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Frobenius n -homomorphisms, transfers and branched coverings

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Abstract

The main purpose is to characterise continuous maps that are n -branched coverings in terms of induced maps on the rings of functions. The special properties of Frobenius n -homomorphisms between two function spaces that correspond to n -branched coverings are determined completely. Several equivalent definitions of a Frobenius n -homomorphism are compared and some of their properties are proved. An axiomatic treatment of n -transfers is given in general and properties of n -branched coverings are studied and compared with those of regular coverings.

1. Introduction

In previous papers [1, 2 and 3] we have studied the relationship between Frobenius n -homomorphisms and symmetric products, in particular we showed that Frobenius n -homomorphisms $C(X) \rightarrow C(Y)$ correspond precisely to continuous maps $f: Y \rightarrow \text{Sym}^n(X)$, thus generalising the classical theorem of Gelfand–Kolmogorov. In this paper we characterise n -branched coverings using transfer maps which we treat as Frobenius n -homomorphisms of a special type. In Section 2 we review and prove the equivalence of a number of definitions of Frobenius n -homomorphisms that were not covered in our previous papers. Section 3 is devoted to showing that the composition of a Frobenius n -homomorphism with a Frobenius m -homomorphism is a Frobenius nm -homomorphism, the

proof depends heavily on a previous result from [1]. A version of this result appeared in a preliminary preprint of the present paper and D. V. Gugin has given an improvement of that result. Section 4 reviews the definition of an n -branched covering due to L. Smith [8] and A. Dold [5] and we prove some new properties and give examples; in particular we make a detailed analysis of n -branched coverings over an interval. Section 5 introduces the notion of an n -transfer for a ring homomorphism as a special kind of Frobenius n -homomorphism. It is shown that the kernel introduced by Gugin [6] for a Frobenius n -homomorphism is trivial for an n -transfer. Section 6 contains the main theorem which characterises n -branched coverings in terms of n -transfers between the relevant function spaces.

2. Frobenius n -homomorphisms

In [1, 2] we introduced the concept of a Frobenius n -homomorphism and studied some of its properties (these papers also contain references to related work by other authors). We recall the basic definition.

Consider a linear map $f: A \rightarrow B$ between two algebras [throughout this paper all algebras will be commutative, associative \mathbb{C} algebras]. The maps $\Phi_n(f): A^{\otimes n} \rightarrow B$ are defined as follows:

Each permutation $\sigma \in \Sigma_n$ the symmetric group on n letters, can be decomposed into a product of disjoint cycles of total length n , say $\sigma = \gamma_1 \gamma_2 \cdots \gamma_r$. If $\gamma = (i_1 \cdots i_m)$ is a cycle, let $f_\gamma(a_1, a_2, \dots, a_n) = f(a_{i_1} a_{i_2} \cdots a_{i_m})$ then

$$\Phi_n(f)(a_1, a_2, \dots, a_n) = \sum_{\sigma \in \Sigma_n} \varepsilon_\sigma f_{\gamma_1}(a_1, a_2, \dots, a_n) f_{\gamma_2}(a_1, a_2, \dots, a_n) \cdots f_{\gamma_r}(a_1, a_2, \dots, a_n)$$

where ε_σ is the sign of the permutation σ .

From this definition it is clear that $\Phi_n(f): A^{\otimes n} \rightarrow B$ is n -linear and symmetric, one can use polarisation to ease the verification of some of its properties; in other words to prove identities it is enough to consider the values of $\Phi_n(f)(a, a, \dots, a)$ for all $a \in A$ (we will sometimes abbreviate this to $\Phi_n(f)(a)$).

There is also an inductive definition (first used by Frobenius in the case of group algebras of finite groups) for the $\Phi_n(f)$ starting with $\Phi_1(f) = f$ and, for $n \geq 1$,

$$\begin{aligned} \Phi_{n+1}(f)(a_0, a_1, \dots, a_n) &= f(a_0) \Phi_n(f)(a_1, a_2, \dots, a_n) \\ &\quad - \sum_{r=1}^n \Phi_n(f)(a_1, a_2, \dots, a_0 a_r, \dots, a_n) \end{aligned}$$

or equivalently, because of the polarisation identities for symmetric multilinear maps,

$$\Phi_{n+1}(f)(a, a, \dots, a) = f(a) \Phi_n(f)(a, a, \dots, a) - n \Phi_n(f)(a^2, a, \dots, a).$$

We will also find it useful to have another equivalent definition: $\Phi_n(f)(a)$ is the determinant of the matrix

$$\begin{pmatrix} f(a) & 1 & 0 & 0 & \cdots & 0 \\ f(a^2) & f(a) & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & & & & & \vdots \\ f(a^{n-1}) & f(a^{n-2}) & f(a^{n-3}) & \cdots & f(a) & n-1 \\ f(a^n) & f(a^{n-1}) & f(a^{n-2}) & \cdots & f(a^2) & f(a) \end{pmatrix}.$$

Definition 2.1. A linear map $f: A \rightarrow B$ of algebras is called a *Frobenius n -homomorphism* if it satisfies $f(1) = n$ and $\Phi_{n+1}(f) \equiv 0$.

We note that, by polarisation, the condition $\Phi_{n+1}(f) \equiv 0$ is equivalent to $\Phi_{n+1}(f)(a) = 0$ for all $a \in A$.

Definition 2.2. An algebra A is *connected* if $a(a - 1) = 0$ only if $a = 0, 1$.

PROPOSITION 2.1. *If A is a connected algebra then, for each $k \geq 1$, the equation $x(x - 1) \cdots (x - k) = 0$ has only the obvious $k + 1$ solutions.*

Proof. Suppose that $a \in A$ satisfies $a(a - 1) \cdots (a - k) = 0$, $a \neq 0, 1, 2, \dots, k$ and that, for this a , k is the smallest such integer, so that if

$$p(x) = \frac{1}{k!}x(x - 1) \cdots (x - k + 1)$$

then, $b = p(a) \neq 0$. The polynomial $p(x)^2 - p(x)$ clearly has $0, 1, 2, \dots, k - 1$ as roots, and as $p(k) = 1$ it also has k as a root. Hence, $b^2 - b = 0$ since $x(x - 1) \cdots (x - k)$ divides $p(x)$. Now suppose $b = 1$, then $a(a - 1) \cdots (a - k + 1) = k!$ and multiplying this equation by $a - k$ gives $0 = k!(a - k)$, hence $a = k$, contradicting the hypothesis. We have found a $b \neq 0, 1$ with $b^2 = b$ and so A is not connected.

The following result is easily proved by induction

LEMMA 2.1. *For any linear map f one has*

$$\Phi_{n+1}(f)(a, 1, \dots, 1) = f(a)(f(1) - 1)(f(1) - 2) \cdots (f(1) - n).$$

By taking $a = 1$ one obtains.

COROLLARY 2.1. *Let B be connected and $f: A \rightarrow B$ be such that $\Phi_{n+1}(f) \equiv 0$ then $f(1) \in \{0, 1, 2, \dots, n\}$.*

These results are closely related to those of [1, corollary 2.5].

The condition $f(1) = n$ plays a crucial rôle [1, proposition 2.7] in the definition of a Frobenius n -homomorphism, as the following indicates.

PROPOSITION 2.2. *If $\Phi_{n+1}(f) \equiv 0$ and $f(1) = k$ then $\Phi_{k+1}(f) \equiv 0$.*

We will use the following result about symmetric polynomials (it is related to some of those in [3, 4]); it helps to explain the determinant expression used above.

PROPOSITION 2.3. *Let $s_k = \beta_1^k + \beta_2^k + \cdots + \beta_n^k$, then the indeterminates β_r ($1 \leq r \leq n$) are the roots of the polynomial $d(t)$ given as the determinant of the matrix*

$$\begin{pmatrix} s_1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ s_2 & s_1 & 2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & & \vdots \\ \vdots & & & & & & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 & n - 1 & 0 \\ s_n & s_{n-1} & s_{n-2} & \cdots & s_2 & s_1 & n \\ t^n & t^{n-1} & t^{n-2} & \cdots & t^2 & t & 1 \end{pmatrix}.$$

Moreover, if $f: A \rightarrow B$ is linear and $f(a^k) = s_k$ then the determinant $d(t)$ is, up to a non-zero constant multiple, equal to

$$t^n - \Phi_1(f)(a)t^{n-1} + \frac{1}{2}\Phi_2(f)(a)t^{n-2} - \dots + \frac{(-1)^n}{n}\Phi_n(f)(a).$$

Proof. As usual, let e_r denote the elementary symmetric polynomial of degree r in β_r ($1 \leq r \leq n$). Denote the columns of the above matrix by $(\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{n+1})$ and replace the first column by the column vector

$$\mathbf{c}_1 - e_1\mathbf{c}_2 + e_2\mathbf{c}_3 - \dots + (-1)^n e_n\mathbf{c}_{n+1}.$$

The determinant is unchanged. Using the standard Newton formulae (see [7, page 20]),

$$s_r - s_{r-1}e_1 + \dots + (-1)^{r-1}s_1e_{r-1} + (-1)^r r e_r = 0$$

we see that the first column of the new matrix has zero entries except for the last entry which equals

$$p(t) = t^n - e_1t^{n-1} + e_2t^{n-2} - \dots + (-1)^n e_n.$$

So $d(t) = (-1)^n n! p(t)$ and hence the roots of $d(t)$ are the same as those of $p(t)$, namely $\{\beta_1, \beta_2, \dots, \beta_n\}$.

Finally, to verify the result about $f: A \rightarrow B$ it is enough to consider the special case $A = \mathbb{C}[a]$, $B = \mathbb{C}[\beta_1, \beta_2, \dots, \beta_n]$. Then, considering the appropriate sub-determinant of the above determinant that defines e_n , we see that $\Phi_n(f)(a) = n!e_n$.

This leads us to yet another characterisation of Frobenius n -homomorphisms (which is closely related to some formulae in [3] and [4]).

PROPOSITION 2.4. *A linear map $f: A \rightarrow B$ is a Frobenius n -homomorphism if and only if for each $a \in A$ there is a polynomial $p_a(t) \in B[t]$ of degree n such that*

$$\sum_{q=0}^{\infty} \frac{f(a^q)}{t^{q+1}} = \frac{d}{dt} \log p_a(t).$$

Proof. Given the Frobenius n -homomorphism $f: A \rightarrow B$ and $a \in A$, consider the polynomial $p_a(t) = \det(M) \in B[t]$ where M is the matrix

$$\begin{pmatrix} f(a) & 1 & 0 & 0 & \dots & \dots & 0 \\ f(a^2) & f(a) & 2 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & & \vdots \\ \vdots & & & & & & \vdots \\ f(a^{n-1}) & f(a^{n-2}) & f(a^{n-3}) & \dots & f(a) & n-1 & 0 \\ f(a^n) & f(a^{n-1}) & f(a^{n-2}) & \dots & f(a^2) & f(a) & n \\ t^n & t^{n-1} & t^{n-2} & \dots & t^2 & t & 1 \end{pmatrix}.$$

Choose an extension \overline{B} of B such that $p_a(t)$ factors completely in $\overline{B}[t]$, say $p_a(t) = n!(t - \beta_1)(t - \beta_2) \cdots (t - \beta_n)$ with $\beta_r \in \overline{B}$. Then

$$\frac{d}{dt} \log p_a(t) = \frac{d}{dt} (\log(t - \beta_1) + \log(t - \beta_2) + \dots + \log(t - \beta_n)).$$

but

$$\frac{d}{dt} \log(t - \beta) = \frac{1}{t - \beta} = \frac{1}{t} \left(1 - \frac{\beta}{t} \right) = \frac{1}{t} + \frac{\beta}{t^2} + \frac{\beta^2}{t^3} + \dots$$

However, by Proposition 2.2, $\beta_1^r + \beta_2^r + \dots + \beta_n^r = f(a^r)$ and the result is proved.

Conversely, if

$$\sum_{q=0}^{\infty} \frac{f(a^q)}{t^{q+1}} = \frac{d}{dt} \log p_a(t)$$

and $p_a(t) = n!(t - \beta_1)(t - \beta_2) \dots (t - \beta_n)$ then $f(a^r) = \beta_1^r + \beta_2^r + \dots + \beta_n^r$ and, with $r = 0$ we have $f(1) = n$; using this and Proposition 2.2 we get that the r th elementary symmetric polynomial in $\beta_1, \beta_2, \dots, \beta_n$ is $\Phi_r(f)(a)$ and hence that $\Phi_{n+1}(f)(a) = 0$.

As in [1] we denote the subalgebra of symmetric tensors in $A^{\otimes n}$ by $S^n A$.

We will find the following fact very useful and it is easy to prove.

LEMMA 2.2. *The diagonal map $\Delta_n: A \rightarrow S^n A$ defined by*

$$\Delta_n(a) = a \otimes 1 \otimes \dots \otimes 1 + 1 \otimes a \otimes \dots \otimes 1 + \dots + 1 \otimes 1 \otimes \dots \otimes a$$

is a Frobenius n -homomorphism.

THEOREM 2.1. *If $f: A \rightarrow B$ is a Frobenius n -homomorphism, then the map defined by*

$$\frac{\Phi_n(f)}{n!}: S^n A \longrightarrow B$$

is a ring homomorphism. Conversely, if $f: A \rightarrow B$ is linear, $f(1) = n$ and $\Phi_n(f)/n!$ is a ring homomorphism, then f is a Frobenius n -homomorphism.

Proof. The first statement is [1, theorem 2.8] so we only need to prove the (easier) converse part.

Since $\Phi_n(f)/n!$ is a ring homomorphism we have that, for $\mathbf{a}, \mathbf{b} \in S^n A$

$$\Phi_n(f)(\mathbf{a})\Phi_n(f)(\mathbf{b}) = n!\Phi_n(f)(\mathbf{ab}).$$

But by the inductive definition and the symmetry of $\Phi_{n+1}(f)$,

$$\Phi_{n+1}(f)(a^{\otimes n+1}) = f(a)\Phi_n(f)(a^{\otimes n}) - n\Phi_n(f)(a^2 \otimes a^{\otimes n-1})$$

and $a^2 \otimes a^{\otimes n-1} + a \otimes a^2 \dots \otimes a + \dots + a^{\otimes n-1} \otimes a^2 = a^{\otimes n} \Delta_n(a)$. By Lemma 2.1, $\Phi_n(f)(\Delta_n(a)) = n!f(a)$. So

$$\begin{aligned} \Phi_{n+1}(f)(a^{\otimes n+1}) &= f(a)\Phi_n(f)(a^{\otimes n}) - \Phi_n(f)(a^{\otimes n} \Delta_n(a)) \\ &= f(a)\Phi_n(f)(a^{\otimes n}) - \Phi_n(f)(a^{\otimes n})\Phi_n(f)(\Delta_n(a))/n! \\ &= \Phi_n(f)(a^{\otimes n})(f(a) - f(a)) = 0. \end{aligned}$$

In an appropriate sense Δ_n is the universal Frobenius n -homomorphism on A as the following shows.

COROLLARY 2.2. *A Frobenius n -homomorphism $f: A \rightarrow B$ factors uniquely as*

$$\tilde{f} \Delta_n: A \longrightarrow S^n A \longrightarrow B$$

where \tilde{f} is a ring homomorphism. Moreover, $\tilde{f} = \Phi_n(f)/n!$.

Proof. By Lemma 2.1, $(\Phi_n(f)/n!)(\Delta_n(a)) = f(a)$ which proves existence.

It remains to prove that \tilde{f} is unique. This is true because of the fact that, as an algebra, $\mathcal{S}^n A$ is generated by the elements $\{\Delta_n(a) : a \in A\}$ since a general element $\sum_{\sigma \in \Sigma_n} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$ of $\mathcal{S}^n A$ equals $\Phi_n(\Delta_n)(a_1, a_2, \dots, a_n)$ which is a polynomial in elements of the form $\Delta_n(a)$ with $a \in A$.

Following an idea due to D. V. Gugin we introduce the appropriate categorical concept of the kernel of a Frobenius n -homomorphism.

Definition 2.3. For a Frobenius n -homomorphism $f: A \rightarrow B$ let $K_f = \{a : f(ax) = 0 \text{ for all } x \in A\}$, it is clearly a subspace of the kernel of the linear map f .

When B has no nilpotent elements, D. V. Gugin has shown that $K_f = \{a : f(a^r) = 0 \text{ for } 1 \leq r \leq n\}$.

3. Compositions

In [1] we showed that the sum of a Frobenius m -homomorphism and a Frobenius n -homomorphism is a Frobenius $m + n$ -homomorphism. The main result of this section is to show that they also behave appropriately under composition.

THEOREM 3.1. *Let A, B, C be associative, commutative \mathbb{C} -algebras. If $f: A \rightarrow B$ and $g: B \rightarrow C$ are Frobenius n - and m -homomorphisms, respectively then $gf: A \rightarrow C$ is a Frobenius mn -homomorphism.*

Proof. We first note that the theorem is trivial if either n or m equals 1 and that we will make constant use of Corollary 2.2.

The map f factors through its universal Frobenius n -homomorphism as

$$\tilde{f} \Delta_n : A \longrightarrow \mathcal{S}^n A \longrightarrow B.$$

Similarly, g factors as

$$\tilde{g} \Delta_m : B \longrightarrow \mathcal{S}^m B \longrightarrow C.$$

Since \tilde{f} is a ring homomorphism and Δ_m is a Frobenius m -homomorphism, the composition

$$\Delta_m \tilde{f} : \mathcal{S}^n A \longrightarrow B \longrightarrow \mathcal{S}^m B$$

is a Frobenius m -homomorphism, so it factors through

$$\Delta_m : \mathcal{S}^n A \longrightarrow \mathcal{S}^m \mathcal{S}^n A.$$

By direct observation, the composition

$$\Delta_m \Delta_n : A \longrightarrow \mathcal{S}^n A \longrightarrow \mathcal{S}^m \mathcal{S}^n A$$

is equal to

$$i \Delta_{mn} : A \longrightarrow \mathcal{S}^{mn} A \longrightarrow \mathcal{S}^m \mathcal{S}^n A$$

where i is the inclusion map. Hence the composition $gf: A \rightarrow C$ factors through

$$\Delta_{mn} : A \longrightarrow \mathcal{S}^{mn} A$$

and so gf is a Frobenius mn -homomorphism.

4. Branched coverings

We consider branched coverings in the sense studied by Smith [8] and by Dold [5] and elaborate on their properties. When X is a topological space or a variety, we denote its n th symmetric product by $\text{Sym}^n(X)$; it is the quotient space of the Cartesian product X^n under the action of the symmetric group Σ_n acting by permuting the factors. We often regard the elements of $\text{Sym}^n(X)$ as n -multisets $[x_1, x_2, \dots, x_n]$ with $x_i \in X$.

Definition 4.1. An n -branched covering $h: X \rightarrow Y$ is a continuous map between two Hausdorff spaces and a continuous map $t: Y \rightarrow \text{Sym}^n(X)$ such that:

- (i) $x \in th(x)$ for every $x \in X$; and
- (ii) $\text{Sym}^n(h)(ty) = ny$ for every $y \in Y$.

Example 4.1. The map $p: \mathbb{C} \rightarrow \mathbb{C}$ defined by a polynomial of degree n is the classic example of an n -branched covering. The map t is defined by $t(w) = [z_1, z_2, \dots, z_n]$ where the z_r are the roots (counted with multiplicities) of the equation $p(z) = w$. It is straightforward to verify the above axioms in this case.

Example 4.2. If G is a finite group acting continuously and effectively on a Hausdorff space X and $h: X \rightarrow Y = X/G$ is the map to the space of orbits, then h is an n -branched covering where n is the cardinality of G and the map $t: Y \rightarrow \text{Sym}^n(X)$ is given by $t(y) =$ the points in the orbit defined by y (counted with multiplicities). More generally (see [5, example 1.4]), if $H \subset G$ is a subgroup of finite index n and X is an effective G -space, then the quotient map $X/H \rightarrow X/G$ is an n -branched covering.

Every 2-branched covering arises from an action of the group with two elements.

PROPOSITION 4.1. *Let $f: X \rightarrow Y$, $s: Y \rightarrow \text{Sym}^n(X)$ and $g: Y \rightarrow Z$, $t: Z \rightarrow \text{Sym}^m(Y)$ be n - and m -branched coverings, then the composition $h = gf: X \rightarrow Z$ is an mn -branched covering with $u: Z \rightarrow \text{Sym}^{mn}(X)$ being the composition*

$$i \text{Sym}^m(s) t: Z \longrightarrow \text{Sym}^m(Y) \longrightarrow \text{Sym}^m(\text{Sym}^n(X)) \longrightarrow \text{Sym}^{mn}(X).$$

Proof. The proof is straightforward.

One defines induced n -branched coverings by taking the obvious pullback: let $h: X \rightarrow Y, t: Y \rightarrow \text{Sym}^n(X)$ be an n -branched covering and $\phi: Z \rightarrow Y$ a continuous map. The usual pullback is $\tilde{X} = \{(z, x) | \phi(z) = h(x)\}$, the projection is $\tilde{h}(z, x) = z$ and $\tilde{t}: Z \rightarrow \text{Sym}^n(\tilde{X})$ is defined by $\tilde{t}(z) = [(z, x_1), (z, x_2), \dots, (z, x_n)]$ where $t\phi(z) = [x_1, x_2, \dots, x_n]$. The properties of an n -branched covering are straightforward to check.

We modify the Ehresmann method [9] which constructs the principal fibration associated with a locally trivial fibration. As a result we get a method to ‘resolve’ an n -branched covering.

An *epimorphism* from the set $[n] = \{1, 2, \dots, n\}$ to an n -multiset is a map which is onto and the counter image of an element with multiplicity m has size m .

If $h: X \rightarrow Y$ is an n -branched covering, let E be the set of all maps $\psi: [n] \rightarrow X$ such that $h\psi(1) = h\psi(2) = \dots = h\psi(n)$ and ψ is an epimorphism onto the multiset $th\psi$. Clearly the symmetric group Σ_n acts on E and the quotient is Y ; moreover, $E \times_{\Sigma_n} [n]$ is isomorphic to X and the projection onto the first factor can then be identified with h . The map ψ can be thought of as a ‘universal’ labeling for the branches of the covering. This proves the following.

THEOREM 4.1 ([5, proposition 1.9]). *Every n -branched covering can be described in the form of a projection $p: E \times_{\Sigma_n} [n] \rightarrow E/\Sigma_n$ for some Σ_n space E .*

Example 4.3. To illustrate all this we consider in some detail the case where the base space is an interval; this is in contrast to the case of regular coverings over an interval in which case they are trivial. The following discussion describes many n -branched coverings of an interval, including all those with finitely many ‘branch points’.

A set partition π of a finite set S consists of a family of disjoint non-empty subsets of S whose union is S . We let $n(\pi)$ denote the number of parts of π .

Let P_S denote the set of all set partitions of S . Two set partitions

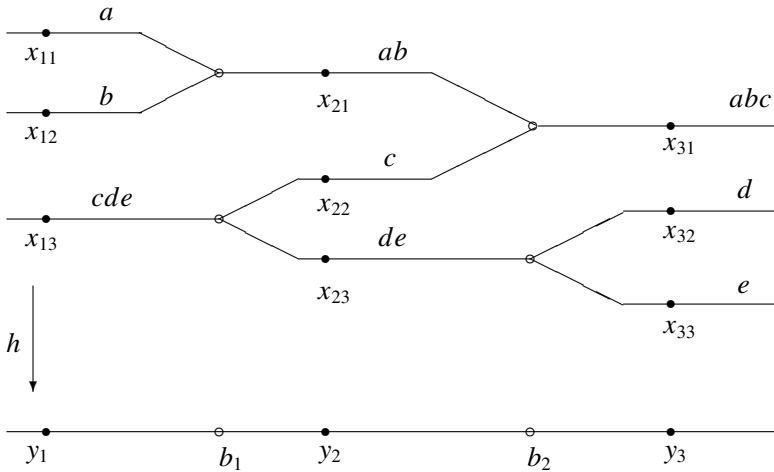
$$\pi_1 = \{A_1, A_2, \dots, A_p\}, \pi_2 = \{B_1, B_2, \dots, B_q\}$$

are called *adjacent* if each A_r is either a union of some of the B 's or is a subset of one of the B 's; this relation is symmetric. [For example, if $S = \{u, v, w\}$ then the two partitions $\{u\}, \{v, w\}$ and $\{u, v\}, \{w\}$ are not adjacent but $\{u\}, \{v, w\}$ and $\{u, v, w\}$ are adjacent.] A map $\phi: [k + 1] \rightarrow P_{[n]}$ is *compatible* if the set partitions $\phi(r), \phi(r + 1)$ are adjacent for $1 \leq r \leq k$.

Let $0 = b_0 < b_1 < b_2 < \dots < b_k < 1 = b_{k+1}$ be a dissection of $I = [0, 1]$. Given a compatible ϕ we can construct an n -branched covering of I that is branched over the points $b_r, 1 \leq r \leq k$ as follows:

Over the interval (b_{r-1}, b_r) there are $n(\phi(r))$ disjoint intervals each labeled by a part of the set partition $\phi(r)$. At the point b_r the two sets of $n(\phi(r))$ and $n(\phi(r + 1))$ intervals are joined according to the adjacency between $\phi(r)$ and $\phi(r + 1)$.

An illustration with $n = 5, k = 2$ and ‘branched’ over two points b_1, b_2 with $\phi(1) = \{a\}, \{b\}, \{cde\}, \phi(2) = \{a, b\}, \{c\}, \{d, e\}, \phi(3) = \{a, b, c\}, \{d\}, \{e\}$ is



The map t has the property that each point $x \in h^{-1}y$ appears in the multiset $t(y)$ with multiplicity equal to the size of the parts by which it is labeled, so for the branched covering of the diagram one has $t(y_1) = [x_{11}, x_{12}, x_{13}, x_{13}, x_{13}]$, $t(y_2) = [x_{21}, x_{21}, x_{22}, x_{23}, x_{23}]$ and $t(y_3) = [x_{31}, x_{31}, x_{31}, x_{32}, x_{33}]$. We note that, despite superficial appearances, this example can only be described as an n -branched covering for $n \geq 5$.

5. Frobenius n -homomorphisms and transfer maps

The aim of this section is to introduce the concept of an n -transfer for a ring homomorphism and to study their algebraic properties as special cases of Frobenius n -homomorphisms.

Definition 5.1. Let A, B be commutative, associative algebras and $f: A \rightarrow B$, a ring homomorphism, then a linear map $\tau: B \rightarrow A$ is an n -transfer for f if:

- (i) τ is a Frobenius n -homomorphism;
- (ii) $\tau(f(a)b) = a\tau(b)$, that is, τ is a map of A -modules and
- (iii) $f\tau: B \rightarrow B$ is the sum of the identity and a Frobenius $(n - 1)$ -homomorphism $g: B \rightarrow B$.

We denote the linear subspace $\{b \in B : g(b) = -b\}$ by L .

PROPOSITION 5.1. *If $f: A \rightarrow B$ is a ring homomorphism, $\tau: B \rightarrow A$ is an n -transfer for f and $g: B \rightarrow B$ is as above then:*

- (i) the composition $\tau f: A \rightarrow A$ is multiplication by n ;
- (ii) there is a split exact sequence of A -modules

$$0 \longrightarrow L \longrightarrow B \xrightarrow{\tau} A \longrightarrow 0;$$

- (iii) $gf = (n - 1)f$ and $\tau g = (n - 1)\tau$.

Proof. Taking $b = 1$ in Definition 5.1(ii) and because $\tau(1) = n$ the result is immediate.

The map τ is split by f/n . If $\tau(b) = 0$ then $0 = f\tau(b) = (1 + g)(b)$ by Definition 5.1(iii) and so the kernel of τ is identified with L .

The equations of (iii) are immediate consequences of the associativity of composition for $f\tau f$ and $\tau f\tau$.

Example 5.1. In the case where A, B are affine algebras, the relations in Proposition 5.1(iii) can be very useful. For example, when $A = B = \mathbb{C}[z]$ and $n = 2$, let $f(z) = p$ and $g(z) = q$ then $gf(z) = p(q(z))$ but this equals $p(z)$ and so g has degree 1.

Example 5.2. This is another application of Proposition 5.1 and we only consider the case $n = 2$. The splitting $B = L \oplus A$ is given by identifying the image of the monomorphism $f: A \rightarrow B$ with A . We show that $xy \in A$ for all $x, y \in L$. From the splitting we obtain $xy = a + z$ where $a \in A, z \in L$; since $g(\ell) = -\ell$ for all $\ell \in L$ and since $n = 2$, Proposition 5.1(iii) gives that $g(a) = a$. Using the fact that g is a ring homomorphism we get that $g(xy) = g(x)g(y) = (-x)(-y) = xy = a + z$ but also $g(xy) = g(a + z) = a - z$. Hence $z = 0$, showing that $xy \in L$.

PROPOSITION 5.2. *Let $f: A \rightarrow B, g: B \rightarrow C$ be ring homomorphisms and $\tau: B \rightarrow A, \sigma: C \rightarrow B$ be n - and m -transfers for f, g respectively. Then $\sigma\tau: C \rightarrow A$ is an nm -transfer for the composition $gf: A \rightarrow C$.*

Proof. We check the conditions of Definition 5.1. The first condition is Theorem 3.1 and the other two follow by a direct calculation.

THEOREM 5.1. *If A, B are algebras with no nilpotent elements and $\tau: B \rightarrow A$ is an n -transfer for the ring homomorphism $f: A \rightarrow B$, then $K_\tau = 0$.*

Proof. Take $b \in K_\tau$, then by Gugin's result mentioned at the end of Section 2 above, $\tau(b) = \tau(b^2) = \dots = \tau(b^n) = 0$. By Proposition 5.1(ii), $g(b^r) = -b^r$ for all $1 \leq r \leq n$.

But since g is a Frobenius $(n-1)$ -homomorphism, $\Phi_n(g)(b) = 0$ and by Definition 2.1 this is equivalent to the vanishing of the determinant of the matrix

$$\begin{pmatrix} -b & 1 & 0 & 0 & \dots & 0 \\ -b^2 & -b & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & & & & & \vdots \\ -b^{n-1} & -b^{n-2} & -b^{n-3} & \dots & -b & n-1 \\ -b^n & -b^{n-1} & -b^{n-2} & \dots & -b^2 & -b \end{pmatrix}.$$

By adding b times the second column and b^2 times the third column etc to the first column one sees that this determinant equals $(-1)^n n! b^n$ and since there are no nilpotent elements, we deduce that $b = 0$.

When X is a compact Hausdorff space, $C(X)$ will denote the algebra of continuous functions $X \rightarrow \mathbb{C}$ with the supremum norm.

Definition 5.2. The direct image $t_1: C(X) \rightarrow C(Y)$ associated with a continuous map $t: Y \rightarrow \text{Sym}^n(X)$ is defined by $(t_1\phi)(y) = \sum \phi(x_r)$ and $t(y) = [x_1, x_2, \dots, x_n]$.

THEOREM 5.2. *If X, Y are compact Hausdorff spaces, then the set of all continuous Frobenius n -homomorphisms $C(X) \rightarrow C(Y)$ can be identified with the space of continuous maps $Y \rightarrow \text{Sym}^n(X)$.*

Proof. The map t_1 is the sum of n ring homomorphisms and so is a Frobenius n -homomorphism.

Conversely, suppose that $f: C(X) \rightarrow C(Y)$ be a Frobenius n -homomorphism and let $\mathcal{E}_y: C(Y) \rightarrow \mathbb{C}$ be evaluation at the point $y \in Y$ then the composition $\mathcal{E}_y f$ is also a Frobenius n -homomorphism and so, by [1, theorem 4.1] corresponds to a multi-set $[x_1, x_2, \dots, x_n]$ in X . This defines the required map $t: Y \rightarrow \text{Sym}^n(X)$.

Remark. In the case $A = C(X)$, the map Δ_n (see Lemma 2.2) corresponds to the identity map on $\text{Sym}^n(X)$ regarded as an n -valued map from $\text{Sym}^n(X)$ to X . If $f: A = C(X) \rightarrow B = C(Y)$ is a Frobenius n -homomorphism, the ring homomorphism $\Phi_n(f)/n!$ corresponds to $t: Y \rightarrow \text{Sym}^n(X)$ (see Theorem 2.1).

Example 5.3. The linear map $C(X) \rightarrow C(X)$ given by $\phi \rightarrow n\phi$ is a Frobenius n -homomorphism and corresponds to the diagonal map $X \rightarrow \text{Sym}^n(X)$.

Example 5.4. If $s: Y \rightarrow \text{Sym}^m(X)$ and $t: Y \rightarrow \text{Sym}^n(X)$ give rise to the Frobenius m, n -homomorphisms $s_1, t_1: C(X) \rightarrow C(Y)$ then the composition

$$Y \longrightarrow \text{Sym}^m(X) \times \text{Sym}^n(X) \longrightarrow \text{Sym}^{m+n}(X)$$

corresponds to $s_1 + t_1: C(X) \rightarrow C(Y)$.

Example 5.5. If $s: Y \rightarrow \text{Sym}^n(X)$ and $t: Z \rightarrow \text{Sym}^m(Y)$ are continuous and $s_1: C(X) \rightarrow C(Y)$, $t_1: C(Y) \rightarrow C(Z)$ are the corresponding n - and m -Frobenius homomorphisms, then the composition

$$Z \longrightarrow \text{Sym}^m(Y) \longrightarrow \text{Sym}^m \text{Sym}^n(X) \longrightarrow \text{Sym}^{mn}(X)$$

corresponds to the composition $t_1 s_1: C(X) \rightarrow C(Z)$ which is a Frobenius mn -homomorphism.

6. Frobenius n -homomorphisms and n -branched coverings

The aim of this section is to characterise n -branched coverings in terms of rings of continuous functions and Frobenius n -homomorphisms.

A continuous map $h: X \rightarrow Y$ induces a ring homomorphism $h^*: C(Y) \rightarrow C(X)$. If h is an n -branched covering, then as above we have a direct image map $t_1: C(X) \rightarrow C(Y)$ which is a Frobenius n -homomorphism. We consider properties of t_1 which will ensure that h is such a covering.

In Example 4.2 of a finite group action on X , the third property of Definition 5.1 becomes very simple: Let $G = \{e = g_1, g_2, \dots, g_n\}$ be the group and $h: X \rightarrow Y = X/G$ is the map to the space of orbits, then the map $h^*t_1: C(X) \rightarrow C(X)$ corresponds geometrically to the map $X \rightarrow X \times \dots \times X \rightarrow X \times \text{Sym}^{n-1}(X)$ given by $x \rightarrow (x, g_2x, g_3x, \dots, g_nx) \rightarrow (x, [g_2x, g_3x, \dots, g_nx])$.

In Theorems 6.1 and 6.2, X, Y will denote compact Hausdorff spaces.

THEOREM 6.1. *Given an n -branched covering $h: X \rightarrow Y, t: Y \rightarrow \text{Sym}^n(X)$, the direct image $t_1: C(X) \rightarrow C(Y)$ is an n -transfer for the ring homomorphism $h^*: C(Y) \rightarrow C(X)$.*

Proof. We check three properties:

As noted in the proof of Theorem 5.2, t_1 is a Frobenius n -homomorphism.

By definition, one has that for $\phi \in C(X)$, $t_1(\phi)(y) = \phi(x_1) + \dots + \phi(x_n)$ where $t(y) = [x_1, x_2, \dots, x_n]$. Therefore

$$\begin{aligned} t_1(h^*(\psi)\phi)(y) &= \psi(h(x_1))\phi(x_1) + \dots + \psi(h(x_n))\phi(x_n) \\ &= \psi(y)\phi(x_1) + \dots + \psi(y)\phi(x_n) = \psi(y)t_1(\phi)(y). \end{aligned}$$

Hence $t_1(h^*(\psi)\phi) = \psi(y)t_1(\phi)$.

The third property follows immediately from Definition 4.1(ii).

The converse of Theorem 6.1 is

THEOREM 6.2. *Given a continuous map $h: X \rightarrow Y$ and a continuous n -transfer τ for $h^*: C(Y) \rightarrow C(X)$, then h is an n -branched covering.*

Proof. By the above, a continuous n -transfer $\tau: C(X) \rightarrow C(Y)$ corresponds to a continuous map $t: Y \rightarrow \text{Sym}^n(X)$ which is such that $th: X \rightarrow \text{Sym}^n(X)$ is the diagonal map and $\text{Sym}^n(h)t: Y \rightarrow \text{Sym}^n(X) \rightarrow \text{Sym}^n(Y)$ is of the form $y \rightarrow [y_1, y_2, \dots, y_n]$ with $y_1 = y$.

More generally, we see that the Frobenius n -homomorphism f corresponding to $t: Y \rightarrow \text{Sym}^n(X)$ is the sum of Frobenius n_1, n_2, \dots, n_k -homomorphisms f_1, f_2, \dots, f_k (where $n = n_1 + n_2 + \dots + n_k$) if and only if t factors as $Y \rightarrow \text{Sym}^{n_1}(X) \times \text{Sym}^{n_2}(X) \times \dots \times \text{Sym}^{n_k}(X) \rightarrow \text{Sym}^n(X)$ where the last map is concatenation.

Using [1, theorem 3.4 and corollary 3.6] which consider the relationship between Frobenius n -homomorphisms on affine algebras and symmetric powers of algebraic varieties, one can in a similar way prove.

THEOREM 6.3. *Let A, B be finitely generated commutative algebras and $f: A \rightarrow B$ a ring homomorphism; let $V(A), V(B)$ be the corresponding varieties and $h: V(B) \rightarrow V(A)$ the map corresponding to f . Then h is an n -fold branched covering if and only if there is an n -transfer $B \rightarrow A$, for h .*

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