THE ALGEBRAIC CONSTRUCTION OF THE NOVIKOV COMPLEX OF A CIRCLE-VALUED MORSE FUNCTION

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Abstract. The Novikov complex of a circle-valued Morse function $f : M \to S^1$ is constructed algebraically from the Morse-Smale complex of the restriction of the real-valued Morse function $\overline{f} : \overline{M} \to \mathbb{R}$ to a fundamental domain of the pullback infinite cyclic cover $\overline{M} = f^*\mathbb{R}$ of M.

Introduction.

The relationship between real-valued Morse functions and chain complexes is well understood. The Morse-Smale complex of a Morse function $f : M \to \mathbb{R}$ on a compact m-dimensional manifold M is defined using a choice of gradient-like vector field v satisfying the transversality condition, to be a based f. g. free $\mathbb{Z}[\pi_1(M)]$-module chain complex $C^{MS}(M,f,v)$ with

$$\mathrm{rank}_{\mathbb{Z}[\pi_1(M)]} C_i^{MS}(M,f,v) \;=\; c_i(f)$$

the number of critical points of f with index i, and the differentials defined by counting the downward v-gradient flow lines in the universal cover $\widetilde{M}$ of M. The pair (f,v) determines a handlebody decomposition on M with one i-handle for each critical point of index i

$$M \;=\; \bigcup_{i=0}^{m} \bigcup_{c_i(f)} D^i \times D^{m-i} \;,$$

and the Morse-Smale complex is the cellular chain complex of $\widetilde{M}$

$$C^{MS}(M,f,v) \;=\; C(\widetilde{M}) \;.$$

The relationship between circle-valued Morse functions $f : M \to S^1$ and chain complexes is more complicated, and not so well understood. A lift of f to the pullback infinite cyclic cover $\overline{M} = f^*\mathbb{R}$ is a real-valued Morse function $\overline{f} : \overline{M} \to \mathbb{R}$ on a non-compact manifold, so traditional Morse theory does not apply directly. The methods developed (by Novikov, Farber, Pajitnov, the author and others) to count the critical points of f use the structure of the fundamental group ring $\mathbb{Z}[\pi_1(M)]$ as a Laurent polynomial extension of $\mathbb{Z}[\pi_1(\overline{M})]$, as well as a completion and a localization of $\mathbb{Z}[\pi_1(M)]$. For any map $f : M \to S^1$ with M and $\overline{M}$ connected the ring $\mathbb{Z}[\pi_1(M)]$ is the α-twisted Laurent polynomial extension

$$\mathbb{Z}[\pi_1(M)] \;=\; \mathbb{Z}[\pi_1(\overline{M})]_\alpha[z,z^{-1}] \;=\; \mathbb{Z}[\pi_1(\overline{M})]_\alpha[z][z^{-1}]$$

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with $\alpha : \pi_1(\overline{M}) \to \pi_1(\overline{M})$ the monodromy automorphism and

$$az \;=\; z\alpha(a) \;(a \in \mathbb{Z}[\pi_1(\overline{M})]) \;.$$

The completion

$$\mathbb{Z}[\widehat{\pi_1(M)}] \;=\; \mathbb{Z}[\pi_1(\overline{M})]_\alpha((z)) \;=\; \mathbb{Z}[\pi_1(\overline{M})]_\alpha[[z]][z^{-1}]$$

is called the *Novikov ring* of $\mathbb{Z}[\pi_1(M)]$. Let Σ be the set of square matrices in $\mathbb{Z}[\pi_1(\overline{M})]_\alpha[z] \subset \mathbb{Z}[\pi_1(M)]$ which become invertible in $\mathbb{Z}[\pi_1(\overline{M})]$ under the augmentation $z \mapsto 0$. The noncommutative localization $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ of $\mathbb{Z}[\pi_1(M)]$ (in the sense of Cohn [C]) is a ring with a morphism $\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ such that any ring morphism $\mathbb{Z}[\pi_1(M)] \to R$ which sends Σ to invertible matrices in R has a unique factorization $\mathbb{Z}[\pi_1(M)] \to \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to R$. The inclusion $\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\widehat{\pi_1(M)}]$ sends Σ to invertible matrices in $\mathbb{Z}[\widehat{\pi_1(M)}]$, so there is a natural ring morphism $\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\widehat{\pi_1(M)}]$.

A vector field on a manifold M is a section of the tangent bundle of M

$$v \;:\; M \to \tau_M \;.$$

The gradient of a Morse function $f : M \to \mathbb{R}$ is a section of the cotangent bundle

$$\nabla f \;=\; (\partial f/\partial x_i) \;:\; M \to \tau_M^* \;=\; \bigcup_{x \in M} \mathrm{Hom}_{\mathbb{R}}(\tau_M(x), \mathbb{R})$$

with zeros the critical points of f. A vector field $v : M \to \tau_M$ is 'gradient-like' for f if there exists a Riemannian metric $\beta : \tau_M \cong \tau_M^*$ on M such that

$$\beta \circ v \;=\; \nabla f \;:\; M \to \tau_M^* \;.$$

A v-gradient flow line $\gamma : \mathbb{R} \to M$ satisfies

$$\gamma'(t) \;=\; -v(\gamma(t)) \in \tau_M(\gamma(t)) \;.$$

The limits

$$\lim_{t \to -\infty} \gamma(t) \;=\; p \;,\; \lim_{t \to \infty} \gamma(t) \;=\; q \in M$$

are critical points of f. For every non-critical point $x \in M$ there is a v-gradient flow line $\gamma_x : \mathbb{R} \to M$ (which is unique up to scaling) such that $\gamma_x(0) = x \in M$. The unstable and stable manifolds of a critical point $p \in M$ of index i are the open manifolds

$$\begin{aligned} W^u(p,v) \;&=\; \{x \in M \mid \lim_{t \to -\infty} \gamma_x(t) = p\} \;, \\ W^s(p,v) \;&=\; \{x \in M \mid \lim_{t \to \infty} \gamma_x(t) = p\} \end{aligned}$$

which are diffeomorphic to $\mathbb{R}^i$, $\mathbb{R}^{m-i}$ respectively. Let $\mathcal{GT}(f)$ denote the space of all gradient-like vector fields v for f which satisfy the Morse-Smale transversality condition that for any distinct critical points $p, q \in M$ the submanifolds $W^u(p,v), W^s(q,v) \subset M$ intersect transversely. For $v \in \mathcal{GT}(f)$ and critical points $p, q \in M$ of index $i, i-1$ there is only a finite number of v-gradient flow lines

$\gamma : \mathbb{R} \to M$ which start at p and terminate at q, and choosing orientations there is obtained an algebraic number $n(p,q) \in \mathbb{Z}$.

A circle-valued Morse function $f : M \to S^1$ lifts to a real-valued Morse function $\overline{f} : \overline{M} \to \mathbb{R}$ on the infinite cyclic cover $\overline{M} = f^*\mathbb{R}$. Let $\mathcal{GT}(f)$ be the space of all vector fields v on M which lift to a gradient-like vector field $\overline{v}$ on $\overline{M}$ satisfying the transversality condition, so that $\overline{v} \in \mathcal{GT}(\overline{f})$.

Novikov [N] and Pajitnov [P1] used the completion $\mathbb{Z}[\widehat{\pi_1(M)}]$ to construct geometrically for any circle-valued Morse function $f : M \to S^1$ and $v \in \mathcal{GT}(f)$ a based f. g. free $\mathbb{Z}[\widehat{\pi_1(M)}]$-module chain complex $C^{Nov}(M,f,v)$ such that

$$\mathrm{rank}_{\mathbb{Z}[\widehat{\pi_1(M)}]} C_i^{Nov}(M,f,v) \;=\; c_i(f)$$

with $c_i(f)$ the number of critical points of f with index i. As in the real-valued case the differentials are defined by counting the $\widetilde{v}$-gradient flow lines in the universal cover $\widetilde{M}$. Moreover, there is a chain equivalence

$$C^{Nov}(M,f,v) \;\simeq\; C(\widetilde{M};\mathbb{Z}[\widehat{\pi_1(M)}])$$

with

$$C(\widetilde{M};\mathbb{Z}[\widehat{\pi_1(M)}]) \;=\; \mathbb{Z}[\widehat{\pi_1(M)}] \otimes_{\mathbb{Z}[\pi_1(M)]} C(\widetilde{M})$$

the $\mathbb{Z}[\widehat{\pi_1(M)}]$-coefficient cellular chain complex of $\widetilde{M}$, for any CW structure on M.

Pajitnov [P2,P3,P4] constructed for any Morse function $f : M \to S^1$ a C^0-dense subspace $\mathcal{G}\mathfrak{C}\mathcal{CT}(f) \subset \mathcal{GT}(f)$, such that for $v \in \mathcal{G}\mathfrak{C}\mathcal{CT}(f)$ the coefficients in the corresponding Novikov complex $C^{Nov}(M,f,v)$ are rational, in the sense that

$$C^{Nov}(M,f,v) \;=\; \mathbb{Z}[\widehat{\pi_1(M)}] \otimes_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]} C^{Paj}(M,f,v)$$

for a based f. g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain complex $C^{Paj}(M,f,v)$, with a chain equivalence

$$C^{Paj}(M,f,v) \;\simeq\; C(\widetilde{M};\Sigma^{-1}\mathbb{Z}[\pi_1(M)])\ .$$

(Strictly speaking, $C^{Paj}(M,f,v)$ was only defined for abelian $\pi_1(M)$, when the natural ring morphism $\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\widehat{\pi_1(M)}]$ is injective, but this was for algebraic convenience rather than out of geometric necessity.)

For a Morse function $f : M \to S^1$ which is transverse regular at $0 \in S^1$ the lift $\overline{f} : \overline{M} \to \mathbb{R}$ is transverse regular at $\mathbb{Z} \subset \mathbb{R}$, and the restriction to a fundamental domain is a Morse function

$$f_N \;=\; \overline{f}| \;:\; (M_N;N,N_1) \;=\; \overline{f}^{-1}(I;\{0\},\{1\}) \to (I;\{0\},\{1\})$$

with $c_i(f_N) = c_i(f)$ critical points of index i, and N_1 a copy of N. Every $v \in \mathcal{GT}(f)$ lifts to $\overline{v} \in \mathcal{GT}(\overline{f})$, and $v_N = \overline{v}| \in \mathcal{GT}(f_N)$ determines a handlebody decomposition

$$M_N \;=\; N \times I \cup \bigcup^{m} \bigcup D^i \times D^{m-i}$$

with one i-handle for each index i critical point of f. Given a CW structure on N with $c_i(N)$ i-cells use the handlebody structure on M_N to define a CW structure on M_N with $c_i(N)+c_i(f)$ i-cells. (In practice, the CW structure will be the one determined by a Morse function $N \to \mathbb{R}$). The inclusion $g: N \to M_N$ induces an inclusion of based f.g. free $\mathbb{Z}[\pi_1(\overline{M})]$-module chain complexes

$$g \ : \ C(\widetilde{N}) \to C(\widetilde{M}_N)$$

with $\widetilde{N}, \widetilde{M}_N$ the covers of N, M_N induced from the universal cover $\widetilde{M}$ of $\overline{M}$. Write the inclusion of N_1 in M_N as $h: N \to M_N$, and note that in general $N_1 = h(N)$ is not a CW subcomplex of M_N. A *chain approximation* for h is a $\mathbb{Z}[\pi_1(\overline{M})]$-module chain map

$$h \ : \ C(\widetilde{N}_1) \ = \ \alpha C(\widetilde{N}) \to C(\widetilde{M}_N)$$

in the chain homotopy class induced by the map $h: N \to M_N$ as given by the cellular approximation theorem. For any choice of chain approximation h Farber and Ranicki [FR] defined algebraically a based f. g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain complex $C^{FR}(M,f,v,h)$ such that

$$\text{rank}_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]} C_i^{FR}(M,f,v,h) \ = \ c_i(f)$$

as a deformation of the $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-coefficient Morse-Smale complex

$$C^{MS}(M_N,f_N,v_N;\Sigma^{-1}\mathbb{Z}[\pi_1(M)]) \ = \ \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(\overline{M})]} C^{MS}(M_N,f_N,v_N)$$

with a chain equivalence

$$C^{FR}(M,f,v,h) \ \simeq \ \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(M)]} C(\widetilde{M}) \ .$$

There is also a $\mathbb{Z}[\widehat{\pi_1(M)}]$-coefficient version

$$C^{FR}(M,f,v,h;\mathbb{Z}[\widehat{\pi_1(M)}]) \ = \ \mathbb{Z}[\widehat{\pi_1(M)}] \otimes_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]} C^{FR}(M,f,v,h) \ .$$

Cokernel Theorem 6.6. *The chain complex of* [FR] *is isomorphic to the cokernel*

$$C^{FR}(M,f,v,h) \ \cong \ \text{coker}(\phi)$$

of the morphism of based f.g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$*-module chain complexes*

$$\phi \ = \ g - zh \ : \ \Sigma^{-1}C(\widetilde{N})_\alpha[z,z^{-1}] \to \Sigma^{-1}C(\widetilde{M}_N)_\alpha[z,z^{-1}]$$

which is a split injection in each degree. $\square$

The expression $C^{FR}(M,f,v,h)$ as a cokernel makes it possible to prove invari-

Invariance Theorem 6.7. *Let $f : M \to S^1$ be a Morse function, and let $v \in \mathcal{GT}(f)$. For any regular values $0, 0' \in S^1$, CW structures on $N = f^{-1}(0)$, $N' = f^{-1}(0')$ and chain approximations*

$$h \ : \ \alpha C(\widetilde{N}) \to C(\widetilde{M}_N) \ , \ h' \ : \ \alpha C(\widetilde{N}') \to C(\widetilde{M}_{N'})$$

there is defined a simple isomorphism of based f. g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain complexes

$$C^{FR}(M, f, v, h) \ \cong \ C^{FR}(M, f, v, h') \ . \qquad \square$$

Here, simple means that the torsion of the isomorphism is in the image of $\{\pm\pi_1(M)\} \subseteq K_1(\Sigma^{-1}\mathbb{Z}[\pi_1(M)])$.

Given $f : M \to S^1$, $v \in \mathcal{GT}(f)$ and a CW structure on $N = f^{-1}(0) \subset M$ we shall say that a chain approximation

$$h^{gra} \ : \ \alpha C(\widetilde{N}) \to C(\widetilde{M}_N)$$

is *gradient-like* if it counts the v-gradient flow lines in the universal cover $\widetilde{M}$. (The precise definition is given in 6.8).

Isomorphism Theorem 6.10 *For $v \in \mathcal{GT}(f)$ with a gradient-like chain approximation h^{gra} there are basis-preserving isomorphisms*

$$C^{Nov}(M, f, v) \ \cong \ C^{FR}(M, f, v, h^{gra}; \mathbb{Z}\widehat{[\pi_1(M)]}) \ ,$$
$$C^{Paj}(M, f, v) \ \cong \ C^{FR}(M, f, v, h^{gra}) \ . \quad \square$$

Pajitnov [P3] (Theorem 7.2) showed that every $v \in \mathcal{GT}(f)$ is C^0-close to $v^{gra} \in \mathcal{G}\mathfrak{C}\mathcal{CT}(f)$ with a gradient-like chain approximation $h^{gra} : C(\widetilde{N}) \to C(\widetilde{M}_N)$. A similar construction was obtained by Hutchings and Lee [HL]. In fact, Cornea and Ranicki [CR] prove that every $v \in \mathcal{GT}(f)$ admits a gradient-like chain approximation h^{gra}.

The plan of the paper is as follows. §1 is purely algebraic in nature, concerning the chain homotopy properties of the algebraic mapping cones and cokernels of chain maps. The glueing properties of the Morse-Smale complex $C^{MS}(M, f, v)$ are described in §2 for finite unions, and in §3 for infinite unions. §4 gives a brief account of the Cohn localization. §5 deals with the cokernel and infinite union construction of chain complexes over a twisted polynomial extension $A_\alpha[z, z^{-1}]$ for any ring A with automorphism $\alpha : A \to A$, the localization $\Sigma^{-1}A_\alpha[z, z^{-1}]$ and the Novikov ring $A_\alpha((z)) = A_\alpha[[z]][z^{-1}]$. The Cokernel, Invariance and Isomorphism Theorems are proved in §6.

The remainder of the Introduction is an outline of the proof of the Cokernel, Invariance and Isomorphism Theorems in the special case when $f_* : \pi_1(M) \to \pi_1(S^1)$ is an isomorphism, so that

$$\pi_1(\overline{M}) \ = \ \{1\} \ , \ \mathbb{Z}[\pi_1(M)] \ = \ \mathbb{Z}[z, z^{-1}] \ ,$$
$$\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \ = \ (1 + z\mathbb{Z}[z])^{-1}\mathbb{Z}[z, z^{-1}] \ ,$$
$$\mathbb{Z}\widehat{[\pi_1(M)]} \ = \ \mathbb{Z}((z))$$

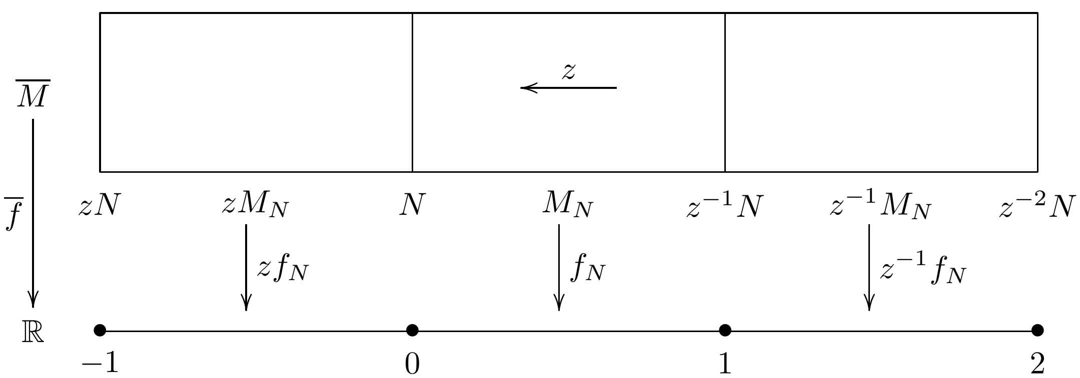
The Novikov complex of a Morse function $f : M \to S^1$ with respect to $v \in \mathcal{GT}(f)$ is the based f. g. free $\mathbb{Z}((z))$-module chain complex

$$C \;=\; C^{Nov}(M,f,v)$$

with

$$d_C \;:\; C_i \;=\; \mathbb{Z}((z))^{c_i(f)} \to C_{i-1} \;=\; \mathbb{Z}((z))^{c_{i-1}(f)} \;;\; \overline{p} \mapsto \sum_{\overline{q}} n(\overline{p},\overline{q})\overline{q}$$

where $n(\overline{p}, z^j\overline{q})$ is the algebraic number of $\overline{v}$-gradient flow lines in $\overline{M}$ from a critical point $\overline{p} \in \overline{M}$ of index i to a critical point $\overline{q} \in \overline{M}$ of index $i-1$, using the transversality property of v to ensure that these numbers are finite. *Here $z : \overline{M} \to \overline{M}$ is the generating covering translation parallel to the v-gradient flow of f.*



If $0 \in S^1$ is a regular value of f the Morse function

$$f_N \;=\; \overline{f}| \;:\; (M_N; N, z^{-1}N) \;=\; \overline{f}^{-1}(I;\{0\},\{1\}) \to (I;\{0\},\{1\})$$

has $c_i(f_N) = c_i(f)$ critical points of index i. Each critical point $p \in M$ of f can be regarded as a critical point $\overline{p} \in M_N$ of f_N. For $v \in \mathcal{GT}(f)$ the Morse-Smale complex of f_N with respect to $v_N \in \mathcal{GT}(f_N)$ is the based f. g. free $\mathbb{Z}$-module chain complex

$$C^{MS}(M_N, f_N, v_N) \;=\; F$$

with

$$d_F \;:\; F_i \;=\; \mathbb{Z}^{c_i(f)} \to F_{i-1} \;=\; \mathbb{Z}^{c_{i-1}(f)} \;;\; \overline{p} \mapsto \sum_{\overline{q}} n(\overline{p},\overline{q})$$

where $n(\overline{p},\overline{q})$ is the algebraic number of v_N-gradient flow lines in M_N from a critical point $\overline{p} \in M_N$ of index i to a critical point $\overline{q} \in M_N$ of index $i-1$.

The Novikov complex $C^{Nov}(M,f,v) = C$ counts the $\overline{v}$-gradient flow lines which start at a critical point $\overline{p} \in z^j M_N \subset \overline{M}$ and terminate at a critical point $\overline{q} \in z^k M_N \subset \overline{M}$ with $k \leq j$. The Morse-Smale complex $C^{MS}(M_N, f_N, v_N) = F$ only counts such flow lines with $j = k = 0$. In order to construct C from F we glue together an infinite number of copies of an 'algebraic fundamental domain' which gives an algebraic picture of the way the v_N-flow lines enter M_N at $z^{-1}N$ and either die at a critical point of f_N in M_N or exit at N.

As above, given an arbitrary CW structure on $N = f^{-1}(0)$ with $c_i(N)$ i-cells use the handlebody decomposition

$$M_N \;=\; N \times I \cup \bigcup_{i=0}^{m} \bigcup_{c_i(f)} D^i \times D^{m-i}~,$$

to give M_N a CW structure with $c_i(N)+c_i(f)$ i-cells, one for each i-cell of N and one for each i-handle of $(M_N;N,z^{-1}N)$. The cellular chain complex of M_N is of the form $C(M_N)=E$ with

$$d_E \;=\; \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} ~:~ E_i \;=\; D_i \oplus F_i \to E_{i-1} \;=\; D_{i-1} \oplus F_{i-1}~,$$

where

$$\begin{array}{l} D \;=\; C(N)~,~F \;=\; C(M_N, N \times I) \;=\; C^{MS}(M_N, f_N, v_N)~, \\ D_i \;=\; \mathbb{Z}^{c_i(N)}~,~E_i \;=\; \mathbb{Z}^{c_i(N)+c_i(f)}~,~F_i \;=\; \mathbb{Z}^{c_i(f)}~, \\ g \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} ~:~ D_i \to E_i \;=\; D_i \oplus F_i~. \end{array}$$

Let $h : \alpha D \to E$ be a chain approximation for the inclusion $h : z^{-1}N \to M$, with components

$$h \;=\; \begin{pmatrix} h_D \\ h_F \end{pmatrix} ~:~ \alpha D_i \to E_i \;=\; D_i \oplus F_i~.$$

The cellular $\mathbb{Z}[z,z^{-1}]$-module chain complex of $\overline{M}$ is the algebraic mapping cone

$$C(\overline{M}) \;=\; \mathcal{C}(g - zh : D[z,z^{-1}] \to E[z,z^{-1}])~.$$

The chain complex of Farber and Ranicki [FR] was defined to be the based f. g. free $\Sigma^{-1}\mathbb{Z}[z,z^{-1}]$-module chain complex

$$C^{FR}(M,f,v,h) \;=\; \widehat{F}$$

with

$$d_{\widehat{F}} \;=\; d_F + z h_F (1 - z h_D)^{-1} c ~:~ \widehat{F}_i \;=\; \Sigma^{-1} F_i[z,z^{-1}] \to \widehat{F}_{i-1} \;=\; \Sigma^{-1} F_{i-1}[z,z^{-1}]$$

a deformation of $\Sigma^{-1}F[z,z^{-1}]$. The inverse of $1 - zh_D$ is defined over the Novikov ring $\mathbb{Z}((z))$ by

$$(1 - z h_D)^{-1} \;=\; 1 + z h_D + z^2 (h_D)^2 + z^3 (h_D)^3 + \dots$$

so the induced $\mathbb{Z}((z))$-module chain complex is

$$C^{FR}\big(M,f,v,h;\mathbb{Z}((z))\big) \;=\; \mathbb{Z}((z)) \otimes_{\Sigma^{-1}\mathbb{Z}[z,z^{-1}]} C^{FR}(M,f,v,h) \;=\; \overline{F}$$

with

$$d_{\overline{F}} \;=\; d_F + \sum^{\infty} z^j h_F (h_D)^{j-1} c ~:~ \overline{F}_i \;=\; F_i((z)) \to \overline{F}_{i-1} \;=\; F_{i-1}((z))~.$$

The chain approximation h is an algebraic model for the $\overline{v}$-gradient flow across a fundamental domain $(M_N; N, z^{-1}N)$ of $\overline{M}$. The formula for $d_{\widehat{F}}$ is interpreted in §5 as the generating function for the number of flow lines in $\overline{M}$ of prescribed length, with $h_F(h_D)^{j-1}c$ counting the flow lines which start in M_N and terminate in z^jM_N, crossing the j walls $N, zN, \dots, z^{j-1}N$ between the adjacent fundamental domains $M_N, zM_N, \dots, z^jM_N$.

The algebraic treatment in §1 of cokernels of chain maps will be used in §6 to prove that the inclusions $\widehat{F}_i \to \Sigma^{-1}E_i[z,z^{-1}]$ induce the isomorphism of the Cokernel Theorem 6.6

$$C^{FR}(M,f,v,h) \cong \operatorname{coker}(\phi)$$

with

$$\phi = g - zh : \Sigma^{-1}D[z,z^{-1}] \to \Sigma^{-1}E[z,z^{-1}] .$$

A chain homotopy $k : h \simeq h' : \alpha D \to E$ determines an isomorphism of $\Sigma^{-1}\mathbb{Z}[z,z^{-1}]$-module chain complexes

$$\operatorname{coker}(\phi) \cong \operatorname{coker}(\phi')$$

(Proposition 5.3) giving the isomorphism of the Invariance Theorem 6.7

$$C^{FR}(M,f,v,h) \cong \operatorname{coker}(\phi) \cong \operatorname{coker}(\phi') \cong C^{FR}(M,f,v,h')$$

in the special case $N = N'$. The general case is proved by an algebraic treatment of handle exchanges (Proposition 5.4).

For a gradient-like chain approximation

$$h^{gra} = \begin{pmatrix} h^{gra}_D \\ h^{gra}_F \end{pmatrix} : \alpha D_i \to E_i = D_i \oplus F_i$$

the algebraic numbers of $\overline{v}$-gradient flow lines between critical points of $\overline{f} : \overline{M} \to \mathbb{R}$ are given by

$$n(\overline{p}, z^j\overline{q}) = \begin{cases} (\overline{p},\overline{q})\text{-coefficient of } d_F : F_i \to F_{i-1} & \text{if } j = 0 \\ (\overline{p},\overline{q})\text{-coefficient of } h^{gra}_F(h^{gra}_D)^{j-1}c : F_i \to F_{i-1} & \text{if } j > 0 \\ 0 & \text{if } j < 0 \end{cases}$$

for any critical points $\overline{p},\overline{q} \in M_N$ of f_N with index $i, i-1$ respectively. It follows that for any $-\infty < j < k < \infty$ the Morse-Smale complex of the real-valued Morse function

$$\begin{aligned} f_N[j,k] &= \overline{f}| : \\ (M_N[j,k]; z^{-j}N, z^{-k}N) &= \overline{f}^{-1}([j,k];\{j\},\{k\}) = (\bigcup_{\ell=-k}^{-j} z^\ell M_N; z^{-j}N, z^{-k}N) \\ &\to ([j,k];\{j\},\{k\}) \end{aligned}$$

is

$$\begin{aligned} & C^{MS}(M_N[j,k], f_N[j,k], v_N[j,k]) \\ &= (\sum_{\ell=-k}^{-j} z^\ell F_i, d_F + zh^{gra}_F(1 + zh^{gra}_D + \dots + z^{k-j-1}(h^{gra}_D)^{k-j-1})c) \\ &\cong \operatorname{coker}\big(g - zh^{gra} : \sum^{-j} z^\ell D \to \sum^{-j} z^\ell E\big) . \end{aligned}$$

Passing to the direct limit as $k \to \infty$ gives the Morse-Smale complex of the proper real-valued Morse function

$$\begin{aligned} f_N[j,\infty) \;=\; \overline{f}| \;:\; (M_N[j,\infty), \partial M_N[j,\infty)) \;&=\; \overline{f}^{\,-1}([j,\infty),\{j\}) \\ &=\; \Big(\bigcup_{\ell=-\infty}^{-j} z^\ell M_N, z^{-j}N\Big) \to ([j,\infty),\{j\}) \end{aligned}$$

to be

$$\begin{aligned} C^{MS}(M_N[j,\infty), f_N[j,\infty), v_N[j,\infty)) \;&=\; \varinjlim_k C^{MS}(M_N[j,k], f_N[j,k], v_N[j,k]) \\ &\cong\; \mathrm{coker}\big(g - zh^{gra} : \sum_{\ell=-\infty}^{-j} z^\ell D \to \sum_{\ell=\infty}^{-j} z^\ell E\big)\ . \end{aligned}$$

Passing to the inverse limit as $j \to -\infty$ gives the Isomorphism Theorem 6.10 for the Novikov complex, with a basis-preserving $\mathbb{Z}((z))$-module isomorphism

$$\begin{aligned} C^{Nov}(M,f,v) \;&=\; \varprojlim_j C^{MS}\big(M_N[j,\infty), f_N[j,\infty), v_N[j,\infty)\big) \\ &\cong\; \mathrm{coker}\big(g - zh^{gra} : D((z)) \to E((z))\big) \\ &\cong\; C^{FR}\big(M,f,v,h^{gra};\mathbb{Z}((z))\big)\ . \end{aligned}$$

The geometric differential in $C = C^{Nov}(M,f,v)$ is just the algebraic differential in $\overline{F} = C^{FR}\big(M,f,v,h^{gra};\mathbb{Z}((z))\big)$, with

$$\begin{aligned} d_C(\overline{p}) \;&=\; \sum_{\overline{q}} \sum_{j \in \mathbb{Z}} n(\overline{p}, z^j\overline{q}) z^j \overline{q} \\ &=\; (d_F + \sum_{j=1}^{\infty} z^j h_F^{gra} (h_D^{gra})^{j-1} c)(\overline{p}) \\ &=\; (d_F + zh_F^{gra}(1 - zh_D^{gra})^{-1}c)(\overline{p}) \\ &=\; d_{\overline{F}}(\overline{p}) \in \overline{F}_{i-1} \;=\; \mathbb{Z}^{c_{i-1}(f)}\ , \end{aligned}$$

so that there is also a basis-preserving $\Sigma^{-1}\mathbb{Z}[z,z^{-1}]$-module isomorphism

$$C^{Paj}(M,f,v) \;\cong\; C^{FR}(M,f,v,h^{gra})\ .$$

The projection

$$\begin{aligned} p \;:\; C\big(\overline{M};\mathbb{Z}((z))\big) \;&=\; \mathcal{C}\big(g - zh^{gra} : D((z)) \to E((z))\big) \\ &\to \mathrm{coker}\big(g - zh^{gra} : D((z)) \to E((z))\big) \;=\; C^{Nov}(M,f,v) \end{aligned}$$

pieces together $\overline{v}$-gradient flow lines in $\overline{M}$ from their intersections with the translates $z^j M_N \subset \overline{M}$ $(j \in \mathbb{Z})$ of the fundamental domain M_N ; p is a chain equivalence with torsion

$$\begin{aligned} \tau(p) \;&=\; \sum_{i=0}^{\infty} (-)^{i+1} \tau\big(1 - zh_D^{gra} : D_i((z)) \to D_i((z))\big) \\ &=\; \prod_{i=0}^{\infty} \det\big(1 - zh_D^{gra} : D_i((z)) \to D_i((z))\big)^{(-)^{i+1}} \\ &\in \widehat{W}(\mathbb{Z}) \subset K_1(\mathbb{Z}((z))) = K_1(\mathbb{Z}) \oplus K_0(\mathbb{Z}) \oplus \widehat{W}(\mathbb{Z}) \end{aligned}$$

with $\widehat{W}(\mathbb{Z}) = 1 + z\mathbb{Z}[[z]]$ under multiplication. The kernel of p corresponds to the closed orbits of the v-gradient flow lines in M, which avoid the critical points of f and so do not contribute to the Novikov complex, and which are counted by the torsion of p.

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1. Cones and cokernels.

The Novikov complex of a circle-valued Morse function $f : M \to S^1$ will be shown in §6 to be isomorphic to the cokernel of a chain map constructed from the Morse-Smale complex of a fundamental domain for the infinite cyclic cover $\overline{M} = f^*\mathbb{R}$ of M; the algebraic mapping cone of the chain map is a cellular chain complex of $\overline{M}$. This section is accordingly devoted to the relationship between the algebraic mapping cone and the cokernel of a chain map. The algebraic mapping cone is a chain homotopy invariant. The cokernel is not a chain homotopy invariant, although it is a homology invariant. The main novelty of this section is the introduction of an equivalence relation on chain maps called 'chain isotopy', which is stronger than chain homotopy, and is such that the cokernels of chain isotopic maps are isomorphic. The chain map with cokernel the Novikov complex depends on a choice of chain map in a chain homotopy class; a different choice will give a chain isotopic chain map, with isomorphic cokernel.

Let A be a ring. The *algebraic mapping cone* of an A-module chain map

$$\phi : D \to E$$

is the A-module chain complex $\mathcal{C}(\phi)$ defined by

$$d_{\mathcal{C}(\phi)} \;=\; \begin{pmatrix} d_E & (-)^{i-1}\phi \\ 0 & d_D \end{pmatrix} \;:\; \mathcal{C}(\phi)_i \;=\; E_i \oplus D_{i-1} \to \mathcal{C}(\phi)_{i-1} \;=\; E_{i-1} \oplus D_{i-2}\ .$$

The natural projections

$$p \;:\; \mathcal{C}(\phi)_i \;=\; E_i \oplus D_{i-1} \to \mathrm{coker}(\phi : D_i \to E_i)\ ;\ (x,y) \mapsto [x]$$

define a chain map

$$p \;:\; \mathcal{C}(\phi) \to \mathrm{coker}(\phi)\ .$$

Proposition 1.1. *For an injective chain map $\phi : D \to E$ the natural projection $p : \mathcal{C}(\phi) \to \mathrm{coker}(\phi)$ induces isomorphisms in homology*

$$p_* \;:\; H_*\big(\mathcal{C}(\phi)\big) \;\cong\; H_*\big(\mathrm{coker}(\phi)\big)\ .$$

Proof. The short exact sequences of A-module chain complexes

$$0 \to E \to \mathcal{C}(\phi) \to D_{*-1} \to 0$$

$$0 \to D \xrightarrow{\phi} E \to \mathrm{coker}(\phi) \to 0$$

induce long exact sequences in homology, which are related by a natural transformation

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_i(D) & \xrightarrow{\phi_*} & H_i(E) & \longrightarrow & H_i(\mathcal{C}(\phi)) & \longrightarrow & H_{i-1}(D) & \longrightarrow \cdots \\
& \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle p_*} & & \Big\downarrow{\scriptstyle 1} & \\
\cdots \longrightarrow & H_i(D) & \xrightarrow{\phi_*} & H_i(E) & \longrightarrow & H_i\big(\mathrm{coker}(\phi)\big) & \longrightarrow & H_{i-1}(D) & \longrightarrow \cdots
\end{array}$$

It now follows from the 5-lemma that the induced morphisms p_* are isomorphisms. □

As usual, a *chain homotopy* between chain maps $\phi, \phi' : D \to E$

$$\theta \ : \ \phi \ \simeq \ \phi' \ : \ D \to E$$

is a collection of A-module morphisms $\theta : D_i \to E_{i+1}$ such that for each i

$$\phi' - \phi \ = \ d_E\theta + \theta d_D \ : \ D_i \to E_i \ .$$

Chain homotopic chain maps have isomorphic algebraic mapping cones:

Proposition 1.2. *A chain homotopy $\theta : \phi \simeq \phi' : D \to E$ determines an isomorphism of the algebraic mapping cones*

$$I \ = \ \begin{pmatrix} 1 & \pm\theta \\ 0 & 1 \end{pmatrix} \ : \ \mathcal{C}(\phi) \to \mathcal{C}(\phi') \ .$$

If D, E are based f. g. free then I is a simple isomorphism.

Proof. By construction. □

In general, the cokernels of chain homotopic chain maps are not isomorphic (or even chain equivalent). The following relation will be convenient in dealing with cokernels of chain maps, in order to avoid this problem.

Definition 1.3 A *chain isotopy* between chain maps $\phi, \phi' : D \to E$

$$\psi \ : \ \phi \sim \phi' \ : \ D \to E$$

is a collection of A-module morphisms $\psi : E_i \to E_{i+1}$ such that

(i) for each i

$$\phi' \ = \ (1 + d_E\psi + \psi d_E)\phi \ : \ D_i \to E_i \ ,$$

defining a chain homotopy

$$\psi\phi \ : \ \phi \ \simeq \ \phi' \ : \ D \to E \ ,$$

(ii) each

$$1 + d_E\psi + \psi d_E \ : \ E_i \to E_i$$

is an automorphism. □

Proposition 1.4. *Chain isotopy is an equivalence relation on chain maps.*

Proof. Reflexivity: every chain map $\phi : D \to E$ is isotopic to itself by $0 : \phi \sim \phi$.
Symmetry: for any chain isotopy $\psi : \phi \sim \phi' : D \to E$ define a chain isotopy $\psi^- : \phi' \sim \phi$ by

$$\psi^- \;=\; -(1 + d_E\psi + \psi d_E)^{-1}\psi \;:\; E_i \to E_{i+1} \;,$$

with

$$1 + d_E\psi^- + \psi^- d_E \;=\; (1 + d_E\psi + \psi d_E)^{-1} \;:\; E_i \to E_i \;.$$

Transitivity: for any chain isotopies $\psi : \phi \sim \phi'$, $\psi' : \phi' \sim \phi'' : D \to E$ define a chain isotopy $\psi'' : \phi \sim \phi''$ by

$$\psi'' \;=\; \psi + \psi'(1 + d_E\psi + \psi d_E) \;:\; E_i \to E_{i+1}$$

with

$$1 + d_E\psi'' + \psi'' d_E \;=\; (1 + d_E\psi' + \psi' d_E)(1 + d_E\psi + \psi d_E) \;:\; E_i \to E_i \;. \qquad \square$$

Isotopic chain maps have isomorphic cokernels:

Proposition 1.5. *A chain isotopy $\psi : \phi \sim \phi' : D \to E$ determines isomorphisms of chain complexes*

$$\begin{aligned} q \;&=\; \begin{pmatrix} 1 & \pm\psi\phi \\ 0 & 1 \end{pmatrix} \;:\; \mathcal{C}(\phi) \to \mathcal{C}(\phi') \;, \\ r \;&=\; [1 + d_E\psi + \psi d_E] \;:\; \mathrm{coker}(\phi) \to \mathrm{coker}(\phi') \end{aligned}$$

and a chain homotopy

$$s \;:\; rp \;\simeq\; p'q \;:\; \mathcal{C}(\phi) \to \mathrm{coker}(\phi')$$

with $p : \mathcal{C}(\phi) \to \mathrm{coker}(\phi)$, $p' : \mathcal{C}(\phi') \to \mathrm{coker}(\phi')$ the projections.

Proof. The isomorphism q is a special case of 1.2.
The isomorphism r is given by the morphism of exact sequences

$$\begin{array}{ccccccc} D & \xrightarrow{\;\phi\;} & E & \longrightarrow & \mathrm{coker}(\phi) & \longrightarrow & 0 \\ \Big\| & & \cong\Big\downarrow 1 + d_E\psi + \psi d_E & & \cong\Big\downarrow r & & \\ D & \xrightarrow{\;\phi'\;} & E & \longrightarrow & \mathrm{coker}(\phi') & \longrightarrow & 0 \end{array}$$

The A-module morphisms

$$s \;:\; \mathcal{C}(\phi)_i \;=\; E_i \oplus D_{i-1} \to \mathrm{coker}(\phi' : D_{i+1} \to E_{i+1}) \;;\; (x, y) \mapsto [\psi(x)]$$

define a chain homotopy

$$s \;:\; rp \;\simeq\; p'q \;:\; \mathcal{C}(\phi) \to \mathrm{coker}(\phi')$$

in the diagram

$$\begin{array}{ccc} \mathcal{C}(\phi) & \xrightarrow{\ p\ } & \operatorname{coker}(\phi) \\ {\scriptstyle q}\downarrow & \overset{s}{\searrow} & \downarrow{\scriptstyle r} \\ \mathcal{C}(\phi') & \xrightarrow{\ p'\ } & \operatorname{coker}(\phi') \ . \end{array}$$

□

Definition 1.6 An *embedding* of chain complexes is a chain map $\phi : D \to E$ such that each $\phi : D_i \to E_i$ is a split injection. □

Proposition 1.7. *Let $\phi : D \to E$ be an embedding of A-module chain complexes.*
(i) *The natural projection $p : \mathcal{C}(\phi) \to \operatorname{coker}(\phi)$ is a chain equivalence.*
(ii) *If each $D_i, E_i, \operatorname{coker}(\phi : D_i \to E_i)$ is a based f.g. free A-module, then p is a chain equivalence with torsion*

$$\tau(p) \ = \ \sum_{i=0}^{\infty} (-)^i \tau(\mathcal{E}_i) \in K_1(A)$$

where $\tau(\mathcal{E}_i) \in K_1(A)$ is the torsion of the short exact sequence of based f.g. free A-modules

$$\mathcal{E}_i \ : \ 0 \longrightarrow D_i \xrightarrow{\ \phi\ } E_i \longrightarrow \operatorname{coker}(\phi : D_i \to E_i) \longrightarrow 0 \ .$$

(iii) *If $\psi : \phi \sim \phi' : D \to E$ is a chain isotopy then $\phi' : D \to E$ is also an embedding. With bases as in* (ii), *and the isomorphism given by 1.5*

$$r \ = \ [1 + d_E \psi + \psi d_E] \ : \ \operatorname{coker}(\phi) \ \cong \ \operatorname{coker}(\phi')$$

has torsion

$$\tau(r) \ = \ \sum_{i=0}^{\infty} (-)^i (\tau(\mathcal{E}'_i) - \tau(\mathcal{E}_i)) \in K_1(A) \ .$$

Proof. (i) Extend each $\phi : D_i \to E_i$ to a direct sum system

$$D_i \underset{e}{\overset{\phi}{\rightleftarrows}} E_i \underset{k}{\overset{j}{\rightleftarrows}} F_i$$

with $F_i = \operatorname{coker}(\phi : D_i \to E_i)$. Let

$$c \ = \ e d_E k \ : \ F_i \to D_{i-1} \ , \ d_F \ = \ j d_E k \ : \ F_i \to F_{i-1} \ ,$$

so that there is defined an isomorphism of chain complexes $(\phi \ k) : E' \to E$ with

$$d_{E'} \ = \ \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \ : \ E'_i \ = \ D_i \oplus F_i \to E'_{i-1} \ = \ D_{i-1} \oplus F_{i-1} \ .$$

The chain map $p : \mathcal{C}(\phi) \to F = \operatorname{coker}(\phi)$ is given by

$$p \ = \ (j \ \ 0) \ : \ \mathcal{C}(\phi)_i \ = \ E_i \oplus D_{i-1} \to F_i$$

The chain map $g : F \to \mathcal{C}(\phi)$ defined by

$$g \;=\; \begin{pmatrix} k \\ -c \end{pmatrix} \;:\; F_i \to \mathcal{C}(\phi)_i \;=\; E_i \oplus D_{i-1}$$

is such that

$$pg \;=\; 1 \;:\; F \to F \;,\; h \;:\; gp \;\simeq\; 1 \;:\; \mathcal{C}(\phi) \to \mathcal{C}(\phi)$$

with h the chain homotopy

$$h \;=\; \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \;:\; \mathcal{C}(\phi)_i \;=\; E_i \oplus D_{i-1} \to \mathcal{C}(\phi)_{i+1} \;=\; E_{i+1} \oplus D_i .$$

Thus $p : \mathcal{C}(\phi) \to F$, $g : F \to \mathcal{C}(\phi)$ are inverse chain equivalences.
(ii) Immediate from the 0-dimensional case, which follows from the sum formula $\tau(fg) = \tau(f) + \tau(g)$.
(iii) By the chain homotopy invariance of torsion, the sum formula, the identity $\tau(q) = 0$ with q as in 1.5 and (ii)

$$\begin{aligned} \tau(r) &= \tau(p') - \tau(p) + \tau(q) \\ &= \tau(p') - \tau(p) \\ &= \sum_{i=0}^{\infty} (-)^i \tau(\mathcal{E}'_i) - \sum_{i=0}^{\infty} (-)^i \tau(\mathcal{E}_i) \in K_1(A) \;. \quad \square \end{aligned}$$

Proposition 1.8. *Let E be an A-module chain complex of the form*

$$d_E \;=\; \begin{pmatrix} d_D & c \\ 0 & d_D \end{pmatrix} \;:\; E_i \;=\; D_i \oplus F_i \to E_{i-1} \;=\; D_{i-1} \oplus F_{i-1} \;.$$

Let $\phi : D \to E$ be a chain map, with

$$\phi \;=\; \begin{pmatrix} \phi_D \\ \phi_F \end{pmatrix} \;:\; D_i \to E_i \;=\; D_i \oplus F_i \;.$$

(i) *If each $\phi_D : D_i \to D_i$ is an automorphism then*

(a) *ϕ is an embedding of chain complexes.*

(b) *The chain complex $\widehat{F}$ defined by*

$$d_{\widehat{F}} \;=\; d_F - \phi_F(\phi_D)^{-1}c \;:\; \widehat{F}_i \;=\; F_i \to \widehat{F}_{i-1} \;=\; F_{i-1}$$

is such that the inclusions $F_i \to E_i$ induce an isomorphism of chain complexes

$$\widehat{F} \;\cong\; \mathrm{coker}(\phi) \;.$$

(c) *The natural projection $p : \mathcal{C}(\phi) \to \mathrm{coker}(\phi)$ is a chain equivalence. The chain complex K defined by*

$$d_K \;=\; \begin{pmatrix} d_D & 0 \\ (-1)^i \phi_D & \phi_D d_D (\phi_D)^{-1} \end{pmatrix} \;:$$

(i.e. the algebraic mapping cone of the isomorphism of chain complexes $\phi_D : D = (D_i, d_D) \to (D_i, \phi_D d_D (\phi_D)^{-1}))$ is contractible, and fits into an exact sequence

$$0 \to K \to \mathcal{C}(\phi) \xrightarrow{\ p\ } \mathrm{coker}(\phi) \to 0$$

with

$$K_i \ = \ D_{i-1} \oplus D_i \to \mathcal{C}(\phi)_i \ = \ D_{i-1} \oplus D_i \oplus E_i \ ; \\ (x,y) \mapsto (x, y, \phi_F(\phi_D)^{-1}(y)) \ .$$

(d) *If D, F are based f. g. free the natural projection*

$$p \ : \ \mathcal{C}(\phi) \to \mathrm{coker}(\phi) \ \cong \ \widehat{F}$$

is a chain equivalence of based f. g. free A-module chain complexes with torsion

$$\tau(p : \mathcal{C}(\phi) \to \widehat{F}) \ = \ -\tau(K) \ = \ -\sum_{i=0}^{\infty} (-)^i \tau(\phi_D : D_i \to D_i) \in K_1(A) \ .$$

(ii) *Given a chain homotopy of chain maps*

$$\theta \ : \ \phi \ \simeq \ \phi' \ : \ D \to E$$

with $\phi_D, \phi'_D : D_i \to D_i$ automorphisms write

$$\theta \ = \ \begin{pmatrix} \theta_D \\ \theta_F \end{pmatrix} \ : \ D_i \to E_{i+1} \ = \ D_{i+1} \oplus F_{i+1} \ .$$

The morphisms defined by

$$\psi \ = \ \begin{pmatrix} \theta_D(\phi_D)^{-1} & 0 \\ \theta_F(\phi_D)^{-1} & 0 \end{pmatrix} \ : \ E_i \ = \ D_i \oplus F_i \to E_i \ = \ D_i \oplus F_i$$

are such that

$$\phi' \ = \ (1 + d_E\psi + \psi d_E)\phi \ : \ D_i \to E_i \ .$$

Thus if each $1 + d_E\psi + \psi d_E : E_i \to E_i$ is an automorphism there is defined a chain isotopy of embeddings

$$\psi \ : \ \phi \ \sim \ \phi' \ : \ D \to E$$

and as in 1.5 there is defined an isomorphism

$$r \ = \ [1 + d_E\psi + \psi d_E] \ : \ \mathrm{coker}(\phi) \ \cong \ \mathrm{coker}(\phi') \ .$$

Moreover, if D, F are based f.g. free the isomorphism

$$r \ : \ \widehat{F} \ \cong \ \mathrm{coker}(\phi) \ \cong \ \mathrm{coker}(\phi') \ \cong \ \widehat{F}'$$

is simple, with

$$\begin{aligned}\tau(r) &= \sum_{i=0}^{\infty}(-)^i(\tau(\phi_D : D_i \to D_i) - \tau(\phi'_D : D'_i \to D'_i)) \\ &= 0 \in K_1(A) .\end{aligned}$$

Proof. (i) (a) Each $\phi : D_i \to E_i$ is a split injection.
(b) It is clear that the inclusion $F \to E$ induces a chain map $\widehat{F} \to \text{coker}(\phi)$. The A-module morphisms

$$(\, -\phi_F(\phi_D)^{-1} \quad 1\,) \;:\; E_i \;=\; D_i \oplus F_i \to \widehat{F}_i \;=\; F_i$$

induce the inverse chain isomorphism $\text{coker}(\phi) \to \widehat{F}$.
(c) Immediate from 1.7 (i).
(d) Apply 1.7 (ii), noting that the short exact sequence

$$\mathcal{E}_i \;:\; 0 \longrightarrow D_i \xrightarrow{\begin{pmatrix}\phi_D \\ \phi_F\end{pmatrix}} D_i \oplus F_i \xrightarrow{(-\phi_F(\phi_D)^{-1}\ 1)} F_i \longrightarrow 0$$

has torsion

$$\tau(\mathcal{E}_i) \;=\; -\tau(\phi_D : D_i \to D_i) \in K_1(A) .$$

(ii) It follows from

$$\begin{aligned}&\begin{pmatrix} d_D & c \\ 0 & d_F\end{pmatrix}\begin{pmatrix}\phi_D \\ \phi_E\end{pmatrix} = \begin{pmatrix}\phi_D \\ \phi_E\end{pmatrix} d_D \;:\; D_i \to E_{i-1} \;=\; D_{i-1} \oplus F_{i-1} , \\ &\begin{pmatrix}\phi'_D \\ \phi'_F\end{pmatrix} - \begin{pmatrix}\phi_D \\ \phi_F\end{pmatrix} = \begin{pmatrix} d_D & c \\ 0 & d_F\end{pmatrix}\begin{pmatrix}\theta_D \\ \theta_E\end{pmatrix} + \begin{pmatrix}\theta_D \\ \theta_E\end{pmatrix} d_D \;:\; D_i \to E_i \;=\; D_i \oplus F_i\end{aligned}$$

that

$$\begin{aligned}(1 + d_E\psi + \psi d_E)\phi \;&=\; \left(\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + \begin{pmatrix} d_D & c \\ 0 & d_F\end{pmatrix}\begin{pmatrix}\theta_D(\phi_D)^{-1} & 0 \\ \theta_F(\phi_D)^{-1} & 0\end{pmatrix}\right. \\ &\qquad \left. + \begin{pmatrix}\theta_D(\phi_D)^{-1} & 0 \\ \theta_F(\phi_D)^{-1} & 0\end{pmatrix}\begin{pmatrix} d_D & c \\ 0 & d_F\end{pmatrix}\right)\begin{pmatrix}\phi_D \\ \phi_E\end{pmatrix} \\ &=\; \begin{pmatrix}\phi'_D \\ \phi'_F\end{pmatrix} \;=\; \phi' \;:\; D_i \to E_i \;=\; D_i \oplus F_i .\end{aligned}$$

If $1+d_E\psi+\psi d_E : E_i \to E_i$ is an automorphism then $\psi : \phi \simeq \phi' : D \to E$ is a chain isotopy, and by 1.7 (iii)

$$\begin{aligned}\tau(r : \widehat{F} \cong \widehat{F}') \;&=\; \sum_{i=0}^{\infty}(-)^i(\tau(\mathcal{E}'_i) - \tau(\mathcal{E}_i)) \\ &=\; \sum_{i=0}^{\infty}(-)^i(\tau(\phi_D : D_i \to D_i) - \tau(\phi'_D : D'_i \to D'_i)) \in K_1(A) .\end{aligned}$$

Now there is defined an isomorphism of short exact sequences

$$\begin{array}{rcccccccc}
\mathcal{E}_i \ : \ 0 & \longrightarrow & D_i & \xrightarrow{\begin{pmatrix}\phi_D \\ \phi_F\end{pmatrix}} & D_i \oplus F_i & \xrightarrow{(-\phi_F(\phi_D)^{-1} \ 1)} & \widehat{F}_i & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow {\scriptstyle 1 + d_E\psi + \psi d_E} & & \Big\downarrow {\scriptstyle r} & & \\
\mathcal{E}'_i \ : \ 0 & \longrightarrow & D_i & \xrightarrow{\begin{pmatrix}\phi'_D \\ \phi'_F\end{pmatrix}} & D_i \oplus F_i & \xrightarrow{(-\phi'_F(\phi'_D)^{-1} \ 1)} & \widehat{F'}_i & \longrightarrow & 0
\end{array}$$

so that

$$\tau(r : \widehat{F}_i \to \widehat{F}'_i) \ = \ \tau(1 + d_E\psi + \psi d_E : E_i \to E_i) \in K_1(A) \ .$$

The chain complex automorphism $1 + d_E\psi + \psi d_E : E \to E$ is chain homotopic to $1 : E \to E$, so

$$\begin{aligned}
\tau(r : \widehat{F} \to \widehat{F}') \ &= \ \sum_{i=0}^{\infty} (-1)^i \tau(r : \widehat{F}_i \to \widehat{F}'_i) \\
&= \ \sum_{i=0}^{\infty} (-1)^i \tau(1 + d_E\psi + \psi d_E : E_i \to E_i) \\
&= \ \tau(1 + d_E\psi + \psi d_E : E \to E) \\
&= \ \tau(1 : E \to E) \ = \ 0 \in K_1(A) \ .
\end{aligned}$$

□

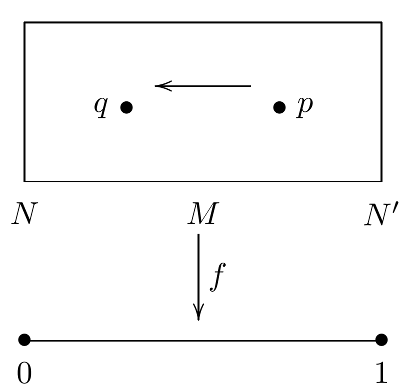
The formula $d_{\widehat{F}} = d_F - \phi_F(\phi_D)^{-1}c : F_i \to F_{i-1}$ was first obtained in [FR,2.3].

2. The Morse-Smale complex $C^{MS}(M, f, v)$.

This section recalls the properties of the Morse-Smale complex $C^{MS}(M, f, v)$ of a real-valued Morse function on an m-dimensional cobordism

$$f \ : \ (M; N, N') \to (I; \{0\}, \{1\})$$

with respect to any $v \in \mathcal{GT}(f)$.



Definition 2.1 The *Morse-Smale complex* of $f : M \to \mathbb{R}$ with respect to $v \in \mathcal{GT}(f)$ is the based f. g. free $\mathbb{Z}[\pi_1(M)]$-module chain complex $C^{MS}(M, f, v)$ with

(i) $\text{rank}_{\mathbb{Z}[\pi_1(M)]} C_i^{MS}(M, f, v) = c_i(f)$, with one basis element $\widetilde{p}$ for each critical point $p \in M$ of index i, corresponding to a choice of lift $\widetilde{p} \in \widetilde{M}$,

(ii) the boundary $\mathbb{Z}[\pi_1(M)]$-module morphisms

$$d \; : \; C^{MS}_i(M,f,v) \;=\; \mathbb{Z}[\pi_1(M)]^{c_i(f)} \to C^{MS}_{i-1}(M,f,v) \;=\; \mathbb{Z}[\pi_1(M)]^{c_{i-1}(f)} \; ;$$
$$\widetilde{p} \mapsto \sum_{\widetilde{q}} n(\widetilde{p},\widetilde{q})\widetilde{q}$$

with $n(\widetilde{p},\widetilde{q}) \in \mathbb{Z}$ the algebraic number of $\widetilde{v}$-gradient flow lines in $\widetilde{M}$ joining $\widetilde{p}$ to $\widetilde{q}$. □

A Morse function on a cobordism

$$f \; : \; (M;N,N') \to (I;\{0\},\{1\})$$

and $v \in \mathcal{GT}(f)$ determine a handlebody decomposition

$$M \;=\; N \times I \cup \bigcup_{i=0}^{m} \bigcup_{c_i(f)} h^i$$

with $c_i(f)$ i-handles $h^i = D^i \times D^{m-i}$. Given a CW structure on N with $c_i(N)$ i-cells $e^i \subset N$ let M have the CW structure with

$$c_i(M) \;=\; c_i(N) + c_i(f)$$

i-cells: there is one i-cell $e^i \times I \subset M$ for each i-cell $e^i \subset N$, and one i-cell $h^i \subset M$ for each critical point of index i. Let $\widetilde{M}$ be the universal cover of M, and let $\widetilde{N},\widetilde{N}'$ be the corresponding covers of N,N'.

Proposition 2.2. *The Morse-Smale complex $C^{MS}(M,f,v)$ is the relative cellular chain complex of $(\widetilde{M},\widetilde{N}\times I)$*

$$C^{MS}(M,f,v) \;=\; C(\widetilde{M},\widetilde{N}\times I) \; .$$

Proof. See Franks [Fr]. □

Definition 2.3 Given a Morse function $f : (M;N,N') \to (I;\{0\},\{1\})$, $v \in \mathcal{GT}(f)$ and CW structures on N and N' write

$$D \;=\; C(\widetilde{N}) \; , \; D' \;=\; C(\widetilde{N}') \; , \; E \;=\; C(\widetilde{M}) \; , \; F \;=\; C^{MS}(M,f,v) \;=\; C(\widetilde{M},\widetilde{N}\times I) \; .$$

(i) The cellular chain complex of $\widetilde{M}$ is of the form

$$d_E \;=\; \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \; : \; E_i \;=\; D_i \oplus F_i \to E_{i-1} \;=\; D_{i-1} \oplus F_{i-1}$$

for a *birth* chain map $c : F_{*+1} \to D$, that is

$$E \;=\; C(\widetilde{M}) \;=\; \mathcal{C}(c)$$

(ii) The natural map $g : N \to M$ is the inclusion of a CW subcomplex, inducing the embedding

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C(\widetilde{N}) = D \to C(\widetilde{M}) = E .$$

A *crossing chain approximation*

$$h : C(\widetilde{N}') = D' \to C(\widetilde{M}) = E$$

is a chain map induced by the inclusion $h : N' \to M$ – in general this is not the inclusion of a CW subcomplex, so the construction requires the cellular approximation theorem, and the CW structures only determine a chain map h up to chain homotopy. The components of

$$h = \begin{pmatrix} h_D \\ h_F \end{pmatrix} : D'_i \to E_i = D_i \oplus F_i$$

are such that

$$d_D h_D + c h_F = h_D d_{D'} : D'_i \to D_{i-1} ,$$
$$d_F h_F = h_F d_{D'} : D'_i \to F_{i-1}$$

defining a *death* chain map

$$h_F : D' \to F$$

and a *survival* chain homotopy

$$h_D : c h_F \simeq 0 : D' \to D_{*-1} . \qquad \square$$

Remark 2.4 For a Morse function

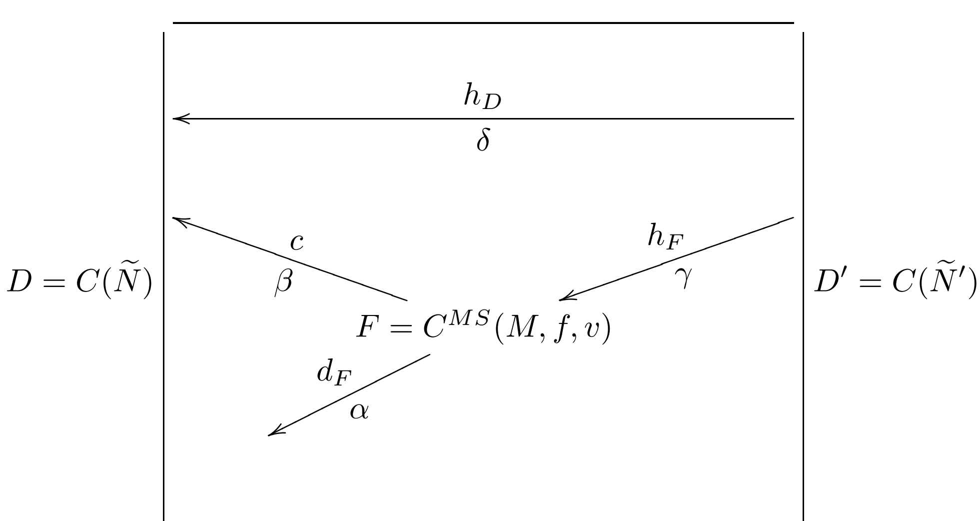
$$f : (M; N, N') \to (I; \{0\}, \{1\})$$

with $v \in \mathcal{GT}(f)$ there are 4 types of v-gradient flow lines, which we shall call $\alpha, \beta, \gamma, \delta$, corresponding to the 4 morphisms

$$d_F : F_i \to F_{i-1} , \ c : F_i \to D_{i-1} , \ h_F : D'_i \to F_i , \ h_D : D'_i \to D_i ,$$

as follows :

(a) complete: α starts at an index i critical point $p \in M$ and terminates at an index $i-1$ critical point $q \in M$.
(b) birth: β starts at an index i critical point $p \in M$ and terminates in N.
(c) death: γ starts in N' and terminates at an index i critical point $p \in M$.
(d) survival: δ starts in N' and terminates in N.



A crossing chain approximation $h : D' \to E$ corresponds to the flow lines which start in N', i.e. those of death and survival type. Pajitnov [P4,§4] obtained an analogue of the cellular approximation theorem for the gradient flow : for any Morse function $f : M \to S^1$ with regular value $0 \in S^1$ it is possible to choose $v \in \mathcal{GT}(f)$ and handlebody structures on N and N' such that

(i) every survival v-gradient flow line in M which starts in an i'-handle of N' ends in an i-handle of N with $i \leq i'$,

(ii) there is a finite number of rel ∂ homology classes of survival flow lines as in (i) with $i = i' - 1$,

(iii) there exist d_F, c, h_F as in 2.3 which actually count the flow lines of type α, β, γ, and h_D counts the rel ∂ homology classes of survival flow lines δ as in (ii),

(iv) the function which sends $y \in M \backslash \mathrm{Crit}(f)$ to the endpoint $\Phi(y) \in N \cup \mathrm{Crit}(f)$ of the flow line of f which starts at y

$$\Phi \ : \ M\backslash \mathrm{Crit}(f) \to N \cup \mathrm{Crit}(f) \ ; \ y \mapsto \Phi(y)$$

restricts to a function

$$N' \cap \Phi^{-1}(N) \to N \cap \Phi(N') \ ; \ x' \mapsto \Phi(x')$$

which is a partially defined map $N' \to N$ inducing the 'partial chain map'

$$h_D \ : \ D' \ = \ C(\widetilde{N}') \to D \ = \ C(\widetilde{N}) \ .$$

(See also Hutchings and Lee [HL]). $\square$

We shall now express the Morse-Smale complex $C^{MS}(M, f, v)$ of a Morse function

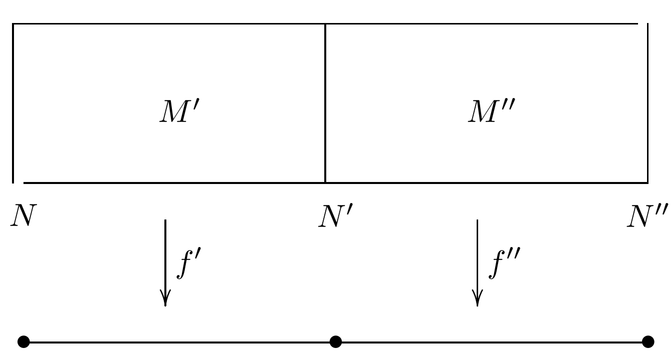
$$f \ : \ (M; N, N'') \to ([0, 2]; 0, 2) \ (v \in \mathcal{GT}(f))$$

which is transverse regular at $1 \in [0, 2]$ in terms of the Morse-Smale complexes $C^{MS}(M', f', v')$, $C^{MS}(M'', f'', v'')$ of the restrictions

$$\begin{aligned} f' \ &= \ f| \ : \ (M'; N, N') \ = \ f^{-1}([0, 1]; \{0\}, \{1\}) \to ([0, 1]; \{0\}, \{1\}) \ , \\ f'' \ &= \ f| \ : \ (M''; N', N'') \ = \ f^{-1}([1, 2]; \{1\}, \{2\}) \to ([1, 2]; \{1\}, \{2\}) \end{aligned}$$

using choices of CW structures for N, N', N'' and crossing chain approximations

$$h' \ : \ C(\widetilde{N}') \to C(\widetilde{M}') \ , \ h'' \ : \ C(\widetilde{N}'') \to C(\widetilde{M}'') \ .$$



If f' (resp. f'') has $c_i(f')$ (resp. $c_i(f'')$) critical points of index i then f has

$$c_i(f) \;=\; c_i(f') + c_i(f'')$$

critical points of index i.

Terminology 2.5 Write the various chain complexes, birth, death, survival and crossing chain approximations for (f', v') and (f'', v'') as

$$\begin{aligned}
&D \;=\; C(\widetilde{N}) \;,\; D' \;=\; C(\widetilde{N}') \;,\; D'' \;=\; C(\widetilde{N}'') \;,\\
&E' \;=\; C(\widetilde{M}') \;,\; E'' \;=\; C(\widetilde{M}'') \;,\\
&F' \;=\; C^{MS}(M', f', v') \;,\; F'' \;=\; C^{MS}(M'', f'', v'') \;,\\
&d_{E'} \;=\; \begin{pmatrix} d_D & c' \\ 0 & d_{F'} \end{pmatrix} \;:\; E'_i \;=\; D_i \oplus F'_i \to E_{i-1} \;=\; D_{i-1} \oplus F'_{i-1} \;,\\
&d_{E''} \;=\; \begin{pmatrix} d_{D'} & c'' \\ 0 & d_{F''} \end{pmatrix} \;:\; E''_i \;=\; D'_i \oplus F''_i \to E''_{i-1} \;=\; D'_{i-1} \oplus F''_{i-1} \;,\\
&h' \;=\; \begin{pmatrix} h'_D \\ h'_{F'} \end{pmatrix} \;:\; D'_i \to E'_i \;=\; D_i \oplus F'_i \;,\\
&h'' \;=\; \begin{pmatrix} h''_{D'} \\ h''_{F''} \end{pmatrix} \;:\; D''_i \to E''_i \;=\; D'_i \oplus F''_i \;.
\end{aligned}$$

Define the chain complexes F, E by

$$\begin{aligned}
&d_F \;=\; \begin{pmatrix} d_{F'} & h'_{F'}c'' \\ 0 & d_{F''} \end{pmatrix} \;:\; F_i \;=\; F'_i \oplus F''_i \to F_{i-1} \;=\; F'_{i-1} \oplus F''_{i-1} \;,\\
&d_E \;=\; \begin{pmatrix} d_D & c' & h'_D c'' \\ 0 & d_{F'} & h'_{F'}c'' \\ 0 & 0 & d_{F''} \end{pmatrix} \;:\\
&\qquad\qquad E_i \;=\; D_i \oplus F'_i \oplus F''_i \to E_{i-1} \;=\; D_{i-1} \oplus F'_{i-1} \oplus F''_{i-1} \;,
\end{aligned}$$

and let

$$c \;:\; F \to D_{*-1} \;,\; h \;=\; \begin{pmatrix} h_D \\ h_F \end{pmatrix} \;:\; D'' \to E$$

be the chain maps defined by

$$\begin{aligned}
&c \;=\; (\, c' \quad h'_D c'' \,) \;:\; F_i \;=\; F'_i \oplus F''_i \to D_{i-1} \;,\\
&h_D \;=\; h'_D h''_{D'} \;:\; D''_i \to D_i \;,\\
&h_F \;=\; \begin{pmatrix} h'_{F'} h''_{D'} \\ h''_{F''} \end{pmatrix} \;:\; D''_i \to F_i \;=\; F'_i \oplus F''_i \;. \quad \square
\end{aligned}$$

Proposition 2.6. (i) *The Morse-Smale complex of*

$$f \;=\; f' \cup f'' \;:\; (M; N, N'') \;=\; (M'; N, N') \cup (M''; N', N'') \to ([0,2]; \{0\}, \{2\})$$

with respect to $v = v' \cup v'' \in \mathcal{GT}(f)$ *is the algebraic mapping cone*

$$C^{MS}(M, f, v) \;=\; \mathcal{C}(g : F'' \to F')$$

of the chain map $a : F''_{*+1} \to F'$ *defined by*

$$a \;:\; F''_{i+1} \;=\; \mathbb{Z}[\pi_1(M)]^{c_{i+1}(f'')} \to F'_i \;=\; \mathbb{Z}[\pi_1(M)]^{c_i(f')} \;;\; \widetilde{p} \mapsto \sum_{\widetilde{q}} \sum_{u \in \pi_1(M)} n(\widetilde{p}, u\widetilde{q})u\widetilde{q}$$

with $n(\widetilde{p}, u\widetilde{q}) \in \mathbb{Z}$ *the algebraic number of* $\widetilde{v}$*-gradient flow lines in* $\widetilde{M}$ *joining* $\widetilde{p} \in \widetilde{M''}$ *to* $u\widetilde{q} \in \widetilde{M'}$.
(ii) *For any death chain map* $h'_{F'} : D' \to F'$ *of* (f', v') *and any birth chain map* $c'' : F''_{*+1} \to D'$ *of* (f'', v'') *there exists a chain homotopy*

$$b \;:\; a \;\simeq\; h'_{F'}c'' \;:\; F''_{*+1} \to F'$$

such that

(a) *b determines a simple isomorphism*

$$I \;=\; \begin{pmatrix} 1 & \pm b \\ 0 & 1 \end{pmatrix} \;:\; \mathcal{C}(a) \;=\; C^{MS}(M, f, v) \to F \;=\; \mathcal{C}(h'_{F'}c'') \;,$$

(b) $cI : C^{MS}(M, f, v)_{*+1} \to D = C(\widetilde{N})$ *is a birth chain map of* (f, v),

(c) *the cellular chain complex of* $\widetilde{M}$ *is*

$$C(\widetilde{M}) \;=\; \mathcal{C}(cI)$$

and there is defined a simple isomorphism

$$1 \oplus I \;:\; E \to C(\widetilde{M}) \;,$$

(d) $(1 \oplus I)h : D'' = C(\widetilde{N}) \to C(\widetilde{M})$ *is a crossing chain approximation for* (f, v), *with components a death chain map*

$$((1 \oplus I)h)_F \;=\; Ih_F \;=\; \begin{pmatrix} h'_{F'}h''_{D'} \pm bh''_{F''} \\ h''_{F''} \end{pmatrix} \;:\; D''_i \to C^{MS}(M, f, v)_i \;=\; F'_i \oplus F''_i$$

and a survival chain map

$$((1 \oplus I)h'')_D \;=\; h_D \;=\; h'_D h''_{D'} \;:\; D''_i \to D_i \;.$$

Proof. (i) This is a direct consequence of the construction of the Morse-Smale complex (2.1).
(ii) Use the handlebody decomposition of M (resp. M', M'') determined by (f, v) (resp. (f', v'), (f'', v'')) to extend the CW structure on N (resp. N, N') to a CW structure on M (resp. M', M''). The existence of a chain homotopy $b : a \simeq h_F c'$ is immediate from the observation that a and $h'_{F'}c''$ are both connecting chain maps for the triad of CW subcomplexes $M \supset M' \supset N$

$$\partial \;:\; F'' \;=\; C(\widetilde{M}, \widetilde{M}') \to C(\widetilde{M}', \widetilde{N}) \;=\; F'$$

which is unique up to chain homotopy. Property (a) is just an application of 1.2. The composite of cellular approximations to the inclusions $N'' \to M''$, $M'' \to M$ is a cellular approximation to the inclusion $N'' \to M$, giving (b),(c) and (d). $\square$

Remark 2.7 Cornea and Ranicki [CR] obtain a sharper version of 2.6 : for any Morse map

$$f \;=\; f' \cup f'' \;:\; (M; N, N'') \;=\; (M; N, N') \cup (M'; N', N'') \to ([0,2]; \{0\}, \{2\})$$

and $v = v' \cup v'' \in \mathcal{GT}(f)$ there exist Morse maps

$$\widehat{f} \;:\; M \to \mathbb{R} \;,\; g \;:\; N' \;=\; f^{-1}(1) \to \mathbb{R}$$

and $\widehat{v} \in \mathcal{GT}(\widehat{f})$, $w \in \mathcal{GT}(g)$ such that

(a) $(\widehat{f}, \widehat{v})$ agrees with (f, v) outside a tubular neighbourhood of N'

$$f^{-1}[1-\epsilon, 1+\epsilon] \;=\; N' \times [1-\epsilon, 1+\epsilon] \subset M$$

for some small $\epsilon > 0$.

(b) $(\widehat{f}, \widehat{v})$ restricts to translates of (g, w)

$$\begin{aligned} (\widehat{f}, \widehat{v})| \;&=\; (g_+, w_+) \;:\; N' \times \{1 + \epsilon/2\} \to \mathbb{R} \;, \\ (\widehat{f}, \widehat{v})| \;&=\; (g_-, w_-) \;:\; N' \times \{1 - \epsilon/2\} \to \mathbb{R} \end{aligned}$$

with

$$\begin{aligned} \mathrm{Crit}_i(g_+) \;&=\; \mathrm{Crit}_i(g) \times \{1 + \epsilon/2\} \;, \\ \mathrm{Crit}_i(g_-) \;&=\; \mathrm{Crit}_i(g) \times \{-\epsilon/2\} \;, \\ \mathrm{Crit}_i(\widehat{f}) \;&=\; \mathrm{Crit}_{i-1}(g_+) \cup \mathrm{Crit}_i(g_-) \cup \mathrm{Crit}_i(f) \;. \end{aligned}$$

(c) The v-gradient flow lines are in one-one correspondence with the broken $\widehat{v}$-gradient flow lines i.e. joined up sequences of $\widehat{v}$-gradient flow lines which start at critical points of f'' and terminate at critical points of f'.

(d) The Morse-Smale complex of $(\widehat{f}, \widehat{v})$ is of the form

$$d_{C^{MS}(M,\widehat{f},\widehat{v})} \;=\; \begin{pmatrix} d_{F'} & h'_{F'} & 0 & 0 \\ 0 & -d_{D'} & 0 & 0 \\ 0 & 1 & d_{D'} & c'' \\ 0 & 0 & 0 & d_{F''} \end{pmatrix} \;:$$

$$\begin{aligned} C^{MS}_i(M, \widehat{f}, \widehat{v}) \;&=\; F'_i \oplus D'_{i-1} \oplus D'_i \oplus F''_i \\ &\to C^{MS}_{i-1}(M, \widehat{f}, \widehat{v}) \;=\; F'_{i-1} \oplus D'_{i-2} \oplus D'_{i-1} \oplus F''_{i-1} \end{aligned}$$

with $D' = C(N', g, w)$, giving choices of 'gradient-like' crossing chain approximations h', h'' such that the chain homotopy in 2.6 (ii) is

$$\begin{aligned} b \;=\; 0 \;:\; a \;\simeq\; h'_{F'} c'' \;: \\ F'' \;=\; C^{MS}(M'', f'', v'') \to F'_{*-1} \;=\; C^{MS}(M', f', v')_{*-1} \end{aligned}$$

(i.e. $a = h'_{F'} c''$) and the simple isomorphism

$$I \;:\; C^{MS}(M, f, v) \;=\; \mathcal{C}(a) \to F \;=\; \mathcal{C}(h'_{F'} c'')$$

is the identity. $\square$

3. The proper Morse-Smale complex.

The construction of $C^{MS}(M,f,v)$ applies just as well to a proper real-valued Morse function on a non-compact manifold :

Definition 3.1 Let $(M,\partial M)$ be a non-compact manifold with compact boundary and a proper real-valued Morse function

$$f\ :\ (M,\partial M)\to([0,\infty),\{0\})\ ,$$

and let $v\in\mathcal{GT}(f)$. The *proper Morse-Smale complex* $C^{MS}(M,f,v)$ is defined exactly as in the compact case, with

(i) $C_i^{MS}(M,f,v)$ the based free $\mathbb{Z}[\pi_1(M)]$-module generated by $\gamma_i(f)$, the set of critical points of f with index i,

(ii) the boundary $\mathbb{Z}[\pi_1(M)]$-module morphisms are given by

$$d\ :\ C_i^{MS}(M,f,v)\ =\ \mathbb{Z}[\pi_1(M)]^{\gamma_i(f)}\to C_{i-1}^{MS}(M,f,v)\ =\ \mathbb{Z}[\pi_1(M)]^{\gamma_{i-1}(f)}\ ;$$

$$\widetilde{p}\mapsto\sum_{\widetilde{q}}\ \sum_{u\in\pi_1(M)}n(\widetilde{p},u\widetilde{q})u\widetilde{q}$$

with $n(\widetilde{p},u\widetilde{q})\in\mathbb{Z}$ the algebraic number of v-gradient flow lines in $\widetilde{M}$ joining $\widetilde{p}$ to $u\widetilde{q}$.

□

Given a proper Morse function $f:(M,\partial M)\to([0,\infty),\{0\})$ and a gradient-like vector field for f and a CW structure for ∂M let M have the CW structure given by the handle decomposition

$$M\ =\ \partial M\times I\cup\bigcup_{i=0}^{m}\ \bigcup_{\gamma_i(f)}h^i\ .$$

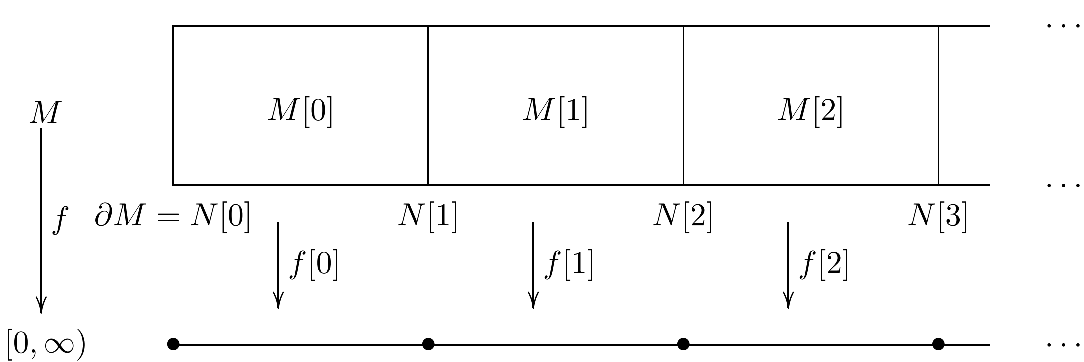
The expression of 2.6 for the Morse-Smale complex of the union of two Morse functions on adjoining compact cobordisms will now be applied to obtain an isomorphism between the Morse-Smale complex $C^{MS}(M,f,v)$ and the relative cellular chain complex of $(\widetilde{M},\partial\widetilde{M})$

$$I\ :\ C^{MS}(M,f,v)\ \cong\ C(\widetilde{M},\partial\widetilde{M})\ .$$

Terminology 3.2 For $j=0,1,2,\dots$ let

$$f[j]\ =\ f|\ :$$
$$(M[j];N[j],N[j+1])\ =\ f^{-1}([j,j+1];\{j\},\{j+1\})\to([j,j+1];\{j\},\{j+1\})$$

be the Morse functions on compact cobordisms given by the restriction of f.



The inclusions

$$g[j]\ :\ N[j]\to M[j]\ ,\ h[j]\ :\ N[j+1]\to M[j]$$

induce embeddings of based f. g. free $\mathbb{Z}[\pi_1(M)]$-module chain complexes

$$g[j]\ :\ C(\widetilde{N}[j])\to C(\widetilde{M}[j])$$

and chain maps

$$h[j]\ :\ C(\widetilde{N}[j+1])\to C(\widetilde{M}[j])\ .$$

Write

$$\begin{aligned} D[j]\ &=\ C(\widetilde{N}[j])\ ,\ E[j]\ =\ C(\widetilde{M}[j])\ ,\ F[j]\ =\ C(\widetilde{M}[j],\widetilde{N}[j])\ ,\\ d_{E[j]}\ &=\ \begin{pmatrix} d_{D[j]} & c[j] \\ 0 & d_{F[j]}\end{pmatrix}\ :\\ &\qquad E_i[j]\ =\ D_i[j]\oplus F_i[j]\to E_{i-1}[j]\ =\ D_{i-1}[j]\oplus F_{i-1}[j]\ ,\\ g[j]\ &=\ \begin{pmatrix} 1 \\ 0\end{pmatrix}\ :\ D_i[j]\to E_i[j]\ =\ D_i[j]\oplus F_i[j]\ ,\\ h[j+1]\ &=\ \begin{pmatrix} h_{D[j+1]} \\ h_{F[j+1]}\end{pmatrix} : D_i[j+1]\to E_i[j]\ =\ D_i[j]\oplus F_i[j]\ .\quad\square \end{aligned}$$

Proposition 3.3. *Let $f:(M,\partial M)\to([0,\infty),\{0\})$ be a proper Morse function, and let $v\in\mathcal{GT}(f)$.*
(i) *The cellular chain complex of $(M,\partial M)$ is of the form*

$$C(\widetilde{M},\partial\widetilde{M})\ =\ \mathrm{coker}\big(g-h:\sum_{j=0}^{\infty}C(\widetilde{N}[j])\to\sum_{j=0}^{\infty}C(\widetilde{M}[j])\big)\ ,$$

a based free $\mathbb{Z}[\pi_1(M)]$-module complex with basis the images of the basis elements in $\sum\limits_{j=0}^{\infty}C(\widetilde{M}[j],\widetilde{N}[j])$, and may be expressed as

$$\begin{aligned} d_{C(\widetilde{M},\partial\widetilde{M})}\ &=\ \begin{pmatrix} d_{F[0]} & h_{F[0]}c[1] & h_{F[0]}h_{D[1]}c[2] & h_{F[0]}h_{D[1]}h_{D[2]}c[3] & \dots \\ 0 & d_{F[1]} & h_{F[1]}c[2] & h_{F[1]}h_{D[2]}c[3] & \dots \\ 0 & 0 & d_{F[2]} & h_{F[2]}c[3] & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\\ &:\ C_i(\widetilde{M},\partial\widetilde{M})\ =\ \sum^{\infty}F_i[k]\to C_{i-1}(\widetilde{M},\partial\widetilde{M})\ =\ \sum^{\infty}F_{i-1}[j]\ . \end{aligned}$$

(ii) *The Morse-Smale complex of* (f,v) *is of the form*

$$d_{C^{MS}(M,f,v)} = \begin{pmatrix} d_{C^{MS}(M[0],f[0],v[0])} & c^{MS}[0,1] & c^{MS}[0,2] & c^{MS}[0,3] & \dots \\ 0 & d_{C^{MS}(M[1],f[1],v[1])} & c^{MS}[1,2] & c^{MS}[1,3] & \dots \\ 0 & 0 & d_{C^{MS}(M[2],f[2],v[2])} & c^{MS}[2,3] & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$: C_i^{MS}(M,f,v) = \sum_{k=0}^{\infty} C_i^{MS}(M[k],f[k],v[k]) \to C_{i-1}^{MS}(M,f,v) = \sum_{j=0}^{\infty} C_{i-1}^{MS}(M[j],f[j],v[j])$$

with

$$c^{MS}[j,k] : C_i^{MS}(M[k],f[k],v[k]) \to C_{i-1}^{MS}(M[j],f[j],v[j]) \ (j<k)$$

counting the $\widetilde{v}$*-gradient flow lines in the universal cover* $\widetilde{M}$ *which start at an index* i *critical point of* $\widetilde{f}[k]$ *and terminate at an index* $i-1$ *critical point of* $\widetilde{f}[j]$.

(iii) *There exists an isomorphism of chain complexes*

$$I : C^{MS}(M,f,v) \cong C(\widetilde{M},\partial\widetilde{M})$$

of the form

$$I = 1 + \sum_{j'<j} b[j',j] : C_i^{MS}(M,f,v) = \sum_{j=0}^{\infty} F_i[j] \to C_i(\widetilde{M},\partial\widetilde{M}) = \sum_{j'=0}^{\infty} F_i[j'] .$$

Proof. (i) The exact sequence

$$0 \to \sum_{j=0}^{\infty} C(\widetilde{N}[j]) \xrightarrow{g-h} \sum_{j=0}^{\infty} C(\widetilde{M}[j]) \to C(\widetilde{M},\partial\widetilde{M}) \to 0$$

is just the chain level Mayer-Vietoris sequence for the union

$$M = M[\text{even}] \cup M[\text{odd}]$$

with

$$M[\text{even}] = \bigcup_{j \text{ even}} M[j] \ , \ M[\text{odd}] = \bigcup_{j \text{ odd}} M[j] \ .$$

The matrix formula for $\text{coker}(g-h)$ is a direct application of 1.8 (i) (b) with

$$\phi = g-h = \begin{pmatrix} 1-h_D \\ h \end{pmatrix} : D_i \to E_i = D_i \oplus F_i \ ,$$

noting that each

$$1-h_D \;=\; \begin{pmatrix} 1 & -h_{D[0]} & 0 & \dots \\ 0 & 1 & -h_{D[1]} & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \;:\; E_i \;=\; \sum_{j=0}^{\infty} F_i[j] \to E_i \;=\; \sum_{j=0}^{\infty} F_i[j]$$

is an automorphism, with inverse

$$(1-h_D)^{-1} \;=\; \begin{pmatrix} 1 & h_{D[0]} & h_{D[0]}h_{D[1]} & \dots \\ 0 & 1 & h_{D[1]} & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \;:$$

$$E_i \;=\; \sum_{j=0}^{\infty} F_i[j] \to E_i \;=\; \sum_{j=0}^{\infty} F_i[j] \;.$$

(ii) By construction.
(iii) For $k=1,2,\dots$ define the Morse function

$$f[0,k] \;=\; \bigcup_{j=1}^{k} f[j] \;=\; f| \;:\; M[0,k] \;=\; \bigcup_{j=1}^{k} M[j] \;=\; f^{-1}[0,k] \to [0,k] \;,$$

and assume inductively that there is given an isomorphism

$$I[0,k] \;:\; C^{MS}(M[0,k],f[0,k],v[0,k]) \;\cong\; C(\widetilde{M}[0,k],\partial\widetilde{M})$$

of the form

$$I[0,k] \;=\; 1+\sum_{j'<j} b[j',j] \;:$$

$$C_i^{MS}(M[0,k],f[0,k],v[0,k]) \;=\; \sum_{j=0}^{k} F_i[j] \to C_i(\widetilde{M}[0,k],\partial\widetilde{M}) \;=\; \sum_{j'=0}^{k} F_i[j'] \;.$$

Now apply 2.6 (i) to the Morse function

$$\begin{aligned} & f[0,k+1] \;=\; f[0,k]\cup f[k] \;: \\ & M[0,k+1] \;=\; M[0,k]\cup M[k] \to [0,k+1] \;=\; [0,k]\cup[k,k+1] \;, \end{aligned}$$

extending $I[0,k]$ to an isomorphism $I[0,k+1]$ of the same form, and pass to the direct limit to obtain an isomorphism

$$\begin{aligned} I \;=\; \varinjlim_k I[0,k] \;:\; C^{MS}(M,f,v) \;&=\; \varinjlim_k C^{MS}(M[0,k],f[0,k],v[0,k]) \\ &\to C(\widetilde{M},\partial\widetilde{M}) \;=\; \varinjlim_k C(\widetilde{M}[0,k],\partial\widetilde{M}) \;. \end{aligned}$$

of the form

$$I \;=\; 1+\sum_{j'<j} b[j',j] \;.$$

□

4. The Cohn noncommutative localization.

We refer to Cohn [C] and Schofield [Scho] for general accounts of the localization $\Sigma^{-1}R$ of a ring R inverting a set Σ of square matrices. The natural morphism $R \to \Sigma^{-1}R$ has the universal property that a morphism of rings $R \to A$ which sends Σ to invertible matrices in A has a unique factorization

$$R \to \Sigma^{-1}R \to A \ .$$

The Gerasimov-Malcolmson normal form expresses every morphism of f. g. free $\Sigma^{-1}R$-modules

$$\phi \ : \ \Sigma^{-1}R^n \to \Sigma^{-1}R^p$$

as a composite

$$\phi \ = \ f\sigma^{-1}g \ : \ \Sigma^{-1}R^n \to \Sigma^{-1}R^m \to \Sigma^{-1}R^m \to \Sigma^{-1}R^p$$

(nonuniquely) for some R-module morphisms

$$f \ : \ R^m \to R^p \ , \ \sigma \ : \ R^m \to R^m \ , \ g \ : \ R^n \to R^m$$

such that σ is $\Sigma^{-1}R$-invertible.

Proposition 4.1. (Sheiham [Sh,3.1]) *Given a ring morphism $\epsilon : R \to A$ let Σ be the set of all square matrices in R which become invertible in A.*
(i) *The ring morphism ϵ extends to a ring morphism*

$$\epsilon \ : \ \Sigma^{-1}R \to A \ .$$

(ii) *An endomorphism of a f. g. free $\Sigma^{-1}R$-module*

$$\phi \ : \ \Sigma^{-1}R^n \to \Sigma^{-1}R^n$$

is an automorphism if and only if $\epsilon(\phi) : A^n \to A^n$ is an A-module automorphism.

Proof. (i) By the universal property of $R \to \Sigma^{-1}R$.
(ii) It is clear that if ϕ is an automorphism then so is $\epsilon(\phi)$.

Conversely, suppose that $\epsilon(\phi)$ is an automorphism. Express ϕ in the Gerasimov-Malcolmson normal form

$$\phi \ = \ f\sigma^{-1}g \ : \ \Sigma^{-1}R^n \to \Sigma^{-1}R^n$$

for some R-module morphisms

$$f \ : \ R^m \to R^n \ , \ \sigma \ : \ R^m \to R^m \ , \ g \ : \ R^n \to R^m$$

such that $\epsilon(\sigma) : A^m \to A^m$ is an automorphism. The R-module endomorphism defined by

$$\theta \ = \ \begin{pmatrix} 0 & -f \\ g & \sigma \end{pmatrix} \ : \ R^n \oplus R^m \to R^n \oplus R^m$$

is A-invertible, since the induced $\Sigma^{-1}R$-module endomorphism

$$\theta \ = \ \begin{pmatrix} \phi & -f \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sigma^{-1}g & 1 \end{pmatrix} \ : \ \Sigma^{-1}(R^n \oplus R^m) \to \Sigma^{-1}(R^n \oplus R^m)$$

is A-invertible. Thus $\theta : \Sigma^{-1}(R^n \oplus R^m) \to \Sigma^{-1}(R^n \oplus R^m)$ is a $\Sigma^{-1}R$-module automorphism, and hence so is ϕ. □

In the application of Cohn localization in §5 below

$$\epsilon \ : \ R \ = \ A_\alpha[z] \to A \ ; \ z \mapsto 0 \ .$$

5. Polynomial extensions.

The relationship between the algebraic mapping cone and cokernel worked out in §1 will now be applied to the algebraic situation arising from a circle-valued Morse function $f : M \to S^1$. The actual application to the Novikov complex $C^{Nov}(M,f,v)$ will be carried out in §6.

Definition 5.1 Let A be a ring with an automorphism $\alpha : A \to A$, and let z be an indeterminate over A such that

$$az \ = \ z\alpha(a) \ (a \in A) \ .$$

(i) The *α-twisted Laurent polynomial extension* of A

$$A_\alpha[z,z^{-1}] \ = \ \sum_{j=-\infty}^{\infty} z^j A \ = \ A_\alpha[z,z^{-1}]$$

is the ring of polynomials $\sum\limits_{j=-\infty}^{\infty} a_j z^j$ $(a_j \in A)$ such that $\{j \in \mathbb{Z} \,|\, a_j \neq 0\}$ is finite.

(ii) The *α-twisted Novikov completion of* $A_\alpha[z,z^{-1}]$

$$A_\alpha((z)) \ = \ \varprojlim_k \sum_{j=-\infty}^{k} z^j A \ = \ A_\alpha[[z]][z^{-1}]$$

is the ring of formal power series $\sum\limits_{j=-\infty}^{\infty} a_j z^j$ $(a_j \in A)$ such that $\{j \leq 0 \,|\, a_j \neq 0\}$ is finite. $\square$

Given an A-module B and $j \in \mathbb{Z}$ let $z^j B$ be the A-module with elements $z^j x$ $(x \in B)$ and

$$a(z^j x) \ = \ z^j \alpha^j(a) x \ , \ z^j x + z^j x' \ = \ z^j(x+x') \ (a \in A, x, x' \in B) \ .$$

The induced $A_\alpha[z,z^{-1}]$-module is then given by

$$A_\alpha[z,z^{-1}] \otimes_A B \ = \ B_\alpha[z,z^{-1}] \ = \ \sum_{j=-\infty}^{\infty} z^j B \ .$$

For any A-module C and $k \in \mathbb{Z}$ the A-module morphisms $z^j B \to z^k C$ are given by

$$z^{k-j}\theta \ : \ z^j B \to z^k C \ ; \ z^j x \mapsto z^k \theta(x)$$

with $\theta : B \to C$ a morphism of the additive groups such that

$$\theta(ax) \ = \ \alpha^{k-j}(a)\theta(x) \in C \ (a \in A, x \in B) \ .$$

We shall write $z^{-1}B$ as αB.

For a f. g. free A-module B and any A-module C every $A_\alpha[z,z^{-1}]$-module morphism

$$\phi \ : \ B_\alpha[z,z^{-1}] \to C_\alpha[z,z^{-1}]$$

is given by

$$\psi \;=\; \sum_{j=-\infty}^{\infty} z^j \psi_j \;:\; \sum_{k=-\infty}^{\infty} z^k x_k \mapsto \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} z^{j+k} \psi_j(x_k)$$

with $\psi_j : B \to z^j C$ A-module morphisms such that $\{j \in \mathbb{Z} \,|\, \psi_j \neq 0\}$ is finite. Similarly, every $A_\alpha((z))$-module morphism

$$\widehat{\psi} \;:\; B_\alpha((z)) \to C_\alpha((z))$$

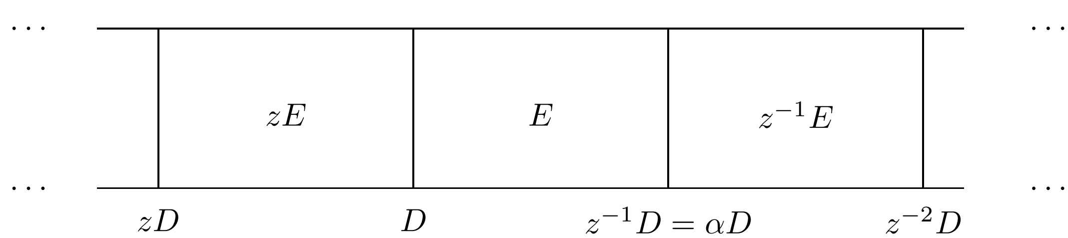
is given by

$$\widehat{\psi} \;=\; \sum_{j=-\infty}^{\infty} z^j \widehat{\psi}_j \;:\; \sum_{k=-\infty}^{\infty} z^k x_k \mapsto \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} z^{j+k} \widehat{\psi}_j(x_k)$$

with $\widehat{\psi}_j : B \to z^j C$ A-module morphisms such that $\{j \leq 0 \,|\, \widehat{\psi}_j \neq 0\}$ is finite.

Definition 5.2 Let $\Sigma^{-1}A_\alpha[z, z^{-1}]$ be the localization of $A_\alpha[z, z^{-1}]$ inverting the set Σ of square matrices in $A_\alpha[z] \subset A_\alpha[z, z^{-1}]$ which become invertible in A under the augmentation

$$\epsilon \;:\; A_\alpha[z] \to A \;;\; z \mapsto 0 \;. \qquad \square$$



Proposition 5.3. *Let D, E be f. g. free A-module chain complexes, and let $g : D \to E$, $h : \alpha D \to E$ be A-module chain maps such that*

$$g \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;:\; D_i \to E_i \;=\; D_i \oplus F_i \;,$$

$$h \;=\; \begin{pmatrix} h_D \\ h_F \end{pmatrix} \;:\; \alpha D_i \to E_i \;=\; D_i \oplus F_i \;,$$

$$d_E \;=\; \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \;:\; E_i \;=\; D_i \oplus F_i \to E_{i-1} \;=\; D_{i-1} \oplus F_{i-1} \;.$$

Given bases for D, F let E have the corresponding basis.
(i) *The $\Sigma^{-1}D_\alpha[z, z^{-1}]$-module morphism*

$$\phi \;=\; g - zh \;:\; \Sigma^{-1}D_\alpha[z, z^{-1}] \to \Sigma^{-1}E_\alpha[z, z^{-1}]$$

is an embedding, the natural projection

$$p \;:\; C \;=\; \mathcal{C}(\phi) \to \widehat{C} \;=\; \mathrm{coker}(\phi)$$

is a chain equivalence, and the inclusions $F_i \to E_i$ induce an isomorphism

$$\widehat{F} \cong \widehat{C}$$

with $\widehat{F}$ *the* $\Sigma^{-1}A_\alpha[z,z^{-1}]$*-module chain complex given by*

$$d_{\widehat{F}} = d_F + zh_F(1-zh_D)^{-1}c \ : \\ \widehat{F}_i \ = \ \Sigma^{-1}(F_i)_\alpha[z,z^{-1}] \to \widehat{F}_{i-1} \ = \ \Sigma^{-1}(F_{i-1})_\alpha[z,z^{-1}] \ .$$

If $\widehat{C}$ *is given the basis determined by the bases of* D,F *and the isomorphism* $\widehat{F}\cong\widehat{C}$ *then*

$$\tau(p:C\simeq\widehat{C}) \ = \ -\sum_{i=0}^{\infty}(-)^i\tau(1-zh_D:\Sigma^{-1}(D_i)_\alpha[z,z^{-1}]\to\Sigma^{-1}(D_i)_\alpha[z,z^{-1}]) \\ \in K_1(\Sigma^{-1}A_\alpha[z,z^{-1}]) \ .$$

(ii) *An* A*-module chain homotopy*

$$k \ : \ h \ \simeq \ h' \ : \ \alpha D\to E$$

determines a $\Sigma^{-1}A_\alpha[z,z^{-1}]$*-module chain isotopy*

$$\psi \ : \ \phi \ = \ g-zh \ \sim \ \phi' \ = \ g-zh' \ : \ \Sigma^{-1}D_\alpha[z,z^{-1}]\to\Sigma^{-1}E_\alpha[z,z^{-1}]$$

and simple isomorphisms of $\Sigma^{-1}A_\alpha[z,z^{-1}]$*-module chain complexes*

$$C \ \cong \ C' \ , \ \widehat{F} \ \cong \ \widehat{F}' \ , \ \widehat{C} \ \cong \ \widehat{C}'$$

where $C=\mathcal{C}(\phi)$, $C'=\mathcal{C}(\phi')$ *etc., with*

$$\begin{aligned} \tau(C\cong C') \ &= \ 0\in K_1(\Sigma^{-1}A_\alpha[z,z^{-1}]) \ , \\ \tau(\widehat{C}\cong\widehat{C}') \ &= \ \sum_{i=0}^{\infty}(-)^i\tau(1-zh_D:\Sigma^{-1}(D_i)_\alpha[z,z^{-1}]\to\Sigma^{-1}(D_i)_\alpha[z,z^{-1}]) \\ &\qquad -\sum_{i=0}^{\infty}(-)^i\tau(1-zh'_D:\Sigma^{-1}(D_i)_\alpha[z,z^{-1}]\to\Sigma^{-1}(D_i)_\alpha[z,z^{-1}]) \\ &= \ 0\in K_1(\Sigma^{-1}A_\alpha[z,z^{-1}]) \ . \end{aligned}$$

(iii) *Let* $\{D(k)\}$, $\{E(k)\}$, $\{F(k)\}$ *be the inverse systems of* A*-module chain complexes defined by*

$$\begin{aligned} D(k) \ &= \ D_\alpha[z,z^{-1}]/\sum_{j=k+1}^{\infty}z^jD \ = \ \sum_{j=-\infty}^{k}z^jD \ , \\ E(k) \ &= \ E_\alpha[z,z^{-1}]/\sum_{j=k+1}^{\infty}z^jE \ = \ \sum_{j=-\infty}^{k}z^jE \ , \\ F(k) \ &= \ \mathrm{coker}\big(g-zh:D(k)\to E(k)\big) \end{aligned}$$

with structure maps the natural projections

$$D(k)\to D(k-1) \ , \ E(k)\to E(k-1) \ , \ F(k)\to F(k-1)$$

The short exact sequence of inverse systems of A-module chain complexes

$$0 \to D(k) \overset{g-zh}{\to} E(k) \to F(k) \to 0$$

induces a short exact sequence of the inverse limit $A_\alpha((z))$-module chain complexes

$$0 \to \varprojlim_k D(k) \overset{g-zh}{\to} \varprojlim_k E(k) \to \varprojlim_k F(k) \to 0$$

with

$$\varprojlim_k D(k) \;=\; D_\alpha((z)) \;,\; \varprojlim_k E(k) \;=\; E_\alpha((z)) \;.$$

Moreover, the inclusions $F_i \to E_i$ induce isomorphisms

$$A_\alpha((z)) \otimes_{\Sigma^{-1}A_\alpha[z,z^{-1}]} \widehat{F} \;\cong\; \varprojlim_k F(k) \;.$$

Proof. (i) Immediate from 1.8 (i), since

$$\phi \;=\; \begin{pmatrix} \phi_D \\ \phi_F \end{pmatrix} \;=\; \begin{pmatrix} 1 - zh_D \\ -zh_F \end{pmatrix} \;: \\ \Sigma^{-1}(D_i)_\alpha[z,z^{-1}] \to \Sigma^{-1}(E_i)_\alpha[z,z^{-1}] \;=\; \Sigma^{-1}(D_i \oplus F_i)_\alpha[z,z^{-1}]$$

with

$$\phi_D \;=\; 1 - zh_D \;:\; \Sigma^{-1}(D_i)_\alpha[z,z^{-1}] \to \Sigma^{-1}(D_i)_\alpha[z,z^{-1}]$$

a $\Sigma^{-1}A_\alpha[z,z^{-1}]$-module automorphism.
(ii) Write the A-module chain homotopy $k : h \simeq h' : \alpha D \to E$ as

$$k \;=\; \begin{pmatrix} k_D \\ k_F \end{pmatrix} \;:\; \alpha D_i \to E_{i+1} \;=\; D_{i+1} \oplus F_{i+1} \;,$$

so that

$$\begin{pmatrix} h'_D \\ h'_F \end{pmatrix} - \begin{pmatrix} h_D \\ h_F \end{pmatrix} \;=\; \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \begin{pmatrix} k_D \\ k_F \end{pmatrix} + \begin{pmatrix} k_D \\ k_F \end{pmatrix} d_D \;:\; \alpha D_i \to E_i \;=\; D_i \oplus F_i \;.$$

Define a $\Sigma^{-1}A_\alpha[z,z^{-1}]$ chain homotopy

$$\theta \;:\; \phi \;\simeq\; \phi' \;:\; \Sigma^{-1}D_\alpha[z,z^{-1}] \to \Sigma^{-1}E_\alpha[z,z^{-1}]$$

by

$$\theta \;=\; \begin{pmatrix} \theta_D \\ \theta_F \end{pmatrix} \;=\; \begin{pmatrix} -zk_D \\ -zk_F \end{pmatrix} \;: \\ \Sigma^{-1}(D_i)_\alpha[z,z^{-1}] \to \Sigma^{-1}(E_{i+1})_\alpha[z,z^{-1}] \;=\; \Sigma^{-1}(D_{i+1} \oplus F_{i+1})_\alpha[z,z^{-1}] \;.$$

As in 1.8 (ii) the $\Sigma^{-1}A_\alpha[z]$-module morphisms

$$\psi \;=\; \begin{pmatrix} \theta_D(\phi_D)^{-1} & 0 \\ \theta_F(\phi_D)^{-1} & 0 \end{pmatrix} \;=\; \begin{pmatrix} -zk_D(1 - zh_D)^{-1} & 0 \\ -zk_F(1 - zh_D)^{-1} & 0 \end{pmatrix} \;:$$

$$\Sigma^{-1}(E_i)_\alpha[z] = \Sigma^{-1}(D_i \oplus F_i)_\alpha[z] \to \Sigma^{-1}(E_{i+1})_\alpha[z] = \Sigma^{-1}(D_{i+1} \oplus F_{i+1})_\alpha[z]$$

are such that

$$\phi' \ = \ (1 + d_E\psi + \psi d_E)\phi \ : \ D \to E \ .$$

The $\Sigma^{-1}A_\alpha[z]$-module endomorphism

$$\begin{aligned} 1 + d_E\psi + \psi d_E \ &= \ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \begin{pmatrix} -zk_D(1-zh_D)^{-1} & 0 \\ -zk_F(1-zh_D)^{-1} & 0 \end{pmatrix} \\ &\qquad + \begin{pmatrix} -zk_D(1-zh_D)^{-1} & 0 \\ -zk_F(1-zh_D)^{-1} & 0 \end{pmatrix} \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \\ &\qquad : \ \Sigma^{-1}(E_i)_\alpha[z] \to \Sigma^{-1}(E_i)_\alpha[z] \end{aligned}$$

has augmentation an A-module automorphism

$$\epsilon(1 + d_E\psi + \psi d_E) \ = \ 1 \ : \ E_i \to E_i \ ,$$

so that

$$1 + d_E\psi + \psi d_E \ : \ \Sigma^{-1}(E_i)_\alpha[z] \to \Sigma^{-1}(E_i)_\alpha[z]$$

is a $\Sigma^{-1}A_\alpha[z]$-module automorphism by 4.1, and ψ defines a chain isotopy

$$\psi \ : \ \phi \ \sim \ \phi' \ : \ \Sigma^{-1}D_\alpha[z] \to \Sigma^{-1}E_\alpha[z] \ .$$

Define the isomorphisms

$$\begin{aligned} q \ &= \ \begin{pmatrix} 1 & \pm\psi\phi \\ 0 & 1 \end{pmatrix} \ : \ C \ = \ \mathcal{C}(\phi) \to C' \ = \ \mathcal{C}(\phi') \ , \\ r \ &= \ [1 + d_E\psi + \psi d_E] \ : \ \widehat{C} \ = \ \text{coker}(\phi) \to \widehat{C}' \ = \ \text{coker}(\phi') \end{aligned}$$

as in 1.5, with q simple. By 1.8 (ii) and (i)

$$\begin{aligned} \tau(r) \ &= \ \tau(p : C \simeq \widehat{C}) - \tau(p' : C' \simeq \widehat{C}') \\ &= \ \sum_{i=0}^{\infty}(-)^i\tau(1 - zh_D : \Sigma^{-1}(D_i)_\alpha[z, z^{-1}] \to \Sigma^{-1}(D_i)_\alpha[z, z^{-1}]) \\ &\qquad - \sum_{i=0}^{\infty}(-)^i\tau(1 - zh'_D : \Sigma^{-1}(D_i)_\alpha[z, z^{-1}] \to \Sigma^{-1}(D_i)_\alpha[z, z^{-1}]) \\ &= \ 0 \in K_1(\Sigma^{-1}A_\alpha[z, z^{-1}]) \ . \end{aligned}$$

(iii) The A-module chain map

$$\phi(k) \ = \ g - zh \ : \ D(k) \to E(k)$$

is of the type considered in 1.8, with the components of

$$\phi(k) \ = \ \begin{pmatrix} \phi_D(k) \\ \phi_F(k) \end{pmatrix} \ : \ D(k)_i \to E(k)_i \ = \ D(k)_i \oplus F(k)_i$$

given by

$$\phi_D(k) = \begin{pmatrix} 1 & -zh_D & 0 & \dots \\ 0 & 1 & -zh_D & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} :$$

$$D(k)_i = \sum_{j=-\infty}^{k} z^j D \to D(k)_i = \sum_{j=-\infty}^{k} z^j D_i \ ,$$

$$\phi_F(k) = \begin{pmatrix} 0 & -zh_F & 0 & \dots \\ 0 & 0 & -zh_F & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} :$$

$$D(k)_i = \sum_{j=-\infty}^{k} z^j D \to E(k)_i = \sum_{j=-\infty}^{k} z^j E_i \ .$$

As in 1.8 $\phi_D(k)$ is an automorphism, with inverse

$$\phi_D(k)^{-1} = \begin{pmatrix} 1 & zh_D & z^2(h_D)^2 & \dots \\ 0 & 1 & zh_D & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} :$$

$$D(k)_i = \sum_{j=-\infty}^{k} z^j D_i \to D(k)_i = \sum_{j=-\infty}^{k} z^j D_i \ ,$$

and the chain complex $\overline{F}(k)$ defined by

$$d_{\overline{F}(k)} = d_F - \phi_F(k)\phi_D(k)^{-1}c = \begin{pmatrix} d_F & zh_F c & z^2 h_F h_D c & \dots \\ 0 & d_F & zh_F c & \dots \\ 0 & 0 & d_F & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} :$$

$$\overline{F}(k)_i = \sum_{j=-\infty}^{k} z^j F_i \to \overline{F}(k)_i = \sum_{j=-\infty}^{k} z^j F_i$$

is such that the inclusions $\overline{F}(k)_i \to E(k)_i$ induce isomorphisms

$$\overline{F}(k) \cong F(k) \ , \ \varprojlim_k \overline{F}(k) \cong \varprojlim_k F(k) \ .$$

The identification $\varprojlim_k D(k) = D_\alpha((z))$ is immediate from the identifications

$$D(k) = \sum^{k} z^j D \to D(k-1) = \sum^{k-1} z^j D \ ; \ \sum^{k} z^j x_j \mapsto \sum^{k-1} z^j x_j \ .$$

Similarly for

$$\varprojlim_k E(k) \;=\; E_\alpha((z))\ ,\ \varprojlim_k \overline{F}(k) \;=\; \overline{F}\ .$$

The short exact sequence of inverse systems

$$0 \to D(k) \xrightarrow{\phi} E(k) \to F(k) \to 0$$

determines an exact sequence of the inverse and derived limits

$$0 \to \varprojlim_k D(k) \xrightarrow{g-zh} \varprojlim_k E(k) \to \varprojlim_k F(k) \to \varprojlim_k{}^1 D(k) \to \dots\ .$$

Since the structure maps $D(k) \to D(k-1)$ are onto the derived limit is

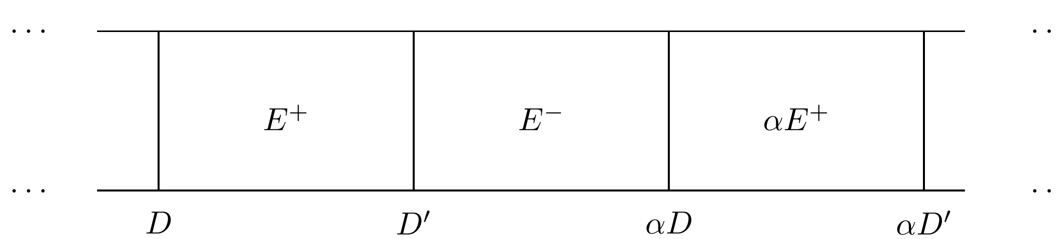
$$\varprojlim_k{}^1 D(k) \;=\; 0$$

and the inverse limits actually fit into a short exact sequence

$$0 \to \varprojlim_k D(k) \xrightarrow{\overline{\phi}} \varprojlim_k E(k) \to \varprojlim_k F(k) \to 0$$

as required, with $\overline{\phi} = g - zh$. Alternatively, identify $\mathrm{coker}(\overline{\phi}) = \overline{F}$ by a direct application of 1.8 (i). □

The following result on algebraic handle exchanges will be required in §6.



Proposition 5.4 *Let D, D', E^+, E^- be f.g. free A-module chain complexes, and let*

$$g^+ \ :\ D \to E^+\ ,\ g^-\ :\ D' \to E^-\ ,\ h^+\ :\ D' \to E^+\ ,\ h^-\ :\ \alpha D \to E^-$$

be A-module chain maps such that

$$g^+ \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ :\ D_i \to E_i^+ \;=\; D_i \oplus F_i^+\ ,$$

$$g^- \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ :\ D'_i \to E_i^- \;=\; D'_i \oplus F_i^-\ ,$$

$$h^+ \;=\; \begin{pmatrix} h_D^+ \\ h_F^+ \end{pmatrix} \ :\ D'_i \to E_i^+ \;=\; D_i \oplus F_i^+\ ,$$

$$h^- \;=\; \begin{pmatrix} h_D^- \\ h_F^- \end{pmatrix} \ :\ \alpha D_i \to E_i^- \;=\; D'_i \oplus F_i^-\ ,$$

$$d_{E^+} \;=\; \begin{pmatrix} d_D & c^+ \\ 0 & d_{F^+} \end{pmatrix} \ :\ E_i^+ \;=\; D_i \oplus F_i^+ \to E_{i-1} \;=\; D_{i-1} \oplus F_{i-1}^+\ ,$$

$$d_{E^-} \;=\; \begin{pmatrix} d_{D'} & c^- \\ 0 & d_{F^-} \end{pmatrix} \ :\ E_i^- \;=\; D'_i \oplus F_i^- \to E_{i-1}^- \;=\; D'_{i-1} \oplus F_{i-1}^-\ .$$

Given bases for D, D', F^+, F^- there are now defined two collections of data as in 5.3

$$(g : D \to E, h : \alpha D \to E, F)\ ,\ (g' : D' \to E', h' : \alpha D' \to E', F')$$

with

$$\begin{aligned}
d_E \ &= \ \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \ : \ E_i \ = \ D_i \oplus F_i \to E_{i-1} \ = \ D_{i-1} \oplus F_{i-1} \ , \\
d_F \ &= \ \begin{pmatrix} d_{F^+} & h_F^+ c^- \\ 0 & d_{F^-} \end{pmatrix} \ : \ F_i \ = \ F_i^+ \oplus F_i^- \to F_{i-1} \ = \ F_{i-1}^+ \oplus F_{i-1}^- \ , \\
c \ &= \ (\, c^+ \quad h_F^+ c^- \,) \ : \ F_i \ = \ F_i^+ \oplus F_i^- \to D_{i-1} \ , \\
g \ &= \ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ : \ D_i \to E_i \ = \ D_i \oplus F_i \ , \\
h_D \ &= \ h_D^+ h_D^- \ : \ \alpha D_i \to D_i \ , \\
h_F \ &= \ \begin{pmatrix} h_F^+ h_D^- \\ h_F^- \end{pmatrix} \ : \ \alpha D_i \to F_i \ = \ F_i^+ \oplus F_i^-
\end{aligned}$$

and

$$\begin{aligned}
d_{E'} \ &= \ \begin{pmatrix} d_{D'} & c' \\ 0 & d_{F'} \end{pmatrix} \ : \ E'_i \ = \ D'_i \oplus F'_i \to E'_{i-1} \ = \ D'_{i-1} \oplus F'_{i-1} \ , \\
d_{F'} \ &= \ \begin{pmatrix} d_{F^+} & 0 \\ h_F^- c^+ & d_{F^-} \end{pmatrix} \ : \ F'_i \ = \ F_i^+ \oplus F_i^- \to F'_{i-1} \ = \ F_{i-1}^+ \oplus F_{i-1}^- \ , \\
c' \ &= \ (\, h_D^- c^+ \quad c^- \,) \ : \ F'_i \ = \ F_i^+ \oplus F_i^- \to D'_{i-1} \ , \\
g' \ &= \ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ : \ D'_i \to E'_i \ = \ D'_i \oplus F'_i \ , \\
h'_D \ &= \ h_D^- h_D^+ \ : \ \alpha D'_i \to D'_i \ , \\
h'_F \ &= \ \begin{pmatrix} h_F^+ \\ h_F^- h_D^+ \end{pmatrix} \ : \ \alpha D'_i \to F'_i \ = \ F_i^+ \oplus F_i^- \ .
\end{aligned}$$

The cokernels of the corresponding embeddings of the based f.g. free $\Sigma^{-1}A_\alpha[z, z^{-1}]$-module chain complexes

$$\begin{aligned}
\phi \ &= \ g - zh \ : \ \Sigma^{-1} D_\alpha[z, z^{-1}] \to \Sigma^{-1} E_\alpha[z, z^{-1}] \ , \\
\phi' \ &= \ g' - zh' \ : \ \Sigma^{-1} D'_\alpha[z, z^{-1}] \to \Sigma^{-1} E'_\alpha[z, z^{-1}]
\end{aligned}$$

are related by an isomorphism of the based f.g. free $\Sigma^{-1}A_\alpha[z, z^{-1}]$-module chain complexes

$$I \ : \ \mathrm{coker}(\phi) \ \cong \ \mathrm{coker}(\phi')$$

which sends basis elements to z^δ(basis elements), with $\delta = 0$ or 1.

Proof. Use 5.3 to identify

$$\mathrm{coker}(\phi) \ = \ \widehat{E} \ , \ \mathrm{coker}(\phi') \ = \ \widehat{E'}$$

with $\widehat{F}, \widehat{F}'$ the based f.g. free $\Sigma^{-1}A_\alpha[z,z^{-1}]$-module chain complexes defined by

$$\begin{aligned} d_{\widehat{F}} &= d_F + zh_F(1-zh_D)^{-1}c \ : \\ &\qquad \widehat{F}_i = \Sigma^{-1}(F_i)_\alpha[z,z^{-1}] \to \widehat{F}_{i-1} = \Sigma^{-1}(F_{i-1})_\alpha[z,z^{-1}] \ , \\ d_{\widehat{F}'} &= d_{F'} + zh'_F(1-zh'_D)^{-1}c' \ : \\ &\qquad \widehat{F}'_i = \Sigma^{-1}(F'_i)_\alpha[z,z^{-1}] \to \widehat{F}'_{i-1} = \Sigma^{-1}(F'_{i-1})_\alpha[z,z^{-1}] \ . \end{aligned}$$

Define an isomorphism of based f.g. free $\Sigma^{-1}A_\alpha[z,z^{-1}]$-module chain complexes

$$I \ : \ \widehat{F} \ \cong \ \widehat{F}'$$

by

$$\begin{aligned} I \ : \ \widehat{F}_i = \Sigma^{-1}(F_i^+ \oplus F_i^-)_\alpha[z,z^{-1}] &\to \widehat{F}'_i = \Sigma^{-1}(F_i^+ \oplus F_i^-)_\alpha[z,z^{-1}] \ ; \\ (x^+,x^-) &\to (x^+, zx^-) \ . \end{aligned}$$

□

6. The Novikov complexes $C^{Nov}(M,f,v)$, $C^{Paj}(M,f,v)$, $C^{FR}(M,f,v,h)$.

This section starts with a review of the geometric constructions of the Novikov complex $C^{Nov}(M,f,v)$ and the Pajitnov complex $C^{Paj}(M,f,v)$ of a circle-valued Morse function $f : M \to S^1$ with respect to $v \in \mathcal{GT}(f)$. Then the proper real-valued Morse function

$$\overline{f} \ : \ \overline{M} \ = \ f^*\mathbb{R} \to \mathbb{R}$$

is used to identify the Novikov complex with the inverse limit of the proper Morse-Smale complexes

$$C^{Nov}(M,f,v) \ = \ \varprojlim_k C^{MS}(M(k),f(k),v(k))$$

of the proper real-valued Morse functions

$$f(k) \ = \ \overline{f}| \ : \ M(k) \ = \ \overline{f}^{-1}[-k,\infty) \to [-k,\infty) \ (k \geq 0)$$

with $v(k) = \overline{v}|$. This is followed by a review of the algebraic construction of the chain complex $C^{FR}(M,f,v,h)$ of Farber and Ranicki [FR]. Finally, all this is put together to prove the Cokernel, Invariance and Isomorphism Theorems already stated in the Introduction. In particular, for $v \in \mathcal{GT}(f)$ with a gradient-like chain approximation h^{gra} there exist basis-preserving isomorphisms

$$\begin{aligned} C^{Nov}(M,f,v) &\cong C^{FR}(M,f,v,h^{gra};\mathbb{Z}[\widehat{\pi_1(M)}]) \ , \\ C^{Paj}(M,f,v) &\cong C^{FR}(M,f,v,h^{gra}) \ . \end{aligned}$$

The infinite cyclic cover of M determined by $f : M \to S^1$ is

$$\overline{M} \ = \ f^*\mathbb{R} \ = \ \{(x,y) \in M \times \mathbb{R} \,|\, f(x) = [y] \in S^1\} \ ,$$

with

$$\overline{f} \ : \ \overline{M} \to \mathbb{R} \ ; \ (x,y) \mapsto y$$

The generating covering translation

$$z \;:\; \overline{M} \to \overline{M}\;;\; (x,y) \mapsto (x, y-1)\;.$$

is parallel to the downward v-gradient flow (and so acts from right to left, or rather from top to bottom). Assume that M and $\overline{M} = f^*\mathbb{R}$ are connected, so that

$$\pi_1(M) \;=\; \pi_1(\overline{M}) \times_\alpha \mathbb{Z}$$

with

$$\alpha \;=\; z_* \;:\; \pi_1(\overline{M}) \to \pi_1(\overline{M})$$

the monodromy automorphism. The group ring of $\pi_1(M)$ is the α-twisted Laurent polynomial extension of $\mathbb{Z}[\pi_1(\overline{M})]$

$$\mathbb{Z}[\pi_1(M)] \;=\; \mathbb{Z}[\pi_1(\overline{M})]_\alpha[z, z^{-1}]\;.$$

Write the Novikov ring of $\mathbb{Z}[\pi_1(M)]$ as

$$\mathbb{Z}[\widehat{\pi_1(M)}] \;=\; \mathbb{Z}[\pi_1(\overline{M})]_\alpha((z)) \;=\; \mathbb{Z}[\pi_1(\overline{M})]_\alpha[[z]][z^{-1}]\;.$$

Definition 6.1 ([N], [P1]) The *Novikov complex* $C^{Nov}(M,f,v)$ of a Morse function $f : M \to S^1$ with respect to $v \in \mathcal{GT}(f)$ is the based f. g. free $\mathbb{Z}[\widehat{\pi_1(M)}]$-module chain complex with

(i) $\mathrm{rank}_{\mathbb{Z}[\widehat{\pi_1(M)}]} C_i^{Nov}(M,f,v) = c_i(f)$, with one basis element $\widetilde{p}$ for each critical point $p \in M$ of index i, corresponding to a choice of lift $\widetilde{p} \in \widetilde{M}$

(ii) the boundary $\mathbb{Z}[\widehat{\pi_1(M)}]$-module morphisms are given by

$$d \;:\; C_i^{Nov}(M,f,v) \;=\; \mathbb{Z}[\widehat{\pi_1(M)}]^{c_i(f)} \to C_{i-1}^{Nov}(M,f,v) \;=\; \mathbb{Z}[\widehat{\pi_1(M)}]^{c_{i-1}(f)}\;;$$
$$\widetilde{p} \mapsto \sum_{\widetilde{q}} \sum_{u \in \pi_1(M)} n(\widetilde{p}, u\widetilde{q})u\widetilde{q}$$

with $n(\widetilde{p}, u\widetilde{q}) \in \mathbb{Z}$ the algebraic number of $\widetilde{v}$-gradient flow lines in $\widetilde{M}$ from $\widetilde{p}$ to $u\widetilde{q}$. $\square$

Let $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ be the localization of $\mathbb{Z}[\pi_1(M)]$ defined in 5.2.

Definition 6.2 (Pajitnov [P2,P3,P4])
For a Morse function $f : M \to S^1$ and $v \in \mathcal{GCCT}(f)$ the Novikov complex is of the form

$$C^{Nov}(M,f,v) \;=\; \mathbb{Z}[\widehat{\pi_1(M)}] \otimes_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]} C^{Paj}(M,f,v)$$

with the *Pajitnov complex* $C^{Paj}(M,f,v)$ a based f. g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain such that

$$\mathrm{rank}_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]} C_i^{Paj}(M,f,v) \;=\; c_i(f)\;. \quad \square$$

From now on it will be assumed that $0 \in S^1$ is a regular value of f, with inverse image a codimension 1 framed submanifold

$$N^{m-1} \;=\; f^{-1}(0) \subset M^m$$

Thus $\overline{f} : \overline{M} \to \mathbb{R}$ is transverse regular at $\mathbb{Z} \subset \mathbb{R}$, and cutting M along N gives a cobordism

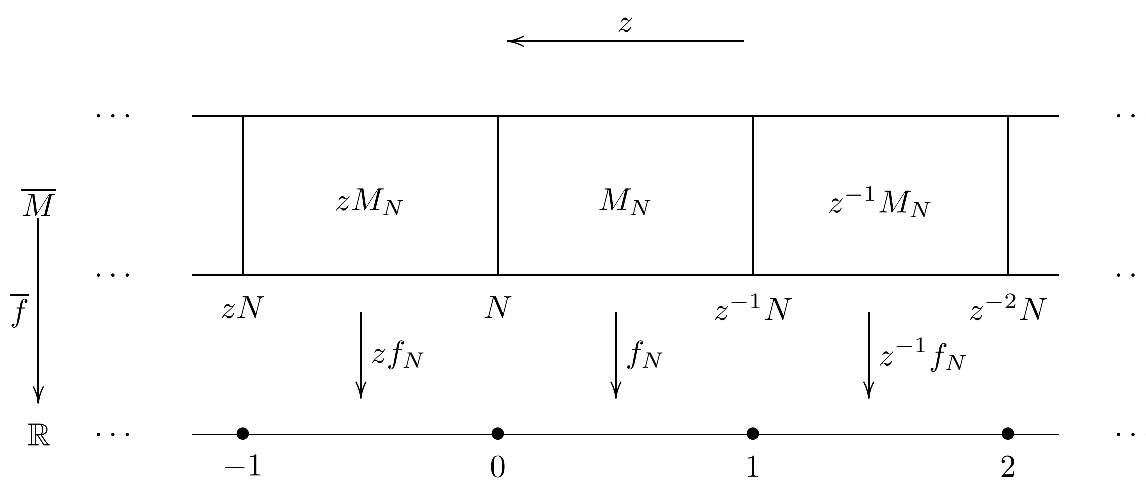
$$(M_N; N, z^{-1}N) \ = \ \overline{f}^{-1}(I; \{0\}, \{1\})$$

which is a fundamental domain for $\overline{M}$

$$\overline{M} \ = \ \bigcup_{j=-\infty}^{\infty} z^j M_N$$

with a Morse function

$$f_N \ = \ \overline{f}| \ : \ (M_N; N, z^{-1}N) \to (I; \{0\}, \{1\}) \ .$$



Proposition 6.3 *The Novikov complex of a Morse function $f : M \to S^1$ with respect to any $v \in \mathcal{GT}(f)$ is the inverse limit*

$$C^{Nov}(M, f, v) \ = \ \varprojlim_k C^{MS}(M(k), f(k), v(k))$$

of the Morse-Smale complexes of the proper real-valued Morse functions

$$f(k) \ = \ \overline{f}| \ : \ M(k) \ = \ \overline{f}^{-1}[-k, \infty) \ = \ \bigcup_{j=-\infty}^{k} z^j M_N \to [-k, \infty)$$

on the non-compact manifolds with boundary

$$(M(k), \partial M(k)) \ = \ \overline{f}^{-1}([-k, \infty), \{-k\}) \ = \ (\bigcup_{j=-\infty}^{k} z^j M_N, z^k N)$$

with respect to the projections

$$C^{MS}(M(k+1), f(k+1), v(k+1)) \ = \ C(\bigcup_{j=-\infty}^{k+1} z^j M_N, z^{k+1}N)$$

$$\to C(\bigcup^{k+1} z^j M_N, z^{k+1}M_N) \ = \ C(\bigcup^{k} z^j M_N, z^k N) \ = \ C^{MS}(M(k), f(k), v(k)) \ .$$

Proof. Identify

$$C^{Nov}(M,f,v) \;=\; \varprojlim_k C^{MS}(M(k),f(k),v(k))$$

using the expression of the Novikov ring as the inverse limit

$$\mathbb{Z}\widehat{[\pi_1(\overline{M})]} \;=\; \varprojlim_k \mathbb{Z}[\pi_1(\overline{M})]_\alpha(k)$$

of the $\mathbb{Z}[\pi_1(\overline{M})]$-modules

$$\mathbb{Z}[\pi_1(\overline{M})]_\alpha(k) \;=\; \sum_{j=-\infty}^{k} z^j\mathbb{Z}[\pi_1(\overline{M})] \subset \mathbb{Z}[\pi_1(M)] \;=\; \mathbb{Z}[\pi_1(\overline{M})]_\alpha[z,z^{-1}]$$

with respect to the natural projections

$$\mathbb{Z}[\pi_1(\overline{M})]_\alpha(k+1) \to \mathbb{Z}[\pi_1(\overline{M})]_\alpha(k) \;;\; \sum_{j=-\infty}^{k} a_j z^j \mapsto \sum_{j=-\infty}^{k-1} a_j z^j \;. \quad \square$$

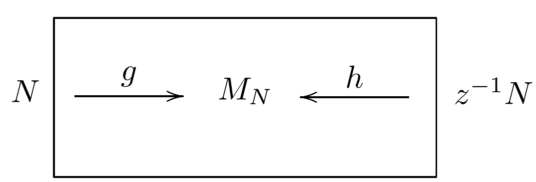
As before, let $f : M \to S^1$ have $c_i(f)$ critical points of index i. The real-valued Morse function

$$f_N \;=\; \overline{f}| \;:\; (M_N; N, z^{-1}N) \to (I;\{0\},\{1\})$$

has $c_i(f_N) = c_i(f)$ critical points of index i, and as in §2 there is a handle decomposition

$$M_N \;=\; N \times I \cup \bigcup_{i=0}^{m} \bigcup_{c_i(f)} h^i \;.$$

Let N have a CW structure with $c_i(N)$ i-cells $e^i \subset N$ and let M_N have the corresponding CW structure with $c_i(N)$ i-cells of type $e^i \times I \subset M_N$ and $c_i(f)$ i-cells of type $h^i \subset M_N$. Let $\widetilde{M}$ be the universal cover of M, and let $\widetilde{M}_N, \widetilde{N}$ be the corresponding covers of M_N, N. The inclusion $g : N \to M_N$ induces an inclusion of the cellular $\mathbb{Z}[\pi_1(\overline{M})]$-module chain complexes $g : C(\widetilde{N}) \to C(\widetilde{M}_N)$. The inclusion $h : z^{-1}N \to M_N$ induces a $\mathbb{Z}[\pi_1(\overline{M})]$-module chain map $h : \alpha C(\widetilde{N}) \to C(\widetilde{M}_N)$.



Now $M = M_N/(N = z^{-1}N)$ has a CW structure with

$$c_i(M) \;=\; c_i(f) + c_i(N) + c_{i-1}(N)$$

i-cells

The following terminology will be used in dealing with the cellular chain complexes associated to the fundamental domain $(M_N; N, z^{-1}N)$ of $\overline{M}$.

Terminology 6.4 As in 2.3 write

$$D = C(\widetilde{N}) \ , \ E = C(\widetilde{M_N}) \ , \ F = C^{MS}(M_N, f_N, v_N) = C(\widetilde{M_N}, \widetilde{N} \times I) \ ,$$
$$d_E = \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} : E_i = D_i \oplus F_i \to E_{i-1} = D_{i-1} \oplus F_{i-1} \ .$$

The inclusions

$$g : N \to M \ , \ h : z^{-1}N \to M$$

induce chain maps

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D_i \to E_i = D_i \oplus F_i \ ,$$
$$h = \begin{pmatrix} h_D \\ h_F \end{pmatrix} : \alpha D_i \to E_i = D_i \oplus F_i \ . \quad \square$$

Definition 6.5 ([FR]) Given a Morse function $f : M \to S^1$ with regular value $0 \in S^1$, $v \in \mathcal{GT}(f)$, a choice of CW structure for $N = f^{-1}(0) \subset M$, and a choice of chain approximation $h : \alpha C(\widetilde{N}) \to C(\widetilde{M_N})$

$$C^{FR}(M, f, v, h) = \widehat{F}$$

be the based f. g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain complex given by

$$\widehat{F}_i = \Sigma^{-1}C_i(\widetilde{M_N}, \widetilde{N})_\alpha[z, z^{-1}] = \Sigma^{-1}\mathbb{Z}[\pi_1(M)]^{c_i(f)} \ ,$$
$$d_{\widehat{F}} = d_{C(\widetilde{M}_N, \widetilde{N})} + zh_F(1 - zh_D)^{-1}c : \widehat{F}_i \to \widehat{F}_{i-1} \ . \quad \square$$

Cokernel Theorem 6.6. (i) *The inclusions $\widehat{F}_i \to \Sigma^{-1}C_i(\widetilde{M_N})_\alpha[z, z^{-1}]$ induce a basis-preserving (and a fortiori simple) isomorphism of based f. g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain complexes*

$$C^{FR}(M, f, v, h) \cong \operatorname{coker}\big(g - zh : \Sigma^{-1}C(\widetilde{N})_\alpha[z, z^{-1}] \to \Sigma^{-1}C(\widetilde{M_N})_\alpha[z, z^{-1}]\big) \ .$$

(ii) *The natural projection*

$$p : C(\widetilde{M}; \Sigma^{-1}\mathbb{Z}[\pi_1(M)]) = \mathcal{C}\big(g - zh : \Sigma^{-1}C(\widetilde{N})_\alpha[z, z^{-1}] \to \Sigma^{-1}C(\widetilde{M_N})_\alpha[z, z^{-1}]\big)$$
$$\to C^{FR}(M, f, v, h)$$

is a chain equivalence of based f. g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain complexes with torsion

$$\tau(p) = -\sum_{i=0}^{\infty}(-)^i\tau\big(1 - zh_D : \Sigma^{-1}C_i(\widetilde{N})_\alpha[z, z^{-1}] \to \Sigma^{-1}C_i(\widetilde{N})_\alpha[z, z^{-1}]\big)$$
$$\in K_1(\Sigma^{-1}\mathbb{Z}[\pi_1(M)])$$

Proof. This is a direct application of 5.3 (i). $\square$

Invariance Theorem 6.7 *Let $f : M \to S^1$ be a Morse function, and let $v \in \mathcal{GT}(f)$. For any regular values $0, 0' \in S^1$, CW structures on $N = f^{-1}(0)$, $N' = f^{-1}(0')$ and chain approximations*

$$h \ : \ \alpha C(\widetilde{N}) \to C(\widetilde{M}_N) \ , \ h' \ : \ \alpha C(\widetilde{N}') \to C(\widetilde{M}_{N'})$$

there is defined a simple isomorphism of based f. g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain complexes

$$C^{FR}(M, f, v, h) \ \cong \ C^{FR}(M, f, v, h') \ .$$

Proof. The case $0 = 0'$, $N = N'$ is given by 5.3 (ii). So assume $0 \neq 0' \in S^1$, and let $I^+, I^- \subset S^1$ be the two arcs joining 0 and $0'$. The restrictions of (f, v)

$$\begin{aligned}(f^+, v^+) \ &= \ (f, v)| \ : \ (M^+; N, N') \ = \ f^{-1}(I^+; \{0\}, \{0'\}) \to (I^+; \{0\}, \{0'\}) \ , \\ (f^-, v^-) \ &= \ (f, v)| \ : \ (M^-; N', N) \ = \ f^{-1}(I^-; \{0'\}, \{0\}) \to (I^-; \{0'\}, \{0\})\end{aligned}$$

are Morse functions with gradient-like vector fields such that

$$\begin{aligned}&(f_N, v_N) \ = \ (f^+, v^+) \cup (f^-, v^-) \ : \\ &(M_N; N, z^{-1}N) \ = \ (M^+; N, N') \cup (M^-; N', N) \to ([0,1]; \{0\}, \{1\}) \ , \\ &(f_{N'}, v_{N'}) \ = \ (f^-, v^-) \cup (f^+, v^+) \ : \\ &(M_{N'}; N', z^{-1}N') \ = \ (M^-; N', N) \cup (M^+; N, N') \to ([0,1]; \{0\}, \{1\}) \ .\end{aligned}$$

Use the handlebody structure on $(M^+; N, N')$ (resp. $(M^-; N', N)$) determined by (f^+, v^+) (resp. (f^-, v^-)) to extend the CW structure on N (resp. N') to a CW structure on M^+ (resp. M^-). The inclusions of CW subcomplexes $g^+ : N \to M^+$, $g^- : N' \to M^-$ induce inclusions of subcomplexes

$$g^+ \ : \ D \ = \ C(\widetilde{N}) \to E^+ \ = \ C(\widetilde{M}^+) \ , \ g^- \ : \ D' \ = \ C(\widetilde{N}') \to E^+ \ = \ C(\widetilde{M}^-) \ .$$

Cellular approximations to the inclusions $h^+ : N' \to M^+$, $h^- : z^{-1}N \to M^-$ induce chain maps

$$h^+ \ : \ D' \ = \ C(\widetilde{N}') \to E^+ \ = \ C(\widetilde{M}^+) \ , \ h^- \ : \ \alpha D \ = \ \alpha C(\widetilde{N}) \to E^- \ = \ C(\widetilde{M}^-) \ .$$

A direct application of 5.4 gives a simple isomorphism of based f.g. free $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain complexes

$$C^{FR}(M, f, v, h) \ = \ \mathrm{coker}(g - zh) \ \cong \ C^{FR}(M, f, v, h') \ = \ \mathrm{coker}(g' - zh') \ . \qquad \square$$

Definition 6.8 A crossing chain approximation $h : \alpha C(\widetilde{N}) \to C(\widetilde{M}_N)$ is *gradient-like* if for any critical points $\widetilde{p}, \widetilde{q} \in \widetilde{M}_N$ of $\widetilde{f_N} : \widetilde{M}_N \to \mathbb{R}$ with index $i, i-1$ the algebraic number of $\widetilde{v}$-gradient flow lines in $\widetilde{M}$ joining $\widetilde{p}$ to $z^j\widetilde{q}$ is

$$n(\widetilde{p}, z^j\widetilde{q}) \ = \ \begin{cases} (\widetilde{p}, \widetilde{q})\text{-coefficient of } d_F : F_i \to F_{i-1} & \text{if } j = 0 \\ (\widetilde{p}, \widetilde{q})\text{-coefficient of } h_F(h_D)^{j-1}c : F_i \to F_{i-1} & \text{if } j > 0 \\ 0 & \text{if } j < 0 \end{cases}$$

A gradient-like chain approximation h will be denoted by h^{gra}. □

Remark 6.9 (i) Pajitnov [P4] proved that for any Morse function $f : M \to S^1$ and $v \in \mathcal{G}\mathfrak{C}\mathcal{T}(f)$ there exists a handlebody structure on N with a gradient-like chain approximation $h^{gra} : \alpha C(\widetilde{N}) \to C(\widetilde{M_N})$ (cf. Remark 2.4).
(ii) Cornea and Ranicki [CR] prove that for any Morse function $f : M \to S^1$ and $v \in \mathcal{GT}(f)$ there exists a CW structure on $N = f^{-1}(0)$ with a gradient-like chain approximation h^{gra} by showing that there exist Morse functions $f' : M \to S^1$, $g : N \to \mathbb{R}$ with $v' \in \mathcal{GT}(f')$, $w \in \mathcal{GT}(g)$ such that

(a) (f', v') agrees with (f, v) outside a tubular neighbourhood of N

$$f^{-1}[-\epsilon, \epsilon] \;=\; N \times [-\epsilon, \epsilon] \subset M$$

for some small $\epsilon > 0$.

(b) (f', v') restricts to translates of (g, w)

$$\begin{aligned} (f', v')| &= (g_+, w_+) \;:\; N \times \{\epsilon/2\} \to S^1 \backslash \{0\} \;, \\ (f', v')| &= (g_-, w_-) \;:\; N \times \{-\epsilon/2\} \to S^1 \backslash \{0\} \end{aligned}$$

with

$$\begin{aligned} \mathrm{Crit}_i(g_+) &= \mathrm{Crit}_i(g) \times \{\epsilon/2\} \;, \\ \mathrm{Crit}_i(g_-) &= \mathrm{Crit}_i(g) \times \{-\epsilon/2\} \;, \\ \mathrm{Crit}_i(f') &= \mathrm{Crit}_{i-1}(g_+) \cup \mathrm{Crit}_i(g_-) \cup \mathrm{Crit}_i(f) \;. \end{aligned}$$

(c) The $\overline{v}$-gradient flow lines are in one-one correspondence with the broken $\overline{v}'$-gradient flow lines i.e. joined up sequences of $\overline{v}'$-gradient flow lines which start and terminate at critical points of $\overline{f}$.

(d) The Morse-Smale complex of $(\overline{f}', \overline{v}')$ is of the form

$$C^{MS}(\overline{M}, \overline{f}', \overline{v}') \;=\; ((D_{i-1} \oplus D_i \oplus F_i)_\alpha[z, z^{-1}], \begin{pmatrix} -d_D & 0 & 0 \\ 1 - zh_D^{gra} & d_D & c \\ -zh_F^{gra} & 0 & d_F \end{pmatrix})$$

for a gradient-like chain approximation h^{gra}, with

$$D \;=\; C^{MS}(N, g, w) \;,\; F \;=\; C^{MS}(M_N, f_N, v_N) \text{ etc.}$$

(e) The cellular chain complex of the universal cover $\widetilde{M_N}$ (or rather the cover of M_N induced from the universal cover $\widetilde{M}$ of $\overline{M}$) is

$$E \;=\; C(\widetilde{M_N}) \;=\; (D_i \oplus F_i, \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix}) \;.$$

(This is the circle-valued analogue of Remark 2.7). Thus the algebraic mapping cone of the $\mathbb{Z}[\pi_1(M)]$-module chain map

$$\phi \;=\; \begin{pmatrix} 1 - zh_D^{gra} \\ -zh_F^{gra} \end{pmatrix} \;:\; D_\alpha[z, z^{-1}] \to E_\alpha[z, z^{-1}]$$

is the Morse-Smale complex of $(\overline{f}', \overline{v}') : \overline{M} \to \mathbb{R}$

$$\mathcal{C}(\phi) \;=\; C^{MS}(\overline{M}, \overline{f}', \overline{v}')$$

and the cokernel of the induced $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-module chain map is

$$\mathrm{coker}(\Sigma^{-1}\phi : \Sigma^{-1}D_\alpha[z,z^{-1}] \to \Sigma^{-1}E_\alpha[z,z^{-1}]) \;=\; C^{FR}(M,f,v,h^{gra}) \;.$$

The kernel of the projection

$$K \;=\; \ker(p : \Sigma^{-1}\mathcal{C}(\phi) \to \mathrm{coker}(\Sigma^{-1}\phi))$$

is an algebraic model for the closed orbits of the v-gradient flow. As in 1.8 (i) (c) identify

$$\begin{aligned} d_K \;&=\; \begin{pmatrix} d_D & 0 \\ (-1)^i(1 - zh_D^{gra}) & (1 - zh_D^{gra})^{-1} d_D (1 - zh_D^{gra}) \end{pmatrix} \;: \\ K_i \;&=\; \Sigma^{-1}(D_{i-1} \oplus D_i)_\alpha[z,z^{-1}] \to K_{i-1} \;=\; \Sigma^{-1}(D_{i-2} \oplus D_{i-1})_\alpha[z,z^{-1}] \;. \end{aligned}$$

(iii) In applying 2.6 and 3.3 to $f : M \to S^1$, $v \in \mathcal{GT}(f)$ with a gradient-like chain approximation h^{gra}, the Morse-Smale complexes of unions of copies of a Morse function

$$f_N \;:\; (M_N; N; z^{-1}N) \to (I; \{0\}, \{1\})$$

and $v_N \in \mathcal{GT}(f_N)$, the chain homotopy b in 2.6 and the higher chain homotopies $b[j,j']$ in 3.3 are 0. In particular, the Morse-Smale complex of the proper real-valued Morse function

$$\overline{f}^+ \;=\; \bigcup_{k=-\infty}^{0} z^k f_N \;=\; \overline{f}| \;:\; \overline{M}^+ \;=\; \bigcup_{k=-\infty}^{0} z^k M_N \;=\; \overline{f}^{-1}[0,\infty) \to [0,\infty)$$

is given as a based free $\mathbb{Z}[\pi_1(\overline{M})]$-module chain complex by

$$C^{MS}(\overline{M}^+, \overline{f}^+, \overline{v}^+) \;=\; (\sum_{k=-\infty}^{0} z^k F_i, d_F + \sum_{j=1}^{\infty} z^j h_F^{gra}(h_D^{gra})^{j-1}c) \;.$$

The coefficients of $h_F^{gra}(h_D^{gra})^{j-1}c$ count the $\widetilde{v}$-gradient flow lines which start at an index i critical point of $\widetilde{M}$, cross j translates of $\widetilde{N} \subset \widetilde{M}$, and terminate at an index $i-1$ critical point of $\widetilde{M}$. $\square$

Isomorphism Theorem 6.10. *Given a Morse function $f : M \to S^1$ with regular value $0 \in S^1$, $v \in \mathcal{GT}(f)$, a choice of CW structure for $N = f^{-1}(0) \subset M$, and a choice of chain approximation $h : \alpha C(\widetilde{N}) \to C(\widetilde{M}_N)$ there is defined a simple isomorphism of based f. g. free R-module chain complexes with $R = \mathbb{Z}[\widehat{\pi_1(M)}]$ (resp. $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$)*

$$I_h \;:\; C^{Nov}(M,f,v) \;\cong\; C^{FR}(M,f,v,h;\mathbb{Z}[\widehat{\pi_1(M)}])$$

$$(\text{resp. } I_h \;:\; C^{Paj}(M,f,v) \;\cong\; C^{FR}(M,f,v,h))$$

For a gradient-like chain approximation $h^{gra} : \alpha C(\widetilde{N}) \to C(\widetilde{M}_N)$ the simple isomorphisms $I_{h^{gra}}$ are basis-preserving.

Proof. By 6.3 the Novikov complex is the inverse limit

$$C^{Nov}(M,f,v) \ = \ \varprojlim_k F(k)$$

of the inverse system

$$F(k) \ = \ C^{MS}(M(k),f(k),v(k))$$

of the Morse-Smale $\mathbb{Z}[\pi_1(\overline{M})]$-module chain complexes of the proper real-valued Morse functions

$$f(k) \ = \ \overline{f}| \ : \ (M(k),\partial M(k)) \ = \ (\bigcup_{j=-\infty}^{k} z^j M_N, z^k N) \to ([-k,\infty),\{-k\}) \ .$$

By 3.3 there is defined an isomorphism of inverse systems

$$I_h(k) \ : \ F(k) \to \mathrm{coker}\big(g - zh : D(k) \to E(k)\big)$$

with

$$D(k) \ = \ \sum_{j=-\infty}^{k} z^j C(\widetilde{N}) \ , \ E(k) \ = \ \sum_{j=-\infty}^{k} z^j C(\widetilde{M}_N) \ .$$

The inverse limits are given by 5.3 (iii) and 6.3

$$\varprojlim_k D(k) \ = \ C(\widetilde{N})_\alpha((z)) \ , \ \varprojlim_k E(k) \ = \ C(\widetilde{M}_N)_\alpha((z))$$

with

$$\begin{aligned}\varprojlim_k \mathrm{coker}\big(g - zh : D(k) \to E(k)\big) \ &= \ \mathrm{coker}\big(g - zh : \varprojlim_k D(k) \to \varprojlim_k E(k)\big) \\ &= \ \mathrm{coker}\big(g - zh : C(\widetilde{N})_\alpha((z)) \to C(\widetilde{M}_N)_\alpha((z))\big) \\ &\cong \ C^{FR}(M,f,v,h;\mathbb{Z}\widehat{[\pi_1(\overline{M})]}) \ .\end{aligned}$$

Define I_h to be the induced isomorphism of inverse limits

$$\begin{aligned}I_h \ = \ \varprojlim_k I_h(k) \ : \ C^{Nov}(M,f,v) \ &= \ \varprojlim_k F(k) \\ \to \varprojlim_k \mathrm{coker}\big(g - zh : D(k) \to E(k)\big) \ &\cong \ C^{FR}(M,f,v,h;\mathbb{Z}\widehat{[\pi_1(\overline{M})]}) \ .\end{aligned}$$

For a gradient-like chain approximation h^{gra} 3.3 gives basis-preserving identifications

$$d_{F(k)} \ = \ d_{C(\widetilde{M}_N,\widetilde{N})} + \sum_{j=1}^{\infty} z^j h_F (h_D^{gra})^{j-1} c \ :$$

$$F(k)_i \ = \ \sum_{j=-\infty}^{k} z^j C_i(\widetilde{M}_N,\widetilde{N}) \to F(k)_{i-1} \ = \ \sum_{j=-\infty}^{k} z^j C_{i-1}(\widetilde{M}_N,\widetilde{N})$$

(as in 6.9 (iii)). Passing to the inverse limit as $k \to \infty$ gives

$$\begin{aligned} d_{C^{Nov}(M,f,v)} &= d_{C(\widetilde{M}_N,\widetilde{N})} + \sum_{j=1}^{\infty} z^j h_F^{gra}(h_D^{gra})^{j-1}c \ : \\ C_i^{Nov}(M,f,v) &= C_i(\widetilde{M}_N,\widetilde{N})_\alpha((z)) \to C_{i-1}^{Nov}(M,f,v) \ = \ C_{i-1}(\widetilde{M}_N,\widetilde{N})_\alpha((z)) \end{aligned}$$

so that

$$I_{h^{gra}} \ : \ C^{Nov}(M,f,v) \cong C^{FR}(M,f,v,h^{gra};\widehat{\mathbb{Z}[\pi_1(M)]})$$

is a basis-preserving isomorphism, with zero torsion. There exists a chain homotopy

$$h \ \simeq \ h^{gra} \ : \ \alpha C(\widetilde{N}) \to C(\widetilde{M}_N)$$

so that by 5.3 (ii)

$$\tau(I_h) \ = \ \tau(I_{h^{gra}}) \ = \ 0 \in K_1(\widehat{\mathbb{Z}[\pi_1(M)]}) \ .$$

Similarly for $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-coefficients. $\square$

Remark 6.11 (i) The formulae given by 6.7 and 6.10

$$\begin{aligned} &\tau\big(p : C(\widetilde{M};\Sigma^{-1}\mathbb{Z}[\pi_1(M)]) \to C^{Paj}(M,f,v)\big) \\ &\quad = \ -\sum_{i=0}^{\infty}(-)^i\tau\big(1 - zh_D^{gra} : \Sigma^{-1}C_i(\widetilde{N})_\alpha[z,z^{-1}] \to \Sigma^{-1}C_i(\widetilde{N})_\alpha[z,z^{-1}]\big) \\ &\qquad\qquad \in K_1(\Sigma^{-1}\mathbb{Z}[\pi_1(M)]) \ , \\ &\tau\big(\widehat{p} : C(\widetilde{M};\widehat{\mathbb{Z}[\pi_1(M)]}) \to C^{Nov}(M,f,v)\big) \\ &\quad = \ -\sum_{i=0}^{\infty}(-)^i\tau\big(1 - zh_D^{gra} : C_i(\widetilde{N};\widehat{\mathbb{Z}[\pi_1(M)]}) \to C_i(\widetilde{N};\widehat{\mathbb{Z}[\pi_1(M)]})\big) \\ &\qquad\qquad \in K_1(\widehat{\mathbb{Z}[\pi_1(M)]}) \end{aligned}$$

are generalizations of the formulae of Hutchings and Lee [HL] and Pajitnov [P5],[P6] counting the critical points of $f : M \to S^1$, the ζ-function of the closed orbits of the gradient flow (corresponding to h_D^{gra}) and the Reidemeister torsion of M. Schütz [Sch1],[Sch2] extended these formulae to the closed orbits of a generic gradient flow of a closed 1-form using Hochschild homology and a chain equivalence of the type $\widehat{p} : C(\widetilde{M};\widehat{\mathbb{Z}[\pi_1(M)]}) \to C^{Nov}(M,f,v)$.
(ii) See Chapters 10,14,15 of [R] for the splitting theorems for the torsion groups $K_1(\widehat{\mathbb{Z}[\pi_1(M)]})$, $K_1(\Sigma^{-1}\mathbb{Z}[\pi_1(M)])$ in the case $\alpha = 1$ (which extend to the case of arbitrary α, Pajitnov and Ranicki [PR]) and for the expressions of the torsions $\tau(1 - zh_D)$ in terms of (noncommutative) characteristic polynomials. For any ring A the classical Bass-Heller-Swan splitting

$$K_1(A[z,z^{-1}]) \ = \ K_1(A) \oplus K_0(A) \oplus \widetilde{\mathrm{Nil}}_0(A) \oplus \widetilde{\mathrm{Nil}}_0(A)$$

generalizes to splittings

$$\begin{aligned} K_1(\Sigma^{-1}A[z,z^{-1}]) &= K_1(A) \oplus K_0(A) \oplus \widetilde{\mathrm{Nil}}_0(A) \oplus V(A) \ , \\ K_1(A((z))) &= K_1(A) \oplus K_0(A) \oplus \widetilde{\mathrm{Nil}}_0(A) \oplus \widehat{V}(A) \end{aligned}$$

with $\widehat{V}(A) \subseteq K_1\big(A((z))\big)$ the image of the multiplicative group of noncommutative Witt vectors $\widehat{W}(A) = 1 + zA[[z]]$, and $V(A) \subseteq K_1(\Sigma^{-1}A[z, z^{-1}])$ the image of the subgroup of the noncommutative rational Witt vectors $W(A) \subseteq \widehat{W}(A)$ generated by $1 + zA[z]$. (In [R] it was claimed that the natural surjections $W(A)^{ab} \to V(A)$, $\widehat{W}(A)^{ab} \to \widehat{V}(A)$ are isomorphisms, but an explicit counterexample was constructed in [PR]). The torsions of the chain equivalences $p, \widehat{p}$ are such that

$$\tau(p) \in V(A) \subseteq K_1(\Sigma^{-1}A[z, \alpha])\ ,\ \tau(\widehat{p}) \in \widehat{V}(A) \subseteq K_1\big(A((z))\big)$$

respectively, with $A = \mathbb{Z}[\pi_1(\overline{M})]$.

(iii) The natural map $\Sigma^{-1}A[z, z^{-1}] \to A((z))$ is injective for a commutative ring A, since in this case $\Sigma^{-1}A[z, z^{-1}]$ is just the localization of $A[z, z^{-1}]$ inverting all the elements of type $1 + az \in A[z, z^{-1}]$ $(a \in A)$. Sheiham [Sh] has constructed an example of a noncommutative ring A such that $\Sigma^{-1}A[z, z^{-1}] \to A((z))$ is not injective.

(iv) Farber [F] extended the construction of Farber and Ranicki [FR] to obtain an algebraic Novikov complex for a closed 1-form, but did not relate it to the geometric Novikov complex. □

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