4.8.
(iv) By construction. □

5. Application to UNil

5.1. **Background.** The topological context for the unitary nilpotent L-groups UNil_* is the following. Let N^n be a closed connected manifold together with a decomposition into n-dimensional connected submanifolds $N_-,N_+\subset N$ such that

$$N = N_-\cup N_+$$

and

$$N_\cap = N_-\cap N_+ = \partial N_- = \partial N_+\subset N$$

is a connected $(n-1)$-manifold with $\pi_1(N_\cap)\to\pi_1(N_\pm)$ injective. Then

$$\pi_1(N) = \pi_1(N_-)*_{\pi_1(N_\cap)}\pi_1(N_+)$$

with $\pi_1(N_\pm)\to\pi_1(N)$ injective. Let M be an n-manifold. A homotopy equivalence $f:M\to N$ is called *splittable along* $N_\cap$ if it is homotopic to a map f', transverse regular to $N_\cap$ (whence $f'^{-1}(N_\cap)$ is an $(n-1)$-dimensional submanifold of M), and whose restriction $f'^{-1}(N_\cap) \to N_\cap$, and a fortiori also $f'^{-1}(N_\pm) \to N_\pm$, is a homotopy equivalence.

We ask the following question : given a simple homotopy equivalence $f : M \to N$, when is M h-cobordant to a manifold M' such that the induced homotopy equivalence $f' : M' \to N$ is splittable along $N_\cap$? The answer is given by Cappell [5], [6] : the problem has a positive solution if and only if a Whitehead torsion obstruction

$$\overline{\Phi}(\tau(f)) \in \widehat{H}^n(\mathbb{Z}_2; \ker(\widetilde{K}_0(A) \to \widetilde{K}_0(B_+) \oplus \widetilde{K}_0(B_-)))$$

(which is 0 if f is simple) and an algebraic L-theory obstruction

$$\chi^h(f) \in \mathrm{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+)$$

vanish, where

$$A \ = \ \mathbb{Z}[\pi_1(N_\cap)] \ , \ B_\pm \ = \ \mathbb{Z}[\pi_1(N_\pm)] \ , \ \mathcal{N}_\pm \ = \ B_\pm - A \ .$$

The groups $\mathrm{UNil}_*(A; \mathcal{N}_-, \mathcal{N}_+)$ are 4-periodic and 2-primary, and vanish if the inclusions $\pi_1(N_\cap) \hookrightarrow \pi_1(N_\pm)$ are square root closed. The groups $\mathrm{UNil}_*(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ arising from the expression of the infinite dihedral group as a free product

$$D_\infty \ = \ \mathbb{Z}_2 * \mathbb{Z}_2$$

are of particular interest. Cappell [3] showed that

$$\mathrm{UNil}_{4k+2}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \ = \ \mathrm{UNil}_{4k+2}(\mathbb{Z}; \mathbb{Z}[\mathbb{Z}_2 - \{1\}], \mathbb{Z}[\mathbb{Z}_2 - \{1\}])$$

contains $(\mathbb{Z}_2)^\infty$, and deduced that there is a manifold homotopy equivalent to the connected sum $\mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}$ which does not have a compatible connected sum decomposition. With

$$B \ = \ \mathbb{Z}[\pi_1(N)] \ = \ B_1 *_A B_2$$

the map

$$\mathrm{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+) \longrightarrow L_{n+1}(B)$$

given by sending the splitting obstruction $\chi^h(f)$ to the surgery obstruction of an $(n+1)$-dimensional normal map between f and a split homotopy equivalence, is a split monomorphism, and

$$L_{n+1}(B) \ = \ L^K_{n+1}(A \to B_+ \cup B_-) \oplus \mathrm{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+)$$

with $K = \ker(\widetilde{K}_0(A) \to \widetilde{K}_0(B_+) \oplus \widetilde{K}_0(B_-))$. Farrell [11] established a factorization of this map as

$$\mathrm{UNil}_{n+1}(A; \mathcal{N}_-, \mathcal{N}_+) \longrightarrow \mathrm{UNil}_{n+1}(B; B, B) \longrightarrow L_{n+1}(B) \ .$$

Thus the groups $\mathrm{UNil}_n(A; A, A)$ for any ring A with involution acquire special importance, and we shall use the usual abbreviation

$$\mathrm{UNil}_n(A) \ = \ \mathrm{UNil}_n(A; A, A) \ .$$

Cappell [3], [4], [5] proved that $\mathrm{UNil}_{4k}(\mathbb{Z}) = 0$ and that $\mathrm{UNil}_{4k+2}(\mathbb{Z})$ is infinitely generated. Farrell [11] showed that for any ring A, $4\mathrm{UNil}_*(A) = 0$. Connolly and Koźniewski [9] obtained $\mathrm{UNil}_{4k+2}(\mathbb{Z}) = \bigoplus_1^\infty \mathbb{Z}_2$.

For any ring with involution A let NL_* denote the L-theoretic analogues of the nilpotent K-groups

$$NK_*(A) \ = \ \ker(K_*(A[x]) \to K_*(A)) \ ,$$

that is

$$NL_*(A) \;=\; \ker(L_*(A[x]) \to L_*(A))$$

where $A[x] \to A$ is the augmentation map $x \mapsto 0$. Ranicki [16, 7.6] used the geometric interpretation of $\mathrm{UNil}_*(A)$ to identify $NL_*(A) = \mathrm{UNil}_*(A)$ in the case when $A = \mathbb{Z}[\pi]$ is the integral group ring of a finitely presented group π. The following was obtained by pure algebra :

Proposition 5.1. (Connolly and Ranicki [10]) *For any ring with involution A*

$$\mathrm{UNil}_*(A) \;\cong\; NL_*(A)\ .$$

□

It was further shown in [10] that $\mathrm{UNil}_1(\mathbb{Z}) = 0$ and $\mathrm{UNil}_3(\mathbb{Z})$ was computed up to extensions, thus showing it to be infinitely generated.

Connolly and Davis [8] related $\mathrm{UNil}_3(\mathbb{Z})$ to quadratic linking forms over $\mathbb{Z}[x]$ and computed the Grothendieck group of the latter. By Proposition 5.1

$$\mathrm{UNil}_3(\mathbb{Z}) \;\cong\; \ker(L_3(\mathbb{Z}[x]) \to L_3(\mathbb{Z})) \;=\; L_3(\mathbb{Z}[x])\ ,$$

using the classical fact $L_3(\mathbb{Z}) = 0$. From a diagram chase one gets

$$L_3(\mathbb{Z}[x]) \;\cong\; \ker(L_0(\mathbb{Z}[x],(2)^\infty) \to L_0(\mathbb{Z},(2)^\infty))\ .$$

By definition, $L_0(\mathbb{Z}[x],(2)^\infty)$ is the Witt group of nonsingular quadratic linking forms (T,λ,μ) over $(\mathbb{Z}[x],(2)^\infty)$, with $2^nT = 0$ for some $n \geqslant 1$. Let $\mathcal{L}(\mathbb{Z}[x],2)$ be a similar Witt group, the difference being that the underlying module T is required to satisfy $2T = 0$. The main results of [8] are

$$L_0(\mathbb{Z}[x],(2)^\infty) \;\cong\; \mathcal{L}(\mathbb{Z}[x],2)$$

and

$$\mathcal{L}(\mathbb{Z}[x],2) \;\cong\; \frac{x\mathbb{Z}_4[x]}{\{2(p^2-p)\,|\,p \in x\mathbb{Z}_4[x]\}} \oplus \mathbb{Z}_2[x]\ .$$

By definition, a ring A is 0-*dimensional* if it is hereditary and noetherian, or equivalently if every submodule of a f.g. projective A-module is f.g. projective. In particular, a Dedekind ring A is 0-dimensional. The symmetric and hyperquadratic L-groups of a 0-dimensional A are 4-periodic

$$L^n(A) \;=\; L^{n+4}(A)\ ,\ \widehat{L}^n(A) \;=\; \widehat{L}^{n+4}(A)\ .$$

Proposition 5.2. (Connolly and Ranicki [10]) *For any 0-dimensional ring A with involution*

$$Q_{n+1}(B^{A[x]},\beta^{A[x]}) \;=\; Q_{n+1}(B^A,\beta^A) \oplus \mathrm{UNil}_n(A)\ (n \in \mathbb{Z})\ .$$

Proof. For any ring with involution A the inclusion $A \to A[x]$ and the augmentation $A[x] \to A; x \mapsto 0$ determine a functorial splitting of the exact sequence

$$\dots \to L_n(A[x]) \to L^n(A[x]) \to \widehat{L}^n(A[x]) \to L_{n-1}(A[x]) \to \dots$$

as a direct sum of the exact sequences

$$\dots \to L_n(A) \to L^n(A) \to \widehat{L}^n(A) \to L_{n-1}(A) \to \dots\ ,$$

$$\dots \to NL_n(A) \to NL^n(A) \to N\widehat{L}^n(A) \to NL_{n-1}(A) \to \dots\ .$$

with $\widehat{L}^{n+4*}(A) = Q_n(B^A, \beta^A)$. It is proved in [10] that for a 0-dimensional A

$$L^n(A[x]) \;=\; L^n(A)\ ,\ NL^n(A) \;=\; 0\ ,\ N\widehat{L}^{n+1}(A) \;=\; NL_n(A) \;=\; \mathrm{UNil}_n(A)\ .$$

□

Example 5.3. Proposition 5.2 applies to $A = \mathbb{Z}$, so that

$$Q_{n+1}(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) \;=\; Q_{n+1}(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) \oplus \mathrm{UNil}_n(\mathbb{Z})$$

with $Q_*(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) = \widehat{L}^*(\mathbb{Z})$ as given by Example 2.32. □

5.2. **The Computation of $Q_*(B^{A[x]}, \beta^{A[x]})$ for 1-even A.** We shall now compute the groups

$$\widehat{L}^n(A[x]) \;=\; Q_n(B^{A[x]}, \beta^{A[x]})\ (n(\mathrm{mod}\,4))$$

for a 1-even ring A. The special case $A = \mathbb{Z}$ computes

$$\widehat{L}^n(\mathbb{Z}[x]) \;=\; Q_n(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) \;=\; \widehat{L}^n(\mathbb{Z}) \oplus \mathrm{UNil}_{n-1}(\mathbb{Z})\ .$$

Proposition 5.4. *The universal chain bundle over $A[x]$ is given by*

$$(B^{A[x]}, \beta^{A[x]}) \;=\; \bigoplus_{i=-\infty}^{\infty} (C(X), \gamma(X))_{*+2i}$$

with $(C(X), \gamma(X))$ the chain bundle over $A[x]$ given by the construction of (2.23) for

$$X \;=\; \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in \mathrm{Sym}_2(A[x])\ .$$

The twisted quadratic Q-groups of $(B^{A[x]}, \beta^{A[x]})$ are

$$Q_n(B^{A[x]}, \beta^{A[x]}) \;=$$

$$\begin{cases} Q_0(C(X), \gamma(X)) \;=\; \dfrac{\{M \in \mathrm{Sym}_2(A[x]) \,|\, M - MXM \in \mathrm{Quad}_2(A[x])\}}{4\mathrm{Quad}_2(A[x]) + \{2(N+N^t) - N^tXN \,|\, N \in M_2(A[x])\}} & \textit{if } n = 0 \\ \mathrm{im}(N_{\gamma(X)} : Q_1(C(X), \gamma(X)) \to Q^1(C(X))) \;=\; \ker(J_{\gamma(X)} : Q^1(C(X)) \to \widehat{Q}^1(C(X))) & \\ = \dfrac{\{N \in M_2(A[x]) \,|\, N + N^t \in 2\mathrm{Sym}_2(A[x]), \frac{1}{2}(N+N^t) - N^tXN \in \mathrm{Quad}_2(A[x])\}}{2M_2(A[x])} & \textit{if } n = 1 \\ 0 & \textit{if } n = 2 \\ Q_{-1}(C(X), \gamma(X)) \;=\; \dfrac{\mathrm{Sym}_2(A[x])}{\mathrm{Quad}_2(A[x]) + \{L - LXL \,|\, L \in \mathrm{Sym}_2(A[x])\}} & \textit{if } n = 3\ . \end{cases}$$

Proof. A special case of Theorem 2.30, noting that by Proposition 2.29 $A[x]$ is 2-even, with $\{1, x\}$ an $A_2[x]$-module basis for $\widehat{H}^0(\mathbb{Z}_2; A[x])$. □

Our strategy for computing $Q_*(B^{A[x]}, \beta^{A[x]})$ will be to first compute $Q_*(C(1), \gamma(1))$, $Q_*(C(x), \gamma(x))$ and then to compute $Q_*(C(X), \gamma(X))$ for

$$(C(X), \gamma(X)) \;=\; (C(1), \gamma(1)) \oplus (C(x), \gamma(x))$$

using the exact sequence given by Proposition 2.8 (ii)

$$\cdots \to H_{n+1}(C(1) \otimes_{A[x]} C(x)) \xrightarrow{\partial} Q_n(C(1), \gamma(1)) \oplus Q_n(C(x), \gamma(x)) \to Q_n(C(X), \gamma(X)) \to H_n(C(1) \otimes_{A[x]} C(x)) \to \dots\ .$$

The connecting maps ∂ have components

$$\begin{aligned}\partial(1) \ &: \ H_{n+1}(C(1)\otimes_{A[x]}C(x)) \to \widehat{Q}^{n+1}(C(1)) \to Q_n(C(1),\gamma(1)) \ ;\\ &\quad (f(1):C(x)^{n+1-*}\to C(1)) \mapsto (0,\widehat{f(1)}^{\%}(S^{n+1}\gamma(x))) \ ,\\ \partial(x) \ &: \ H_{n+1}(C(1)\otimes_{A[x]}C(x)) \to \widehat{Q}^{n+1}(C(x)) \to Q_n(C(x),\gamma(x)) \ ;\\ &\quad (f(x):C(1)^{n+1-*}\to C(x)) \mapsto (0,\widehat{f(x)}^{\%}(S^{n+1}\gamma(1))) \ .\end{aligned}$$

Proposition 5.5. (i) *The twisted quadratic Q-groups*

$$Q_n(C(1),\gamma(1)) \ = \begin{cases} \dfrac{A[x]}{2A[x]+\{a-a^2\,|\,a\in A[x]\}} & \text{if } n=-1\\ \dfrac{\{a\in A[x]\,|\,a-a^2\in 2A[x]\}}{8A[x]+\{4b-4b^2\,|\,b\in A[x]\}} & \text{if } n=0\\ \dfrac{\{a\in A[x]\,|\,a-a^2\in 2A[x]\}}{2A[x]} & \text{if } n=1\end{cases}$$

(*as given by Theorem 2.24*) *are such that*

$$Q_n(C(1),\gamma(1)) \ \cong \ \begin{cases} A_2[x] & \text{if } n=-1\\ A_8\oplus A_4[x]\oplus A_2[x] & \text{if } n=0\\ A_2 & \text{if } n=1\end{cases}$$

with isomorphisms

$$\begin{aligned} f_{-1}(1) \ &: \ Q_{-1}(C(1),\gamma(1)) \to A_2[x] \ ; \ \sum_{i=0}^{\infty} a_i x^i \mapsto a_0 + \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{(2i+1)2^j})x^{i+1} \ ,\\ f_0(1) \ &: \ Q_0(C(1),\gamma(1)) \to A_8\oplus A_4[x]\oplus A_2[x] \ ;\\ &\qquad \sum_{i=0}^{\infty} a_i x^i \mapsto (a_0, \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{(2i+1)2^j}/2)x^i, \sum_{k=0}^{\infty}(a_{2k+2}/2)x^k)\\ f_1(1) \ &: \ Q_1(C(1),\gamma(1)) \to A_2 \ ; \ a \ = \ \sum_{i=0}^{\infty} a_i x^i \mapsto a_0 \ .\end{aligned}$$

The connecting map components $\partial(1)$ are given by

$$\begin{aligned}\partial(1) \ &: \ H_1(C(1)\otimes_{A[x]}C(x)) \ = \ A_2[x] \to Q_0(C(1),\gamma(1)) \ ; \ c\mapsto (0,2c,0) \ ,\\ \partial(1) \ &: \ H_0(C(1)\otimes_{A[x]}C(x)) \ = \ A_2[x] \to Q_{-1}(C(1),\gamma(1)); \ c\mapsto cx.\end{aligned}$$

(ii) *The twisted quadratic Q-groups*

$$Q_n(C(x),\gamma(x)) \ = \begin{cases} \dfrac{A[x]}{2A[x]+\{a-a^2x\,|\,a\in A[x]\}} & \text{if } n=-1\\ \dfrac{\{a\in A[x]\,|\,a-a^2x\in 2A[x]\}}{8A[x]+\{4b-4b^2x\,|\,b\in A[x]\}} & \text{if } n=0\\ \dfrac{\{a\in A[x]\,|\,a-a^2x\in 2A[x]\}}{2A[x]} & \text{if } n=1\end{cases}$$

(*as given by Theorem 2.24*) *are such that*

$$Q_n(C(x),\gamma(x)) \cong \begin{cases} A_2[x] & \text{if } n=-1 \\ A_4[x]\oplus A_2[x] & \text{if } n=0 \\ 0 & \text{if } n=1 \end{cases}$$

with isomorphisms

$$f_{-1}(x) : Q_{-1}(C(x),\gamma(x)) \to A_2[x] \; ; \; a = \sum_{i=0}^{\infty} a_i x^i \mapsto \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{(2i+1)2^j-1})x^i$$

$$f_0(x) : Q_0(C(x),\gamma(x)) \to A_4[x]\oplus A_2[x] \; ;$$

$$\sum_{i=0}^{\infty} a_i x^i \mapsto (\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{(2i+1)2^j-1}/2)x^i, \sum_{k=0}^{\infty}(a_{2k+1}/2)x^k) \; .$$

The connecting map components $\partial(x)$ are given by

$$\partial(x) : H_1(C(1)\otimes_{A[x]} C(x)) = A_2[x] \to Q_0(C(x),\gamma(x)) \; ; \; c \mapsto (2c,0) \; ,$$

$$\partial(x) : H_0(C(1)\otimes_{A[x]} C(x)) = A_2[x] \to Q_{-1}(C(x),\gamma(x)) \; ; \; c \mapsto c \; .$$

Proof. (i) We start with $Q_1(C(1),\gamma(1))$. A polynomial $a(x) = \sum_{i=0}^{\infty} a_i x^i \in A[x]$ is such that $a(x) - a(x)^2 \in 2A[x]$ if and only if

$$a_{2i+1} \; , \; a_{2i+2} - (a_{i+1})^2 \in 2A \; (i \geqslant 0) \; ,$$

if and only if $a_k \in 2A$ for all $k \geqslant 1$, so that $f_1(1)$ is an isomorphism.

Next, we consider $Q_{-1}(C(1),\gamma(1))$. A polynomial $a(x) = \sum_{i=0}^{\infty} a_i x^i \in A[x]$ is such that

$$a(x) \in 2A[x] + \{b(x) - b(x)^2 \,|\, b(x) \in A[x]\}$$

if and only if there exist $b_1, b_2, \dots \in A$ such that

$$a_0 = 0 \; , \; a_1 = b_1 \; , \; a_2 = b_2 - b_1 \; , \; a_3 = b_3 \; , \; a_4 = b_4 - b_2 \; , \; \dots \in A_2 \; ,$$

if and only if

$$a_0 = \sum_{j=0}^{\infty} a_{(2i+1)2^j} = 0 \in A_2 \; (i \geqslant 0)$$

(with $b_{(2i+1)2^j} = \sum_{k=0}^{j} a_{(2i+1)2^k} \in A_2$ for any $i, j \geqslant 0$). Thus $f_{-1}(1)$ is well-defined and injective. The morphism $f_{-1}(1)$ is surjective, since

$$\sum_{i=0}^{\infty} c_i x^i = f_{-1}(1)(c_0 + \sum_{i=0}^{\infty} c_{i+1} x^{2i+1}) \in A_2[x] \; (c_i \in A) \; .$$

The map $\widehat{Q}^1(C(1)) \to Q_0(C(1),\gamma(1))$ is given by

$$\widehat{Q}^1(C(1)) = A_2[x] \to Q_0(C(1),\gamma(1)) = A_8 \oplus A_4[x] \oplus A_2[x] \; ;$$

$$a = \sum_{i=0}^{\infty} a_i x^i \mapsto (4a_0, \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} 2a_{(2i+1)2^j})x^i, 0) \; .$$

If $a=c^2x$ for $c=\sum\limits_{i=0}^{\infty}c_ix^i\in A_2[x]$ then

$$(4a_0,\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty}2a_{(2i+1)2^j})x^i)~=~(0,2c)\in A_8\oplus A_4[x]~,$$

so that the composite

$$\partial(1)~:~H_1(C(1)\otimes_{A[x]}C(x))=A_2[x]\to\widehat{Q}^1(C(1))\to Q_0(C(1),\gamma(1))=A_8\oplus A_4[x]\oplus\mathbb{Z}_2[x]$$

is given by $c\mapsto(0,2c,0)$.

Next, we consider $Q_0(C(1),\gamma(1))$. A polynomial $a(x)\in A\oplus 2xA[x]$ is such that

$$a(x)\in 8A[x]+\{4(b(x)-b(x)^2)\,|\,b(x)\in A[x]\}$$

if and only if there exist $b_1,b_2,\dots\in A$ such that

$$a_0~=~0~,~a_1~=~4b_1~,~a_2~=~4(b_2-b_1)~,~a_3~=~4b_3~,~a_4~=~4(b_4-b_2)~,~\dots\in A_8~,$$

if and only if

$$\begin{aligned}a_1~=~a_2~=~a_3~=~a_4~=~\dots~=~0\in A_4~,\\ a_0~=~\sum_{j=0}^{\infty}a_{(2i+1)2^j}~=~0\in A_8~(i\geqslant 0)~.\end{aligned}$$

Thus $f_0(1)$ is well-defined and injective. The morphism $f_0(1)$ is surjective, since

$$\begin{aligned}(a_0,\sum_{i=0}^{\infty}b_ix^i,\sum_{i=0}^{\infty}c_ix^i)~&=~f_0(1)(a_0+2\sum_{i=0}^{\infty}b_ix^{2i+1}+2\sum_{i=0}^{\infty}c_ix^{2i+2})\\ &\in A_8\oplus A_4[x]\oplus A_2[x]~(a_0,b_i,c_i\in A)~.\end{aligned}$$

The map $\widehat{Q}^0(C(1))\to Q_{-1}(C(1),\gamma(1))$ is given by

$$\begin{aligned}\widehat{Q}^0(C(1))~=~A_2[x]\to Q_{-1}(C(1),\gamma(1))~=~A_2[x]~;\\ a=\sum_{i=0}^{\infty}a_ix^i\mapsto a_0+\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty}a_{(2i+1)2^j})x^{i+1}~.\end{aligned}$$

If $a=c^2x$ for $c=\sum\limits_{i=0}^{\infty}c_ix^i\in A_2[x]$ then

$$a_0+\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty}a_{(2i+1)2^j})x^{i+1}~=~cx\in A_2[x]~,$$

so that the composite

$$\partial(1)~:~H_0(C(1)\otimes_{A[x]}C(x))=A_2[x]\to\widehat{Q}^0(C(1))\to Q_{-1}(C(1),\gamma(1))=A_2[x]$$

is given by $c\mapsto cx$.

(ii) We start with $Q_1(C(x),\gamma(x))$. For any polynomial $a=\sum\limits_{i=0}^{\infty}a_ix^i\in A[x]$

$$a-a^2x~=~\sum_{i=0}^{\infty}a_ix^i-\sum_{i=0}^{\infty}a_ix^{2i+1}\in A_2[x]~.$$

Now $a-a^2x\in 2A[x]$ if and only if the coefficients $a_0,a_1,\dots\in A$ are such that

$$a_0~=~a_1-a_0~=~a_2~=~a_3-a_1~=~\dots~=~0\in A_2~,$$

if and only if

$$a_0~=~a_1~=~a_2~=~a_3~=~\dots~=~0\in A_2~.$$

It follows that $Q_1(C(x),\gamma(x)) = 0$.

Next, $Q_{-1}(C(x),\gamma(x))$. A polynomial $a = \sum\limits_{i=0}^{\infty} a_i x^i \in A[x]$ is such that

$$a \in 2A[x] + \{b - b^2 x \,|\, v \in A[x]\}$$

if and only if there exist $b_0, b_1, \dots \in A$ such that

$$a_0 = b_0\ ,\ a_1 = b_1 - b_0\ ,\ a_2 = b_2\ ,\ a_3 = b_3 - b_1\ ,\ a_4 = b_4\ ,\ \dots \in A_2\ ,$$

if and only if

$$\sum_{j=0}^{\infty} a_{(2i+1)2^j - 1} = 0 \in A_2\ (i \geqslant 0)\ .$$

Thus $f_{-1}(x)$ is well-defined and injective. The morphism $f_{-1}(x)$ is surjective, since

$$\sum_{i=0}^{\infty} c_i x^i = f_{-1}(x)(\sum_{i=0}^{\infty} c_i x^{2i}) \in A_2[x]\ (c_i \in A)\ .$$

The map $\widehat{Q}^0(C(x)) \to Q_{-1}(C(x),\gamma(x))$ is given by

$$\widehat{Q}^0(C(x)) = A_2[x] \to Q_{-1}(C(x),\gamma(x)) = A_2[x]\ ;$$
$$b = \sum_{i=0}^{\infty} b_i x^i \mapsto \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} b_{(2i+1)2^j-1})x^i\ .$$

If $b = c^2$ for $c = \sum\limits_{i=0}^{\infty} c_i x^i \in A_2[x]$ then

$$\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} b_{(2i+1)2^j-1})x^i = c \in A_2[x]\ ,$$

so that the composite

$$\partial(x)\ :\ H_0(C(1) \otimes_{A[x]} C(x)) = A_2[x] \to \widehat{Q}^0(C(x)) \to Q_{-1}(C(x),\gamma(x)) = A_2[x]$$

is just the identity $c \mapsto c$.

Next, $Q_0(C(x),\gamma(x))$. For any $a \in A[x]$

$$a \in 8A[x] + \{4(b - b^2 x) \,|\, b \in A[x]\}$$

if and only there exist $b_0, b_1, \dots \in A$ such that

$$a_0 = 4b_0\ ,\ a_1 = 4(b_1 - b_0)\ ,\ a_2 = 4b_2\ ,\ a_3 = 4(b_3 - b_1)\ ,\ \dots \in A_8,$$

if and only if

$$a_0 = a_1 = a_2 = a_3 = \dots = 0 \in A_4\ ,$$
$$\sum_{j=0}^{\infty} a_{(2i+1)2^j-1} = 0 \in A_8\ (i \geqslant 0)\ .$$

Thus $f_0(x)$ is well-defined and injective. The morphism $f_0(x)$ is surjective, since

$$(\sum_{i=0}^{\infty} c_i x^i, \sum_{i=0}^{\infty} d_i x^i) = f_0(x)(\sum_{i=0}^{\infty} 2c_i x^{2i} + \sum_{i=0}^{\infty} 2d_i x^{2i+1}) \in A_4[x] \oplus A_2[x]\ (c_i, d_i \in A)\ .$$

The map $\widehat{Q}^1(C(x)) \to Q_0(C(x),\gamma(x))$ is given by

$$\widehat{Q}^1(C(x)) \;=\; A_2[x] \to Q_0(C(x),\gamma(x)) \;=\; A_4[x]\oplus A_2[x] \;;$$
$$b=\sum_{i=0}^{\infty} b_i x^i \mapsto (\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} 2b_{(2i+1)2^j-1})x^i,0) \;.$$

If $b=c^2$ for $c=\sum\limits_{i=0}^{\infty} c_i x^i \in A_2[x]$ then

$$\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} 2a_{(2i+1)2^j})x^i) \;=\; 2c\in A_4[x] \;,$$

so that the composite

$$\partial(x) \;:\; H_1(C(1)\otimes_{A[x]} C(x)) = A_2[x] \to \widehat{Q}^1(C(x)) \to Q_0(C(x),\gamma(x)) = A_4[x]\oplus A_2[x]$$

is given by $c\mapsto (2c,0)$. ☐

We can now prove Theorem 0.2 :

Theorem 5.6. *The hyperquadratic L-groups of $A[x]$ for a 1-even A are given by*

$$\widehat{L}^n(A[x]) \;=\; Q_n(B^{A[x]},\beta^{A[x]}) \;=\; \begin{cases} A_8\oplus A_4[x]\oplus A_2[x]^3 & \text{if } n\equiv 0 (\text{mod } 4)\\ A_2 & \text{if } n\equiv 1 (\text{mod } 4)\\ 0 & \text{if } n\equiv 2 (\text{mod } 4)\\ A_2[x] & \text{if } n\equiv 3 (\text{mod } 4)\;. \end{cases}$$

(i) *For $n=0$*

$$Q_0(B^{A[x]},\beta^{A[x]}) \;=\; \frac{\{M\in \mathrm{Sym}_2(A[x])\,|\,M-MXM\in \mathrm{Quad}_2(A[x])\}}{4\mathrm{Quad}_2(A[x])+\{2(N+N^t)-4N^tXN\,|\,N\in M_2(A[x])\}} \;.$$

An element $M\in Q_0(B^{A[x]},\beta^{A[x]})$ is represented by a matrix

$$M \;=\; \begin{pmatrix} a & b\\ b & c\end{pmatrix}\in \mathrm{Sym}_2(A[x]) \;(a=\sum_{i=0}^{\infty} a_i x^i,\, c=\sum_{i=0}^{\infty} c_i x^i\in A[x])$$

with $a-a_0,b,c\in 2A[x]$. The isomorphism

$$Q_0(B^{A[x]},\beta^{A[x]}) \xrightarrow{\;\cong\;} \widehat{L}^0(A[x]) \;=\; \widehat{L}^1(A[x],(2)^\infty) \;;\; M\mapsto (T_M,\lambda_M,\mu_M;U_M)$$

sends M to the Witt class of the nonsingular quadratic linking form (T_M,λ_M,μ_M) over $(A[x],(2)^\infty)$ with a lagrangian $U_M\subset T_M$ for (T_M,λ_M) corresponding to the (-1)-quadratic $(2)^\infty$-formation over $A[x]$

$$\partial(M) \;=\; (H_-(A[x]^4);A[x]^4,\mathrm{im}\Big(\begin{pmatrix}\begin{pmatrix} I & 0\\ -2X & I-XM\end{pmatrix}\\[1em] \begin{pmatrix} 0 & 2I\\ 2I & M\end{pmatrix}\end{pmatrix} : A[x]^4\to A[x]^4\oplus A[x]^4\Big))$$

(4.10), *with*

$$\partial \;:\; Q_0(B^{A[x]},\beta^{A[x]}) \;=\; \widehat{L}^0(A[x]) \to L_{-1}(A[x]) \;;\; M\mapsto \partial(M) \;.$$

The inverse isomorphism is defined by the linking Arf invariant (4.9).

Writing

$$2\Delta \ : \ A_2[x] \to A_4[x] \oplus A_4[x] \ ; \ d \mapsto (2d, 2d)$$

there are defined isomorphisms

$$\begin{array}{l} Q_0(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} A_8 \oplus \mathrm{coker}(2\Delta) \oplus A_2[x] \oplus A_2[x] \ ; \\ M \ = \ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \ = \ \begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix} \ (c' = c - b^2) \\ \mapsto \big(a_0, [\sum\limits_{i=0}^{\infty}(\sum\limits_{j=0}^{\infty} a_{(2i+1)2^j}/2)x^i, \sum\limits_{i=0}^{\infty}(\sum\limits_{j=0}^{\infty} c'_{(2i+1)2^j-1}/2)x^i], \sum\limits_{k=0}^{\infty}(a_{2k+2}/2)x^k, \sum\limits_{k=0}^{\infty}(c'_{2k+1}/2)x^k\big) \ , \\ \mathrm{coker}(2\Delta) \xrightarrow{\cong} A_4[x] \oplus A_2[x] \ ; \ [d, e] \mapsto (d - e, d) \ . \end{array}$$

In particular $M \in Q_0(B^{A[x]}, \beta^{A[x]})$ *can be represented by a diagonal matrix* $\begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix}$.

(ii) *For* $n = 1$

$$\begin{array}{l} Q_1(B^{A[x]}, \beta^{A[x]}) \\ = \ \dfrac{\{N \in M_2(A[x]) \,|\, N + N^t \in 2\mathrm{Sym}_2(A[x]), \frac{1}{2}(N + N^t) - N^t X N \in \mathrm{Quad}_2(A[x])\}}{2M_2(A[x])} \end{array}$$

and there is defined an isomorphism

$$Q_1(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} Q_1(B^A, \beta^A) \ = \ A_2 \ ; \ N \ = \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a_0 \ ,$$

with

$$\begin{array}{l} \partial \ : \ Q_1(B^{A[x]}, \beta^{A[x]}) \ = \ \widehat{L}^1(A[x]) \ = \ A_2 \to L_0(A[x]) \ ; \\ a_0 \mapsto A[x] \otimes_A (A \oplus A, \begin{pmatrix} a_0(a_0 - 1)/2 & 1 - 2a_0 \\ 0 & -2 \end{pmatrix}) \ . \end{array}$$

(iii) *For* $n = 2$

$$Q_2(B^{A[x]}, \beta^{A[x]}) \ = \ 0 \ .$$

(iv) *For* $n = 3$

$$Q_3(B^{A[x]}, \beta^{A[x]}) \ = \ \frac{\mathrm{Sym}_2(A[x])}{\mathrm{Quad}_2(A[x]) + \{M - MXM \,|\, M \in \mathrm{Sym}_2(A[x])\}} \ .$$

There is defined an isomorphism

$$\begin{array}{l} Q_3(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} A_2[x] \ ; \\ M \ = \ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \ = \ \begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix} \mapsto d_0 + \sum\limits_{i=0}^{\infty}(\sum\limits_{j=0}^{\infty} d_{(2i+1)2^j})x^{i+1} \\ (a' = a - b^2 x \ , \ c' = c - b^2 \in A[x] \ , \ d = a' + c'x = a + cx \in A_2[x]) \ . \end{array}$$

The isomorphism

$$Q_3(B^{A[x]}, \beta^{A[x]}) \xrightarrow{\cong} \widehat{L}^0(A[x]) \ = \ \widehat{L}^3(A[x]) \ ; \ M \ = \ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto (K_M, \psi_M; L_M)$$

sends M to the Witt class of the nonsingular (-1)-quadratic form over $A[x]$

$$(K_M,\psi_M) \;=\; (A[x]^2\oplus A[x]^2, \begin{pmatrix} X & 1 \\ 0 & M\end{pmatrix})$$

with a lagrangian $L_M = A[x]^2\oplus 0\subset K_M$ for $(K_M,\psi_M-\psi_M^*)$ (3.8), and

$$\partial \;:\; Q_3(B^{A[x]},\beta^{A[x]}) \;=\; \widehat{L}^3(A[x])\to L_2(A[x])\;;\; M\mapsto (K_M,\psi_M)\;.$$

In particular $M\in Q_3(B^{A[x]},\beta^{A[x]})$ can be represented by a diagonal matrix $\begin{pmatrix} a & 0\\ 0 & c'\end{pmatrix}$. The inverse isomorphism is defined by the generalized Arf invariant (3.4).

Proof. Proposition 5.4 expresses $Q_n(B^{A[x]},\beta^{A[x]})$ in terms of 2×2 matrices. We deal with the four cases separately.
(i) Let $n=0$. Proposition 5.5 gives an exact sequence

$$0\to H_1(C(1)\otimes_{A[x]}C(x)) \xrightarrow{\;\partial\;} Q_0(C(1),\gamma(1))\oplus Q_0(C(x),\gamma(x))\to Q_0(C(X),\gamma(X))\to 0$$

with

$$\begin{aligned} &H_1(C(1)\otimes_{A[x]}C(x)) = A_2[x]\to \\ &Q_0(C(1),\gamma(1))\oplus Q_0(C(x),\gamma(x)) = (A_8\oplus A_4[x]\oplus A_2[x])\oplus(A_4[x]\oplus A_2[x])\;;\\ &\qquad\qquad x\mapsto ((0,2c,0),(2c,0))\end{aligned}$$

so that there is defined an isomorphism

$$\mathrm{coker}(\partial)\xrightarrow{\;\cong\;} A_8\oplus \mathrm{coker}(2\Delta)\oplus A_2[x]\oplus A_2[x]\;;\;(s,t,u,v,w)\mapsto (s,[t,v],u,w)\;.$$

We shall define an isomorphism $Q_0(C(X),\gamma(X))\cong \mathrm{coker}(\partial)$ by constructing a splitting map

$$Q_0(C(X),\gamma(X))\to Q_0(C(1),\gamma(1))\oplus Q_0(C(x),\gamma(x))\;.$$

An element in $Q_0(C(X),\gamma(X))$ is represented by a symmetric matrix

$$M \;=\; \begin{pmatrix} a & b\\ b & c\end{pmatrix}\in \mathrm{Sym}_2(A[x])$$

such that

$$M-MXM \;=\; \begin{pmatrix} a-a^2-b^2x & b-ab-bcx\\ b-ab-bcx & c-b^2-c^2x\end{pmatrix}\in \mathrm{Quad}_2(A[x])\;,$$

so that

$$a-a^2-b^2x\;,\; c-b^2-c^2x\in 2A[x]\;.$$

Given $a=\sum\limits_{i=0}^{\infty}a_ix^i\in A[x]$ let

$$d \;=\; \max\{i\geqslant 0\,|\,a_i\notin 2A\}\ (=0 \text{ if } a\in 2A[x])$$

so that $a\in A_2[x]$ has degree $d\geqslant 0$,

$$(a_d)^2 \;=\; a_d\neq 0\in A_2$$

and $a-a^2\in A_2[x]$ has degree $2d$. Thus if $b\neq 0\in A_2[x]$ the degree of $a-a^2=b^2x\in A_2[x]$ is both even and odd, so $b\in 2A[x]$ and hence also $a-a^2, c-c^2x\in 2A[x]$. It follows from $a(1-a)=0\in A_2[x]$ that $a=0$ or $1\in A_2[x]$, so $a-a_0\in 2A[x]$. Similarly, it follows from $c(1-cx)=0\in A_2[x]$ that $c=0\in A_2[x]$, so $c\in 2A[x]$. The matrices defined by

$$N \;=\; \begin{pmatrix} 0 & -b/2 \\ 0 & 0 \end{pmatrix} \in M_2(A[x])\ ,\ M' \;=\; \begin{pmatrix} a & 0 \\ 0 & c-b^2 \end{pmatrix} \in \mathrm{Sym}_2(A[x])$$

are such that

$$M+2(N+N^t)-4N^tXN \;=\; M' \in \mathrm{Sym}_2(A[x])$$

and so $M=M'\in Q_0(C(X),\gamma(X))$. The explicit splitting map is given by

$$Q_0(C(X),\gamma(X)) \to Q_0(C(1),\gamma(1))\oplus Q_0(C(x),\gamma(x))\ ;\ M=M'\mapsto (a,c-b^2)\ .$$

The isomorphism

$$Q_0(C(X),\gamma(X)) \xrightarrow{\cong} \mathrm{coker}(\partial)\ ;\ M\mapsto (a,c-b^2)$$

may now be composed with the isomorphisms given in the proof of Proposition 5.5 (i)

$$\begin{aligned} &Q_0(C(1),\gamma(1)) \xrightarrow{\cong} A_8\oplus A_4[x]\oplus A_2[x]\ ;\\ &\qquad \sum_{i=0}^{\infty} d_ix^i \mapsto (d_0, \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} d_{(2i+1)2^j}/2)x^i, \sum_{k=0}^{\infty}(d_{2k+2}/2)x^k)\ ,\\ &Q_0(C(x),\gamma(x)) \xrightarrow{\cong} A_4[x]\oplus A_2[x]\ ;\\ &\qquad \sum_{i=0}^{\infty} e_ix^i \mapsto (\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} e_{(2i+1)2^j-1}/2)x^i, \sum_{k=0}^{\infty}(e_{2k+1}/2)x^k)\ . \end{aligned}$$

(ii) Let $n=1$. If $N=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in M_2(A[x])$ represents an element $N\in Q_1(B^{A[x]},\beta^{A[x]})$

$$\begin{aligned} N+N^t \;&=\; \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} \in 2\mathrm{Sym}_2(A[x])\ ,\\ \frac{1}{2}(N+N^t)-N^tXN \;&=\; \begin{pmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{pmatrix} - \begin{pmatrix} a^2+c^2x & ab+cdx \\ ab+cdx & b^2+d^2x \end{pmatrix}\\ &\qquad \in \mathrm{Quad}_2(A[x]) \end{aligned}$$

then

$$b+c\ ,\ a-a^2-c^2x\ ,\ d-b^2-d^2x\in 2A[x]\ .$$

If $d\notin 2A[x]$ then the degree of $d-d^2x=b^2\in A_2[x]$ is both even and odd, so that $d\in 2A[x]$ and hence $b,c\in 2A[x]$. Thus $a-a^2\in 2A[x]$ and so (as above) $a-a_0\in 2A[x]$. It follows that

$$Q_1(B^{A[x]},\beta^{A[x]}) \;=\; Q_1(B^A,\beta^A) \;=\; A_2\ .$$

(iii) Let $n=2$. $Q_2(B^{A[x]},\beta^{A[x]})=0$ by 5.5.
(iv) Let $n=3$. Proposition 5.5 gives an exact sequence

$$0\to H_0(C(1)\otimes_{A[x]}C(x)) \xrightarrow{\partial} Q_{-1}(C(1),\gamma(1))\oplus Q_{-1}(C(x),\gamma(x)) \to Q_3(C(X),\gamma(X))\to 0$$

with

$$\begin{aligned} &\partial\ :\ H_0(C(1)\otimes_{A[x]}C(x)) = A_2[x]\to\\ &Q_{-1}(C(1),\gamma(1))\oplus Q_{-1}(C(x),\gamma(x)) = A_2[x]\oplus A_2[x]\ ;\ c\mapsto (cx,c)\ , \end{aligned}$$

so that there is defined an isomorphism

$$\mathrm{coker}(\partial) \xrightarrow{\cong} A_2[x]\ ;\ (a,b)\mapsto a+bx\ .$$

We shall define an isomorphism $Q_3(C(X), \gamma(X)) \cong \mathrm{coker}(\partial)$ by constructing a splitting map

$$Q_3(C(X), \gamma(X)) \to Q_{-1}(C(1), \gamma(1)) \oplus Q_{-1}(C(x), \gamma(x)) \ .$$

For any $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathrm{Sym}_2(A[x])$ the matrices

$$L \ = \ \begin{pmatrix} 0 & -b \\ -b & 0 \end{pmatrix} \ , \ M' \ = \ \begin{pmatrix} a - b^2x & 0 \\ 0 & c - b^2 \end{pmatrix} \in \mathrm{Sym}_2(A[x])$$

are such that

$$M' \ = \ M + L - LXL \in \mathrm{Sym}_2(A[x])$$

so $M = M' \in Q_3(C(X), \gamma(X))$. The explicit splitting map is given by

$$Q_3(C(X), \gamma(X)) \to Q_{-1}(C(1), \gamma(1)) \oplus Q_{-1}(C(x), \gamma(x)) \ ; \ M = M' \mapsto (a - b^2x, c - b^2) \ .$$

The isomorphism

$$Q_3(C(X), \gamma(X)) \xrightarrow{\ \cong\ } Q_{-1}(C(1), \gamma(1)) \ ; \ M \mapsto (a - b^2x) + (c - b^2)x \ = \ a + cx$$

may now be composed with the isomorphism given in the proof of Proposition 5.5 (ii)

$$Q_{-1}(C(1), \gamma(1)) \xrightarrow{\ \cong\ } A_2[x] \ ; \ d = \sum_{i=0}^{\infty} d_i x^i \mapsto d_0 + \sum_{i=0}^{\infty} (\sum_{j=0}^{\infty} d_{(2i+1)2^j}) x^{i+1} \ .$$

□

Remark 5.7. (i) Substituting the computation of $Q_*(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]})$ given by Theorem 5.6 in the formula

$$Q_{n+1}(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) \ = \ Q_{n+1}(B^{\mathbb{Z}}, \beta^{\mathbb{Z}}) \oplus \mathrm{UNil}_n(\mathbb{Z})$$

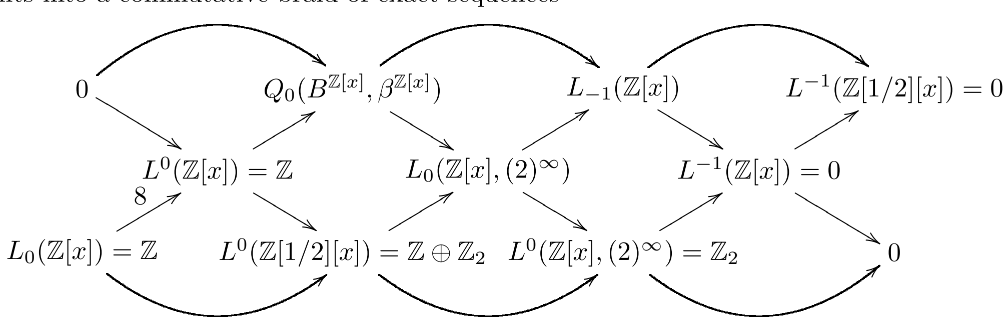
recovers the computations

$$\mathrm{UNil}_n(\mathbb{Z}) \ = \ NL_n(\mathbb{Z}) \ = \begin{cases} 0 & \text{if } n \equiv 0, 1 (\mathrm{mod}\ 4) \\ \mathbb{Z}_2[x] & \text{if } n \equiv 2 (\mathrm{mod}\ 4) \\ \mathbb{Z}_4[x] \oplus \mathbb{Z}_2[x]^3 & \text{if } n \equiv 3 (\mathrm{mod}\ 4) \end{cases}$$

of Connolly and Ranicki [10] and Connolly and Davis [8].
(ii) The twisted quadratic Q-group

$$Q_0(B^{\mathbb{Z}[x]}, \beta^{\mathbb{Z}[x]}) \ = \ \mathbb{Z}_8 \oplus L_{-1}(\mathbb{Z}[x]) \ = \ \mathbb{Z}_8 \oplus \mathrm{UNil}_3(\mathbb{Z})$$

fits into a commutative braid of exact sequences



with $L_0(\mathbb{Z}[x],(2)^\infty)$ (resp. $L^0(\mathbb{Z}[x],(2)^\infty)$) the Witt group of nonsingular quadratic (resp. symmetric) linking forms over $(\mathbb{Z}[x],(2)^\infty)$, and

$$L^0(\mathbb{Z}[x],(2)^\infty) \xrightarrow{\cong} \mathbb{Z}_2 \ ; \ (T,\lambda) \mapsto n \text{ if } |\mathbb{Z}\otimes_{\mathbb{Z}[x]} T| = 2^n \ .$$

The twisted quadratic Q-group $Q_0(B^{\mathbb{Z}[x]},\beta^{\mathbb{Z}[x]})$ is thus the Witt group of nonsingular quadratic linking forms (T,λ,μ) over $(\mathbb{Z}[x],(2)^\infty)$ with $|\mathbb{Z}\otimes_{\mathbb{Z}[x]} T| = 4^m$ for some $m \geqslant 0$. $Q_0(B^{\mathbb{Z}[x]},\beta^{\mathbb{Z}[x]})$ can also be regarded as the Witt group of nonsingular quadratic linking forms (T,λ,μ) over $(\mathbb{Z}[x],(2)^\infty)$ together with a lagrangian $U\subset T$ for the symmetric linking form (T,λ). The isomorphism class of any such quadruple $(T,\lambda,\mu;U)$ is an element $\phi\in Q_0(B,\beta)$. The chain bundle β is classified by a chain bundle map

$$(f,\chi) \ : \ (B,\beta) \to (B^{\mathbb{Z}[x]},\beta^{\mathbb{Z}[x]})$$

and the Witt class is given by the linking Arf invariant

$$(T,\lambda,\mu;U) \ = \ (f,\chi)_\%(\phi) \in Q_0(B^{\mathbb{Z}[x]},\beta^{\mathbb{Z}[x]}) \ = \ \mathbb{Z}_8\oplus\mathbb{Z}_4[x]\oplus\mathbb{Z}_2[x]^3 \ .$$

(iii) Here is an explicit procedure obtaining the generalized linking Arf invariant

$$(T,\lambda,\mu;U)\in Q_0(B^{A[x]},\beta^{A[x]}) \ = \ A_8\oplus A_4[x]\oplus A_2[x]^3$$

for a nonsingular quadratic linking form (T,λ,μ) over $(A[x],(2)^\infty)$ together with a lagrangian $U\subset T$ for the symmetric linking form (T,λ) such that $[U]=0\in \widetilde{K}_0(A[x])$, for any 1-even ring A.

Use a set of $A[x]$-module generators $\{g_1,g_2,\dots,g_u\}\subset U$ to obtain a f.g. free $A[x]$-module resolution

$$0\to B_1 \xrightarrow{d} B_0 = A[x]^u \xrightarrow{(g_1,g_2,\dots,g_u)} U \to 0 \ .$$

Let $(p_i,q_i)\in A_2[x]\oplus A_2[x]$ be the unique elements such that

$$\mu(g_i) \ = \ (p_i)^2 + x(q_i)^2 \in \widehat{H}^0(\mathbb{Z}_2;A[x]) \ = \ A_2[x] \ (1\leqslant i\leqslant u) \ ,$$

and use arbitrary lifts $(p_i,q_i)\in A[x]\oplus A[x]$ to define

$$\begin{aligned} b_i \ &= \ (p_i)^2 + x(q_i)^2 \in A[x] \ , \\ p \ &= \ (p_1,p_2,\dots,p_u) \ , \ q \ = \ (q_1,q_2,\dots,q_u)\in A[x]^u \ . \end{aligned}$$

The diagonal symmetric form on B_0

$$\beta \ = \ \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_u \end{pmatrix} \in \mathrm{Sym}(B_0)$$

is such that

$$d^*\beta d \in \mathrm{Quad}(B_1)\subset \mathrm{Sym}(B_1) \ ,$$

and represents the chain bundle

$$\beta \ = \ \mu|_U \in \widehat{Q}^0(B^{-*}) \ = \ \mathrm{Hom}_A(U,\widehat{H}^0(\mathbb{Z}_2;A[x])) \ .$$

The $A[x]$-module morphisms

$$f_0 = \begin{pmatrix} p \\ q \end{pmatrix} : B_0 = A[x]^u \to B_0^{A[x]} = A[x] \oplus A[x] ; (a_1, a_2, \dots, a_u) \mapsto \sum_{i=1}^{u} a_i(p_i, q_i) ,$$

$$f_1 : B_1 = A[x]^u \to B_1^{A[x]} = A[x] \oplus A[x] ; a = (a_1, a_2, \dots, a_u) \mapsto \frac{f_0 d(a)}{2}$$

define a chain bundle map

$$(f, 0) : (B, \beta) \to (B^{A[x]}, \beta^{A[x]}) ,$$

with

$$\beta_0^{A[x]} = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} : B_0^{A[x]} = A[x] \oplus A[x] \to (B_0^{A[x]})^* = A[x] \oplus A[x] .$$

The $(2)^\infty$-torsion dual of U has f.g. free $A[x]$-module resolution

$$0 \to B^0 = A[x]^u \xrightarrow{d^*} B^1 \to U\widehat{\ } \to 0 .$$

Lift a set of $A[x]$-module generators $\{h_1, h_2, \dots, h_u\} \subset U\widehat{\ }$ to obtain a basis for B^1, and hence an identification $B^1 = A[x]^u$. Also, lift these generators to elements $\{h_1, h_2, \dots, h_u\} \subset T$, so that $\{g_1, g_2, \dots, g_u, h_1, h_2, \dots, h_u\} \subset T$ is a set of $A[x]$-module generators such that

$$d^{-1} = (\lambda(g_i, h_j)) \in \frac{\mathrm{Hom}_{A[1/2][x]}(B_0[1/2], B_1[1/2])}{\mathrm{Hom}_{A[x]}(B_0, B_1)} .$$

Lift the symmetric $u \times u$ matrix $(\lambda(h_i, h_j))$ with entries in $A[1/2][x]/A[x]$ to a symmetric form on the f.g. free $A[1/2][x]$-module $B^1[1/2] = A[1/2][x]^u$

$$\Lambda = (\lambda_{ij}) \in \mathrm{Sym}(B^1[1/2])$$

such that $\lambda_{ii} \in A[1/2][x]$ has image $\mu(h_i) \in A[1/2][x]/2A[x]$. Let $\phi = (\phi_{ij})$ be the symmetric form on $B^0 = A[x]^u$ defined by

$$\phi = d\Lambda d^* \in \mathrm{Sym}(B^0) \subset \mathrm{Sym}(B^0[1/2]) .$$

Then T has a f.g. free $A[x]$-module resolution

$$0 \to B_1 \oplus B^0 \xrightarrow{\begin{pmatrix} 0 & d^* \\ d & \phi \end{pmatrix}} B^1 \oplus B_0 \xrightarrow{(g_1, \dots, g_u, h_1, \dots, h_u)} T \to 0 ,$$

and

$$\phi_{ii} - \sum_{j=1}^{u} (\phi_{ij})^2 b_j \in 2A[x] .$$

The symmetric form on $(B_0^{A[x]})^* = A[x] \oplus A[x]$ defined by

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = f_0 \phi f_0^* = \begin{pmatrix} \phi(p,p) & \phi(p,q) \\ \phi(q,p) & \phi(q,q) \end{pmatrix} \in \mathrm{Sym}((B_0^{A[x]})^*)$$

$$(p = (p_1, p_2, \dots, p_u), q = (q_1, q_2, \dots, q_u) \in B^0 = A[x]^u)$$

is of the type considered in the proof of Theorem 5.6 (i), with

$$a - a^2 = b^2 x , \ c - c^2 x = b^2 \in A_2[x] , \ b \in 2A[x] .$$

The Witt class is

$$\begin{aligned}(T,\lambda,\mu;U) &= (f,0)_\%(\phi) \\ &= \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix} \in Q_0(B^{A[x]},\beta^{A[x]})\ (c' = c - b^2)\ ,\end{aligned}$$

with isomorphisms

$$Q_0(B^{A[x]},\beta^{A[x]}) \xrightarrow{\cong} A_8 \oplus \operatorname{coker}(2\Delta) \oplus A_2[x] \oplus A_2[x]\ ;$$

$$\begin{pmatrix} a & 0 \\ 0 & c' \end{pmatrix} \mapsto \big(a_0, \big[\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{(2i+1)2^j}/2)x^i, \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} c'_{(2i+1)2^j-1}/2)x^i\big], \sum_{k=0}^{\infty}(a_{2k+2}/2)x^k, \sum_{k=0}^{\infty}(c'_{2k+1}/2)x^k\big)\ ,$$

$$\operatorname{coker}(2\Delta) \xrightarrow{\cong} A_4[x] \oplus A_2[x]\ ;\ [m,n] \mapsto (m-n,m)\ ,$$

where

$$2\Delta\ :\ A_2[x] \to A_4[x] \oplus A_4[x]\ ;\ m \mapsto (2m,2m)$$

as in Theorem 5.6, and

$$Q_0(B^{A[x]},\beta^{A[x]}) = A_8 \oplus A_4[x] \oplus A_2[x]^3\ .$$

For Dedekind A the splitting formula of [10] gives

$$\mathrm{UNil}_3(A) \cong Q_0(B^{A[x]},\beta^{A[x]})/A_8 \cong A_4[x] \oplus A_2[x]^3\ .$$

□

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