Spectral Families and Geometry Of Banach Spaces

Danilo Blagojevic

Doctor of Philosophy
University of Edinburgh
2007

To my aunty Teka

Acknowledgements

I would like to thank first and foremost my supervisor Prof. Alastair Gillespie, who has invested considerable time and energy into guiding me through my PhD. His professional help and pastoral support have been most invaluable. I would like to thank Prof. Jim Wright for his help and ideas during a critical period. I am also grateful to my fellow PhD friends for their support and sharing in the ups and downs of the research experience.

Last but not least, I want to thank my family, especially my aunty Diana, for their unwavering emotional support and firm belief in me throughout my research.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Danilo Blagojevic*)

Abstract

The principal objects of study in this thesis are arbitrary spectral families, E, on a complex Banach space X. The central theme is the relationship between the geometry of X and the $p-$variation of E. We show that provided X is super-reflexive, then given any E, there exists a value $1 \leq p < \infty$, depending only on E and X, such that $var_p(E) < \infty$. If X is uniformly smooth we provide an explicit range of such values p, which depends only on E and the modulus of convexity of X^*, $\delta_{X^*}(\cdot)$.

We show that given a trigonometrically well bounded operator T on a super-reflexive X, there exist constants $C > 0$ and $0 < \alpha < 1$, depending on T and X, such that for all $n \in \mathbb{Z} \setminus \{0\}$, we have $\|T^n\| \leq C|n|^\alpha$. This is an improvement on the previously known upper growth bound of $O(|n|)$.

We show that in a Hilbert space H, a spectral family E arises from a spectral measure if and only if both $var_2(E) < \infty$ and $var_2(E^*) < \infty$. Further, given any k-fold logarithmic power growth, $\log^{(k)}(|n|)$, there exists a Hilbert space H and trigonometrically well-bounded T such that $\|T^n\| \sim \log^{(k)}(|n|)$ and the spectral family E of T satisfies $var_2(E) = \infty$. This contrasts with the power-bounded case, where $\sup_n \|T^n\| \leq K$ implies $var_2(E) < \infty$.

We prove that BV_q spectral integration is possible with respect to any trigonometrically well bounded operator, provided the space X is super-reflexive. In other words, we dispense with the previous requirement that T be power bounded. We also prove a BV_q multiplier theorem for UMD spaces, and indicate under which conditions an $\mathfrak{M}_q$ multiplier theorem also holds.

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Common Notation

There is a comprehensive list of notation at the back of this thesis. Much of it is specific to the work undertaken herein. The following notation, however, relates to objects of study in many areas of analysis and will be required from the outset.

(i) The letter X always denotes a complex Banach space, and H a complex Hilbert space.
(ii) B_X is the closed unit ball in X.
(iii) By $\mathcal{B}(X)$ and $\mathcal{B}(H)$ we denote the spaces of bounded linear operators on X and H respectively.
(iv) For an invertible operator $T \in \mathcal{B}(X)$, $\|T^n\| \sim |n|^\alpha$ means that there exists a constant $C_T > 0$ such that for all $n \in \mathbb{Z} \setminus \{0\}$ we have $\|T^n\| \leq C_T|n|^\alpha$.
(v) For $a < b \in \mathbb{R}$, $\mathcal{P}_{[a,b]}$ denotes the set of all partitions $u = \{a = \lambda_0 < ... < \lambda_N = b\}$ of the interval $[a, b]$ partially ordered and directed to increase by refinement.
(vi) C_c^∞ and $C_c^\infty(\mathbb{R})$ denote the space of smooth complex-valued functions on $\mathbb{R}$ with compact support.
(vii) $C_c^\infty(X)$ and $C_c^\infty(\mathbb{R}, X)$ denote the space of smooth X-valued functions on $\mathbb{R}$ of compact support.

Introduction

This thesis is devoted to the study of spectral families $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ on a complex Banach space X, and the way they interact with the geometry of X. In chapter 1 we outline the origins of the study of spectral families and trigonometrically well bounded operators. T is trigonometrically well bounded if, for some (essentially) unique spectral family E concentrated on $[0, 2\pi]$, $T = \int_{0^-}^{2\pi} e^{i\lambda} dE(\lambda)$. We give an account of key results characterizing these operators, and describe how spectral families can be used to construct continuous algebra homomorphisms from various function algebras $\mathcal{A}$ into $\mathcal{B}(X)$. We then define the geometric UMD property and state a key result linking it to the Hilbert transform on vector valued L^p_X spaces. This leads us finally to give a summary of the current state of knowledge in vector valued multiplier theory.

In chapter 2 we introduce new geometric notions, those of super-reflexivity, uniform convexity and uniform smoothness. We prove some background results on the geometry of X. We then define the p variation of an arbitrary spectral family E on X, as

$$var_p(E) = \sup_{\|x\|\leq 1} \sup_{u\in\mathcal{P}_{[-a,a]}} \sup_{a>0} \Big\{ \sum_1^N \|E(\lambda_j)x - E(\lambda_{j-1})x\|^p \Big\}^{1/p}, \tag{1}$$

where $u = \{-a = \lambda_0 < ... < \lambda_N = a\}$ are partitions of the interval $[-a, a]$. The central result states that, provided X is super-reflexive, there exists a number $1 < p < \infty$, depending on only X and $\sup_\lambda \|E(\lambda)\| \equiv \|E\|_\infty$, such that $var_p(E) < \infty$. We go even further when X is uniformly smooth. In this setting we provide an explicit range of such values p for which $var_p(E) < \infty$. This range depends only on $\|E\|_\infty$ and the modulus of convexity of X^*, δ_{X^*}.

Using these results on p-variation we prove power growth estimates for trigonometrically well bounded operators. We show that, provided X is super-reflexive and T is trigonometrically well bounded, there are constants $C > 0$ and $0 < \alpha < 1$, depending on T and X, such that for all $n \in \mathbb{Z} \setminus \{0\}$, we have $\|T^n\| \leq C|n|^\alpha$. This is an improvement on the previously known upper estimate of $O(|n|)$.

In chapter 3 we turn to Hilbert space, H. We show that, given a spectral family

E on H the joint condition $var_2(E) < \infty$ and $var_2(E^*) < \infty$ is equivalent to E arising from a spectral measure $\mathcal{E}$ on the Borel σ-algebra on $\mathbb{R}$. We show that this does not occur with all spectral families: we produce, for any given $s \geq 2$, a Hilbert space H and a spectral family E such that $var_s(E) = \infty$. We finish by showing that given any k-fold composition of the logarithm, $\ln^{(k)}(|n|)$, there is a trigonometrically well-bounded $T \in \mathcal{B}(H)$ such that $\|T^n\| \sim \ln^{(k)}(|n|)$, and its spectral family E satisfies $var_2(E) = \infty$. This shows a sharp departure from *power bounded* operators T, for which it is known that the spectral family E arises from a spectral measure $\mathcal{E}$, and so must satisfy $var_2(E) < \infty$.

In chapter 4 we concentrate on spectral integration. It is known that on a UMD space X, both a power-bounded operator T, and a uniformly bounded, strongly continuous operator group $\{U_t\}$, have representations in terms of a spectral family E. The current theory asserts that, for a special type of UMD space, these associated families allow for spectral integration of certain function algebras. We now show that if X is super-reflexive, the power boundedness assumption on T can be dropped. Specifically, we show that given *any* trigonometrically well bounded T on a super-reflexive X, there is some $s_T > 1$ such that the map $BV_s(\mathbb{T}) \to \mathcal{B}(X)$, $\phi \mapsto \int_{0^-}^{2\pi} \phi(e^{i\lambda}) dE(\lambda)$ is a well defined continuous algebra homomorphism for all $1 < s < s_T$. We then proceed to prove a vector-valued $BV_s(\mathbb{R})$ multiplier theorem for UMD spaces. This is accomplished using p-variation properties of the spectral family E of the right shift group $\{R_t\}$ on $L^p_X(\mathbb{R})$. Finally, we show that this result extends to a Marcinkiewicz $\mathfrak{M}_s$ multiplier theorem, provided that for each $\psi \in \mathfrak{M}_s(\mathbb{R})$ the set $\left\{ \int_{I_j} \psi \, dE \right\}_{j \in \mathbb{Z}}$ is R-bounded (where I_j are the dyadic intervals of $\mathbb{R}$).

Chapter 5 deals with densely defined operator groups $\{U_t\}_{t \in \mathbb{R}}$ and $\{U^k\}_{k \in \mathbb{Z}}$. We say that a densely defined group $\{U_t\}$ with domains $\mathcal{D}(U_t)$ has a densely defined spectral decomposition if for some unique spectral family E we have

$$U_t x = \lim_{a \to \infty} \int_{-a}^{a} e^{i\lambda t} dE(\lambda) x \; dt \quad \text{for } x \in \mathcal{D}(U_t). \tag{2}$$

We show that an arbitrary spectral family E gives rise to a densely defined one parameter operator group $\{U_t\}$. We prove a partial converse to this: a given densely defined operator group $\{U_t\}$ satisfies (2) on a certain *subspace* of $\cap_{t \in \mathbb{R}} \mathcal{D}(U_t)$ provided that, for each $\phi \in C_c^\infty$, the operator $\int_{\mathbb{R}} \phi(t) U_{-t} dt$ is well defined and satisfies $\left\| \int_{\mathbb{R}} \phi(t) U_{-t} dt \right\| \leq \gamma \|\hat{\phi}\|_{BV}$ for some $\gamma > 0$ independent of ϕ. To illustrate these phenomena we examine weighted $L^p(\mathbb{R})$ spaces, where the weight function $w(t)$ is an A_p weight.

CHAPTER 1

1.1 Spectral Theory of Trigonometrically well-bounded operators

The key motivation for this work is the study of trigonometrically well-bounded operators. Over the years various Banach space analogues of self-adjointness of operators have been developed, to mirror some aspect of the theory in a Hilbert space. One key concept is the idea of well-bounded operators, introduced and studied by Smart and Ringrose in [28], [29] and [30]. By definition, $T \in \mathcal{B}(X)$ is **well-bounded** if it has a functional calculus on the Banach algebra of absolutely continuous functions on a compact interval. Berkson and Dowson [2] refined this notion further; they defined T to be **well bounded (B)** if this functional calculus is weakly compact. It is readily verifiable that if X is reflexive then every well bounded T is automatically well bounded (B).

It is easy to check that, in a Hilbert space H, any scalar-type operator is well bounded. That the two notions do not coincide was neatly dealt with by Gillespie in [21]; it is known that for two commuting scalar type operators on a Hilbert space H, both the sum and the product are also scalar-type. Gillespie showed that in the case of two commuting *well bounded* operators on H, neither the sum nor the product need be well-bounded.

Returning to self-adjointness, an alternative approach is motivated by the Spectral Theorem for a normal operator N on a Hilbert space. The theorem gives a projection-valued measure $\mathcal{E}(\cdot)$ on the Borel σ-algebra of its spectrum, $\mathfrak{B}(\sigma(N))$. The measure $\mathcal{E}(\cdot)$ is strongly countably additive and $N = \int_{\sigma(N)} z\ \mathcal{E}(dz)$. Motivated by this representation, Dunford introduced the idea of a scalar-type operator on a Banach space, $S \in \mathcal{B}(X)$ which, by definition, can be written as $S = \int_{\sigma(S)} z\ \mathcal{E}(dz)$, for some projection-valued measure $\mathcal{E}(\cdot)$ on $\mathfrak{B}(\sigma(S))$. Now, one way to mimic the idea of self-adjointness was to stipulate that $\sigma(S) \subset [a, b]$, for some $-\infty < a < b < \infty$. Then $\mathcal{E}(\cdot)$ can be extended to all of $\mathbb{R}$ by setting it to be zero on $\mathfrak{B}(\mathbb{R} \setminus \sigma(S))$. By defining $E(\lambda) \equiv \mathcal{E}((-\infty, \lambda])$, $\lambda \in \mathbb{R}$, the representation of S becomes

$$S = aE(a) + \int_a^b \lambda dE(\lambda) \tag{1.1}$$

where the integral exists as a strong Riemann-Stieltjes limit. But now it becomes clear that the integral on the right side of (1.1) makes sense for a wider class of projection-valued functions than only those arising from a spectral measure. Such a function is called a *spectral family*, and is defined as follows.

Definition 1.1. *Let X be a Banach space. A* ***spectral family*** *on X is a projection-valued function $E : \mathbb{R} \to \mathcal{B}(X)$ having the following properties:*
(i) $\sup_{\mathbb{R}} \|E(\lambda)\| \equiv \|E\|_\infty < \infty$;
(ii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda \wedge \mu)$;
(iii) for each $x \in X$, $\lim_{\mu\to\lambda^+} E(\mu)x = E(\lambda)x$ *and* $\lim_{\mu\to\lambda^-} E(\mu)x = E(\lambda^-)x$ *exists;*
(iv) for each $x \in X$, $\lim_{\mu\to\infty} E(\mu)x = x$ *and* $\lim_{\mu\to-\infty} E(\mu)x = 0$.
We say that E is ***concentrated on*** $[a, b]$ *if* $E(\lambda) = I$ *for all* $\lambda \geq b$ *and* $E(\lambda) = 0$ *for all* $\lambda < a$.

Any spectral family E gives rise to a Riemann-Stieltjes notion of spectral integration with respect to E; see [36] for a rigorous treatment, or [4] for a succinct account. Briefly, let $\mathcal{P}_{\mathbb{R}}$ (respectively $\mathcal{P}_{[a,b]}$) denote the collection of partitions of $\mathbb{R}$ (respectively $[a, b]$), directed to increase by refinement (so that $u \leq v$ means that v is a refinement of u). For a bounded function $f : [a, b] \to \mathbb{C}$ and a partition $u = \{-\infty < a = \lambda_0 < ... < \lambda_N = b < \infty\}$, we define

$$\mathcal{S}(u, f) = \sum_{k=1}^{N} f(\lambda_k)\{E(\lambda_k) - E(\lambda_{k-1})\}.$$

If the net $\{\mathcal{S}(u, f)\}$ converges in the strong-operator topology as u increases through $\mathcal{P}_{[a,b]}$, we denote the strong limit by $\int_a^b f(\lambda)dE(\lambda)$ and define

$$\int_{a^-}^b f(\lambda)dE(\lambda) \equiv f(a)E(a) + \int_a^b f(\lambda)dE(\lambda). \tag{1.2}$$

If f is defined on all of $\mathbb{R}$ we proceed in the same manner: $\mathcal{S}(u, f)$ is defined as above, and if the net $\{\mathcal{S}(u, f)\}$ converges in the strong-operator topology as u increases through $\mathcal{P}_{\mathbb{R}}$, we denote the limit by $\int_{\mathbb{R}} f(\lambda)dE(\lambda)$. Notice that such f is automatically integrable with respect to E over any interval $[a, b]$:

$$\int_a^b f dE = \int_{\mathbb{R}} f dE\{E(b) - E(a)\} \quad \text{and} \quad \text{st}\lim_{a\to\infty} \int_a^a f dE = \int_{\mathbb{R}} f dE.$$

The Banach algebras $BV(J)$ and $AC(J)$ are central in the study of well bounded operators, so let us give their formal definitions. Let $J = [a, b]$ be a compact interval. $BV(J)$ is the space of functions $f : [a, b] \to \mathbb{C}$ of bounded variation:

$$var(f, [a, b]) = \sup_{u \in \mathcal{P}_{[a,b]}} \sum_{1}^{N} |f(\lambda_k) - f(\lambda_{k-1})| < \infty.$$

The quantity $\|f\|_{BV(J)} = \sup_J |f(t)| + var(f, J)$ defines a complete algebra norm on $BV(J)$. It is also a complete algebra norm on $AC(J) \subset BV(J)$, the space of absolutely continuous functions on J. We can similarly define $BV(\mathbb{T})$ to be the Banach algebra of functions $f : \mathbb{T} \to \mathbb{C}$ satisfying

$$\|f\|_{BV(\mathbb{T})} = |f(1)| + var(f, \mathbb{T}) < \infty.$$

Similarly, $AC(\mathbb{T}) \subset BV(\mathbb{T})$ is the Banach subalgebra of absolutely continuous functions on $\mathbb{T}$. Both $AC(\mathbb{R})$ and $AC(\mathbb{T})$ are the $\|\cdot\|_{BV}$-closures of the polynomial and trigonometric polynomial spaces in $BV(\mathbb{R})$ and $BV(\mathbb{T})$ respectively.

Let E be a spectral family concentrated on a compact interval J. If $f \in BV(J)$ then the operator $\int_{a^-}^{b} f(\lambda) dE(\lambda)$ exists in the sense described in (1.2). Moreover, $\| \int_{a^-}^{b} f(\lambda) dE(\lambda)\| \leq \|f\|_{BV(J)} \|E\|_\infty$ (see [4]), so we have a continuous algebra homomorphism

$$\Theta_E : \ BV(J) \to \mathcal{B}(X), \qquad f \mapsto \int_{a^-}^{b} f(\lambda) dE(\lambda). \tag{1.3}$$

Equally, if J is replaced by $\mathbb{R}$ we have a continuous algebra homomorphism $f \in BV(\mathbb{R}) \mapsto \int_{\mathbb{R}} f dE$ with $\| \int_{\mathbb{R}} f dE\| \leq \|f\|_{BV(\mathbb{R})} \|E\|_\infty$.

Now, from (1.3) it can be readily deduced that if X is reflexive and $\{E(\lambda)\}$ is a spectral family concentrated on $[0, 2\pi]$, then the map $AC[0, 2\pi] \to \mathcal{B}(X)$, $f \mapsto \int_{0^-}^{2\pi} f(\lambda) dE(\lambda)$ is a weakly compact functional calculus for the operator $A \equiv \int_{0^-}^{2\pi} \lambda \ dE(\lambda)$. So in particular, A is well bounded (B). Furthermore, the operator $T \equiv \int_{0^-}^{2\pi} e^{i\lambda} dE(\lambda)$ satisfies $T = e^{iA}$ and has a $AC(\mathbb{T})$ functional calculus $\phi \mapsto \int_{0^-}^{2\pi} \phi(e^{i\lambda}) dE(\lambda)$. Gillespie and Berkson [3] formalized these ideas by defining a new type of operator.

Definition 1.2. *Let X be a Banach space. We say that T is **trigonometrically well-bounded** if it has a weakly compact $AC(\mathbb{T})$ functional calculus.*

At this juncture Gillespie and Berkson established the following key result ([3]) which ties in the ideas of spectral families and well bounded (B) operators.

Theorem 1.1. *Let $T \in \mathcal{B}(X)$. T is trigonometrically well-bounded if and only if there exists a spectral family on X concentrated on $[0, 2\pi]$ such that*

$$T = \int_{0^-}^{2\pi} e^{i\lambda} dE(\lambda).$$

In this case it is possible to arrange matters so that $E(2\pi^-) = I$. With this extra property E is uniquely determined and is called the ***spectral family of*** *T.*

There is a useful characterization of trigonometrically well-bounded operators. Let $\mathfrak{P}(\mathbb{T})$ denote the space of trigonometric polynomials. So if $q \in \mathfrak{P}(\mathbb{T})$ then $q(e^{i\lambda t}) = \sum_{-N}^{N} \hat{q}_k e^{itk}$, where $\hat{q}_k$ are the usual Fourier coefficients of q. The space $\mathfrak{P}(\mathbb{T})$ is dense in $AC(\mathbb{T})$. For a trigonometric polynomial q and an invertible T it is natural to define

$$q(T) = \sum_{-N}^{N} \hat{q}_k T^k.$$

[3] establishes the following useful result.

Proposition 1.1. *T is trigonometrically well-bounded if and only if it is invertible, and*

(i) there exists $K > 0$ such that

$$\|q(T)\| \leq K \|q\|_{BV(\mathbb{T})} \quad \text{for all} \quad q \in \mathfrak{P}(\mathbb{T}),$$

(ii) the set $\{q(T) \ : \ q \in \mathfrak{P}(\mathbb{T}), \ \ \|q\|_{BV(\mathbb{T})} \leq 1\}$ has compact closure in the weak operator topology.

If X is reflexive condition (ii) can be dispensed with.

1.2 UMD Spaces

The geometry of a Banach space affects the nature of operators acting on it. One key property is the Unconditional Martingale Difference property. To define this notion, let $([0, 1], \mathcal{F}, \mu)$ be a probability measure space, and let $\{\mathcal{F}_n\}$ be a filtration for $\mathcal{F}$. We say that $\{d_k\}_{k \geq 0}$ is an X-valued martingale difference sequence if each $d_k : (\Omega, \mathcal{F}, \mu) \to X$ is strongly measurable and $Z_0 \equiv d_0$, $Z_k \equiv \sum_0^k d_j$ for $k > 0$ is an X-valued martingale with respect to the filtration $\{\mathcal{F}_k\}$ on Ω.

Definition 1.3. *A Banach space X is said to be UMD if, for some, or equivalently each, $1 < p < \infty$, there exists a constant C_p such that, for all martingale difference sequences $\{d_k\}_{k \in \mathbb{Z}} \in L_X^p([0, 1], \mu)$ and all $\{\epsilon_k\}_{k \in \mathbb{Z}} \in \{0, 1\}^\infty$ we have*

$$\Big\| \sum_{k=0}^{N} \epsilon_k d_k \Big\|_{L_X^p([0,1],\mu)} \leq C_p \Big\| \sum_{k=0}^{N} d_k \Big\|_{L_X^p([0,1],\mu)}.$$

The list of UMD spaces includes many of the more common spaces encountered in functional analysis: any finite dimensional and any Hilbert space; the scalar-valued $L^p(\Omega, \mu)$ spaces, for $1 < p < \infty$ and (Ω, μ) an arbitrary measure space; the Schatten-p spaces C_p, for $1 < p < \infty$. If X is UMD, then so are the X valued function spaces in Definition 1.4 below. Burkholder and Bourgain established an alternative characterization of the UMD property. It involves the Hilbert transform operator on spaces $L^p(\mathbb{R}, X)$, $L^p(\mathbb{T}, X)$ and $l_p(X)$. These function spaces have natural definitions, which we now give. Throughout the rest of the section we assume that $1 < p < \infty$.

Definition 1.4. *(i) For $G = \mathbb{R}$ or $\mathbb{T}$, let dt denote Haar measure on G. $L^p(G, X) \equiv L^p_X(G)$ is the space of X-valued strongly measurable, Bochner-integrable functions f such that*

$$\|f\|_{L^p(G,X)} \equiv \left\{ \int_G \|f(t)\|_X^p dt \right\}^{1/p} < \infty.$$

(ii) $l_p(X)$ is the space of X-valued sequences $\{x_k\}$ such that

$$\|x\|_{l_p(X)} \equiv \{ \sum_{-\infty}^{\infty} \|x_k\|^p \}^{1/p} < \infty.$$

(iii) For $G = \mathbb{R}$ or $\mathbb{T}$, $L^0_X(G)$ is the space of strongly measurable X-valued functions on G. l^0_X is the space of finitely supported X-valued sequences.

We shall use both notations $L^p(G, X)$ and $L^p_X(G)$, as they are both used widely in literature.

Just as in the classical case, we can define the Hilbert transform operator of X-valued functions. This is done via their truncated versions. For $\epsilon > 0$ and $N \geq 1$ we define

$$\begin{aligned} H_\epsilon f(e^{it}) &= \frac{1}{2\pi} \int_{\epsilon \leq |s| \leq \pi} \cot(s/2) f(e^{i(t-s)}) ds, \\ (H_N\, x)_m &= \sum_{|k| \leq N,\ k \neq 0} \frac{1}{k} x_{m-k}, \\ H_\epsilon f(x) &= \int_{\epsilon \leq |s| < 1/\epsilon} \frac{1}{s} f(x-s) ds. \end{aligned}$$

Then, provided the following limits exist in X, we define

Definition 1.5. *$Hf(e^{it}) = \lim_{\epsilon \to 0} H_\epsilon f(e^{it})$, $Hx_m = \lim_{N \to \infty} (H_N x)_m$, and $Hf(x) = \lim_{\epsilon \to 0} H_\epsilon f(x)$.*

The following result by Burkholder and Bourgain([17] and [16]) is very important because it shows that the geometric property UMD is very closely linked to the behaviour of the Hilbert transform. It links the *geometry* of a Banach space X to properties of an *operator* on X-valued functions.

Theorem 1.2. *The following conditions are equivalent:*
(i) X is UMD.
(ii) For some (or equivalently all) $p \in (1, \infty)$, the Hilbert transform is a bounded operator on any one of $L^p_X(\mathbb{T})$, $l_p(X)$ and $L^p_X(\mathbb{R})$.

A key result in [15] states that an invertible power-bounded operator T on a UMD space is necessarily trigonometrically well-bounded. By Theorem 1.1 there is an associated spectral family E, which can be used in special cases to define spectral integrals of function algebras $BV_q(\mathbb{T})$ and $\mathfrak{M}_q(\mathbb{T})$ (see section 1.3). The technique used is the Coifman-Weiss transference principle (see [10] for details), and the power-boundedness of T plays an indispensable role. One central theme of this thesis is to show that if we replace the UMD property with a different geometric notion, *any* trigonometrically well-bounded T gives rise to such spectral integration (for appropriate values of q).

Finally, the UMD property is used to establish a variety of multiplier theorems, which we now address.

1.3 L^p Multipliers and Spectral Integration

We start with the classical setting of scalar-valued $L^p(G)$ spaces ($G = \mathbb{Z}$, $\mathbb{T}$ or $\mathbb{R}$) with $1 < p < \infty$. For brevity of exposition, let us use $G = \mathbb{R}$. For $f \in L^1(\mathbb{R})$ the usual Fourier Transform of f is denoted by $\hat{f}$, so $\hat{f}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} f(x) dx$. The inverse transform is denoted by $\check{f}$. A complex bounded measurable $\psi \in L^\infty(\mathbb{R})$ is an **L-p multiplier** if the map

$$T_\psi : C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R}), \qquad f \mapsto (\psi \hat{f})^\vee$$

in fact maps into $L^p(\mathbb{R})$ and extends to a bounded linear map on all of $L^p(\mathbb{R})$. In this case we denote the extension with T_ψ and write $\psi \in M_p(\mathbb{R})$. We can define Fourier Multipliers of vector valued functions too, in the following manner. Let G be any of the above groups and let m be the usual Haar measure. $\hat{G}$ denotes the dual group of G. Now consider the algebraic tensor product $L^1 \cap L^\infty(G) \otimes X$ consisting of finite sums $\sum f_i x_i$; this space is dense in $L^p(G, X)$ under the norm in Definition 1.4. By Parseval's Theorem, any $\psi \in L^\infty(\hat{G})$ is a bounded multiplier transform on $L^2(G)$, denoted by T_ψ. For a function $F = \sum f_i x_i \in L^1 \cap L^\infty(G) \otimes X$ we define

$$T_\psi F = \sum T_\psi(f_i) x_i.$$

Thus $T_\psi F \in L^2(G, X)$. We say that ψ is an $L^p(G, X)$ multiplier if T_ψ extends to a bounded linear map from all of $L^p(G, X)$ into $L^p(G, X)$. In that case we write $\psi \in M_{p,X}(\hat{G})$. The classical space of multipliers on $L^p(G)$ is denoted simply by $M_p(\hat{G})$. Note that, unless $X = \{0\}$, we have the strict inclusion $M_{p,X}(\hat{G}) \subset M_p(\hat{G})$. The reverse inclusion, however, is certainly not true in general. For example, if X is not UMD, then the Hilbert transform is not bounded on $L^p(G, X)$. But the Hilbert transform arises as a multiplier from the sign function on $\mathbb{R}$, $\sigma(t)$. Thus we have $\sigma \in M_p(\hat{G})$, but $\sigma \notin M_{p,X}(\hat{G})$.

Remark Somewhat abusing the terminology, we shall use the term multiplier for both the function ψ and the operator T_ψ. This will not however cause any ambiguities or conflicts.

A key task in vector-valued multiplier theory is to establish which algebras of functions give rise to elements of $M_{p,X}(\hat{G})$. The theory is well developed in the scalar setting, where the main result is the Strong Marcinkiewicz Theorem in [38]. One aim of this thesis is to find which algebras of functions belong to $M_{p,X}(\hat{G})$, for a given space X. Let us define the algebras of concern to us.
We have already defined the spaces $BV(\mathbb{R})$ and $BV(\mathbb{T})$, as well as $BV(J)$, for an arbitrary compact interval. These spaces have a natural generalization, by considering the p variation, $1 < p < \infty$, of a given function $f : [a, b] \to \mathbb{C}$. More precisely, we define

$$var_p(f, [a, b]) = \sup_{u \in \mathcal{P}_{[a,b]}} \Big(\sum_{1}^{N} |f(\lambda_k) - f(\lambda_{k-1})|^p \Big)^{1/p}. \tag{1.4}$$

If we replace the compact interval $[a, b]$ with $\mathbb{R}$ (or $\mathbb{T}$), the definition remains the same, with the difference that we consider partitions $u \in \mathcal{P}_{\mathbb{R}}$ ($\mathcal{P}_{\mathbb{T}}$ respectively)

Definition 1.6. *For* $1 \leq p < \infty$, $BV_p(\mathbb{R})$ *is the Banach algebra of functions* $f : \mathbb{R} \to \mathbb{C}$ *with the norm*

$$\|f\|_{BV_p} = \sup_{t \in \mathbb{R}} |f(t)| + var_p(f) < \infty.$$

For $1 \leq p < \infty$, $BV_p(\mathbb{T})$ *is the Banach algebra of functions* $f : \mathbb{T} \to \mathbb{C}$ *with the norm*

$$\|f\|_{BV_p} = |f(1)| + var_p(f) < \infty.$$

Of particular interest are functions whose p-variation on the dyadic intervals is uniformly bounded. More precisely, let $\{s_n\}_{n \in \mathbb{Z}}$ be the dyadic points of $\mathbb{R}$ given by: $s_n = 2^{n-1}$ for $n > 0$ and $s_n = \frac{-1}{2^n}$ for $n \leq 0$. The dyadic intervals I_n of $\mathbb{R}$ are given by $I_n = [s_n, s_{n+1})$ for $n > 0$, $I_n = (s_n, s_{n+1}]$ for $n < 0$ and $I_0 = (s_0, s_1)$. We treat $\mathbb{T}$ analogously: let $t_n = 2^{n-1}\pi$ for $n \leq 0$ and $t_n = 2\pi - \frac{\pi}{2^n}$ for $n > 0$.

Then the dyadic intervals on $\mathbb{T}$ are given by $J_n = \{e^{it} : \ t_n \le t < t_{n+1}\}$ for $n > 0$, and $J_n = \{e^{it} : \ t_n < t \le t_{n+1}\}$ for $n \le 0$. For $1 \le p < \infty$ and a bounded measurable function $f : \mathbb{T} \to \mathbb{C}$ we have, using (1.4), $var_p(f, \bar{J}_n) = \sup_{u \in \mathcal{P}_{\bar{J}_n}} \left(\sum_1^N |f(\lambda_{j+1}) - f(\lambda_j)|^p \right)^{\frac{1}{p}}$, so we can define

$$\|f\|_{\mathfrak{M}_p(\mathbb{T})} = |f(1)| + \sup_n var_p(f, \bar{J}_n).$$

Similarly for a bounded measurable $f : \mathbb{R} \to \mathbb{C}$, we define $\|f\|_{\mathfrak{M}_p(\mathbb{R})} = \sup_{\mathbb{R}} |f(t)| + \sup_n var_p(f, \bar{I}_n)$.

The situation is somewhat simpler when $G = \mathbb{Z}$. Here the dyadic intervals are $\tilde{I}_n = I_n \cap \mathbb{Z}$. So, given $x \in l_\infty$, we have simply $var_p(x, \tilde{I}_n) = \left\{ \sum_{k=s_n}^{s_{n+1}-1} |x_{k+1} - x_k|^p \right\}^{1/p}$, and then $\|x\|_{\mathfrak{M}_p(\mathbb{Z})} = \sup_{\mathbb{Z}} |x_k| + \sup_n var_p(x, \tilde{I}_n)$.

Definition 1.7. *For $1 \le p < \infty$, let $\mathfrak{M}_p(\mathbb{Z}) = \{ \ x \in l_\infty : \ \|x\|_{\mathfrak{M}_p(\mathbb{Z})} < \infty\}$, $\mathfrak{M}_p(\mathbb{T}) = \{ \ f \in L^\infty(\mathbb{T}) : \ \|f\|_{\mathfrak{M}_p(\mathbb{T})} < \infty\}$ and $\mathfrak{M}_p(\mathbb{R}) = \{ \ f \in L^\infty(\mathbb{R}) : \ \|f\|_{\mathfrak{M}_p(\mathbb{R})} < \infty\}$. These are Banach algebras, collectively called the Marcinkiewicz-p classes.*

Classical multiplier theory is well developed. The Strong Marcinkiewicz Multiplier Theorem asserts that for all $1 < p < \infty$, $\phi \in \mathfrak{M}_1(\mathbb{R})$ is a multiplier for $L^p(\mathbb{R})$, with the norm not exceeding $C_p\|\phi\|_{\mathfrak{M}_1}$ for some constant $C_p > 0$. More recently, the following extension has been established in [18].

Theorem 1.3. *Let $1 < p < \infty$. Provided $q \in [1, \infty)$ satisfies $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$, then every $\phi \in \mathfrak{M}_q(\mathbb{R})$ gives rise to a multiplier $T_\phi \in M_p(\mathbb{R})$, with norm not exceeding $C_{p,q}\|\phi\|_{\mathfrak{M}_q}$ for some constant $C_{p,q}$ independent of ϕ.*

The vector-valued analogues of multiplier theorems are not as fully understood; but there has been a substantial breakthrough, by Berkson and Gillespie, for a special type of UMD space. Specifically, they define the class $\mathfrak{I}$ of UMD spaces as follows. For a Hilbert space Y_0 and a UMD space Y_1, which are compatible, we let $[Y_0, Y_1]_t$, $0 < t < 1$, denote the usual complex-interpolated space. Then $\mathfrak{I}$ consists of those X which are isomorphic to some $[Y_0, Y_1]_t$ space. (It is in fact an open question whether *all* UMD spaces belong to $\mathfrak{I}$). The main result, to be found in [14], is the following.

Proposition 1.2. *Suppose $X \in \mathfrak{I}$. Then there is a real number $s_X \in (1, \infty)$, depending only on X, such that for $1 \le q < s_X$, each $\phi \in \mathfrak{M}_q(\mathbb{R})$ gives a multiplier $T_\phi \in M_{p,X}(\mathbb{R})$ with $\|T_\phi\| \le A_{p,q}\|\phi\|_{\mathfrak{M}_q}$ for some $A_{p,q} > 0$ independent of ϕ.*

We have already addressed spectral families associated with trigonometrically well bounded operators. In fact, spectral families can arise from a wider class of operators, namely strongly continuous one-parameter operator groups.

Definition 1.8. *Let $(G, \circ)$ be one of $(\mathbb{R}, +)$, $(\mathbb{T}, \cdot)$ or $(\mathbb{Z}, +)$, with identity element e. A family of bounded operators $\{U_\gamma\}_{\gamma\in G}$ is a **strongly continuous operator group** on X if*
(i) $U_e = I_X$;
(ii) for all $\beta,\ \gamma \in G$, $U_\beta U_\gamma = U_\gamma U_\beta = U_{\beta\circ\gamma}$;
(iii) for each $x \in X$ the map $\gamma \mapsto U_\gamma x$ is continuous from G into X.

Berkson and Gillespie have shown in [10] that, provided X is UMD and $\{U_\gamma\}$ is *uniformly bounded*, then the group has a spectral family associated with it. Of particular interest to us is the right translation group $\{R_\gamma\}_{\gamma\in G}$ on the classical scalar spaces $L^p(G)$ and the vector-valued analogues $L^p_X(G)$ (of course, $\{R_\gamma\}$ is not in general a strongly continuous group; for example, if X is a certain weighted $L^p(\mathbb{R})$ space - see chapter 4).

Now, both scalar and vector valued multipliers (with UMD X in the latter case) can be viewed as special types of spectral integrals. This integration is with respect to the spectral family of $\{R_\gamma\}_{\gamma\in G}$ on $L^p_X(G)$. (The existence of the spectral family is guaranteed by the fact the X is UMD). Thus multiplier problems are linked to the following key question in spectral integration theory. *For a given spectral family E on a given space X, which Banach algebras give rise to spectral integrals?*

There are many important results in this area, again due to Berkson and Gillespie. They are to be found in [11], [8] and [13]. Here is a selection which gives the general flavour. The results, stated here for $\mathbb{T}$, are equally proved by the authors for $\mathbb{Z}$ and $\mathbb{R}$. Proposition (b) relates to *mean-2 bounded* operators T on a Hilbert space. These are bounded invertible operators characterized by the existence of a constant $C > 0$ such that for all $x \in H$ and $N \geq 1$, $\frac{1}{2N+1}\sum_{-N}^{N}\|T^k x\|^2 \leq C\|x\|^2$.

Proposition 1.3. *(a) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $X \subset L^p(\mu)$, $1 < p < \infty$, a closed subspace. Let U be a power bounded (and hence trigonometrically well-bounded) operator on X, with spectral family $\{E(\lambda)\}$. Then if either*
(i) $\phi \in BV(\mathbb{T})$, or
(ii) $\phi \in \mathfrak{M}_q(\mathbb{T})$ where $q \in [1\infty)$ satisfies $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$,
we have a bounded operator $T_\phi := \int_{0^-}^{2\pi}\phi(e^{i\lambda})dE(\lambda)$ such that in (i) the bound is $\|T_\phi\| \leq C_p\|\phi\|_{BV}$ for some $C_p > 0$ independent of ϕ, and in (ii) we have $\|T_\phi\| \leq K_{p,q}\|\phi\|_{\mathfrak{M}_q}$ for some $K_{p,q} > 0$ independent of ϕ.
(b) Let H be a Hilbert space and $T \in \mathcal{B}(H)$ a mean-2 bounded operator such that the family $\{T^k(T^*)^k\}$ commutes. Then T is trigonometrically well bounded and

there is a number $s > 2$ such that provided $1 \leq q < s$, each $\phi \in \mathfrak{M}_q(\mathbb{T})$ gives a bounded operator $T_\phi \equiv \int_{0^-}^{2\pi} \phi(e^{i\lambda}) dE(\lambda)$ with $\|T_\phi\| \leq C_{p,q} \|\phi\|_{\mathfrak{M}_q}$ for some $C_{p,q} > 0$ independent of ϕ.

In chapter 4 we prove a $BV_q(\mathbb{R})$ version of the multiplier theorem for *any* UMD space. We further show that, provided certain conditions hold, there is also a $\mathfrak{M}_q(\mathbb{R})$ multiplier theorem for an arbitrary UMD space X.

1.4 A_p weights

An important class of Banach spaces consists of weighted analogues of l_p, $L^p(\mathbb{R})$ and $L^p(\mathbb{T})$. We shall deal in chapters 2 and 5 with the first two, so let us treat them in some detail here. $L^p(\mathbb{T})$ is similar in many ways to $L^p(\mathbb{R})$, to which we turn now.
A measurable function $w : \mathbb{R} \to [0, \infty]$ which satisfies $w(t) \in (0, \infty)$ $a.e.(t)$, is called a weight function. The space $L^p(w)$ is then the Banach space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ for which $\left\{ \int_{\mathbb{R}} w(t) |f(t)|^p dt \right\}^{1/p} < \infty$. This quantity is indeed a complete norm on $L^p(w)$.
An important class of weights are those which satisfy the A_p condition of Muckenhoupt [25] and [26]. We say that w belongs to the $A_p(\mathbb{R})$ class, $1 < p < \infty$, if there is a constant $C_p > 0$ such that for all compact intervals K of length $|K| > 0$,

$$\left\{ \frac{1}{|K|} \int_K w(t) dt \right\} \left\{ \frac{1}{|K|} \int_K w(t)^{\frac{-1}{p-1}} dt \right\}^{p-1} \leq C_p. \tag{1.5}$$

The theory of A_p weights is rich and well understood, and we refer the reader to [25] and [7] for a full discussion of the main results and properties. We do, however, record a couple of features, both of which are stated and proved in the above references.

Proposition 1.4. *(a) Let $w(t)$ be a weight function and $1 < p < \infty$. Then the following three statements are equivalent:*
(i) $w \in A_p(\mathbb{R})$.
(ii) The Hilbert transform is a bounded operator from $L^p(w)$ into itself.
(b) If $-1 < \alpha < p - 1$, the function $w(t) = |t|^\alpha$ belongs to $A_p(\mathbb{R})$.

The treatment of $l_p(w)$ is entirely analogous, and somewhat easier to deal with, as we avoid any measure-theoretic issues. A scalar sequence $\{w_k\}_{k \in \mathbb{Z}}$ satisfies the A_p condition, $1 < p < \infty$, or equivalently belongs to $A_p(\mathbb{Z})$ if there exists a constant $B_p > 0$ such that for all finite intervals $K \subset \mathbb{Z}$,

$$\left\{ \frac{1}{|K|} \sum_{k \in K} w_k \right\} \left\{ \frac{1}{|K|} \sum_{k \in K} w_k^{\frac{-1}{p-1}} \right\}^{p-1} \leq B_p. \tag{1.6}$$

Importantly, Proposition 1.4 remains true for $l_p(w)$ spaces. In particular, if $-1 < \alpha < p - 1$, the sequence $w_k = |k|^\alpha$, $w_0 = 1$, is an A_p weight.
The right translation operator R plays a central role in the study of $l_p(w)$ spaces. For $j, k \in \mathbb{Z}$ we have $(R^k x)_j = x_{j-k}$, and an easy computation shows that $\|R^k\| = \sup_n \frac{w_{n+k}}{w_n}$. Crucially, *the operator R is trigonometrically well bounded on $l_p(w)$ if and only if H is a bounded operator from $l_p(w)$ into itself* (see [7]).

There is an analogue of R in $L^p(\mathbb{R})$ spaces. It is the *right translation group* $\{R_t\}_{t\in\mathbb{R}}$, where for a measurable function $f \in L^0(\mathbb{R})$, $R_t f(s) = f(s-t)$ *a.e.*(s). However, we defer its treatment until Chapter 5.

The aim of this chapter was to give an outline of the theory pertaining to this thesis. We now turn to a detailed treatment of the geometry of a Banach space X and the way this interacts with spectral families on X.

CHAPTER 2

2.1 Geometric properties of a Banach space

The central theme of this report is the interplay between the geometric properties of a Banach space and operators acting on it. The UMD property, as explained, has been shown to yield a rich array of results, centering on the Hilbert transform. In this chapter we shall investigate several geometric properties, namely the notions of super-reflexivity, uniform convexity and uniform smoothness. These properties are extensively studied in [33]. James and Gurarii ([22], [23] and [24]) have produced a series of results concerning sequences in a space enjoying these properties. We shall use some of their ideas to build new results about spectral families, and thence prove new properties of trigonometrically well bounded operators. We begin with a few definitions.

Definition 2.1. *Let Y and X be Banach spaces.*
(i) We say that Y is ***finitely representable*** *in X if for each finite dimensional $F \subset Y$ and $\epsilon > 0$ there exists a (finite dimensional) subspace $E \subset X$ and an isomorphism $T : E \to F$ such that $\|T\| \, \|T^{-1}\| \leq (1 + \epsilon)$.*
(ii) Let P be a geometric property. We say that X ***has super-P*** *if every Y finitely representable in X has property P*

Thus we immediately get our first key definition: a Banach space X is **super-reflexive** if every Y finitely representable in X is reflexive.

Definition 2.2. *Let X be a Banach space.*
(i) Let $0 < \theta < 1$ and $N \geq 1$. We say that the ***matrix condition holds for*** *(N, θ) if there exist $\{x_1, ..., x_N\} \in B_X$ and $\{\xi_1, ..., \xi_N\} \in B_{X^*}$ satisfying*

$$\begin{pmatrix} (x_1, \xi_1) & \dots & (x_1, \xi_N) \\ \vdots & & \vdots \\ (x_N, \xi_1) & \dots & (x_N, \xi_N) \end{pmatrix} = \begin{pmatrix} \theta & \dots & \theta \\ 0 & \ddots & \theta \\ 0 & 0 & \theta \end{pmatrix}. \tag{2.1}$$

(ii) For $0 < \theta < 1$ define

$$N_X(\theta) = \min\{ \ N : \ \text{the matrix condition fails for } (N, \theta) \ \}.$$

The definition of super-reflexivity is rather abstract to deal with in concrete situations. There is a more useful characterization of in terms of the matrix condition.

Proposition 2.1. *Let X be a Banach space. The following are equivalent:*
(i) X is not super-reflexive.
(ii) For some, or equivalently all, $\theta \in (0,1)$, the matrix condition holds for all (N,θ), $N \geq 1$.

Proof. See 4.I.3 in [33]. □

Observe that this says that X is super-reflexive if and only if for some, or equivalently all, $0 < \theta < 1$, $N_X(\theta) < \infty$. We shall rephrase Proposition 2.1 in the following manner, for later use in Theorem 2.1.

Proposition 2.2. *The following two conditions are equivalent:*
(A) For some $\theta \in (0,1)$, the matrix condition holds for all (N,θ), $N \geq 1$;
(B) there exist constants $0 < \alpha < \beta$ such that for all integers $N > 1$ we can find a sequence $\{x_1, ..., x_N\} \in X$ satisfying

$$(i) \qquad \Big\| \sum_{i=1}^{k} x_i \Big\| < \beta \ \ \textit{for all } k \leq N,$$

$$(ii) \qquad \Big\| \sum_{i=1}^{N} x_i a_i \Big\| \geq \alpha \sup |a_i| \ \ \textit{for all sequences } \{a_1, ..., a_N\} \subset \mathbb{C}.$$

Proof. Suppose that (A) holds. So, there is some $0 < \theta < 1$ such that $N_X(\theta) = \infty$. Let $N > 1$ be given and suppose $\{z_i\}$ and $\{\xi_i\}$ satisfy the matrix condition for (N,θ). Define

$$x_1 = z_1 \quad x_i = z_i - z_{i-1} \ \text{ for } \ i = 2, ..., N.$$

Then $\langle x_i, \xi_j \rangle = \delta_{ij}\theta$, so that, for each $k \leq N$,

$$\theta|a_k| = |\langle \sum_{j=1}^{N} a_j x_j, \xi_k \rangle| \leq \Big\| \sum_{j=1}^{N} a_j x_j \Big\| \ \text{ and } \ \Big\| \sum_{j=1}^{k} x_j \Big\| = \|z_k\| \leq 1.$$

Hence (i) and (ii) are satisfied with $\alpha = \theta$ and any $\beta > 1$.

Conversely, suppose there exist constants $0 < \alpha < \beta$ as in the statement of (B). Let $N \geq 1$ be given and let $\{x_k\}_{k=1,...,N}$ be the chosen sequence satisfying (i) and (ii). Define $z_k = \frac{1}{\beta} \sum_1^k x_j$ and define ξ_i on $lin\{x_1, ..., x_N\}$ by $\langle x_i, \xi_j \rangle = \alpha \delta_{ij}$. Then $\|z_k\| < 1$ and

$$|\langle \sum_{j=1}^{N} a_j x_j, \xi_k \rangle| = \alpha |a_k| \leq \| \sum_{j=1}^{N} a_j x_j \|$$

So we can extend each ξ_k to all of X with $\|\xi_k\| \leq 1$. Also, $\{\xi_i\}$ and $\{z_j\}$ satisfy the matrix condition for $(N, \frac{\alpha}{\beta})$. □

Remark

Observe that Proposition 2.2 gives the following result: X is super-reflexive if and only if, for all constants $0 < \alpha < \beta$, we have $N_X(\alpha/\beta) < \infty$. We shall use this fact in the proof of Theorem 2.1 later on. For now we continue with more geometric properties.

Definition 2.3. *(i) Let $\theta \in (0, 2]$. A $(1, \theta)$* ***branch*** *is a pair of points $\{x_1, x_2\} \subset B_X$ satisfying $\|x_2 - x_1\| \geq \theta$.*
Suppose $N \geq 1$ and that an (N, θ) branch has been defined. An $(N+1, \theta)$ ***branch*** *is a collection of points $\{x_1, ..., x_{2^{N+1}}\} \subset B_X$ such that $\|x_{2i} - x_{2i-1}\| \geq \theta$ for $i = 1, ..., 2^N$ and $\left\{\frac{1}{2}(x_{2i} + x_{2i-1})\right\}_{i=1,...,2^N}$ is an (N, θ) branch.*
(ii) For $0 < \theta < 2$ define

$$M_X(\theta) = \min\{\ M :\ B_X \textit{ does not contain an } (M, \theta) \textit{ branch } \}.$$

(iii) X has the ***Finite Tree Property*** *if, given $\theta \in (0, 2]$ and $N \geq 1$, there is an (N, θ) branch in B_X.*
(iv) X has the ***Infinite Tree Property*** *if, given $\theta \in (0, 2]$, there is a sequence $\{x_i\}_{i\geq 1} \subset B_X$ such that for each $N \geq 1$, the set $\{x_1, ..., x_{2^N}\}$ is an (N, θ) branch in B_X.*

Akin to Proposition 2.2, there is another useful way of characterizing super-reflexivity.

Proposition 2.3. *Let X be a Banach space. The following are equivalent.*
(i) X is super-reflexive.
(ii) For some, or equivalently all, $0 < \theta \leq 2$, $M_X(\theta) < \infty$.

Proof. Theorem 2 in 4.(i)3 in [33] says that X is super-reflexive if and only if it does not have the Finite Tree Property. But this is precisely equivalent to saying $M_X(\theta) < \infty$ for all $0 < \theta \leq 2$. □

Definition 2.4. *Let S_X denote the unit sphere of X. For $\epsilon \in [0, 2]$, the* ***modulus of convexity*** *of X is*

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\|\ :\ x, y \in B_X \textit{ with } \|x - y\| \geq \epsilon\ \}.$$

For $\tau > 0$ the ***modulus of smoothness*** *of X is*

$$\rho_X(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1\ :\ x \in S_X,\ \|y\| \leq \tau\}.$$

X is ***uniformly convex*** *if $\delta_X(\epsilon) > 0$ whenever $\epsilon > 0$. It is* ***uniformly smooth*** *if $\lim_{\tau\to 0} \frac{\rho_X(\tau)}{\tau} = 0$.*

An important fact is that X *is uniformly convex if and only if* X^* *is uniformly smooth.* A proof is provided in [33]. We shall use this property in section 2.3.

The standard properties of $\delta_X : [0,2) \to [0,1)$, are treated in [39] and [33] and we refer the reader there for the details. It is noteworthy that $\delta_X(\cdot)$ is not easy to calculate for an arbitrary Banach space. There is a formula for $X = L^p(\mu)$, for $2 \leq p < \infty$, but even for $1 < p < 2$ there is no explicit form (see [39] for an approximate behaviour). The functions $M_X(\cdot)$ and $N_X(\cdot)$ are similarly non-trivial to calculate. We can, however, see how they relate to each other, and to $\delta_X(\cdot)$.

Proposition 2.4. *Let X be a uniformly convex space, with modulus of convexity $\delta(\cdot)$. Let $0 < \eta \leq 2$, $0 < \theta < 1$ and $K \geq 1$. Then provided $1-\theta \leq \delta(\eta\theta^K)$ we have*

$$M_X(\eta\theta^K) \leq M_X(\eta) + K.$$

Proof. Let $\eta,\ \theta,\ K$ satisfy the hypothesis. Then we have $\eta\theta^K < \eta\theta^{K-1} < ... < \eta\theta$. Since δ is a strictly increasing function, it follows that

$$1-\theta < \delta(\eta\theta^K) < \delta(\eta\theta^{K-1}) < ... < \delta(\eta\theta). \tag{2.2}$$

We shall first show:

Claim

If B_X contains an $(N+1, \eta\theta)$ branch, then it contains an (N, η) branch.

Proof of Claim

Let $\{x_1^0, ..., x_{2^{N+1}}^0\}$ be an $(N+1, \eta\theta)$ branch in B_X. Then the mid-points $x_i^1 = \frac{1}{2}(x_{2i}^0 + x_{2i-1}^0)$, $i = 1, ..., 2^N$, form an $(N, \eta\theta)$ branch. Note that

$$\|x_{2i}^0 - x_{2i-1}^0\| \geq \eta\theta \ \Rightarrow \ \|x_i^1\| \leq 1 - \delta(\eta\theta) \quad i = 1, ..., 2^N.$$

Let $y_i^0 = \frac{1}{\theta}x_i^1$ for $i = 1, ..., 2^N$. Then $\{y_i^0\}$ form an (N, η) branch in B_X. To see this, note that $\|y_i^0\| \leq \frac{1-\delta(\eta\theta)}{\theta} \leq 1$, so that $y_i^0 \in B_X$. Also,

$$\|y_{2i}^0 - y_{2i-1}^0\| = \frac{1}{\theta}\|x_{2i}^1 - x_{2i-1}^1\| \geq \frac{1}{\theta}\eta\theta = \eta.$$

Continuing inductively we set, for $M = 0, ..., N-1$, $y_i^M = \frac{1}{\theta}x_i^{M+1} = \frac{1}{2\theta}(x_{2i}^M + x_{2i-1}^M)$ for $i = 1, ..., 2^{N-M}$. Then

$$\|x_{2i}^M - x_{2i-1}^M\| \geq \eta\theta \ \Rightarrow \ \|y_i^M\| \leq \frac{1-\delta(\eta\theta)}{\theta} \leq 1 \ \text{ so that } \ y_i^M \in B_X.$$

Also,

$$\|y_{2i}^M - y_{2i-1}^M\| = \frac{1}{\theta}\|x_{2i}^{M+1} - x_{2i-1}^{M+1}\| \geq \frac{1}{\theta}\eta\theta = \eta.$$

Thus $\{y_i^0\}$ satisfy the definition of an (N, η) branch in B_X, and hence the claim follows. Observe that we have used only $1 - \theta < \delta(\eta\theta)$ in the proof. In other words we have: provided $1 - \theta \leq \delta(\eta\theta)$

$$\nexists (N, \eta) \text{ branch } \Rightarrow \nexists (N+1, \eta\theta) \text{ branch.}$$

By replacing η with $\eta\theta^n$, for $n = 1, ..., K-1$, we deduce: provided $1-\theta \leq \delta(\eta\theta^{n+1})$

$$\nexists (N, \eta\theta^n) \text{ branch } \Rightarrow \nexists (N+1, \eta\theta^{n+1}) \textit{ branch.}$$

But these conditions are simultaneously guaranteed for all $n \in \{1, ..., K-1\}$ by (2.2), so we deduce that provided $1 - \theta \leq \delta(\eta\theta^K)$

$$\nexists (N, \eta) \text{ branch } \Rightarrow \nexists (N+K, \eta\theta^K) \text{ branch.}$$

Finally, by applying this to $N = M_X(\eta)$ we see that $M_X(\eta) + K$ is in the minimizing set for $\eta\theta^K$ in Definition 2.3 (ii). But the minimum of that set is precisely $M_X(\eta\theta^K)$, so that

$$M_X(\eta\theta^K) \leq M_X(\eta) + K.$$

□

The relationship between the Finite Tree Property and the matrix condition is quite explicit, and we describe it with the aid of the following lemma.

Lemma 2.1. *Let X be a Banach space and suppose the matrix condition holds for $(2^N, \theta)$, for some $N \geq 1$ and $0 < \theta < 1$. Then there exists an (N, θ) branch in B_X.*

Proof. Let $\{x_i\} \in B_X$ and $\{\xi_i\} \in B_{X^*}$, $i = 1, ..., 2^N$, satisfy the matrix condition. Then, for each $0 < k < 2^N$ we have

$$dist\big(\text{conv}(x_1, ..., x_k), \text{conv}(x_{k+1}, ..., x_{2^N})\big) \geq \theta. \tag{2.3}$$

To see this, let $u = \sum_1^k \alpha_i x_i$ and $v = \sum_{k+1}^{2^N} \beta_i x_i$ with $\alpha_i,\ \beta_i \geq 0$ and $\sum_1^k \alpha_i = \sum_{k+1}^{2^N} \beta_i = 1$. Then, as $\langle x_i, \xi_k \rangle = 0$ for $i > k$, we have

$$\theta = \Big|\sum_1^k \theta\alpha_i\Big| = \Big|\Big\langle \sum_1^k \alpha_i x_i - \sum_{k+1}^{2^N} \beta_i x_i, \xi_{k+1} \Big\rangle\Big| \leq \Big\| \sum_1^k \alpha_i x_i - \sum_{k+1}^{2^N} \beta_i x_i \Big\|.$$

As this holds for all such u and v, (2.3) follows.

Hence, for each $k = 0, ..., N-1$ we have 2^{N-k-1} pairs of points of the form $\frac{1}{2^k}(x_{2^k l+1} + ... + x_{2^k(l+1)})$ and $\frac{1}{2^k}(x_{2^k(l+1)+1} + ... + x_{2^k(l+2)})$ (with $l = 0, ..., 2^{N-k} - 2$), which satisfy

$$\Big\| \frac{x_{2^k l+1} + ... + x_{2^k(l+1)}}{2^k} - \frac{x_{2^k(l+1)+1} + ... + x_{2^k(l+2)}}{2^k} \Big\| \geq \theta.$$

This shows that $\{x_1, ..., x_{2^N}\}$ form an (N, θ) branch. □

Corollary 2.1. *Let $0 < \theta < 1$. Then we have $N_X(\theta) \leq 2^{M_X(\theta)}$.*

Proof. Let $N \geq 1$. From Lemma 2.1 we deduce that if B_X does not contain an (N, θ) branch, then the pair $(2^N, \theta)$ does not satisfy the matrix condition. Hence, taking $N = M_X(\theta)$, we see that $2^{M_X(\theta)}$ belongs to the minimizing set in Definition 2.2 (ii). But $N_X(\theta)$ is the minimum of this set, so the result follows. □

It is worth pausing here to consider what happens in a Hilbert space H. Here we can in fact calculate exactly $N_H(\theta)$. Provided $\theta \in (2/\sqrt{5}, 1]$, we have $N_H(\theta) = 2$. It follows from the following Lemma.

Lemma 2.2. *Let H be a Hilbert space. If $\theta \in (2/\sqrt{5}, 1]$ then there are no vectors $\{x_1, x_2, y_1, y_2\} \in B_H$ such that*

$$\begin{pmatrix} (y_1, x_1) & (y_1, x_2) \\ (y_2, x_1) & (y_2, x_2) \end{pmatrix} = \begin{pmatrix} \theta & \theta \\ 0 & \theta \end{pmatrix}. \tag{2.4}$$

Proof. Suppose on the contrary that such $\{x_1, x_2, y_1, y_2\} \in B_H$ exist. Then x_1, x_2 are linearly independent, for

$$ax_1 + bx_2 = 0 \quad \Rightarrow 0 = \langle y_2, ax_1 + bx_2 \rangle = \bar{b}\theta \quad \Rightarrow b = 0 \quad \Rightarrow a = 0.$$

Similarly y_1, y_2 are linearly independent. Now let $\{e_1, e_2\}$ be the orthonormal basis of $lin\{x_1, x_2\}$ obtained from x_1, x_2 by applying Gramm-Schmidt. Thus

$$\begin{aligned} e_1 &= \frac{x_1}{\|x_1\|}, \\ e_2 &= \frac{\|x_1\|^2 x_2 - \langle x_2, x_1 \rangle x_1}{\big\| \|x_1\|^2 x_2 - \langle x_2, x_1 \rangle x_1 \big\|}. \end{aligned}$$

Let $P : H \to lin\{x_1, x_2\}$ be the usual projection. Since $\langle Py_i, x_j \rangle = \langle y_i, x_j \rangle$, we may assume without loss that $y_1, y_2 \in lin\{x_1, x_2\}$. Then, with respect to the basis $\{e_1, e_2\}$ we can write

$$x_1 = (d, 0) \quad x_2 = (a, b),$$
$$y_1 = (s, t) \quad y_2 = (0, g).$$

But, we know that

$$x_2 = \frac{\langle x_2, x_1 \rangle}{\|x_1\|} e_1 + \frac{\big\| \|x_1\|^2 x_2 - \langle x_2, x_1 \rangle x_1 \big\|}{\|x_1\|^2} e_2$$

so that $b \in \mathbb{R}$ and $d = \|x_1\| \in \mathbb{R}$. We now have $\langle y_2, x_2 \rangle = \theta \in \mathbb{R}$ so that $\theta = g\bar{b} = \bar{g}b$. Hence $0 \neq b \in \mathbb{R}$ implies that $\bar{g}b = gb$ and $\frac{\theta}{b} = g \in \mathbb{R}$. Similarly,

$\theta = \langle y_1, x_1\rangle = s\overline{d}$, which implies that $sd = s\overline{d} = \overline{s}d$. But $d \neq 0$ and so $s = \overline{s} = \frac{\theta}{d}$. So now we can write

$$x_2 = (a_1 + ia_2,\ b), \quad y_1 = (\frac{\theta}{d},\ t_1 + it_2), \quad y_2 = (0,\ \frac{\theta}{b}).$$

Now, for $i = 1, 2$ we have $\theta \leq \|x_i\|,\ \|y_i\| \leq 1$. So $\|y_2\| \leq 1$ implies that $\theta \leq |b|$, $\theta \leq \|y_1\|$ gives $\theta \leq s$ and $\|y_1\| \leq 1$ gives $\theta \leq d$. Hence

$$\begin{aligned} \|y_1\| \leq 1 \quad &\Rightarrow \quad t_1^2 + t_2^2 + s^2 \leq 1 \\ &\Rightarrow \quad t_1^2 + t_2^2 \leq 1 - \theta^2 \\ \text{and} \quad \|x_2\| \leq 1 \quad &\Rightarrow \quad a_1^2 + a_2^2 \leq 1 - \theta^2. \end{aligned} \tag{2.5}$$

Now $\theta = \langle y_1, x_2\rangle = s(a_1 - ia_2) + (t_1 + it_2)b \quad \in \mathbb{R}$ implies that $t_2 b = sa_2$ and so $t_2 = \frac{a_2\theta}{db}$. Also, $\theta = a_1 s + bt_1$ gives $t_1 = \frac{\theta}{b}(1 - \frac{a_1}{d})$.
Putting all this into (2.5) gives $\left[1 - \frac{a_1}{d}\right]^2 + \frac{a_2^2}{d^2} \leq (1-\theta^2)\frac{b^2}{\theta^2}$. But, since $1 \geq \theta > 2/\sqrt{5}$, it follows that $(1-\theta^2)/\theta^2 < 1/4$. Thus

$$\left[1 - \frac{a_1}{d}\right]^2 + \frac{a_2^2}{d^2} < \frac{b^2}{4} \leq \frac{1}{4}.$$

From this we have $\left\{1 - \frac{a_1}{d}\right\}^2 < \frac{1}{4}$, and so $0 < \frac{d}{2} < a_1 < \frac{3d}{2}$. Hence $\frac{d^2}{4} < a_1^2$. Putting all this together we have

$$1 \geq \|x_2\|^2 = a_1^2 + a_2^2 + b^2 \geq a_1^2 + b^2 > \frac{\theta^2}{4} + \theta^2 > \left(\frac{1}{4}\right)\left(\frac{4}{5}\right) + \frac{4}{5} = 1,$$

which is a contradiction. $\square$

Finally, a key point to note is that there is a hierarchical relationship between super-reflexivity, uniform smoothness, and the UMD property. All uniformly smooth spaces and all UMD spaces are super-reflexive ([19]). However, Pisier has shown ([27], [19]) that there is a super-reflexive space which is not UMD.

2.2 Spectral Families on a Super-reflexive Space

We begin this section by extending the notion of p-variation, $var_p(E)$, to spectral families. It turns out that the geometry of the space X determines the nature of $var_p(E)$, for a given spectral family $\{E(\lambda)\}$.
We begin by extending the idea of $p-$variation of a scalar function to the vector-valued case. Let us recall that $\mathcal{P}_{[a,b]}$ is the collection of partitions $u = \{a = \lambda_0 < ... < \lambda_N = b\}$ of the interval $[a, b]$, directed to increase by refinement.

Definition 2.5. *Let E be a spectral family on X and let $x \in X$. For $q \in [1, \infty)$ we define the* ***q-variation of*** *$E(\cdot)x$ as*

$$var_q(E(\cdot)x) = \sup_{a>0} \sup_{u \in \mathcal{P}_{[-a,a]}} \big(\sum_{j=1}^{N} \|E(\lambda_j)x - E(\lambda_{j-1})x\|^q \big)^{\frac{1}{q}}.$$

The ***q-variation of E*** *is*

$$var_q(E) = \sup_{\|x\| \le 1} var_q(E(\cdot)x). \tag{2.6}$$

We say that E has ***bounded q-variation*** *if $var_q(E) < \infty$.*

Observe the following property of a spectral family, which follows directly from the definition: for $x \in X$ and $d < c < b < a$ we have
$\{E(b) - E(c)\}\{E(a) - E(d)\}x = \{E(b) - E(c)\}x$. Hence

$$\|\{E(b) - E(c)\}x\| \le 2\|E\|_\infty \|\{E(a) - E(d)\}x\| \text{ for all } x \in X, \quad d < c < b < a$$

This property allows us to make the following definition.

$$\begin{aligned} char(E) = \sup\{K > 0 : \ K\|\{E(b) - E(c)\}x\| &\le \|\{E(a) - E(b)\}x\| \\ &\text{for all } x \in X, \quad d < c < b < a\}. \end{aligned} \tag{2.7}$$

In particular we always have $char(E) \ge \frac{1}{2\|E\|_\infty}$. Let us prove a useful lemma.

Lemma 2.3. $\frac{1}{char(E)} = \sup_{\mu<\lambda} \|E(\lambda) - E(\mu)\|$

Proof. Let $d < c < b < a$ and $x \in X$. Then we have

$$\|E(b)x - E(c)x\| \le \|E(b) - E(c)\| \, \|E(a)x - E(d)x\| \le \sup_{\mu<\lambda} \|E(\lambda) - E(\mu)\| \, \|E(a)x - E(d)x\|.$$

Thus we have

$$\frac{1}{char(E)} \le \sup_{\mu<\lambda} \|E(\lambda) - E(\mu)\|.$$

Conversely, fix $\lambda > \mu$ and $x \in S_X$, the unit sphere in X. Then we have

$$\|E(\lambda)x - E(\mu)x\| \le \frac{1}{char(E)} \|E(\nu)x - E(-\nu)x\| \text{ for all } \nu > |\lambda| \vee |\mu|.$$

So, letting $\nu \to \infty$, we get

$$\|E(\lambda)x - E(\mu)x\| \le \frac{1}{char(E)} \|x\|.$$

Now take $\sup_{S_X}$ to obtain $\|E(\lambda) - E(\mu)\| \le \frac{1}{char(E)}$. But this holds for all $\lambda > \mu$ and so we get

$$\sup_{\mu<\lambda} \|E(\lambda) - E(\mu)\| \le \frac{1}{char(E)}.$$

as required. □

Now, by a *two-sided partition* of $\mathbb{R}$ we mean a sequence $u = \{s_j\}_{j\in\mathbb{Z}}$, with $s_j \to \pm\infty$ as $j \to \pm\infty$. We denote the set of such partitions with $\mathcal{P}_{\mathbb{R}}$. Given a spectral family $\{E(\lambda)\}$ and a partition $u \in \mathcal{P}_{\mathbb{R}}$ let $\{\Delta_j\} = \big\{[E(s_j) - E(s_{j-1})]\big\}_{j\in\mathbb{Z}}$ be the associated **Schauder decomposition**. That is, $\sum_{-\infty}^{\infty} \Delta_j = I$ (with strong operator convergence) and $\Delta_j\Delta_k = \delta_{jk}\Delta_j$, where δ_{jk} is the Kronecker delta. That this is so follows immediately from the definition of a spectral family. If we wish to distinguish between two decompositions arising from two different partitions, u and v say, we shall use the superscript notation $\{\Delta_k^{(u)}\}$ and $\{\Delta_k^{(v)}\}$. Observe also the following properties of $var_q(E)$.

Lemma 2.4. *Let E be a spectral family on X and suppose $var_q(E) < \infty$ for some $1 \le q < \infty$. Then*
(i) $var_q(E(\cdot)x) \le \|x\|\ var_q(E)$ for any $x \in X$;
(ii) $var_q(E) = \inf\{K : \ \{\sum_{-\infty}^{\infty} \|\Delta_k x\|^q\}^{1/q} \le \ K\|x\|, \quad x \in X, \quad u \in \mathcal{P}_{\mathbb{R}}\ \}$.

Proof. (i) Let $a > 0$ and $u = \{-a = \lambda_{-N} < ... < \lambda_N = a\} \in \mathcal{P}_{[-a,a]}$. Let $\{\Delta_k^{(u)}\}$ be the associated Schauder decomposition. Then

$$\begin{aligned}\Big\{\sum_{-N+1}^{N} \|\Delta_k^{(u)}x\|^q\Big\}^{1/q} &= \|x\|\Big\{\sum_{-N+1}^{N} \Big\|\Delta_k^{(u)}\big(x/\|x\|\big)\Big\|^q\Big\}^{1/q} \\ &\le \|x\| \sup_{\|y\|\le 1} \Big\{\sum_{-N+1}^{N} \|\Delta_k^{(u)}y\|^q\Big\}^{1/q} \\ &\le \|x\| \sup_{a>0}\ \sup_{\pi\in\mathcal{P}_{[-a,a]}}\ \sup_{\|y\|\le 1} \Big\{\sum_{-N+1}^{N} \|\Delta_k^{(\pi)}y\|^q\Big\}^{1/q} = \|x\|\ var_q(E).\end{aligned}$$

Taking $\sup_{a>0} \sup_{u\in\mathcal{P}_{[-a,a]}}$ on the left hand side gives the desired result.
(ii) Let $\mathfrak{S}$ be the set on the right hand side of (ii), and let $S = \inf \mathfrak{S}$. From (i) it follows that $S \le var_q(E)$. Suppose, though, that $S < var_q(E)$. Then there is an $\epsilon > 0$ such that $(S + \epsilon/2) < var_q(E)$ and $(S + \epsilon/2) \in \mathfrak{S}$. But then by definition of $\mathfrak{S}$, we have for all $x \in X$ and $u \in \mathcal{P}_{\mathbb{R}}$, $\Big\{\sum_{-\infty}^{\infty} \|\Delta_k^{(u)}x\|^q\Big\}^{1/q} \le (S + \epsilon/2)\|x\|$. So taking $\sup_{\|x\|\le 1}$ and $\sup_{u\in\mathcal{P}_{\mathbb{R}}}$ on the left gives $var_q(E) \le (S + \epsilon/2)$ which is a contradiction. □

We can now state and prove the central result. It asserts that if X is super-reflexive, then every spectral family on X has bounded q variation for some $1 < q < \infty$.

Theorem 2.1. *Let X be a super-reflexive Banach space and let E be a spectral family on X. Given any $0 < \phi < \frac{1}{4\|E\|_\infty}$, there exists $q \in (1, \infty)$, depending only on ϕ and X, such that*

$$var_q(E) \le \frac{1}{\phi}. \tag{2.8}$$

Theorem 2.1 is more substantial when stated in the following manner.

Corollary 2.2. *Let X be a super-reflexive Banach space and let E be a spectral family on X. Then there exists $q \in (1, \infty)$, depending only on X and $\|E\|_\infty$, such that*

$$var_q(E) < \infty. \tag{2.9}$$

The proof of Theorem 2.1 is rather involved, and requires several technical results. The following Lemma is implicitly used by James in [24]. We provide an original proof herein.

Lemma 2.5. *Let $\{a_{-M}, ..., a_M\}$ be a sequence in $\mathbb{R}^+$ with $M \geq 3$. Let $1 < N < M$ and $0 < \theta < 1$. Suppose we have*

$$a_i < \frac{\theta}{2N+1} \sum_{-M}^{M} a_k \quad \textit{for each} \quad a \in \{-M, ...M\}.$$

(i) There is a subsequence $\{0 = M_0 < M_1 < ... < M_N = M\} \subseteq \{1, ..., M\}$ such that

$$\Big| \sum_{-M_j}^{M_j} a_k - \frac{j}{N} \sum_{-M}^{M} a_k \Big| < \frac{\theta}{2N} \sum_{-M}^{M} a_k \quad \textit{for} \quad j = 0, ...N. \tag{2.10}$$

(ii) Furthermore, for $j = 1, ..., N$ we have

$$\Big| \sum_{M_{j-1} < |k| \leq M_j} a_k - \frac{1}{N} \sum_{-M}^{M} a_k \Big| < \frac{\theta}{N} \sum_{-M}^{M} a_k. \tag{2.11}$$

Proof. (i) The statement is trivially true for $j = 0$. Let us write $b_0 = a_0$ and for $j = 1, ..., N$, $b_j = \sum_{-j}^{j} a_k$. Then we have

$$b_j - b_{j-1} = a_{-j} + a_j < \frac{2\theta}{2N+1} b_M < \frac{\theta}{N} b_M.$$

Let us divide $[0, b_M]$ into N intervals

$$K_j = [\frac{j-1}{N} b_M, \frac{j}{N} b_M) \quad \text{for } j = 1, ..., N-1, \qquad K_N = [\frac{N-1}{N} b_M, b_M].$$

Note that $b_0 < \theta b_M / N$ and $\theta b_M / N < b_M / N$. Hence $b_0 \in K_1$, and each K_j contains an element of $\{b_1, ..., b_M\}$. Therefore, the following are well-defined:

$$U_j = max\{i \ : \ b_i \in K_j\}, \qquad j = 1, ..., N$$
$$L_j = min\{i \ : \ b_i \in K_{j+1}\}, \qquad j = 0, ..., N-1.$$

Note that $U_j + 1 = L_j$ for $j = 1, ..., N-1$. (†)
Further, we set $M_N = M$ and for $j = 1, ..., N-1$

$$M_j = U_j \quad \text{if } \frac{j}{N} b_M - b_{U_j} < b_{L_j} - \frac{j}{N} b_M,$$
$$M_j = L_j \quad \text{otherwise.}$$

The sequence $\{M_j\}$, $j = 0, ..., N-1$ is strictly increasing. For, if $M_j = U_j$, then $M_{j+1} \geq U_{j+1} \geq L_j > U_j = M_j$; and, if $M_j = L_j$, then $b_{1+L_j} \in K_{j+1}$, so $U_{j+1} \geq L_j + 1$ and so $M_{j+1} \geq L_j + 1 > M_j$. Moreover, we have, for each $j = 1, ..., N$

$$\left|b_{M_j} - \frac{j}{N}b_M\right| < \frac{\theta}{2N}b_M.$$

To see this, suppose, on the contrary, that

$$\left|b_{M_j} - \frac{j}{N}b_M\right| \geq \frac{\theta}{2N}b_M \quad \text{for some } j. \tag{2.12}$$

Then, using (†), in the case $M_j = L_j$ we have $\left|b_{M_j} - \frac{j}{N}b_M\right| = b_{L_j} - \frac{j}{N}b_M \geq 0$ and so

$$\frac{j}{N}b_M - b_{U_j} = (b_{L_j} - b_{U_j}) + (\frac{j}{N}b_M - b_{L_j}) \; < \frac{\theta b_M}{N} - \frac{\theta b_M}{2N} = \frac{\theta b_M}{2N} \leq b_{L_j} - \frac{j}{N}b_M.$$

But this is precisely the condition for $M_j = U_j$, giving a contradiction. Similarly, in the other case, namely $M_j = U_j$, (2.12) would give $\left|b_{M_j} - \frac{j}{N}b_M\right| = \frac{j}{N}b_M - b_{U_j} \geq 0$ and

$$b_{L_j} - \frac{j}{N}b_M = (b_{L_j} - b_{U_j}) - (\frac{j}{N}b_M - b_{U_j}) \; < \frac{\theta b_M}{N} - \frac{\theta b_M}{2N} = \frac{\theta b_M}{2N} \leq \frac{j}{N}b_M - b_{U_j}.$$

But this is precisely the condition for $M_j = L_j$, again giving a contradiction. Finally note that the construction of the sequence $\{b_j\}$ implies that (2.12) is precisely (2.10) in the statement of (i). Thus we have constructed the desired subsequence $\{M_j\}$ of $\{1, ..., M\}$.

(ii) Inequality (2.11) now follows immediately. For, if $j = 1, ..., N$, the left hand side can be rewritten as

$$\begin{aligned}
& \Bigg| \quad \Big\{ \sum_{|k| \leq M_j} a_k - \frac{j}{N} \sum_{-M}^{M} a_k \Big\} - \Big\{ \sum_{|k| \leq M_{j-1}} a_k - \frac{j-1}{N} \sum_{-M}^{M} a_k \Big\} \Bigg| \\
\leq & \quad \Bigg| \Big\{ \sum_{|k| \leq M_j} a_k - \frac{j}{N} \sum_{-M}^{M} a_k \Big\} \Bigg| + \Bigg| \Big\{ \sum_{|k| \leq M_{j-1}} a_k - \frac{j-1}{N} \sum_{-M}^{M} a_k \Big\} \Bigg| \\
< & \quad \frac{\theta}{2N} \sum_{-M}^{M} a_k + \frac{\theta}{2N} \sum_{-M}^{M} a_k = \frac{\theta}{N} \sum_{-M}^{M} a_k
\end{aligned}$$

□

Let us now prove Theorem 2.1.

Proof of Theorem 2.1 For ease of notation let us write $\epsilon = \frac{1}{2\|E\|_\infty}$. Observe that $char(E) \geq \epsilon$. We shall prove the following claim.

Claim

There exists $1 < q < \infty$, independent of the partition $\{s_j\}_{j\in\mathbb{Z}} \in \mathcal{P}_{\mathbb{R}}$, such that, for all $x \in X$,

$$\phi\Big(\sum_{j\in\mathbb{Z}} \|\Delta_j x\|^q\Big)^{1/q} \leq \Big\|\sum_{j\in\mathbb{Z}} \Delta_j x\Big\| = \|x\|.$$

This claim immediately gives the desired result, first by taking the supremum over partitions $\{s_j\} \cap [-a, a]$, then over $\{a > 0\}$ and finally over $\{x \in B_X\}$.

Proof of the claim

Suppose the claim is false. We shall establish the existence of $\alpha \leq \frac{2\phi^2}{\epsilon}$ and $\beta > \frac{2}{\epsilon}$ such that $N_X(\alpha/\beta) = \infty$. This will contradict the super-reflexivity of X, by the remark following Proposition 2.2.

So, let $N > 2$ be any integer. Pick $0 < \lambda < 1$ such that $2\phi < \lambda^2\epsilon$. Now pick $q > 1$ sufficiently large such that $(1 + 2N)^{1/q} < \frac{1}{\lambda}(1 - \lambda)^{1/q}$. Then, if $\beta_i \geq 0$ for $i \in \mathbb{Z}$, we have

$$\big(\sum_{i=1}^{N} \beta_i\big)^{1/q} < \frac{1}{\lambda} \sup_i \beta_i^{1/q}. \tag{2.13}$$

Now, if the claim is wrong then there is $z_0 \in X$ and a partition (λ_j) of $\mathbb{R}$ such that

$$\|z_0\| < \phi\big(\sum_{-\infty}^{\infty} \|\Delta_j z_0\|^q\big)^{1/q}.$$

Let $0 < \eta < 2\|z_0\|$ be such that

$$\|z_0\| + \eta < \phi\big(\sum_{-\infty}^{\infty} \|\Delta_j z_0\|^q\big)^{1/q}.$$

Then there are positive integers U_0, L_0 such that

$$\|z_0\| + \frac{\eta}{2} < \phi\big(\sum_{-L}^{U} \|\Delta_j z_0\|^q\big)^{1/q} \quad \text{for all } \; U \geq U_0, \quad L \geq L_0.$$

Also, there are positive integers A_0 and B_0 such that

$$\|z_0\| - \frac{\eta}{2} < \big\|\sum_{-B}^{A} \Delta_j z_0\big\| < \|z_0\| + \frac{\eta}{2} \quad \text{for all } \; A \geq A_0, \quad B \geq B_0.$$

Now let $M_0 = \max\{U_0, L_0, A_0, B_0, N + 1\}$. Then for all $K, L \geq M_0$ we have

$$\big\|\sum_{-L}^{K} \Delta_j z_0\big\| < \|z_0\| + \frac{\eta}{2} < \phi\big(\sum_{-M_0}^{M_0} \|\Delta_j z_0\|^q\big)^{1/q}.$$

Consider now $0 \neq y_0 = \{E(\lambda_{M_0}) - E(\lambda_{-M_0-1})\}z_0$. We then have $\Delta_j y_0 = \Delta_j z_0$ for $j = -M_0, ..., M_0$ and $\Delta_j y_0 = 0$ otherwise. Thus we have, for all $K, L \geq M_0$

$$\sum_{-\infty}^{\infty} \|\Delta_j y_0\|^q = \sum_{-M_0}^{M_0} \|\Delta_j y_0\|^q = \sum_{-M_0}^{M_0} \|\Delta_j z_0\|^q > \frac{1}{\phi^q}\big\|\sum_{-L}^{K} \Delta_j z_0\big\|^q.$$

Taking $K = L = M_0$, we obtain

$$\Big(\sum_{-M_0}^{M_0} \|\Delta_j y_0\|^q\Big)^{1/q} > \frac{1}{\phi}\Big\|\sum_{-M_0}^{M_0} \Delta_j y_0\Big\| = \frac{1}{\phi}\|y_0\|.$$

In other words, we have found an $M \equiv M_0$ and $0 \neq y \in \{E(\lambda_M) - E(\lambda_{-M-1})\}X$ such that

$$\frac{\|y_0\|}{\big(\sum_{|j|\le M} \|\Delta_j y_0\|^q\big)^{1/q}} < \phi.$$

Now let

$$A = \inf\left\{\frac{\|x\|}{\big(\sum_{|j|\le M} \|\Delta_j x\|^q\big)^{1/q}} \quad : \quad 0 \neq x \in \{E(\lambda_M) - E(\lambda_{-M-1})\}X\right\}. \quad (2.14)$$

The calculations above imply that A is well-defined, and moreover $0 \le A < \phi$. To show that $A > 0$, suppose on the contrary that $A = 0$. Then for each $\delta > 0$, we can find an $x \in X$ such that

$$\|\{E(\lambda_M) - E(\lambda_{-M-1})\}x\| < \delta\Big(\sum_{|j|\le M} \|\Delta_j x\|^q\Big)^{1/q}.$$

But, since $char(E) \ge \epsilon$, we have for each $k \in \{-M, ..., M\}$

$$\Big\|\sum_{-M}^{M} \Delta_j x\Big\| \ge \epsilon\|\Delta_k x\|.$$

Thus for each k we have $\epsilon^q\|\Delta_k x\|^q < \delta^q \sum_{|j|\le M} \|\Delta_j x\|^q$
So, summing over $\{k : \ |k| \le M\}$, we get

$$\epsilon^q \sum_{|k|\le M} \|\Delta_k x\|^q < (2M+1)\delta^q \sum_{|k|\le M} \|\Delta_k x\|^q,$$

and so $\epsilon < (2M+1)^{1/q}\delta$. But $\delta > 0$ was arbitrary and so we get the contradiction $\epsilon = 0$. Hence $A > 0$.
Observing that $A < A/\lambda$, we choose $x_0 \in \{E(\lambda_M) - E(\lambda_{-M-1})\}X$ with $\|x_0\| = 1$ and

$$\frac{\|x_0\|}{\big(\sum_{|j|\le M} \|\Delta_j x_0\|^q\big)^{1/q}} < \frac{A}{\lambda}. \quad (2.15)$$

Again, as $char(E) \ge \epsilon$, we know that for all $y \in X$ and $k \in \{-M, ..., M\}$

$$\Big\|\sum_{-M}^{M} \Delta_j y\Big\| \ge \epsilon\|\Delta_k y\|.$$

Hence for each k we have

$$\|\Delta_k x_0\| \le \frac{A}{\epsilon\lambda}\Big(\sum_{|j|\le M} \|\Delta_j x_0\|^q\Big)^{1/q}.$$

Now, the fact that $A < \phi < \frac{\lambda^2 \epsilon}{2}$, and $\frac{\lambda^q}{2^q} < \frac{(1-\lambda)}{2N+1}$ gives

$$\|\Delta_k x_0\|^q \leq \lambda^q \sum_{-M}^{M} \|\Delta_j x_0\|^q < \frac{1-\lambda}{2N+1} \sum_{-M}^{M} \|\Delta_j x_0\|^q.$$

Then Lemma 2.5 gives a subsequence $\{0 = M_0, M_1, ..., M_N = M\} \subseteq \{1, ..., M\}$ such that for $j = 0, ..., N$ we have

$$\left| \sum_{-M_j}^{Mj} \|\Delta_k x_0\|^q - \frac{j}{N} \sum_{-M}^{M} \|\Delta_k x_0\|^q \right| < \frac{1-\lambda}{2N} \sum_{-M}^{M} \|\Delta_k x_0\|^q. \tag{2.16}$$

Now let

$$\begin{aligned} u_j &= \sum_{M_{(j-1)}+1}^{M_j} \Delta_j x_0 + \sum_{-M_j}^{-M_{(j-1)}-1} \Delta_j x_0 \quad j = 2, ..., N, \quad u_1 = \sum_{-M_1}^{M_1} \Delta_j x_0, \\ v_j &= \sum_{M_{(j-1)}+1}^{M_j} \|\Delta_j x_0\|^q + \sum_{-M_j}^{-M_{(j-1)}-1} \|\Delta_j x_0\|^r \quad j = 2, ..., N, \quad v_1 = \sum_{-M_1}^{M_1} \|\Delta_j x_0\|^q. \end{aligned}$$

Note that $x_0 = \sum_{j=1}^N u_j$ and $u_j \in \{E(\lambda_M) - E(\lambda_{-M-1})\}X$ for $j = 1, ..., N$. Then by Lemma 2.5 (ii) we have

$$\left| v_j - \frac{1}{N} \sum_{-M}^{M} \|\Delta_k x_0\|^q \right| < \frac{1-\lambda}{N} \sum_{-M}^{M} \|\Delta_k x_0\|^q.$$

Hence for each $j \in \{1, ..., N\}$

$$\frac{\lambda}{N} \sum_{-M}^{M} \|\Delta_k x_0\|^q < v_j < \frac{2-\lambda}{N} \sum_{-M}^{M} \|\Delta_k x_0\|^q < \frac{1}{N\lambda} \sum_{-M}^{M} \|\Delta_k x_0\|^q,$$

and this implies:

$$v_j > \lambda^2 \sup_{1 \leq k \leq N} v_k \quad \text{for } j = 1, ..., N. \tag{2.17}$$

Thus we have, for each j,

$$\begin{aligned} \frac{\lambda}{\left(\sum_{-M}^{M} \|\Delta_k x_0\|^q \right)^{1/q}} &< A \leq \frac{\|u_j\|}{v_j^{1/q}} \\ &< \frac{\|u_j\|}{\lambda^{2/q} \sup\limits_{1 \leq k \leq N} v_k^{1/q}} < \frac{\|u_j\|}{\lambda^3 \left(\sum_{-M}^{M} \|\Delta_k x_0\|^q \right)^{1/q}}. \end{aligned}$$

These inequalities follow by (2.15), (2.14), (2.17) and (2.13) respectively. Using the terms on the extreme left and right we obtain

$$\|u_j\| > \lambda^4 \ \text{ for each } j = 1, ..., N.$$

But we also know that $char(E) \geq \epsilon$ and $\lambda^4 > \frac{4\phi^2}{\epsilon^2}$ so that for any sequence $\{a_i\}$ of scalars and $k = 1, ..., N$

$$\Big\| \sum_1^N a_j u_j \Big\| \geq \frac{\epsilon}{2} \|a_k u_k\| \geq \frac{\epsilon \lambda^4}{2} |a_k| \geq \frac{2\phi^2}{\epsilon} |a_k|.$$

Let $\alpha = \frac{2\phi^2}{\epsilon} = 4\|E\|_\infty \phi^2$. Observe also

$$1 = \|x_0\| = \Big\| \sum_1^N u_j \Big\| \geq \frac{\epsilon}{2} \Big\| \sum_1^k u_j \Big\| \quad \text{for all } k \leq N.$$

Hence $\left\| \sum_1^k u_j \right\| \leq \frac{2}{\epsilon}$ for all $k \leq N$. Now we choose any $\beta > \frac{2}{\epsilon}$ satisfying $\alpha < \beta$. In particular, for example, $\beta = \frac{4}{\epsilon} = \frac{2}{\|E\|_\infty}$. Hence, using Proposition 2.2, we have constructed $\alpha < \beta$ such that the matrix condition holds for $(N, \frac{\alpha}{\beta})$, so that $N_X(\frac{\alpha}{\beta}) > N$. Since N was arbitrary, and α and β are independent of N, it follows that $N_X(\frac{\alpha}{\beta}) = \infty$. By the remark following Proposition 2.2 this contradicts the super-reflexivity of X.$\Box$.

Theorem 2.1, or equivalently Corollary 2.2, both have important applications both in vector valued multiplier theory and in the study of trigonometrically well bounded operators. We turn to the latter now, and deal with multipliers in chapter 4.

The notion of trigonometric well-boundedness was discussed in detail in chapter 1. Recall that $T \in \mathcal{B}(X)$ is trigonometrically well-bounded precisely if there is a spectral family E on X, concentrated on $[0, 2\pi]$ such that

$$T = \int_{0^-}^{2\pi} e^{i\lambda} dE(\lambda) \equiv \; E(0) + \int_0^{2\pi} e^{i\lambda} dE(\lambda).$$

The resulting $AC(\mathbb{T})$ functional calculus automatically gives a representation for powers of T, $T^n = \int_{0^-}^{2\pi} e^{in\lambda} dE(\lambda)$, for $n \in \mathbb{Z}$. Using integration by parts we calculate

$$\begin{aligned} T^n &= \int_{0^-}^{2\pi} e^{in\lambda} dE(\lambda) \\ &= E(0) + \left[e^{in\lambda} E(\lambda) \right]_0^{2\pi} - \; in \int_0^{2\pi} e^{in\lambda} E(\lambda) \; d\lambda \\ &= I - \; in \int_0^{2\pi} e^{in\lambda} E(\lambda) \; d\lambda. \end{aligned}$$

Hence an easy estimate for the power growth of T is

$$\|T^n\| \leq 1 + 2\pi \|E\|_\infty |n|. \tag{2.18}$$

In other words, T has at most $\sim |n|$ power growth. We can now show that if X is super-reflexive, then every trigonometrically well-bounded T has a *slower* power growth. $\|T^n\|$ is dominated by $C|n|^\alpha$ where $C > 0$ and $\alpha \in (0,1)$ are constants depending on X and T. In fact, the dependence on T is only through $\|E\|_\infty$.

Theorem 2.2. *Let X be a super-reflexive Banach space and suppose $T \in \mathcal{B}(X)$ is trigonometrically well-bounded, with spectral family $\{E(\lambda)\}$. Then there exist constants $C > 0$ and $\alpha \in (0,1)$, both depending on X and $\|E\|_\infty$, such that for all $n \in \mathbb{Z} \setminus \{0\}$*

$$\|T^n\| \le C|n|^\alpha. \tag{2.19}$$

The two key ingredients involved here are Theorem 2.1, and the following Holder-type inequality, due to L.C. Young, [32].

Lemma 2.6. *Let J be a compact interval and let $1 < p,\ q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} > 1$. If $f \in BV_p(J)$ and $g \in BV_q(J)$ have no common discontinuities, then*

$$\Big| \int_J f(t)\ dg(t) \Big| \le K_{p,q} \|f\|_{BV_p}\ var_q(g).$$

where $K_{p,q} = 1 + \zeta(1/p + 1/q)$ and $\zeta(s) = \sum_1^\infty \frac{1}{n^s}$ is the zeta function.

In order to apply Lemma 2.6 in the proof of Theorem 2.2, we need the following estimate on the p-variation of the functions $\epsilon_n(\lambda) = e^{in\lambda}$, $\lambda \in [0, 2\pi]$.

Lemma 2.7. *For $n \in \mathbb{Z}$ let*

$$\epsilon_n : [0, 2\pi] \to \mathbb{C}, \quad \lambda \to e^{in\lambda}.$$

Then for all $p \in [1, \infty)$

$$var_p(\epsilon_n) \le (2\pi + 13)|n|^{\frac{1}{p}}.$$

Proof. Let $n \in \mathbb{Z}$ be fixed. In line with previous notation, let $\mathcal{P}_{[0,2\pi]}$ be the set of partitions of $[0, 2\pi]$. For $u = \{0 = \lambda_0 < ... < \lambda_N = 2\pi\} \in \mathcal{P}_{[0,2\pi]}$, define

$$var_p(\epsilon_n, u) = \left\{ \sum_{j=1}^{N} |\epsilon_n(\lambda_j) - \epsilon_n(\lambda_{j-1})|^p \right\}^{1/p}.$$

Hence $var_p(\epsilon_n) = \sup_{u \in \mathcal{P}_{[0,2\pi]}} var_p(\epsilon_n, u)$. Observe also that

$$var_1(\epsilon_n) = \int_0^{2\pi} |\epsilon_n'(\lambda)| d\lambda = 2\pi |n|.$$

Now, for any $u \in \mathcal{P}_{[0,2\pi]}$, let

$$G_u = \{ j \ge 1 \ : \ 0 < |n|(\lambda_j - \lambda_{j-1}) < \frac{1}{3}\pi \ \}$$
$$B_u = \{ j \ge 1 \ : \ \frac{1}{3}\pi \le |n|(\lambda_j - \lambda_{j-1}) \ \}.$$

Then

$$var_p(\epsilon_n, u)^p = \sum_{j\in G_u} |\epsilon_n(\lambda_j) - \epsilon_n(\lambda_{j-1})|^p + \sum_{j\in B_u} |\epsilon_n(\lambda_j) - \epsilon_n(\lambda_{j-1})|^p. \qquad (2.20)$$

Let $K = |B_u|$. Since for $j \in B_u$ we have $(\lambda_j - \lambda_{j-1}) \geq \frac{\pi}{3|n|}$, it follows that we also have $K \leq \frac{2\pi}{\pi/3|n|} = 6|n|$. Also, for any $j \in G_u$ we have $|\epsilon_n(\lambda_j) - \epsilon_n(\lambda_{j-1})| < 1$ so that

$$|\epsilon_n(\lambda_j) - \epsilon_n(\lambda_{j-1})|^p < |\epsilon_n(\lambda_j) - \epsilon_n(\lambda_{j-1})| \quad \text{for } j \in G_u.$$

Hence we have from (2.20)

$$\begin{aligned} var_p(\epsilon_n, u)^p &\leq \sum_{j\in G_u} |\epsilon_n(\lambda_j) - \epsilon_n(\lambda_{j-1})| + 2^p.6|n| \\ &\leq var_1(\epsilon_n, u) + 2^p|n| \\ &\leq 2\pi|n| + 2^p6|n| = (2\pi + 2^p6)|n|. \end{aligned}$$

This is true for all $u \in \mathcal{P}_{[0,2\pi]}$ and so

$$var_p(\epsilon_n) \leq (2\pi + 12)|n|^{\frac{1}{p}}.$$

□

Now we are ready to prove Theorem 2.2.

Proof. We have at hand a super-reflexive Banach space X and a trigonometrically well-bounded $T \in \mathcal{B}(X)$. Let E be its spectral family, concentrated on the interval $[0, 2\pi]$. In the statement of Theorem 2.1 let us take $\phi = \frac{1}{8\|E\|_\infty}$. We also had $\epsilon = \frac{1}{2\|E\|_\infty}$. The conclusion of Theorem 2.1 now gives a value $q \in (1, \infty)$, depending on ϕ and X, such that

$$var_q(E) \leq \frac{1}{\phi} \equiv 8\|E\|_\infty.$$

Now, we have $T^n = \int_{0^-}^{2\pi} \epsilon_n(\lambda) dE(\lambda)$ for $n \in \mathbb{Z} \setminus \{0\}$ ($n = 0$ simply gives I_X). Let $x \in X$ and $\xi \in X^*$. Let also $p \in (1, q^{'})$, where $\frac{1}{q} + \frac{1}{q^{'}} = 1$, so that $1/p + 1/q > 1$. Then we have

$$\langle T^n x, \xi\rangle = \int_{0^-}^{2\pi} \epsilon_n(\lambda)\langle dE(\lambda)x, \xi\rangle.$$

The scalar functions ϵ_n and $\langle \xi, E(.)x\rangle$ have no common discontinuities in $[0, 2\pi]$. So, applying Lemma 2.6 we have

$$|\langle T^n x, \xi\rangle| \leq \big\{1 + \zeta(1/p + 1/q)\big\}\big\{1 + var_p(\epsilon_n)\big\}\ var_q\big(\langle \xi, E(.)x\rangle\big).$$

Observe that $1 + \zeta(1/p + 1/q) \leq \frac{2(q+p)-pq}{(q+p)-pq}$. Further, by Lemma 2.7 (i), $1 + var_p(\epsilon_n) \leq (2\pi + 13)|n|^{1/p}$ for $n \neq 0$ and by Lemma 2.4 (i),

$var_q\big(\langle \xi, E(.)x\rangle\big) \leq \; var_q(E)\|x\| \; \|\xi\| \leq \; 8\|E\|_\infty \|x\| \; \|\xi\|$. Putting this information together, we get

$$|\langle T^n x, \xi\rangle| \leq \; 8\|E\|_\infty (2\pi + 13) \; \frac{2(q+p) - pq}{(q+p) - pq} \; |n|^{1/p} \; \|x\| \; \|\xi\|. \tag{2.21}$$

Now, this holds for any $p \in (1, q^{'})$, so in particular it holds for $p = \frac{1+q^{'}}{2} = \frac{2q-1}{2(q-1)}$. In this case we also have $\frac{2(q+p)-pq}{(q+p)-pq} = \frac{2q^2}{q-1}$. Hence

$$|\langle T^n x, \xi\rangle| \leq \; 8\|E\|_\infty (2\pi + 7)\frac{2q^2}{q-1} \; |n|^{\frac{2(q-1)}{2q-1}} \|x\| \; \|\xi\|.$$

Since this holds for any $x \in X$ and $\xi \in X^*$, we finally have

$$\|T^n\| \leq \left\{ 8\|E\|_\infty (2\pi + 13)\frac{2q^2}{q-1} \right\} |n|^{\frac{2(q-1)}{2q-1}} \quad \text{for} \quad n \neq 0.$$

As the value q depends on $\phi \equiv \frac{1}{\|E\|_\infty}$ and X only, it follows that the constants

$$C \equiv 8\|E\|_\infty (2\pi + 13)\frac{2q^2}{q-1} \qquad \text{and} \qquad \alpha \equiv \frac{2(q-1)}{2q-1}$$

do as well. Hence we have established (2.19), noting that $0 < \alpha < 1$. □

Remarks

(i) In Theorem 2.1 we have shown that for any $\phi \in (0, \frac{1}{4\|E\|_\infty})$ there is a $q_\phi \in (1, \infty)$ such that $var_{q_\phi}(E) < \infty$. The proof, however, does not give an explicit value of q_ϕ, it only asserts the existence of one.

(ii) In the proof of Theorem 2.2 we make a choice of $\phi \in (0, \frac{1}{4\|E\|_\infty})$, namely $\phi = \frac{1}{8\|E\|_\infty}$. This clearly impacts the value of q. However, since we do not have an explicit relationship between ϕ and q, it would be futile trying to pick an "optimal" ϕ, which minimizes q.

(iii) Note that inequality (2.21) holds for any $p \in (1, q^{'})$. Hence we can in fact obtain power growth arbitrarily close to $\sim |n|^{1/q^{'}}$. The problem is, however, that $\lim_{p \nearrow q^{'}} \frac{2(q+p)-pq}{(q+p)-pq} = \infty$, so we cannot say that there is a constant C_q such that $\|T^n\| \leq C_q |n|^{1/q^{'}}$ for all $n \neq 0$.

The point argued in these remarks is that we have proved the *existence* of $q \in (1, \infty)$ and $\alpha \in (0, 1)$ for which (2.9) and (2.19) hold, rather than calculating explicit values. The situation is different if we are dealing with *uniformly smooth spaces*, as we shall see in the next section.

But before closing off this part, let us illustrate that super-reflexivity is genuinely needed for Theorem 2.2 to hold. It cannot be relaxed to reflexivity.

Example

There exist a reflexive, non-super-reflexive Banach space X, and a trigonometrically well-bounded $T \in \mathcal{B}(X)$ such that

$$\|T^n\| = 1 + 2\pi|n| \qquad \text{for all } n \in \mathbb{Z}.$$

Let $\mathbb{Q}_{\mathbb{T}}^n$ be the set of dyadic points in $[0, 2\pi)$ of order n. That is $\mathbb{Q}_{\mathbb{T}}^n = \{0 = s_1 < ... < s_{2^n}\}$ where $s_k = \frac{\pi(k-1)}{2^{n-1}}$. For a function $f : \mathbb{Q}_{\mathbb{T}}^n \to \mathbb{C}$ let

$$\|f\|_{BV(n)} = |f(0)| + \sum_{j=2}^{2^n} |f(s_j) - f(s_{j-1})|.$$

Let $X_n = \{\ f : \mathbb{Q}_{\mathbb{T}}^n \to \mathbb{C}\}$ equipped with the norm $\| \cdot \|_{BV(n)}$. Define $M_k : X_k \to X_k$, by $M_k f(t) = e^{it} f(t)$. Then, for a trigonometric polynomial $p(e^{it})$, we have $p(M_k)f(t) = p(e^{it})f(t)$. Hence $\|p(M_k)\| = \|p\|_{BV(k)}$, where of course $\|p\|_{BV(k)} = |p(1)| + \sum_{j=2}^{2^n} |p(e^{is_j}) - p(e^{is_{j-1}})|$. Now let

$$X = l_2(X_k) \ \text{ and } \ M = \oplus_k M_k,$$

observing that X is reflexive. Since $\cup_{n\geq 1}\mathbb{Q}_{\mathbb{T}}^n \cap\ [0, 2\pi)$ is dense in $[0, 2\pi)$, we have $\sup_k \|p\|_{BV(k)} = \|p\|_{BV}$ and so

$$\|p(M)\| = \sup_k \|p(M_k)\| = \|p\|_{BV}.$$

So by Proposition 1.1 in Chapter 1, M is trigonometrically well bounded. But, we also have

$$\|M^N\| \equiv \sup_k \|(M_k)^N\| = \|e^{iN(\cdot)}\|_{BV} = 1 + 2\pi|N|.$$

so M has exact power growth of order N.

That X is not super-reflexive follows from Proposition 2.3, because $M_X(2) = \infty$. This is quite straightforward to see, by finding an $(n, 2)$ branch in B_X, for any $n \geq 1$. First note that each X_n is isometrically isomorphic to $l_1^{(2^n)}$ via the map

$$\begin{aligned} U_n : X_n \ &\to \ l_1^{(2^n)}, \\ f \ &\mapsto \ \big(f(s_1), f(s_2) - f(s_1), ..., f(s_{2^n}) - f(s_{2^n - 1})\big). \end{aligned}$$

The inverse is given by $\big(U_n^{-1}x\big)(s_k) = x_1 + ... + x_k$. Hence the map

$$U = \oplus_k U_k : X \to l_2\big(l_1^{(2^n)}\big).$$

is an isometric isomorphism. Now, $l_1^{(2^n)}$ contains an $(n, 2)$ branch in its unit ball, given by the canonical basis $\{e_1, ..., e_{2^n}\}$. For,

$$\begin{aligned} \|e_1 - e_2\| \ &= \ \|e_3 - e_4\| = ... = \|e_{2^n-1} - e_{2^n}\| = 2, \\ \Big\|\frac{e_1 + e_2}{2} - \frac{e_3 + e_4}{2}\Big\| \ &= \ ... = \Big\|\frac{e_{2^n-3} + e_{2^n-2}}{2} - \frac{e_{2^n-1} + e_{2^n}}{2}\Big\| = 2. \end{aligned}$$

and so on. Hence $M_{l_2\left(l_1^{(2^n)}\right)}(2) = \infty$. Now U, being an isometric isomorphism, transfers this branch to X. $\square$

2.3 Spectral Families on a Uniformly Smooth Space

The situation is somewhat clearer when dealing with a uniformly smooth space (recall this property from Definition 2.4). In this case, given an arbitrary spectral family E we can, in fact, find an explicit range of values q for which $var_q(E) < \infty$. This range depends only on $\delta_{X^*}(\phi(E^*))$, where δ_{X^*} is the modulus of convexity of X^* and $\phi(E^*)$ is defined in Definition 2.6.
At the end of this chapter we return to trigonometrically well bounded operators, in particular the shift operator on weighted sequence spaces. Motivated by its power growth we pose the following question: if T is a trigonometrically well-bounded operator on an $L^p(\mu)$ space (with $1 < p < \infty$), is the power growth of T^n at most $\sim |n|^{1/p}$?

Before dealing with uniformly smooth spaces, let us turn to uniformly convex ones. The reason for this is that *a space X is uniformly convex if and only if its dual X^* is uniformly smooth.* We shall use duality arguments to deduce results about a spectral family E by looking at its dual E^*.

To start with, let E be a spectral family on a uniformly convex space X. As before, let $\{s_j\}_{j\in\mathbb{Z}}$ be a two-sided partition of $\mathbb{R}$ and let $\{\Delta_k\}$ be the associated Schauder decomposition of E. For $j > i \geq 0$ we define subspaces

$$X_{ij} = Im\left(\sum_{k=i+1}^{j} \Delta_k + \sum_{k=-j+1}^{-i} \Delta_k\right) = \big[E(s_j) - E(s_i) + E(s_{-i}) - E(s_{-j})\big]X.$$

Definition 2.6. *Let S_X be the unit sphere in X.*
(i) For $0 \neq x, y \in X$, $\phi(x,y) = \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$.
(ii) For subspaces $P, Q \subset X$, with $P \cap Q = \{0\}$, let

$$\phi(P,Q) = \inf_{0\neq x\in P \;\; 0\neq y\in Q} \phi(x,y) = dist(S_P, S_Q).$$

(iii) For a spectral family E and $u \in \mathcal{P}_\mathbb{R}$ let

$$\begin{aligned}\phi_u(E) &= \inf_{j>i>0} \phi(X_{0,i}, X_{i,j}) \equiv \inf_{j>i} dist(S_{X_{0,i}}, S_{X_{i,j}})\\ \phi(E) &= \inf_{u\in\mathcal{P}_\mathbb{R}} \phi_u(E).\end{aligned}$$

Note that we always have $\phi_u(E) \geq \phi(E) \geq \frac{1}{2\|E\|_\infty}$. Let us record two Lemmas required for future use.

Lemma 2.8. *Let X be a uniformly convex Banach space. Let $\epsilon > 0$ and $\lambda = 2[1 - \delta_X(\epsilon)]$.*
(i) If $\lambda \leq 1$, then for all $t > 0$, $p > 1$ and $x, y \in S_X$ with $\|x - y\| \geq \epsilon$ we have

$$\|x + ty\|^p < 1 + t^p.$$

(ii) If $\lambda > 1$, let $p_0 = \log_\lambda 2$. In this case, for any $1 < p < p_0$ there exists a $\Delta(\lambda, p) > 0$, such that for all $x, y \in S_X$ with $\|x - y\| \geq \epsilon$ and $|1 - t| < \Delta(\lambda, p)$

$$\|x + ty\|^p < 1 + t^p. \tag{2.22}$$

Proof. Observe that if $x, y \in S_X$ with $\|x - y\| \geq \epsilon$, then $1 - \|\frac{1}{2}(x + y)\| \geq \delta(\epsilon)$, i.e. $\|x + y\| \leq \lambda$.
(i) This part is easy. Note that $\lambda \leq 1$ implies $\|x + y\| \leq 1$ for all x, y as above. Suppose, contrary to the claim, that there are some $t_0 > 0$, $p > 1$ and x, y such that $1 + t_0^p \leq \|x + t_0 y\|^p$. If $t_0 \geq 1$ then we have

$$\|x + t_0 y\| \ \leq \ \|x + y\| \ + \ (t_0 - 1)\|y\| \ \leq 1 \ + \ (t_0 - 1) \ = \ t_0.$$

Then $1 + t_0^p \leq \|x + t_0 y\|^p \leq t_0^p$, which is a contradiction. Otherwise, if $t_0 < 1$, then

$$\|x + t_0 y\| \ \leq \ t_0\|x + y\| \ + \ (1 - t_0)\|x\| \ \leq t_0 \ + \ (1 - t_0) \ = \ 1.$$

This gives $1 + t_0^p \leq \|x + t_0 y\|^p \leq 1$, that is $t_0 = 0$ which is again a contradiction. Hence no such $t_0 > 0$ and $p > 1$ exist.
(ii) Observe that $p < p_0$ implies that $\lambda^p < \lambda^{p_0} = 2$. Hence inequality (2.22) is true at $t = 1$.
But, the family $\{f_{x,y}\}_{x,y \in S_X}$ given by $f_{x,y}(t) = \|x + ty\|^p - 1 - t^p$ is equi-continuous in t with $f_{x,y}(1) < 0$. Hence there exists $\Delta > 0$ such that, for all $x, y \in S_X$ and $|1 - t| < \Delta$ we have $f_{x,y}(t) < 0$. □

Remark
Observe that uniform convexity is genuinely needed in this Lemma. If $\epsilon > 0$ then $\lambda < 2$. The latter is required for (ii).

Lemma 2.9. *Let (a_i) and (b_i), $i = 1, ..., N$, be two sequences of non-negative reals. Suppose there exists $\epsilon > 0$ such that*

$$a_1 - b_1 > 0, \quad a_N - b_N < 0, \quad |a_{i+1} - a_i| < \epsilon \ \text{ and } |b_{i+1} - b_i| < \epsilon \ \text{ for } \ i = 1, ..., N.$$

Then there exists $1 \leq i_0 \leq N$ such that $|a_{i_0} - b_{i_0}| < \epsilon$.

Proof. Suppose, on the contrary, that no such i_0 exists, i.e. that $|a_i - b_i| \geq \epsilon$ for all i. Then we have

$$a_1 - b_1 \geq \epsilon \text{ and } b_N - a_N \geq \epsilon.$$

So there exists an i_0 such that

$$a_1 - b_1 \geq \epsilon \;\; ... \;\; a_{i_0-1} - b_{i_0-1} \geq \epsilon \text{ and } b_{i_0} - a_{i_0} \geq \epsilon.$$

So, if $a_{i_0} \leq a_{i_0-1}$, then

$$b_{i_0-1} \leq a_{i_0-1} - \epsilon < a_{i_0} \leq b_{i_0} - \epsilon \quad \Rightarrow |b_{i_0} - b_{i_0-1}| \geq \epsilon,$$

which is a contradiction. On the other hand, if $a_{i_0-1} \leq a_{i_0}$, then

$$b_{i_0-1} \leq a_{i_0-1} - \epsilon \leq a_{i_0} - \epsilon \leq b_{i_0} - 2\epsilon \quad \Rightarrow |b_{i_0} - b_{i_0-1}| \geq 2\epsilon.$$

This is again a contradiction. □

We shall now prove a lower bound q-variation result for a spectral family on a *uniformly convex* space X. Observe that we already know that for such a family E, there is some $q \in (1, \infty)$ such that $var_q(E) < \infty$. This follows from the fact that a uniformly convex space is super-reflexive (see [33]).

Proposition 2.5. *Let X be a uniformly convex space and E a spectral family on X. Let $0 < d \leq \phi(E)$ and set $\lambda = 2[1 - \delta_X(d)]$.*
(i) If $\lambda \leq 1$, then for any $p > 1$ there exists a constant $A_p > 0$ such that for all $x \in X$ and $u \in \mathcal{P}_{\mathbb{R}}$ we have

$$\|x\| \leq A_p \Big(\sum_{-\infty}^{\infty} \|\Delta_k x\|^p \Big)^{1/p}.$$

(ii) If $\lambda > 1$ then for any $1 < p < \log_\lambda 2$ there exists a constant $A_p > 0$ such that for all $x \in X$ and $u \in \mathcal{P}_{\mathbb{R}}$ we have

$$\|x\| \leq A_p \Big(\sum_{-\infty}^{\infty} \|\Delta_k x\|^p \Big)^{1/p}. \tag{2.23}$$

Proof. Let $u = \{\lambda_k\} \in \mathcal{P}_{\mathbb{R}}$ be fixed. As before, $\{\Delta_k\} = \{(E(\lambda_k) - E(\lambda_{k-1}))\}$ is the associated Schauder decomposition. Define

$$X_0 = \{x \in X \;:\; \Delta_k x = 0 \quad \text{for all but finitely many } k \;\}.$$

We shall prove (i) and (ii) for $x \in X_0$, using induction on $K \equiv$ number of non-zero terms in the sum $\sum_{-\infty}^{\infty} \Delta_k x$. Then density of X_0 in X will imply that (i) and (ii) hold for all $x \in X$. We shall show that in case of (ii), $A_p = \frac{4}{\Delta(p,\lambda)}$, where

$\Delta(p,\lambda)$ is the number appearing in Lemma 2.8, chosen so that $\Delta(p,\lambda) < 2$. For (i) we shall simply fix $\Delta(p,\lambda) = \frac{1}{2}$ and appeal to the same argument as for (ii). Observe that both choices of A_p are independent of the particular partition u.

So, let us prove (ii). Clearly inequality (2.23) is true for $K = 1$, for then we have $x = \Delta_k x$ for some $k \in \mathbb{Z}$.
Suppose now (2.23) holds, with $A_p = \frac{4}{\Delta(p,\lambda)}$, for all x with K non-zero terms in $\sum_{-\infty}^{\infty} \Delta_k x$.
Now let $x \in X$ have $K+1$ non-zero terms. Pick the smallest positive integer M such that $\Delta_k x = 0$ for $k > M$ and $k < -M+1$, so that $x = \sum_{-M+1}^{M} \Delta_k x$. Let us now define

$$x_k = \sum_{-k+1}^{k} \Delta_j x \;\text{ for } k = 1, ..., M; \quad x_0 = 0$$

and

$$y_k = \sum_{j=k+1}^{M} \Delta_j x + \sum_{j=-M+1}^{-k} \Delta_j x, \;\text{ for } k = 0, ..., M-1; \quad y_M = 0.$$

Note that we then have $x_M = x = x_k + y_k$ for $k = 0, ..., M$. We now have two cases to consider.

Case I: $\|\Delta_j x\| \leq \frac{\Delta(p,\lambda)}{4}\|x_M\|$ for all $j \in \{-M+1, ..., M\}$.
Then we have $\|x_0\| - \|y_0\| < 0$, $\|x_M\| - \|y_M\| > 0$ and for $i \in \{1, ..., M-1\}$

$$\Big|\|x_{i+1}\| - \|x_i\|\Big| < \frac{\Delta(p,\lambda)}{2}\|x_M\|, \qquad \Big|\|y_{i+1}\| - \|y_i\|\Big| < \frac{\Delta(p,\lambda)}{2}\|x_M\|.$$

The last two follow from, for example,

$$\|x_{i+1}\| - \|x_i\| \;\leq\; \|x_{i+1} - x_i\| \;=\; \|\Delta_{i+1}x + \Delta_{-i}x\| \;\leq\; \frac{\Delta(p,\lambda)}{2}\|x_M\|.$$

Hence, by Lemma 2.9, there exists $1 \leq r \leq M$ such that

$$\big|\|x_r\| - \|y_r\|\big| < \frac{\Delta(p,\lambda)}{2}\|x_M\|.$$

(Note that $r \neq 0$ or M because $\|y_0\| = \|x_M\| = \|x\| \not< \frac{\Delta(p,\lambda)}{2}\|x_M\|$). By homogeneity of the last equation, we may assume that $\|x_r\| = 1$. For, if this is not the case, we merely set $\tilde{x_k} = \frac{x_k}{\|x_r\|}$ and $\tilde{y_k} = \frac{y_k}{\|x_r\|}$ for $k = 0, ..., M$. Then all the inequalities under Case I still hold with $\tilde{x_k}, \tilde{y_k}$ in place of x_k, y_k.
Moreover, without loss of generality we may assume that $\|x_r\| \geq \|y_r\|$ (otherwise we interchange the two). Note that

$$x = x_M = x_r + y_r \;\text{ implies }\; \|x\| \leq \|x_r\| + \|y_r\| \leq 2.$$

Hence

$$\Big|1 - \|y_r\|\Big| = \Big|\|x_r\| - \|y_r\|\Big| \le \frac{\Delta(p,\lambda)}{2}\|x\| \le \Delta(p,\lambda). \tag{2.24}$$

Now set $\hat{y_r} = y_r/\|y_r\|$ and observe that $x_r \in S_{X_{0,r}}$ and $\hat{y_r} \in S_{X_{r,M}}$. Since $\phi(X_{0,r}, X_{r,M}) \ge \phi(E) \ge d$, we have $\|x_r - \hat{y_r}\| \ge d$. So, using (2.24) and Lemma 2.8 with $t = \|y_r\|$, we have

$$\|x\|^p = \|x_M\|^p = \|x_r + t\hat{y_r}\|^p \le 1 + t^p = \|x_r\|^p + \|y_r\|^p. \tag{2.25}$$

But, x_r and y_r have at most K non-zero terms in their defining sums, and so by the inductive hypothesis we have, with $A_p = \frac{4}{\Delta(p,\lambda)}$,

$$\|x_r\| \le A_p\Big(\sum_{-r+1}^{r} \|\Delta_k x\|^p\Big)^{1/p} \quad \|y_r\| \le A_p\Big(\sum_{k=-M+1}^{-r} \|\Delta_k x\|^p + \sum_{k=r+1}^{M} \|\Delta_k x\|^p\Big)^{1/p}.$$

Putting this into (2.25), we obtain $\|x\|^p = \|x_M\|^p \le A_p^p \sum_{-M+1}^{M} \|\Delta_k x\|^p$, which completes the inductive step.

Case II: There exists some $1 \le r \le M$ such that

$$\|\Delta_r x\| > \frac{\Delta(p,\lambda)}{4}\|x\|.$$

But then we immediately have, on rearranging,

$$\|x\| = \|x_M\| < \frac{4}{\Delta(p,\lambda)}\|\Delta_r x\| \le A_p\Big(\sum_{-M+1}^{M} \|\Delta_k x\|^p\Big)^{1/p}, \text{ again with } A_p = \frac{4}{\Delta(p,\lambda)}.$$

So, in either case we see that inequality (2.23) holds for all x with $K+1$ non-zero terms in $\sum_{-\infty}^{\infty} \Delta_k x$. Hence by induction it holds for all $x \in X_0$. The latter is dense in X, and this establishes (ii).

As noted above, case (i) is proved in exactly the same manner. But this time, Lemma 2.8 holds with any $\Delta(p,\lambda) > 0$, so we can fix it to be $\frac{1}{2}$, say. Finally, observe that in both (i) and (ii) the constant $A_p = \frac{4}{\Delta(p,\lambda)}$ is independent of the particular partition u, and so the proposition is proved. □

We are now ready to turn to uniformly smooth spaces. We know that X is uniformly smooth if and only if the dual X^* is uniformly convex (see, for example, [33]). Furthermore, given a spectral family E on such X, the operators $E^*(\lambda) \equiv E(\lambda)^*$ define a spectral family on X^*. This follows from the reflexivity of X, as this Lemma shows.

Lemma 2.10. *Let E be a spectral family on a reflexive space X. The family E^* satisfies the following properties:*
(i) $E^*(\mu)E^*(\lambda) = E^*(\mu \wedge \lambda)$*;*
(ii) $\|E^*\|_\infty = \|E\|_\infty$*;*
(iii) E is right continuous and has left limits in the strong-operator topology of X^*;
(iv) $\lim_{\lambda\to\infty} E^*(\lambda) = I$ *and* $\lim_{\lambda\to-\infty} E^*(\lambda) = 0$ *in the strong-operator topology of X^*.*

Proof. (i) and (ii) are straightforward. (iii) and (iv) use the same following argument. It is a general fact that a family of bounded projections on a reflexive space, which is increasing in the sense of (i), has strong right and left limits. Hence we know that for each $\xi \in X^*$ and $\lambda \in \mathbb{R}$, $E^*(\lambda^+)\xi$ and $E^*(\lambda^-)\xi$ exist in the strong operator topology. But we also know that these limits exist in the weak-* topology because

$$\begin{aligned} |\langle x, E^*(\lambda)\xi\rangle - \langle x, E^*(\mu)\xi\rangle| &= |\langle E(\lambda)x, \xi\rangle - \langle E(\mu)x\xi\rangle| \\ &\leq \|\xi\|.\|E(\lambda)x - E(\mu)x\|. \end{aligned}$$

So, if we fix $\lambda \in \mathbb{R}$ and let $\mu \searrow \lambda$ or $\mu \nearrow \lambda$, we see that, in weak-* convergence $E^*(\lambda^+)\xi = E^*(\lambda)\xi$ and the limit $E^*(\lambda^-)\xi$ exists. But, as X is reflexive, weak-* limits are precisely the weak limits, and so the strong limits must coincide with them. Exactly the same reasoning works for (iv). □

Now we are ready to state and prove our main result. Recall that for $r \in (1, \infty)$, r' is the conjugate index, i.e. $\frac{1}{r} + \frac{1}{r'} = 1$.

Theorem 2.3. *Let X be a uniformly smooth space, and let E be a spectral family on X. Let $d = \phi(E^*)$ and suppose $\lambda \equiv 2[1 - \delta_{X^*}(d)] > 1$. Then for any $1 < r < \log_\lambda 2$, we have $var_{r'}(E) < \infty$, where $1/r + 1/r' = 1$.*

Proof. Let $r \in (1, \log_\lambda 2)$. Our aim is to find a constant $B_r > 0$ such that for any $x \in X$, $u = \{s_j\}_{j\in\mathbb{Z}} \in \mathcal{P}_\mathbb{R}$ and $N \geq 1$ we have

$$\left(\sum_{-N}^{N} \|\Delta_k x\|^{r'}\right)^{1/r'} \leq B_r\|x\|. \tag{2.26}$$

Then we can take suprema over $N \geq 1$, $u \in \mathcal{P}_\mathbb{R}$, and $x \in S_X$ respectively, to obtain $var_{r'}(E) \leq B_r$. As before, let $\{\Delta_j\} = \big\{(E(s_j) - E(s_{j-1}))\big\}$ be the Schauder decomposition associated with $u = \{s_j\}$. Let us also write $\Delta_j^* = E^*(s_j) - E^*(s_{j-1})$. Observe that by Lemma 2.10 E^* is a spectral family on the uniformly convex space X^*. The key tool is to apply Proposition 2.5 to E^* in place of E.

Let us show (2.26). So, let $x = \sum_{-\infty}^{\infty} \Delta_k x \in X$. Fix $N \in \mathbb{N}$ and for each $|i| \leq N$ choose $\phi_i \in X^*$ such that $\langle \Delta_i x, \phi_i \rangle = \|\Delta_i x\|$ and $\|\phi_i\| = 1$. Let $F_i = \|\Delta_i x\|^{1/(r-1)} \phi_i$. Then we have $\langle \Delta_i x, F_i \rangle = \|\Delta_i x\|^{r'}$. Also, for any $y \in X$ we have

$$\begin{aligned} |\langle y, \Delta_i^* F_i \rangle|^r = |\langle \Delta_i y, F_i \rangle|^r &= \|\Delta_i x\|^{r'} |\langle \Delta_i y, \phi_i \rangle|^r \\ &\leq 2^r \|\Delta_i x\|^{r'} \|E\|_\infty^r \|y\|^r. \end{aligned}$$

Hence we have $\|\Delta_i^* F_i\|^r \leq 2^r \|\Delta_i x\|^{r'} \|E\|_\infty^r$. Now set $F = \sum_{-N}^{N} \Delta_i^* F_i \in X^*$ and observe that, since $\Delta_j^* F = \Delta_j^* F_j$ for all $|j| \leq N$, we still have

$$\langle \Delta_j x, F \rangle = \|\Delta_j x\|^{r'} \quad \text{and} \quad \|\Delta_j^* F\|^r \leq 2^r \|\Delta_j x\|^{r'} \|E\|_\infty^r. \tag{2.27}$$

Now, by Proposition 2.5 there is a constant $A_r > 0$, independent of the partition u, such that

$$\|F\| \leq A_r \Big(\sum_{j=-N}^{N} \|\Delta_j^* F\|^r \Big)^{1/r}.$$

Observe also that $|\langle x, F \rangle| \leq \|F\|.\|x\|$. Hence

$$\|x\| \geq \frac{|\langle x, F \rangle|}{\|F\|} \geq \frac{\frac{1}{A_r} |\langle x, F \rangle|}{\Big(\sum_{-N}^{N} \|\Delta_j^* F\|^r \Big)^{1/r}}. \tag{2.28}$$

But, using the first equation in (2.27), and the fact that $\langle \Delta_k x, \Delta_j^* F \rangle = \delta_{jk} F(\Delta_j x)$, where δ_{jk} is the Kronecker delta, we have

$$\langle x, F \rangle \equiv \sum_{j=-N}^{N} \Big\langle \sum_{k \in \mathbb{Z}} \Delta_k x, \Delta_j^* F \Big\rangle = \sum_{j=-N}^{N} \langle \Delta_j x, F \rangle = \sum_{j=-N}^{N} \|\Delta_j x\|^{r'}. \tag{2.29}$$

Now let $B_r = \frac{1}{2\|E\|_\infty A_r}$, and observe that it is independent of the partition u. Substituting (2.27) and (2.29) into (2.28) gives

$$\begin{aligned} \|x\| \geq \frac{\frac{1}{A_r} \sum_{-N}^{N} \|\Delta_j x\|^{r'}}{\Big(\sum_{-N}^{N} \|\Delta_j^* F\|^r \Big)^{1/r}} &\geq \frac{B_r \sum_{-N}^{N} \|\Delta_j x\|^{r'}}{\Big(\sum_{-N}^{N} \|\Delta_j x\|^{r'} \Big)^{1/r}} \\ &= B_r \Big(\sum_{-N}^{N} \|\Delta_j x\|^{r'} \Big)^{1/r'}. \end{aligned}$$

This last inequality holds for any $N \geq 1$, and so we have proved (2.26), as required. $\square$

Remark

Let us return to power growth of trigonometrically well bounded operators on a given space X. Motivated by Theorem 2.2, we define the quantity α_X,

$$\alpha_X \;=\; \inf\big\{\alpha:\ \|T^n\| \sim |n|^\alpha \ \text{ for all trig well bounded } T \text{ and } n \in \mathbb{Z}\setminus\{0\}\big\}.$$

Equation (2.18) shows that $\alpha_X \leq 1$. We can in fact calculate a lower bound too, for certain weighted $l_p(w)$ sequence spaces.

Proposition 2.6. *Let $0 < \theta < 1$. Let $l_p(w)$ be the scalar weighted sequence space, with the weight $\{w_k^{(\theta)}\}$ given by $w_0^{(\theta)} = 1$ and $w_k^{(\theta)} = |k|^\theta$ for $k \neq 0$. Then $\alpha_{l_p(w)} \geq \frac{\theta}{p}$.*

Proof. Suppose, on the contrary, that this is not so. Then there exists $\alpha_0 \in (\alpha_{l_p(w)}, \frac{\theta}{p})$ such that $\|T^n\| \sim |n|^{\alpha_0}$ for all trigonometrically well-bounded T. However, this is not satisfied by the right shift R operator: an easy calculation shows that, for any weight sequence $\{w_k\}$, $\|R^n\| = \sup_k \frac{w_{n+k}}{w_k}$. Hence, for the weight $\{w_k^{(\theta)}\}$, we have $\|R^n\| = (1+|n|)^{\frac{\theta}{p}}$. But we also know that this weight sequence is an A_p weight, and that therefore R is trigonometrically well bounded (see section 1.4 in chapter 1). Hence no such α_0 exists, giving the desired contradiction. $\square$

Observe that θ can be chosen arbitrarily close to 1. So, motivated by this example, we pose the following question.

Let $1 < p < \infty$, let $(\Omega, \mathcal{F}, \mu)$ be an arbitrary measure space and $X = L^p(\mu)$. If $T \in \mathcal{B}(X)$ is trigonometrically well bounded, does there exists a constant K_T such that for all $n \in \mathbb{Z}\setminus\{0\}$, $\|T^n\| \leq K_T |n|^{1/p}$?

This question is still open to solve.

We now turn to Hilbert spaces, where the 2-variation of a spectral family E, and its adjoint E^*, play a key role.

CHAPTER 3

The results of Chapter 2 are all equally valid when $X \equiv H$ is a Hilbert space. However, two extra features are noteworthy. First, if E is a spectral family on H, then E^* is a spectral family on the same space; so we shall consider their behaviour simultaneously. Secondly, since any Hilbert space can be realized as some $L^2(\mu)$ space, the index $q = 2$ is of particular interest.

Some surprising features occur in a Hilbert space. On the one hand, the joint condition $var_2(E) < \infty$ and $var_2(E^*) < \infty$ is equivalent to E arising from a spectral measure on $\mathbb{R}$. On the other hand, given any $s \in [2, \infty)$ (in particular $s = 2$!), it is possible to construct a Hilbert space H and a spectral family E on H such that $var_s(E) = \infty$. Further still, given any k-fold composition $\ln^{(k)}$ of the logarithm, E can be chosen so that the resulting spectral integral $\int_{\mathbb{R}} e^{i\lambda} dE(\lambda)$ is a bounded, trigonometrically well bounded operator and has power growth $\sim \ln^{(k)}(|n|)$. This is in sharp contrast with the power bounded operators. For, if $T \in \mathcal{B}(X)$ is power bounded, its spectral family automatically arises from a spectral measure and so $var_s(E) < \infty$ for all $s \in [2, \infty)$.

3.1 Spectral Measures and Spectral Families

Let us first address the significance of $var_2(E)$ and $var_2(E^*)$. As before we denote with $\mathcal{P}_{\mathbb{R}}$ the set of partitions of $\mathbb{R}$, namely $u = \{\lambda_k\}_{k \in \mathbb{Z}}$ such that $\lambda_k \to \pm\infty$ as $k \to \pm\infty$. For a given $u \in \mathcal{P}_{\mathbb{R}}$ we denote the associated Schauder decomposition with $\Delta_k = \{E(\lambda_k) - E(\lambda_{k-1})\} \in \mathcal{B}(H)$. Further, since H is reflexive, the dual family defined by $E^*(\lambda) \equiv \big(E(\lambda)\big)^*$ is also a spectral family (see Proposition 2.10). We denote the group $\{\pm 1\}^\infty$ by D^∞ and a typical element $\{\epsilon_k\}_{k \in \mathbb{Z}}$ by ϵ.

Proposition 3.1. *Let H be a complex Hilbert space, and E a spectral family on H. The following two conditions are equivalent.*
(i) $K_1 = var_2(E) < \infty$ and $K_2 = var_2(E^*) < \infty$;
(ii) For each $x \in H$, $u \in \mathcal{P}_{\mathbb{R}}$, the series $\sum_{k \in \mathbb{Z}} \Delta_k x$ converges unconditionally, ***uniformly in u***

Remarks

(i) We say that $\sum_{-\infty}^{\infty} \Delta_k x$ converges unconditionally if, for each $\epsilon \in D^\infty$, the balanced partial sums $\sum_{-N}^{N} \epsilon_k \Delta_k x$ converge to an element $x_{u,\epsilon} \in H$ as $N \to \infty$.
(ii) By uniform convergence in u we mean that for each $x \in H$ there exists a constant $M_x < \infty$ such that $\|x_{u,\epsilon}\| \le M_x$ for all $u \in \mathcal{P}_{\mathbb{R}}$ and $\epsilon \in D^\infty$.

Lemma 3.1. *Let $2 \le p < \infty$ and suppose that $K = var_p(E^*) < \infty$. Then*

$$\|x\| \le K\Big(\sum_{\infty}^{\infty} \|\Delta_k x\|^{p'}\Big)^{1/p'} \text{ for each } x \in H \quad u \in \mathcal{P}_{\mathbb{R}}.$$

Proof. Let $a > 0$ and $x \in \{E(a) - E(-a)\}H$ and write $\xi = \frac{x}{\|x\|}$ so that $(x, \xi) = \|x\|$. Let $u \in \mathcal{P}_{\mathbb{R}}$. Then there exists some $N \ge 1$ such that $x = \sum_{-N}^{N} \Delta_k x$. Hence we have

$$\begin{aligned} \|x\| = (\xi, \sum_{|k|\le N} \Delta_k x) &= \sum_{|k|\le N} (\xi, \Delta_k x) \\ &= \sum_{|k|\le N} (\xi, \Delta_k^2 x) = \sum_{|k|\le N} (\Delta_k^* \xi, \Delta_k x). \end{aligned}$$

So, using Holder's inequality in the last term we have

$$\|x\| \le \Big(\sum_{|k|\le N} \|\Delta_k x\|^{p'}\Big)^{1/p'} \Big(\sum_{-\infty}^{\infty} \|\Delta_k^* \xi\|^p\Big)^{1/p}.$$

But we know from Lemma 2.4 that

$$\Big(\sum_{-\infty}^{\infty} \|\Delta_k^* \xi\|^p\Big)^{1/p} \le var_p(E^*)\|\xi\| = K.$$

Hence

$$\|x\| \le K\Big(\sum_{-\infty}^{\infty} \|\Delta_k x\|^{p'}\Big)^{1/p'}. \tag{3.1}$$

Since $\bigcup_{a\ge 1}\{E(a) - E(-a)\}H$ is dense in H, the last inequality holds for all $x \in H$. For suppose not; then there exists $x \in H$ and $\epsilon > 0$ such that $\|x\| = \epsilon + K\big(\sum_{-\infty}^{\infty} \|\Delta_k x\|^{p'}\big)^{1/p'}$. We can pick $M \ge 1$ such that $\|x - \sum_{-M}^{M} \Delta_k x\| < \frac{\epsilon}{2}$. Then we have

$$K\Big(\sum_{-\infty}^{\infty} \|\Delta_k x\|^{p'}\Big)^{1/p'} + \frac{\epsilon}{2} = \|x\| - \frac{\epsilon}{2} < \|x\| - \|x - \sum_{-M}^{M} \Delta_k x\| \le \|\sum_{-M}^{M} \Delta_k x\|.$$

But $\|\sum_{-M}^{M} \Delta_k x\| \le K\big(\sum_{-M}^{M} \|\Delta_k x\|^{p'}\big)^{1/p'}$. Hence we get $\big(\sum_{-\infty}^{\infty} \|\Delta_k x\|^{p'}\big)^{1/p'} + \frac{\epsilon}{2K} < \big(\sum_{-\infty}^{\infty} \|\Delta_k x\|^{p'}\big)^{1/p'}$ which is a contradiction. □

Proof of Proposition 3.1

Let us show (i)$\Rightarrow$ (ii). Let $K_1 = var_2(E)$ and $K_2 = var_2(E^*)$. Using Lemma 3.1 with $p = 2$ we have, for each $x \in H$ and $u \in \mathcal{P}_{\mathbb{R}}$,

$$\frac{1}{K_1}\big(\sum_{-\infty}^{\infty} \|\Delta_k x\|^2\big)^{1/2} \leq \|x\| \leq K_2\big(\sum_{-\infty}^{\infty} \|\Delta_k x\|^2\big)^{1/2}.$$

This shows that for each $x \in H$ and $\{\lambda_k\} \in \mathcal{P}_{\mathbb{R}}$, $\sum_{-\infty}^{\infty} \Delta_k x$ converges unconditionally. To see this, let $\epsilon \in D^\infty$ and $\delta > 0$. Since $\{E(\lambda_N) - E(\lambda_{-N})\}x \to x$ as $N \to \infty$, we can find $N_0 \geq 1$ such that

$$\left(\sum_{N<|k|<M} \|\Delta_k x\|^2\right)^{1/2} \leq \frac{\delta}{K_2} \quad \text{for all} \quad N_0 \leq N < M. \tag{3.2}$$

Then we have

$$\big\|\sum_{|k|\leq N} \epsilon_k \Delta_k x\big\| \leq K_2\big(\sum_{k=-\infty}^{\infty} \|\Delta_k x\|^2\big)^{1/2} \leq K_1 K_2 \|x\| \ \text{ for all } N \geq 1, \tag{3.3}$$

and

$$\big\|\sum_{N<|k|<M} \epsilon_k \Delta_k x\big\| \leq K_2\left(\sum_{N<|k|} \|\Delta_k x\|^2\right)^{1/2} \leq \delta \ \text{ for all } N_0 \leq N < M.$$

Hence $\displaystyle\sum_{-\infty}^{\infty} \epsilon_k \Delta_k x$ is Cauchy and so converges to some $x_{u,\epsilon} \in H$. Furthermore, (3.3) shows that the convergence is uniform in u, in the sense defined in the above Remark.

To show (ii) $\Rightarrow$ (i), suppose that, for each $x \in H$, the series $\sum \Delta_k x$ converges unconditionally, uniformly in u. So there exists a constant $M_x > 0$ such that

$$\big\|\sum_{k=-\infty}^{\infty} \epsilon_k \Delta_k x\big\| \leq M_x \quad \text{for all} \quad u \in \mathcal{P}_{\mathbb{R}}, \quad \epsilon \in D^\infty. \tag{3.4}$$

But this means that all the balanced partial sums are also bounded:

$$\big\|\sum_{|k|\leq K} \epsilon_k \Delta_k x\big\| \leq M_x \ \text{ for all } \ K \geq 1, u \in \mathcal{P}_{\mathbb{R}} \ \ \epsilon \in D^\infty.$$

To see this, let us fix $\{\epsilon_k\}$. Define $\epsilon_k^{'} = \epsilon_k^{''} = \epsilon_k$ for $|k| \leq K$ and $\epsilon_k^{'} = -\epsilon_k^{''} = 1$ for $|k| > K$. Then we have

$$\begin{aligned} 2\big\|\sum_{|k|\leq K} \epsilon_k \Delta_k x\big\| &= \big\|\sum_{k=-\infty}^{\infty} \epsilon_k^{'} \Delta_k x + \sum_{k=-\infty}^{\infty} \epsilon_k^{''} \Delta_k x\big\| \\ &\leq \big\|\sum_{k=-\infty}^{\infty} \epsilon_k^{'} \Delta_k x\big\| + \big\|\sum_{k=-\infty}^{\infty} \epsilon_k^{''} \Delta_k x\big\| \leq 2M_x. \end{aligned}$$

Hence we can apply the Uniform Boundedness Principle to the collection of operators $\{\sum_{|k|\le K}\epsilon_k\Delta_k \;:\; K\ge 1, u\in \mathcal{P}_{\mathbb{R}}\;\; \epsilon\in D^\infty\}$ to deduce the existence of a constant $C>0$ such that $\|\sum_{|k|\le K}\epsilon_k\Delta_k\|\le C$ for all $K\ge 1$, $u\in\mathcal{P}_{\mathbb{R}}$ and $\epsilon\in D^\infty$. Hence $\big\|\sum_{-\infty}^{\infty}\epsilon_k\Delta_k x\big\|\le C\|x\|$ for all $x\in H$. So the operators $A_{\epsilon,u}\equiv\sum_{-\infty}^{\infty}\epsilon_k\Delta_k$ are uniformly bounded by C, i.e. the collection $G=\{A_{\epsilon,u}\;:\; u\in\mathcal{P}_{\mathbb{R}}\;\;\epsilon\in D^\infty\}$ is bounded above by C. Observe that this is in fact a well-defined Abelian group. For, let $\Delta^{(u)}$ and $\Delta^{(v)}$ correspond to partitions u and $v\in\mathcal{P}_{\mathbb{R}}$ respectively, and let $A_{\epsilon,u}\equiv\sum_{-\infty}^{\infty}\epsilon_k\Delta_k^{(u)}$ and $A_{\delta,v}\equiv\sum_{-\infty}^{\infty}\delta_k\Delta_k^{(v)}$. Let w be the union of the points of u and v (so that it refines both). Now rewrite $A_{\epsilon,u}=\sum_{-\infty}^{\infty}\tilde\epsilon_j\Delta_j^{(w)}$, where for each j, we define $\tilde\epsilon_j=\epsilon_k$, with k being the unique integer for which $\Delta_j^{(w)}\Delta_k^{(u)}\ne 0$. We can rewrite $A_{\delta,v}=\sum_{-\infty}^{\infty}\tilde\delta_j\Delta_j^{(w)}$, in terms of w in an exactly analogous manner. Then the product of the two operators is uniquely given by

$$A_{\epsilon,u}A_{\delta,v}=\sum_{-\infty}^{\infty}\tilde\epsilon_j\tilde\delta_j\Delta_j^{(w)}.$$

We now apply a result of B. Sz. Nagy: there exists an inner product $\langle\;,\;\rangle$ on H, equivalent to the original $(\;,\;)$, with respect to which all $A_{\epsilon,u}$ are unitary. That is, there exist constants C_1 and C_2 such that

$$\begin{aligned}\langle A_{\epsilon,u}x, A_{\epsilon,u}y\rangle &= \langle x,y\rangle \;\text{ for all }\; x,y\in H\\ C_1\langle x,x\rangle &\le (x,x)\le C_2\langle x,x\rangle \;\text{ for all } x\in H.\end{aligned}$$

Now, for any partition u, the operators $\{\Delta_k\}$ are orthogonal with respect to this new inner product, in the sense that $\langle\Delta_k x,\Delta_j x\rangle=0$ if $k\ne j$. To see this, let us fix a partition u and $k\ne j$. Choose $\epsilon\in D^\infty$ such that $\epsilon_j\epsilon_k=-1$. Then, using the unitary property, we have $\langle A_{\epsilon,u}\Delta_k x, A_{\epsilon,u}\Delta_j x\rangle=\langle\Delta_k x,\Delta_j x\rangle$. But we also have $A_{\epsilon,u}\Delta_i x=\epsilon_i\Delta_i x$ for any $i\in\mathbb{Z}$ so that

$$\langle A_{\epsilon,u}\Delta_k x, A_{\epsilon,u}\Delta_j x\rangle=\epsilon_k\epsilon_j\langle\Delta_k x,\Delta_j x\rangle=-\langle\Delta_k x,\Delta_j x\rangle.$$

Combining the last two equations we get $\langle\Delta_k x,\Delta_j x\rangle=-\langle\Delta_k x,\Delta_j x\rangle=0$, as claimed.

Now let $a>0$ and $x\in\{E(a)-E(-a)\}H$. So $x=\sum_{-M}^{M}\Delta_k x$ for some $M\ge 1$. Then, denoting the new norm by $\|x\|_N^2\equiv\langle x,x\rangle$,

$$\langle x,x\rangle=\langle\sum_{-M}^{M}\Delta_k x,\sum_{-M}^{M}\Delta_j x\rangle=\big(\sum_{-M}^{M}\|\Delta_k x\|_N^2\big)^{1/2}.$$

Hence we have

$$C_1\sum_{-M}^{M}\|\Delta_k x\|_N^2\le\|x\|^2\le C_2\sum_{-M}^{M}\|\Delta_k x\|_N^2.$$

But we also have $\frac{1}{C_2}\|\Delta_k x\|^2 \leq \|\Delta_k x\|_N^2 \leq \frac{1}{C_1}\|\Delta_k x\|^2$. Combining these two we obtain

$$\sqrt{\frac{C_1}{C_2}}\Big(\sum_{-M}^{M}\|\Delta_k x\|^2\Big)^{1/2} \leq \|x\| \leq \sqrt{\frac{C_2}{C_1}}\Big(\sum_{-M}^{M}\|\Delta_k x\|^2\Big)^{1/2}. \qquad (\dagger)$$

This holds for any $x \in \{E(a)-E(-a)\}H$, and any partition u. But $\bigcup_{a>0}\{E(a)-E(-a)\}H$ is dense in H, so ($\dagger$) holds for all $x \in H$. So, taking the supremum over $u \in \mathcal{P}_{\mathbb{R}}$ and $x \in B_H$ in the left inequality, we deduce that $var_2(E) \leq \sqrt{\frac{C_2}{C_1}}$.

Let us show that $var_2(E^*) \leq 2\|E\|_\infty\sqrt{\frac{C_2}{C_1}}$. Again, let $\xi \in \{E^*(a) - E^*(-a)\}H$ for some fixed $a > 0$ and let $u \in \mathcal{P}_{\mathbb{R}}$. So there exists some $N \geq 1$ such that $\xi = \sum_{-N}^{N}\Delta_k^*\xi$. Set $z_i = \Delta_i^*\xi$ for $i = -N, ..., N$. Then we have for each i

$$\|\Delta_i z_i\| = \|\Delta_i\Delta_i^*\xi\| \leq 2\|E\|_\infty\|\Delta_i^*\xi\| \qquad (\ddagger)$$

Now set $z = \sum_{-N}^{N}\Delta_i z_i$. Then

$$\Delta_k z = \Delta_k z_k \quad \text{for} \quad k = -N, ..., N, \qquad (3.5)$$

$$(z, \Delta_k^*\xi) = (\Delta_k z_k, \xi) = (z_k, \Delta_k^*\xi) = \|\Delta_k^*\xi\|^2. \qquad (3.6)$$

Now, by ($\dagger$) we have $\|z\| \leq \sqrt{\frac{C_2}{C_1}}\big(\sum_{-N}^{N}\|\Delta_k z\|^2\big)^{1/2}$ and also $|(z,\xi)| \leq \|z\|\ \|\xi\|$, so that

$$\|\xi\| \geq \frac{|(z,\xi)|}{\|z\|} \geq \frac{|(z,\xi)|}{\sqrt{\frac{C_2}{C_1}}\big(\sum_{-N}^{N}\|\Delta_k z\|^2\big)^{1/2}}$$

But now we have from ($\ddagger$) and (3.5)

$$\Big(\sum_{|k|\leq N}\|\Delta_k z\|^2\Big)^{1/2} \leq 2\|E\|_\infty\Big(\sum_{|k|\leq N}\|\Delta_k^*\xi\|^2\Big)^{1/2},$$

$$(z,\xi) = \sum_{|k|\leq N}\|\Delta_k^*\xi\|^2.$$

Hence

$$\|\xi\| \geq \frac{\sum_{-N}^{N}\|\Delta_k^*\xi\|^2}{2\|E\|_\infty\sqrt{\frac{C_2}{C_1}}\big(\sum_{-N}^{N}\|\Delta_k^*\xi\|^2\big)^{1/2}} = \frac{1}{2\|E\|_\infty\sqrt{\frac{C_2}{C_1}}}\Big(\sum_{-N}^{N}\|\Delta_k^*\xi\|^2\Big)^{1/2}.$$

So we have

$$\Big(\sum_{-N}^{N}\|\Delta_k^*\xi\|^2\Big)^{1/2} \leq 2\|E\|_\infty\sqrt{\frac{C_2}{C_1}}\|\xi\|.$$

This holds for any $\xi \in \bigcup_{a>0}\{E^*(a) - E^*(-a)\}H$ and any partition $u \in \mathcal{P}_{\mathbb{R}}$, so taking suprema we obtain

$$var_2(E^*) \leq 2\|E\|_\infty\sqrt{\frac{C_2}{C_1}}, \quad \text{as required.} \qquad \square$$

This Proposition helps establish the main result of this chapter.

Theorem 3.1. *Let E be a spectral family on a complex Hilbert space H. If both $var_2(E) = K_1 < \infty$ and $var_2(E^*) = K_2 < \infty$, then E gives rise to a spectral measure on $\mathfrak{B}$, the Borel $\sigma-$algebra on $\mathbb{R}$. That is there exists a spectral measure $\mathcal{E}$ on $\mathfrak{B}$ such that for any $A = (a, b] \subset \mathbb{R}$, $\mathcal{E}(A) = \{E(b) - E(a)\}$.*

Proof. Let us use the notation from the statement and proof of Proposition 3.1. We have shown therein that provided $var_2(E) < \infty$ and $var_2(E^*) < \infty$, the operators $A_{\epsilon,u} \equiv \sum_{-\infty}^{\infty} \epsilon_k \Delta_k$ are well-defined and bounded, and moreover the Abelian group $G \equiv \{A_{\epsilon,u} \ : \ u \in \mathcal{P}_\mathbb{R} \ \ \epsilon \in D^\infty\}$ is uniformly well-bounded.

Now, by XV 6.1 in [37], there exists an invertible self-adjoint $S \in \mathcal{B}(H)$ such that for every $A_{\epsilon,u} \in G$, the operator $B_{\epsilon,u} \equiv S^{-1} A_{\epsilon,u} S$ is unitary. Observe that, since $A_{\epsilon,u}^2 = I_H$, we have $B_{\epsilon,u}^2 = I_H = B_{\epsilon,u} B_{\epsilon,u}^*$, so that each $B_{\epsilon,u}$ is self-adjoint.

Now, observe that for any $\mu \in \mathbb{R}$, $E(\mu) \in G$. To see this, simply define $u \in \mathcal{P}_\mathbb{R}$ to be $\mathbb{Z}$, with the exception $\lambda_0 = \mu$. Then choose $\epsilon \in D^\infty$ to be $e_j = -1$ for $j \leq 0$ and $e_j = 1$ for $j > 0$. Then we have $A_{\epsilon,u} = I - 2E(\mu)$ and hence

$$E(\mu) = \frac{1}{2}(I - A_{\epsilon,u}). \tag{3.7}$$

Now let $F(\mu) = S^{-1} E(\mu) S$. This is a well-defined spectral family in H and (3.7) gives $F(\mu) = \frac{1}{2}(I - B_{\epsilon,u})$ so that F is in fact self-adjoint. Let us now write $H_n = \{F(n) - F(n-1)\}H$ and $x_n = \{F(n) - F(n-1)\}x$ for $x \in H$. Then H is a direct-sum decomposition $H = \bigoplus_{-\infty}^{\infty} H_n$. Let

$$T_n = \int_\mathbb{R} \lambda dF(\lambda)\big|_{H_n}.$$

Then T_n is a *bounded* self-adjoint operator on H_n. Hence by the classical Spectral Theorem there exists a spectral measure $\mathcal{F}_n$ on the Borel σ-algebra $\mathfrak{B}\big((n-1, n]\big)$ such that $T_n = \int_{n-1}^{n} \lambda \mathcal{F}_n(d\lambda)$. Now define an operator-valued set function

$$\mathcal{F}(A) \equiv \bigoplus_{-\infty}^{\infty} \mathcal{F}_n(A \cap (n-1, n]) \qquad A \in \mathfrak{B}(\mathbb{R}).$$

First observe that $\mathcal{F}(A)$ is well-defined, since $\mathcal{F}_n(A \cap (n-1, n])$ is a bounded operator from H_n into itself and so $\mathcal{F}(A)x = \bigoplus_{-\infty}^{\infty} \mathcal{F}_n(A \cap (n-1, n]) x_n$ is well-defined. In fact, $\mathcal{F}$ defines a projection-valued measure, for it satisfies the following three properties:

(i) $\mathcal{F}(\mathbb{R}) = I_H$;

(ii) if $A, B \in \mathfrak{B}(\mathbb{R})$ then $\mathcal{F}(A \cap B) = \mathcal{F}(A)\mathcal{F}(B)$;

(iii) if $\{A_k\} \subset \mathfrak{B}(\mathbb{R})$ is a sequence of pairwise disjoint sets, then for each $x \in H$, $\mathcal{F}\big(\cup_k A_k\big)x = \sum_{k=1}^{\infty} \mathcal{F}(A_k)x$.

(i) is trivially true, as $\mathcal{F}_n((n-1,n]) = I_{H_n}$ for all n. (ii) is equally easy. For if $A, B \in \mathfrak{B}(\mathbb{R})$, then $A \cap (n-1,n], B \cap (n-1,n] \in \mathfrak{B}\big((n-1,n]\big)$ for each n. So, $\mathcal{F}_n$ being a spectral measure, we have

$$\mathcal{F}_n\big(A \cap B \cap (n-1,n]\big) = \mathcal{F}_n\big(A \cap (n-1,n]\big)\mathcal{F}_n\big(B \cap (n-1,n]\big).$$

Hence

$$\begin{aligned}
\mathcal{F}(A \cap B) &= \bigoplus_{-\infty}^{\infty} \mathcal{F}_n\big(A \cap B \cap (n-1,n]\big) \\
&= \Big\{\bigoplus_{-\infty}^{\infty} \mathcal{F}_n(A \cap (n-1])\Big\}\Big\{\bigoplus_{-\infty}^{\infty} \mathcal{F}_n(B \cap (n-1])\Big\} \qquad (3.8) \\
&= \mathcal{F}(A)\mathcal{F}(B).
\end{aligned}$$

(The equality in (3.8) is just the definition of the product of direct-sum operators.) Finally, to check (iii), let $\{A_k\} \subset \mathfrak{B}(\mathbb{R})$ be a sequence of disjoint Borel sets, set $A = \bigcup_{k=1}^{\infty} A_k$ and let $x \in H$. Using orthogonality of the spaces H_n we have

$$\langle \mathcal{F}(A)x, x\rangle = \Big\langle \bigoplus_{n=-\infty}^{\infty} \mathcal{F}_n\big(A \cap (n-1,n]\big)x, x\Big\rangle = \sum_{n=-\infty}^{\infty} \langle \mathcal{F}_n\big(A \cap (n-1,n]\big)x_n, x_n\rangle. \qquad (3.9)$$

Now, since each $\mathcal{F}_n$ is a spectral measure on $(n-1,n]$, we have

$$\langle \mathcal{F}_n\big(A \cap (n-1,n]\big)x_n, x_n\rangle = \sum_{k=1}^{\infty} \langle \mathcal{F}_n\big(A_k \cap (n-1,n]\big)x_n, x_n\rangle.$$

Moreover, for every $n \in \mathbb{Z}$, $\langle \mathcal{F}_n(\cdot)x_n, x_n\rangle$ is a *positive* Borel measure. Hence we can swap the order of summation in line (3.10) below.

$$\begin{aligned}
\langle \mathcal{F}(A)x, x\rangle &= \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} \langle \mathcal{F}_n\big(A_k \cap (n-1,n]\big)x_n, x_n\rangle \\
&= \sum_{k=1}^{\infty} \Big\{ \sum_{n=-\infty}^{\infty} \langle \mathcal{F}_n\big(A_k \cap (n-1,n]\big)x_n, x_n\rangle \Big\} \qquad (3.10) \\
&= \sum_{k=1}^{\infty} \Big\langle \bigoplus_{n=-\infty}^{\infty} \mathcal{F}_n\big(A_k \cap (n-1,n]\big)x_n,\ x\Big\rangle \\
&= \sum_{k=1}^{\infty} \langle \mathcal{F}(A_k)x, x\rangle.
\end{aligned}$$

To put this another way, $\lim_{N\to\infty} \big\langle \big\{ \mathcal{F}(A) - \sum_{k=1}^{N} \mathcal{F}(A_k) \big\} x, x \big\rangle = 0$. Now, the operators $\mathcal{F}(A)$ and $\{\sum_{k=1}^{N} \mathcal{F}(A_k)\}_{N\geq 1}$ are self-adjoint, so by polarization we have

$$\lim_{N\to\infty} \big\langle \big\{ \mathcal{F}(A) - \sum_{k=1}^{N} \mathcal{F}(A_k) \big\} x, y \big\rangle = 0 \qquad \text{for all } x,\ y \in H.$$

Hence $\sum_{k=1}^{\infty} \mathcal{F}(A_k)x$ converges weakly, and so strongly, to $\mathcal{F}(A)x$, and this establishes (iii). Thus $\mathcal{F}$ is a genuine spectral measure.

Now, suppose $A = (a, b]$ is an interval such that $n - 1 \leq a < n \leq b < n + 1$ for some integer $n \in \mathbb{Z}$. Then $\mathcal{F}_k(A \cap (k-1, k]) = 0$ if $k \neq n$ or $n + 1$. Furthermore,

$$\begin{aligned} \mathcal{F}_n(A \cap (n-1, n]) &= \mathcal{F}_n\big((a, n]\big) = \int_{n-1}^{n} \mathbb{I}_{(a,n]}(\lambda) \mathcal{F}_n(d\lambda) \\ &= \int_{n-1}^{n} \mathbb{I}_{(a,n]}(\lambda) dF(\lambda)\big|_{H_n} = \{F(n) - F(a)\}\big|_{H_n}. \end{aligned}$$

Similarly $\mathcal{F}_{n+1}(A \cap (n, n+1]) = \{F(b) - F(n)\}\big|_{H_{n+1}}$, and so writing somewhat clumsily,

$$\mathcal{F}(A) = \bigoplus_{k=-\infty}^{n-1} 0\big|_{H_k} \oplus \{F(n) - F(a)\}\big|_{H_n} \oplus \{F(b) - F(n)\}\big|_{H_{n+1}} \oplus \bigoplus_{k=n+2}^{\infty} 0\big|_{H_k}.$$

But this says precisely that $\mathcal{F}(A) = F(b) - F(a)$. In a similar manner we can show that $\mathcal{F}\big((c, d]\big) = F(d) - F(c)$ for *any* interval $(c, d]$. So, we finally define

$$\mathcal{E}(A) = S\mathcal{F}(A)S^{-1} \qquad \text{for } A \in \mathfrak{B}(\mathbb{R}).$$

$\mathcal{E}(\cdot)$ is then a well defined spectral measure on $\mathfrak{B}(\mathbb{R})$ and the last calculation shows that satisfies $\mathcal{E}(A) = \{E(b) - E(a)\}$ for a subset $A = (a, b] \in \mathbb{R}$. $\square$

3.2 An Example of $var_s(E) = \infty$

Proposition 3.1 clearly shows that $var_2(E) < \infty$ and $var_2(E^*) < \infty$ is a very restrictive condition: it is equivalent to E being a spectral measure. It is of interest, therefore, to establish that not all spectral families on a Hilbert space exhibit this phenomenon. In fact, we can show more. Given any $s \geq 2$, there exists a Hilbert space H and a spectral family E on H such that $var_s(E) = \infty$. To achieve this, we shall construct a conditional basic sequence $\{e_k\}_{k\geq 1}$ in $L^2(\mathbb{T})$ and let $H = \overline{lin}\{e_k\}$. Then we shall define a spectral family E and an $x \in H$, dependant on the given value of s, such that for all sufficiently fine partitions $u \in \mathcal{P}_{\mathbb{R}}$, $\sum_{-\infty}^{\infty} \|\Delta_k x\|_H^s = \infty$. The search for a suitable conditional basic sequence is motivated by [23], in particular the following theorem therein.

Theorem 3.2. *Let $0 < b < 1/2$ and $1 < p < \infty$ satisfy $\frac{1}{p} > \frac{1}{2} - b$. Let $\{a_k\}_{k\geq 0}$ be a positive monotonic decreasing sequence such that, $\sum_{k=0}^{\infty} a_k^p < \infty$. Then the series $\sum_{k=0}^{\infty} a_k |t|^b \cos kt$ converges in $L^2(\mathbb{T})$.*

Proof. See [23]. □

It is necessary for our basic sequence to be bounded below, and the following Lemma ensures that is the case.

Lemma 3.2. *Let $0 < b < \frac{1}{2}$ and define functions $e_k \in L^2[-\pi, \pi]$ by $e_k(t) = |t|^b \cos kt$ for $k \geq 0$ Then there exits a constant $M_b > 0$ such that $\|e_k\|_{L^2} \geq M_b$ for all $k \geq 0$.*

Proof. For any $k \geq 0$ we have $\|e_k\|^2 = \int_{-\pi}^{\pi} |t|^{2b} \cos^2 kt\ dt = 2\int_0^{\pi} t^{2b} \cos^2 kt\ dt$. Let us consider $k \geq 3$. Let

$$\begin{aligned} I_0 &= \Big[0, \frac{\pi}{3k}\Big], \\ I_{j+1} &= \Big[\frac{(2+3j)\pi}{3k}, \frac{(4+3j)\pi}{3k}\Big] \qquad j = 0, ..., k-2. \end{aligned}$$

Then $\bigcup_{0\leq j\leq k-2} I_j \subset [0, 2\pi]$ and on each I_j we have $\cos^2 kt \geq \frac{1}{4}$. Hence

$$\begin{aligned} \|e_k\|^2 &\geq 2\sum_{j=0}^{k-2} \int_{I_j} t^{2b} \cos^2 kt\ dt \\ &\geq \frac{1}{2}\sum_{j=0}^{k-2} \int_{I_j} |t|^{2b} dt \\ &= \frac{1}{2}\sum_{j=0}^{k-2} \frac{1}{1+2b}\Big\{\Big[\frac{4+3j}{3k}\pi\Big]^{1+2b} - \Big[\frac{2+3j}{3k}\pi\Big]^{1+2b}\Big\} \\ &= \frac{1}{2(1+2b)}\Big(\frac{\pi}{3k}\Big)^{1+2b} \sum_{j=0}^{k-2}\Big\{(4+3j)^{1+2b} - (2+3j)^{1+2b}\Big\}. \end{aligned} \tag{3.11}$$

Now, the function $f(x) = (4+3x)^{1+2b} - (2+3x)^{1+2b}$ is increasing and concave on $x \geq 0$, so that

$$\int_0^{k-2} f(x)dx \leq \sum_0^{k-2}\Big\{(4+3j)^{1+2b} - (2+3j)^{1+2b}\Big\}.$$

Hence

$$\frac{1}{2+2b}\Big\{(3k-2)^{2+2b} - (3k-4)^{2+2b} - 4^{2+2b} + 2^{2+2b}\Big\} \leq \sum_0^{k-2}\Big\{(4+3j)^{1+2b} - (2+3j)^{1+2b}\Big\}.$$

Substituting this into (3.11) we have, for $k \geq 3$,

$$\|e_k\|^2 \geq \frac{\pi^{1+2b}}{2(1+2b)(2+2b)}\Big\{\frac{(3k-2)^{2+2b} - (3k-4)^{2+2b} - 4^{2+2b} + 2^{2+2b}}{(3k)^{1+2b}}\Big\}.$$

Now the right hand side of this inequality is increasing as $k \to \infty$. Moreover, at $k = 3$ the right side is equal to

$$\frac{\pi^{1+2b}}{2(1+2b)(2+2b)} \left\{ \frac{7^{2+2b} - 5^{2+2b} - 4^{2+2b} + 2^{2+2b}}{9^{1+2b}} \right\} > 0.$$

Thus, for we have

$$\|e_k\|^2 \geq \frac{\pi^{1+2b}}{2(1+2b)(2+2b)} \left\{ \frac{7^{2+2b} - 5^{2+2b} - 4^{2+2b} + 2^{2+2b}}{9^{1+2b}} \right\} \quad \text{for all} \quad k \geq 3.$$

Further, we can let $m = \min\{\|e_j\|^2 \ : \ j = 0, 1, 2\} > 0$ and then set

$$M_b^2 = \min\{ \ m, \ \frac{\pi^{1+2b}}{2(1+2b)(2+2b)} \left\{ \frac{7^{2+2b} - 5^{2+2b} - 4^{2+2b} + 2^{2+2b}}{9^{1+2b}} \right\} \}.$$

And from this it follows that $\|e_k\| \geq M_b$ for all $k \geq 0$ as required. □

Proposition 3.2. *For any $s \geq 2$ there exists a Hilbert space H and a spectral family $E\{(\lambda)\}_{\lambda \in \mathbb{R}}$ on H such that $var_s(E) = \infty$.*

Proof. Let $\mathcal{P}_{\mathbb{R}} \cap [-a, a]$ denote the set of partitions of $\mathbb{R}$, restricted to the interval $[-a, a]$. As, before, if $\{E(\lambda)\}$ is a spectral family and $u = (\mu_k)_{k \in \mathbb{Z}} \in \mathcal{P}_{\mathbb{R}}$, then $\{\Delta_k\} = \{(E(\mu_k) - E(\mu_{k-1}))\}$ is the associated Schauder decomposition. Then, by definition, $var_s(E) = \sup_{\|x\| \leq 1} \sup_{a > 0} \sup_{u \in \mathcal{P}_{\mathbb{R}} \cap [-a,a]} \big(\sum_{k=-\infty}^{\infty} \|\Delta_k x\|^s \big)^{1/s}$.

Therefore, given $2 \leq s < \infty$, it will suffice to construct a spectral family E and $x \in H$ such that

$$\sup_{u \in \mathcal{P}_{\mathbb{R}}} \big(\sum_{k=-\infty}^{\infty} \|\Delta_k x\|^s \big)^{1/s} = \infty.$$

So, let $s \geq 2$ be given. Choose $-\frac{1}{2} < a < 0$ and $s_1 > s$ such that

$$0 < \frac{1}{2} + a < \frac{1}{s_1} < \frac{1}{s} \leq \frac{1}{2}.$$

Let $e_k(t) = |t|^{-a} \cos kt \ \in L^2[-\pi, \pi]$ for $k \geq 0$. By Lemma 3.2 there exists a constant $M_a > 0$ such that $\|e_k\|_{L^2} \geq M_a$ for all k. This is a conditional basic sequence in $L^2[-\pi, \pi]$ (see [1], so the space $H = \overline{lin}\{e_k : \ k \geq 0\}$ is a Hilbert space.

Let $\{a_k\} \in l_{s_1}$ be given by $a_0 = 1$ and $a_k = \frac{1}{k^{1/s}}$ for $k \geq 1$. Now the basis $\{e_k\}$, the sequence $\{a_k\}$, and s_1 satisfy the conditions of Theorem 3.2, so that the series $\sum_0^\infty a_k e_k$ converges in H. But we also note that $\big(\sum_0^\infty |a_k|^s \big)^{1/s} = \infty$.

Now we are ready to construct the required spectral family on H. Let $\{\lambda_k\}$

be a monotone strictly increasing sequence with $\lambda_0 = 0$ and $\lambda_k \nearrow 2\pi$. Let $\{\xi_k\}$ be the bi-orthogonal functionals associated with $\{e_k\}$ in the sense that $\langle e_k, \xi_j \rangle = \int_{-\pi}^{\pi} e_k(t)\overline{\xi_j(t)}dt = 0$ for $k \neq j$. Define

$$P_k : H \to H \quad y \to \langle y, \xi_k \rangle e_k, \quad \text{for } k \geq 0.$$

Now define $E(\mu)$ as follows

$$\begin{aligned} E(\mu) &= 0 \quad \text{for } \mu \in (-\infty, 0), \\ E(\mu) &= \sum_{j=0}^{k} P_j \ \text{ for } \mu \in [\lambda_k, \lambda_{k+1}), \quad k \geq 0, \\ E(\mu) &= I \quad \text{for } \mu \in [2\pi, \infty). \end{aligned} \tag{3.12}$$

E is now a spectral family on H and is concentrated on $[0, 2\pi]$. Note that, in particular, $E(\lambda_k) - E(\lambda_{k-1}) = P_k$. Let $x = \sum_0^\infty a_k e_k$, with $\{a_k\}$ as defined above. Since this sum converges in L^2 norm, x is a genuine element of H.

Claim

$$\sup_{u \in \mathcal{P}_\mathbb{R}} \Big\{ \sum_{k=-\infty}^{\infty} \|\Delta_k x\|^s \Big\}^{1/s} = \infty.$$

It suffices to show that for each $N \geq 1$ there exists a partition $u_N \in \mathcal{P}_\mathbb{R}$ such that $\big(\sum_{-\infty}^{\infty} \|\Delta_k x\|^s\big)^{1/s} \geq N$. So let $N \geq 1$ be given. Since $\big(\sum_{-\infty}^{\infty} |a_k|^s\big)^{1/s} = \infty$, we can pick J_N such that $M_a\big(\sum_0^{J_N} |a_k|^s\big)^{1/s} > N$.
Let us define u_N as follows: let $\{\mu_k\}_{k \leq 0}$ be any partition of $(-\infty, 0]$ with $\mu_0 = 0$. Let $\mu_j = \lambda_j$ for $j = 1, ..., J_N$ and, without loss of generality, let $\mu_K = 2\pi$ for some $K > J_N$.
Finally, let $\{\mu_j\}_{j \geq K}$ be any partition of $[2\pi, \infty)$.

Thus, for any $y \in H$, we have $\Delta_k y = 0$ for $k < 0$ and $k > K$, and $\Delta_k y = P_k y$ for $0 \leq k \leq J_N$. Hence

$$\sum_{-\infty}^{\infty} \|\Delta_k y\|^s = \sum_{k=0}^{J_N} \|P_k y\|^s \ + \sum_{k=J_N+1}^{K} \|\Delta_k y\|^s \ \geq \ \sum_{k=0}^{J_N} \|P_k y\|^s.$$

We now apply this to $y = x$ and note that $P_k x = a_k e_k$ for $k \geq 0$. Thus we have

$$\begin{aligned} \Big\{ \sum_{-\infty}^{\infty} \|\Delta_k x\|^s \Big\}^{1/s} &\geq \Big\{ \sum_{k=0}^{J_N} |a_k|^s \|e_k\|^s \Big\}^{1/s} \\ &\geq M_a \Big\{ \sum_{k=0}^{J_N} |a_k|^s \Big\}^{1/s} > N. \end{aligned}$$

This proves the Claim, and hence the Proposition. □

Thus we have settled the question of existence of a spectral family on a Hilbert space, which does not arise from a spectral measure. In fact the above construction gives a trigonometrically well bounded operator $S_H = \int_{0^-}^{2\pi} e^{i\lambda} dE(\lambda)$ with interesting power growth properties.

3.2.1 Power Growth Revisited

In chapter 2 we used the spectral family $E(\lambda)$ of a trigonometrically well-bounded operator $T \in \mathcal{B}(X)$ on a super-reflexive space X, to estimate the power growth of T. Specifically, we showed that if $var_q(E) < \infty$ for some $1 < q < \infty$, then for every $1 < p < q'$ there is a constant K_p such that $\|T^n\| \leq K_p\ |n|^{1/p}$. The vehicle for this proof was Young's inequality for Riemann-Stieltjes integrals (Lemma 2.6). However, we can now demonstrate that in a Hilbert space the condition $var_q(E) < \infty$ is not necessary for this power growth, it is merely sufficient. More specifically, given any $2 \leq q < \infty$ there is a trigonometrically well-bounded operator T on a Hilbert space H such that $var_q(E) = \infty$ and yet $\|T^n\| \leq K \ln(|n|)$ for all $n \neq 0, 1$.
To find such a T, we shall exploit the spectral family in Section 3.2. The following Lemma will help define the sequence $\{\lambda_k\}$ used therein.

Lemma 3.3. *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be the sequence $\lambda_0 = 0$ and $\lambda_k = 2\pi(1 - 2^{-|k|})$ for $k \neq 0$. For $n \in \mathbb{Z}$ define $\psi^{(n)} = \{e^{in\lambda_j}\}_{j\in\mathbb{Z}}$. Then for each $n \in \mathbb{Z}$, we have $\psi^{(n)} \in \mathfrak{M}_1(\mathbb{Z})$. Further, $\|\psi^{(1)}\|_{\mathfrak{M}_1} \leq 3$ and*

$$\|\psi^{(n)}\|_{\mathfrak{M}_1} \leq 16 \ln(|n|) \quad \text{for } |n| \geq 2.$$

Proof. Trivially $\psi^{(0)} \in \mathfrak{M}_1(\mathbb{Z})$ and $\|\psi^{(n)}\|_{\mathfrak{M}_1} = 1$. So let $n \neq 0$ be fixed. Let I_l, $l \in \mathbb{Z}$, denote the l-th dyadic interval on $\mathbb{Z}$ (see section 1.3). We wish to calculate

$$\sup_l \sum_{k\in I_l} |\psi^{(n)}_{k+1} - \psi^{(n)}_k| \equiv \sup_l \sum_{k\in I_l} |e^{in\lambda_{k+1}} - e^{in\lambda_k}|.$$

Since the sequence $\{\psi^{(n)}_k\}_{k\in\mathbb{Z}}$ is symmetric (in that $\psi^{(n)}_k = \psi^{(n)}_{-k}$ for $k \geq 1$), we need only deal with $l \geq 1$. Note that for each $k \geq 1$

$$|e^{in\lambda_{k+1}} - e^{in\lambda_k}| \leq 2 \wedge |n|(\lambda_{k+1} - \lambda_k) = 2 \wedge \frac{\pi|n|}{2^k}.$$

Now, we can split $\mathbb{Z}^+$ into a disjoint union

$$\begin{aligned} \mathbb{Z}^+ &= \{l :\ 2 < \frac{\pi|n|}{2^k}\ \forall\ k \in I_l\} \cup \{l_n\} \cup \{l :\ 2 \geq \frac{\pi|n|}{2^k}\ \forall\ k \in I_l\} \\ &= L_0\ \cup\ \{l_n\}\ \cup\ L_1, \quad \text{where } l_n \text{ is determined by } (\dagger) \text{ below.} \end{aligned}$$

Suppose $l \in L_1$. Then $\sum_{k\in I_l} |e^{in\lambda_{k+1}} - e^{in\lambda_k}| \le \sum_{k\in I_l} \frac{\pi|n|}{2^k} \le \frac{1}{2^{2^{l-1}}}\pi|n|$. But since $2^{l-1} \in I_l = \{2^{l-1}, ..., 2^l - 1\}$, we have $\frac{\pi|n|}{2^{2^{l-1}}} \le 2$, so that $\sum_{k\in I_l} |e^{in\lambda_{k+1}} - e^{in\lambda_k}| \le 2$.

Next, if $l \in L_0$, then $\sum_{k\in I_l} |e^{in\lambda_{k+1}} - e^{in\lambda_k}| < 2|I_l| = 2\ 2^{l-1}$. But this time $2^{l-1} \in I_l$ implies $2^{2^{l-1}} < \frac{\pi|n|}{2}$, so $2^{l-1} < \frac{1}{\ln 2}(\ln\frac{\pi}{2} + \ln|n|) < 3\ln|n|$ for $|n| \ge 2$.

Finally, for l_n we can write $I_{l_n} = \{2^{l_n-1}, ..., k_0, ..., 2^{l_n} - 1\}$ (†), where $2 < \frac{\pi|n|}{2^k}$ for $2^{l_n-1} \le k \le k_0$ and $2 \ge \frac{\pi|n|}{2^k}$ for $k_0 < k < 2^{l_n}$. Then

$$\begin{aligned}\sum_{k\in I_{l_n}} |e^{in\lambda_{k+1}} - e^{in\lambda_k}| &= \sum_{2^{(l_n-1)}}^{k_0} |e^{in\lambda_{k+1}} - e^{in\lambda_k}| + \sum_{k_0+1}^{2^{l_n}-1} |e^{in\lambda_{k+1}} - e^{in\lambda_k}| \\ &< 2|I_{l_n}| + \frac{\pi|n|}{2^{k_0+1}}\sum_0^\infty \frac{1}{2^k} = 2.2^{l_n-1} + \frac{2\pi|n|}{2^{k_0+1}}. \qquad (3.13)\end{aligned}$$

Again, note that $\frac{\pi|n|}{2^{k_0+1}} \le 2$ and $2 < \frac{\pi|n|}{2^{2^{(l_n-1)}}}$. Hence $2^{(l_n-1)} < \frac{1}{\ln 2}(\ln\frac{\pi}{2} + \ln|n|) < 3\ln|n|$ for $|n| \ge 2$. Using this on the right side of (3.13) gives $\sum_{k\in I_l} |e^{in\lambda_{k+1}} - e^{in\lambda_k}| < 6\ln|n| + 4 < 14\ln|n|$ for $|n| \ge 2$. Hence we have shown that for $|n| \ge 1$,

$$\sup_{l\ge1}\sum_{k\in I_l} |e^{in\lambda_{k+1}} - e^{in\lambda_k}| \le \max\{2, 14\ln|n|\}.$$

Finally, since $|\psi_k^{(n)}| = 1$ for all $n, k \in \mathbb{Z}$, and $1 < 2\ln|n|$ for $|n| \ge 2$, we get $\|\psi^{(n)}\|_{\mathfrak{M}_1} \le 16\ln|n|$ for $|n| \ge 2$ and $\|\psi^{(1)}\|_{\mathfrak{M}_1} \le 3$ as claimed. □

The Hilbert space H in Section 3.2 can be viewed as a weighted space. Given a number $0 < b < 1/2$, we define a weight on $L^2(\mathbb{T})$, by $w(t) = |t|^{2b}$ for $-\pi \le t \le \pi$. Then $L^2_w(\mathbb{T})$ is the space of functions $f : \mathbb{T} \to \mathbb{C}$ with the norm $\int_{-\pi}^{\pi} |t|^{2b}|f(e^{it})|^2 dt = \|f\|^2_{L^2_w} < \infty$. Now, the sequence $\{|t|^b e^{ikt}\}_{k\in\mathbb{Z}}$ is a conditional basis for $L^2(\mathbb{T})$ (see [1]). This is precisely equivalent to saying that $\{e^{ikt}\}_{k\in\mathbb{Z}}$ is a conditional basis for $L^2_w(\mathbb{T})$. That is, if $f \in L^2_w(\mathbb{T})$, then $\lim_{N\to\infty} \|f - \sum_{-N}^{N} \hat{f}_k e^{ikt}\|_{L^2_w} = 0$.
This weighted space context is useful in that there is a rich theory of multipliers in this setting. In particular, the following Theorem in [9] provides the key ingredient.

Theorem 3.3. *Let $1 < p < \infty$ and let $w(t) \in A_p(\mathbb{T})$. Then, for any $1 < q < \infty$ satisfying $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$, $\mathfrak{M}_q(\mathbb{Z}) \subset M_{p,w}(\mathbb{T})$. Moreover, there exists a constant $K_{p,q}$, such that if T_ψ is the multiplier associated with $\psi \in \mathfrak{M}_q(\mathbb{Z})$, then*

$$\|T_\phi\| \le K_{p,q}\|\psi\|_{\mathfrak{M}_q}.$$

Collecting all these ideas together we have the following Theorem.

Theorem 3.4. *Given any $s \geq 2$ there exists a Hilbert space H and a trigonometrically well-bounded $T \in \mathcal{B}(H)$ with spectral family $E(\lambda)$ such that $\|T^n\| \sim \ln|n|$ and yet $var_s(E) = \infty$.*

Proof. Recalling the discussion of weighted spaces in section 1.4, we know that provided $-1 < \alpha < p-1$, the function $w(t) = |t|^\alpha$, $t \in [-\pi, \pi]$, is an $A_p(\mathbb{T})$ weight. Let us choose $p = 2$ and $q = 1$ in Theorem 3.3 and fix $0 < b < 1/2$. Observe that this choice of p and b ensures that $w(t) = |t|^{2b}$ is an $A_2(\mathbb{T})$ weight. Further, Lemma 3.3 gives that $\psi^{(n)} \in \mathfrak{M}_1(\mathbb{Z})$ for all $n \in \mathbb{Z}$. Thus Theorem 3.3 ensures that the associated multipliers $T_{\psi^{(n)}} : f \mapsto (\psi^{(n)}\hat{f})^\vee$ are bounded from $L^2_w(\mathbb{T})$ into itself. Note that for $f = \sum_{-\infty}^{\infty} \hat{f}_k e^{ikt} \in L^2_w(\mathbb{T})$, we have $T_{\psi^{(n)}} f = \sum_{-\infty}^{\infty} \psi_k^{(n)} \hat{f}_k e^{ikt}$, with conditional convergence in $\|\cdot\|_{L^2_w}$ norm. Moreover, since $\left(\psi_k^{(1)}\right)^n = \psi_k^{(n)}$ and $\left(\psi_k^{(1)}\right)^{-1} = \psi_k^{(-1)}$, it follows that $\left(T_{\psi^{(1)}}\right)^{-1} = T_{\psi^{(-1)}}$ and $\left(T_{\psi^{(1)}}\right)^n = T_{\psi^{(n)}}$ for all $n \in \mathbb{Z}$.

As in Section 3.2, let $H = \overline{lin}\ \{|t|^b \cos kt\}_{k \geq 0} \subset L^2(\mathbb{T})$ (the closure is in the $\|\cdot\|_{L^2(\mathbb{T})}$ norm). Note that for $f \in H$,

$$f = \lim_{N\to\infty} \sum_{k=0}^{N} \alpha_k |t|^b \cos kt = \lim_{N\to\infty} \left\{ \sum_{-N}^{0} \frac{a_{-k}}{2} |t|^b e^{ikt} + \sum_{0}^{N} \frac{a_k}{2} |t|^b e^{ikt} \right\}, \qquad (3.14)$$

these being the $\|\cdot\|_{L^2(\mathbb{T})}$ limits. Thus if also $f = \lim_{N\to\infty} \sum_{-N}^{N} \beta_k |t|^b e^{ikt}$, it must follow that $\beta_0 = \alpha_0$ and $\beta_k = \beta_{-k} = \frac{1}{2}\alpha_k$ for $k \neq 0$. This comes simply by pairing both expressions against $\langle \cdot, \xi_k \rangle$, where $\{\xi_k\}_{k\in\mathbb{Z}}$ are bi-orthogonal to $\{|t|^b e^{ikt}\}$ in $L^2(\mathbb{T})$. Conversely, if $f = \lim_{N\to\infty} \sum_{-N}^{N} \beta_k |t|^b e^{ikt}$ satisfies $\beta_k = \beta_{-k}$ for all $k \neq 0$, it follows trivially that $f = \beta_0 |t|^b + \lim_{N\to\infty} 2 \sum_1^N \beta_k |t|^b \cos\ kt$, that is $f \in H$.

Now, let us fix s such that $\frac{1}{2} - b < \frac{1}{s} \leq \frac{1}{2}$, and let $\{\lambda_k\} = \psi^{(1)}$ be the sequence in Lemma 3.3. Observe that $0 = \lambda_0 < \lambda_1 < \lambda_2 < ...$ and $\lambda_k \nearrow 2\pi$. Now let $E(\lambda)$ be the spectral family constructed in the manner of Section 3.2, and define

$$S_H : H \to H, \quad S_H = \int_{0^-}^{2\pi} e^{i\mu} dE(\mu).$$

Clearly S_H is automatically trigonometrically well-bounded. Let us also define, for $n \in \mathbb{Z}$, the operators

$$S_{(n)} : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \qquad \sum_{-\infty}^{\infty} \beta_k |t|^b e^{ikt} \mapsto \sum_{-\infty}^{\infty} e^{in\lambda_k} \beta_k |t|^b e^{ikt}.$$

We can check directly that $S_{(n)}$ are indeed bounded operators on $L^2(\mathbb{T})$. For, if $f \equiv \sum_{-\infty}^{\infty} \beta_k |t|^b e^{ikt} \in L^2(\mathbb{T})$, then $g \equiv \sum_{-\infty}^{\infty} \beta_k e^{ikt} \in L^2_w(\mathbb{T})$, and $\|f\|_{L^2} = \|g\|_{L^2_w}$. Furthermore, for each $n \in \mathbb{Z}$,

$$\|S_{(n)} f\|^2_{L^2(\mathbb{T})} \;=\; \int_{-\pi}^{\pi} |t|^{2b} \Big| \sum_{-\infty}^{\infty} e^{in\lambda_k} \beta_k e^{ikt} \Big|^2 dt \;=\; \|T_{\psi^{(n)}} g\|^2_{L^2_w}.$$

Hence

$$\|S_{(n)} f\|_{L^2(\mathbb{T})} = \|T_{\psi^{(n)}} g\|_{L^2_w} \le \|T_{\psi^{(n)}}\|_{\mathcal{B}(L^2_w)} \; \|g\|_{L^2_w} = \|T_{\psi^{(n)}}\|_{\mathcal{B}(L^2_w)} \; \|f\|_{L^2(\mathbb{T})},$$

and so every $S_{(n)}$ is bounded. It remains to verify that for any $n \in \mathbb{Z}$,

$$S_{(n)} f = \big(S_H\big)^n f, \qquad f \in H. \tag{3.15}$$

To show this we note the simple form which S_H actually takes.

Claim

For $f = \sum_{k=0}^{\infty} \alpha_k |t|^b \cos kt \in H$, we have $\big(S_H\big)^n f = \sum_{k=0}^{\infty} e^{in\lambda_k} \alpha_k |t|^b \cos kt$.

Proof of Claim: By definition $\big(S_H\big)^n$ is the strong limit

$$\big(S_H\big)^n = \lim_{u} \; E(0) + \sum_{1}^{N} e^{in\mu_k} \{E(\mu_k) - E(\mu_{k-1})\},$$

where $u = \{0 = \mu_0 < ... < \mu_N = 2\pi\}$. Let $\mathcal{S}(u)$ denote the approximating sum on the right. Without loss of generality we can assume that for some $K \ge 1$, $\{\lambda_1, ..., \lambda_K\} \subset u$, and that $\lambda_K = \mu_{N-1} < \mu_N = 2\pi$. Moreover, using the definition of $E(\mu)$ in (3.12) we see that $E(0) = P_0$ and $\mathcal{S}(u) = P_0 + \sum_{k=1}^{K} e^{in\lambda_k} P_k + \{I - \sum_{k=1}^{K} P_k\}$. Hence we have, in $\|\cdot\|_{L^2(\mathbb{T})}$ norm,

$$\lim_{u} \mathcal{S}(u) f = P_0 f + \lim_{K\to\infty} \sum_{k=1}^{K} e^{in\lambda_k} P_k f \;+\; \lim_{K\to\infty} \big\{I - \sum_{k=1}^{K} P_k\big\} f.$$

Considering that $P_k f = \alpha_k |t|^b \cos kt$ (for $k \ge 0$) and $\|f - \sum_{k=1}^{K} \alpha_k |t|^b \cos kt\|_{L^2(\mathbb{T})} \to 0$, we deduce that $\big(S_H\big)^n f = \sum_{k=0}^{\infty} e^{in\lambda_k} \alpha_k |t|^b \cos kt$ as required to prove the Claim.

Returning to the proof of (3.15), let $f = \sum_{-\infty}^{\infty} \beta_k |t|^b e^{ikt} \in H$. By (3.14) and the discussion following it, we deduce that $f = \beta_0 |t|^b + 2\sum_{1}^{\infty} \beta_k |t|^b \cos kt$. So,

$$\begin{aligned} S_{(n)} f &= \lim_{N\to\infty} \sum_{-N}^{N} e^{in\lambda_k} \beta_k |t|^b e^{ikt} \\ &= \beta_0 |t|^b + \lim_{N\to\infty} 2 \sum_{1}^{N} e^{in\lambda_k} \beta_k |t|^b \cos kt, \quad \text{since } \; e^{in\lambda_k}\beta_k = e^{in\lambda_{-k}} \beta_{-k} \; \text{ for } \; k > 0, \\ &= \big(S_H\big)^n f, \end{aligned}$$

with all the limits in the $\|\cdot\|_{L^2(\mathbb{T})}$ norm. The last equality follows by defining $\alpha_0 = \beta_0$, $\alpha_k = 2\beta_k$ for $k \geq 1$, and appealing to the Claim. Using this, Theorem 3.3 and Lemma 3.3, we have for all $|n| \geq 2$,

$$\begin{aligned} \|\big(S_H\big)^n\|_{\mathcal{B}(L^2(\mathbb{T}))} = \|S_{(n)}\|_{\mathcal{B}(L^2(\mathbb{T}))} &\leq \|T_{\psi^{(n)}}\|_{\mathcal{B}(L^2_w)} \\ &\leq K_{2,1}\|\psi^{(n)}\|_{\mathfrak{M}_1} \leq 16K_{2,1}\ln|n|. \end{aligned}$$

Hence $\big(S_H\big)^n$ has power growth $\sim \ln|n|$, and yet $var_s(E) = \infty$. So this construction shows that $var_s(E) < \infty$ is not necessary for a slow power growth of a trigonometrically well-bounded operator. □

The machinery set up in this section can be used to produce trigonometrically well bounded operators with arbitrary $N-$fold logarithmic power growth.

Corollary 3.1. *Let $s \geq 2$ and $0 < b < 1/2$ satisfy the conditions in Theorem 3.4. Let $H = \overline{lin}\ \{|t|^b \cos\ kt\}_{k\geq 0} \subset L^2(\mathbb{T})$. Given any $N \geq 1$ there exists a trigonometrically well bounded operator $T \in \mathcal{B}(H)$ and a constant $K_T > 0$ such that for all $n \neq 0$, $\|T^n\| \leq K_T\ \ln^{(N)}(|n|)$, where $\ln^{(N)}(\cdot)$ is the N-fold composition of $\ln(\cdot)$. Furthermore, if E is the spectral family of T, then $var_s(E) = \infty$.*

Proof. Let $N \geq 1$ be fixed. In Lemma 3.3 we choose $\lambda_0 = 0$ and $\lambda_k = 2\pi(1 - 2^{2^{\cdots^{-|k|}}})$ for $k \neq 0$, where the term in the brackets contains an N-fold power of 2. For $n \in \mathbb{Z}$ define $\psi^{(n)} = \{e^{in\lambda_j}\}_{j\in\mathbb{Z}}$. Then, arguing in the same manner as in Lemma 3.3, there is a constant $K_N > 0$ such that for each $n \neq 0, 1$, we have $\psi^{(n)} \in \mathfrak{M}_1(\mathbb{Z})$ and $\|\psi^{(n)}\|_{\mathfrak{M}_1} \leq K_N \ln^{(N)}(|n|)$. Feeding this into Theorem 3.4 immediately gives the desired result. □

As explained at the beginning of the chapter, it is interesting to contrast this result with the situation where the operator $T \in \mathcal{B}(H)$ is *power bounded*. For, we know that if $T \in \mathcal{B}(H)$ is invertible and $\|T^n\| \leq K$ for all $n \in \mathbb{Z}$, then it is equivalent to a unitary operator. In particular it is trigonometrically well-bounded and its spectral family $\{E(\lambda)\}$ comes from a spectral measure $\mathcal{E}$. But any such spectral family automatically has the property $var_2(E) < \infty$ and $var_2(E^*) < \infty$. Here we see that relaxing power-boundedness of T to *arbitrarily slow* power growth immediately destroys this property: we simply choose $s = 2$ in Corollary 3.1.

CHAPTER 4

4.1 $BV_q(\mathbb{T})$ Functional Calculus

We turn to the second key application of results in chapter 2, namely in the theory of spectral integration and vector valued multipliers. The following question is a recurring task in vector valued harmonic analysis.

Given a space X and a spectral family on X, for which function algebras $\mathcal{A}$ is the map $\psi \mapsto \int_{\mathbb{R}} \psi(\lambda)dE(\lambda)$ a well-defined continuous algebra homomorphism into $\mathcal{B}(X)$?

An argument in [4] shows that, given any space X and a trigonometrically well-bounded T, with spectral family E, the algebra $BV(\mathbb{T})$ fits the bill. Other function algebras give rise to spectral integration too: see [8], [6] and [14]. These results deal with three types of spectral family:

(i) the spectral family of the right translation operator group $\{R_\gamma\}_{\gamma\in G}$ on $L^p_X(G)$, where $1 < p < \infty$, $G = \mathbb{Z}, \mathbb{T}$ and $\mathbb{R}$ and X is UMD;

(ii) the spectral family of a power bounded operator T on a UMD X.

(iii) the spectral family of a uniformly bounded, strongly continuous operator group $\{U_\gamma\}_{\gamma\in G}$ on a UMD X.

In (ii) and (iii) the space X is of a special interpolated kind (see chapter 1 section 1.3 for details). Furthermore, in both (ii) and (iii), the *power boundedness* of T and $\{U_\gamma\}_{\gamma\in G}$ is a necessary requirement. We are now in a position to dispense with this restriction. We can formulate a $BV_q(\mathbb{T})$ spectral theorem for *any* trigonometrically well bounded T, provided the space X is *super-reflexive.*

Theorem 4.1. *Let X be a super-reflexive space and let $T \in \mathcal{B}(X)$ be a trigonometrically well bounded operator with spectral family E. Then there exists $1 \leq r_E < \infty$, depending only on E, such that for all $1 \leq q < r_E'$, the map*

$$\Theta_{T,q} : BV_q(\mathbb{T}) \to \mathcal{B}(X), \quad \psi \mapsto \int_{0^-}^{2\pi} \psi(e^{i\lambda})dE(\lambda),$$

is a well defined, norm continuous algebra homomorphism.

Proof. By Theorem 2.1 there exists some $1 < p < \infty$ such that $var_p(E) < \infty$. So the number $r_E \equiv \inf\{r : \ var_r(E) < \infty\}$ is finite. Let r_E' denote its conjugate index and observe that $1 < r_E' \leq \infty$. Suppose now $1 \leq q < r_E'$ is fixed. We choose any number $r \in (r_E, q')$. So $q < r' < r_E'$ and $var_r(E) < \infty$. Now let $\psi \in BV_q(\mathbb{T}) \cap BV_1(\mathbb{T})$, noting that this is a dense subspace of $BV_q(\mathbb{T})$. Define $\tilde{\psi} \in BV_q[0, 2\pi] \cap BV_1[0, 2\pi]$ by $\tilde{\psi}(\lambda) = \psi(e^{i\lambda})$. Observe that $\|\tilde{\psi}\|_{BV_q[0,2\pi]} \leq 2\|\psi\|_{BV_q(\mathbb{T})}$. Since $\psi \in BV_1(\mathbb{T})$, the operator $\int_{0^-}^{2\pi} \psi(e^{i\lambda}) dE(\lambda)$ is well-defined (see [4]). Furthermore, for each $x \in X$ and $\xi \in X^*$, we have, by Lemma 2.6,

$$\begin{aligned} \left| \int_{0^-}^{2\pi} \tilde{\psi}(\lambda) \langle dE(\lambda)x, \xi \rangle \right| &\leq \|\tilde{\psi}\|_{BV_q} |\langle E(0)x, \xi \rangle| \\ &+ K_{rq} \, \|\tilde{\psi}\|_{BV_q} \, var_r(E) \|x\| \, \|\xi\|. \end{aligned}$$

where $K_{rq} = 1 + \zeta(1/r + 1/q)$ and ζ is Euler's zeta function. Hence

$$\Big\| \int_{0^-}^{2\pi} \tilde{\psi}(\lambda) dE(\lambda) \Big\| \leq 2\|\psi\|_{BV_q} \{\|E\|_\infty + K_{rq} \, var_r(E)\} < \infty.$$

Thus the map $\Theta_{T,q} : \psi \mapsto \int_{0^-}^{2\pi} \psi(e^{i\lambda}) dE(\lambda)$ is a norm-continuous linear map from $BV_q(\mathbb{T}) \cap BV_1(\mathbb{T})$ into $\mathcal{B}(X)$, and so has a continuous extension to all of $BV_q(\mathbb{T})$.

It remains to show that $\Theta_{T,q}$ is multiplicative. So, let $u = \{0 = \lambda_0 < ... < \lambda_N = 2\pi\} \in \mathcal{P}_{[0,2\pi]}$ and write $\mathcal{S}(\psi, u) = \psi(1)E(0) + \sum_{j=1}^N \psi(e^{i\lambda_j})\{E(\lambda_j) - E(\lambda_{j-1})\}$. Then, for $\psi \in BV_q(\mathbb{T})$, $\Theta_{T,q}(\psi) = \displaystyle\int_{0^-}^{2\pi} \psi(e^{i\lambda}) dE(\lambda) = \text{st} \lim_{u \in \mathcal{P}_{[0,2\pi]}} \mathcal{S}(\psi, u)$. Observe that the net $\{\mathcal{S}(\psi, u)\}_{u \in \mathcal{P}_{[0,2\pi]}}$ is uniformly bounded, by

$$2\{\|E\|_\infty + K_{rq} \, var_r(E)\} \|\psi\|_{BV_q} \;\equiv\; C_{rq} \|\psi\|_{BV_q}.$$

Suppose now $\psi, \ \phi \in BV_q(\mathbb{T})$ and $x \in X$. Observe that $\mathcal{S}(\psi, u)\mathcal{S}(\phi, u) = \mathcal{S}(\psi\phi, u)$. Then, suppressing the limits 0^- and 2π in the integrals,

$$\begin{aligned} &\Big\| \ \mathcal{S}(\psi\phi, u)x - \int_{0^-}^{2\pi} \psi dE \int_{0^-}^{2\pi} \phi dE \ x \Big\| = \Big\| \mathcal{S}(\psi, u)\mathcal{S}(\phi, u)x - \int \psi dE \int \phi dE \ x \Big\| \\ &\leq \ \Big\| \mathcal{S}(\psi, u)\mathcal{S}(\phi, u)x - \mathcal{S}(\psi, u) \int \phi dE \ x \Big\| + \Big\| \mathcal{S}(\psi, u) \int \phi dE \ x - \int \psi dE \int \phi dE \ x \Big\| \\ &\leq \ C_{rq} \|\psi\|_{BV_q} \Big\| \mathcal{S}(\phi, u)x - \int_{0^-}^{2\pi} \phi \ dE \ x \Big\| + \Big\| \mathcal{S}(\psi, u)y - \int \psi dE \ y \Big\|, \end{aligned}$$

where $y = \int_{0^-}^{2\pi} \phi \ dE \ x$. So, given $\epsilon > 0$, we can choose a partition u_0 such that $\|\mathcal{S}(\phi, u)x - \int_{0^-}^{2\pi} \phi \ dE \ x\| < \frac{\epsilon}{4C_{rq}\|\psi\|_{BV_q}}$ and $\|\mathcal{S}(\psi, u)y - \int \psi dE \ y\| < \frac{\epsilon}{4}$ for all $u \supseteq u_0$. Then we choose u_1 such that $\| \int \psi\phi \ dE \ x - \mathcal{S}(\psi\phi, u)x\| < \frac{\epsilon}{2}$ for all $u \supseteq u_2$. Hence, for all $u \supseteq u_1 \cup u_2$, we have $\| \int \psi\phi \ dE \ x - \int \psi \ dE \int \phi \ dE \ x\| < \epsilon$. Since ϵ is arbitrary, this gives $\int \psi\phi \ dE \ x \ = \ \int \psi \ dE \int \phi \ dE \ x$. □

4.2 $BV_q(\mathbb{R})$ Multiplier Theorem

As we remarked in section 1.3 in chapter 1, multipliers on $L^p_X(G)$, ($G = \mathbb{R}, \mathbb{Z}$ or $\mathbb{T}$), can be viewed essentially as special examples of spectral integrals. The integration is with respect to the spectral family (when it exists) of the right translation (or shift) group $\{R_\gamma\}_{\gamma \in G}$. We shall concentrate on $G = \mathbb{R}$ in this section. Let us recall the definition of the space of multipliers, $M_{p,X}(\mathbb{R})$. We say that a bounded complex measurable function $\psi : \mathbb{R} \to \mathbb{C}$ is a member of $M_{p,X}(\mathbb{R})$ if the linear map

$$S_\psi : \ C_c^\infty \otimes X \to L^p_X, \quad \sum_1^N f_k x_k \mapsto \sum_1^N (\psi \hat{f}_k)^\vee x_k$$

is a bounded linear map into L^p_X, with $\|S_\psi f\|_{L^p_X} \leq K \|f\|_{L^p_X}$ for all $f \in C_c^\infty \otimes X$. Suppose we fix $1 < q < \infty$ and $\psi \in BV_q(\mathbb{R})$. Then, in particular, $\psi \in L^\infty(\mathbb{R})$ so that for $f \in C_c^\infty \otimes X$, $S_\psi f$ makes sense pointwise almost everywhere. It is shown in [18] that, provided $p \in (1, \infty)$ and $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$, then $\psi \in M_{p,\mathbb{C}}(\mathbb{R})$ (that is, ψ is a scalar-valued multiplier). Here we give an analogous result in the vector-valued setting.

Theorem 4.2. *Let X be a UMD space and let $1 < p < \infty$. Then there exists $q_p > 1$ such that for all $1 < q < q_p$ we have*

$$BV_q(\mathbb{R}) \subset M_{p,X}(\mathbb{R}).$$

Before proving the Theorem, let us state and prove some relevant Lemmas.

Lemma 4.1. *Let $k \in L^1(\mathbb{R})$ satisfy $\int_{\mathbb{R}} |k(s)| ds = 1$, and let $\psi \in BV_q(\mathbb{R})$, with $1 \leq q < \infty$. Then $var_q(k * \psi) \leq var_q(\psi)$.*

Proof. This is easy to show. Let $u = \{\lambda_0 < \lambda_1 < ... < \lambda_N\} \in \mathcal{P}_{\mathbb{R}}$. Then we simply compute, using Minkowski's inequality in the third line:

$$\Big\{\sum_{j=1}^{N}|k*\psi(\lambda_{j+1})-k*\psi(\lambda_j)|^q\Big\}^{1/q}=$$

$$\Big\{\sum_{j=1}^{N}\Big|\int_{\mathbb{R}}k(s)[\psi(\lambda_{j+1}-s)-\psi(\lambda_j-s)]ds\Big|^q\Big\}^{1/q}$$

$$\leq\Big\{\sum_{j=1}^{N}\Big(\int_{\mathbb{R}}|k(s)||\psi(\lambda_{j+1}-s)-\psi(\lambda_j-s)|ds\Big)^q\Big\}^{1/q}$$

$$\leq\int_{\mathbb{R}}\Big\{\sum_{j=1}^{N}|k(s)|^q|\psi(\lambda_{j+1}-s)-\psi(\lambda_j-s)|^q\Big\}^{1/q}ds$$

$$=\int_{\mathbb{R}}|k(s)|\Big\{\sum_{j=1}^{N}|\psi(\lambda_{j+1}-s)-\psi(\lambda_j-s)|^q\Big\}^{1/q}ds$$

$$\leq var_q(\psi)\int_{\mathbb{R}}|k(s)|ds \ = \ var_q(\psi).$$

□

Lemma 4.2. *Let $f\in BV_r[a,b]$ and $g\in BV_s[a,b]$ with $\frac{1}{r}+\frac{1}{s}>1$. Write $K_{r,s}=1+\zeta(\frac{1}{r}+\frac{1}{s})\leq\frac{2rs-r-s}{rs-r-s}$, where $\zeta(\cdot)$ is Euler's zeta function. Let $\eta>0$ and let $u=\{a=\lambda_0<\lambda_1<...<\lambda_N=b\}$ be a partition of $[a,b]$. Writing $I_k=[\lambda_{k-1},\lambda_k]$, we define*

$$osc(f,I_k)\equiv\sup_{x,y\in I_k}|f(x)-f(y)|.$$

Let $r_1>r$ and $s_1>s$ satisfy $\frac{1}{r_1}+\frac{1}{s_1}>1$. Then:
(i) for any $\xi\in(\lambda_{k-1},\lambda_k)$,

$$\Big|\int_{I_k}\{f(t)-f(\xi)\}dg(t)\Big|\leq K_{r_1,s_1}var_{r_1}(f,I_k)\ var_{s_1}(g,I_k),$$

(ii) suppose further that u satisfies $osc(g,I_k)<\eta$ for all $k=1,...,N$; then

$$\sum_{k=1}^{N}var_{s_1}(g,I_k)\ var_{r_1}(f,I_k)\ \leq\eta^{\frac{s_1-s}{s_1}}(var_s\ g)^{\frac{s}{s_1}}\ var_{r_1}(f).$$

Proof. See 10.8 and 10.9 in [32]. □

Lemma 4.3. *Let $1<r,s<\infty$ be such that $\frac{1}{r}+\frac{1}{s}>1$ and let $-\infty<a<b<\infty$. Let $\{f_n\}\in BV_r([a,b])$ be a collection of functions satisfying*
(i) $f_n(x)\to 0$ for $x\in[a,b]$;
(ii) $var_r(f_n)\leq M_1$ for all $n\geq 1$.
Let $\{g_\alpha\}_{\alpha\in A}\in BV_s([a,b])$ be a collection indexed by an arbitrary set A, such that
(i) $var_s(g_\alpha)\leq M_2$ for each $\alpha\in A$,

(ii) $\{g_\alpha\}$ *are equi-continuous: for each* $x \in [a,b]$ *and* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for all* $\alpha \in A$ *and* $y \in [a,b] \cap (x-\delta, x+\delta)$*,* $|g_\alpha(x) - g_\alpha(y)| < \epsilon$*.*
Then

$$\lim_{n\to\infty} \int_a^b f_n(x) dg_\alpha(x) = 0 \quad \textit{uniformly in } \alpha \in A.$$

Proof. Since $[a,b]$ is compact, each g_α is uniformly continuous, and hence $\{g_\alpha\}_{\alpha\in A}$ are uniformly equi-continuous. Suppose $\epsilon > 0$ is given. We can choose $r_1 > r$ and $s_1 > s$ such that $\frac{1}{r_1} + \frac{1}{s_1} > 1$ and then $0 < \eta < 1$ small enough so that $\{K_{r_1,s_1} M_2^{\frac{s}{s_1}} M_1\}\eta^{\frac{s_1-s}{s_1}} < \frac{\epsilon}{2}$. By uniform equi-continuity of $\{g_\alpha\}_{\alpha\in A}$, we can choose a fine enough partition $u = \{a = \lambda_0 < \lambda_1 < ... < \lambda_N = b\}$ of $[a,b]$ such that $osc(g_\alpha, I_k) < \eta$ for all $\alpha \in A$ and $k = 1, ..., N$ (here $I_k = [\lambda_{k-1}, \lambda_k]$). We now fix any points points $\xi_k \in (\lambda_{k-1}, \lambda_k)$, $k = 1, ..., N$. So $f_n(\xi_k) \to 0$ for each k and $\max_{1\le k\le N} |f_n(\xi_k)| \to 0$. So there exists $N_\eta \ge 1$ such that $\max_{1\le k\le N} |f_n(\xi_k)| < \frac{\epsilon}{2N}$ for all $n \ge N_\eta$. Hence,

$$\Big|\sum_{k=1}^N f_n(\xi_k)\{g_\alpha(\lambda_k) - g_\alpha(\lambda_{k-1})\}\Big| \le N\eta \max_{1\le r\le N} |f_n(\xi_k)| < \frac{\epsilon}{2} \quad \text{for all } \alpha \in A, \ \ n \ge M_\eta.$$

Also, with the aid of Lemma 4.2 we have for all $\alpha \in A$,

$$\begin{aligned} \Big|\sum_{k=1}^N \int_{I_k} \{f_n(\lambda) - f_n(\xi_k)\} dg_\alpha(\lambda)\Big| &\le K_{r_1,s_1} \sum_{k=1}^N var_{s_1}(g_\alpha, I_k)\ var_{r_1}(f_n, I_k) \\ &\le K_{r_1,s_1} \eta^{\frac{s_1-s}{s_1}} var_s(g_\alpha)^{\frac{s}{s_1}} var_{r_1}(f_n) \\ &\le \{K_{r_1,s_1} M_2^{\frac{s}{s_1}} M_1\}\eta^{\frac{s_1-s}{s_1}} < \frac{\epsilon}{2}. \end{aligned}$$

Now we can write

$$\int_a^b f_n(\lambda) dg_\alpha(\lambda) = \sum_{k=1}^N f_n(\xi_k)\{g_\alpha(\lambda_k) - g_\alpha(\lambda_{k-1})\} + \sum_{k=1}^N \int_{I_k} \{f_n(\lambda) - f_n(\xi_k)\} dg_\alpha(\lambda).$$

So, using the last two inequalities we deduce that for all $n > N_\eta$ and all $\alpha \in A$,

$$\Big|\int_a^b f_n(\lambda) dg_\alpha(\lambda)\Big| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This holds for all $\alpha \in A$ and hence the desired result. □

Let us now prove Theorem 4.2

Proof. Keeping in line with previous notation, let $C_c^\infty(X)$ denote the space of functions $f : \mathbb{R} \to X$ which are smooth and of compact support. This is of course a dense subspace in $L_X^p(\mathbb{R}) \equiv L_X^p$. Since X is UMD, so is L_X^p, (as $1 < p < \infty$).

Hence by Theorem 3.8 in [5] we deduce that $\{R_t\}_{t\in\mathbb{R}}$ has a spectral decomposition. Namely there exists a spectral family E on L^p_X such that

$$R_t = \text{strong} \lim_{a\to\infty} \int_{-a}^{a} e^{it\lambda} dE(\lambda).$$

Moreover, this family E is strongly *continuous*, not merely right-continuous (this is a consequence of Corollary 5.1 in chapter 5 below). Now, X being UMD, and so super-reflexive, we know that there exists a number $1 < r_p < \infty$ such that $var_{r_p}(E) < \infty$. Let q_p be its conjugate index. We can now argue as in the proof of Theorem 2.2, to deduce that for each $1 < q < q_p$ and $\psi \in BV_q(\mathbb{R})$, the operator $\int \psi(-\lambda)dE(\lambda)$ is well-defined and bounded on $L^p_X(\mathbb{R})$ (note that the function $\tilde{\psi}$ defined by $\tilde{\psi}(\lambda) := \psi(-\lambda)$ is also in $BV_q(\mathbb{R})$ and $\|\tilde{\psi}\|_{BV_q} = \|\psi\|_{BV_q}$). We further have

$$\Big\| \int \psi(-\lambda)dE(\lambda)\Big\| \le C_{p,q} var_{r_p}(E)\ \|\psi\|_{BV_q}, \quad \text{for some constant } C_{p,q} \tag{4.1}$$

Henceforth let us fix $q \in (1, q_p)$.
The first aim is to show that for each $\psi \in BV_q \cap L^1(\mathbb{R})$ the (well-defined) linear map

$$S_\psi : C^\infty_c(X) \to L^\infty_X(\mathbb{R}), \quad f \mapsto (\psi\hat{f})^\vee,$$

is a bounded map into $L^p_X(\mathbb{R})$, with a norm not exceeding $K\|\psi\|_{BV_q}$ for some $K > 0$ independent of ψ. We shall then deal with general $\psi \in BV_q(\mathbb{R})$ by approximating with a sequence $\{\psi_n\} \subset L^1 \cap BV_q(\mathbb{R})$. So, let $\psi \in L^1 \cap BV_q(\mathbb{R})$, so that $\hat{\psi}$ is uniformly continuous on $\mathbb{R}$. Let $f \in C^\infty_c(X)$. We claim that, for each $s \in \mathbb{R}$, we have $\int_{\mathbb{R}} \hat{\psi}(t)R_{-t}f(s)dt = (\psi\hat{f})^\vee(s)$. To see this, first note that $\int_{\mathbb{R}} \hat{\psi}(t)R_{-t}f(s)dt$ makes sense pointwise ar each $s \in \mathbb{R}$ because the function $t \mapsto \hat{\psi}(t)f(t+s)$ (with s fixed) is continuous and of compact support. We now have

$$\begin{aligned}
\int_{t\in\mathbb{R}} \hat{\psi}(t)R_{-t}f(s)dt &= \int_{t\in\mathbb{R}} \hat{\psi}(t)f(s+t)dt \\
&= \frac{1}{2\pi}\int_t\int_u e^{-iut}\psi(u)f(s+t)du\ dt \\
&= \frac{1}{2\pi}\int_u \Big\{ \int_t e^{-iu(s+t)}f(s+t)dt\Big\} e^{ius}\psi(u)du \\
&= \int_u e^{ius}\psi(u)\hat{f}(u)du \\
&= (\psi\hat{f})^\vee(s) \ = \ S_\psi f(s).
\end{aligned} \tag{4.2}$$

The use of Fubini in the third line is justified because

$$\int_u\int_t \|f(s+t)\|_X|\psi(u)|dtdu \le \|f\|_{L^1}\|\psi\|_{L^1} < \infty.$$

Now let k_n denote the n^{th} Fejer kernel on $\mathbb{R}$. Let us also define

$$\Psi(\lambda) = \frac{1}{2}\{\psi(\lambda^+) + \psi(\lambda^-)\} \quad \lambda \in \mathbb{R}.$$

Then $k_n * \psi(\lambda) \to \Psi(\lambda)$ pointwise, and from Lemma 4.1 it follows that for any $n \geq 1$, $var_q(k_n * \psi) \leq var_q(\psi)$. In particular, $k_n * \psi \in BV_q(\mathbb{R})$, so that the operators $\int_{\mathbb{R}} k_n * \psi(-\lambda) dE(\lambda)$ are well-defined members of $\mathcal{B}(L^p_X)$.

Claim

$$\int_{\mathbb{R}} \Psi(-\lambda) dE(\lambda) = \text{st-}\lim_{n\to\infty} \int_{\mathbb{R}} k_n * \psi(-\lambda) dE(\lambda).$$

To prove this Claim, let us fix $a > 0$ and let $f \in \{E(a) - E(-a)\} L^p_X(\mathbb{R})$. Let us also take $\xi \in L^{p'}_{X^*}(\mathbb{R})$ with $\|\xi\| \leq 1$. Then

$$\begin{aligned} \int_{\mathbb{R}} k_n * \psi(-\lambda) dE(\lambda) f &= \int_{-a}^{a} k_n * \psi(-\lambda) dE(\lambda) f, \qquad (4.3) \\ \text{and so } \Big\langle \int_{\mathbb{R}} k_n * \psi(-\lambda) dE(\lambda) f,\ \xi \Big\rangle &= \int_{-a}^{a} k_n * \psi(-\lambda) \langle dE(\lambda) f, \xi \rangle. \end{aligned}$$

Now the scalar function $\lambda \mapsto \langle E(\lambda) f, \xi \rangle$ is continuous and is a member of $BV_{r_p}([-a, a])$. Hence we can apply Lemma 4.3 with $(r, s) \equiv (q, r_p)$,
$\{f_n(\lambda)\}_{n\geq 1} \equiv \{(k_n * \psi)(-\lambda) - \Psi(-\lambda)\}_{n\geq 1}$ and $\{g_\alpha\}_{\alpha\in A} \equiv \{\langle E(\cdot) f, \xi \rangle\}_{\xi \in B(L^{p'}_{X^*})}$.
The Lemma then gives

$$\begin{aligned} \lim_{n\to\infty} \int_{-a}^{a} k_n * \psi(-\lambda) \langle dE(\lambda) f,\ \xi \rangle &= \int_{-a}^{a} \Psi(-\lambda) \langle dE(\lambda) f, \xi \rangle \qquad (4.4) \\ &= \Big\langle \int_{-a}^{a} \Psi(-\lambda) dE(\lambda) f, \xi \Big\rangle = \Big\langle \int_{\mathbb{R}} \Psi(-\lambda) dE(\lambda) f, \xi \Big\rangle. \end{aligned}$$

But in fact Lemma 4.3 shows that this convergence is *uniform* in $\xi \in L^{p'}_{X^*}(\mathbb{R})$, $\|\xi\| \leq 1$. So, using (4.3) and (4.4), we deduce that for each $f \in \bigcup_{a\geq 1} \{E(a) - E(-a)\} L^p_X(\mathbb{R})$ we have

$$\lim_{n\to\infty} \Big\| \int_{\mathbb{R}} \Psi(-\lambda) dE(\lambda) f - \int_{\mathbb{R}} k_n * \psi(-\lambda) dE(\lambda) f \Big\|_{L^p_X(\mathbb{R})} = 0. \qquad (4.5)$$

However, the family of operators $\big\{ \int k_n * \psi(-\lambda) dE(\lambda),\ \int \Psi(-\lambda) dE(\lambda) \big\}_{k\geq 1}$ is uniformly bounded, so the convergence in equation (4.5) holds for all $f \in L^p_X(\mathbb{R})$.

In fact, the operators $\int k_n * \psi(-\lambda) dE(\lambda)$ are precisely $\frac{1}{2\pi} \int_{-n}^{n} \hat{k}\hat{\psi}(t) R_{-t} dt$. To show this let us again fix $f \in \{E(a) - E(-a)\} L^p_X(\mathbb{R})$ and $\xi \in L^{p'}_{X^*}(\mathbb{R})$. We have

$$\int_{-n}^{n} \hat{k_n}(t) \hat{\psi}(t) R_{-t} f dt = \int_{|t|\leq n} (k_n * \psi)^\wedge(t) \Big\{ \int_{|\lambda|\leq a} e^{-i\lambda t} dE(\lambda) f \Big\} dt. \qquad (4.6)$$

Now, using integration by parts,

$$\begin{aligned}\int_{|\lambda|\le a} e^{-i\lambda t}\langle dE(\lambda)f,\xi\rangle &= \Big[e^{-i\lambda t}\langle E(\lambda)f,\xi\rangle\Big]_{-a}^{a} \ + \ it\int_{|\lambda|\le a} e^{-i\lambda t}\langle E(\lambda)f,\xi\rangle d\lambda \\ &= e^{-iat}\langle f,\xi\rangle \ + \ it\int_{|\lambda|\le a} e^{-i\lambda t}\langle E(\lambda)f,\xi\rangle d\lambda.\end{aligned}$$

Substituting this into (4.6),

$$\begin{aligned}\Big\langle \int_{-n}^{n}(k_n*\psi)^\wedge(t)R_{-t}f\ dt,\ \xi\Big\rangle &= \int_{|t|\le n} e^{-iat}(k_n*\psi)^\wedge(t)\langle f,\xi\rangle dt \\ &= +i\int_{|t|\le n} t(k_n*\psi)^\wedge(t)\int_{|\lambda|\le a} e^{-i\lambda t}\langle E(\lambda)f,\xi\rangle d\lambda\ dt \\ &= 2\pi k_n*\psi(-a)\langle f,\xi\rangle \ + \ 2\pi\int_{|\lambda|\le a}(k_n*\psi)'(-\lambda)\langle E(\lambda)f,\xi\rangle d\lambda \\ &= 2\pi\int_{|\lambda|\le a}(k_n*\psi)(-\lambda)\langle dE(\lambda)f,\xi\rangle, \quad \text{integrating by parts,} \\ &= 2\pi\Big\langle \int_{|\lambda|\le a}(k_n*\psi)(-\lambda)dE(\lambda)f,\xi\Big\rangle.\end{aligned}$$

As this holds for all $f\in\bigcup_{a\ge1}\{E(a)-E(-a)\}L^p_X(\mathbb{R})$ and $\xi\in L^{p'}_{X^*}(\mathbb{R})$ we deduce that for all $n\ge1$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{k_n}\hat{\psi}(t)R_{-t}dt = \int_{-\infty}^{\infty}k_n*\psi(-\lambda)dE(\lambda). \tag{4.7}$$

Combining this with equation (4.5) we deduce that

$$\Big\|\int\Psi(-\lambda)dE(\lambda)f - \frac{1}{2\pi}\int\hat{k_n}\hat{\psi}(t)R_{-t}f\ dt\Big\|_{L^p_X}\to 0 \ \text{ for all } f\in L^p_X. \tag{4.8}$$

Now let $f\in C_c^\infty(X)\subset L^p_X(\mathbb{R})$. Since the $\|\cdot\|_{L^p_X}$ convergence in (4.8) holds, there exists a subsequence $\{n_j\}$ such that, *a.e.*(s),

$$\Big\|\int\Psi(-\lambda)dE(\lambda)f(s) - \frac{1}{2\pi}\int\hat{k_n}\hat{\psi}(t)R_{-t}f(s)dt\Big\|_X\to 0. \tag{4.9}$$

However, for any $s\in\mathbb{R}$,

$$\begin{aligned}&\Big\| \ \int(k_n*\psi)^\wedge(t)R_{-t}f(s)dt - \int\hat{\psi}(t)R_{-t}f(s)dt\Big\|_X \\ = \ &\Big\|\int\big\{(k_n*\psi)^\wedge(t)-\hat{\psi}(t)\big\}f(s+t)dt\Big\|_X \le \sup_t|(k_n*\psi)^\wedge(t)-\hat{\psi}(t)|\ \|f\|_{L^1_X} \\ \le \ &\|k_n*\psi-\psi\|_{L^1(\mathbb{R})}\|f\|_{L^1_X}\to 0.\end{aligned}$$

The last convergence is due to $\{k_n\}$ being an approximate identity. Combining this with (4.9) we deduce that

$$\frac{1}{2\pi}\int\hat{\psi}(t)R_{-t}f dt = \int\Psi(-\lambda)dE(\lambda)f \ \text{ for } f\in C_c^\infty(X).$$

But as noted in (4.1), $\int \Psi(-\lambda)dE(\lambda)$ is indeed a bounded operator, and so $\int \hat{\psi}(t)R_{-t}dt$ extends to a bounded linear operator on all of $L^p_X(\mathbb{R})$, with $\frac{1}{2\pi}\int \hat{\psi}(t)R_{-t}dt = \int \Psi(-\lambda)dE(\lambda)$. But we have already shown in (4.2) that $\int \hat{\psi}(t)R_{-t}dt = S_\psi$ on $C_c^\infty(X)$, so that S_ψ is also a bounded operator on $L^p_X(\mathbb{R})$ and

$$\frac{1}{2\pi}S_\psi = \int_{\mathbb{R}} \Psi(-\lambda)dE(\lambda). \tag{4.10}$$

Let T_ψ denote the bounded map on the right side of (**??**). These calculations show that $T_\psi = S_\psi$ for $\psi \in BV_q \cap L^1(\mathbb{R})$. Now let $\psi \in BV_q(\mathbb{R})$. We wish to show that for $f \in C_c^\infty \otimes X$, $T_\psi f = (\psi\hat{f})^\vee$. Thence $T_\psi = S_\psi$ on a dense subspace of L^p_X, so that S_ψ is a bounded linear map, i.e. $\psi \in M_{p,X}(\mathbb{R})$. To this end, let us go back to the definition of S_ψ. For $f \in C_c^\infty \otimes X$, $f = \sum_1^N f_k x_k$ we had $S_\psi(f) = \sum_1^N (\psi\hat{f}_k)^\vee x_k$. Let us take a sequence $\{\psi_n\} \in BV_q \cap L^1(\mathbb{R})$ such that $\|\psi_n - \psi\|_{BV_q} \to 0$. Then the first part of the proof shows that for each $f \in C_c^\infty \otimes X$ we have

$$T_{\psi_n} f = \int \Psi_n(-\lambda)dE(\lambda)f = \frac{1}{2\pi}S_{\psi_n} f = \frac{1}{2\pi}\sum_1^N (\psi_n\hat{f}_k)^\vee x_k. \tag{4.11}$$

Claim For each $f \in C_c^\infty \otimes X$,

$$\sum_1^N (\psi_n\hat{f}_k)^\vee x_k(s) \to \sum_1^N (\psi\hat{f}_k)^\vee x_k(s) \quad a.e.(s).$$

To prove the Claim, let us fix $1 < r < \infty$ such that $|\frac{1}{r} - \frac{1}{2}| < \frac{1}{q}$. For each $k = 1, ..., N$, we have $f_k \in C_c^\infty \subset L^r(\mathbb{R})$. Since [18] says that $BV_q(\mathbb{R}) \subset M_{r,\mathbb{C}}(\mathbb{R})$, we know that $\{\psi_n\}, \psi \in M_{r,\mathbb{C}}(\mathbb{R})$, and that $\|\psi_n - \psi\|_{M_r} \le C_{r,q}\|\psi_n - \psi\|_{BV_q}$, for some constant $C_{r,q}$. Hence, for each $k \ge 1$, $\|(\psi_n\hat{f}_k)^\vee - (\psi_n\hat{f}_k)^\vee\|_{L^r(\mathbb{R})} \le C_{r,q}\|\psi_n - \psi\|_{BV_q} \to 0$. So there exist for each $k = 1, ...N$ subsets $\Omega_k \subset \mathbb{R}$ of full measure such that

$$(\psi_n\hat{f}_k)^\vee(s) \to (\psi_n\hat{f}_k)^\vee(s) \text{ for all } s \in \Omega_k.$$

Hence for $s \in \bigcap_{k=1}^N \Omega_k$ we have $\sum_1^N (\psi_n\hat{f}_k)^\vee(s) \to \sum_1^N (\psi\hat{f}_k)^\vee(s)$, and the Claim is proved. Combining this with (4.11) shows that $T_{\psi_n} f$ converges $a.e$ to $S_\psi f$.

Now, since each $\psi_n \in BV_q \cap L^1(\mathbb{R})$, the first part of the proof gives that $(\psi_n\hat{f})^\vee = T_{\psi_n} f$ and $\|T_{\psi_n} f - T_\psi f\|_{L^p_X} \to 0$. Thus $\{T_{\psi_n} f\}$ converges $a.e.$ to $S_\psi f$ and to $T_\psi f$ in $\|\cdot\|_{L^p_X}$ norm, so that $T_\psi f = S_\psi f$. Hence S_ψ extends to a bounded linear operator on all of L^p_X, and $\|S_\psi\| = \|2\pi \int \Psi(-\lambda)dE(\lambda)\| \le 4\pi C_{p,q} var_{r_p}(E)\|\psi\|_{BV_q}$. □

An Alternative Proof

It is worth noting that there is a shorter proof of Theorem 4.2. However, it does not give the explicit description of S_ψ as $\int \Psi(-\lambda)dE(\lambda)$.

Proof. In [18] Coifman and Rubio introduced the spaces R_s, $1 < s < \infty$, which are defined as follows. Let us write $\mathbb{I}_A$ for the indicator function of a half-open interval $A = (a, b]$.

Definition 4.1. *For* $1 < s < \infty$ *let*

$$\mathcal{R}_s = \{f = \sum_{k\geq 1} \lambda_k \mathbb{I}_{A_k} \ : \ \sum_{k\geq 1} |\lambda_k|^s \leq 1, \quad \{A_k\} \textit{ are disjoint } \}.$$

The space

$$R_s = \{f = \sum_{j\geq 1} \alpha_j g_j \ : \ g_j \in \mathcal{R}_s, \ \ \sum_{j\geq 1} |\alpha_j| < \infty\},$$

is a Banach space under the norm $\|f\|_{R_s} = \inf\{\sum |\alpha| \ : \ f = \sum_j \alpha_j g_j\}$.

Note that in particular, if $f \in \mathcal{R}_s$, then $\|f\|_{R_s} \leq 1$. In [18] it is proved that

$$BV_q \subset R_s \quad \text{for} \ \ 1 < q < s < \infty. \tag{4.12}$$

We are ready to start the proof. As before, let $1 < r_p < \infty$ be such that $var_{r_p}(E) < \infty$, and suppose $\frac{1}{r_p} + \frac{1}{q_p} = 1$. Let $\psi = \sum_k \lambda_k \mathbb{I}_{A_k} \in \mathcal{R}_{q_p}$, with $A_k = (a_k, b_k]$ being disjoint intervals satisfying $b_k < a_{k+1}$. Note that for $A = (a, b]$, the associated multiplier on L^p_X is $S_{\mathbb{I}_A} = E(b) - E(a)$, where E is the spectral family of the right translation $\{R_t\}$. Hence $S_\psi = \sum_k \lambda_k\{E(b_k) - E(a_k)\}$. This is indeed a bounded operator, since, using Holder's inequality we have

$$\begin{aligned} \|S_\psi f\|_{L^p_X} \ &\leq \ \Big\{\sum_k |\lambda|^{q_p}\Big\}^{1/q_p} \Big\{\sum_k \|\{E(b_k) - E(a_k)\}f\|^{r_p}_{L^p_X}\Big\}^{1/r_p} \\ &\leq \ \ var_{r_p}(E(\cdot)f) \ \ \leq \ \|f\|_{L^p_X} \ var_{r_p}(E). \end{aligned}$$

Hence $\|S_\psi\| \leq var_{r_p}(E)$. Now suppose $\phi = \sum_1^N \alpha_k \psi_k \in R_{q_p}$. Then

$$S_\phi = \sum_1^N \alpha_k S_{\psi_k}, \quad \text{and so } \|S_\phi\| \leq \ var_{r_p}(E) \sum_1^N |\alpha_k|.$$

Since this holds for any such representation of ϕ it follows that $\|S_\phi\| \leq \ var_{r_p}(E)\|\phi\|_{R_{q_p}}$. Hence every $\phi \in R_{q_p}$ is an $L^p_X(\mathbb{R})$-multiplier. In other words, $R_{q_p} \subset M_{p,X}(\mathbb{R})$. Now using inclusion (4.12) we deduce that

$$BV_q(\mathbb{R}) \subset R_{q_p} \subset M_{p,X}(\mathbb{R}) \ \text{ for all } \ q \in (1, q_p).$$

As noted earlier, this is a shorter proof of Theorem 4.2, but does not establish the description of S_ψ as $2\pi \int \Psi(-\lambda)dE(\lambda)$. □

4.3 $\mathfrak{M}_q(\mathbb{R})$ Multiplier Theorem and a Conjecture

In the scalar valued setting, the $BV_q(\mathbb{R})$ multiplier theorem for $L^p(\mathbb{R})$ combines with Littlewood-Paley theory to give rise to $\mathfrak{M}_q$ multipliers (for values of $q \in (1, \infty)$ satisfying $|1/p - 1/2| < 1/q$); we refer the reader to [18] for the full details. Furthermore, the account in [14] extends the $\mathfrak{M}_q$ multiplier theorem to vector valued spaces L^p_X, provided X is of class $\mathfrak{I}$ (see the definition in chapter 1). This is a special type of UMD space.

We can now show that, provided certain conditions hold, the $\mathfrak{M}_q$ multiplier theorem will hold for any UMD space X. The condition in question relates to *R-boundedness*, defined herein.

Definition 4.2. *Let X be a Banach space, $\mathcal{T} \subset \mathcal{B}(X)$, and let $\{r_k\}_{k\geq 0}$ be the Rademacher functions on $[0,1]$. $\mathcal{T}$ is **R-bounded** if there exists a constant $K > 0$ such that for all finite sequences $x_1, ..., x_N \in X$ and $T_1, ..., T_N \in \mathcal{T}$ the following inequality holds:*

$$\int_0^1 \Big\| \sum_1^N r_k(t) T_k x_k \Big\| dt \leq K \int_0^1 \Big\| \sum_1^N r_k(t) x_k \Big\| dt.$$

*K is called **an R-bound** for $\mathcal{T}$. The infimum over such K is **the R-bound**.*

Proposition 4.1. *Let $\mathcal{T} \subset \mathcal{B}(X)$. For $1 \leq p < \infty$ define*

$$aco_p(\mathcal{T}) = \Big\{ \sum_1^N \alpha_k T_k \ : \ \{T_k\} \subset \mathcal{T}, \ \ \sum_1^N |\alpha|^p \leq 1 \Big\}.$$

(i) If $\mathcal{T}$ is R-bounded, then so is $aco_1(\mathcal{T})$. Moreover, if K is an R-bound of $\mathcal{T}$, then $2K$ is an R-bound of $aco_1(\mathcal{T})$.

(ii) If $\mathcal{T}$ is R-bounded, then so is $\overline{\mathcal{T}}$, the closure of $\mathcal{T}$ in the strong operator topology.

Proof. See [31] □

Now let X be UMD and $L^p_X \equiv L^p_X(\mathbb{R})$, where $1 < p < \infty$. Let $\{I_j\}$ denote the dyadic intervals on $\mathbb{R}$ under the usual enumeration. That is, $s_j = 2^{j-1}$ for $j > 0$ and $s_j = -\frac{1}{2^j}$ for $j \leq 0$. Then $I_j = [s_j, s_{j+1})$ for $j > 0$, $I_j = (s_j, s_{j+1}]$ for $j < 0$ and $I_0 = (s_0, s_1)$. Let $\mathfrak{M}_q(\mathbb{R})$ be the Marcinkiewicz q-class of functions on $\mathbb{R}$. Note that if $\psi \in \mathfrak{M}_q(\mathbb{R})$, then $\psi_k \equiv \mathbb{I}_k \psi \in BV_q(\bar{I}_k)$, where $\mathbb{I}_k$ is the indicator function of the *closed* interval $\bar{I}_k$. Let S_j denote the multiplier operator on L^p_X associated with $\mathbb{I}_j$. Thus, if $\{E(\lambda)\}$ is the spectral family of the right translation group $\{R_t\}_{t\in\mathbb{R}}$, then $S_j = \{E(s_{j+1}) - E(s_j)\}$ and for every $f \in L^p_X$, we have $f = \sum_{-\infty}^{\infty} S_j f$ unconditionally (see, for example, [6]).

Proposition 4.2. *Let X be UMD and $1 < p < \infty$. Let $\{E(\lambda)\}$ be the spectral family of the right translation group on L^p_X. Then $\mathcal{T} \equiv \{ E(\lambda) : \lambda \in \mathbb{R} \}$ is R-bounded.*

Proof. See [6]. □

Proposition 4.3. *Let $1 < p < \infty$. There exists a constant $C_p > 0$ such that for all $f \in L^p_X$*

$$C_p \|f\|_{L^p_X} \;\le\; \int_0^1 \Big\| \sum_{-\infty}^{\infty} r_k(t) S_k f \Big\|_{L^p_X} dt \;\le\; \frac{1}{C_p} \|f\|_{L^p_X}.$$

Proof. See [31]. □

In trying to extend the $BV_q(\mathbb{R})$ multiplier theorem to $\mathfrak{M}_q(\mathbb{R})$ we require R-boundedness of a certain subset of $aco_q(\mathcal{T})$, which we now describe. Let $\psi \in \mathfrak{M}_q(\mathbb{R})$ with $\|\psi\|_{\mathfrak{M}_q} \le \frac{1}{2}$, and let $u \in \mathcal{P}_{\mathbb{R}}$ be a partition which includes the dyadic points $\{s_k\}$. Writing $u_k = \{s_k = \lambda_0 < ... < \lambda_N = s_{k+1}\} \in \mathcal{P}_{[s_k, s_{k+1}]}$, we define $\mathcal{S}(\psi_k, u_k) = \sum_1^N \psi(\lambda_j)\{E(\lambda_j) - E(\lambda_{j-1})\}$. In order to establish an $\mathfrak{M}_q(\mathbb{R})$ multiplier result, we require that the set $\{ \mathcal{S}(\psi_k, u_k) : k \in \mathbb{Z},\ u \in \mathcal{P}_{\mathbb{R}}\}$ be R-bounded. Now, by rearrangement we can write $\mathcal{S}(\psi_k, u_k) \equiv \psi(s_{k+1})E(s_{k+1}) - \psi(s_k)E(s_k) - \sum_1^N \{\psi(\lambda_{j+1}) - \psi(\lambda_j)\}E(\lambda_j)$. Hence $\mathcal{S}(\psi_k, u_k)$ is a member of $aco_q\Big(\{E(\lambda)\}\Big)$, because

$$\begin{aligned} |\psi(s_k)|^q + |\psi(s_{k+1})|^q \;&+\; \sum_1^N |\psi(\lambda_{j+1}) - \psi(\lambda_j)|^q \\ &\le\; 2 \sup_\lambda |\psi(\lambda)|^q + var_q(\psi, \bar{I}_k)^q \\ &\le\; 2\big\{ \sup_\lambda |\psi(\lambda)| + var_q(\psi, \bar{I}_k) \big\}^q \;\le\; 2\|\psi\|_{\mathfrak{M}_q(\mathbb{R})} \;\le\; 1. \end{aligned}$$

It may not be the case that $aco_q\big(\{E(\lambda)\}\big)$ is in general R-bounded (for appropriate values $1 < q < \infty$). However, at least in a Hilbert space, we can show that the set $\{ \mathcal{S}(\psi_k, u_k) : k \in \mathbb{Z},\ u \in \mathcal{P}_{\mathbb{R}}\}$ is R-bounded. The Proposition below is a rather more straightforward result than at first seems, because in a Hilbert space R-boundedness is equivalent to uniform boundedness. The reason for the formal-looking setup and proof is that it motivates the Conjecture which follows.

Proposition 4.4. *Let H be a Hilbert space, and let $E(\lambda)$ be the spectral family of the right translation group $\{R_t\}_{t\in\mathbb{R}}$ on L^2_H, so that $var_2(E) < \infty$. As before let $\mathcal{T} \equiv \{ E(\lambda) : \lambda \in \mathbb{R} \}$. Then for any $1 < q < 2$, and $\psi \in \mathfrak{M}_q(\mathbb{R})$, the set $\mathcal{C}_\psi = \{ \mathcal{S}(\psi_k, u_k) : k \in \mathbb{Z},\ u \in \mathcal{P}_{\mathbb{R}}\}$ is R-bounded.*

Proof. Without loss of generality, assume that $\|\psi\|_{\mathfrak{M}_q} \leq 1$. Due to Kahane's inequality, it will suffice to find a constant $C > 0$ such that for all $x_1, ..., x_N \in L^2_H$ and $W_1, ..., W_N \in \mathcal{C}_\psi$ we have

$$\Big\{ \int_0^1 \Big\| \sum_1^N r_k(t) W_k x_k \Big\|^2_{L^2_H} dt \Big\}^{1/2} \leq C \Big\{ \int_0^1 \Big\| \sum_1^N r_k(t) x_k \Big\|^2_{L^2_H} \Big\}^{1/2} dt. \tag{4.13}$$

However, since L^2_H is a Hilbert space itself, (4.13) reduces to showing

$$\sum_1^N \|W_k x_k\|^2_{L^2_H} \leq C^2 \sum_1^N \|x_k\|^2_{L^2_H}.$$

This in turn is equivalent to showing that $\mathcal{C}_\psi$ is uniformly bounded, that is $\|\mathcal{S}(\psi_k, u_k)\| \leq C$ for all $\mathcal{S}(\psi_k, u_k) \in \mathcal{C}_\psi$. But this follows easily from Young's inequality. Let $x, y \in L^2_H$. From the definition of $\mathcal{S}(\psi_k, u_k)$ we have

$$\langle \mathcal{S}(\psi_k, u_k)x, y\rangle = \sum_1^N \psi(\lambda_j) \langle \{E(\lambda_j) - E(\lambda_{j-1})\}x, y\rangle.$$

Writing $K_q = \big\{1 + \zeta(1/2 + 1/q)\big\}$, Young's inequality ([32], 6.2) gives

$$\begin{aligned} \big|\langle \mathcal{S}(\psi_k, u_k)x, y\rangle \;&-\; \langle E(s_{k+1})x, y\rangle \{\psi(s_{k+1}) - \psi(s_k)\}\big| \\ &\leq\; K_q var_q(\psi_k)\ var_2(\langle E(\cdot)x, y\rangle) \\ &\leq\; K_q\ var_q(\psi_k)\ var_2(E) \|x\|_{L^2_H}\ \|y\|_{L^2_H}. \end{aligned}$$

Hence, using the triangle inequality in the left term, and the fact that $var_q(\psi_k) \leq \|\psi\|_{\mathfrak{M}_q} \leq 1$,

$$\begin{aligned} |\langle \mathcal{S}(\psi_k, u_k)x, y\rangle| \;&\leq\; \|x\|_{L^2_H}\ \|y\|_{L^2_H} \{\ \|\psi\|_{\mathfrak{M}_q} \|E\|_\infty + K_q \|\psi\|_{\mathfrak{M}_q} var_2(E)\} \\ &\leq\; \|x\|_{L^2_H}\ \|y\|_{L^2_H} \{\ \|E\|_\infty + K_q var_2(E)\}. \end{aligned}$$

Hence $\|\mathcal{S}(\psi_k, u_k)\| \leq \{\ \|E\|_\infty + K_q var_2(E)\}$ for any $\mathcal{S}(\psi_k, u_k) \in \mathcal{C}_\psi$, and so (4.13) is satisfied with $C = \{\|E\|_\infty + K_q var_2(E)\}$. □

Proposition 4.4 motivates us to conjecture that a similar result is true for any UMD space X and any $1 < p < \infty$. More precisely, we conjecture the following.

Conjecture 4.1. *Let X be UMD and $1 < p < \infty$. Let $E(\lambda)$ be the spectral family of the right translation group on L^p_X and set $\mathcal{T} \equiv \{\ E(\lambda) : \ \lambda \in \mathbb{R}\ \}$. Suppose $1 < r_p < \infty$ is such that $var_{r_p}(E) < \infty$. Then for any $1 < q < r_p'$ and $\psi \in \mathfrak{M}_q(\mathbb{R})$ with $\|\psi\|_{\mathfrak{M}_q(\mathbb{R})} \leq 1$, the set $\mathcal{C}_\psi = \{\ \mathcal{S}(\psi_k, u_k) : \ k \in \mathbb{Z},\ u \in \mathcal{P}_\mathbb{R}\}$ is R-bounded, with an R-bound depending only on q and X.*

Using Proposition 4.1 (ii) we can immediately deduce the following from Conjecture 4.1

Conjecture 4.2. *Let the setup be the same as in Conjecture 4.1. Then the set* $\{ \int_{\mathbb{R}} \psi_k(\lambda) dE(\lambda) : \ k \in \mathbb{Z} \}$ *is R-bounded, with the same R-bound as* $\mathcal{C}_\psi$.

That this follows from Conjecture 4.1, simply note that the set $\{ \int_{\mathbb{R}} \psi_k(\lambda) dE(\lambda) : k \in \mathbb{Z} \}$ is the closure in the strong operator topology of $\mathcal{C}_\psi$. Proposition 4.1 (ii) now yields the desired conclusion. With this at hand we can state an $\mathfrak{M}_q(\mathbb{R})$ multiplier result.

Proposition 4.5. *Let X be a UMD space and $1 < p < \infty$.* ***Suppose that Conjecture 4.2 holds****. Then there exists $1 < q_p < \infty$ such that for all $1 < q < q_p$*

$$\mathfrak{M}_q(\mathbb{R}) \subset M_{p,X}(\mathbb{R}).$$

The proof of Proposition 4.5 requires the following technical Lemma.

Lemma 4.4. *Let $\{F(\lambda)\}$ be any spectral family on an arbitrary reflexive space Y. Let $\phi \in BV(\mathbb{R})$. Suppose F is strongly continuous at $c \in (a, b)$. Then* $\int_a^b \phi(\lambda) dF(\lambda) = \int_a^c \phi(\lambda) dF(\lambda) + \int_c^b \phi(\lambda) dF(\lambda)$.

Proof. Let $\mathcal{P}_{[a,b]}$ be the set of partitions of $[a, b]$ containing c, so $u \in \mathcal{P}_{[a,b]}$ looks like $u = \{a = \lambda_0 < ... < c = \lambda_M < ... < \lambda_N\}$. For any $x \in Y$, $\displaystyle\int_a^b \phi(\lambda) dF(\lambda)x = \lim_{u \in \mathcal{P}_{[a,b]}} \Big\{ \sum_1^N \phi(\lambda_j)\{F(\lambda_j) - F(\lambda_{j-1})\}x \Big\} = \lim_{u \in \mathcal{P}_{[a,b]}} \mathcal{S}(\psi, u)x$. We can now split this up:

$$\mathcal{S}(\psi, u) = \sum_1^M \phi(\lambda_j)\{F(\lambda_j) - F(\lambda_{j-1})\}x \tag{4.14}$$

$$+ \sum_M^N \phi(\lambda_j)\{F(\lambda_j) - F(\lambda_{j-1})\}x \ - \ \phi(c)\{F(c) - F(\lambda_{M-1})\}x \tag{4.15}$$

Due to strong continuity of F at c we have $\lim_{u \in \mathcal{P}_{[a,b]}} F(\lambda_{M-1})x = \lim_{\lambda \to c^-} F(\lambda)x = F(c)x$. Hence, taking limits in (4.14) as u runs through $\mathcal{P}_{[a,b]}$ gives $\int_a^b \phi(\lambda) dF(\lambda)x = \int_a^c \phi(\lambda) dF(\lambda)x + \int_c^b \phi(\lambda) dF(\lambda)x$. □

Proof of Proposition 4.5

Since X is UMD, and so super-reflexive, we know that the spectral family $\{E(\lambda)\}$ of $\{R_t\}_{t \in \mathbb{R}}$ has bounded r_p-variation, for some $1 < r_p < \infty$. Let q_p be its conjugate index, and let us fix $q \in (1, q_p)$. It will suffice to show that for each $\psi \in B_{\mathfrak{M}_q(\mathbb{R})}$ and $f \in L^p_X$,

$$\lim_{b \to \infty} \int_{-b}^b \psi(\lambda) dE(\lambda) \equiv T_\psi f \ \text{ exists.}$$

and $\|T_\psi f\|_{L^p_X} \le \gamma \|f\|_{L^p_X}$ for some $\gamma > 0$. Let $T_{\psi_k} = \int_{I_k} \psi(\lambda) dE(\lambda) = \int_{\mathbb{R}} \psi_k(\lambda) dE(\lambda)$. Then $T_{\psi_k} \in \mathcal{B}(L^p_X)$ by the $BV_q(\mathbb{R})$ multiplier Theorem 4.2. Note that

$$T_{\psi_k} S_j = S_j T_{\psi_k} = \begin{matrix} T_{\psi_k} & \text{for } k = j \\ 0 & \text{for } k \neq j \end{matrix} . \tag{4.16}$$

So for $f = \sum_{-\infty}^{\infty} S_j f = \sum_{-\infty}^{\infty} f_j$, we have $T_{\psi_k} f = T_{\psi_k} f_k$. Now, if Conjecture 4.2 holds, let K_q be an R-bound of the set $\{T_{\psi_k}\}_{k \in \mathbb{Z}}$. We shall first show that $\sum_{-\infty}^{\infty} T_{\psi_k} f$ converges. It suffices to show that the balanced partial sums form a Cauchy sequence. To that end we apply (4.16), Conjecture 4.2 and Proposition 4.3 twice.

$$\begin{aligned} \Big\| \sum_{M<|k|\le L} T_{\psi_k} f \Big\|_{L^p_X} &\le \frac{1}{C_p} \int_0^1 \Big\| \sum_{j=-\infty}^{\infty} r_j(t) S_j \Big\{ \sum_{M<|k|\le L} T_{\psi_k} f \Big\} \Big\|_{L^p_X} dt \\ = \frac{1}{C_p} \int_0^1 \Big\| \sum_{M<|j|\le L} r_j(t) T_{\psi_j} f_j \Big\|_{L^p_X} dt &\le \frac{K_q}{C_p} \int_0^1 \Big\| \sum_{M<|j|\le L} r_j(t) f_j \Big\|_{L^p_X} dt \\ = \frac{K_q}{C_p} \int_0^1 \Big\| \sum_{M<|j|\le L} r_j(t) S_j f \Big\|_{L^p_X} dt &\le \frac{K_q}{C_p^2} \Big\| \sum_{M<|j|\le L} S_j f \Big\|_{L^p_X}. \end{aligned} \tag{4.17}$$

But the balanced partial sums $\sum_{-N}^{N} S_j f$ converge to f and so form a Cauchy sequence. Hence $\Big\| \sum_{M<|j|\le L} S_j f \Big\|_{L^p_X} \to 0$ as $L > M \to \infty$. Thus $\Big\{ \sum_{-N}^{N} T_{\psi_k} f \Big\}$ is also Cauchy and so converges. Moreover, we see from (4.17) that $\Big\| \sum_{-\infty}^{\infty} T_{\psi_k} f \Big\|_{L^p_X} \le \frac{K_q}{C_p^2} \|f\|_{L^p_X}$.

Next we show that the limit $T_\psi f$ exists and $T_\psi f = \sum_{-\infty}^{\infty} T_{\psi_k} f$. To show this, let $b > s_1$ and pick N such that $s_N < b \le s_{N+1}$ and $s_{-N} \le -b < s_{-N+1}$ (such N exists uniquely, because $s_{-k} = -s_{k+1}$ for $k \ge 1$). Since E is strongly continuous (see Corollary 5.1 in chapter 5), we can use Lemma 4.4 to write

$$\int_{-b}^{b} \psi \, dEf = \int_{s_N}^{b} \psi \, dEf + \int_{s_{-N+1}}^{s_N} \psi \, dEf + \int_{-b}^{s_{-N+1}} \psi \, dEf. \tag{4.18}$$

Applying Lemma 4.4 to the dyadic points $\{s_k\}$, we have

$$\int_{s_{-N+1}}^{s_N} \psi \, dEf = \sum_{k=-N+1}^{N-1} \int_{\mathbb{R}} \psi_k \, dEf = \sum_{k=-N+1}^{N-1} T_{\psi_k} f$$

Substituting this into (4.18) gives

$$\begin{aligned} \Big\| \int_{-b}^{b} \psi \, dE - \sum_{k=-N+1}^{N-1} T_{\psi_k} f \Big\|_{L^p_X} &\le \Big\| \int_{s_N}^{b} \psi \, dEf + \int_{-b}^{s_{-N+1}} \psi \, dEf \Big\|_{L^p_X} \\ &= \Big\| \int_{s_N}^{b} \psi \, dEf_N + \int_{-b}^{s_{-N+1}} \psi \, dEf_{-N} \Big\|_{L^p_X} \\ &\le K_{r_p,q} \, var_{r_p}(E) \, \{ \|f_N\|_{L^p_X} + \|f_{-N}\|_{L^p_X} \}. \end{aligned}$$

Here $K_{r_p,q} = 1 + \zeta(1/r_p + 1/q)$. Note that $N \to \infty$ as $b \to \infty$, and also $\|f_N\|_{L^p_X}$, $\|f_{-N}\|_{L^p_X} \to 0$ as $N \to \infty$. Hence, letting $b \to \infty$ in the last inequality we get $T_\psi f = \sum_{-\infty}^{\infty} T_{\psi_k} f$ as claimed. Finally, (4.17) gives $\|T_\psi f\|_{L^p_X} \le \frac{K_q}{C_p^2} \|f\|_{L^p_X}$ $\Box$.

CHAPTER 5

In this Chapter we shall address the relationship between spectral families and densely defined one-parameter operator groups. We shall, in particular, look at
(i) X valued function spaces S_X with the right translation group $\{R_t\}_{t\in\mathbb{R}}$;
(ii) X valued sequence spaces with the right shift operators $\{R^k\}_{k\in\mathbb{Z}}$.
However, we shall first treat more general spaces X and densely defined groups $\{U_t\}_{t\in\mathbb{R}}$.

5.1 Unbounded Operator Groups

It is shown in [5] that, provided $\{U_t\}_{t\in\mathbb{R}}$ is a strongly continuous operator group, it has an associated spectral family if and only if there is a constant $\gamma > 0$ such that $\| \int_{\mathbb{R}} \phi(t)U_{-t}dt\| \leq \gamma\|\hat{\phi}\|_{BV}$ for every $\phi \in C_c^\infty$ (see Theorem 5.1 below). This result does not generalize if we drop the strong continuity of $\{U_t\}_{t\in\mathbb{R}}$, but instead merely stipulate that each U_t is densely defined. In fact, in this situation the result is not true, and we shall give an example to demonstrate this. We begin with some definitions.

Definition 5.1. *We say that $(U_t)_{t\in\mathbb{R}}$ is a **densely defined one-parameter group** of operators on X if:*
(i) $U_0 = I_X$;
(ii) $X_M \equiv \{x \in \bigcap_{t\in\mathbb{R}} \mathcal{D}(U_t) : \ t \mapsto U_t x$ is continuous$\}$ is dense in X;
(iii) for all $s, t \in \mathbb{R}$, we have $U_sU_t \subset U_{s+t}$ and $\mathcal{D}(U_sU_t) = \mathcal{D}(U_{s+t}) \cap \ \mathcal{D}(U_t)$.

Definition 5.2. *We say that $(U_t)_{t\in\mathbb{R}}$ has a **densely defined spectral decomposition** if there exists a spectral family E on X such that for each $t \in \mathbb{R}$ and $x \in \mathcal{D}(U_t)$ we have*

$$U_t x = \lim_{a\to\infty} \int_{-a}^{a} e^{i\lambda t} dE(\lambda)x. \tag{5.1}$$

If it happens that $\mathcal{D}(U_t) = X$ for each t, we drop the 'densely defined' phrase, and say simply that $\{U_t\}$ has a spectral decomposition. As mentioned above, the motivating result in this chapter is the following Theorem in [5]. We state it precisely:

Theorem 5.1. *Let $\{U_t\}_{t\in\mathbb{R}}$ be a strongly continuous one-parameter group of operators. Then the following two statements are equivalent:*
(i) $\{U_t\}$ has a spectral decomposition.
(ii) There exists a constant $\gamma > 0$ such that:

$$\Big\| \int \phi(t)U_{-t}dt \Big\| \leq \|\hat{\phi}\|_{BV} \quad \textit{for all} \ \ \phi \in C_c^\infty(\mathbb{R}),$$

and the set

$$\{ \int \phi(t)U_{-t}dt \ : \ \phi \in C_c^\infty(\mathbb{R}), \ \|\hat{\phi}\|_{BV} \leq 1 \}$$

is relatively compact in the weak-operator topology.

If the space X is reflexive, the second condition can be dropped.

In view of Definitions 5.1 and 5.2, it is natural to ask in what sense Theorem 5.1 can be generalized to densely defined operator groups of unbounded operators. The first point to note is that if we drop the notion of strong continuity of $\{U_t\}$, it does not even make sense to talk about $\int \phi(t)U_{-t}dt$. However, Definition 5.1 (ii) does imply that $\int \phi(t)U_{-t}x\ dt \in X$ is well defined if $x \in X_M$. Then, if it happens that there is a constant $A > 0$ such that $\| \int \phi(t)U_{-t}x\ dt\| \leq A\|x\|$ for all $x \in X_M$, we denote the linear extension of $\int \phi(t)U_{-t}dt$ to all of X with $\hat{\phi}(U)$. We state in two separate parts the weaker result which holds for densely defined operator groups.

Theorem 5.2. *Let E be a spectral family on a reflexive space X. Define*

$$\mathcal{D}(U_t) \ = \{ \ x \in X \ : \ \lim_{a\to\infty} \int_{-a}^{a} e^{i\lambda t} dE(\lambda)x \ \textit{exists} \ \}; \qquad (5.2)$$

$$\textit{and for } x \in \ \mathcal{D}(U_t) \ \ , \quad U_t x = \lim_{a\to\infty} \int_{-a}^{a} e^{i\lambda t} dE(\lambda)x.$$

Then $(U_t)_{t\in\mathbb{R}}$ is a densely defined one-parameter group of closed operators on X with $X_M \supseteq X_0$, where $X_0 = \bigcup_{n\geq 1}\{E(n) - E(-n)\}X$.
Further, there exists a constant $\gamma > 0$ such that for each $\phi \in C_c^\infty(\mathbb{R})$ the operator $\int \phi(t)U_{-t}dt : X_0 \to X$ has a bounded extension $\hat{\phi}(U)$ on all of X with

$$\|\hat{\phi}(U)\| \leq \gamma \|\hat{\phi}\|_{BV}. \qquad (5.3)$$

Remark
Observe that the convergence in (5.2) can in fact be restricted to intervals $[-n, n]$ with $n \in \mathbb{N}$. To show this, let $\lfloor a \rfloor$ denote the floor of any $a \in \mathbb{R}^+$ and $\lceil a \rceil$ the ceiling. So suppose we know that $\lim_{n\to\infty} \int_{-n}^{n} e^{i\lambda t} dE(\lambda)x$ exists, with $n \in \mathbb{N}$. Then

$\lim_{n\to\infty}\int_n^{n+1} e^{i\lambda t}dE(\lambda)x + \int_{-n-1}^{-n} e^{i\lambda t}dE(\lambda)x = 0$. Then, writing $\int_{\lfloor a\rfloor}^{\lceil a\rceil} e^{i\lambda t}dE(\lambda)x = (I-E(0))\{\int_{\lfloor a\rfloor}^{\lceil a\rceil} e^{i\lambda t}dE(\lambda)x + \int_{-\lceil a\rceil}^{-\lfloor a\rfloor} e^{i\lambda t}dE(\lambda)x\}$ it follows that $\lim_{a\to\infty}\int_{\lfloor a\rfloor}^{\lceil a\rceil} e^{i\lambda t}dE(\lambda)x = 0$. Similarly, using $E(0)$ in place of $(I-E(0))$, $\lim_{a\to\infty}\int_{-\lceil a\rceil}^{-\lfloor a\rfloor} e^{i\lambda t}dE(\lambda)x = 0$. Now, $\int_{\lfloor a\rfloor}^{a} e^{i\lambda t}dE(\lambda)x = E(a)\int_{\lfloor a\rfloor}^{\lceil a\rceil} e^{i\lambda t}dE(\lambda)x$. Since the family $\{E(\lambda)\}$ is uniformly bounded it follows that $\lim_{a\to\infty}\int_{\lfloor a\rfloor}^{a} e^{i\lambda t}dE(\lambda)x = 0$.

Similarly $\lim_{a\to\infty}\int_{-a}^{-\lfloor a\rfloor} e^{i\lambda t}dE(\lambda)x = 0$. Hence, as

$$\int_{-a}^{a} e^{i\lambda t}dE(\lambda)x = \int_{-a}^{-\lfloor a\rfloor} e^{i\lambda t}dE(\lambda)x \; + \int_{-\lfloor a\rfloor}^{\lfloor a\rfloor} e^{i\lambda t}dE(\lambda)x \; + \int_{\lfloor a\rfloor}^{a} e^{i\lambda t}dE(\lambda)x,$$

it follows that $\lim_{a\to\infty}\int_{-a}^{a} e^{i\lambda t}dE(\lambda)x$ exists, because the outer two terms on the right side vanish.

Theorem 5.3. *Let $(U_t)_{t\in\mathbb{R}}$ be a densely defined one-parameter group of operators on X, satisfying (i) to (iii) of Definition 5.1. Suppose there exists $\gamma > 0$ such that for all $\phi \in C_c^\infty(\mathbb{R})$, the linear map $\int \phi(t)U_{-t}dt$, defined on X_M, has a continuous extension $\hat{\phi}(U)$ which satisfies inequality (5.3). Then there exists a unique spectral family E on X such that*

$$U_t x = \lim_{a\to\infty}\int_{-a}^{a} e^{it\lambda}dE(\lambda)x \quad \textit{for all} \quad x \in X_M. \tag{5.4}$$

We shall need the following simple Lemma in proving Theorem 5.2

Lemma 5.1. *Let $\{y_{mn}\} \in X$ be a doubly indexed sequence such that*
(i) $\lim_{m\to\infty}\lim_{n\to\infty} y_{mn} = y$ exists,
(ii) $\lim_{n\to\infty} y_{mn}$ exists for each m,
(iii) $\lim_{m\to\infty} y_{mn}$ exists, uniformly in n.

Then $\lim_{n\to\infty}\lim_{m\to\infty} y_{mn}$ exists and equals y.

Proof. Let $\lim_{n\to\infty} y_{mn} = y_m$ and $\lim_{m\to\infty} y_{mn} = z_n$. We have to show that $\lim_{n\to\infty} z_n = y$. So, let $\epsilon > 0$ be given. Choose $M \geq 1$ such that $\|y - y_M\| < \frac{\epsilon}{3}$ and $\|y_{Mn} - z_n\| < \frac{\epsilon}{3}$ for all $n \geq 1$. Then choose $N_M \geq 1$ such that $\|y_M - y_{Mn}\| < \frac{\epsilon}{3}$ for all $n \geq N_M$. Hence, for all $n \geq N_M$ we have

$$\|y - z_n\| \leq \|y - y_M\| + \|y_M - y_{Mn}\| + \|y_{Mn} - z_n\| < \epsilon.$$

□

Proof of Theorem 5.2. Let us prove (i) to (iii) in Definition 5.1.
Property (i) is obvious, since, for any $x \in X$ we have $\int_{-a}^{a} dE(\lambda)x = \{E(a) - E(-a)\}x \ \to \ x$.
For (ii), it suffices to show that X_0 satisfies the defining property of X_M. So, let $t \in \mathbb{R}$ and $x \in X_0$, so that $x = \{E(n) - E(-n)\}y$ for some fixed $n \in \mathbb{N}$ and $y \in X$. Then for any $a > n$ we have

$$\int_{-a}^{a} e^{i\lambda t} dE(\lambda)x = \int_{-n}^{n} e^{i\lambda t} dE(\lambda)x.$$

Hence $\lim_{a\to\infty} \int_{-a}^{a} e^{i\lambda t} dE(\lambda)x$ exists, so $x \in \mathcal{D}(U_t)$. Thus $X_0 \subset \mathcal{D}(U_t)$ and this also shows that each U_t is densely defined.
To prove the strong continuity assertion in (ii), again let $x \in X_0$, with $x = \{E(n) - E(-n)\}y$. Then $U_t x = \int_{-n}^{n} e^{i\lambda t} dE(\lambda)x$. Let $\xi \in X^*$. Using integration by parts we have

$$\begin{aligned}
\langle U_s x \ &- \ U_t x, \xi\rangle = \int_{-n}^{n} (e^{i\lambda s} - e^{i\lambda t})\langle dE(\lambda)x, \xi\rangle \\
&= \ \left[(e^{i\lambda s} - e^{i\lambda t})\langle \ E(\lambda)x, \xi\rangle\right]_{-n}^{n} - i\int_{-n}^{n} (se^{i\lambda s} - te^{i\lambda t})\langle \ E(\lambda)x, \xi\rangle d\lambda \\
&= \ (e^{isn} - e^{itn})\langle x, \xi\rangle - i\int_{-n}^{n} (se^{i\lambda s} - te^{i\lambda t})\langle E(\lambda)x, \xi\rangle d\lambda.
\end{aligned}$$

Now, given $\epsilon \geq 0$, we can find $\delta > 0$ such that $|t - s| < \delta$ implies $\sup_{|\lambda|\leq n} |se^{i\lambda s} - te^{i\lambda t}| < \frac{\epsilon}{4n\|E\|_\infty\|x\|}$ and $|e^{isn} - e^{itn}| < \frac{\epsilon}{2\|E\|_\infty\|x\|}$. Then $|t - s| < \delta$ implies $|\langle U_s x - U_t x, \xi\rangle| < 2\epsilon\|x\|.\|\xi\|$ and hence $\|U_s x - U_t x\| < \epsilon$. Thus $\{U_t\}$ is strongly continuous for $x \in X_0$.
To prove (iii) requires more work. Recall that $\mathcal{P}_{[a,b]}$ denotes the collection of partitions of the interval $[a, b]$, directed and partially ordered by inclusion. For $u, v \in \mathcal{P}_{[a,b]}$ we write $u \leq v$ if v refines u. For $f \in BV[a, b]$ and $u = \{a = \lambda_0 < ... < \lambda_N = b\} \in \mathcal{P}[a, b]$ we write

$$\mathcal{S}(f, u, [a, b])x = f(a)E(a)x + \sum_{i=1}^{N} f(\lambda_i)\{E(\lambda_i) - E(\lambda_{i-1})\}x.$$

If $f(\lambda) = e^{is\lambda}$, and $[a, b] = [-a, a]$, we shall use the shorter notation $\mathcal{S}(s, u_a)x$. Also, to make the calculations easier to follow, let us write

$$U_t^{(a)} = \int_{-a}^{a} e^{i\lambda t} dE(\lambda).$$

Note that each $U_t^{(a)}$ is a bounded operator on X with $\|U_t^{(a)}\| \leq 2\|E\|_\infty(1 + a|t|)$.
Let $0 < a < b$ be fixed. For $v_b \in \mathcal{P}[-b, b]$, let v_a denote its restriction to $[-a, a]$ (without loss of generality we can assume that $v_b = \{-b = \lambda_0 < ... < \lambda_K =$

$-a < ... < \lambda_N = a < ... < \lambda_M = b\}$). Let $x \in \mathcal{D}(U_sU_t)$. We have to show that $x \in \mathcal{D}(U_{s+t})$. Since $\mathcal{S}(s, v_a)$ and $\mathcal{S}(t, v_b)$ commute, we have

$$\begin{aligned}
\mathcal{S}(s,v_a)\mathcal{S}(t,v_b)x &= \Big[e^{-itb}E(-b) + \sum_{j=1}^{M} e^{it\lambda_j}\{E(\lambda_j) - E(\lambda_{j-1})\}\Big] \\
&\Big[\ e^{-isa}E(-a) + \sum_{j=K+1}^{N} e^{is\lambda_j}\{E(\lambda_j) - E(\lambda_{j-1})\}\Big]x \\
&= e^{-itb}e^{-isa}E(-b)x + e^{-isa}\sum_{j=1}^{K} e^{it\lambda_j}\{E(\lambda_j) - E(\lambda_{j-1})\}x \\
&+ \sum_{j=K+1}^{N} e^{i(t+s)\lambda_j}\{E(\lambda_j) - E(\lambda_{j-1})\}x \\
&= \mathcal{S}(t+s, v_a)x \ + \ e^{-isa}\sum_{j=1}^{K} e^{it\lambda_j}\{E(\lambda_j) - E(\lambda_{j-1})\}x \\
&- e^{-i(s+t)a}E(-a)x + e^{-isa}e^{-itb}E(-b)x.
\end{aligned}$$

Since the family $\left\{\mathcal{S}(s, v_a) \ : \ v_a \in \mathcal{P}_{[-a,a]}\right\}$ is uniformly bounded, we can take $\lim_{v_b \in \mathcal{P}[-b,b]}$ on both sides to obtain

$$U_s^{(a)}U_t^{(b)}x = U_{s+t}^{(a)}x \ + \ e^{-isa}\int_{-b}^{-a} e^{it\lambda}dE(\lambda)x - e^{-i(s+t)a}E(-a)x. \tag{5.5}$$

Now, since $U_s^{(a)}$ is continuous and $x \in \mathcal{D}(U_t)$, we can let $b \to \infty$ on the left to obtain $U_s^{(a)}U_tx$. But this implies that the limit of the right side must exist as an element of X. Hence $\int_{-\infty}^{-a} e^{it\lambda}dE(\lambda)x$ is a well-defined element of X and we have

$$U_s^{(a)}U_tx = U_{s+t}^{(a)}x \ + \ e^{-isa}\int_{-\infty}^{-a} e^{it\lambda}dE(\lambda)x - e^{-i(s+t)a}E(-a)x. \tag{5.6}$$

Further, $U_tx \in \mathcal{D}(U_s)$ so we can let $a \to \infty$ on the left hand side of (5.6) to obtain U_sU_tx. But

$$\lim_{a\to\infty}\int_{-\infty}^{-a} e^{it\lambda}dE(\lambda)x = 0, \quad \text{and} \quad \lim_{a\to\infty} E(-a)x = 0.$$

Using this in (5.6) implies that $\lim_{a\to\infty} U_{s+t}^{(a)}x$ exists. In other words $x \in \mathcal{D}(U_{s+t})$ and $U_sU_tx = U_{s+t}x$. This says precisely that $U_sU_t \subset U_{s+t}$. Note that we have just established $\mathcal{D}(U_{s+t}) \cap \mathcal{D}(U_t) \supset \mathcal{D}(U_sU_t)$. We can use (5.6) to show the reverse inclusion too. For, if $x \in \mathcal{D}(U_{s+t}) \cap \mathcal{D}(U_t)$, we just need to show that $\lim_{a\to\infty} U_s^{(a)}U_tx$ exists. But this follows immediately from (5.6): we know that $\lim_{a\to\infty} U_{s+t}^{(a)}x$ exists, and the other two terms on the right side of (5.6) vanish. Thus $\lim_{a\to\infty} U_s^{(a)}U_tx$ exists, and so $x \in \mathcal{D}(U_sU_t)$. Thus we have

$$\mathcal{D}(U_{s+t}) \cap \mathcal{D}(U_t) = \mathcal{D}(U_sU_t).$$

We have verified (i)-(iii) in Definition 5.1. Finally let us check that the operators $\{U_t\}$ are closed. Let t be fixed and let $\{x_m\}_{m\geq 1} \in \mathcal{D}(U_t)$ be a sequence such that $(x_m, U_t x_m) \to (x, y)$. We need to show that $x \in \mathcal{D}(U_t)$ and $y = U_t x$. Observe that by the Remark following Theorem 5.2, we can say that for each $m \geq 1$, $U_t x_m = \lim_{n\to\infty} U_t^{(n)} x_m$, with $n \in \mathbb{N}$. We thus have the following

$$\begin{aligned} y &= \lim_{m\to\infty} U_t x_m \\ &= \lim_{m\to\infty} \lim_{n\to\infty} U_t^{(n)} x_m \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \{E(n) - E(-n)\} U_t x_m. \end{aligned}$$

Now, the operators $\{E(n) - E(-n)\}_{n\geq 1}$ are uniformly bounded, so that

$$\lim_{m\to\infty} \{E(n) - E(-n)\} U_t x_m = \{E(n) - E(-n)\} y \quad \text{uniformly in } n.$$

Hence, by Lemma 5.1, we can interchange the limits above. This, together with each $U_t^{(n)}$ being bounded, yields

$$\begin{aligned} y &= \lim_{n\to\infty} \lim_{m\to\infty} \{E(n) - E(-n)\} U_t x_m \\ &= \lim_{n\to\infty} \lim_{m\to\infty} \{E(n) - E(-n)\} U_t^{(n)} x_m \\ &= \lim_{n\to\infty} \{E(n) - E(-n)\} U_t^{(n)} x \\ &= \lim_{n\to\infty} U_t^{(n)} x. \end{aligned}$$

In other words $x \in \mathcal{D}(U_t)$ and $y = U_t x$ as claimed.

It remains to establish inequality (5.3). For $\phi \in C_c^\infty(\mathbb{R})$ and $x \in X_0$, the Bochner integral $\int \phi(t) U_{-t} x dt$ is a well defined element of X because $U_{(.)} x$ is a continuous map on the compact support of ϕ into X. Moreover, we have a linear map

$$\int \phi(t) U_{-t} dt : \; X_0 \to \; X, \quad x \mapsto \int \phi(t) U_{-t} x dt.$$

We shall find $\gamma > 0$, independent of ϕ, such that for any $x \in X_0$ we have $\left\| \int \phi(t) U_{-t} x dt \right\| \leq \gamma \|\hat{\phi}\|_{BV} \|x\|$. This will imply that $\int \phi(t) U_{-t} dt$ has a continuous extension to all of X, denoted with $\hat{\phi}(U)$, with $\|\hat{\phi}(U)\| \leq \gamma \|\hat{\phi}\|_{BV}$.
So, let $x \in X_0$, say $x \in \{E(n) - E(-n)\} X$ for some fixed n. Then we have

$$U_{-t}^{(b)} x = U_{-t}^{(n)} x \quad \text{for all } b \geq n.$$

Let $\xi \in X^*$. Then, integrating by parts, we have

$$\langle U_{-t} x, \xi \rangle = e^{-int} \langle x, \xi \rangle + \int_{-n}^{n} ite^{-i\lambda t} \langle E(\lambda) x, \xi \rangle d\lambda.$$

Since the map $t \mapsto \phi(t)\langle U_t x, \xi\rangle$ is continuous on $\mathrm{supp}(\phi)$,

$$\begin{aligned}\Big\langle \int \phi(t)U_{-t}x\ dt, \xi\Big\rangle &= 2\pi\hat{\phi}(n)\langle x,\xi\rangle + i\int t\phi(t)\Big\{\int_{-n}^{n} e^{-i\lambda t}\langle E(\lambda)x,\xi\rangle d\lambda\Big\}dt \\ &= 2\pi\hat{\phi}(n)\langle x,\xi\rangle - 2\pi\int_{-n}^{n}(\hat{\phi})'(\lambda)\langle E(\lambda)x,\xi\rangle d\lambda.\end{aligned}$$

Hence $\left|\langle \int \phi(t)U_{-t}x\ \ dt, \xi\rangle\right| \le 2\pi\|x\|\ \|\xi\|\ \|E\|_\infty\|\hat{\phi}\|_{BV}$. But $\xi \in X^*$ is arbitrary, so $\left\|\int \phi(t)U_{-t}x\ \ dt\right\| \le 2\pi\|E\|_\infty\|\hat{\phi}\|_{BV}\|x\|$ and this establishes (5.3) with $\gamma = 2\pi\|E\|_\infty$. $\square$

Before giving a full proof of Theorem 5.3, let us prove a Lemma used therein.

Lemma 5.2. *Let $\{U_t\}$ be an operator group satisfying the conditions of Theorem 5.3. Let $\widehat{C_c^\infty} = \{\hat{f} :\ f \in C_c^\infty\}$.*
(i) For every $s \in \mathbb{R}$, $x \in X_M$ and $g \in C_c^\infty(\mathbb{R})$, we have $\hat{g}(U)x \in X_M$ and $U_s\hat{g}(U)x = \hat{g}(U)U_s x$.
(ii) The map $\Psi : \widehat{C_c^\infty} \to \mathcal{B}(X)$, $\hat{f} \mapsto \hat{f}(U)$ is an algebra homomorphism.

Proof. (i) To show that $\hat{g}(U)x \in \mathcal{D}(U_s)$, observe that $\hat{g}(U)x = \int_{-K}^{K} g(t)U_{-t}x\ dt$ where $\mathrm{supp}(g) \subset [-K,K]$. Let $\pi = \{-K = t_1, ..., t_N = K\}$ denote a partition of $[-K,K]$ with rational points. Then

$$\hat{g}(U)x = st\ \lim_{\pi}\sum_{k=1}^{N}(t_{k+1}-t_k)g(t_k)U_{-t_k}x. \tag{5.7}$$

Now, for any k, $x \in \mathcal{D}(U_{-t_k}) \cap \mathcal{D}(U_{s-t_k}) = \mathcal{D}(U_sU_{-t_k})$, and $U_sU_{-t_k}x = U_{s-t_k}x$. Hence,

$$U_s\Big(\sum_{k=1}^{N}(t_{k+1}-t_k)g(t_k)U_{-t_k}x\Big) = \sum_{k=1}^{N}(t_{k+1}-t_k)g(t_k)U_{s-t_k}x. \tag{5.8}$$

But $st\lim_\pi \sum_{k=1}^{N}(t_{k+1}-t_k)g(t_k)U_{s-t_k}x = \int_{-K}^{K} g(t)U_{s-t}x\ dt$, and since U_s is closed it follows from (5.7) and (5.8) that $\hat{g}(U)x \equiv \int_{-K}^{K} g(t)U_{-t}x\ dt\ \in \mathcal{D}(U_s)$ and

$$U_s\hat{g}(U)x = \int_{-K}^{K} g(t)U_{s-t}x\ dt. \tag{5.9}$$

To show that $U_sx \in X_M$, it suffices to verify that the map $t \mapsto U_tU_sx$ is continuous and $U_sx \in \bigcap_{t\in\mathbb{R}}\mathcal{D}(U_t)$. But both of these are trivial. For, given any t, $x \in \mathcal{D}(U_{s+t}) \cap \mathcal{D}(U_s) = \mathcal{D}(U_tU_s)$ which means $U_sx \in \mathcal{D}(U_t)$. Furthermore, $U_tU_sx = U_{t+s}x$, and the map $t \mapsto U_{t+s}x$ is certainly continuous as $x \in X_M$. Thus $U_sx \in X_M$. Now, using the definition of $\hat{g}(U)$ on X_M, we have $\hat{g}(U)U_sx = \int_{-K}^{K} g(t)U_{-t}U_sx\ dt$. But again, $x \in \mathcal{D}(U_{-t+s}) \cap \mathcal{D}(U_s) = \mathcal{D}(U_{-t}U_s)$ for all $t \in [-K, K]$, and $U_{-t}U_s x = U_{-t+s}x$, and hence $\hat{g}(U)U_s x = \int_{-K}^{K} g(t)U_{-t+s}x\ dt$. Comparing this with (5.9) gives $U_s\hat{g}(U)x = \hat{g}(U)U_s x$. Finally, since the map $s \mapsto \hat{g}(U)U_s x$ is continuous and $\hat{g}(U)x \in \bigcap_{s\in\mathbb{R}} \mathcal{D}(U_s)$, it follows that $\hat{g}(U)x \in X_M$.

(ii) The map Ψ defined in (ii) is clearly well defined and linear. We have to check it is multiplicative. So, let $f, g \in C_c^\infty(\mathbb{R})$. We claim that, as operators on X, $(f*g)^\wedge(U) = \hat{g}(U)\hat{f}(U)$. It suffices to show they agree on X_M. So, let $x \in X_M$ be fixed. Then

$$\begin{aligned}(f * g)^\wedge(U)x &= \int_t (f*g)(t)U_{-t}x\ dt \\ &= \int_t\int_s f(t-s)g(s)U_{-t}x\ ds\ dt. \end{aligned} \tag{5.10}$$

Since $x \in \mathcal{D}(U_{-t}) \cap \mathcal{D}(U_{-(t-s)}) = \mathcal{D}(U_{-s}U_{-(t-s)})$ for all s, t, we can write $U_{-t}x = U_{-s}U_{-(t-s)}x$, and so (5.10) becomes

$$\int_t\int_s f(t-s)g(s)U_{-s}U_{-(t-s)}x\ ds\ dt. \tag{5.11}$$

By pairing with an arbitrary $\xi \in X^*$, (5.11) equals $\int_s g(s)\{\int_t f(t-s)U_{-s}U_{-(t-s)}x\ dt\}\ ds$. Now, by part (i), $\int_t f(t-s)U_{-s}U_{-(t-s)}x\ dt = U_{-s}\Big(\int_t f(t-s)U_{-(t-s)}x\ dt\Big)$ The latter is precisely $U_{-s}\hat{f}(U)x$, because $x \in X_M$. Hence $(f*g)^\wedge(U)x = \int_s g(s)U_{-s}\hat{f}(U)x\ ds$. But again by (i) $\hat{f}(U)x \in X_M$ and so this is precisely $\hat{g}(U)\hat{f}(U)x$ as claimed. □

Proof of Theorem 5.3 . This argument is a mild adaptation Theorem 2.4 in [5]. Let

$$AC_0(\mathbb{R}) = \{f \in AC(\mathbb{R}) :\ \lim_{|t|\to\infty} f(t) = 0\}, \quad AC_0(\mathbb{T}) = \{f \in AC(\mathbb{T}) :\ f(1) = 0\}.$$

The set $\widehat{C_c^\infty}$ is $\|\cdot\|_{BV}$ dense in $AC_0(\mathbb{R})$. The map $\Psi : \widehat{C_c^\infty} \to \mathcal{B}(X)$, $\hat{f} \mapsto \hat{f}(U)$ is a well-defined algebra homomorphism, by Lemma 5.2. Ψ is norm continuous because $\{U_t\}$ satisfies (5.3), and so extends to a continuous linear map on all of $AC_0(\mathbb{R})$.

Now, fix $\theta \in C^\infty(\mathbb{R})$ such that for all $t \in \mathbb{R}$, $\theta'(t) > 0$ and θ is surjective onto $(0, 2\pi)$. Let $\Theta : AC_0(\mathbb{T}) \to AC_0(\mathbb{R})$, $f \mapsto \Theta_f$ be given by $\Theta_f(t) = f(e^{i\theta(t)})$. Then Θ is an isometric algebra isomorphism. Define $\Phi_0 = \Psi\circ\Theta : AC_0(\mathbb{T}) \to \mathcal{B}(X)$, so that $\|\Phi_0\| = \|\Psi\| \le \gamma$. Since $AC(\mathbb{T}) = \{f + \alpha :\ f \in AC_0(\mathbb{T})\ \ \alpha \in \mathbb{C}\}$, Φ_0 can be extended to all of $AC(\mathbb{T})$ by defining $\Phi(f + \alpha) = \Phi_0(f) + \alpha I$. Clearly Φ is an algebra homomorphism from $AC(\mathbb{T})$ into $\mathcal{B}(X)$ and $\|\Phi\| \le \gamma \vee 1$.

This set-up now implies that there exists a spectral family $\{F(\lambda)\}_{\lambda\in\mathbb{R}}$, concentrated on $[0, 2\pi]$, with $F(2\pi^-) = I$ and

$$\Phi(h) = \int_{0^-}^{2\pi} h(\lambda)dF(\lambda) \quad \text{for} \quad h \in AC(\mathbb{T}). \tag{5.12}$$

We now claim that $F(0) = 0$. Let $f \in AC_0(\mathbb{T})$, so that $f(1)F(0) = 0$. Then for any $u = \{0 = \lambda_0 < ... < \lambda_N = 2\pi\}$ we have

$$\mathcal{S}(u, f)F(0) = \sum_1^N f(\lambda_j)\ \{F(\lambda_j) - F(\lambda_{j-1})\}F(0) = 0.$$

Since this holds for any partition of $[0, 2\pi]$, it follows that $\int_{0^-}^{2\pi} f(\lambda)dF(\lambda)F(0) = 0$. Hence $\Phi(f)F(0) = 0$ for all $f \in AC_0(\mathbb{T})$, and so $\Psi(g)F(0) \equiv \Phi(\Theta^{-1}g)F(0) = 0$ for all $g \in AC_0(\mathbb{R})$. (†)

Now, if $\phi \in C_c^\infty$, then $\Psi(\hat{\phi}) \equiv \hat{\phi}(U)$. Also, $\{F(\lambda)\}$ commutes with $\Phi(g)$ for any $g \in AC(\mathbb{T})$. Using this, (†), and the fact that $\hat{\phi}(U)x = \int \phi(t)U_{-t}x\ dt$ for $x \in X_M$, we have, for each $x \in X_M$

$$\begin{aligned} 0 = \Psi(\hat{\phi})F(0)x &= F(0)\Psi(\hat{\phi})x = F(0)\hat{\phi}(U)x \\ &= F(0)\int \phi(t)U_{-t}x\ dt \\ &= \int \phi(t)F(0)U_{-t}x\ dt, \qquad x \in X_M. \end{aligned}$$

This holds for all $\phi \in C_c^\infty$. Also $U_{(\cdot)}x$ is strongly continuous for $x \in X_M$, so the last equation implies that $F(0)U_tx = 0$ for all t. In particular $F(0)x = 0$. Hence the density of X_M in X implies $F(0) \equiv 0$, as claimed.

Now define $E(\lambda) = F\big(\theta(\lambda)\big)$ for $\lambda \in \mathbb{R}$. Then E is a spectral family and we have, in strong operator topology, $\lim_{\lambda\to\infty} E(\lambda) = F(2\pi^-) = I$ and $\lim_{\lambda\to-\infty} E(\lambda) = F(0) = 0$. Further, for $x \in X$ and $\phi \in C_c^\infty$,

$$\begin{aligned} \hat{\phi}(U)x &= \Psi(\hat{\phi})x = \Phi_0(\Theta^{-1}\hat{\phi})x \\ &= \int_0^{2\pi} \Theta^{-1}\hat{\phi}(e^{i\lambda})dF(\lambda)x \quad \text{since } F(0) = 0 \\ &= \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} \Theta^{-1}\hat{\phi}(e^{i\lambda})dF(\lambda)x \\ &= \lim_{a\to\infty}\int_{-a}^{a} \hat{\phi}(\mu)dE(\mu)x. \end{aligned} \tag{5.13}$$

Now let $y \in \{E(a) - E(-a)\}X_M$, say $y = \{E(a) - E(-a)\}x$ with $x \in X_M$. Since $E(\pm a)$ both commute with $\hat{\phi}(U) \equiv \Psi(\hat{\phi})$, we have

$$\hat{\phi}(U)y = \{E(a) - E(-a)\}\hat{\phi}(U)x = \{E(a) - E(-a)\}\int_{\mathbb{R}} \phi(t)U_{-t}xdt$$

Also, for any $b \geq a$, $\int_{-b}^{b} \hat{\phi}(\lambda)dE(\lambda)y = \int_{-a}^{a} \hat{\phi}(\lambda)dE(\lambda)y = \int_{-a}^{a} \hat{\phi}(\lambda)dE(\lambda)x$. Hence (5.13) gives $\hat{\phi}(U)y = \int_{-a}^{a} \hat{\phi}(\lambda)dE(\lambda)y$.
Now let $\xi \in X^*$. Then, using integration by parts, we have

$$\begin{aligned}
\Big\langle \{E(a) \; - \; E(-a)\} \int_{\mathbb{R}} \phi(t)U_{-t}x \; dt, \; \xi \Big\rangle &= \Big\langle \{E(a) - E(-a)\} \int_{-a}^{a} \hat{\phi}(\lambda)dE(\lambda)x, \xi \Big\rangle \\
&= \hat{\phi}(a)\langle y, \xi \rangle \; - \; \int_{-a}^{a} \hat{\phi}'(\lambda)\langle E(\lambda)x, \xi \rangle \; d\lambda \\
&= \int_{\mathbb{R}} e^{-iat}\phi(t) \; dt \langle y, \xi \rangle \; + i \int_{-a}^{a} \int_{t} te^{-it\lambda}\phi(t)\langle E(\lambda)x, \xi \rangle \; dt \; d\lambda \\
&= \int_{\mathbb{R}} \phi(t) \Big\{ e^{-iat}\langle y, \xi \rangle \; + \; i \int_{-a}^{a} te^{-it\lambda}\langle E(\lambda)x, \xi \rangle \; d\lambda \Big\} \; dt \\
&= \int_{t} \phi(t) \Big\{ \int_{-a}^{a} e^{-it\lambda}\langle dE(\lambda)x, \xi \rangle \Big\} \; dt \\
&= \Big\langle \int_{t} \phi(t) \Big\{ \int_{-a}^{a} e^{-it\lambda} dE(\lambda)x \Big\} \; dt, \; \xi \Big\rangle.
\end{aligned}$$

Hence

$$\{E(a) - E(-a)\} \int_{\mathbb{R}} \phi(t)U_{-t}x \; dt = \int_{\mathbb{R}} \phi(t) \Big\{ \int_{-a}^{a} e^{-it\lambda} dE(\lambda)x \Big\} \; dt.$$

This holds for all $\phi \in C_c^\infty$. Since the function $t \mapsto \{E(a) - E(-a)\}U_{-t}x$ is continuous (as $x \in X_M$), then $\{E(a) - E(-a)\}U_{-t}x = \int_{-a}^{a} e^{-it\lambda}dE(\lambda)x$ for all $t \in \mathbb{R}$. Letting $a \to \infty$ we obtain

$$U_{-t}x = \lim_{a \to \infty} \int_{-a}^{a} e^{-it\lambda} dE(\lambda)x \quad \text{for} \;\; x \in X_M,$$

and this establishes (5.4).

Finally, let us address the uniqueness of E. So, suppose there are two spectral families E and $\tilde{E}$ satisfying (5.4). Let $x \in X_M$ and $\xi \in X^*$. Then by (5.13) and integration by parts, we have

$$\int_{-\infty}^{\infty} \hat{\phi}'(\lambda)\langle E(\lambda)x, \xi \rangle \;\; = \;\; \hat{\phi}(U)x = \int_{-\infty}^{\infty} \hat{\phi}'(\lambda)\langle \tilde{E}(\lambda)x, \xi \rangle.$$

This holds for all $\phi \in C_c^\infty(\mathbb{R})$. Following the argument in [5], $\hat{\phi}'$ can be replaced by any ψ' where ψ belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$. But this implies that the function $\lambda \mapsto \langle \{E(\lambda) - \tilde{E}(\lambda)\}x, \xi \rangle$ is constant $a.e(\lambda)$. Since both E and $\tilde{E}$ are right continuous and have left limits, the above function is constant for all $\lambda \in \mathbb{R}$. Finally, letting $\lambda \to -\infty$ gives that this constant is zero. Hence $E(\lambda) = \tilde{E}(\lambda)$ for all $\lambda \in \mathbb{R}$. □

Let us observe that Theorems 5.2 and 5.3 are not complete converses of each other. For, equation (5.4) holds only for $x \in X_M$, not necessarily all $x \in \mathcal{D}(U_t)$, as in (5.2). The spectral family $\{E(\lambda)\}$ derived in Theorem 5.3 can be used to define a new densely defined operator group as in Theorem 5.2, $\{W_t\}$ say. The latter need not necessarily satisfy $\mathcal{D}(W_t) = \mathcal{D}(U_t)$. Informally speaking, $\{E(\lambda)\}$ need not recover the original $\{U_t\}$. However, it is the case that for each $\phi \in C_c^\infty$, the two operators $\int \phi(t)W_{-t}\, dt : X_0 \to X$ and $\int \phi(t)U_{-t}\, dt : X_M \to X$ have the same linear extension to all of X, that is $\hat{\phi}(U) = \hat{\phi}(W)$.

Weighted Space Example

The motivation for examining unbounded operator groups is the right translation group $\{R_t\}$ on weighted $L^p(w)$ spaces. These spaces were discussed in Chapter 1 Section 1.4. We concentrate here on $1 < p < \infty$. The right translation group $\{R_t\}_{t\in\mathbb{R}}$ is the natural one parameter group to examine in $L^p(w)$. Indeed, it provides an example which illustrates the phenomenon in Theorems 5.2 and 5.3. We shall show by direct calculation that if the weight $w(t)$ is chosen appropriately, then $\{R_t\}$ satisfies the conditions and conclusion of Theorem 5.3.
Let $\alpha \in (-1, p-1)$ and define $w(t) = |t|^\alpha$. Following the discussion in Chapter 1 Section 1.4, $w(t)$ is an A_p weight, and it is easy to verify that $\{R_t\}$ satisfies Definition 5.1. For the domains $\mathcal{D}(R_t)$ are naturally defined as

$$\mathcal{D}(R_t) = \{\ f \in L^p(w) : \int_{\mathbb{R}} w(s)|f(s-t)|^p ds < \infty\ \}.$$

It follows automatically that $\mathcal{D}(R_sR_t) = \mathcal{D}(R_{s+t}) \cap \mathcal{D}(R_t)$. The following lemma helps prove that $\{R_t\}$ is only densely defined.

Lemma 5.3. *Let $w(t)$ be an arbitrary weight function on $\mathbb{R}$. Then for all $s \neq 0$, R_s is bounded from $L^p(w)$ into $L^p(w)$ if and only if $\frac{w(\cdot\ +s)}{w(\cdot)} \in L^\infty(\mathbb{R})$. In this case, $\|R_s\| = \left\|\frac{w(\cdot\ +s)}{w(\cdot)}\right\|_\infty^{1/p}$ where $\|\cdot\|_\infty$ is the essential supremum over $\mathbb{R}$.*

Proof. Observe that if $f \in L^p(w)$, then $g \equiv w\ |f|^p \in L^1(\mathbb{R})$. Then

$$\begin{aligned}\|R_sf\|^p_{L^p(w)} &= \int_{\mathbb{R}} w(t)|R_sf(t)|^p dt = \int_{\mathbb{R}} w(t)|f(t-s)|^p dt \\ &= \int_{\mathbb{R}} w(t+s)|f(t)|^p dt \ = \int_{\mathbb{R}} w(t)|f(t)|^p \frac{w(t+s)}{w(t)} dt \\ &= \int_{\mathbb{R}} |g(t)| \frac{w(t+s)}{w(t)} dt\end{aligned}$$

Hence

$$\begin{aligned}\|R_s\|^p &= \sup\{\|R_sf\|^p_{L^p(w)} :\ \|f\|_{L^p(w)} \le 1\} \\ &= \sup\{\int_{\mathbb{R}} |g(t)| \frac{w(t+s)}{w(t)} dt :\ \|g\|_{L^1(\mathbb{R})} \le 1\} \ = \left\|\frac{w(\cdot\ +s)}{w(\cdot)}\right\|_\infty.\end{aligned}$$

The last equality follows from the fact that $L^\infty(\mathbb{R})$ is the dual of $L^1(\mathbb{R})$. $\square$

It can readily be seen from Lemma 5.3 that if $-1 < \alpha < p-1$ and $w(t) = |t|^\alpha$, then $\|R_s\| = \sup_{t_0} |\frac{t_0-s}{t_0}|^\alpha = \infty$ for all $s \neq 0$. Let us make a further restriction on α , by specifying $\alpha \in (0, 1\wedge(p-1))$. This ensures that for all $f \in L^p(w)\cap L^p(\mathbb{R})$,

$$\|R_s f\|^p_{L^p(w)} \leq |s|^\alpha \|f\|^p_{L^p(\mathbb{R})} + \|f\|^p_{L^p(w)}. \tag{5.14}$$

Hence $L^p(w)\cap L^p(\mathbb{R}) \subset \bigcap_{s\in\mathbb{R}} \mathcal{D}(R_s)$ and so $\{R_t\}$ is densely defined.

To show that $\{R_t\}$ satisfies the other conditions of Theorem 5.3, it remains to establish inequality (5.3). But this follows from the boundedness of the Hilbert transform. For, Theorem 5.4, in section 5.2.2 below, shows that if the Hilbert transform is bounded on $L^p(w)$ (or indeed a more general X-valued function space S_X), then the right translation group $\{R_t\}$ satisfies inequality (5.3) for every $\phi \in C_c^\infty(\mathbb{R})$. But in the current setting, $w(t)$ being an A_p weight ensures that the Hilbert transform is indeed bounded.

To illustrate the conclusion of Theorem 5.3, we shall explicitly describe the spectral family $\{E^w(\lambda)\}$ which has the property that for all $f \in L^p(\mathbb{R})\cap L^p(w)$, $R_t f = \lim_{a\to\infty}\int_{-a}^a e^{it\lambda} dE^w(\lambda)f$ in $\|\cdot\|_{L^p(w)}$ norm. To avoid confusion, let $\{S_t\}$ denote the right translation on the unweighted space $L^p(\mathbb{R})$. The spectral family $\{E^u(\lambda)\}$ of $\{S_t\}$ is just the family of multipliers associated with the characteristic functions $\mathbb{I}_{(-\infty,\lambda]}$. So $E^u(\lambda)f = (\mathbb{I}_{(-\infty,\lambda]}\hat{f})^\vee$ for $f \in L^p(\mathbb{R})\cap L^2(\mathbb{R})$.

Proposition 5.1. *The family* $\{\mathbb{I}_{(-\infty,\lambda]}\}_{\lambda\in\mathbb{R}}$ *gives rise to bounded multipliers on* $L^p(w)$*, denoted with* $\{E^w(\lambda)\}$*. This is a spectral family on* $L^p(w)$ *and, if* $f \in L^p(\mathbb{R})\cap L^p(w)$*, then*

$$\lim_{a\to\infty} \left\|R_t f - \int_{-a}^a e^{it\lambda} dE^w(\lambda) f\right\|_{L^p(w)} = 0.$$

Proof. Let H^w and H^u denote the Hilbert transform on $L^p(w)$ and $L^p(\mathbb{R})$ respectively. They are both bounded, the former because $w(t) = |t|^\alpha$ is an A_p weight, the latter by the classical result of M.Riesz. Let $\sigma(t) = i\{2\mathbb{I}_{(-\infty,0]} - 1\}$. Then H^w and H^u are precisely the multipliers associated with σ: So, if $f \in L^p(w)\cap L^p\cap L^2(\mathbb{R})$, then $H^w f(s) = H^u f(s) = (\sigma\hat{f})^\vee(s)$ *a.e.*(s). Let us also write $M_\lambda : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, $M_\lambda f(s) = e^{i\lambda s} f(s)$ and let $E^w(\lambda)$ and $E^u(\lambda)$ be the multipliers on $L^p(w)\cap L^2(\mathbb{R})$ and $L^p(\mathbb{R})\cap L^2(\mathbb{R})$, associated with $\mathbb{I}_{(-\infty,\lambda]}$. Then

$$E^w(\lambda) = \frac{1}{2} M_\lambda (I - iH^w) M_{-\lambda}.$$

so each $E^w(\lambda)$ is bounded on $L^p(w)$. Moreover, an argument analogous to that in [15] shows that $\{E^w(\lambda)\}$ is a spectral family on $L^p(w)$. The calculations in [15] also show that $\{E^u(\lambda)\}$ is the spectral family of $\{S_t\}$; that is, for all $f \in L^p(\mathbb{R})$

$$S_t f = \lim_{a\to\infty} \int_{-a}^{a} e^{i\lambda t} dE^u(\lambda) f \quad \text{in } \| \ \|_{L^p(\mathbb{R})} \text{ norm.} \tag{5.15}$$

Claim: For all $f \in L^p(\mathbb{R}) \cap L^p(w)$, $\int_{-a}^{a} e^{i\lambda t} dE^u(\lambda) f = \int_{-a}^{a} e^{i\lambda t} dE^w(\lambda) f$.

Proof of Claim

First observe that for any $f \in L^p(w) \cap L^p(\mathbb{R}) \cap L^2(\mathbb{R})$,
$E^w(\lambda) f(s) = E^u(\lambda) f(s)$ $a.e.(s)$. This subspace is $\|\cdot\|_{L^p(\mathbb{R})}$ and $\|\cdot\|_{L^p(w)}$ dense in $L^p(w) \cap L^p(\mathbb{R})$. So, given $f \in L^p(w) \cap L^p(\mathbb{R})$, we can choose an appropriate sequence $\{f_n\}$ in the first subspace to show that $E^w(\lambda) f(s) = E^u(\lambda) f(s)$ $a.e.(s)$. $(*)$

So, let $f \in L^p(w) \cap L^p(\mathbb{R})$ be fixed. Let $\mathcal{P}_a^{\mathbb{Q}}$ be the (countable) set of partitions of $[-a, a]$ with rational points. Since both E^w and E^u are strongly continuous (by Corollary 5.1 in section 5.2.1 below), we have

$$\int_{-a}^{a} e^{i\lambda t}(\lambda) dE^w(\lambda) f = \lim_{v\in\mathcal{P}_a^{\mathbb{Q}}} \{e^{-ita} E^w(-a) f + \sum_{1}^{N} e^{i\lambda_j t}[E^w(\lambda_j) - E^w(\lambda_{j-1})] f\}$$

$$\int_{-a}^{a} e^{i\lambda t} dE^u(\lambda) f = \lim_{v\in\mathcal{P}_a^{\mathbb{Q}}} \{e^{-ita} E^u(-a) f + \sum_{1}^{N} e^{i\lambda_j t}[E^u(\lambda_j) - E^u(\lambda_{j-1})] f\} \tag{5.16}$$

Again, these limits are in the $\|\cdot\|_{L^p(w)}$ and $\|\cdot\|_{L^p(\mathbb{R})}$ norms respectively. (Usually, the limits would involve $\mathcal{P}_{[-a,a]}$, but the strong continuity of E^u and E^w ensures we can use $\mathcal{P}_a^{\mathbb{Q}}$). Let $\mathcal{S}(v, E^w) f$ and $\mathcal{S}(v, E^u) f$ denote the sums on the right. By observation $(*)$, there exists for each partition $v \in \mathcal{P}_a^{\mathbb{Q}}$ a full-measure set $\Omega_v \subset \mathbb{R}$ such that for all $t \in \Omega_v$, $\mathcal{S}(v, E^w) f(t) = \mathcal{S}(v, E^u) f(t)$. Further, by (5.16) we can pass to a (countable) subset $\mathcal{T} \subset \mathcal{P}_a^{\mathbb{Q}}$ such that for all $s \in A_f \subset \mathbb{R}$, a set of full measure,

$$\lim_{v\in\mathcal{T}} \mathcal{S}(v, E^w) f(s) = \int_{-a}^{a} e^{i\lambda t}(\lambda) dE^w(\lambda) f(s)$$

$$\lim_{v\in\mathcal{T}} \mathcal{S}(v, E^u) f(s) = \int_{-a}^{a} e^{i\lambda t}(\lambda) dE^u(\lambda) f(s).$$

So for $s \in \bigcap_{v\in\mathcal{T}} \Omega_v \cap A_f$,

$$\int_{-a}^{a} e^{i\lambda t} dE^w(\lambda) f(s) = \int_{-a}^{a} e^{i\lambda t} dE^u(\lambda) f(s).$$

But the set $\cap_{v\in\mathcal{T}} \Omega_v \cap A_f$ has full measure, so the last equation says precisely that for $f \in L^p(\mathbb{R}) \cap L^p(w)$,

$$\int_{-a}^{a} e^{i\lambda t}(\lambda) dE^u(\lambda) f = \int_{-a}^{a} e^{i\lambda t}(\lambda) dE^w(\lambda) f. \tag{5.17}$$

Hence the claim is proved.
Now fix $a > 0$ and let $f \in L^p(\mathbb{R}) \cap L^p(w)$. Writing $g_a = \{E^u(a) - E^u(-a)\}f \equiv \{E^w(a) - E^w(-a)\}f$,

$$\int_{-b}^{b} e^{i\lambda t}(\lambda)dE^w(\lambda)g_a = \int_{-a}^{a} e^{i\lambda t}(\lambda)dE^w(\lambda)f \quad \text{for all} \quad b \geq a.$$

The analogous expression holds for E^u as well, and so (5.15) implies that $S_t g_a = \int_{-a}^{a} e^{i\lambda t}(\lambda)dE^u(\lambda)f$. Combining this with the Claim, it follows that

$$R_t g_a(s) = \int_{-a}^{a} e^{i\lambda t}(\lambda)dE^w(\lambda)f(s) \quad a.e.(s). \tag{5.18}$$

Now using (5.14),

$$\begin{aligned} \|R_t g_a - R_t f\|_{L^p(w)} &= \|R_t\{E^w(a) - E^w(-a)\}f - R_t f\|_{L^p(w)} \\ &\leq |t|^\alpha \|\{E^w(a) - E^w(-a)\}f - f\|_{L^p(\mathbb{R})} \\ &+ \|\{E^w(a) - E^w(-a)\}f - f\|_{L^p(w)}. \end{aligned}$$

Since E^w is a spectral family, the right side tends to zero as $a \to \infty$. Hence substituting (5.18) into the left hand side gives

$$\lim_{a\to\infty} \Big\| \int_{-a}^{a} e^{i\lambda t}(\lambda)dE^w(\lambda)f - R_t f \Big\|_{L^p(w)} = 0 \ ,$$

as was required to prove. □

5.2 Function Spaces

5.2.1 $\mathbb{R}$-function Spaces

$L^p(w)$ is an example of a scalar valued function space on $\mathbb{R}$. A detailed treatment of function spaces is provided in [39] and [34], and we refer the reader there for a full exposition. Here we collect only the definitions and properties of concern to us. Let $\mathfrak{S}(\mathbb{R})$ temporarily denote the space of simple functions on $\mathbb{R}$. Let $\|\cdot\|_S$ be a non-negative, sub-additive functional on $\mathfrak{S}(\mathbb{R})$ satisfying:
(i) $\|\alpha f\|_S = |\alpha| \ \|f\|_S$ for $\alpha \in \mathbb{C}$;
(ii) $\|f\|_S = 0$ implies $f(s) = 0 \quad a.e.(s)$;
(iii) $\|\mathbb{I}_A\|_S < \infty$ for every Borel set $A \subset \mathbb{R}$ with finite Lebesgue measure.
We define $\mathcal{S}$ to be the completion of $\mathfrak{S}(\mathbb{R})$ under $\|\cdot\|_S$ and define S to be the quotient Banach space $\mathcal{S}/\{\text{null functions}\}$.

Definition 5.3. *Let S^* be the dual of S and let $\langle , \rangle_S$ denote the usual Banach space pairing between S and S^*. The Banach space S is called a* ***scalar $\mathbb{R}$-function space*** *if the following conditions hold.*

(i) S has the ***Fatou Property****: for each non-negative sequence $\{f_n\} \subset S$, $f_n(t) \nearrow f(t)$ a.e.(t) and $\sup_n \|f_n\|_S < \infty$ together imply that $f \in S$ and $\|f\|_S = \lim_n \|f_n\|_S$.*
(ii) $\|\cdot\|_S$ is a ***lattice norm****: for $f, g \in S$*

$$|f(t)| \leq |g(t)| \ \ a.e.(t) \ \ implies \ \ \|f\|_S \leq \|g\|_S.$$

(iii) $\|\cdot\|_S$ is ***absolutely continuous****: given a decreasing sequence of measurable sets $\{E_n\}$ such that $E_n \searrow 0$ a.e., then $\|f\mathbb{I}_{E_n}\|_S \searrow 0$.*

Given such a space S we define the **space of integrals on** S by

$$S^{'} = \{g : \mathbb{R} \to \mathbb{C} : g \text{ measurable}, \ \ gf \in L^1(\mathbb{R}) \ \text{ for all } f \in S\}.$$

To complete the list of terminology, a subset $\mathcal{T} \subset S^*$ is a **norming subset** if, given $f \in S$, $\|f\|_S = \sup\{ \langle f, g\rangle_S : \ g \in \mathcal{T}, \ \|g\|_{S^*} \leq 1\}$. The following results are proved in [39].

Lemma 5.4. *(i) $S^{'} \subset S^*$ and $\langle f, g\rangle_S = \int_{\mathbb{R}} f(t)g(t)dt$ for all $f \in S$ and $g \in S^{'}$.*
(ii) S is reflexive if and only if both S and $S^{'}$ have absolutely continuous norms.
(iii) $S^{'}$ is canonically isometrically isomorphic to S^* if and only if S is reflexive.

Scalar valued function spaces have vector valued analogues. Let X be a Banach space and let L^0_X be the space of strongly measurable X valued functions on $\mathbb{R}$. For $f \in L^0_X$ define $\|f\|_X : \mathbb{R} \to \mathbb{C}$, $t \mapsto \|f(t)\|_X$. This is a measurable function. Define

$$S_X(\mathbb{R}) = \{f : \mathbb{R} \to X \ : \ \|f\|_X \in S\}.$$

For brevity we suppress $\mathbb{R}$ in the notation and write S_X. This is a Banach space under the norm given by $\|f\|_{S_X} \equiv \big\| \ \|f\|_X \ \big\|_S$. It is, in fact, the completion, under this norm, of the algebraic tensor product $S \otimes X$. The latter consists of finite formal sums $\sum_k f_k x_k$, $f_k \in S$, $x_k \in X$. Let also $C^\infty_c(X)$ denote the space of X-valued smooth functions of compact support. *Motivated by the scalar case, we assume that this is a dense subspace of S_X.*

Now, since $S^{'}$ is also a function space, $S^{'}_{X^*}$ is formed in an entirely analogous manner to S_X. It can readily be identified as a subspace of $\big(S_X\big)^*$: the action of $g \in S^{'}_{X^*}$ on $f \in S_X$ is given by $\langle f, g\rangle_{S_X} = \int_{\mathbb{R}} \langle f(t), g(t)\rangle_X \, dt$. It may happen that $S^{'}_{X^*}$ is a *proper* subspace of $\big(S_X\big)^*$. For example, for $1 \leq p < \infty$, $\big(L^p_X\big)^* = L^{p^{'}}_{X^*}$ if and only if X^* has the Radon-Nikodym property (see [35]). However $S^{'}_{X^*}$ is always a norming subset of $\big(S_X\big)^*$. Moreover, if both X and S are reflexive, then so is S_X.

With all the background set up, let us turn to the question of concern. Theorem 5.1 proves a result for strongly continuous operator groups $\{U_t\}$ on an arbitrary Banach space Y. In Section 5.1 we saw that if $\{U_t\}$ is just densely defined then it has only a densely defined spectral decomposition (as defined in Definition 5.2). In fact, the strong continuity of $\{U_t\}$ (not merely the fact that all U_t are bounded), is key in establishing Theorem 5.1. Our main result here is that this condition can be relaxed in the special case $Y = S_X$ and $\{U_t\} = \{R_t\}$, where $\{R_t\}$ is the right translation group. In this case, strong continuity can be relaxed to *local boundedness* for Theorem 5.1 still to hold.

Definition 5.4. *An operator group $\{U_t\}$ on a Banach space Y is* ***locally bounded*** *if $U_t \in \mathcal{B}(Y)$ for all $t \in \mathbb{R}$ and for any $K > 0$, $\sup_{|t| \leq K} \|U_t\| < \infty$.*

The following Proposition shows that Theorem 5.1 holds if the original assumption on $\{R_t\}$ is relaxed from strong continuity to local boundedness.

Proposition 5.2. *Let S and X be reflexive and let S satisfy Definition 5.3. Suppose that $\{R_t\}$ is locally bounded.*
(i) If $\{R_t\}$ has a spectral decomposition $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$, as in Definition 5.2, then R is strongly continuous.
(ii) Suppose there exists $\gamma > 0$ such that for each $\phi \in C_c^\infty(\mathbb{R})$ the operator $\int \phi(t) R_{-t} dt : C_c^\infty(X) \to C_c^\infty(X)$ extends to a bounded operator $\hat{\phi}(R)$ on all of S_X and satisfies

$$\|\hat{\phi}(R)\| \leq \gamma \|\hat{\phi}\|_{BV}. \tag{5.19}$$

Then R is strongly continuous.

Observe that $\int \phi(t) R_{-t} dt$ is a well defined linear map on $C_c^\infty(X)$. For, given $f \in C_c^\infty(X)$ with $\mathrm{supp}(f) \cap \mathrm{supp}(\phi) = A$, then $s \mapsto \int \phi(t) R_{-t} f(s)\, dt \equiv \int_A \phi(t) f(s - t) dt$ is also a well defined element of $C_c^\infty(X)$.
Before setting out the proof of Proposition 5.2, let us prove some preliminary results first.

Lemma 5.5. *Let $\{U_t\}_{t \in \mathbb{R}}$ be a locally bounded operator group on a Banach space V. Let $Y \subset V$ be a dense subspace. If $\{U_t\}$ is strongly continuous on Y then it is strongly continuous on V.*

Proof. Let $x \in V$, $t \in \mathbb{R}$ and $\epsilon > 0$. We wish to find $\delta > 0$ such that $\|U_t x - U_s x\| < \epsilon$ whenever $s \in B(t, \delta)$. Let $\Lambda = [t-1, t+1]$. Since $\{U_s\}$ is locally bounded, $\sup_{s \in \Lambda} \|U_s\| = \gamma < \infty$. Since Y is dense in V, there exists $y \in Y$ such that $\|y - x\| < \epsilon/3\gamma$.
Now choose $0 < \delta < 1$ such that $\|U_s y - U_t y\| < \epsilon/3$ whenever $s \in B(t, \delta)$. Then

$s \in B(t,\delta)$ implies $s,t \in \Lambda$ and so $\|U_s y - U_s x\| \leq \|U_s\|.\|y-x\| < \epsilon/3$. Hence, whenever $s \in B(x,\delta)$ we have

$$\begin{aligned} \|U_s x - U_t x\| &\leq \|U_t y - U_t x\| + \|U_s y - U_t y\| + \|U_s y - U_s x\| \\ &\leq \|U_t\|.\|x-y\| + \epsilon/3 + \|U_s\|.\|x-y\| \\ &\leq \gamma\ (\epsilon/3\gamma) + \epsilon/3 + \gamma\ (\epsilon/3\gamma)\ =\ \epsilon. \end{aligned}$$

□

Lemma 5.6. *Let $f \in C_c^\infty(\mathbb{R}, X)$ and let $\phi \in C_c^\infty(\mathbb{R})$ be even and non-negative, with $\int \phi(t)dt = \|\phi\|_{L^1(\mathbb{R})} = 1$. Define*

$$\phi_\epsilon(u) = \frac{1}{\epsilon}\phi(\frac{u}{\epsilon}) \quad \text{and } \psi_\epsilon^{(t)}(u) = \frac{1}{\epsilon}\phi(\frac{u-t}{\epsilon}).$$

Then:
(i) $\lim_{\epsilon\to 0}\|\phi_\epsilon * f(t) - f(t)\|_X = 0$ *uniformly in* $t \in \mathbb{R}$.
(ii) $\lim_{\epsilon\to 0}\|\phi_\epsilon * f - f\|_{S_X} = 0$.
(iii) Given $t \in \mathbb{R}$ *and* $r > 0$, $\lim_{\epsilon\to 0}\|\psi_\epsilon^{(s)} * f - R_s f\|_{S_X} = 0$ *uniformly for* $s \in B(t,r)$.

Proof. Recall that $B(a,r) = \{x :\ |a-x| < r\}$.
First choose $K > 0$ such that supp$(f)\cup$ supp$(\phi) \subset [-K,K]$.
(i) Observe that

$$\|\phi_\epsilon * f(t) - f(t)\|_X = \Big\|\int \frac{1}{\epsilon}\phi(\frac{s}{\epsilon})f(t-s)ds - f(t)\Big\|_X = \Big\|\int_{|s|\leq K}\phi(s)[f(t-\epsilon s) - f(t)]ds\Big\|_X.$$

Let $\eta > 0$ be given. Since f is uniformly continuous, we can pick $\delta > 0$ such that $\|f(t-s) - f(t)\|_X < \eta$ whenever $|s| < \delta$. Then if $\epsilon < \delta/K$ we have $|\epsilon s| < \delta$ for all $s \in [-K,K]$ so

$$\|\phi_\epsilon * f(t) - f(t)\|_X \leq \eta\|\phi\|_{L^1(\mathbb{R})} = \eta.$$

The δ is independent of $t \in \mathbb{R}$ and so (i) follows.
(ii) Note that for $0 < \epsilon < 1$, supp$(\phi_\epsilon * f) \subset [-2K, 2K]$. Let $M_\epsilon = \sup_{|t|\leq 2K}\|\phi_\epsilon * f(t) - f(t)\|_X$. By (i) $\lim_{\epsilon\to 0} M_\epsilon = 0$. Let $\theta \in C_c^\infty(\mathbb{R})$ be a bump function identically 1 on $[-2K, 2K]$. Then

$$\|\phi_\epsilon * f(t) - f(t)\|_X \leq M_\epsilon\theta(t) \quad \text{for all } t \in \mathbb{R}.$$

Since $\|\phi_\epsilon * f - f\|_X$ and $M_\epsilon\theta$ are in S, and $\|\cdot\|_S$ is a lattice norm, it follows that $\|\phi_\epsilon * f - f\|_{S_X} \leq M_\epsilon\|\theta\|_{S_X}$. Hence letting $\epsilon \to 0$ gives (ii).
(iii) Let $t \in \mathbb{R}$ and $r > 0$ be given. We have, uniformly in $\lambda \in \mathbb{R}$ and $s \in B(t,r)$,

$$\lim_{\epsilon\to 0}\|\psi_\epsilon^{(s)} * f(\lambda) - R_s f(\lambda)\|_X = 0.$$

To show this first note that

$$\begin{aligned}\|\psi_\epsilon^{(s)} * f(\lambda) - R_s f(\lambda)\|_X &= \Big\| \int \frac{1}{\epsilon}\phi(\frac{\mu - s}{\epsilon}) f(\lambda-\mu)d\mu - f(\lambda - s)\Big\|_X \\ &= \Big\| \int \phi(\mu) f(\lambda - \epsilon\mu - s)d\mu - f(\lambda - s)\Big\|_X \\ &= \Big\| \int_{|\mu|\le K} \phi(\mu)[f(\lambda - \epsilon\mu - s) - f(\lambda - s)]d\mu\Big\|_X. \quad (5.20)\end{aligned}$$

Now let $\eta > 0$. Pick $0 < \delta < K$ such that $\|f(x) - f(y)\|_X < \eta$ whenever $|x - y| < \delta$. Then, if $\epsilon < \delta/K$ we have $|\epsilon\mu| < \delta$ for all $\mu \in [-K, K]$. So

$$\|f(\lambda - \epsilon\mu - s) - f(\lambda - s)\|_X < \eta \text{ for all } \lambda \in \mathbb{R},\ s \in B(t, r) \text{ and } \mu \in [-K, K].$$

Hence from (5.20) we have, for $\epsilon < \delta/K$, $\lambda \in \mathbb{R}$ and $s \in B(t, r)$,

$$\|\psi_\epsilon^{(s)} * f(\lambda) - R_s f(\lambda)\|_X < \eta.$$

Now let $M_\epsilon = \sup_{s\in B(t,r)} \sup_{\lambda\in B(s,K)} \|\psi_\epsilon^{(s)} * f(\lambda) - R_s f(\lambda)\|_X$. The preceding argument shows that $\lim_{\epsilon\to 0} M_\epsilon = 0$. Let $\theta \in C_c^\infty(\mathbb{R})$ be a bump function such that $\theta(\lambda) = 1$ for $\lambda \in B(t, 2K + r)$. Then

$$\|\psi_\epsilon^{(s)} * f(\lambda) - R_s f(\lambda)\|_X \le M_\epsilon \theta(\lambda) \quad \text{for all } \lambda \in \mathbb{R} \text{ and } s \in B(t, r).$$

Again, the left hand side and $M_\epsilon\theta$ are both members of S. But $\|\cdot\|_S$ is a lattice norm and hence $\|\psi_\epsilon^{(s)} * f - R_s f\|_{S_X} \le M_\epsilon\|\theta\|_S$. Letting $\epsilon \to 0$ completes the proof of (iii). □

Proof of Proposition 5.2.
(i) The space $\bigcup_1^\infty\{E(n) - E(-n)\}S_X \equiv Y_0$ is dense in S_X. So, it suffices to show that for a fixed $n \ge 1$ and $f \in \{E(n) - E(-n)\}S_X$, the map $t \mapsto R_t f$ is continuous. Observe that for all $a \ge n$ and $t \in \mathbb{R}$, we have

$$\int_{-a}^{a} e^{i\lambda t}dE(\lambda)f = \int_{-n}^{n} e^{i\lambda t}dE(\lambda)f.$$

Since $\{E(\lambda)\}$ is the spectral family of $\{R_t\}$, equation (5.1) in Definition 5.2 gives $R_t f = \int_{-n}^{n} e^{i\lambda t}dE(\lambda)f$ for all $t \in \mathbb{R}$. Now let t be fixed and let $\xi \in S'_{X^*}$. Then, integrating by parts,

$$\begin{aligned}\big|\langle(R_t - R_s)f, \xi\rangle\big| &= \Big|\int_{-n}^{n}(e^{i\lambda t} - e^{i\lambda s})\langle dE(\lambda)f, \xi\rangle\Big| \\ &\le \|E\|_\infty\|f\|_{S_X}\|\xi\|\Big\{|e^{int} - e^{ins}| + \int_{-n}^{n}|te^{i\lambda t} - se^{i\lambda s}|d\lambda\Big\}.\end{aligned}$$

Since t and n are fixed, it is clear that the term in the braces vanishes as $s \to t$. Thus the right side vanishes uniformly in $\xi \in B_{X^*}$, which implies that

$\|(R_t - R_s)f\| \to 0$ as $s \to t$. Hence by Lemma 5.5 $\{R_t\}$ is strongly continuous on all of S_X.

To prove (ii), let $\phi \in C_c^\infty(\mathbb{R})$ be an even, non-negative function such that $\int \phi(t)dt = \|\phi\|_{L^1(\mathbb{R})} = 1$. Define ϕ_ϵ and $\psi_\epsilon^{(t)}$ as in Lemma 5.6. As before, for any $f \in C_c^\infty(X)$ let $\phi_\epsilon * f$ denote the usual convolution. Since $\phi(s) = \phi(-s)$ for all $s \in \mathbb{R}$, $\phi_\epsilon * f(t) = \int_{\mathbb{R}} \phi_\epsilon(-s)f(t-s)ds = \int_{\mathbb{R}} \phi_\epsilon(s)f(t+s)ds$. But the right hand side is precisely $\int_{\mathbb{R}} \phi_\epsilon(s)R_{-s}f ds(t)$ and so $\phi_\epsilon * f(t) = \hat{\phi}_\epsilon(R)f(t)$ pointwise for all $t \in \mathbb{R}$. Now let $f \in C_c^\infty(X)$ and choose $K > 0$ such that supp$(f)\cup$ supp$(\phi) \subset [-K, K]$. Then, by Lemma 5.6 (ii), $\lim_{\epsilon\to 0} \|\phi_\epsilon * f - f\|_{S_X} = 0$. In other words, $\lim_{\epsilon\to 0} \|\hat{\phi}_\epsilon(R)f - f\|_{S_X} = 0$. Furthermore, for any fixed t we have the following convergence, uniform in $s \in B(t, 1)$:

$$\lim_{\epsilon\to 0} \|(\psi_\epsilon^{(s)})^\wedge(R)f - R_s f\|_{S_X} \equiv \lim_{\epsilon\to 0} \|\psi_\epsilon^{(s)} * f - R_s f\|_{S_X} = 0. \tag{5.21}$$

This is again using Lemma 5.6 (iii) with $\delta = 1$. Suppose now $t \in \mathbb{R}$ and $0 < \eta < 1$ are given. For any $\epsilon > 0$ and $s \in B(t, 1)$, we have $\psi_\epsilon^{(t)} - \psi_\epsilon^{(s)} \in C_c^\infty(X)$ and so (5.19) gives

$$\|\psi_\epsilon^{(t)}(R)f - \psi_\epsilon^{(s)}(R)f\|_{S_X} \le \gamma\|(\psi_\epsilon^{(t)} - \psi_\epsilon^{(s)})^\wedge\|_{BV}\|f\|_{S_X}.$$

Using Fourier inversion we have,

$$\begin{aligned} 2\pi(\psi_\epsilon^{(t)}(\xi) - \psi_\epsilon^{(s)})^\wedge(\xi) &= \frac{1}{\epsilon}\int e^{-ix\xi}\phi(\frac{x-t}{\epsilon})dx - \frac{1}{\epsilon}\int e^{-ix\xi}\phi(\frac{x-s}{\epsilon})dx \\ &= \frac{e^{-it\xi}}{\epsilon}\int e^{-i(x-t)\xi}\phi(\frac{x-t}{\epsilon})dx - \frac{e^{-is\xi}}{\epsilon}\int e^{-i(x-s)\xi}\phi(\frac{x-s}{\epsilon})dx \\ &= 2\pi(e^{-it\xi} - e^{-is\xi})\hat{\phi}(\epsilon\xi). \end{aligned} \tag{5.22}$$

Hence

$$\frac{d}{d\xi}(\psi_\epsilon^{(t)} - \psi_\epsilon^{(s)})^\wedge(\xi) = -(ite^{-it\xi} - ise^{-is\xi})\hat{\phi}(\epsilon\xi) + \epsilon(e^{-it\xi} - e^{-is\xi})(\hat{\phi})'(\epsilon\xi).$$

So, for $s \in B(t, 1)$,

$$\Big|\frac{d}{d\xi}(\psi_\epsilon^{(t)} - \psi_\epsilon^{(s)})^\wedge(\xi)\Big| \le (2|t|+1)|\hat{\phi}(\epsilon\xi)| + 2|(\hat{\phi})'(\epsilon\xi)|.$$

Further, from (5.22), $\lim_{s\to t} \frac{d}{d\xi}(\psi_\epsilon^{(t)}(\xi) - \psi_\epsilon^{(s)})^\wedge(\xi) = 0$ pointwise. Observing that $\hat{\phi}$ and $(\hat{\phi})'$ are in $L^1(\mathbb{R})$, Dominated Convergence Theorem gives $\lim_{s\to t} \int \big|\frac{d}{d\xi}(\psi_\epsilon^{(t)} - \psi_\epsilon^{(s)})^\wedge(\xi)\big|d\xi = 0$, and so $\lim_{s\to t} \big\|(\psi_\epsilon^{(t)} - \psi_\epsilon^{(s)})^\wedge\big\|_{BV} = 0$. Now we can write

$$\begin{aligned} \|R_t f - R_s f\|_{S_X} &\le \|R_t f - \psi_\epsilon^{(t)}(R)f\|_{S_X} \\ &+ \|\psi_\epsilon^{(t)}(R)f - \psi_\epsilon^{(s)}(R)f\|_{S_X} + \|R_s f - \psi_\epsilon^{(s)}(R)f\|_{S_X}. \end{aligned}$$

Using (5.21) we can choose $\epsilon > 0$ such that $\|R_s f - \psi_\epsilon^{(s)}(R)f\|_{S_X} < \frac{\eta}{3}$ for all $s \in B(t,1)$. Then we pick $\delta < 1$ such that $s \in B(t,\delta)$ implies $\|(\psi_\epsilon^{(t)} - \psi_\epsilon^{(s)})^\wedge\|_{BV} < \frac{\eta}{3\gamma\|f\|}$. Then $s \in B(t,\delta)$ implies that $\|R_t f - R_s f\|_{S_X} < \eta$.
Hence $\{R_t\}$ is strongly continuous for all $f \in C_c^\infty(\mathbb{R}, X)$. So, by Lemma 5.5 it is strongly continuous for all $f \in S_X$. □

Corollary 5.1. *Let X be a reflexive Banach space and S a scalar valued function space satisfying Definition 5.3. Suppose $\{R_t\}$ is locally bounded on S_X. Then the conclusion of Theorem 5.1 still holds. Moreover, the spectral family $\{E(\lambda)\}$ of $\{R_t\}$ is strongly continuous on the left.*

The following Lemma helps prove the left continuity of E.

Lemma 5.7. *Let S, and X be as in Corollary 5.1. Let $a \in \mathbb{R}$ be fixed. Suppose $g : \mathbb{R} \to X$ is a strongly measurable function such that for all $s \in \mathbb{R}$ we have $R_s g \in S_X$ and $R_s g = e^{isa} g$. Then $g \equiv 0$.*

Proof. Since $\|.\|_{S_X}$ is an absolutely continuous norm we know that $\|g\mathbb{I}_{[n,\infty)}\|_{S_X} \searrow 0$ as $n \to \infty$. But now, taking $s = 1$ we have

$$\|g\mathbb{I}_{[n,\infty)}\|_{S_X} = \|e^{-ia} g(\cdot - 1)\mathbb{I}_{[n,\infty)}\|_{S_X} = \|g\mathbb{I}_{[n+1,\infty)}\|_{S_X}.$$

Hence

$$\|g\mathbb{I}_{[n,\infty)}\|_{S_X} = \|g\mathbb{I}_{[m,\infty)}\|_{S_X} \quad \text{for all } n, m \in \mathbb{R}.$$

So these must all be zero. Hence $g\mathbb{I}_{[n,\infty)} \equiv 0$ for all n, so we must have $g \equiv 0$. □

Proof of Corollary 5.1. Let us first show that the conclusion of Theorem 5.1 holds. Since S_X is reflexive, we need only show that statements 5.1 (i) and the first part of 5.1 (ii) are equivalent. Now, if (i) is true, then by Proposition 5.2 (i) $\{R_t\}$ is strongly continuous. Hence it satisfies the full conditions of Theorem 5.1 (i) and so (ii) of Theorem 5.1 holds.
Conversely, if Theorem 5.1 (ii) is true, then by Proposition 5.2 (ii), $\{R_t\}$ is again strongly continuous. Thus the full conditions of Theorem 5.1 (ii) hold and so Theorem 5.1 (i) follows.

Now let us show that the spectral family of $\{R_t\}$ is left-continuous. Suppose, on the contrary, that it is not. Then there exists $a \in \mathbb{R}$ such that $E(a^-) \neq E(a)$. So there is some $f \in C_c^\infty(X)$ such that $g \equiv \{E(a) - E(a^-)\}f \neq 0$ and $g \in S_X$. Then, for any $s \in \mathbb{R}$, $R_s g \neq 0$ and

$$R_s g = \lim_{b\to\infty} \int_{-b}^{b} e^{is\lambda} dE(\lambda) g.$$

Observe that, by definition of g, $\int_{-b}^{b} e^{is\lambda} dE(\lambda)g = \int_{-a}^{a} e^{is\lambda} dE(\lambda)g$ for any $b \geq a$. Hence $R_s g = \int_{-a}^{a} e^{is\lambda} dE(\lambda)g$. Now, for a partition $u = \{-a = \lambda_0 < ... < \lambda_N = a\} \in \mathcal{P}_{[-a,a]}$, let

$$\mathcal{S}(u,s)g = e^{isa}E(a)g - \sum_{j=1}^{N}\{e^{is\lambda_j} - e^{is\lambda_{j-1}}\}E(\lambda_{j-1})g.$$

Then

$$R_s g = \lim_{u \in \mathcal{P}_{[-a,a]}} \mathcal{S}(u,s)g \qquad \text{in } \|\cdot\|_{S_X} \text{ norm.} \tag{5.23}$$

Now note that $E(a)g = g$ and that for any $\lambda \in (-\infty, a)$, $E(\lambda)g = 0$. This is because $E(\lambda)g = \lim_{c \nearrow a} E(\lambda)\{E(a) - E(c)\}f$. But the right hand side is zero, since
$E(\lambda)\{E(a) - E(c)\} = 0$ for all $c \in (\lambda, a)$.
Hence, for any $u \in \mathcal{P}_{[-a,a]}$, $\mathcal{S}(u,s)g = e^{isa}g$ and so (5.23) gives that $R_s g = e^{isa}g$. But this now holds for all $s \in \mathbb{R}$ and so by Lemma 5.7 $g \equiv 0$. But this contradicts our original assumption on a and f and so there is no $a \in \mathbb{R}$ at which $E(a) \neq E(a^-)$. □

5.2.2 The Hilbert Transform

Let us tie the ideas from the Section 5.2 with the Hilbert transform. Recall from chapter 1 the operators H_ϵ and H on the scalar-valued $L^p(\mathbb{R})$ spaces:

$$H_\epsilon f(t) = \int_{\epsilon \leq |s| \leq 1/\epsilon} \frac{1}{s} f(t-s)\, ds, \qquad \text{and} \quad Hf(t) = \lim_{\epsilon \to 0} H_\epsilon f(t).$$

We know from [38] that for $1 < p < \infty$, H is bounded from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ and $\sup_{\epsilon>0} \|H_\epsilon\| \leq 2\|H\|$. This state of affairs remains the same when we replace $L^p(\mathbb{R})$ with $L^p_X(\mathbb{R})$, with X UMD. It is therefore of interest to examine those function spaces S_X which have the property that both $\|H\| < \infty$ and $\sup_{\epsilon>0} \|H_\epsilon\| < \infty$. In fact, if S_X has the Fatou property then the latter condition implies the former. In this case, we obtain the same final conclusion as in Theorem 5.2 (inequality (5.3)), with the right translations $\{R_t\}$ replacing the arbitrary group $\{U_t\}$.

Theorem 5.4. *Let S_X be a function space as in Definition 5.3, and let $\{R_t\}$ be the right translation group on S_X. Suppose $\sup_{\epsilon>0} \|H_\epsilon\|_{S_X \to S_X} = K < \infty$. Then there exists a constant $\gamma > 0$ such that for each $\phi \in C_c^\infty(\mathbb{R})$,*

$$\Big\| \int_{\mathbb{R}} \phi(\lambda) R_{-\lambda} f \, d\lambda \Big\|_{S_X} \leq \gamma \|\hat{\phi}\|_{BV} \; \|f\|_{S_X} \quad \textit{for all } f \in C_c^\infty(X).$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R})$ be fixed and let $f \in C_c^\infty(X)$. Let $\epsilon > 0$ be small enough so that $\text{supp}(\phi)\cup \text{ supp}(f) \subseteq [-1/\epsilon, 1/\epsilon]$ and write $\phi_\epsilon = \phi \mathbb{I}_{\{|\lambda|\le\epsilon\}}$. Note that for $t \in \mathbb{R}$, the function $\lambda \mapsto \phi(\lambda)f(t+\lambda)$ is continuous on $[-1/\epsilon, 1/\epsilon]$, so that $\int_{\mathbb{R}} \phi(\lambda)R_{-\lambda}f(t)\, d\lambda \equiv \int_{|\lambda|\le 1/\epsilon} \phi(\lambda)f(t+\lambda)d\lambda$ makes sense pointwise at each $t \in \mathbb{R}$. Splitting this integral gives

$$\int_{\mathbb{R}} \phi(\lambda)R_{-\lambda}f(t)\, d\lambda = \int_{\epsilon\le|\lambda|\le 1/\epsilon} \phi(\lambda)f(t+\lambda)d\lambda \;+\; \int_{|\lambda|\le\epsilon} \phi(\lambda)f(t+\lambda)d\lambda. \quad (5.24)$$

In the first integral we use Fourier inversion on ϕ, integration by parts and Fubini's theorem respectively to obtain

$$\begin{aligned}
\int_{\epsilon\le|\lambda|\le 1/\epsilon} \phi(\lambda)f(t+\lambda)d\lambda &= \int_{\epsilon\le|\lambda|\le 1/\epsilon} \Big\{ \int_{s\in\mathbb{R}} \hat{\phi}(s)e^{is\lambda} f(t+\lambda)ds \Big\} d\lambda \\
&= i\int_{\epsilon\le|\lambda|\le 1/\epsilon} \Big\{ \int_{s\in\mathbb{R}} \frac{1}{\lambda}(\hat{\phi})'(s)e^{-ist}e^{is(\lambda+t)} f(t+\lambda)ds \Big\} d\lambda \\
&= i\int_{s\in\mathbb{R}} \int_{\epsilon\le|\lambda|\le 1/\epsilon} \frac{1}{\lambda}(\hat{\phi})'(s)e^{-ist}e^{is(\lambda+t)} f(t+\lambda)d\lambda\; ds \\
&= i\int_{s\in\mathbb{R}} (\hat{\phi})'(s)M_{-s}H_\epsilon M_s f(t)ds \qquad (5.25)
\end{aligned}$$

where, as before, $M_s f(t) = e^{ist}f(t)$. The use of Fubini's theorem is justified because $\int_{s\in\mathbb{R}} \Big\{ \int_{\epsilon\le|\lambda|\le 1/\epsilon} \frac{1}{|\lambda|} |(\hat{\phi})'(s)| d\lambda \Big\} ds \le 4\|\phi\|_{BV} \log(1/\epsilon) < \infty$.

Let us write Q_ϵ for the convolution by the function $\phi_\epsilon(-t)$. Substituting this and (5.25) into (5.24) gives

$$\int_{\mathbb{R}} \phi(\lambda)R_{-\lambda}f\, d\lambda = i\int_{s\in\mathbb{R}} (\hat{\phi})'(s)M_{-s}H_\epsilon M_s f\, ds \;+\; Q_\epsilon f$$

for all $\epsilon > 0$ small enough. We shall show that $\lim_{\epsilon\to 0} \|Q_\epsilon f\|_{S_X} = 0$. To that end, we fix $K > 0$ such that $\text{supp}(\phi)\cup \text{supp}(f) \subseteq [-K, K]$ and consider only $\epsilon \in (0, K)$. Observe that if $|t+\lambda| > K$ for all $\lambda \in (-\epsilon, \epsilon)$, we have $\int_{-\epsilon}^{\epsilon} \phi(\lambda)f(t+\lambda)\, d\lambda = 0$. Hence, $\text{supp}(Q_\epsilon f) \subset [-2K, 2K]$. Let

$$M_\epsilon = \sup_{t\in\mathbb{R}} \int_{-\epsilon}^{\epsilon} \|\phi(\lambda)f(t+\lambda)\|_X\, d\lambda \;\le\; 2\epsilon\, \|\phi\|_\infty\, \|f\|_\infty.$$

Observe that $\lim_{\epsilon\to 0} M_\epsilon = 0$. Let $\theta \in C_c^\infty$ be a bump function which is identically 1 on $[-2K, 2K]$. Then $\big\| \int_{-\epsilon}^{\epsilon} \phi(\lambda)f(t+\lambda)\, d\lambda \big\|_X \le\; M_\epsilon\theta(t)$ for all $t \in \mathbb{R}$. Hence $\big\|Q_\epsilon f\big\|_{S_X} \le\; M_\epsilon\|\theta\|_S$. But the latter tends to zero as $\epsilon \to \infty$, and this gives the required convergence for $\|Q_\epsilon f\|_{S_X} \to 0$.

Hence, using (5.24) and (5.25), we have for $f \in C_c^\infty(X)$ and all $\epsilon > 0$ small enough,

$$\begin{aligned}
\big\| \int_{\mathbb{R}} \phi(\lambda)R_{-\lambda}f\, d\lambda \big\|_{S_X} &\le \big\| \int_{s\in\mathbb{R}} (\hat{\phi})'(s)M_{-s}H_\epsilon M_s f\, ds \big\|_{S_X} \;+\; \|Q_\epsilon f\|_{S_X} \\
&\le \|\hat{\phi}\|_{BV}\, \sup_{s\in\mathbb{R}} \|M_{-s}H_\epsilon M_s f\|_{S_X} \;+\; \|Q_\epsilon f\|_{S_X} \\
&\le \|\hat{\phi}\|_{BV}\, \|H_\epsilon\|\, \|f\|_{S_X} \;+\; \|Q_\epsilon f\|_{S_X}. \qquad (5.26)
\end{aligned}$$

Now, $\sup_{\epsilon>0} \|H_\epsilon\| = K < \infty$, so letting ϵ tend to zero in (5.26) gives

$$\Big\| \int_{\mathbb{R}} \phi(\lambda) R_{-\lambda} f \; d\lambda \Big\|_{S_X} \leq K \|\hat{\phi}\|_{BV} \; \|f\|_{S_X}.$$

Hence, as $C_c^\infty(X)$ is dense in S_X, there exists a linear extension $\hat{\phi}(R)$ of $\int_{\mathbb{R}} \phi(\lambda) R_{-\lambda} d\lambda$ to all of S_X satisfying $\|\hat{\phi}(R)\| \leq K \|\hat{\phi}\|_{BV}$ as claimed. $\square$

This theorem has implications for the right translation group $\{R_t\}$ on S_X. The domain of each R_t is naturally defined as $\mathcal{D}(R_t) = \{f : \; f(\cdot - t) \in S_X\}$, and the space $C_c^\infty(X) \subset \cap_{t\in\mathbb{R}} \mathcal{D}(R_t)$ is dense in S_X. Furthermore, by Proposition 5.2 (ii), $\{R_t\}$ is strongly continuous on $C_c^\infty(X)$. Hence $\{R_t\}$ is a densely defined operator group on S_X, satisfying Definition 5.1 and the conditions of Theorem 5.3. Hence we have the following Corollary.

Corollary 5.2. *Let S_X be a function space as in Theorem 5.4. Then the right translation group $\{R_t\}$ has a densely defined spectral decomposition, in the sense that there exists a spectral family $\{E(\lambda)\}$ on S_X such that for each $t \in \mathbb{R}$,*

$$R_t f = \lim_{a\to\infty} \int_{-a}^{a} e^{i\lambda t} dE(\lambda), \qquad \textit{for all } f \in C_c^\infty(X).$$

5.2.3 Sequence Spaces

Note that Theorem 5.4 gives a one-way implication: it does not show that H is bounded on S_X if $\{R_t\}$ satisfies inequality (5.3). However, if we replace $(\mathbb{R}, dt)$ with $(\mathbb{Z}, dn)$, then the converse of Theorem 5.4 is also true. The setup for scalar sequence spaces is somewhat easier than that of $\mathbb{R}$-function spaces as there are no measure theoretic issues to deal with. So, we begin with a scalar valued sequence space $S \subset l_\infty$ equipped with a complete *lattice norm* $\|\cdot\|_S$ which has the *Fatou property.* Furthermore, l_0, the space of finitely supported sequences, is norm dense in S.

Definition 5.5. *Let S be a scalar sequence space as described above. We call $S_X(\mathbb{Z}) \subset l_\infty(X)$ an **X-valued sequence space** if:*
(i) $\{x_k\}_{k\in\mathbb{Z}} \in S_X(\mathbb{Z})$ if and only if $\big\{\|x_k\|_X\big\}_{k\in\mathbb{Z}} \in S$ and define $\|\{x_k\}\|_{S_X(\mathbb{Z})} = \big\|\{\|x_k\|_X\}\big\|_S$.
(ii) l_X^0, the space of finitely supported X-valued sequences, is norm dense in $S_X(\mathbb{Z})$.
(iii) The projection operators $P_k : S_X(\mathbb{Z}) \to S_X(\mathbb{Z}), \quad \{x_j\} \mapsto (...0, x_k, 0...)$ are all bounded.

Analogous to $\{R_t\}_{t\in\mathbb{R}}$ we now have the right shift group $\{R^k\}_{k\in\mathbb{Z}}$ generated by the right shift operator R. So, for $x \in S_X(\mathbb{Z})$, $(R^k x)_j = x_{j-k}$. We can now formulate the ideas of Section 5.2 entirely analogously. Recall that the algebra $\{\hat{\phi} : \phi \in C_c^\infty\}$ is $\|\cdot\|_{BV}$-dense in $AC_0(\mathbb{R})$. Similarly, $\{p(e^{it}) \in C(\mathbb{T}) : \hat{p} \in l_0\}$ is $\|\cdot\|_{BV}$-dense in $AC(\mathbb{T})$. But this is precisely the collection of trigonometric polynomials $p(e^{it}) = \sum_{-N}^{N} \hat{p}_k e^{ikt}$. So the operators $\int_{\mathbb{R}} \phi(t) R_{-t} dt$ now get replaced by $p(R) = \sum_{-N}^{N} \hat{p}_k R^k$. Moreover, inequality (5.3) is now replaced by

$$\|p(R)\| \leq \gamma \|p\|_{BV(\mathbb{T})}. \tag{5.27}$$

The picture we now have contrasts with that in Section 5.2, because we now have a complete relationship between the Hilbert transform and inequality (5.27).

Theorem 5.5. *Let S be a sequence space satisfying Definition 5.5 and X be a reflexive Banach space. Let $S_X(\mathbb{Z})$ be defined as above. Then the following two are equivalent.*
(i) The discrete Hilbert transform is a bounded operator from $S_X(\mathbb{Z})$ into itself.
(ii) The right shift operator on $S_X(\mathbb{Z})$ is trigonometrically well-bounded.

Proof. To show that (i) implies (ii), it suffices, by Proposition 1.1, to find a $\gamma > 0$ such that for all trigonometric polynomials $q(e^{it})$, inequality (5.27) is satisfied. To fix notation, let $f * g$ denote the usual convolution of sequences. This is well-defined pointwise if $f \in l_0$ and g is an arbitrary X valued sequence. Now, for any $x \in S_X(\mathbb{Z})$, $q(R)x = \hat{q} * x$, because

$$(q * x)_n = \sum_{|k| \leq N} \hat{q}_k x_{n-k} = \sum_{|k| \leq N} \hat{q}_k (R^k(x))_n = \Big(q(R)(x)\Big)_n. \tag{5.28}$$

We define as before the operator $M_t : S_X(\mathbb{Z}) \to S_X(\mathbb{Z})$ by $(x_k) \mapsto (x_k e^{ikt})$. This operator is bounded, by definition of $S_X(\mathbb{Z})$. Let $x \in l_X^0$ and let us compute $\frac{i}{2\pi} \int_{\mathbb{T}} q^{'}(e^{it}) M_{-t} H M_t(x)\, dt$.
Pick N such that $x_k = \hat{q}_k = 0$ for $|k| > N$. Then we have

$$\begin{aligned}
\frac{i}{2\pi} \int_{\mathbb{T}} q'(e^{it}) \big(M_{-t} H M_t(x)\big)_n \, dt \;&=\; \frac{i}{2\pi} \int_{\mathbb{T}} q'(e^{it}) e^{-int} \big(H M_t(x)\big)_n \, dt \\
&= \frac{i}{2\pi} \int_{\mathbb{T}} q'(e^{it}) e^{-int} \sum_{k=n-N, k\neq 0}^{n+N} \frac{1}{k} e^{i(n-k)t} x_{n-k} \, dt \\
&= \sum_{|k| \leq N,\; k \neq 0} \hat{q}_k x_{n-k} \;+ \frac{i}{2\pi} \sum_{N+1}^{N+n} \int_{\mathbb{T}} q'(e^{it}) \frac{1}{k} e^{-ikt} x_{n-k} \, dt \\
&= \sum_{|k| \leq N,\; k \neq 0} \hat{q}_k x_{n-k} + \sum_{N+1}^{N+n} \hat{q}_k \frac{1}{k} x_{n-k} \;=\; \sum_{|k| \leq N,\; k \neq 0} \hat{q}_k x_{n-k}.
\end{aligned}$$

Now, since the coordinate projections are bounded, we have,

$$\int_{\mathbb{T}} q'(e^{it}) \big(M_{-t} H M_t(x)\big)_n \, dt = \Big(\int_{\mathbb{T}} q'(e^{it}) M_{-t} H M_t(x) dt \Big)_n. \tag{5.29}$$

So, using (5.28) and the preceding calculation we have
$\Big(q(R)(x)\Big)_n = \hat{q}_0 x_n + \frac{i}{2\pi}\Big(\int_{\mathbb{T}} q'(e^{it}) M_{-t} H M_t(x) dt\Big)_n$. Thus for $x \in l^0_X$,
$q(R)x = \hat{q}_0 x + \frac{i}{2\pi}\int_{\mathbb{T}} q'(e^{it}) M_{-t} H M_t x\ dt$.
Hence

$$\begin{aligned} \|q(R)x\| &\leq \Big\{|\hat{q}_0| + \frac{1}{2\pi}\ var_{\mathbb{T}}(q)\ \|H\|\Big\}\ \|x\| \\ &\leq \Big\{1 + \frac{1}{2\pi}\ \|H\|\Big\}\ \|x\|\ \|q\|_{BV}. \end{aligned}$$

Since l^0_X is dense in $S_X(\mathbb{Z})$, it follows that $\|q(R)\| \leq \Big\{1 + \frac{1}{2\pi}\ \|H\|\Big\}\ \|q\|_{BV}$. Hence inequality (5.27) is established with $\gamma = \Big\{1 + \frac{1}{2\pi}\ \|H\|\Big\}$.

Let us prove the converse. For $x \in l^0_X$ the Fourier transform is well-defined pointwise for all $e^{it} \in \mathbb{T}$, as $\hat{x}(e^{ikt}) = \frac{1}{2\pi}\sum_k x_k e^{-ikt}$ (note that this is a finite sum). For an integrable X-valued function $f : \mathbb{T} \to X$, the inverse Fourier transform is given by $\check{f}_k = \int_{\mathbb{T}} f(e^{it}) e^{ikt} dt$. This is again well-defined pointwise. It is easily seen that for any integrable scalar-valued g and X-valued f, $\check{f} * \check{g} = (fg)^{\vee}$. Now let $\{h_k\}$ denote the discrete Hilbert kernel, so $h_k = \frac{1}{k}$ for $k \neq 0$ and $h_0 = 0$. Let $\psi : \mathbb{T} \to \mathbb{C}, \quad e^{it} \mapsto i(\pi - t)$, so that $\check{\psi}_k = h_k$. Then for $x \in l^0_X$, we have $Hx = h * x = \check{\psi} * x = (\psi\hat{x})^{\vee}$. Now let κ_n denote the n^{th} Fejer kernel on $\mathbb{T}$ and let $p^{(n)} = \kappa_n * \psi$. Then $\{p^{(n)}\}$ is a sequence of trigonometric polynomials such that $\lim_{n\to\infty} |p^{(n)}(e^{it}) - \psi(e^{it})| = 0$ pointwise and $\sup_n \|p^{(n)}\|_{BV} \leq \|\psi\|_{BV} < \infty$. Then, by bounded convergence,

$$\lim_{n\to\infty} |(p^{(n)})^{\vee}_j - h_j| = 0 \qquad \text{for each } j \in \mathbb{Z}. \tag{5.30}$$

Now fix $x \in l^0_X$ and let M be an integer such that $x_k = 0$ for all $|k| > M$. Then, since $\big((p^{(n)})^{\vee} * x\big)_k = \sum_{j=k-M}^{k+M} (p^{(n)})^{\vee}_j x_{k-j}$, we have that

$$\lim_{n\to\infty} \big\|\big((p^{(n)})^{\vee} * x\big)_k - (h * x)_k\big\|_X = 0 \ \text{ for each } k \in \mathbb{Z}.$$

This follows by applying (5.30) to each $j \in \{k - M, ..., k + M\}$. Hence
$\lim_{n\to\infty} \big\|\big((p^{(n)})^{\vee} * x\big)_k\big\|_X - \big\|(h * x)_k\big\|_X = 0$. Now, using the definition of $\|\cdot\|_{S_X(\mathbb{Z})}$ and the fact that $\|\cdot\|_S$ has the Fatou property, we have

$$\begin{aligned} \|h * x\|_{S_X} &= \Big\|\ \{\|(h * x)_k\|_X\}\ \Big\|_S \leq \liminf_n \Big\|\ \{\|\big((p^{(n)})^{\vee} * x\big)_k\|_X\}\ \Big\|_S \\ &= \liminf_n \Big\|\ \{\big((p^{(n)})^{\vee} * x\big)_k\}\ \Big\|_{S_X(\mathbb{Z})} = \liminf_n \|p^{(n)}(R)x\|_{S_X(\mathbb{Z})}. \end{aligned}$$

Now, since R is trigonometrically well bounded, there is a $\gamma > 0$ such that $\|p^{(n)}(R)\| \leq \gamma\|p^{(n)}\|_{BV} \leq \gamma\|\psi\|_{BV}$ for all $n \geq 1$ (see Proposition 1.1). Observing that $\|\psi_{BV} = (1 + 2\pi)$ we have for $x \in l^0_X$,

$$\|h * x\|_{S_X(\mathbb{Z})} = \|Hx\|_{S_X(\mathbb{Z})} \leq \gamma(1 + 2\pi)\|x\|_{S_X(\mathbb{Z})}.$$

Finally, l_X^0 is dense in $S_X(\mathbb{Z})$ and so H is a bounded operator from $S_X(\mathbb{Z})$ into itself, and $\|H\| \leq \gamma(1+2\pi)$. □

We have already seen an example of an $\mathbb{R}$-function space (the $L^p(w)$ in section 5.1) where $\|H\| < \infty$ and yet $\{R_t\}$ does not have an everywhere defined spectral decomposition. This anomaly does not occur in scalar valued sequence spaces $l_p(w)$. Here the boundedness of the Hilbert transform is equivalent to R, the right shift operator on $l_p(w)$, being trigonometrically well bounded (so that, in particular, the group $\{R^k\}$ has an everywhere defined spectral decomposition).

The accounts in sections 5.2.2 and 5.2.3 show that this equivalence extends to more general X valued sequence spaces $S_X(\mathbb{Z})$, but not $S_X(\mathbb{R})$.

5.3 Closing Remarks

The main thrust of this thesis is in the spirit of vector valued harmonic analysis and operator theory; its aim is to establish new connections between geometric properties of a space X and operators acting on it. This has been a key idea behind previous work on UMD spaces and the Hilbert transform. In view of this, chapters 2 and 4 make a useful contribution. Chapter 3 essentially shows that the extra geometric features in a Hilbert space do not yield a significant improvement on the super-reflexive setting.

Some questions do, however, remain open to further investigation.
1. Conjecture 4.1 in chapter 4 remains elusive. It may be that extra geometric conditions have to be imposed on X for it to hold. One such property worth investigating is Pisier's property (α).

2. Section 2.3 on uniformly smooth spaces was closed off with a question about $L^p(\mu)$ spaces for an arbitrary measure space (Ω, μ) and $1 < p < \infty$. It conjectures that any trigonometrically well bounded operator T on $L^p(\mu)$ can have at most $O(|n|^{1/p})$ power growth. Again, this remains open to settle.

As we have just seen in chapter 5, if the Hilbert transform is bounded from $S_X(\mathbb{R})$ into itself, then the right translation group $\{R_t\}$ has a densely defined spectral decomposition in terms of a spectral family. Indeed, if X is UMD, and $S_X(\mathbb{R}) = L_X^p$ $(1 < p < \infty)$, the converse is also true. But it remains open for more general function spaces.

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LIST OF NOTATION

The following is a list of notation used in this thesis.

$AC(J)$ Banach algebra of absolutely continuous functions on the compact interval J. The same definition holds with $\mathbb{T}$ or $\mathbb{R}$ in place of J.

$\mathcal{B}(X)$ The space of bounded linear operators on a Banach space X.

B_X The unit ball of a space X, $\{x \in X : \|x\| \leq 1\}$.

$\mathfrak{B}(\Omega)$ The Borel σ algebra of a topological space Ω.

$B(t, \delta)$ The unit ball of radius δ, centered at t.

$BV_p(J)$ Banach algebra of complex-valued functions of bounded p-variation, $1 \leq p < \infty$, on the interval J. Analogous definition holds with $\mathbb{T}$ or $\mathbb{R}$ in place of J.

$char(E)$ The characteristic of a spectral family E, given by

$$char(E) = \sup\{K > 0 : K\|\{E(b) - E(c)\}x\| \leq \|\{E(a) - E(b)\}x\| \text{ for all } x \in X, \quad d < c < b < a\}$$

$\mathcal{D}(T)$ The domain of an operator T on a space X.

$\delta_X(\epsilon)$ The modulus of convexity of a Banach space X, defined for $\epsilon \in [0, 2]$ as

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| \ : \ \|x\| = \|y\| = 1 \text{ with } \|x - y\| \geq \epsilon \ \}$$

$\|E\|_\infty$ For a given spectral family $\{E(\lambda)\}$, this is $\sup_{\lambda \in \mathbb{R}} \|E(\lambda)\|$.

H The Hilbert transform operator. See Definition 1.5.

$\mathbb{I}_{[a,b]}$ The indicator function of the interval $[a, b]$.

$L^p_X(G)$ The space of X valued strongly measurable p-Bochner integrable functions on the Haar group G.

l^0_X The space of finitely supported X valued sequences.

$M_{p,X}(G)$ The space of multipliers on $L^p_X(\hat{G})$.

$\mathfrak{M}_p(\mathbb{R})$ Banach algebra of complex-valued functions on $\mathbb{R}$ for which $\sup_{I_k} var_p(f, I_k) < \infty$, where I_k are the dyadic intervals of $\mathbb{R}$.

$\mathcal{P}_{[a,b]}$ The set of all partitions $u = \{a = \lambda_0 < ... < \lambda_N = b\}$ of the interval $[a, b]$ partially ordered and directed to increase by refinement.

$S_X(\mathbb{R})$, S_X A general X valued function space on $\mathbb{R}$. See Definition 5.3.

$S_X(\mathbb{Z})$ A general X valued sequence space. See Definition 5.5.

st-lim The limit in the strong operator topology; for example, $T = \text{st-}\lim_{n\to\infty} T_n$ means $\lim_{n\to\infty} \|T_n x - Tx\| = 0$ for all $x \in X$.

T_ψ The multiplier operator associated with the function ψ, namely the map $f \in C_c^\infty(X) \mapsto (\psi \hat{f})^\vee$.

$U_s^{(a)}$ The truncated operator , $\int_{-a}^{a} e^{i\lambda t} dE(\lambda)$, associated with a spectral family $\{E(\lambda)\}$.

$\Delta_k, \Delta_k^{(u)}$ The Schauder decomposition operator $\{E(\lambda_{k+1}) - E(\lambda_k)\}$ associated with a partition $u = \{\lambda_k\}$ of an interval J or $\mathbb{R}$.

$\phi(E)$ See Definition 2.6.

$\hat{\phi}(U)$ The continuous extension (if it exists) of the operator $\int \phi(t) U_{-t} dt$ defined on the dense subspace $X_M \subset \cap_{t\in\mathbb{R}} \mathcal{D}(U_t)$.