On k-normality and Regularity of Normal Projective Toric Varieties

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Bach Le Tran*)

“Where words fail, music speaks.”
- Hans Christian Andersen

Tặng Ngoại và Mẹ

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Abstract

We study the relationship between geometric properties of toric varieties and combinatorial properties of the corresponding lattice polytopes. In particular, we give a bound for a very ample lattice polytope to be k-normal. Equivalently, we give a new combinatorial bound for the Castelnuovo-Mumford regularity of normal projective toric varieties. We also give a new combinatorial proof for a special case of Reider's Theorem for smooth toric surfaces.

Contents

List of Figures

Chapter 1

Introduction

This thesis contains the work I have been working on during my Ph.D. program at the University of Edinburgh. The main body of the thesis will be divided into three main parts, namely:

1. On k-normality of Very Ample Lattice Polytopes.
2. The Castelnuovo-Mumford Regularity of Projective Toric Varieties.
3. Reider's Theorem for Smooth Toric Surfaces.

1.1 On k-normality and Regularity of Normal Projective Toric Varieties

Let L be a very ample line bundle on an irreducible projective variety X defining an embedding $X \to \mathbb{P}(\mathrm{H}^0(X, L)) \cong \mathbb{P}^r$. We say that (the embedding of) X is k-normal if the restriction map

$$\mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to \mathrm{H}^0(X, \mathcal{O}_X(k))$$

is surjective. We define the k-normality of X to be the smallest positive integer k_X such that X is k-normal for all $k \geq k_X$. The k-normality of X is closely related to its Castelnuovo-Mumford regularity. In particular, X is $(k+1)$-regular if and only if X is k-normal and $\mathcal{O}_X$ is k-regular (Proposition 4.1.4).

Now suppose that X is a normal projective toric variety and let L be a very ample line bundle on X. Then $L = \mathcal{O}_X(D)$ for some torus invariant divisor D. Hence, L corresponds to a lattice polytope $P = P_D$. We say that P is k-normal if the map

$$\underbrace{P \cap M + \cdots + P \cap M}_{k \text{ times}} \to kP \cap M$$

is surjective. We also define the k-normality of P to be the smallest positive integer k_P such that P is k-normal for all $k \geq k_P$. Then X is k-normal if and only if P is k-normal. Hence, k_X and k_P coincide.

In this part, we will give a new combinatorial bound of the k-normality of very ample lattice polytopes. First of all, define d_P to be the smallest positive integer such that the map

$$P \cap M + kP \cap M \to (k+1)P \cap M$$

is surjective for all $k \geq d_P$. Such a d_P always exists by Lemma 2.2.7. Since P is very ample, for every vertex $v \in P$, the semigroup $\mathbb{R}_{\geq 0}(P - v) \cap M$ is generated by $(P - v) \cap M$. Thus, for any lattice point $x \in d_P \cdot P \cap M$ and vertex $d_P \cdot v$ of $d_P \cdot P$, we have

$$x - d_P \cdot v = \sum_{i=1}^{m} (w_i - v),$$

for some $m < +\infty$, $w_i \in P \cap M$. For such a pair $(x, d_P v)$, we define

$$\sigma(x, d_P v) = \min \left\{ m \in \mathbb{N} \;\middle|\; x - d_P v = \sum_{i=1}^{m} (w_i - v) \text{ for some } w_i \in P \cap M \right\}.$$

Let

$$m_P = \max \left\{ \sigma(x, d_P v) \;\middle|\; x \in (d_P P) \cap M,\ v \text{ a vertex of } P \right\}.$$

We now state the most important corollary of our main result, Theorem 3.2.1, as follows.

Corollary 1.1.1. *Suppose that P is a very ample lattice polytope that has n vertices. Then*

$$k_P \leq (m_P - d_P) \cdot n + 1.$$

It is then natural to ask for an upper bound of m_P. If P is a smooth polytope, we obtained the following result.

Corollary 1.1.2. *Let P be a smooth d-dimensional lattice polytope (cf. Definition 2.4.2) that has n vertices. Let γ be the smallest integer such that $P \subseteq C_{v,\gamma} := \operatorname{conv}(v, v + \gamma \cdot (w_{E_1} - v), \dots, v + \gamma \cdot (w_{E_d} - v))$ for any vertex $v \in P$, where the $(w_{E_i} - v)$'s are the primitive ray generators of the edges of P coming from v. Then P is k-normal for all*

$$k \geq (\gamma - 1) \cdot (d - 1) \cdot n + 1.$$

Finding the explicit value of k_P is a really hard question in general. Beck et. al. ([BDGM15]) showed that the k-normality of the polytope in Example 4.3.1 is $s - 1$. With their conventions, we have $k_P = \gamma(P) + 1$ where $\gamma(P)$ is the largest height that contains gaps in M_P. There are also results by Higashitani ([Hig14]), Lasoń and Michałek ([LM17]) that give k_P for some classes of lattice polytopes. Oda ([Oda08]) asked if P is smooth, is it always the case that $k_P = 1$? Despite the simple statement, it is still an open question at the time of writing. For bounds of k-normality, Ogata ([Oga05, Theorem 2]) proved that any projective toric variety of dimension $n \geq 4$ which is a quotient of the projective n-space by a finite abelian group embedded by a very ample line bundle in $\mathbb{P}^r$ is k-normal for every $k \geq n - 1 + [n/2]$. Equivalently, any n-dimensional very ample lattice simplex is k-normal for $k \geq n - 1 + [n/2]$.

For some potential applications of k-normality, it would be interesting to study the k-normality of toric degenerations of non-toric varieties, toric Fano varieties with arbitrary polarizations, hypersurfaces in toric varieties. Furthermore, it would be nice to understand the behavior of k-normality under deformations of some families of (Fano) varieties.

1.2 The Castelnuovo-Mumford Regularity of Projective Toric Varieties

The main motivation for the study of k-normality is its relation to the Castelnuovo - Mumford regularity, an important invariant in algebraic geometry. First of all, the regularity measures the complexity of the ideal sheaf $\mathcal{I}_X$ from the perspective of free resolutions and gives a bound for the maximal degree of the defining equations of projective varieties. It also gives a bound of complexity for algorithms calculating minimal free resolution of ideals generated by finitely many homogeneous polynomials ([MM82, Buc83]). There has been a big focus on finding upper bounds for the Castelnuovo-Mumford regularity of varieties in general. Mumford ([BM92]) proved that if $X \subset \mathbb{P}^r$ is a reduced smooth subscheme purely of dimension d in characteristics 0, then $\operatorname{reg}(X) \leq (d + 1)(\deg(X) - 2)) + 2$. Kwak ([Kwa98]) proved that if X is a smooth variety of dimension d in $\mathbb{P}^r$ then $\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 2$ if $d = 3$ and $\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 5$ if $d = 4$. Recently Kwak and Park ([KP14]) obtained an upper bound for the regularity of non-degenerate smooth projective varieties; however, it is very hard to find explicit bounds for particular cases.

For toric varieties, Peeva and Sturmfels proved that for a projective toric variety X of codimension 2 in $\mathbb{P}^{d-1}$, not contained in any hyperplane, $\operatorname{reg}(X) \leq \deg(X) - 1$ ([PS98], [Stu95, Theorem 4.2]). Sturmfels also proved that if X is a projective toric variety in $\mathbb{P}^{d-1}$ then $\operatorname{reg}(X) \leq d \cdot \deg(X) \cdot \operatorname{codim}(X)$ ([Stu95, Theorem 4.5]).

The most well-known question in finding upper bounds for the regularity of projective varieties is the Eisenbud-Goto ([EG84]) conjecture which says that if X is irreducible and reduced then

$$\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 1.$$

Even though the conjecture fails in general ([MP17]), it has motivated many results on regularity. In some particular cases, the Eisenbud-Goto conjecture is proven to be true: smooth surfaces in characteristic zero ([Laz87]), connected reduced curves ([Gia05]), etc. Furthermore, Eisenbud and Goto ([EG84]) proved that their conjecture holds when X is arithmetically Cohen-Macaulay. Therefore, it holds for projectively normal toric varieties since they are arithmetically Cohen-Macaulay ([Hoc72]). For a more detailed list of cases where the Eisenbud-Goto conjecture holds, refer to [Nit14]. Note that the Eisenbud-Goto conjecture is still open for toric varieties.

Combinatorially, for a normal projective toric variety X embedded in $\mathbb{P}^r$ via a very ample line bundle whose corresponding lattice polytope is P, $\deg(X) = \operatorname{Vol}(P)$, the normalized volume of P, and $\operatorname{codim}(X) = |P \cap M| - \dim P - 1$. We define the degree of P, denoted by $\deg(P)$, as follows. If P has no interior lattice points, let $\deg(P)$ be the smallest non-negative integer i such that kP contains no interior lattice points for $1 \leq k \leq d - i$. If P has interior lattice point(s) then we define $\deg P = d$. By Proposition 4.1.7, we have $\operatorname{reg}(X) = \max\{k_P, \deg(P)\} + 1$. Hence, the Eisenbud-Goto conjecture can be translated as if

$$\max\{k_P, \deg(P)\} \leq \operatorname{Vol}(P) - |P \cap M| + \dim P + 1.$$

Note that by [HKN17, Proposition 2.2],

$$\deg(P) \leq \operatorname{Vol}(P) - |P \cap M| + \dim P + 1.$$

Hence, it remains to verify if

$$k_P \leq \operatorname{Vol}(P) - |P \cap M| + \dim P + 1.$$

In particular, if $k_P \leq \deg(P)$ then the Eisenbud-Goto conjecture holds in this case. Our bound in Theorem 3.2.1 proves the conjecture for cases; in particular, the case $s = 4$ in Example 4.3.1.

1.3 Reider-type Theorem for Toric Surfaces

The problem of determining whether a line bundle is nef or (very) ample is an important question in algebraic geometry. The Nakai-Moishezon criterion ([Nak63, Moi64]) states that a Cartier divisor D on a proper scheme X over an algebraically closed field is ample if and only if $D^{\dim(Y)} \cdot Y > 0$ for every closed integral subscheme Y of X. For toric varieties, a special form of the criterion holds: if $D \cdot C > 0$ for every torus-invariant curve $C \subset X$ then D is ample. Furthermore, if $D \cdot C \geq 0$ for every torus-invariant curve $C \subset X$ then D is globally generated ([Lat96, Mav00, Mus02]). However, the question is more complicated when we consider the adjoint bundle $D + K_X$. Namely, are there numerical conditions for $D \cdot C$ so that $D + K_X$ is globally generated or ample? Fujita conjectured the following:

Conjecture 1.3.1 ([Fuj85])**.** *Let X be an n-dimensional projective algebraic variety, smooth or with mild singularities, and D an ample divisor on X. Then*

(1) For $t \geq n+1$, $tD + K_X$ is basepoint free.

(2) For $t \geq n+2$, $tD + K_X$ is very ample.

The conjecture is true for toric varieties ([Fuj03, Pay06]). For smooth surfaces, Fujita's conjecture follows from Reider's theorem ([Rei88]) as follows.

Theorem 1.3.2 ([Rei88])**.** *Let L be a nef line bundle on a smooth projective surface X. Let K_X be the canonical divisor of X.*

1. *If $L^2 \geq 5$ and p is a base point of $|K_X + L|$, then there exists an effective divisor E passing through p such that*

$$L \cdot E = 0, E^2 = -1, \text{ or}$$
$$L \cdot E = 1, E^2 = 0.$$

2. *If $L^2 \geq 10$ and p, q are two points of X (possibly infinitely near) which fail to be separated by $|K_X + L|$, then there exists an effective divisor E passing through p and q such that*

$$\text{either } L \cdot E = 0 \text{ and } E^2 = -1 \text{ or } -2,$$
$$\text{or } L \cdot E = 1 \text{ and } E^2 = 0 \text{ or } -1,$$
$$\text{or } L \cdot E = 2 \text{ and } E^2 = 0.$$

In Chapter 5, we will present a combinatorial proof for a special case of the theorem when X is a smooth projective toric surface and L is ample.

Chapter 2

Preliminaries

2.1 Toric Varieties

Let K be an algebraically closed field. The affine variety $(K^*)^n$ is a group under componentwise multiplication. We define a torus T to be an affine variety isomorphic to $(K^*)^n$ such that T has a group structure induced from that of $(K^*)^n$.

Definition 2.1.1. A toric variety is a normal algebraic variety V containing a torus $T_N \cong (K^*)^n$ as a dense Zariski open subset, such that the action of T_N on itself can be extended to an action of T_N on V; i.e., $T_N \times V \to V$ is given by a morphism.

Example 2.1.2.

- $X = Z(x^3 - y^2) \subset \mathbb{C}^2$ is a toric variety with the torus
$$X \cap \mathbb{C}^2 = \{(t^2, t^3) | t \in \mathbb{C}^*\} \simeq \mathbb{C}^*.$$
- $X = Z(xy - zw) \subset \mathbb{C}^4$ is a toric variety with the torus
$$X \cap \mathbb{C}^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1}) | t_1, t_2, t_3 \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^3.$$

Definition 2.1.3. Let T be a torus. A character of T is a morphism $\chi : T \to K^*$ that is a group homomorphism.

Let $N \cong \mathbb{Z}^n$ be a lattice and $M := \mathrm{Hom}(N, \mathbb{Z})$ be its dual lattice. Let $T \cong (K^*)^n$ be a torus. Then an element $u = (u_1, \dots, u_n) \in M$ gives a character $\chi^u : T \to K[T] = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by
$$(u_1, \dots, u_n) \mapsto x_1^{u_1} \cdots x_n^{u_n}.$$

Definition 2.1.4. Let $N_{\mathbb{R}} := N \otimes \mathbb{R}$ and $S \subseteq N_{\mathbb{R}}$ a finite subset. A convex polyhedral cone in $N_{\mathbb{R}}$ is a set of the form
$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u u \;\middle|\; \lambda_u \geq 0 \right\} \subseteq N_{\mathbb{R}}.$$
As a convention, $\mathrm{Cone}(\emptyset) := \{0\}$. We say that σ is generated by S. If $S \subseteq N$, we say that σ is a rational cone.

Definition 2.1.5. Let $M_{\mathbb{R}} = M \otimes \mathbb{R}$. The dual cone of a cone $\sigma \subseteq N_{\mathbb{R}}$, denoted by $\sigma^\vee$, is defined by
$$\sigma^\vee = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Example 2.1.6. Take $N = \mathbb{Z}^2$. The cone $\sigma = \mathrm{conv}(e_2, e_1 - 2e_2)$ and its dual cone $\sigma^\vee$ are given by the following figure:

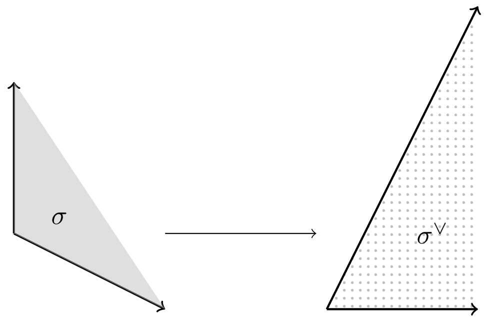

Figure 2.1: $\mathrm{conv}(e_2, e_1 - 2e_2)$ and its dual cone

Definition 2.1.7. For any $0 \neq m \in M_\mathbb{R}$, we have a hyperplane

$$H_m = \{u \in N_\mathbb{R} \mid \langle m, u\rangle = 0\} \subseteq N_\mathbb{R}.$$

Let $\tau, \sigma \subseteq N_\mathbb{R}$ be convex cones. We say that τ is a face of σ, denoted by $\tau \preceq \sigma$, if $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$.

Definition 2.1.8. A fan $\Sigma \subset N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ such that

1. Any $\sigma \in \Sigma$ is a convex rational cone.
2. For any $\sigma \in \Sigma$, its faces are also in Σ.
3. For all σ_1, σ_2 in Σ, the intersection $\sigma_1 \cap \sigma_2$ is a face of each cone.

Define the support of Σ by

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_\mathbb{R}.$$

We say that the fan Σ is complete if $|\Sigma| = N_\mathbb{R}$.

Now let $S_\sigma = \sigma^\vee \cap M$. By Gordan's lemma ([Ful93, Page 12]), S_σ is a finitely generated semigroup. Then $K[S_\sigma] = \{\chi^u \mid u \in S_\sigma\}$ is a finitely generated K-algebra. We obtain an affine algebraic variety

$$U_\sigma = \mathrm{Spec}(K[S_\sigma])$$

of dimension n. Suppose now that $\Sigma \subseteq N_\mathbb{R}$ is a rational polyhedral fan. For any cones σ_1 and σ_2 in Σ, let $\tau = \sigma_1 \cap \sigma_2$. Then $\tau = \sigma_1 \cap H_m = \sigma_2 \cap H_m$ for some $m \in (\sigma_1)^\vee \cap (-\sigma_2)^\vee \cap M$. We have an isomorphism

$$g_{\sigma_1,\sigma_2} : (U_{\sigma_1})_{\chi^m} \simeq (U_{\sigma_2})_{\chi^{-m}},$$

which is the identity in U_τ. Hence, we can glue the U_σ with $\sigma \in \Sigma$ to obtain a toric variety $X(\sigma)$. In fact, all toric varieties arise this way ([CLS11, Corollary 3.1.8]).

Theorem 2.1.9 ([CLS11, Theorem 3.4.6])**.** *X_Σ is a complete variety if and only if the fan Σ is complete.*

Since all toric varieties in this thesis are projective, they are all complete. Therefore, as a convention, from now on all fans in this thesis will be complete unless mentioned otherwise.

2.2 Lattice Polytopes

In this section, we will quickly review some basic material on lattice polytopes and then prove some important lemmas for our main result in Chapter 3.

2.2.1 Polytopes

Let $M \cong \mathbb{Z}^n$ be a lattice and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. We define a polytope in $M_\mathbb{R}$ as follows.

Definition 2.2.1. A polytope $P \subseteq M_\mathbb{R}$ is the convex hull of a finite set $S = \{v_1, \ldots, v_s\} \subseteq M_\mathbb{R}$ given by

$$\operatorname{conv}(S) := \left\{ \sum_{i=1}^{s} \lambda_i v_i \;\middle|\; \sum_{i=1}^{s} \lambda_i = 1,\ \lambda_i \geq 0 \text{ for all } 1 \leq i \leq s \right\}.$$

The dimension of a polytope $P \subseteq M_\mathbb{R}$ is the dimension of the smallest affine subspace of $M_\mathbb{R}$ containing P. Given a non-zero vector $u \in N_\mathbb{R}$ and $b \in \mathbb{R}$, we can define a affine hyperlane $H_{u,b}$ and a closed half-space $H_{u,b}^+$ by

$$H_{u,b} = \{m \in M_\mathbb{R} \mid \langle m, u\rangle = b\} \text{ and } H_{u,b}^+ = \{m \in M_\mathbb{R} \mid \langle m, u\rangle \geq b\}. \tag{2.1}$$

A subset $Q \subseteq P$ is a face of P if there exist $u \in N_\mathbb{R} \backslash \{0\}$ and $b \in \mathbb{R}$ such that

$$Q = H_{u,b} \cap P \text{ and } P \subseteq H_{u,b}^+.$$

Definition 2.2.2. Let P be a polytope. We define the vertices and the edges of P to be the faces of dimension 0 and 1, respectively.

Proposition 2.2.3 ([CLS11, Proposition 2.2.1])**.** *Let $P \subset M_\mathbb{R}$ be a polytope. Then*

1. *P is the convex hull of its vertices.*
2. *If $P = \operatorname{conv}(S)$ then every vertex of P lies in S.*

Definition 2.2.4. Let P be a polytope. We denote by P^0 and ∂P the set of interior and boundary points of P, respectively.

2.2.2 Lattice Polytopes

Definition 2.2.5. A polytope $P \subseteq M_\mathbb{R}$ is called a lattice polytope if all of its vertices are lattice points in M. A lattice polytope P is empty if it has no lattice points except its vertices.

We define the Minkowski sum of two polytopes as follows.

Definition 2.2.6. Suppose that $P \subseteq M_\mathbb{R}$ and $Q \subseteq M_\mathbb{R}$ are two polytopes, then their Minkowski sum is defined to be

$$P + Q := \{x + y \mid x \in P, y \in Q\}.$$

For $c \in \mathbb{R}_{>0}$, we define

$$cP := \{cx \mid x \in P\}.$$

Lemma 2.2.7. *Let P be a d-dimensional lattice polytope that has n vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$.*

(a) For any $k \geq n-1$,

$$(k+1)P \cap M = \mathcal{V} + kP \cap M.$$

(b) [EW91, LTZ93, BGT97] For $k \geq d-1$, we have

$$(k+1)P \cap M = P \cap M + kP \cap M.$$

Proof. We follow the argument in [EW91] to give a proof for (a). Let x be a lattice point in $(k+1)P \cap M$. Then $x = \sum_{i=1}^n \lambda_i v_i$ for some $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = k+1$. Since $k+1 \geq n$, there must be an i such that $\lambda_i \geq 1$. Then

$$x = v_i + \left(\sum_{\substack{1 \leq j \leq n \\ j \neq i}} \lambda_j v_j + (\lambda_i - 1) v_i \right),$$

where $\left(\sum_{\substack{1\le j\le n\\ j\neq i}} \lambda_j v_j + (\lambda_i - 1)v_i\right) \in kP \cap M$. The conclusion follows. □

From the above lemma, we obtain two well-defined invariants of P as follows.

Definition 2.2.8. Let P be a lattice polytope whose set of vertices is $\mathcal{V} = \{v_1, \dots, v_n\}$. We define d_P to be the smallest positive integer such that for every $k \ge d_P$,

$$(k+1)P \cap M = P \cap M + kP \cap M.$$

We also define ν_P to be the smallest positive integer such that for any $k \ge \nu_P$,

$$(k+1)P \cap M = \mathcal{V} + kP \cap M.$$

Example 2.2.9. Let P_C be the convex polytope given by

$$P_C := \operatorname{conv}(0, e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4 + 2e_5) \subset \mathbb{R}^5.$$

The only lattice points of P_C are its vertices. We have $P_C \cap M + P_C \cap M = 2P_C \cap M$. However, the point $(1, 1, 1, 1, 1)$ lies in $3P_C \cap M$ but not in $2P_C \cap M + P_C \cap M$. Hence, by Lemma 2.2.7, $d_{P_C} = 4$.

It is clear from the definitions that $d_P \le \nu_P \le n - 1$. Also, if P is an empty lattice polytope then $\nu_P = d_P$. In general, this is not true.

Example 2.2.10. Let P be the polygon given by $P = \operatorname{conv}(0, e_1, e_2, 3e_1 + e_2)$.

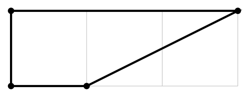

Figure 2.2: An example of P in which $d_P \neq \nu_P$

Then we can check directly that $d_P = 1$ while $\nu_P = 2$, and $(5, 3)$ is the only point in $2P \cap M$ that cannot be written as a sum of a point in $P \cap M$ and a vertex of P.

For a lattice polytope in dimension 2, we have the following result:

Lemma 2.2.11 ([Ark80, Lemma 1])**.** *Every lattice polygon with at least 5 edges has at least an interior lattice point.*

Proof. Suppose that our polygon contains a pentagon P whose vertices are $v_1, \dots, v_5$. Since the sum of any two adjacent angles of P cannot all be less than or equal to π, there must be two adjacent angles such that their sum is greater than π. Without loss of generality, we assume that $\angle v_2 + \angle v_3 > \pi$ and that the Euclidean distance from v_1 is not bigger than the distance from v_4 to the edge v_2v_3. Then the ray from v_3 parallel with v_1v_2 and the ray from v_1 parallel with v_2v_3 meet at a point x inside our polygon P. Then x is a lattice point in the interior of P. The conclusion follows.

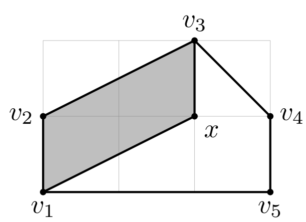

Figure 2.3: A lattice pentagon

□

2.2.3 Volume of Polytopes

We define the normalized volume of a polytope as follows.

Definition 2.2.12. Let P be a lattice polytope of dimension d. The normalized volume of P, denoted by $\mathrm{Vol}(P)$, is defined to be

$$\mathrm{Vol}(P) = d! \cdot \{\text{Euclidean volume of } P\}.$$

The following classical lemma gives a straightforward way to calculate the normalized volume of any lattice polytope given the coordinates of its vertices.

Lemma 2.2.13. *Let P be a d-simplex whose vertices are $\{v_0, \dots, v_d\}$. Then the normalized volume of P is given by*

$$\mathrm{Vol}(P) = |\det(v_1 - v_0, \dots, v_d - v_0)|.$$

Furthermore, we have an equation between the self-intersection number of a divisor D on a toric variety X with the normalized volume of the corresponding polytope P_D as follows.

Lemma 2.2.14 ([Ful93, Page 111])**.** *Let P_D be the polytope corresponding to a divisor D on a toric variety X of dimension d. Then*

$$\mathrm{Vol}(P_D) = D^d.$$

2.2.4 Ehrhart's Theory of Lattice Polytopes

For any lattice polytope $P \subset M_{\mathbb{R}}$, we define the Ehrhart function $\mathrm{ehr}_P(k) = |kP \cap M|$, the number of lattice points in kP. The Ehrhart series of P is defined by

$$\mathrm{Ehr}_P(t) = 1 + \sum_{k=1}^{\infty} \mathrm{ehr}_P(k) t^k.$$

By [Ehr62] and [Sta80, Theorem 2.1],

$$\mathrm{Ehr}_P(t) = \frac{h_P^*(t)}{(1-t)^{d+1}}$$

for some polynomial $h_P^* \in \mathbb{Z}_{\geq 0}[t]$ of degree less than or equal to d. Let $h_P^*(t) = \sum_{i=0}^{d} h_i^* t^i$. We call $h_P^* = (h_0^*, \dots, h_d^*)$ the Ehrhart delta vector of P and have

$$\begin{aligned}
h_0^* &= 1, \\
h_1^* &= |P \cap M| - d - 1, \\
h_h^* &= |P^0 \cap M|, \\
\sum_{i=0}^{d} h_i^* &= \mathrm{Vol}(P).
\end{aligned}$$

We can define the degree of a lattice polytope P as follows.

Definition 2.2.15 ([Bat07, Remark 2.6])**.** Let P be a lattice polytope of dimension d. We define the degree of P, denoted by $\deg(P)$, to be the degree of $h_P^*(t)$. Equivalently,

$$\deg(P) = \begin{cases} d & \text{if } |P^0 \cap M| \neq 0. \\ \min\left\{ i \in \mathbb{Z}_{\geq 0} | (kP)^0 \cap M = \emptyset \text{ for all } 1 \leq k \leq d - i \right\} & \text{otherwise.} \end{cases}$$

Proposition 2.2.16 ([Sta93, Theorem 3.3])**.** *Let $Q \subseteq P$ be lattice polytope with Erhart delta vectors h_Q^* and h_P^*, respectively. Then for all $0 \leq i \leq \dim P$,*

$$h_{Q,i}^* \leq h_{P,i}^*.$$

As a consequence, if $Q \subseteq P$ then $\deg(Q) \leq \deg(P)$.

Lemma 2.2.17. *Let $Q \subseteq P$ be lattice polytopes. Then $\deg(P) \geq \deg(Q)$.*

Proof. Let $h^*_P = \left(h^*_{P,0}, \ldots, h^*_{P,\deg(P)}\right)$ and $h^*_Q = \left(h^*_{Q,0}, \ldots, h^*_{Q,\deg(Q)}\right)$ be their h^*-vectors, respectively. By [Sta93, Theorem 3.3], $h^*_{P,i} \geq h^*_{Q,i}$ for all $0 \leq i \leq \deg P$. In particular,

$$h^*_{P,\deg Q} \geq h^*_{Q,\deg Q} > 0.$$

Therefore, $\deg(P) \geq \deg(Q)$. □

2.3 The Correspondence Between Lattice Polytopes and Toric Varieties

Let $X = X(\Sigma)$ be a toric variety corresponding to a fan $\Sigma \subseteq N_{\mathbb{R}}$ of rank n. Let $\Sigma(1)$ be the set of rays of Σ and for every $\rho \in \Sigma(1)$ let u_ρ be the first lattice point along ρ_i. Then each $\rho \in \Sigma(1)$ determines a prime divisor $D_\rho = V(\rho)$ on X, where $V(\rho)$ is the Zariski closure of $\mathrm{Spec}(K[\rho^\perp \cap M])$ in X (cf. [CLS11, Section 4.1]). We have that the D_i are invariant under the torus action.

Proposition 2.3.1 ([CLS11, Proposition 4.1.2])**.** *Let M be a lattice and $\Sigma \subseteq N_{\mathbb{R}}$ a fan. Let X_Σ be the toric variety given by Σ. For any $m \in M$, the character χ^m is a rational function on X_Σ, and its divisor is given by*

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho,$$

where u_ρ is the minimal generator of the ray $\rho \in \Sigma(1)$.

By [CLS11, Theorem 4.1.3], the D_i generate the class group $\mathrm{Cl}(X)$, so any divisor D on X can be represented as $D \sim \sum_\rho a_\rho D_\rho$ where $\rho \in \Sigma(1)$. Now consider a torus invariant divisor $D = \sum_\rho a_\rho D_\rho$ on X. By [CLS11, Proposition 4.3.8], we can define a convex polytope P_D by

$$P_D := \left\{ u \in M_{\mathbb{R}} \mid \langle u, u_\rho \rangle \geq -a_\rho \right\}.$$

Then by [CLS11, Proposition 4.3.3],

$$\Gamma(X, \mathcal{O}_X(D)) = \bigoplus_{u \in P_D \cap M} K \cdot \chi^u. \tag{2.2}$$

Conversely, let $P \subseteq M_R$ be a lattice polytope with the facet presentation

$$P = \left\{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \prec P \right\}.$$

We construct the fan Σ_P whose cones are in inclusion reversing correspondence with the faces of P as follows. For any vertex v of P, we have a cone $\mathrm{Cone}(P \cap M - v) \subseteq M_{\mathbb{R}}$, which gives us the dual cone $\sigma_v = \mathrm{Cone}(P \cap M - v)^\vee \subseteq N_{\mathbb{R}}$. For a facet $Q \prec P$, we define $\sigma_Q = \mathrm{Cone}(u_F | F \text{ contains } Q)$, and

$$\Sigma_P = \left\{ \sigma_Q \mid Q \preceq P \right\}.$$

By [CLS11, Theorem 2.3.2] Σ_P is a fan, which we call the normal fan of P. The ray generators of the normal fan Σ_P are the facet normals u_F. We denote by D_F their corresponding prime divisors. From [CLS11, Proposition 2.3.7], if Q, R are two faces of P, then $Q \preceq R$ if and only if $\sigma_R \subseteq \sigma_Q$. Now we can define a toric variety $X_P = X_{\Sigma_P}$ and a divisor

$$D_P = \sum_{F \text{ facet of } P} a_F D_F.$$

The above constructions give us bijections between the set of polytopes

$$\{P \subseteq M_{\mathbb{R}} \mid P \text{ is a full dimensional lattice polytope}\}$$

and the set of pairs

$$\{(X_\Sigma, D) \mid \Sigma \text{ is a complete fan in } N_{\mathbb{R}}, D \text{ a torus-invariant ample divisor on } X_\Sigma\}$$

by [CLS11, Theorem 6.2.1].

Example 2.3.2 ([CLS11, Example 2.3.16]). Let $P_{a,b}$ be the polygon given by

$$P_{a,b} = \operatorname{conv}(0, ae_1, e_2, be_1 + e_2) \subseteq \mathbb{R}^2,$$

where $a, b \in \mathbb{N}$ and $1 \leq a \leq b$. Let $r = b - a$.

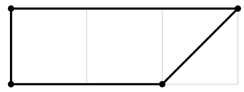

Figure 2.4: The polygon of a Hirzebruch surface

Then the image of the normal fan $\Sigma = \Sigma_{P_{a,b}}$ is given by Figure 2.3.2. The corresponding toric variety X_Σ of P is obtained by gluing the open affine subsets

$$\begin{aligned}
U_{\sigma_1} &= \operatorname{Spec}(\mathbb{C}[x, y]) \cong \mathbb{C}^2 \\
U_{\sigma_2} &= \operatorname{Spec}(\mathbb{C}[x, y^{-1}]) \cong \mathbb{C}^2 \\
U_{\sigma_3} &= \operatorname{Spec}(\mathbb{C}[x^{-1}, x^{-r}y^{-1}]) \cong \mathbb{C}^2 \\
U_{\sigma_4} &= \operatorname{Spec}(\mathbb{C}[x^{-1}, x^{r}y]) \cong \mathbb{C}^2.
\end{aligned}$$

We call X_Σ the Hirzebruch surface and denote it by $\mathscr{F}_r$. We have $\mathscr{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and for $r \geq 1$, $\mathscr{F}_r \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(r))$ [CLS11, Example 7.3.4].

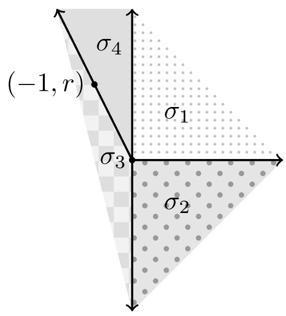

Figure 2.5: The Hirzebruch fan

2.4 Very Ample Lattice Polytopes and k-normality

We start this section with the definition of very ample lattice polytopes.

Definition 2.4.1. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is very ample if for every vertex $v \in P$, the semigroup $S_{P,v} = \mathbb{N}(P \cap M - v)$ generated by the set

$$P \cap M - v = \{u - v | u \in P \cap M\}$$

is saturated in M; i.e., if $c \cdot u \in S_{P,v}$ for some $c \in \mathbb{N}^*$ then $u \in S_{P,v}$. Equivalently, P is very ample if the affine monoid $\mathbb{R}_{\geq 0}(P - v) \cap M$ is generated by the set $P \cap M - v$.

Definition 2.4.2. We say that P is smooth if for every vertex v, the set $w_E - v$, where E is an edge of P containing v and w_E is the first lattice point of E different from v encountered as one transverses E starting at v, form a subset of a basis of M.

Example 2.4.3. The standard 3-simplex $P = \operatorname{conv}(0, e_1, e_2, e_3)$ is very ample. For example,

$$P \cap M - e_1 = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

generates $\mathbb{R}_{\geq 0}(P - e_1) \cap M$.

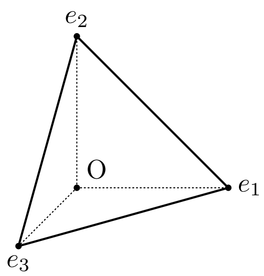

Figure 2.6: The standard 3-simplex

Example 2.4.4. The polytope $P = \operatorname{conv}(0, e_1, e_2, e_1 + 2e_2 + 3e_3) \subset \mathbb{R}^3$ is not very ample.

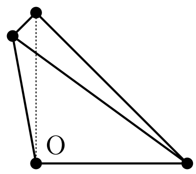

Figure 2.7: $P = \operatorname{conv}(0, e_1, e_2, e_1 + 2e_2 + 3e_3)$

Definition 2.4.5. A lattice polytope P is called k-normal if for a fixed $k \in \mathbb{Z}_{\geq 1}$, the map

$$\underbrace{(P \cap M) + \cdots + (P \cap M)}_{k \text{ times}} \to kP \cap M$$

is surjective. In other words, every lattice point in kP can be written as a sum of k lattice points in P. We also define its k-normality, denoted by k_P, to be the minimum number such that P is k-normal for all $k \geq k_P$; i.e.,

$$k_P = \min\{l \in \mathbb{Z}_{\geq 1} \mid P \text{ is } k\text{-normal for all } k \geq l\}.$$

The polytope P is normal if P is k-normal for every $k \geq 2$.

Example 2.4.6. Consider the polytope P_C given in Example 2.2.9. Then P_C is 2-normal but not 3-normal. Furthermore, $3P_C$ is a very ample and normal polytope.

By the example above and as noted in the introduction, P is k-normal does not imply that P is also $(k+1)$-normal. However, if P is k-normal for some $k \geq d_P$ then it is $(k+1)$-normal.

Lemma 2.4.7. *Let P be a lattice polytope. If P is k-normal for some positive integer $k \geq d_P$ then P is $(k+1)$-normal.*

Proof. Suppose that P is k-normal; i.e.,

$$\underbrace{P \cap M + \cdots + P \cap M}_{k} \twoheadrightarrow kP \cap M,$$

then by the definition of d_P and since $k \geq d_P$, we have

$$\underbrace{P \cap M + \cdots + P \cap M}_{k} + P \cap M \twoheadrightarrow kP \cap M + P \cap M \twoheadrightarrow (k+1)P \cap M.$$

In other words, P is $(k+1)$-normal. $\square$

Lemma 2.4.8 ([LTZ93, EW91, BGT97])**.** *Let P be a lattice polytope of dimension d. Then $(d-1)P$ is normal.*

Proof. For any $k \geq 1$,

$$\begin{aligned}(d-1)P \cap M + \underbrace{P \cap M + \cdots + P \cap M}_{(k-1)(d-1) \text{ times}} &\subseteq \underbrace{(d-1)P \cap M + \cdots + (d-1)P \cap M}_{k \text{ times}} \\ &\subseteq k(d-1)P \cap M.\end{aligned}$$

On the other hand, by Lemma 2.2.7,

$$(d-1)P \cap M + \underbrace{P \cap M + \cdots + P \cap M}_{(d-1)(d-1) times} = k(d-1)P \cap M.$$

Hence,

$$\underbrace{(d-1)P \cap M + \cdots + (d-1)P \cap M}_{k \text{ times}} = k(d-1)P \cap M.$$

Therefore, $(d-1)P$ is normal. $\square$

2.5 Vanishing Theorems of Toric Varieties

In this section, we quickly list all vanishing theorems that are used in this thesis. We begin with the classical Serre vanishing.

Proposition 2.5.1 ([CLS11, Theorem 9.0.6])**.** *Let L be an ample line bundle on a projective variety X. Then for any cohenrent sheaf $\mathscr{F}$ on X, we have*

$$\mathrm{H}^p\left(X, \mathscr{F} \otimes_X L^{\otimes k}\right) = 0$$

for all $p > 0$ and k sufficiently large.

For toric varieties, we have the Demazure vanhishing:

Proposition 2.5.2 ([CLS11, Theorem 9.2.3])**.** *Let D be a $\mathbb{Q}$-Cartier divisor on X_Σ. If $|\Sigma|$ is convex and D is globally generated, then*

$$H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = 0$$

for all $p > 0$.

We also make use of the Batyrev-Borisov vanishing.

Proposition 2.5.3 ([BB96, Theorem 2.5])**.** *Let D be a globally generated Cartier divisor on a projective variety X. Let P be the polytope associated to D. Then*

- $\mathrm{H}^p(X, \mathcal{O}_X(-D)) = 0$ *for all* $p \neq \dim P$.

- *If* $p = \dim P$, $\mathrm{H}^p(X, \mathcal{O}_X(-D)) \cong \bigoplus_{m \in \mathrm{Relint}(P) \cap M} \mathbb{C} \cdot \chi^{-m}$, *where* Relint$(P)$ *denotes the relative interior of* P *, the set of interior points of* P *which is considered as a subset of the minimal* $\mathbb{R}$*-linear affine subspace containing* P*.*

The following lemma follows directly from Proposition 2.5.2 and 2.5.3.

Lemma 2.5.4 ([CLS11, Example 9.2.8])**.** *Let* Δ_n *denote the standard* n*-simplex*

$$\Delta_n = \mathrm{conv}(0, e_1, \dots, e_n) \subset \mathbb{R}^n.$$

For any $p > 0$*,*

$$\dim \mathrm{H}^p(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a)) = \begin{cases} 0 & p \neq n \text{ or if } p = n \text{ and } a \geq 0 \\ |(-a\Delta_n)^0 \cap M| & p = n \text{ and } a < 0. \end{cases}$$

2.6 Very Ample Divisors Versus Very Ample Lattice Polytopes

We have a correspondence between very ample divisors on projective toric varieties and very ample lattice polytopes as follows.

Proposition 2.6.1 ([CLS11, Proposition 6.1.10])**.** *Let* P *be a lattice polytope and define* X_P *and* D_P *as in Section 2.3. Then* D_P *is ample and globally generated. Moreover,* D_P *is very ample if and only if* P *is a very ample polytope.*

Furthermore, we have equality between the k-normality of P and D_P. In fact, let $L = \mathcal{O}_X(D_P)$, then the map

$$\Gamma(X, D^m) \otimes \Gamma(X, D^n) \to \Gamma(X, D^{m+n})$$

is surjective if and only if

$$mP \cap M + nP \cap M \to (m+n)P \cap M$$

is also surjective. This follows directly from (2.2).

Proposition 2.6.2. *Let* P *be a lattice polytope. Then* P *is very ample if and only if* P *is* k*-normal for some* k *sufficiently large.*

Proof. We first show that if P is k-normal for all k big enough then P is very ample. By our hypothesis, k_P exists. For any vertex v of P, suppose that $cu \in \mathbb{N}(P \cap M - v)$; i.e.,

$$cu = \sum_{i=1}^{n} (w_i - v)$$

for some $w_i \in P \cap M$. Take $k = \max\{k_P, n\}$, we have

$$cu + ckv = \sum_{i=1}^{n} w_i + (ck - n)v \in ckP \cap M.$$

Therefore, $u + kv \in kP \cap M$. It follows from the definition of k_P that $u + kv = \sum_{i=1}^{k} w_i'$ for some $w_i' \in P \cap M$. In other words,

$$u = \sum_{i=1}^{k_P} (w_i' - v) \in \mathbb{N}(P \cap M - v).$$

This shows that P is very ample.

Now suppose that P is very ample. We will show that $X = X_P$ (embedded in $\mathbb{P}^r \cong \mathbb{P}(\mathrm{H}^0(X, \mathcal{O}_X(D_P)))$ is k-normal for some k big enough. Consider the exact sequence of coherent sheaves

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_X \to 0,$$

where $\mathcal{I}_X = \ker(\mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_X)$. Twisting the sequence by tensoring with $\mathcal{O}_{\mathbb{P}^r}(n)$ yields a long exact sequence

$$0 \to H^0(\mathbb{P}^r, \mathcal{I}_X(n)) \to H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to H^0(X, \mathcal{O}_X(n)) \to H^1(\mathbb{P}^r, \mathcal{I}_X(n)) \to \cdots .$$

By Serre vanishing theorem for projective varieties (Proposition 2.5.1) there exists an integer $k \gg 0$ such that

$$H^1(\mathbb{P}^r, \mathcal{I}_X(k)) = 0$$

for all $k \geq N$. Thus

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(X, \mathcal{O}_X(k))$$

is surjective for all $k \geq N$. The conclusion follows. □

Chapter 3

On k-normality of Very Ample Lattice Polytopes

In this chapter, we will give a bound for k-normality of very ample lattice polytopes.

3.1 On Some Invariants of Lattice Polytopes

We begin by recalling the definition of the invariant m_P of a very ample lattice polytope P.

Definition 3.1.1. Let P be a very ample lattice polytope. For any lattice point $x \in d_P P \cap M$ and vertex $d_P v$ of $d_P P$, since P is very ample,

$$x - d_P \cdot v = \sum_{i=1}^{m} (w_i - v),$$

for some $m < +\infty$, $w_i \in P \cap M$. For such a pair $(x, d_P v)$, we define

$$\sigma(x, d_P v) = \min \left\{ m \in \mathbb{N} \;\middle|\; x - d_P v = \sum_{i=1}^{m} (w_i - v) \text{ for some } w_i \in P \cap M \right\}.$$

We define m_P to be

$$m_P = \max \left\{ \sigma(x, d_P v) \;\middle|\; x \in (d_P P) \cap M,\ v \text{ a vertex of } P \right\}.$$

From the definition, for any $x \in d_P P$ and v a vertex of P, it follows that

$$x + (m_P - d_P) v = \sum_{i=1}^{m_P} w_i \in m_P P \cap M$$

for some $w_i \in P \cap M$. Hence, in some senses, $m_P - d_P$ measures a "minimum" number of lattice points of P we need to add to x to make it expressible as a sum of lattice points in P. We guess that $k_P = m_P$ for every very ample lattice polytope P. It is clear that if P is normal then $k_P = m_P$. For P not normal, in both Example 4.3.1 and Example 4.3.5, $k_P = m_P$.

However, we are yet to find a proof for this guess and will prove a weaker result in this chapter. In order to do so, let us show some properties of the invariants of lattice polytopes defined in Section 2.2.

Lemma 3.1.2. *Let P be a d-dimensional lattice polytope of dimension d with n vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$.*

(a) For any $k \geq d_P$ and $u \in kP \cap M$, we can write u as

$$u = x + \sum_{i=1}^{k-d_P} u_i,$$

where $x \in d_P P \cap M$ and $u_i \in P \cap M$ for all $1 \leq i \leq k - d_P$.

(b) If $k \geq \nu_P$, for any $u \in kP \cap M$,

$$u = x + \sum_{i=1}^{\nu_P - d_P} u_i + \sum_{i=1}^{n} \lambda_i v_i$$

for some $x \in d_P P \cap M$, $u_i \in P \cap M$, $\lambda_i \in \mathbb{N}$ such that $\sum_{i=1}^{n} \lambda_i = k - \nu_P$.

(c) $d_P \leq m_P \leq k_P$.

(d) P is normal $\Leftrightarrow d_P = k_P \Leftrightarrow d_P = m_P$. Therefore, if P is not normal then $k_P \geq m_P \geq d_P + 1$.

Proof. (a) By the definition of d_P, we have a surjective map

$$d_P P \cap M + \underbrace{P \cap M + \cdots + P \cap M}_{k - d_P} \twoheadrightarrow kP \cap M.$$

Hence, for any $k \geq d_P$ and $u \in kP \cap M$, we can write u as

$$u = x + \sum_{i=1}^{k - d_P} u_i,$$

where $x \in d_P P \cap M$ and $u_i \in P \cap M$ for all $1 \leq i \leq k - d_P$.

(b) Similarly, for $k \geq \nu_P$, we have a surjection

$$d_P P \cap M + \underbrace{P \cap M + \cdots + P \cap M}_{\nu_P - d_P} + \underbrace{\mathcal{V} + \cdots + \mathcal{V}}_{k - \nu_P} \twoheadrightarrow kP \cap M,$$

which yields the conclusion.

(c) It follows from the definitions that $k_P \geq d_P$. Now let x be any lattice point in $d_P P \cap M$ and v a vertex of P. Then since $x + (k_P - d_P)v \in kP \cap M$, there exists $w_i \in P \cap M$, $i = 1, \ldots, k$ such that $x + (k_P - d_P)v = \sum_{i=1}^{k_P} w_i$. In other words,

$$x - d_P v = \sum_{i=1}^{k_P} (w_i - v).$$

Therefore, $\sigma(x, d_P v) \leq k_P$. Hence, $m_P \leq k_P$.

For any vertex v of P, let $w \in P \cap M$ be a point with maximal distance from v. We have

$$d_P w - d_P v = \sum_{i=1}^{\sigma(d_P w, d_P v)} (w_i - v)$$

for some $w_i \in P \cap M$. Then

$$\|d_P w - d_P v\| = \left\| \sum_{i=1}^{\sigma(d_P w, d_P v)} (w_i - v) \right\| \leq \sum_{i=1}^{\sigma(d_P w, d_P v)} \|(w_i - v)\| \leq \sigma(d_P w, d_P v) \cdot \|w - v\|.$$

It follows that $\sigma(d_P w, d_P v) \geq d_P$. Therefore, $m_P \geq d_P$.

(d) • P is normal $\Leftrightarrow d_P = k_P$: if P is normal then $k_P = d_P = 1$. Conversely, suppose that $k = d_P = k_P \geq 2$. Then by the definitions of k_P and d_P, we have a surjection

$$\underbrace{P \cap M + \cdots + P \cap M}_{k} \twoheadrightarrow kP \cap M,$$

while the map

$$(k-1)P \cap M + P \cap M \to kP \cap M$$

is not surjective. This is a contradiction because

$$\underbrace{P \cap M + \cdots + P \cap M}_{k} \subseteq (k-1)P \cap M + P \cap M.$$

Hence, P must be normal in this case.

• P is normal $\Leftrightarrow m_P = d_P$: if P is normal then it is clear that $m_P = d_P = 1$. Suppose conversely that $m_P = d_P$. Then for any vertex $v \in P \cap M$ and any $x \in d_P P \cap M$, $x - d_P v = \sum_{i=1}^{d_P}(w_i - v)$ which implies $x = \sum_{i=1}^{d_P} w_i$ for some $w_i \in P \cap M$. Then P is d_P-normal, so $d_P \geq k_P$ by part (a). It follows that $d_P = k_P$ by the first part of this lemma. Thus P is normal by the second equivalence.

□

Remark 3.1.3.

- The above lemma plays a crucial role in our main result in this section. Notice that $m_P = k_P$ does not implies P is normal. A counterexample is given by the case $s = 4$ in Example 4.3.1.
- From the definition, for any $x \in d_P P$ and v a vertex of P, it follows that

  $$x + (m_P - d_P)v = \sum_{i=1}^{m_P} w_i \in m_P P \cap M$$

  for some $w_i \in P \cap M$. Hence, in some senses, $m_P - d_P$ measures a "minimum" number of lattice points of P we need to add to x to make it expressible as a sum of lattice points in P. We guess that $k_P = m_P$ for every very ample lattice polytope P. It is clear that if P is normal then $k_P = m_P$. For P not normal, in both Example 4.3.1 and Example 4.3.5, $k_P = m_P$.

Remark 3.1.4. Let P be a d-dimensional lattice polytope. Let $\mathcal{LD}_P(n)$ be the property that for any $k \geq n$ and $u \in kP \cap M$, we can write u as

$$u = x + \sum_{i=1}^{k-n} u_i,$$

where $x \in nP \cap M$ and $u_i \in P \cap M$ for all $1 \leq i \leq k-n$. Then

$$d_P = \min\{n \in \mathbb{N} \mid \mathcal{LD}_P(n) \text{ holds}\}.$$

Indeed, suppose that $N = \min\{n \in \mathbb{N} \mid \mathcal{LD}_P(n) \text{ holds}\}$. Then we have a surjection

$$kP \cap M + P \cap M \twoheadrightarrow (k+1)P \cap M$$

for all $k \geq N$. Therefore, $N \geq d_P$. On the other hand, by Lemma 3.1.2, $\mathcal{LD}_P(d_P)$ holds so $N \leq d_P$ because of the minimality of N. Hence, $N = d_P$. The conclusion follows.

3.2 Main Theorem

In this section, we will show our main result.

Theorem 3.2.1. *Suppose that P is a very ample lattice polytope that has n vertices. Then*

$$k_P \leq (m_P - d_P) \cdot n + 1.$$

The equality occurs if and only if P is normal. Furthermore, if P is not normal then

$$k_P \leq (m_P - d_P - 1) \cdot n + \nu_P + 1.$$

Proof. If P is normal then $m_P = d_P = k_P = 1$ by Lemma 3.1.2 (d). Assume that P is not normal, since $\nu_P \leq n-1$, it is enough to show that

$$k_P \leq (m_P - d_P - 1) \cdot n + \nu_P + 1.$$

By Lemma 3.1.2 (d), $m_P \geq d_P + 1$. Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be the set of vertices of P. Let $k \geq (m_P - d_P - 1) \cdot n + \nu_P + 1$ and $p \in kP \cap M$. Notice that $(m_P - d_P - 1) \cdot n + \nu_P + 1 \geq \nu_P + 1$, by Lemma 3.1.2 (b) the lattice point p of kP can be written as

$$p = x + \sum_{i=1}^{\nu_P - d_P} u_i + \sum_{i=1}^{n} \lambda_i v_i \tag{3.1}$$

for some $x \in d_P P \cap M$, $u_i \in P \cap M$, and $\lambda_i \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{n} \lambda_i = k - \nu_P$. Now $k \geq (m_P - d_P - 1) \cdot n + \nu_P + 1$ implies that $k - \nu_P \geq (m_P - d_P - 1)n + 1$. Thus, by the pigeonhole principle, there must be an i such that $\lambda_i \geq m_P - d_P$. Without loss of generality, assume that $\lambda_1 \geq m_P - d_P$. Since P is very ample, we can write

$$x - d_P v_1 = \sum_{i \in I} a_i (w_i - v_1) \tag{3.2}$$

for some $a_i \in \mathbb{N}$ and $w_i \in P \cap M$ such that $\sum_{i \in I} a_i \leq m_P$. Substituting Equation (3.2) into Equation (3.1) yields

$$\begin{aligned} p &= d_P v_1 + \sum_{i \in I} a_i (w_i - v_1) + \sum_{i=1}^{\nu_P - d_P} u_i + \sum_{i=1}^{n} \lambda_i v_i \\ &= \left(d_P + \lambda_1 - \sum_{i \in I} a_i \right) v_1 + \sum_{i \in I} a_i w_i + \sum_{i=1}^{\nu_P - d_P} u_i + \sum_{i=2}^{n} \lambda_i v_i. \end{aligned}$$

The sum of the coefficients in the last line is k and each of them is non-negative since $d_P + \lambda_1 \geq m_P \geq \sum_{i \in I} a_i$. Hence, p can be written as a sum of k lattice points in P; i.e., P is k-normal. Therefore, $k_P \leq (m_P - d_P - 1) \cdot n + \nu_P + 1$.

Now suppose that $k_P = (m_P - d_P) \cdot n + 1$ but P is not normal. Then

$$(m_P - d_P) \cdot n + 1 \leq (m_P - d_P - 1) \cdot n + \nu_P + 1$$

which implies $\nu_P \geq n$, a contradiction. Hence, P must be normal. □

Remark 3.2.2. If P is normal, then $m_P = d_P = 1$ and $(m_P - d_P) \cdot n + 1 = 1$. Our bound is sharp for this case. Another case where our bound is sharp is given in Example 4.3.1.

The following example gives a comparison between known results on k-normality of polytopes and our result in Theorem 3.2.1 for the case of unit hypercubes.

Example 3.2.3. Consider the unit d-dimensional hypercube P. Let X be the toric variety obtained from P. Then we know that $d_P = 1$ and it follows that $m_P = 1$. Our bound in Theorem 3.2.1 implies that $k_P = 1$. This bound is sharp. We have the following table of other known bounds of k_P.

k_P	Mumford ([BM92])	Sturmfels ([Stu95])	Eisenbud-Goto ([EG84])
1	$(d+1)(d!-2)+1$	$2^d \cdot (d!)(2^d - d - 1) - 1$	$d! - 2^d + d + 1$

The only occasion where we need very-ampleness in the proof of Theorem 3.2.1 is to define m_P. Thus, if we assume m_P is defined for an arbitrary lattice polytope P, it follows that P is k-normal for k big enough. We obtain the following criterion for a lattice polytope to be very ample.

Proposition 3.2.4. *Let P be a lattice polytope. Then P is very ample if and only if there exists $r \geq d_p$ such that for any $x \in rP \cap M$ and v a vertex of P*

$$x - rv = \sum_{i=1}^{n} (w_i - v)$$

for some $n < \infty$ and $w_i \in P \cap M$.

Proof. The "only if" part follows directly from the definition of very ample polytopes. We now prove the "if" part. For any $r \geq d_P$, let

$$m = \max \left\{ \sigma(x, rv) \middle| x \in rP \cap M, v \text{ a vertex of } P \right\}.$$

It follows from the proofs of Lemma 3.1.2 (c) and Theorem 3.2.1 that $m \geq d_P$ and

$$k_P \leq (m - d_P) \cdot n + 1.$$

Then P is k-normal for $k \gg 0$, which implies that P is very ample. The conclusion follows. □

3.3 Bounds of m_P and Aplications on Smooth Polytopes

In this section, we will give some bounds for m_P depending on the combinatorial data of any smooth lattice polytope P. Using normalized volume, we obtain our first bound of m_P:

Proposition 3.3.1. *Let P be a smooth d-dimensional lattice polytope. Then for every $x \in (d_P \cdot P) \cap M$ and v a vertex of P,*

$$m_P \leq d \cdot d_P^d \cdot \mathrm{Vol}(P).$$

Proof. Let $\mathcal{V} = \{v_1, \dots, v_n\}$ be the set of vertices of P. For any $x \in d_P \cdot P \cap M$, and $v \in \mathcal{V}$, since P is smooth,

$$x - d_P \cdot v = \sum_{i=1}^{d} a_{E_i}(w_{E_i} - v),$$

for some $a_i \in \mathbb{Z}_{\geq 0}$, with $w_{E_i} - v$ is the primitive ray generator of $\mathrm{Cone}(v_{E_i} - v)$, where v_{E_i} is a neighbor of v. By Cramer's rule

$$a_{E_i} = \frac{\Delta_{E_i}}{\Delta},$$

where

$$\Delta = \det(w_{E_1} - v, \dots, w_{E_d} - v) = 1, \text{ and}$$

$$\Delta_{E_i} = \det(w_{E_1} - v, \dots, w_{E_{i-1}} - v, x - d_P \cdot v, w_{E_{i+1}} - v, \dots, w_{E_d} - v).$$

By Lemma 2.2.13, Δ_{E_i} is the normalized volume of the simplex

$$\mathrm{conv}(d_P v, w_{E_1} + (d_P - 1)v, \dots, w_{E_{i-1}} + (d_P - 1)v, x, w_{E_{i+1}} + (d_P - 1)v, \dots, w_{E_d} + (d_P - 1)v),$$

which lies inside the polytope $d_P \cdot P$. Thus, $\Delta_{E_i} \leq \mathrm{Vol}(d_P \cdot P) = d_P^d \,\mathrm{Vol}(P)$. Therefore,

$$\sigma(x, d_P \cdot v) = \sum_{i=1}^{d} a_{E_i} \leq d \cdot d_P^d \,\mathrm{Vol}(P).$$

□

Definition 3.3.2. Let P be a d-dimensional smooth lattice polytope. Then for each vertex v of P, there exist d neighbor vertices to v, say $v_{E_i}, \dots, v_{E_d}$, where E_i is the edge of joining v with v_{E_i}. Let $w_{E_i} - v$ be the primitive ray generator of $\mathrm{Cone}(v_{E_i} - v)$. We define the corner of P at v, a vertex of P, to be

$$C_v := \mathrm{conv}\left(v, w_{E_i}, \dots, w_{E_d}\right),$$

and the γ-scaling of C_v to be

$$C_{v,\gamma} = \mathrm{conv}(v, v + \gamma \cdot (w_{E_1} - v), \ldots, v + \gamma \cdot (w_{E_d} - v)).$$

Then for some γ big enough, $C_{v,\gamma}$ contains the whole polytope P.

Example 3.3.3. Consider the polytope P given by

$$P = \mathrm{conv}\begin{pmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{pmatrix}.$$

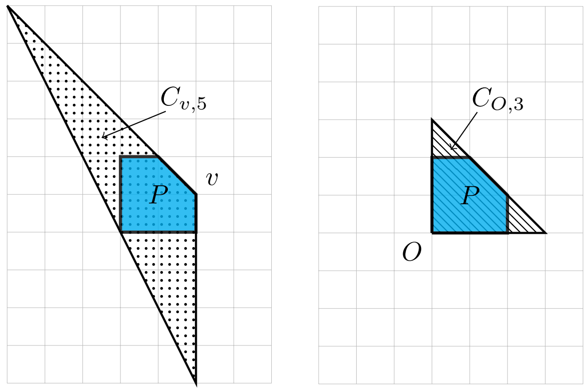

Figure 3.1: The γ-scaling

Let O be the origin and $v = (1,1)^T$. Then

$$C_O = \mathrm{conv}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } C_v = \mathrm{conv}\begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

We have $C_{v,5} \supseteq P$ and $C_{O,3} \supseteq P$.

Proposition 3.3.4. *Let P be a smooth d-dimensional lattice polytope, $\mathcal{V} = \{v_1, \ldots, v_n\}$ the set of its vertices. Let γ be the miminum interger such that $P \subseteq C_{v_i,\gamma}$ for every $1 \leq i \leq n$. Then for any $u \in d_P \cdot P \cap M$, $v \in \mathcal{V}$,*

$$\sigma(u, d_P \cdot v) \leq d_P \cdot \gamma,$$

which implies $m_P \leq d_P \cdot \gamma$.

Proof. For any lattice point $u \in mP \cap M$ ($m \in \mathbb{Z}_{\geq 1}$) and vertex v of mP, u lies inside the d-simplex formed by scaling the corner at v by $m\gamma$. Precisely,

$$u \in C_{v,m\gamma} = \mathrm{conv}(v, v + m\gamma \cdot (w_{E_1} - v), \ldots, v + m\gamma \cdot (w_{E_d} - v)),$$

where $w_{E_i} - v$ is the primitive ray generator of $v_{E_i} - v$. Equivalently, there exist $\lambda_i \geq 0$ such that $\sum_{i=0}^{d} \lambda_i = 1$ and

$$u = \lambda_0 v + \sum_{i=1}^{d} \lambda_i \cdot (v + m\gamma \cdot (w_{E_i} - v)) = v + \sum_{i=1}^{d} \lambda_i \cdot m \cdot \gamma \cdot (w_{E_i} - v). \tag{3.3}$$

Hence,

$$u - v = \sum_{i=1}^{d} \lambda_i \cdot m \cdot \gamma \cdot (w_{E_i} - v). \tag{3.4}$$

Since mP is smooth at v, $C_{v,m\gamma}$ is also smooth at v, and we can express $u-v$ uniquely in the form

$$u - v = \sum_{i=1}^{d} a_i(w_{E_i} - v), \tag{3.5}$$

where $a_i \in \mathbb{N}$ for $1 \le i \le d$. Comparing the coefficients in the equations (3.4) and (3.5) yields

$$a_i = \lambda_i \cdot m \cdot \gamma. \tag{†}$$

Applying (†) for $m = d_P$ and $u \in d_P P \cap M$,

$$\sum_{i=1}^{d} a_i = \sum_{i=1}^{d} d_P \cdot \lambda_i \cdot \gamma = d_P \cdot \gamma \cdot \sum_{i=1}^{d} \lambda_i \le d_P \cdot \gamma.$$

In other words,

$$\sigma(u, d_P \cdot v) \le d_P \cdot \gamma.$$

In particular, since m_P is the maximum of the $\sigma(u, d_P \cdot v)$, $m_P \le d_P \cdot \gamma$. □

As a corollary, we obtain a bound for smooth lattice polytopes as follows.

Corollary 3.3.5. *Let P be a smooth d-dimensional lattice polytope that has n vertices, γ is the minimum integer such that $P \subseteq C_{v,\gamma}$ for every vertex v of P. Then P is k-normal for all*

$$k \ge \min \left\{ \begin{array}{l} d_P(\gamma - 1) \cdot n + 1, \\ \left(d \cdot d_P^d \cdot \mathrm{Vol}(P) - d_P\right) \cdot n + 1 \end{array} \right\}.$$

Proof. This follows directly from Theorem 3.2.1, Proposition 3.3.1, and Proposition 3.3.4. □

Corollary 1.1.2 follows since $d_P \le d - 1$.

Remark 3.3.6. As a final remark to this session, suppose that P is a d-dimensional smooth lattice polytope. Then for any lattice point $u \in P \cap M$ and any vertex $v \in \mathcal{V}$,

$$u - v = \sum_{i=1}^{d} a_i(w_{E_i} - v), \tag{3.6}$$

where $a_i \in \mathbb{Z}_{\ge 0}$ and $w_{E_i} - v$ is the primitive generator of $\mathrm{Cone}(v_{E_i} - v)$. Take m' to be the maximal of all such a_i; i.e.,

$$m' = \max_{u \in P \cap M, v \in \mathcal{V}} \left\{ a_i \;\middle|\; u - v = \sum_{i=1}^{d} a_i(w_{E_i} - v) \right\}.$$

Then m' is well-defined because P is a smooth polytope and $|P \cap M| < \infty$. We have $P \cap M \subseteq C_{v,m'}$ for every $v \in \mathcal{V}$. In other words, $\gamma \le m'$.

3.4 A Survey On d_P and Normal Polytopes

We will give a short survey on d_P in this section. We first begin with some upper bounds of d_P:

Proposition 3.4.1. *Let P be a lattice polytope. Then*

1. *[Her06, Proposition IV.10] If P is not a standard simplex, then*

$$d_P \le \deg P.$$

2. *[HKN17, Proposition 2.2] If P is spanning, in particular if P is very ample, then*

$$d_P \leq \mathrm{Vol}(P) + d + 1 - |P \cap M|.$$

Remark 3.4.2. Unfortunately, unlike Lemma 3.1.2, P is normal does not imply that $d_P = \deg(P)$, with the standard simplex is a counterexample. In this case, $d_P = 1$ while $\deg(P) = 0$. In addition, $d_P = \deg(P)$ does not imply that P is normal, with a counterexample given in Example 4.3.1, where $d_P = \deg(P) = 2$.

Since $d_P = 1$ if and only if P is normal, we obtain a simple combinatorial proof for part of [BSV15, Proposition 6.9].

Corollary 3.4.3 ([BSV15, Proposition 6.9])**.** *Any lattice polytope of degree* 0 *or* 1 *is normal.*

Proof. If $\deg P = 0$, then P is a basic simplex, so it is normal. Now suppose that $\deg P = 1$. By Proposition 3.4.1, $d_P \leq \deg P \leq 1$, which implies that P is normal. □

From Part (2) of Proposition 3.4.1, we may ask if it is also true that $d_P \leq \mathrm{Vol}(P)$ even if P is not very ample. This is obviously true if $\mathrm{Vol}(P) \geq d - 1$. The interesting case is when $\mathrm{Vol}(P) \leq d - 2$; i.e, P is a "small" polytope.

Example 3.4.4. Let P be a lattice polytope such that $\dim P \leq 3$. Then

$$d_P \leq \mathrm{Vol}\, P.$$

Also, for any d-dimensional lattice polytope P, if $d_P \leq 2$, then $d_P \leq \mathrm{Vol}(P)$.

Proof. In dimension 2, since $d_P = 1$ always, $d_P \leq \mathrm{Vol}(P)$. If $\dim P = 3$, then either $d_P = 1$, so $d_P \leq \mathrm{Vol}(P)$ trivially, or $d_P = 2$. If $d_P = 2$, then $\mathrm{Vol}(P) \geq d_P$. This is because if $\mathrm{Vol}(P) = 1$ then P is a standard simplex and so P would be normal and $d_P = 1$, a contradiction. Hence, $d_P \leq \mathrm{Vol}(P)$ for the case $\dim P = 3$ as well. □

However, d_P is not always bounded above by $\mathrm{Vol}(P)$.

Example 3.4.5. Consider the polytope P_C in $\mathbb{R}^5$ as in Example 2.2.9

$$P_C = \mathrm{conv}(0, e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4 + 2e_5).$$

Then $\mathrm{Vol}(P_C) = 2$ and the point $(1, 1, 1, 1, 1)$ lies in $3P_C \cap M$ but not in $2P_C \cap M + P_C \cap M$. Hence, $d_{P_C} = 4 > \mathrm{Vol}(P_C)$.

In general, the difference $k_P - d_P$ cannot be bounded by any polynomial of $\dim(P)$, with an example again given by Example 4.3.1.

Proposition 3.4.6 ([BDGM15, Theorem 3.3])**.** *For any non-negative integer n, there exists a* 3*-dimensional very ample lattice polytope P such that $k_P - d_P = n$.*

Proof. For $n = 0$, any normal polytope P would give the desired result. For $n \geq 1$, take P to be the polytope in Example 4.3.1 with $s = n + 3$. Then $d_P = 2$ and by [BDGM15, Theorem 3.3], $k_P = n + 2$. The conclusion follows. □

However, for very ample lattice simplices we have the following result.

Proposition 3.4.7 ([BGT97, Theorem 1.3.3 (a)], [Oga05, Proposition 2.4])**.** *Let P be a very ample lattice simplex. Then*

$$k_P - d_P \leq \dim P - 1.$$

Proof. If $\dim P = 2$ then P is normal by [BGT97, Theorem 1.3.3 (a)], so $0 = k_P - d_P \leq \dim P - 1 = 1$. The case $\dim P \geq 3$ follows from the proof of [Oga05, Proposition 2.4]. □

The ultimate goal in bounding $k_P - d_P$ is to prove or disprove that for any smooth lattice polytope P, $k_P - d_P = 0$. This is another interpretation of Oda's question [Oda08].

Another question is what dilations of P are normal. It is well-known that if P is a d-dimensional lattice polytope then $(d-1)P$ is normal (Lemma 2.4.8). The following lemma, which follows easily from the definition of d_P, gives a slightly improved result, which also implies [Her06, Proposition IV.10] because of Proposition 3.4.1.

Proposition 3.4.8. *Let P be a lattice polytope. Then mP is normal for every $m \geq d_P$.*

As a corollary of Proposition 3.4.1 and Proposition 3.4.8, we have:

Corollary 3.4.9. *For any very ample lattice polytope P, $\mathrm{Vol}(P) \cdot P$ is normal.*

Furthermore, d_P is a natural candidate for the minimum number γ_P with the property that kP is normal for every $k \geq \gamma_P$. This is true in case $\dim P = 2$ or $\dim P = 3$.

Example 3.4.10. Let P be a lattice polytope such that $\dim(P) \leq 3$. Then $d_P = \gamma_P$.

Proof. The case $\dim P = 2$ is trivial because any polygon is normal by [BGT97, Theorem 1.3.3 (a)]. If $\dim P = 3$, either $d_P = 1$ or $d_P = 2$. For the case $d_P = 1$, P is then normal; hence, the statement is true in this case. If $d_P = 2$, P is not normal but kP is normal for all $k \geq 2$ by Proposition 3.4.8. The conclusion follows. □

Unfortunately, d_p is not equal to γ_P in general.

Example 3.4.11. Again, consider the polytope $P_C = \mathrm{conv}(0, e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4 + 2e_5)$ as in Example 2.2.9. We know that $d_{P_C} = 4$. However, $3P_C$ is normal. By Proposition 3.4.8, kP_C is normal for all $k \geq 4$. Hence, $4 = d_{P_C} > \gamma_{P_C} = 3$.

In general, nP is normal does not imply $(n + 1)P$ is also normal. Lasoń and Michałek ([LM17, Section 4]) found an example of a polytope P such that $2P$ and $3P$ are normal but $5P$ is not. For such a P we must have $5 < d_P$; otherwise, $5P$ must be normal by Proposition 3.4.8. This example also implies that the inequalities $k_{P+Q} \leq \max\{k_P, k_Q\}$ and $d_{P+Q} \leq \max\{d_P, d_Q\}$ do not hold in general.

The following corollary gives a criterion for normality of polytopes with small degree.

Corollary 3.4.12. *Let P be a lattice polytope. Suppose that P is k-normal for all $k \leq \dim P - \deg P$. If $2 \deg P \leq \dim P$ then P is normal.*

Proof. Since $d_P \leq \deg(P)$ and $2 \deg P \leq \dim P$, $\dim P - \deg P \geq d_P$. Thus, P is normal because of the definition of d_P. □

Since P is k-normal does not imply P is also $(k + 1)$-normal, it makes sense to verify k-normality for $k \leq \dim P - \deg P$ in the above corollary.

Chapter 4

The Castelnuovo-Mumford Regularity of Projective Toric Varieties

4.1 The Castelnuovo-Mumford Regularity of Normal Toric Varieties

In this section, we will give a survey on combinatorial interpretations of the Eisenbud-Goto conjecture.

Let L be a very ample line bundle on an irreducible projective variety X defining an embedding $i : X \to \mathbb{P}(\mathrm{H}^0(X, L)) \cong \mathbb{P}^r$. Define $\mathcal{O}_X(1)$ to be the pullback of the embedding $\mathcal{O}_X(1) = i_*\mathcal{O}_{\mathbb{P}^r}(1)$. Then for any coherent sheaf $\mathscr{F}$ on X, we denote by $\mathscr{F}(k)$ the twisted sheaf

$$\mathscr{F} \otimes \underbrace{\mathcal{O}_X(1) \otimes \cdots \otimes \mathcal{O}_X(1)}_{k \text{ times}}.$$

Let us now recall the definition of Castelnuovo-Mumford regularity.

Definition 4.1.1. Let $X \subset \mathbb{P}^r$ be a projective variety and $\mathscr{F}$ a coherent sheaf over X. We say that $\mathscr{F}$ is k-regular if

$$\mathrm{H}^i(X, \mathscr{F}(k-i)) = 0$$

for all $i > 0$. The regularity of $\mathscr{F}$, denoted by $\mathrm{reg}(\mathscr{F})$, is the minimum number k such that $\mathscr{F}$ is k-regular. We also say that X is k-regular if the ideal sheaf $\mathcal{I}_X$ of X is k-regular and use $\mathrm{reg}(X)$ to denote the regularity of X (or of $\mathcal{I}_X$).

By a result is attributed to Castelnuovo by Mumford:

Lemma 4.1.2 ([Mum66, Lecture 14]). *If $\mathscr{F}$ is k-regular then it is also $(k+1)$-regular.*

We also recall the definition of k-normality.

Definition 4.1.3. Let $X \subset \mathbb{P}^r$ be an irreducible projective variety X. We say that (the embedding of) X is k-normal if the restriction map

$$\mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to \mathrm{H}^0(X, \mathcal{O}_X(k))$$

is surjective. We define the k-normality of X to be the smallest positive integer k_X such that X is k-normal for all $k \geq k_X$.

Regularity and k-normality are closely related by the well-known fact as noted, for example, in [Kwa98, Page 195] as follows.

Proposition 4.1.4. *Let $X \subseteq \mathbb{P}^r$ be an irreducible projective variety. Then for $k \in \mathbb{Z}_{\geq 1}$, X is $(k+1)$-regular if and only if X is k-normal and $\mathcal{O}_X$ is k-regular.*

Proof. The exact sequence
$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_X \to 0$$
twisted by $\mathcal{O}_X(k+1-i)$ yields a long exact sequence

$$\begin{aligned}
0 \longrightarrow \mathrm{H}^0(\mathcal{I}_X(k+1-i)) &\longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(k+1-i)) \longrightarrow \mathrm{H}^0(\mathcal{O}_X(k+1-i)) \\
\longrightarrow \mathrm{H}^1(\mathcal{I}_X(k+1-i)) &\longrightarrow \mathrm{H}^1(\mathcal{O}_{\mathbb{P}^r}(k+1-i)) \longrightarrow \mathrm{H}^1(\mathcal{O}_X(k+1-i)) \\
\dashrightarrow \mathrm{H}^i(\mathcal{I}_X(k+1-i)) &\longrightarrow \mathrm{H}^i(\mathcal{O}_{\mathbb{P}^r}(k+1-i)) \longrightarrow \mathrm{H}^i(\mathcal{O}_X(k+1-i)) \longrightarrow \cdots
\end{aligned}$$

for any integer i. Suppose that X is $(k+1)$-regular; i.e., $\mathrm{H}^i(\mathcal{I}_X(k+1-i)) = 0$ for all $i \geq 1$. Then for $i = 1$, it follows that $\mathrm{H}^1(\mathcal{I}_X(k)) = 0$; i.e., X is k-normal. For $i \geq 2$, we have an exact sequence
$$\cdots \to \mathrm{H}^{i-1}(\mathcal{O}_{\mathbb{P}^r}(k+1-i)) \to \mathrm{H}^{i-1}(\mathcal{O}_X(k+1-i)) \to \mathrm{H}^i(\mathcal{I}_X(k+1-i)) \to \cdots.$$
By Lemma 2.5.4, $\mathrm{H}^{i-1}(\mathcal{O}_{\mathbb{P}^r}(k+1-i)) = 0$ for all $i \geq 2$. Since $\mathrm{H}^i(\mathcal{I}_X(k+1-i)) = 0$, it follows that $\mathrm{H}^{i-1}(\mathcal{O}_{\mathbb{P}^r}(k+1-i)) = \mathrm{H}^{i-1}(\mathcal{O}_{\mathbb{P}^r}(k-(i-1)) = 0$ for all $i \geq 2$. In other words, $\mathcal{O}_X$ is k-regular.

Conversely, suppose that X is k-normal and $\mathcal{O}_X$ is k-regular. For $i \geq 2$, we have an exact sequence
$$\cdots \to \mathrm{H}^{i-1}(\mathcal{O}_X(k+1-i)) \to \mathrm{H}^i(\mathcal{I}_X(k+1-i)) \to \mathrm{H}^i(\mathcal{O}_{\mathbb{P}^r}(k+1-i)) \to \cdots.$$
Now since $\mathrm{H}^i(\mathcal{O}_{\mathbb{P}^r}(k+1-i)) = 0$ for all $i \geq 2$ by Lemma 2.5.4 and $\mathrm{H}^{i-1}(\mathcal{O}_X(k+1-i)) = 0$ by the hypothesis that $\mathcal{O}_X$ is k-regular, $\mathrm{H}^i(\mathcal{I}_X(k+1-i)) = 0$ for all $i \geq 2$. Also, $\mathrm{H}^1(\mathcal{I}_X(k)) = 0$ since X is k-normal. Hence, X is $(k+1)$-regular. The conclusion follows. □

As a corollary, we obtain an equation of $\mathrm{reg}(X)$ in terms of $\mathrm{reg}(\mathcal{O}_X)$ and k_X for any irreducible projective variety X.

Proposition 4.1.5. *Let $X \subseteq \mathbb{P}^r$ be an irreducible projective variety. Then*
$$\mathrm{reg}(X) = \max\{\mathrm{reg}(\mathcal{O}_X), k_X\} + 1.$$

Proof. For $k = \max\{\mathrm{reg}(\mathcal{O}_X), k_X\}$, X is k-normal and $\mathcal{O}_X$ is k-regular. By Proposition 4.1.4, X is $(k+1)$-regular. Hence,
$$\mathrm{reg}(X) \leq \max\{\mathrm{reg}(\mathcal{O}_X), k_X\} + 1.$$
Now suppose that $k \leq \max\{\mathrm{reg}(\mathcal{O}_X), k_X\} - 1$. Then either X is not k-normal or $\mathcal{O}_X$ is not k-regular. Hence, $\mathcal{I}_X$ is not $(k+1)$-regular by Proposition 4.1.4. Therefore,
$$\mathrm{reg}(X) \geq \max\{\mathrm{reg}(\mathcal{O}_X), k_X\} + 1.$$
The conclusion follows. □

It follows from Batyrev-Borisov vanishing (cf. Proposition 2.5.3) that $\mathrm{reg}(\mathcal{O}_X)$ coincides with the degree of the polytope associated to the embedding.

Proposition 4.1.6 ([Her06, Theorem IV.5])**.** *Let $X \subset \mathbb{P}^r$ be a normal projective toric variety and let P be the corresponding lattice polytope of the embedding. Then $\mathrm{reg}(\mathcal{O}_X) = \deg(P)$, where $\deg(P)$ is defined as in Definition 2.2.15.*

Proof. Suppose that $\dim X = d$. First, we show that $\mathcal{O}_X$ is $\deg(P)$-regular. Let $k = \deg(P)$. If $1 \leq i \leq k$, by Proposition 2.5.2,

$$\mathrm{H}^i(X, \mathcal{O}_X(k-i)) = 0.$$

If $i > k$ and $i \neq d$, by Proposition 2.5.3, we also have

$$\mathrm{H}^i(X, \mathcal{O}_X(k-i)) = 0.$$

Now suppose that $i = d > k$, then by the definition of $\deg(P)$ and Proposition 2.5.3,

$$\dim \mathrm{H}^i(X, \mathcal{O}_X(k-d)) = |((d-k)P)^0 \cap M| = 0.$$

Therefore, $\mathcal{O}_X$ is k-regular. Now suppose that $k \leq \deg(P) - 1$. Then

$$\dim \mathrm{H}^d(X, \mathcal{O}_X(k - \dim X)) = |((d-k)P)^0 \cap M| \neq 0$$

since $d - k \geq d - \deg(P) + 1$. Hence, $\mathcal{O}_X$ is not k-regular. The conclusion follows. □

Combining Propositions 4.1.5 and 4.1.6, we obtain a combinatorial relation between $\mathrm{reg}(X)$, k_P, and $\deg(P)$, the degree of P.

Proposition 4.1.7. *Let $X \subset \mathbb{P}^r$ be a d-dimensional normal projective toric variety X and let P be the corresponding very ample lattice polytope of the embedding of X. Then*

$$\mathrm{reg}(X) = \max\{k_P, \deg(P)\} + 1.$$

Proof. This follows directly from Propositions 4.1.5 and 4.1.6. □

Notice that $\deg(P) \leq d$. Thus, using the upper bound of k_P we obtained in Theorem 3.2.1 for Proposition 4.1.7, we obtain an upper bound for $\mathrm{reg}(X)$:

Corollary 4.1.8. *Let $X \subset \mathbb{P}^r$ be a normal projective toric variety and let P be the corresponding lattice polytope of the embedding. Suppose that P has n vertices. Then*

$$\mathrm{reg}(X) \leq \max\{(m_P - d_P)n, \dim P\} + 1,$$

where m_P and d_P are defined as in Theorem 3.2.1.

Proof. This follows directly from Propositions 3.2.1, 4.1.7, and the fact that $\deg(P) \leq \dim(P) \leq n - 1$. □

By [HKN17, Proposition 2.2], we have that for P a very ample lattice polytope,

$$\deg(P) \leq \mathrm{Vol}(P) - |P \cap M| + d + 1. \tag{4.1}$$

Combining this with Proposition 4.1.7, we have the following corollary.

Corollary 4.1.9. *Let $X \subset \mathbb{P}^r$ be a normal projective toric variety and let P be the corresponding lattice polytope of the embedding. Suppose that $k_P \leq \dim(X)$; i.e., P is $\dim(X)$-normal, then*

$$\mathrm{reg}(X) \leq \dim(X) + 1.$$

Furthermore, if $k_P \leq \deg(P)$, then

$$\mathrm{reg}(X) \leq \min\{\dim(X) + 1, \deg(X) - \mathrm{codim}(X) + 1\}.$$

Proof. Let $d = \dim(X) = \dim(P)$. If $k_P \leq d$, then since $\deg(P) \leq d$ by definition,

$$\max\{k_P, \deg(P)\} \leq d.$$

Hence, by Proposition 4.1.7,

$$\mathrm{reg}(X) \leq \dim(X) + 1.$$

Now suppose that $\deg(P) \geq k_P$. Then $k_P \leq d$, so we only need to show that $\text{reg}(X) \leq \deg(X) - \text{codim}(X) + 1\}$ by the argument above. By Proposition 4.1.7, $\text{reg}(X) = \deg(P) + 1$. By (4.1), it follows that

$$\text{reg}(X) \leq \deg(X) - \text{codim}(X) + 1\}.$$

The conclusion follows. □

Proposition 4.1.10 ([EG84,Hoc72])**.** *The Eisenbud-Goto conjecture holds for projectively normal toric varieties.*

Proof. Let P be the corresponding polytope of a projectively normal toric varieties $X \subset \mathbb{P}^r$. Then P is normal and $k_P = 1$. The conclusion follows from Corollary 4.1.9. □

Remark 4.1.11. By Propositions 4.1.7 and (4.1), we can now restate the Eisenbud-Goto conjecture combinatorially as follows: if P is a non-normal very ample d-dimensional lattice polytope, then

$$k_P \leq \text{Vol}(P) - |P \cap M| + d + 1.$$

4.2 The k-normality and Regularity of Torus-invariant Subvarieties

In this section, we will give a proof that for any torus-invariant subvariety Y of $X \subset \mathbb{P}^r$, $\text{reg}(X) \geq \text{reg}(Y)$. Similar to the definition of k-normality of $X \subset \mathbb{P}^r$, we define the k-normality of polarized toric varieties as follows.

Definition 4.2.1. A polarized toric variety is a pair (X, L) of a projective toric variety and an ample line bundle L. If L is very ample, we have an embedding $X \to \mathbb{P}(\text{H}^0(X, L)) \cong \mathbb{P}^r$. The k-normality of (X, L), denoted by $k_{X,L}$ (or k_X if there is no confusions), is the smallest integer such that the restriction map

$$\text{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to \text{H}^0(X, \mathcal{O}_X(k))$$

is surjective for every $k \geq k_{X,L}$.

Now suppose that (X, L) is a polarized toric variety and P the corresponding polytope. Then the set of faces of P is 1-to-1 corresponds to the set of T-invariant subvarieties $(Y, L|_Y)$ of X by [Ale15, Theorem 2.1.3].

Lemma 4.2.2. *Let P be a very ample lattice polytope and Q a face of P. Then Q is very ample and $k_Q \leq k_P$.*

Proof. Suppose that $Q = H_{u,b} \cap P$ is a face of P for some $u \in N_{\mathbb{R}}$ and $b \in \mathbb{R}$ such that $P \subseteq H^+_{u,b}$; i.e., $\langle p, u \rangle \geq b$ for every $p \in P$ (cf. Equation (2.1)). For any $k \geq k_P$ and $x \in kQ \cap M$, by the definition of k_P, it follows that

$$x = \sum_{i=1}^{k} w_i$$

for some $w_i \in P \cap M$. Then

$$kb = \langle x, u \rangle = \left\langle \sum_{i=1}^{k} w_i, u \right\rangle = \sum_{i=1}^{k} \langle w_i, u \rangle \geq kb.$$

The equality occurs if and only if $\langle w_i, u \rangle = b$ for $1 \leq i \leq k$; i.e, each w_i lies in Q. Therefore, Q is k-normal and $k_Q \leq k_P$. It follows that Q is very ample. □

Corollary 4.2.3. *Let (X, L) be a polarized toric variety such that L is very ample and $(Y, L|_Y)$ a T-invariant subvariety of X. Then $L|_Y$ is very ample and $k_X \geq k_Y$. Furthermore,* $\text{reg}(X) \geq \text{reg}(Y)$.

Proof. Let P be the polytope associated to (X, L) and Q the face associated to $(Y, L|_Y)$. Since P is very ample, Q is also very ample by Lemma 4.2.2 and $k_Q \leq k_P$. It follows that $L|_Y$ is very ample and $k_X \geq k_Y$ (cf. Proposition 2.6.1).

The inequality $\mathrm{reg}(X) \geq \mathrm{reg}(Y)$ follows directly from the fact that $k_X \geq k_Y$, Lemma 2.2.17, and Proposition 4.1.7. □

Remark 4.2.4. To conclude this section, as noted before, we have a polytope P such that $2P$ and $3P$ are normal but $5P$ is not ([LM17, Section 4]). From this polytope P, we obtain a polarized toric variety (X, L) such that the $k_{X,2L} = k_{X,3L} = 1$, while $k_{X,5L} \gneqq 1$. We know that the $k_{X,(\dim(X)-1)L} = 1$ by Lemma 2.4.8. Hence, k-normality behaves wildly under different polarizations.

However, if (X, L) is a toric Fano projective variety such that L is very ample, we expect that the $k_{X,L} = 1$ always. The case where X is smooth is a sub-case of the Oda's question.

4.3 The Regularity of Some Non-Normal Very Ample Polytopes

In this section, we will show that the Eisenbud-Goto conjecture holds for some known examples of non-normal very ample polytopes. We first consider the following example by Gubeladze and Bruns.

Example 4.3.1 ([GB09]). Consider the polytope P which is the convex hull of the vertices given by the columns of the following matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & s & s+1 \end{pmatrix},$$

where $s \geq 4$.

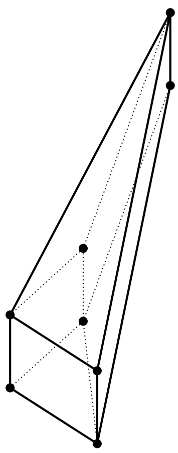

Figure 4.1: An example of Gubeladze and Bruns

We can verify directly that P is not $(s-2)$-normal for $s \geq 4$. Indeed, let $v = (1, 1, s-1)^T$.

Then $v \in (s-2)P \cap M$ since

$$\begin{pmatrix} 1 \\ 1 \\ s-1 \end{pmatrix} = \frac{s(s-2)-s-1}{s(s-2)} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{s(s-2)} \begin{pmatrix} s-2 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{s(s-2)} \begin{pmatrix} 0 \\ s-2 \\ 0 \end{pmatrix} + \frac{s-1}{s(s-2)} \begin{pmatrix} s-2 \\ s-2 \\ s(s-2) \end{pmatrix},$$

and

$$1 = \frac{s(s-2)-s-1}{s(s-2)} + \frac{1}{s(s-2)} + \frac{1}{s(s-2)} + \frac{s-1}{s(s-2)}.$$

But $v \notin \underbrace{P \cap M + \cdots + P \cap M}_{s-2}$.

Now $\mathrm{Vol}(P) = s+6$, $|P \cap \mathbb{Z}^3| = 8$, $\dim(P) = 3$, so let X be the toric variety associated to P,

$$\deg(X) = s+6 \text{ and } \operatorname{codim}(X) = |P \cap \mathbb{Z}^3| - (\dim(P)+1) = 4.$$

The Eisenbud-Goto conjecture says that $\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 1 = s+3$. By [BDGM15, Theorem 3.3], $k_P = s-1 \geq 3 \geq \deg(P)$. Hence, by Proposition 4.1.9, $\operatorname{reg}(X) = s$ and the Eisenbud-Goto conjecture holds for this example.

To compare this to the bound of Theorem 3.2.1, we have $d_P = \nu_P = 2$, $d_P + 1 \leq m_P \leq s-1$ by Lemma 3.1.2, and $k_P \leq 8(s-4)+3 = 8s-29$. We obtain the following table of the known bounds of the Castelnuovo-Mumford regularity of X:

$\operatorname{reg}(X)$	Theorem 3.2.1	Sturmfels ([Stu95])	Eisenbud-Goto ([EG84])
s	$8s-28$	$24(s+6)$	$s+3$

For $s = 4$, $8s-28 = 4$, so the bound in Theorem 3.2.1 is sharp. For $s \geq 5$, since $8s-28 > s+3$, our bound does not imply the Eisenbud-Goto conjecture.

This example is interesting in many ways. First of all, it gives an example of non-normal very ample polytopes. In addition, since P is not $(s-2)$-normal and by Proposition 4.1.7, one cannot bound the k-normality and Castelnuovo-Mumford regularity of X by any polynomial of $\dim X$. Furthermore, the polytope P gives an example of very ample polytopes that cannot be covered by very ample lattice simplices. To show this, we need the following lemma.

Lemma 4.3.2. *Let $P_1, \ldots, P_n$ be very ample lattice polytopes such that $P = \bigcup_{i=1}^n P_i$ is a convex polytope. Then P is very ample and*

$$k_P \leq \max\{k_{P_i} | i = 1, \ldots, n\}.$$

Proof. Suppose that $x \in kP$ for some $k \geq \max\{k_{P_i} | i = 1, \ldots, n\}$. Then $x \in kP_i$ for some $i = 1, \ldots, n$. Hence, x can be expressed as a sum of k lattice points in $P_i \subseteq P$. Therefore, P is k-normal for all $k \geq \max\{k_{P_i} | i = 1, \ldots, n\}$; i.e., $k_P \leq \max\{k_{P_i} | i = 1, \ldots, n\}$ and P is very ample since we know that a polytope is very ample if and only if it is k-normal for some k big enough. □

Proposition 4.3.3. *Any 3-dimensional very ample non-normal lattice polytope P cannot be covered by very ample lattice 3-simplices.*

Proof. Suppose that P can be covered by very ample 3-simplices $P = \cup_{i=1}^n P_i$. Then each P_i is normal and $k_{P_i} = 1$ by [Oga05, Proposition 2.2]. Hence, $k_P = 1$ by Lemma 4.3.2. This contradicts the assumption that P is non-normal; i.e., $k_P \geq 2$. Therefore, P cannot be covered by very ample simplices. □

From Proposition 4.3.3, it follows that the polytope P defined in Example 4.3.1 cannot be covered by very ample lattice 3-simplices.

Definition 4.3.4. For a very ample lattice polytope P, we define its holes to be the lattice points in kP that cannot be expressed at a sum of k lattice points in P, where k runs from 2 to $k_P - 1$.

Example 4.3.5. For $d \geq 3$ and $h \geq 1$, Higashitani contructed a class of d-dimensional very ample lattice polytopes $\mathcal{P}_{d,h}$ with exactly h holes ([Hig14, Theorem 1.]), as follows. Let

$$\begin{aligned} u_1 &= 0, \\ u_2 &= e_d, \\ u_3 &= e_2 + \cdots + e_{d-1}, \\ u_4 &= h(e_2 + \cdots + e_{d-1} + e_d), \\ u_5 &= (h-1)(e_2 + \cdots + e_{d-1}) + he_d, \\ u_6 &= h(e_2 + \cdots + e_{d-1}) + (h-1)e_d, \\ u_7 &= e_1 + 4e_d, \\ u_8 &= e_1 + 5e_d, \\ u_9 &= e_1 + e_2 + \cdots + e_{d-1}, \\ u_{10} &= e_1 + e_2 + \cdots + e_{d-1} + e_d, \end{aligned}$$

and

$$\begin{aligned} v_i &= e_i, & i &= 2, \ldots, d-1 \\ v_i' &= e_i + e_d, & i &= 2, \ldots, d-1, \end{aligned}$$

where $e_1, \ldots, e_d$ are the unit coordinate vectors of $\mathbb{R}^d$. Then define $\mathcal{P}_{d,h}$ to be the convex hull of

$$\{u_1, \ldots, u_{10}\} \cup \{v_i, v_i' | i = 2, \ldots, d-1\}.$$

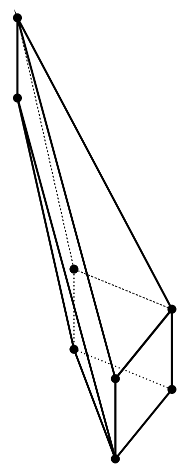

Figure 4.2: $\mathcal{P}_{3,1}$

We have $\mathcal{P}_{d,h}$ is very ample and $k_{\mathcal{P}_{h,d}} = 3$ ([Hig14, Theorem 1]). Furthermore, $\mathcal{P}_{h,d}$ has h holes and its facets are all normal ([Hig14, Lemma 5 & 6]), so the holes are interior lattice points of $\mathcal{P}_{h,d}$. Thus $\deg(\mathcal{P}_{h,d}) = \dim(\mathcal{P}_{h,d}) = d \geq 3 = k_{\mathcal{P}_{h,d}}$.

By Proposition 4.1.9, $\operatorname{reg}(X) = \deg(\mathcal{P}_{h,d}) + 1$, where $X \subseteq \mathbb{P}^r$ is the toric variety obtained from $\mathcal{P}_{d,h}$. The Eisenbud-Goto conjecture holds for X because of (4.1).

Since P is not normal, $1 < d_{\mathcal{P}_{h,d}} = 2 < m_{\mathcal{P}_{h,d}} \leq k_{\mathcal{P}_{h,d}} = 3$ by Lemma 3.1.2. Hence $d_{\mathcal{P}_{h,d}} = 2$ and $m_{\mathcal{P}_{h,d}} = k_{\mathcal{P}_{h,d}} = 3$. Theorem 3.2.1 yields

$$k_{\mathcal{P}_{h,d}} \leq (m_P - d_P - 1)n + \nu_P + 1 = \nu_P + 1 \leq n \leq 2d + 8.$$

This is stronger than the Sturmfels' bound ([Stu95]). Indeed, $|P \cap M| \geq n$, $\mathrm{Vol}(P) \geq 2$ since P is not a standard simplex, and $|P \cap M| - d - 1 > 1$, Sturmfels' result yields

$$k_P \leq |P \cap M| \cdot \mathrm{Vol}(P) \cdot (|P \cap M| - d - 1) - 1,$$

while

$$|P \cap M| \cdot \mathrm{Vol}(P) \cdot (|P \cap M| - d - 1) - 1 \geq 2n \geq 4d + 16.$$

Chapter 5

Reider's Theorem for Smooth Toric Surfaces

5.1 Nef and Ample Divisors

Let Σ be the fan of a smooth toric surface $X = X_\Sigma$. By [CLS11, Proposition 4.2.6], any Weil divisor on X is also Cartier. As in Section 2.3, any divisor D on X is of the form $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$. By [CLS11, Theorem 4.2.8], for each $\sigma \in \Sigma$, there exists $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho\rangle = -a_\rho$ for all $\rho \in \sigma$, where u_ρ is the minimal generator of the ray $\rho \in \Sigma(1)$. We can then define the support function of D as follows

$$\begin{aligned} \varphi_D : |\Sigma| &\to \mathbb{R} \\ u &\mapsto \varphi_D(u) = \langle m_\sigma, u\rangle \text{ when } u \in \sigma. \end{aligned}$$

By [CLS11, Theorem 4.2.12], this function is well defined and is integral with respect to N.

Lemma 5.1.1 ([CLS11, Lemma 6.1.13, Theorem 6.1.14]). *Assume that φ_D is the support function of a Cartier divisor $D = \sum_\rho a_\rho D_\rho$ on a complete toric variety X_Σ of dimension n. Then the following are equivalent:*

1. *D is ample;*
2. *φ_D is strictly convex;*
3. *$\langle m_\sigma, u_\rho\rangle > -a_\rho$ for all $\rho \in \Sigma(1)\backslash\sigma(1)$ and $\sigma \in \Sigma(n)$.*

Lemma 5.1.2 ([CLS11, Theorem 6.1.15]). *On a smooth complete toric variety X, a divisor D is ample if and only if it is very ample.*

Definition 5.1.3. Let X be a normal variety. Then a Cartier divisor D on X is nef if $D \cdot C \geq 0$ for all irreducible complete curve $C \subseteq X$.

Proposition 5.1.4 ([CLS11, Theorem 6.3.12]). *Let D be a Cartier divisor on a toric variety X_Σ whose fan Σ has a convex support of full dimension. Then D is nef if and only if D is basepoint free.*

For X a complete toric variety, we have a numerical criterion to check whether a Cartier divisor is ample or nef as follows.

Theorem 5.1.5 ([Mus02, Theorem 3.1 and 3.2]). *Let D be a Cartier divisor on a complete toric variety X. Then*

1. *D is nef if and only if $D \cdot C \geq 0$ for all torus invariant curves $C \subseteq X$.*
2. *D is ample if and only if $D \cdot C > 0$ for all torus invariant curves $C \subseteq X$.*

From the theorem, it is natural to ask for a combinatorial meaning of the intersection numbers $D \cdot C$. We have the following result due to Laterveer.

Lemma 5.1.6 ([Lat96, (1.4) and Page 457])**.** *Let A be an ample line bundle on a projective variety X corresponding to a polytope P. For a torus invariant curve C, let E be the corresponding edge on P. Then $A \cdot C$ is equal to the lattice length of E, i.e.,*

$$A \cdot C = |E \cap M| - 1.$$

Example 5.1.7. Consider the Hirzebruch surface $\mathscr{F}_r = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$, $r \geq 1$, whose fan Σ given by Figure 2.3.2. The ray generators of Σ are $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (-1,r)$, and $v_4 = (0,-1)$. Let the associated divisors be D_1, D_2, D_3, and D_4, respectively. By Proposition 2.3.1,

$$0 \sim \operatorname{div}(\chi^{e_1}) = \sum_{i=1}^{4} \langle e_1, v_i \rangle D_i = D_1 - D_3$$
$$0 \sim \operatorname{div}(\chi^{e_2}) = \sum_{i=1}^{4} \langle e_2, v_i \rangle D_i = D_2 + aD_3 - D_4.$$

Thus $D_3 \sim D_1$, $D_4 \sim D_2 + aD_3$, and

$$\operatorname{Pic}(\mathscr{F}_r) \simeq \{aD_3 + bD_4 \mid a, b \in \mathbb{Z}\}.$$

The maximal cones of Σ are σ_1, σ_2, σ_3 and σ_4 as in Figure 2.3.2. Let $D = mD_3 + nD_4$. We compute the m_{σ_i} to be

$$m_1 = (-a, 0), \quad m_2 = (-a, b), \quad m_3 = (rb, b), \quad m_4 = (0, 0).$$

Then by Lemma 5.1.1(3), D is very ample if and only if $a, b > 0$. The nef cone of $\mathscr{F}_r$ is given by

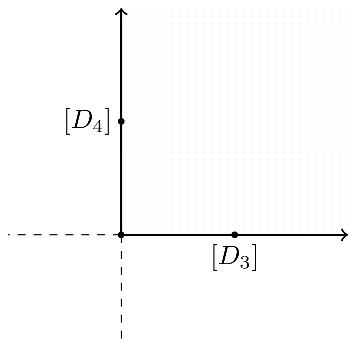

Figure 5.1: The nef cone of $\mathscr{F}_r$

The D_1 and D_3 are fibers and D_2 is the unique irreducible curve that has negative self-intersection number, which is the zero section in the total space of $\mathcal{O}_{\mathbb{P}^1}(r)$.

5.2 Smooth Toric Surfaces Revisited

We have a correspondence between smooth toric varieties and smooth lattice polytopes as follows.

Proposition 5.2.1 ([CLS11, Theorem 2.4.3])**.** *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. Then the following are equivalent:*

1. *X_P is a smooth projective toric variety.*

2. *The fan Σ is smooth: each cone $\sigma \in \Sigma_P$ is smooth; i.e., the minimal generators of σ form a part of a $\mathbb{Z}$-basis of N.*

3. *P is a smooth polytope (cf. Definition 2.4.2).*

Theorem 5.2.2 ([CLS11, Theorem 10.4.3])**.** *Every smooth complete toric surfaces is a finite blowup of either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or the Hirzebruch surface $\mathscr{F}_a$, where $a \geq 2$.*

Smooth toric surfaces are interesting objects to work with; partially because of their computability. For example, we have the following lemma.

Lemma 5.2.3 ([CLS11, Lemma 10.4.1 and Page 499])**.** *Let $u_0, \dots, u_r$ be ray generators of a smooth complete fan Σ in $N_{\mathbb{R}} \cong \mathbb{R}^2$. There exist integers $b_1, \dots, b_{r-1}$ such that*

$$u_{i-1} + u_{i+1} = b_i u_i,$$

where $u_{-1} = u_r$ and $u_{r+1} = u_0$. Let $D_i = V(u_i)$ be the prime divisors on X_Σ. The intersection matrix $(D_i \cdot D_j)_{1 \leq i,j \leq r}$ is given as follows.

$$D_i \cdot D_j = \begin{cases} -b_i & \text{if } j = i \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

As a straight corollary, we have the following lemma.

Lemma 5.2.4 ([CLS11, Proposition 10.4.11])**.** *Let $u_0, \dots, u_r$ be ray generators of a smooth complete fan Σ in $N_{\mathbb{R}} \cong \mathbb{R}^2$. Let $X = X_\Sigma$ be the smooth projective toric surface from Σ and $D_i = V(u_i)$ for $0 \leq i \leq r$. Let K_X be the canonical divisor $K_X = -\sum_{i=0}^r D_i$. Then*

$$K_X \cdot D_i = D_i^2 - 2,$$

where the $b_1, \dots, b_{r-1}$ are integers such that $u_{i-1} + u_{i+1} = b_i u_i$ for all $0 \leq i \leq r$, where $u_{-1} = u_r$ and $u_{r+1} = u_0$.

The following corollary follows directly from Lemma 5.2.3 and Lemma 5.2.4.

Corollary 5.2.5. *Let $u_0, \dots, u_r$ be ray generators of a smooth complete fan Σ in $N_{\mathbb{R}} \cong \mathbb{R}^2$. Let $X = X_\Sigma$ be the smooth projective toric surface from Σ and $D_i = V(u_i)$ for $0 \leq i \leq r$. Let K_X be the canonical divisor $K_X = -\sum_{i=0}^r D_i$. Then for $0 \leq i \leq r$,*

$$(L + K_X) \cdot D_i = L \cdot D_i - D_i^2 - 2.$$

We also know that the blowup of a toric variety corresponds to a subdivision of fan. Thus the number of generating rays of the fan corresponding to a toric surface increases after a blowup. This is because of the following proposition.

Proposition 5.2.6 ([CLS11, Proposition 3.3.15])**.** *If Δ is a fan in a lattice N, and $\sigma \in \Delta$ is a cone, the star subdivision of Δ along σ, call it Δ', is a refinement of Δ. Then the morphism $X(\Delta') \to X(\Delta)$ of toric varieties induced by identity map of N exhibits $X(\Delta')$ as the blowup of $X(\Delta)$ at the distinguished point x_σ (fixed point of the torus action).*

Example 5.2.7. We have that $\mathbb{C}^2$ is given by the fan

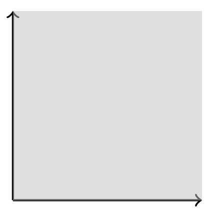

Figure 5.2: The fan of $\mathbb{C}^2$

Then the blow up of $\mathbb{C}^2$ at the origin is given by

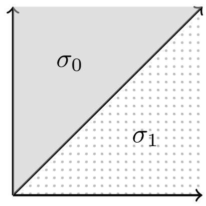

Figure 5.3: The blowup of $\mathbb{C}^2$ at the origin

Finally, we will make use of the Hodge's Index Theorem:

Lemma 5.2.8 ([Har77, Theorem V.1.9])**.** *Let D be an ample divisor on a smooth projective surface S. If E is a divisor such that $D \cdot E = 0$, then $E^2 \leq 0$. The equality occurs if and only if E is numerically equivalent to 0.*

Corollary 5.2.9 ([Har77, Exercise V.1.9])**.** *Let D be an ample divisor on a smooth projective surface S and E an arbitrary divisor. Then*

$$(D \cdot E)^2 \geq D^2 E^2.$$

Proof. Since D is ample, $D^2 > 0$. Let $H = (D^2)E - (D \cdot E)D$. We have

$$D \cdot H = (D^2)E \cdot D - (D \cdot E)D^2 = 0.$$

Then by Lemma 5.2.8, we must have $H^2 \leq 0$. In other words,

$$\begin{aligned} 0 \geq &\big((D^2)E - (D \cdot E)D\big) \cdot \big((D^2)E - (D \cdot E)D\big) \\ =&D^4E^2 - 2(D \cdot E)^2(D^2) + D^2(D \cdot E)^2 \\ =&D^2\left(D^2E^2 - (D \cdot E)^2\right). \end{aligned}$$

Since $D^2 > 0$, it follows that $(D \cdot E)^2 \geq D^2E^2$. □

5.3 Reider-type Theorem for Toric Surfaces

We will devote this section to prove the following proposition:

Proposition 5.3.1. *Let X be a smooth projective toric surface not isomorphic to $\mathbb{P}^2$, and let L be an ample line bundle on X.*

1. *Assume that the adjoint series $|K_X + L|$ is not base point free. Then there exists an effective torus-invariant divisor $D \subset X$ such that*

   $$D \cdot L = 1 \text{ and } D^2 = 0.$$

2. *Assume that the adjoint series $|K_X + L|$ is not ample. Then there exists an effective torus-invariant divisor $D \subset X$ such that either*

   $$\begin{aligned} &D \cdot L = 1 \text{ and } D^2 = -1 \text{ or } D^2 = 0; \text{ or} \\ &D \cdot L = 2 \text{ and } D^2 = 0; \text{ or} \\ &D \cdot L = 3 \text{ and } D^2 = 1. \end{aligned}$$

   Furthermore, if $L^2 \geq 10$, then there exists an effective torus-invariant divisor $D \subset X$

such that either

$$D \cdot L = 1 \text{ and } D^2 = -1 \text{ or } D^2 = 0; \text{ or}$$
$$D \cdot L = 2 \text{ and } D^2 = 0.$$

By Lemma 5.2.2, we have to show that the proposition holds for $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surfaces $\mathscr{F}_a$ ($a \geq 1$), and any blowups of them (the blowup of $\mathbb{P}^2$ is $\mathscr{F}_1$, and any sequential blowup is then a result from the blowup of $\mathscr{F}_1$). First of all, it is true for $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 5.3.2. *Proposition 5.3.1 holds for* $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Let Σ be the fan of $X = \mathbb{P}^1 \times \mathbb{P}^1$ as follows.

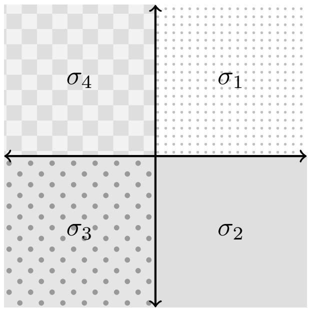

Figure 5.4: The fan of $\mathbb{P}^1 \times \mathbb{P}^1$

By Lemma 5.2.3, $D_\rho^2 = 0$ for all $\rho \in \Sigma(1)$. Thus, we need to show that there exists ρ such that $L \cdot D_\rho = 1$ in the first part and $L \cdot D_\rho \leq 2$ in the second part.

For any ample bundle L on X, if $L + K_X$ is not basepoint free, then there exists $\rho \in \Sigma(1)$ such that $(L + K_X) \cdot D_\rho < 0$. Then By lemma 5.2.4,

$$(L + K_X) \cdot D_\rho = L \cdot D_\rho - D_\rho^2 - 2 < 0.$$

This implies $0 < L \cdot D_\rho < D_\rho^2 + 2 = 2$, so that $L \cdot D_\rho = 1$.

Now suppose that $L + K_X$ is not ample and $(L + K_X) \cdot D_\rho \leq 0$. Then By lemma 5.2.4,

$$(L + K_X) \cdot D_\rho = L \cdot D_\rho - D_\rho^2 - 2 \leq 0.$$

This implies $1 \leq L \cdot D_\rho \leq D_\rho^2 + 2 = 2$. Hence, either $L \cdot D_\rho = 1$ and $D_\rho^2 = 0$ or $L \cdot D_\rho = 2$ and $D_\rho^2 = 0$. The conclusion follows. □

Secondly, we show that Proposition 5.3.1 holds for Hirzebruch surfaces.

Lemma 5.3.3. *Proposition 5.3.1 holds for* $X \cong \mathscr{F}_a$, $a \geq 1$.

Proof. Consider the Hirzebruch surface $X = \mathscr{F}_r = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$, $r \geq 1$ as in Example 5.1.7. We have

$$\operatorname{Pic}(\mathscr{F}_r) \simeq \{aD_3 + bD_4 \mid a, b \in \mathbb{Z}\}.$$

The canonical divisor of X is given by

$$K_X = -(D_1 + D_2 + D_3 + D_4) \sim -(2 - a)D_3 - 2D_4.$$

Recall that $D_1^2 = D_3^2 = 0$, $D_2^2 = -a$, $D_4^2 = a$, $D_1 \cdot D_2 = D_2 \cdot D_3 = D_3 \cdot D_4 = D_4 \cdot D_1 = 1$, and $D_1 \cdot D_3 = D_2 \cdot D_4 = 0$ (cf. Lemma 5.2.3).

Let L be an ample line bundle over $\mathscr{F}_r$. Then $L^2 > 0$. We have two cases as follows.

- If $r = 1$ then $K_X = -D_3 - 2D_4$. For L to be ample while $L + K_X$ is not nef, L has to be of the form $L \sim cD_3 + D_4$, $c > 0$. If $L \sim cD_3 + D_4$, take $D = D_3$, then

$$L \cdot D = 1 \text{ and } D^2 = 0.$$

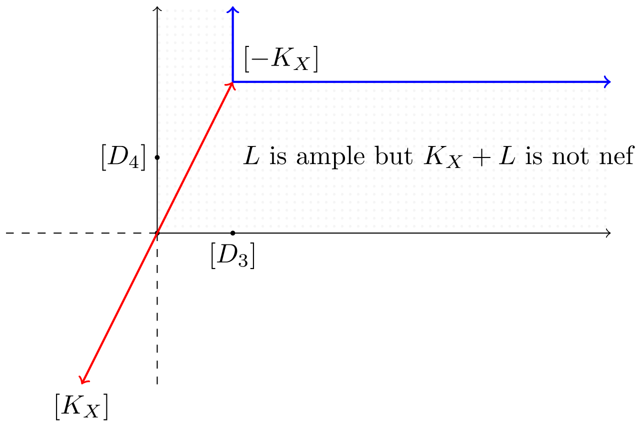

Figure 5.5: $\mathscr{F}_1$

For L to be ample while $L + K_X$ is not ample, L has to be of the form $L \sim D_3 + cD_4$, or $L \sim cD_3 + D_4$, or $L \sim cD_3 + 2D_4$, where $c \geq 1$.

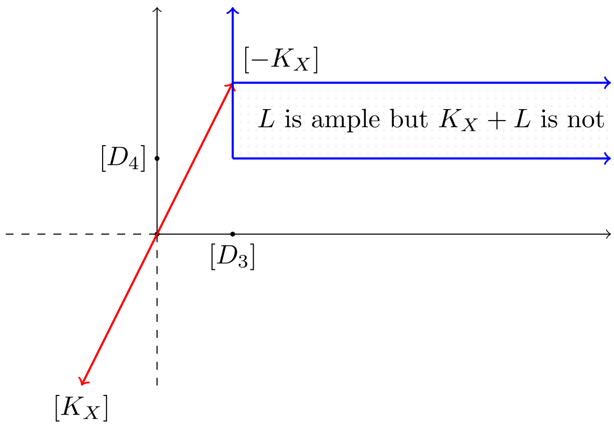

Figure 5.6: $\mathscr{F}_1$

If $L \sim D_3 + cD_4$, take $D = D_2$, then

$$L \cdot D = 1 \text{ and } D^2 = -1.$$

If $L \sim cD_3 + D_4$, take $D = D_3$, then

$$L \cdot D = 1 \text{ and } D^2 = 0.$$

If $L \sim cD_3 + 2D_4$, take $D = D_3$, then

$$L \cdot D = 2 \text{ and } D^2 = 0.$$

- $r \geq 2$: For L to be ample but $K_X + L$ is not nef, L has the form

$$L \sim D_4 + cD_3 \quad (c \geq 0).$$

Take $D = D_3$, then $L \cdot D = 1$ and $D^2 = 0$.

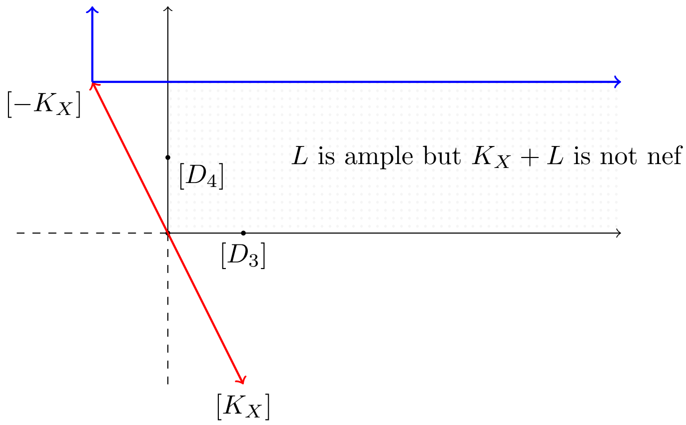

Figure 5.7: $\mathscr{F}_a$, $a \geq 2$

For L to be ample but $K_X + L$ is not, L has the form $L \sim cD_3 + D_4$ or $L \sim cD_3 + 2D_4$, where $c \geq 1$.

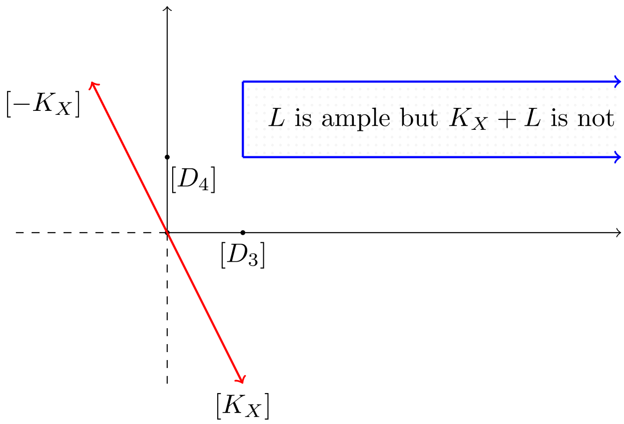

Figure 5.8: $\mathscr{F}_a$, $a \geq 2$

If $L \sim cD_3 + D_4$, take $D = D_3$, then

$$L \cdot D = 1 \text{ and } D^2 = 0.$$

If $L \sim cD_3 + 2D_4$, take $D = D_3$, then

$$L \cdot D = 2 \text{ and } D^2 = 0.$$

□

We will need the following lemmas for the proof of Proposition 5.3.1.

Lemma 5.3.4. *Let L be an ample line bundle over a smooth projective toric surface X. Let Σ be the fan of X. Suppose that Σ has $n \geq 5$ rays $\rho_1, \ldots, \rho_n$. Then for any integer $1 \leq i \leq n$,*

$$L^2 \geq L \cdot D_{\rho_i} + 4.$$

Proof. Let P be the polytope associated to L. By Pick's theorem ([Pic99]) and since L is ample so that $L \cdot D_{\rho_i} \geq 1$ for all i,

$$\mathrm{vol}(P) = \frac{L^2}{2} = \frac{|\partial P \cap M|}{2} + |P^0 \cap M| - 1, \tag{5.1}$$

where ∂P and P^0 are the sets of all boundary points and interior points of P, respectively. By Lemma 5.1.6,

$$|\partial P \cap M| = \sum_{j=1}^{n} L \cdot D_{\rho_j}. \tag{5.2}$$

Hence, combining (5.1) and (5.2) gives

$$L^2 = \sum_{j=1}^{n} L \cdot D_{\rho_j} + 2|P^0 \cap M| - 2 \geq L \cdot D_{\rho_i} + (n-1) + 2|P^0 \cap M| - 2.$$

Since $n \geq 5$, by Lemma 2.2.11, $|P^0 \cap M| \geq 1$. Therefore,

$$L^2 \geq L \cdot D_{\rho_i} + 4.$$

□

Lemma 5.3.5. *Let $v_1, \ldots, v_5$ be lattice points such that no three points are collinear. Then there exists a lattice point in* $\mathrm{conv}(v_1, \ldots, v_5) \backslash \{v_1, \ldots, v_5\}$.

Proof. Let the coordinates of v_i be (x_i, y_i) for $i = 1, \ldots, 5$. By the pigeonhole principle, there must be $i \neq j$ such that $x_i \equiv x_j \pmod 2$ and $y_i \equiv y_j \pmod 2$. Then the midpoint m of $v_i v_j$ is a lattice point. Since no three points in $\{v_1, \ldots, v_5\}$ are collinear, it follows that $m \in \mathrm{conv}(v_1, \ldots, v_5) \backslash \{v_1, \ldots, v_5\}$. □

Lemma 5.3.6. *Let P be a lattice polygon that has at least 5 vertices and assume that one of its edges has lattice length 4. Then* $\mathrm{Vol}(P) \geq 9$.

Proof. It suffices to prove the lemma when P is a lattice pentagon. Let $P = \mathrm{conv}(v_1, \ldots, v_5)$, where $v_1, \ldots, v_5$ are ordered clockwise in M. Without loss of generality suppose that the lattice length of the edge joining v_1 and v_5 is 4; i.e., there are 3 other lattice points y_1, y_2, y_3 in between v_1 and v_5.

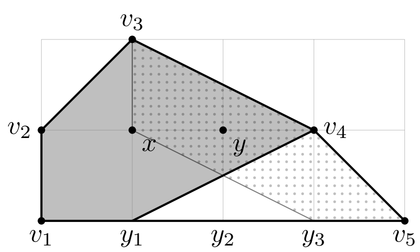

Figure 5.9: A lattice pentagon that has an edge whose lattice length is 4

Consider the polytope $Q = \mathrm{conv}(v_1, v_2, v_3, v_4, y_1)$. Then by Lemma 2.2.11, there must be a lattice point x in the interior of Q. Then x lies in at most one of the segments v_1v_3, y_1v_3, y_2v_3, y_3v_3, v_5v_3. If x lies in v_1v_3 or if x does not lie in any mentioned segments, consider the set of 5 points $\{x, v_3, v_4, v_5, y_1\}$. By Lemma 5.3.5, there must be another lattice point y in P that is not the same as anyone listed before. If $y \in \partial P$, then $|\partial P \cap M| \geq 9$ and $|P^0 \cap M| \geq 1$. By Pick's theorem ([Pic99]),

$$\mathrm{Vol}(P) = |\partial P \cap M| + 2|P^0 \cap M| - 2 \geq 9.$$

If $y \in P^0$, then $|\partial P \cap M| \geq 8$ and $|P^0 \cap M| \geq 2$. By Pick's theorem ([Pic99]),

$$\mathrm{Vol}(P) = |\partial P \cap M| + 2|P^0 \cap M| - 2 \geq 10.$$

If x lies in v_3y_1 or v_3y_2 then we get such a point y from $\mathrm{conv}(x, v_3, v_4, v_5, y_3)$. If x lies in v_3y_3 or v_3v_5 then we get y from $\mathrm{conv}(v_1, v_2, v_3, x, y_2)$. The same argument follows and we proved the lemma. □

Finally, we will give the proof for the final case of Proposition 5.3.1, when X is an arbitrary blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface.

Proof of Proposition 5.3.1. By the classification of smooth projective toric surfaces, Lemma 5.2.6, and the proofs for $\mathbb{P}^1 \times \mathbb{P}^1$ (Lemma 5.3.2) and $\mathscr{F}_a$ (Lemma 5.3.3), it suffices to prove the proposition in the case that the fan Σ of X has at least 5 rays.

We first prove part 1. Suppose that $K_X + L$ is not basepoint free. Then there exists $\rho \in \Sigma(1)$ such that $(K_X + L) \cdot D_\rho < 0$. Take $D = D_\rho$. By Lemma 5.2.4,

$$(L + K_X) \cdot D = L \cdot D - D^2 - 2 < 0.$$

This implies $L \cdot D < D^2 + 2$, so since L is ample,

$$0 \leq L \cdot D - 1 \leq D^2. \tag{5.3}$$

- If $D^2 \leq -1$, then $L \cdot D \leq 0$, which is a contradiction to the hypothesis that L is ample.
- If $D^2 = 0$, either $D \cdot L = 0$ or $D \cdot L = 1$. But $D \cdot L > 0$ since L is ample. Thus $D \cdot L = 1$. The proposition holds for this case.

It remains to show that D^2 cannot be positive. Since the fan of X contains at least 5 rays, by Lemma 5.3.4,

$$L^2 \geq L \cdot D + 4. \tag{5.4}$$

In addition, it follows from Lemma 5.2.9 that

$$(L \cdot D)^2 \geq L^2 \cdot D^2. \tag{5.5}$$

Combining (5.5) with (5.3) and (5.4) yields

$$(L \cdot D)^2 \geq (L \cdot D - 1)(L \cdot D + 4) = (L \cdot D)^2 + 3L \cdot D - 4.$$

This implies $L \cdot D \leq 1$. The only possibility is $L \cdot D = 1$. Then by (5.5), $D^2 = L^2 = 1$, which is impossible since $L^2 \geq L \cdot D + 4 = 5$. Therefore, it cannot be the case that $D^2 > 0$.

We now prove the second part of the proposition. Suppose that $K_X + L$ is not ample, so there exists $\rho \in \Sigma(1)$ such that $(K_X + L) \cdot D_\rho \leq 0$. Let $D = D_i$. By Lemma 5.2.5,

$$(L + K_X) \cdot D = L \cdot D - D^2 - 2 \leq 0.$$

This implies $L \cdot D \leq D^2 + 2$; hence,

$$1 \leq L \cdot D \leq D^2 + 2. \tag{5.6}$$

- If $D^2 = -1$, then $1 \leq L \cdot D \leq 1$, so $L \cdot D = 1$.
- If $D^2 = 0$, either $D \cdot L = 1$ or $D \cdot L = 2$.

Now we consider the case that $D^2 \geq 1$. Since the fan of X contains at least 5 rays, by Lemma 5.3.4,

$$L^2 \geq L \cdot D + 4. \tag{5.7}$$

By Lemma 5.2.9,

$$(L \cdot D)^2 \geq L^2 \cdot D^2 \tag{5.8}$$

Since $D^2 \geq 1$, then by (5.7), $L^2 \geq 5$. Thus by (5.8), $(L \cdot D)^2 \geq L^2 \cdot D^2 \geq 5$, so $L \cdot D > 2$. It follows that $L \cdot D \geq 3$. Hence, $L \cdot D - 2 \geq 1$. This inequality combining with (5.6) and (5.7) yields

$$(L \cdot D)^2 \geq (L \cdot D - 2)(L \cdot D + 4) = (L \cdot D)^2 + 2L \cdot D - 8.$$

This implies $L \cdot D \leq 4$. The only possibilities are $L \cdot D = 3$ or $L \cdot D = 4$.

- If $D^2 = 1$ then $L \cdot D \leq 3$ by (5.6). Since $L \cdot D$ can only be either 3 or 4, $L \cdot D = 3$ in this case. Furthermore, suppose that $L^2 \geq 10$. If $L \cdot D = 3$ and $D^2 = 1$ then $9 = (L \cdot D)^2 < 10 \leq L^2 \cdot D^2$, a contradiction to (5.8).
- Now assume that $D^2 \geq 2$. If $L \cdot D = 3$, then $L^2 \geq 7$ by (5.7), and $L^2 \cdot D^2 \geq 7 \cdot 2 = 14 > 9 = (L \cdot D)^2$, a contradiction to (5.8). Now assume that $L \cdot D = 4$. Then the polygon P_L associated to L has at least 5 vertices and one of its edges has lattice length 4 by Lemma 5.1.6. Hence, $L^2 \geq 9$ by Lemma 5.3.6. It follows that $16 = (L \cdot D)^2 < 18 \leq L^2 \cdot D^2$, a contradiction to (5.8).

The proposition follows. □

Remark 5.3.7. The converse of Proposition 5.3.1 trivially holds by Corollary 5.2.5.

The following corollary gives an affirmative answer for a stronger form of Fujita's conjecture (Conjecture 1.3.1) in case of smooth complete toric surfaces. Note that for n-dimensional toric varieties, the Fujita's conjecture is in fact a corollary of [Fuj03, Corollary 0.2] and [Pay06, Theorem 1].

Corollary 5.3.8 ([Fuj03, Pay06])**.** *Let X be a smooth complete surface not isomorphic to $\mathbb{P}^2$. Let L be an ample line bundle on X such that $L \cdot C \geq 2$ for all toric invariant curve $C \subset X$. Then $\mathcal{O}_X(K_X + L)$ is globally generated. If $L^2 \geq 10$ and $L \cdot C \geq 3$ for all toric invariant curve $C \subset X$, then $\mathcal{O}_X(K_X + L)$ is very ample.*

Proof. Suppose that $\mathcal{O}_X(K_X + L)$ is not globally generated. By Proposition 5.3.1, there exists a toric invariant curve C such that $L \cdot C = 0$ or $L \cdot C = 1$, a contradiction. □

As a corollary, we have a stronger form of [Laz94, Corollary 2.7] for smooth toric surfaces as follows.

Corollary 5.3.9. *If A is an ample line bundle on a smooth complete toric surface X not isomorphic to $\mathbb{P}^2$, then $|K_X + 2A|$ is nef, and $|K_X + 4A|$ is very ample.*

Proof. Take $L = 2A$, then for any toric invariant curve $C \subset X$, $L \cdot C = 2A \cdot C \geq 2$. By Proposition 5.3.1, $|K_X + 2A|$ is nef. Similarly, take $L' = 4A$, then $(L')^2 = 16A^2 > 10$, and $L \cdot C = 4A \cdot C \geq 4$. By Proposition 5.3.1, $|K_X + 4A|$ is very ample. □

Remark 5.3.10. It would be interesting to see if we can apply the classification in Proposition 5.3.1 to the study of Iskovskikh-Shokurov conjecture ([IS05]) for conic bundles over smooth toric surfaces.

Appendix A

Counterexamples of the Eisenbud-Goto Conjecture

With some counterexamples of the Eisenbud-Goto conjecture recently given by McCullough and Peeva ([MP17]), we hope to produce some toric counterexamples based on those. The idea is to calculate the Hilbert series (or the Hilbert polynomial) of the counterexamples and try to construct some lattice polytopes with the same Hilbert series (resp., polynomial).

Example A.0.1 ([MP17, Example 4.7]). Let $S = k[u, v, w, x, y, z]$ and

$$I = \langle u^6, v^6, u^2w^4 + v^2x^4 + uvwy^3 + uvxz^3 \rangle.$$

Consider the ideal $M \subset W = S[w_1, w_2, w_3]$ of the Rees algebra $S[It]$, with $\deg(w_i) = 1$ for $i = 1, 2, 3$. Then using Macaulay2 ([GS]), one can see that $\operatorname{maxdeg}(M) = 38$, $\deg(W/M) = 31$, and $\operatorname{pd}(W/M) = 5$. Since $\dim(W) = 9$, Bertini's Theorem ([Fle77]) yields a projective threefold in $\mathbb{P}^5$ whose degree is 31 but its regularity is 38.

We used Macaulay2 to calculate the Hilbert series of M, but it is too complicated to be written out and we highly doubt if there exists any lattice polytope with the same Hilbert series. The other counterexamples in [MP17] are rather too complicated to be calculated. Therefore, it would be really interesting if one can produce a toric counterexample for the Eisenbud-Goto conjecture.

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Index