Mapping properties of multi-parameter multipliers

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. The work has not been submitted for any other degree or professional qualification.

(*Odysseas Bakas*)

Publications

This thesis includes material adapted from work of the author which has been submitted for publication. In particular, Chapters 3, 4, and 5 are based on the author's work appeared in [3], [4], and [2] respectively.

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To the memory of my father, Nikolaos Bakas

Lay Summary

This thesis focuses on problems in harmonic analysis related to mapping properties of certain classes of operators initially defined in the so-called frequency space.

Broadly speaking, one can think of an operator as a "system" whose "inputs" and "outputs" are functions. The problems studied in this thesis can be roughly stated as follows. *Given a class of "systems", find the appropriate classes of "inputs" such that the corresponding "outputs" have some desired "properties"*. Usually, in harmonic analysis or in analysis in general, these desired "properties" are encoded in terms of the magnitude of the "input" and "output" functions.

This thesis is motivated by a problem concerning the study of mapping properties of a class of operators acting on functions of several variables, known as multi-parameter Marcinkiewicz multiplier operators. It is well-known that the corresponding one-dimensional problem is connected with some classical results in harmonic analysis involving functions whose "frequency portrait" is sparse or, in other words, functions whose Fourier transforms are supported in thin sets of integers, such as geometric sequences. Motivated by this fact, we obtain higher-dimensional variants of the aforementioned results involving thin sets. Furthermore, we establish sharp endpoint bounds of operators that can essentially be regarded as prototypical Marcinkiewicz multiplier operators.

Abstract

This thesis is motivated by the problem of understanding the endpoint mapping properties of higher-dimensional Marcinkiewicz multipliers. The one-dimensional case was definitively characterised by Tao and Wright. In particular, they proved that Marcinkiewicz multipliers acting on functions over the real line map the Hardy space $H^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$ and they locally map $L \log^{1/2} L$ to $L^{1,\infty}$ and that these results are sharp.

The classical inequalities of Paley and Zygmund involving lacunary sequences can be regarded as rudimentary prototypes of the aforementioned results of Tao and Wright on the behaviour of Marcinkiewicz multipliers "near" $L^1(\mathbb{R})$. Motivated by this fact, in Chapter 3 we obtain higher-dimensional variants of these two inequalities and we establish sharp multiplier inclusion theorems on the torus and on the real line. In Chapter 4 we extend the multiplier inclusion theorem on $\mathbb{T}$ of Chapter 3 to higher dimensions.

In the last chapter of this thesis, we study endpoint mapping properties of the classical Littlewood-Paley square function which can essentially be regarded as a model Marcinkiewicz multiplier. More specifically, we give a new proof to a theorem due to Bourgain on the growth of the operator norm of the Littlewood-Paley square function as $p \to 1^+$ and then extend this result to higher dimensions. We also obtain sharp weak-type inequalities for the multi-parameter Littlewood-Paley square function and prove that the two-parameter Littlewood-Paley square function does not map the product Hardy space H^1 to $L^{1,\infty}$.

Notation

1. We denote by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0$ the set of non-negative integers. The set of real numbers is denoted by $\mathbb{R}$ and the set of complex numbers is denoted by $\mathbb{C}$.

2. The unit disc in $\mathbb{C}$ is denoted by $\mathbb{D}$; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

3. We denote the torus $\mathbb{R}/\mathbb{Z}$ by $\mathbb{T}$. We identify functions on $\mathbb{T}$ with functions on $[0, 1)$ in the usual way.

4. For $n \geqslant 2$, we use the notation $\underline{x} = (x_1, \cdots, x_n)$ for elements in n-dimensional euclidean space $\mathbb{R}^n$ or in $\mathbb{T}^n$.

5. If G is a locally compact abelian group, we denote its dual by $\widehat{G}$. Let m_G be a Haar measure on G. If $f \in L^1(G, m_G)$, then its Fourier transform $\widehat{f} : \widehat{G} \to \mathbb{C}$ is given by $\widehat{f}(\gamma) = \int_G f(x)\gamma(-x)dm_G(x)$, $\gamma \in \widehat{G}$. If G is compact, we consider the normalised Haar measure $m_G(G) = 1$ and, in this case, we simply write $\widehat{f}(\gamma) = \int_G f(x)\gamma(-x)dx$.

6. Let $(X, \mathcal{A}, \mu)$ be a measure space. If f is $\mathcal{A}$-measurable, then for $0 < p < \infty$, we write $\|f\|_{L^p(X)} = (\int_X |f(x)|^p d\mu(x))^{1/p}$. For $p = \infty$, we define $\|f\|_{L^\infty(X)} = \inf\{C \geqslant 0 : |f(x)| \leqslant C \text{ for } \mu - \text{a.e.}\}$. For $0 < p \leqslant \infty$, $L^p(X)$ denotes the space of all $\mathcal{A}$-measurable functions f such that $\|f\|_{L^p(X)} < \infty$.

7. If X is a locally compact Hausdorff space and μ is a measure on $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ denotes the Borel σ-algebra over X, then its total variation measure $|\mu|$ is given by

$$|\mu|(E) = \sup\{\sum_{n=1}^{N} |\mu(E_n)| : (E_n)_{n=1}^N,\ E_n \in \mathcal{B}(X) \text{ and } \cup_{n=1}^N E_n = E\}$$

   for $E \in \mathcal{B}(X)$. We denote by $M(X)$ the class of all complex-valued regular Borel measures μ on $(X, \mathcal{B}(X))$ such that $\|\mu\| = |\mu|(X) < \infty$.

8. If G is a locally compact abelian group and $\mu \in M(G)$, then its Fourier-Stieltjes transform $\widehat{\mu} : \widehat{G} \to \mathbb{C}$ is given by $\widehat{\mu}(\gamma) = \int_G \gamma(-x)d\mu(x)$, $\gamma \in \widehat{G}$.

9. Let G be a compact abelian group and let X be a subspace of $L^1(G)$. We say that $m \in L^\infty(\widehat{G})$ is a multiplier from X to $L^2(G)$ if for every $f \in X$ one

has $\sum_{\gamma\in\widehat{G}} |m(\gamma)\widehat{f}(\gamma)|^2 < \infty$. The class of all multipliers from X to $L^2(G)$ is denoted by $\mathcal{M}_{X\to L^2(G)}$.

10. Let X be a subspace of $L^1(\mathbb{R}^n)$. A function $m \in L^\infty(\mathbb{R}^n)$ is said to be a multiplier from X to $L^2(\mathbb{R}^n)$ if for every $f \in X$ one has $\int_{\mathbb{R}^n} |m(\underline{\xi})\widehat{f}(\underline{\xi})|^2 d\underline{\xi} < \infty$.

11. Given a non-empty set X and a function $f : X \to \mathbb{C}$, we denote by $\mathrm{supp}(f)$ the support of f, namely $\mathrm{supp}(f) = \{x \in X : f(x) \neq 0\}$.

12. We denote the characteristic function of a set A by χ_A; $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.

13. If A is a subset of B, we write $A \subset B$. If A is a proper subset of B, i.e. $A \subset B$ and $B\backslash A \neq \varnothing$, then we write $A \subsetneq B$.

14. If A is a subset of $\mathbb{R}^n$ (or $\mathbb{T}^n$), then $|A|$ denotes its Lebesgue measure. We use the same symbol to denote the length of a vector in $\mathbb{R}^n$ or the modulus of a complex number.

15. If G is a locally compact abelian group, then $C(G)$ denotes the class of all continuous functions on G. In the euclidean setting, we denote by $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ the space of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support. The class of Schwartz functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$.

16. For $n \in \mathbb{N}$, the Fejér kernel of order n is the trigonometric polynomial given by $K_n(x) = \sum_{j=-n}^{n}[1 - |j|/(n+1)]e^{i2\pi jx}$, $x \in \mathbb{T}$.

17. The expression $X \lesssim Y$ means that there exists a positive constant C such that $X \leqslant CY$. To specify the dependence of this constant on additional parameters $\alpha_1, \cdots, \alpha_n$ we write $X \lesssim_{\alpha_1,\cdots,\alpha_n} Y$. If $X \lesssim Y$ and $Y \lesssim X$, we write $X \sim Y$.

Contents

Chapter 1

Introduction

1.1 Marcinkiewicz multipliers

By the L^2-theory of Fourier series, every function $m \in \ell^\infty(\mathbb{Z})$ corresponds to a bounded, linear operator on $L^2(\mathbb{T})$ defined as follows. If $f \in L^2(\mathbb{T})$, then

$$\sum_{n\in\mathbb{Z}} |m(n)\widehat{f}(n)|^2 \leqslant \|m\|^2_{\ell^\infty(\mathbb{Z})} \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 = \|m\|^2_{\ell^\infty(\mathbb{Z})}\|f\|^2_{L^2(\mathbb{T})}$$

and so, $(m(n)\widehat{f}(n))_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Hence, there exists a function $g = T_m(f)$ such that $\|g\|_{L^2(\mathbb{T})} \leqslant \|m\|_{\ell^\infty(\mathbb{Z})}\|f\|_{L^2(\mathbb{T})}$ and $\widehat{g}(n) = m(n)\widehat{f}(n)$, $n \in \mathbb{Z}$. In such a way, given $m \in \ell^\infty(\mathbb{Z})$, we define a bounded, linear operator T_m on $L^2(\mathbb{T})$ and we say that T_m is a Fourier multiplier operator, or simply a multiplier operator whose corresponding multiplier is m. Moreover, it can easily be seen that $\|T_m\|_{L^2(\mathbb{T})\to L^2(\mathbb{T})} = \|m\|_{\ell^\infty(\mathbb{Z})}$. Similarly, for any locally compact abelian group G, one defines Fourier multiplier operators on $L^2(G)$.

In [38], Marcinkiewicz showed that if we assume that $m \in \ell^\infty(\mathbb{Z})$ moreover satisfies

$$\sup_{n\in\mathbb{N}_0}\left\{\sum_{k=2^n-1}^{2^{n+1}} |m(k+1)-m(k)| + \sum_{k=-2^{n+1}+1}^{-2^n} |m(k+1)-m(k)|\right\} < \infty, \tag{1.1}$$

then the corresponding multiplier operator T_m can be extended as a bounded operator on $L^p(\mathbb{T})$ for all $1 < p < \infty$. Multiplier operators T_m whose associated multipliers m satisfy (1.1) are referred to as periodic Marcinkiewicz multiplier operators or simply Marcinkiewicz multipliers on $\mathbb{T}$.

Analogously, on the real line, a function $m \in L^\infty(\mathbb{R})$ is said to be a Marcinkiewicz multiplier on $\mathbb{R}$ if m is of bounded variation uniformly over all dyadic intervals, namely

$$\sup_{n\in\mathbb{Z}} \int_{(-2^{n+1},-2^n]\cup[2^n,2^{n+1})} |dm(\xi)| < \infty. \tag{1.2}$$

As in the periodic case, if m is a Marcinkiewicz multiplier, then the corresponding multiplier operator T_m is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$, see, e.g., Theorem 6, Chapter IV in [59].

Marcinkiewicz multiplier operators are not L^1-bounded. In [62], Tao and Wright studied the mapping properties of the class of Marcinkiewicz multiplier operators "near" $L^1(\mathbb{R})$. In particular, Tao and Wright proved that Marcinkiewicz multipliers map the (real) Hardy space $H^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$ and they locally map $L\log^{1/2} L$ to $L^{1,\infty}$. Moreover, it is shown in [62] that these results are sharp[1]. By adapting the argument of Tao and Wright to the periodic setting, one can show that Marcinkiewicz multiplier operators acting on functions defined over the torus map the (real) Hardy space $H^1(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$ and $L\log^{1/2} L(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$. The classical inequality of Paley

$$\Big(\sum_{n\in\mathbb{N}_0} |\widehat{f}(2^n)|^2\Big)^{1/2} \lesssim \|f\|_{H^1(\mathbb{T})} \tag{1.3}$$

and the following inequality of Zygmund

$$\Big(\sum_{n\in\mathbb{N}_0} |\widehat{f}(2^n)|^2\Big)^{1/2} \lesssim 1 + \int_{\mathbb{T}} |f(x)| \log^{1/2}(1+|f(x)|)dx \tag{1.4}$$

can be regarded as rudimentary prototypes of the aforementioned endpoint results of Tao and Wright on Marcinkiewicz multiplier operators. More precisely, as we will see in Chapter 3, (1.3) and (1.4) can be obtained by using the square function characterisations of $H^1(\mathbb{T})$ and $L\log^{1/2} L(\mathbb{T})$, the latter being established by Tao and Wright in [62].

In the two-dimensional case, the class of two-parameter Marcinkiewicz multipliers on $\mathbb{R}^2$ is defined as follows.

Definition 1.1.1 (Two-parameter Marcinkiewicz multipliers)**.** *Denote by $\mathcal{I}$ the class of all intervals of the form $\pm[2^k, 2^{k+1})$, $k \in \mathbb{Z}$, where $-[2^k, 2^{k+1})$ is the interval $(-2^{k+1}, -2^k]$.*

Assume that m is a bounded function on $\mathbb{R}^2$ that is C^2 in all rectangles $I \times J$, where $I, J \in \mathcal{I}$. Moreover, if we suppose that there is an absolute constant $A > 0$ such that

- $\sup_{\eta\in\mathbb{R}} \sup_{I\in\mathcal{I}} \int_I |\partial_1 m(\xi,\eta)|d\xi \leqslant A,$
- $\sup_{\xi\in\mathbb{R}} \sup_{J\in\mathcal{I}} \int_J |\partial_2 m(\xi,\eta)|d\eta \leqslant A,$ *and*
- $\sup_{I\in\mathcal{I}} \sup_{J\in\mathcal{I}} \int_{I\times J} |\partial_1\partial_2 m(\xi,\eta)|d\xi d\eta \leqslant A,$

then m is called a two-parameter Marcinkiewicz multiplier on $\mathbb{R}^2$.

Similarly, one defines the class of n-parameter Marcinkiewicz multipliers over $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n}$, $n \geqslant 2$. It is well-known that multi-parameter Marcinkiewicz multiplier operators are bounded on L^p for all $1 < p < \infty$, see[2] e.g. Theorem 6' in Chapter IV of [59].

[1] In particular, there exist Marcinkiewicz multiplier operators on $\mathbb{R}$ that do not map $H^1(\mathbb{R})$ to $L^{1,q}(\mathbb{R})$ for any $q < \infty$ and there are Marcinkiewicz multiplier operators on $\mathbb{R}$ that do not locally map $L\log^r L$ to $L^{1,\infty}$ for $r < 1/2$. See section 3 in [62].

[2] Originally this theorem was stated and proved by Marcinkiewicz for multipliers acting on functions defined over the two-torus, see [38, Theorem 2].

1.2 Overview of the thesis

The present thesis is motivated by the problem of understanding the mapping properties of two-parameter Marcinkiewicz multipliers "near" L^1. More precisely, two-parameter Marcinkiewicz multiplier operators do not map $L^1(\mathbb{R}^2)$ to $L^{1,\infty}(\mathbb{R}^2)$ and the main question is to understand for which subspaces X of L^1, one has

$$\|T_m(f)\|_{L^{1,\infty}} \leqslant C_m \|f\|_X$$

globally or locally, where T_m is a two-parameter Marcinkiewicz multiplier operator. As it was briefly mentioned above, the endpoint mapping properties of one-dimensional Marcinkiewicz multiplier operators are connected with the classical inequalities of Paley and Zygmund. Hence, motivated by this fact, in Chapters 3 and 4 we study variants of (1.3) and (1.4) and in Chapter 5 we study mapping properties of the classical multi-parameter Littlewood-Paley square function, which can essentially be regarded as a model Marcinkiewicz multiplier operator. More specifically, this thesis is organised as follows.

- In Chapter 2 we give some background and preliminairies needed for the main chapters of the thesis. In section 2.2 we briefly present Hardy spaces and some of their characterisations that will be used in Chapters 3, 4, and 5. In section 2.3 we give some basic facts about the spaces $L\log^r L$ and $\exp L^{1/r}$ that appear in Chapters 3, 4, and 5 and in section 2.4 we present basic definitions and results regarding thin sets in harmonic analysis, such as $\Lambda(p)$ sets and Sidon sets.

- Chapter 3 is based on the author's work [3] and it focuses on variants of the classical inequalities of Paley and Zygmund. In particular, in section 3.2 the notion of Sidon weights is introduced and it is shown that tensor products of Sidon weights are multipliers from $L\log^{n/2} L(G_1 \times \cdots \times G_n)$ to $L^2(G_1 \times \cdots \times G_n)$, G_j being compact abelian groups, $j = 1, \cdots, n$. In section 3.3 we prove that the class of all multipliers from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$ is properly contained in the class of all multipliers from $L\log^{1/2} L(\mathbb{T})$ to $L^2(\mathbb{T})$ and as a consequence of this multiplier inclusion theorem we exhibit a multiplier from $L\log^{1/2} L(\mathbb{T})$ to $L^2(\mathbb{T})$ which is not a Sidon weight. In section 3.4 we prove an analogous inclusion theorem on the real line and in the last section we obtain higher-dimensional extensions of a theorem due to Bonami.

- In Chapter 4 we extend the multiplier inclusion theorem of Chapter 3 to higher dimensions. In particular, the main result of Chapter 4 is that the class of all multipliers from the n-parameter Hardy space $H^1_{\text{prod}}(\mathbb{T}^n)$ to $L^2(\mathbb{T}^n)$ is properly contained in the class of all multipliers from $L\log^{n/2} L(\mathbb{T}^n)$ to $L^2(\mathbb{T}^n)$. Chapter 4 is based on the author's work [4].

- In Chapter 5 we study mapping properties of the Littlewood-Paley square function "near" L^1. More specifically, in section 5.2 we give a new proof to a theorem of Bourgain concerning the behaviour of the operator norm of the classical Littlewood-Paley square function as $p \to 1^+$ and in section 5.3 we extend this result to higher dimensions. In sections 5.4 and 5.5 we obtain sharp weak-type inequalities for the Littlewood-Paley square function in the periodic and in the euclidean setting, respectively and in section 5.6 we prove that the Littlewood-Paley square function does not map the product Hardy space to $L^{1,\infty}$. This chapter is based on the author's work [2].

Chapter 2

Background and Preliminairies

2.1 introduction

This chapter is devoted to the background needed for Chapters 3, 4, and 5. More specifically, in section 2.2 we briefly present Hardy spaces and state their characterisations in terms of Littlewood-Paley square functions and atomic decompositions. Section 2.3 focuses on the Zygmund classes $L \log^r L$ and $\exp L^{1/r}$, $r > 0$. As these classes of functions can be realised either as Orlicz spaces or as Lorentz-Zygmund spaces, in subsections 2.3.1 and 2.3.2 we present basic definitions and facts concerning Orlicz spaces and Lorentz-Zygmund spaces, respectively. In the last section of this chapter we review some standard facts about Sidon sets.

2.2 Hardy spaces

In [30], Hardy showed that given $p > 0$ and a holomorphic function f in the unit disc $\mathbb{D} = \{z = re^{2\pi i\theta} : 0 \leqslant r < 1, \theta \in \mathbb{T}\}$, $h_p(f,r) = \int_{\mathbb{T}} |f(re^{i2\pi\theta})|^p d\theta$ is an increasing function of $r \in [0,1)$. Motivated by this result of Hardy, F. Riesz introduced in [51] the Hardy space $H^p(\mathbb{D})$ over $\mathbb{D}$ ($p > 0$) as the class of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{0\leqslant r<1} h_p(f,r) = \sup_{0\leqslant r<1} \int_{\mathbb{T}} |f(re^{i2\pi\theta})|^p d\theta < \infty.$$

For $f \in H^p(\mathbb{D})$, one sets $\|f\|_{H^p(\mathbb{D})} = [\sup_{0\leqslant r<1} h_p(f,r)]^{1/p}$, $p > 0$. For $p \geqslant 1$, the Hardy space $H^p(\mathbb{D})$ can be identified with the class of all $f \in L^p(\mathbb{T})$ such that $\widehat{f}(n) = 0$ for all $n < 0$, see e.g. Chapter III in [34]. For $p \geqslant 1$, we shall refer to the latter space as the analytic Hardy space H^p on the torus and denote it by $H^p_A(\mathbb{T})$, namely $H^p_A(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : \widehat{f}(n) = 0 \text{ for all } n < 0\}$. In the multi-parameter case, one defines $H^p(\mathbb{D}^n)$ analogously. If $p \geqslant 1$, then $H^p(\mathbb{D}^n)$ may be identified with the space of all functions $f \in L^p(\mathbb{T}^n)$ such that $\mathrm{supp}(\widehat{f}) \subset \mathbb{N}_0^n$. We denote this space by $H^p_A(\mathbb{T}^n)$. For $p \geqslant 1$, it can easily be seen that $(H^p_A(\mathbb{T}^n), \|\cdot\|_{L^p(\mathbb{T}^n)})$ is a closed subspace of $(L^p(\mathbb{T}^n), \|\cdot\|_{L^p(\mathbb{T}^n)})$ and hence it is a Banach space.

Similarly, if $\mathbb{R}^2_+ = \{x + iy \in \mathbb{C} : y > 0\}$ is the upper-half plane, then for $p > 0$ one defines the analytic Hardy spaces $H^p((\mathbb{R}^2_+)^n)$ as the space of all holomorphic functions f on $(\mathbb{R}^2_+)^n$ such that

$$\sup_{y_1,\cdots,y_n>0}\int_{\mathbb{R}^n}|f(x_1+iy_1,\cdots,x_n+iy_n)|^p dx_1\cdots dx_n<\infty.$$

The present thesis mainly focuses on the so-called real-variable Hardy spaces H^1 or simply real Hardy spaces H^1 on $\mathbb{T}^n$ and $\mathbb{R}^n$. For $t>0$ and $\underline{x}\in\mathbb{R}^d$, the Poisson kernel on $\mathbb{R}^d$ is given by

$$P_t^{(d)}(\underline{x})=c_d\frac{t}{(|\underline{x}|^2+t^2)^{(d+1)/2}},$$

where $c_d=\Gamma[(d+1)/2]/(\pi^{(d+1)/2})$. For $\underline{x}\in\mathbb{R}^d$, let $\Gamma(\underline{x})$ denote the cone $\Gamma(\underline{x})=\{(\underline{y},t)\in\mathbb{R}^d\times(0,\infty):|\underline{x}-\underline{y}|\leqslant t\}$. Then, following [24], the n-parameter Hardy space $H^1_{\text{prod}}(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n})$ over $\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n}$ is the space of all functions f in $L^1(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n})$ such that $f^*\in L^1(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n})$, where

$$f^*(\underline{x_1},\cdots,\underline{x_n})=\sup_{(\underline{y_1},t_1)\in\Gamma(\underline{x_1}),\cdots,(\underline{y_n},t_n)\in\Gamma(\underline{x_n})}|f*(P_{t_1}^{(d_1)}\otimes\cdots\otimes P_{t_n}^{(d_n)})(\underline{y_1},\cdots,\underline{y_n})|.$$

We set $\|f\|_{H^1_{\text{prod}}(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n})}=\|f^*\|_{L^1(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n})}$. If $d_1=\cdots=d_n=1$, we simply write $H^1_{\text{prod}}(\mathbb{R}^n)$.

Similarly, in the periodic setting, the n-parameter Hardy space $H^1(\mathbb{T}^n)$, $n\in\mathbb{N}$, is defined to be the space of all integrable functions f on the n-torus such that $f^*\in L^1(\mathbb{T}^n)$, where

$$f^*(x_1,\cdots,x_n)=\sup_{r_1e^{i2\pi y_1}\in\Gamma(x_1),\cdots,r_ne^{i2\pi y_n}\in\Gamma(x_n)}|f*(P_{r_1}\otimes\cdots\otimes P_{r_n})(y_1,\cdots,y_n)|,$$

for $\underline{x}=(x_1,\cdots,x_n)\in\mathbb{T}^n$. In the periodic setting, $P_r(x)$ denotes the Poisson kernel on $\mathbb{T}$ given by

$$P_r(x)=\frac{1-r^2}{1-2r\cos(2\pi x)+r^2},$$

$x\in\mathbb{T}$, $0<r<1$ and for $x\in\mathbb{T}$, $\Gamma(x)=\{z\in\mathbb{D}:|z-e^{i2\pi x}|\leqslant 2(1-|z|)\}$. One then defines $\|f\|_{H^1(\mathbb{T}^n)}=\|f^*\|_{L^1(\mathbb{T}^n)}$.

It is well-known that in the above definitions of the real Hardy spaces in euclidean spaces[1] one can replace the Poisson kernel by any approximation of the identity $\phi_\epsilon(\underline{x})=\epsilon^{-d}\phi(\epsilon^{-1}x)$, where $\phi\in C_c^\infty(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d}\phi=1$. Moreover, it can easily be seen that the analytic Hardy spaces can be regarded as subspaces of the corresponding real Hardy spaces, see, e.g., p.132 in [60] for the case of the real line and for the multi-parameter setting, see [24] and [29].

There are several characterisations of Hardy spaces. In the following two subsections we briefly present the Littlewood-Paley square function characterisation as well as the atomic decomposition of Hardy spaces. For more details on characterisations of one-parameter Hardy spaces, we refer the reader to Chapters III and IV of [60], Chapter 6 of [27], and Chapters I and III of [26]. For characterisations of multi-parameter Hardy spaces, see [18], [24], and [29].

[1] A similar remark holds in the periodic setting, see [29].

2.2.1 Square function characterisation of Hardy spaces

The Littlewood-Paley square function characterisation of Hardy spaces was obtained by Stein in [58] as a consequence of the following multiplier theorem [58, Théorème 4].

Theorem 2.2.1 (Stein's multiplier theorem for $p = 1$, [58]). *Let η be a $C_c^\infty(\mathbb{R})$ function supported away from the origin and such that $\sum_{n\in\mathbb{Z}}[\eta(2^{-n}\xi)]^2 = 1$ for every $\xi \neq 0$.*

For $(a_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, define $m : \mathbb{R} \to \mathbb{C}$ by

$$m(\xi) = \sum_{n\in\mathbb{Z}} a_n \eta(2^{-n}\xi)$$

and denote its restriction to the integers by $\widetilde{m}$.

Then, the multiplier operator T_m acting on functions on $\mathbb{R}$ is bounded on $H^1(\mathbb{R})$ and the corresponding periodic multiplier operator $T_{\widetilde{m}}$ is bounded on $H^1(\mathbb{T})$.

The above result was originally stated for η being even, supported in $\pm[1,3]$ such that $\eta|_{[3/2,2]} \equiv 1$ and affine on $[1,3/2]$ and on $[3/2,2]$ and for analytic Hardy spaces. In such a case, the multipliers are in fact H^p_{A}-bounded for all $p > 2/3$. The proof of Theorem 2.2.1 is based on characterisations of Hardy spaces in terms of area integrals. For more details see [58] and Chapter VII in [59]. An alternative proof of Theorem 2.2.1 by using the atomic decomposition of Hardy spaces can be found in [19].

For $\eta \in C_c^\infty(\mathbb{R})$ as in the statement of Theorem 2.2.1 and $n \in \mathbb{Z}$, the corresponding smooth Littlewood-Paley projection $\widetilde{\Delta}_n$ is given by

$$(\widetilde{\Delta}_n(f))\widehat{\ }(\xi) = \eta(2^{-n}\xi)\widehat{f}(\xi)$$

and the smooth Littlewood-Paley square function on $\mathbb{R}$ is defined by

$$\widetilde{S}_{\mathbb{R}}(f) = \big(\sum_{n\in\mathbb{Z}} |\widetilde{\Delta}_n(f)|^2\big)^{1/2}$$

for all f in $\mathcal{S}(\mathbb{R})$. Similarly, in the periodic setting, for $n \in \mathbb{N}_0$, the n-th "smooth" Littlewood-Paley projection $\widetilde{D}_n(f)$ of f is the trigonometric polynomial given by

$$\widetilde{D}_n(f)(x) = \sum_{m\in\mathbb{Z}} \eta(2^{-n}m)\widehat{f}(m)e^{i2\pi mx}$$

and the corresponding square function is defined by

$$\widetilde{S}_{\mathbb{T}}(f) = \big(\sum_{n\in\mathbb{N}_0} |\widetilde{D}_n(f)|^2\big)^{1/2}.$$

A corollary of the aforementioned multiplier theorem of Stein is the smooth Littlewood-Paley square function characterisation of H^1, see [58, Théorème 5]. More specifically, a function $f \in L^1(\mathbb{R})$ with zero integral belongs to $H^1(\mathbb{R})$ if and only if, $\widetilde{S}_{\mathbb{R}}(f) \in L^1(\mathbb{R})$ and moreover,

$$\|f\|_{H^1(\mathbb{R})} \sim \|\widetilde{S}_{\mathbb{R}}(f)\|_{L^1(\mathbb{R})}.$$

Similarly, in the periodic setting, one has

$$\|f\|_{H^1(\mathbb{T})} \sim \|\widetilde{S}_{\mathbb{T}}(f)\|_{L^1(\mathbb{T})}.$$

Analogous characterisations hold in the multi-parameter setting. The smooth n-parameter Littlewood-Paley square function on $\mathbb{R}^n$ is given by

$$\widetilde{S}_{\mathbb{R}^n}(f) = \Big(\sum_{k_1,\cdots,k_n \in \mathbb{Z}} |\widetilde{\Delta}_{k_1} \otimes \cdots \otimes \widetilde{\Delta}_{k_n}(f)|^2 \Big)^{1/2}$$

and it is initially defined for $f \in \mathcal{S}(\mathbb{R}^n)$. In the periodic case, if g is a trigonometric polynomial on $\mathbb{T}^n$, then we define

$$\widetilde{S}_{\mathbb{T}^n}(g) = \Big(\sum_{k_1,\cdots,k_n \in \mathbb{N}_0} |\widetilde{D}_{k_1} \otimes \cdots \otimes \widetilde{D}_{k_n}(g)|^2 \Big)^{1/2}.$$

Then, a function $f \in L^1(\mathbb{R}^n)$ with zero integral with respect to each variable belongs to $H^1_{\text{prod}}(\mathbb{R}^n)$ if and only if $\widetilde{S}_{\mathbb{R}^n}(f) \in L^1(\mathbb{R}^n)$ and, moreover,

$$\|f\|_{H^1_{\text{prod}}(\mathbb{R}^n)} \sim \|\widetilde{S}_{\mathbb{R}^n}(f)\|_{L^1(\mathbb{R}^n)}. \tag{2.1}$$

Similarly, one has

$$\|f\|_{H^1_{\text{prod}}(\mathbb{T}^n)} \sim \|\widetilde{S}_{\mathbb{T}^n}(f)\|_{L^1(\mathbb{T}^n)}. \tag{2.2}$$

2.2.2 Atomic decomposition of Hardy spaces

In the one-parameter case, we say that a_Q is an $H^1(\mathbb{R}^n)$-atom associated to a cube Q in $\mathbb{R}^n$ if and only if, $\text{supp}(a_Q) \subset Q$, $\int_Q a_Q = 0$, and $\|a_Q\|_{L^2(Q)} \leqslant |Q|^{-1/2}$. It is well-known that a function f belongs to $H^1(\mathbb{R}^n)$ if and only if there is a sequence of scalars $(\lambda_k)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and a sequence of $H^1(\mathbb{R}^n)$-atoms $(a_{Q_k})_{k\in\mathbb{N}}$ such that $f = \sum_{k\in\mathbb{N}} \lambda_k a_{Q_k}$, where the convergence is with respect to the H^1-norm and it is unconditional, see e.g. Chapter III in [60].

Let $R = I \times J$ be a dyadic rectangle in $\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, where I is a dyadic cube in $\mathbb{R}^{k_1}$ and J is a dyadic cube in $\mathbb{R}^{k_2}$. Following [22], we say that a function a_R is a rectangle atom associated to R, if a_R is supported in R, $\|a_R\|_{L^2(\mathbb{R}^2)} \leqslant |R|^{-1/2}$, $\int_I a_R(x', y)dx' = 0$ for every $y \in J$ and $\int_J a_R(x, y')dy' = 0$ for every $x \in I$. We define $H^1_{\text{rect}}(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2})$ to be the space spanned by the class of all rectangle atoms on $\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, namely

$$H^1_{\text{rect}}(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}) = \Big\{ \sum_{n\in\mathbb{N}} \lambda_{R_n} a_{R_n} : a_{R_n} \text{ is a rectangle atom and } \sum_{n\in\mathbb{N}} |\lambda_{R_n}| < \infty \Big\}.$$

It had been conjectured that the product Hardy space $H^1_{\text{prod}}(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2})$ can be generated by using rectangle atoms. In [14] Carleson disproved that and some years later, in [16] and [17], Chang and R. Fefferman showed that $H^1_{\text{prod}}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})$ is indeed an atomic space but the atoms needed to span the product Hardy space $H^1_{\text{prod}}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})$ are more involved.

Definition 2.2.2 ($H^1_{\text{prod}}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})$-atoms, [22])**.** *A measurable function a_Ω is said to be an $H^1_{\text{prod}}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})$-atom associated to some open set $\Omega\subset\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}$ of finite measure if and only if:*

1. *a_Ω is supported in $\Omega^*=\{(x,y)\in\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}: M_S\chi_\Omega(x,y)>1/2^{k_1+k_2}\}$, where M_S denotes the strong maximal function.*

2. *$\|a_\Omega\|_{L^2(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})}\leqslant C|\Omega|^{-1/2}$, where $C>0$ is an absolute constant.*

3. *a_Ω can be further decomposed as $a_\Omega=\sum_{R\in\mathcal{M}(\Omega)}a_R$, where $\mathcal{M}(\Omega)$ is the set of all maximal dyadic rectangles in Ω and each a_R has the following properties:*

    - *a_R is supported in $2R$, where $2R$ denotes the rectangle concentric to R and dilated by 2.*
    - *$\int_{\mathbb{R}^{k_1}}a_R(x,y)dx=0$ for each $y\in\mathbb{R}^{k_2}$ and $\int_{\mathbb{R}^{k_2}}a_R(x,y)dy=0$ for each $x\in\mathbb{R}^{k_1}$.*
    - *$\big[\sum_{R\in\mathcal{M}(\Omega)}\|a_R\|^2_{L^2(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})}\big]^{1/2}\leqslant|\Omega|^{-1/2}$.*

In [16] and [17], Chang and R. Fefferman proved that $H^1_{\text{prod}}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})$ admits an atomic decomposition in terms of $H^1_{\text{prod}}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})$-atoms, namely $f\in H^1_{\text{prod}}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})$ if and only if there exists a sequence of $H^1_{\text{prod}}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})$-atoms $(a_k)_{k\in\mathbb{N}}$ and a sequence of scalars $(\lambda_k)_{k\in\mathbb{N}}\in\ell^1(\mathbb{N})$ such that $f=\sum_{k\in\mathbb{N}}\lambda_k a_k$ and moreover,

$$\|f\|_{H^1_{\text{prod}}(\mathbb{R}^{k_1}\times\mathbb{R}^{k_2})}\sim\inf\Big\{\sum_{k\in\mathbb{N}}|\lambda_k|:f=\sum_{k\in\mathbb{N}}\lambda_k a_k\Big\}.$$

In the periodic setting, a function a is said to be an $H^1(\mathbb{T})$-atom if it is either a constant function[2] on $\mathbb{T}$ or if a is supported in a dyadic interval $I\subset\mathbb{T}$, has mean zero and $\|a_I\|_{L^2(\mathbb{T})}\leqslant|I|^{-1/2}$. See, e.g., [19], [26], [35]. In the two-parameter case, one needs to consider constant functions on $\mathbb{T}^2$ and "essentially one-dimensional atoms", that is functions defined on $\mathbb{T}^2$ that are constant in one variable $a(x,y)=a_I(x)$ (or $a(x,y)=a_I(y)$), a_I is supported in a dyadic interval $I\subset\mathbb{T}$, has mean zero and $\|a_I\|_{L^2(\mathbb{T})}\leqslant|I|^{-1/2}$. We define $H^1_{\text{rect}}(\mathbb{T}^2)$ to be the space spanned by rectangle atoms associated to dyadic rectangles in $\mathbb{T}^2$, constant functions on $\mathbb{T}^2$ and "essentially one-dimensional atoms" on $\mathbb{T}^2$. One can show that the product Hardy space $H^1_{\text{prod}}(\mathbb{T}^2)$ admits an atomic decomposition analogous to the one presented above for $H^1_{\text{prod}}(\mathbb{R}^2)$ and $H^1_{\text{rect}}(\mathbb{T}^2)$ is a proper subspace of $H^1_{\text{prod}}(\mathbb{T}^2)$.

[2]Note that, unlike the euclidean case, functions in $H^1(\mathbb{T})$ need not have integral equal to 0.

2.2.3 Paley's theorem and multipliers from Hardy spaces to L^2

Let $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ be a lacunary sequence of positive integers, namely $\Lambda \subset \mathbb{N}$ and $\inf_{n\in\mathbb{N}}\{\lambda_{n+1}/\lambda_n\} > 1$.

In [46], Paley proved that for every function f in the analytic Hardy space $H^1_{\mathrm{A}}(\mathbb{T})$, $(\widehat{f}(n))_{n\in\Lambda}$ is square summable. Since $(H^1_{\mathrm{A}}(\mathbb{T}), \|\cdot\|_{L^1(\mathbb{T})})$ is a Banach space, by using the closed graph theorem, it follows that Paley's theorem is equivalent to the fact that there exists a constant $C_\Lambda > 0$ such that for every $f \in H^1_{\mathrm{A}}(\mathbb{T})$ one has

$$\Big(\sum_{n\in\Lambda} |\widehat{f}(n)|^2\Big)^{1/2} \leqslant C_\Lambda \|f\|_{L^1(\mathbb{T})}. \tag{2.3}$$

A version of Paley's theorem for the real Hardy space $H^1(\mathbb{T})$ states that if Λ is lacunary sequence in $\mathbb{N}$, then there is a constant $C'_\Lambda > 0$ such that

$$\Big(\sum_{n\in\Lambda} |\widehat{f}(n)|^2\Big)^{1/2} \leqslant C'_\Lambda \|f\|_{H^1(\mathbb{T})}. \tag{2.4}$$

Both of these versions of Paley's theorem, i.e. (2.3) and (2.4) can be obtained either by using the atomic decomposition or by using the square function characterisation of the corresponding Hardy spaces.

The aforementioned result of Paley can also be stated as follows. if $\Lambda \subset \mathbb{N}$ is lacunary, then χ_Λ is a multiplier from $H^1_{\mathrm{A}}(\mathbb{T})$ to $L^2(\mathbb{T})$. In 1937, in [31], Hardy and Littlewood extended Paley's theorem proving that if $M = (m(n))_{n\in\mathbb{N}_0}$ satisfies the property

$$\sup_{N\in\mathbb{N}_0} \sum_{N\leqslant n\leqslant 2N} |m(n)|^2 < \infty, \tag{2.5}$$

then $M = (m(n))_{n\in\mathbb{N}_0}$ is a multiplier from $H^1_{\mathrm{A}}(\mathbb{T})$ to $L^2(\mathbb{T})$. In the opposite direction, in 1957, Rudin proved in [52] that if χ_Λ is a multiplier from $H^1_{\mathrm{A}}(\mathbb{T})$ to $L^2(\mathbb{T})$, then χ_Λ satisfies (2.5). Namely, Rudin proved that $\chi_\Lambda \in \mathcal{M}_{H^1_{\mathrm{A}}(\mathbb{T})\to L^2(\mathbb{T})}$ if and only if,

$$\sup_{N\in\mathbb{N}_0} \#\{\Lambda \cap [N, 2N]\} < \infty. \tag{2.6}$$

Note that $\Lambda \subset \mathbb{N}$ satisfies condition (2.6) if and only if Λ can be written as a finite union of lacunary sequences. In [21], Duren and Shields extended Rudin's result showing that in fact every multiplier $M = (m(n))_{n\in\mathbb{N}_0}$ from $H^1_{\mathrm{A}}(\mathbb{T})$ to $L^2(\mathbb{T})$ necessarily satisfies (2.5). An analogous characterisation holds for the class of all multipliers from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$, namely

$$\mathcal{M}_{H^1(\mathbb{T})\to L^2(\mathbb{T})} = \Big\{M = (m(n))_{n\in\mathbb{Z}} : \sup_{N\in\mathbb{N}_0} \sum_{N\leqslant |n|\leqslant 2N} |m(n)|^2 < \infty\Big\}.$$

In 1979, D. Oberlin extended the aforementioned results to higher dimensions, [43, Theorem 1]. In particular, it follows by the work of D. Oberlin[3] that a

[3] In fact, in [43] the characterisation of the class $\mathcal{M}_{H^1_{\mathrm{A}}(\mathbb{T}^2)\to L^2(\mathbb{T}^2)}$ is given, but the argument can be readily extended to the class $\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^n)\to L^2(\mathbb{T}^n)}$.

function $m : \mathbb{Z}^n \to \mathbb{C}$ is a multiplier from $H^1_{\text{prod}}(\mathbb{T}^n)$ to $L^2(\mathbb{T}^n)$ if and only if,

$$\sup_{N_1,\cdots,N_n\in\mathbb{N}_0} \sum_{N_1\leqslant|k_1|\leqslant 2N_1} \cdots \sum_{N_n\leqslant|k_n|\leqslant 2N_n} |m(k_1,\cdots,k_n)|^2 < \infty. \tag{2.7}$$

D. Oberlin's proof makes use of the square function characterisation of H^1 and the argument of Duren and Shields, given in [21], suitably adapted to the product setting.

A corollary of the above characterisation of the class $\mathcal{M}_{H^1_{\text{prod}}(\mathbb{T}^n)\to L^2(\mathbb{T}^n)}$ is the following higher-dimensional version of Paley's inequality, [43, Corollary of Theorem 1]; if $\Lambda_1,\cdots,\Lambda_n$ are lacunary sequences in $\mathbb{N}$, then there exists a constant $C_{\Lambda_1,\cdots,\Lambda_n} > 0$, depending only on $\Lambda_1,\cdots,\Lambda_n$, such that

$$\Big(\sum_{(k_1,\cdots,k_n)\in\Lambda_1\times\cdots\times\Lambda_n} |\widehat{f}(k_1,\cdots,k_n)|^2\Big)^{1/2} \leqslant C_{\Lambda_1,\cdots,\Lambda_n}\|f\|_{H^1_{\text{prod}}(\mathbb{T}^n)}.$$

For other variants of Paley's inequality see, e.g., [8], [25], [37], [54, Theorem 8.6], [63], and the references therein.

2.3 The spaces $L\log^r L$ and $\exp L^{1/r}$

Let $(X,\mathcal{A},\mu)$ be a measure space with $\mu(X) < \infty$. In this thesis we shall work with the classes $L\log^r L(X)$ and $\exp L^{1/r}(X)$, $r > 0$ defined as follows. The class $L\log^r L(X)$ consists of all $\mathcal{A}$-measurable functions f such that

$$\int_X |f(x)|\log^r(1+|f(x)|)d\mu(x) < \infty$$

and $\exp^{1/r} L(X)$ is the class of all $\mathcal{A}$-measurable functions f such that there exists a $\lambda = \lambda(f) > 0$ so that $\exp(|f/\lambda|^{1/r})$ is μ-integrable. The class $L\log^r L(X)$ is often referred to as the Zygmund class in the literature[4].

2.3.1 Orlicz spaces

Following [6], if $\phi : [0,\infty) \to [0,\infty)$ is an increasing, left-continuous function with $\phi(0) = 0$ and such that $\lim_{x\to\infty}\phi(x) = \infty$, then the function $\Phi : [0,\infty) \to [0,\infty)$ given by $\Phi(x) = \int_0^x \phi(x')dx'$ is said to be a Young's function[5].

Given a Young's function Φ and a measure space $(X,\mathcal{A},\mu)$ with $\mu(X) < \infty$, we define the Orlicz class $\Phi(X)$ associated to Φ as the space of all $\mathcal{A}$-measurable functions f such that

$$\int_X \Phi(|f(x)|/\lambda)d\mu(x) < \infty$$

[4] Strictly speaking, the class $L\log L(X)$ is originally referred to as Zygmund's class.

[5] In literature, Young's functions are also called N-functions, see [36] for example. Also, instead of left-continuous we may consider right-continuous functions.

for some $\lambda = \lambda(f) > 0$. If we equip $\Phi(X)$ with the norm

$$\|f\|_{\Phi(X)} = \inf\{\lambda > 0 : \int_X \Phi(|f(x)|/\lambda) d\mu(x) \leqslant 1\},$$

then $(\Phi(X), \|\cdot\|_{\Phi(X)})$ becomes a Banach space, see e.g. Theorem 8.9 in Chapter 4 of [6]. Moreover, if f is not μ-a.e. zero, then the infimum is attained, namely one has in this case

$$\|f\|_{\Phi(X)} = \min\{\lambda > 0 : \int_X \Phi(|f(x)|/\lambda) d\mu(x) \leqslant 1\}.$$

Let $\Phi(x) = \int_0^x \phi(x')dx'$ be a Young's function. Define $\widetilde{\phi} : [0, \infty) \to [0, \infty)$ to be the left-continuous inverse of ϕ, namely $\widetilde{\phi}(y) = \inf\{x \geqslant 0 : \phi(x) \geqslant y\}$ for $y \geqslant 0$. One can easily check that $\widetilde{\widetilde{\phi}} = \phi$ and $\widetilde{\Phi}(y) = \int_0^y \widetilde{\phi}(y')dy'$ is a Young's function. Given a Young's function Φ, we say that $\widetilde{\Phi}$ is the complementary Young's function of Φ. Furthermore, Young's inequality, see, e.g. Theorem 8.12 in Chapter 4 of [6], asserts that

$$xy \leqslant \Phi(x) + \widetilde{\Phi}(y) \tag{2.8}$$

for all $x, y \geqslant 0$, whenever Φ and $\widetilde{\Phi}$ are Young's functions as above. In (2.8) equality holds if and only if $x = \widetilde{\phi}(y)$ or $y = \phi(x)$. This result implies that

$$\|f\|_{\Phi(X)} \sim \sup_{\|g\|_{\widetilde{\Phi}(X)} \leqslant 1} \Big| \int_X f(x)g(x)d\mu(x) \Big|. \tag{2.9}$$

See Theorem 8.12 in Chapter 4 of [6].

The aforementioned classes $L\log^r L(X)$ and $\exp L^{1/r}(X)$, $r > 0$ can be realised as Orlicz spaces associated to Young's functions growing like $\Phi_r(x) \sim x \log^r x$ and $\Psi_r(x) \sim \exp(x^{1/r})$ as $x \to \infty$, respectively. The precise definition of Φ_r and Ψ_r is of no importance. Indeed, if F_1 and F_2 are two Young's functions such that $F_1(x) \lesssim F_2(x)$ and $F_1(x) \lesssim F_2(x)$ for all sufficiently large x, then $F_1(X) = F_2(X)$ and $\|f\|_{F_1(X)} \sim \|f\|_{F_2(X)}$, see, e.g., (13.7) in [36]. For instance, for $L\log^r L(X)$ we consider the Young's function $\Phi_r(x) = x[1 + \log(x + 1)]^r$, $x \geqslant 0$. Note that for $r \geqslant 1$ one may consider the Young's function $x \mapsto x\log^r(x + 1)$ instead of Φ_r. For the exponential class $\exp L^{1/r}(X)$, if $0 < r \leqslant 1$ we set $\Psi_r(x) = \exp(x^{1/r}) - 1$, $x \geqslant 0$. Note that for $r > 1$, the function $x \mapsto \exp(x^{1/r}) - 1$ is not convex in the whole of $[0, \infty)$ and so, it is not a Young's function. For $r > 1$, we may take

$$\Psi_r(x) = \begin{cases} 0, \text{ if } 0 \leqslant x \leqslant (r-1)^r \\ \exp(x^{1/r}) - r^{-1}(r-1)^{1-r}\exp(r-1)x - r^{-1}\exp(r-1), \text{ if } x > (r-1)^r. \end{cases}$$

Notice that, as $\Phi_r(x) = x[1 + \log(x + 1)]^r$, $x \geqslant 0$ satisfies the doubling condition $\Phi_r(2x) \leqslant c_r \Phi_r(x) < \infty$ for all sufficiently large $x > 0$, the class

$$\{f : \int_X \Phi_r(|f(x)|)d\mu(x) < \infty\}$$

is a linear function space, see Proposition 8.5 in Chapter 4 of [6], and hence, it is equal to the corresponding Orlicz class

$$\Phi_r(X) = \{f : \int_X \Phi_r(|f(x)|/\lambda)dx < \infty \text{ for some } \lambda > 0\}.$$

Therefore, the function space $\Phi_r(X)$ equals to the Zygmund class[6]

$$L\log^r L(X) = \{f : \int_X |f(x)| \log^r(1+|f(x)|)d\mu(x) < \infty\}.$$

Hence, instead of $(\Phi_r(X), \|\cdot\|_{\Phi_r(X)})$, we may write $(L\log^r L(X), \|\cdot\|_{\Phi_r(X)})$.

It is well-known that $(L\log^r L(X), \|\cdot\|_{\Phi_r(X)})$ and $(\exp L^{1/r}(X), \|\cdot\|_{\Psi_r(X)})$ can be put in duality. More specifically, if $\widetilde{\Psi}_r$ denotes the complementary Young's function of Ψ_r, then it is a standard fact that

$$\widetilde{\Psi}_r(x) \sim x[1+\log(1+x)]^r \tag{2.10}$$

for all sufficiently large x and so, by [36, Theorem 3.2],

$$\widetilde{\Phi}_r(x) \sim \Psi_r(x) \tag{2.11}$$

for all sufficiently large x. Therefore, by (2.10) and (2.11) one deduces that

$$\|f\|_{\widetilde{\Psi}_r(X)} \sim \|f\|_{\Phi_r(X)} \tag{2.12}$$

and

$$\|f\|_{\widetilde{\Phi}_r(X)} \sim \|f\|_{\Psi_r(X)}, \tag{2.13}$$

respectively. Hence, by using (2.12), (2.13), and (2.9), one obtains that

$$\|f\|_{\Phi_r(X)} \sim \sup_{\|g\|_{\Psi_r(X)}\leqslant 1} \Big| \int_X f(x)g(x)d\mu(x) \Big| \tag{2.14}$$

and

$$\|f\|_{\Psi_r(X)} \sim \sup_{\|g\|_{\Phi_r(X)}\leqslant 1} \Big| \int_X f(x)g(x)d\mu(x) \Big|. \tag{2.15}$$

The proof of (2.10) is elementary. Assume that $r > 1$, the case $0 < r \leqslant 1$ is treated similarly. For $r > 1$, if Ψ_r is defined as above, then for $x > (r-1)^r$ we have

$$\psi_r(x) = \Psi_r'(x) = r^{-1}x^{-1+1/r}\exp(x^{1/r}) - r^{-1}(r-1)^{1-r}\exp(r-1)$$

and so, there exists an $a_r > (r-1)^r$ such that

$$1 < (2r)^{-1}x^{-1+1/r}\exp(x^{1/r}) \leqslant \psi_r(x) \leqslant r^{-1}x^{-1+1/r}\exp(x^{1/r})$$

[6] Note that for all $r > 0$ one has $\int_X |f(x)|[1+\log(1+|f(x)|)]^r d\mu(x) < \infty$ if and only if, $\int_X |f(x)|\log^r(1+|f(x)|)d\mu(x) < \infty$ and, moreover, for $r \geqslant 1$ the function $x \mapsto x\log^r(x+1)$, $x \geqslant 0$ is a Young's function.

for all $x \geqslant a_r$. Hence,

$$0 < -\log(2r)+(-1+1/r)\log x+x^{1/r} \leqslant \log(\psi_r(x)) \leqslant -\log(r)+(-1+1/r)\log x+x^{1/r}$$

for all $x \geqslant a_r$ and so, there is a positive real number $b_r > a_r$ such that

$$2^{-1}x^{1/r} \leqslant \log(\psi_r(x)) \leqslant x^{1/r} \tag{2.16}$$

for all $x \geqslant b_r$. To show the right-hand side of (2.10), namely to prove that $\widetilde{\Psi}_r(x) \lesssim x[1+\log(x+1)]^r$ for all sufficiently large x, let $\widetilde{\psi}_r$ denote the left-continuous inverse of ψ_r and let y_r be such that $\widetilde{\psi}_r(y_r) = b_r$, b_r being as above. Then, by (2.16), we have

$$\widetilde{\psi}_r(y) = x \leqslant 2^r \log^r y \leqslant 2^r[\log^r y + r\log^{r-1} y]$$

for every $y \geqslant y_r$. Therefore, there exists a $y_r' > y_r$ and a positive constant $C_r > 0$ such that

$$\begin{aligned} \widetilde{\Psi}_r(y) = \int_0^y \widetilde{\psi}_r(y')dy' &\leqslant \int_0^{y_r} \widetilde{\psi}_r(y')dy' + 2^r \int_{y_r}^{y} [\log^r(y') + r\log^{r-1}(y')]dy' \\ &= 2^r y \log^r y + \int_0^{y_r} \widetilde{\psi}_r(y')dy' - y_r \log^r(y_r) \\ &\leqslant C_r \Phi_r(y) \end{aligned}$$

for all $y \geqslant y_r'$, where $\Phi_r(x) = x[1+\log(x+1)]^r$, $x \geqslant 0$. The proof of the other estimate in (2.10), i.e. $\widetilde{\Psi}_r(x) \gtrsim \Phi_r(x)$ for sufficiently large x, is similar. For more details on Orlicz spaces, we refer the reader to the books [6], [36], and [65].

2.3.2 Lorentz-Zygmund spaces

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. The decreasing rearrangement of an $\mathcal{A}$-measurable function f is the function $f^* : [0,\infty) \to [0,\infty]$ given by

$$f^*(t) = \inf\left\{\lambda > 0 : \mu(\{x \in X : |f(x)| > \lambda\}) \leqslant t\right\}$$

with the convention that $\inf \varnothing = \infty$.

For $0 < p < \infty$ and $0 < q < \infty$, the Lorentz space $L^{p,q}(X)$ is defined as the space of all $\mathcal{A}$-measurable functions f such that

$$\|f\|_{L^{p,q}(X)} = \Big(\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\Big)^{1/q} < \infty.$$

Note that for $p = q \geqslant 1$ we recover the usual L^p spaces; $\|f\|_{L^{p,p}(X)} = \|f\|_{L^p(X)}$. For $q = \infty$, $L^{p,\infty}(X)$ is the space of all $\mathcal{A}$-measurable functions f satisfying

$$\|f\|_{L^{p,\infty}(X)} = \sup_{t>0}\{t^{1/p} f^*(t)\} < \infty.$$

For $p = q = \infty$, we make the convention that $\|f\|_{L^{\infty,\infty}(X)} = \|f\|_{L^\infty(X)}$.

Assume now that $\mu(X) = 1$. For $0 < p < \infty$, $0 < q < \infty$ and $-\infty < r < \infty$, the Lorentz-Zygmund space $L^{p,q}\log^r L(X)$ is the class of all $\mathcal{A}$-measurable functions f such that

$$\|f\|_{L^{p,q}\log^r L(X)} = \Big(\int_0^1 [t^{1/p}(1-\log t)^r f^*(t)]^q \frac{1}{t}\Big)^{1/q} < \infty$$

and for $q = \infty$, $L^{p,\infty}\log^r L(X)$ is the space of all $\mathcal{A}$-measurable functions f such that

$$\|f\|_{L^{p,\infty}\log^r L(X)} = \sup_{t\in[0,1]} \{t^{1/p}(1-\log t)^r f^*(t)\} < \infty.$$

In the case where $p = q = 1$ and $r > 0$, we have $\|f\|_{L^{1,1}\log^r L(X)} < \infty$ if and only if, $f \in L\log^r L(X)$ and hence, we simply write $L\log^r L(X)$. Moreover, the norm

$$\|f\|_{L\log^r L(X)} = \int_0^1 \log^r(1/t) f^*(t)dt$$

is equivalent to the corresponding one given above, that is $\|f\|_{L^{1,1}\log^r L(X)} \sim \|f\|_{L\log^r L(X)}$ and the space $(L\log^r L(x), \|\cdot\|_{L\log^r L(X)})$ is Banach, see [5, Theorem 8.3]. Furthermore, one has, see the proof of [5, Lemma 10.1] that if $\|f\|_{L\log^r L(X)} = 1$, then

$$\int_X |f(x)|[1+\log(1+|f(x)|)]^r d\mu(x) \lesssim_r 1$$

and if $\|f\|_{\Phi_r(X)} = 1$, then

$$\int_0^1 f^*(t)\log^r(1/t)dt \lesssim_r 1.$$

One can easily see that these two facts imply that $\|f\|_{\Phi_r(X)} \sim \|f\|_{L\log^r L(X)}$. So, $(L\log^r L(X), \|\cdot\|_{L\log^r L(X)})$ and $(L\log^r L(X), \|\cdot\|_{\Phi_r(X)})$ are equivalent. Thus, the Zygmund class $L\log^r L(X) = \{f : \int_X |f(x)|\log^r(1+|f(x)|)d\mu(x) < \infty\}$ can be regarded either as a Lorentz-Zygmund space or as an Orlicz space and the corresponding Banach spaces $(L\log^r L(X), \|\cdot\|_{L\log^r L(X)})$, $(L\log^r L(X), \|\cdot\|_{\Phi_r(X)})$ are equivalent.

2.4 Thin sets in Harmonic Analysis

Let G be a compact abelian group whose dual is denoted by $\widehat{G}$.

In this thesis, we focus on compact abelian groups G such that $\widehat{G}$ is countable. For basic facts on harmonic analysis on groups, see [54].

If a function $f \in L^1(G)$ is such that $\operatorname{supp}(\widehat{f})$ is finite, then f is said to be a trigonometric polynomial on G.

If Λ is an non-empty subset of $\widehat{G}$, then every trigonometric polynomial f on G such that $\operatorname{supp}(\widehat{f}) \subset \Lambda$ is called a Λ-polynomial.

Definition 2.4.1 ($\Lambda(p)$ sets)**.** *Let $p > 2$. We say that $\Lambda \subset \widehat{G}$ is a $\Lambda(p)$ set if and* *only if, there exists a constant* $A(p,\Lambda) > 0$ *such that*

$$\|f\|_{L^p(G)} \leqslant A(p,\Lambda)\|f\|_{L^2(G)}$$

for every Λ*-polynomial* f*. The smallest constant* $A(p,\Lambda)$ *such that the above inequality holds is called the* $\Lambda(p)$ *constant of* Λ*.*

Recall from subsection 2.2.3 that a sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ is called lacunary if $\rho_\Lambda = \inf_{n\in\mathbb{N}}\{\lambda_{n+1}/\lambda_n\} > 1$. The number ρ_Λ is said to be the ratio of the lacunary sequence Λ.

A classical result of Zygmund [64] asserts that lacunary sequences in $\mathbb{N}$ are $\Lambda(p)$ sets for all $p > 2$ and their $\Lambda(p)$ constants grow like $p^{1/2}$ as $p \to \infty$. Namely, Zygmund proved in [64] that if $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ is a lacunary sequence in $\mathbb{N}$, then there exists a constant $C_\Lambda > 0$ such that

$$\|f\|_{L^p(\mathbb{T})} \leqslant C_\Lambda p^{1/2}\|f\|_{L^2(\mathbb{T})} \tag{2.17}$$

for every Λ-polynomial f. By duality (and properties of $\exp L^{1/2}(\mathbb{T})$), see appendix A, this is equivalent to the fact that there are positive constants A_Λ and B_Λ so that

$$\Big(\sum_{n\in\Lambda}|\widehat{f}(n)|^2\Big)^{1/2} \leqslant A_\Lambda \int_{\mathbb{T}} |f(x)|\log^{1/2}(1+|f(x)|)dx + B_\Lambda, \tag{2.18}$$

see Theorem 7.6 in Chapter XII of [65]. In what follows, we shall refer to (2.18) as Zygmund's inequality. Moreover, the constants C_Λ, A_Λ and B_Λ in (2.17), (2.18) respectively depend only on the ratio $\rho_\Lambda = \inf_{n\in\mathbb{N}}\{\lambda_{n+1}/\lambda_n\}$ of Λ.

Remark 2.4.2. *By using a scaling argument one can show that* (2.18) *is equivalent to the inequality*

$$\Big(\sum_{n\in\Lambda}|\widehat{f}(n)|^2\Big)^{1/2} \leqslant D_\Lambda\|f\|_{\Phi_{1/2}(\mathbb{T})}, \tag{2.19}$$

where, for instance, one may take $\Phi_{1/2}(x) = x[1+\log(x+1)]^{1/2}$*,* $x \geqslant 0$*.*

In [57], Sidon showed that if $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ is a lacunary sequence in $\mathbb{N}$, then there exists a constant $S_\Lambda > 0$ such that

$$\sum_{n\in\mathbb{N}}|\widehat{f}(\lambda_n)| \leqslant S_\Lambda\|f\|_{L^\infty(\mathbb{T})} \tag{2.20}$$

for every Λ-polynomial f on $\mathbb{T}$. As in the theorem of Zygmund mentioned above, the constant S_Λ depends only the ratio ρ_Λ of Λ.

Motivated by the aforementioned theorem of Sidon, Rudin in his celebrated paper [53] defined the notion of Sidon sets in $\mathbb{Z}$ and in [54] Rudin defined the notion of Sidon sets in the dual of any compact abelian group. The definition of Sidon sets in the dual of a compact abelian group is as follows.

Definition 2.4.3 (Sidon sets)**.** *Let* G *be a compact abelian group.*

An infinite set Λ in $\widehat{G}$ is said to be a Sidon set in $\widehat{G}$ if there exists a constant $S_\Lambda > 0$ such that

$$\sum_{\gamma\in\Lambda} |\widehat{f}(\gamma)| \leqslant S_\Lambda \|f\|_{L^\infty(G)} \tag{2.21}$$

for every Λ-polynomial f.

The smallest constant S_Λ such that (2.21) *holds for every Λ-polynomial f is called the Sidon constant of Λ.*

By Sidon's theorem, every lacunary sequence is a Sidon set in $\mathbb{Z}$. Moreover, one can construct Sidon sets in $\mathbb{Z}$ that are not finite unions of lacunary sequences, see e.g. [53, Remark 2.5(3)] and therefore, the class of finite unions of lacunary sequences is properly contained in the class of Sidon sets in $\mathbb{Z}$. In [53], Rudin extended Zygmund's inequality from lacunary sequences to Sidon sets in $\mathbb{Z}$ and, in [54], he extended Zygmund's inequality to Sidon sets in the dual of any compact abelian group. In other words, Rudin proved in [54] that if Λ is a Sidon set, then it is a $\Lambda(p)$ set for all $p > 2$ with $\Lambda(p)$ constant growing like $p^{1/2}$ as $p \to \infty$. Rudin also conjectured that this property characterises Sidon sets, namely if an infinite set Λ is a $\Lambda(p)$ set for all $p > 2$ with $\Lambda(p)$ constant growing like $p^{1/2}$, then it is necessarily a Sidon set. In 1978, in [48], Pisier proved that this is indeed the case. Therefore, by combining the aforementioned results of Rudin and Pisier, it follows by duality that a set Λ is Sidon if and only if χ_Λ is a multiplier from $L\log^{1/2} L(G)$ to $L^2(G)$. Furthermore, one can actually obtain a characterisation of the class of all multipliers from $L\log^{1/2} L(G)$ to $L^2(G)$ in the case where G is a compact group, not necessarily abelian, see [39]. For other proofs of Pisier's theorem, see [11] and [13]. For more details on Sidon sets, we refer the reader to the book [28].

Another class of thin sets is that of q-Rider sets defined as follows. For $q \geqslant 1$, a set $\Lambda \subset \widehat{G}$ is said to be q-Rider if there is a constant $R_{\Lambda,q} > 0$ such that

$$\Big(\sum_{\gamma\in\Lambda} |\widehat{f}(\gamma)|^q\Big)^{1/q} \leqslant R_{\Lambda,q}[|f|] \tag{2.22}$$

for every Λ-polynomial f. Here, we use the notation

$$[|f|] = \mathbb{E}\Big[\Big\| \sum_{\gamma\in\widehat{G}} r_\gamma \widehat{f}(\gamma)\gamma \Big\|_{L^\infty(G)}\Big],$$

where $(r_\gamma)_\gamma$ denotes the set of Rademacher functions, see, e.g., Appendix B.

Note that every Sidon set is automatically a 1-Rider set. A classical theorem of Rider [50] shows that the converse is also true, namely every 1-Rider set is also a Sidon set.[7]

It is also well-known that if Λ is a $\Lambda(p)$ set for all $p > 2$ with $\Lambda(p)$ constant

[7]Originally, Rider stated his result in [50] using Steinhaus trigonometric series instead of Rademacher trigonometric series. See also Corollary 6.23 in [42]. Moreover, if one defines $[|f|]_1 = \mathbb{E}\big[\big\| \sum_{\gamma\in\widehat{G}} g_\gamma \widehat{f}(\gamma)\gamma \big\|_{L^\infty(G)}\big]$, where $(g_\gamma)_\gamma$ is a collection of independent gaussian random variables with mean 0 and variance 1, then $[|f|] \sim [|f|]_1$, see Theorem 3.5(i) in Chapter V in [39].

growing like $p^{k/2}$ as $p \to \infty$, $k \in \mathbb{N}$, then Λ is a q-Rider set with $q = 2k/(k+1)$, see [49, Théorème 6.3].

Chapter 3

Variants of the classical inequality of Zygmund

3.1 Introduction

As mentioned in section 2.4, Rudin, in [53], extended Zygmund's inequality from lacunary sequences in $\mathbb{N}$ to Sidon sets in the dual of any compact abelian group. Furthermore, higher-dimensional versions of Rudin's extension of Zygmund's inequality are well-known. Namely, if Λ_j is a Sidon set in the dual of a compact abelian group G_j, then $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ satisfies the following "n-dimensional" version of Zygmund's inequality

$$\Big(\sum_{\gamma \in \Lambda} |\widehat{f}(\gamma)|^2 \Big)^{1/2} \leqslant A_\Lambda \int_G |f(x)| \log^{n/2}(1 + |f(x)|) dx + B_\Lambda, \tag{3.1}$$

where $G = G_1 \times \cdots \times G_n$, see e.g. [9, Chapter VII] or [49, Remarque, p. 24].

The present chapter focuses on variants of Zygmund's inequality for functions of one and several variables. To be more specific, in section 3.2 we introduce the notion of Sidon weights and then we extend (3.1) from products of Sidon sets to products of Sidon weights. In particular, Sidon weights are multipliers from $L \log^{1/2} L$ to L^2 and hence one can naturally ask whether the converse is true, namely whether all multipliers from $L \log^{1/2} L$ to L^2 are Sidon weights. In section 3.3 we give a negative answer to this question, based on a sharp multiplier inclusion theorem for functions defined over the torus. More specifically, in section 3.3 we show that the class of all multipliers from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$ is properly contained in the class of all multipliers from $L \log^{1/2} L(\mathbb{T})$ to $L^2(\mathbb{T})$. Moreover, the inclusion is sharp in the sense that the exponent $r = 1/2$ in $L \log^{1/2} L(\mathbb{T})$ cannot be improved. As a corollary of this multiplier inclusion theorem, we give an example of a multiplier in $\mathcal{M}_{H^1(\mathbb{T}) \to L^2(\mathbb{T})}$ and hence in $\mathcal{M}_{L \log^{1/2} L(\mathbb{T}) \to L^2(\mathbb{T})}$ which is not a Sidon weight. In section 3.4 we obtain an analogous inclusion theorem for functions defined on the real line by using a result of Tao and Wright on a Littlewood-Paley inequality for compactly supported functions in $L \log^{1/2} L$ with mean zero. In the last section we study some further variants of Zygmund's inequality in higher dimensions. In particular, we obtain higher-dimensional extensions of a classical result of Bonami [10] and as a corollary of our results we get a special case of (3.1) for products of lacunary sequences in $\mathbb{Z}$.

This chapter is based on the author's work [3], submitted for publication.

3.2 Higher-dimensional variants of Zygmund's inequality

Our goal in this section is to examine weighted versions of (3.1). More specifically, given compact abelian groups $G_1, \cdots, G_n$, we shall obtain a class of multipliers from $L\log^{n/2} L(G_1 \times \cdots \times G_n)$ to $L^2(G_1 \times \cdots \times G_n)$ that properly contains multipliers of the form $\chi_{\Lambda_1 \times \cdots \times \Lambda_n}$, where Λ_j is a Sidon set in the dual of G_j $(j = 1, \cdots, n)$.

As mentioned in the previous chapter, we shall focus on compact abelian groups G whose duals $\widehat{G}$ are countable.

We begin by defining the notion of Sidon weights which is a weighted analogue of the notion of Sidon sets.

Definition 3.2.1 (Sidon weights)**.** *A function $m : \widehat{G} \to \mathbb{C}$ is said to be a Sidon weight on $\widehat{G}$ if there exists a positive constant S_m such that*

$$\sum_{\gamma \in \mathrm{supp}(m)} |m(\gamma)\widehat{f}(\gamma)| \leqslant S_m \|f\|_{L^\infty(G)} \tag{3.2}$$

for every trigonometric polynomial f on G whose spectrum lies in $\mathrm{supp}(m) = \{\gamma \in \widehat{G} : m(\gamma) \neq 0\}$.

Note that, by (3.2), every Sidon weight is a bounded function on $\widehat{G}$, in particular, $|m(\gamma)| \leqslant S_m$ for every $\gamma \in \mathrm{supp}(m)$. Moreover, if Λ is a Sidon set in the dual of G, then every bounded function supported in Λ is a Sidon weight. Furthermore, note that the sequence $M = (m(n))_{n \in \mathbb{Z}}$ given by $m(0) = 0$ and $m(n) = n^{-2}$ for $n \neq 0$ is a Sidon weight on $\mathbb{Z}$. Therefore, the notion of Sidon weights extends that of Sidon sets.

In [39] it is shown that a bounded function is a multiplier from $L\log^{1/2} L(G)$ to $L^2(G)$, if and only if,

$$\sum_{\gamma \in \widehat{G}} |m(\gamma)\widehat{f}(\gamma)| \leqslant C_m[|f|] \tag{3.3}$$

for all $f \in C_{\text{a.s.}}(G) = \{f \in L^2(G) : \sum_{\gamma \in \widehat{G}} r_\gamma(\omega)\widehat{f}(\gamma)\gamma \text{ is } \omega\text{−almost surely in } C(G)\}$, where $(r_\gamma)_\gamma$ denotes the set of Rademacher functions and $[|f|]$ is as in section 2.4. It is clear that every Sidon weight on $\widehat{G}$ automatically satisfies (3.3) and hence, Sidon weights are multipliers from $L\log^{1/2} L(G)$ to $L^2(G)$. As mentioned in section 2.4, in the unweighted setting, a classical result of Rider [50] asserts that if for every Λ-polynomial f one has

$$\sum_{\gamma \in \Lambda} |\widehat{f}(\gamma)| \leqslant R_\Lambda[|f|],$$

then Λ is a Sidon set. Therefore, the following question arises. *Does Rider's result hold in the weighted setting? In other words, is it true that every multiplier* *from $L\log^{1/2} L(G)$ to $L^2(G)$ is necessarily a Sidon weight?* In the next section we shall give a negative answer to this question.

In the rest of this section we focus on weighted variants of (3.1). In the "one-dimensional" case, by adapting the argument of Rudin [53] that extends Zygmund's inequality to Sidon sets, one can show that Sidon weights are multipliers from $L\log^{1/2} L(G)$ to $L^2(G)$ without appealing to the aforementioned characterisation of the class $\mathcal{M}_{L\log^{1/2} L(G)\to L^2(G)}$. Indeed, towards this aim, the first step is to obtain the following proposition, which is a weighted version of a well-known characterisation of Sidon sets [53, Theorem 5.7.3].

Proposition 3.2.2 (Characterisation of Sidon weights)**.** *Let G be a compact abelian group and let $m : \widehat{G} \to \mathbb{C}$ be a function. Put $\Lambda_m = \{\gamma \in \widehat{G} : m(\gamma) \neq 0\}$.*

The following are equivalent:

1. *m is a Sidon weight.*
2. *For every function $f \in C(G)$ with $\mathrm{supp}(\widehat{f}) \subset \Lambda_m$ one has*
$$\sum_{\gamma\in\Lambda_m} |m(\gamma)\widehat{f}(\gamma)| < \infty.$$
3. *For every bounded function $b : \widehat{G} \to \mathbb{C}$ supported in a subset of Λ_m there exists a measure $\nu_b \in M(G)$ such that:*
   - *$\widehat{\nu_b}(\gamma) = b(\gamma)|m(\gamma)|$ for every $\gamma \in \Lambda_m$ and*
   - *$\|\nu_b\| \leqslant C_m\|b\|_{L^\infty(\Lambda_m)}$, where $C_m > 0$ is a constant that depends only on m and not on b.*

Proof. The proof of this proposition is a straightforward adaptation of the corresponding one given by Rudin in [53] and [54] and we include it here for the convenience of the reader.

Denote by $C_{\Lambda_m}(G)$ the space of all continuous functions on G with spectrum in Λ_m, i.e. $C_{\Lambda_m}(G) = \{f \in C(G) : \mathrm{supp}(\widehat{f}) \subset \Lambda_m\}$.

<u>(1) $\implies$ (2)</u>. Consider an $f \in C_{\Lambda_m}(G)$ and take $\gamma_1, \cdots, \gamma_N \in \widehat{G}$. Let $\epsilon > 0$. Then, by [54, Theorem 2.6.8], there exists a trigonometric polynomial k on G such that $\|k\|_{L^1(G)} < 1 + \epsilon$ and $\widehat{k}(\gamma_n) = 1$ for $n = 1, \cdots, N$. We thus have

$$\sum_{n=1}^{N} |m(\gamma_n)\widehat{f}(\gamma_n)| = \sum_{n=1}^{N} |m(\gamma_n)\widehat{f}(\gamma_n)\widehat{k}(\gamma_n)| \leqslant S_m\|f * k\|_{L^\infty(G)}.$$

Since $\|f * k\|_{L^\infty(G)} \leqslant \|f\|_{L^\infty(G)}\|k\|_{L^1(G)} < (1 + \epsilon)\|f\|_{L^\infty(G)}$, it follows that for every finite set $\{\gamma_1, \cdots, \gamma_N\}$ in Λ_m, one has

$$\sum_{n=1}^{N} |m(\gamma_n)\widehat{f}(\gamma_n)| \leqslant S_m\|f\|_{L^\infty(G)}$$

and we thus deduce that (2) holds.

<u>(2) $\implies$ (1)</u>. First of all note that $C_{\Lambda_m}(G)$ endowed with the $\|\cdot\|_{L^\infty(G)}$-norm is a

closed subspace of $C(G)$ and hence, $(C_{\Lambda_m}(G), \|\cdot\|_{L^\infty(G)})$ is a Banach space. Indeed, to see that $C_{\Lambda_m}(G)$ is a closed subspace of $C(G)$ with respect to the$\|\cdot\|_{L^\infty(G)}$-norm, let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $C_{\Lambda_m}(G)$ that converges to some f with respect to $\|\cdot\|_{L^\infty(G)}$. So, $f \in C(G)$ and for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|f - f_k\|_{L^\infty(G)} < \epsilon$ whenever $k > N$. We need to show that $f \in C_{\Lambda_m}(G)$, namely, for every $\gamma \notin \Lambda$ we have $\widehat{f}(\gamma) = 0$. For this, note that if $\gamma \notin \Lambda_m$, then

$$|\widehat{f}(\gamma)| = |\widehat{f}(\gamma) - \widehat{f}_k(\gamma)| \leqslant \|f - f_k\|_{L^1(G)} \leqslant \|f - f_k\|_{L^\infty(G)} < \epsilon$$

for every $\epsilon > 0$. Therefore, $\widehat{f}(\gamma) = 0$ for each $\gamma \notin \Lambda_m$ and thus, $f \in C_{\Lambda_m}(G)$.

Assume that (2) holds. Define $L^1_{\Lambda_m}(m)$ to be the set of all complex-valued functions α on $\widehat{G}$ such that:

- $\operatorname{supp}(\alpha) \subset \Lambda_m$,
- $\sum_{\gamma\in\Lambda_m} |\alpha(\gamma)m(\gamma)| < \infty$.

If we equip $L^1_{\Lambda_m}(m)$ with the norm $\|\cdot\|_{L^1_{\Lambda_m}(m)}$, i.e. $\|\alpha\|_{L^1_{\Lambda_m}(m)} = \sum_{\gamma\in\Lambda_m} |\alpha(\gamma)m(\gamma)|$, then $(L^1_{\Lambda_m}(m), \|\cdot\|_{L^1_{\Lambda_m}(m)})$ is a Banach space. Next, consider the linear map $T : C_{\Lambda_m}(G) \to L^1_{\Lambda_m}(m)$ by $f \mapsto (\widehat{f}(\gamma))_{\gamma\in\Lambda_m}$, which is well-defined thanks to (2). We shall prove that T is bounded from $(C_{\Lambda_m}(G), \|\cdot\|_{L^\infty(G)})$ to $(L^1_{\Lambda_m}(m), \|\cdot\|_{L^1_{\Lambda_m}(m)})$ by using the closed graph theorem. For this, let $(f_k)_{k\in\mathbb{N}} \subset C_{\Lambda_m}(G)$ be such that:

- $f_k \to 0$ in $\|\cdot\|_{L^\infty(G)}$-norm and
- $\|\widehat{f}_k - c\|_{L^1_{\Lambda_m}(m)} \to 0$ for some $c \in L^1_{\Lambda_m}(m)$.

We need to show that $c \equiv 0$. To this end, note that for each $\gamma \in \Lambda_m$ one has

$$|c(\gamma)m(\gamma)| \leqslant |c(\gamma) - \widehat{f}_k(\gamma)||m(\gamma)| + |\widehat{f}_k(\gamma)||m(\gamma)| \leqslant \sum_{\gamma\in\Lambda_m} |c(\gamma) - \widehat{f}_k(\gamma)||m(\gamma)| + |m(\gamma)|\|f_k\|_{L^\infty(G)},$$

As $m(\gamma) \neq 0$ when $\gamma \in \Lambda_m = \{\gamma \in \widehat{G} : m(\gamma) \neq 0\}$, then, by our assumptions, we deduce that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $k > N$, $\|f_k\|_{L^\infty(G)} < (2|m(\gamma)|)^{-1}\epsilon$ and $\|\widehat{f}_k - c\|_{L^1_{\Lambda_m}(m)} < \epsilon/2$. Hence, $|c(\gamma)m(\gamma)| < \epsilon$ for every $\epsilon > 0$ and since $m(\gamma) \neq 0$, we conclude that $c(\gamma) = 0$.

Therefore, T is bounded and so, there exists a constant $B > 0$, depending only on m, such that $\|T(f)\|_{L^1_{\Lambda_m}(m)} \leqslant B\|f\|_{L^\infty(G)}$ for every $f \in C_{\Lambda_m}(G)$. In particular, for every Λ_m-polynomial f one has

$$\sum_{\gamma\in\Lambda_m} |m(\gamma)\widehat{f}(\gamma)| \leqslant B\|f\|_{L^\infty(G)}.$$

(3) $\Longrightarrow$ (2). Let $f \in C_{\Lambda_m}(G)$. Consider the bounded function $b : \widehat{G} \to \mathbb{C}$ given by $b(\gamma) = e^{i\arg\{\widehat{f}(\gamma)\}}$, $\gamma \in \operatorname{supp}(\widehat{f})$ and $b(\gamma) = 0$ otherwise. It follows by (3) that there exists a measure $\nu_b \in M(G)$ such that $\widehat{\nu_b}(\gamma) = b(\gamma)|m(\gamma)|$ for all $\gamma \in \Lambda_m$.

Let $\{\gamma_1, \cdots, \gamma_N\}$ be a finite subset of $\mathrm{supp}(\widehat{f})$ and let $\epsilon > 0$. As above, there is a trigonometric polynomial k on G such that $\widehat{k}(\gamma_n) = 1$ for $n = 1, \cdots, N$ and $\|k\|_{L^1(G)} < 1 + \epsilon$. Then

$$\begin{aligned}\sum_{n=1}^{N} |m(\gamma_n)\widehat{f}(\gamma_n)| = \sum_{n=1}^{N} \overline{\widehat{f}(\gamma_n)} b(\gamma_n)|m(\gamma_n)| &= \sum_{n=1}^{N} \overline{\widehat{f}(\gamma_n)\widehat{k}(\gamma_n)}\widehat{\nu_b}(\gamma_n) \\ &= \sum_{n=1}^{N} \overline{\widehat{f * k}(\gamma_n)}\widehat{\nu_b}(\gamma_n) \\ &= \int_G \overline{f * k(x)} d\nu_b(x).\end{aligned}$$

Hence,

$$\sum_{n=1}^{N} |m(\gamma_n)\widehat{f}(\gamma_n)| \leqslant \|f * k\|_{L^\infty(G)}\|\nu_b\| \leqslant (1+\epsilon)C_m\|f\|_{L^\infty(G)}.$$

and so, we deduce that

$$\sum_{\gamma \in \widehat{G}} |m(\gamma)\widehat{f}(\gamma)| \leqslant C_m\|f\|_{L^\infty(G)} < \infty,$$

as desired.

<u>(1) $\implies$ (3)</u>. Let b be a bounded function on $\widehat{G}$ supported in Λ_m. Let Λ_b denote the linear functional defined over the space of Λ_m-polynomials by

$$\Lambda_b(f) = \sum_{\gamma \in \Lambda_m} \widehat{f}(\gamma)b(\gamma)|m(\gamma)|.$$

If (1) holds, then Λ_b is well-defined and satisfies $|\Lambda_b(f)| \leqslant S_m\|b\|_{L^\infty(\widehat{G})}\|f\|_{L^\infty(G)}$ for every f lying in the subspace of $C(G)$ consisting of all Λ_m-polynomials. Hence, by the Hahn-Banach theorem, there exists an extension of Λ_b to a continuous linear functional on all of $C(G)$ with norm at most $S_m\|b\|_{L^\infty(\widehat{G})}$. Hence, there exists a measure $\nu \in M(G)$ with $\|\nu\| \leqslant S_m\|b\|_{L^\infty(\widehat{G})}$ and such that

$$\int_G f(x)d\nu(x) = \Lambda_b(f) = \sum_{\gamma \in \Lambda} \widehat{f}(\gamma)b(\gamma)|m(\gamma)|$$

for all Λ_m-polynomials f. If $\gamma \in \Lambda_m$, then, by using the above equation for $f(x) = \gamma(x)$, we deduce that $\widehat{\nu}(\gamma) = b(\gamma)|m(\gamma)|$. □

Having obtained the above characterisation of Sidon weights, by using a standard adaptation of Rudin's argument to higher dimensions, we prove the following weighted variant of (3.1).

Proposition 3.2.3. *Let $G_1, \cdots, G_n$ be compact abelian groups. For $j = 1, \cdots, n$, let $m_j : \widehat{G_j} \to \mathbb{C}$ be Sidon weights.*

Set $G = G_1 \times \cdots \times G_n$ and $m = m_1 \otimes \cdots \otimes m_n$. Then there are positive constants A_m and B_m, depending on m, such that

$$\Big(\sum_{\gamma\in\widehat{G}}|m(\gamma)\widehat{f}(\gamma)|^2\Big)^{1/2}\leqslant A_m\int_G|f(x)|\log^{n/2}(1+|f(x)|)dx+B_m. \tag{3.4}$$

In particular, $m=m_1\otimes\cdots\otimes m_n$ *is a multiplier from* $L\log^{n/2}L(G)$ *to* $L^2(G)$.

Proof. The argument is standard. For the unweighted multi-dimensional case, see e.g. [9, Chapter VII]. The idea is to use duality (see Appendix A), multi-dimensional Khintchine's inequality (see, e.g., (B.2) in Appendix B) and Proposition 3.2.2.

To be more specific, by duality, to prove (3.4), it suffices to show that for every finite sequence of complex-valued scalars $(a_\gamma)_{\gamma\in\widehat{G}}$ the trigonometric polynomial g on G given by

$$g(x)=\sum_{\gamma\in\widehat{G}}a_\gamma|m(\gamma)|^2\gamma(x)$$

satisfies

$$\|g\|_{L^p(G)}\leqslant C_m p^{n/2}\Big(\sum_{\gamma\in\widehat{G}}|m(\gamma)a_\gamma|^2\Big)^{1/2}$$

for all $p>2$, where $C_m>0$ is an absolute constant that depends only on m and not on $(a_\gamma)_{\gamma\in\widehat{G}}$. For this, fix a $p>2$ and a finite sequence of scalars $(a_\gamma)_{\gamma\in\widehat{G}}$. Let $\Lambda_j=\mathrm{supp}(m_j)\subset\widehat{G_j}$, $j=1,\cdots,n$. Hence, $\mathrm{supp}(m)=\Lambda_1\times\cdots\times\Lambda_n$ and we write

$$g(x_1,\cdots,x_n)=\sum_{(\gamma_1,\cdots,\gamma_n)\in\Lambda_1\times\cdots\times\Lambda_n}a_{(\gamma_1,\cdots,\gamma_n)}\Big[\prod_{j=1}^n|m_j(\gamma_j)|^2\Big]\gamma_1(x_1)\cdots\gamma_n(x_n).$$

For $\underline{\omega}\in[0,1]^n$ consider the following "randomised" trigonometric polynomial $g_{\underline{\omega}}$ associated to g

$$g_{\underline{\omega}}(x_1,\cdots,x_n)=\sum_{(\gamma_1,\cdots,\gamma_n)\in\Lambda_1\times\cdots\times\Lambda_n}a_{(\gamma_1,\cdots,\gamma_n)}\Big[\prod_{j=1}^n r_{\gamma_j}(\omega_j)|m_j(\gamma_j)|\Big]\gamma_1(x_1)\cdots\gamma_n(x_n),$$

where, for $j=1,\cdots,n$, $(r_{\gamma_j})_{\gamma_j\in\Lambda_j}$ denotes the set of Rademacher functions on $[0,1]$ indexed by Λ_j. Notice that for every $\underline{\omega}=(\omega_1,\cdots,\omega_n)\in[0,1]^n$ we may write

$$g(x_1,\cdots,x_n)=$$
$$\sum_{(\gamma_1,\cdots,\gamma_n)\in\Lambda_1\times\cdots\times\Lambda_n}\widehat{g_{\underline{\omega}}}(\gamma_1,\cdots,\gamma_n)\prod_{j=1}^n\big[r_{\gamma_j}(\omega_j)|m_j(\gamma_j)|\big]\gamma_1(x_1)\cdots\gamma_n(x_n).$$

For $j=1,\cdots,n$, define $b^{(j)}_{\omega_j}:\Lambda_j\to\mathbb{R}$ by $b^{(j)}_{\omega_j}(\gamma_j)=r_{\gamma_j}(\omega_j)$, $\gamma_j\in\Lambda_j$. Then, by Proposition 3.2.2, there exists a measure $\nu^{(j)}_{\omega_j}\in M(G_j)$ such that $(\nu^{(j)}_{\omega_j})\widehat{\ }(\gamma_j)=b^{(j)}_{\omega_j}(\gamma_j)|m_j(\gamma_j)|=r_{\gamma_j}(\omega_j)|m_j(\gamma_j)|$ for each $\gamma_j\in\Lambda_j$ with $\|\nu^{(j)}_{\omega_j}\|\leqslant C_{m_j}$, where C_{m_j} is a positive constant that depends only on m_j and not on ω_j. Note that for every

$(\gamma_1, \cdots, \gamma_n) \in \Lambda_1 \times \cdots \times \Lambda_n$ one has

$$\prod_{j=1}^{n} b_{\gamma_j}^{(j)}(\omega_j)|m(\gamma_j)| = \prod_{j=1}^{n} (\nu_{\omega_j}^{(j)})\widehat{\ }(\gamma_j)(\gamma_j) = (\nu_{\omega_1}^{(1)} \otimes \cdots \otimes \nu_{\omega_n}^{(n)})\widehat{\ }(\gamma_1, \cdots, \gamma_n),$$

where $\nu_{\omega_1}^{(1)} \otimes \cdots \otimes \nu_{\omega_n}^{(n)}$ denotes the product measure of $\nu_{\omega_1}^{(1)}, \cdots, \nu_{\omega_n}^{(n)}$ on $(G, \mathcal{B}(G))$ given by

$$d(\nu_{\omega_1}^{(1)} \otimes \cdots \otimes \nu_{\omega_n}^{(n)})(x_1, \cdots, x_n) = h_1(x_1) \cdots h_n(x_n) d(|\nu_{\omega_1}^{(1)}| \otimes \cdots \otimes |\nu_{\omega_n}^{(n)}|)(x_1, \cdots, x_n),$$

where h_j is such that $d\nu_{\omega_j}^{(j)} = h_j d|\nu_{\omega_j}^{(j)}|$ and $|h_j| = 1$ on G_j (for $j = 1, \cdots, n$), see [55, Theorem 6.12]. It thus follows by [55, Theorem 6.13] that $\|\nu_{\omega_1}^{(1)} \otimes \cdots \otimes \nu_{\omega_n}^{(n)}\| = \|\nu_{\omega_1}^{(1)}\| \cdots \|\nu_{\omega_n}^{(n)}\|$. Hence, one has

$$g = (g_{\underline{\omega}}) * (\nu_{\omega_1}^{(1)} \otimes \cdots \otimes \nu_{\omega_n}^{(n)})$$

and so, by Young's inequality [54, Theorem 1.3.2 (c)] one obtains

$$\|g\|_{L^p(G)} \leqslant \|g_{\underline{\omega}}\|_{L^p(G)} \|\nu_{\omega_1}^{(1)} \otimes \cdots \otimes \nu_{\omega_n}^{(n)}\|.$$

Since $\|\nu_{\omega_1}^{(1)} \otimes \cdots \otimes \nu_{\omega_n}^{(n)}\| \leqslant C_{m_1} \cdots C_{m_n}$, one has

$$\|g\|_{L^p(G)}^p \leqslant \big(C_{m_1} \cdots C_{m_n}\big)^p \int_{[0,1]^n} \|g_{\underline{\omega}}\|_{L^p(G)}^p d\omega_1 \cdots d\omega_n. \tag{3.5}$$

By Fubini's theorem, one deduces

$$\int_{[0,1]^n} \|g_{\underline{\omega}}\|_{L^p(G)}^p d\omega_1 \cdots d\omega_n = \int_G \Big(\int_{[0,1]^n} |g_{\underline{\omega}}(x_1, \cdots, x_n)|^p d\omega_1 \cdots d\omega_n \Big) dx_1 \cdots dx_n.$$

If, for fixed $(x_1, \cdots, x_n) \in G$, we regard $g_{\underline{\omega}}(x_1, \cdots, x_n)$ as a function of $\underline{\omega} \in [0,1]^n$,

$$g_{\underline{\omega}}(x_1, \cdots, x_n) = \sum_{(\gamma_1, \cdots, \gamma_n) \in \Lambda} a_{\gamma_1, \cdots, \gamma_n} \prod_{j=1}^{n} \big[|m_j(\gamma_j)| \gamma_j(x_j)\big] r_{\gamma_1}(\omega_1) \cdots r_{\gamma_n}(\omega_n)$$

then, by using multi-dimensional Khintchine's inequality, see e.g. (B.2), we deduce that

$$\int_{[0,1]^n} |g_{\underline{\omega}}(x_1, \cdots, x_n)|^p d\omega_1 \cdots d\omega_n \leqslant B^p p^{np/2} \Big(\sum_{\gamma \in \Lambda} |m(\gamma) a_\gamma|^2 \Big)^{p/2},$$

where $B > 0$ is an absolute constant. Therefore, the last inequality and (3.5) give the desired estimate with $C_m \leqslant B C_{m_1} \cdots C_{m_n}$. □

Since Proposition 3.2.2 characterises Sidon weights, it doesn't seem that the argument of Rudin can be adapted to the case where $m_j : \widehat{G_j} \to \mathbb{C}$ are just multipliers from $L \log^{1/2} L(G_j)$ to $L^2(G_j)$, as we will see that the class of all multipliers from $L \log^{1/2} L$ to L^2 is strictly larger than Sidon weights. Therefore, the converse of Proposition 3.2.3 is not true, even in the "one-dimensional" case. However, in the classical setting, namely in the unweighted case, the converse holds.

Proposition 3.2.4. *Let $n \geqslant 2$ be given. Let $G_1, \cdots, G_n$ be compact abelian groups and let $\Lambda_j \subset \widehat{G}_j$, $j \in \{1, \cdots, n\}$. Put $G = G_1 \times \cdots \times G_n$, $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$, and $S = \{j \in \{1, \cdots, n\} : \#\{\Lambda_j\} = \infty\}$.*

Assume that $S \neq \varnothing$. Then, the inequality

$$\Big(\sum_{\gamma \in \Lambda} |\widehat{f}(\gamma)|^2\Big)^{1/2} \leqslant A_\Lambda \int_G |f(x)| \log^{|S|/2}(1 + |f(x)|)dx + B_\Lambda, \tag{3.6}$$

where A_Λ and B_Λ are positive constants depending on Λ, holds if and only if, Λ_j is a Sidon set for each $j \in S$.

Proof. Suppose first that Λ_j is a Sidon set for each $j \in S$. If $S = \{1, \cdots, n\}$, then (3.6) coincides with (3.1). So, let us assume that $S \neq \{1, \cdots, n\}$. Without loss of generality, we may suppose that $S = \{1, \cdots, k\}$, $k < n$. That is, $\Lambda_1, \cdots, \Lambda_k$ are infinite countable sets, whereas the sets $\Lambda_{k+1}, \cdots, \Lambda_n$ are finite. Set $\Lambda_S = \Lambda_1 \times \cdots \times \Lambda_k$. By duality, it is enough to show that for every Λ-polynomial f one has

$$\|f\|_{L^p(G)} \lesssim p^{|S|/2} \|f\|_{L^2(G)}, \tag{3.7}$$

where the implied constant depends only on $\Lambda_1, \cdots, \Lambda_n$ and not on f. The main idea is to prove (3.7) first for the special case of $\Lambda_S \times \{\gamma'_{k+1}\} \times \cdots \times \{\gamma'_n\}$-polynomials, where $\{\gamma'_j\}$ is an arbitrary element of Λ_j, $j \in \{k+1, \cdots, n\}$. To this end, fix $\gamma'_j \in \Lambda_j$, $j \in \{k+1, \cdots, n\}$. If we consider a $\Lambda_S \times \{\gamma'_{k+1}\} \times \cdots \times \{\gamma'_n\}$-polynomial f, then, by using (3.1), one can easily check that (3.7) holds for f. Observe now that every Λ-polynomial can be written as a sum of at most $(\#\{\Lambda_{k+1}\}) \cdots (\#\{\Lambda_n\})$ Λ-polynomials of the special form studied in the previous step. Therefore, by using the triangle inequality one deduces that

$$\|f\|_{L^p(G)} \leqslant (\#\{\Lambda_{k+1}\}) \cdots (\#\{\Lambda_n\}) A_{\Lambda_S} p^{|S|/2} \|f\|_{L^2(G)},$$

where A_{Λ_S} is a constant that depends only on Λ_S.

To obtain the opposite direction, by Pisier's characterisation of Sidon sets, it suffices to prove that if f is a Λ_j-polynomial, then

$$\|f\|_{L^p(G_j)} \lesssim p^{1/2} \|f\|_{L^2(G_j)} \tag{3.8}$$

for all $p > 2$, $j \in S$. Towards this aim, take $p > 2$ and let $j \in S$ be fixed. Consider an arbitrary Λ_j-polynomial f. Without loss of generality, we may assume that $S \backslash \{j\} \neq \varnothing$. Note that if f_l are Λ_l-polynomials, $l \in \{1, \cdots, n\} \backslash \{j\}$, then the function F on G given by

$$F(x_1, \cdots, x_n) = f(x_j) \cdot \Big(\prod_{l \in \{1, \cdots, n\} \backslash \{j\}} f_l(x_l)\Big)$$

is a Λ-polynomial. We define the Λ_l-polynomials f_l as follows,

- if $l \in S\backslash\{j\}$, then choose f_l to be a Λ_l-polynomial satisfying

$$\|f_l\|_{L^p(G_l)} \gtrsim p^{1/2}\|f_l\|_{L^2(G_l)}.$$

  This is possible thanks to a construction due to Rudin [53, Theorem 3.4].

- If $l \notin S$, then put $f_l(x_l) = \gamma_l'(x_l)$ for some $\gamma_l' \in \Lambda_l$. In this case, for all $q > 0$, $\|f_l\|_{L^q(G_l)} = 1$.

Next, note that for each $q > 0$ one has

$$\|F\|_{L^q(G)} = \|f\|_{L^q(G_j)} \cdot \Big(\prod_{l \in S\backslash\{j\}} \|f_l\|_{L^q(G_l)}\Big) \tag{3.9}$$

Since F is a Λ-polynomial, it follows by hypothesis that

$$\|F\|_{L^p(G)} \lesssim p^{|S|/2}\|F\|_{L^2(G)}$$

and so, by (3.9) for $q = p$ and $q = 2$ one obtains

$$\|f\|_{L^p(G_j)} \cdot \Big(\prod_{l \in S\backslash\{j\}} \|f_l\|_{L^p(G_l)}\Big) \lesssim p^{|S|/2}\|f\|_{L^2(G_j)} \cdot \Big(\prod_{l \in S\backslash\{j\}} \|f_l\|_{L^2(G_l)}\Big).$$

By our construction

$$\prod_{l \in S\backslash\{j\}} \|f_l\|_{L^p(G_l)} \gtrsim p^{(|S|-1)/2} \cdot \prod_{l \in S\backslash\{j\}} \|f_l\|_{L^2(G_l)}$$

and so, it follows that

$$p^{(|S|-1)/2}\|f\|_{L^p(G_j)} \lesssim p^{|S|/2}\|f\|_{L^2(G_j)}$$

and hence, (3.8) holds. Therefore, by Pisier's characterisation of Sidon sets, Λ_j is a Sidon set and the proof is complete. □

3.3 A multiplier inclusion theorem

By Rudin's characterisation of the class of spectral sets satisfying classical Paley's inequality it follows that $\chi_\Lambda \in \mathcal{M}_{H^1(\mathbb{T}) \to L^2(\mathbb{T})}$ if and only if, Λ is a finite union of lacunary sequences. Since finite unions of lacunary sequences are Sidon sets, one deduces that $\chi_\Lambda \in \mathcal{M}_{H^1(\mathbb{T}) \to L^2(\mathbb{T})}$ implies that $\chi_\Lambda \in \mathcal{M}_{L \log^{1/2} L(\mathbb{T}) \to L^2(\mathbb{T})}$. Motivated by this observation, one can naturally ask whether the class $\mathcal{M}_{H^1(\mathbb{T}) \to L^2(\mathbb{T})}$ is contained[1] in the class $\mathcal{M}_{L \log^{1/2} L(\mathbb{T}) \to L^2(\mathbb{T})}$. Our main goal in this section is to show that this is indeed the case.

[1]Note that since $\|H(f)\|_{L^1(\mathbb{T})} \lesssim 1 + \int_{\mathbb{T}} |f| \log(1 + |f|)$, where H is the periodic Hilbert transform, one deduces that $L \log L(\mathbb{T}) \subset H^1(\mathbb{T})$ and hence, one trivially has $\mathcal{M}_{H^1(\mathbb{T}) \to L^2(\mathbb{T})} \subset \mathcal{M}_{L \log L(\mathbb{T}) \to L^2(\mathbb{T})}$.

Theorem 3.3.1. *The class of all multipliers from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$ is contained in the class of all multipliers from $L\log^{1/2}L(\mathbb{T})$ to $L^2(\mathbb{T})$, i.e.*

$$\mathcal{M}_{H^1(\mathbb{T})\to L^2(\mathbb{T})} \subset \mathcal{M}_{L\log^{1/2}L(\mathbb{T})\to L^2(\mathbb{T})} \tag{3.10}$$

and moreover, the inclusion is proper and it is sharp in the sense that the exponent $r = 1/2$ in $L\log^{1/2}L(\mathbb{T})$ cannot be improved.

Proof. Recall from subsection 2.2.3 that it follows by the work of Hardy and Littlewood [31] and the work of Duren and Shields [21] that $M = (m(n))_{n\in\mathbb{Z}}$ belongs to the class $\mathcal{M}_{H^1(\mathbb{T})\to L^2(\mathbb{T})}$ if and only if,

$$\sup_{N\in\mathbb{N}} \sum_{N\leqslant|n|\leqslant 2N} |m(n)|^2 < \infty. \tag{3.11}$$

To prove (3.10) the main idea is that one can rule out the multipliers in the class $\mathcal{M}_{H^1(\mathbb{T})\to L^2(\mathbb{T})}$ with "large" support in the sets of the form $I_K = \pm[K, 2K) \cap \mathbb{Z}$ (for example $m(n) = 1/\sqrt{|n|}$ for $n \neq 0$) and focus only on multipliers of the form $M = \chi_\Lambda$, where Λ is a subset of integers satisfying $\sup_{K\in\mathbb{N}} \#\{\pm[K, 2K) \cap \Lambda\} \lesssim 1$ which can be handled by classical Zygmund's inequality.

To be more specific, let $M = (m(n))_{n\in\mathbb{Z}}$ be a given multiplier from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$. We may assume without loss of generality that $\operatorname{supp}(M) \subset \mathbb{N}_0$. We need to show that for every $f \in L\log^{1/2}L(\mathbb{T})$ one has

$$\sum_{n\in\mathbb{N}_0} |m(n)\widehat{f}(n)|^2 < \infty.$$

For this, fix an arbitrary function $f \in L\log^{1/2}L(\mathbb{T})$ and notice that

$$\Big(\sum_{n\in\mathbb{N}_0} |m(n)\widehat{f}(n)|^2\Big)^{1/2} = \Big(\sum_{k\in\mathbb{N}_0} \sum_{2^k-1\leqslant n\leqslant 2^{k+1}-2} |m(n)\widehat{f}(n)|^2\Big)^{1/2}$$

is majorised by

$$\Big(\sum_{k\in\mathbb{N}_0} \max_{2^k-1\leqslant n\leqslant 2^{k+1}-2} |\widehat{f}(n)|^2 \sum_{2^k-1\leqslant n\leqslant 2^{k+1}-2} |m(n)|^2\Big)^{1/2}.$$

Since $M = (m(n))_{n\in\mathbb{Z}}$ is a multiplier from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$ we have by (3.11),

$$A_M = \sup_{k\in\mathbb{N}_0} \sum_{2^k-1\leqslant n\leqslant 2^{k+1}-2} |m(n)|^2 < \infty$$

and so,

$$\Big(\sum_{n\in\mathbb{N}_0} |m(n)\widehat{f}(n)|^2\Big)^{1/2} \leqslant A_M^{1/2}\Big(\sum_{k\in\mathbb{N}_0} \max_{2^k-1\leqslant n\leqslant 2^{k+1}-2} |\widehat{f}(n)|^2\Big)^{1/2}.$$

Hence, it suffices to prove that

$$\sum_{k\in\mathbb{N}_0} \max_{2^k-1\leqslant n\leqslant 2^{k+1}-2} |\widehat{f}(n)|^2 < \infty. \tag{3.12}$$

For $k \in \mathbb{N}_0$, denote the "dyadic" interval of integers $[2^k - 1, 2^{k+1} - 2] \cap \mathbb{N}_0$ by I_k. For each $k \in \mathbb{N}_0$ choose a $\lambda_k \in I_k$ such that

$$\max_{n\in I_k} |\widehat{f}(n)| = |\widehat{f}(\lambda_k)|.$$

In such a way we construct a sequence of positive integers $(\lambda_k)_{k\in\mathbb{N}_0}$, depending on f, such that

- $\lambda_k \in I_k$ for every $k \in \mathbb{N}_0$ and
- $\sum_{k\in\mathbb{N}_0} \max_{n\in I_k} |\widehat{f}(n)|^2 = \sum_{k\in\mathbb{N}_0} |\widehat{f}(\lambda_k)|^2$.

Therefore, it is enough to show that $\sum_{k\in\mathbb{N}_0} |\widehat{f}(\lambda_k)|^2 < \infty$. Notice that the first property listed above does not necessarily imply that $(\lambda_k)_{k\in\mathbb{N}_0}$ is lacunary and so one cannot make use of Zygmund's inequality directly. However, if we decompose $(\lambda_k)_{k\in\mathbb{N}_0} = \Lambda_1 \cup \Lambda_2$, where $\Lambda_1 = (\lambda_{2k})_{k\in\mathbb{N}_0}$ and $\Lambda_2 = (\lambda_{2k+1})_{k\in\mathbb{N}_0}$, then Λ_1 and Λ_2 are lacunary sequences with $2 \leqslant \rho_{\Lambda_i} \leqslant 16$, $i = 1, 2$. We thus deduce by (2.18) that

$$\sum_{k\in\mathbb{N}_0} |\widehat{f}(\lambda_k)|^2 = \sum_{n\in\Lambda_1} |\widehat{f}(n)|^2 + \sum_{n\in\Lambda_2} |\widehat{f}(n)|^2 < \infty$$

and this completes the proof of the inclusion (3.10).

To prove that the inclusion is proper, take a Sidon set Λ in $\mathbb{Z}$ which cannot be written as a finite union of lacunary sequences. Then, by Rudin's characterisation of all spectral sets satisfying Paley's inequality and Rudin's extension of Zygmund's inequality, $\chi_\Lambda \in \mathcal{M}_{L\log^{1/2} L(\mathbb{T})\to L^2(\mathbb{T})} \backslash \mathcal{M}_{H^1(\mathbb{T})\to L^2(\mathbb{T})}$.

We shall prove that the exponent of $r = 1/2$ in $L\log^{1/2} L(\mathbb{T})$ is sharp in the next subsection. □

3.3.1 Sharpness of Theorem 3.3.1

If Λ is a lacunary sequence of positive integers with ratio $\rho_\Lambda \geqslant 2$, then the Sidon constant S_Λ of Λ is independent of ρ_Λ, see, e.g., [28]. Also, it can be shown that the constants A_Λ and B_Λ in (2.18), actually, depend only on S_Λ and hence, if $\rho_\Lambda \geqslant 2$, the constants A_Λ and B_Λ can be taken to be independent of Λ. Therefore, the argument in the proof of (3.10) implies, in fact, that if $M = (m(n))_{n\in\mathbb{Z}}$ is a multiplier from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$, then

$$\Big(\sum_{n\in\mathbb{Z}} |m(n)\widehat{f}(n)|^2\Big)^{1/2} \leqslant C_M\big[1 + \int_{\mathbb{T}} |f| \log^{1/2}(1 + |f|)\big], \tag{3.13}$$

where $C_M > 0$ is a constant that depends only on $M = (m(n))_{n\in\mathbb{Z}}$.

To see that the multiplier inclusion (3.10) is sharp in the sense that the exponent $r = 1/2$ in $L\log^{1/2} L(\mathbb{T})$ cannot be improved, assume that there exists an $r \geqslant 0$ such that every multiplier from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$ is a multiplier from

$L\log^r L(\mathbb{T})$ to $L^2(\mathbb{T})$. We need to show that $r \geqslant 1/2$. For this, let N be a large positive integer to be chosen later. Let $V_{2^N} = 2K_{2^{N+1}+1} - K_{2^N}$ be the de la Vallée Poussin kernel of order 2^N, where K_n denotes the Fejér kernel of order n. Since for every $n \in \mathbb{N}$ one has $\|K_n\|_{L^1(\mathbb{T})} = 1$ and $\|K_n\|_{L^\infty(\mathbb{T})} \lesssim n$, we obtain

$$\int_{\mathbb{T}} |V_{2^N}(x)| \log^r(1 + |V_{2^N}(x)|)dx \lesssim N^r.$$

Take $M = (m(n))_{n\in\mathbb{Z}}$ to be $m(n) = 1/\sqrt{|n|}$ for $n \neq 0$ and $m(0) = 0$. Then M satisfies (3.11) and so, it is a multiplier from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$. Hence, by (3.13), we necessarily have that

$$\Big(\sum_{n\in\mathbb{Z}} |m(n)\widehat{V_{2^N}}(n)|^2\Big)^{1/2} \lesssim N^r.$$

Since $\widehat{V_{2^N}}(n) = 1$ for each $|n| \leqslant 2^N + 1$, we have

$$\Big(\sum_{n\in\mathbb{Z}} |m(n)\widehat{V_{2^N}}(n)|^2\Big)^{1/2} \geqslant \Big(\sum_{1\leqslant n\leqslant 2^N} \frac{1}{n}\Big)^{1/2} \sim N^{1/2}.$$

Therefore, if we choose N to be large enough, we deduce that we must have $r \geqslant 1/2$.

As a corollary of Theorem 3.3.1, we obtain the following inequality.

Corollary 3.3.2. *There are positive constants A and B such that*

$$\Big(\sum_{n\neq 0} \frac{|\widehat{f}(n)|^2}{|n|}\Big)^{1/2} \leqslant A\int_{\mathbb{T}} |f(x)| \log^{1/2}(1 + |f(x)|)dx + B$$

and the exponent $r = 1/2$ in $L\log^{1/2} L(\mathbb{T})$ cannot be improved.

3.3.2 An example of a multiplier from $L\log^{1/2} L(\mathbb{T})$ to $L^2(\mathbb{T})$ which is not a Sidon weight

As in the previous subsection, consider the bounded sequence $M = (m(n))_{n\in\mathbb{Z}}$ given by $m(n) = 1/\sqrt{|n|}$ for $n \neq 0$ and $m(0) = 0$. If we take $0 < \gamma < 1$ and $c \in ((\gamma + 1)/2, 1]$, then the series

$$\sum_{n\geqslant 2} \frac{e^{i2\pi n(\log n)^\gamma}}{n^{1/2}(\log n)^c} e^{i2\pi nx}$$

converges uniformly to some $f \in C(\mathbb{T})$, see [32]. However, since $0 < c \leqslant 1$,

$$\sum_{n\in\mathbb{Z}} |m(n)\widehat{f}(n)| = \sum_{n\geqslant 2} \frac{1}{n(\log n)^c} = \infty$$

and hence, $M = (m(n))_{n\in\mathbb{Z}}$ cannot be a Sidon weight. See also [44] and [45].

3.4 Variants of Zygmund's inequality on the real line

Our objective in this section is to prove a real-line analogue of the multiplier inclusion theorem presented in the previous section. In order to state our main result, we need to revisit Paley's inequality for functions defined on $\mathbb{R}$ first. Then, by using a result of Tao and Wright on a Littlewood-Paley inequality for compactly supported functions in $L\log^{1/2} L$ with mean zero, we show that essentially all non-negative measures satisfying Paley's inequality on $\mathbb{R}$ also satisfy a version of Zygmund's inequality for functions supported on compact sets in the real line.

3.4.1 Variants of Paley's inequality on $\mathbb{R}$

To formulate our main result on a real-line version of Zygmund's inequality, we first examine variants of Paley's inequality on $\mathbb{R}$. Characterisations of the classes of multipliers from $H^p(\mathbb{R})$ to $L^q(\mathbb{R})$ for $0 < p \leqslant q < \infty$ are well-known, see [40]. However, as we will see in the next subsection, it is more natural to state our variant of Zygmund's inequality on $\mathbb{R}$ in terms of measures. Hence, in this subsection, we also study versions of Paley's inequality with respect to non-negative measures on $\mathbb{R}$.

Definition 3.4.1 (Paley measures). *Let $\mathcal{I}$ denote the set of all dyadic intervals in $\mathbb{R}$ of the form $\pm[2^k, 2^{k+1})$ for $k \in \mathbb{Z}$.*

A non-negative measure μ on the real line is said to be a Paley measure if

$$\sup_{I\in\mathcal{I}} \mu(I) < \infty.$$

Proposition 3.4.2. *A non-negative measure μ on $\mathbb{R}$ satisfies*

$$\|\widehat{f}\|_{L^2(d\mu)} \leqslant C_\mu \|f\|_{H^1(\mathbb{R})} \tag{3.14}$$

if and only if, μ is a Paley measure.

Proof. Assume first that μ is a Paley measure. To prove that μ satisfies (3.14), consider the set $I_k = (-2^{k+1}, -2^k] \cup [2^k, 2^{k+1})$ for $k \in \mathbb{Z}$ and write

$$\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\mu(\xi) = \sum_{k\in\mathbb{Z}} \int_{I_k} |\widehat{f}(\xi)|^2 d\mu(\xi).$$

Note that for every $\xi \in I_k$ one has

$$\begin{aligned} |\widehat{f}(\xi)| \lesssim |\eta(2^{-k+1}\xi)\widehat{f}(\xi)| + |\eta(2^{-k}\xi)\widehat{f}(\xi)| &\leqslant \|(\widetilde{\Delta}_{k-1}(f))\widehat{\ }\|_{L^\infty(\mathbb{R})} + \|(\widetilde{\Delta}_k(f))\widehat{\ }\|_{L^\infty(\mathbb{R})} \\ &\leqslant \|\widetilde{\Delta}_{k-1}(f)\|_{L^1(\mathbb{R})} + \|\widetilde{\Delta}_k(f)\|_{L^1(\mathbb{R})}, \end{aligned}$$

where η and $\widetilde{\Delta}_k$ are as in subsection 2.2.1. Therefore,

$$\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\mu(\xi) \lesssim \sum_{k\in\mathbb{Z}} \mu(I_k)\big(\|\widetilde{\Delta}_{k-1}(f)\|^2_{L^1(\mathbb{R})} + \|\widetilde{\Delta}_k(f)\|^2_{L^1(\mathbb{R})}\big).$$

Hence, by using our assumption that μ is a Paley measure, it follows that

$$\Big(\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\mu(\xi)\Big)^{1/2} \lesssim \big[\sup_{I\in\mathcal{I}} \mu(I)\big]^{1/2}\Big(\sum_{k\in\mathbb{Z}} \|\widetilde{\Delta}_k(f)\|^2_{L^1(\mathbb{R})}\Big)^{1/2}.$$

By Minkowski's integral inequality and the square function characterisation of $H^1(\mathbb{R})$, we deduce that

$$\Big(\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\mu(\xi)\Big)^{1/2} \lesssim_{\mu} \|f\|_{H^1(\mathbb{R})},$$

as desired. An analogous argument was used in the proof of [43, Theorem 1].

For the opposite direction, we shall adapt a construction of Rudin [52] to the euclidean setting. More precisely, suppose that μ is a non-negative measure that is not a Paley measure, namely

$$\sup_{k\in\mathbb{Z}} \mu(I_k) = \infty,$$

where I_k is as above. In such a case, either there exists an increasing subsequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that $\mu(I_{k_j}) \to \infty$ or there exists a decreasing subsequence $(k_j)_{j\in\mathbb{N}}$ of negative integers such that $\mu(I_{k_j}) \to \infty$. Without loss of generality, we may assume that we have an increasing subsequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ with $\mu(I_{k_j}) \to \infty$, and passing to a further subsequence if necessary, we may assume that $k_{j+1} > 5k_j$ and $\mu(I_{k_j}) \geqslant j^4$. Consider the function

$$f(x) = \sum_{j\in\mathbb{N}} \frac{1}{j^2}\widecheck{\eta_{k_j}}(x),$$

where $\eta_k(\xi) = \eta(2^{-k}\xi)$, η being as above. Since $\|\widecheck{\eta_k}\|_{H^1(\mathbb{R})} \lesssim 1$, where the implied constant does not depend on k, we see that $f \in H^1(\mathbb{R})$. For every $j \in \mathbb{N}$, we have

$$\int_{I_{k_j}} |\widehat{f}(\xi)|^2 d\mu(\xi) \gtrsim j^{-4}\mu(I_{k_j}) \geqslant 1$$

and therefore $\|\widehat{f}\|_{L^2(d\mu)} = \infty$, completing the proof of the proposition. □

Remark 3.4.3. *Since every function $m \in L^\infty(\mathbb{R})$ induces an absolutely continuous, non-negative measure μ on $\mathbb{R}^n$ given by $d\mu(\xi) = |m(\xi)|^2 d\xi$, one deduces that a function $m \in L^\infty(\mathbb{R})$ is a multiplier from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$ if and only if, $\sup_{I\in\mathcal{I}} \int_I |m(\xi)|^2 d\xi < \infty$, where $\mathcal{I}$ is as in Definition* 3.4.1.

We thus recover [40, Theorem A] for the case where $p = 1$ and $q = 2$. Moreover, our method is different than the one used in [40].

We remark that the argument presented above can be adapted to the multiparameter case in a straightforward way. We thus obtain the euclidean analogue of D. Oberlin's theorem [43, Theorem 1] stated in the introduction.

Proposition 3.4.4. *A non-negative measure μ on $\mathbb{R}^n$ satisfies*

$$\|\widehat{f}\|_{L^2(d\mu)} \leqslant C_{\mu,n}\|f\|_{H^1_{\text{prod}}(\mathbb{R}^n)}$$

if and only if, $\sup_{I_1,\cdots,I_n\in\mathcal{I}} \mu(I_1\times\cdots\times I_n)<\infty$*, where* $\mathcal{I}$ *is as in definition* 3.4.1.

3.4.2 A real-line version of Zygmund's inequality

In the previous subsection we obtained a real-line version of Paley's inequality based on the square function characterisation of $H^1(\mathbb{R})$. A similar argument can be used for compactly supported functions in $L\log^{1/2}L$ with zero mean thanks to the following deep result of Tao and Wright [62, Proposition 4.1].

Theorem 3.4.5 (Tao and Wright)**.** *Let* $K\subset\mathbb{R}$ *be a compact set. Let* f *be a function in* $L\log^{1/2}L(K)$ *with zero integral.*

Then for every $k\in\mathbb{Z}$ *there exists a non-negative function* F_k *such that*

$$|\widetilde{\Delta}_k(f)(x)|\lesssim F_k*\phi_k(x)$$

for all $x\in\mathbb{R}$ *and*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|F_k|^2\Big)^{1/2}\Big\|_{L^1(\mathbb{R})}\leqslant A_K\|f\|_{L\log^{1/2}L(K)}.$$

Here $\phi_k(x)=2^k(1+2^{2k}|x|^2)^{-3/4}$.

We are now ready to establish a real-line analogue of Theorem 3.3.1.

Theorem 3.4.6 (Weighted Zygmund's inequality on $\mathbb{R}$)**.** *Let* μ *be a Paley measure such that* $\mu([-\delta,\delta])=0$ *for some* $\delta>0$.

For every compact set $K\subset\mathbb{R}$ *there is a constant* $C=C(\mu,K)>0$ *such that whenever* $\operatorname{supp}(f)\subset K$ *one has*

$$\|\widehat{f}\|_{L^2(d\mu)}\leqslant C(\mu,K)\Big[\int_K|f(x)|\log^{1/2}(1+|f(x)|)dx+1\Big]. \tag{3.15}$$

Proof. Let $K\subset\mathbb{R}$ be a fixed compact set and let f be a function supported in K. Assume first that $\int_K f=0$.

The proof of (3.15) proceeds in the same way as the proof of weighted Paley's inequality (3.14). By the aforementioned result of Tao and Wright, for each $k\in\mathbb{Z}$ there is a function F_k such that $|\widetilde{\Delta}_k(f)|\leqslant F_k*\phi_k$ and

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|F_k|^2\Big)^{1/2}\Big\|_{L^1(\mathbb{R})}\lesssim_K 1+\int_K|f(x)|\log^{1/2}(1+|f(x)|)dx.$$

Since $\|\phi_k\|_{L^1(\mathbb{R})}\lesssim 1$, it follows that $\|\widetilde{\Delta}_k(f)\|_{L^1(\mathbb{R})}\lesssim\|F_k\|_{L^1(\mathbb{R})}$. Hence,

$$\begin{aligned}\|\widehat{f}\|_{L^2(d\mu)}=\Big(\sum_{k\in\mathbb{Z}}\int_{I_k}|\widehat{f}(\xi)|^2d\mu(\xi)\Big)^{1/2}&\lesssim\big[\sup_{I\in\mathcal{I}}\mu(I)\big]^{1/2}\Big(\sum_{k\in\mathbb{Z}}\|F_k\|^2_{L^1(\mathbb{R})}\Big)^{1/2}\\&\leqslant\big[\sup_{I\in\mathcal{I}}\mu(I)\big]^{1/2}\Big\|\Big(\sum_{k\in\mathbb{Z}}|F_k|^2\Big)^{1/2}\Big\|_{L^1(\mathbb{R})}\\&\lesssim_{\mu,K}1+\int_K|f(x)|\log^{1/2}(1+|f(x)|)dx,\end{aligned}$$

where the sets I_k are is in the proof of Proposition 3.4.2 and $\mathcal{I}$ is as in definition 3.4.1.

We now show why we can remove the condition that f has mean zero when the measure μ vanishes on a neighbourhood of 0. For our function f supported in K, we may assume, without loss of generality, that

$$\int_K |f(x)| \log^{1/2}(1+|f(x)|)dx \leqslant 1.$$

Hence, if we set $I = \int_K f$, then $|I| \lesssim 1$. Consider

$$g(x) = f(x) - I\psi(x),$$

where ψ is a smooth function, supported in K and such that $\int \psi = 1$. Then g is supported in K, has mean zero and

$$\int_K |g(x)| \log^{1/2}(1+|g(x)|)dx \lesssim 1.$$

Hence, (3.15) holds for g, as μ is a Paley measure. But

$$\|\widehat{f}\|_{L^2(d\mu)} \leqslant \|\widehat{g}\|_{L^2(d\mu)} + |I|\|\widehat{\psi}\|_{L^2(d\mu)}$$

and if μ vanishes in a neighbourhood of the origin, we have

$$\int_{\mathbb{R}} |\widehat{\psi}(\xi)|^2 d\mu(\xi) = \int_{|\xi|>\delta} |\widehat{\psi}(\xi)|^2 d\mu(\xi) \leqslant [\sup_{\xi\in\mathbb{R}} |\xi\widehat{\psi}(\xi)|^2] \sum_{k\geqslant -M} 2^{-2k}\mu(I_k) \lesssim 1,$$

where the sets I_k are as in the proof of Proposition 3.4.2. Note that M depends on $\delta > 0$ and the implicit constant in the last inequality also depends on M. Hence, the implicit constant depends on δ, i.e. on μ. Thus, (3.15) also holds for f. □

Remark 3.4.7. *Compared to weighted Paley's inequality on $\mathbb{R}$, in the previous theorem we imposed the extra hypothesis that μ vanishes on a neighbourhood of 0. To see that this condition is necessary, consider the Paley measure $d\mu(\xi) = |\xi|^{-1}d\xi$ and take f to be in the class $L\log^{1/2} L(K)$ with $\widehat{f}(0) = \int_K f \neq 0$, for some compact set $K \subset \mathbb{R}$. Since $\widehat{f}$ is continuous,*

$$\|\widehat{f}\|^2_{L^2(d\mu)} = \int_{\mathbb{R}} \frac{|\widehat{f}(\xi)|^2}{|\xi|} d\xi = \infty.$$

Note that for every $g \in H^1(\mathbb{R})$, one automatically has $\int_{\mathbb{R}} g = 0$.

Note that if Λ is a Sidon set in $\mathbb{N}$ that cannot be written as a finite union of lacunary sequences, then it follows by Rudin's extension of classical Zygmund's inequality that the discrete measure $\mu = \sum_{n\in\Lambda} \delta_n$ satisfies (3.15), but it is not a Paley measure. Here, δ_n denotes the dirac measure supported on $\{n\}$. Therefore, there are non-negative measures satisfying weighted Zygmund's inequality (3.15) that are not Paley measures. It is an interesting problem to characterise the class of all non-negative measures μ on $\mathbb{R}$ satisfying (3.15).

3.4.3 A second proof of Theorem 3.3.1

We now show how by adapting the argument in the proof of Theorem 3.4.6 to the periodic setting, one can give an alternative proof to Theorem 3.3.1. To see this, let $M = (m(n))_{n\in\mathbb{Z}}$ be a multiplier from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$. We shall prove that $M \in \mathcal{M}_{L\log^{1/2}L(\mathbb{T})\to L^2(\mathbb{T})}$

Assume first that $f \in L\log^{1/2}L(\mathbb{T})$ has mean zero. Using the result of Tao and Wright as in the proof of Theorem 3.4.6, we deduce that there is a constant $C_M > 0$, depending only on M, such that

$$\Big(\sum_{n\in\mathbb{Z}} |m(n)\widehat{f}(n)|^2\Big)^{1/2} \leqslant C_M\Big[1 + \int_{\mathbb{T}} |f(x)|\log^{1/2}(1+|f(x)|)dx\Big].$$

Alternatively, one can directly apply Theorem 3.4.6 to the discrete Paley measure measure $\sum_{n\in\mathbb{Z}} |m(n)|^2\delta_n$.

Suppose now that $\widehat{f}(0) = \int_{\mathbb{T}} f \neq 0$. Hence, from the previous step, $g = f - \widehat{f}(0)$ satisfies

$$\Big(\sum_{n\in\mathbb{Z}} |m(n)\widehat{g}(n)|^2\Big)^{1/2} \leqslant C_M\Big[1 + \int_{\mathbb{T}} |g(x)|\log^{1/2}(1+|g(x)|)dx\Big].$$

Moreover, $\int_{\mathbb{T}} |g(x)|\log^{1/2}(1+|g(x)|)dx \lesssim 1 + \int_{\mathbb{T}} |f(x)|\log^{1/2}(1+|f(x)|)dx$ and $\widehat{g}(n) = \widehat{f}(n)$ for all non-zero integers n. Hence,

$$\begin{aligned}\Big(\sum_{n\in\mathbb{Z}} |m(n)\widehat{f}(n)|^2\Big)^{1/2} &\leqslant \Big(\sum_{n\neq 0} |m(n)\widehat{g}(n)|^2\Big)^{1/2} + |m(0)\widehat{f}(0)| \\ &\lesssim_M 1 + \int_{\mathbb{T}} |f(x)|\log^{1/2}(1+|f(x)|)dx + \|m\|_{\ell^\infty(\mathbb{Z})}\|f\|_{L^1(\mathbb{T})} \\ &\lesssim_M 1 + \int_{\mathbb{T}} |f(x)|\log^{1/2}(1+|f(x)|)dx.\end{aligned}$$

Therefore, $M = (m(n))_{n\in\mathbb{Z}}$ is a multiplier from $L\log^{1/2}L(\mathbb{T})$ to $L^2(\mathbb{T})$.

3.5 Higher-dimensional versions of Zygmund's inequality by using a theorem of Bonami

In the final section of this chapter we obtain further extensions of Zygmund's inequality for spectral sets in $\mathbb{Z}^n$ of product-type by using a classical theorem of Bonami.

Let $n \geqslant 1$ be a fixed integer. It follows by duality that (3.1), in the case where $G_i = \mathbb{T}$ $(i = 1, \cdots, n)$, is equivalent to the fact that for every $\Lambda_1 \times \cdots \times \Lambda_n$-

polynomial f one has

$$\|f\|_{L^p(\mathbb{T}^n)} \lesssim_{\Lambda_1,\cdots,\Lambda_n} p^{n/2}\|f\|_{L^2(\mathbb{T}^n)} \tag{3.16}$$

for all $p > 2$, where the implied constant depends only on $\Lambda_1, \cdots, \Lambda_n$ and not on f, p. In what follows we shall focus on the case where Λ_i is a lacunary sequence in $\mathbb{N}$ with ratio $\rho_{\Lambda_i} \geqslant 2$, $i = 1, \cdots, n$. Recall that, in particular, the classical inequality of Zygmund (2.18) is equivalent to the fact that every lacunary sequence in $\mathbb{Z}$ is a $\Lambda(p)$ set for all $p > 2$ with $\Lambda(p)$ constant growing like $p^{1/2}$ as $p \to \infty$.

Consider the case $n = 2$ in (3.16) first. In order to prove (3.16) (for $n = 2$), a plausible idea is to try to iterate the one-dimensional result. To be more specific, to prove (3.16) in the case of the two-torus ($n = 2$), consider a $\Lambda_1 \times \Lambda_2$-polynomial f and write

$$f(x, y) = \sum_{m\in\Lambda_1} f_y(m)e^{i2\pi mx},$$

where $f_y(m) = \sum_{n\in\Lambda_2} \widehat{f}(m, n)e^{i2\pi ny}$. Hence, fixing $y \in \mathbb{T}$, we may regard $f(x, y)$ as a Λ_1-polynomial. By using (3.16) for $n = 1$ (i.e. classical Zygmund's inequality) one deduces that for all $p > 2$

$$\int_{\mathbb{T}} \Big| \sum_{m\in\Lambda_1} f_y(m)e^{i2\pi mx}\Big|^p dx \leqslant A^p_{\Lambda_1} p^{p/2}\Big(\sum_{m\in\Lambda_1} |f_y(m)|^2\Big)^{p/2} \tag{3.17}$$

for each fixed $y \in \mathbb{T}$. Observe now that

$$\sum_{m\in\Lambda_1} |f_y(m)|^2 = \sum_{n,n'\in\Lambda_2} E_{n,n'}e^{i2\pi(n-n')y},$$

where $E_{n,n'} = \sum_{m\in\Lambda_1} \widehat{f}(m, n)\overline{\widehat{f}(m, n')}$. Therefore, by integrating both sides of (3.17) with respect to $y \in \mathbb{T}$, one deduces

$$\|f\|^p_{L^p(\mathbb{T}^2)} \leqslant A^p_{\Lambda_1} p^{p/2} \int_{\mathbb{T}} \Big| \sum_{n,n'\in\Lambda_2} E_{n,n'}e^{i2\pi(n-n')y}\Big|^{p/2} dy.$$

Note that in the right-hand side of the last inequality we have a trigonometric polynomial on $\mathbb{T}$ frequency supported in the set $\{n - n' : n, n' \in \Lambda_2\}$. As Zygmund's inequality handles only lacunary sequences, to obtain (3.16) for $n = 2$, one cannot just iterate Zygmund's inequality twice. However, one can surpass this difficulty by using the following classical result of Bonami [10, Corollaire 4].

Theorem 3.5.1 (Bonami). *Let $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ be a lacunary sequence of positive integers with ratio $\rho_\Lambda \geqslant 2$. For some $k \geqslant 1$, consider the sumset*

$$\Lambda^{(k)} := \{\pm\lambda_{n_1} \pm \cdots \pm \lambda_{n_k} : n_1 > \cdots > n_k\}.$$

Then, $\Lambda^{(k)}$ is a $\Lambda(p)$ set for every $p > 2$ with $\Lambda(p)$ constant growing like $p^{k/2}$ and in particular, there exists a constant $A(k)$ such that for every $\Lambda^{(k)}$-polynomial f on $\mathbb{T}$ one has $\|f\|_{L^p(\mathbb{T})} \leqslant A(k)p^{k/2}\|f\|_{L^2(\mathbb{T})}$ for all $p > 2$.

In fact, the $\Lambda(p)$ constant of $\Lambda^{(k)}$ grows exactly like $p^{k/2}$ as $p \to \infty$, see [10, Théorème 5].

To see how we can employ Bonami's result to our problem of establishing (3.16) for $n = 2$, write

$$\sum_{n,n' \in \Lambda_1} E_{n,n'} e^{2\pi i (n-n')y} = \Big(\sum_{n=n'} + \sum_{n<n'} + \sum_{n>n'} \Big) E_{n,n'} e^{2\pi i (n-n')y}$$

and note that the diagonal term satisfies

$$\sum_{n=n'} E_{n,n'} e^{2\pi i (n-n')y} = \sum_{(m,n) \in \Lambda_1 \times \Lambda_2} |\widehat{f}(m,n)|^2 = \|f\|^2_{L^2(\mathbb{T}^2)}.$$

Since $p > 2$, the function $x \mapsto x^{p/2}$ ($x \geqslant 0$) is convex and hence,

$$\Big| \sum_{n,n' \in \Lambda_1} E_{n,n'} e^{2\pi i (n-n')y} \Big|^{p/2} \leqslant$$
$$\frac{3^{p/2}}{3} \Big(\|f\|^p_{L^2(\mathbb{T}^2)} + \Big| \sum_{n<n'} E_{n,n'} e^{2\pi i (n-n')y} \Big|^{p/2} + \Big| \sum_{n>n'} E_{n,n'} e^{2\pi i (n-n')y} \Big|^{p/2} \Big).$$

Thus,

$$\|f\|^p_{L^p(\mathbb{T}^2)} \leqslant A^p_{\Lambda_1} p^{p/2} \int_{\mathbb{T}} \Big| \sum_{n,n' \in \Lambda_2} E_{n,n'} e^{2\pi i (n-n')y} \Big|^{p/2} dy \leqslant$$
$$(3A_{\Lambda_1})^p p^{p/2} \Big(\|f\|^p_{L^2(\mathbb{T}^2)} + \int_{\mathbb{T}} \Big| \sum_{n<n'} E_{n,n'} e^{2\pi i (n-n')y} \Big|^{p/2} dy + \int_{\mathbb{T}} \Big| \sum_{n>n'} E_{n,n'} e^{2\pi i (n-n')y} \Big|^{p/2} dy \Big)$$

and so, by using Bonami's result to bound the off-diagonal terms, the last quantity is majorised by

$$B^p \Big(p^{p/2} \|f\|^p_{L^2(\mathbb{T}^2)} + p^p \big(\sum_{n<n'} |E_{n,n'}|^2 \big)^{p/4} + p^p \big(\sum_{n>n'} |E_{n,n'}|^2 \big)^{p/4} \Big),$$

where B depends only on Λ_1, Λ_2. By using the Cauchy-Schwarz inequality, one gets

$$\sum_{n,n' \in \Lambda_2} |E_{n,n'}|^2 \leqslant \|f\|^4_{L^2(\mathbb{T}^2)}$$

and hence, $\|f\|_{L^p(\mathbb{T}^2)} \leqslant B(p^{p/2} + 2p^p)^{1/p} \|f\|_{L^2(\mathbb{T}^2)} \leqslant 3Bp\|f\|_{L^2(\mathbb{T}^2)}$. Therefore, the proof of (3.16) for $n = 2$ is complete.

As one can easily observe, in fact the above method can be used to obtain variants of Zygmund's inequality for spectral sets of the form $\Lambda_1^{(k_1)} \times \cdots \times \Lambda_n^{(k_n)} \subset \mathbb{Z}^n$, where $\Lambda_j^{(k_j)} = \big\{ \pm \lambda_{j,n_1} \pm \cdots \pm \lambda_{j,n_{k_j}} : n_1 < \cdots < n_{k_j} \big\}$ and $\Lambda_j = (\lambda_{j,n})_{n \in \mathbb{N}}$ is a lacunary sequence with ratio at least 2, for all $j = 1, \cdots, n$. In other words, using the above method one obtains the following higher-dimensional extension of Bonami's result.

Proposition 3.5.2. *Let $\Lambda_j = (\lambda_{j,m})_{m \in \mathbb{N}}$ be lacunary sequences with $\rho_{\Lambda_j} \geqslant 2$ for*

$j = 1, \cdots, n$. *Let* $(k_1, \cdots, k_n)$ *be a given* n*-tuple of positive integers. Then, there are positive constants* $A_{\Lambda_1^{(k_1)}, \cdots, \Lambda_n^{(k_n)}}$ *and* $B_{\Lambda_1^{(k_1)}, \cdots, \Lambda_n^{(k_n)}}$ *such that*

$$\Big(\sum_{(m_1, \cdots, m_n) \in \Lambda_1^{(k_1)} \times \cdots \times \Lambda_n^{(k_n)}} |\widehat{f}(m_1, \cdots, m_n)|^2 \Big)^{1/2} \leqslant$$

$$A_{\Lambda_1^{(k_1)}, \cdots, \Lambda_n^{(k_n)}} \int_{\mathbb{T}^n} |f(\underline{x})| \log^{K_n/2}(1 + |f(\underline{x})|) d\underline{x} + B_{\Lambda_1^{(k_1)}, \cdots, \Lambda_n^{(k_n)}},$$

where $K_n = k_1 + \cdots + k_n$.

To prove Proposition 3.5.2, the main idea is to induct on the dimension $n \in \mathbb{N}$. To use induction, one needs the following lemma.

Lemma 3.5.3. *Let* $n \geqslant 1$ *be a given integer. Let* E *be a subset of* $\mathbb{Z}^n$*, such that there are constants* $C_E > 0$ *and* $N_E \in \mathbb{N}$ *so that for every* E*-polynomial* g *one has* $\|g\|_{L^p(\mathbb{T}^n)} \leqslant C_E p^{N_E/2} \|g\|_{L^2(\mathbb{T}^n)}$ *for every* $p > 2$.

Then, for every lacunary sequence $\Lambda = (\lambda_n)_{n \in \mathbb{N}}$ *with ratio at least* 2 *and for each* $k \in \mathbb{N}$*, there are positive constants* $A_{E,\Lambda,k}$ *and* $B_{E,\Lambda,k}$ *such that*

$$\Big(\sum_{(m,n) \in E \times \Lambda^{(k)}} |\widehat{f}(m,n)|^2 \Big)^{1/2} \leqslant A_{E,\Lambda,k} \int_{\mathbb{T}^{n+1}} |f(\underline{x})| \log^{(N_E+k)/2}(1+|f(\underline{x})|) d\underline{x} + B_{E,\Lambda,k},$$

where $\Lambda^{(k)} = \big\{ \pm \lambda_{n_1} \pm \cdots \pm \lambda_{n_k} : n_1 < \cdots < n_k \big\}$.

Proof. The proof of the lemma is a variant of the argument given above. More precisely, take an $E \times \Lambda^{(k)}$-polynomial g over $\mathbb{T}^{n+1}$ and for $x \in \mathbb{T}^n$, $y \in \mathbb{T}$ write

$$g(x,y) = \sum_{m \in E} \Big(\sum_{n \in \Lambda^{(k)}} \widehat{g}(m,n) e^{2\pi i n y} \Big) e^{2\pi i (m \cdot x)},$$

where $m \cdot x$ denotes the dot product of m and x, i.e. $m \cdot x = m_1 x_1 + \cdots + m_n x_n$, in the case where $n > 1$. Otherwise, $m \cdot x$ is just the scalar multiplication of $m \in \mathbb{Z}$ with $x \in \mathbb{T}$. For $p > 2$, using our hypothesis, one has

$$\|g\|^p_{L^p(\mathbb{T}^{n+1})} \leqslant C_E^p p^{pN_E/2} \int_{\mathbb{T}} \Big(\sum_{m \in E} \Big| \sum_{n \in \Lambda^{(k)}} \widehat{g}(m,n) e^{2\pi i n y} \Big|^2 \Big)^{p/2} dy.$$

We write

$$\sum_{m \in E} \Big| \sum_{n \in \Lambda^{(k)}} \widehat{g}(m,n) e^{2\pi i n y} \Big|^2 = \sum_{n, n' \in \Lambda^{(k)}} \Big(\sum_{m \in E} \widehat{g}(m,n) \overline{\widehat{g}(m,n')} \Big) e^{2\pi i (n-n') y}$$

and then split the off-diagonal part of the last sum into terms of the form

$$\sum_{j_1 < \cdots < j_{2l}} \Big(\sum_{m \in E} \widehat{g}(m, \pm\lambda_{j_1} \pm \cdots \pm \lambda_{j_l}) \overline{\widehat{g}(m, \pm\lambda_{j_{l+1}} \pm \cdots \pm \lambda_{j_{2l}})} \Big) e^{2\pi i (\pm\lambda_{j_1} \pm \cdots \pm \lambda_{j_{2l}}) y}$$

where $l \in \{1, \cdots, k\}$, in order to use Bonami's theorem. The diagonal term is treated as before. The number of the above subsums depends only on k and not on g, Λ, p and so, exactly as above, one shows that

$$\|g\|^p_{L^p(\mathbb{T}^{n+1})} \leqslant D^p_{E,\Lambda,k} p^{p(N_E+k)/2} \Big(\sum_{n,n' \in \Lambda^{(k)}} |E_{n,n'}|^2 \Big)^{p/4},$$

where $E_{n,n'} = \sum_{m \in E} \widehat{g}(m,n)\overline{\widehat{g}(m,n')}$. By using the Cauchy-Schwarz inequality, the desired estimate follows. □

3.5.1 Proof of Proposition 3.5.2

To prove Proposition 3.5.2, assume that we are given a sequence of lacunary sequences $(\Lambda_j)_{j\in\mathbb{N}}$ with $\rho_{\Lambda_j} \geqslant 2$ and a sequence of positive integers $(k_j)_{j\in\mathbb{N}}$. For each $j \in \mathbb{N}$ form the sumsets $\Lambda_j^{(k_j)}$. We shall use induction on the dimension $n \in \mathbb{N}$. Note that the one-dimensional case is Bonami's theorem. Suppose now that for $n \in \mathbb{N}$, Proposition 3.5.2 holds. To obtain the $(n+1)$-dimensional case, we set $E = \Lambda_1^{(k_1)} \times \cdots \times \Lambda_n^{(k_n)}$ and $\Lambda = \Lambda_{n+1}$. By the inductive step, it follows that E satisfies the assumptions of Lemma 3.5.3 for $N_E = k_1 + \cdots + k_n$. Therefore, by using that lemma, the $(n+1)$-dimensional case follows and hence, the proof of Proposition 3.5.2 is complete.

3.5.2 Some further remarks

Since $L\log L(\mathbb{T}) \subset L\log^{1/2} L(\mathbb{T})$, it automatically follows that $\mathcal{M}_{L\log^{1/2} L(\mathbb{T}) \to L^2(\mathbb{T})}$ is a subclass of $\mathcal{M}_{L\log L(\mathbb{T})\to L^2(\mathbb{T})}$. The aforementioned theorem of Bonami gives examples of spectral sets $\Lambda \subset \mathbb{Z}$ such that $\chi_\Lambda \in \mathcal{M}_{L\log L(\mathbb{T})\to L^2(\mathbb{T})} \backslash \mathcal{M}_{L\log^{1/2} L(\mathbb{T})\to L^2(\mathbb{T})}$. Indeed, if $\Lambda = \{\pm 3^j \pm 3^k : j < k\}$, then Λ is a $\Lambda(p)$ set whose constant grows like p as $p \to \infty$, see[2] [10, Théorème 5]. Hence, by Pisier's characterisation of Sidon sets, Λ is not a Sidon set and so, $\chi_\Lambda \notin \mathcal{M}_{L\log^{1/2} L(\mathbb{T})\to L^2(\mathbb{T})}$. Consequently, we may write

$$\mathcal{M}_{H^1(\mathbb{T})\to L^2(\mathbb{T})} \subsetneq \mathcal{M}_{L\log^{1/2} L(\mathbb{T})\to L^2(\mathbb{T})} \subsetneq \mathcal{M}_{L\log L(\mathbb{T})\to L^2(\mathbb{T})}.$$

Note that the fact that $\chi_\Lambda \notin \mathcal{M}_{H^1(\mathbb{T})\to L^2(\mathbb{T})}$ also follows from the observation that the spectral set $\Lambda = \{\pm 3^j \pm 3^k : j < k\}$ does not satisfy the condition

$$\sup_{N\in\mathbb{N}_0} \#\big\{\Lambda \cap ([N,2N] \cup [-2N,-N])\big\} < \infty.$$

Therefore, by Rudin's characterisation of the class of spectral sets satisfying Paley's inequality, χ_Λ cannot be a member of the class $\mathcal{M}_{H^1(\mathbb{T})\to L^2(\mathbb{T})}$. In particular, by Bonami's theorem and duality, there exist positive constants A and B such that

$$\Big(\sum_{j<k} |\widehat{f}(\pm 3^j \pm 3^k)|^2\Big)^{1/2} \leqslant A \int |f(x)| \log(1 + |f(x)|) dx + B,$$

[2]Moreover, note that if A, B are infinite subsets of $\mathbb{Z}$, then $A + B$ is not a Sidon set, see e.g. Chapter 6 in [33].

whereas, by Rudin's characterisation of all spectral sets satisfying Paley's inequality, there exists an $f \in H^1(\mathbb{T})\backslash L\log L(\mathbb{T})$ such that

$$\sum_{j<k} |\widehat{f}(\pm 3^j \pm 3^k)|^2 = \infty.$$

Chapter 4

A multiplier inclusion theorem in higher dimensions

4.1 Introduction

Our goal in this chapter is to extend the multiplier inclusion theorem of the previous chapter, namely Theorem 3.3.1, to the multi-parameter setting. In particular, the main result of this chapter is the following sharp multiplier inclusion theorem on the n-torus.

Theorem 4.1.1. *One has the inclusion*

$$\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^n)\to L^2(\mathbb{T}^n)} \subset \mathcal{M}_{L\log^{n/2}L(\mathbb{T}^n)\to L^2(\mathbb{T}^n)}. \tag{4.1}$$

Moreover, the above inclusion is proper and it is sharp, in the sense that the exponent $r = n/2$ in $L\log^{n/2}L(\mathbb{T}^n)$ cannot be improved.

As we saw in the previous chapter one can obtain Theorem 3.3.1 either by using classical Zygmund's inequality or by using a theorem of Tao and Wright on a Littlewood-Paley inequality for compactly supported functions in $L\log^{1/2}L$ with mean zero. As a higher-dimensional version of the aforementioned result of Tao and Wright is not yet available, we shall argue as in section 3.3 and reduce the proof of Theorem 4.1.1 to a higher-dimensional version of Zygmund's inequality for spectral sets in $\mathbb{Z}^n$ that are not necessarily of product type. In particular, as we will see in section 4.3, the multiplier inclusion (4.1) is a consequence of the following proposition, which is a result of independent interest. To state this version of Zygmund's inequality on $\mathbb{T}^n$, let $\mathcal{J}$ denote the set of all "intervals" of integers of the form $\pm\{2^n-1,\cdots,2^{n+1}-2\}$, $n\in\mathbb{N}_0$, namely $\mathcal{J}$ consists of sets of the form $\{2^k-1,\cdots,2^{k+1}-2\}$, $k\in\mathbb{N}_0$ and $\{-2^{l+1}+2,\cdots,-2^l+1\}$, $l\in\mathbb{N}_0$.

Proposition 4.1.2. *Let $\mathcal{J}$ be as above.*

If $E\subset\mathbb{Z}^n$ is a non-empty set satisfying the condition

$$D_E = \sup_{I_1,\cdots,I_n\in\mathcal{J}} \#\big\{E\cap(I_1\times\cdots\times I_n)\big\} < \infty, \tag{4.2}$$

then there exists a positive constant A_{D_E}, depending only on D_E, such that

$$\Big(\sum_{(k_1,\cdots,k_n)\in E} |\widehat{f}(k_1,\cdots,k_n)|^2\Big)^{1/2} \leqslant A_{D_E}[1+\int_{\mathbb{T}^n} |f|\log^{n/2}(1+|f|)]. \tag{4.3}$$

Let $E \subset \mathbb{Z}^n$ be an infinite set satisfying (4.2). To prove that (4.3) holds, it suffices by duality to show that E is a $\Lambda(p)$ set in $\mathbb{Z}^n$ for any $p > 2$ with $\Lambda(p)$ constant growing like $p^{n/2}$ as $p \to \infty$. As E is not necessarily of product type, it does not seem that one can make use of the methods presented in the previous chapter[1]. Hence, in order to exploit the "sparsity" of E and prove that it satisfies (4.3), one needs to follow a different approach. The crucial observation is that, in view of (4.2), if f is an E-polynomial, then its "smooth" Littlewood-Paley square function is essentially controlled by the L^2-norm of f. More precisely, the aforementioned version of Zygmund's inequality on the n-torus will be a corollary of a higher-dimensional extension of a result due to Seeger and Trebels [56] concerning sharp bounds of sums involving "smooth" Littlewood-Paley projections on $\mathbb{T}^n$. To state our result, for a fixed Schwartz function η supported in $(-2,2)$ such that $\eta|_{[-1,1]} \equiv 1$, consider $\phi(\xi) = \eta(\xi) - \eta(2\xi)$. For $k \in \mathbb{N}$, set $\phi_k(\xi) = \phi(2^{-k}\xi)$ and for $k = 0$, set $\phi_0 = \eta$. One can easily see that $\sum_{k\in\mathbb{N}_0} \phi_k(\xi) = 1$ for every $\xi \in \mathbb{R}$. Then, for $k \in \mathbb{N}_0$, the corresponding "smooth" Littlewood-Paley projection in the periodic setting is defined by[2]

$$\widetilde{D}_k(f)(x) = \sum_{r\in\mathbb{Z}} \phi_k(r)\widehat{f}(r)e^{i2\pi rx}$$

for any, say, trigonometric polynomial f on $\mathbb{T}$. On the n-torus we put

$$\begin{aligned}\widetilde{D}_{k_1,\cdots,k_n}(f)(x_1,\cdots,x_n) &= \widetilde{D}_{k_1}\otimes\cdots\otimes\widetilde{D}_{k_n}(f)(x_1,\cdots,k_n)\\ &= \sum_{r_1,\cdots,r_n\in\mathbb{Z}} \phi_{k_1}(r_1)\cdots\phi_{k_n}(r_n)\widehat{f}(r_1,\cdots,r_n)e^{i2\pi(r_1x_1+\cdots+r_nx_n)}\end{aligned}$$

initially defined over trigonometric polynomials f on $\mathbb{T}^n$. Then, Proposition 4.1.2 is a consequence of the following result.

Proposition 4.1.3. *There exists a constant $C_n > 0$, depending only on our choice of ϕ and on n, such that the following inequality holds*

$$\|f\|_{L^p(\mathbb{T}^n)} \leqslant Cp^{n/2}\Big(\sum_{k_1,\cdots,k_n\in\mathbb{N}_0} \|\widetilde{D}_{k_1,\cdots,k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2} \tag{4.4}$$

for every trigonometric polynomial f on $\mathbb{T}^n$ and for each $p > 2$.

The proof of Proposition 4.1.3 is an adaptation of the work of Seeger and Trebels [56] to the product setting combined with a well-known inequality on

[1] For instance, in the two-dimensional case, note that one can construct a set $E \subset \mathbb{Z}^2$ such that E satisfies (4.2) for $n = 2$ and $\pi_x(E) = \pi_y(E) = \mathbb{Z}$, where $\pi_x(E)$ ($\pi_y(E)$) denotes the projection of E onto the x-axis (y-axis, respectively).

[2] Note that here we used a partition of unity, $\sum_{k\in\mathbb{N}_0} \phi_k(\xi) = 1$ different than the one defined in subsection 2.2.1.

multiple martingales, see section 4.2.

This chapter presents the details of the author's work [4], submitted for publication, and it is organised as follows. In section 4.2 we present a well-known inequality involving square functions of multiple martingales which will be used in the proof of Proposition 4.1.3. In section 4.3 we show how the proof of our multiplier inclusion theorem follows from Proposition 4.1.2 and in section 4.4, we prove that Proposition 4.1.3 implies Proposition 4.1.2. Then, in section 4.5 we give a proof of Proposition 4.1.3. In the last section we briefly present some further applications of our results.

4.2 Inequalities for dyadic square functions

If $f \in L^1(\mathbb{T})$ and $m \in \mathbb{N}_0$, then the m-th conditional expectation of f is given by

$$\mathbb{E}_m(f)(x) = 2^m \int_I f(x')dx',$$

where I is the unique dyadic interval in $\mathbb{T}$ of the form $I = [s2^{-m}, (s+1)2^{-m})$, $s = 0, 1, \cdots, 2^m - 1$ such that $x \in I$.

For $m \in \mathbb{N}$, let $\mathbb{D}_m = \mathbb{E}_m - \mathbb{E}_{m-1}$ denote the martingale differences acting on functions defined on $\mathbb{T}$. For $m = 0$, we set $\mathbb{D}_0 = \mathbb{E}_0$.

For a given n-tuple $(m_1, \cdots, m_n)$ of non-negative integers, we define the corresponding operators acting on functions on the n-torus by

$$\mathbb{E}_{m_1,\cdots,m_n} = \mathbb{E}_{m_1} \otimes \cdots \otimes \mathbb{E}_{m_n}$$

and

$$\mathbb{D}_{m_1,\cdots,m_n} = \mathbb{D}_{m_1} \otimes \cdots \otimes \mathbb{D}_{m_n}.$$

More precisely, if $n > 1$ then, given $\mathbb{D}_{m_1,\cdots,m_{n-1}}$, we define

$$\mathbb{D}_{m_1,\cdots,m_n} = \mathbb{D}_{m_1,\cdots,m_{n-1}} \otimes \mathbb{D}_{m_n}$$

and so, if $m_n = 0$ then we set $\mathbb{D}_{m_1,\cdots,m_n} = \mathbb{D}_{m_1,\cdots,m_{n-1}} \otimes \mathbb{E}_0$ and if $m_n \geqslant 1$, then $\mathbb{D}_{m_1,\cdots,m_n} = \mathbb{D}_{m_1,\cdots,m_{n-1}} \otimes (\mathbb{E}_{m_n} - \mathbb{E}_{m_{n-1}})$.

It follows by the work of Chang, Wilson, and Wolff [15], in particular by corollary 3.1 in [15], that

$$\|f\|_{L^p(\mathbb{T})} \leqslant Cp^{1/2} \Big\| \Big(\sum_{m \in \mathbb{N}_0} |\mathbb{D}_m(f)|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{T})} \tag{4.5}$$

for all $p > 2$, where $C > 0$ is an absolute constant, see also, e.g., p. 152 in [56]. Moreover, Chang, Wilson, and Wolff obtained in [15] an inequality analogous to (4.5) involving Lusin area integrals. See also [1] and the references therein. In [47], Pipher extended (4.5) and its analogous version on Lusin area integrals to the two-parameter setting and in [23], R. Fefferman and Pipher extended the aforementioned inequality of Chang, Wilson, and Wolff involving Lusin area integrals to ℓ^2-valued functions. Their arguments can easily be adapted to obtain an ℓ^2-valued extension of (4.5), see [20]. By using this ℓ^2-valued extension of (4.5) together with induction on n, one deduces that there exists a constant $C_n > 0$, depending only on the dimension $n \in \mathbb{N}$, such that

$$\|f\|_{L^p(\mathbb{T}^n)} \leqslant C_n p^{n/2} \Big\| \Big(\sum_{m_1, \cdots m_n \in \mathbb{N}_0} |\mathbb{D}_{m_1,\cdots,m_n}(f)|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{T}^n)} \tag{4.6}$$

for every $p > 2$, see also, e.g., [20, Proposition 4.5] and [7].

4.3 Proposition 4.1.2 implies Theorem 4.1.1

To prove that Proposition 4.1.2 implies Theorem 4.1.1, we adapt the argument given in Chapter 3 to the multi-parameter setting by using the characterisation of $\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^n) \to L^2(\mathbb{T}^n)}$.

Assume that Proposition 4.1.2 holds. To prove Theorem 4.1.1, take an arbitrary m in the class $\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^n) \to L^2(\mathbb{T}^n)}$. Then, by definition, we need to show that for every $f \in L \log^{n/2} L(\mathbb{T}^n)$ one has

$$\sum_{(k_1,\cdots,k_n) \in \mathbb{Z}^n} |m(k_1,\cdots,k_n)\widehat{f}(k_1,\cdots,k_n)|^2 < \infty.$$

Towards this aim, fix an $f \in L \log^{n/2} L(\mathbb{T}^n)$ and, arguing as in the previous chapter, note that the sum

$$\sum_{(k_1,\cdots,k_n) \in \mathbb{Z}^n} |m(k_1,\cdots,k_n)\widehat{f}(k_1,\cdots,k_n)|^2$$

is bounded by

$$\sum_{I_1,\cdots,I_n \in \mathcal{J}} \max_{(k_1,\cdots,k_n) \in I_1 \times \cdots \times I_n} |\widehat{f}(k_1,\cdots,k_n)|^2 \Big(\sum_{k_1 \in I_1} \cdots \sum_{k_n \in I_n} |m(k_1,\cdots,k_n)|^2 \Big).$$

Hence, by (2.7), it follows that

$$\sum_{(k_1,\cdots,k_n) \in \mathbb{Z}^n} |m(k_1,\cdots,k_n)\widehat{f}(k_1,\cdots,k_n)|^2 \lesssim_m \sum_{(\widetilde{k}_1,\cdots,\widetilde{k}_n) \in E_f} |\widehat{f}(\widetilde{k}_1,\cdots,\widetilde{k}_n)|^2,$$

where E_f is a set in $\mathbb{Z}^n$ defined as follows. Given $I_1,\cdots,I_n \in \mathcal{J}$, choose $(\widetilde{k}_1,\cdots,\widetilde{k}_n)$ in $I_1 \times \cdots \times I_n$ such that

$$|\widehat{f}(\widetilde{k}_1,\cdots,\widetilde{k}_n)| = \max_{(k_1,\cdots,k_n) \in I_1 \times \cdots \times I_n} |\widehat{f}(k_1,\cdots,k_n)|.$$

Then, having chosen a set of n-tuples $(\widetilde{k}_1,\cdots,\widetilde{k}_n)$ as above, we define

$$E_f = \{(\widetilde{k}_1,\cdots,\widetilde{k}_n) \in \mathbb{Z}^n : \text{ for } I_1,\cdots,I_n \in \mathcal{J},\ (\widetilde{k}_1,\cdots,\widetilde{k}_n) \in I_1 \times \cdots \times I_n \text{ being as above}\}.$$

Notice that as the choice of n-tuples $(\widetilde{k}_1,\cdots,\widetilde{k}_n)$ is not necessarily unique, there might be several choices of sets E_f. We just choose one of them to write

$$\sum_{I_1,\cdots,I_n\in\mathcal{J}} \max_{(k_1,\cdots,k_n)\in I_1\cdots\times I_n} |\widehat{f}(k_1,\cdots,k_n)|^2 = \sum_{(\widetilde{k}_1,\cdots,\widetilde{k}_n)\in E_f} |\widehat{f}(\widetilde{k}_1,\cdots,\widetilde{k}_n)|^2.$$

Note that any such set E_f satisfies condition (4.2) in Proposition 4.1.2 with $D_{E_f}=1$. Therefore, as $f\in L\log^{n/2}L(\mathbb{T}^n)$, it follows by (4.3) that

$$\sum_{(k_1,\cdots,k_n)\in\mathbb{Z}^n} |m(k_1,\cdots,k_n)\widehat{f}(k_1,\cdots,k_n)|^2<\infty.$$

4.3.1 Sharpness of (4.1)

As in the one-dimensional case presented in the previous chapter, we remark that, in fact, the above argument shows that if $m\in\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^n)\to L^2(\mathbb{T}^n)}$, then there is a constant $C_m>0$, depending only on m, such that

$$\Big(\sum_{(k_1,\cdots,k_n)\in\mathbb{Z}^n} |m(k_1,\cdots,k_n)\widehat{f}(k_1,\cdots,k_n)|^2\Big)^{1/2}\leqslant C_m\big[1+\int_{\mathbb{T}^n}|f|\log^{n/2}(1+|f|)\big].$$

Therefore, to prove that the exponent $r=n/2$ in $L\log^{n/2}L(\mathbb{T}^n)$ cannot be improved, we argue as in the one-dimensional case. More specifically, assume that every multiplier from $H^1_{\mathrm{prod}}(\mathbb{T}^n)$ to $L^2(\mathbb{T}^n)$ is a multiplier from $L\log^r L(\mathbb{T}^2)$ to $L^2(\mathbb{T}^2)$. We shall prove that $r\geqslant n/2$. To this end, for a large positive integer N, take f to be a trigonometric polynomial on $\mathbb{T}^n$ given by $f=V_{2^N}\otimes\cdots\otimes V_{2^N}$, where, as in Chapter 3, V_{2^N} denotes the de la Vallée Poussin kernel of order 2^N. Since $\|V_{2^N}\|_{L^1(\mathbb{T})}\lesssim 1$ and $\|V_{2^N}\|_{L^\infty(\mathbb{T})}\lesssim 2^N$, we deduce that

$$\int_{\mathbb{T}^n}|f(x_1,\cdots,x_n)|\log^r(1+|f(x_1,\cdots,x_n)|)dx_1\cdots dx_n\lesssim N^r.$$

So, if we take $M=(m(k_1,\cdots,k_n))_{k_1,\cdots,k_n\in\mathbb{Z}}$ with $m(k_1,\cdots,k_n)=1/\sqrt{k_1\cdots k_n}$ for $k_1>0,\cdots,k_n>0$ and $m(k_1,\cdots,k_n)=0$ otherwise (i.e. when at least one of the coordinates is less or equal to 0), then $M\in\mathcal{M}_{H^1_{\mathrm{prod}}(\mathbb{T}^n)\to L^2(\mathbb{T}^n)}$ and hence,

$$\Big(\sum_{(k_1,\cdots,k_n)\in\mathbb{Z}^n} |m(k_1,\cdots,k_n)\widehat{f}(k_1,\cdots,k_n)|^2\Big)^{1/2}\lesssim N^r.$$

Since

$$\begin{aligned}\Big(\sum_{(k_1,\cdots,k_n)\in\mathbb{Z}^n} |m(k_1,\cdots,k_n)\widehat{f}(k_1,\cdots,k_n)|^2\Big)^{1/2} &\geqslant \Big(\sum_{1\leqslant k_1,\cdots,k_n\leqslant 2^N}\frac{1}{k_1\cdots k_n}\Big)^{1/2}\\ &=\prod_{i=1}^n\Big(\sum_{1\leqslant k_i\leqslant 2^N}\frac{1}{k_i}\Big)^{1/2}\\ &\sim N^{n/2},\end{aligned}$$

we see that, by choosing N to be large enough, we necessarily have $r \geqslant n/2$.

4.3.2 Sharpness of (4.3)

Let E be a set satisfying (4.2). Then $M = \chi_E$ satisfies (2.7) and so, it is a multiplier from $H^1_{\text{prod}}(\mathbb{T}^n)$ to $L^2(\mathbb{T}^n)$. Moreover, if we make use of the argument of the previous subsection for $M = \chi_E$, we see that the Orlicz space $L \log^{n/2} L(\mathbb{T}^n)$ in the right-hand side of higher-dimensional Zygmund's inequality (4.3) cannot be improved.

4.3.3 The inclusion (4.1) is proper

To see that the inclusion (4.1) is proper, as in the previous chapter, take Λ to be a Sidon set in $\mathbb{Z}$ that cannot be written as a finite union of lacunary sequences. Then $M = \chi_{\Lambda \times \cdots \times \Lambda}$ belongs to the class $\mathcal{M}_{L \log^{n/2} L(\mathbb{T}^n) \to L^2(\mathbb{T}^n)}$, see, e.g., (3.1) or Proposition 3.2.3. However, $M = \chi_{\Lambda \times \cdots \times \Lambda}$ does not satisfy (2.7) and hence, we deduce that $\chi_{\Lambda \times \cdots \times \Lambda} \notin \mathcal{M}_{H^1_{\text{prod}}(\mathbb{T}^n) \to L^2(\mathbb{T}^n)}$.

4.4 Proposition 4.1.3 implies Proposition 4.1.2

Our goal in this section is to prove that Proposition 4.1.3 implies Proposition 4.1.2. Towards this aim, take $E \subset \mathbb{Z}^n$ to be a set satisfying the assumption of Proposition 4.1.2, i.e. condition (4.2). Assume first that E satisfies (4.2) with $D_E = 1$. By duality, to prove (4.3), it suffices to show that E is a $\Lambda(p)$ set in $\mathbb{Z}^n$ for every $p > 2$ with $\Lambda(p)$ constant growing like $p^{n/2}$ as $p \to \infty$. Namely, it is enough to show that for every E-polynomial f one has for every $p > 2$,

$$\|f\|_{L^p(\mathbb{T}^n)} \leqslant A_E p^{n/2} \|f\|_{L^2(\mathbb{T}^n)}, \tag{4.7}$$

where A_E is an absolute constant, independent of p and f. As we will see momentarily, if $D_E = 1$, then, in fact, A_E depends only on n and in particular, it can be taken to be independent of E.

To prove (4.7), fix an E-polynomial f and note that for every $(k_1, \cdots, k_n) \in \mathbb{N}_0^n$ one has by the triangle inequality

$$\begin{aligned}\|\widetilde{D}_{k_1,\cdots,k_n}(f)\|_{L^\infty(\mathbb{T}^n)} &\leqslant \sum_{(r_1,\cdots,r_n) \in E \cap (I_{k_1} \times \cdots \times I_{k_n})} |\phi_{k_1}(r_1) \cdots \phi_{k_n}(r_n) \widehat{f}(r_1, \cdots, r_n)| \\ &\lesssim_{\phi,n} \sum_{(r_1,\cdots,r_n) \in E \cap (I_{k_1} \times \cdots \times I_{k_n})} |\widehat{f}(r_1, \cdots, r_n)|,\end{aligned}$$

where I_{k_l} denotes the set $\mathbb{Z} \cap \{(-2^{k_l+1}, -2^{k_l-1}] \cap [2^{k_l-1}, 2^{k_l+1})\}$, $l = 1, \cdots, n$. Observe that, thanks to condition (4.2) for $D_E = 1$, the sum

$$\sum_{(r_1,\cdots,r_n) \in E \cap (I_{k_1} \times \cdots \times I_{k_n})} |\widehat{f}(r_1, \cdots, r_n)|$$

consists of at most 6^n terms. Hence,

$$\|\widetilde{D}_{k_1,\cdots,k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)} \lesssim_{n,\phi} \sum_{(r_1,\cdots,r_n)\in E\cap(I_{k_1}\times\cdots\times I_{k_n})} |\widehat{f}(r_1,\cdots,r_n)|^2$$

and we thus deduce that

$$\Big(\sum_{k_1,\cdots,k_n\in\mathbb{N}_0} \|\widetilde{D}_{k_1,\cdots,k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2} \lesssim_{n,\phi} \Big(\sum_{(r_1,\cdots,r_n)\in E} |\widehat{f}(r_1,\cdots,r_n)|^2\Big)^{1/2}. \quad (4.8)$$

Note that the quantity on the right-hand side of the last inequality equals to $\|f\|_{L^2(\mathbb{T}^n)}$, as $\mathrm{supp}(\widehat{f}) \subset E$. Hence, (4.7) follows from (4.4) and (4.8) in the case where $D_E = 1$. Moreover, note that, in the case where $D_E = 1$, the implied constant in (4.8) depends only on the dimension n and on our choice of ϕ and it is independent of E.

In the case where $D_E > 1$, write $f = \sum_{i=1}^{D_E} f_i$, where f_i are trigonometric polynomials on $\mathbb{T}^n$ such that $\mathrm{supp}(\widehat{f_i}) \subset E_i$, where $E = \cup_{i=1}^{D_E} E_i$ and $D_{E_i} = 1$. Then, by the triangle inequality and the previous step we have

$$\|f\|_{L^p(\mathbb{T}^n)} \leqslant \sum_{i=1}^{D_E} \|f_i\|_{L^p(\mathbb{T}^n)} \leqslant Ap^{n/2} \sum_{i=1}^{D_E} \|f_i\|_{L^2(\mathbb{T}^n)} \leqslant AD_E p^{n/2} \|f\|_{L^2(\mathbb{T}^n)},$$

since, by our construction and the L^2-theory, $\|f_i\|_{L^2(\mathbb{T}^n)} \leqslant \|f\|_{L^2(\mathbb{T}^n)}$ for all $i = 1,\cdots,D_E$.

4.5 Proof of Proposition 4.1.3

To prove Proposition 4.1.3, note that, as $p > 2$, it follows by Minkowski's inequality that

$$\Big\|\Big(\sum_{m_1,\cdots,m_n\in\mathbb{N}_0} |\mathbb{D}_{m_1,\cdots,m_n}(f)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{T}^n)} \leqslant \Big(\sum_{m_1,\cdots,m_n\in\mathbb{N}_0} \|\mathbb{D}_{m_1,\cdots,m_n}(f)\|^2_{L^p(\mathbb{T}^n)}\Big)^{1/2}.$$

Moreover, since one trivially has

$$\Big(\sum_{m_1,\cdots,m_n\in\mathbb{N}_0} \|\mathbb{D}_{m_1,\cdots,m_n}(f)\|^2_{L^p(\mathbb{T}^n)}\Big)^{1/2} \leqslant \Big(\sum_{m_1,\cdots,m_n\in\mathbb{N}_0} \|\mathbb{D}_{m_1,\cdots,m_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2},$$

we deduce by (4.6) that

$$\|f\|_{L^p(\mathbb{T}^n)} \leqslant C_n p^{n/2} \Big(\sum_{m_1,\cdots,m_n\in\mathbb{N}_0} \|\mathbb{D}_{m_1,\cdots,m_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2} \quad (4.9)$$

for all $p > 2$. Hence, to prove that (4.4) holds, it suffices, in view of (4.9), to show that

$$\Big(\sum_{m_1,\cdots,m_n\in\mathbb{N}_0}\|\mathbb{D}_{m_1,\cdots,m_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2}\lesssim_n\Big(\sum_{k_1,\cdots,k_n\in\mathbb{N}_0}\|\widetilde{D}_{k_1,\cdots,k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2}.$$

This last inequality follows from the next lemma which is an n-dimensional analogue of [56, Lemma 2.3].

Lemma 4.5.1. *Let δ be a Schwartz function that is even, supported in $(-4,4)$ and such that $\delta|_{[-2,2]}\equiv 1$.*

Define $\psi(\xi)=\delta(\xi)-\delta(2\xi)$. For $k\in\mathbb{N}$, put $\psi_k(\xi)=\psi(2^{-k}\xi)$ and for $k=0$, put $\psi_0=\delta$. Consider the operator

$$\Psi_k(f)(x)=\sum_{r\in\mathbb{Z}}\psi_k(r)\widehat{f}(r)e^{i2\pi rx}$$

acting on functions defined over the torus. For $k_1,\cdots,k_n\in\mathbb{N}_0$ we write $\Psi_{k_1,\cdots,k_n}=\Psi_{k_1}\otimes\cdots\otimes\Psi_{k_n}$.

There exists a constant $C_n>0$, depending only on the dimension n and on ψ, such that for all n-tuples of non-negative integers $(m_1,\cdots,m_n)$ and $(k_1,\cdots,k_n)$ one has

$$\|\mathbb{D}_{m_1,\cdots,m_n}\Psi_{k_1,\cdots,k_n}\|_{L^\infty(\mathbb{T}^n)\to L^\infty(\mathbb{T}^n)}\leqslant C_n\prod_{j=1}^n 2^{-|k_j-m_j|}. \tag{4.10}$$

The proof of Lemma 4.5.1 will be given in the next subsection. By using (4.10) in the lemma above, one can easily complete the proof of Proposition 4.1.3. Indeed, towards this aim, we argue as in the proof of [56, Proposition 2.2]. More precisely, we consider a trigonometric polynomial f on $\mathbb{T}^n$ and write $f=\sum_{k_1,\cdots,k_n\in\mathbb{N}_0}\widetilde{D}_{k_1,\cdots,k_n}(f)$. For fixed η (and hence ϕ), if ψ is as in the statement of Lemma 4.5.1, then $\psi\phi=\phi$ and hence, $\Psi_{k_1,\cdots,k_n}\widetilde{D}_{k_1,\cdots,k_n}=\widetilde{D}_{k_1,\cdots,k_n}$. So, by using (4.10), we obtain

$$\begin{aligned}\|\mathbb{D}_{m_1,\cdots,m_n}(f)\|_{L^\infty(\mathbb{T}^n)}&\leqslant\sum_{k_1,\cdots,k_n\in\mathbb{N}_0}\|\mathbb{D}_{m_1,\cdots,m_n}[\widetilde{D}_{k_1,\cdots,k_n}(f)]\|_{L^\infty(\mathbb{T}^n)}\\&\leqslant\sum_{k_1,\cdots,k_n\in\mathbb{N}_0}\|\mathbb{D}_{m_1,\cdots,m_n}\Psi_{k_1,\cdots,k_n}\|_{L^\infty(\mathbb{T}^n)\to L^\infty(\mathbb{T}^n)}\|\widetilde{D}_{k_1,\cdots,k_n}(f)\|_{L^\infty(\mathbb{T}^n)}\\&\lesssim_n\sum_{k_1,\cdots,k_n\in\mathbb{N}_0}\big(\prod_{j=1}^n 2^{-|m_j-k_j|}\big)\|\widetilde{D}_{k_1,\cdots,k_n}(f)\|_{L^\infty(\mathbb{T}^n)}\end{aligned}$$

and it thus follows that

$$\Big(\sum_{m_1,\cdots,m_n\in\mathbb{N}_0}\|\mathbb{D}_{m_1,\cdots,m_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2}\lesssim_n$$
$$\Big[\sum_{m_1,\cdots,m_n\in\mathbb{N}_0}\Big(\sum_{k_1,\cdots,k_n\in\mathbb{N}_0}\big(\prod_{j=1}^n 2^{-|m_j-k_j|}\big)\|\widetilde{D}_{k_1,\cdots,k_n}(f)\|_{L^\infty(\mathbb{T}^n)}\Big)^2\Big]^{1/2},$$

where the implied constant depends only on n. By Minkowski's integral inequality,

$$\Big[\sum_{m_1,\cdots,m_n\in\mathbb{N}_0}\Big(\sum_{k_1,\cdots,k_n\in\mathbb{N}_0}\big(\prod_{j=1}^{n}2^{-|m_j-k_j|}\big)\|\widetilde{D}_{k_1,\cdots,k_n}(f)\|_{L^\infty(\mathbb{T}^n)}\Big)^2\Big]^{1/2}\leqslant$$

$$\sum_{m_1,\cdots,m_n\in\mathbb{Z}}\big(\prod_{j=1}^{n}2^{-|m_j|}\big)\Big(\sum_{k_1\geqslant -m_1}\cdots\sum_{k_n\geqslant -m_n}\|\widetilde{D}_{k_1+m_1,\cdots,m_n+k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2}.$$

Since we have

$$\sum_{m_1,\cdots,m_n\in\mathbb{Z}}\big(\prod_{j=1}^{n}2^{-|m_j|}\big)\Big(\sum_{k_1\geqslant -m_1}\cdots\sum_{k_n\geqslant -m_n}\|\widetilde{D}_{k_1+m_1,\cdots,m_n+k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2}\lesssim$$

$$\Big(\sum_{k_1,\cdots,k_n\in\mathbb{N}_0}\|\widetilde{D}_{k_1,\cdots,k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2},$$

the proof of Proposition 4.1.3 will be complete once we prove Lemma 4.5.1. This will be done in the following subsection.

4.5.1 Proof of Lemma 4.5.1

The proof of Lemma 4.5.1 can easily be obtained by iterating the corresponding one-dimensional result of Seeger and Trebels [56, Lemma 2.3] which, in particular, asserts that for all $m,k\in\mathbb{N}_0$ one has

$$\|\mathbb{D}_m\Psi_k\|_{L^\infty(\mathbb{T})\to L^\infty(\mathbb{T})}\leqslant C2^{-|m-k|},\tag{4.11}$$

where $C>0$ is a constant that depends only on ψ. More precisely, to prove Lemma 4.5.1 we shall induct on the dimension $n\in\mathbb{N}$. Note that the one-dimensional case is (4.11). Assume now that for some $n>1$ the estimate (4.10) holds for the $(n-1)$-dimensional case, namely for all $(n-1)$-tuples of non-negative integers $(m_1,\cdots,m_{n-1})$ and $(k_1,\cdots,k_{n-1})$ one has

$$\|\mathbb{D}_{m_1,\cdots,m_{n-1}}\Psi_{k_1,\cdots,k_{n-1}}\|_{L^\infty(\mathbb{T}^{n-1})\to L^\infty(\mathbb{T}^{n-1})}\leqslant C_{n-1}\prod_{j=1}^{n-1}2^{-|k_j-m_j|}.\tag{4.12}$$

Consider two n-tuples of non-negative integers $(m_1,\cdots,m_n)$ and $(k_1,\cdots,k_n)$ and let f be a trigonometric polynomial on $\mathbb{T}^n$. By definition, we have $\mathbb{D}_{m_1,\cdots,m_n}=\mathbb{D}_{m_1,\cdots,m_{n-1}}\otimes\mathbb{D}_{m_n}$. If $m_n\geqslant 1$, then we may write

$$\mathbb{D}_{m_1,\cdots,m_n}=\mathbb{D}_{m_1,\cdots,m_{n-1}}\otimes\mathbb{E}_{m_n}-\mathbb{D}_{m_1,\cdots,m_{n-1}}\otimes\mathbb{E}_{m_{n-1}}$$

and hence, for $(x_1,\cdots,x_n)\in\mathbb{T}^n$, one can write

$$\mathbb{D}_{m_1,\cdots,m_n}[\Psi_{k_1,\cdots,k_n}(f)](x_1,\cdots,x_n)$$

as

$$2^{m_n}\int_{I_n}\sum_{r_n\in\mathbb{Z}}\psi(2^{-k_n}r_n)\mathbb{D}_{m_1,\cdots,m_{n-1}}[\Psi_{k_1,\cdots,k_{n-1}}(f_{r_n})](x_1,\cdots,x_{n-1})e^{i2\pi r_n x_n'}dx_n'-$$
$$2^{m_n-1}\int_{\widetilde{I}_n}\sum_{r_n\in\mathbb{Z}}\psi(2^{-k_n}r_n)\mathbb{D}_{m_1,\cdots,m_{n-1}}[\Psi_{k_1,\cdots,k_{n-1}}(f_{r_n})](x_1,\cdots,x_{n-1})e^{i2\pi r_n x_n'}dx_n',$$

where I_n is the unique interval in $\mathbb{T}$ of length 2^{-m_n} containing x_n, $\widetilde{I}_n$ is the unique interval in $\mathbb{T}$ of length $2^{-(m_n-1)}$ containing x_n, and for $r_n\in\mathbb{Z}$ we use the notation

$$f_{r_n}(x_1,\cdots,x_{n-1})=\sum_{r_1,\cdots,r_{n-1}\in\mathbb{Z}}\widehat{f}(r_1,\cdots,r_n)e^{i2\pi(r_1x_1+\cdots+r_{n-1}x_{n-1})}.$$

By applying (4.11) to the n-th variable, for fixed $(x_1,\cdots,x_{n-1})\in\mathbb{T}^{n-1}$, we obtain

$$|\mathbb{D}_{m_1,\cdots,m_n}[\Psi_{k_1,\cdots,k_n}(f)](x_1,\cdots,x_n)|\leqslant$$
$$C2^{-|m_n-k_n|}\sup_{x_n\in\mathbb{T}}\Big|\sum_{r_n\in\mathbb{Z}}\mathbb{D}_{m_1,\cdots,m_{n-1}}[\Psi_{k_1,\cdots,k_{n-1}}(f_{r_n})](x_1,\cdots,x_{n-1})e^{i2\pi r_n x_n}\Big|$$

for all $x_n\in\mathbb{T}$. Note that we may write

$$\sum_{r_n\in\mathbb{Z}}\mathbb{D}_{m_1,\cdots,m_{n-1}}[\Psi_{k_1,\cdots,k_{n-1}}(f_{r_n})](x_1,\cdots,x_{n-1})e^{i2\pi r_n x_n}$$

as

$$\mathbb{D}_{m_1,\cdots,m_{n-1}}[\Psi_{k_1,\cdots,k_{n-1}}(f_{x_n})](x_1,\cdots,x_{n-1}),$$

where, for fixed $x_n\in\mathbb{T}$, we use the notation $f_{x_n}(x_1,\cdots,x_{n-1})=f(x_1,\cdots,x_n)$. Hence, by using the inductive hypothesis (4.12), for fixed $x_n\in\mathbb{T}$, we have

$$\Big|\sum_{r_n\in\mathbb{Z}}\mathbb{D}_{m_1,\cdots,m_{n-1}}[\Psi_{k_1,\cdots,k_{n-1}}(f_{r_n})](x_1,\cdots,x_{n-1})e^{i2\pi r_n x_n}\Big|\leqslant$$
$$C_{n-1}\prod_{j=1}^{n-1}2^{-|m_j-k_j|}\sup_{(x_1,\cdots,x_{n-1})\in\mathbb{T}^{n-1}}|f_{x_n}(x_1,\cdots,x_{n-1})|$$

for all $(x_1,\cdots,x_{n-1})\in\mathbb{T}^{n-1}$. We thus deduce that

$$|\mathbb{D}_{m_1,\cdots,m_n}[\Psi_{k_1,\cdots,k_n}(f)](x_1,\cdots,x_n)|\leqslant CC_{n-1}\prod_{j=1}^{n}2^{-|m_j-k_j|}\|f\|_{L^\infty(\mathbb{T}^n)}$$

for all $(x_1,\cdots,x_n)\in\mathbb{T}^n$ and this implies the desired result when $m_n\geqslant 1$. A similar argument establishes the case where $m_n=0$. Hence, the proof of the lemma is complete.

Note that the argument above gives $C_n=C^n$, where $C>0$ is the constant in (4.11).

Remark 4.5.2. *We remark that one can give an alternative proof to Lemma* 4.5.1 *by adapting the argument in the proof of [56, Lemma 2.3] to higher dimensions.*

4.6 Some Further remarks and applications

4.6.1 Applications in thin sets

A special case of (3.1) for spectral sets of the form $\Lambda_1 \times \cdots \times \Lambda_n$, where Λ_i is a lacunary sequence in $\mathbb{Z}$ ($i = 1, \cdots, n$) can be obtained as a corollary of Proposition 4.1.2. Furthermore, Proposition 4.1.2 gives examples of $\Lambda(p)$ sets in $\mathbb{Z}^n$ whose corresponding $\Lambda(p)$ constant grows like $p^{n/2}$ as $p \to \infty$ and they cannot be written as products of Sidon sets. Therefore, those sets, i.e. the sets $E \subset \mathbb{Z}^n$ that cannot be written as n-fold products of sets in $\mathbb{Z}$ and satisfy condition $\sup_{I_1, \cdots, I_n \in \mathcal{J}} \#\big\{E \cap (I_1 \times \cdots \times I_n)\big\} < \infty$ are examples of $2n/(n+1)$-Rider sets in $\mathbb{Z}^n$ that cannot be written as products of Sidon sets in $\mathbb{Z}$.

Notice, however, that Proposition 4.1.2 cannot handle spectral sets of the form $\Lambda_1 \times \cdots \times \Lambda_n$, where Λ_j is a Sidon set that cannot be written as a finite union of lacunary sequences ($j = 1, \cdots, n$).

4.6.2 A version of (4.4) for "rough" projections

For $k \in \mathbb{N}$ consider the corresponding classical Littlewood-Paley projections

$$D_k(f)(x) = \sum_{n=2^{k-1}}^{2^k - 1} \widehat{f}(n) e^{i2\pi nx} \text{ and } D_{-k}(f)(x) = \sum_{n=-2^k+1}^{-2^{k-1}} \widehat{f}(n) e^{i2\pi nx}.$$

For $k = 0$, set $D_0(f)(x) = \widehat{f}(0)$. For $k_1, \cdots, k_n \in \mathbb{Z}$ we write

$$D_{k_1, \cdots, k_n} = D_{k_1} \otimes \cdots \otimes D_{k_n}.$$

Since for every trigonometric polynomial f on the n-torus we may write $f = \sum_{m_1, \cdots, m_n \in \mathbb{Z}} D_{m_1, \cdots, m_n}(f)$, we have

$$\widetilde{D}_{k_1, \cdots, k_n}(f) = \sum_{m_1, \cdots, m_n \in \mathbb{Z}} \widetilde{D}_{k_1, \cdots, k_n} D_{m_1, \cdots, m_n}(f).$$

Since $\widetilde{D}_{k_1, \cdots, k_n} D_{m_1, \cdots, m_n} = 0$ if there exists an index $j_0 \in \{1, \cdots, n\}$ such that $|k_{j_0} - |m_{j_0}|| > 1$, we deduce that

$$\begin{aligned} \|\widetilde{D}_{k_1, \cdots, k_n}(f)\|_{L^\infty(\mathbb{T}^n)} &\leqslant \sum_{\substack{(m_1, \cdots, m_n) \in \mathbb{Z}^n: \\ |k_j - |m_j|| \leqslant 1 \text{ for all } j \in \{1, \cdots, n\}}} \|\widetilde{D}_{k_1, \cdots, k_n} D_{m_1, \cdots, m_n}(f)\|_{L^\infty(\mathbb{T}^n)} \\ &\lesssim_n \sum_{\substack{(m_1, \cdots, m_n) \in \mathbb{Z}^n: \\ |k_j - |m_j|| \leqslant 1 \text{ for all } j \in \{1, \cdots, n\}}} \|D_{m_1, \cdots, m_n}(f)\|_{L^\infty(\mathbb{T}^n)}. \end{aligned}$$

Therefore,

$$\Big(\sum_{k_1, \cdots, k_n \in \mathbb{N}_0} \|\widetilde{D}_{k_1, \cdots, k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)} \Big)^{1/2} \lesssim_n \Big(\sum_{k_1, \cdots, k_n \in \mathbb{Z}} \|D_{k_1, \cdots, k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)} \Big)^{1/2}$$

and hence, it follows by (4.4) that for every trigonometric polynomial f on $\mathbb{T}^n$ one has

$$\|f\|_{L^p(\mathbb{T}^n)} \lesssim p^{n/2}\Big(\sum_{k_1,\cdots,k_n\in\mathbb{Z}} \|D_{k_1,\cdots,k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\Big)^{1/2} \tag{4.13}$$

for every $p > 2$. Estimate (4.13) is a multi-parameter version of an inequality due to C.N. Moore [41]. In particular, we obtain the following multi-parameter extension of [41, Theorem, p.30].

Corollary 4.6.1. *There exist positive constants $c_1(n)$ and $c_2(n)$, depending only on the dimension n, such that whenever*

$$\sum_{k_1,\cdots,k_n\in\mathbb{Z}} \|D_{k_1,\cdots,k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)} < \infty$$

one has

$$\int_{\mathbb{T}^n} \exp\Big\{c_1(n)\Big[\frac{|f(x_1,\cdots,x_n)|}{\big(\sum_{k_1,\cdots,k_n\in\mathbb{Z}}\|D_{k_1,\cdots,k_n}(f)\|^2_{L^\infty(\mathbb{T}^n)}\big)^{1/2}}\Big]^{2/n}\Big\}dx_1\cdots dx_n < c_2(n).$$

Chapter 5

Mapping properties of the Littlewood-Paley square function

5.1 Introduction

In this final chapter, we study mapping properties "near" L^1 of the Littlewood-Paley square function which can essentially be regarded as a prototypical Marcinkiewicz multiplier. Recall from the last section of the previous chapter that if f is a trigonometric polynomial on $\mathbb{T}$, then for $k \in \mathbb{N}$ one defines

$$D_k(f)(x) = \sum_{n=2^{k-1}}^{2^k-1} \widehat{f}(n)e^{i2\pi nx}, \ D_{-k}(f)(x) = \sum_{n=-2^k+1}^{-2^{k-1}} \widehat{f}(n)e^{i2\pi nx},$$

and for $k = 0$, $D_0(f)(x) = \widehat{f}(0)$, $x \in \mathbb{T}$. The classical Littlewood-Paley square function $S_{\mathbb{T}}(f)$ of f is given by[1]

$$S_{\mathbb{T}}(f)(x) = \Big(\sum_{k\in\mathbb{Z}} |D_k(f)(x)|^2 \Big)^{1/2}.$$

A celebrated theorem of Littlewood and Paley asserts that the square function $S_{\mathbb{T}}$ can be extended as a bounded operator on $L^p(\mathbb{T})$ for all $1 < p < \infty$, namely for each $1 < p < \infty$ there is a constant $C(p) > 0$ so that

$$\|S_{\mathbb{T}}(f)\|_{L^p(\mathbb{T})} \leqslant C(p)\|f\|_{L^p(\mathbb{T})}. \tag{5.1}$$

In [12, Theorem 1], Bourgain determined the behaviour of $C(p)$ in (5.1) as $p \to 1^+$. In particular, he showed that there exist absolute constants $c_1, c_2 > 0$ such that

$$c_1(p-1)^{-3/2} < C(p) < c_2(p-1)^{-3/2} \tag{5.2}$$

[1] All the results of this chapter also hold for the "modified" version of the classical Littlewood-Paley square function given by $S'_{\mathbb{T}}(f) = \big(\sum_{k\in\mathbb{N}_0} |D'_k(f)|^2\big)^{1/2}$, where $D'_0(f) = \widehat{f}(0)$ and for $k \in \mathbb{N}$, $D'_k(f) = \sum_{n=2^{k-1}}^{2^k-1} \widehat{f}(n)e^{i2\pi nx} + \sum_{n=-2^k+1}^{-2^{k-1}} \widehat{f}(n)e^{i2\pi nx}$, and its corresponding multi-parameter versions $S'_{\mathbb{T}^n}$. Note that $S'_{\mathbb{T}^n}(f) \lesssim_n S_{\mathbb{T}^n}(f)$ pointwise. However, the converse inequality does not hold pointwise.

for every $1 < p \leqslant 2$.

In this chapter we give a simple proof to the upper estimate in (5.2) and extend it to higher dimensions. Moreover, we establish sharp weak-type inequalities for the multi-parameter Littlewood-Paley square function and study its boundedness on product Hardy spaces. More precisely, in section 5.2 we give a new proof of the upper estimate in (5.2) based on results of Tao and Wright on mapping properties of Marcinkiewicz multipliers [62] and on Tao's converse extrapolation [61]. More specifically, using the fact that Marcinkiewicz multipliers locally map $L\log^{1/2} L$ to $L^{1,\infty}$ [62, Theorem 1.2], together with interpolation and Tao's converse extrapolation [61], one deduces that $\| \sum_{k\in\mathbb{Z}} \pm D_k\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \lesssim (p-1)^{-3/2}$, which is essentially the upper estimate in (5.2). Furthermore, we extend (5.2) to higher dimensions. Indeed, by using $\| \sum_{k\in\mathbb{Z}} \pm D_k\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \lesssim (p-1)^{-3/2}$ and iteration, we obtain higher-dimensional extensions of (5.2) in section 5.3. In section 5.4 we prove sharp weak-type inequalities for the multi-parameter Littlewood-Paley square function on $\mathbb{T}^n$ and in section 5.5 we establish the corresponding weak-type endpoint results in euclidean spaces. It is well-known that the Littlewood-Paley square function maps $H^1(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$. Motivated by this fact, a natural question is whether the two-parameter Littlewood-Paley square function maps the product real Hardy space $H^1_{\text{prod}}(\mathbb{T}^2)$ to $L^{1,\infty}(\mathbb{T}^2)$. In section 5.6 we show that this is not the case and in the last section of this chapter we conclude with some remarks on the endpoint mapping properties of general multi-parameter Marcinkiewicz multipliers.

This chapter presents the details of the author's work [2], submitted for publication.

5.2 A new proof of the upper estimate in (5.2)

As mentioned in Chapter 1, in [62], Tao and Wright proved that if T_m is a Marcinkiewicz multiplier operator acting on functions defined over $\mathbb{R}$, then it locally maps $L\log^{1/2} L$ to $L^{1,\infty}$. In particular, for every compact set $K \subset \mathbb{R}$ there is a constant $C > 0$, depending on the measure of K and on $\|m\|_{L^\infty(\mathbb{R})} + \sup_{k\in\mathbb{Z}} \int_{\pm[2^k,2^{k+1})} |dm|$, such that

$$\|T_m(f)\|_{L^{1,\infty}(K)} \leqslant C\|f\|_{L\log^{1/2} L(K)} \tag{5.3}$$

for all measurable functions f supported in K. By adapting the argument of Tao and Wright to the periodic setting, one shows that for every $\omega \in [0,1]$ the prototypical Marcinkiewicz multiplier operator

$$T_\omega = \sum_{k\in\mathbb{Z}} r_k(\omega) D_k$$

acting on functions defined over $\mathbb{T}$ maps $L\log^{1/2} L(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$, where $(r_k)_{k\in\mathbb{Z}}$ denotes the set of Rademacher functions indexed by $\mathbb{Z}$. In particular, for any trigonometric polynomial f on $\mathbb{T}$, one has

$$\|T_\omega(f)\|_{L^{1,\infty}(\mathbb{T})} \leqslant C\|f\|_{L\log^{1/2} L(\mathbb{T})}, \tag{5.4}$$

where $C > 0$ is an absolute constant independent of ω.

Using (5.4) and the fact that T_ω is bounded on $L^2(\mathbb{T})$ with operator norm equal to 1, one can easily show, by using a Marcinkiewicz-type interpolation argument, that T_ω is bounded from $L\log^{3/2} L(\mathbb{T})$ to $L^1(\mathbb{T})$. In fact, we have the following interpolation lemma.

Lemma 5.2.1. *Let T be a sublinear operator acting on functions defined over some measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) = 1$.*

If T is bounded on $L^2(X)$ and bounded from $L\log^{1/2} L(X)$ to $L^{1,\infty}(X)$, then it is bounded from $L\log^{r+3/2} L(X)$ to $L\log^r L(X)$ for every $r \geqslant 0$. In particular, one has

$$\|T(f)\|_{L\log^r L(X)} \lesssim \|f\|_{L\log^{r+3/2} L(X)}. \tag{5.5}$$

Proof. Let $r \geqslant 0$. Fix $f \in L\log^{r+3/2} L(X)$ and for $\alpha > 0$, we write $f^\alpha = f\chi_{\{|f|>\alpha\}}$ and $f_\alpha = f\chi_{\{|f|\leqslant\alpha\}}$.

Observe that, see e.g. the proof of [5, Lemma 10.1],

$$\|T(f)\|_{L\log^r L(X)} \lesssim_r 1 + \int_X |T(f)| \log^r(1 + |T(f)|)d\mu.$$

We shall prove that

$$\int_X |T(f)| \log^r(1 + |T(f)|)d\mu \lesssim_r 1 + \int_X |f| \log^{r+3/2}(1 + |f|)d\mu. \tag{5.6}$$

For this, note that one trivially has

$$\int_X |T(f)| \log^r(1 + |T(f)|)d\mu \leqslant \int_X |T(f)| \log^r(2 + |T(f)|)d\mu$$

and write

$$\int_X |T(f)| \log^r(2 + |T(f)|)d\mu =$$
$$\int_0^\infty \big[\log^r(2 + \alpha) + r\frac{\alpha}{2 + \alpha} \log^{r-1}(2 + \alpha)\big]\mu(\{x \in X : |T(f)(x)| > \alpha\})d\alpha.$$

Since T is sublinear, we have

$$\mu(\{x \in X : |T(f)(x)| > \alpha\}) \leqslant$$
$$\mu(\{x \in X : |T(f^\alpha)(x)| > \alpha/2\}) + \mu(\{x \in X : |T(f_\alpha)(x)| > \alpha/2\})$$

and we thus estimate

$$\int_X |T(f)| \log^r(2 + |T(f)|)d\mu \leqslant I_1 + I_2,$$

where

$$I_1 = \int_0^\infty \big[\log^r(2 + \alpha) + r\frac{\alpha}{2 + \alpha} \log^{r-1}(2 + \alpha)\big]\mu(\{x \in X : |T(f^\alpha)(x)| > \alpha/2\})d\alpha$$

and

$$I_2 = \int_0^\infty \big[\log^r(2+\alpha) + r\frac{\alpha}{2+\alpha}\log^{r-1}(2+\alpha)\big]\mu(\{x \in X : |T(f_\alpha)(x)| > \alpha/2\})d\alpha.$$

We shall bound I_1 and I_2 separately.

For the first one, we use the $L\log^{1/2} L$ to $L^{1,\infty}$ boundedness of T and Fubini's theorem as follows,

$$\begin{aligned}
I_1 &\leqslant \int_0^\infty \big[\log^r(2+\alpha) + r\log^{r-1}(2+\alpha)\big]\mu(\{x \in X : |T(f^\alpha)(x)| > \alpha/2\})d\alpha \\
&\lesssim_r 1 + \int_1^\infty \big[\log^r(2+\alpha) + r\log^{r-1}(2+\alpha)\big]\mu(\{x \in X : |T(f^\alpha)(x)| > \alpha/2\})d\alpha \\
&\lesssim 1 + \int_1^\infty \big[\log^r(2+\alpha) + r\log^{r-1}(2+\alpha)\big]\frac{1}{\alpha}\big[\int_X |f^\alpha|\log^{1/2}(1+|f^\alpha|)d\mu\big]d\alpha \\
&\leqslant 1 + \int_X |f(x)|\log^{1/2}(1+|f(x)|)\big[\int_1^{|f(x)|}\frac{\log^r(2+\alpha) + r\log^{r-1}(2+\alpha)}{\alpha}d\alpha\big]d\mu(x) \\
&\lesssim_r 1 + \int_X |f(x)|\log^{r+3/2}(1+|f(x)|)d\mu(x).
\end{aligned}$$

To estimate I_2, we use the L^2-boundedness of T and the fact that there is an absolute constant $A_r > 1$ such that the map $x \mapsto x[\log^{r+3/2} x]^{-1}$ is increasing on $[A_r, \infty)$, for instance one may take $A_r = e^{r+3/2}$. More specifically,

$$\begin{aligned}
I_2 &\leqslant \int_0^\infty \big[\log^r(2+\alpha) + r\log^{r-1}(2+\alpha)\big]\mu(\{x \in X : |T(f_\alpha)(x)| > \alpha/2\})d\alpha \\
&\lesssim_r 1 + \int_{A_r}^\infty \big[\log^r(2+\alpha) + r\log^{r-1}(2+\alpha)\big]\mu(\{x \in X : |T(f_\alpha)(x)| > \alpha/2\})d\alpha \\
&\lesssim 1 + \int_{A_r}^\infty \big[\log^r(2+\alpha) + r\log^{r-1}(2+\alpha)\big]\frac{1}{\alpha^2}\big[\int_{A_r \leqslant |f| \leqslant \alpha} |f(x)|^2 d\mu(x)\big]d\alpha \\
&\leqslant 1 + \int_{A_r}^\infty \big[\log^r(2+\alpha) + r\log^{r-1}(2+\alpha)\big]\frac{1}{\alpha^2}\big[\int_{A_r \leqslant |f| \leqslant \alpha} |f(x)|\frac{\alpha\log^{r+3/2}|f(x)|}{\log^{r+3/2}\alpha}d\mu(x)\big]d\alpha \\
&\lesssim 1 + \big[\int_{A_r}^\infty \big(\frac{1}{\alpha\log^{3/2}\alpha} + \frac{1}{\alpha\log^{5/2}\alpha}\big)d\alpha\big]\big[\int_X |f(x)|\log^{r+3/2}(1+|f(x)|)d\mu(x)\big] \\
&\lesssim_r 1 + \int_X |f(x)|\log^{r+3/2}(1+|f(x)|)d\mu(x).
\end{aligned}$$

Therefore, (5.6) holds. In particular, we deduce that

$$\|T(f)\|_{L\log^r L(X)} \lesssim 1 + \int_X |f|\log^{r+3/2}(1+|f|)d\mu. \tag{5.7}$$

The desired estimate (5.5) easily follows from the last inequality by a scaling argument. Indeed, first of all, one can easily see that

$$\int_X |f|\log^k(1+|f|)d\mu \lesssim_k 1 + \|f\|_{L\log^k L(X)} + \|f\|_{L^1(X)}\log^k(\|f\|_{L^1(X)}) \tag{5.8}$$

for every $k > 0$, see, e.g., the proof of [5, Lemma 10.1]. Hence, if $\|f\|_{L\log^{r+3/2}L(X)} = 1$, then it follows by (5.7) and (5.8) that there exists a constant $C_r > 0$ such that $\|T(f)\|_{L\log^r L(X)} \leqslant C_r$. In the general case, if f is non-zero, then the function $g = \|f\|^{-1}_{L\log^{r+3/2}L(X)} f$ satisfies $\|g\|_{L\log^{r+3/2}L(X)} = 1$. We thus deduce that

$$\|T(\|f\|^{-1}_{L\log^{r+3/2}L(X)} f)\|_{L\log^r L(X)} \leqslant C_r$$

and so, (5.5) follows. □

By using Lemma 5.2.1, we obtain

$$\|T_\omega(f)\|_{L^1(\mathbb{T})} \lesssim \|f\|_{L\log^{3/2}L(\mathbb{T})},$$

where the implied constant is independent of ω. By Tao's converse extrapolation theorem [61], it follows that

$$\|T_\omega\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \leqslant \frac{A}{(p-1)^{3/2}} \quad (\text{as } p \to 1^+) \tag{5.9}$$

where $A > 0$ is a positive constant independent of ω. To complete the proof of the upper estimate in (5.2), we use (5.9) and Khintchine's inequality. More precisely, let $p > 1$ be close to 1 and let f be a trigonometric polynomial. Then, by Khintchine's inequality, we have for every $x \in \mathbb{T}$

$$S(f)(x) \lesssim \int_{[0,1]} |T_\omega(f)(x)| d\omega,$$

where the implied constant is independent of $x \in \mathbb{T}$ and f. Therefore, by integrating over $\mathbb{T}$ and using Minkowski's inequality, we get

$$\begin{aligned}\|S(f)\|_{L^p(\mathbb{T})} \lesssim \Big(\int_{\mathbb{T}} \Big|\int_{[0,1]} |T_\omega(f)(x)| d\omega\Big|^p dx\Big)^{1/p} &\leqslant \int_{[0,1]} \|T_\omega(f)\|_{L^p(\mathbb{T})} d\omega \\ &\lesssim \int_{[0,1]} \frac{1}{(p-1)^{3/2}} \|f\|_{L^p(\mathbb{T})} d\omega \\ &= \frac{1}{(p-1)^{3/2}} \|f\|_{L^p(\mathbb{T})},\end{aligned}$$

which is the upper estimate in (5.2).

5.3 Higher-dimensional extension of (5.2)

For $n \in \mathbb{N}$, let $S_{\mathbb{T}^n}$ denote the n-parameter Littlewood-Paley square function on $\mathbb{T}^n$ initially defined over trigonometric polynomials on $\mathbb{T}^n$ by

$$S_{\mathbb{T}^n}(f)(x) = \Big(\sum_{k_1,\cdots,k_n \in \mathbb{Z}} |D_{k_1,\cdots,k_n}(f)(x)|^2\Big)^{1/2},$$

where, as in the previous chapter, we set $D_{k_1,\cdots,k_n} = D_{k_1} \otimes \cdots \otimes D_{k_n}$. The corresponding n-parameter Littlewood-Paley inequality is

$$\|S_{\mathbb{T}^n}(f)\|_{L^p(\mathbb{T}^n)} \leqslant C_p(n)\|f\|_{L^p(\mathbb{T}^n)}. \tag{5.10}$$

Our goal in this section is to show that

$$C_p(n) \sim (p-1)^{-3n/2}.$$

As mentioned in the introduction, this can be done quite easily by iteration thanks to the fact that $\|T_\omega\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \lesssim (p-1)^{-3/2}$.

Proposition 5.3.1. *There exist positive constants $c_1(n), c_2(n)$, depending only on the dimension n, such that*

$$\frac{c_1(n)}{(p-1)^{3n/2}} < C_p(n) < \frac{c_2(n)}{(p-1)^{3n/2}}, \tag{5.11}$$

where $C_p(n)$ is the constant in (5.10).

Proof. To obtain the upper estimate in (5.11), let $\omega_1, \cdots, \omega_n$ be arbitrary numbers in $[0,1]$. Then, by using (5.9) and iteration, we deduce that

$$\|T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}\|_{L^p(\mathbb{T}^n)\to L^p(\mathbb{T}^n)} \leqslant \frac{A^n}{(p-1)^{3n/2}}.$$

As in the one-dimensional case, by using multi-dimensional Khintchine's inequality and Minkowski's inequality, we obtain

$$\|S_{\mathbb{T}^n}(f)\|_{L^p(\mathbb{T}^n)} \leqslant \frac{c_2(n)}{(p-1)^{3n/2}}\|f\|_{L^p(\mathbb{T}^n)},$$

where $c_2(n)$ is a constant that depends only on $n \in \mathbb{N}$.

To prove the lower estimate, we use the corresponding argument of Bourgain that shows the lower estimate in (5.2). As in [12], given $p > 1$, take $N \in \mathbb{N}$ to be such that $\log N \sim (p-1)^{-1}$ and set $f = V_N$, where V_N denotes the de la Vallée Poussin kernel of order N. Since

$$\|S_{\mathbb{T}}(V_N)\|_{L^p(\mathbb{T})} \gtrsim (p-1)^{-3/2},$$

we have

$$\|S_{\mathbb{T}^n}(V_N \otimes \cdots \otimes V_N)\|_{L^p(\mathbb{T}^n)} = \|S_{\mathbb{T}}(V_N)\|_{L^p(\mathbb{T})} \cdots \|S_{\mathbb{T}}(V_N)\|_{L^p(\mathbb{T})} \gtrsim_n (p-1)^{-3n/2},$$

as desired. ☐

It is worth noting that by adapting the method presented in section 5.2 one can give an alternative proof to the upper estimate in (5.11). In particular, one can first study the endpoint mapping properties of n-dimensional Marcinkiewicz multiplier operators of the form $T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}$ and then, one can use converse extrapolation to deduce the growth of $C_p(n)$ as $p \to 1^+$ (see also remark 5.4.4).

The advantage of this indirect approach is that it motivates the study of sharp weak-type inequalities for $S_{\mathbb{T}^n}$ and, roughly speaking, $S_{\mathbb{T}^n}$ can be viewed as a model case for general multi-parameter Marcinkiewicz multiplier operators.

5.4 Sharp weak-type estimates for the Littlewood-Paley square function on $\mathbb{T}^n$

5.4.1 The one-dimensional case

Assume that for some $r \geqslant 0$ the Littlewood-Paley square function $S_{\mathbb{T}}$ satisfies a weak-type inequality of the form

$$\|S_{\mathbb{T}}(f)\|_{L^{1,\infty}(\mathbb{T})} \leqslant C\|f\|_{L\log^r L(\mathbb{T})}$$

over all trigonometric polynomials f on $\mathbb{T}$, where $C > 0$ is some absolute constant. We shall prove that necessarily $r \geqslant 1/2$. For this, note that by using the above inequality and the fact that $S_{\mathbb{T}}$ is bounded on $L^2(\mathbb{T})$, arguing as in the proof of Lemma 5.2.1, we deduce that

$$\|S_{\mathbb{T}}(f)\|_{L^1(\mathbb{T})} \lesssim \|f\|_{L\log^{r+1} L(\mathbb{T})}$$

for all trigonometric polynomials f on $\mathbb{T}$. However, if we take $f = V_{2^N}$, then we have $\|f\|_{L\log^r L(\mathbb{T})} \lesssim 1+\int_{\mathbb{T}} |f|\log^{r+1}(1+|f|) \lesssim N^{r+1}$ and, moreover, by Minkowski's inequality,

$$\|S_{\mathbb{T}}(f)\|_{L^1(\mathbb{T})} \geqslant \Big\| \Big(\sum_{k=1}^{N} |D_k(V_{2^N})|^2 \Big)^{1/2} \Big\|_{L^1(\mathbb{T})} \geqslant \Big(\sum_{k=1}^{N} \|D_k(V_{2^N})\|_{L^1(\mathbb{T})}^2 \Big)^{1/2} \gtrsim \big(\sum_{k=1}^{N} k^2 \big)^{1/2} \gtrsim N^{3/2}.$$

We thus get $N^{3/2} \lesssim N^{r+1}$ and hence, by letting $N \to \infty$, it follows that the best we can expect is $r \geqslant 1/2$.

Proposition 5.4.1. *There exists an absolute constant $C > 0$ such that the classical Littlewood-Paley square function $S_{\mathbb{T}}$ satisfies the weak-type inequality*

$$\|S_{\mathbb{T}}(f)\|_{L^{1,\infty}(\mathbb{T})} \leqslant C\|f\|_{L\log^{1/2} L(\mathbb{T})} \tag{5.12}$$

for every trigonometric polynomial f on $\mathbb{T}$.

Proof. The weak-type inequality (5.12) follows immediately from the work of Tao and Wright [62] and it can be regarded as a vector-valued version of (5.4). To be more specific, the main idea is to make use of the fact that for every measure space $(X, \mathcal{A}, \mu)$ one has

$$\|g\|_{L^{1,\infty}(X)} \sim \sup_{\substack{E\subset X:\\ 0<\mu(E)<\infty}} \frac{\|g\|_{L^{1/2}(E)}}{|E|}, \tag{5.13}$$

see, e.g., page 485 in [26]. Then for every fixed trigonometric polynomial f on $\mathbb{T}$ and for every measurable subset E of $\mathbb{T}$ with $|E| > 0$ Khintchine's inequality implies that for all $x \in E$ one has

$$\int_{[0,1]} |T_\omega(f)(x)|^{1/2} d\omega \gtrsim \left(S_{\mathbb{T}}(f)(x)\right)^{1/2},$$

where the implied constant is independent of E and f. Therefore, by integrating with respect to $x \in E$ and by using Fubini's theorem, we get

$$\int_{[0,1]} \|T_\omega(f)\|_{L^{1/2}(E)}^{1/2} d\omega \gtrsim \|S_{\mathbb{T}}(f)\|_{L^{1/2}(E)}^{1/2}.$$

Hence, there exists a choice of $\omega' \in [0,1]$, depending on E and f, such that

$$\|T_{\omega'}(f)\|_{L^{1/2}(E)}^{1/2} \gtrsim \|S_{\mathbb{T}}(f)\|_{L^{1/2}(E)}^{1/2}$$

and thus,

$$\frac{\|T_{\omega'}(f)\|_{L^{1/2}(E)}}{|E|} \gtrsim \frac{\|S_{\mathbb{T}}(f)\|_{L^{1/2}(E)}}{|E|}.$$

By (5.13), the last inequality implies that

$$\|T_{\omega'}(f)\|_{L^{1,\infty}(\mathbb{T})} \gtrsim \frac{\|S_{\mathbb{T}}(f)\|_{L^{1/2}(E)}}{|E|}$$

and hence, by (5.4),

$$\frac{\|S_{\mathbb{T}}(f)\|_{L^{1/2}(E)}}{|E|} \lesssim \|f\|_{L \log^{1/2} L(\mathbb{T})}.$$

By taking the supremum with respect to all $E \subset \mathbb{T}$ with $|E| > 0$ and then by using (5.13), we obtain (5.12). □

5.4.2 The higher-dimensional case

In this subsection we extend (5.12) to higher dimensions, namely we obtain weak-type estimates for the n-parameter Littlewood-Paley square function $S_{\mathbb{T}^n}$. To do this, as in the one-dimensional case, we reduce the problem to the study of the corresponding mapping properties of certain randomised analogues of $S_{\mathbb{T}^n}$, namely we study first the mapping properties of Marcinkiewicz multiplier operators of the form $T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}$ on $\mathbb{T}^n$, where $\omega_i \in [0,1]$.

Having obtained lemma 5.2.1, it is now an easy task to get sharp weak-type estimates for Marcinkiewicz multiplier operators of the form $T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}$ on $\mathbb{T}^n$, $\omega_i \in [0,1]$. Indeed, by using (5.6) and induction, one can easily establish sharp weak-type estimates for Marcinkiewicz multiplier operators of the form $T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}$ on $\mathbb{T}^n$, $\omega_i \in [0,1]$.

Lemma 5.4.2. *Let $n \in \mathbb{N}$ be a given dimension.*

For $\omega_1, \cdots, \omega_n \in [0,1]$ consider the n-dimensional Marcinkiewicz multiplier operator $T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}$, where T_{ω_i} is as in section 5.2.

Then the operator $T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}$ *maps* $L\log^{a_n} L(\mathbb{T}^n)$ *to* $L^{1,\infty}(\mathbb{T}^n)$*, where* $a_n = 1/2 + 3(n-1)/2$*, and in particular,*

$$\|T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}(f)\|_{L^{1,\infty}(\mathbb{T}^n)} \lesssim_n \|f\|_{L\log^{a_n} L(\mathbb{T}^n)}. \tag{5.14}$$

Proof. First of all, note that if T is as in the statement of Lemma 5.2.1, then T satisfies (5.6).

To prove (5.14), we proceed by induction on $n \in \mathbb{N}$. The case $n = 1$ corresponds to (5.4). Assume now that for some integer $n > 1$ the desired inequality (5.14) holds. To obtain the $(n+1)$-dimensional case, fix an arbitrary $\alpha > 0$ and some f in $L\log^{a_{n+1}} L(\mathbb{T}^{n+1})$. Then, by using Fubini's theorem, we may write

$$|\{(x_1, \cdots, x_{n+1}) \in \mathbb{T}^{n+1} : |T_{\omega_1} \otimes \cdots \otimes T_{\omega_{n+1}}(f)(x_1, \cdots, x_{n+1})| > \alpha\}| =$$
$$\int_{\mathbb{T}} |\{(x_1, \cdots x_n) \in \mathbb{T}^n : |T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}(T_{\omega_{n+1}}(f))(x_1, \cdots, x_{n+1})| > \alpha\}| dx_{n+1}.$$

Hence, by our inductive hypothesis and Fubini's theorem,

$$\alpha|\{(x_1, \cdots, x_{n+1}) \in \mathbb{T}^{n+1} : |T_{\omega_1} \otimes \cdots \otimes T_{\omega_{n+1}}(f)(x_1, \cdots, x_{n+1})| > \alpha\}| \lesssim_n$$
$$1 + \int_{\mathbb{T}^n} \Big[\int_{\mathbb{T}} |T_{\omega_{n+1}}(f)(x_1, \cdots, x_{n+1})| \log^{a_n}(1 + |T_{\omega_{n+1}}(f)(x_1, \cdots, x_{n+1})|) dx_{n+1} \Big] dx_1 \cdots dx_n$$

and so, by (5.6) and Fubini's theorem,

$$\alpha|\{(x_1, \cdots, x_{n+1}) \in \mathbb{T}^{n+1} : |T_{\omega_1} \otimes \cdots \otimes T_{\omega_{n+1}}(f)(x_1, \cdots, x_{n+1})| > \alpha\}| \lesssim_n$$
$$1 + \int_{\mathbb{T}^n} \Big[\int_{\mathbb{T}} |f(x_1, \cdots, x_{n+1})| \log^{a_n + 3/2}(1 + |f(x_1, \cdots, x_{n+1})|) dx_{n+1} \Big] dx_1 \cdots dx_n$$

and the last quantity equals to

$$1 + \int_{\mathbb{T}^{n+1}} |f(x_1, \cdots, x_{n+1})| \log^{a_n + 3/2}(1 + |f(x_1, \cdots, x_{n+1})|) dx_1 \cdots dx_{n+1}.$$

Since $a_{n+1} = a_n + 3/2$, we obtain

$$\|T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}(f)\|_{L^{1,\infty}(\mathbb{T}^n)} \lesssim_n 1 + \int_{\mathbb{T}^n} |f| \log^{a_n}(1 + |f|). \tag{5.15}$$

The desired estimate (5.14) follows from (5.15) by a scaling argument similar to the one used in the proof of Lemma 5.2.1. Hence, the proof is complete. □

Arguing as in the one-dimensional case, we obtain the following result.

Proposition 5.4.3. *For any given* $n \in \mathbb{N}$*, there is a constant* $C_n > 0$ *such that the* n*-parameter Littlewood-Paley square function satisfies the weak-type inequality*

$$\|S_{\mathbb{T}^n}(f)\|_{L^{1,\infty}(\mathbb{T}^n)} \leqslant C_n \|f\|_{L\log^{a_n} L(\mathbb{T}^n)}, \tag{5.16}$$

for all trigonometric polynomials f *on* $\mathbb{T}^n$*, where* $a_n = 1/2 + 3(n-1)/2$*. Moreover, the exponent* a_n *in* (5.16) *is sharp.*

Proof. As in the one-dimensional case, we use Khintchine's inequality and (5.13) to show that there exists a choice of $\omega_1', \cdots, \omega_n' \in [0,1]$ such that

$$\|S_{\mathbb{T}^n}(f)\|_{L^{1,\infty}(\mathbb{T}^n)} \lesssim_n \|T_{\omega_1'} \otimes \cdots \otimes T_{\omega_n'}(f)\|_{L^{1,\infty}(\mathbb{T}^n)}.$$

Hence, by using (5.14), we deduce that $S_{\mathbb{T}^n}$ satisfies (5.16).

To prove that the exponent a_n in (5.16) cannot be improved, assume that the inequality holds for some $r \geqslant 0$. Since $S_{\mathbb{T}^n}$ is bounded on $L^2(\mathbb{T}^n)$, by interpolation and the fact that $\|f\|_{L\log^{r+1}L(\mathbb{T}^n)} \lesssim_{n,r} 1 + \int_{\mathbb{T}^n} |f| \log^{r+1}(1+|f|)$, it follows that $S_{\mathbb{T}^n}$ satisfies

$$\|S_{\mathbb{T}^n}(f)\|_{L^1(\mathbb{T}^n)} \lesssim_{n,r} 1 + \int_{\mathbb{T}^n} |f(x_1, \cdots, x_n)| \log^{r+1}(1 + |f(x_1, \cdots, x_n)|) dx_1 \cdots dx_n.$$

If we take f to be $V_{2^N} \otimes \cdots \otimes V_{2^N}$, then

$$\|S_{\mathbb{T}^n}(f)\|_{L^1(\mathbb{T}^n)} = \|S_{\mathbb{T}}(V_{2^N})\|_{L^1(\mathbb{T})} \cdots \|S_{\mathbb{T}}(V_{2^N})\|_{L^1(\mathbb{T})} \gtrsim N^{3n/2}$$

but $\int_{\mathbb{T}^n} |f| \log^{r+1}(1+|f|) \lesssim N^{r+1}$. Hence, by letting $N \to \infty$, we see that we must have $r \geqslant -1 + 3n/2 = a_n$. $\square$

Remark 5.4.4. *As mentioned in section* 5.3*, by using Lemma* 5.4.2*, interpolation, and converse extrapolation exactly as in the one-dimensional case, one can give an alternative proof to Proposition* 5.3.1*.*

5.5 Endpoint mapping properties of the rough Littlewood-Paley square function in the euclidean case

If f is a Schwartz function on $\mathbb{R}$, we define its rough Littlewood-Paley square function $S_{\mathbb{R}}(f)$ by

$$S_{\mathbb{R}}(f)(x) = \Big(\sum_{k \in \mathbb{Z}} |\Delta_k(f)(x)|^2 \Big)^{1/2},$$

where $(\Delta_k f)\hat{}(\xi) = \chi_{[2^k, 2^{k+1})}(\xi)\widehat{f}(\xi) + \chi_{(-2^{k+1}, -2^k]}(\xi)\widehat{f}(\xi)$ is the rough Littlewood-Paley projection at frequencies $|\xi| \sim 2^k$, $k \in \mathbb{Z}$. For $n \in \mathbb{N}$, the n-parameter rough Littlewood-Paley square function is given by

$$S_{\mathbb{R}^n}(f)(x) = \Big(\sum_{k_1, \cdots, k_n \in \mathbb{Z}} |\Delta_{k_1} \otimes \cdots \otimes \Delta_{k_n}(f)(x)|^2 \Big)^{1/2}$$

for f initially belonging to the class of Schwartz functions on $\mathbb{R}^n$.

In the following proposition we show that the n-parameter rough Littlewood-Paley square function on $\mathbb{R}^n$ satisfies sharp weak-type inequalities analogous to the ones obtained in the previous section, if we restrict ourselves to compact subsets of $\mathbb{R}^n$.

Proposition 5.5.1. *For any given $n \in \mathbb{N}$ and each compact set K in $\mathbb{R}^n$, there is a constant $C_{K,n} > 0$ such that the n-parameter Littlewood-Paley square function satisfies the weak-type inequality*

$$\|S_{\mathbb{R}^n}(f)\|_{L^{1,\infty}(K)} \leqslant C_{K,n} \|f\|_{L \log^{a_n} L(K)} \tag{5.17}$$

for each measurable function f supported in K, where $a_n = 1/2 + 3(n-1)/2$. Moreover, the exponent a_n in (5.17) is sharp.

Proof. The argument that establishes (5.17) is similar to the one given in the previous section, where one uses (5.3) instead of (5.4).

It remains to prove sharpness. Consider the one-dimensional case first. For this, assume that for some $r \geqslant 0$ one has

$$\|S_{\mathbb{R}}(f)\|_{L^{1,\infty}([-1,1])} \lesssim 1 + \int_{[-1,1]} |f| \log^r(1 + |f|)$$

for every measurable function f supported in $K = [-1, 1]$. Arguing as in the proof of Lemma 5.2.1, we deduce that

$$\|S_{\mathbb{R}}(f)\|_{L^1([-1,1])} \lesssim 1 + \int_{[-1,1]} |f| \log^{r+1}(1 + |f|) \tag{5.18}$$

for all measurable functions f with $\mathrm{supp}(f) \subset [-1, 1]$.

To show that $r \geqslant a_1 = 1/2$, we shall test (5.18) against a "continuous" analogue of the de la Vallée Poussin kernel used in the periodic case. More precisely, let N be a large positive integer to be chosen later and let ζ be a fixed Schwartz function such that $\mathrm{supp}(\zeta) \subset [-2, 2]$ and $\zeta|_{[-1,1]} \equiv 1$. Define $g(x) = 2^N \check{\zeta}(2^N x)$, $x \in \mathbb{R}$. Then g is a Schwartz function satisfying $\|g\|_{L^1(\mathbb{R})} \sim 1$, $\|g\|_{L^\infty(\mathbb{R})} \lesssim 2^N$, where the implied constants depend only on ζ and not on N. Hence,

$$\int_{[-1,1]} |g| \log^{r+1}(1 + |g|) \lesssim N^{r+1}. \tag{5.19}$$

Using Minkowski's inequality and the fact that $\widehat{g}|_{[-2^N, 2^N]} \equiv 1$ we get

$$\begin{aligned} \|S_{\mathbb{R}}(g)\|_{L^1([-1,1])} \geqslant \Big(\sum_{k \in \mathbb{Z}} \|\Delta_k(g)\|^2_{L^1([-1,1])} \Big)^{1/2} &\geqslant \Big(\sum_{k=1}^{N} \|\Delta_k(g)\|^2_{L^1([-1,1])} \Big)^{1/2} \\ &\gtrsim \Big(\sum_{k=1}^{N} k^2 \Big)^{1/2} \\ &\gtrsim N^{3/2}. \end{aligned}$$

However, g is not compactly supported. To consider an appropriate function f supported in $[-1, 1]$ which enables us to show that the exponent r in (5.18) is necessarily greater or equal than $1/2$, define $f = g\chi_{[-1,1]}$ and $e = g - f$. One can easily check that, by the construction of g, the "error" satisfies $\|e\|_{L^2(\mathbb{R})} \lesssim 1$. Moreover, f is supported in $[-1, 1]$ and

$$\|S_{\mathbb{R}}(g)\|_{L^1([-1,1])} \leqslant \sqrt{2}[\|S_{\mathbb{R}}(f)\|_{L^1([-1,1])} + \|S_{\mathbb{R}}(e)\|_{L^1([-1,1])}].$$

By using the Cauchy-Schwarz inequality,

$$\|S_{\mathbb{R}}(e)\|_{L^1([-1,1])} \leqslant \sqrt{2}\|S_{\mathbb{R}}(e)\|_{L^2([-1,1])} \leqslant \sqrt{2}\|S_{\mathbb{R}}(e)\|_{L^2(\mathbb{R})}$$

and since, by Plancherel's theorem, $\|S_{\mathbb{R}}(e)\|_{L^2(\mathbb{R})} = \|e\|_{L^2(\mathbb{R})} \lesssim 1$, we deduce that

$$\|S_{\mathbb{R}}(f)\|_{L^1([-1,1])} \gtrsim N^{3/2}. \tag{5.20}$$

Since $|f| \leqslant |g|$, (5.19) implies that

$$\int_{[-1,1]} |f| \log^{r+1}(1+|f|) \lesssim N^{r+1}. \tag{5.21}$$

Combining (5.18), (5.20) and (5.21), we get $N^{3/2} \lesssim N^{r+1}$. Letting $N \to \infty$, it follows that $r \geqslant a_1 = 1/2$, as desired.

To prove sharpness in the n-dimensional case, assume that (5.17) holds for some $r \geqslant 0$ and for f being as above, take $h = f \otimes \cdots \otimes f$. Then h is supported in $[-1,1]^n$, $\int_{[-1,1]^n} |h| \log^{r+1}(1+|h|) \lesssim N^{r+1}$ and

$$\|S_{\mathbb{R}^n}(h)\|_{L^1([-1,1]^n)} = \|S_{\mathbb{R}}(f)\|_{L^1([-1,1])} \cdots \|S_{\mathbb{R}}(f)\|_{L^1([-1,1])} \gtrsim N^{3n/2}.$$

Therefore, we must have $r \geqslant a_n = 1/2 + 3(n-1)/2$. □

Remark 5.5.2. *Tao's converse extrapolation theorem can also be stated for translation invariant operators acting on functions defined on $\mathbb{R}^n$, if we restrict our attention to compact sets in $\mathbb{R}^n$.*

In particular, exactly as in the periodic case presented in sections 5.2 *and* 5.3, *by using* (5.3) *for some given compact set $K \subset \mathbb{R}$, interpolation, and converse extrapolation, it follows that there are positive constants $d_1(n,K), d_2(n,K)$, depending on n and K, such that*

$$\frac{d_1(n,K)}{(p-1)^{3n/2}} < \|S_{\mathbb{R}^n}\|_{L^p(K^n)\to L^p(K^n)} < \frac{d_2(n,K)}{(p-1)^{3n/2}},$$

as $p \to 1^+$, where

$$\|S_{\mathbb{R}^n}\|_{L^p(K^n)\to L^p(K^n)} = \sup_{\substack{f \text{ measurable:} \\ \operatorname{supp}(f)\subset K^n, \\ \|f\|_{L^p(K^n)}=1}} \|S_{\mathbb{R}^n}(f)\|_{L^p(K^n)}.$$

5.6 Negative results

It is well-known that $S_{\mathbb{R}}$ maps $H^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$. Indeed, one may write

$$S_{\mathbb{R}}(f)(x) = \Big(\sum_{k\in\mathbb{Z}} |\Delta_k(f_k)(x)|^2\Big)^{1/2}, \tag{5.22}$$

where $f_k = \widetilde{\Delta}_k(f)$ and $\widetilde{\Delta}_k$ denotes an appropriate smoothed-out version of Δ_k. By Corollary 2.13 on page 488 of [26], one has

$$\Big|\Big\{x \in \mathbb{R} : \Big(\sum_{k\in\mathbb{Z}} |\Delta_k(f_k)(x)|^2\Big)^{1/2} > \alpha\Big\}\Big| \leqslant \frac{C}{\alpha}\Big\|\Big(\sum_{k\in\mathbb{Z}} |f_k|^2\Big)^{1/2}\Big\|_{L^1(\mathbb{R})} \tag{5.23}$$

for every $\alpha > 0$. Hence, the estimate $\|S_{\mathbb{R}}(f)\|_{L^{1,\infty}(\mathbb{R})} \lesssim \|f\|_{H^1(\mathbb{R})}$ follows from (5.22), (5.23), and the square function characterisation of $H^1(\mathbb{R})$. Similarly, the Littlewood-Paley square function $S_{\mathbb{T}}$ maps $H^1(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$.

A natural question is whether an analogous weak-type estimate holds for the two-parameter rough Littlewood-Paley square function. In the two-parameter setting, a candidate endpoint function space is the product Hardy space $H^1_{\text{prod}}(\mathbb{R}^2)$. Our next result shows that such an estimate is not possible in the product setting, as $S_{\mathbb{R}^2}$ does not even locally map $H^1_{\text{rect}}(\mathbb{R}^2)$ to $L^{1,\infty}(\mathbb{R}^2)$.

Proposition 5.6.1. *The two-parameter rough Littlewood-Paley square function does not locally map $H^1_{\text{rect}}(\mathbb{R}^2)$ to $L^{1,\infty}(\mathbb{R}^2)$.*

Proof. To show that $S_{\mathbb{R}^2}$ does not locally map $H^1_{\text{prod}}(\mathbb{R}^2)$ to $L^{1,\infty}(\mathbb{R}^2)$, it is enough to prove that, there is a compact set $K \subset \mathbb{R}^2$ such that for every $N \in \mathbb{N}$ there exists a function f_N with $\operatorname{supp}(f_N) \subset K$ and such that

$$\|S_{\mathbb{R}^2}(f_N)\|_{L^{1,\infty}(K)}/\|f_N\|_{H^1_{\text{prod}}(\mathbb{R}^2)} \gtrsim N.$$

Towards this aim, observe that if a function g is supported in $K \subset \mathbb{R}^2$, belongs to $L\log^2 L(K)$, and has zero integral with respect to each variable, then $g \in H^1_{\text{prod}}(\mathbb{R}^2)$ and, in particular, we have $\|g\|_{H^1_{\text{prod}}(\mathbb{R}^2)} \lesssim 1 + \int_K |g|\log^2(1+|g|)$. Moreover, as we saw in the previous section, $S_{\mathbb{R}^2}$ locally maps $L\log^2 L$ to $L^{1,\infty}$ and hence

$$\frac{\|S_{\mathbb{R}^2}(g)\|_{L^{1,\infty}(K)}}{1 + \int_K |g|\log^2(1+|g|)} \lesssim_K 1,$$

whenever g is supported in K (not necessarily with mean zero in each variable). Therefore, in order to exhibit a suitable function f_N that is supported in $K \subset \mathbb{R}^2$, such that the ratio $\|S_{\mathbb{R}^2}(f_N)\|_{L^{1,\infty}(K)}/\|f_N\|_{H^1_{\text{prod}}(\mathbb{R}^2)}$ is "large", we need to find an f such that $\int |f_N|\log^2(1+|f|)$ is "large", whereas $\|f_N\|_{H^1_{\text{prod}}(\mathbb{R}^2)}$ is "small".

Motivated by the above remarks, let $K = [0,1]^2$ and for a given large positive integer $N \geqslant 5$, we consider the function $a_N(x) = 2^{N-1}e^{i2\pi 2^{N-1}x}\chi_{[0,2^{-(N-1)})}(x)$. Note that if we set $f_N = a_N \otimes a_N$, then f_N is an atom in $H^1_{\text{rect}}(\mathbb{R}^2)$ and thus $\|f_N\|_{H^1_{\text{prod}}(\mathbb{R}^2)} \lesssim \|f_N\|_{H^1_{\text{rect}}(\mathbb{R}^2)} \sim 1$, whereas $\int |f_N|\log^r(1+|f_N|) \sim N^r$, for $r \geqslant 0$.

We shall prove that $\|S_{\mathbb{R}^2}(f_N)\|_{L^{1,\infty}([0,1]^2)} \gtrsim N$, with $f_N = a_N \otimes a_N$ being as above. For this, observe that for $x \neq 0$ the kernel of Δ_{N-1} is given by

$$\int_{2^{N-1}}^{2^N} e^{i2\pi\xi x}d\xi + \int_{-2^N}^{-2^{N-1}} e^{i2\pi\xi x}d\xi = \frac{e^{i2\pi 2^N x} - e^{i2\pi 2^{N-1}x}}{i2\pi x} + \frac{e^{-i2\pi 2^{N-1}x} - e^{-i2\pi 2^N x}}{i2\pi x}.$$

Hence, for $8 \cdot 2^{-(N-1)} \leqslant x \leqslant 1$ one can write

$$\Delta_{N-1}(a_N)(x) = I_1^{(N)}(x) + I_2^{(N)}(x) + I_3^{(N)}(x) + I_4^{(N)}(x),$$

where

- $I_1^{(N)}(x) = -\frac{2^{N-1}e^{i2\pi 2^{N-1}x}}{i2\pi}\int_{[0,2^{-(N-1)})}\frac{1}{x-y}dy$,
- $I_2^{(N)}(x) = \frac{2^{N-1}e^{i2\pi 2^{N}x}}{i2\pi}\int_{[0,2^{-(N-1)})}\frac{e^{-i2\pi 2^{N-1}y}}{x-y}dy$,
- $I_3^{(N)}(x) = \frac{2^{N-1}e^{-i2\pi 2^{N-1}x}}{i2\pi}\int_{[0,2^{-(N-1)})}\frac{e^{-i2\pi 2^{N}y}}{x-y}dy$ and
- $I_4^{(N)}(x) = -\frac{2^{N-1}e^{-i2\pi 2^{N}x}}{i2\pi}\int_{[0,2^{-(N-1)})}\frac{e^{-i6\pi 2^{N-1}y}}{x-y}dy$.

Note that for each $8\cdot 2^{-(N-1)} \leqslant x \leqslant 1$ one has

$$|I_1^{(N)}(x)| = \frac{2^{N-1}}{2\pi}\int_{[0,2^{-(N-1)})}\frac{1}{x-y}dy \geqslant \frac{1}{2\pi x}.$$

We shall bound $|I_2^{(N)}(x)|$, $|I_3^{(N)}(x)|$ and $|I_4^{(N)}(x)|$ from above. To bound $|I_2^{(N)}(x)|$, we make use of the cancellation of $e^{-i2\pi 2^{N-1}y}$ over $[0,2^{-(N-1)})$,

$$\begin{aligned}|I_2^{(N)}(x)| &= \frac{2^{N-1}}{2\pi}\Big|\int_{[0,2^{-(N-1)})} e^{-i2\pi 2^{N-1}y}\big(\frac{1}{x-y} - \frac{1}{x-2^{-1}\cdot 2^{-(N-1)}}\big)dy\Big| \\ &\leqslant \frac{2^{N-1}}{2\pi}\int_{[0,2^{-(N-1)})}\Big|\frac{2^{-1}\cdot 2^{-(N-1)}-y}{(x-y)(x-2^{-1}\cdot 2^{-(N-1)})}\Big|dy \\ &\leqslant \frac{2}{15\pi x},\end{aligned}$$

since $x-y \geqslant x/2$ for all $y \in [0,2^{-(N-1)})$ and $x-2^{-1}\cdot 2^{-(N-1)} \geqslant 15x/16$. Similarly, $|I_3^{(N)}(x)| \leqslant 2/(15\pi x)$ and $|I_4^{(N)}(x)| \leqslant 2/(15\pi x)$. Therefore,

$$|\Delta_{N-1}(a_N)(x)| \geqslant |I_1^{(N)}(x)| - |I_2^{(N)}(x)| - |I_3^{(N)}(x)| - |I_4^{(N)}(x)| \geqslant \frac{1}{10\pi x}$$

for all $8\cdot 2^{-(N-1)} \leqslant x \leqslant 1$ and hence,

$$S_{\mathbb{R}^2}(a_N\otimes a_N)(x,y) \geqslant |(\Delta_{N-1}\otimes\Delta_{N-1})(a_N\otimes a_N)(x,y)| \geqslant \frac{1}{100\pi^2 xy}$$

for $(x,y)\in[8\cdot 2^{-(N-1)},1]^2$. It thus follows that

$$\|S_{\mathbb{R}^2}(a_N\otimes a_N)\|_{L^{1,\infty}([0,1]^2)} \gtrsim N.$$

Since $\|a_N\otimes a_N\|_{H^1_{\mathrm{rect}}(\mathbb{R}^2)} \lesssim 1$, by letting $N\to\infty$, one deduces that $S_{\mathbb{R}^2}$ does not locally map $H^1_{\mathrm{rect}}(\mathbb{R}^2)$ to $L^{1,\infty}(\mathbb{R}^2)$. □

By adapting the argument of the previous proposition to the periodic setting, we obtain a corresponding negative result for functions on the two-torus.

Proposition 5.6.2. *The two-parameter Littlewood-Paley square function $S_{\mathbb{T}^2}$ does not map $H^1_{\mathrm{rect}}(\mathbb{T}^2)$ to $L^{1,\infty}(\mathbb{T}^2)$.*

Proof. Let $N \geqslant 9$ be an integer to be chosen later. We decompose the kernel of D_N as

$$\sum_{n=2^{N-1}}^{2^N-1} e^{i2\pi nx} = \frac{e^{i2\pi 2^N x} - e^{i2\pi 2^{N-1}x}}{e^{i2\pi x}-1} = \beta_N(x) + \gamma_N(x),$$

where for $x \in (0,1)$ one has

$$\beta_N(x) = (e^{i2\pi 2^N x} - e^{i2\pi 2^{N-1}x})\Big(\frac{1}{e^{i2\pi x}-1} - \frac{1}{i2\pi x}\Big) \text{ and } \gamma_N(x) = \frac{e^{i2\pi 2^N x} - e^{i2\pi 2^{N-1}x}}{i2\pi x}$$

and $\beta_N(0) = 0$, $\gamma_N(0) = 2^{N-1}$. Define $a_N(x) = 2^{N-1} e^{i2\pi 2^{N-1}x} \chi_{[0,2^{-(N-1)})}(x)$ for $x \in [0,1)$. Arguing as in the proof of Proposition 5.6.1, one shows that

$$|\gamma_N * a_N(x)| \geqslant \frac{11}{30\pi x}$$

for all $8 \cdot 2^{-(N-1)} \leqslant x < 1$. Using the series expansion of $e^{i2\pi x}$ and the fact that $\sin(2\pi x) \geqslant 4x$ for every $0 \leqslant x \leqslant 2^{-2}$, one obtains $|\beta_N(x)| \leqslant \pi e^{\pi/2}$ for all $0 \leqslant x \leqslant 2^{-2}$. Since $\|a_N\|_{L^1(\mathbb{T})} = 1$, it follows that $|\beta_N * a_N(x)| \leqslant \pi e^{\pi/2}$ for every $2^{-(N-1)} \leqslant x \leqslant 2^{-2}$. Therefore, for each $8 \cdot 2^{-(N-1)} \leqslant x \leqslant 2^{-8}$ one has

$$|D_N(a_N)(x)| \geqslant |\gamma_N * a_N(x)| - |\beta_N * a_N(x)| \geqslant \frac{11}{30\pi x} - \pi e^{\pi/2} \geqslant \frac{11}{30\pi x} - \frac{1}{16x} \sim \frac{1}{x}.$$

Since we may regard $a_N \otimes a_N$ as an atom of $H^1_{\text{rect}}(\mathbb{T}^2)$, by letting $N \to \infty$, we deduce that $S_{\mathbb{T}^2}$ does not map $H^1_{\text{rect}}(\mathbb{T}^2)$ to $L^{1,\infty}(\mathbb{T}^2)$. □

Remark 5.6.3. *The negative results of this section suggest that two-dimensional Marcinkiewicz multipliers do not map $H^1_{\text{rect}}(\mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$. This fact can also be obtained by making use of the construction of section* 3.1 *of [62].*

5.7 Remarks on the mapping properties of general two-parameter Marcinkiewicz multipliers

It follows by the arguments of sections 5.4 and 5.5 that n-fold tensor products of one-dimensional Marcinkiewicz multiplier operators on $\mathbb{R}^n$ locally map $L\log^{a_n} L$ to $L^{1,\infty}$, where $a_n = 1/2 + 3(n-1)/2$ and that the exponent a_n cannot be improved[2]. Therefore, it seems natural to conjecture that two-parameter Marcinkiewicz multiplier operators locally map $L\log^2 L$ to $L^{1,\infty}$, i.e. for every two-parameter Marcinkiewicz multiplier operator T_m and every compact set $K \subset \mathbb{R}^2$ there exists a constant $C_{m,K} > 0$ such that

$$\|T_m(f)\|_{L^{1,\infty}(K)} \leqslant C_{m,K} \|f\|_{L\log^2 L(K)} \tag{5.24}$$

for each measurable function f supported in K.

[2]Note that sharpness can also be obtained by using the example given in section 3.2 of [62].

It is worth noting that in the two-dimensional case, the expected endpoint space $L\log^2 L$ in (5.24) is strictly smaller than the space $L\log L$ appeared in the two-dimensional Zygmund's inequality involving two-fold products of lacunary sequences in $\mathbb{Z}$. Hence, one expects that to prove the above conjecture, one cannot just iterate the arguments used in the corresponding one-dimensional problem.

Appendix A

Equivalent formulations of Zygmund-type inequalities

Let G be a compact abelian group such that $\widehat{G}$ is countable. Fix $r > 0$ and let Λ be a infinite set in $\widehat{G}$.

In Chapters 3 and 4 we studied inequalities of the form

$$\Big(\sum_{\gamma\in\Lambda}|\widehat{f}(\gamma)|^2\Big)^{1/2} \leqslant B_{\Lambda,r} + A_{\Lambda,r}\int_G |f(x)|\log^r(1+|f(x)|)dx, \tag{A.1}$$

where the positive constants $A_{\Lambda,r}$ and $B_{\Lambda,r}$ depend only on Λ and r.

Our goal in this appendix is to give a proof of the well-known fact that (A.1) holds if and only if, Λ is a $\Lambda(p)$ set for all $p > 2$ with $\Lambda(p)$ constant $A(p,\Lambda) = C_{\Lambda,r}p^r$, $p > 2$, where $C_{\Lambda,r} > 0$ is a constant independent of p. Namely, we shall prove in this appendix that (A.1) is equivalent to the fact that for every Λ-polynomial g one has

$$\|g\|_{L^p(G)} \leqslant C_{\Lambda,r}p^r\|g\|_{L^2(G)} \tag{A.2}$$

for all $p > 2$, where $C_{\Lambda,r} > 0$ is a constant that depends on Λ and r and not on p, g. The presentation here is based on the corresponding ones given in [10] and [65].

Assume first that (A.1) holds. To prove (A.2), take a Λ-polynomial g in the unit ball of $L^2(G)$ and note that by (2.15) one has

$$\|g\|_{\Psi_r(G)} \leqslant D_r \sup_{\|f\|_{\Phi_r(G)}\leqslant 1}\Big|\int_G g(x)\overline{f(x)}dx\Big|,$$

where $D_r > 0$ is a constant that depends on r (and on the particular choices of Ψ_r and Φ_r). Hence, by using Parseval's formula, the Cauchy-Schwarz inequality, and (A.1), we obtain

$$
\begin{aligned}
\|g\|_{\Psi_r(G)} &\leqslant D_r \sup_{\|f\|_{\Phi_r(G)}\leqslant 1} \Big|\sum_{\gamma\in\Lambda} \widehat{g}(\gamma)\overline{\widehat{f}(\gamma)}\Big| \\
&\leqslant D_r \sup_{\|f\|_{\Phi_r(G)}\leqslant 1} \Big(\sum_{\gamma\in\Lambda} |\widehat{g}(\gamma)|^2\Big)^{1/2}\Big(\sum_{\gamma\in\Lambda} |\widehat{f}(\gamma)|^2\Big)^{1/2} \\
&\leqslant D_r \sup_{\|f\|_{\Phi_r(G)}\leqslant 1} \Big[B_{\Lambda,r} + A_{\Lambda,r}\int_G |f(x)| \log^r(1+|f(x)|)dx\Big] \\
&\leqslant K_{\Lambda,r},
\end{aligned}
$$

where $K_{\Lambda,r} > 0$ is a constant that depends only on D_r, $A_{\Lambda,r}$, and $B_{\Lambda,r}$. In the above chain of inequalities we used our assumption that $\big(\sum_{\gamma\in\Lambda} |\widehat{g}(\gamma)|^2\big)^{1/2} = \|g\|_{L^2(G)} = 1$ and the fact that if $\|f\|_{\Phi_r(G)} \leqslant 1$, then $\int_G \Phi_r(|f(x)|)dx \leqslant 1$. So, it follows that $\|K_{\Lambda,r}^{-1} g\|_{\Psi_r(G)} \leqslant 1$ and thus,

$$
\int_G \Psi_r(K_{\Lambda,r}^{-1}|g(x)|)dx \leqslant 1.
$$

By considering the definition of Ψ_r and by using the series expansion of $\exp(x^{1/r})$ together with Tonelli's theorem, we deduce that there exists a constant[1] L_r, depending on $K_{\Lambda,r}$ and not on g, such that

$$
\sum_{n\in\mathbb{N}} \frac{1}{n!} K_{\Lambda,r}^{-n/r} \|g\|_{L^{n/r}(G)}^{n/r} \leqslant L_r.
$$

Hence, by using Stirling's approximation formula,

$$
\|g\|_{L^{n/r}(G)} \leqslant \widetilde{L}_r (n/r)^r
$$

for all $n \in \mathbb{N}$, where $\widetilde{L}_r > 0$ is a constant depending only on $K_{\Lambda,r}$, L_r and hence only on Λ and r. Now, for $p > 2$ there exists an integer $n \geqslant 1$ such that $n/r < p \leqslant (n+1)/r$. Since $\|g\|_{L^p(G)} \leqslant \|g\|_{L^{(n+1)/r}(G)}$, one obtains the desired inequality (A.2) with constant $C_{\Lambda,r} = 2^r \widetilde{L}_r$, $\widetilde{L}_r$ being as above.

To prove the opposite direction, assume that (A.2) holds and let μ be a positive constant to be chosen later. For each Λ-polynomial g one has

$$
\int_G e^{(\mu|g(x)|)^{1/r}} dx = \sum_{n\in\mathbb{N}_0} \frac{\mu^{n/r}}{n!} \int_G |g(x)|^{n/r} dx \leqslant \sum_{n\in\mathbb{N}_0} \frac{n^n}{n!} \Big(\frac{[\mu C_{\Lambda,r} \|g\|_{L^2(G)}]^{1/r}}{r}\Big)^n.
$$

Take a Λ-polynomial g with $\|g\|_{L^2(G)} = 1$. Since for every $n \in \mathbb{N}$ one has

$$
n^n/n! \leqslant \sum_{k\in\mathbb{N}_0} n^k/k! = e^n,
$$

[1] If $0 < r \leqslant 1$ and $\Psi_r(x) = \exp(x^{1/r}) - 1$, then one can take $L_r = 1$. If $r > 1$ and $\Psi_r(x) = \exp(x^{1/r}) - B_r x - A_r$ for $x > (r-1)^r$ and $\Psi_r(x) = 0$ for $0 \leqslant x \leqslant (r-1)^r$, where $B_r = r^{-1}(r-1)^{1-r}\exp(r-1)$ and $A_r = r^{-1}\exp(r-1)$, then one can take $L_r = K_{\Lambda,r}^{-1} B_r + A_r + \exp(r-1)$.

it follows that

$$\int_G e^{(\mu|g(x)|)^{1/r}}dx \leqslant \sum_{n\in\mathbb{N}_0} \Big(\frac{[\mu C_{\Lambda,r}]^{1/r}e}{r}\Big)^n$$

and the series converges when $\mu < \mu_0(r,\Lambda) = r^r/C_{\Lambda,r}e^r$. So, if we set

$$\delta(r,\Lambda;\mu) = \frac{r}{r-(\mu C_{\Lambda,r})^{1/r}e}$$

for $0 < \mu < \mu_0(r,\Lambda)$, then

$$\int_G e^{(\mu|g(x)|)^{1/r}}dx \leqslant \delta(r,\Lambda;\mu)$$

for every Λ-polynomial g in the unit ball of $L^2(G)$. In particular,

$$\int_G e^{(\mu_1|g(x)|)^{1/r}}dx \leqslant \delta_1 \tag{A.3}$$

where $\mu_1 = \mu_0(r,\Lambda)/2$ and $\delta_1 = \delta(r,\Lambda;\mu_1)$. In order to show that (A.1) holds, take a finite set $\{e_1,\cdots,e_N\}$ of points in Λ. Then, by using duality and Parseval's identity, we have

$$\begin{aligned}\Big(\sum_{n=1}^N |\widehat{f}(e_n)|^2\Big)^{1/2} &= \sup_{\|(c_n)_{n=1}^N\|_{\ell^2}=1} \Big|\sum_{n=1}^N \widehat{f}(e_n)\overline{c_n}\Big| \\ &= \sup_{\|(c_n)_{n=1}^N\|_{\ell^2}=1} \Big|\int_G f(x)\overline{g_N(x)}dx\Big|,\end{aligned}$$

where, given a finite sequence $(c_n)_{n=1}^N$ with $\|(c_n)_{n=1}^N\|_{\ell^2} = 1$, g_N is the Λ-polynomial whose Fourier coefficients are given by $\widehat{g_N}(e_n) = c_n$ for $n = 1,\cdots,N$. Since, by construction, $\|g_N\|_{L^2(G)} = \|(c_n)_{n=1}^N\|_{\ell^2} = 1$, we have by using (2.8), (2.10), (A.3) as well as the fact that $x[1+\log(x+1)]^r \sim_r 1 + x\log^r(x+1)$ for all $x \geqslant 0$,

$$\begin{aligned}\Big|\int_G f(x)\overline{g_N(x)}dx\Big| &\leqslant \frac{1}{\mu_1}\Big[U_r\int_G |f(x)|\log^r(|f(x)|+1)dx + C_r\int_G e^{(\mu_1|g_N(x)|)^{1/r}}dx + V_r\Big] \\ &\leqslant \frac{1}{\mu_1}\Big[U_r\int_G |f(x)|\log^r(|f(x)|+1)dx + C_r\delta_1 + V_r\Big],\end{aligned}$$

where U_r, V_r, C_r are positive constants depending only on r and $\mu_1 = \mu_0(r,\Lambda)/2$, $\delta_1 = \delta(r,\Lambda;\mu_1)$ are as above. Therefore, for every finite subset $\{e_1,\cdots,e_N\}$ of Λ we have

$$\Big(\sum_{n=1}^N |\widehat{f}(e_n)|^2\Big)^{1/2} \leqslant \frac{U_r}{\mu_1}\int_G |f(x)|\log^r(|f(x)|+1)dx + \frac{C_r\delta_1+V_r}{\mu_1}$$

and so, it follows that (A.1) holds with constants $A_{\Lambda,r} = U_r/\mu_1$ and $B_{\Lambda,r} = (C_r\delta_1+V_r)/\mu_1$.

Appendix B

Khintchine's inequality

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $(r_n)_{n\in F}$ be a countable collection of independent random variables over $(\Omega, \mathcal{A}, \mathbb{P})$ such that

$$\mathbb{P}(\{\omega \in \Omega : r_n(\omega) = 1\}) = \mathbb{P}(\{\omega \in \Omega : r_n(\omega) = -1\}) = 1/2$$

for all $n \in F$. Khintchine's inequality asserts that for every $p > 0$ and for all finite sets of complex numbers $(a_n)_{n\in F'}$, $F' \subset F$ finite, one has

$$\Big\| \sum_{n\in F'} a_n r_n \Big\|_{L^p(\Omega)} \sim_p \Big(\sum_{n\in F'} |a_n|^2 \Big)^{1/2}$$

where the implied constant depends only on p and not on $(a_n)_{n\in F'}$. In particular, there exists an absolute constant C such that for every finite set of complex numbers $(a_n)_{n\in F'}$ one has

$$\Big\| \sum_{n\in F'} a_n r_n \Big\|_{L^p(\Omega)} \leqslant C p^{1/2} \Big(\sum_{n\in F'} |a_n|^2 \Big)^{1/2} \tag{B.1}$$

for all $p > 2$.

A multi-dimensional version of (B.1) states that for all finite sets of complex numbers $(a_{k_1,\cdots,k_n})_{k_1,\cdots,k_n\in F'}$, $F' \subset F$ finite, one has for all $p > 0$

$$\Big\| \sum_{k_1,\cdots,k_n\in F'} a_{k_1,\cdots,k_n} r_{k_1} \cdots r_{k_n} \Big\|_{L^p(\Omega^n)} \sim_{n,p} \Big(\sum_{k_1,\cdots,k_n\in F'} |a_{k_1,\cdots,k_n}|^2 \Big)^{1/2}$$

and in particular, there exists a constant $C_n > 0$ such that

$$\Big\| \sum_{k_1,\cdots,k_n\in F'} a_{k_1,\cdots,k_n} r_{k_1} \cdots r_{k_n} \Big\|_{L^p(\Omega^n)} \leqslant C_n p^{n/2} \Big(\sum_{k_1,\cdots,k_n\in F'} |a_{k_1,\cdots,k_n}|^2 \Big)^{1/2} \tag{B.2}$$

for all $p > 2$. For proofs of (B.1) and (B.2) see, e.g., Appendix D in [59].

Remark B.0.1. *A concrete example of a sequence of random variable satisfying the properties above is that of Rademacher functions $(r_k)_{k\in\mathbb{N}_0}$ defined over $[0,1]$ by $r_k(\omega) = \mathrm{sign}\{\sin(2\pi(2^k\omega))\}$, $\omega \in [0,1]$. Note that automatically $\|r_k\|_{L^2([0,1])} = 1$ for all $k \in \mathbb{N}_0$ and, moreover, for each finite set of indices $j_1 < \cdots < j_n$ one has* *the property*

$$\int_{[0,1]} \prod_{k=1}^{n} r_{j_k}(\omega) d\omega = 0.$$

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