Birational geometry of Fano fibrations

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Contents

Abstract

An algebraic variety is called rationally connected if two generic points can be connected by a curve isomorphic to the projective line. The output of the minimal model program applied to rationally connected variety is variety admitting Mori fiber spaces over a rationally connected base. In this thesis we study the birational geometry of a particular class of rationally connected Mori fiber spaces: Fano fibrations over the projective line.

We construct examples of Fano fibrations with a unique Mori fiber space in their birational classes. We prove that these examples are not birational to varieties of Fano type, thus answering the question of Cascini and Gongyo. That is we prove that the classes of rationally connected varieties and varieties of Fano type are not birationally equivalent. To construct the examples we use the techniques of birational rigidity.

A Mori fiber space is called birationally rigid if there is a unique Mori fiber space structure in its birational class. The birational rigidity of smooth varieties admitting a del Pezzo fibration of degrees 1 and 2 is a well studied question. Unfortunately it is not enough to study smooth del Pezzo fibrations as there are fibrations which do not have smooth or even smoothable minimal models. In the case of fibrations of degree 2 we know that there is a minimal model with 2-Gorenstein singularities. These singularities are degenerations of the simplest terminal quotient singularity: singular points of the type $\frac{1}{2}(1,1,1)$. We give first examples of birationally rigid del Pezzo fibrations with 2-Gorenstein singularities. We then apply this result to study finite subgroups of the Cremona group of rank three.

We then study the birational geometry of Fano fibrations from a different side. Using the reduction to characteristic 2 method we prove that double covers of $\mathbb{P}^n$-bundles over $\mathbb{P}^m$ branched over a divisor of sufficiently high degree are not stably rational. For a del Pezzo fibration $Y \to \mathbb{P}^1$ of degree 2 such that X is smooth there is a double cover $Y \to X$, where X is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. In this case a stronger result holds: a very general Y with $\mathrm{Pic}(Y) \cong \mathbb{Z} \oplus \mathbb{Z}$ is not stably rational. We discuss the proof of this statement.

Chapter 1

Introduction

The rationality question is an old problem in algebraic geometry. The question was solved in dimension 2 by Castelnuovo, who discovered "if and only if" condition for a surface to be rational. In higher dimensions there is no such criterion.

Rational varieties are rationally connected, thus we only need to study the question for rationally connected varieties. The rationally connected varieties have good representatives in their birational classes: Mori fiber spaces with rationally connected base. In dimension 2 these are Hirzebruch surfaces and $\mathbb{P}^2$, which are rational. In dimension 3 these are Fano varieties, conic bundles over rational surfaces, and del Pezzo fibrations over the projective line, many of these are not rational.

All results in this thesis touch on birational geometry of del Pezzo fibrations, rationality in particular.

1.1 Birational rigidity

One way to show that a variety is not rational is to prove that it has a "unique" Mori fiber space in its birational class. Such varieties are called birationally rigid.

Definition 1.1 ([54])**.** We say that a variety X is in the *Mori category* if X a $\mathbb{Q}$-factorial variety with terminal singularities.

Definition 1.2 ([54])**.** Let X be a variety in Mori category. We say that $\pi : X \to Z$ is a *Mori fiber space* if

- $\dim Z < \dim X$,
- $\operatorname{rk}(\operatorname{Pic} X / \operatorname{Pic} Z) = 1$, and
- a generic fiber of π is a Fano variety.

We say that a variety X admits a Mori fiber space if X is in the Mori category and there exists a Mori fiber space $\pi : X \to Z$ for some Z.

If $\dim Z = 0$, that is Z is a point, we say that X is a *Fano variety.* Note that whenever we say that X is a Fano variety we assume that X is in a Mori category and $\operatorname{Pic}(X) \cong \mathbb{Z}$. If $\dim X = \dim Z + 1$, then any fiber of π is isomorphic to a plane conic. Then we say that $\pi : X \to Z$ is a *conic bundle.* If $\dim X = \dim Z + 2$, then a generic fiber is a del Pezzo surface. In this case we say that a Mori fiber space $\pi : X \to Z$ is a *del Pezzo fibration.* We say that $\pi : X \to Z$ is a del Pezzo fibration of degree n if a general fiber F of π satisfies $K_F^2 = n$.

Definition 1.3 ([29])**.** We say that a Fano variety X is *birationally superrigid* any birational map to a variety admitting a Mori fiber space is an isomorphism. Note that when we say that a variety admits a Mori fiber space we imply that it is in the Mori category. In particular we assume that X is in the Mori category.

We say that a Fano variety X is *birationally rigid* if it is a unique variety admitting a Mori fiber space in its birational class, that is for any birational map $\chi : X \dashrightarrow Y$ to a variety Y admitting a Mori fiber space we have $X \cong Y$.

We say that a Mori fiber space $\pi : X \to \mathbb{P}^1$ is *birationally rigid* (*superrigid*) if for any birational map $\chi : X \dashrightarrow Y$ to a variety admitting a Mori fiber space $\pi_Y : Y \to Z$, the base Z is the projective line, the following diagram is commutative

$$\begin{array}{ccc} X & \stackrel{\chi}{\dashrightarrow} & Y \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ \mathbb{P}^1 & = & Z, \end{array}$$

and the general fibers of X and Y are isomorphic (χ is an isomorphism on a general fiber of X).

The anticanonical class of birationally rigid Mori fiber spaces over non-trivial bases satisfies certain negativity conditions, for example the K^2-condition.

Definition 1.4 ([10, Definition 3])**.** Let $\pi : X \to \mathbb{P}^1$ be a del Pezzo fibration. We say that X satisfies K^2*-condition* if K_X^2 is not in the interior of the closure of the cone of effective curves $\overline{NE}(X)$.

The following theorem is the main result of Chapter 3.

Theorem 1.5. *Let $\pi : X \to \mathbb{P}^1$ be a del Pezzo fibration of degree 2 with only $\frac{1}{2}(1,1,1)$-singularities (simplest terminal quotient singularities). Suppose X is a quasismooth hypersurface in a toric $\mathbb{P}(1,1,1,2)$-scroll over $\mathbb{P}^1$. Suppose X satisfies K^2-condition. Suppose also that the equations in $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$ of the fibers of π containing the singularities of X are of the form $q_4(x,y,z) = 0$. Then X is birationally rigid, in particular not rational.*

Example 1.6. Let X be a hypersurface of bi-degree $(4,l)$ in $\mathbb{P}(1_x, 1_y, 1_z, 2_w) \times \mathbb{P}^1_{u,v}$. Then $\pi : X \to \mathbb{P}^1_{u,v}$ is a del Pezzo fibration of degree 2. Since $-K_X$ is of bi-degree $(1, 2-l)$, the variety X satisfies the K^2-condition when $l \geqslant 2$. In particular it is not satisfied for $l = 0$, that is when X is a direct product. Let the equation of X be $p_l(u,v)w^2 = q_{4,l}(x,y,z;u,v)$, where p and q are generic polynomials of degree l in u, v and degree 4 in x, y, z. Then the fibers containing singularities are quartic cones and X satisfies the assumptions of Theorem 1.5.

The motivation behind Theorem 1.5 is two-fold. Rationality questions appear naturally in the study of finite subgroups of the Cremona group of rank 3. We need to consider the rationality of certain singular del Pezzo fibrations in order to classify embeddings of $\mathrm{PSL}_2(7)$ into the Cremona group. We will discuss it more in the next section. On the other hand, the rationality question was considered mostly for smooth del Pezzo fibrations. Theorem 1.5 generalizes the following theorem.

Theorem 1.7 ([70])**.** *Let $\pi : X \to \mathbb{P}^1$ be a del Pezzo fibration of degree 1 or 2. Suppose X is smooth and satisfies the K^2-condition. Then the variety X is birationally rigid, in particular X is not rational.*

We would like to extend the rationality results to the entire Mori category but sometimes we do not have to. Let $\pi : X \to S$ be a conic bundle over a rational surface. Then X birational to a smooth variety X' such that there is conic bundle $\pi' : X' \to S'$. Thus we can study the rationality of this smooth model X'.

On the other hand sometime we cannot use birational maps to improve the singularities. For example, a quintic X in $\mathbb{P}(1,1,1,1,2)$ is a Fano variety and has a singular point of the type $\frac{1}{2}(1,1,1)$. The variety X belongs to the 95 families [43, List 16.6] of index 1 Fano hypersurfaces. It was shown that these varieties are birationally rigid ([30], [16]), hence have a unique Mori fiber space in their birational class. Thus we cannot find a smooth model which also admits a Mori fiber space in the birational class of X. Recently more examples of Fano varieties satisfying this property have been found by Ahmadinezhad, Okada and Zucconi ([55], [2], [3]).

Del Pezzo fibrations are in-between: there are "good" models, but they are not necessarily smooth or even Gorenstein [28]. The "good" models of del Pezzo fibrations over $\mathbb{P}^1$ of degree 2 are 2-Gorenstein, that is double canonical class is a Cartier divisor. Hence good models may not admit smoothing since the singularities of 2-Gorenstein varieties are degenerations of the $\frac{1}{2}(1,1,1)$-singularity which does not admit smoothing. Varieties with only $\frac{1}{2}(1,1,1)$-singularities in a family of 2-Gorenstein varieties are the equivalent of smooth varieties in a family of Gorenstein varieties. Thus we see that it is naturally to consider the rationality of the del Pezzo fibrations with only $\frac{1}{2}(1,1,1)$-singularities.

It is expected that del Pezzo fibrations of degree 1 also have models with good singularities, these models have to be 6-Gorenstein. Therefore for del Pezzo fibrations of degree 1 it is natural to work with $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$-singularities [28]. The following is a well-known and widely believed conjecture.

Conjecture 1.8. *Let $\pi : X \to \mathbb{P}^1$ be a del Pezzo fibration of degree 1 or 2. Suppose X has only singularities of the type $\frac{1}{2}(1,1,1)$ if π is a del Pezzo fibration of degree 2 and only singularities of the type $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$ if π is of degree 1. Suppose $\mathrm{Pic}(X) = \mathbb{Z} \oplus \mathbb{Z}$, and suppose X satisfies the K^2-condition, then X is birationally rigid.*

Theorem 1.5 is a step towards proving this conjecture.

Smooth del Pezzo surfaces of degree 2 can all be embedded as quartics into $\mathbb{P}(1,1,1,2)$, therefore the requirement of Theorem 1.5 that the varieties are embedded into toric $\mathbb{P}(1,1,1,2)$-bundles is quite natural. In particular it is always satisfied for smooth varieties admitting a del Pezzo fibration of degree 2.

Example 1.9. Every smooth variety X admitting a del Pezzo fibration of degree 2 can be embedded into a toric $\mathbb{P}(1,1,1,2)$-scroll. Indeed, it was shown in [70] that there is a double cover $\sigma : X \to V$, where V is an $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. We can express V as $\mathrm{Proj}\left(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(a) \oplus \mathcal{O}_{\mathbb{P}_1}(b)\right)$ for some $a, b \in \mathbb{Z}$, $0 \leqslant a \leqslant b$. Let $f(x,y,z,u,v) = 0$ be the equation of the branching divisor of σ in V and let $\deg f = (4, 2d)$. Let Y be the toric $\mathbb{P}(1,1,1,2)$-bundle given as a quotient $\left(\mathbb{C}^6 \setminus Z(I)\right)/(\mathbb{C}^*)^2$, where $I = \langle x,y,z,w \rangle \cap \langle u,v \rangle$, the action of $(\mathbb{C}^*)^2$ is given by the matrix

$$\begin{pmatrix} u & v & x & y & z & w \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & a & b & d \end{pmatrix}$$

Then we can embed X into Y as a hypersurface given by the equation $w^2 = f(x,y,z,u,v)$. Consider a projection $\sigma_Y : Y \dashrightarrow V$ defined as

$$(x,y,z,w,u,v) \mapsto (x,y,z,u,v).$$

Then $\sigma_Y|_X$ is a double cover branched over the hypersurface given by $f(x,y,z,u,v) = 0$, that is $\sigma_Y|_X = \sigma$.

We now discuss the assumption on the structure of the fibers containing singularities. Let $\pi : X \to \mathbb{P}^1$ be a del Pezzo fibration and suppose X is embedded into a toric $\mathbb{P}(1,1,1,2)$-bundle. Then the fiber F_t of π over the point $t \in \mathbb{P}^1$ has the equation

$$p(t)w^2 + q_2(x,y,z;t)w + q_4(x,y,z;t) = 0.$$

The fiber F_t contains a $\frac{1}{2}$-point of X if and only if $p(t) = 0$. Therefore the equation of a fiber containing a singularity of X is of the form

$$q_4(x,y,z) + wq_2(x,y,z) = 0.$$

In Theorem 1.5 we impose the condition $q_2(x,y,z) \equiv 0$ for every fiber containing singularity. Thus varieties we consider are special, our motivation to study them comes from the Cremona group as we see in the next section. The specialty condition is there only for technical reasons, we expect the same results to hold for a general del Pezzo fibration of degree 2 which can be embedded into a toric $\mathbb{P}(1,1,1,2)$-scrolls. We expect that one can prove Conjecture 1.8 using the same technique, possibly with some generality assumptions.

For varieties embedded into toric $\mathbb{P}(1,1,1,2)$-bundles the K^2-condition has an easy numerical interpretation.

Proposition 1.10. *Let Y be the toric $\mathbb{P}(1,1,1,2)$-bundle given as a quotient $\Big(\mathbb{C}^6 \backslash Z(I)\Big)/(\mathbb{C}^*)^2$, where $I = \langle x,y,z,w\rangle \cap \langle u,v\rangle$, the action of $(\mathbb{C}^*)^2$ is given by the matrix*

$$\begin{pmatrix} u & v & x & y & z & w \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & a & b & c \end{pmatrix}$$

and $0 \leqslant a \leqslant b$. Let $X \subset Y$ be a quasismooth hypersurface of bi-degree $(4,d)$. Suppose $c \geqslant 2a$, then X satisfies the K^2-condition if and only if

$$2d \geqslant 2b + 2c + a + 4.$$

Suppose $c < 2a$, then X satisfies the K^2-condition if and only if

$$2d \geqslant 2a + 2b + \frac{3}{2}c + 4.$$

See [10, Section 4.1] for a similar treatment of cubic fibrations.

It is well known that del Pezzo fibrations over $\mathbb{P}^1$ of degree $\geqslant 5$ are rational. The rationality of del Pezzo fibrations $\pi : X \to \mathbb{P}^1$ of lower degree has been studied extensively, as a result a nearly complete solution to the rationality problem has been obtained for smooth X. See [4] and [76] for rationality of fibrations of degree 4. Rationality for degrees 1, 2, and 3 has been studied in [70], [35], [36], and [37]. In the papers of Pukhlikov and Grinenko, the primary way of study of the rationality question is the birational rigidity. There are also new results on stable rationality which we discuss in Section 1.4.

Let $\pi : X \to \mathbb{P}^1$ be a del Pezzo fibration of degree 1 or 2. We say that the fiber $F_t = \pi^{-1}(t)$ is birationally smoothable if there exists a fiberpreserving map $X \dashrightarrow Y$ to another del Pezzo fibration $\pi_Y : Y \to \mathbb{P}^1$ such that the fiber $F_{Y,t} = \pi_Y^{-1}(t)$ is smooth. It was shown that del Pezzo fibrations of degree 1 and 2 with sufficiently many singular birationally non-smoothable fibers are weakly rigid [23, Corollary 7.4]. It is unclear how to verify this condition, but we believe that a hypersurface X of bi-degree $(4,d)$ in a toric $\mathbb{P}(1,1,1,2)$-scroll satisfies it for sufficiently big d.

1.2 Conjugacy classes of subgroups of Cremona group

We apply the rationality result from the previous section to study the embeddings of the Klein simple group into the Cremona group of rank 3.

Definition 1.11. The *Cremona group* $\mathrm{Cr}_n(\mathbb{C})$ of rank n is a group of birational transformations of $\mathbb{P}^n$.

Definition 1.12 ([64])**.** Suppose a group G acts faithfully on a variety X, where X is in a Mori category. We say that $\pi : X \to Z$ is a $G\mathbb{Q}$-*Mori fiber space* if it satisfies the following conditions:

- the variety X is $G\mathbb{Q}$-factorial, that is every G-invariant Weil divisor on X is $\mathbb{Q}$-Cartier;
- the morphism π is flat, G-equivariant and, the invariant relative Picard rank $\rho^G(X/Z) = 1$;
- a generic fiber of π is a Fano variety.

Example 1.13. Let $Y_n = \Big(\mathbb{C}^6 \setminus Z(I)\Big)/(\mathbb{C}^*)^2$, where $I = \langle x,y,z,w\rangle \cap \langle u,v\rangle$ and the action of $(\mathbb{C}^*)^2$ is given by the matrix:

$$\begin{pmatrix} u & v & x & y & z & w \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & -n \end{pmatrix}$$

The variety Y_n is a 4-dimensional toric variety with $\mathrm{Cox}(Y_n) = \mathbb{C}[x, y, z, w, u, v]$, the grading of this ring is defined by the matrix above. Clearly, Y_0 is a direct product $\mathbb{P}^1 \times \mathbb{P}(1, 1, 1, 2)$. Let X_n be the hypersurface of bi-degree $(4, 0)$ in Y_n given by the equation

$$a_{2n}(u, v)w^2 = x^3y + y^3z + z^3x,$$

where a_{2n} is a homogeneous polynomial of degree $2n$ without multiple roots. Consider the restriction π to X_n of the projection

$$(u : v : x : y : z : w) \mapsto (u : v).$$

Its general fiber is a del Pezzo surface of degree 2. Indeed, the general fiber is a smooth quartic

$$w^2 = x^3y + y^3z + z^3x$$

in $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$ which is a del Pezzo surface of degree 2. We claim that $\pi : X_n \to \mathbb{P}^1$ is a $\mathrm{PSL}_2(7)$-Mori fiber space.

The varieties X_n admit a faithful action of the group $\mathrm{PSL}_2(7)$. It is enough to prove that the general scheme fiber of π admits a faithful action of $\mathrm{PSL}_2(7)$, then X_n admits a faithful action and π is $\mathrm{PSL}_2(7)$-equivariant assuming trivial action on the base $\mathbb{P}^1$. Indeed, there is an irreducible representation of the group $\mathrm{PSL}_2(7)$ in $\mathrm{GL}_3(\mathbb{C})$, therefore $\mathrm{PSL}_2(7)$ acts on $\mathbb{P}^2$ faithfully. We may choose coordinates x, y, z on $\mathbb{P}^2$ in such a way that the polynomial $x^3y + y^3z + z^3x$ is $\mathrm{PSL}_2(7)$-invariant. The same argument works for $\mathbb{P}^2_{\mathbb{C}(t)}$ over the field $\mathbb{C}(t)$. Let $\bar{X}$ be the surface given by the equation

$$w^2 = a_{2n}(t, 1)(x^3y + y^3z + z^3x)$$

in $\mathbb{P}^2(1, 1, 1, 2)$ over $\mathbb{C}(t)$. There exists a double cover $\bar{X} \to \mathbb{P}^2_{\mathbb{C}(t)}$ branched over the quartic $x^3y+y^3z+z^3x = 0$. This equation is $\mathrm{PSL}_2(7)$-invariant, therefore the action of $\mathrm{PSL}_2(7)$ extends to the faithful action on $\bar{X}$. The general scheme fiber of π is isomorphic to $\bar{X}$, hence X_n admits faithful $\mathrm{PSL}_2(7)$-action.

The variety X_n has $2n$ points of the type $\frac{1}{2}(1, 1, 1)$ and has no other singularities. Thus X_n is in the Mori category. The relative Picard rank $\mathrm{Pic}(X_n/\mathbb{P}^1)$ is one for $n \geqslant 1$, thus we see that $\pi : X_n \to \mathbb{P}^1$ is a Mori fiber space with a faithful group action preserving fibers of π. Thus $\pi : X_n \to \mathbb{P}^1$ is $\mathrm{PSL}_2(7)\mathbb{Q}$-Mori fiber space. Now suppose $n = 0$, then X_0 is a direct product of a unique $\mathrm{PSL}_2(7)$-del Pezzo surface S_2 and $\mathbb{P}^1$. The $\mathrm{PSL}_2(7)$-invariant Picard rank of S_2 is 1, therefore

$$\mathrm{Pic}^{\mathrm{PSL}_2(7)}(X_0/\mathbb{P}^1) = \mathrm{Pic}^{\mathrm{PSL}_2(7)}(S_2) = 1$$

.

We classify $\mathrm{PSL}_2(7)\mathbb{Q}$-del Pezzo fibrations.

Theorem 1.14 ([1, Conjecture 3.3]). *Let X be a $\mathrm{PSL}_2(7)$ $\mathbb{Q}$-del Pezzo fibration over $\mathbb{P}^1$. Let S_2 be the double cover of $\mathbb{P}^2$ branched over the Klein quartic. Then*

- *either a generic fiber is $\mathbb{P}^2$ and $X \cong \mathbb{P}^2 \times \mathbb{P}^1$,*
- *or a generic fiber is S_2 and $X \cong X_n$.*

Okada has proven that a very general X_n is not rational for $n \geqslant 5$ using the reduction to characteristic 2 [56]. Applying Theorem 1.5 to varieties X_n we get a better result.

Corollary 1.15. *The varieties X_n are birationally rigid for $n \geqslant 3$, in particular they are not rational.*

We prove Theorem 1.14 in Chapter 5 and Corollary 1.15 in Section 2.4. Both results are motivated by the study of conjugacy classes of the Cremona group of rank 3. The following lemma implies that the classification of the subgroups of $\mathrm{Cr}_n(\mathbb{C})$ up to conjugation is equivalent to the classification of the rational $G\mathbb{Q}$-Mori fiber spaces up to G-equivariant birational equivalence.

Lemma 1.16 ([64, Proposition 1.2]). *Let X be a rationally connected variety and let G be a finite subgroup of* $\operatorname{Bir} X$. *Then there exists a $G\mathbb{Q}$-Mori fiber space $Y \to Z$ and a G-equivariant birational map $\varphi : X \dashrightarrow Y$.*

This approach worked very well for the Cremona group of rank 2. Finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$ have been classified up to isomorphism in [32]. We know more about simple non-abelian finite subgroups, their embeddings into $\mathrm{Cr}_2(\mathbb{C})$ have been classified up to conjugacy [32].

For a while very little was known about finite subgroups of Cremona group of higher rank. Serre even asked whether every group can be embedded into the Cr_3 [77, Question 6.0]. However several striking results have been proven recently, among them is the following theorem of Prokhorov answering the question of Serre.

Theorem 1.17 ([63, Theorem 1.3]). *Let G be a finite simple non-abelian group. Then* $\mathrm{Cr}_3(\mathbb{C})$ *has a subgroup isomorphic to G if and only if G is one of the following groups:* $\mathcal{A}_5$, $\mathrm{PSL}_2(7)$, $\mathcal{A}_6$, $\mathcal{A}_7$, $\mathrm{PSL}_2(8)$, *or* $\mathrm{PSU}_4(2)$.

Here $\mathcal{A}_n$ is the group of even permutations. The theorem has been proven using the same idea: transfer the problem into the language of birational geometry and work with good models of the action. This is of course a lot harder to do in dimension 3 than in dimension 2. Already the rationality question for 3-fold Mori fiber spaces has several unresolved aspects.

There is no hope to classify finite subgroups of Cr_3, but we expect that it is possible to classify finite simple non-abelian subgroups up to conjugacy. The $\mathcal{A}_7\mathbb{Q}$-, $\mathrm{PSL}_2(8)\mathbb{Q}$-, and $\mathrm{PSU}_4(2)\mathbb{Q}$-Mori fiber spaces have already been classified, they are $G\mathbb{Q}$-Fano varieties since these groups cannot be embedded into Cr_2. The case of Fano varieties with group actions is the most studied case so far: [20], [21], [64], [65], and [66]. For $G\mathbb{Q}$-conic bundles all we know is that the process of standardizing a conic bundle can be carried over equivariantly [5]. We focus on $G\mathbb{Q}$-del Pezzo fibrations. The groups which could act on del Pezzo fibrations must be embedded into Cr_2, these are: $\mathcal{A}_6$, $\mathrm{PSL}_2(7)$, and $\mathcal{A}_5$. The $\mathcal{A}_6\mathbb{Q}$-del Pezzo fibrations have already been classified.

Theorem 1.18 ([21, Appendix B.]). *Let X be an $\mathcal{A}_6\mathbb{Q}$-del Pezzo fibration over $\mathbb{P}^1$, then $X \cong \mathbb{P}^2 \times \mathbb{P}^1$.*

We can see that for $\mathrm{PSL}_2(7)$ there is a series of families of del Pezzo fibrations and most of them have moduli, which makes the study and results more exciting compared to the case of $\mathcal{A}_6$. For $\mathcal{A}_5$ the problem is a lot harder, because $\mathcal{A}_5$ may act nontrivially on $\mathbb{P}^1$.

Note that X_0 and X_1 are unique, while there is a $2n-3$-dimensional family of X_n. We expect the following to hold.

Conjecture 1.19. *Let G be a finite simple non-abelian group. Then there are finitely many embeddings of G into* Cr_n *up to conjugation.*

For $n \geqslant 2$ the varieties X_n have moduli. It is also can be shown that the varieties X_n are not birational $\mathrm{PSL}_2(7)\mathbb{Q}$-Mori fibrations other than themselves for $n \geqslant 2$. Suppose the varieties X_n are rational. Then it means that there exists a family of embeddings of $\mathrm{PSL}_2(7)$ embeddings into the Cremona group of rank 3. As we expect that this does not occur we get the following conjecture.

Conjecture 1.20 ([1, Conjecture 3.5]). *The varieties X_n are not rational for $n \geqslant 2$.*

Remark 1.21. Corollary 1.15 proves this conjecture for $n \geqslant 3$. The varieties X_2 do not satisfy the K^2-condition, therefore we cannot apply Theorem 1.5. We expect that X_2 is birationally rigid, despite it not satisfying K^2-condition. But it is very close to not being rigid, thus our technique is very hard to apply.

Let W_4 be a quartic threefold with a $\mathrm{PSL}_2(7)$-action, then in some coordinates W_4 is given by the equation

$$x^3y + y^3z + z^3x + p_4(u, v) = 0,$$

where p_4 is a general homogeneous polynomial of degree 4. Let $\tau : (x : y : z : u : v) \mapsto (x : y : z : -u : -v)$ be an involution of $\mathbb{P}^4$. The set of $\mathrm{PSL}_2(7)$-fixed points consists of 4 isolated fixed points given as $W_4 \cap \mathbb{P}^1_{u,v}$ and points of the fixed curve $W_4 \cap \mathbb{P}^2_{x,y,z}$. Let V be the quotient W_4/τ. Then V has 4 singular points of the type $\frac{1}{2}(1,1,1)$, these points are the images of the 4 isolated fixed points. Variety V also has an ordinary double point along a curve C which is the image of a $\mathrm{PSL}_2(7)$-fixed curve, thus V has canonical singularities. It is easy to see that $-K_V$ is ample, hence V is a weak Fano. Let $\sigma : X \to V$ be the blow up at C, then X is X_2. We can acquire every X_2 in this manner, therefore we see that the varieties X_2 are birational to a weak Fano, and thus are very close to not being rigid.

We develop a new technique to prove Theorem 1.14 and to reprove Theorem 1.18.

Definition 1.22. We say that a birational map $f : X \dashrightarrow Y$ between fibrations $\pi_X : X \to B$ and $\pi_Y : Y \to B$ is *fiberpreserving* if the following diagram is commutative

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ \mathbb{P}^1 & = & \mathbb{P}^1 \end{array}$$

and f is an isomorphism on a generic fiber of π_X or, equivalently, induces an isomorphism between the general (scheme) fibers X/B and Y/B.

Consider a $\mathbb{P}^1$-bundle $f : X \to \mathbb{P}^1$. We can blow up a point on a fiber F of f and then contract the proper transform of F to get another $\mathbb{P}^1$-bundle $f_Y : Y \to \mathbb{P}^1$. This is an elementary transformation of $\mathbb{P}^1$-bundles and this is the simplest example of a fiberpreserving map.

To prove Theorem 1.14 we first prove that for any $\mathrm{PSL}_2(7)\mathbb{Q}$-del Pezzo fibration there exists a fiberpreserving $\mathrm{PSL}_2(7)$-equivariant map to $\mathbb{P}^2 \times \mathbb{P}^1$ or X_n (Proposition 5.4). Then we prove that the only fiberpreserving $\mathrm{PSL}_2(7)$-equivariant map from $\mathbb{P}^2 \times \mathbb{P}^1$ or X_n to another $\mathrm{PSL}_2(7)\mathbb{Q}$-del Pezzo fibration is an isomorphism. Thus the fibration we started with must be one of these. We reprove Theorem 1.18 similarly. The original proof uses the classification of terminal singularities to show that every fiber is smooth and from there Prokhorov deduced that the $\mathcal{A}_6\mathbb{Q}$-del Pezzo fibration is $\mathbb{P}^2 \times \mathbb{P}^1$.

1.3 Rationally connected non Fano type varieties

We use the techniques of birational rigidity to study the birational equivalence of birational classes of varieties of Fano type and rationally connected varieties.

Definition 1.23. We say that a normal projective variety X is of *Fano type* if there is an effective $\mathbb{Q}$-divisor Δ on X such that the pair (X, Δ) is Kawamata log terminal and $-(K_X + \Delta)$ is ample.

Varieties of Fano type have been introduced by Shokurov and Prokhorov in [68]. They behave very well with regards to the log minimal model program. If X is a variety of Fano type then we can run the D-MMP on it for any divisor D on X [68, Corollary 2.7]. The Fano type property is preserved under flips and contractions ([68, Lemma 2.8]). Thus if we run the MMP on a variety of Fano type, then on every step we have a variety Fano type. In particular the result of running the MMP on a variety of Fano type is a variety of Fano type. It would be nice if we could say if in a birational class of a given rationally connected variety there is a variety which behaves well with respect to the D-MMP for any divisor D.

Example 1.24. The quotient $Y = X/G$ of a Fano variety by a finite group is a variety of Fano type. Let $f : X \to Y$ be the quotient morphism, then $K_X = f^*(K_Y + \frac{R}{|G|})$, where R is the ramification divisor. Thus $-(K_Y + \frac{R}{|G|})$ is ample. The pair $(Y, \frac{R}{|G|})$ is klt [48, Proposition 3.16] and hence Y is a variety of Fano type.

Example 1.25. Suppose (S, D_S) is a klt del Pezzo surface. Naturally it is of Fano type. Let $f : X \to S$ be a minimal resolution of singularities of S and let D be a proper transform of D_S.

Then X is also of Fano type. Indeed we can write $K_X + D = f^*(K_S + D_S) + \sum a_i E_i$, where $-1 < a_i \leqslant 0$, thus $F = -(K_X + D + \sum a_i E_i)$ is a pullback of an ample divisor. Note that F is not ample because $F \cdot E_i = 0$. Consider the pair $(X, D + \sum (1-\varepsilon_i) a_i E_i)$, for some $\varepsilon_i << 0$ the pair is still klt and $-K_X + D + \sum(1-\varepsilon_i)a_i E_i$ is ample.

Example 1.26. Suppose a surface S is of Fano type. Then there is a divisor D such that $-(K_S + D)$ is ample. Thus $-K_S$ is the sum of an ample and an effective divisor, therefore it is big. If S is a blow up of 10 points on $\mathbb{P}^2$ in general position then $-K_S$ is not big [22, Lemma 4.7], therefore such S is not of Fano type. Note that if points are not in a general position then $-K_S$ might still be big [79].

The varieties of Fano type are rationally connected [82, Theorem 1]. Thus it is natural to ask if the converse is true. Even in dimension 2 it is not as we see in Example 1.26. We may broaden the converse statement, however.

Question 1.27 ([12, Question 5.2])**.** *Let X be a rationally connected variety. Is X birationally equivalent to varieties of Fano type?*

In dimension 2 the answer is positive since every rationally connected surface is birational to $\mathbb{P}^2$. We apply techniques of birational rigidity to prove that the answer is negative in dimension $\geqslant 3$. Namely we prove the following two theorems.

Theorem 1.28. *(i) Let Y be a generic smooth divisor of degree $(2M, 2l)$ on $\mathbb{P}^M \times \mathbb{P}^1$, $M \geqslant 3$, $l \geqslant 3$. Let $\sigma : V \to \mathbb{P}^m \times \mathbb{P}^1$ be the double cover branched over Y. Then every birational map from V to a variety admitting a Mori fiber space is an isomorphism.*

(ii) Let $Y = (\mathbb{C}^6 \setminus Z(\langle u, v\rangle \cap \langle x, y, z, w\rangle))/(\mathbb{C}^*)^2$, where $(\mathbb{C}^*)^2$-action is given by the matrix

$$\begin{pmatrix} u & v & x & y & z & w \\ 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & -3 & -3 & 0 & 0 \end{pmatrix}$$

and let $X \subset Y$ be the hypersurface given by the equation

$$Q = w^2 + z^3 + (u^{12} + v^{12}) M_4(x, y) z + R_{18}(u, v) x^2 y^2 (x-y)^2,$$

where M_4 and R_{18} are generic homogeneous polynomials of degrees 4 and 18 respectively. Then every birational map from X to a variety admitting a Mori fiber space is an isomorphism.

Theorem 1.29. *The varieties V and X, described in Theorem 1.28, are not birationally equivalent to varieties of Fano type.*

Proof of Theorem 1.29. Suppose X (or V) is birational to a variety of Fano type. Then by Proposition 4.1 there exists a Mori fiber $\pi_Y Y \to Z$ such that $-K_Y$ big and Y is birational to X. Theorem 1.28 implies $Y \cong X$ (resp. $Y \cong V$) but $-K_X$ (resp. $-K_V$) is not big by Lemma 4.15 (resp. Lemma 4.9), contradiction. □

Since V and X are rationally connected (Lemma 4.19) we conclude that the answer to Question 1.27 is negative for these varieties.

To prove Theorem 1.28 we use the techniques of birational rigidity. The birational superrigidity is usually proven using Noether-Fano inequality (Proposition 4.3), its origin is the theorem on generation of the Cremona group $\mathrm{Cr}_2 = \mathrm{Bir}(\mathbb{P}^2)$ by $\mathrm{PGL}_3(\mathbb{C})$ and the standard quadratic transformation. If there is a birational map $X \dashrightarrow Y$ between varieties admitting Mori fiber spaces $\pi : X \to Z$ and $\pi_Y \to Z_Y$, then there is a linear system $\mathcal{M}$ on X such that for a generic fiber F we have the equivalence $\mathcal{M} \sim -nK_X$ and the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical.

We now sketch the proof of Theorem 1.28. For the part (i) we use the techniques developed in [73] and [74]. We impose generality conditions on the branching divisor W. These conditions allow us to control singularities of linear systems on fibers of π (Theorem 4.7) and hence on V. Together with Noether-Fano inequality (Proposition 4.3) this is enough.

For the part (ii) we suppose there is an exceptional divisor E over X with a negative discrepancy $a(E, X, \mathcal{M})$. Then we go over the possible images of E on X. Due to [70, proof of Theorem 2.1] the image of E must be a point and it was shown in [59] that this point must be a cusp of a curve of anticanonical degree 1 in a fiber of π. We show that all such points are ordinary double points of X or of a fiber of the del Pezzo fibration (Lemma 4.11). Then we show that $(X, \frac{1}{n}\mathcal{M})$ is always canonical at these points. Thus we get a contradiction with the Noether-Fano inequality.

1.4 Stable rationality of double covers

Recent breakthroughs by Voisin [81], expanded by Colliot-Thélène, Pirutka, and Totaro [25], [80], have changed the landscape of the study of stable rationality. Consequently failure of stable rationality was proven for large classes of rationally connected varieties: covers of $\mathbb{P}^n$ with the branching divisor of sufficiently high degree ([6], [81], [26], [57]), hypersurfaces of high enough degree ([25], [80]). All of these results concern Fano varieties, but the techniques may be applied to study Fano fibrations as well ([38], [39]).

Definition 1.30. We say that X is *stably rational* if $X \times \mathbb{P}^n$ is rational for some n.

We use the reduction to characteristic 2 method to study double covers of $\mathbb{P}^n$-bundles over $\mathbb{P}^m$. Let X be a toric $\mathbb{P}^n$-bundle over $\mathbb{P}^m$. Then we can write

$$X = \operatorname{Proj}\big(\mathcal{O}_{\mathbb{P}^m} \oplus \mathcal{O}_{\mathbb{P}^m}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^m}(a_n)\big),$$

where for some integers $0 \leqslant a_1 \leqslant \cdots \leqslant a_n$. There is a natural grading on $\mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ corresponding to X defining homogeneous coordinates on X given by

$$\deg x_i = (1, a_i) \quad \text{and} \quad \deg y_i = (0, 1).$$

Theorem 1.31. *Let X be a $\mathbb{P}^n$-bundle over $\mathbb{P}^m$ given as*

$$X = \operatorname{Proj}\big(\mathcal{O}_{\mathbb{P}^m} \oplus \mathcal{O}_{\mathbb{P}^m}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^m}(a_n)\big).$$

Let $P \in \mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ be a very general homogeneous polynomial of bi-degree (r, d). If the dimension of X is even,

$$2r \geqslant n + 1, \quad \textit{and}$$
$$2d \geqslant \max(2ra_n + 2, m + 1 + \sum a_i),$$

then the double cover of X branched over the divisor R given by $P = 0$ is not stably rational.

If the dimension of X is odd,

$$2r \geqslant \max(3, n + 1), \quad \textit{and}$$
$$2d \geqslant \max(2ra_n + 3, m + 1 + \sum a_i),$$

then the double cover of X branched over the divisor R given by $P = 0$ is not stably rational.

Del Pezzo fibrations of degree 2 are double covers of $\mathbb{P}^2$-bundles over $\mathbb{P}^1$. We may apply Theorem 1.31 for this case, but the result is not quite satisfactory. For example the double cover of $\mathbb{P}^2 \times \mathbb{P}^1$ branched over a smooth divisor of bidegree $(4, 2)$ is known to be not rational but is not covered by the theorem.

We strengthen Theorem 1.31 for the case of del Pezzo fibrations of degree 2 over $\mathbb{P}^1$. Let $\pi : Y \to \mathbb{P}^1$ be a del Pezzo fibration of degree 2. Suppose Y is smooth, then there is a double cover $\sigma : Y \to X$, where X is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ and the branching divisor is a quartic in every fiber of π. We say that X is very general if the branching divisor of σ is very general.

Theorem 1.32 ([53, Theorem 1.3]). *Let $\pi : Y \to \mathbb{P}^1$ be a del Pezzo fibration of degree 2. Suppose Y is smooth and very general then Y is not stably rational.*

Note that Y cannot be a direct product $S_2 \times \mathbb{P}^1$, where S_2 is a del Pezzo surface of degree 2, since then the relative Picard rank is > 1.

The study of stable rationality has been particularly successful in dimension three. Hassett, Tschinkel, and Kresch have shown in [40] how to apply Voisin's method for families of threefold conic bundles. Hassett and Tschinkel have also proven, that very general Fano threefolds which are not rational and not birational to a cubic threefold are not stably rational [41].

Thus among threefolds only del Pezzo fibrations are left to consider. Del Pezzo fibrations of degree 5 are rational, therefore stably rational. Fibrations of degree 4 can be transformed into conic bundles and have been considered in [41]. Thus it remains to study the stable rationality of fibrations of degree 1, 2, and 3.

Chapter 2

Preliminaries

This chapter is aimed to provide the background necessary for the purpose of this thesis. In Section 2.1 we recall the definition of canonical singularities of pairs. This notion is critical for the study of rigidity of algebraic varieties, which we do in Chapters 3 and 4. In Sections 2.2 and 2.3 we go through technical statements related to blow ups and singularities of pairs. Section 2.4 is about Cox rings of toric varieties and the corresponding homogeneous coordinates.

We write $\equiv$ for numerical equivalence of $\mathbb{Q}$-divisors and $\mathbb{Q}$-cycles and $\sim$ for linear equivalence of $\mathbb{Q}$-divisors. We denote the symmetric group by $\mathcal{S}_n$ and its subgroup of even permutations by $\mathcal{A}_n$. All varieties are assumed to be irreducible, normal, projective, $\mathbb{Q}$-factorial, and defined over $\mathbb{C}$ unless stated otherwise.

2.1 Singularities

Let X be an algebraic variety, possibly non-projective and singular. Let E be a prime divisor on X. Then there is a discrete valuation ν_E of $\mathbb{C}(X)$ corresponding to E defined as $\nu_E(f) = \mathrm{mult}_E(f)$.

Definition 2.1 ([69])**.** Let $\varphi : \widetilde{X} \to X$ be a projective birational morphism and let ν be a discrete valuation of $\mathbb{C}(X)$. We say that a triple $(\widetilde{X}, \varphi, E)$ is a *realization* of the discrete valuation ν if E is a prime divisor on $\widetilde{X}$ and $\nu_E = \nu$. We say that $\varphi(E)$ is the *center* of the discrete valuation ν_E on X.

Note that if X is projective, then every discrete valuation of the field $\mathbb{C}(X)$ has the center on X which does not depend on a realization.

Definition 2.2. Let D be a divisor on X. We define the *multiplicity* of a discrete valuation ν at D by the number

$$\nu(D) = \mathrm{mult}_E\, \varphi^*(D)$$

for some realization $(\widetilde{X}, \varphi, E)$ of ν.

If the center of ν on X is of codimension $\geqslant 2$, then we can write

$$\varphi^*(D) = \varphi^{-1}(D) + \nu(D)E + \sum a_i E_i,$$

where E_i are the other exceptional divisors of φ and a_i are some rational numbers. The multiplicity does not depend on the realization.

Definition 2.3 ([61, p. 6])**.** Let D be a divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $\pi : \widetilde{X} \to X$ be a birational morphism and let $\widetilde{D} = \pi^{-1}(D)$ be the proper transform of D. Then

$$K_{\widetilde{X}} + \widetilde{D} \sim \pi^*(K_X + D) + \sum_E a(E, X, D)E,$$

where E runs through all the distinct exceptional divisors of π on $\widetilde{X}$ and $a(E, X, D)$ is a rational number. The number $a(E, X, D)$ $\big(=a(\nu_E, X, D)\big)$ is called the *discrepancy* of a divisor E (discrete valuation ν_E) with respect to the pair (X, D).

Definition 2.4 ([61, p. 6])**.** Let D be a divisor on X. We say that the pair (X, D) is *terminal* (resp. *canonical*, *log terminal*, *log canonical*) *at the discrete valuation* ν with a center on X if $a(E, X, \mathcal{M}) > 0$ (resp. $a(E, X, \mathcal{M}) \geqslant 0$, $a(E, X, \mathcal{M}) > -1$, $a(E, X, \mathcal{M}) \geqslant -1$) for some realization $(\widetilde{X}, \varphi, E)$ of ν. We say that the pair (X, D) is terminal (resp. canonical, log terminal, log canonical) *at a subvariety* Z if it is terminal (resp. resp. canonical, log terminal, log canonical) at every discrete valuation ν on $K(X)$ such that a center of ν on X is Z. We say that the pair (X, D) is terminal (resp. canonical, *purely log terminal*, log canonical) if it is (resp. canonical, log terminal, log canonical) at every subvariety of codimension $\geqslant 2$. If $D = 0$, we simply say that X has only terminal (resp. canonical, log terminal, log canonical).

Note that there is a difference between being canonical along a curve and at a curve. In the first case we ask for the pair to be canonical at a curve and at every point of the curve. While in the latter case we require for the pair to be canonical at the generic scheme point of the curve. If the pair is not canonical at a curve, it is sometimes said that the curve is a *center* of non-canonical singularity.

Definition 2.5. Let D_i be prime divisors on X and let $D = \sum a_i D_i$ be an effective divisor on X. We say that the pair (X, D), is klt if it is purely log terminal and $a_i < 1$ for all i.

Definition 2.6. Let $\mathcal{M}$ be a linear system, not necessarily mobile, on X. We say that the pair $(X, \lambda\mathcal{M})$ is terminal (resp. canonical, purely log terminal, log canonical) if it the pair $(X, \lambda D)$ is terminal (resp. canonical, log terminal, log canonical) at every subvariety of codimension $\geqslant 2$ for a general $D \in \mathcal{M}$.

Example 2.7.

- Consider the pair $(S, \lambda C)$, where S is a smooth surface and $C \subset S$ is a smooth curve. If $\lambda < 1$ then the pair is terminal everywhere. If $\lambda = 1$, then the pair is canonical at every point of C and is terminal at every other point. If $\lambda > 1$ then the pair is not log canonical at every point $P \in C$.
- Consider the pair $(\mathbb{P}^2, \lambda|L|)$, where L is a line. Then the pair is terminal at every point $P \in \mathbb{P}^2$ since a generic line does not pass through P.
- Let $\mathcal{L}$ be the linear system of lines on $\mathbb{P}^2$ passing through a point P. Consider the pair $(\mathbb{P}^2, \lambda\mathcal{L})$, it is terminal at P if $\lambda < 1$, it is canonical at P if $\lambda = 1$ and is not log canonical if $\lambda > 1$. The pair is terminal at every point $Q \neq P$.
- Let D be a simple normal crossing divisor on a surface S. Then the pair (S, D) is log canonical. In particular, it is strictly log canonical at the points of intersection of components of D.

Remark 2.8. Consider the pair $(X, \mathcal{M})$. Let $f: Y \to X$ be a projective birational morphism, let E_i be the exceptional divisors of f and let $\widetilde{\mathcal{M}}$ be the proper transform of $\mathcal{M}$ on Y. Then

$$K_Y + \widetilde{\mathcal{M}} - \sum a(E_i, X, \mathcal{M})E_i \sim f^*(K_X + \mathcal{M})$$

The pair

$$\big(Y, \widetilde{\mathcal{M}} - \sum a(E_i, X, \mathcal{M})E_i\big)$$

is called the *log pullback* of the pair $(X, \mathcal{M})$. It follows from the definition that the log pullback of the pair has the same singularities as the pair.

Theorem 2.9 (Inversion of adjunction, [47, Theorem 17.7])**.** *Let $S \subset X$ be an irreducible divisor and let D be an effective divisor on X. Assume that $K_X + S$ is $\mathbb{Q}$-Cartier and that the pair (X, S) is purely log terminal. Then the pair $(X, S + D)$ is log canonical in a neighborhood of S if and only if the pair $\big(S, \mathrm{Diff}(D)\big)$ is log canonical.*

Remark 2.10. For the definition of a *different* we refer to [47, Chapter 16]. If X is smooth in codimension 2, then $\mathrm{Diff}(D) = D\big|_S$ by [47, Corollary 16.7] (case 16.6.3, $m = 1$).

We use Inversion of adjunction to study singularities of pairs as follows. Let F be a prime Cartier divisor on a variety X. Let D be an effective divisor such that $K_X + D$ is not canonical at some $Z \subset F$, then $(X, D + F)$ is not log canonical at Z. Suppose X is smooth in codimension 2, then by Inversion of adjunction $(F, D\big|_F)$ is not log canonical at Z.

We generalize [15, Theorem 1.5] for our purposes.

Theorem 2.11. *Let $\pi : V \to Z$ be a $G\mathbb{Q}$-Mori fiber space over a curve Z and let F be an irreducible fiber of π. Suppose the pair $(V, \frac{1}{n}\mathcal{M})$ is canonical at any subset of F for any G-invariant mobile linear system $\mathcal{M} \subset \big| -nK_V + lF\big|$. Suppose $\varphi : V \dashrightarrow \bar{V}$ is a G-equivariant birational map to a variety admitting a $G\mathbb{Q}$-Mori fiber space $\bar{\pi} : \bar{V} \to Z$ such that*

$$\begin{array}{ccc} V & \overset{\varphi}{\dashrightarrow} & \bar{V} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ Z & \overset{g}{=\!=\!=} & Z \end{array}$$

is commutative. Suppose φ is an isomorphism outside F, then φ is an isomorphism.

Proof. Let D be a very ample G-invariant divisor on Z such that $-K_V + \pi^*(D)$ and $-K_{\bar{V}} + \bar{\pi}^*(D)$ are ample. Put

$$\Lambda = \big| -nK_V + \pi^*(nD)\big|, \ \Gamma = \big| -nK_{\bar{V}} + \bar{\pi}^*(nD)\big|, \ \bar{\Lambda} = \varphi(\Lambda), \ \bar{\Gamma} = \varphi^{-1}(\Gamma),$$

where n is a natural number such that Λ and Γ have no base points. Put

$$M_V = \frac{2\varepsilon}{n}\Lambda + \frac{1-\varepsilon}{n}\bar{\Gamma}, \ M_{\bar{V}} = \frac{2\varepsilon}{n}\bar{\Lambda} + \frac{1-\varepsilon}{n}\Gamma,$$

where ε is a positive rational number. Note that the linear systems $\big|K_V + M_V\big|$ and $\big|K_{\bar{V}} + M_{\bar{V}}\big|$ are ample and G-invariant for $\varepsilon \ll 1$. If the singularities of both log pairs (V, M_V) and $(V, M_{\bar{V}})$ are canonical, then φ is an isomorphism by the uniqueness of canonical model [13, Theorem 1.3.20].

The linear system Γ does not have base points, therefore for small enough ε the pair $(\bar{V}, M_{\bar{V}})$ is canonical. By the assumption of the theorem the pair $(V, \frac{1}{n}\bar{\Gamma})$ is canonical, thus $(V, \frac{1-\varepsilon}{n}\bar{\Gamma})$ is canonical. Therefore (V, M_V) is also canonical since Λ has no base points. □

Corollary 2.12. *Let $\pi : V \to Z$ be a $G\mathbb{Q}$-More fiber space over a curve Z and let F be a fiber of π. Suppose F is irreducible and log terminal. Suppose the pair $(F, \frac{1}{n}\mathcal{M}_F)$ is log canonical for any G-invariant linear system $\mathcal{M}_F \subset \big| -nK_F\big|$. Let $\varphi : V \dashrightarrow \bar{V}$ be a G-equivariant birational map to a variety admitting a $G\mathbb{Q}$-Mori fiber space $\bar{\pi} : \bar{V} \to Z$ such that*

$$\begin{array}{ccc} V & \overset{\varphi}{\dashrightarrow} & \bar{V} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ Z & \overset{g}{=\!=\!=} & Z \end{array}$$

is commutative. Suppose φ is an isomorphism outside F, then φ is an isomorphism.

Proof. Suppose there is a G-equivariant mobile linear system

$$\mathcal{M} \subset \big| -nK_V + lF\big|$$

such that the pair $(V, \frac{1}{n}\mathcal{M})$ is not canonical at some subset B of a fiber F. Then by Inversion of adjunction the pair $(F, \frac{1}{n}\mathcal{M}\big|_F)$ is not log canonical, contradiction. Thus V and φ satisfy the conditions of Theorem 2.11, and φ is an isomorphism. □

2.2 Divisors and curves on blow ups

Lemma 2.13 ([44, Lemma 2.2.14]). *Let $g : Y \to X$ be the blow up of a smooth curve $C \subset X$ on a smooth threefold X. Let E be the exceptional divisor of g, then E is the projectivization of the normal bundle $N_{C/X}$. Let $f \in A^2(Y)$ be the class of a fiber of the ruled surface E, then the following equalities hold*

(i) $E^2 = -g^*(C) + \deg(N_{C/X})f$,

(ii) $E^3 = -\deg(N_{C/X})$,

(iii) $E \cdot f = -1$,

(iv) $E \cdot g^*(D) = (C \cdot D)f$,

(v) $f \cdot g^*(D) = 0$,

(vi) $E \cdot g^*(Z) = f \cdot g^*(Z) = 0$,

(vii) $\deg(N_{C/X}) = 2g(C) - 2 - K_X \cdot C$.

Let $\sigma : \mathbb{C}^n \to X$ be the quotient of $\mathbb{C}^n$ by the involution $x_i \mapsto -x_i$. Let P be the image of zero on X, then we say that P is a singular point of the type $\frac{1}{2}(1, \dots, 1)$, or simply that P is $\frac{1}{2}(1, \dots, 1)$. If a singularity Q is analytically isomorphic to P we also say that Q is $\frac{1}{2}(1, \dots, 1)$.

We call the singularities of the type $\frac{1}{2}(1, \dots, 1)$ *half-points*. Half-point is the simplest terminal quotient singularity. We state the following results only for half-points on threefolds for the sake of simplicity.

Lemma 2.14 ([62, Lemma 4.10]). *Let $\sigma : \mathbb{C}^3 \to X$ be the quotient by the involution $x_i \mapsto -x_i$. Let Q be the half point on X, that is Q is the image of the origin. Let $f : Y \to X$ be the blow up of Q and let E be the exceptional divisor of f. Then*

(i) $K_Y \sim f^*(K_X) + \frac{1}{2}E$,

(ii) if the local equation of $\sigma^*(D)$ at 0 is $x_i = 0$, then $f^{-1}(D) = f^*(D) - \frac{1}{2}E$,

(iii) $\mathcal{O}_E(E)\big|_E = \mathcal{O}_E(-2)$.

The morphism $f : Y \to X$ is a resolution of singularities of X. Indeed, consider the variety V acquired by blowing up the origin on $\mathbb{C}^3$. There is a double cover $\sigma_V : V \to Y$. Clearly, E is the branching of this double cover. Thus Y is a quotient of a smooth variety by the involution and the set of fixed points of this involution is the divisor $\sigma_V^{-1}(E)$, therefore Y is smooth.

$$\begin{array}{ccc} V & \xrightarrow{\sigma_V} & Y \\ \downarrow & & \downarrow f \\ \mathbb{C}^3 & \xrightarrow{\sigma} & X \end{array}$$

The blow up of a half point is also known as Kawamata blow up due to the following result.

Proposition 2.15 ([46]). *Let $f : \widetilde{X} \to X$ be the blow up of a half-point Q and let E be the exceptional divisor of f. Then a pair (X, D) is canonical at Q if and only if it is canonical at E, that is $a(E, X, D) < 0$.*

Corollary 2.16. *Suppose a pair (X, D) is canonical at a half-point Q. Then it is canonical at every curve passing through Q.*

Proof. Suppose (X, D) is not canonical at C passing through Q, then $m = \operatorname{mult}_C D > 1$. Let $f : \widetilde{X} \to X$ be the blow up of the point Q and let E be the exceptional divisor of f. Then $\nu_E(D) \geqslant \frac{m}{2} > \frac{1}{2}$. On the other hand $a(E, X, 0) = \frac{1}{2}$ by Lemma 2.14, thus $a(E, X, D) < 0$ and the pair is not canonical at Q, contradiction. $\square$

Lemma 2.17. *Let $Q \in X$ be a half-point. Suppose D is a divisor passing through the point Q. Suppose also that there is a curve C passing through the point Q such that $C \cdot D = \frac{1}{2}$ and $C \not\subset \operatorname{Supp} D$. Then the pair (X, D) is canonical at Q.*

Proof. Let $f : \widetilde{X} \to X$ be the blow up of the point Q. Let $\widetilde{C}$ and $\widetilde{D}$ be the proper transforms on $\widetilde{X}$ of C and D respectively. By the projection formula

$$0 \leqslant \widetilde{D} \cdot \widetilde{C} = D \cdot C - \nu_E(D) E \cdot C = \frac{1}{2} - \nu_E(D),$$

thus $\nu_E(D) \leqslant \frac{1}{2}$. By Lemma 2.14 and Proposition 2.15 the pair (X, D) is canonical at Q. □

The following statements on the behavior of cycles on threefolds are well known, see [70] for applications. First we recall the notion of multiplicity of a curve at a point. Let Z be a curve on a variety X and let $|A|$ be a very ample linear system. We define the multiplicity of Z at a point $P \in X$ by the number

$$\operatorname{mult}_P Z = \big(Z \cdot D\big)_P,$$

where D is a general divisor in $|A|$ passing through the point P.

Lemma 2.18. *Let Z be a 1-cycle on a threefold X. Let $\sigma : \widetilde{X} \to X$ be the blow up along a subvariety B of codimension $\geqslant 2$ and let E be the exceptional divisor. Then $\sigma^* Z = \sigma^{-1} Z + Z_E$, where* $\operatorname{Supp} Z_E \subset E$ *and*

- *if B is a nonsingular point, then $E \cong \mathbb{P}^2$ and $\deg Z_E = \operatorname{mult}_B Z$, where $\deg Z_E$ is the degree of the plane curve $Z_E \subset E \cong \mathbb{P}^2$.*
- *if B is a smooth curve then $Z_E \equiv (C \cdot B)_S f$, where f is the class of a fiber of the ruled surface E and S is some surface containing C and B which is smooth at every point of $C \cap B$.*

Note that the smoothness of S is important. For example, consider a cone S in $\mathbb{P}^3$ given by the equation $xy = z^2$. The lines in its ruling intersect at $x = y = z = 0$ and for the lemma to work the intersection number of these lines in $\mathbb{P}^3$ has to be equal to 1. But on the cone the intersection of these lines is equal to $\frac{1}{2}$.

Lemma 2.19. *Let F be the hyperplane given by the equation $z = 0$ in $\mathbb{C}^3$. Let L be a curve in F and let C be an irreducible curve which does not lie in F. Let $\sigma : X \to \mathbb{C}^3$ be the blow up of L and let E be the exceptional divisor. Let $f \in A^2(X)$ be the class of a fiber of the ruled surface E. Then $\sigma^* C \equiv \sigma^{-1} C + kf$, where $k \leqslant C \cdot F$.*

Proof. By the projection formula

$$k = E \cdot \sigma^* C - kE \cdot f = E \cdot \sigma^{-1} C \leqslant \sigma^* F \cdot \sigma^{-1} C = F \cdot C.$$

□

Lemma 2.20. *Let D_1 and D_2 be generic divisors in a mobile linear system $\mathcal{M}$ on a smooth variety X and let $Z = D_1 \cdot D_2$. Let $\sigma : \widetilde{X} \to X$ be the blow up at some subvariety B of codimension $\geqslant 2$ and let E be the exceptional divisor. Let $\widetilde{D}_i$ be the proper transform of D_i on $\widetilde{X}$. Then*

$$\widetilde{D}_1 \cdot \widetilde{D}_2 \equiv \sigma^*(Z) + Z_E,$$

where $\operatorname{Supp} Z_E \subset E$.

Suppose also that B is a smooth curve. Let $m = \nu_E(\mathcal{M})$ and let $f \in A^2(\widetilde{X})$ be the class of a fiber of a ruled surface E. Then

$$Z_E \equiv m^2 E^2 - 2m(D_1 \cdot B) f.$$

Proof. We only prove the second part. Since $\sigma^* D_i = \widetilde{D}_i + mE$, we have

$$\widetilde{D}_1 \cdot \widetilde{D}_2 \equiv D_1 \cdot D_2 + m^2 E^2 - 2m\sigma^*(D_1 \cdot E) \equiv \sigma^*(Z) + m^2 E^2 - 2m(D_1 \cdot B)f$$

by Lemma 2.13. □

Let ν be a discrete valuation of $K(X)$ with a center on a surface X. Let

$$X_K \xrightarrow{\sigma_k} X_{K-1} \xrightarrow{\sigma_{k-1}} \dots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = X.$$

be the resolution of ν [69, Definition 10]. That is

- σ_i is the blow up at the center B_{i-1} of ν on X_{i-1},
- E_i is the exceptional divisor of σ_i and $\nu_{E_K} = \nu$.

Let E^i_j be the proper transform of E_j on X_i. There is a natural structure of an oriented graph on the set of indices $\{1, \dots, K\}$: vertices i and j are connected by the arrow if $i > j$ and $B_{i-1} \subset E^{i-1}_j$. Let p_i be the number of paths from K to i if $i < K$ and let $p_K = 1$. Let $\mathcal{M}$ be the linear system on X, $\mathcal{M}^i$ be the proper transform of $\mathcal{M}$ on X_i, and $\nu_i = \mathrm{mult}_{B_{i-1}} \mathcal{M}^{i-1}$. It is well known, how to express discrepancy of ν and multiplicity of a $\mathcal{M}$ at ν in terms of p_i and ν_i.

Lemma 2.21. *Let ν be a discrete valuation of $\mathbb{C}(X)$ with a center on a smooth surface X. Let*

$$X_K \xrightarrow{\sigma_k} X_{K-1} \xrightarrow{\sigma_{k-1}} \dots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = X.$$

be the resolution of ν, then

$$a(\nu, X) = \sum_{i=1}^{K} p_i, \quad \text{and} \quad \nu(\mathcal{M}) = \sum_{i=1}^{K} \nu_i p_i.$$

Proof. Suppose resolution of ν consists of a single blow up, then obviously $a(\nu, X) = 1 = p_1$ and $\nu(\mathcal{M}) = \nu_1 = p_1 \nu_1$.

Now suppose the resolution consists of K blow ups. Then

$$K_{X_K} = \sigma^*_K(X_{K-1}) + E_K = \sigma^*_K \circ \dots \circ \sigma^*_1(K_X) + \sum_{j=1}^{K-1} a(E_j, X)\sigma^*_K E^{K-1}_j + E_K.$$

Let p_{ji} be the number of paths from j to i, in particular, p_i is p_{Ki}. Then by assumption of induction $a(E_j, X) = \sum_{i=1}^{j} p_{ji}$. Let $\gamma_{ij} = 1$ if there is an arrow from i to j, and let $\gamma_{ij} = 0$ otherwise. Then $p_i = \sum_{j=i}^{K-1} \gamma_{Kj} p_{ji}$ and $\sigma^*_K E^{K-1}_j = E^K_j + \gamma_{Kj}$. Thus

$$a(E_K, X) = \sum_{j=1}^{K-1} \gamma_{Kj} a(E_j, X) = \sum_{j=1}^{K-1} \gamma_{Kj} \Big(\sum_{i=1}^{j} p_{ji} \Big) + 1 = \sum_{j \geqslant i \geqslant 1}^{K-1} \gamma_{Kj} p_{ji} + p_{KK} =$$

$$= \sum_{i=1}^{K-1} \sum_{j=i}^{K-1} \gamma_{Kj} p_{ji} + p_K = \sum_{i=1}^{K-1} p_i + p_K = \sum_{i=1}^{K} p_i.$$

The proof of the step of induction for the second equality is analogous. □

We could make the similar statement for higher dimensions, but we need to also keep track of the dimension of the center since discrepancies of the exceptional divisors of blow ups of points and curves differ.

2.3 Local inequalities

There are many powerful techniques for working with log pairs. Almost all of them, however, stop working when the boundary has negative coefficients. A remarkable thing about the inequalities in this section is that they work very well for negative boundaries.

Lemma 2.22 ([70]). *Let $\mathcal{M}$ be a movable linear system on $\mathbb{C}^2$ and let C_i be curves passing through the origin. Let ν be a discrete valuation such that its center is $0 \in \mathbb{C}^2$ and let*

$$e = -a\big(\nu, \mathbb{C}^2, \frac{1}{n}\mathcal{M} - \sum \alpha_i C_i\big),$$

where $\alpha_i > 0$. Then there exists a positive integer S such that for generic divisors $D_1, D_2 \in \mathcal{M}$ and for any j

$$\operatorname{mult}_0 D_1 \cdot D_2 \geqslant n^2 \frac{(S + \sum \alpha_i \nu(C_i) + e)^2}{\nu(C_j)S}.$$

Proof. Let $\sigma_i : X_i \to X_{i-1}$ be the resolution of ν. Denote $\sum p_i = S$ and note that $a(\nu, \mathbb{C}^2) = S$. By Lemma 2.21 We have the following expression for the discrepancy in terms of the graph of resolution and multiplicities

$$a\Big(\nu, \mathbb{C}^2, \frac{1}{n}\mathcal{M} - \sum_{i=1}^{K} \alpha_i C_i\Big) = a(\nu, \mathbb{C}^2) - \frac{1}{n}\sum_{i=1}^{K} p_i \nu_i + \sum_{i=1}^{K} \alpha_i \nu(C_i)$$

since $\nu(\mathcal{M}) = \sum p_i \nu_i$. Thus

$$\sum_{i=1}^{K} p_i \nu_i = n\Big(e + S + \sum_{i=1}^{K} \alpha_i \nu(C_i)\Big)$$

it is classically known that

$$\operatorname{mult}_0 D_1 \cdot D_2 \geqslant \sum_{i=1}^{K} \nu_i^2.$$

Let us find the minimum of the right hand side assuming the condition

$$\sum_{i=1}^{K} p_i \nu_i = n\Big(e + S + \sum \alpha_i \nu(C_i)\Big) = c.$$

Using the Lagrange method

$$\mathcal{L} = \sum_{i=0}^{K} \nu_i^2 - \lambda\Big(\sum_{i=0}^{K} \nu_i p_i - c\Big)$$

We get the conditions $2\nu_i = \lambda p_i$ and $\frac{\lambda}{2}\sum_{i=0}^{K} p_i^2 = c$. Thus $\lambda = \frac{2c}{\sum_{i=0}^{K} p_i}$, substituting it all back we get the inequality

$$\operatorname{mult}_0 D_1 \cdot D_2 \geqslant \frac{(\sum p_i \nu_i)^2}{\sum p_i^2} \geqslant n^2 \frac{(S + \sum \alpha_i \nu(C_i) + e)^2}{\sum p_i^2}.$$

Note that $p_i \leqslant p_1 \leqslant \nu(C_j)$ for every i and j, thus $\sum p_i^2 \leqslant p_1 S \leqslant \nu(C_j)S$. Adding this bound we get the statement of the lemma. □

This corollary is a weaker version of [18, Theorem 1.6].

Corollary 2.23. *Let $\mathcal{M}$ be a mobile linear system on $\mathbb{C}^2$. Let C be a curve passing through* *the origin. Suppose the pair* $(\mathbb{C}^2, \frac{1}{n}\mathcal{M} - \alpha C)$ *is not terminal at* 0 *then*

$$\operatorname{mult}_0 D_1 \cdot D_2 \geqslant 4n^2\alpha.$$

Proof. Since the pair is not terminal, there exists a discrete valuation ν centered at 0 such that $e = -a(\nu, \mathbb{C}^2, \frac{1}{n}\mathcal{M} - \alpha C) \geqslant 0$. Using previous lemma for the discrete valuation ν we get

$$\operatorname{mult}_0 D_1 \cdot D_2 \geqslant n^2 \frac{(S + \alpha\nu(C) + e)^2}{\nu(C)S}.$$

Since $e \geqslant 0$, we can get rid of it and inequality will only get stronger. Also since $(a+b)^2 \geqslant 4ab$

$$\operatorname{mult}_0 D_1 \cdot D_2 \geqslant n^2 \frac{4S\alpha\nu(C)}{\nu(C)S} = 4n^2\alpha.$$

□

Theorem 2.24 (Pukhlikov's inequality, [70]). *Let* $F_1, \ldots, F_n \in \mathbb{C}^3$ *be irreducible surfaces passing through the origin. Let* $\mathcal{M}$ *be a mobile linear system on* $\mathbb{C}^3$ *and let* $Z = D_1 \cdot D_2$ *be the intersection of general divisors* $D_1, D_2 \in \mathcal{M}$. *Write* $Z = Z_h + \sum Z_i$, *where the support of* Z_i *is contained in* F_i *and* Z_h *intersects* $\sum F_i$ *properly. Let* ν *be a discrete valuation such that its center is* $0 \in \mathbb{C}^3$. *Let* $e = -a(\nu, \mathbb{C}^3, \frac{1}{n}\mathcal{M} - \sum \alpha_i F_i)$ *for some* $r > 0$ *and* $\alpha_i > 0$. *Then there are positive numbers* S_0 *and* S_1 *such that*

$$S_0 \operatorname{mult}_0 Z_h + \sum_{i=1}^{n} \frac{\nu(F_i)}{\operatorname{mult}_0 F_i} \operatorname{mult}_0 Z_i \geqslant n^2 \frac{\left(e + 2S_0 + S_1 + \sum_{i=1}^{n} \alpha_i \nu(F_i)\right)^2}{S_0 + S_1}.$$

The idea of the proof is similar to that of Lemma 2.21 but there are complications arising from the increasing dimensions.

Remark 2.25. Decomposition $Z = Z_h + \sum Z_i$ may not be unique, but the inequality holds for any choice of the decomposition. Also note that we do not care if $\sum F_i$ is a normal crossing divisor, or if surfaces F_i are smooth or not.

The main application of this inequality is for the case when the pair $(X, \frac{1}{n}\mathcal{M} - \sum \alpha_i F_i)$ is strictly canonical. Then $e \geqslant 0$ and we have the lower bound on multiplicities.

Theorem 2.26 (Corti's inequality, [29, Theorem 3.12]). *Let* $F_1, \ldots, F_n \in \mathbb{C}^3$ *be irreducible surfaces passing through the origin. Let* $\mathcal{M}$ *be a mobile linear system on* $\mathbb{C}^3$ *and let* $Z = D_1 \cdot D_2$ *be the intersection of general divisors* $D_1, D_2 \in \mathcal{M}$. *Write* $Z = Z_h + \sum Z_i$, *where the support of* Z_i *is contained in* F_i *and* Z_h *intersects* $\sum F_i$ *properly. Let* $\alpha_i \geqslant 0$ *be rational numbers such that the pair* $(\mathbb{C}^3, \frac{1}{n}\mathcal{M} - \sum \alpha_i F_i)$ *is not terminal at* 0. *Then there are rational numbers* $0 < t_i \leqslant 1$ *such that*

$$\operatorname{mult}_0 Z_h + \sum t_i \operatorname{mult}_0 Z_i \geqslant 4n^2(1 + \sum \alpha_i t_i \operatorname{mult}_0 F_i).$$

Using Pukhlikov's inequality involves a lot of calculations. Corti's inequality looks prettier and is easier to use. On the other hand it is weaker. In fact we are going to derive Corti's inequality from Pukhlikov's inequality. The original proof is different and uses mathematical induction.

Proof. Since the pair is not terminal, there is a discrete valuation ν such that its center is 0 and

$$-e = (\nu, X, \frac{1}{n}\mathcal{M} - \sum \alpha_i F_i) \leqslant 0.$$

Applying Pukhlikov's inequality and using the inequality $e \geqslant 0$ we see that

$$\operatorname{mult}_0 Z_h S_0 + \sum_{i=1}^{n} \operatorname{mult}_0 Z_i \frac{\nu(F_i)}{\operatorname{mult}_0 F_i} \geqslant n^2 \frac{\left(2S_0 + S_1 + \sum_{i=1}^{n} \alpha_i \nu(F_i)\right)^2}{S_0 + S_1}.$$

Set

$$t_i = \frac{\nu(F_i)}{S_0 \operatorname{mult}_0 F_i}$$

and let us divide both parts of Pukhlikov's inequality by S_0, then

$$\operatorname{mult}_0 Z_h + \sum_{i=1}^{n} t_i \operatorname{mult}_0 Z_i \geqslant n^2 \frac{\left(2S_0 + S_1 + \sum_{i=1}^{n} \alpha_i \nu(F_i)\right)^2}{S_0(S_0 + S_1)}.$$

Let us open the brackets, use the equalities $\nu(F_i) = t_i S_0 \operatorname{mult}_0 F_i$, and rearrange the terms:

$$\operatorname{mult}_0 Z_h + \sum_{i=1}^{n} t_i \operatorname{mult}_0 Z_i \geqslant n^2 \Bigg(4 + \frac{4S_0^2(\sum t_i \operatorname{mult}_0 F_i \alpha_i)}{S_0(S_0 + S_1)} + \\ + \frac{S_1^2 + 2S_1 S_0(\sum t_i \operatorname{mult}_0 F_i \alpha_i) + S_0^2(\sum t_i \operatorname{mult}_0 F_i \alpha_i)^2}{S_0(S_0 + S_1)} \Bigg).$$

Clearly

$$S_1^2 + S_0^2\left(\sum t_i \alpha_i \operatorname{mult}_0 F_i\right)^2 \geqslant 2S_1 S_0 \left(\sum t_i \alpha_i \operatorname{mult}_0 F_i\right),$$

Therefore after simplifying we get

$$\operatorname{mult}_0 Z_h + \sum_{i=1}^{n} t_i \operatorname{mult}_0 Z_i \geqslant 4n^2 + 4n^2 \left(\sum t_i \alpha_i \operatorname{mult}_0 F_i\right),$$

thus Corti's inequality holds. □

Note that if we do not have a negative boundary, that is all $t_i = 0$, then the Corti's inequality turns into the $4n^2$-inequality.

Corollary 2.27 ($4n^2$-inequality, [71, Corollary 7.3]). *Let $\mathcal{M}$ be the linear system on $\mathbb{C}^3$ and suppose the pair $(\mathbb{C}^3, \frac{1}{n}\mathcal{M})$ is not canonical at 0. Then for generic divisors $D_1, D_2 \in \mathcal{M}$*

$$\operatorname{mult}_0(D_1 \cdot D_2) \geqslant 4n^2$$

This inequality was used implicitly by Iskovskih and Manin to prove non-rationality of quartic threefold. The idea of their proof is to get the lower bound on multiplicity of $D_1 \cdot D_2$ from non canonicity of the linear system, which contradicts the upper bound derived from the degree of $D_1 \cdot D_2$. We describe it in more details in Section 3.

2.4 Homogeneous coordinates on toric varieties

In this section we recall the construction of the toric varieties as quotients and the notion of the Cox ring of a toric variety. We also introduce homogeneous coordinates and consider a few examples, relevant for us later. For more details see [31, Chapter 5].

Definition 2.28. Let $A = \mathbb{C}[x_0, \dots, x_n]$ be a graded ring with $\deg(x_i) = a_i$, where $a_i \in \mathbb{Z}_{>0}$, and let

$$\mathbb{P}(a_0, \dots, a_n) = \operatorname{Proj}(A)$$

We say that $\mathbb{P}(a_0, \ldots, a_n)$ is the weighted projective space with weights $a_0, \ldots, a_n$.

Weighted projective spaces are the simplest examples of toric varieties. The fan of a weighted projective space is given by primitive vectors $v_0, \ldots, v_n$ in $\mathbb{Z}^n$, satisfying $\sum v_i a_i = 0$. The ring $A = \mathbb{C}[x_0, \ldots, x_n]$ with the grading as a above is called a Cox ring of a weighted projective space. The divisors in $\mathbb{P}(a_0, \ldots, a_n)$ are given by the equations $f = 0$, where $f \in A$ is a homogeneous polynomial. Clearly x_i are the homogeneous coordinates. Also the torus invariant divisors are given by the equations $x_i = 0$.

We can also construct $\mathbb{P}(a_0, \ldots, a_n)$ as a quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$, where the $\mathbb{C}^*$-action is defined as follows:

$$\lambda \cdot (x_0, \ldots, x_n) = (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n).$$

Clearly, the quotient map sends hyperplanes $x_i = 0$ into the torus invariant divisors. The homogeneous polynomials $f \in A$ are the semi-invariants of $\mathbb{C}[x_0, \ldots x_n]$.

Example 2.29. Weighted projective spaces give us a convenient way of constructing double covers. Smooth del Pezzo surfaces of degree 2 are double covers of $\mathbb{P}^2$ branched over a quartic $q_4(x, y, z) = 0$. Thus we may see them as a quartics in $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$ given by equations

$$w^2 = q_4(x, y, z).$$

It is known that smooth del Pezzo surfaces of degree 1 are double covers of the quadratic cone branched over cubic sections. Quadratic cone is $\mathbb{P}(1_x, 1_y, 2_z)$ and a cubic section on it is given by a homogeneous equation of degree 6: $f_6(x, y, z) = 0$. Thus, smooth del Pezzo surfaces of degree 1 are sextics of the form $w^2 = f_6(x, y, z)$ in $\mathbb{P}(1_x, 1_y, 2_z, 3_w)$.

The quotient construction of weighted projective spaces can be generalized. Every toric variety X may be constructed as a quotient $\big(\mathbb{C}^N \setminus Z(I)\big)/(\mathbb{C}^*)^k$ for some ideal I and some $(\mathbb{C}^*)^m$-action. The ring $\mathbb{C}[x_0, \ldots, x_{N-1}]$ with the $\mathbb{Z}^m$-grading corresponding to the $(\mathbb{C}^*)^m$-action is the Cox ring of X, and $x_0, \ldots, x_{N-1}$ are the homogeneous coordinates.

Example 2.30. Choose coordinates $y_0, \ldots, y_m, x_0, \ldots, x_n$ on $\mathbb{C}^{n+m+2}$, and let

$$I = \langle y_0, \ldots, x_y \rangle \cap \langle x_0, \ldots, x_n \rangle.$$

Consider the quotient $X = \big(\mathbb{C}^{n+m+2} \setminus Z(I)\big)/(\mathbb{C}^*)^2$, where the action of $(\mathbb{C}^*)^2$ is given by the matrix

$$\begin{pmatrix} y_0 & \ldots & y_m & x_0 & x_1 & \ldots & x_n \\ 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ 1 & \ldots & 1 & 0 & a_1 & \ldots & a_n \end{pmatrix}$$

that is, for $(\lambda, \mu) \in (\mathbb{C}^*)^2$

$$(\lambda, \mu) \cdot (y_0, \ldots, y_m, x_0, \ldots, x_n) = (\mu y_0, \ldots, \mu y_m, \lambda x_0, \lambda \mu^{a_1} x_1, \ldots, \lambda \mu^{a_n} x_n).$$

The Cox ring of X is $A = \mathbb{C}[y_0, \ldots, y_m, x_0, \ldots, x_n]$, it is bi-graded with $\mathrm{wt}(y_i) = (0, 1)$ and $\mathrm{wt}(x_i) = (1, a_i)$. Divisors in X are given by the polynomials in A which are homogeneous with respect to both gradings. We say that the bi-degree of a divisor on X is the bi-degree of a polynomial defining X.

Consider the projection map $\pi : X \to \mathbb{P}^m$ given by

$$(y_0, \ldots, y_m, x_0, \ldots, x_n) \mapsto (y_0, \ldots, y_m).$$

It is easy to see that the fiber of π is $\mathbb{P}^n$, hence X is a $\mathbb{P}^n$-bundle over $\mathbb{P}^m$. We may also construct it as

$$X = \mathrm{Proj}\,\big(\mathcal{O}_{\mathbb{P}^m} \oplus \mathcal{O}_{\mathbb{P}^m}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^m}(a_n)\big).$$

Note that if $a_i = 0$ for all i, X is a direct product. And if $n = m = 1$, then X is a Hirzebruch surface $X \cong \mathbb{F}_{a_1}$.

Example 2.31. We may generalize the previous construction by adding weights to the fibers. Change the action in the previous example to the action given by the matrix

$$\begin{pmatrix} y_0 & \dots & y_m & x_0 & x_1 & \dots & x_n \\ 0 & \dots & 0 & b_0 & b_1 & \dots & b_n \\ 1 & \dots & 1 & a_0 & a_1 & \dots & a_n \end{pmatrix}$$

The projection onto $(y_0, \dots, y_m)$ defines the structure of a $\mathbb{P}(b_0, \dots, b_n)$-bundle on X. We can define the bidegree of polynomials and divisors on X in the same way we did in the previous example.

Consider a $\mathbb{P}(1,1,1,2)$-bundle $\pi_Y : Y \to \mathbb{P}^m$, that Y is a variety defined above with $n = 3$, $b_0 = b_1 = b_2 = 1$, and $b_3 = 2$. Let X be a hypersurface of bi-degree $(4, d)$ in Y and let $\pi = \pi_Y|_X$. Suppose a generic fiber of π is smooth. Then fiber of π is a quartic in $\mathbb{P}(1,1,1,2)$, therefore it is a del Pezzo surface of degree 2. Thus if X is quasismooth and ample, then $\pi : X \to \mathbb{P}^1$ is a del Pezzo fibration. Similarly a quasismooth ample irreducible divisor of bi-degree $(6, d)$ in a $\mathbb{P}(1,1,2,3)$-bundle admits a del Pezzo fibration of degree 1.

Proof of Proposition 1.10. Let D_s, where $s \in \{x, y, z, w, u, v\}$ be the torus-invariant divisor on Y given by the equation $s = 0$. Then

$$-K_Y = D_x + D_y + D_z + D_w + D_u + D_v$$

and a $\deg K_Y = (-5, -2-a-b-c)$. The definition of degree can naturally be extended from Y to X, thus by adjunction $\deg -K_X = (1, 2+a+b+c-d)$. Denote $D = D_x\big|_X$ and $F = D_u\big|_X$, clearly $\deg D = (1, 0)$, and $\deg F = (0, 1)$. Thus

$$K_X^2 = D^2 + (2a + 2b + 2c + 4 - 2d) D \cdot F$$

Let f be the class of a line in a fiber of the del Pezzo fibration, then $F \cdot D = 2f$.

Suppose $c \geqslant 2a$, let $D' = D_y\big|_X$, and denote $s = D' \cdot D$. Then the classes s and f generate the cone of effective 1-cycles on X. Hence

$$D^2 = D \cdot D' - D \cdot aF = s - aD \cdot F$$

and

$$K_X^2 = s + (a + 2b + 2c + 4 - 2d) D \cdot F.$$

Therefore X satisfies the K^2-condition if an only if $2d \geqslant a + 2b + 2c + 4$.

Suppose $c \leqslant 2a$, let $D' = \frac{1}{2} D_w\big|_X$, and denote $s = D' \cdot D$. Then the casses s and f generate the cone of effective 1-cycles on X. Hence

$$D^2 = D \cdot D' - D \cdot \frac{c}{2} F = s - \frac{c}{2} D \cdot F$$

and

$$K_X^2 = s + (2a + 2b + \frac{3}{2} c + 4 - 2d) D \cdot F.$$

Therefore X satisfies the K^2-condition if an only if $2d \geqslant 2a + 2b + \frac{3}{2}c + 4$. □

Proof of Corollary 1.15. In the notations of Proposition 1.10 we have the varieties X_n and Y_n satisfy $d = a = b = 0$ and $c = -n$. By Proposition 1.10 the variety X_n satisfies the K^2-condition if and only if $0 \geqslant -\frac{3}{2}n + 4$, that is if $n \geqslant 3$. Hence by Theorem 1.5 the variety X_n is birationally rigid and hence not rational if $n \geqslant 3$.

□

Chapter 3

Birational rigidity

In this chapter we discuss the proof of birational rigidity of smooth del Pezzo fibrations and prove the rigidity of certain singular ones, that is we prove Theorem 1.5

3.1 Rigidity of Fano varieties

This proposition is the primary way of showing birational superrigidity of a Fano variety.

Proposition 3.1 (Noether-Fano inequality, [27, Theorem 4.2])**.** *Let V be a Fano variety with $\operatorname{Pic} V = \mathbb{Z}$. Then V is birationally superrigid if and only if there are no movable linear systems $\mathcal{M}$ on V such that $\lambda\mathcal{M} + K_V \sim_{\mathbb{Q}} 0$ and the pair $(V, \lambda\mathcal{M})$ is not canonical for some $\lambda \in \mathbb{Q}$.*

To show the superrigidity of Fano variety it is enough to prove that all linear systems after appropriate scaling have canonical singularities. For this we assume the system exists and derive a contradiction from one of the local inequalities. The following is the first known example of superrigid varieties.

Theorem 3.2 ([69, Theorem 3])**.** *A smooth quartic $X \in \mathbb{P}^4$ is birationally superrigid.*

Proof. Suppose X is not superrigid, then by Noether-Fano inequality there exists a mobile linear system $\mathcal{M} \subset \left| -nK_X \right|$ such that the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical. Applying [69, Theorem 2] we conclude that $\operatorname{mult}_C \mathcal{M} \leqslant n$ for any curve C, thus the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at curves. Suppose the pair is not canonical at some point P. Then by $4n^2$-inequality (Corollary 2.27) for generic divisors $D_1, D_2 \in \mathcal{M}$ and generic hyperplane H in $\mathbb{P}^4$

$$4n^2 = D_1 \cdot D_2 \cdot H \geqslant \operatorname{mult}_P D_1 \cdot D_2 > 4n^2,$$

contradiction. □

The proof of this theorem highlights the essence of the method used to prove rigidity. There is a system which is not canonical at some subvariety B, thus its multiplicity at B is bounded from below. But on the other hand the system has a degree which bounds the multiplicity from above. If these bounds contradict we can show superrigidity. The method is harder to apply to del Pezzo fibrations and conic bundles because of fiberpreserving maps.

3.2 Rigidity of del Pezzo fibrations

In this section we recall the proof of rigidity of smooth del Pezzo fibrations of degree 2 and see what is the difficulty in the singular case. First, we recall a version of Noether-Fano inequality for Fano fibrations.

Recall that a Fano fibration is birationally superrigid if and only if any birational map to a variety admitting a Mori fiber space is fiberpreserving. For birationally rigid varieties we may only require for birational maps to varieties admitting Mori fiber spaces to fit into the commutative diagram in Definition 1.3, it automatically implies that the general fibers are isomorphic.

Theorem 3.3 (Noether-Fano inequality, [70]). *Suppose $\pi : X \to \mathbb{P}^1$ is a del Pezzo fibration and let F be a fiber of π. Suppose it satisfies the K^2-condition. Suppose $g : X \dashrightarrow Y$ is a birational map which is not a morphism. If Y admits a Mori fiber space, then there exists a linear system $\mathcal{M}$ and positive rational numbers λ and γ such that $\lambda\mathcal{M} + K_X \sim \gamma F$ and the pair $(X, \lambda\mathcal{M})$ is not canonical.*

Suppose the pair $(X, \lambda\mathcal{M})$ is not canonical at some valuation ν. Let (Y, f, E) be a realization of ν, that is $f : Y \to X$ and $\nu_E = \nu$. Denote $a = a(X, E)$ and define m by the number $f^*(\mathcal{M}) = f^{-1}(\mathcal{M}) + mE$. Then the following inequality holds

$$0 \geqslant a(X, E, \lambda\mathcal{M}) = a - \lambda m$$

or, equivalently $a \geqslant \lambda m$. The previous theorem is called Noether-Fano inequality because non canonicity of the pair implies this inequality.

Consider the double cover $\sigma : X \to \mathbb{P}^2 \times \mathbb{P}^1$ branched over a smooth divisor of bi-degree $(4, 2l)$. If $l = 0$ then the variety is a direct product of a del Pezzo surface and $\mathbb{P}^1$, and therefore clearly is rational. If $l \geqslant 1$, then X admits a Mori fiber space. Indeed, a generic fiber of a composition π of σ and the projection onto $\mathbb{P}^1$ satisfies is a Mori is a del Pezzo surface of degree 2. Variety X is smooth since the branching divisor is smooth and $\operatorname{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ by Lefschetz hyperplane section theorem. We have $K_X = \sigma^*(D)$, where D is a divisor of bi-degree $(1, l-2)$, thus X satisfies the K^2-condition if and only if $l \geqslant 2$.

To prove birational rigidity we would like to show that there are no systems like in Noether-Fano inequality. Unfortunately in our case there are such systems. There is a birational involution χ_C for each section C which does not pass through the ramification divisor on a general fiber. In particular we can see that del Pezzo fibrations of degree 2 are not super-rigid. But this is the least of our difficulties. fiberpreserving transformations also give systems with non canonical singularities even without K^2-condition (Theorem 2.11). The method of supermaximal singularities was developed in [70] to deal with this difficulty.

Proposition 3.4 ([70, Proposition 2.1]). *Let $\pi : X \to \mathbb{P}^1$ be a del Pezzo fibration of degree 2 satisfying the K^2-condition. Let $g : X \dashrightarrow V$ be a birational map to a variety admitting a Mori fiber space $\bar{\pi} : V \to Z$. Suppose also that a map g is not fiberpreserving if $Z \cong \mathbb{P}^1$. Let $\mathcal{M} = g^{-1}|\bar{\pi}^*H|$, where H is a very ample divisor on Z. Then there are numbers $n > 0$, $\gamma \geqslant 0$ such that $\mathcal{M} \subset |-nK_X + \gamma nF|$, where F is a fiber of π and one of the following holds.*

(i) There is a discrete valuation ν of the field $K(X)$ such that its center on X is a curve and the pair $\left(X, \frac{1}{n}\mathcal{M}\right)$ is not canonical at ν;

(ii) There are finitely many discrete valuations ν_i of $K(X)$ such that

- *the centers P_i of ν_i are the points which all lie in different fibers F_i,*
- *the pair $\left(X, \frac{1}{n}\mathcal{M}\right)$ is not canonical at every ν_i,*
- *and the following inequality holds*

$$\gamma < -\sum \frac{a(\nu_i, X, \frac{1}{n}\mathcal{M})}{\nu_i(F_i)}.$$

This proposition was stated in [70] only for smooth varieties but the proof can be applied to varieties in the Mori category without any changes.

We now assume that X satisfies the requirements of Theorem 1.5. Suppose X is a quasismooth hypersurface of bi-degree $(4, d)$ in a toric $\mathbb{P}(1, 1, 1, 2)$-bundle Y. In our case it means that X is smooth outside of $X \cap \operatorname{Sing}(Y)$ and singularities of X are only $\frac{1}{2}(1, 1, 1)$-points. Suppose also that $\operatorname{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ this is true, for example, when X is an ample divisor on Y. And we assume the that every fiber containing a singularity is given by the equation $q_4(x, y, z) = 0$ in $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$.

Proposition 3.5. *Let C be a curve on X which is not a section of π Then the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at C. Let Q be a half-point on X, then $(X, \frac{1}{n}\mathcal{M})$ is canonical at Q.*

Proof. It has been shown in [70] that the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at C if it is not a section and does not lie in a fiber containing a half-point.

Let F be a fiber containing the half-point Q, then F is a cone by assumption. The curves from the ruling of F intersect $-K_X$ by $\frac{1}{2}$, therefore the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at half-points by Lemma 2.17. Thus by Corollary 2.16 the pair is also canonical at every curve passing through the half-point.

Suppose that a curve C does not pass through Q. Let L be a generic curve such that $L \cdot K_F = -\frac{1}{2}$. Then $L \not\subset \operatorname{Supp} C$ and the intersection $C \cdot L$ is an integer $\geqslant 1$. On the other hand $C \equiv rL$ for some r and

$$C \cdot L = rL^2 = \frac{r}{8} \geqslant 1.$$

Suppose the pair is not canonical at C. Then for a generic $D \in \mathcal{M}$ there is the decomposition $D\big|_F = kC + D'$, where $k > 1$ and D' is an effective divisor on F which does not contain L. It is impossible since

$$rkL + D' \equiv D\big|_F \equiv -K_F \equiv 4L.$$

□

Note that while proving the second assertion we used the geometry of the fiber containing the half-point Q of X.

The general fiber $X_{/\mathbb{P}^1}$ is a del Pezzo surface of degree 2, thus there is a double cover $\sigma_{/\mathbb{P}^1} : X_{/\mathbb{P}^1} \to \mathbb{P}^2_{\mathbb{C}(t)}$. Let $R_{/\mathbb{P}^1} \subset X_{/\mathbb{P}^1}$ be the ramification divisor of $\sigma_{/\mathbb{P}^1}$. Note that there is a $1:1$ correspondence between sections of π and points on $X_{/\mathbb{P}^1}$. Let us denote the point on $X_{/\mathbb{P}^1}$ corresponding to a section C as P_C.

Proposition 3.6 ([70, Section 3])**.** *Let C be a section of π. Suppose $P_C \in R_{/\mathbb{P}^1}$, then the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at C. Suppose $P_C \notin R_{/\mathbb{P}^1}$, then there exists a birational involution χ_C such that*

- *the following diagram is commutative*

$$\begin{array}{ccc} X & \overset{\chi_C}{\dashrightarrow} & X \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}^1 & = & \mathbb{P}^1, \end{array}$$

- *there are numbers $n' < n$ and γ' such that $\chi_C(\mathcal{M}) \subset \big| -n'K_X + n'\gamma'\big|$, and*
- *the pair $\big(X, \frac{1}{n'}\chi_C(\mathcal{M})\big)$ is canonical at C.*

Consider a del Pezzo surface S of degree 2 over the field $\mathbb{C}(t)$. Let $P_C \in S$ be a point on S. Let $\widetilde{S}$ be the del Pezzo surface of degree one which we acquired by the blow σ_C of the point P_C. The map given by the linear system $\big| -2K_{\widetilde{S}}\big|$ is a double cover of a quadratic cone. Let $\tau : \widetilde{S} \to \widetilde{S}$ be the associated biregular involution. Let $\chi_C = \sigma_C \circ \tau \circ \sigma_C^{-1}$, it is a birational involution of S known as Geiser involution. This is the map which induces the birational involution of the fibration in the Proposition 3.6.

The maps χ_C are the kind which we allow in the definition of rigid varieties. We prove that every map to a variety admitting a Mori fiber space is a composition of χ_C and fiberpreserving maps. Using Proposition 3.6 we can *untwist* the curve C, that is we replace the map f with $f \circ \chi_C$ and gain less singular linear system. Thus we may assume that the pair $(X, \frac{1}{n})$ is no canonical only at nonsingular points of X. This is the difficult case.

3.3 Supermaximal singularities

Suppose we are in the case (ii) of Proposition 3.4. By Proposition 3.5 we may assume that the points, where the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical, are not singular. Let $D_1, D_2 \in \mathcal{M}$

be generic divisors and let $Z = D_1 \cdot D_2$. For some point P we bound $\mathrm{mult}_P Z$ from above using the degrees and from below using Corti's inequality. We show that these bounds contradict each other if the fiber containing P does not pass through half-points.

Let $C \subset X$ be an irreducible curve. We say that C is *horizontal* if $\pi(C) = \mathbb{P}^1$ and that it is *vertical* if $\pi(C)$ is a point. We say that a cycle C is vertical (horizontal) if every curve irreducible curve in $\mathrm{Supp}(C)$ is vertical (horizontal). We decompose

$$Z = Z^v + Z^h$$

into the vertical and the horizontal components. For the vertical component we have the decomposition

$$Z^v = \sum_{t \in \mathbb{P}^1} Z_t^v,$$

where the support of Z_t^v is in the fiber $F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$. Define the degree of a vertical 1-cycle C^v by the number $\deg C^v = C^v \cdot (-K_X)$ and degree of a horizontal 1-cycle C^h by the number $\deg C^h = C^h \cdot F$. Let $f \in A^2(X)$ be the class of a line in a fiber, clearly $\deg f = 1$.

Lemma 3.7 ([70]). *Let C be an irreducible curve on X and let $P \in X$ be a nonsingular point such that $P \in C$. Then*

(i) if C is horizontal then $\mathrm{mult}_P C \leqslant \deg C$,

(ii) if C lies in a fiber F, which does not contain a half-point then $\mathrm{mult}_P C \leqslant \deg C$.

Proof. Suppose C is horizontal. Then $\mathrm{mult}_P F \geqslant 1$ implies

$$\mathrm{mult}_P C \leqslant C \cdot F = \deg C.$$

Suppose C is a curve in a fiber. For $n >> 0$ the linear system $\big| - K_X + nF\big|$ is ample, thus a general $D \in \big| - K_X + nF\big|$ does not contain C. Since general D is smooth

$$\mathrm{mult}_P C \leqslant C \cdot D = -K_X \cdot C = \deg C.$$

□

Lemma 3.8 ([70]). *The following holds for the degrees of Z^h and Z^v*

$$\deg Z^h = 2n^2 \quad \textit{and} \quad \sum_{t \in \mathbb{P}^1} \deg Z_t^v \leqslant 4n^2\gamma.$$

Proof. The class of the cycle Z is

$$(-nK_X + \gamma nF)^2 \equiv n^2 K_X^2 + 4n^2\gamma f.$$

It follows from the K^2-condition and the effectiveness of the cycle Z^h that the class of Z^v is lf, where $l \leqslant 4n^2\gamma$. We intersect the cycle Z with F to find the degree of Z^h

$$\deg Z^h = F \cdot (n^2 K_X^2 + 4n\gamma f) = n^2(F \cdot K_X^2) = n^2\big(2f \cdot (-K_X)\big) = 2n^2.$$

□

Now we need to find which pair and which point do we apply Corti's inequality to. Let F_i be the fibers containing the centers P_i of ν_i. Let Z_i^v be the part of vertical cycle which is contained in F_i.

Lemma 3.9. *There are numbers γ_i such that the pair $(X, \frac{1}{n}\mathcal{M} - \sum \gamma_i F_i)$ is strictly canonical at each ν_i and $\sum \gamma_i > \gamma$.*

Proof. Set $\gamma_i = -\frac{a(\nu_i, X, \frac{1}{n}\mathcal{M})}{\nu_i(F)}$, these numbers satisfy the inequality by Proposition 3.4. The pair is obviously canonical at every ν_i. □

Corollary 3.10. *There is an index i such that*

$$\deg Z_i^v < 4n^2\gamma_i.$$

Proof. By Lemma 3.8 and Lemma 3.9 we have

$$\sum \deg Z_i^v \leqslant 4n^2\gamma < 4n^2 \sum \gamma_i.$$

If inequality holds for the sums, then it must hold for at least one i. □

We say that ν_i a *supermaximal singularity* if $\deg Z_i^v < 4n^2\gamma_i$. Fix a supermaximal singularity ν_i. To simplify the notations, from now on denote F_i as F, γ_i as γ, P_i as P, Z_i^v as Z_v, Z^h as Z_h, and ν_i as ν.

Proposition 3.11 ([70]). *Suppose a fiber F does not contain a singular point of the type $\frac{1}{2}(1,1,1)$, then there are no supermaximal singularities with a center on F.*

Proof. The pair $(X, \frac{1}{n}\mathcal{M} - \gamma F)$ is strictly canonical at the point P. Hence by Corti's inequality there is a number $0 < t \leqslant 1$ such that

$$\operatorname{mult}_P Z_h + t \operatorname{mult}_P Z_v \geqslant 4n^2(1 + \gamma t \operatorname{mult}_P F) \geqslant 4(1+\gamma t)n^2.$$

On the other hand Lemma 3.7, Lemma 3.8, and the definition of supermaximal singularity imply

$$\operatorname{mult}_P Z_h + t \operatorname{mult}_P Z_v < 2n^2 + 4t\gamma n^2,$$

contradiction. □

Corollary 3.12 ([70]). *Suppose X is a smooth, then X is birationally rigid.*

Proof. Suppose X is not birationally rigid, that is there is a birational map g to a variety admitting a Mori fiber space which does not fit into commutative diagram in Definition 1.3. Then by Proposition 3.4 there is a system $\mathcal{M} \subset \left|-nK_X + \gamma nF\right|$ such that one of the conditions (i) and (ii) holds.

Suppose the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical at a curve C. By Proposition 3.5 the curve C is a section. Then by Proposition 3.6 there is a birational involution χ_C such that $g \circ \chi_C$ does not fit into the commutative diagram in Definition 1.3. Now we consider the map $g \circ \chi_C$ instead of g and the linear system

$$\chi_C(\mathcal{M}) \subset \left| -n'K_X + n'\gamma' \right|$$

instead of $\mathcal{M}$. We repeat this process until either $n' = 0$, that is the new map $g' = g \circ \chi_{C_1} \circ \cdots \circ \chi_{C_k}$ is fiberpreserving (contradiction), or until the pair, corresponding to g' is canonical at all curves.

Now we are in the situation (ii) of Proposition 3.4. Hence by Corollary 3.10 there is a supermaximal singularity, but it contradicts Proposition 3.11. □

Remark 3.13. Note that there are difficulties when F contains the half-point. There will be "half-lines": curves of degree $\frac{1}{2}$. Thus the bound on multiplicity (Lemma 3.8) becomes $\operatorname{mult}_P Z_v \leqslant 8\gamma n^2$ and it no longer contradicts Corti's inequality.

Pukhlikov has encountered similar problem when he was proving birational rigidity of cubic fibrations. Lines on cubic surfaces have low degree and high multiplicity, thus the bound was only $\operatorname{mult}_P Z_v \leqslant 6\gamma n^2$. To deal with this Pukhlikov constructed the "ladder": a sequence of blow ups of curves on X starting with the problematic curve. Then he applied Pukhlikov's inequality upstairs. We use this method and combine it with Corti inequality.

3.4 Construction of the ladder

Let $X^{(0)}$ be a smooth threefold, let $F^{(0)} \in X^{(0)}$ be a smooth surface, and let $L_0 \subset F^{(0)}$ be a smooth rational curve. Let us associate the following construction to L_0 which we call the *ladder*. The ladder associated to L_0 is the following chain of morphisms which we define inductively

$$X^{(M)} \xrightarrow{\sigma_M} X^{(M-1)} \xrightarrow{\sigma_{M-1}} \dots \xrightarrow{\sigma_1} X^{(0)}.$$

The morphism $\sigma_i : X^{(i)} \to X^{(i-1)}$ is the blow up of L_{i-1} and we denote $E^{(i)}$ to be its exceptional divisor. Clearly $E^{(i)} \cong \mathbb{F}_m$ for some m, we assume $m > 0$ for every i. Let L_i be the exceptional section of $E^{(i)}$. Denote the proper transform of $E^{(i)}$ on $X^{(j)}$, as $E^{(i,j)}$ and the proper transform of $F^{(0)}$ on $X^{(j)}$ as $F^{(j)}$.

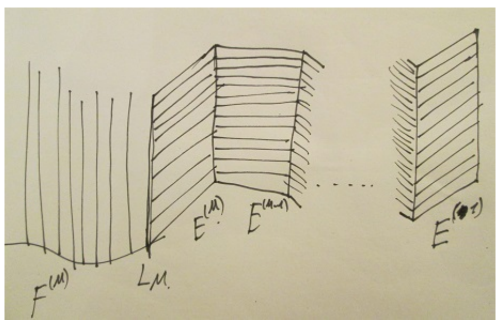
Theorem 3.14. *Suppose $L_0 \cdot K_{X^{(0)}} = 0$ and $L_0 \cdot F^{(0)} = -2$. Then the following assertions are true for the ladder associated to L_0 for all $i > 0$:*

(i) $E^{(i)} \cong \mathbb{F}_2$,

(ii) $\sigma_i^*(L_{i-1}) \equiv L_i$,

(iii) $\nu_{E^{(i)}}(F^{(0)}) = i$,

(iv) $E^{(i,i+1)}\big|_{E^{(i+1)}}$ *is disjoint from L_{i+1}, in particular the dual graph associated to the exceptional divisors of the ladder is a simple chain.*



To prove this theorem we need the following two lemmas.

Lemma 3.15. *Let $\sigma : \widetilde{X} \to X$ be the blow up of a smooth rational curve L on a smooth threefolds and let E be the exceptional divisor of σ. Suppose $L \cdot K_X = 0$ and suppose that there is a smooth surface F such that $L \cdot F = -2$, then $E \cong \mathbb{F}_2$.*

Proof. By Lemma 2.13

$$\deg N_{L/X} = 2g(L) - 2 - K_X \cdot L = -2. \tag{3.1}$$

The equality $F \cdot L = -2$ implies

$$N_{F/X}\big|_L = \mathcal{O}_L(-2). \tag{3.2}$$

There is an exact sequence of normal sheaves

$$0 \to N_{L/F} \to N_{L/X} \to (N_{F/X})\big|_L \to 0.$$

Clearly $N_{L/X} = \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)$ for some a and b, and (3.1) implies that $a + b = -2$. Without loss of generality we may assume that $a \leqslant b$. The inequality $a \leqslant -2$ follows from (3.2). On the other hand (3.1), (3.2), and the exact sequence imply $N_{L/F} = \mathcal{O}_L$, therefore $b \geqslant 0$. Hence $a = -2$ and $b = 0$. Thus $E = \operatorname{Proj}\big(\mathcal{O}_L(-2) \oplus \mathcal{O}_L\big) \cong \mathbb{F}_2$. □

Lemma 3.16. *Let $f_i, s_i \in A^2(X^{(i)})$ be the classes of a fiber and of the exceptional section respectively of a ruled surface $E^{(i)}$. Suppose $E^{(i)} \cong \mathbb{F}_2$, $K_{X^{(i-1)}} \cdot L_{i-1} = 0$, and $F^{(i-1)} \cdot L_{i-1} = -2$. Then* *(i)* $E^{(i+1)} \cong \mathbb{F}_2$,

(ii) $\sigma_i^*(L_{i-1}) \equiv L_i$,

(iii) $K_{X^{(i)}} \cdot L_i = 0$,

(iv) $F^{(i)} \cdot L_i = -2$.

Proof. By Lemma 2.13

$$0 = E^{(i)} \cdot \sigma_i^*(L_{i-1}) = E^{(i)}\big|_{E^{(i)}} \cdot \sigma_i^*(L_{i-1}) = \big(\sigma_i^*(L_{i-1}) - 2f_i\big) \cdot \sigma_i^*(L_{i-1}).$$

Thus $\sigma_i^*(L_{i-1})^2 = -2f \cdot \sigma_i^*(L_{i-1})$. Clearly $\sigma_i^*(L_{i-1})$ must be a section. Indeed, by Lemma 2.13

$$2 = \big(E^{(i)}\big)^3 = \Big((\sigma_i^*(L_{i-1}) - 2f_i\Big)^2 = 2f_i \cdot \sigma_i^*(L_{i-1}),$$

therefore $f_i \cdot \sigma_i^*(L_{i-1}) = 1$, that is $\sigma_i^*(L_{i-1})$ is a section. Since $\sigma_i^*(L_{i-1})^2 = -2$, as computed above, it is the exceptional section L_i.

It follows from *(ii)*, that $L_i \cdot E^{(i)} = 0$. Thus

$$K_{X^{(i)}} \cdot L_i = K_{X^{(i-1)}} \cdot L_{i-1} + E^{(i)} \cdot L_i = 0.$$

Similarly

$$F^{(i)} \cdot L_i = F^{(i-1)} \cdot L_{i-1} + E^{(i)} \cdot L_i = F^{(i-1)} \cdot L_{i-1} = -2.$$

By Lemma 3.15 assertions *(iii)* and *(iv)* imply *(i)*. □

Proof of Theorem 3.14. Lemma 3.16 implies *(i)* and *(ii)*.

Clearly $\nu_{E^{(1)}}(F^{(0)}) = 1$. On the other hand $L_{M-1} \subset F^{(M-1)}$ since $L_{M-1} \cdot F^{(M-1)} < 0$. Hence $\nu_{E^{(M)}}(F^{(0)}) = \nu_{E^{(M-1)}}(F^{(0)}) + 1$ and *(iii)* holds.

By Lemma 2.13 and *(i)*

$$E^{(i-1,i)}\big|_{E^{(i)}} = \big(L_{i-1} \cdot E^{(i-1)}\big)f - E^{(i)}\big|_{E^{(i)}} = s_i + 2f_i.$$

Therefore $E^{(i-1,i)}\big|_{E^{(i)}} \cdot L_i = 0$ and *(iv)* holds. □

Suppose $\pi : X \to \mathbb{P}^1$ is a del Pezzo fibration of degree 2. Suppose $Q \in X$ is a $\frac{1}{2}(1,1,1)$ point and F is a fiber containing Q. Suppose that the fiber F can be embedded into $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$ as a cone $q_4(x,y,z) = 0$. Let $\sigma_Q : X^{(0)} \to X$ be the blow up of X at Q and let E_Q be the exceptional divisor of σ_Q. Clearly, $X^{(0)}$ is smooth in a neighborhood of F.

Let $L \subset F$ be a "half-line", that is a curve L such that $L \cdot (-K_F) = \frac{1}{2}$. Note that every curve in a ruling of F is a half-line and that every half-line is such. Denote the proper transforms of L and F on $X^{(0)}$ as L_0 and $F^{(0)}$ respectively . Since F is a cone, its blow up $F^{(0)}$ is a ruled surface over a curve of genus 3. Clearly the curve is a plane quartic $q_4(x,y,z) = 0$. The curve is smooth, since X has only $\frac{1}{2}(1,1,1)$-singularities. In the next section we work with the ladder associated to L_0. We say that the ladder is also associated to the half-line L. Now we show that $X^{(0)}$, $F^{(0)}$, and L_0 satisfy the assumptions of Theorem 3.14.

Lemma 3.17. *The following equalities hold*

(i) $L_0 \cdot E_Q = 1$,

(ii) $L_0 \cdot F^{(0)} = -2$,

(iii) $K_{X^{(0)}} \cdot L_0 = 0$.

Proof. Since $L_0 \subset F^{(0)}$ we have $L_0 \cdot E_Q = L_0 \cdot E_Q\big|_{F^{(0)}}$. Let $\mathbb{P} = \mathbb{P}(1,1,1,2)$, and consider the embedding of F into $\mathbb{P}$. We can describe $L \subset F$ in $\mathbb{P}$ as the intersection $H_1 \cdot H_2$, for some $H_i \in \big|\mathcal{O}_{\mathbb{P}}(1)\big|$. Let $\sigma_{\mathbb{P}} : \widetilde{\mathbb{P}} \to \mathbb{P}$ be the blow up of the point Q and let $E_{\mathbb{P}}$ be its exceptional divisor. Clearly $\sigma_{\mathbb{P}} : \sigma_{\mathbb{P}}^{-1}(F) \to F$ is the blow up of Q thus without any confusion we may identify $\sigma_{\mathbb{P}}^{-1}(F)$ with $F^{(0)}$ and $\sigma_{\mathbb{P}}^{-1}(L)$ with L_0. Let $\widetilde{H}_i$ be the proper transform of H_i on $\widetilde{\mathbb{P}}$, then $L_0 = \widetilde{H}_1 \cdot \widetilde{H}_2$. Denote the exceptional divisor of $\sigma_{\mathbb{P}}$ as $E_{\mathbb{P}}$, then

$$L_0 \cdot E_Q\big|_{F^{(0)}} = L_0 \cdot E_{\mathbb{P}}\big|_{F^{(0)}} = L_0 \cdot E_{\mathbb{P}} = \widetilde{H}_1 \cdot \widetilde{H}_2 \cdot E_{\mathbb{P}} = \widetilde{H}_1\big|_{\widetilde{H}_2} \cdot E_{\mathbb{P}}\big|_{\widetilde{H}_2}.$$

Clearly H_2 is isomorphic to $\mathbb{P}(1,1,2)$ and $\sigma_{\mathbb{P}}\big|_{\widetilde{H}_2}$ is the blow up of a singular point. Thus $\widetilde{H}_2 \cong \mathbb{F}_2$, $E_{\mathbb{P}}\big|_{\widetilde{H}_2}$ is the exceptional section, and $\widetilde{H}_1\big|_{\widetilde{H}_2}$ is a fiber of $\widetilde{H}_2$. Hence

$$1 = \widetilde{H}_1\big|_{\widetilde{H}_2} \cdot E_{\mathbb{P}}\big|_{\widetilde{H}_2} = L_0 \cdot E_Q.$$

Consider the affine open subset $U \in \mathbb{P}$ given by $w \neq 0$. Clearly $U = \mathbb{C}^3/\langle -I_3 \rangle$ and the local equation of F at Q on U is $q_4(x,y,z) = 0$. Thus Lemma 2.14 implies $F^{(0)} = \sigma_{\mathbb{P}}^*(F) - 2E_Q$ and we find the intersection

$$L_0 \cdot F^{(0)} = -2L_0 \cdot E_Q = -2.$$

The equality (iii) follows from (i) and Lemma 2.14

$$K_{X^{(0)}} \cdot L_0 = (\sigma_Q^* K_X + \frac{1}{2}E_Q) \cdot L_0 = K_X \cdot L + \frac{1}{2}E_Q \cdot L_0 = 0.$$

□

3.5 Multiplicities on the ladder

The plan is to associate a ladder to a half-line, to apply Corti's inequality upstairs, and to derive a contradiction. Thus we need to find bounds on multiplicities of the cycles upstairs.

Let A be a cycle, a divisor, or a linear system on X. We denote its proper transform on $X^{(i)}$ as $A^{(i)}$. For divisors and cycles on $X^{(j)}$ we add upper index. For example, $E^{(1,3)}$ is the proper transform of $E^{(1)}$ on $X^{(3)}$. By σ^* we mean the appropriate composition of σ_i^*. For example, $E^{(1,3)} = \sigma^*(E^{(1)}) - E^{(2,3)} - E^{(3)}$, here $\sigma^* = \sigma_2^* \circ \sigma_1^*$.

Recall that there is a mobile linear system $\mathcal{M} \subset \big| -nK_X + lF \big|$ and discrete valuations ν_i such that the pair $(X, \frac{1}{n}\mathcal{M})$ is strictly canonical at a discrete valuation ν and ν is a supermaximal singularity. The center of ν is a nonsingular point $P \in X$. Let F be a fiber containing P, we have shown before that F contains half-point. We want to track the center of ν as we go up the ladder.

Proposition 3.18 ([70])**.** *Let $X^{(0)}$ be a threefold and let $F^{(0)}$ be a surface in it. Suppose L_0 is a smooth rational curve in $F^{(0)}$. Let $\sigma_i : X^{(i)} \to X^{(i-1)}$ be the associated ladder. Let ν be a discrete valuation of $K(X^{(0)})$ and suppose that a center of ν on $X^{(0)}$ is a point on L_0. Then there is a positive integer M such that for every $i < M$ the center of ν on $X^{(i)}$ is a point on the exceptional section L_i and the center of ν on $X^{(M)}$ is one of the following*

A) a fiber of a ruled surface $E^{(M)}$,

B) a point not on L_M and not on $E^{(M-1,M)}$, or

C) a point on $E^{(M)} \cap E^{(M-1,M)}$.

Recall the notations of Section 3.3. Let $Z = D_1 \cdot D_2$ for generic divisors $D_1, D_2 \in \mathcal{M}$. Let Z_h be the horizontal part of Z and let Z_v be the part of Z which lies in F. Let γ be the number such that the pair $(X, \frac{1}{n}\mathcal{M} - \gamma F)$ is strictly canonical at ν. Let L be a unique curve from the ruling of F passing through P. The cycle Z_v can be decomposed as $Z_v = kL + C$, where $k \geqslant 0$ and C does not contain L.

Lemma 3.19. *The inequality $C \cdot L \leqslant \gamma n^2$ holds.*

Proof. Note that $\left(\operatorname{Div} F/\equiv\right) \cong \mathbb{Z}$. Thus $-K_F \equiv 4L$ and that $C \equiv rL$ for some r. By Corollary 3.10 there is a bound $\deg(kL + C) \leqslant 4\gamma n^2$, therefore $r \leqslant 8\gamma n^2 - k$. Hence

$$C \cdot L = rL^2 = \frac{r}{8} \leqslant \gamma n^2 - \frac{k}{8} \leqslant \gamma n^2.$$

□

Lemma 3.20. *Let ν_Q be the discrete valuation corresponding to the exceptional divisor E_Q of the blow up of the half-point Q. Then for a generic $D \in \mathcal{M}$*

$$D^{(i)} \cdot L_i = \frac{n}{2} - \nu_Q(D).$$

Proof. Since $\sigma_i^* L_{i-1} = L_i$ by Theorem 3.14, the equality $D^{(0)} \cdot L_0 = D^{(i)} \cdot L_i$ holds for all i. On the other hand $E_Q \cdot L_0 = 1$ by Lemma 3.17, hence

$$D^{(0)} \cdot L_0 = \sigma_Q^*(D) \cdot L_0 - \nu_Q(D) E_Q \cdot L_0 = \sigma_Q^*(D) \cdot L_0 - \nu_Q(D),$$

and by projection formula

$$\sigma_Q^*(D) \cdot L_0 = D \cdot L = -nK_X \cdot L = \frac{n}{2}.$$

Combining the equalities we get the statement of the lemma. □

Denote $Z_i = D_1^{(i)} \cdot D_2^{(i)}$, then by Lemma 2.20 and Lemma 2.18 there is the decomposition

$$Z_0 = Z_v^{(0)} + Z_h^{(0)} + Z_Q,$$

where Z_Q is the part of the cycle which lives on the exceptional divisor. We disregard the part $Z_Q^{(i)}$ in further computations since it is away from from the center of ν.

For every $i > 0$ there is a part C_i of the cycle $Z^{(i)}$ which is contained in $E^{(i)}$. Recall that $E^{(i)}$ is a ruled surface $\mathbb{F}_2$ and $\sigma_i\big|_{E^{(i)}}$ is a $\mathbb{P}^1$-fibration. We say that a curve B on $E^{(i)}$ is vertical if $\sigma_i(B)$ is a point and horizontal otherwise. The cycle C_i can be decomposed into the sum of the exceptional section with multiplicity, the rest of the horizontal part, and the vertical part:

$$C_i = k_i L_i + C_h^{(i)} + C_v^{(i)}.$$

Note that $\sigma^* C^{(i-1,i)} = C^{(i-1,i+k)}$ for any $k > 0$, $i > 1$ since $E^{(i-1,i)}$ is disjoint from L_i. Thus there are the decompositions

$$\begin{aligned}
Z_0 &= Z_h^{(0)} + Z_v^{(0)} = Z_h^{(0)} + C^{(0)} + k_0 L_0,\\
Z_1 &= Z_h^{(1)} + C^{(1)} + C_h^{(1)} + C_v^{(1)} + k_1 L_1,\\
Z_2 &= Z_h^{(2)} + C^{(2)} + C_h^{(1,2)} + C_v^{(1,2)} + C_h^{(2)} + C_v^{(2)} + k_2 L_2,\\
Z_i &= Z_h^{(i)} + C^{(i)} + \sigma^* C_h^{(1,2)} + \sigma^* C_v^{(1,2)} + \cdots + C_h^{(i-1,i)} + C_v^{(i-1,i)} + C_h^{(i)} + C_v^{(i)} + k_i L_i.
\end{aligned}$$

Let $\lambda_i = \operatorname{mult}_{L_{i-1}} \mathcal{M}^{(i-1)}$ and recall that $f_i, s_i \in A^2(X^{(i)})$ are the classes of a fiber and of the exceptional section of $E^{(i)}$ respectively. Thus $C_v^{(i)} \equiv d_v^{(i)} f_i$ and $C_h^{(i)} \equiv d_h^{(i)} s_i + \beta_i f_i$ for some $d_v^{(i)}$, $d_h^{(i)}$, and β_i. Also $2d_h^{(i)} \leqslant \beta_i$ because $C_h^{(i)}$ does not contain the exceptional section.

Lemma 3.21. *The following relations for the proper transforms and the pullbacks of the cycles* *hold*

$$\begin{aligned}
C_h^{(i,i+1)} &\equiv d_h^{(i)} s^i + \beta_i f_i - (\beta_i - 2d_h^{(i)}) f_{i+1},\\
C_v^{(i,i+1)} &\equiv d_v^{(i)}(f_i - f_{i+1}),\\
Z_h^{(i+1)} &\equiv Z_h^{(i)} - \alpha_{i+1} f_{i+1},\\
C^{(i+1)} &\equiv \sigma^* C^{(i)} - (C^{(0)} \cdot L_0)_{F^{(0)}} f_{i+1},
\end{aligned}$$

where $\alpha_i \leqslant 2n^2$.

Proof. The equalities follow from Lemma 2.18 and computations of intersections

$$\begin{aligned}
(C_h^{(i)} \cdot L_i)_{E^{(i)}} &= \beta_i - 2d_h^{(i)},\\
(C_v^{(i)} \cdot L_i)_{E^{(i)}} &= d_v^{(i)},\\
(C^{(i)} \cdot L_i)_{F^{(i)}} &= (C^{(0)} \cdot L_0)_{F^{(0)}}.
\end{aligned}$$

The bound on α_i follows from Lemma 2.19 and the equality $Z_h^{(i)} \cdot \sigma^*(F) = 2n^2$. □

Lemma 3.22. *The vertical degrees* β_i *and* $d_v^{(i)}$ *satisfy the following relations. For* $i = 1$

$$\beta_1 + d_v^{(1)} = \alpha_1 + (C^{(0)} \cdot L_0) - \lambda_1\big(n - 2\nu_Q(D)\big) - 2\lambda_1^2,$$

and for $i \geqslant 2$

$$\beta_i + d_v^{(i)} = \alpha_i + (C^{(0)} \cdot L_0) - \lambda_i\big(n - 2\nu_Q(D)\big) - 2\lambda_i^2 + d_v^{(i-1)} + \big(\beta_{i-1} - 2d_h^{(i-1)}\big).$$

Proof. By Lemma 2.20 and Lemma 3.20

$$z_1 \equiv \sigma_1^*(z_0) + \lambda_1^2\big(E^{(1)}\big)^2 - 2\lambda_1(D_1^{(0)} \cdot L_0)f \equiv \sigma_1^*(z_0) - \lambda_1^2\sigma_1^*(L_0) - \Big(\lambda_1\big(n - 2\nu_Q(D)\big) + 2\lambda_1^2\Big) f_1.$$

On the other hand the decomposition of Z_1 and Lemma 3.21 imply

$$\begin{aligned}
Z_1 = Z_h^{(1)} + C^{(1)} + C_h^{(1)} + C_v^{(1)} + k_1 L_1 \equiv\\
\equiv \sigma_1^*\big(Z_h^{(0)} + C^{(0)} + k_1 L_0\big) + C_h^{(1)} + C_v^{(1)} - (\alpha_1 + C^{(0)} \cdot L_0) f_1.
\end{aligned}$$

Combining these equivalences we find that the following holds modulo pullback of a cycle

$$(\beta_1 + d_v^{(1)}) f_1 \equiv C_h^{(1)} + C_v^{(1)} \equiv -\Big(\lambda_1\big(n - 2\nu_Q(D)\big) + 2\lambda_1^2\Big) f_1 + \big(\alpha_1 + C^{(0)} \cdot L_0\big) f_1.$$

Similarly by Lemma 2.20 and Lemma 3.20

$$\begin{aligned}
z_i \equiv \sigma_i^*\big(Z^{(i-1)}\big) + \lambda_i^2\big(E^{(i)}\big)^2 - 2\lambda_i\big(D_1^{(i-1)} \cdot L_{i-1}\big) f \equiv\\
\equiv \sigma_i^*\big(Z^{(i-1)}\big) - \lambda_i^2\sigma^*(s_{i-1}) - \Big(\lambda_i\big(n - 2\nu_Q(D)\big) - 2\lambda_i^2\Big) f_i.
\end{aligned}$$

Once again from the decomposition of $Z^{(i)}$ and Lemma 3.21 we see that

$$\begin{aligned}
Z^{(i)} = Z_h^{(i)} + C^{(i)} + \sigma^* C_h^{(1,2)} + \sigma^* C_v^{(1,2)} + \dots + C_h^{(i-1,i)} + C_v^{(i-1,i)} + C_h^{(i)} + C_v^{(i)} + k_i L_i \equiv\\
\equiv \sigma^*(\dots) + C_h^{(i)} + C_v^{(i)} - \Big(\alpha_i + C^{(0)} \cdot L_0 + \big(\beta_{i-1} - 2d_h^{(i-1)}\big) + d_v^{(i-1)}\Big) f_i.
\end{aligned}$$

Combining these equivalences and considering them modulo pullbacks of the cycles we conclude that

$$d_v^{(i)} + \beta_i = \Big(\alpha_i + C^{(0)} \cdot L_0 + \big(\beta_i - 2d_h^{(i)}\big) + d_v^{(i-1)}\Big) - \Big(\lambda_i\big(n - 2\nu_Q(D)\big) + 2\lambda_i^2\Big).$$

□

Corollary 3.23. *The vertical degrees are bounded as follows*

$$\beta_i + d_v^{(i)} < \sum_{j=1}^{i} \left(2n^2 + n^2\gamma - 2\lambda_i^2\right)$$

Proof. The inequality $n \geqslant 2\nu_Q(D)$ holds since the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at Q. By Lemma 3.22 for $i = 1$

$$\beta_1 + d_v^{(1)} = \alpha_1 + (C^{(0)} \cdot L_0) - \lambda_1\big(n - 2\nu_Q(D)\big) - 2\lambda_1^2.$$

Combining it with the bounds $\alpha_1 \leqslant 2n^2$ and $C^{(0)} \cdot L_0 \leqslant C \cdot L \leqslant \gamma n^2$ we get

$$\beta_1 + d_v^{(1)} < 2n^2 + \gamma n^2 - 2\lambda_1^2.$$

Now suppose the inequality holds for $i - 1$. Then using the same bounds we get

$$\begin{aligned}\beta_i + d_v^{(i)} &= \alpha_i + C^{(0)} \cdot L_0 + d_v^{(i-1)} + \big(\beta_{i-1} - 2d_h^{(i-1)}\big) - \\ &- \lambda_i\big(n - 2\nu_Q(D)\big) - 2\lambda_i^2 \leqslant (2n^2 + \gamma n^2 - 2\lambda_i^2) + (d_v^{(i-1)} + \beta_{i-1}).\end{aligned}$$

□

Corollary 3.24. *(i) Let B be a fiber of the ruled surface $E^{(i)}$ then*

$$\operatorname{mult}_B Z^{(i)} = \operatorname{mult}_B C_v^{(i)} \leqslant \sum_{j=1}^{i} \left(2n^2 - 2\lambda_i^2 + n^2\gamma\right)$$

(ii) Let B be a point on $E^{(i)}$ then

$$\operatorname{mult}_B (C_v^{(i)} + C_h^{(i)}) \leqslant \sum_{j=1}^{i} \left(2n^2 - 2\lambda_i^2 + n^2\gamma\right).$$

Proof. Clearly $\operatorname{mult}_B C_v^{(i)}$ is bounded by a vertical degree $d_v^{(i)}$ whether B is a point or a curve. Thus the inequality holds if B is a curve.

Similarly, if B is a point, then $\operatorname{mult}_B C_h^{(i)} \leqslant d_h^{(i)}$, hence $\operatorname{mult}_B(C_v^{(i)} + C_h^{(i)}) \leqslant d_v^{(i)} + d_h^{(i)}$. Since $C_h^{(i)}$ does not contain the exceptional section $d_h^{(i)} \leqslant \beta_i$. Therefore by Corollary 3.23 the inequalities hold. □

3.6 Supermaximal singularities upstairs

In the previous section we found an upper bound on the multiplicity of components of $Z^{(M)}$ at the center of ν on $X^{(M)}$. In this section we show that it contradicts Corti's inequality.

Lemma 3.25. *The pair*

$$\left(X^{(M)}, \frac{1}{n}\mathcal{M}^{(M)} - \left(1 - \frac{\lambda_1}{n}\right)E^{(1,M)} - \cdots - \left(M - \sum \frac{\lambda_i}{n} + M\gamma\right)E^{(M)} - \gamma F^{(M)}\right)$$

is strictly canonical at ν.

Proof. Since the dual graph of the divisors $E^{(i)}$ is a simple chain by Theorem 3.14

$$K_{X^{(M)}} - \sum_{i=1}^{M} iE^{(i)} - \frac{1}{2}E_Q \sim \sigma^*(K_X).$$

We disregard E_Q in all equivalences since E_Q is away from the center of ν. For a generic divisor $D \in \mathcal{M}$

$$D^{(M)} + \sum_{i=1}^{M} \Big(\sum_{j=1}^{i} \lambda_j \Big) E^{(i)} = \sigma^*(D)$$

and by Theorem 3.14

$$F^{(M)} + \sum_{i=1}^{M} iE^{(i)} = \sigma^*(F).$$

Thus the pair in the statement of the lemma is a log pullback of the pair $\Big(X, \frac{1}{n}\mathcal{M} - \gamma F\Big)$. Hence by Remark 2.8 the pair is strictly canonical at ν. □

Clearly ν and $X^{(0)}$ satisfy the requirements of Proposition 3.18. Let M be the minimal integer such that the center of ν on $X^{(M)}$ is not a point on the exceptional section of $E^{(M)}$. We consider the three possibilities for the center of ν.

3.6.1 Case A

Suppose the center B of ν on $X^{(M)}$ is a fiber of the ruled surface $E^{(M)}$. Then the only divisor in the boundary which contains B is $E^{(M)}$. Thus the pair

$$\Big(X^{(M)}, \frac{1}{n}\mathcal{M}^{(M)} - \big(M - \sum \frac{\lambda_i}{n} + M\gamma\big) E^{(M)} \Big)$$

is strictly canonical at ν. By Corollary 2.23

$$\operatorname{mult}_B Z^{(M)} \geqslant 4n^2 \frac{Mn - \sum \lambda_i}{n} + 4n^2 M\gamma.$$

Combining this inequality with Corollary 3.24 we get

$$2Mn^2 + Mn^2\gamma - 2\sum_{i=1}^{M} \lambda_i^2 > 4\big(Mn^2 - n\sum \lambda_i + Mn^2\gamma\big)$$

or, equivalently,

$$0 > 3Mn^2\gamma + 2\sum_{i=1}^{M} \big(n^2 - 2n\lambda_i + \lambda_i^2\big),$$

contradiction.

3.6.2 Case B

Suppose the center B of ν on $X^{(M)}$ is a point which is not on $E^{(M-1)}$. Then the only divisor in the boundary containing B is $E^{(M)}$. Thus the pair

$$\Big(X^{(M)}, \frac{1}{n}\mathcal{M}^{(M)} - \big(M - \sum \frac{\lambda_i}{n} + M\gamma\big) E^{(M)} \Big)$$

is strictly canonical at ν. The components of $Z^{(M)}$ which may pass through B are $Z_h^{(M)}$, $C_v^{(M)}$, and $C_h^{(M)}$. By Corti's inequality there is a number $0 < t \leqslant 1$ such that

$$\operatorname{mult}_B Z_h^{(M)} + t \operatorname{mult}_B\big(C_v^{(M)} + C_h^{(M)}\big) > 4n^2\Big(1 + t\frac{Mn - \sum \lambda_i}{n} + Mtn^2\gamma\Big) = \\ = 4n^2 + 4tMn^2 + 4tMn^2\gamma - 4tn\sum \lambda_i.$$

On the other hand $\operatorname{mult}_B Z_h^{(M)} \leqslant Z_h \cdot F = 2n^2$ and we have a bound on $\operatorname{mult}_B(C_h^{(M)} + C_v^{(M)})$ by Corollary 3.24. Combining the bounds we get

$$2n^2 + 2tMn^2 + tMn^2\gamma - 2t\sum_{i=1}^{M} \lambda_i^2 > 4n^2 + 4tMn^2 + 4tMn^2\gamma - 4tn\sum_{i=1}^{M} \lambda_i.$$

Rearranging the terms we find an equivalent inequality:

$$0 > 2n^2 + 3tMn^2\gamma + 2t\sum_{i=1}^{M}(n - \lambda_i)^2,$$

contradiction.

3.6.3 Case C

Suppose the center B of ν on $X^{(M)}$ is a point on the intersection $E^{(M)} \cap E^{(M-1)}$. Clearly these are the only divisors of the boundary containing B. Denote $M^- = M - 1$ for the compactness of formulas. Then the pair

$$\Big(X^{(M)}, \frac{1}{n}\mathcal{M}^{(M)} - \big(M - \sum \frac{\lambda_i}{n} + M\gamma\big)E^{(M)} - \big(M^- - \sum \frac{\lambda_i}{n} + M^-\gamma\big)E^{(M^-)}\Big)$$

is strictly canonical at ν. Hence, compared to the last case there are 2 more cycles which may contain B: $C_h^{(M^-,M)}$ and $C_v^{(M^-,M)}$. By Corti's inequality there are numbers $0 < t, t^- \leqslant 1$ such that

$$\operatorname{mult}_B Z_h^{(M)} + t \operatorname{mult}_B\big(C_v^{(M)} + C_h^{(M)}\big) + t^- \operatorname{mult}_B\big(C_v^{(M^-,M)} + C_h^{(M^-,M)}\big) \geqslant \\ \geqslant 4n^2 + 4tMn^2 + 4tMn^2\gamma - 4tn\sum_{i=1}^{M}\lambda_i + 4t^-M^-n^2 + 4t^-M^-n^2\gamma - 4t^-n\sum_{i=1}^{M^-}\lambda_i.$$

On the other hand we have the bounds on multiplicities from Corollary 3.24. After combining the inequalities and rearranging the terms we get

$$0 > 2n^2 + 3tMn^2\gamma + 3t^-M^-n^2\gamma + 2t\sum_{i=1}^{M}(n - \lambda_i)^2 + 2t^-\sum_{i=1}^{M^-}(n - \lambda_i)^2,$$

contradiction.

3.6.4 Proof of Theorem 1.5

Suppose X is not birationally rigid, that is there is a birational map g to a variety admitting a Mori fiber space which does not fit into the commutative diagram of Definition 1.3. Then by Proposition 3.4 there is a system $\mathcal{M} \subset \big|-nK_X + \gamma nF\big|$ such that one of the conditions (i) and (ii) holds.

Suppose the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical at a curve C. By Proposition 3.5 the curve C is a section. Then by Proposition 3.6 there is a Geiser involution $\chi_C \in \operatorname{Bir} X$ such that $g \circ \chi_C$ does not fit into the commutative diagram of Definition 1.3. Now we consider the map $g \circ \chi_C$

instead of g and the linear system

$$\chi_C(\mathcal{M}) \subset \big| - n'K_X + n'\gamma' \big|$$

instead of $\mathcal{M}$. We repeat this process until either $n' = 0$, that is the new map $g' = g \circ \chi_{C_1} \circ \cdots \circ \chi_{C_k}$ is fiberpreserving (contradiction), or until the pair, corresponding to g' is canonical at all curves.

Now we are in the situation (ii) of Proposition 3.4. Hence by Corollary 3.10 there is a supermaximal singularity. By Proposition 3.11 and Proposition 3.5 its center is a nonsingular point in the fiber containing half-point. There is a unique half-line passing through the center of ν. Let us consider the associated ladder $\sigma_i : X^{(i)} \to X^{(i-1)}$. Let $D_1^{(i)}$ and $D_2^{(i)}$ be the proper transforms on $X^{(i-1)}$ of generic divisors $D_1, D_2 \in \mathcal{M}$ and let $Z_i = D^{(i)} \cdot D^{(i)}$. Then as we have shown in Section 3.4, there are bounds on multiplicities on components of the cycle Z_i. In Section 3.5 we have proven that these bounds contradict Corti's inequality. Thus there are no supermaximal singularities and we have arrived to a contradiction, thus X is birationally rigid.

Chapter 4

Rationally connected non Fano type varieties

In this chapter we construct examples of rationally connected which are not birational to varieties of Fano type. These are Fano fibrations over $\mathbb{P}^1$ with a unique Mori fiber space in their birational class. The obstruction for a variety to be birational to a variety of Fano type is provided by the following proposition.

Proposition 4.1. *Let X be a variety of Fano type. Then there is a variety V admitting a Mori fiber space such that V is birational to X and $-K_V$ is big.*

Proof. There is an effective divisor D on X such that $-(K_X + D)$ is ample and (X, D) is klt. Since $-K_X$ is a sum of an effective and an ample divisor it is big. If X is not $\mathbb{Q}$-factorial we may replace it by its $\mathbb{Q}$-factorization Y ([8, Corollary 1.4.3]). Note that there is a morphism $g : Y \to X$ which is an isomorphism in codimension 1, therefore $-K_Y$ is big and (Y, D_Y) is klt, where D_Y is the proper transform of D on Y. If X is $\mathbb{Q}$-factorial, set $Y = X$. Since $-K_Y$ is $\mathbb{Q}$-Cartier and (Y, D_Y) is klt, the variety Y has log terminal singularities. There is terminalization morphism $f : Z \to Y$, that is a birational morphism such that Z has terminal singularities and all exceptional divisors E_i of f satisfy $a(E_i, Y) \leqslant 0$ ([8, Corollary 1.4.3]). The anticanonical class of Z is big since

$$-K_Z = -f^*K_Y - \sum a(E_i, Y)E_i.$$

The variety X is rationally connected by [82, Theorem 1], therefore the output of the MMP is a Mori fiber space $V \to Z$. We claim that $-K_V$ is big. Indeed, for a divisorial contraction $h : W \to U$ with the exceptional divisor E we can write

$$-h^*K_U = -K_W + aE, \quad a > 0,$$

hence $-K_U$ is big if $-K_W$ is big. Isomorphisms in codimension 1 preserve the property of divisors being big, therefore the anticanonical class is big on every step of the MMP, in particular $-K_V$ is big. □

To prove that the varieties X and V from Theorem 1.28 are examples we use Noether-Fano inequality. We have already mentioned 2 versions (Proposition 3.2 and Proposition 3.3 of this inequality, this is another well known form of this idea.

Definition 4.2. We say that a Mori fiber space $\pi : X \to \mathbb{P}^1$ satisfies the K-condition if $-K_X$ is not in the interior of the closure of the cone of mobile divisors.

Note that if $-K_X$ is not big, that is $-K_X$ is not in the interior of the closure of the cone of effective divisors, then X satisfies the K-condition.

Proposition 4.3 (Noether-Fano inequality)**.** *Let $\pi : X \to \mathbb{P}^1$ be a Fano fibration such that X is terminal and* $\operatorname{Pic} X = \mathbb{Z} \oplus \mathbb{Z}$. *Suppose X satisfies the K-condition. Suppose that there is a* *birational map $\chi : X \dashrightarrow Y$ which is not an isomorphism and suppose Y admits a Mori fiber space. Then there is a linear system $\mathcal{M}$ on X and numbers $s, \lambda \in \mathbb{Q}_{>0}$ such that $\lambda\mathcal{M}+K_X \sim sF$, where F is a fiber of π, and the pair $(X, \lambda\mathcal{M})$ is not canonical.*

Proof. Let $\mathcal{M}_Y$ be a base point free linear system on Y and let $\mathcal{M}$ be its proper transform on X. Let

$$\begin{array}{ccc} & W & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ X & \overset{\chi}{\dashrightarrow} & Y \end{array}$$

be a resolution of the birational map χ. Let $\mathcal{M}_W = \varphi^{-1}(\mathcal{M})$ then for any $\lambda \in \mathbb{Q}$

$$\varphi^*(K_X + \lambda\mathcal{M}) + \sum a_i E_i \sim K_W + \lambda\mathcal{M}_W \sim \psi^*(K_Y + \lambda\mathcal{M}_Y) + \sum b_j E_j, \qquad (4.1)$$

where $b_j > 0$ since Y is terminal and $\mathcal{M}_Y$ is base point free.

Suppose Y is a Fano variety with $\mathrm{Pic}(Y) \cong \mathbb{Z}$. Let $\mathcal{M}_Y$ be a very ample linear system on Y. Let λ and s be rational numbers such that $K_X + \lambda\mathcal{M} \sim sF$, where F is a fiber of π. Such $\lambda > 0$ exists since $\mathcal{M}_Y$ is ample and the K-condition implies $s > 0$.

Suppose the pair $(X, \lambda\mathcal{M})$ is canonical, then $a_i \geqslant 0$. Suppose $K_Y + \lambda\mathcal{M}_Y$ is ample, then χ is an isomorphism by the uniqueness of canonical model [13, Theorem 1.3.20]. Thus $K_Y + \lambda\mathcal{M}_Y$ is not ample. Let H be a very ample divisor on Y and let us intersect both sides of equivalence (1) by $\psi^*(H)$, then

$$\psi^*(H) \cdot \varphi^*(sF) + \sum a_i \psi^*(H) \cdot E_i \equiv \psi^*\big(H \cdot (K_Y + \lambda\mathcal{M}_Y)\big).$$

The righthand side is not effective, on the other hand $\psi^*(H) \cdot \varphi^*(sF)$ is effective and not zero while $\psi^*(H) \cdot E_i$ is effective. Thus there must be some $a_i < 0$.

Now suppose there is a Mori fiber space $\pi_Y : Y \to Z$ with $\dim Z > 0$. Let H be a very ample divisor on Z and let $\mathcal{M}_Y = \big|\pi_Y^*(H)\big|$. Then there are rational numbers $n, l \geqslant 0$ such that $\mathcal{M} \subset \big| -nK_X + lF\big|$. If $n = 0$, then $Z = \mathbb{P}^1$ and χ is fiberpreserving, hence by Theorem 2.11 there exists a linear system we are looking for. Thus we may assume $n > 0$. Set $\lambda = \frac{1}{n}$ and $s = \frac{l}{n}$ then $K_X + \lambda\mathcal{M} \sim sF$, and the K-condition implies $s > 0$. Let f be the class of a fiber of π_Y and note that a generic fiber of π_Y does not pass through the centers of the exceptional divisors of ψ. Let us intersect (1) with $\psi^*(f)$, then

$$s\psi^*(f) \cdot \varphi^*(F) + \sum a_i \psi^*(f) \cdot E_i \equiv \psi^*(f \cdot K_Y).$$

Clearly the righthand side is not effective hence there must be $a_i < 0$. $\square$

Example 4.4. Consider a hypersurface $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ given by an equation of bi-degree $(3, l)$. Let $\pi : X \to \mathbb{P}^1$ be the restriction to X of the projection onto $\mathbb{P}^1$, then it is a cubic fibration. Clearly, nK_X, $n \geqslant 1$, is effective if an only if $l \leqslant 1$. Thus the K-condition holds if and only if $l \geqslant 2$.

4.1 Example in dimension $\geqslant$4

Let W be a generic smooth divisor of bi-degree $(2M, 2l)$ on $\mathbb{P}^M \times \mathbb{P}^1$, $M \geqslant 3$, $l \geqslant 3$. Let $\sigma : V \to \mathbb{P}^M \times \mathbb{P}^1$ be the double cover branched over W. The variety V is smooth because W is smooth and $\mathrm{Pic}(V) = \mathbb{Z} \oplus \mathbb{Z}$ by Lefschetz hyperplane section theorem. Let $\pi : V \to \mathbb{P}^1$ be the composition of σ and the projection onto $\mathbb{P}^1$. Clearly, the restriction of σ to any fiber of π is the double cover of $\mathbb{P}^M$ branched over a hypersurface of degree $2M$. Since W is generic we may also assume that a generic fiber of π is smooth and every special fiber has a unique singularity, which is an ordinary double point. For a smooth fiber F by Hurwitz formula and Lefschetz hyperplane section theorem $\mathrm{Pic}(F) = -K_F\mathbb{Z}$. For a special fiber F, if $M \geqslant 4$ then by [14, Lemma 28] $\mathrm{Cl}(F) = \mathrm{Pic}(F) = -K_F\mathbb{Z}$ and if $M = 3$, then the same is due to [24] (or we could use [78, Theorem 4.1]).

The superrigidity of varieties like V has been considered in [72, Theorem 1]. That is, it was shown that if V satisfies the generality conditions from [72], then for every variety Y admitting a Mori fiber space $\bar{\pi} : Y \to Z$ and birational to V and every birational map $\chi : V \dashrightarrow Y$, we must have $Z = \mathbb{P}^1$ and χ is fiberpreserving with respect to π and $\bar{\pi}$. But we need more than this as we do not know how do fiberpreserving maps affect the canonical class.

We now describe some sufficient conditions on the branching divisor (modification of those in [73, p. 22-23]) which let us control the singularities of linear systems on double covers of $\mathbb{P}^M$. Then we show that the locus of hypersurfaces of degree $2M$ not satisfying these conditions is of codimension at least 2. This implies that for a general W, the restriction $W\big|_{\mathbb{P}^M}$ on every fiber of a projection onto $\mathbb{P}^1$ satisfies these conditions.

Definition 4.5. Let $W \subset \mathbb{P}^M$ be a hypersurface of degree $2M$, $M \geqslant 3$. For a nonsingular point $x \in W$ fix a system of affine coordinates $z_1, \dots, z_M$ on $\mathbb{P}^M$ with the origin at x and set

$$q_1 + q_2 + \dots + q_{2M} = 0$$

to be the equation of the hypersurface W, where $q_i = q_i(z)$ are homogeneous polynomials of degree $\deg q_i = i$. We can assume that $q_1 = z_1$ since W is smooth at x. Denote

$$\bar{q}_i = \bar{q}_i(z_2, \dots, z_M) = q_i\big|_{\{z_1=0\}} = q_i(0, z_2, \dots, z_M).$$

Suppose $M \geqslant 5$. We say that W satisfies the regularity conditions at a smooth point x if the rank of the quadratic form $\bar{q}_2$ is at least 2.

For $M = 4$ we need either

(i) The rank of the quadratic form form $\bar{q}_2$ is at least 2

(ii) or the rank of $\bar{q}_2$ is 1 and the following additional condition is satisfied. Without loss of generality we assume that $\bar{q}_2 = z_2^2$. We require that one of the polynomials $\bar{q}_3(0, z_3, 0) = q_3(0, 0, z_3, 0)$ or $\bar{q}_4(0, z_3, 0)$ is not zero.

Suppose $M = 3$. Then we require either

(i) The rank of the quadratic form form $\bar{q}_2$ is at least 2,

(ii) the rank of $\bar{q}_2$ is 1 and the following additional condition is satisfied. Without loss of generality we assume that $\bar{q}_2 = z_2^2$. We require that at least one of the polynomials $\bar{q}_3(0, z_3)$, $\bar{q}_4(0, z_3)$, or $\bar{q}_5(0, z_3)$ is not zero,

(iii) or the rank of $\bar{q}_2$ is 0 and the polynomial $\bar{q}_3(1, t)$ has 3 distinct roots.

Definition 4.6. Let $W \subset \mathbb{P}^M$ be a hypersurface of degree $2M$ satisfying the regularity conditions at every nonsingular point. Suppose $\mathrm{Sing}(W)$ is empty or is a unique ordinary double point, then we call W a regular hypersurface. Let $\sigma : F \to \mathbb{P}^M$, be the double cover branched over a regular hypersurface $W \subset \mathbb{P}^M$ then we say X is a regular double space. Note that X is smooth or has a unique ordinary double point, which is a preimage of the ordinary double point on the branching divisor.

The following theorem is a stronger version of [74, Theorem 4].

Theorem 4.7. *Let X be a regular double space. Then for every effective divisor $D \in \big|-nK_X\big|$ the pair $\big(X, \frac{1}{n}D\big)$ is canonical.*

Proof. Note that if the pairs (X, F) and (X, F') are canonical than so is the pair

$$\big(X, \alpha F + (1-\alpha)F'\big)$$

for $0 \leqslant \alpha \leqslant 1$. Hence we may assume that D is irreducible since $\mathrm{Cl}(X) = \mathrm{Pic}(X) \cong \mathbb{Z}$.

It follows from [74, Proof of Theorem 4, page 11] that $\big(X, \frac{1}{n}D\big)$ is canonical at the ordinary double point. By [73, p. 23-24] the pair $\big(X, \frac{1}{n}D\big)$ may only be singular at the nonsingular points on ramification divisor. Let $x \in X$ and let A be the hyperplane in $\mathbb{P}^M$ tangent to W at the point $\sigma(x) \in W$. Denote $\sigma^{-1}(A)$ as H. It was shown in [73, Proof of Lemma 5, p. 29-30] that the pair $\left(X, \frac{1}{n}D\right)$ is canonical at x, unless $D = H$. In this case $n = 1$, since by Hurwitz formula $H \in \left| -K_X \right|$.

We have not used the regularity conditions yet. If $\operatorname{rank}(\bar{q}_2) \geqslant 2$, then by [73, p. 29-30] the pair (X, H) is canonical at x. This proves the theorem for $M \geqslant 5$.

Suppose $M = 4$ and $\operatorname{rank} \bar{q}_2 = 1$, then the local equation of H at x is

$$y^2 = z_2^2 + c_1 z_3^3 + c_2 z_3^4 + z_4(\dots),$$

As either c_1 or c_2 is not zero, $x \in H$ is a singularity of the type cA_3 at worst. Thus H has terminal singularities and (X, H) is canonical.

Now suppose $M = 3$ and $\bar{q}_2 = z_2^2$. It follows from the regularity conditions that the local equation of H is $y^2 = z_2^2 + z_3^3$, $y^2 = z_2^2 + z_3^4$, $y^2 = z_2^2 + z_3^5$, or $y^2 = z_2(z_2^2 + z_3^2)$ thus x is a Du Val singularity, H has canonical singularities, and therefore the pair (X, H) is canonical. □

It was proven in [73, Proposition 5], that a generic hypersurface of degree $2M$ is regular. Thus a generic fiber of a generic fibration is a regular double space. We want every fiber to be regular.

Proposition 4.8. *Let $\mathcal{W}$ be the space of all hypersurfaces of degree $2M$ in $\mathbb{P}^M$. Denote the space of all regular hypersurfaces by $\mathcal{W}_{reg} \subset \mathcal{W}$. If $M \geqslant 3$ then the codimension of $\mathcal{W} \backslash \mathcal{W}_{reg}$ is 2 or more.*

Proof. It is classically known that codimension of the locus of hypersurfaces with 2 or more ordinary double points or with worse singularities is 2. Thus it is enough to prove that the space of hypersurfaces which do not satisfy the regularity conditions at nonsingular points has codimension $\geqslant 2$.

Clearly, $\mathcal{W} = H^0\big(\mathbb{P}^M, \mathcal{O}(2M)\big)$. Let $V = \mathbb{P}^M \times \mathcal{W}$ and let I be the incidence hypersurface in it

$$I = \{(x, F) \in V \mid F(x) = 0\}.$$

Let p and q be the natural projections $p : I \to \mathbb{P}^M$ and $q : I \to \mathcal{W}$. Let Y be the subset of "bad" pairs, that is

$$Y = \{(x, F) \in I \mid \text{F is not regular at } x \text{ if } x \text{ is smooth}\}.$$

To prove the proposition it is enough to show that $\operatorname{codim} q(Y) \geqslant 2$. To show this it is sufficient to prove that

$$\operatorname{codim}_{p^{-1}(x)} p^{-1}(x) \cap Y = \operatorname{codim}_I Y \geqslant M + 1$$

since the dimension of a fiber of q is $M - 1$.

Consider the equation of W in affine coordinates in the neighborhood of a point x

$$q_1 + q_2 + \dots + q_{2M} = 0,$$

where q_i is a homogeneous polynomial of degree i. The hypersurface W is smooth at x if and only if $q_1 \neq 0$. Thus we may assume that $q_1 = z_1$ and take $\bar{q}_2 = q_2(0, z_2, \dots, z_M)$. The set of quadratic forms of rank $\leqslant 1$ in the variables $z_2, \dots, z_M$ is of codimension

$$c(M) = \frac{(M-1)(M-2)}{2}.$$

When $M \geqslant 5$ we have $c(M) \geqslant M + 1$. Suppose $M = 4$. As the conditions $\bar{q}_3(0, z_3, 0) = 0$ and $\bar{q}_4(0, z_3, 0) = 0$ add 2 to the codimension, the codimension is $c(4)+2 = 5$. Similarly if $M = 3$ the variety of hypersurfaces with $\operatorname{rank}(\bar{q}_2) = 1$ satisfying the conditions: $\bar{q}_3(0, z_3) = 0$, $\bar{q}_4(0, z_3) = 0$ and $\bar{q}_5(0, z_3) = 0$, is of codimension 4. On the other hand, the variety of hypersurfaces with $\bar{q}_2 = 0$ is of codimension 3 and the condition on $\bar{q}_3(1, t) = 0$ having a multiple root, adds 1 to the codimension. □

Lemma 4.9. *Suppose V is the variety from Theorem 1.28 (i). Then the anticanonical class of the variety V is not big and V satisfies the K-condition.*

Proof. We compute

$$-K_V = g^*(L),$$

where L is a divisor of bidegree $(1, 2-l)$, $l \geqslant 3$. Thus the K-condition holds and $-K_V$ is not big. □

Proof of Theorem 1.28 (i). Let $\pi : V \to \mathbb{P}^1$ be the composition of the double cover and the projection onto $\mathbb{P}^1$. There is a map $\bar{\pi} : \mathbb{P}^1 \to \mathcal{W}$ corresponding to the fibration π: $\bar{\pi}$ maps $t \in \mathbb{P}$ to the branching divisor of the double cover $F_t \to \mathbb{P}^M$, where F_t is the fiber over t. The image $\bar{\pi}(\mathbb{P}^1)$ is a curve of degree $2l$ since W is a divisor of bi-degree $(2M, 2l)$. By Proposition 4.8 the codimension of the set of non regular hypersurfaces is 2, therefore a generic rational curve of degree $2l$ does not intersect this set. Thus for a generic divisor W every fiber of π is a regular double space.

Let χ be a birational map to a variety admitting a Mori fiber space. Suppose χ is not an isomorphism. By Lemma 4.9 the variety V satisfies the K-condition, hence Proposition 4.3 is applicable. Thus there is a linear system $\mathcal{M}$ and rational numbers λ, s such that

$$K_V + \lambda\mathcal{M} \sim sF$$

and the pair $(V, \lambda\mathcal{M})$ is not canonical. Suppose the pair is not canonical at a subvariety Z. Suppose $Z \subset F$ for some fiber F of π. Then the pair

$$(F, \lambda\mathcal{M}\big|_F)$$

is not log canonical at Z by Inversion of adjunction (Remark 2.10). Since $\lambda\mathcal{M}\big|_F \sim -K_F$ it contradicts Theorem 4.7. Thus Z is not in any fiber of π. Consider a generic fiber F of π, then $(F, \lambda\mathcal{M}\big|_F)$ is not canonical at $Z \cap F$. This also contradicts Theorem 4.7, thus χ is an isomorphism. □

4.2 Example in dimension 3

In this section we construct an example in dimension 3 and prove the part (ii) of Theorem 1.28.

Let $Y = (\mathbb{C}^6 \setminus Z(\langle u, v\rangle \cap \langle x, y, z, w\rangle)/(\mathbb{C}^*)^2$, where the $(\mathbb{C}^*)^2$-action is given by the matrix

$$\begin{pmatrix} u & v & x & y & z & w \\ 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & -3 & -3 & 0 & 0 \end{pmatrix}$$

and let

$$Q = w^2 + z^3 + (u^{12} + v^{12})M_4(x, y)z + R_{18}(u, v)x^2y^2(x-y)^2,$$

where M_4 and R_{18} are homogeneous polynomials of degrees 4 and 18 respectively. Let $\mathcal{L}$ be the set of hypersurfaces given by $Q = 0$ for all the different M_4 and R_{18}. Clearly $\mathcal{L}$ is a linear system of divisors. Let X be a generic variety in $\mathcal{L}$. Let $\pi_Y : Y \to \mathbb{P}^1$ be defined by

$$(u : v : x : y : z : w) \mapsto (u : v),$$

clearly, this map is a $\mathbb{P}(1, 1, 2, 3)$-fibration. Let $\pi = \pi_Y|_X$, as we discussed in Example 2.32 a fiber of π is del Pezzo surface of degree 1.

Denote the torus-invariant divisor given by $s = 0$, $s \in \{u, v, x, y, z, w\}$, as D_s, and note that $D_u \sim D_v \sim F_Y$, where F_Y is a fiber of π_Y. It is easy to see that $\mathrm{Cl}(Y) = D_y\mathbb{Z} \oplus F\mathbb{Z}$.

Lemma 4.10. *Let F be a fiber of $\pi_Y : Y \to \mathbb{P}^1$ and suppose $X \in \mathcal{L}$. Then* *(i)* $|X| = |6D_y + 18F_Y|$ *is base point free,*

(ii) $X + F_Y$ *is ample,*

(iii) X *is big.*

Proof. The equation of $D \in |X|$ may contain monomials: w^2, z^3, x^6u^{18}, x^6v^{18}, y^6u^{18} and y^6v^{18} which are not all equal to zero at the same time at any point on Y, thus (i) holds.

Suppose C is a curve in a fiber then

$$C \cdot (X + F_Y) = C \cdot X = C \cdot X\big|_{F_Y} = \deg\left(\mathcal{O}_{\mathbb{P}(1,1,2,3)}(6)\right)\big|_C > 0.$$

Suppose a curve C is not in a fiber. Then since $|X|$ is base point free

$$C \cdot (X + F_Y) \geqslant C \cdot F_Y > 0.$$

Thus $X + F$ is ample by Nakai-Moishezon criterion.

Clearly $2X \sim (X + F_Y) + (6D_y + 17F_Y)$, hence (iii) follows from (ii). □

Lemma 4.11. *Let X be a generic divisor in the linear system $\mathcal{L}$. Then the following assertions hold:*

(i) There are 108 *cuspidal curves of anticanonical degree* 1 *in fibers of π:* 72 *of them are given by the equations $R_{18} = M_4 = 0$, the other* 36 *curves are given by $u^{12} + v^{12} = 0$ and $x = 0$, $y = 0$, or $x = y$.*

(ii) Let F be a fiber of π and let $C \in F$ be one of the 72 *curves. Let P be the cusp of C, then P is an ordinary double point of F.*

(iii) Let C be one of the 36 *curves and let P be the cusp of C. Then P is an ordinary double point of X.*

(iv) The variety X is smooth outside of the 36 *ordinary double points described in* (iii).

Proof. A fiber of π is defined by the ratio $(u : v)$ and curves of degree 1 in it by $(x : y)$. The coefficient at z^2 is zero for every fiber, therefore the curve is cuspidal if and only if the coefficients at z and at the free term are 0, that is

$$R_{18}(u, v)x^2y^2(x - y)^2 = 0,$$
$$(u^{12} + v^{12})M_4(x, y) = 0.$$

As R_{18} and M_4 are generic we must have $R_{18} = 0$ and $M_4 = 0$ (72 curves) or $u^{12} + v^{12} = 0$ and $x^2y^2(x - y)^2 = 0$ (36 curves), thus (i) holds. Note that the cusps of these curves are at $w = z = 0$.

The local equation of X at the cusp of one of the 72 curves is

$$w^2 + z^3 + zs + t = 0,$$

where t is a prime factor of R_{18} and s is a prime factor of M_4. Clearly the fiber $t = 0$ has an ordinary double point at $w = z = s = 0$.

If C is one of the 36 curves, then the local equation of X at the cursp of C is

$$w^2 + z^3 + zt + s^2 = 0,$$

where $s = x$, $s = y$, or $s = x - y$ and t is a prime factor of $u^{12} + v^{12}$. Clearly X has an ordinary double point at $w = z = t = s = 0$.

Note that X does not pass through the singular locus of Y. Suppose X is singular at the point P with coordinates (x, y, z, w, u, v). Then, clearly $w = 0$. By Bertini's theorem [48, Theorem 4.1] P is a base point of $\mathcal{L}$.

Clearly, there is a linear function $t(u, v)$ such that $t \neq 0$ at P. Let $X_t \in \mathcal{L}$ be a variety with $R_{18} = t^{18}$. Suppose $xy(x - y) \neq 0$ then for some $c \in \mathbb{C}$ the point P does not lie on X_{ct}. Thus $x = 0$, $y = 0$, or $x = y$. Note that it means that generic $M_4 \neq 0$ at P.

Suppose $z(u^{12}+v^{12}) \neq 0$ at P. Then repeating the argument for some linear function $s(x,y)$ and $M_4 = (cs)^4$ we conclude that P is not a base point of $\mathcal{L}$. Thus $z = 0$ or $(u^{12}+v^{12}) = 0$. We have $(u^{12}+v^{12}) = 0$ if and only if $z = 0$, because

$$\frac{\partial Q_P}{\partial z} = 3z^2 + (u^{12}+v^{12})M_4(x,y) = 0.$$

Thus every singular point must satisfy $w = z = u^{12}+v^{12} = xy(x-y) = 0$. There are 36 points satisfying these equations and they are described in (*iii*). Note that there are 3 ordinary double points in the fiber $t = 0$. □

We now prove that the del Pezzo fibration $\pi : X \to \mathbb{P}^1$ is a Mori fiber space. We already know that X is terminal, thus we only need to show that it is $\mathbb{Q}$-factorial and that the relative Picard rank $\rho(X/\mathbb{P}^1) = 1$. We intend to use the following theorem.

Theorem 4.12 ([78, Theorem 4.1])**.** *Let Y be a complete simplicial toric fourfold and let $X \subset Y$ be a hypersurface such that $\mathrm{Sing}(Y) \cap X = \varnothing$ and $\mathrm{Sing}(X)$ consists only of ordinary double points. Let $\widetilde{X}$ be a resolution of X obtained by blowing up the ordinary double points. Let μ_X be the number of singular points on X and let $\mathcal{I}_{\mathrm{Sing}(X)}$ be the sheaf of ideals on Y of singular points of X. Assume that the following conditions are satisfied:*

(i) $H^i\big(Y, \mathcal{O}_Y(-2X)\big) = 0$ *for* $i = 1,2,3$;

(ii) $H^i\big(Y, \mathcal{O}_Y(-X)\big) = 0$ *for* $i = 1,2,3$;

(iii) $H^i\big(Y, \bar{\Omega}^1_Y \otimes \mathcal{O}_Y(-X)\big) = 0$ *for* $i = 1,2,3$;

Denote $\delta_X = H^0\big(Y, \mathcal{O}(K_Y + 2X) \otimes \mathcal{I}_{\mathrm{Sing}(X)}\big) - \Big(H^0\big(Y, \mathcal{O}(K_Y+2X)\big) - \mu_X\Big)$. *Then*

$$h^{1,1}(\widetilde{X}) = h^1(\bar{\Omega}^1_Y) + \mu_X + \delta_X.$$

Let $U = Y \setminus \mathrm{Sing}(Y)$ and let $i : U \to Y$ be the inclusion map, then denote $\bar{\Omega}^1_Y = i_*(\Omega^1_U)$. The number δ_X is the difference between the expected and the actual dimension of the space and is always non-negative, it is positive when the singular points are not in a general position. It is known as the defect of a hypersurface [78].

Lemma 4.13. *For a generic $X \in \mathcal{L}$ the defect $\delta_X = 0$.*

Proof. It is easy to see that

$$\deg K_Y = (-1-1-2-3, -1-1+3+3) = (-7,4),$$

hence $K_Y \sim -7D_y - 17F$ as $\deg D_y = (1,-3)$. Let P_i, $i = 1,\ldots,36$, be the ordinary double points of X. It is enough show that there are effective divisors

$$H_i \in \big|2X + K_Y\big| = \big|5D_x + 19F\big|, \quad i = 1\ldots 36,$$

such that $P_j \in H_i$ for any $j \neq i$ and $P_i \notin H_i$. Indeed, suppose $\mathcal{W}$ is the space of polynomials of bidegree $(5,-1) = \deg 2X + K_Y$. Let $Q_i \in \mathcal{W}$ be the polynomials defining divisors H_i. Suppose $Q \in \mathcal{W}$ is a polynomial which vanishes at all P_j. Then the polynomial $Q + \sum \alpha_i Q_i$ vanishes at P_i if an only if $\alpha_i = 0$. Thus the codimension of the subspace of polynomials in $\mathcal{W}$ which vanish at P_i is 36.

Let F_k be the fibers containing the singular points of X, there are 12 of them, since there are 3 ordinary double points in each F_k. Let D_{x-y} be the divisor defined by $y = x$. By choosing 11 fibers F_k and 2 divisors out of D_x, D_y, D_{x-y} we get a divisor H'_i linearly equivalent to $2D_x + 11F$ passing through any 35 singular points out of 36. The divisor A given by the equation $(x-2y)^3u^8$ does not pass through the singular points of X, therefore we may set $H_i = H'_i + A$. □

Proposition 4.14. *Suppose $X \in \mathcal{L}$ is generic. Then X is $\mathbb{Q}$-factorial and*

$$\operatorname{Pic} X \otimes \mathbb{Q} = \big(D_y\big|_X\big)\mathbb{Q} \oplus F\mathbb{Q},$$

where F is a fiber of π.

Proof. Let us show that X satisfies the conditions of Theorem 4.12. By Lemma 4.10 the divisor X is big and nef, therefore the toric Kawamata-Viehweg vanishing theorem ([31, Theorem 9.3.10]) implies $H^i\big(Y, \mathcal{O}_Y(K_Y + 2X)\big) = 0$, $i > 0$. Thus by the toric Serre Duality theorem ([31, Theorem 9.2.10]) we have (i). Using the same argument we get (ii).

As $\operatorname{Cl}(Y) \cong \mathbb{Z} \oplus \mathbb{Z}$ the following sequence is exact by [31, Theorem 8.1.6].

$$0 \to \bar{\Omega}^1_Y(-X) \to \bigoplus_{s \in Z} \mathcal{O}_Y(-D_s - X) \to \mathcal{O}_Y(-X) \oplus \mathcal{O}_Y(-X) \to 0.$$

By taking associated long exact sequence of cohomologies and applying (ii) we get

$$0 \to H^i\big(Y, \bar{\Omega}^1_Y(-X)\big) \to \bigoplus_{s \in Z} H^i\big(Y, \mathcal{O}_Y(-D_s - X)\big), \quad i = 1, 2, 3.$$

It is enough to prove that $H^i\big(Y, \mathcal{O}_Y(-D_s - X)\big) = 0$ for all $s \in Z$, $i = 1, 2, 3$. If $s \in u, v, z, w$ then $D_s + X$ is big and nef, therefore by Serre duality and Kawamata-Viehweg vanishing $H^i\big(Y, \mathcal{O}_Y(-D_s - X)\big) = 0$. Let F be a fiber of π. Then $\mathcal{O}_F(F) \cong \mathcal{O}_F$ and the sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(F) \to \mathcal{O}_F \to 0$$

is exact. Hence the sequences

$$H^i(Y, \mathcal{O}_Y) \to H^i\big(Y, \mathcal{O}_Y(F)\big) \to H^i(F, \mathcal{O}_F), \quad i = 1, 2, 3$$

are exact. Applying [31, Theorem 9.3.2] we see that

$$H^i(Y, \mathcal{O}_Y) = H^i(F, \mathcal{O}_F) = 0, \quad i = 1, 2, 3,$$

hence $H^i\big(Y, \mathcal{O}_Y(F)\big) = 0$. As $-X - D_y \sim -X - D_y \sim K_Y - F$ by toric Serre duality

$$H^i\big(Y, \mathcal{O}_Y(-D_x - X)\big) \cong H^{4-i}\big(Y, \mathcal{O}_Y(F)\big) = 0.$$

Let $\widetilde{X}$ be the resolution of X acquired by blowing up the 36 double points. We have shown X and Y satisfy the requirements of Theorem 4.12 therefore $\operatorname{rk}\operatorname{Pic}(\widetilde{X}) = 38$. On the other hand $\operatorname{rk}\operatorname{Pic}(\widetilde{X}) \geqslant \operatorname{rk}\operatorname{Cl}(X) + 36$, thus $\operatorname{rk}\operatorname{Pic}(X) = \operatorname{rk}\operatorname{Cl}(X) = 2$ and X is $\mathbb{Q}$-factorial. Clearly, divisors $D_y\big|_X$ and $F\big|_X$ are generators. ∎

Lemma 4.15. *The variety X satisfies the K-condition and $-K_X$ is not big.*

Proof. We have $K_Y = -7D_y - 17F$, therefore by adjunction

$$-K_X = D_y\big|_X - F\big|_X.$$

Thus $\big| - nK_X\big| = \varnothing$ for any $n > 0$. ∎

Remark 4.16 ([59, Proposition 3.2])**.** Let F be a del Pezzo surface of degree 1 and let $C \in \big| - K_F\big|$ be an irreducible curve. Suppose (F, C) is not log canonical at P and F is smooth at P. Then C is a cuspidal rational curve.

Lemma 4.17 ([29, proof of Theorem 3.10])**.** *Let X be a 3-dimensional variety and let D be an effective $\mathbb{Q}$-Cartier divisor on X. Suppose $P \in X$ is an ordinary double point, let $f : X^+ \to X$ be the blow up of P, and let E be the exceptional divisor. Then the pair (X, D) is not canonical at the point P if an only if $a(E, X, D) < 0$.*

Proof of Theorem 1.28 (ii). Let χ be a birational map to a variety admitting a Mori fiber space and suppose χ is not an isomorphism. Let F be a fiber of π. Then by Proposition 4.3 there is a linear system $\mathcal{M}$ and numbers $\lambda, s \in \mathbb{Q}_{>0}$ such that $K_X + \lambda\mathcal{M} \sim sF$ and $(X, \lambda\mathcal{M})$ is not canonical.

Suppose the pair is not canonical at a curve C, then $\operatorname{mult}_C \lambda\mathcal{M} > 1$. Suppose $C \in F$ for some fiber F, then for a generic $D \in \mathcal{M}$,

$$1 = \lambda D\big|_F \cdot (-K_F) > C \cdot (-K_F) \geqslant 1,$$

where the equality holds because F is a del Pezzo surface of degree 1 and the first inequality because $\big|-K_F\big|$ is mobile. Suppose C is not in any fiber, then for a generic fiber F let $P \in F \cap C$. Since F is generic and $\mathcal{M}$ is mobile, the system $\mathcal{M}\big|_F$ does not have fixed components. Clearly $\operatorname{mult}_P \lambda\mathcal{M}\big|_F > 1$, but for generic curves $Z_1, Z_2 \in \mathcal{M}\big|_F$

$$1 = \lambda^2 Z_1 \cdot Z_2 \geqslant \lambda^2 (Z_1 \cdot Z_2)_P \geqslant \left(\operatorname{mult}_P \lambda\mathcal{M}\big|_F\right)^2 > 1,$$

where the equality holds since F is a del Pezzo surface of degree 1.

Thus the pair is not canonical at some point P. Let F be the fiber containing P, let $f : X^+ \to X$ be the blow up of P, let E be the exceptional divisor of f, let D^+ be the proper transform of D, and let a be the number defined by the equality $f^*(\lambda D) = \lambda D^+ + aE$. Let C be a generic curve in $\big|-2K_F\big|$ passing through P and let C^+ be its proper transform on X^+. Note that $\lambda D \cdot C = -2K_F^2 = 2$.

Suppose P is an ordinary double point of X. Then

$$K_{X^+} + \lambda D^+ (a-1)E \sim f^*(K_X + \lambda D),$$

and $a > 1$ by Lemma 4.17. The surface F is singular at P since F is a Cartier divisor on X. The equality $C^+ \cdot E \geqslant 2$ holds since F is singular at P and C is a Cartier divisor on F, therefore

$$0 \leqslant \lambda D^+ \cdot C^+ = f^*(\lambda D) \cdot C^+ - aC^+ \cdot E = \lambda D \cdot C - 2a = 2 - 2a < 0$$

contradiction.

Thus X is smooth at P. Then $a = \operatorname{mult}_P \lambda D > 1$ since the pair $(X, \lambda D)$ is not canonical at P. Suppose P is a singular point of F. The equality $C^+ \cdot E \geqslant 2$ holds again, therefore

$$0 \leqslant \lambda D^+ \cdot C^+ = f^*(\lambda D) \cdot C^+ - aC^+ \cdot E \leqslant \lambda D \cdot C - 2a = 2 - 2a < 0,$$

contradiction.

Thus we may assume that F is also smooth at P. Let $D \in \mathcal{M}$ be a generic divisor in a linear system. Then the pair $(F, \lambda D\big|_F)$ is not log canonical at P by Inversion of adjunction (Remark 2.10). Let $C \in \big|-K_F\big|$ be the curve passing through P. By construction C is smooth or nodal, therefore (F, C) is log canonical. Let $A \in \big|-nK_F\big|$ be a curve such that $\operatorname{Supp}(A)$ does not contain C. Then

$$\operatorname{mult}_P A \leqslant A \cdot C = n,$$

hence the pair $(F, \frac{1}{n}A)$ is log canonical at P. Therefore $\left(F, \alpha C + \frac{1-\alpha}{n}A\right)$ is log canonical at P for any $0 \leqslant \alpha \leqslant 1$. In particular $(F, \lambda D\big|_F)$ is log canonical, contradiction. □

Remark 4.18. It was shown in [23, Corollary 7.4] that a del Pezzo fibration $\pi : X \to \mathbb{P}^1$ of degree 1 is birationally rigid if for every del Pezzo fibration $\pi_Y : Y \to \mathbb{P}^1$, such that there is a fiberpreserving birational map $\chi : X \to Y$, the variety Y satisfies the K-condition. It is unclear how to check the requirements of the corollary and there are no known examples of varieties satisfying this property. The variety X from Theorem 1.28 satisfies this property trivially, however: $\pi : X \to \mathbb{P}^1$ is a unique del Pezzo fibration in the birational class and X satisfies the K-condition.

4.3 Non Fano type varieties

Lemma 4.19. *The varieties V and X described in Theorem 1.28 are rationally connected.*

Proof. Since V and X are Fano fibrations over $\mathbb{P}^1$ they are rationally connected by [52, Theorem 0.1] and [34, Corollary 1.3]. □

Thus the varieties V and X are examples of rationally connected varieties which are not birational to varieties of Fano type.

Remark 4.20. We could also construct an example by using fibrations onto Fano hypersurfaces of index one. But it is more tiresome and provides examples only for dimension $\geqslant 9$. The approach is the same: we use Pukhlikov's generality conditions and make them a little bit weaker. Then we could prove that there is a fibration over $\mathbb{P}^1$ such that every fiber is a hypersurface satisfying these generality conditions.

Remark 4.21. Other examples in dimension three have been constructed recently in [51]. The examples are the conic bundles with sufficiently big discriminant curve. More precisely, Kollár gives examples of rationally connected threefolds which are not birational to Calabi-Yau pairs.

Remark 4.22. We can run D-MMP on a Mori dream space for any divisor D. Thus, we have another class of varieties which behave very well under the D-MMP. It has been proven in [8, Corollary 1.3.1], that every $\mathbb{Q}$-factorial variety of Fano type is a Mori dream space. The converse is not true even for smooth Mori dream spaces: there exists a smooth rational Mori dream space of dimension 2 which is not of Fano type [79, Section 3]. In fact, a $\mathbb{Q}$-factorial normal projective variety is of Fano type if and only if it is a Mori dream space and spectrum of its Cox ring has only log terminal singularities [33, Theorem 1.1]. One could ask Question 1 for Mori dream spaces instead of varieties of Fano type.

Chapter 5

The Klein simple group and the Cremona group

In this chapter we reprove Theorem 1.18 and prove Theorem 1.14.

By Lemma 1.16 there is a rational $G\mathbb{Q}$-Mori fiber space corresponding to every embedding of G into Cr_n. By a $G\mathbb{Q}$-del Pezzo fibration X we mean a three dimensional $G\mathbb{Q}$-Mori fiber space over the projective line. In this case there are groups G_F, $G_{\mathbb{P}^1}$, and an exact sequence

$$1 \to G_F \to G \to G_{\mathbb{P}^1} \to 1,$$

such that G_F acts faithfully on the general fiber of the del Pezzo fibrations and $G_{\mathbb{P}^1}$ acts faithfully on $\mathbb{P}^1$. Note that $\mathrm{PSL}_2(7)$ and $\mathcal{A}_6$ are simple groups and that they cannot act on $\mathbb{P}^1$ faithfully, thus a generic fiber of $\mathrm{PSL}_2(7)\mathbb{Q}$- or $\mathcal{A}_6$-del Pezzo fibration is a $\mathrm{PSL}_2(7)\mathbb{Q}$- or $\mathcal{A}_6$-del Pezzo surface respectively.

The group $\mathcal{A}_6$ has a unique central extension with an irreducible 3-dimensional representation. Thus there is a unique faithful action of $\mathcal{A}_6$ on $\mathbb{P}^2$. The group $\mathrm{PSL}_2(7)$ has two 3-dimensional representations. The representations are conjugate by the outer automorphism and the representations of central extensions induce the same action, hence, up to conjugation, there is a unique subgroup of $\mathrm{PGL}_2(\mathbb{C})$ isomorphic to $\mathrm{PSL}_2(7)$. There is also a unique del Pezzo surface S_2 of degree 2 with a faithful action of $\mathrm{PSL}_2(7)$, it is the double cover $r : S_2 \to \mathbb{P}^2$ branched over the Klein quartic. There are no other smooth del Pezzo surfaces admitting the action of $\mathcal{A}_6$ or $\mathrm{PSL}_2(7)$. In fact these are the only del Pezzo surfaces with mild singularities admitting a faithful $\mathrm{PSL}_2(7)$ or $\mathcal{A}_6$-action.

Theorem 5.1 ([7, Theorem 1.4]). *Let S be a del Pezzo surface with log terminal singularities.*

- *Suppose S admits a $\mathrm{PSL}_2(7)$-action. Then S is $\mathbb{P}^2$ or S_2.*
- *Suppose S admits an $\mathcal{A}_6$-action. Then S is $\mathbb{P}^2$.*

Thus a generic fiber of an $\mathcal{A}_6$-del Pezzo fibration is $\mathbb{P}^2$ and of a $\mathrm{PSL}_2(7)\mathbb{Q}$-del Pezzo fibration is S_2 or $\mathbb{P}^2$. To prove Theorem 1.18 we show that an $\mathcal{A}_6$-del Pezzo fibration, can be transformed by an $\mathcal{A}_6$-equivariant fiberpreserving map into $\mathbb{P}^2 \times \mathbb{P}^1$. Then we show that $\mathbb{P}^2 \times \mathbb{P}^1$ does not have $\mathcal{A}_6$-equivariant fiberpreserving maps to $\mathcal{A}_6$-del Pezzo fibrations other than itself. We prove Theorem 1.14 similarly.

Lemma 5.2. *Let $\pi : X \to B$ and $\pi' : Y \to B$ be $G\mathbb{Q}$-del Pezzo fibrations such that G acts trivially on the base. Suppose that the general fibers $X_{/B}$ and $Y_{/B}$ of π and π' are isomorphic as surfaces over $\mathbb{C}(B)$ and suppose that surface $X_{/B}$ admits a unique G-action up to an isomorphism. Then there exists a G-equivariant fiberpreserving map $X \dashrightarrow Y$.*

Proof. An isomorphism map $X_{/B} \to Y_{/B}$ induces the isomorphism of fields $\chi^* : \mathbb{C}(X) \to \mathbb{C}(Y)$. Thus birational map corresponding to χ^* is a fiberpreserving map by definition. The group G acts trivially on $\mathbb{C}(B)$, hence the G-action on X and Y induces the action on $X_{/B}$ and $Y_{/B}$. Since the G-action on $X_{/B}$ is unique, we may choose the isomorphism χ_B in such a way that it is G-equivariant. Then χ^* and the corresponding fiberpreserving maps are G-equivariant as well. □

Given a del Pezzo fibration we can consider its general fiber as a quartic in $\mathbb{P}(1,1,1,2)_{\mathbb{C}(t)}$. Algebraic operations we do on the equation of the general fiber correspond to the fiberpreserving transformations of the del Pezzo fibration.

Example 5.3. Consider, the double cover $X \to \mathbb{C}^1 \times \mathbb{P}^2$ branched over the central fiber $\{t = 0\}$ and a divisor which is a Klein quartic in every fiber. Its general fiber is defined by the equation $w^2 = t(x^3y + y^3z + z^3x)$ in $\mathbb{P}_{\mathbb{C}(t)}(1,1,1,2)$.

Clearly, X is canonical along preimage of a Klein quartic in the central fiber F. We may blow up this curve and then we can contract the proper transform of F into a singular point of the type $\frac{1}{2}(1,1,1)$. Let $\widetilde{X}$ be the variety we acquire after performing these operations. The equation of its general fiber is $t(w')^2 = x^3y + y^3z + z^3x$. Here we have made the coordinate change $wt = w'$, and have divided both sides of the equation by t.

Lemma 5.4. *Let $\pi : X \to \mathbb{P}^1$ be a* $\mathrm{PSL}_2(7)\mathbb{Q}$*-del Pezzo fibration of degree* 2*. Then there is a fiberpreserving* $\mathrm{PSL}_2(7)$*-birational map to X_n for some n.*

Proof. The general fiber $X_{/B}$ of π is a del Pezzo surface of degree 2 over $\mathbb{C}(t)$, that is it is a double cover of $\mathbb{P}^2_{\mathbb{C}(t)}$ branched over a quartic $q_4 \in \mathbb{C}[x,y,z](t)$. Thus $X_{/B}$ is a quartic given by an equation

$$w^2 = q_4(x,y,z)$$

in $\mathbb{P}(1_x, 1_y, 1_z)$ over $\mathbb{C}(t)$. We may multiply the equation by the denominators of the coefficients at monomials of q_4. Thus we get the equation

$$r(t)w^2 = \sum_{i=0}^{\infty} p_i(x,y,z)t^i,$$

where $r(t) \in \mathbb{C}[t]$, p_i are the $\mathrm{PSL}_2(7)$-invariant quartics, and $p_i = 0$ for $i >> 0$, that is the sum on the righthand side is finite. In suitable coordinates we have $p_i = 0$ or $p_i = x^3y + y^3z + z^3x$ since there is a unique $\mathrm{PSL}_2(7)$-invariant of degree 4. Thus the equation of $X_{/B}$ becomes

$$r(t)w^2 = q(t)(x^3y + y^3z + z^3x),$$

where $r, q \in \mathbb{C}[t]$. Let us change the coordinate $w = q\bar{w}$, then

$$r(t)q(t)\bar{w}^2 = x^3y + y^3z + z^3x.$$

We may also assume that rq does not have multiple roots, since we can change $\bar{w}$ again to get rid of them. Let n be an integer such that $\deg rq = 2n$ or $\deg(rq) = 2n - 1$. Consider a variety X_n defined by the equation

$$v^{2n} r\left(\frac{u}{v}\right) q\left(\frac{u}{v}\right) \bar{w}^2 = x^3y + y^3z + z^3x.$$

Since the general fibers $(X_n)_{/\mathbb{P}^1}$ and $X_{/\mathbb{P}^1}$ have the same equations there is fiberpreserving $\mathrm{PSL}_2(7)$-equivariant map from X to X_n by Lemma 5.2. □

Lemma 5.5 ([60, Lemma 4])**.** *Suppose G acts faithfully on a variety X and P is a G-invariant point. Then G acts faithfully on T_PX.*

Corollary 5.6. *The following assertions hold.*

(i) Suppose $\mathcal{A}_6$ acts on a surface S and let Σ be an orbit of a nonsingular point, then $|\Sigma| \geqslant 10$.

(ii) Let Σ be a $\mathrm{PSL}_2(7)$*-orbit on $\mathbb{P}^2$, then $|\Sigma| \geqslant 14$.*

Proof. Let H be the stabilizer of $P \in \Sigma$. Then by Lemma 5.5 there is an induced irreducible representation of H on $T_P S \cong \mathbb{C}^2$. The subgroups of $\mathcal{A}_6$ of index < 10 are isomorphic to $\mathcal{A}_5$ [GAP] but $\mathcal{A}_5$ does not have irreducible 2-dimensional representations, thus (i) holds.

The subgroups of $\mathrm{PSL}_2(7)$ of index < 14 are F_{21} and $\mathcal{S}_4$ ([GAP]). The group F_{21} does not have irreducible 2-dimensional representations. On the other hand, the induced action of S_4 on $\mathbb{P}^2$ is base point free ([GAP]). Thus S_4 cannot be a stabilizer and (ii) holds. □

Lemma 5.7. *Let $r > 0$ be an integer.*

(i) Suppose $C \in \big|\mathcal{O}_{\mathbb{P}^2}(r)\big|$ is an $\mathcal{A}_6$-invariant curve on $\mathbb{P}^2$, then $r \geqslant 6$.

(ii) Suppose $C \in \big|\mathcal{O}_{\mathbb{P}^2}(r)\big|$ is a $\mathrm{PSL}_2(7)$-invariant curve on $\mathbb{P}^2$, then $r \geqslant 4$.

(iii) Suppose $C \in \big| -rK_{S_2}\big|$ is a $\mathrm{PSL}_2(7)$-invariant curve on S_2, then $r \geqslant 2$.

(iv) Let $S \in \mathbb{P}(1_x, 1_y, 1_z, 2_w)$ be a surface given by the equation $x^3y + y^3z + z^3x = 0$. Suppose $C \in \big| -rK_S\big|$ is a $\mathrm{PSL}_2(7)$-invariant curve on S, then $r \geqslant 2$.

Proof. The G-actions on $\mathbb{P}^2$ is induced by a 3-dimensional representation of a central extension $\overline{G}$ of G. This representation induces the representation of $\overline{G}$ on polynomials of degree k as $\mathrm{Sym}^k(\mathbb{C}^3)$. Therefore every G-invariant curve of degree k corresponds to a 1-dimensional $\overline{G}$-invariant subspace of $\mathrm{Sym}^k(\mathbb{C}^3)$. The minimal k such that $\mathrm{Sym}^k(\mathbb{C}^3)$ has 1-dimensional $\overline{G}$-invariant representations is 6 for $\mathcal{A}_6$ and 4 for $\mathrm{PSL}_2(7)$ [GAP], this proves (i) and (ii)

Suppose $C \in \big|-rK_{S_2}\big|$ is a $\mathrm{PSL}_2(7)$-invariant curve on S_2. Without loss of generality we may assume that C is $\mathrm{PSL}_2(7)$-irreducible. Consider the double cover $\pi : S_2 \to \mathbb{P}^2$ branched over Klein quartic C_4. Since π is a canonical morphism, it is $\mathrm{PSL}_2(7)$-equivariant. Invariant Picard group $\mathrm{Pic}^G(S) = \mathbb{Z}$, therefore C is a pullback of a curve from $\mathbb{P}^2$ or C is the ramification divisor. In the first case $r \geqslant 4$, since $-K_{S_2}$ is a pullback of a line and the $\mathrm{PSL}_2(7)$-invariant curve of the lowest degree is a quartic. Clearly, in the latter case $r = 2$.

There is a G-invariant curve $C_0 \in \big| -2K_S\big|$ given by the equation $w = 0$, note that C_0 does not pass through a singular point, C_0 is isomorphic to the Klein quartic and C_0 is $\mathrm{PSL}_2(7)$-invariant. Suppose $C \neq C_0$, then $C \cap C_0$ is a union of orbits on C_0. Let $\Sigma \subset C \cap C_0$ be an orbit and let $H \subset \mathrm{PSL}_2(7)$ be a stabilizer of Σ. Then by Lemma 5.5 the group H is cyclic. The cyclic subgroup of $\mathrm{PSL}_2(7)$ of maximal size has 7 elements [GAP], therefore $\big|\Sigma\big| \geqslant 24$. On the other hand $\big|C \cap C_0\big| \leqslant C \cdot C_0 = 4r$, thus $4r \geqslant 24$. Hence $r \geqslant 6$ unless $C = C_0$, in which case $r = 2$. □

Lemma 5.8. *Let $\mathcal{M} \subset \big|\mathcal{O}_{\mathbb{P}^2}(r)\big|$ be a G-invariant linear system on $\mathbb{P}^2$. Suppose G is $\mathcal{A}_6$ or $\mathrm{PSL}_2(7)$ then the pair $(\mathbb{P}^2, \frac{3}{r}\mathcal{M})$ is log canonical.*

Proof. Let $\mathcal{M} = F + \mathcal{M}'$, where $F \in \big|\mathcal{O}_{\mathbb{P}^2}(r_1)\big|$ is a fixed part and $\mathcal{M}' \subset \big|\mathcal{O}_{\mathbb{P}^2}(r_2)\big|$ is a mobile linear system. If both $(X, \frac{3}{r_1}F)$ and $(X, \frac{3}{r_2}\mathcal{M}')$ are log canonical, then $(\mathbb{P}^2, \frac{3}{r}\mathcal{M})$ is log canonical.

Suppose the pair $(\mathbb{P}^2, \frac{3}{r_2}\mathcal{M}')$ is not log canonical. It is log canonical at curves since it is mobile, hence it is not log canonical at some point P. Then it is not log canonical at the orbit Σ of P. Consider generic divisors $D_1, D_2 \in \mathcal{M}'$, clearly $\mathrm{mult}_\Sigma D_i > \frac{r_2}{3}$. By Corollary 5.6

$$\frac{10r_2^2}{9} \leqslant \frac{r_2^2}{9}\big|\Sigma\big| < D_1 \cdot D_2 = r_2^2,$$

contradiction.

The pair $(X, \frac{3}{r_1}F)$ is log canonical at curves by Lemma 5.7. Suppose the pair $(X, \frac{3}{r_1}F)$ is not log canonical at some points. We may scale the pair with $\alpha < 1$ in such a way that the pair is $(X, \frac{3\alpha}{r_1}F)$ is log canonical and is strictly log canonical at some points P_i. The divisor class $-K_X + \frac{3\alpha}{r_1}F$ is ample, therefore such point is unique, since the locus at which the pair is not log canonical must be connected [47, Theorem 17.4]. As a fixed part of a G-invariant linear system the divisor F is also G-invariant, therefore the point P_1 is G-invariant. Then by Lemma 5.5 there is an irreducible representation of G on $T_{P_1}X \cong \mathbb{C}^2$, contradiction. □

Lemma 5.9. *Let S be a del Pezzo surface of degree 2 with a* $\mathrm{PSL}_2(7)$*-action. Let* $\mathcal{M} \subset \left|-rK_S\right|$ *be a* $\mathrm{PSL}_2(7)$*-invariant linear system on S. Then the pair* $(\mathbb{P}^2, \frac{1}{r}\mathcal{M})$ *is log canonical.*

Proof. The proof is analogous to the proof of Lemma 5.8. □

Lemma 5.10. *Let* $S \subset \mathbb{P}(1_x, 1_y, 1_z, 2_w)$ *be a surface given by the equation* $x^3y+y^3z+z^3x=0$*. Let* $\mathcal{M} \subset \left|-rK_S\right|$ *be a* $\mathrm{PSL}_2(7)$*-invariant linear system on S. Then the pair* $(\mathbb{P}^2, \frac{1}{r}\mathcal{M})$ *is log canonical outside of the singular point of S.*

Proof. The pair is log canonical at curves by Lemma 5.7. Suppose it is not log canonical at a point P. Let L be the curve from the ruling of the cone passing through the point P, then $L \cdot (-K_S) = \frac{1}{2}$. There is a number $k \geqslant 0$ such that $\mathcal{M}\big|_S = kL + \mathcal{M}'$, where $\mathcal{M}'$ does not contain L. Since the pair is log canonical at curves $k \leqslant r$. Consider a generic divisor $D \in \mathcal{M}'$. The pair $(S, \frac{1}{r}D + \frac{k}{r}L)$ is not log canonical at P, therefore by [17, Theorem 7]

$$r < (D \cdot L)_P \leqslant D \cdot L = r - \frac{k}{2}.$$

□

Proposition 5.11. *Suppose $\pi : X \to \mathbb{P}^1$ is a $G\mathbb{Q}$-del Pezzo fibration. Suppose f is a G-equivariant fiberpreserving birational map to a $G\mathbb{Q}$-del Pezzo fibration.*

(i) If G is $\mathcal{A}_6$ and X is $\mathbb{P}^2 \times \mathbb{P}^1$ then f is an isomorphism.

(ii) If G is $\mathrm{PSL}_2(7)$ *and X is $\mathbb{P}^2 \times \mathbb{P}^1$ or X_n, then f is an isomorphism.*

Proof. Suppose the map f is not an isomorphism. Then by Theorem 2.11 there is a G-invariant linear system $\mathcal{M} \subset \left| -nK_X + lF\right|$ such that the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical at $B \subset F$. Lemma 2.14 implies that B cannot be a singular point of the type $\frac{1}{2}(1,1,1)$ since curves from the ruling of the cone F intersect divisors in $\mathcal{M}$ by $\frac{n}{2}$. Thus B is a nonsingular point of X or a curve. Then by Inversion of Adjunction ([47, Theorem 17.7]) the pair $(F, \mathcal{M}\big|_F)$ is not log canonical at B which contradicts Lemma 5.8, Lemma 5.9, or Lemma 5.10. □

Proof of Theorem 1.18. Suppose $\pi : X \to \mathbb{P}^1$ is an $\mathcal{A}_6$-del Pezzo fibration, then the general fiber $X/\mathbb{P}^1$ of π is $\mathbb{P}^2_{\mathbb{C}(t)}$ by Theorem 5.1. By Lemma 5.2 there exists an $\mathcal{A}_6$-equivariant fiberpreserving map from X to $\mathbb{P}^2 \times \mathbb{P}^1$. By Proposition 5.11 this map must be an isomorphism, thus $X \cong \mathbb{P}^2 \times \mathbb{P}^1$. □

Proof of Theorem 1.14 is analogous.

Thus Theorem 1.5 is applicable and Corollary 1.15 holds.

Chapter 6

Stable rationality of del Pezzo fibrations

In this chapter we prove Theorem 1.31 and discuss the proof of Theorem 1.32. All fields are algebraically closed unless stated otherwise but fields may be of any characteristic. All varieties are assumed to be of dimension $\geqslant 3$. For a line bundle L we denote the complete linear system of divisors corresponding to L by $|L|$.

6.1 Method

Definition 6.1 ([57, Definition 1.2]). (i) A variety X defined over a field $\Bbbk$ is *universally* CH_0*-trivial* if for any field F containing $\Bbbk$, the degree map $\mathrm{CH}_0(X_F) \to \mathbb{Z}$ is an isomorphism.

(ii) A projective morphism $\varphi : Y \to X$ defined over a field $\Bbbk$ is *universally* CH_0*-trivial* if for any field F containing $\Bbbk$, the push-forward map $\varphi_* : \mathrm{CH}_0(Y_F) \to \mathrm{CH}_0(X_F)$ is an isomorphism.

The property of universal CH_0-triviality is closely related to the hodge numbers and the stable rationality.

Lemma 6.2 ([25, Lemma 1.5]). *If X is not universally* CH_0*-trivial, then it is not stably rational.*

Lemma 6.3 ([80, Lemma 2.2]). *Let X be a smooth projective variety. If $H^0(X, \Omega^i)$ is not zero for some $i > 0$, then X is not universally* CH_0*-trivial.*

If X is a rationally connected variety over $\mathbb{C}$, then it is well known that $H^0(X, \Omega^i) = 0$ for any $i > 0$. But over fields of finite characteristic this is no longer true (Proposition 6.5).

Definition 6.4 ([57, Definition 3.7]). Let X be a smooth variety over a field of characteristic 2 and let L be a line bundle on X. We say that a section $s \in H^0(X, L)$ has an *almost nondegenerate* critical point at $P \in X$ if

- the dimension of X is even and in suitable local coordinates $x_1, \ldots, x_n$ the section s can be written as

$$s = x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n + f,$$

where f consists of monomials of the degree at least 3, or if

- the dimension of X is odd and in suitable local coordinates $x_1, \ldots, x_n$ the section s can be written as

$$s = x_2x_3 + x_4x_5 + \cdots + x_{n-1}x_n + \gamma x_1^3 + f,$$

where $\gamma \neq 0$ and f consists of monomials of the degree at least 3 except x_1^3.

We provide a specific case of the following proposition for double covers in characteristic 2.

Proposition 6.5 ([49, p. 246-249]). *Let X be a smooth variety of dimension n over a field $\Bbbk$ of characteristic 2, L a line bundle on X. Let $s \in H^0(X, L^{\otimes 2})$ be a section with only almost nondegenerate critical points. Let $f : Y \to X$ be a double cover branched over $s = 0$, let $g : Y' \to Y$ be the blow up of Y along $f^{-1}(\operatorname{Crit} s)$, and let $\pi = f \circ g$. Then there is an injection*

$$\pi^*(\Omega_X^n \otimes L^{\otimes 2}) \to \Omega^{n-1}(Y').$$

In particular if $\left|\Omega_X^n \otimes L^{\otimes 2}\right| \neq \varnothing$ then $H^0\big(\Omega^{n-1}(Y')\big) \neq 0$. Hence Y' is not universally CH_0-trivial by Lemma 6.3. Note that the variety Y is always singular at the preimages of critical points of s. This phenomenon arises when the characteristic of the base field divides the degree of cover. This is why we have to consider the resolution rather than variety itself. Also note, that Y does not have to be universally CH_0-trivial even if Y' is. By the following proposition the morphism g is a universally CH_0-trivial resolution of singularities, hence Y is also not universally CH_0-trivial.

Proposition 6.6 ([57, Proposition 4.1]). *Let X be a smooth variety over a field of characteristic 2 and let $f : Y \to X$ be a double cover of branched over $s = 0$, where $s \in H^0(X, L^{\otimes 2})$. Suppose s has only almost nondegenerate critical points, then the blow up $g : Y' \to Y$ of Y at $f^{-1}(\operatorname{Crit} s)$ is a universally CH_0-trivial resolution of singularities.*

The following theorem allows us to lift the results on CH_0-triviality from characteristic 2 back to $\mathbb{C}$.

Theorem 6.7 ([25, Theorem 1.12]). *Let A be a discrete valuation ring with fraction field $\mathbb{K}$ and residue field $\Bbbk$, with $\Bbbk$ algebraically closed. Let $\mathcal{X}$ be a flat proper scheme over A with geometrically integral fibers. Let X be the general fiber $\mathcal{X} \times_A \mathbb{K}$ and Y the special fiber $\mathcal{X} \times_A \Bbbk$. Assume that there is a proper birational morphism $Y' \to Y$ with Y' smooth over $\Bbbk$ such that $\mathrm{CH}_0(Y') \to \mathrm{CH}_0(Y)$ is universally an isomorphism. Let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$. Assume that there is a proper birational morphism $X' \to X$ with X' smooth over $\mathbb{K}$ such that $X'_{\overline{\mathbb{K}}}$ is universally CH_0-trivial. Then Y' is universally CH_0-trivial.*

Thus to prove that a generic double cover in the family is not universally CH_0-trivial (and therefore is not stably rational), we only need to find one such double cover in the family in characteristic 2. To do this we find a double cover $f : Y \to X$ such that Y is in the family of varieties we consider, X is the smooth, the double cover f is branched over $s = 0$, where $s \in H^0(X, L^{\otimes 2})$ has only almost nondegenerate critical points, and $\left|L^{\otimes 2} \otimes \Omega_X^n\right| \neq \varnothing$. Then Y is not universally CH_0-trivial. Indeed, by Proposition 6.6 the blow up $g : Y' \to Y$ of Y at $f^{-1}(\operatorname{Crit} s)$ is a universally CH_0-trivial resolution of singularities. Then by Proposition 6.5 there exists an injection $g^* \circ f^*(\Omega_X^n \otimes L^{\otimes 2}) \to \Omega^{n-1}(Y')$. Then by Lemma 6.3 the variety Y' is not universally CH_0-trivial and so is Y since the resolution g is universally CH_0-trivial. To prove the existence of sections with almost nondegenerate singularities we use the following lemma.

Lemma 6.8 ([50, Proposition 2.2]). *Let X be a smooth variety over a field of characteristic 2. Suppose L is a line bundle on X such that the restriction map*

$$H^0(X, L^{\otimes 2}) \to L^{\otimes 2} \otimes \big(\mathcal{O}_{X,P}/\mathfrak{m}_P^l\big)$$

is surjective at every point $P \in X$, where $l = 3$ if the dimension of X is even and $l = 4$ if it is odd. Then a generic divisor $R \in \left|L^{\otimes 2}\right|$ has only admissible singularities.

Combining these results we get the following theorem.

Theorem 6.9 (Colliot-Thélène, Pirutka, Totaro, Kollár). *Let X be a smooth variety of dimension n over $\mathbb{C}$. Let L be a line bundle on X such that*

(i) the restriction map

$$H^0(X, L^{\otimes 2}) \to L^{\otimes 2} \otimes \mathcal{O}_{X,P}/\mathfrak{m}_P^l$$

is surjective at every point $P \in X$, where $l = 3$ if $\dim X$ is even and $l = 4$ if it is odd;

(ii) the linear system $\left|L^{\otimes 2} \otimes \Omega_X^n\right| \neq \varnothing$.

Then for a very general $R \in \left|L^{\otimes 2}\right|$ the double cover of X branched over R is not stably rational.

Let $\mathcal{M}$ be a family of algebraic varieties. We say that a property holds for a very general $X \in \mathcal{M}$ if it holds for every X in $U \subset \mathcal{M}$, where U is the complement to a countable union of Zariski closed subsets.

Remark 6.10. The condition (ii) is quite restrictive and we can do better in many situations. In the next section we do it for del Pezzo fibrations of degree 2 over $\mathbb{P}^1$.

6.2 Double covers of $\mathbb{P}^n$-bundles over $\mathbb{P}^m$

We now apply the method above to double covers of toric $\mathbb{P}^n$-bundles over $\mathbb{P}^m$.

Remark 6.11. Let X be a $\mathbb{P}^n$-bundle over $\mathbb{P}^m$. Then it X is a projectivization of some $\mathbb{C}^{n+1}$-bundle over $\mathbb{P}^m$. The variety X is toric if and only if the $\mathbb{C}^{n+1}$-bundle splits into the sum of line bundles.

We now introduce the notations and coordinates for $\mathbb{P}^n$-bundles over $\mathbb{P}^m$ for the rest of the chapter. If X is a toric $\mathbb{P}^n$-bundle over $\mathbb{P}^m$, then we may view X as

$$X = \operatorname{Proj}\left(\mathcal{O}_{\mathbb{P}^m} \oplus \mathcal{O}_{\mathbb{P}^m}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^m}(a_n)\right),$$

where $0 \leqslant a_1 \leqslant \cdots \leqslant a_n$. As we discussed in Section 2.4, there are homogeneous coordinates $y_0, \ldots, y_m, x_0, \ldots x_n$ such that every divisor is given by a homogeneous equation in these coordinates. The coordinates are bi-graded as follows: $\operatorname{wt}(y_j) = (0,1)$ and $\operatorname{wt}(x_i) = (1, a_i)$, where $a_0 = 0$. Clearly, the coordinates y_j are the pullbacks of coordinates from the base.

We now use the homogeneous coordinates to reformulate the conditions (i) and (ii) in terms of the numbers a_i and the bi-degree of the line bundle L.

Lemma 6.12. *Let $X = \operatorname{Proj}\left(\mathcal{O}_{\mathbb{P}^m} \oplus \mathcal{O}_{\mathbb{P}^m}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^m}(a_n)\right)$ and let L be a line bundle of bi-degree (r,d). Then*

(i) the restriction map

$$H^0(X, L^{\otimes 2}) \to L^{\otimes 2} \otimes \mathcal{O}_{X,P}/\mathfrak{m}_P^l$$

is surjective for every $P \in X$ if and only if $2r \geqslant l-1$ and $2d \geqslant 2ra_n + l - 1$,

(ii) $\left|L^{\otimes 2} \otimes \Omega_X^n\right| \neq \varnothing$ if and only if $2r \geqslant n+1$ and $2d \geqslant m + 1 + \sum a_i$.

Proof. We prove the lemma for the case $l = 2$ for clarity, other cases are analogous. Consider the affine map $x_k \neq 0$, $y_s \neq 0$. For the map to be surjective we need the monomials $x_i x_k^{2r-1} y_s^{2d-(2r-1)a_k - a_i}$ and $y_i x_k^{2r} y_s^{2d-2ra_k-1}$ contributing to $H^0(X, L^{\otimes 2})$. This happens if and only if all powers in all monomials above are nonnegative for all i, k, s: $2r \geqslant 1$ and $2d \geqslant 2ra_n + 1$ ($2r \geqslant l-1$ and $2d \geqslant 2ra_n + l - 1$ in general).

The bi-degree of K_X is $(-n-1, -m-1-\sum a_i)$, thus the assertion (ii) holds. ☐

Combining this statement with Theorem 6.9 we get Theorem 1.31.

Since every vector bundle over $\mathbb{P}^1$ splits into the sum of line bundles this theorem is particularly efficient for the study of double covers of $\mathbb{P}^n$-bundles over $\mathbb{P}^1$.

6.2.1 Del Pezzo fibrations of degree 2

It is well known that smooth del Pezzo fibrations are double covers of $\mathbb{P}^2$-bundles over $\mathbb{P}^1$ hence it is natural to apply Theorem 1.31 to them. Unfortunately this theorem does not cover many families which we expect to be not stably rational. Consider a very general double cover Y of $\mathbb{P}^2 \times \mathbb{P}^1$ branched over a divisor $(4,2)$. It is a del Pezzo fibration over $\mathbb{P}^1$ of degree

2 and it also is a conic bundle over $\mathbb{P}^2$ with a discriminant curve of degree 8. We expect that Y is not stably rational by [40], but the theorem above is only applicable when $2d \geqslant 3$.

In the rest of the section when we say that X is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ we use the following representation

$$X = \operatorname{Proj}\big(\mathcal{O}_{\mathbb{P}^m} \oplus \mathcal{O}_{\mathbb{P}^m}(a_1) \oplus \mathcal{O}_{\mathbb{P}^m}(a_2)\big),$$

where $a_2 \geqslant a_1 \geqslant 0$.

Lemma 6.13 ([53, Lemma 5.3]). *Suppose the base field is of characteristic* 0*. Let X be a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. Let L be a line bundle over X of bi-degree $(2, d)$. Then there exists a smooth divisor R in $|L^{\otimes 2}|$ if and only if one the following holds*

(i) $2d \geqslant 3a_2 + a_1$ *or*

(ii) $2d = 3a_2 \geqslant 4a_1$.

Proof. Suppose the condition (i) holds. Consider a divisor $R \in |L^{\otimes 2}|$ given by the equation

$$s = p_{2d}(y_0, y_1)x_0^4 + p_{2d-4a_1}(y_0, y_1)x_1^4 + p_{2d-3a_2}(y_0, y_1)x_0x_2^3 + p_{2d-3a_2-a_1}(y_0, y_1)x_1x_2^3 = 0,$$

where p_k are homogeneous polynomials of degree k. Note that all degrees are nonnegative since (i) holds. By Bertini theorem R may only be singular at the points of a curve $x_0 = x_1 = 0$. This means that $\frac{\partial}{\partial x_2} = 0$ automatically. Thus any singular point satisfies the equations

$$\frac{\partial s}{\partial x_0} = 4p_{2d}(y_0, y_1)x_0^3 + p_{2d-3a_2}(y_0, y_1)x_2^3 = 0, \text{ that is, } p_{2d-3a_2}(y_0, y_1) = 0, \text{and}$$
$$\frac{\partial s}{\partial x_1} = 4p_{2d-4a}(y_0, y_1)x_1^3 + p_{2d-3a_2-a}(y_0, y_1)x_2^3 = 0, \text{ that is, } p_{2d-3a_2-a_1}(y_0, y_1) = 0.$$

Clearly, if the polynomials p_k are general then there are no solutions, hence R is smooth.

For the case $2d = 3a_2 \geqslant 4a_1$ consider a divisor $R \in |L^{\otimes 2}|$ given by the equation

$$s = p_{2d}(y_0, y_1)x_0^4 + p_{2d-4a_1}(y_0, y_1)x_1^4 + x_0x_2^3 = 0,$$

One again by Bertini's theorem R may only be singular at the points satisfying $x_0 = x_1 = 0$, but clearly

$$\frac{\partial s}{\partial x_0}\Big|_{x_0=x_1=0} = x_2^3 \neq 0.$$

On the other hand, suppose $2d < 3a_2 + a_1$. Let R be given by the equation $s = 0$. If the degree of f in x_2 is less than 3, then R is singular along $x_0 = x_1 = 0$. If the degree is 3, then we can write the degree 3 part of f as $p_{2d-3a_2}(y_0, y_1)x_0x_2^3$. It is easy to see that in this case R is singular at $x_0 = x_1 = p_{2d-3a_2}(y_0, y_1) = 0$ unless $2d = 3a_2$. Suppose $2d < 4a_1$, then s is divisible by x_0, and hence R is singular. □

Let X_2 be the $\mathbb{P}^2$-bundle over a field of characteristic 2 with the same a_1 and a_2 as X. To complete the proof of Theorem 1.32 we need to find $s \in H^0(X_2, L^{\otimes 2})$ with only only almost nondegenerate critical points. To do this we consider several cases and I give examples of a few cases in this thesis.

Lemma 6.14 ([53, pages 19-21]). *Suppose the characteristic of the base field is* 2*. Let X be a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. Let L be a line bundle of bi-degree $(2, d)$. Suppose d satisfies $2d \geqslant 3a_2 + a_1$ or $2d = 3a_2 > 4a_1$, then a general $s \in H^0(X, L^{\otimes 2})$ has only almost non degenerate critical points.*

Proof. In this thesis we prove the lemma for only a few cases. First suppose $a_2 = 0$, that is $X \cong \mathbb{P}^2 \times \mathbb{P}^1$. If $d \geqslant 2$, then by Lemma 6.12 the restriction map

$$\phi_{P,l} : H^0(X, L^{\otimes 2}) \to L^{\otimes 2} \otimes \mathcal{O}_{X,P}/\mathfrak{m}_P^l$$

is surjective for $l \leqslant$ every $P \in X$. Thus general s has only almost nondegenerate critical points. Suppose $d = 1$, then by Lemma 6.12 the map $\phi_{P,2}$ is surjective for every $P \in X$. Thus, for a given point P, the codimension of all s which have a critical point at P is 3. Without loss of generality we may assume that $P = (1 : 0 : 0; 1 : 0)$. Consider a section of the form

$$s = y_0 y_1 x_0^3 x_1 + y_0^2 x_0 x_2^3 + \dots,$$

it has an almost nondegenerate critical point at P. This shows that additional conditions are imposed on a section to have a critical point at P which is worse than almost nondegenerate critical points. Thus we conclude that the codimension of all s which have a worse than an almost nondegenrate critical point at P is at least 4. Therefore a general s has only almost nondegenerate critical points.

Now we may assume that $a_2 \geqslant 1$ and therefore $d \geqslant 2$. Then repeating the proof of Lemma 6.12 we can see that the map $\phi_{P,4}$ is surjective for any $P \in X \setminus \{x_0 = 0\}$. Thus a general s has only nondegenerate critical points on $X \setminus \{x_0 = 0\}$. If $d > 2a_2$, then $\phi P, 2$ is surjective on $\{x_0 = 0\}$, therefore a general s is smooth on $\{x_0 = 0\}$.

If $3a_2 \leqslant 2d \leqslant 4a_2$, then we once again conclude that a general s has only nondegenerate critical points on $X \setminus \{x_0 = 0\}$. For the set $\{x_0 = 0\}$ we have to consider which monomials exist and we look at the possible equations to prove the lemma. For the remainder of the proof we refer to [53, pages 19-21]. ☐

There is one case which is not covered by the lemma above. In the case $2d = 3a_2 = 4a_1$ there is a smooth divisor in $|L^{\otimes 2}|$ and hence smooth del Pezzo fibration, but when we consider reduction to characteristic 2 every $s \in H^0(X, L^{\otimes 2})$ has critical points along a curve. Thus we cannot apply our method for the double cover. We reduce to characteristic 3 and use a triple cover instead to solve this problem. We refer to [53, page 21] for details.

Proposition 6.15. *Let $X = \operatorname{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2))$. Let L be a line bundle of bidegree $(2, d)$, where d satisfies the requirements of Lemma 6.14 and suppose $2d \geqslant 2 + a_1 + a_2$. Then a smooth del Pezzo fibration of degree 2 which is a double cover of X branched over a very general divisor in $R \in |L^{\otimes 2}|$ is not stably rational.*

Proof. We consider a reduction to characteristic 2. Let X_2 be the $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ over the field of characteristic 2 with the same a_1 and a_2, and let L_2 be a line bundle of bi-degree $(2, d)$ on X_2.

By Lemma 6.14 there is a section $s \in H^0(X_2, L_2^{\otimes 2}$ with only nondegenerate critical points. Let R_2 be the divisor given by $s = 0$ then Lemma 6.12 $|K_{X_2} + R_2| \neq \varnothing$. Thus the double cover branched over R_2 is not universally CH_0-trivial by Lemma 6.3 and Proposition 6.5. Hence by Theorem 6.9 for a very general $R \in |D|$ the double cover of X branched over R is not stably rational. ☐

Proof of Theorem 1.32. There exists a $\mathbb{P}^2$-bundle X over $\mathbb{P}^1$ such that there is a double cover $f : Y \to X$ branched over a smooth divisor R of degree $(4, 2d)$ for some d. Then the smoothness of R implies that

1. $2d \geqslant 3a_2 + a_1$,
2. $2d = 3a_2 > 4a_1$, or
3. $2d = 3a_2 = 4a_1$.

For the case (iii) we need to consider triple cover instead and we refer to [53, page 21]. Thus suppose (i) or (ii) holds, then $2d \geqslant 3a_2$. We show that $d \geqslant 2 + a_2 + a_1$. Suppose $d < 2 + a_2 + a_1$, then $3a_2 < 2 + a_2 + a_1$, hence $a_2 = 0$ or $a_2 = 1$. If $a_2 = 0$ then $d = 0$, therefore Y is a direct product. Then $\operatorname{Pic} Y \cong \mathbb{Z}^9$, contradiction. Suppose $a_2 = 1$, then $3 \leqslant 2d < 3 + a_1$ which is impossible. Thus $d \geqslant 2 + a_2 + a_1$ and therefore Proposition 6.15 is applicable. Therefore very general variety in the family of Y is not stably rational. ☐

Example 6.16. Let V be a very general quartic double solid, that is a variety such that there is a double cover $\sigma : V \to \mathbb{P}^3$ branched over a very general quartic surface S_4. In [81] it is was shown that V is not stably rational. Consider a pencil of a hyperplanes $\mathcal{H}$ passing through a line on $\mathbb{P}^3$. The preimage $\sigma^{-1}\mathcal{H}$ of this pencil is a pencil of del Pezzo surfaces of degree 2. Let L be a base line of $\mathcal{H}$ and let $\phi : Y \to V$ be a blow up $\sigma^{-1}(L)$. As we have blown up the base curve of $\sigma^{-1}(\mathcal{H})$, the variety Y admits a del Pezzo fibration of degree 2. By Theorem 1.32, Y is not stably rational, hence so is V. Let us look at this example closer. There is a double cover $\sigma_Y : Y \to X$, where X is acquired by blowing up L on $\mathbb{P}^3$ and the branching divisor is a proper transform of S_4. Thus we have the following commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\sigma_Y} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ V & \xrightarrow{\sigma} & \mathbb{P}^3 \end{array}$$

where $X = \mathrm{Proj}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right)$. The branching divisor of σ_Y is a divisor of bidegree $(4, 4)$, thus a_1, a_2, d satisfy the requirements of Proposition 6.15 and Y is not stably rational.

Thus we have an alternative proof of non stable rationality of V.

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