Geometric and Non-Geometric Backgrounds of String Theory

George Moutsopoulos

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*George Moutsopoulos*)

Abstract

This thesis explores the geometry of string theory backgrounds and the non-geometric features of string theory that arise due to T-duality. For this reason, it is divided into two complementary parts. Part I deals with the superalgebras of symmetries of string theory and M-theory backgrounds, the so-called Killing superalgebras. It is shown that one can define a Lie superalgebra consisting of the infinitesimal field-preserving isometries and the supersymmetries of the background. We also explore the extension of a Killing superalgebra with brane charges. Part II deals with non-geometric backgrounds. In particular, we adopt the framework of the doubled geometry, also known as the doubled torus. We analyze the hamiltonian dynamics of the system and quantize a model T-fold. Finally we extended the doubled torus system to include worldsheet supersymmetry. Throughout part II, we focus on the equivalence, classical and quantum, of the doubled formalism with the conventional formulation.

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Dedicated to my family

Contents

Chapter 1

Introduction

String theory is currently the most prominent theory put forward to describe all fundamental interactions, including gravity. In principal, it can be defined as the study of the quantum relativistic string. In this chapter we describe some of its main ingredients, before we set out our perspective and aim of the thesis.

The theory originates in the guise of the so-called dual models, which were put forward in the '60s to explain hadronic scattering [1]. The dual models, though, were abandoned with the successes of quantum chromodynamics. It was later found [2, 3] that a massless helicity-two particle in the theory has the properties of the graviton. The discovery of the graviton in the string spectrum remains one of the most striking and prominent features of string theory. By the late '80s there were five consistent string theories in 10 spacetime dimensions, which were considered inherently different.

One can appreciate the advantage of quantum gravity derived from string theory in the following way. The low energy dynamics of the five string theories are described by various supergravity theories of critical dimension 10. The graviton fluctuates around its classical value as a massless string oscillation mode. Whereas a canonical quantization of gravity is in principle non-renormalizable, string theory smooths the ultraviolet divergences by replacing graviton Feynman interactions with conformal worldsheet diagrams. This suggests that string theory can describe concisely quantum gravity with no ultraviolet divergences.

One might ask how these extra dimensions are related to a 4-dimensional world. It is interesting to take the extra dimensions literally [4] and there are in principle two ways of deriving an effective 4-dimensional theory. In the first approach [5], the extra dimensions are compact and of relatively small size. Assuming a typical radius of the compact dimensions R, the massless fields can be decomposed as a sum of Fourier modes around the compact directions, thereby giving a tower of effective masses n/R [6]. For low energies, one would only observe the massless states. In general, the massless spectrum of the so-called Kaluza-Klein compactification depends on the internal, compact manifold that is used [7]. A different approach is to combine the ideas of brane-world scenarios [8] with the D-brane dynamics of string theory [9, 10]. In this approach the 4-dimensional universe is a 3-brane floating in an ambient space.

The quantum dynamics of string theory are further enriched with the dynamics of D-branes. These are surfaces on which open strings can end and couple to the gauge fields of a supersymmetric field theory [9]. The AdS/CFT correspondence is a widely held and tested conjecture, that conformal field theories, defined on typically 4 dimensions, are dual to string theories defined on $\mathrm{AdS}_5 \times S^5$ spaces [11]. A further exciting result is a microscopic explanation of black hole entropy [12]. These developments point towards a precise holographic principle of quantum gravity in the context of string theory.

Nevertheless, it is difficult to extract definite experimental signatures from string theory. One expects the string length to be of order of the Planck scale and, therefore, the first string corrections are expected at very high energies. String phenomenology is an active area of research. Interesting topics that are pursued include large extra dimensions phenomenology [13], black hole production at accelerators [14] and various string inspired field theories [15]. The next half a decade will be very fruitful in the field of particle physics. With the LHC probing energies of the TeV scale, it is very possible that new physics beyond the standard model will emerge.

At the same time, string theory remains an active area of research with many of its non-perturbative features unknown. A promising feature is the realization that the five different string theories are limiting cases of a unifying membrane theory. It is known that the five different theories are related, at certain regimes of their moduli space, by quantum dualities [16, 17, 18]. The so-called M-theory is a conjectured non-perturbative theory that exists in 11 dimensions and gives the various string theories as limits. 11-dimensional supergravity would then be a low energy limit of M-theory.

This thesis examines two theoretical aspects of string theory. The first aspect involves the geometry of string backgrounds, described at low energies by supergravity theories. In particular, we prove that supergravity backgrounds come with a Lie superalgebra structure, where the bosonic elements are the field-preserving Killing vectors of the background and the fermionic elements are its supersymmetries.

The second aspect involves a feature unique to string theory, that of T-duality. This duality is non-geometric, in the sense that one cannot describe it with smooth maps. This allows for the exotic case of a T-fold, that is a consistent string background that does not have a smooth geometry. In particular, we will study the "non-geometricity" of string theory by using a duality-symmetric formulation [19] that allows us to lift the non-geometric background to a globally defined smooth manifold.

The following sections introduce some material that allow us to establish a link with the overall picture. Firstly, we describe how the massless modes of string theory correspond to supergravity fields. We then introduce T-duality and motivate the duality-symmetric formulation. We conclude with an outline of the thesis.

1.1 String Theory Backgrounds

An important feature of string theory is that the consistency of the worldsheet dynamics determines the geometry of spacetime. At the same time the massless states describe the quantum fluctuations around the classical background. Furthermore, we will see that a consistent supersymmetric string determines a geometry described by supergravity.

A bosonic string can couple to a metric g, a 2-form b and a scalar field, the dilaton ϕ, with dynamics prescribed by the Polyakov action

$$S_P = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\big(\sqrt{\eta}\eta^{\alpha\beta}g_{mn}+\epsilon^{\alpha\beta}b_{mn}\big)\partial_\alpha X^m\partial_\beta X^n+\frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{\eta}R^{(2)}\phi\ , \quad (1.1)$$

where $R^{(2)}$ is the scalar curvature of the worldsheet metric $\eta_{\alpha\beta}$. For ϕ constant, $g_s = \exp(-\phi)$ can be taken as a worldsheet coupling constant [20]. The sigma model will fail to preserve the conformal invariance unless the beta functions vanish [21]. The beta functions can be calculated perturbatively in α' and to zeroth order we get the condition that the dimension is fixed to be $d = 26$. At higher orders in α', we get differential conditions on the data (g, b, ϕ), which can be equivalently encoded in the spacetime action [22]

$$S_s = \int_{\mathscr{M}} d^dx\sqrt{|g|}e^{-\phi}\left(R+|d\phi|_g^2-\frac{1}{12}|H|_g^2\right)+\mathcal{O}(\alpha')\ .$$

At the same time, the massless string states describe the quantum fluctuations around the classical background. For definiteness let us take a flat spacetime with vanishing b-field and dilaton, so the solution that minimizes (1.1), in the light cone gauge, is $X^i = X^i_R + X^i_L$ with

$$\begin{aligned} X^i_R(\sigma^-) =& \frac{1}{2}(X^i_0+p^i_0\sigma^-)+\frac{1}{\sqrt{2}}\sum_{k\neq 0}\frac{ia^i_k}{k}e^{-ik\sigma^-} \\ X^i_L(\sigma^+) =& \frac{1}{2}(X^i_0+p^i_0\sigma^+)+\frac{1}{\sqrt{2}}\sum_{k\neq 0}\frac{i\tilde{a}^i_k}{k}e^{-ik\sigma^+}\ , \end{aligned} \quad (1.2)$$

for $i = 1, \ldots 8$. The massless states in this case are $a^i_{-1}\tilde{a}^j_{-1}\,|0; p_0 = 0\rangle$, $i, j = 1\ldots 8$, acting on a Fock space vacuum. Under the isotropy group SO(24) of the light cone, the massless fields reduce according to $24 \otimes 24 = 299 \oplus 276 \oplus 1$, that correspond to fluctuations around the background fields, respectively of g_{mn}, b_{mn} and ϕ. Coherent states of this type, can further correct the effective lagrangian of (1.1).

There are two shortcomings of the bosonic string, namely the vacuum $|0\rangle$ is tachyonic and the theory contains no spacetime fermions. It is for these reasons

Sector	spacetime	SO(8) irreps	massless fields
NS-NS	boson	$8_v \otimes 8_v = 35 \oplus 28 \oplus 1$	g_{mn}, b_{mn}, ϕ
NS-R	fermion	$8_v \otimes 8_s = 8_s \oplus 56_s$	Ψ_m, λ
R-NS	fermion	$8_s \otimes 8_v = 8_s \oplus 56_s$	Ψ'_m, λ'
R-R	boson	$8_s \otimes 8_s = p$-forms	Ramond-Ramond fields

Table 1.1: The massless states of the closed string. The fields in the last column are identified according to their light-cone SO(8) irreducible representation and a subscript v (s) denotes a vector (spinor) representation.

that we consider supersymmetric string theory. For simplicity, let us take a closed string that couples to a flat metric g. In the superconformal gauge, the Polyakov action generalizes to

$$S_{II} = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{\eta} g_{mn} \left(\eta^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n + i\overline{\psi}^m \ \partial\!\!\!/ \psi^n \right)$$

and is invariant under a global worldsheet supersymmetry, [23].

Similarly to the bosonic closed string, the solutions can be split into left-handed and right-handed parts, which are respectively given by their chiral components. But now for a closed string, the chiral components of the fermion fields $\psi = (\psi_+, \psi_-)$ can be either periodic (Ramond) or antiperiodic (Neveu-Schwarz) along the string circumference. Furthermore, the zero modes of the Ramond sector furnish on-shell a representation of the Clifford algebra $C\ell(8)$. The Ramond vacuum is thus an SO(8) spinor. Conformal invariance requires that we consider all left-handed times right-handed sectors: NS-NS, R-R, NS-R and R-NS. The massless sector is represented in table 1.1 .

The vacuum of the NS sector is tachyonic, but there is a consistent truncation of the spectrum, the so-called Gliozzi-Scherk-Olivie (GSO) projection [24], that projects out the tachyon. The NS sector is projected with $1 - (-1)^F$, where the operator $(-1)^F$ anticommutes with the NS fermions. The R sector is projected under $1 \pm (-1)^F$, where now the operator $(-1)^F$ also acts on the spinorial SO(8) Ramond vacuum as the chirality operator. Choosing the opposite sign of the GSO projection for the left- versus the right-handed Ramond vacuum defines the type IIa string and the massless fields are the field content of type IIa supergravity. Choosing the same sign for the GSO projection is the type IIb string and the massless field content is that of type IIb supergravity [25].

The type I string can be obtained as an orientifold of the type IIb theory, whereby we keep the diagonal sum of the type IIb chiral gravitini. The massless field content is that of type I supergravity. The heterotic string takes a left-handed fermionic superstring (d=10) and a right-handed bosonic string (d=26). Conformal invariance is guaranteed, if the extra 16 dimensions are those of an even, self-dual lattice, corresponding to the weight lattices of the groups SO(32) and $E_8 \times E_8$ [26]. The massless field content is that of heterotic supergravity.

Supergravity theories are very interesting in their own right. They are the extensions of Einstein gravity which have spacetime supersymmetry. More precisely, they reproduce the Poincaré supergroup with either $(1,0)$, $(1,1)$ or $(2,0)$ chiral supersymmetries, so that the graviton is the highest helicity representation within the field content. Although we shall focus on the supergravity theories, we have shown here how these give consistent backgrounds for superstring theory.

One of the supergravity theories we will describe is the maximal d=11 supergravity. This holds a primary role in supergravity theory. By dimensional reduction we get type IIa supergravity that is in turn T-dual to type IIb supergravity [27]. Classically, a type IIa background lifts to a d=11 background compactified on a circle of radius $R_{11} = \exp(2\phi/3)$. Therefore, the strong coupling of type IIa string theory suggests a theory of one extra dimension, whose low energy limit is 11-dimensional supergravity.

In the next section, we focus on T-duality, which relates type IIa with type IIb string theory. It is an introduction that will motivate the second part of the thesis, namely a T-dual formulation of string theory.

1.2 T-duality

String theory possesses a duality-symmetry that does not have an analogue in quantum point-particle mechanics. Two string theories are equivalent on respective manifolds that are related by a non-diffeomorphic map. In fact, for the case of the type II string, it relates two theories with different field content.

To illustrate this, let us first consider a bosonic closed string on a flat manifold, whose 26th dimension $X \equiv X^{26}$ is a circle of radius R. Because the coordinate is identified under a translation of $2\pi R$, we can allow a winding mode around the spacetime circle

$$X(\sigma + 2\pi, \tau) = X(\sigma, \tau) + 2\pi m R \ , \quad m \in \mathbb{Z} \ .$$

Furthermore, because of quantization, the momentum of the center of mass[1] is quantized. The solution of (1.2) generalizes to

$$\begin{aligned} X_R(\sigma^-) =& \frac{1}{2} X_0 + p_R \sigma^- + \frac{1}{\sqrt{2}} \sum_{k \neq 0} \frac{i a_k}{k} e^{-ik\sigma^-} \\ X_L(\sigma^+) =& \frac{1}{2} X_0 + p_L \sigma^+ + \frac{1}{\sqrt{2}} \sum_{k \neq 0} \frac{i \tilde{a}_k}{k} e^{-ik\sigma^+} \ , \end{aligned} \tag{1.3}$$

with

$$\begin{aligned} p_L + p_R &\in \frac{2\pi}{R} n \ , \qquad && n \in \mathbb{Z} \\ p_L - p_R &\in 2\pi R m \ , && m \in \mathbb{Z} \ . \end{aligned}$$

T-duality in this case is the map from the bosonic string defined on a circle of radius R to a bosonic string defined on a circle of radius $1/R$, whereby the ordinary derivatives of the maps tranform as $\dot{X}_L \to \dot{X}_L$ and $\dot{X}_R \to -\dot{X}_R$. In particular, the map exchanges the winding mode m with the momentum integer n. We notice that T-duality is a map between two points in the moduli space of the bosonic string, respectively defined at R and $1/R$. That is, the map on the worldsheet operators is accompanied with a map of the geometry. We refer to the latter as the Buscher rules.

One can check that the two theories are equivalent by comparing the energy, momentum and the spin of a given configuration under T-duality. The corresponding values match exactly. More importantly, one can show that the

[1] we set henceforth $\alpha' = 1$

partition function is invariant under T-duality. It is computed [20] to be

$$Z(\tau; R) = |\eta(\tau)|^2 \sum_{m,n\in\mathbb{Z}} \exp\left(-\pi\tau_2\left(\frac{n^2}{R^2} + m^2R^2\right) + 2\pi i\tau_1 mn\right) ,$$

where η is the Dedekind function. We see that the partition function is invariant, by simply replacing the dummy variables $m \leftrightarrow n$. The partition function captures, to first order in the string coupling constant, the spectrum of the bosonic string. One can show that the partition function defined on worldsheets of higher genus is again invariant.

The case of T-duality on a circle is a general feature of bosonic string theory on a manifold where there is a compact field-preserving isometry [28, 29]. The dual geometry is given, for vanishing dilaton, by the Buscher rules

$$\begin{aligned}
\tilde{g}_{kk} &= \frac{1}{g_{kk}} \\
\tilde{g}_{ki} &= \frac{b_{ki}}{g_{kk}} \\
\tilde{g}_{ij} &= g_{ij} + \frac{g_{ik}g_{jk} - b_{ik}b_{jk}}{g_{kk}} \\
\tilde{b}_{ki} &= \frac{g_{ki}}{g_{kk}} \\
\tilde{b}_{ij} &= b_{ij} + \frac{g_{ik}b_{kj} - g_{jk}b_{ki}}{g_{kk}} \ .
\end{aligned}$$

In the formulation of [30], the two theories are shown to be equivalent as conformal field theories, by expressing them as descendants of a one-dimension higher parent CFT where the isometry is represented by a chiral symmetry.

T-duality is also a perturbative duality of the superstring theories. On the type II string it reverses the sign of the chirality operator, thereby exchanging the type IIa with the type IIb string. T-duality will also act on the open string spectrum, exchanging Dirichlet with Neumann conditions. Not surprisingly, the Buscher rules express a symmetry map between type IIa and type IIb supergravity. By Kaluza-Klein reduction on a circle, the two type II supergravity theories reduce to the unique $N = 2$ supergravity theory in 9 dimensions [27].

We will be interested mainly in bosonic T-duality and a useful concept is that of the Narrain lattice. A compactification on an n-dimensional torus leads to the grouping of the winding and momentum modes into a $2n$ vector, so that (p_R, p_L) takes values in a self-dual even lattice Λ, defined by the radii of the torus. One can show [31] that T-duality acts as the group $O(n, n, \mathbb{Z})$, which is generated by

- Large diffeomorphisms of the torus in $\mathrm{GL}(n, \mathbb{Z})$,
- Gauge shifts of the b-field $\delta b \in \mathrm{Mat}(n, \mathbb{Z})$,
- Factorized T-duality that are inversions on each of the compactified radii of the torus.

The group $O(n, n, \mathbb{Z})$ effectively acts on the Narrain lattice Λ. Nevertheless, it is known that even self-dual lattices are connected under $O(n, n, \mathbb{R})$ transformations and equivalent under $O(n) \times O(n)$ transformations, thereby giving the moduli space $O(n, n)/ \left(O(n) \times O(n)\right)$. But as we have described, the moduli space of string theory should be further identified under T-duality, giving the space

$$\frac{O(n, n)/ \left(O(n) \times O(n)\right)}{O(n, n, \mathbb{Z})} \, .$$

This is to be considered as the moduli space of equivalent, that is T-dual, string theories compactified on a torus T^n.

The Narrain lattice provides us with a conceptual tool to identify the moduli space of tori compactifications. Furthermore, it hints towards the necessity of enlarging the space of string compactifications to accommodate for T-duality. A typical scattering process involves the addition of a vertex operator, e^{ipx}, at some moment on the worldsheet. As T-duality is a symmetry of string theory, one might envision the need to consider separate conjugate momenta to the left and right-handed momenta. This is related to dualizing the theory, so that both $X = X_L + X_R$ and its T-dual coordinate $\tilde{X} = X_L - X_R$ are considered. It is conjectured that string field theory would necessitate such a formulation and allow the flow of the renormalization group into directions that are hidden by T-duality.

Considering a duality-symmetric formulation of T-duality has appeared at the level of equations of motion in [32, 33] and as a lagrangian system with no manifest Lorentz symmetry in [34]. These formulations are related to the so-called doubled geometry, or doubled torus formulation, which was proposed by C. Hull in [19]. T-duality allows for the construction of well-defined string backgrounds that do not have a smooth spacetime interpretation, which are called T-folds. Such T-folds can arise either as Scherk-Schwarz reductions with T-duality twists [35], or even from a conventional string background after a nontrivial T-duality is performed [36]. The doubled geometry provides a framework for studying T-folds.

1.3 Outline of Thesis

The thesis is logically partitioned into two parts. Part I deals with smooth features of string theory backgrounds. We mainly study the Killing superalgebra of supergravity solutions and its superalgebra extensions. Part II deals with the non-smooth features that arise due to T-duality. We study the duality-symmetric formulation of string theory known as the doubled torus. In particular, we shall analyze the system in its hamiltonian formulation and quantize a model T-fold.

In chapter 2 we give an overall presentation of supersymmetry in d=11 supergravity. The theory is characterized by 32 supersymmetries, a solution generating symmetry modeled on the d=11 Poincaré superalgebra. We find that the superalgebra of charges capture some further charges that extend the super-Poincaré algebra. The Killing superalgebra of the background is also associated to every background. This consists of the supersymmetries that leave the background invariant and the field-preserving isometries of the background. Most of chapter 2 contains well known concepts. However, we present some formulae[2] explicitly. The extension of the charge superalgebra motivates the question of when and how a Killing superalgebra can be extended. We return to this question in chapter 5.

[2]such as the curvature of the superconnection and the ADM superalgebra written in a universal and covariant way.

In chapter 3 we construct the Killing superalgebra of type IIb backgrounds. We explain the construction in detail and present the only nontrivial calculations one has to perform. These are the proof that the odd-odd bracket closes on the even subalgebra and that the odd-odd-odd Jacobi identity is satisfied. This last calculation requires an intricate interplay of the supergravity Killing equations, verifying that supersymmetry in supergravity has such a tight structure that allows this construction. A theorem is also replicated from [37], which shows that enough supersymmetry guarantees local homogeneity of the background.

Chapter 4 is devoted to heterotic string backgrounds. The construction of the Killing superalgebra is repeated for this theory. Whereas many aspects are similar, we find that heterotic backgrounds are often much simpler. In particular, the holonomy of the supersymmetry transformations is contained in the spin group of the geometry, thereby on the one hand simplifying the calculations and on the other hand providing some stronger results concerning the homogeneity of heterotic backgrounds.

In chapter 5 we return to the problem of extending the Killing superagebra with additional geometric objects. In particular, we restrict our attention to the so-called minimally full extensions. We show that whereas the Killing superalgebras of the maximally supersymmetric flat and the Freund-Rubin spaces admit full extensions, the maximally supersymmetric pp-wave does not.

Part II deals with the T-dual symmetric formulation of the string [19]. Chapter 6 is a review of the doubled geometry formalism in its lagrangian formulation. In particular, we introduce a conceptual framework, whereby a doubled geometry is a target spacetime manifold obeying some restrictions and a doubled torus system is a constrained pseudo-lagrangian system. We discuss T-duality in this framework.

In chapter 7 we analyze the doubled torus system using Dirac's theory of constraints. Most importantly, we find that the constraints are of primary and second class. We investigate the Dirac dynamics and discuss the energy momentum tensor. We conclude with an interesting discussion of "what is T-duality?" from the point of view of the hamiltonian formulation of the T-dual string. We contrast this picture with the lagrangian formulation of chapter 6.

Chapter 8 uses the tools developed in the previous chapters, to study T-folds. In particular, we specify a model T-fold and quantize the string canonically. We investigate the Hilbert space and most importantly write the partition function. The partition function is found to be modular invariant and equivalent to the calculations of the model, when treated as an asymmetric orbifold in the conventional formulation.

Finally, chapter 9 deals with the supersymmetric extension of the doubled torus. The pseudo-lagrangian of the doubled torus can be extended with supersymmetric fields in the usual fashion. The constraint of the doubled torus system, though, requires special attention. We propose a supersymmetric extension of the constraint and provide arguments that show the classical equivalence to the conventional formulation.

We conclude the thesis with a short discussion of both parts. Appendix A contains some generic conventions used. Appendix C contains the Killing superagebra of the maximally supersymmetric wave. The results of chapters 3 and 4 have been presented in collaboration with E. Hackett-Jones and J. M. Figueroa O'Farrill in [38]. The results of chapters 7, 8 and 9 have been presented in collaboration with E. Hackett-Jones in [39].

Part I

Geometry

Chapter 2

Supersymmetry in M-theory

In this chapter we review 11-dimensional supergravity and elaborate on three different but closely related superalgebra structures.

In section §2.1 we set up our conventions and introduce the theory. In section §2.2 we talk about the supersymmetry of the theory, that is a superalgebra of infinitesimal variations that preserve (on-shell) the equations of motion. As a gauge symmetry that incorporates diffeomorphisms, the superalgebra of variations gives rise to a superalgebra of charges only under certain conditions. The superalgebra of ADM supercharges, given in section §2.3, is precisely such a superalgebra, and can be defined whenever the solution is asymptotically flat. Finally, we talk about the Killing superalgebra, which is the superalgebra of infinitesimal variations that leave the background invariant.

This chapter serves as an introduction to supersymmetry in string theory and M-theory. The following chapters will concentrate on the Killing superalgebra of type IIb and heterotic string backgrounds. We shall finally return to M-theory Killing superalgebras and examine its extension in chapter 5.

2.1 M-theory Backgrounds

11-dimensional supergravity is the unique field theory in 11 dimensions with 32 supercharges such that the highest spin field it contains is the graviton. The theory was constructed using Noether's method to complete a theory with supersymmetry [40]. The theory itself has a fascinating history and until the early '80s was considered in its own right a candidate for 'the' fundamental quantum field theory of nature [7]. Various non-renormalizability and chirality issues resulted in string theory research superseding realistic supergravity models. Nevertheless, the theory surfaced again as the low energy limit of M-theory.

The field content of 11-dimensional supergravity is the mostly minus metric g, a closed 4-form field strength F, which can locally be written as $F = dA$, and a Rarita-Schwinger field, the gravitino Ψ_M of odd Grassman parity. The dynamics are given by the lagrangian

$$\begin{aligned}\mathcal{L} =& -\frac{1}{4}R\,(\omega)\,\mathrm{dvol} - \frac{1}{2}\bar{\Psi}_M\Gamma^{MNP}D_N\left(\frac{\omega+\hat{\omega}}{2}\right)\Psi_P\,\mathrm{dvol} - \frac{1}{2}F\wedge *F\\ &- \frac{1}{3!2^5}\bar{\Psi}_M\left(\Gamma^{MNABCD} + 12\Gamma^{CD}e^{AN}e^{BM}\right)\Psi_N\left(F+\hat{F}\right)_{ABCD}\mathrm{dvol}\\ &+ \frac{1}{3}rF\wedge F\wedge A\ ,\end{aligned}$$

where r is the sign of the volume element in a representation of the Clifford algebra $\mathrm{C}\ell(1,10)$, which is generated by the gamma matrices that obbey

$$\{\Gamma^A, \Gamma^B\} = -2\eta^{AB}\ .$$

In the lagrangian we have defined two 'supercovariant' tensors

$$\begin{aligned}\hat{\omega}_{MAB} =& \omega_{MAB} - \frac{1}{4}\bar{\Psi}_C\Gamma_{MAB}{}^{CD}\Psi_D\\ \hat{F} =& F + \frac{3}{4!}\bar{\Psi}\Gamma_{AB}\wedge\Psi\wedge e^A\wedge e^B\end{aligned} \tag{2.1}$$

They are called supercovariant, as they transform under supersymmetry in a covariant way. In particular, the equation of motion of the gravitino is by construction supercovariant

$$\Gamma^{MNP}\mathcal{D}_N\Psi_N = 0\ ,$$

with $\mathcal{D}$ the supercovariant derivative

$$\mathcal{D}_M = D_M(\hat{\omega}) - \frac{1}{3!4!}\left(8\Gamma^{BCD}e^A_M + \Gamma^{ABCD}{}_M\right)\hat{F}_{ABCD}$$

and D is the exterior spin derivative.

As supergravity is considered a classical limit of a quantum theory, it is most natural to take solutions for which the gravitino vanishes. Such backgrounds are called bosonic M-theory backgrounds and are defined by the manifold $\mathscr{M}$ with metric g, the spinor bundle S and the field strength F. The bosonic part of the lagrangian is just

$$\mathcal{L} = -\frac{1}{4}R\,\mathrm{dvol} - \frac{\lambda^2}{2}F \wedge *F + \frac{\lambda^3}{3}F \wedge F \wedge A \ . \tag{2.2}$$

We use $\lambda = 1$, but it is often useful to let the normalization of F vary, in order to make contact with other conventions.

2.2 M-theory Supersymmetry

The supergravity action is invariant [41] under the following transformations

$$\begin{aligned}\delta e^A =& \bar{\varepsilon}\Gamma^A\Psi \\ \delta\Psi =& D(\hat{\omega})\varepsilon - \frac{1}{3!4!}\left(8\Gamma^{BCD}e^A_M + \Gamma^{ABCD}{}_M\right)\hat{F}_{ABCD}\varepsilon\, dx^M = \mathcal{D}\varepsilon \\ \delta A =& -\frac{1}{4}\bar{\varepsilon}\Gamma_{AB}\Psi \wedge e^A \wedge e^B \ ,\end{aligned} \tag{2.3}$$

where D is the spin covariant exterior derivative and ε is a spinor field of odd Grassman parity. The commutator algebra of supersymmetry variations generate on-shell diffeomorphisms, spin rotations and gauge transformations [40]. Supersymmetry is a solution-generating symmetry of supergravity solutions. Much more can be said though. As an example, consider the well-known fact that in supersymmetric field theories, the fermionic equations of motion transform under supersymmetry into the bosonic equations of motion.

Indeed, let us define the curvature of the supercovariant derivative $R^{\mathcal{D}}{}_{MN} = [\mathcal{D}_M, \mathcal{D}_N]$, where the supercovariant derivative is, when $\Psi = 0$, in any conventions[1]

$$\mathcal{D}_M = D_M(\omega) + aF_{MABC}\Gamma^{ABC} + bF_{ABCD}\Gamma_M{}^{ABCD} ,$$

with $a = 8b$. The term $\Omega = \mathcal{D} - D$ can be written in the convenient form

$$\Omega_X = \lambda\left(\frac{1}{12}X \cdot F - \frac{1}{4}F \cdot X\right) ,$$

for all vector fields X. We present the curvature $R^{\mathcal{D}}{}_{MN}$ on page 18. By varying the gravitino equation of motion, with respect to an arbitrary supersymmetry transformation, and then setting $\Psi_M = 0$, we get $\Gamma^{MNP}R^{\mathcal{D}}{}_{MN} = 0$. Contracting with Γ_P, we *equivalently* have $\Gamma^M R^{\mathcal{D}}{}_{MN} = 0$.

We find that the transformation of the gravitino equation is not just onto the whole bosonic sector of equations of motion, but quite curiously there is 'room' for more information. It is tantalizing to view this as a sign for hidden symmetries that are related to the holonomy of the supercovariant connection [42, 43, 44, 45]. In any case, the equations of motion we get from $\Gamma^M R^{\mathcal{D}}{}_{MN} = 0$ are

$$\begin{aligned} R_{NC} =& - \lambda^2(2F^2_{NC} - \frac{2}{3}F^2\eta_{NC}) \\ d * F =& \lambda F \wedge F \\ dF =& 0 , \end{aligned}$$

which are precisely those from the lagrangian (2.2) with $\lambda = -r\frac{3!4!a}{8}$. This calculation provides an immediate argument that enough supersymmetry invariance of a background can guarantee most of the bosonic equations of motion.

[1]fixing $\lambda = -r\frac{3!4!a}{8}$ is related to the so-called trombone symmetry, which amounts to a redefinition of fields.

Expanding $R^{\mathcal{D}}{}_{MN}$ gives

$$\begin{aligned}
-R^{\mathcal{D}}_{MN} =& - R^{\nabla}_{MN} + a(\nabla_N F_{MA_1\ldots A_3})\Gamma^{A_1\ldots A_3} + b(\nabla_N F_{A_1\ldots A_4})\Gamma_M{}^{A_1\ldots A_3} \\
& - a(\nabla_M F_{NA_1\ldots A_3})\Gamma^{A_1\ldots A_3} - b(\nabla_M F_{A_1\ldots A_4})\Gamma_N{}^{A_1\ldots A_4} \\
& + a_{11} F_{A_1\ldots A_4} F_{B_1\ldots B_4} \varepsilon_{MN}{}^{A_1\ldots A_4 B_1\ldots B_4}{}_C \Gamma^C \\
& + a_{21} F_{MA_1A_2C_1} F_N{}^{A_1A_2}{}_{C_2} \Gamma^{C_1C_2} \\
& + a_{22} F_M{}^{A_1\ldots A_3} F_{A_1\ldots A_3 C_1} \Gamma^{C_1}{}_N \\
& - a_{22} F_N{}^{A_1\ldots A_3} F_{A_1\ldots A_3 C_1} \Gamma^{C_1}{}_M \\
& + a_{23} F_{A_1\ldots A_4} F^{A_1\ldots A_4} \Gamma_{MN} \\
& + a_{31} F_{MA_1\ldots A_3} F_{B_1\ldots B_4} \varepsilon^{A_1\ldots A_3 B_1\ldots B_4}{}_{NC_1\ldots C_3} \Gamma^{C_1\ldots C_3} \\
& - a_{31} F_{NA_1\ldots A_3} F_{B_1\ldots B_4} \varepsilon^{A_1\ldots A_3 B_1\ldots B_4}{}_{MC_1\ldots C_3} \Gamma^{C_1\ldots C_3} \\
& + a_{41} F_{MA_1A_2C_1} F^{A_1A_2}{}_{C_2C_3} \Gamma^{C_1\ldots C_3}{}_N \\
& - a_{41} F_{NA_1A_2C_1} F^{A_1A_2}{}_{C_2C_3} \Gamma^{C_1\ldots C_3}{}_M \\
& + a_{42} F_{MNA_1C_1} F^{A_1}{}_{C_2C_3C_4} \Gamma^{C_1\ldots C_4} \\
& + a_{51} F_{MA_1\ldots A_3} F_{NA_4\ldots A_6} \varepsilon^{A_1\ldots A_6}{}_{C_1\ldots C_5} \Gamma^{C_1\ldots C_5} \\
& + a_{52} F_N{}^{A_1}{}_{B_1\ldots B_2} F_{A_1B_3\ldots B_5} \varepsilon_M{}^{B_1\ldots B_5}{}_{C_1\ldots C_5} \Gamma^{C_1\ldots C_5} \\
& - a_{52} F_M{}^{A_1}{}_{B_1\ldots B_2} F_{A_1B_3\ldots B_5} \varepsilon_N{}^{B_1\ldots B_5}{}_{C_1\ldots C_5} \Gamma^{C_1\ldots C_5} \\
& + a_{53} F^{A_1A_2}{}_{B_1B_2} F_{A_1A_2B_3B_4} \varepsilon_{MN}{}^{B_1\ldots B_4}{}_{C_1\ldots C_5} \Gamma^{C_1\ldots C_5}
\end{aligned}$$

The coefficients are given by

$$\begin{aligned}
a_{11} &= 2b^2 r & a_{21} &= 3(4!)^2 b^2 & a_{22} &= -2^3 4! b^2 & a_{23} &= -4! 2 b^2 \\
a_{31} &= -\frac{8}{3} b^2 r & a_{41} &= (4!)^2 b^2 & a_{42} &= 4! 2^5 b^2 & a_{51} &= \frac{4}{5} b^2 r \\
a_{52} &= -\frac{4}{5} b^2 r & a_{53} &= \frac{6}{5} b^2 r
\end{aligned}$$

The coefficients $a = 8b$ are usually given by $b = -\frac{r}{4!3!}$, while r is the sign of the volume element $\mathrm{dvol} \in \mathrm{C}\ell(\mathscr{M}, g)$ in the representation.

2.3 Superalgebra of Charges

In equation (2.3), we gave the supersymmetry variations that transform the action into a boundary term. It is a pedestal of field theory that symmetries give rise to conserved charges and vice versa. Since supersymmetry generates diffeomorphisms, the existence of conserved currents is rather subtle. Indeed, if we were able to find a local expression for the energy-momentum of a gravitating system, this would contradict the equivalence principle.

Nevertheless under certain conditions, one can define conserved Noether quantities corresponding to spacetime symmetries [46, 47, 48]. We shall use the lagrangian Noether method of [49, 50, 51] to find the supercharges that correspond to supersymmetry. For this we need the boundary term into which the action transforms. This is

$$\begin{aligned}\delta_{\text{susy}}\mathcal{L} = &+ \frac{1}{2\times 3!} d(\bar{\Psi}\Gamma_{ABC} \wedge (D(\hat{\omega}) + \Omega(\hat{F}))\varepsilon \wedge *e^{ABC}) \\ &- \frac{1}{3!} d(\bar{\varepsilon}\Gamma_{ABC} \wedge (D(\hat{\omega}) + \Omega(\hat{F})\wedge)\Psi \wedge *e^{ABC}) \\ &+ d(\delta_{\text{susy}} A \frac{\partial \mathcal{L}}{\partial F}) \ .\end{aligned} \tag{2.4}$$

The terms proportional to ε cancel, because of supersymmetry. Now on the other hand, if we denote the Euler-Lagrange equations of any field $\phi = g, A, \Psi$ by $\delta S/\delta\phi$, the variation of the lagrangian under any variation, and in particular the supersymmetry variation, gives

$$\begin{aligned}\delta_{\text{susy}}\mathcal{L} = &+ \frac{1}{2\times 3!} d(\bar{\Psi}\Gamma_{ABC} \wedge \delta_{\text{susy}}\Psi \wedge *e^{ABC}) + d(\delta_{\text{susy}} A \frac{\partial \mathcal{L}}{\partial F}) \\ &+ \delta_{\text{susy}}\bar{\Psi}\frac{\delta S}{\delta \bar{\Psi}} + \delta_{\text{susy}} A \frac{\delta S}{\delta A} + \delta_{\text{susy}} e \frac{\delta S}{\delta e} \ .\end{aligned} \tag{2.5}$$

By subtracting the two variations of the lagrangian, equations (2.4) and (2.5), one obtains the so-called Noether cascade equations

$$\begin{aligned}&- \frac{1}{3!} d(\bar{\varepsilon}\Gamma_{ABC} \wedge (D(\hat{\omega}) + \Omega(\hat{F})\wedge)\Psi \wedge *e^{ABC}) \\ &\qquad = \delta_{\text{susy}}\bar{\Psi}\frac{\delta S}{\delta \bar{\Psi}} + \delta_{\text{susy}} A \frac{\delta S}{\delta A} + \delta_{\text{susy}} e \frac{\delta S}{\delta e} \ .\end{aligned} \tag{2.6}$$

We partially integrate the left-hand side of (2.6) and write down the equivalent

$$-\frac{1}{3!}d(\bar{\varepsilon}\Gamma_{ABC}\wedge(\Omega(\hat{F})\wedge\Psi\wedge+\Psi\wedge D(\hat{\omega}))*e^{ABC})+\frac{1}{3!}d(D(\hat{\omega})\bar{\varepsilon}\,\Gamma_{ABC}\wedge\Psi\wedge *e^{ABC})$$
$$=\delta_{\rm susy}\bar{\Psi}\frac{\delta S}{\delta\bar{\Psi}}+\delta_{\rm susy}A\frac{\delta S}{\delta A}+\delta_{\rm susy}e\frac{\delta S}{\delta e}\ . \quad (2.7)$$

We can now define a Noether supercurrent and supercharge as follows. First let us define the *bare* Noether current $J\in S\otimes T^*\mathscr{M}$ as

$$*J=-\frac{1}{3!}\Gamma_{ABC}\wedge(\Omega(\hat{F})\wedge\Psi\wedge+\Psi\wedge D(\hat{\omega}))*e^{ABC}$$

and the *bare* superpotential $U\in S\otimes\Lambda^2T^*\mathscr{M}$ as

$$*U=+\frac{1}{3!}\Gamma_{ABC}\wedge\Psi\wedge *e^{ABC}\ .$$

Equation (2.7) becomes

$$d(\bar{\varepsilon}*J)+d(D(\hat{\omega})\bar{\varepsilon}\,*U)=\delta_{\rm susy}\bar{\Psi}\frac{\delta S}{\delta\bar{\Psi}}+\delta_{\rm susy}A\frac{\delta S}{\delta A}+\delta_{\rm susy}e\frac{\delta S}{\delta e}\ . \quad (2.8)$$

If we use a set of spinors ε^i, $i=1,\dots 32$, we can expand the spinor ε as $\varepsilon=\rho_i\,\varepsilon^i$. We then define the covariant quantities, the Noether currents $J^i\in T^*\mathscr{M}$ and the superpotentials $U^i\in\Lambda^2T^*\mathscr{M}$:

$$\begin{aligned}*J^i&=*\,\bar{\varepsilon}^iJ+*(D(\hat{\omega})\bar{\varepsilon}^i*U)\\ *U^i&=-\,\bar{\varepsilon}^iU=-\frac{1}{3!}\bar{\varepsilon}^i\Gamma_{ABC}\wedge\Psi\wedge *e^{ABC}\ .\end{aligned}$$

Since ρ_i and $d\rho_i$ are arbitrary, using equation (2.8) and keeping terms proportional to ρ_i we get the first Noether conservation law, which says

$$d*J^i=0\ ,$$

when Ψ, A and g are on-shell. The terms proportional to $d\rho_i$ give the second

Noether conservation law, which says

$$*J^i + d * U^i = 0 \ ,$$

when Ψ is on-shell. This implies $d * J^i = 0$ when Ψ is on-shell alone. Both conservation laws are true irrespective of the choice of the ε^i.

We define the quantities Q^i as

$$Q^i = \int_\Sigma *J^i \ .$$

By using the second Noether conservation law, we can write them as

$$Q^i = -\int_{\partial\Sigma} *U^i \ .$$

We thus see that the quantities Q^i depend on the boundary of the $(d-1)$-surface Σ and any two surfaces Σ and Σ' with the same boundary, $\partial\Sigma = \partial\Sigma'$, yields the same charge. Furthermore, since U^i is proportional to ε^i, the charges depend only on the value of the spinors ε^i on the boundary.

Although the spinors ϵ^i were chosen arbitrarily we can be more specific. Let us choose Σ to be a space-like surface that extends to the *asymptotic* boundary of the manifold and $S = \partial\Sigma$ is then spatial infinity. If the surface is asymptotically flat, then the ϵ^i approach at spatial infinity constant spinors, that are the asymptotic Killing spinors of the background. Their fall off rate can be further constrained with a differential constraint $\nabla\!\!\!\!/\,\varepsilon = 0$. Such a constraint seems natural to preserve a gauge choice of the gravitino. It is also necessary for the superalgebra to close, as we shall now see. The fall off rate however does not influence the vale of the Q^i.

If we vary the supercharge with respect to a supersymmetry ε^j, we obtain

$$\delta_{\varepsilon^j} Q^i = \int_{\partial\Sigma} \frac{1}{3!} \bar{\varepsilon}^i \Gamma_{ABC} (D(\hat{\omega}) + \Omega(\hat{F})) \varepsilon^j \wedge *e^{ABC} \ .$$

Using

$$\Gamma^{ABC}\Omega_B(\hat{F}) = \frac{1}{4}\Gamma_{CD}\hat{F}^{ACDB} + \frac{1}{48}\Gamma^{ACC_1\ldots C_4}\hat{F}_{C_1\ldots C_4} \ ,$$

we can write the right-hand side as

$$\delta_{\varepsilon^j} Q^i = P + Q_{\rm e} + Q_{\rm m} \ , \tag{2.9}$$

with

$$P = \int_{\partial\Sigma} \frac{1}{3!}\bar{\varepsilon}^i\Gamma_{ABC}D(\hat{\omega})\varepsilon^j \wedge *e^{ABC} \tag{2.10}$$

$$Q_{\rm e} = -\frac{1}{2}\int_{\partial\Sigma}\left(\frac{1}{2!}\bar{\varepsilon}^i\Gamma_{AB}\varepsilon^j e^{AB}\right) \wedge *F \tag{2.11}$$

$$Q_{\rm m} = +r\frac{1}{2}\int_{\partial\Sigma}\left(\frac{1}{5!}\bar{\varepsilon}^i\Gamma_{A_1\cdots A_5}\varepsilon^j e^{A_1\cdots A_5}\right) \wedge F \ . \tag{2.12}$$

Note that these equations hold irrespective of $S = \partial\Sigma$ and the spinors ε^i.

Equation (2.9) is an extended version of the Poincaré superalgebra. Let us assume that the spacetime is asymptotically flat and the spinors ε^i approach constant spinors at spatial infinity $S = \partial\Sigma$. By expanding the term P, we find the Nester-Witten form [52, 53], which gives the ADM momentum for asymptoticaly flat spacetimes, along with a term that corresponds to the charge of a KK monopole [54]

$$P = \int_{\partial\Sigma}\frac{1}{4}\omega_{AB}\bar{\varepsilon}^i\Gamma_C\varepsilon^j \wedge *e^{ABC} - \int_{\partial\Sigma}\frac{1}{4\times 3!}\bar{\varepsilon}^i\Gamma_{ABCDE}\varepsilon^j\omega^{DE} \wedge *e^{ABC} \ .$$

The Nester-Witten form has well-known origins in supergravity [55, 56]. Besides the additional gravitational charge, the odd-odd bracket of charges is extended to include an electric charge Q_e and a magnetic charge Q_m.

The superalgebra of charges foretells the existence of electric and magnetic sources of charge and could be derived from a supersymmetric worldvolume action with a Wess-Zumino term [57].

2.4 Killing Superalgebras

Given an M-theory background $(\mathscr{M}, S, g, F)$, one can ask the question, what are the isometries of the background. That is, which diffeomorphisms of the manifold leave the metric and the field F invariant. An infinitesimal symmetry K of the background is a vector field that satisfies $\mathscr{L}_K g = \mathscr{L}_K F = 0$. We shall call such vector fields **(supergravity) Killing vectors**. In particular, its derivative $A = -\nabla K \in \mathrm{End}(T\mathscr{M})$, should belong to the Lie algebra of $so(T\mathscr{M}, g)$, or equivalently via the metric should reduce to a two-form.

Since a Killing vector K is a pseudo-riemannian Killing vector, it is defined by its value and the value of its first derivative at a point. Therefore, supergravity Killing vectors span a finite-dimensional space $\mathfrak{k}_0$, whose dimesnion can range from 0 to $11(11+1)/2 = 66$. Indeed, for two Killing vectors K^i, $i = 1, 2$, one can prove an integrability condition, Killing's identity

$$\nabla_X A_{K^i} = R(K^i, X) \tag{2.13}$$

and the commutation property

$$[A_{K^i}, A_{K^j}] = A_{[K^i, K^j]} + R(K^i, K^j) \ . \tag{2.14}$$

Proof of (2.13) and (2.14). We shall prove (2.13) using coordinates. The identity translates into $K_{m;np} - R_{mnpq}K^q = 0$. Define the tensor $\Omega_{mnp} = K_{m;np} - R_{mnpq}K^q$ and compute that

$$\Omega_{mpn} = K_{m;pn} - R_{mpnq}K^q = K_{m;np} - R_{pnmq}K^q - R_{mpnq}K^q = \Omega_{mnp} \ ,$$

where in the last equation we used the algebraic Bianchi identity. But note also that $\Omega_{mnp} = -\Omega_{nmp}$, due to the Killing property and the antisymmetry properties of the curvature. Hence $\Omega_{mnp} = \Omega_{mpn} = -\Omega_{nmp} = 0$. The proof of (2.14) is faster without the use of coordinates. We have

$$\begin{aligned}
\nabla_X[K^i, K^j] &= \nabla_X(\nabla_{K^i} K^j - \nabla_{K^j} K^i) \\
&= \nabla_X(-A_{K^j} K^i + A_{K^i} K^j) \\
&= -\nabla_X(A_{K^j}) K^i + \nabla_X(A_{K^i}) K^j - A_{K^j} \nabla_X K^i + A_{K^i} \nabla_X K^j \\
&= -R(K^j, X) K^i + R(K^i, X) K^j + A_{K^j} A_{K^i} X - A_{K^i} A_{K^j} X \\
&= \big(-[A_{K^i}, A_{K^j}] + R(K^i, K^j)\big)\, X \ ,
\end{aligned}$$

which yields (2.14). □

It is because of Killing's identity that a pseudo-riemannian Killing vector is defined by its value and the value of its first derivative at a point. This can be equivalently understood as a correspondence of pseudo-riemannian Killing vectors with parallel sections of the vector bundle $\mathcal{E} = T\mathscr{M} \oplus so(T\mathscr{M})$ with respect to the connection

$$\mathcal{D}^{\mathcal{E}}_X \begin{pmatrix} Y \\ A \end{pmatrix} = \begin{pmatrix} \nabla_X Y + A(X) \\ \nabla_X A - R(X, Y) \end{pmatrix} \ ,$$

where $Y + A \in T\mathscr{M} \oplus so(T\mathscr{M})$ and $X \in T\mathscr{M}$. Supergravity Killing vectors will also form a Lie algebra, since $[\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{L}_{[X,Y]}$. Therefore, if X and Y preserve the metric and the field F, so will $[X, Y]$. Since this Lie algebra is a finite-dimensional subalgebra of the Lie algebra of vector fields, the Jacobi identity is automatically satisfied.

A different question one can ask is how much supersymmetry a background preserves. That is to say, how many spinor fields leave it invariant under the supersymmetry transformations of (2.3). Since the gravitino is set to zero, such a spinor field ε will satisfy the equation $\mathcal{D}_M \varepsilon = 0$. We call such spinor fields **(supergravity) Killing spinors**. The equation defining them is a first order $\mathbb{R}$-linear differential equation. Therefore, they span a real finite-dimensional space $\mathfrak{k}_1$ and are determined by their value at a point. The dimension of $\mathfrak{k}_1$ can vary from 0 to 32. Equivalently, Killing spinors can be defined as the finite-dimensional space of spinor fields that are left invariant under the holonomy group of the connection

$\mathcal{D}$ [58].

The $\mathbb{Z}_2$-graded vector space $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_0$ can be given the structure of a Lie superalgebra [37]. Indeed, it is well known that Killing spinors square to Killing vectors, which also preserve F. The odd-odd bracket of the Killing superalgebra is therefore the map

$$\begin{aligned} [-,-] : S^2\mathfrak{k}_1 &\longrightarrow \mathfrak{k}_0 \\ \varepsilon &\mapsto \bar{\varepsilon}\gamma^m\varepsilon \ , \end{aligned} \tag{2.15}$$

while the even-even bracket is the Lie bracket of vector fields. The even-odd bracket of a Killing vector field K and a Killing spinor ε is defined as the action of the spinorial Lie derivative

$$\mathscr{L}_K\varepsilon = \nabla_K\varepsilon - \frac{1}{4}\nabla_M K_N \Gamma^{MN}\varepsilon \ .$$

The action closes on Killing spinors, because K preserves F and so $[\mathscr{L}_K, \mathcal{D}_X] = \mathcal{D}_{[K,X]}$. That is, if $\mathcal{D}\varepsilon = 0$ and so ε is a Killing spinor, then the same holds for $\mathscr{L}_K\varepsilon$. Note that in this construction, we mod out the Grasmann odd parity of the spinors. The bracket in (2.15) is therefore symmetric.

The even-odd-odd Jacobi identity of the Killing superalgebra holds, because the spinorial Lie derivative is compatible with the spinorial inner product

$$\mathscr{L}_X\left(\overline{\varepsilon}\Gamma^{A_1\cdots A_n}\varepsilon'\right) = \overline{\mathscr{L}_X\varepsilon}\Gamma^{A_1\cdots A_n}\varepsilon' + \overline{\varepsilon}\Gamma^{A_1\cdots A_n}\mathscr{L}_X\varepsilon' \ .$$

The even-even-odd Jacobi identity is the geometric identity $[\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{L}_{[X,Y]}$ acting on Killing spinors. The odd-odd-odd Jacobi identity reduces to an algebraic identity, which was shown to hold in [37] using symbolic computing.

Yet another question one can ask is whether the supersymmetry of a background can determine the geometry of the space. In [37], J. Figueroa O'Farrill proved a useful theorem, which says that enough supersymmetry guarantees local homogeneity of the background. In particular, if the amount of supersymmetry is larger than the critical fraction of 24/32, then the tangent bundle is spanned pointwise by a local frame of supergravity Killing vectors. In the following two chapters, we shall establish the same results for type IIb and heterotic supergravity, namely the existence of a Killing superalgebra for the type IIb and heterotic string backgrounds and prove the equivalent homogeneity theorems.

The type IIa results follow from those in [37] and show that 24+ implies local homogeneity. Indeed, a 24+ IIa background oxidises to a 24+ background of eleven-dimensional supergravity, which is therefore locally homogeneous. The eleven-dimensional geometry is the total space of a circle bundle over the IIa geometry. Furthermore, the Killing spinors of the eleven-dimensional supergravity background are constant on the fibers and so are the Killing vectors obtained by squaring them. By dimensional reduction, this shows that the IIa background is locally homogeneous.

The existence of the Killing superalgebra of M-theory backgrounds is a consequence of the supersymmetry variations (2.3). That is to say, for Killing spinors, the infinitesimal variations will reproduce the Killing superalgebra of the background. However we saw that under certain asymptotic conditions, the ADM superalgebra of charges contains additional conserved quantities. The symmetric square of a Killing spinor gives likewise a Killing one-form along with a two-form and five-form. This suggests that the Killing superalgebra might be extended, for certain backgrounds, with higher-degree (supergravity) Killing forms coming from the square of Killing spinors. We shall return to this problem in chapter 5.

Chapter 3

Type IIb Killing Superalgebra

In this chapter we repeat the construction made for M-theory in [37] for the case of type IIb supergravity. That is, the infinitesimal bosonic and fermionic symmetries of a type IIb bosonic background have the structure of a Lie superalgebra, which is termed the Killing superalgebra of the background. The bosonic symmetries are the Killing vectors that leave invariant all the fields of the type IIb background. The fermionic symmetries are the Killing spinors of the background.

In section §3.1 we set up our spinorial conventions and prove a useful Fierz identity. In §3.2 we set up our conventions for type IIb supergravity. In §3.3 we show that a Killing Lie superalgebra can indeed be constructed. There are only two calculations that are involved with respect to the construction. The first is to show that Killing spinors square to vectors that preserve all type IIb fields. The second is to prove the validity of the odd-odd-odd Jacobi identity. Having shown that the structure is indeed that of a Lie superalgebra, we discuss in §3.4 some type IIb Killing superalgebras. We prove a useful theorem, which shows that enough supersymmetry implies, at least locally, the homogeneity of the background.

3.1 Spinors in 1+9 Dimensions

In this section we set up our conventions for spinors in $1+9$ dimensions. At first, we describe the spinor module, the spin invariant inner product and the bispinor representation as a reducible representation of the orthogonal group. We then describe bispinors as spinor endomorphisms and deduce a useful Fierz identity.

3.1.1 Spin modules

The conventions we use are those of a mostly positive lorentzian metric η and the Clifford algebra $\mathrm{C}\ell(1,9)$, which is generated by vectors X in the ideal

$$X \cdot X = +|X|_\eta^2 \ ,$$

where $|X|_\eta^2$ the squared length of the vector. A representation of the Clifford algebra is generated by the gamma matrices γ^m that satisfy

$$\gamma^m \gamma^n + \gamma^n \gamma^m = 2\eta^{mn} \mathbb{1} \ . \tag{3.1}$$

As a real associative algebra [59], the Clifford algebra is isomorphic to a matrix algebra, $\mathrm{C}\ell(1,9) = \mathrm{C}\ell(9,1) = \mathrm{Mat}(32,\mathbb{R})$. Hence, there is only one irreducible representation given by real 32×32 matrices. These act on the Clifford module of spinors Δ, which is a real vector space of dimension 32. The representation further reduces under the spin group $\mathrm{Spin}(1,9)$ and the module splits into two real 16-dimensional spaces $\Delta_+ \oplus \Delta_-$ that are chiral. That is to say, they are the eigenspaces of the volume form in the Clifford algebra

$$\mathrm{dvol} \cdot \Delta_\pm = \pm \Delta_\pm \ .$$

There is a symplectic inner product on Δ, given by the charge conjugation matrix $C = \gamma^0$. We shall use the following notation

$$\begin{aligned} &\Delta \otimes \Delta \longrightarrow \mathbb{R} \\ &(\varepsilon_1, \varepsilon_2) \mapsto \langle \varepsilon_1, \varepsilon_2 \rangle \equiv \bar{\varepsilon}_1 \varepsilon_2 := \varepsilon_1^t C \varepsilon_2 \ . \end{aligned}$$

Besides being symplectic $C^t = -C$, the inner product satisfies

$$\langle X-, -\rangle = -\langle -, X-\rangle \ , \tag{3.2}$$

for any vector X, or equivalently $C\gamma_m = -\gamma_m^t C$. The symplectic adjoint of any p-form c is thus

$$c^* = (-1)^{p(p+1)/2} c \ .$$

In particular, the inner product is $\mathrm{Spin}(1,9)$ invariant. Furthermore, the spin modules $\Delta_\pm$ are lagrangian with respect to the inner product

$$\langle \mathrm{dvol}\, -, -\rangle = -\langle -, \mathrm{dvol}\, -\rangle \ .$$

The symplectic inner product C is non-degenerate and pairs Δ_+ with Δ_-. We can, therefore, identify Δ_- with the dual of Δ_+.

As representations of the spin group, the tensor square $\Delta^{\otimes 2}$ is isomorphic to the exterior algebra of the vector representation. The tensor square of the chiral representation $\Delta_+^{\otimes 2}$, though, is smaller and can be deduced as follows. A bispinor $(\varepsilon, \varepsilon') \in \Delta \otimes \Delta$ squares for any n to the n-form ξ_n

$$\xi_n(\varepsilon, \varepsilon') = \frac{1}{n!}\, \bar{\varepsilon}\gamma_{a_1 \ldots a_n}\varepsilon'\, e^{a_1 \ldots a_n} \ . \tag{3.3}$$

The property in (3.2), along with the symplectic property $C = -C^t$, show that the ξ_n are symmetric in $\varepsilon_1, \varepsilon_2$ for $n = 1, 2 \mod 4$ and antisymmetric otherwise

$$\xi_n(\varepsilon, \varepsilon') = -(-1)^{\frac{n}{2}(n+1)} \xi_n(\varepsilon', \varepsilon) \ .$$

Also one can show that for positive chirality spinors the n-forms produced are identically zero for $n = 0 \mod 2$, as can be seen by the equality

$$\langle \mathrm{dvol}\, \varepsilon, \gamma_{a_1 \cdots a_n}\varepsilon'\rangle = -(-1)^n \langle \varepsilon, \gamma_{a_1 \cdots a_n}\, \mathrm{dvol}\, \varepsilon'\rangle \ ,$$

and they are also related by Hodge duality up to a sign

$$\xi_n = (-1)^{\frac{n}{2}(n+1)} * \xi_{10-n} \ .$$

This last equation holds, since with $\bar{n} = 10 - n$ and any $\varepsilon, \varepsilon' \in \Delta$ we have

$$\begin{aligned} *\xi_{\bar{n}} &= \frac{1}{n!\bar{n}!}\bar{\varepsilon}\gamma_{\mu_1\dots\mu_{\bar{n}}}\varepsilon' \, \epsilon^{\mu_1\dots\mu_{\bar{n}}}{}_{\nu_1\dots\nu_n} e^{\nu_1\dots\nu_n} \\ &= (-1)^{\frac{n}{2}(n+1)}\frac{1}{n!}\bar{\varepsilon}\gamma_{\nu_1\dots\nu_n}\,\mathrm{dvol}\,\varepsilon' \, e^{\nu_1\dots\nu_n} \ . \end{aligned}$$

By counting dimensions, we conclude that the symmetric and antisymmetric square of the chiral spinor representation Δ_+ is

$$S^2\Delta_+ \cong \bigwedge^1 \mathbb{R}^{1,9} \oplus \bigwedge^{5-} \mathbb{R}^{1,9} \tag{3.4a}$$

$$\bigwedge^2 \Delta_+ \cong \bigwedge^3 \mathbb{R}^{1,9} \ . \tag{3.4b}$$

A single chiral spinor ε squares to a 1-form K and an anti-self-dual 5-form Ξ. Both of these forms annihilate the spinor ε for the following reason. The stabilizer of a nonzero chiral spinor ε is isomorphic to $\mathrm{Spin}(7) \ltimes \mathbb{R}^8 \subset \mathrm{Spin}(1,9)$ (see, e.g. , [60]). As the inner product is spin invariant, the stabilizer also leaves the forms K and Ξ invariant. The $\mathbb{R}^8$-subgroup consists of null rotations around a light-like direction spanned by the one-form K. A typical null rotation is thus proportional to $Y \cdot K\varepsilon = 0$, where Y is a space-like vector orthogonal to K. Since Y is invertible, we get $K^m\gamma_m\varepsilon = 0$. The anti-self-dual 5-form takes the form $\Xi = K \wedge \Phi$, with Φ a Cayley 4-form on the transverse space-like dimensions to K. Similarly, $\Xi \cdot \varepsilon = 0$ because $K \cdot \varepsilon = 0$. We shall also prove this using a Fierz identity in the next section, §3.1.2.

Globalising on a $1+9$ spin manifold $(\mathscr{M}, g, S)$ with fiber $S_p \cong \Delta$, we obtain a squaring map from spinor fields to differential forms. In IIb supergravity the relevant spinor fields are doublets of chiral spinors and we denote the spinor bundle by $\mathbb{S}_+ = S_+ \oplus S_+$. The symplectic structure C extends to $C \otimes \mathbb{1}$ on $\mathbb{S} = S \oplus S$, relative to which $\mathbb{S}_+$ and $\mathbb{S}_- := S_- \oplus S_-$ are complementary lagrangian subspaces and hence naturally dual. There is a natural action of $\mathrm{Spin}(9,1) \times$

GL$(2,\mathbb{R})$ on $\mathbb{S}_+$, but only Spin$(9,1) \times$ SO(2) preserves the symplectic structure. A typical endomorphism is of the form $c \otimes \lambda$, where c is a p-form and λ a 2×2 matrix, and its symplectic adjoint is $c^* \otimes \lambda^t$.

3.1.2 A Fierz identity

Our aim here is to produce a Fierz identity that will be useful in proving the odd-odd-odd Jacobi identity for a type IIb Killing superalgebra.

We have reduced the tensor product of the spinor module $\Delta^{\otimes 2}$ into irreducible representations of the orthogonal group. These are the various forms ξ_n in (3.3). One can do more, since $\Delta^{\otimes 2}$ is isomorphic as an associative algebra, via the spinor pairing, to the endomorphisms of spinors End(Δ). Indeed, a bispinor is a rank one endomorphism

$$\varepsilon_1 \times \varepsilon_2 \mapsto \varepsilon_1\bar{\varepsilon}_2 : (\varepsilon_1\bar{\varepsilon}_2)\varepsilon_3 = (\bar{\varepsilon}_2\varepsilon_3)\varepsilon_1 \equiv \langle \varepsilon_2, \varepsilon_3 \rangle \varepsilon_1 \ .$$

Since $\bigwedge^* \mathbb{R}^{1,9} = C\ell(1,9) = \mathrm{Mat}(32,\mathbb{R})$, the set of antisymmetric gamma matrices $\{\gamma^{a_1 \cdots a_n}\}_n$, along with the identity matrix, comprises a basis for the endomorphisms of the Clifford module. The bispinor $\varepsilon_1\bar{\varepsilon}_2$ can be decomposed in the gamma basis as

$$\varepsilon_{2\alpha}\bar{\varepsilon}_1{}^{\beta} = \sum_{k=0}^{10} (-1)^{\frac{k}{2}(k-1)} \frac{1}{32k!} \bar{\varepsilon}_1 \gamma_{c_1 \ldots c_k} \varepsilon_2 \left(\gamma^{c_1 \ldots c_k}\right)_{\alpha}{}^{\beta} \ . \qquad (3.5)$$

The coefficients above are found using the traces

$$\mathrm{Tr}(\mathbb{1}_{32 \times 32}) = 32$$

and

$$\mathrm{Tr}(\gamma^{c_1 \ldots c_n}) = 0 \ ,$$

which can be easily derived from the defining relation (3.1).

We can adapt equation (3.5) for spinors of definite chirality. First note the identity

$$\frac{1}{n!}\left(\gamma^{m_1\ldots m_n}\right)_\alpha{}^\beta\left(\gamma_{m_1\ldots m_n}\right)_\gamma{}^\delta=-\frac{1}{\bar{n}!}\left(\gamma^{m_1\ldots m_{\bar{n}}}\cdot \mathrm{dvol}\right)_\alpha{}^\beta\left(\gamma_{m_1\ldots m_{\bar{n}}}\cdot \mathrm{dvol}\right)_\gamma{}^\delta\ , \quad (3.6)$$

where $\bar{n}=10-n$. Assume three spinors α_i, $i=1,2,3$, with definite chiralities χ_1,χ_2,χ_3 and $\chi_2=-\chi_1$. The master Fierz identity for chiral spinors is

$$(\overline{\alpha}_1\alpha_2)\alpha_3=\tfrac{1}{32}\sum_{n=0}^{5}a_{n(\chi_1\chi_3)}\tfrac{1}{n!}\left(\overline{\alpha}_1\gamma_{c_1\cdots c_n}\alpha_3\right)\gamma^{c_1\cdots c_n}\alpha_2\ , \quad (3.7)$$

where the coefficients $a_{n(\chi_1\chi_3)}$ are computed to be

n	$a_{n(\pm\pm)}$	$a_{n(\pm\mp)}$
0	0	2
1	2	0
2	0	-2
3	-2	0
4	0	2
5	1	0

When 'fierzing' we shall find useful the following Clifford operation. For $c_n\in\bigwedge^n\mathbb{R}^{1,9}$ we have

$$\gamma^m c_n\gamma_m \quad = \quad (-1)^n(10-2n)c_n\ . \quad (3.8)$$

In particular, $\gamma^m c_5\gamma_m=0$.

As a warm up in fierzing, let us prove using the master Fierz identity that the one-form and five-form of the square of a positive chirality spinor ε leave it fixed

$$\bar{\varepsilon}\gamma_m\varepsilon\gamma^m\varepsilon=0$$
$$\frac{1}{5!}\bar{\varepsilon}\gamma_{m_1\ldots m_5}\varepsilon\gamma^{m_1\ldots m_5}\varepsilon=0\ .$$

We use the Fierz identity (3.7), with $\bar{\alpha}_1=\bar{\varepsilon}\gamma_m$, $\alpha_2=\varepsilon$ and $\alpha_3=\gamma^m\varepsilon$. Their chiralities are $\chi_1 = -1$ and $\chi_3 = -1$, while we also employ equation (3.8)

$$\begin{aligned}(\bar{\varepsilon}\gamma_m \cdot \varepsilon)\,\gamma^m\varepsilon = \frac{1}{32}\big(&2(-1)(10-2\times 1)\bar{\varepsilon}\gamma_c\varepsilon\gamma^c\varepsilon \\ &+\frac{-2}{3!}(-1)(10-2\times 3)\bar{\varepsilon}\gamma_{c_1\ldots c_3}\varepsilon\gamma^{c_1\ldots c_3}\varepsilon \\ &+\frac{1}{5!}(-1)(10-2\times 5)\bar{\varepsilon}\gamma_{c_1\ldots c_5}\varepsilon\gamma^{c_1\ldots c_5}\varepsilon\big)\ .\end{aligned}$$

Collecting terms and since $\bar{\varepsilon}\gamma^{(3)}\varepsilon = 0$ we have $\bar{\varepsilon}\gamma_m\varepsilon\gamma^m\varepsilon = 0$. The action of the five-form on the chiral spinor

$$\frac{1}{5!}\bar{\varepsilon}\gamma_{m_1\ldots m_5}\varepsilon\gamma^{m_1\ldots m_5}\varepsilon$$

is shown to be zero by using equation (3.6). In fact, any anti-self-dual 5-form annihilates a positive chirality spinor. This is a consequence of the following equality in the Clifford algebra

$$*c_n = (-1)^{\frac{n}{2}(n-1)}c_n \cdot \mathrm{dvol}\ , \tag{3.9}$$

with $c_n \in \bigwedge^n \mathbb{R}^{1,9}$. We are now ready to derive the main Fierz identity of this section. To prove the odd-odd-odd Jacobi identity of the Killing superalgebra we require

Lemma 1. *For two positive chirality spinors ε_1, ε_2 and any k-form $G^{(k)}$ we have*

$$\begin{aligned}&(\bar{\varepsilon}_1\gamma_m G^{(k)}\gamma_n\varepsilon_2)\gamma^{nm}\varepsilon_1 + (\bar{\varepsilon}_1\gamma_m\varepsilon_1)G^{(k)}\gamma^m\varepsilon_2 = \\ &-(-1)^k(10-2k)(\bar{\varepsilon}_1 G^{(k)}\varepsilon_2)\varepsilon_1 + \tfrac{1}{2}(-1)^k(10-2k)(\bar{\varepsilon}_1\gamma_m\varepsilon_1)\gamma^m G^{(k)}\varepsilon_2\end{aligned} \tag{3.10}$$

Proof. We begin by fierzing

$$\bar{\varepsilon}_1\gamma_m \cdot G^{(k)}\gamma_n\varepsilon_2(\gamma^n)\cdot\gamma^m\varepsilon_1\ ,$$

with $\bar{\alpha}_1\gamma_m = \bar{\varepsilon}_1\gamma_m$, $\alpha_2 = G^{(k)}\gamma_n\varepsilon_2$ and $\alpha_3 = \gamma^n\gamma^m\varepsilon_1$. Their chiralities are $\chi_1 = -1$

and $\chi_3 = +1$. A straightforward application of the master Fierz identity gives

$$\begin{aligned}&\left(\bar\varepsilon_1\gamma_m \cdot G^{(k)}\gamma_n\varepsilon_2\right)\gamma^n\gamma^m\varepsilon_1\\ &\qquad = \tfrac{1}{32}\big(2(\bar\varepsilon_1\gamma_m\gamma^n\gamma^m\varepsilon_1)G^{(k)}\gamma_n\varepsilon_2 - \tfrac{2}{2!}(\bar\varepsilon_1\gamma_m\gamma_{c_1c_2}\gamma^n\gamma^m\varepsilon_1)\gamma^{c_1c_2}G^{(k)}\gamma_n\varepsilon_2\\ &\qquad\qquad \tfrac{2}{4!}(\bar\varepsilon_1\gamma_m\gamma_{c_1\ldots c_4}\gamma^n\gamma^m\varepsilon_1)\gamma^{c_1\ldots c_4}G^{(k)}\gamma_n\varepsilon_2\big)\end{aligned} .$$

The third term is zero since $\gamma^m\gamma^{(5)}\gamma_m = 0$ and $\bar\varepsilon_1\gamma^{(3)}\varepsilon_1 = 0$. Then, from the second term, we keep the contraction of γ^n with $\gamma_{c_1c_2}$ in the spinor inner product - the wedge part is zero, because $\bar\varepsilon_1\gamma^n{}_{c_1c_2}\varepsilon_1 = 0$. Using (3.8), we arrive at

$$(\bar\varepsilon_1\gamma_m G^{(k)}\gamma_n\varepsilon_2)\gamma^n\gamma^m\varepsilon_1 = -\frac{1}{2}(\bar\varepsilon_1\gamma_m\varepsilon_1)G^{(k)}\gamma^m\varepsilon_2 + \frac{1}{2}(\bar\varepsilon_1\gamma_m\varepsilon_1)\gamma^{mn}G^{(k)}\gamma_n\varepsilon_2$$

Using $\gamma^{mn} = \gamma^m\gamma^n - \eta^{mn}$ on both sides of the equation and again equation (3.8) we arrive at the wanted Fierz identity. □

3.2 Type IIb Supergravity and Supersymmetry

In this section we set up our conventions for type IIb supergravity and introduce its two main ingredients, namely supersymmetry and S-duality.

3.2.1 The theory

Type IIb supergravity [61, 62, 63] is the unique 10-dimensional chiral supergravity theory with 32 supercharges. It is also the field theory limit of type IIb superstring theory.

The bosonic fields are a ten-dimensional lorentzian, mostly plus, metric g, the dilaton ϕ, the Ramond-Ramond (RR) gauge potentials $C^{(0)}$, $C^{(2)}$ and $C^{(4)}$ and the NS-NS 2-form gauge potential B. The axion $C^{(0)}$ and dilaton ϕ combine into the axi-dilaton $\tau = C^{(0)} + ie^{-\phi}$, taking values in the upper half-plane. The fermionic fields in the theory are often described as a complex chiral gravitino ψ_m and a complex anti-chiral axi-dilatino λ.

It is interesting to describe the so called 'gauge fixed' structure of the theory.

The axi-dilaton is a map from the manifold into the upper half plane

$$\tau : M \longrightarrow SL(2,\mathbb{R})/SO(2).$$

The fermions are sections of a $U(1)$ bundle with appropriate weights. A $U(1)$ bundle L_τ is defined as the pullback of $SL(2,\mathbb{R})$ by τ

$$\begin{array}{ccc} L_\tau & \xrightarrow{\tau_*} & SL(2,\mathbb{R}) \\ \big\downarrow{\scriptstyle \pi_\tau} & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{M} & \xrightarrow[\tau]{} & SL(2,\mathbb{R})/SO(2) \end{array}$$

The gravitino is a section then of $T^*\mathscr{M} \otimes L_\tau^{3/2} \otimes S_+$ and the dilatino is a section of $L_\tau^{1/2} \otimes S_-$.

The theory has a global $\mathrm{SL}(2,\mathbb{R})$ symmetry under which g (in the Einstein frame) and $C^{(4)}$ are inert, whereas $C^{(2)}$ and B transform as a doublet, and τ transforms via fractional linear transformations on the upper-half plane. That is, for a global group element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

they transform as

$$\begin{aligned} g\,\tau &= \frac{a\tau + b}{c\tau + d} \\ g \begin{pmatrix} C^{(2)} \\ B \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C^{(2)} \\ B \end{pmatrix} \quad . \end{aligned}$$

Under the global $\mathrm{SL}(2,\mathbb{R})$ symmetry, all fermionic fields transform by appropriate phases [64, 65], which correspond to the local SO(2) automorphism of the L_τ bundle. String theory will preserve only the $\mathrm{SL}(2,\mathbb{Z})$ subgroup and it relates the strong coupling regime with the weak coupling regime of the type IIb string. We will call the supergravity symmetry **S-duality**, athough note that strictly-speaking the duality refers to its integer subgroup.

We are interested in bosonic backgrounds; that is, solutions of the theory where the fermions are set to zero. The equations of motion for the bosonic fields can be found by varying an $\mathrm{SL}(2,\mathbb{R})$ invariant pseudo-action [66]. In order to write it, let us combine the potentials into the following field strengths

$$\begin{aligned} H &= dB \\ G^{(1)} &= dC^{(0)} \\ G^{(3)} &= dC^{(2)} - C^{(0)}H \\ G^{(5)} &= dC^{(4)} - \tfrac{1}{2}dB \wedge C^{(2)} + \tfrac{1}{2}dC^{(2)} \wedge B \ . \end{aligned} \tag{3.11}$$

The RR potential $C^{(4)}$ is constrained so that $G^{(5)}$ is anti-self-dual. In the string frame the pseudo-action is

$$\begin{aligned} I_{\mathrm{IIb}} = \int \mathrm{dvol} \left\{ e^{-2\phi} \left(R + 4|d\phi|^2 - \tfrac{1}{2}|H|^2 \right) \right. \\ \left. -\tfrac{1}{2} \left(|G^{(1)}|^2 + |G^{(3)}|^2 + \tfrac{1}{2}|G^{(5)}|^2 \right) \right\} \\ - \tfrac{1}{2} \int C^{(4)} \wedge dC^{(2)} \wedge dB \end{aligned} \tag{3.12}$$

and the anti-self-duality $G^{(5)} = - * G^{(5)}$ must then be imposed by hand.

The pseudo-action in (3.12) is S-duality invariant, although one has to go to the Einstein frame, $g_{\mathrm{E}} = e^{-\frac{\phi}{2}} g_{\mathrm{s}}$, to see a manifest symmetry. Whereas the terms involving the 5-form and Cern-Simons term are already manifestly $SL(2,\mathbb{R})$ invariant, the curvature plus axi-dilaton terms become

$$\int \mathrm{dvol}[g_{\mathrm{E}}] \left(R[g_{\mathrm{E}}] - \frac{1}{2}\frac{1}{\tau_2^2}|d\tau|^2_{g_{\mathrm{E}}} \right)$$

and the 3-form terms become

$$-\frac{1}{2} \int \mathrm{dvol}[g_{\mathrm{E}}] |\tilde{G}|^2_{g_{\mathrm{E}}} \ ,$$

where the complex 3-form

$$\tilde{G} = \frac{1}{\sqrt{\tau_2}} \left(\tau dB - dC^{(2)}\right)$$

can be shown to transform under S duality with a phase.

3.2.2 Supersymmetry

When discussing type IIb supersymmetry, it is convenient to think of the complex fermions as doublets of real fermions, because the supersymmetric derivative and dilatino variation do not act complex-linearly. The supersymmetry parameters are an SO(2) doublet of real chiral spinors $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$. A spinor is a **supergravity Killing spinor** of a bosonic type IIb background, if the corresponding supersymmetry variations of the fermionic fields vanish. Those of the bosonic fields are automatically zero, because the fermions have been put to zero. The variation of the gravitino gives rise to a differential equation, whereas the variation of the dilatino is an algebraic equation [67, 68]

$$\begin{aligned} \mathcal{D}_m \boldsymbol{\varepsilon} &= \nabla_m \boldsymbol{\varepsilon} + \tfrac{1}{8} H_{mnp} \gamma^{np} \otimes \lambda_3 \boldsymbol{\varepsilon} + \widetilde{\Omega} \gamma_m \boldsymbol{\varepsilon} = 0 \\ P \boldsymbol{\varepsilon} &= \left(d\phi + \tfrac{1}{2} H \otimes \lambda_3 + \Omega\right) \boldsymbol{\varepsilon} = 0 \ , \end{aligned} \tag{3.13}$$

where

$$\widetilde{\Omega} = \tfrac{1}{8} e^{\phi} \left(G^{(1)} \otimes \lambda_2 - G^{(3)} \otimes \lambda_1 + \frac{1}{2} G^{(5)} \otimes \lambda_2 \right) \ , \tag{3.14}$$

and

$$\Omega = \gamma^m \widetilde{\Omega} \gamma_m = e^{\phi} \left(\frac{1}{2} G^{(3)} \otimes \lambda_1 - G^{(1)} \otimes \lambda_2 \right) \ ,$$

while the 2×2 matrices λ_a, given by

$$\lambda_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\lambda_2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\lambda_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

span $\mathfrak{sl}(2, \mathbb{R})$.

In both $\mathcal{D}$ and P we have isolated the terms $\widetilde{\Omega}$ and Ω that come from the RR fields. In the forthcoming sections we will see that this is a useful separation to make. They are related by $\Omega = \gamma^m \tilde{\Omega} \gamma_m$. Also note how the RR terms are all self-adjoint with respect to the symplectic inner product $C \otimes \mathbb{1}$, as explained at the end of section §3.1.1. We have

$$\begin{aligned} \left(G^{(1)} \otimes \lambda_2\right)^* &= G^{(1)} \otimes \lambda_2 \\ \left(G^{(3)} \otimes \lambda_1\right)^* &= G^{(3)} \otimes \lambda_1 \\ \left(G^{(5)} \otimes \lambda_2\right)^* &= G^{(5)} \otimes \lambda_2 \\ \left(H \otimes \lambda_3\right)^* &= H \otimes \lambda_3 \end{aligned}$$

and therefore $\widetilde{\Omega}^* = \widetilde{\Omega}$ and $\Omega^* = \Omega$. For the purpose of condensing notation in some of the calculations that follow, we introduce the following mixed degree forms

$$\mathbb{G}^{\pm} = G^{(1)} \pm G^{(3)} + \frac{1}{2} G^{(5)} .$$

They are related by $\mathbb{G}^{\pm *} = -\mathbb{G}^{\mp}$. The RR part of the connection in $\mathcal{D}\boldsymbol{\varepsilon}$ can then be written as

$$\widetilde{\Omega} \gamma_m \boldsymbol{\varepsilon} = \frac{1}{8} e^{\phi} \left(\mathbb{G}^- \gamma_m \varepsilon_2 , \; -\mathbb{G}^+ \gamma_m \varepsilon_1 \right) .$$

The equations defining the Killing spinors are $\mathbb{R}$-linear. Therefore, Killing spinors span a real vector space, denoted by $\mathfrak{k}_1$, the space of Killing spinors of a background. Since Killing spinors are parallel with respect to the connection

$\mathcal{D}$, the dimension of $\mathfrak{k}_1$ can range from 0 to 32 and is usually expressed as 32ν, where ν is the fraction of the supersymmetry preserved. More precisely, 32ν is the multiplicity of the singlets that appear under the decomposition of $\mathbb{S}_+$ into $\mathrm{Hol}(\mathcal{D})$ irreducible representations and that are also annihilated by P.

Killing spinors are related to Killing vectors, the vector fields that are infinitesimal symmetries of the background. The aim of the next section is to construct a geometric superalgebra that contains all the infinitesimal symmetries of a bosonic background, namely Killing vector fields and Killing spinors.

3.3 Killing Superalgebra

A lot of effort has been put into classifying bosonic backgrounds. The program usually takes into account the fraction of supersymmetry it preserves, denoted by ν. At the high end of the supersymmetry fraction, there is a complete classification of maximally supersymmetric backgrounds [69] as well as non existence results [70, 71, 72] for so-called preonic backgrounds preserving a fraction $\nu = \frac{31}{32}$. At the low end of the fraction, e.g., $\nu = \frac{1}{32}$, local expressions for the metric and fluxes have been derived using the various interplays between Killing spinors and differential forms: either by considering the reduction of the structure group of the manifold brought about by the existence of differential forms built out of Killing spinors (the so-called "G-structure" approach) or else by thinking of spinors themselves as differential forms (in the so-called "spinorial geometry" approach of [73, 74, 75]).

A geometric Killing spinor (see chapter 5) on a spin manifold squares to a Killing vector field. Twistor spinors are known to square to conformal Killing vectors [76, 77] and are related to odd Killing vectors on an associated supergeometry [78]. The supersymmetry connection $\mathcal{D}$ is indeed more complicated. Nevertheless, two supergravity Killing spinors

$$\boldsymbol{\varepsilon} =(\varepsilon_1, \varepsilon_2) \ , \quad \boldsymbol{\varepsilon}' =(\varepsilon'_1, \varepsilon'_2) \ ,$$

also square to a metric preserving vector field k [68], where k is given by

$$k^m = \bar{\boldsymbol{\varepsilon}}\gamma^m \otimes \mathbb{1}\boldsymbol{\varepsilon}' = (\bar{\varepsilon}_1\gamma^m\varepsilon'_1 + \bar{\varepsilon}_2\gamma^m\varepsilon'_2) \ .$$

More can be said of the vector field k. We shall show that it preserves, besides the metric, all other fields of the theory. It is thus an infinitesimal symmetry of the background. We shall call *(pseudo)-riemannian* Killing vector fields that leave invariant all the fields of a type IIb supergravity background **the Killing vectors** of the background. Killing vectors, when they exist, are determined uniquely by their value and the value of their first derivative at a point, as explained in §2.4. Therefore Killing vectors span an $\mathbb{R}$-linear space that we denote $\mathfrak{k}_0$, the space of Killing vectors of a background, whose dimension can range from 0 to $10(10+1)/2 = 55$. Note that, by using the metric, we identify a Killing vector field with its dual one-form.

Perhaps it is not surprising that Killing spinors square to Killing vectors. Type IIb supergravity is the unique gravity theory modeled on the chiral $(2,0)$ super Poincaré algebra

$$\{Q^i_\alpha, Q^j_\beta\} = \delta^{ij}(C\gamma^m)_{\alpha\beta}P_m \ .$$

This has a nice realization on a supergravity background, whereby a supersymmetry transformation along the direction of a Killing spinor acts trivially on the background. Likewise, its square is a Killing vector that preserves the background. Were we to take this picture literally, the translation P_m above would be replaced by a Lie derivative. This is precisely the way that the IIb equations of motion were found in [62], by using the so-called Noether's method.

We also expect a Killing spinor to give one and only one Killing vector. Note that a Killing spinor is a chiral doublet $\boldsymbol{\varepsilon}$ and one can construct the following three vector fields

$$\begin{aligned}
k^m &= \bar{\boldsymbol{\varepsilon}}\gamma^m \otimes \mathbb{1}\boldsymbol{\varepsilon} = \bar{\varepsilon}_1\gamma^m\varepsilon_1 + \bar{\varepsilon}_2\gamma^m\varepsilon_2 \\
l^m &= \bar{\boldsymbol{\varepsilon}}\gamma^m \otimes \lambda_3\boldsymbol{\varepsilon} = \bar{\varepsilon}_1\gamma^m\varepsilon_1 - \bar{\varepsilon}_2\gamma^m\varepsilon_2 \\
k_{12}^m &= \frac{1}{2}\bar{\boldsymbol{\varepsilon}}\gamma^m \otimes \lambda_2\boldsymbol{\varepsilon} = \bar{\varepsilon}_1\gamma^m\varepsilon_2 \ .
\end{aligned}$$

Under S-duality, the metric (in the Einstein frame) and RR 4-form gauge field $C^{(4)}$ remain fixed. The Killing spinors transform under an $SO(2)$ transformation. Only the vector field k remains fixed under S-duality. Thus if there is such a Killing vector field, it should be k.

The vector space $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ can be equipped with a Lie superalgebra structure, which we call the **Killing superalgebra** of the type IIb background. The construction for specific backgrounds is well-known and has been explained in [79, 37]. The bracket between two Killing vector fields is the usual Lie derivative of vector fields. The bracket of a Killing vector field with a Killing spinor will be the action of the spinorial Lie derivative [80] on the spinor along the direction of the Killing vector field. The bracket between two Killing spinors is the squaring map we mentioned above. The brackets are collectively

$$\begin{aligned}
&[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0 && [k, k'] = -[k', k] = \mathscr{L}_k k' \\
&[\mathfrak{k}_0, \mathfrak{k}_1] \subseteq \mathfrak{k}_1 && [k, \boldsymbol{\varepsilon}] = -[k, \boldsymbol{\varepsilon}] = \mathscr{L}_k \boldsymbol{\varepsilon} \\
&[\mathfrak{k}_1, \mathfrak{k}_1] \subseteq \mathfrak{k}_0 && [\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'] = [\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}] = \bar{\boldsymbol{\varepsilon}}\gamma_m \otimes \mathbb{1}\boldsymbol{\varepsilon}'
\end{aligned}$$

We first show that the odd-odd bracket indeed closes. That is, Killing spinors square to supergravity Killing vectors. We then complete the picture by showing that all brackets close. We also check the Jacobi identities. Most of these are satisfied for trivial geometric reasons. The odd-odd-odd Jacobi identity, though, is satisfied through a delicate interplay of the supersymmetry conditions (3.13) and for this we shall need the Fierz identity in §3.1.2.

3.3.1 Closure

Closure of odd-odd bracket

As often in such calculations, we can polarize for a Killing vector that comes from the square of a *single* Killing spinor $k^m = \overline{\boldsymbol{\varepsilon}}\gamma^m \otimes \mathbb{1}\boldsymbol{\varepsilon}$. If we show that k^m is a supergravity Killing vector, then closure of the odd-odd bracket follows from "depolarizing", that is replacing $\boldsymbol{\varepsilon} \to \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}'$.

To show that k preserves the metric $\mathscr{L}_k g = 0$, it suffices to show that in components $\nabla_m k_n$ is antisymmetric in its indices. In what follows we shall also use the vector

$$l^m = \bar{\varepsilon}\gamma^m\varepsilon_1 - \bar{\varepsilon}_2\gamma^m\varepsilon_2 \ . \tag{3.15}$$

The derivative of k^m is

$$\nabla_m k_n = \overline{\nabla_m\varepsilon}\gamma_n\varepsilon + \bar{\varepsilon}\gamma_n\nabla_m\varepsilon \ ,$$

whereas using the differential Killing spinor equation in the form

$$\nabla_m\varepsilon = -\frac{1}{8}H_{mnp}\gamma^{np}\lambda_3\varepsilon - \tilde{\Omega}\gamma_m\varepsilon$$

and that $\tilde{\Omega}$ is self-conjugate (3.2.2), the expression becomes

$$\nabla_m k_n = -\frac{1}{2}H_{mnp}l^p + \bar{\varepsilon}\gamma_m\tilde{\Omega}\gamma_n\varepsilon - \bar{\varepsilon}\gamma_n\tilde{\Omega}\gamma_m\varepsilon \ .$$

This expression is clearly antisymmetric in its indices m, n, thus showing that k preserves the metric.

Using similar steps, the derivative of l is found to be

$$\nabla_m l_n = -\frac{1}{2}H_{mnp}k^p - 2\bar{\varepsilon}\gamma_n\lambda_3\tilde{\Omega}\gamma_m\varepsilon \ .$$

The second term is *symmetric* in the indices m,n. Indeed, by using $\lambda_3\tilde{\Omega} = -\tilde{\Omega}\lambda_3$, we see that $\lambda_3\tilde{\Omega}$ is anti-self-conjugate and thus

$$\bar{\varepsilon}\gamma_n\lambda_3\tilde{\Omega}\gamma_m\varepsilon = -\overline{\gamma_n\varepsilon}\lambda_3\tilde{\Omega}\gamma_m\varepsilon = \overline{\gamma_m\varepsilon}(\lambda_3\tilde{\Omega})^*\gamma_n\varepsilon = -\overline{\gamma_m\varepsilon}\lambda_3\tilde{\Omega}\gamma_n\varepsilon \ .$$

As already stated, the vector field l cannot be a Killing vector. Its exterior derivative is $dl = -\iota_k H$, from which we can show that k conserves H. Indeed, since H is closed, we have

$$\mathscr{L}_k H = d\iota_k H = 0 \ .$$

To show that k conserves the dilaton, we take the inner product $\bar{\varepsilon}P\varepsilon$, which is zero by the algebraic Killing equation. All the terms in P but the dilatino term are self-adjoint, therefore, the only term that survives is $\bar{\varepsilon}d\phi \otimes \mathbb{1}\varepsilon$, or

$$k(\phi) = \iota_K d\phi = 0 \ .$$

Conservation by k of the axion $C^{(0)}$ follows by considering the inner product $\bar{\varepsilon}\lambda_2 P\varepsilon = 0$. All terms vanish due to self-adjointness apart from $e^{\phi}\bar{\varepsilon}G^{(1)} \otimes \mathbb{1}\varepsilon$. With $G^{(1)} = dC^{(0)}$ we, therefore, have

$$k(C^{(0)}) = \iota_K dC^{(0)} = 0 \ .$$

The calculations which show that k preserves $G^{(3)}$ and $G^{(5)}$ are more involved. In guessing how to go about, we rely on a list of calculations for the forms produced by type IIb Killing spinors in [68]. For the case of $G^{(3)}$ we focus on the vector field

$$k_{12}^m = \frac{1}{2}\bar{\boldsymbol{\varepsilon}}\gamma^m \otimes \lambda_2\boldsymbol{\varepsilon} = \bar{\varepsilon}_1\gamma^m\varepsilon_2 \ ,$$

whose derivative is

$$\nabla_m k_{12n} = \frac{1}{8}\bar{\varepsilon}_1 H_{mpq}\gamma^{pq}\gamma_n\varepsilon_2 + \frac{1}{8}\bar{\varepsilon}_1\gamma_n H_{mpq}\gamma^{pq}\varepsilon_2 - \frac{1}{8}e^{\phi}\bar{\varepsilon}_2\gamma_m\mathbb{G}^+\gamma_n\varepsilon_2 + \frac{1}{8}e^{\phi}\bar{\varepsilon}_1\gamma_n\mathbb{G}^+\gamma_m\varepsilon_1 \ .$$

The last two terms are not antisymmetric in the indices m, n. In fact the contribution from symmetrizing in m, n come from the forms $G^{(1)}$ and $G^{(5)}$ in $\mathbb{G}^+$. It thus fails to preserve the metric. On the other hand, if we antisymmetrize, we get the exterior derivative of k_{12}

$$dk_{12} = \frac{1}{4} H_{mpq} \bar{\varepsilon}_1 \gamma_n{}^{pq} \varepsilon_2 dx^m \wedge dx^n + \frac{1}{4} e^{\phi} \left(\iota_k G^{(3)} + \iota_{G^{(3)}} \Xi \right) \ , \tag{3.16}$$

where Ξ the 5 form from the Killing spinors $\Xi_{mnopq} = \bar{\boldsymbol{\varepsilon}} \gamma_{mnopq} \otimes \mathbb{1} \boldsymbol{\varepsilon}$. The contraction of the three-form with the five-form has the following convention

$$\iota_{G^{(3)}} \Xi = \frac{1}{3!2} G^{(3)}_{c_1 c_2 c_3} \Xi_{mn}{}^{c_1 c_2 c_3} dx^m \wedge dx^n \ .$$

We can manipulate the identity for dk_{12} by using the algebraic Killing spinor equation. In particular, the identity $\bar{\boldsymbol{\varepsilon}} \gamma_{mn} \otimes \lambda_1 P \boldsymbol{\varepsilon} = 0$ gives

$$\begin{aligned} 2k_{12} \wedge d\phi + \frac{1}{2} \bar{\varepsilon}_1 H_{m c_1 c_2} \gamma_n{}^{c_1 c_2} \varepsilon_2 dx^m \wedge dx^n & \\ + \frac{1}{2} e^{\phi} \iota_{G^{(3)}} \Xi - \frac{1}{2} e^{\phi} \iota_k G^{(3)} + e^{\phi} l \wedge G^{(1)} &= 0 \ , \end{aligned}$$

which can be combined with (3.16) to yield

$$2d(e^{-\phi} k_{12}) = -l \wedge G^{(1)} + \iota_k G^{(3)} \ .$$

If we differentiate this, we get

$$d(\iota_k G^{(3)}) = dl \wedge G^{(1)} \ .$$

On the other hand, using the definition of $G^{(3)}$ and the fact that $\iota_k G^{(1)} = 0$, we also have

$$\iota_k dG^{(3)} = \iota_k (H \wedge G^{(1)}) = \iota_k H \wedge G^{(1)} = -dl \wedge G^{(1)} \ .$$

Therefore the two results give

$$\mathscr{L}_k G^{(3)} = \iota_k dG^{(3)} + d\iota_k G^{(3)} = 0$$

and the vector field k preserves $G^{(3)}$.

We follow a similar procedure for $G^{(5)}$. The mathematica package GAMMA [81] has been helpful in simplifying the various gamma contractions. We focus on the 3 form produced by the Killing spinors

$$\xi^{(3)}_{mno} = \overline{\varepsilon}_1 \gamma_{mno} \varepsilon_2 \ ,$$

whose derivative is

$$\nabla_m \xi^{(3)}_{nop} = \frac{1}{8} \overline{\varepsilon}_1 H_{mqr} \gamma^{qr} \gamma_{nop} \varepsilon_2 + \frac{1}{8} \overline{\varepsilon}_1 \gamma_{nop} H_{mqr} \gamma^{qr} \varepsilon_2 - \frac{1}{8} e^{\phi} \overline{\varepsilon}_2 \gamma_m \mathbb{G}^+ \gamma_{nop} \varepsilon_2 + \frac{1}{8} e^{\phi} \overline{\varepsilon}_1 \gamma_{nop} \mathbb{G}^+ \gamma_m \varepsilon_1 \ .$$

We then compute its exterior derivative. We also take into account that $G^{(5)}$ is anti-self-dual and thus annihilates chiral spinors (3.9). The result is

$$d\xi^{(3)} = \frac{1}{4!} H_{mc_1c_2} \overline{\varepsilon}_1 \gamma_{nop}{}^{c_1c_2} \varepsilon_2 dx^{mnop} + \frac{3}{2} k_{12} \wedge H + \frac{1}{2} e^{\phi} \iota_{G^{(1)}} \Xi + \frac{1}{4} e^{\phi} l \wedge G^{(3)} - \frac{1}{2} e^{\phi} \iota_k G^{(5)} - \frac{1}{2\,4!} e^{\phi} G^{(3)}_{mc_1c_2} \Theta_{nop}{}^{c_1c_2} dx^{mnop} \ ,$$

where we used the additional 5-form from Killing spinors $\Theta_{mnopq} = \overline{\boldsymbol{\varepsilon}} \gamma_{mnopq} \otimes \lambda_3 \boldsymbol{\varepsilon}$. We can simplify this using the algebraic Killing spinor equation. We take the equation $\overline{\boldsymbol{\varepsilon}} \gamma_{mnpq} \otimes \lambda_2 P \boldsymbol{\varepsilon} = 0$ which is expanded into

$$2k_{12} \wedge d\phi - \frac{2}{4!} \overline{\varepsilon}_1 \gamma_{mno}{}^{c_1c_2} \varepsilon_2 H_{pc_1c_2} dx^{mnop} + k_{12} \wedge H - e^{\phi} \frac{1}{4!} G^{(3)}_{mc_1c_2} \Theta_{nop}{}^{c_1c_2} dx^{mnop} - \frac{1}{2} e^{\phi} l \wedge G^{(3)} + e^{\phi} \iota_{G^{(1)}} \Xi = 0$$

and can be used to rewrite the above derivative $d\xi^{(3)}$ as

$$d\xi^{(3)} = k_{12} \wedge H + d\phi \wedge \xi^{(3)} + \frac{1}{2} e^{\phi} l \wedge G^{(3)} - \frac{1}{2} e^{\phi} \iota_k G^{(5)} \ .$$

Differentiating and resubstituting the expressions for $d\xi^{(3)}$ and dk_{12}, we have

$$dl \wedge G^{(3)} - d(\iota_k G^{(5)}) + \iota_k G^{(3)} \wedge H = 0 \ .$$

Now, we also have from the defining relation (3.11)

$$\iota_k dG^{(5)} = \iota_k(H \wedge G^{(3)}) = -dl \wedge G^{(3)} - H \wedge \iota_k G^{(3)} \ ,$$

where we have used $\iota_k H = -dl$. Combining the above two equations gives

$$\mathscr{L}_k G^{(5)} = \iota_k dG^{(5)} + d\iota_k G^{(5)} = 0 \ .$$

We have shown that the Killing vector k that comes from the square of a Killing spinor is a supergravity Killing spinor. That is, the odd-odd bracket on Killing spinors indeed closes on supergravity Killing vectors, $[\mathfrak{k}_1, \mathfrak{k}_1] \subset \mathfrak{k}_0$.

Closure of even odd/even bracket

We have established that the bracket closes on $\mathfrak{k}_1 \times \mathfrak{k}_1$. The rest of the brackets close for reasons that are geometrically obvious.

Assume two Killing vectors K and K'. Then the commutator of the Lie derivative along the two Killing vectors is the Lie derivative along the commutator of the vector fields, $[\mathscr{L}_K, \mathscr{L}_{K'}] = \mathscr{L}_{[K,K']}$. Therefore, since the left-hand side preserves the type IIb background, so will the right-hand side. That is, $[K, K'] = \mathscr{L}_K K'$ is a Killing vector of the background and so the bracket $[\mathfrak{k}_0, \mathfrak{k}_0]$ closes on $\mathfrak{k}_0$.

For the even-odd bracket, assume a Killing vector K and a Killing spinor $\boldsymbol{\varepsilon}$. The even-odd bracket is the action of the Lie derivative along K that acts diagonally on the two chiral components of $\boldsymbol{\varepsilon}$

$$[K, \boldsymbol{\varepsilon}] = \mathscr{L}_K \boldsymbol{\varepsilon} = (\nabla_K + \frac{1}{4}(\nabla K)_{mn}\gamma^{mn}) \otimes \mathbb{1}\boldsymbol{\varepsilon} \ .$$

To show that $[K, \boldsymbol{\varepsilon}]$ is a Killing vector, let us consider the identities $[\mathscr{L}_K, \mathcal{D}_X] = \mathcal{D}_{[K,X]}$ and $[\mathscr{L}_K, P] = 0$. Indeed these are true, because K preserves all fields in the connection $\mathcal{D}$ and the dilatino variation P, while $[\mathscr{L}_K, \nabla_X] = \nabla_{[K,X]}$. Then when we let these equations act on a Killing spinor $\boldsymbol{\varepsilon}$, we get $D_X \mathscr{L}_K \boldsymbol{\varepsilon} = 0$ and $P \mathscr{L}_K \boldsymbol{\varepsilon} = 0$. Therefore, $\mathscr{L}_K \boldsymbol{\varepsilon}$ is a Killing spinor and the bracket $[\mathfrak{k}_0, \mathfrak{k}_1]$ closes on

$\mathfrak{k}_1$.

For flat Minkowski we have the N=2 chiral Poincaré superalgebra, where $\mathfrak{k}_0 = \mathbb{R}^{1,9} \oplus \mathfrak{so}(1,9)$ and $\mathfrak{k}_1 = \Delta_+ \oplus \Delta_+$ are constant chiral spinors. By construction, the odd derived algebra is just the translation part $[\mathfrak{k}_1, \mathfrak{k}_1] = \mathbb{R}^{1,9}$, the elements of which have zero derivative. The translations $\mathbb{R}^{1,9}$ act trivially on $\mathfrak{k}_1$, because for parallel vectors the spinorial Lie derivative acts through the Levi-Civita connection, whereas the Killing spinors are for this background constant. Similarly the $\mathfrak{so}(1,9)$ generators are the Killing vectors

$$K_{ab} = x_a \partial_b - x_b \partial_a \ ,$$

but on parallel vectors and parallel spinors, they act through their derivative $L_{ab} = \frac{1}{2}\gamma_{ab}$. This is precisely the structure of the Poincaré superalgebra.

An interesting remark is that, although the even part $\mathfrak{k}_0$ acts diagonally on the two chiral components, the two chiral components of $\mathfrak{k}_1$ mix in the odd-odd bracket. This might not seem to be the case when looking at the equation for k, but let us remember that the Killing vector is given by its value, at a point, and the value of its first derivative. It is the derivative ∇k that mixes the two chiral components and thus the Killing superalgebra is not a trivial extension of a single chiral superalgebra.

3.3.2 Jacobi identities

We have shown closure of the brackets and what remains is to check that the Jacobi identities are satisfied. The even-even-odd Jacobi is satisfied because of the equality $[\mathscr{L}_K, \mathscr{L}_{K'}] = \mathscr{L}_{[K,K']}$ acting on a spinor doublet. The even-even-even Jacobi identity is satisfied as the Jacobi identity of the Lie algebra of vector fields. The even-odd-odd Jacobi identity is satisfied because the Lie derivative is compatible with the spinor pairing. That is to say

$$\mathscr{L}_K \left(\bar{\boldsymbol{\varepsilon}} \gamma_m \otimes \mathbb{1} \boldsymbol{\varepsilon}' \right) = \overline{\mathscr{L}_K \boldsymbol{\varepsilon}} \gamma_m \otimes \mathbb{1} \boldsymbol{\varepsilon}' + \bar{\boldsymbol{\varepsilon}} \gamma_m \otimes \mathbb{1} \mathscr{L}_K \boldsymbol{\varepsilon}' \ ,$$

because the connection in $\mathscr{L}_K$ is in $\mathfrak{spin}(g)$, where g is the background metric.

In what follows we shall show that the odd-odd-odd Jacobi identity is satisfied. We shall see that the identity is satisfied by the tight structure that supersymmetry provides, whereby both the Killing spinor equations and the Fierz identity found in §3.1.2 are used. We can polarize the identity, so that it suffices to show that $[[\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}],\boldsymbol{\varepsilon}] = \mathscr{L}_k\boldsymbol{\varepsilon} = 0$. The Jacobi identity then follows by depolarizing, that is setting $\boldsymbol{\varepsilon} \to \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}' + \boldsymbol{\varepsilon}''$.

Assume a Killing spinor $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ and its associated Killing form $k_m = \bar{\varepsilon}_1\gamma_m\varepsilon_1 + \bar{\varepsilon}_2\gamma_m\varepsilon_2$. The Lie derivative acts diagonally on $\boldsymbol{\varepsilon}$ as

$$\mathscr{L}_k\boldsymbol{\varepsilon} = k^m\nabla_m\boldsymbol{\varepsilon} + \frac{1}{4}\nabla_m k_n\gamma^{mn}\boldsymbol{\varepsilon} \ .$$

We can turn the right-hand side, from a differential to a polynomial equation in $\boldsymbol{\varepsilon}$. To this end, we use the differential Killing equation

$$\nabla_m\boldsymbol{\varepsilon} = -\frac{1}{8}H_{mnp}\gamma^{np}\otimes\lambda_3\boldsymbol{\varepsilon} - \tilde{\Omega}\gamma_m\boldsymbol{\varepsilon}$$

and re-express $\nabla_m k_n$ like

$$\nabla_m k_n = -\frac{1}{2}H_{mnp}l^p - 2\bar{\boldsymbol{\varepsilon}}\gamma_n\tilde{\Omega}\gamma_m\boldsymbol{\varepsilon} \ ,$$

where l was defined in (3.15). In both cases let us condense the R-R dependent part of the connection to the form

$$\tilde{\Omega}\gamma_m\boldsymbol{\varepsilon} = \frac{1}{8}e^{\phi}\left(\mathbb{G}^-\gamma_m\varepsilon_2, -\mathbb{G}^+\gamma_m\varepsilon_1\right) \ ,$$

where $\mathbb{G}^\pm = G^{(1)} \pm G^{(3)} + \frac{1}{2}G^{(5)}$.

We now break up $\mathscr{L}_k\boldsymbol{\varepsilon}$ into a part $\mathscr{I}_1$ that depends on the NS-NS form H and a piece $\mathscr{I}_2$ that depends on the R-R fields, $\mathscr{L}_k\boldsymbol{\varepsilon} = \mathscr{I}_1 + \mathscr{I}_2$. The H-dependent term is

$$\mathscr{I}_1 = \frac{1}{4}e^{\phi}\begin{pmatrix} -H_{\mu\nu\rho}\bar{\varepsilon}_1\gamma^m\varepsilon_1\gamma^{np}\epsilon_1 \\ +H_{\mu\nu\rho}\bar{\varepsilon}_2\gamma^m\varepsilon_2\gamma^{np}\epsilon_2 \end{pmatrix} \ . \tag{3.17}$$

The G-dependent part becomes

$$\mathscr{I}_2 = \frac{1}{8}e^{\phi}\begin{pmatrix} -\bar{\varepsilon}_1\gamma^m\varepsilon_1\mathbb{G}^+\gamma_m\varepsilon_2 + \bar{\varepsilon}_1\gamma_m\mathbb{G}^-\gamma_n\varepsilon_2\gamma^{mn}\varepsilon_1 \\ \bar{\varepsilon}_2\gamma^m\varepsilon_2\mathbb{G}^+\gamma_m\varepsilon_1 - \bar{\varepsilon}_2\gamma_m\mathbb{G}^+\gamma_n\varepsilon_1\gamma^{mn}\varepsilon_2 \end{pmatrix} , \tag{3.18}$$

where we used $\bar{\varepsilon}_i\gamma_m\varepsilon_i\gamma^m\varepsilon_i = 0$, for $i = 1, 2$, to get rid of those terms.

We shall show that the two terms add up to zero, after fierzing and using the algebraic Killing property. The term $\mathscr{I}_2$ can be simplified further using the Fierz identity

$$\begin{aligned} &- \bar{\varepsilon}_1\gamma_m G^{(k)}\gamma_n\varepsilon_2\gamma^{mn}\varepsilon_1 + \bar{\varepsilon}_1\gamma_m\varepsilon_1 G^{(k)}\gamma^m\varepsilon_2 \\ &\qquad\qquad = (10-2k)\left(\bar{\varepsilon}_1 G^{(k)}\varepsilon_2\varepsilon_1 - \frac{1}{2}\bar{\varepsilon}_1\gamma^m\varepsilon_1\gamma_m G^{(k)}\varepsilon_2\right) , \end{aligned}$$

for any two positive chirality spinors ε_i, proved as lemma 1 (page 33). The terms involving the R-R 5-form vanish identically, whereas the rest become after some simplification

$$e^{\phi}\begin{pmatrix} \bar{\varepsilon}_1\left(\frac{1}{2}G^{(3)} - G^{(1)}\right)\varepsilon_2\varepsilon_1 - \frac{1}{2}\bar{\varepsilon}_1\gamma^m\varepsilon_1\gamma_m\left(\frac{1}{2}G^{(3)} - G^{(1)}\right)\varepsilon_2 \\ \bar{\varepsilon}_2\left(\frac{1}{2}G^{(3)} + G^{(1)}\right)\varepsilon_1\varepsilon_2 - \frac{1}{2}\bar{\varepsilon}_2\gamma^m\varepsilon_2\gamma_m\left(\frac{1}{2}G^{(3)} + G^{(1)}\right)\varepsilon_1 \end{pmatrix} . \tag{3.19}$$

The algebraic Killing spinor equation in components reads

$$\begin{aligned} e^{\phi}\left(\frac{1}{2}G^{(3)} - G^{(1)}\right)\varepsilon_2 &= -\left(\frac{1}{2}H + d\phi\right)\cdot\varepsilon_1 \\ e^{\phi}\left(\frac{1}{2}G^{(3)} + G^{(1)}\right)\varepsilon_1 &= \left(\frac{1}{2}H - d\phi\right)\cdot\varepsilon_2 \end{aligned}$$

and so the expression in (3.19) becomes

$$\mathscr{I}_2 = \begin{pmatrix} -\bar{\varepsilon}_1 d\phi\varepsilon_1\varepsilon_1 + \frac{1}{2}\bar{\varepsilon}_1\gamma^m\varepsilon_1\gamma_m d\phi\varepsilon_1 + \frac{1}{4}\bar{\varepsilon}_1\gamma^m\varepsilon_1\gamma_m H\varepsilon_1 \\ -\bar{\varepsilon}_2 d\phi\varepsilon_2\varepsilon_2 + \frac{1}{2}\bar{\varepsilon}_2\gamma^m\varepsilon_2\gamma_m d\phi\varepsilon_2 - \frac{1}{4}\bar{\varepsilon}_2\gamma^m\varepsilon_2\gamma_m H\varepsilon_2 \end{pmatrix} ,$$

where we get rid of the terms $\bar{\varepsilon}_i H\varepsilon_i = 0$, for $i = 1, 2$. It is easy to see that the dilaton-dependent terms vanish, since for any positive-chirality spinor ε,

$$-\bar{\varepsilon} d\phi \varepsilon \varepsilon + \frac{1}{2} \bar{\varepsilon} \gamma^m \varepsilon \gamma_m d\phi \varepsilon = 0 \ ,$$

where we use the Clifford algebra $\gamma_m d\phi + d\phi \gamma_m = 2(d\phi)_m$ together with the identity $\bar{\varepsilon}\gamma^m \varepsilon \gamma_m \varepsilon = 0$, which holds for chiral spinors.

By adding the contributions of $\mathscr{I}_1$ and $\mathscr{I}_2$, we find

$$\mathscr{L}_K \boldsymbol{\varepsilon} = \frac{1}{4} \begin{pmatrix} \bar{\varepsilon}_1 \gamma^m \varepsilon_1 \gamma_m H \varepsilon_1 - \bar{\varepsilon}_1 \gamma^m \varepsilon_1 H_{mnp} \gamma^{np} \varepsilon_1 \\ -\bar{\varepsilon}_2 \gamma^m \varepsilon_2 \gamma_m H \varepsilon_2 - \bar{\varepsilon}_2 \gamma^m \varepsilon_2 H_{mnp} \gamma^{np} \varepsilon_2 \end{pmatrix} \ ,$$

which is again seen to cancel after using the Clifford algebra and the identity $\bar{\varepsilon}_i \gamma^m \varepsilon_i \gamma_m \varepsilon_i = 0$, for $i = 1, 2$.

3.4 Homogeneity

We have verified that the Killing vectors and Killing spinors of a type IIb supergravity background belong to a Lie superalgebra. We now ask the question that was answered in [37] for the M-theory Killing superalgebra. That is, since the derived algebra $[\mathfrak{k}_1, \mathfrak{k}_1]$ is a Lie algebra of Killing vectors[1]: "is there a minimum amount of supersymmetry ν_c above which local homogeneity is ensured?". **Local homogeneity** is the property for which every point p of the manifold $\mathscr{M}$ has a local frame made up of Killing vectors. Equivalently, any two points $p, q \in \mathscr{M}$ have neighborhoods U_p, U_q, for which there is a background preserving isometry ϕ, with $\phi(p) = q$, see for instance the proof in [37].

For $\nu = 1$, the case of the maximally supersymmetric IIb backgrounds [69], local homogeneity follows from representation theory alone. The nonexistence of preonic solutions [72] puts $\nu_c \leq \frac{15}{16}$. On the low end, the existence of the cohomogeneity-one $\frac{1}{2}$-BPS D3-brane background shows that if there is such a critical fraction, then $\nu_c > 16/32$. We find that for $\nu > 3/4$, the odd derived

[1] By using the even-odd-odd Jacobi identity and closure, the derived algebra is in fact an ideal of $\mathfrak{k}_0$.

subalgebra $[\mathfrak{k}_1, \mathfrak{k}_1]$ spans pointwise $T_p\mathscr{M}$ thus ensuring local homogeneity. The proof is identical to that in 11 dimensions [37], with a mere alteration due to the fact that there is no symplectic inner product on $\mathbb{S}_+$.

To begin with, fix a point $p \in \mathscr{M}$. Then the tangent bundle defines a vector space with metric of signature $(1,9)$, $T_p\mathscr{M} \approx \mathbb{R}^{1,9}$, and the spinor bundle defines a spinor module of chiral doublets, $(\mathbb{S}_+)_p \approx \Delta_+ \oplus \Delta_+$. In particular, the Killing spinors of the background define a subspace $W \subseteq \Delta_+ \oplus \Delta_+$ and we assume that $\dim W > 16$. If the squaring map on W does not surject T_pM, then there is a vector $v \in T_p\mathscr{M}$ such that

$$\overline{\boldsymbol{\varepsilon}}'\gamma_a \otimes \mathbb{1}\boldsymbol{\varepsilon}\, v^a = 0 \ ,$$

for all $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in W$. Therefore, the Clifford operation

$$v \otimes \mathbb{1} : \Delta_+ \oplus \Delta_+ \longrightarrow \Delta_- \oplus \Delta_- \overset{C\otimes\mathbb{1}}{\approx} (\Delta_+ \oplus \Delta_+)^*$$

sends W to its annihilator W^0 in the dual space[2] of $\Delta_+ \oplus \Delta_+$. Since $\dim W > 16$ and $\dim W + \dim W^0 = 32$, we have $\dim W^0 < 16$ and thus $v \otimes \mathbb{1}$ must have a kernel in W on dimensional grounds. Since $v \cdot v = |v|^2$, $v \otimes \mathbb{1}$ can only have a kernel if v is null. A null vector though has rank equal to the dimension of its kernel. Indeed one can write v in a suitable basis and up to a constant as $\gamma_0(1 + \gamma^{01})$ and since γ_0 is invertible the kernel of v is that of a half-projection. Therefore, the map $v \otimes \mathbb{1}$ on $\Delta_+ \oplus \Delta_+$ has rank 16. Equivalently, the symmetric bilinear $\beta = \langle -, v \otimes \mathbb{1}\, - \rangle$ has rank 16.

Let us split the two copies of chiral spinors $\Delta_+ \oplus \Delta_+$ into $W \oplus U$, where U is any complementary space. Then β has the following matrix form

$$\beta = \begin{pmatrix} 0 & A \\ A^t & B \end{pmatrix} \ ,$$

where $A : U \to W^*$ and $B : U \to U^*$. The kernel of β are those $w + u \in W \oplus U$ for which $Au = 0$ and $A^t w + Bu = 0$. Its rank is $\dim \operatorname{im} \beta = 32 - \dim \ker \beta$.

[2] the spinor inner product pairs chiral doublets with anti-chiral doublets and so maps anti-chiral doublets to the dual space of chiral doublets, see e.g., pages 29 and 30.

Evidently

$$\operatorname{im}\beta = \operatorname{im}\beta|_{\emptyset\oplus U} + \operatorname{im}\beta|_{W\oplus\emptyset} = \operatorname{im}(A\oplus B) + \emptyset\oplus\operatorname{im}A^t \subseteq \overline{U}\oplus U^* \ ,$$

where $\overline{U}$ is any vector subspace in W^* of dimension $\dim U$, obtained by completing the image of A in W^*. In particular $\operatorname{rank}\beta \leq 2\dim U$. The upper bound can be achieved when $\ker A = 0$ and so A^t is onto. Indeed, the kernel of β is in this case $0{+}w$, with $w \in \ker A^t$ and so $\dim\ker\beta = \dim W - \dim U = 32 - 2\dim U$. We thus have an upper bound for the rank of β, which we know to be 16. Equivalently $\dim U \geq 8$ or $\dim W \leq 24$. Conversely, if $\dim W > 24$, then no such v can exist and hence $[\mathfrak{k}_1, \mathfrak{k}_1]$ surjects pointwise the tangent space.

The result would be made sharp, if we could find an $\nu = 24/32$ supersymmetric background that is not locally homogeneous. Type IIb backgrounds with 24 supersymmetries have been given in [82, 83, 84, 85, 85, 86] . In [84] a 24-supersymmetric background was constructed by discrete quotients of the maximally supersymmetric pp-wave [87] that breaks a quarter of the supersymmetries. The resulting background inherits the local homogeneity of the maximally supersymmetric pp-wave. In [82] (see also [83] and [85]) a homogeneous pp-wave with 24 supersymmetries is given, which can be obtained as a Penrose limit of a D3/D3 intersection. Only the 5-form is turned on, but it has a different structure to the maximally supersymmetric one. In [86] there are two families of homogeneous pp-waves with 24 supersymmetries. The above examples are all homogeneous plane waves, which were classified in [88]. The metric of a plane wave in Brinkmann coordinates is

$$g = 2dx^+dx^- + A_{ij}z^iz^j(dx^+)^2 + d\vec{z}\cdot d\vec{z} \ ,$$

where the z^i, $i = 1, \ldots, 8$ label the transverse coordinates.

It is proved in [89, 3.1] that eleven-dimensional plane waves admitting more than 16 supersymmetries are automatically locally homogeneous. Their argument also applies to the IIb plane waves with more than 16 supersymmetries, like the ones described above. This is because plane waves always have 16 Killing spinors $\varepsilon_i, i = 1, \ldots 16$ which are annihilated by the null direction of the wave γ^+ and hence

$$\boldsymbol{\varepsilon}_i = -\frac{1}{2}\gamma^+\gamma^-\boldsymbol{\varepsilon}_i \ .$$

They were named in [82] basic.

Let us essentially reduce[3] the problem to the square of single chiral basic spinors $\varepsilon_i \in S_+, i = 1 \dots 8$. The square of basic spinors spans the null direction of the wave. Indeed,

$$\bar{\varepsilon}_i\gamma^m\varepsilon_j = -\frac{1}{2}\bar{\varepsilon}_i\gamma^-\gamma^+\gamma^m\varepsilon_j = \begin{cases} 0 & m = i, + \\ -\sqrt{2}\varepsilon^T\varepsilon' & m = - \end{cases} \tag{3.20}$$

A basis for the single chiral basic spinors can be given as $\epsilon_i = \gamma_i\chi$, where χ is an anti-chiral basic spinor[4]. If a plane wave has more than 16 supersymmetries, then the extra Killing spinors, which are called "supernumerary", are annihilated by γ^-. A similar calculation to that of equation (3.20) shows that the square of supernumerary spinors span e_-. The square of basic spinors with a supernumerary spinor, though, will span the remaining transverse space. Indeed, let us take a supernumerary chiral spinor ε' and the 8 basic spinors in the basis $\varepsilon^i = \gamma^i\chi$. If we consider the vector $\bar{\varepsilon}^i\gamma^m\varepsilon'$, then for the $m = \pm$ directions we have

$$\bar{\varepsilon}^i\gamma^m\varepsilon' = -\frac{1}{2}\bar{\varepsilon}^i\gamma^-\gamma^+\gamma^m\varepsilon' = 0 \ ,$$

because $\gamma^-\epsilon' = 0$ and $(\gamma^+)^2 = 0$. For $m = j$ we have $\bar{\varepsilon}^i\gamma^j\varepsilon' = -\bar{\chi}\gamma^i\gamma^j\varepsilon'$. If we define the bilinear map

$$\begin{aligned} \gamma : \mathbb{R}^8 \times \mathbb{R}^8 &\longrightarrow \mathbb{R} \\ (x_i, y_j) &\mapsto -\bar{\chi}x_i\gamma^i y_j\gamma^j\varepsilon' \ , \end{aligned}$$

we see that γ is non-degenerate. Therefore, there is a one-to-one map from the space of basic spinors onto the transverse directions of the tangent space, given by the square of the chiral spinors with a supernumerary spinor. In summary, a

[3]for this we can use S-duality, which acts on the doublets like SO(2).

[4]to show linear independence assume $\lambda_i\epsilon^i = 0$. Then by acting on it with $\lambda_i\gamma^i\cdot$ we get $\vec{\lambda}^2 = 0$ or $\chi = 0$

plane wave with more than 16 supersymmetries will be locally homogeneous.

Chapter 4

Heterotic and Type I Killing Superalgebra

In this chapter we construct a Lie superalgebra structure on the infinitesimal $\mathbb{Z}_2$-graded symmetries of a heterotic supergravity background. The bosonic symmetries are the Killing vectors that leave invariant all the fields of the background. The fermionic symmetries are the Killing spinors of the background.

We use the same spinor conventions we used for type IIb supergravity, which were explained in section §3.1. We first set up our conventions for heterotic supergravity. We then show how the superalgebra is constructed. In particular, we show closure of the algebra and that the Jacobi identities are satisfied. We then discuss how supersymmetry can imply local homogeneity. The results can be applied to type I supergravity, by setting the Yang-Mills fields to zero.

4.1 Theory and Supersymmetry

The field theory limit of the heterotic string [90] is given by ten-dimensional $N = 1$ supergravity [91] coupled to $N = 1$ supersymmetric Yang-Mills [92]. The theory was constructed in [93] by generalizing the construction in [94] for the abelian case.

A heterotic background is a spin manifold $(\mathscr{M}, g, S)$ with bosonic field content, besides the metric, the dilaton ϕ , the NS-NS 3-form H and a gauge field-strength

F. In heterotic string theory the gauge group is constrained, but we will work with arbitrary gauge group in what follows.

The fields are subject to the equations of motion derived from the following action (in string frame)

$$I_{\mathrm{I}} = \int_{\mathcal{M}} \mathrm{dvol}_g \, e^{-2\phi}(R + 4|d\phi|^2 - \frac{1}{2}|H|^2 - \frac{1}{2}|F|^2).$$

The 3-form H is not closed in the supergravity limit but instead satisfies

$$dH = \frac{1}{2}\mathrm{Tr}\,(F \wedge F) \ . \tag{4.1}$$

The theory has a single chiral supersymmetry, generated by single chiral spinor fields $\varepsilon \in S_+$. **Killing spinors** of a bosonic background are spinors that obey a differential equation, which comes from the variation of the gravitino, and two algebraic equations, which come from varying the dilatino and the gaugino. They are respectively

$$\begin{aligned} \nabla^+_m \varepsilon &= \left(\nabla_m - \tfrac{1}{8} H_{mnp}\gamma^{np}\right)\varepsilon = 0 \\ P\varepsilon &= \left(\partial_m\phi\gamma^m - \tfrac{1}{12} H_{mnp}\gamma^{mnp}\right)\varepsilon = 0 \\ Q\varepsilon &= \frac{1}{2} F_{mn}\gamma^{mn}\varepsilon = 0 \ . \end{aligned} \tag{4.2}$$

The connection ∇^+ is the metric compatible connection with torsion H. Note that both the gravitino and gaugino variation lie in $\mathfrak{spin}(g)$. As these equations are linear, the set of Killing spinors is a real vector space, denoted $\mathfrak{k}_1$ and of dimension 16ν, where $0 \leq \nu \leq 1$ is the fraction of supersymmetry preserved by the background $(\mathscr{M}, g, \phi, H, F)$. A lot of progress has been made on the classification of supersymmetric heterotic backgrounds [95, 96, 69, 60].

A Killing vector field which preserves not only the metric but also ϕ, H and F, up to a gauge transformation, is said to be a **Killing vector** of the background. It is clear that Killing vectors form a vector space $\mathfrak{k}_0$, with real dimension ranging from 0 to $10(10+1)/2 = 55$. Moreover, they close under the Lie bracket of vector fields, that is they form a Lie algebra. In the next section we will show that, just as in the case of M-theory and type IIb supergravity, the vector superspace

$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ carries the structure of a Lie superalgebra extending the Lie algebra structure of $\mathfrak{k}_0$. Note that type I supergravity is identical to the heterotic case, but ignoring the gauge field strengths.

4.2 Killing Superalgebra

The vector space $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ can be equipped with a Lie superalgebra structure, which we call the **Killing superalgebra** of the heterotic background. The bracket between two Killing vector fields is the usual Lie derivative of vector fields restricted to Killing vectors. The bracket of a Killing vector field with a Killing spinor will be the action of the spinorial Lie derivative [80] on the spinor along the direction of the Killing vector field. The bracket between two Killing spinors is the squaring map. The brackets are collectively given by

$$\begin{aligned} &[\mathfrak{k}_0, \mathfrak{k}_0] \subset \mathfrak{k}_0 && [k, k'] = -[k', k] = \mathscr{L}_k k' \\ &[\mathfrak{k}_0, \mathfrak{k}_1] \subset \mathfrak{k}_1 && [k, \varepsilon] = -[k, \varepsilon] = \mathscr{L}_k \varepsilon \\ &[\mathfrak{k}_1, \mathfrak{k}_1] \subset \mathfrak{k}_0 && [\varepsilon, \varepsilon'] = [\varepsilon', \varepsilon] = \bar{\varepsilon}\gamma_m \varepsilon' \ . \end{aligned}$$

In the following we show closure of the bracket and that the Jacobi identities are satisfied.

Closure

We begin by showing that Killing spinors square to field-preserving Killing vectors. Let us take k^m defined by squaring the Killing spinors ϵ, ϵ' and compute its ∇^+-derivative

$$\nabla^+_m K^n = \nabla^+_m(\bar{\epsilon}\gamma^n\epsilon') = \bar{\epsilon}'\gamma^n\nabla^+_m\epsilon + \bar{\epsilon}\gamma^n\nabla^+_m\epsilon' = 0.$$

We, therefore, find that k is ∇^+-parallel. Symmetrizing in m and n we get $\mathscr{L}_k g = 0$, that is, k preserves g. Antisymmetrizing in m and n we get $\iota_k H = dk$. In view of (4.1), we have $\mathscr{L}_k H = \iota_k dH = \mathrm{Tr}(\iota_k F \wedge F)$.

By using the dilatino variation, we obtain

$$0 = \bar{\varepsilon}(d\phi - \frac{1}{2}H)\varepsilon' = \bar{\varepsilon}d\phi\varepsilon' - \frac{1}{2}\bar{\varepsilon}H\varepsilon'.$$

The two terms in this sum should vanish separately, being respectively symmetric and anti-symmetric in ϵ, ϵ'. Thus $\mathscr{L}_k\phi = \iota_K d\phi = \bar{\epsilon}d\phi\epsilon' = 0$ and k preserves the dilaton.

Using the gaugino variation now, we find for any vector X

$$0 = \bar{\varepsilon}X \cdot F\varepsilon' = \bar{\varepsilon}(X \wedge F + \iota_X F)\varepsilon'$$

Again the two terms should vanish separately, being respectively antisymmetric and symmetric in $\varepsilon, \varepsilon'$. The second term is in fact $F(X, k) = -(\iota_k F)(X) = 0$ and since X is arbitrary we gain $\iota_k F = 0$. Using the Bianchi identity for the gauge field A whose field strength is F, $dF = -[A, F]$, and $\iota_k F = 0$ repeatedly, we obtain

$$\mathscr{L}_k F = d\iota_k F + \iota_k dF = -[\iota_k A, F] \ .$$

That is, F is invariant up to a gauge transformation with gauge parameter $-\iota_k A$. It is always possible to choose the "temporal" gauge $\iota_k A = 0$, in which F is truly invariant under k. Finally, since $\iota_k F = 0$, we have $\mathscr{L}_k H = N\,\mathrm{Tr}(\iota_k F \wedge F) = 0$. That is, k preserves H.

We have shown that the square of a Killing spinor is a Killing vector of the background, that is the Lie bracket indeed closes $[\mathfrak{k}_1, \mathfrak{k}_1] \subseteq \mathfrak{k}_0$. The rest of the brackets close similarly to the M-theory and type IIb case. Indeed, the commutator of the Lie derivative along two Killing vectors K, K' is the Lie derivative along the commutator of the vector fields, $[\mathscr{L}_K, \mathscr{L}_{K'}] = \mathscr{L}_{[K,K']}$. Since the right-hand side preserves the background, so will the left-hand side. The even-odd bracket closes because a Killing vector K preserves the Killing spinor equations. That is, since K preserves all fields we have $[\mathscr{L}_K, \nabla^+_X] = \nabla^+_{[K,X]}$, $[\mathscr{L}_K, P] = [\mathscr{L}_K, Q] = 0$ and if ε satisfies the Killing spinor equations (4.2), so will $\mathscr{L}_K\varepsilon$.

Jacobi identities

The odd-even-odd Jacobi identity is the statement that the Lie derivative is $\mathfrak{spin}(g)$ compatible. The even-even-even Jacobi identity is the even-even-even Jacobi identity of vector fields restricted to Killing vectors. It is interesting to see how the odd-odd-odd Jacobi identity is satisfied. For type IIb this involved a nontrivial Fierz transformation and using both Killing equations. The type I case is relatively easier, as no Fierz identity is involved. In particular, we shall use the fact that a chiral spinor ε is annihilated by the vector produced by its square $\bar{\varepsilon}\gamma_m\varepsilon\gamma^m\varepsilon$.

We shall use the polarization technique, whereby we show that $[[\varepsilon,\varepsilon],\varepsilon] = 0$. If $k = [\varepsilon,\varepsilon]$ this amounts to $\mathscr{L}_k\varepsilon = 0$. First note that k is ∇^+-parallel and so

$$\begin{aligned}\mathscr{L}_k\varepsilon &= \nabla_k\varepsilon + \frac{1}{4}(\nabla k)_{ab}\gamma^{ab}\varepsilon \\ &= \frac{1}{8}k^m H_{mnp}\gamma^{ab}\varepsilon - \frac{1}{4}k^m H_{mnp}\gamma^{ab}\varepsilon\end{aligned}$$

and all we need to show is

$$k^m H_{mnp}\gamma^{np}\varepsilon = 0 \ .$$

We can rewrite the left-hand side as

$$\tfrac{1}{6}k^l H_{mnp}\left(\gamma_l\gamma^{mnp} + \gamma^{mnp}\gamma_l\right)\varepsilon \ ,$$

which, using the dilatino variation $P\varepsilon = 0$, can be rewritten further as

$$2k^l\partial_m\phi\gamma_l\gamma^m\varepsilon + \tfrac{1}{6}k^l H_{mnp}\gamma^{mnp}\gamma_l\varepsilon \ .$$

Since $k^m\partial_m\phi = 0$, we can further rewrite this as

$$-2k^l\partial_m\phi\gamma^m\gamma_l\varepsilon + \tfrac{1}{6}k^l H_{mnp}\gamma^{mnp}\gamma_l\varepsilon \ .$$

which vanishes, because $k^l\gamma_l\varepsilon = 0$.

4.2.1 Homogeneity

Similarly to the M-theory and type IIb case, we can ask whether there is a critical fraction of supersymmetry ν_c, above which local homogeneity is ensured. We know that there are half-BPS heterotic backgrounds of cohomogeneity 1 ([97]), therefore $\nu_c > \frac{1}{2}$. We can apply the homogeneity theorem of §3.4 to show that if $\nu > \frac{3}{4}$, then the Killing spinors squared span pointwise the tangent space.

Indeed, let us fix a point $p \in \mathscr{M}$. The Killing spinors define a space $W \subseteq \Delta_+$ and we assume that $\dim W > 8$. If the squaring map on W does not surject T_pM, then there is a vector v such that $\overline{\varepsilon}' v \varepsilon = 0$, for all $\varepsilon, \varepsilon' \in W$. Therefore, v sends W to its annihilator $W^0 \subseteq (\Delta_+)^* = \Delta_-$. Since $\dim W > 8$ and $\dim W + \dim W^0 = 16$, v must have a kernel in Δ_+ and since $v \cdot v = |v|^2$ it must be null. Its rank is therefore 8. Equivalently, the symmetric bilinear defined by $\beta(-,-) = \langle -, v- \rangle$ on chiral spinors has rank 8.

Let us split $\Delta_+ = W \oplus U$, where U is any complementary space. Then β has the following matrix form

$$\beta = \begin{pmatrix} 0 & A \\ A^t & B \end{pmatrix} ,$$

where $A : U \to W^*$ and $B : U \to U^*$. But β has maximal rank $2 \dim U$, that is the case when A has zero kernel. We know that the rank of β is 8, so $\dim U \geq 4$ or $\dim W \leq 12$. Conversely, if $\dim W > 12$, then no such v can exist and hence W surjects pointwise the vector space. That is, for $\nu_c > 3/4$ the Killing vectors in $[\mathfrak{k}_1, \mathfrak{k}_1]$ provide a frame of infinitesimal symmetries and the background is locally homogeneous.

Type I and heterotic supergravity, though, present some novel features compared to M-theory and type IIb. The differential Killing equation is given by a metric compatible connection with torsion, the holonomy group of which lies in $\mathrm{Spin}(1,9)$. Furthermore, Killing vectors constructed from Killing spinors are parallel with respect to the connection.

Suppose, therefore, that we have a background with $\nu > \frac{1}{2}$. In particular, this means that the space of ∇^+-parallel spinors must have dimension $d > 8$. This means that the holonomy group of ∇^+ must be contained in the subgroup of

Spin$(1,9)$ which fixes $d > 8$ linearly-independent spinors. The possible stabilizer subgroups of spinors come in two families [95, 98, 60, 96], whose Lie algebras are

$$\mathfrak{spin}(7) \ltimes \mathbb{R}^8 \supset \mathfrak{g}_2 \supset \mathfrak{su}(3) \supset \mathfrak{sp}(1)$$
$$\mathfrak{spin}(7) \ltimes \mathbb{R}^8 \supset \mathfrak{su}(4) \ltimes \mathbb{R}^8 \supset \mathfrak{sp}(2) \ltimes \mathbb{R}^8 \supset (\mathfrak{sp}(1) \times \mathfrak{sp}(1)) \ltimes \mathbb{R}^8 \supset \mathbb{R}^8 ,$$

along which the dimension of the subspace of invariant spinors increases from left to right: being 1 for $\mathfrak{spin}(7) \ltimes \mathbb{R}^8$ and 8 for either $\mathfrak{sp}(1)$ or $\mathbb{R}^8$. Whence, if $d > 8$ the holonomy algebra of ∇^+ must be trivial and ∇^+ must be flat. The same group-theoretical argument shows that the kernel of the gaugino variation Q can be at most 8-dimensional, unless the gauge field is flat. Therefore, if the space of Killing spinors has dimension bigger than 8, then $F = 0$. By the results of [95] the background must have constant dilaton and it follows from [99] that it must be locally isometric to a Lie group with a bi-invariant lorentzian metric, which in particular is locally homogeneous. We conclude that $\nu_c = \frac{1}{2}$ for heterotic backgrounds. It is not clear though that a frame of local symmetries is provided from the square of Killing spinors $[\mathfrak{k}_1, \mathfrak{k}_1]$.

Type I is formally identical to the heterotic case, but ignoring the gauge fields. In particular, the differential Killing spinor equation is the same and so the same result applies: local homogeneity is guaranteed for backgrounds admitting more than 8 supersymmetries.

Chapter 5

Maximally Extended Killing Superalgebra

The most general supertranslation algebra can be extended with bosonic charges on the right-hand side of the odd-odd bracket [100]. These charges can be generically realized from the spacetime global supersymmetries of an extended object coupled to a Wess-Zumino term [101]. It is also understood that the charges do not always arise as Kaluza-Klein modes of some higher-dimensional theory [102]. Indeed, in the maximally supersymmetric flat 10+1 spacetime, the Poincaré superalgebra is extended with the charges coming from an electric (M2) 2-brane and a magnetic (M5) 5-brane [103, 104, 54].

The fundamental M2 and M5 branes can also be studied from the spacetime perspective and in the low energy limit of M-theory, there exist background configurations that realize the back-reaction of branes [105]. For a general review of branes in supergravity see [106]. Not surprisingly, we came across such charges from the spacetime perspective in section §2.3. We found that the odd-odd bracket of the Noether charges, when they can be defined, generate besides the ADM momentum, the electric and magnetic charge, along with a gravitational charge [54].

Similarly, in the context of supergravity, we raise the question: "when can a Killing superalgebra be extended with additional charges?". For the case of the maximally supersymmetric flat spacetime, the Poincaré superalgebra is extended with a 2-form and a 5-form. These forms are central with respect to the super-translational part and are realized as the constant 2-forms and 5-forms of flat spacetime. The 'M' in the term 'M-theory algebra' of [103] stands in our perspective for the 'M' in Minkowski. For the general case, we shall see that extending *maximally* a Killing superalgebra is not always possible.

We restrict our attention to maximally supersymmetric backgrounds of M-theory [69]. In section §5.1 we treat the problem algebraically and define a maximal extension of a superalgebra. We shall use the term 'minimally full' superalgebra, when the bosonic elements are equal to the symmetric square of the odd elements and are generated by them. The definition is used to classify the minimally full superalgebras, of which the maximally extended Poincaré superalgebra of M-theory is an extreme case. We then argue that the Killing superalgebra of the maximally supersymmetric plane wave does not admit a full extension. We are left to examine the maximal extension of the Killing superalgebra of the Freund-Rubin backgrounds.

In §5.2 we explore some geometric properties of Killing spinors and Killing-Yano forms. The latter are a natural generalization of Killing one-forms, the metric duals of Killing vectors. We prove different properties that we will find useful later.

We focus on the $\mathrm{AdS}_4 \times S^7$ maximally supersymmetric background. Its Killing superalgebra is known to be $\mathfrak{osp}(8|2)$ [79] and its maximally supersymmetric extension is $\mathfrak{osp}(1|32)$. In section §5.3 we find a geometric realization of $\mathfrak{osp}(1|32)$. We conclude with an interpretation of the supergravity Killing forms.

5.1 Superalgebra Extensions

In this section we develop a notion of a Lie superalgebra extension and obtain some useful results in the context of M-theory. We begin with the following definition

Definition. *A* ***superalgebra extension*** *of* $(\mathfrak{l}, [\,,\,]_{\mathfrak{l}})$*, where* $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$*, is a superalgebra* $(\mathfrak{m}, [\,,\,]_{\mathfrak{m}})$*, on* $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1$*, such that*

- $\mathfrak{m}_1 = \mathfrak{l}_1$
- $\mathfrak{l}_0 < \mathfrak{m}_0$ *is an even subalgebra of the extension*
- *the restriction of the Lie bracket* $[\,,\,]_{\mathfrak{m}}$ *on* $\mathfrak{l}_0 \times \mathfrak{l}_1$ *should agree with the even-odd bracket of* $[\,,\,]_{\mathfrak{l}}$.

If we split the even part of the extension as $\mathfrak{m}_0 = \mathfrak{l}_0 \oplus \mathfrak{z}$, where $\mathfrak{z}$ is any complement of $\mathfrak{l}_0$, then the extension is defined essentially by the bracket restricted to $\mathfrak{m} \times \mathfrak{z}$ and the odd-odd bracket, which generates the additional elements in $\mathfrak{z}$.

The definition agrees with what we want to achieve. The space $\mathfrak{l}_1$ will be the vector space of Killing spinors and $\mathfrak{l}_0$ will be the vector space of Killing vectors. By extending we do not enlarge the space of Killing spinors and, furthermore, the Killing vectors should obey the same algebra. The last clause essentially preserves the action of Killing vectors on Killing spinors.

We shall concentrate on odd generated superalgebras, for which the odd elements generate the even part $\mathfrak{m}_0 = [\mathfrak{m}_1, \mathfrak{m}_1]$. Then, the size of the extension can be characterized by the odd-odd bracket. When the span of the odd-odd bracket of a superalgebra is isomorphic to the symmetric square $S^2\mathfrak{m}_1$, we call the superalgebra full. A full superalgebra is dimension-wise the largest odd generated extension we can construct. Such an extension will be called a **minimally full extension**. This agrees with the notion of a maximal extension, the extension though is minimally full in the sense that it can be further enlarged with bosonic elements. Since $[\mathfrak{m}_1, \mathfrak{m}_1] \oplus \mathfrak{m}_1$ is an ideal of $\mathfrak{m}_0 \oplus \mathfrak{m}_1$, the additional bosonic elements (when they exist) act as outer endomorphisms, the typical example being the spin algebra of the (extended) Poincaré superalgebra.

The Poincaré superalgebra, without the Lorentz generators, extended with the central constant 2-forms and 5-forms is minimally full. The $\mathfrak{osp}(1|32)$ algebra is another minimally full superalgebra. It contains the spinor representation Δ of 11 dimensions and the endomorphisms of Δ that are skewsymmetric with respect to a symplectic bilinear on Δ. These two superalgebras are in fact the two extremal cases of the following classification [107]

Theorem 1. *A minimally full superalgebra* $(\mathfrak{m}, [\,,\,])$ *is determined by an anti-symmetric bilinear* $\omega \in \bigwedge^2 \mathfrak{m}_1^*$.

Proof. Let us choose a basis Q_α of the N-dimensional odd vector space $\mathfrak{m}_1$ and define the odd generated elements $Z_{\alpha\beta} = [Q_\alpha, Q_\beta]$. Then the even-odd bracket is defined by the structure coefficients

$$[Z_{\alpha\beta}, Q_\gamma] = \Omega_{\alpha\beta\gamma}{}^{\epsilon} Q_\epsilon \ ,$$

while the odd-odd-odd Jacobi identity implies that the cyclic sum $\sum_{\alpha,\beta,\gamma} \Omega_{\alpha\beta\gamma}{}^{\epsilon}$ is zero. The superalgebra being full implies the even-even bracket. From the even-odd-odd Jacobi identity we have

$$[Z_{\alpha\beta}, Z_{\gamma\delta}] = 2\Omega_{\alpha\beta(\gamma}{}^{\epsilon} Z_{\delta)\epsilon} = -2\Omega_{\gamma\delta(\alpha}{}^{\epsilon} Z_{\beta)\epsilon} \ ,$$

which implies

$$\Omega_{\alpha\beta(\gamma}{}^{\epsilon} \delta^{\zeta}_{\delta)} + \Omega_{\gamma\delta(\alpha}{}^{\epsilon} \delta^{\zeta}_{\beta)} + \epsilon \leftrightarrow \zeta = 0 \ .$$

By contracting the ζ, δ indices, we get

$$(N+1)\Omega_{\alpha\beta\gamma}{}^{\epsilon} + \Omega_{\alpha\beta\delta}{}^{\delta} \delta^{\epsilon}_{\gamma} + 2\Omega_{\gamma\delta(\alpha}{}^{\delta} \delta^{\epsilon}_{\beta)} = 0 \ , \tag{5.1}$$

where we used the cyclic identity. Contracting again with γ, ϵ and using again the cyclic identity, we find $\Omega_{\alpha\beta\gamma}{}^{\gamma} = 0$. Therefore, equation (5.1) becomes

$$\Omega_{\alpha\beta\gamma}{}^{\delta} = 2\omega_{\gamma(\alpha} \delta^{\delta}_{\beta)} \ ,$$

where $\omega_{\alpha\beta} = -\frac{1}{N+1} \Omega_{\delta\alpha\beta}{}^{\delta}$. The cyclic property implies that ω is antisymmetric and the superalgebra coefficients become

$$\begin{aligned}
[Q_\alpha, Q_\beta] &= Z_{\alpha\beta} \\
[Z_{\alpha\beta}, Q_\gamma] &= \omega_{\gamma\alpha} Q_\beta + \omega_{\gamma\beta} Q_\alpha \\
[Z_{\alpha\beta}, Z_{\gamma\delta}] &= \omega_{\gamma\alpha} Z_{\beta\delta} + \omega_{\gamma\beta} Z_{\alpha\delta} + \omega_{\delta\alpha} Z_{\gamma\beta} + \omega_{\delta\beta} Z_{\gamma\alpha} \ .
\end{aligned}$$

□

A simple exercise shows that ω is invariant under $\mathfrak{m}_0$. If a minimally full superalgebra $(\mathfrak{m}, [\,,\,]_{\mathfrak{m}})$ is an extension of $(\mathfrak{l}, [\,,\,]_{\mathfrak{l}})$, then the bilinear ω is invariant under $\mathfrak{l}_0$. An antisymmetric bilinear is given up to isomorphism by its rank and so, minimally full superalgebras are classified up to isomorphism by the rank of ω, which is always an even number $n = 0, \ldots 2\left\lfloor \frac{N}{2} \right\rfloor$. For the case $N = 32$ and $n = 0$, the superalgebra is the extended Poincaré superalgebra while for $n = 32$, which is the case when ω is non-degenerate, the superalgebra is $\mathfrak{osp}(1|32)$.

We now turn our attention to Lie algebra contractions. We first give the following definition of a Lie algebra isomorphism

Definition. *A* ***Lie algebra isomorphism*** $\phi : (\mathfrak{m}, [\,,\,]) \to (\mathfrak{m}', [\,,\,]')$ *is a vector space isomorphism* $\phi : \mathfrak{m} \to \mathfrak{m}'$ *and the Lie algebra brackets satisfy* $[X, Y] = \phi^{-1}([\phi(X), \phi(Y)]')$ *for all* $X, Y \in \mathfrak{m}$.

If we have a one-parameter Lie superalgebra isomorphism ϕ_ϵ from $(\mathfrak{m}, [\,,\,])$ onto $(\mathfrak{m}, [\,,\,]_\epsilon)$, for $\epsilon \in \mathbb{R}^+$, then one obtains a limiting superalgebra whenever the limiting bracket $[X, Y]_0 = \lim_{\epsilon \to 0}[X, Y]_\epsilon$ exists for all $X, Y \in \mathfrak{m}$. This defines a contraction whenever the limiting superalgebra is not isomorphic to the original superalgebra.

In fact all minimally full superalgebras can be obtained from contractions of the extremal one, whose bilinear is non-degenerate. Indeed, let us split the odd vector space into $\mathfrak{m}_1 = \mathfrak{m}_1^+ \oplus \mathfrak{m}_1^-$, which induces a split of the even part $\mathfrak{m}_0 = \mathfrak{m}_0^{++} \oplus \mathfrak{m}_0^{+-} \oplus \mathfrak{m}_0^{--}$, with $\mathfrak{m}_0^{\pm\pm} = [\mathfrak{m}_1^\pm, \mathfrak{m}_1^\pm]$. If we define the vector space isomorphism $\phi_\epsilon(X) = \epsilon^{w(X)} X$, for $X \in \mathfrak{m}_1$ where the weights are given by

$$w(X) = \begin{cases} 1 & \text{if } X \in \mathfrak{m}_1^+ \\ 0 & \text{if } X \in \mathfrak{m}_1^- \end{cases},$$

then the map ϕ_ϵ can be extended on $\mathfrak{m}_0$, so that the contraction will remain minimally full. For this to be the case, we define the superalgebra isomorphism

$\phi_\epsilon : \mathfrak{m} \to \mathfrak{m}$ with the additional weights

$$w(X) = \begin{cases} 2 & \text{if } X \in \mathfrak{m}_0^{++} \\ 1 & \text{if } X \in \mathfrak{m}_0^{+-} \\ 0 & \text{if } X \in \mathfrak{m}_0^{--} \end{cases} .$$

It is easy to see that the contracted superalgebra is no other than the minimally full superalgebra defined by the bilinear

$$\omega_0(X,Y) = \begin{cases} 0 & \text{if } X \text{ or } Y \in \mathfrak{m}_1^+ \\ \omega(X,Y) & \text{if } X, Y \in \mathfrak{m}_1^- \end{cases} .$$

The rank of ω_0 is evidently $2\left\lfloor \frac{\mathfrak{m}_1^-}{2} \right\rfloor$.

Flat Minkowski space, with $F = 0$, is a maximally supersymmetric solution of M-theory. Its supersymmetries are the constant spinors that square to the constant translations of the Poincaré group. The supertranslational group can be further extended with the constant 2-forms and 5-forms from the symmetric square of the spinors. With

$$\{Q, Q\} = P + Z_2 + Z_5 ,$$

all bosonic charges on the right-hand side are central. This defines a minimally full superalgebra for which the form ω is identically zero. The spin representation acts in the usual way on the spinorial indices and the full superalgebra, with the spin generators included, is the so-called M-theory algebra. The same construction cannot be repeated for the other maximally supersymmetric backgrounds. In fact as we shall now see, the maximally supersymmetric wave does not admit a full extension.

Let us define a superalgebra on the vector space $\mathfrak{l}$, spanned by the odd elements Q_α in $\mathfrak{l}_1$ and the even elements K_m in $\mathfrak{l}_0$. Suppose $(\mathfrak{l}, [,]_\mathfrak{l})$ is extended to a minimally full superalgebra $(\mathfrak{m}, [,]_\mathfrak{m})$, with Z_i spanning the complement ζ of $\mathfrak{l}_0$ in $\mathfrak{m}_0$, that is $\mathfrak{m}_0 = \mathfrak{l}_0 \oplus \zeta$. With $Z_{\alpha\beta} = [Q_\alpha, Q_\beta]$, we write $K_m = \phi_m^{\alpha\beta} Z_{\alpha\beta}$. Both brackets enjoy

$$[K_m, Q_\alpha] = \phi_m^{\beta\gamma} \omega_{\alpha\beta} Q_\gamma \ . \tag{5.2}$$

Let

$$\mathfrak{l}_1^\perp = \{X \in \mathfrak{l}_1 : \omega(X,Y) = 0 \text{ for all } Y \in \mathfrak{l}_1\}$$

denote the radical of ω. Then if $Q \in \mathfrak{l}_1^\perp$, we have $[K,Q] = 0$ for all $K \in \mathfrak{l}_0$ and so $\mathfrak{l}_1^\perp \subset \mathfrak{l}_1^{\mathfrak{l}_0}$, where $\mathfrak{l}_1^{\mathfrak{l}_0}$ contains the $\mathfrak{l}_0$-invariant elements of $\mathfrak{l}_1$. Therefore, if $\mathfrak{l}_1^{\mathfrak{l}_0}$ is trivial, then the bilinear ω is symplectic. On the other hand, if ω is symplectic, then (from equation (5.2)) we see that no element in $\mathfrak{l}_0$ can act trivially on the whole of $\mathfrak{l}_1$. Therefore, if a Lie superalgebra $(\mathfrak{l}, [,]_\mathfrak{l})$ is such that there exists an element $K \in \mathfrak{l}_0$ that commutes with all odd elements, but there are no $\mathfrak{l}_0$-invariant odd elements, then no minimally full extension for that superalgebra exists.

This is the case for the the maximally supersymmetric wave of M-theory. This is the Kowalski-Glikman wave, with Cahen-Wallach metric

$$g = 2dx^+ dx^- + A_{ij} x^i x^j (dx^-)^2 - d\vec{x}^2$$

and field strength $F = \mu dx^- \wedge dx^{123}$. The matrix A is given by

$$A = \begin{cases} -\frac{4}{9}\mu^2 \delta_{ij} & i = 1,2,3 \\ -\frac{4}{36}\mu^2 \delta_{ij} & i = 4 \ldots 9 \ . \end{cases}$$

We find the Killing superalgebra, originally computed in [108], in appendix C. We note that the orthonormal frame in the direction of the wave, e^+, acts trivially on the supersymmetries, whereas there is no supersymmetry that is left invariant by all Killing vectors. The maximally supersymmmetric wave, therefore, does not admit a full extension of its Killing superalgebra.

The maximally supersymmetric solutions of 11d supergravity are flat space, the Kowalski-Glikman wave and the two Freund-Rubin solutions, $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ [69]. The Killing superalgebras of the Freund-Rubin backgrounds admit a minimally full extension. We shall show that the extension is $\mathfrak{osp}(1|32)$, which corresponds to ω being the symplectic spinor inner product. In particular, we shall take as an example the maximally supersymmetric background $\mathrm{AdS}_4 \times S^7$. We shall eventually find a geometric realization of the algebra. We make, though, a detour in the following section, in order to develop familiarity with the so-called Killing-Yano forms.

5.2 Killing Objects

The problem of the Killing superalgebras of the Freund-Rubin solutions and their extension reduces to the study of geometric Killing spinors and Killing-Yano forms, which are defined on the AdS space and the sphere. Killing-Yano forms are the natural generalization of the metric dual of Killing vectors to higher degree forms and we shall refer to them simply as Killing forms[1]. In this section we suspend our focus on supergravity, in order to study the properties of these geometric, in this sense purely riemannian, Killing objects.

For the rest of the section we assume a spin manifold $(\mathscr{M}, g, S)$ of dimension d and positive scalar curvature R. When appropriate, we will specialize to AdS_4 and S^7.

5.2.1 The nature of Killing

With only geometric data on a manifold $\mathscr{M}$ the metric g and a spin structure S, we can define

Definition. *A spinor field ϵ is called a* ***geometric Killing spinor****, if it satisfies the differential equation*

$$\nabla_X \epsilon = \lambda X \cdot \epsilon \ ,$$

for all vector fields X.

[1] There is also a suitable generalization of conformal Killing vectors to higher degree forms, of which Killing forms are a special case. Related to Killing 2-forms are also the Killing-Stackel tensors; they are symmetric $(2, 0)$ tensors that roughly correspond to the square of a Killing 2-form [109].

The integrability condition for the spinor is

$$-\frac{1}{4}R_{ABCD}\gamma^{CD}\cdot\epsilon = -2\lambda^2\Gamma_{AB}\cdot\epsilon \ , \tag{5.3}$$

from which we find that the scalar curvature R is constant and related to the Killing number λ via

$$R = \lambda^2 4d(d-1) \ .$$

Since spinors do not sense[2] a change in signature from (s,t) to (t,s), we can always take $R > 0$. In this case, the Killing number is real and we focus on the positive value. It is evident that Killing spinors define a real finite-dimensional space, which we denote $\mathcal{K}_1$. For AdS spaces and spheres, the curvature two-form is

$$R(X,Y) = 4\lambda^2 X^\flat \wedge Y^\flat \ ,$$

where $X^\flat$ is the metric dual of X, $X^\flat = g(X,-)$. Therefore, the integrability condition is automatically satisfied. By the cone construction, we shall show that only quadrics in flat spacetime admit the maximal dimension of $\mathcal{K}_1$. More generally, riemannian manifolds admitting *real* Killing spinors have been completely classified [110] by using the holonomy classification of indecomposable metrics and the cone construction. Partial results exist for the lorentzian case [111, 112, 113].

Whereas parallel spinors square to parallel forms, Killing spinors square to Killing one-forms and their higher-degree analogues: Killing n-forms. For the riemannian case, there are many results concerning Killing forms and their conformal analogues [114, 115]. In particular, the riemannian space must also admit a Killing spinor. For our purposes, we shall show that Killing spinors square to what we shall call *square* Killing forms. We first define Killing forms

Definition. *An n-form b is called a **Killing form**, if its derivative ∇b reduces to an $(n+1)$-form*

$$\nabla b = \frac{1}{n+1} db \ .$$

The above definition is an immediate generalization of (the metric dual of) a

[2] the spin groups $\mathrm{Spin}(t,s)$ and $\mathrm{Spin}(s,t)$ are isomorphic and so are their modules.

Killing vector. It is easy to see that they define a conserved $(n-1)$-form along the trajectory of a geodesic. Indeed, consider a geodesic $\gamma(\tau)$, so that $\nabla_{\dot\gamma}\dot\gamma = 0$. Then for a Killing n-form b, the $(n-1)$-form $\phi = \iota_{\dot\gamma} b$ is conserved along γ. That is, $\nabla_{\dot\gamma}\phi = 0$. Killing forms are useful in finding integrals of motion, see for instance [109] and references therein.

The Killing forms satisfy an integrability condition, which is the generalization of Killing's identity of section §2.4, equation (2.13). One way to write the condition is to extend the riemannian curvature tensor, defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, to

$$\begin{array}{rcl} R : TM \otimes \bigwedge^p T^*M & \longrightarrow & \bigwedge^2 TM \otimes \bigwedge^{p-1} T^*M \\ (X,b) & \mapsto & \sum_{a,b} \eta^{ab} R(X,e_a) \otimes \iota_{e_b} b \end{array} ,$$

where e_a is an orthonormal frame. Also consider the projection from the tensor algebra to the exterior algebra $p_\wedge : \otimes^{p+1} T^*M \to \bigwedge^{p+1} T^*M$. Then Killing's identity for Killing forms is

Theorem 2. *A Killing p-form b satisfies the integrability condition*

$$\nabla_X \nabla b = -\frac{n+1}{2} p_\wedge R(X,b) \ .$$

Proof. In components we want to prove

$$\nabla_\mu a_{\nu\nu_2\cdots\nu_p} = -\frac{p+1}{2} R_{[\nu\mu_1|\mu}{}^\sigma b_{\sigma|\nu_2\cdots\nu_p]} \ ,$$

where $a = \nabla b$. Taking advantage of the antisymmetry of a, we have

$$\begin{aligned} \nabla_\mu a_{\nu\nu_1\nu_2\cdots\nu_p} - \nabla_\nu a_{\mu\nu_1\nu_2\cdots\nu_p} &= n R_{\mu\nu[\nu_1}{}^\sigma K_{\sigma\nu_2\cdots\nu_p]} \\ \nabla_\nu a_{\mu\nu_1\nu_2\cdots\nu_p} - \nabla_{\nu_1} a_{\mu\nu\nu_2\cdots\nu_p} &= -n R_{\nu\nu_1[\mu}{}^\sigma K_{\sigma\nu_2\cdots\nu_p]} \\ \nabla_{\nu_1} a_{\mu\nu\nu_2\cdots\nu_p} + \nabla_\mu a_{\nu\nu_1\nu_2\cdots\nu_p} &= n R_{\nu_1\mu[\nu}{}^\sigma K_{\sigma\nu_2\cdots\nu_p]} \ . \end{aligned}$$

By adding the three equations, we obtain

$$2\nabla_\mu a_{\nu\nu_1\nu_2\cdots\nu_p} = nR_{\mu\nu[\nu_1}{}^\sigma K_{\sigma\nu_2\cdots\nu_p]} - nR_{\nu\nu_1[\mu}{}^\sigma K_{\sigma\nu_2\cdots\nu_p]} + nR_{\nu_1\mu[\nu}{}^\sigma K_{\sigma\nu_2\cdots\nu_p]} \ .$$

We notice that the right-hand side is antisymmetric in $\nu\nu_1\cdots\nu_p$. If we manifest the antisymmetry, we have

$$\begin{aligned}2\nabla_\mu a_{\nu\nu_1\nu_2\cdots\nu_p} e^{\nu\nu_1\nu_2\cdots\nu_p} = n(&R_{\mu\nu\nu_1}{}^\sigma K_{\sigma\nu_2\cdots\nu_p} - \frac{1}{n} R_{\nu\nu_1\mu}{}^\sigma K_{\sigma\nu_2\cdots\nu_p} \\ &+ \frac{n-1}{n} R_{\nu\nu_1\nu_2}{}^\sigma K_{\sigma\mu\nu_3\cdots\nu_p} + R_{\nu_1\mu\nu}{}^\sigma K_{\sigma\nu_2\cdots\nu_p}) e^{\nu\nu_1\nu_2\cdots\nu_p} \ .\end{aligned}$$

The third summand on the right-hand side vanishes because of the Bianchi identity. Similarly the first and last term combine through the Bianchi identity to yield Killing's identity. □

As a result of the integrability condition, a Killing p-form has

$$\binom{d}{p} + \binom{d}{p+1} = \binom{d+1}{p+1}$$

degrees of freedom. Of special importance are special Killing forms, for which there is an integrability condition stronger than Killing's identity

Definition. *A* ***special Killing form*** *is a Killing form b, for which its derivative satisfies the integrability condition*

$$\nabla_X \nabla b = -4\lambda'^2 X^\flat \wedge b \ ,$$

for some constant λ' and any vector X, where $X^\flat = g(X, -)$.

Special Killing forms on a constant scalar curvature space, for which the constant λ' is precisely Killing's constant λ, are 'doubly' special. We shall call them **square Killing forms** and denote the space of odd square Killing forms by $\mathcal{K}_0$. For AdS spaces and spheres, one can show that all Killing forms are square. By the cone construction, we shall show that quadrics have the maximal dimension of Killing forms.

We now show that the square of Killing spinors are square Killing forms. The spinor pairing $\langle , \rangle$, with property

$$\langle X \cdot -, - \rangle = -\langle -, X \cdot - \rangle \ ,$$

is for AdS_4 (S^7) antisymmetric (symmetric). Any two spinors $\varepsilon, \varepsilon'$ square to an n-form ξ_n

$$\xi_n = \frac{1}{n!} \langle \epsilon, \gamma_{a_1 \cdots a_n} \epsilon' \rangle e^{a_1 \cdots a_n} \ . \tag{5.4}$$

If the spinors are Killing, then the odd n-forms are square Killing and the even forms are their derivatives

$$\nabla_X \xi_n = \begin{cases} -2\lambda \, \iota_X \xi_{n+1} & n = \text{odd} \\ -2\lambda \, \xi_{n-1} \wedge X^\flat & n = \text{even} \ . \end{cases} \tag{5.5}$$

In particular, the scalar ξ_0 is constant.

The even forms produced can be shown to be Hodge dual to square Killing forms

$$\nabla_X * \xi_n = \begin{cases} -2\lambda \, (-1)^n X^\flat \wedge *\xi_{n+1} & n = \text{odd} \\ -2\lambda \, \iota_X * \xi_{n-1} & n = \text{even} \ . \end{cases} \tag{5.6}$$

For d odd, we only get odd Killing forms from Killing spinors. When the dimension of the manifold d is even, we get square Killing forms of all degrees. Notice that, when the dimension d of the manifold is odd, the volume element dvol in the Clifford module is trivial and one can show that

$$\mathrm{dvol} \cdot \xi_n = (-1)^{\frac{n}{2}(n+1)} * \xi_{d-n} \ . \tag{5.7}$$

Killing spinors and square Killing forms lift to parallel spinors and parallel forms on the cone of the manifold, which we now describe.

5.2.2 Killing on the Cone

The cone of a constant and positive scalar curvature R spin manifold $(\mathscr{M}, g, S)$, is the manifold $\hat{\mathscr{M}} = \mathscr{M} \times \mathbb{R}^+$ with metric

$$\hat{g} = dr^2 + 4r^2\lambda^2 g \ ,$$

where λ is the Killing number. The cones of AdS and spheres are independent of λ and flat. This is evident, since the AdS spaces and spheres are precisely defined by their respective quadrics. In stereographic coordinates the cone metric of a sphere (+) and an AdS space (−) are

$$\hat{g} = dr^2 + 4r^2\lambda^2 \frac{1}{(1 \pm \lambda^2 x^2)^2} dx^2 \ .$$

Let us choose the vielbein of the cone $(\hat{e}^a, \hat{e}^r) = (dr,\, 2\lambda r e^a)$, where e^a is a vielbein of the base. The spin coefficients are found to be

$$\begin{aligned} \hat{\omega}^a{}_r &= 2\lambda e^a \\ \hat{\omega}^a{}_b &= \omega^a{}_b \ , \end{aligned} \tag{5.8}$$

while the Christoffel coefficients can be conveniently written as

$$\begin{aligned} \hat{\nabla}_r dr = 0 \qquad & \hat{\nabla}_\mu dr = 4\lambda^2 r g_{\mu\nu} dx^n \\ \hat{\nabla}_r (r dx^\mu) = 0 \qquad & \hat{\nabla}_\mu dx^\nu = \nabla_\mu dx^\nu - \frac{1}{r}\delta^\nu_\mu dr \ . \end{aligned} \tag{5.9}$$

We have the correspondence

Theorem 3. *A square Killing $(p-1)$-form b on a manifold $\mathscr{M}$ is in one-to-one correspondence with a parallel p-form $\hat{b}$ defined on the cone of the manifold. The correspondence is given (up to a convenient factor) by*

$$\hat{b} = (2\lambda)^{p-1}(r^p \nabla b + r^{p-1} dr \wedge b) = (2\lambda)^{p-1} d(\frac{r^p}{p} b) \ .$$

Proof. We decompose a p-form on the cone like

$$\hat{b} = r^p a + r^{p-1} dr \wedge b$$

and consider the condition $\hat{\nabla}\hat{b} = 0$. The component $\hat{\nabla}_r \hat{b} = 0$ gives no r dependence of a and b. The component $\hat{\nabla}_\mu \hat{b} = 0$ expands to

$$r^p \nabla_\mu a - r^{n-1} dr \wedge i_\mu a + 4\lambda^2 r^p g_{\mu\nu} dx^\nu \wedge b + r^{p-1} dr \wedge \nabla_\mu b = 0 \ .$$

Thus b is a square Killing $(p-1)$-form and a is its derivative

$$\nabla_X b = \iota_X a$$
$$\nabla_X a = -\frac{R}{d(d-1)} X^\flat \wedge b \ .$$

□

We now show that Killing spinors lift to parallel spinors on the cone. The even part of the Clifford algebra of the cone is isomorphic to the Clifford algebra of the base

$$\mathfrak{spin}(s+1,t) \subset \mathrm{C}\ell^0(s+1,t) = \mathrm{C}\ell(s,t) \ .$$

For the dimensions that interest us, a spinor in signature $(7,0)$ lifts to a chiral spinor on $(8,0)$ and a spinor in $(1,3)$ lifts to a spinor in $(2,3)$. We elaborate on this in the appendix B. We can find suitable representations, so that the spin generators on the cone are in terms of the Clifford generators of the base

$$\Gamma^{ab} = \gamma^{ab}$$
$$\Gamma^{ar} = \gamma^a \ .$$

The spin generators, therefore, act as

$$\hat{s}_{ar} = -\frac{1}{2}\gamma_a$$
$$\hat{s}_{ab} = -\frac{1}{2}\gamma_{ab}$$

and so the cone spin connection, equation (5.8), on spinors on the cone is

$$\hat{\nabla}_X = \nabla_X - \lambda X \cdot$$
$$\hat{\nabla}_r = \partial_r \ ,$$

where on the right-hand side we have the Clifford action on the base. We thus have a one-to-one correspondence of Killing spinors on the base with parallel spinors on the cone. Furthermore, we can easily show that the operations of squaring a Killing spinor and lifting to, or lowering from, the cone commute.

The Killing spinors on S^7 lift to parallel chiral spinors on $\mathbb{R}^8$ and Killing spinors on AdS_4 lift to parallel spinors on $\mathbb{R}^{2,3}$. The cone metrics are flat and so we get the maximal dimension of Killing spinors. For the same reason we get the maximal dimension of square Killing forms.

We can combine the above results and construct a Killing algebra that is induced by the Clifford algebra on the cone. This is a closed algebra of Killing forms, whereas Killing forms also act on Killing spinors. We briefly describe this and specialize for the spaces AdS_4 and S^7.

5.2.3 Clifford-Killing algebra

Let us turn our attention to the endomorphisms of Killing spinors. Since Killing spinors lift to parallel spinors on the cone, of definite chirality if the cone is even-dimensional, the endomorphisms of Killing spinors correspond to parallel even forms on the cone. Indeed, when the dimension of the cone is odd, the volume element is trivial and so the constant forms act on the constant spinors up to Hodge duality, as in equation (5.7). When the dimension of the cone is even, the endomorphisms of chiral constant spinors are again the even constant forms that preserve their chirality but they also act up to Hodge duality, again due to (5.7). Therefore, the endomorphisms on the space of Killing spinors $\mathrm{End}(\mathcal{K}_1)$ are the odd square Killing forms in $\mathcal{K}_0$ of degree n, where in quite generality $n = 1, 3, \ldots 2m-1$ with $d = 2m$ or $d = 4m \pm 1$. Obviously constant functions on the base, which lift to constant functions on the cone, also act trivially on Killing spinors. With this correspondence, the algebra of endomorphisms of Killing spinors is that of the Clifford algebra of the corresponding parallel forms on the cone.

In the above discussion we considered the Hodge dual of constant forms on the cone. Hodge duality on the cone relates the square Killing forms on the base as follows

$$\hat{*}(r^p a + r^{p-1} dr \wedge b) = (2\lambda)^{d-2p} \left((2\lambda)^2 r^{d+1-p} * b + r^{d-p} dr \wedge *a\right) \ .$$

This is illustrated in figure 5.1.

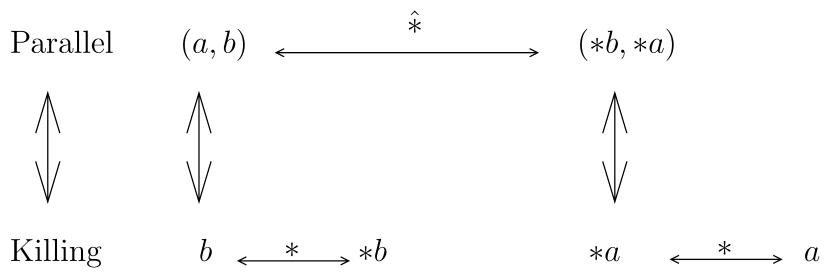

Figure 5.1: The fate of Killing forms under Hodge duality

One can derive expressions for the Killing algebra and its action on Killing spinors, entirely in terms of the base. The image of an even parallel form on the cone of a manifold into the Clifford algebra of the base manifold is

$$\begin{gathered} \mu : \mathrm{C}\ell^0(T^* \hat{\mathscr{M}}) \rightarrow \mathrm{End}(\mathcal{K}_1) \\ (2\lambda)^{p-1}(r^b \nabla b + r^{p-1} dr \wedge b) \mapsto (2\lambda)^{-1} \nabla b - b \ . \end{gathered}$$

The action of a Killing one-form b on a Killing spinor ε is proportional to the spinorial Lie derivative

$$((2\lambda)^{-1} \nabla b - b)\varepsilon = ((2\lambda)^{-1} \nabla b - (\lambda)^{-1} \nabla_b)\varepsilon = -(\lambda)^{-1} \mathscr{L}_b \varepsilon \ .$$

The symmetry operator of a Killing p-form b on a Killing spinor ε is generalized from the spinorial Lie derivative to

$$\mathscr{L}_b \varepsilon = \left(\frac{1}{p!} b_{a_1 \cdots a_{p-1}}{}^{a_p} \gamma^{a_1 \cdots a_{p-1}} \nabla_{a_p} - \frac{1}{2} (\nabla b) \cdot \right) \varepsilon \ .$$

The symmetry operator along square odd Killing forms will send Killing spinors to Killing spinors. It can be further generalized to a symmetry operator of the Dirac operator [116].

Clifford multiplication on the cone gives, in general, a sum of forms by contraction. The map μ induces an algebra of odd square Killing forms. For two odd square Killing forms b and b' of degree p and q respectively, the one-contraction is given up to a factor by the Schouten-Nijenhuis bracket of the forms on the base

$$[b, b'] = (-1)^p g^{\mu\nu} \iota_\mu b \wedge \nabla_\nu b' - (-1)^{(p-1)q} g^{\mu\nu} \iota_\mu b' \wedge \nabla_\nu b \ .$$

In [117] the authors show that the Schouten-Nijenhuis bracket is closed when the metric is of constant sectional curvature. We generalize their result for odd square Killing forms on a constant and positive scalar curvature space. As noted in [115], all examples of Killing forms that we have are these special square Killing forms.

Let us now specialize for the spaces AdS_4 and S^7. We denote the space of Killing spinors on these spaces respectively by Δ_A and Δ_S. By the cone construction, Δ_A is a Clifford module of $\mathrm{C}\ell(2,3)$ and Δ_S is a chiral Clifford module of $\mathrm{C}\ell^0(8,0)$. The endomorphisms of parallel spinors on the cones of both AdS_4 and S^7 are the constant even forms and equivalent up to Hodge duality on their cone

$$\begin{aligned} \mathrm{End}(\Delta_A) &= \textstyle\bigwedge^0_A \oplus \bigwedge^2_A \oplus \bigwedge^4_A \\ \mathrm{End}(\Delta_S) &= \textstyle\bigwedge^0_S \oplus \bigwedge^2_S \oplus \bigwedge^{4+}_S \ , \end{aligned}$$

where for convenience we define $\bigwedge^n_A = \bigwedge^n \mathbb{R}^{2,3}$ and $\bigwedge^n_S = \bigwedge^n \mathbb{R}^8$. Similarly, we establish that the symmetric and antisymmetric square of the Killing spinors in

Δ_A or Δ_S span their endomorphisms as follows

$$
\begin{aligned}
S^2\Delta_A &= \textstyle\bigwedge^2_A \\
\textstyle\bigwedge^2 \Delta_A &= \textstyle\bigwedge^0_A \oplus \bigwedge^4_A \\
S^2\Delta_S &= \textstyle\bigwedge^0_S \oplus \bigwedge^{4+}_S \\
\textstyle\bigwedge^2 \Delta_S &= \textstyle\bigwedge^2_S \ .
\end{aligned}
$$

5.3 Freund-Rubin Backgrounds

In this section we set up the maximally supersymmetric solution $\mathrm{AdS}_4 \times S^7$. The conventions we use for a supergravity background (g, F) are those of chapter 2 with $\lambda = 1$. The background $\mathrm{AdS}_4 \times S^7$ is one of the two Freund-Rubin solutions of supergravity, with metric g and field strength F:

$$
\begin{aligned}
g &= g_A - g_S \\
F &= \sqrt{\frac{3R}{2}}\,\mathrm{dvol}(\mathrm{AdS}_4) \ .
\end{aligned}
$$

The AdS_4 space has metric g_A with scalar curvature $8R$ and the seven-sphere has metric g_S with scalar curvature $7R$, where $R > 0$ is the total scalar curvature.

We choose a Clifford module by fixing the sign of the volume element. Supergravity Killing spinors are in the kernel of the superderivative $\mathcal{D}$. We denote the space of supergravity Killing spinors $\mathfrak{k}_1$. We shall show they belong to the product of the geometric Killing spinor spaces of each factor AdS_4 and S^7.

There is an isomorphism

$$
\begin{aligned}
\mathrm{C}\ell(1,10) &\xrightarrow{\equiv} \mathrm{C}\ell(1,3) \otimes \mathrm{C}\ell(7,0) \\
a + b &\mapsto \mathrm{dvol}(1,3)a \otimes \mathbb{1} + \mathrm{dvol}(1,3) \otimes b \ ,
\end{aligned}
$$

where $a + b \in T^*(\mathrm{AdS}_4) \oplus T^*(S^7)$ is an orthogonal split. Any vector Z can be split into tangential parts $Z = X + Y \in T(\mathrm{AdS}_4) \oplus T(S^7)$, but note how $g(-, Y) = -g_S(-, Y)$. The spinor bundle of the background is isomorphic to the product of the spinor bundles over each factor $S_A \otimes S_S$. If we choose a 'pure' spinor $\epsilon = \alpha \otimes \beta$, we can solve for the Killing equation on each factor

$$\begin{aligned} \nabla_X \alpha &= \sqrt{\frac{R}{6}} X \cdot \alpha \quad \text{on } AdS_4(8R) \\ \nabla_Y \beta &= \sqrt{\frac{R}{4!}} Y \cdot \beta \quad \text{on } S^7(7R) \ . \end{aligned}$$

The space of supergravity Killing spinors $\mathfrak{m}_1$ has maximal dimension 32. The space of geometric Killing spinors on AdS_4, which we denote by Δ_A, has real dimension 4 and on S^7, which we denote by Δ_S, has real dimension 8. By splitting the spinor we obtain all solutions

$$\mathfrak{k}_1 = \Delta_A \otimes \Delta_S \ .$$

From representation theory alone, the symmetric square of the Killing spinors will span the tangent bundle. Furthermore, it is evident that it will span the tensor product of the Killing algebra of each factor, which are the constant two-forms on their respective cones. The Killing superalgebra $(\mathfrak{k}, [,]_{\mathfrak{k}})$, with $\mathfrak{k}_0 = [\mathfrak{k}_1, \mathfrak{k}_1]_{\mathfrak{k}}$ has

$$\mathfrak{k}_0 = \textstyle\bigwedge^0_A \otimes \bigwedge^2_S \oplus \bigwedge^2_A \otimes \bigwedge^0_S \equiv \mathfrak{spin}(2,3) \oplus \mathfrak{spin}(8) \ .$$

Let us assume that the Killing superalgebra extends to a minimally full superalgebra $(\mathfrak{m}, [,]_{\mathfrak{m}})$. Then the form $\omega \in \bigwedge^2 \mathfrak{k}_1^*$ that defines the extension should be invariant under $\mathfrak{k}_0$. The dual of $\mathfrak{k}_1$ is naturally associated to $\mathfrak{k}_1$ using the spinor inner product C. We can decompose the two-forms in $\bigwedge^2 \mathfrak{k}_1$ as a representation of $\mathfrak{k}_0$

$$\begin{aligned} \textstyle\bigwedge^2 \mathfrak{k}_1 &= S^2\Delta_A \otimes \textstyle\bigwedge^2 \Delta_S \oplus \bigwedge^2 \Delta_A \otimes S^2 \Delta_S \\ &= \textstyle\bigwedge^2_A \otimes \bigwedge^2_S \oplus \left(\bigwedge^0_A \oplus \bigwedge^4_A\right) \otimes \left(\bigwedge^0_S \oplus \bigwedge^{4+}_S\right) \ . \end{aligned}$$

Since $\mathfrak{k}_0$ is non-abelian, we conclude that the minimally full extension of the Killing superalgebra is defined up to isomorphism by the unique $\mathfrak{k}_0$-invariant two-form C, which defines the inner product in $C\ell(1,10)$. It is symplectic and defines $\mathfrak{osp}(1|32)$.

Let us elaborate on the structure of $\mathfrak{osp}(1|32)$ on $\mathrm{AdS}_4 \times S^7$. The endomorphisms of $\mathfrak{m}_1$ have a geometric realization. They are in the tensor product of parallel even forms on the cone of each factor

$$\mathrm{End}(\mathfrak{m}_1) = \mathrm{End}(\Delta_A) \otimes \mathrm{End}(\Delta_S) = (\textstyle\bigwedge_A^0 \oplus \bigwedge_A^2 \oplus \bigwedge_A^4) \otimes (\bigwedge_S^0 \oplus \bigwedge_S^2 \oplus \bigwedge_S^{4+}) \ .$$

Killing bispinors $\mathfrak{m}_1 \otimes \mathfrak{m}_1$ are isomorphic to the endomorphisms of Killing spinors $\mathrm{End}(\mathfrak{m}_1)$, via the symplectic inner product C. That is $(\varepsilon \otimes \varepsilon')\varepsilon'' = \langle \varepsilon', \varepsilon'' \rangle \varepsilon$. We are interested in the symmetric square

Theorem 4. *The symmetric square is onto the endomorphisms that are skew with respect to the spinor pairing*

Proof. Polarizing for $\epsilon\bar{\epsilon} \in S^2\mathfrak{m}_1 \subset \mathrm{End}(\mathfrak{m}_1)$ we have

$$\begin{aligned}\langle \epsilon\bar{\epsilon}\epsilon_1, \epsilon_2 \rangle &= \langle \epsilon, \epsilon_1 \rangle \langle \epsilon, \epsilon_2 \rangle \\ &= -\langle \epsilon_1, \epsilon\bar{\epsilon}\epsilon_2 \rangle\end{aligned}$$

so $\epsilon\bar{\epsilon} \in \mathrm{SkewEnd}(\mathfrak{m}_1)$ and thus $S^2\mathfrak{m}_1 \subseteq \mathrm{SkewEnd}(\mathfrak{m}_1)$. By counting dimensions, we find that $S^2\mathfrak{m}_1 = \mathrm{SkewEnd}(\mathfrak{m}_1)$. □

Furthermore the commutator of endomorphisms is closed on $\mathrm{SkewEnd}(\mathfrak{m}_1)$. We can show this again by polarizing for two endomorphisms

$$\epsilon\bar{\epsilon}, \epsilon'\overline{\epsilon'} \in \mathrm{SkewEnd}(\mathfrak{m}_1)$$

$$\epsilon\bar{\epsilon}\epsilon'\overline{\epsilon'} - \epsilon'\overline{\epsilon'}\epsilon\bar{\epsilon} = \langle \epsilon, \epsilon' \rangle (\epsilon\overline{\epsilon'} + \epsilon'\bar{\epsilon}) \in \mathrm{SkewEnd}(\mathfrak{m}_1) \ .$$

In terms of the classification of theorem 1 in §5.1, we have $\omega = -C$.

We then have a Lie superalgebra on $\mathfrak{m}_1 \oplus [\mathfrak{m}_1, \mathfrak{m}_1]$ with Lie bracket

- $[\mathfrak{m}_1, \mathfrak{m}_1] \to S^2\mathfrak{m}_1$ the natural symmetric squaring.
- $[S^2\mathfrak{m}_1, \mathfrak{m}_1] \to \mathfrak{m}_1$ acting as endomorphisms
- $[S^2\mathfrak{m}_1, S^2\mathfrak{m}_1] \to S^2\mathfrak{m}_1$ the commutator of endomorphisms .

The Jacobi identities are satisfied because of the antisymmetry of the spinor inner product (theorem 1).

We conclude by realizing $(\mathfrak{m}, [,]_{\mathfrak{m}}) = \mathfrak{osp}(1|32)$ in terms of geometric objects. The odd part consists of the Killing spinors, $\mathfrak{m}_1 = \Delta_A \otimes \Delta_S$. We decompose the even elements $\mathfrak{m}_0 = [\mathfrak{m}_1, \mathfrak{m}_1]$ as representations of $\mathfrak{k}_0$, which is the isometry algebra of the background. The symmetric tensor square of $\mathfrak{m}_1$ is

$$\begin{aligned} S^2\mathfrak{k}_1 &= S^2\Delta_A \otimes S^2\Delta_S \oplus \bigwedge^2 \Delta_A \otimes \bigwedge^2 \Delta_S \\ &= \bigwedge_A^2 \otimes (\bigwedge_S^0 \oplus \bigwedge_S^{4+}) \oplus (\bigwedge_A^0 \oplus \bigwedge_A^4) \otimes \bigwedge_S^2 \\ &= (\bigwedge_A^2 \otimes \bigwedge_S^0) \oplus (\bigwedge_A^0 \otimes \bigwedge_S^2) \oplus (\bigwedge_A^2 \otimes \bigwedge_S^{4+}) \oplus (\bigwedge_A^4 \otimes \bigwedge_S^2) , \end{aligned}$$

where we recognize the terms on the right as the tensor product of definite degree Killing forms on each factor. The results of this chapter make it seem natural to use a double-cone description for the embeddings of branes in the Freund-Rubin backgrounds.

Part II

Non-Geometry

Chapter 6

The Doubled Torus

Let us consider a string model, $\phi : \Sigma \to \mathscr{M}^{(26)}$, where the target space $\mathscr{M}^{(26)}$ is a bundle with fibers some compact directions $\{x^i\} \in T^n$. We assume that the compact directions are generated by isometries and so T-duality can act on the maps $X^i = x^i \circ \phi$. The aim of the doubled torus formalism is to dualize the X^i and use both dual descriptions in the same framework. In this chapter we introduce the doubled torus in the lagrangian formalism of [19] and investigate some of its properties.

In string theory one first defines a background on which the string will propagate. Then one solves for the dynamics of string propagation. The quantization of string propagation depends on the background at hand, while the background is constrained to satisfy certain conditions in order to preserve conformal invariance. Similarly for the doubled torus, we shall first introduce doubled torus backgrounds and then introduce the dynamics of string propagation on such backgrounds.

In section §6.1 we define a doubled torus background. The target space is now a bundle with fiber the doubled torus T^{2n}, which has local coordinates x^I. If one trivializes the T^{2n} as $T^n \oplus T^n$, the geometric data of the doubled torus splits and gives the original conventional data. This is achieved through a constant metric L of signature (n, n), which fixes a subgroup $O(n, n, \mathbb{Z})$ of the large diffeomorphism group of the doubled torus. The group $O(n, n, \mathbb{Z})$ is the T-duality group that manifests itself geometrically. This is how we will recover the Buscher rules.

In section §6.2 we introduce the doubled torus system [19], which is a pseudo-action plus a constraint. In doubling the torus of the target space, we increase the degrees of freedom of string propagation. In order to recover the conventional formalism, we impose a constraint that removes the extra degrees of freedom. The lagrangian and constraint are covariant under the geometric group $O(n,n,\mathbb{Z})$.

Solving the constraint amounts to splitting the string coordinates of T^{2n} as $X^I = (X^i, \tilde{X}_i)$. In section §6.3 we solve the dual coordinates $\tilde{X}_i$ in terms of the X^i. The resulting equations of motion for X^i can be identified with the conventional string. If we use the dual solution of the $\tilde{X}_i$ in terms of X^i, we recover the conventional T-dual model. We will show classical equivalence with the conventional model.

The choice of which directions are the physical (independent) ones is called a polarization $\mathcal{P}$. T-duality manifests itself as a geometric group of transformations, where a covariant choice $\mathcal{P}$ can, but not necessarily, be given. In section §6.4 we comment on this and interpret conventional T-duality as a geometric group in the doubled torus formalism. This chapter is a necessary introduction for the succeeding chapters.

6.1 Doubled Torus Backgrounds

We give the following formal definition for a doubled torus background

Definition. *A* ***doubled torus background*** *is a T^{2n} fiber bundle $\mathscr{E}$, with base space $\mathscr{N}$, such that n and the dimension of $\mathscr{N}$ add up to the critical dimension. The bundle $\mathscr{E}$ has transition functions in $O(n,n,\mathbb{Z}) \times \mathrm{U}(1)^{2n}$ that preserve a constant metric L on the fiber.*

$$\begin{array}{l} \mathscr{E} \overset{loc.}{\approx} \mathscr{N} \times T^{2n} \\ \downarrow \pi \\ \mathscr{N} \end{array}$$

The background is further specified with the geometric data (H, A, G, B)

- *A local trivialization of the bundle $\mathscr{E}$ is $T^{2n} \times \mathscr{N}$ with coordinates (x^I, y^m).*

The triplet $(H, A, \tilde{G})$ *make up the components of a metric on the total space*

- H *is a positive definite metric on the fiber with values* H_{IJ}*,*
- A *are the mixed components of the metric with values* A_{Im}*,*
- $\tilde{G}$ *a lorentzian signature metric on the base space with values* $\tilde{G}_{mn}$*.*

and $\tilde{B}$ *is a b-field on the base space defined up to closed terms.*

- *Locally* H *is restricted to be a coset metric for* $O(n,n)/(O(n) \times O(n))$*.*
- *The data* $(H, A, \tilde{G}, \tilde{B})$ *depend only on the base.*

Conformal invariance of the sigma model will impose further differential conditions upon quantization. Such an analysis was undertaken in [118, 119]. In this section we shall work with the above definition. In the following we discuss the transition functions of the bundle, the coset metric H and we will see how to rewrite the Buscher rules.

6.1.1 Transition functions

The transition functions of any T^{2n}-bundle belong to the large diffeomorphism group of the torus $GL(2n, \mathbb{Z})$ times the periodic translations $U(1)^{2n}$. It is easy to see that the group $GL(2n, \mathbb{Z})$ factors as $\mathbb{Z}_2 \times SL(2n, \mathbb{Z})$. The transition functions of the bundle $\mathscr{E}$ are the elements of $GL(2n, \mathbb{Z})$ that fix the constant metric L of signature (n, n). The isometry group of L is $O(n, n)$, so the transition functions belong to the subgroup

$$O(n, n, \mathbb{Z}) \subset GL(2n, \mathbb{Z}) \ .$$

Say for instance H is defined locally on the patches U_α and U_β of the base space $\mathscr{N}$. On the intersection $U_{\alpha\beta}$, the metric H will transform, under the transition functions $g_{\alpha\beta} \in O(n, n, \mathbb{Z})$, as

$$H \longrightarrow H' = g^t_{\alpha\beta} H g_{\alpha\beta}. \tag{6.1}$$

Let us choose a basis so that the metric L has components

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

We shall call such a basis and the form of the metric **canonical**. This fixes the group $O(n,n)$ as a subgroup of $GL(2n,\mathbb{R})$. As the transition functions are chosen to be in the integer subgroup of this, the form of L remains canonical in all patches.

We shall produce two sets of equations for elements of $O(n,n)$. A typical element g in the canonical basis

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ,$$

satisfies the defining relation $g^t L g = L$, which is expanded as

$$\begin{aligned} a^t c + c^t a &= 0 \\ b^t d + d^t b &= 0 \\ a^t d + c^t b &= 1 \ . \end{aligned} \tag{6.2}$$

The relations can be 'reversed' in the following way. Since

$$g^{-1} = L^{-1} g^t L = \begin{pmatrix} d^t & b^t \\ c^t & a^t \end{pmatrix} \ ,$$

we use $gg^{-1} = \mathbb{1}_{2n}$ to obtain a second set of equations

$$\begin{aligned} ab^t + ba^t &= 0 \\ cd^t + dc^t &= 0 \ . \\ ad^t + bc^t &= 1 \end{aligned} \tag{6.3}$$

We shall make use of these equations later on. Note that for the group $O(n,n,\mathbb{Z})$, the element g should furthermore have determinant ± 1 and integer entries.

6.1.2 The coset metric

The metric H takes values of a metric for the coset space

$$O(n,n)/(O(n)\times O(n))\ .$$

To achieve this we first define a *vielbein* V so that for a patch $U_\alpha \subset \mathscr{N}$, V is a local section

$$V : U_\alpha \longrightarrow O(n,n)\ . \tag{6.4}$$

We then fix the maximal compact subgroup $O(n)\times O(n)$ by specifying a metric Δ of signature $(2n,0)$. The metric H is defined as

$$H = V^T \Delta V\ .$$

The vielbein in (6.4) can be identified under the left $O(n)\times O(n)$ action

$$V \equiv hV \quad \text{for all } h \in O(n)\times O(n) \subset O(n,n)\ .$$

The transition functions $g \in O(n,n,\mathbb{Z})$ act on the representative of the coset V on the right. Therefore, the metric transforms like $H \to g^T H g$, while V may transform with a compensating factor $V \to hVg$.

It might seem that there is a choice of the metric Δ, but this is not quite so. Were we to choose a conjugate maximal compact subgroup $O(n)\times O(n)$, we would change the metric Δ by the adjoint action of $O(n,n)$. This would be in effect a redefinition of V.

In a canonical basis a convenient choice for Δ is

$$\Delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ .$$

The $O(n)\times O(n) \subset O(n,n)$ subgroup is given by precisely those elements in

$O(n,n)$ that preserve Δ. In the chosen basis they have the form

$$g = \frac{1}{2}\begin{pmatrix} a+d & a-d \\ a-d & a+d \end{pmatrix} \in O(n)\times O(n) \subset O(n,n) \ ,$$

with $(a,d) \in O(n)\times O(n)$.

Let us denote the inverse of L_{IJ} as L^{IJ} and define $S^I{}_J = L^{IK}H_{KJ}$. In section §6.2 we shall need to use the relation $S^2 = 1$. The endomorphism S is an almost product (or almost real) structure with respect to both L and H. We show the equivalent

Lemma 2. *The coset metric H commutes with the metric L*

$$\begin{array}{ccc} T(T^{2n}) & \xrightarrow{H} & T^\star(T^{2n}) \\ \big\downarrow L & & \big\downarrow L^{-1} \\ T^\star(T^{2n}) & \xrightarrow{H^{-1}} & T(T^{2n}) \end{array}$$

.

Proof. In the canonical basis it is clear that $\Delta^{-1}L = L^{-1}\Delta$ and so if $S = L^{-1}H$ then

$$S^2 = L^{-1}V^T\Delta\underbrace{VL^{-1}V^T}_{L^{-1}}\Delta V = L^{-1}V^T\underbrace{\Delta L^{-1}\Delta}_{L}V = L^{-1}\underbrace{V^TLV}_{L} = L^{-1}L = 1_{2n} \ .$$

Since it is true in one basis, it will be true in all. □

It is convenient to define a new object $\tilde{A}$ by

$$\tilde{A}_{Im} = -H_{IJ}L^{JK}A_{Km} \ .$$

By using $S^2 = 1$, we can consider this as a self-dual constraint on the doublet $(A, \tilde{A})$.

6.1.3 Buscher rules

The action of $O(n,n,\mathbb{Z})$ on the coset metric H reproduces the Buscher rules. To see this, we shall first find a canonical form of H. In particular, in a patch U_α, we can use the $O(n)\times O(n)$ gauge freedom $V\equiv Vh$ to bring the vielbein into the gauge fixed form

$$V=\begin{pmatrix} e^T & 0 \\ -e^{-1}B & e^{-1}\end{pmatrix} . \tag{6.5}$$

Proof of (6.5). Assume the element $h\in O(n)\times O(n)$ acts on $\tilde{V}$ to give $V=h\tilde{V}$. Let us write these as

$$h=\frac{1}{2}\begin{pmatrix} a+b & a-b \\ a-b & a+b\end{pmatrix} , \qquad V=\begin{pmatrix} A & B \\ C & D\end{pmatrix} , \qquad \tilde{V}=\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D}\end{pmatrix} ,$$

with $a,b\in O(n)$. Then the action gives the following relations

$$\begin{aligned} A&=(a+d)\tilde{A}+(a-d)\tilde{C}, & B&=(a+d)\tilde{B}+(a-d)\tilde{D},\\ C&=(a-d)\tilde{A}+(a+d)\tilde{C}, & D&=(a-d)\tilde{B}+(a+d)\tilde{D} .\end{aligned}$$

Setting $s=a^Td\in O(n)$ and demanding $B=0$, we get the condition

$$s(\tilde{D}-\tilde{B})=(\tilde{B}+\tilde{D}) .$$

But $\tilde{D}-\tilde{B}$ is invertible. Were it not, then there would be a $v\in\mathbb{R}^{2n}$ such that $(\tilde{D}-\tilde{B})v=0$. The defining conditions for $\tilde{V}$ are, from equations (6.3),

$$\begin{gathered} \tilde{C}^T\tilde{A}+\tilde{A}^T\tilde{C}=\tilde{B}^T\tilde{D}+\tilde{D}^T\tilde{B}=0 \\ \tilde{C}^T\tilde{B}+\tilde{A}^T\tilde{D}=1 . \end{gathered} \tag{6.6}$$

Hence, $(\tilde{D}-\tilde{B})^T(\tilde{D}-\tilde{B}) = \tilde{B}^T\tilde{B}+\tilde{D}^T\tilde{D}$ implies $\tilde{B}v=\tilde{D}v=0$, but from $\tilde{C}^T\tilde{B}+\tilde{A}^T\tilde{D}=1$ we get $v=0$. Thus, $\tilde{D}-\tilde{B}$ is invertible and with $s=(\tilde{B}+\tilde{D})(\tilde{D}-\tilde{B})^{-1}$, V assumes the form of (6.5). The remaining components are fixed as V has to satisfy (6.6): $B\in\Lambda^2T^n$ and e is a vielbein on T^n. □

There is a residual gauge freedom in $O(n) \times O(n)$, namely elements with $a = d \in O(n)$. These correspond to an identification $e \equiv ea$ for $a \in O(n)$. Having fixed V, H is given locally by

$$H = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} ,$$

with $G = e^T e$.

By using the transformation of H in (6.1) and making use of the relations in (6.3), we validate the equality

$$(1 + G'^{-1}B')(b^t(G+B) + d^t) = G'^{-1}(a^t(G+B) + c^t) .$$

This means that by defining the matrix $E = G + B$, we can express the transformation of (G, B) in (6.1) in the compact form

$$E \longrightarrow E' = (a^t E + c^t)(b^t E + d^t)^{-1} . \tag{6.7}$$

The above transformations are the celebrated **Buscher rules**, in the absence of dilatonic and RR fields. In our formalism they appear as transition functions (figure 6.1). They allow for the construction of non-geometric backgrounds.

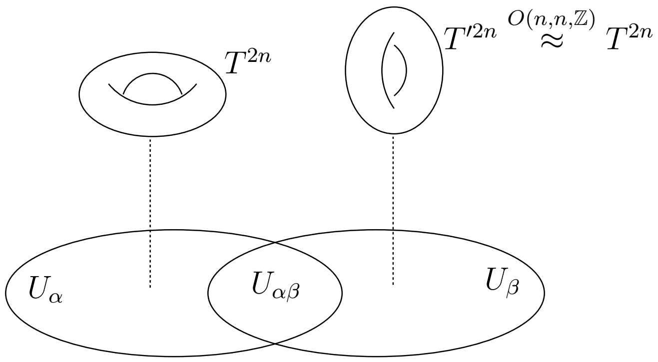

Figure 6.1: Two tori that are patched together through $O(n, n, \mathbb{Z})$.

On the other hand, a global transformation on all patches by a constant $O(n, n, \mathbb{Z})$ element will leave the doubled torus data invariant, because the transition functions stay the same. It will imply, though, a different parametrization in terms of the moduli E. In the conventional formalism this would manifest itself by working in the dual geometry. In the doubled formalism, the two 'dual geometries' are *geometrically* equivalent.

We can identify the subgroup of $O(n,n,\mathbb{Z})$ that in the conventional formalism corresponds to geometric (diffeomorphism and gauge) transformations. We need to set $b=0$, so that in equation (6.7) E does not transform in terms of its inverse. Thus $ad^t = \mathbb{1}_n$ and the transformation reduces to

$$E \to E' = a^t E a + c^t a \ ,$$

where $a \in GL(n,\mathbb{Z})$ and $c^t a$ is a constant $\mathbb{Z}$-valued antisymmetric matrix. This means that the metric G transforms under $a \in GL(n,\mathbb{Z})$, the linear transformations that preserve the T^n lattice, while B transforms under the same transformation plus a shift by a constant antisymmetric matrix.

Although we found the Buscher rules as a geometric transformation, we still have not introduced any string dynamics. In the next section we present the doubled torus system. We will show that the above transformations are the transformations that correspond to T-duality.

6.2 Doubled Torus System

In section §6.1 we defined a doubled torus background as a T^{2n} fiber bundle, with data $(\mathscr{E}, H, A, G, B)$ that obey certain conditions. We now define the **doubled torus system**, which is a string propagating on the doubled torus background under a constraint. The system is defined by a pseudo-lagrangian plus a covariant constraint, that were introduced in the work of [19]

Definition. *The* ***doubled torus system*** *is a sigma model $\phi : \Sigma \to \mathscr{E}$, from a worldsheet Σ into a doubled torus background with data $(\mathscr{E}, H, A, \tilde{G}, \tilde{B})$. If we trivialize the background using coordinates (x^I, y^m) and set $X^I = x^I \circ \phi$, $Y^m = y^m \circ \phi$, then the dynamics are obtained by*

- *varying the doubled torus pseudo-action*

$$S = \int_\Sigma \big(\frac{1}{2}H_{IJ}(Y)dX^I \wedge \star dX^J + dX^I \wedge \star J_I(Y) + \frac{1}{2}\tilde{G}_{mn}(Y)dY^m \wedge \star dY^n + \frac{1}{2}\tilde{B}_{mn}(Y)dY^m \wedge dY^n\big) \ , \quad (6.8)$$

where J is the current

$$J_I = A_{Im}(Y)dY^m + \tilde{A}_{Im}(Y) \star dY^m \ ,$$

- *imposing the doubled torus constraint*

$$\star dX^I = L^{IJ}(H_{JK}(Y)\, dX^K + J_J(Y)) \ . \quad (6.9)$$

In the above d, $\wedge$ and $\star$ are operations on the worldsheet, while the dual fields A and $\tilde{A}$ are related as $\tilde{A}_{Im} = -H_{IJ}L^{JK}A_{Km}$. Locally the worldsheet is conformally flat and we can write the lagrangian as

$$\mathcal{L} = \frac{1}{2}H_{IJ}\partial_a X^I \partial_b X^J \eta^{ab} + A_{In}\partial_a X^I \partial_b Y^n \eta^{ab} - \tilde{A}_{In}\partial_a X^I \partial_b Y^n \epsilon^{ab} + \frac{1}{2}\tilde{G}_{mn}\partial_a Y^m \partial_b Y^n \eta^{ab} - \frac{1}{2}\tilde{B}_{mn}\partial_a Y^m \partial_b Y^n \epsilon^{ab} \ , \quad (6.10)$$

where $\sigma^{a,b} = \tau, \sigma$ are the worldsheet coordinates, $\eta = \text{diag}(+1, -1)$ is the flat worldsheet metric and $\epsilon_{01} = +1$. In the following we discuss the invariance of the system and we give a first description of the degrees of freedom. We will solve the system in section §6.3.

6.2.1 Symmetries

An important feature of the *doubled torus system* is the manifest $O(n, n; \mathbb{Z})$ invariance. The pseudo-action is invariant under $GL(2n, \mathbb{R})$. Because the constraint contains the metric L explicitly, the group elements are restricted in $O(n, n)$. The combined system is invariant under a global $g \in O(n, n)$, with the X, H and J

transforming as

$$\begin{aligned} X &\to g^{-1}X \\ H &\to g^T H g \\ J &\to g^T J \ . \end{aligned} \tag{6.11}$$

However, only the discrete subgroup $O(n,n;\mathbb{Z})$ will leave the lattice of T^{2n} invariant. This does not mean that all radii in the T^{2n} are equal, but rather that the radii enter through the metric H_{IJ}.

The transition functions of the *doubled torus background* are in the large diffeomorphism group of T^{2n} that preserve the metric L, which is $O(n,n,\mathbb{Z})$. For a nontrivial background, the $O(n,n,\mathbb{Z})$ symmetry works as a monodromy condition for the appropriate solutions of the system. This is similar to the Scherk-Schwarz reductions in the context of supergravity, when there is a global symmetry group [120]. If the structure group does not reduce, the solutions obtained are nontrivial with respect to T-duality. Therefore, the doubled formalism can describe backgrounds that would appear non-geometric in the conventional formalism. Such duality twists were the focus of [121] and we shall return to them in chapter 8.

6.2.2 Degrees of freedom

We now investigate the constraint in relation to the Euler-Lagrange equations. By varying X^I one obtains its equation of motion

$$d \star (HP + J) = 0 \ , \tag{6.12}$$

where $P = dX$, or more explicitly

$$\eta^{ab}\partial_a(H_{IJ}\partial_b X^J + A_{In}\partial_b Y^n) - \epsilon^{ab}\partial_m \tilde{A}_{In}\partial_a Y^m \partial_b Y^n = 0 \ .$$

The constraint (6.9) is

$$P^I = \star\left(S^I{}_J P^J + L^{IJ} J_J\right) \ , \tag{6.13}$$

where we wrote $S^I{}_J = L^{IK} H_{KJ}$. Note that

$$A_{In} = -H_{IJ} L^{JK} \tilde{A}_{Kn}$$

and because $S^2 = 1$, we have $\star SLJ = -LJ$. Since $\star^2 = 1$ on worldsheet one-forms, the constraint is self-dual. Moreover, for a coset metric $H = V^T V$, we have that $\mathrm{Tr}(S) = \mathrm{Tr}(V^T V L^{-1}) = \mathrm{Tr}(L^{-1}) = 0$, so S has (pointwise) an equal number of ± 1 eigenvalues. We effectively have n rather than $2n$ constraints.

The n constraints reduce the degrees of freedom from $2n + \dim(\mathscr{N})$ to $n + \dim(\mathscr{N})$. That is, one arrives at a number of degrees of freedom equal to the critical dimension. To elucidate this, we write the constraint in its two components

$$\Psi_1 = P_\tau - SP_\sigma - LJ_\sigma \approx 0$$
$$\Psi_2 = P_\sigma - SP_\tau - LJ_\tau \approx 0 \ ,$$

where we omit I-indices for brevity. Taking sums and differences of these two equations one finds

$$\frac{1}{2}(1 \pm S)(P_\tau \mp P_\sigma) = \frac{1}{2} L(J_\sigma \mp J_\tau) \ .$$

Now since $S^2 = 1$ and $\mathrm{Tr}\, S = 0$, the $(1 \pm S)/2$ are projectors onto two orthogonal n-dimensional subspaces. Therefore, the constraint is forcing half of the X^Is to be purely left-moving, and half to be purely right-moving.

The constraint is manifestly consistent with the dynamics of X^I. By acting on (6.13) with an exterior derivative, we get the Bianchi identity on the left-hand side and the equation of motion on the right. This anticipates the solution of the constraint in the following section, whereby n 'unphysical' coordinates are solved in terms of n 'physical' ones. The choice of physical ones is related to the choice of an L-lagrangian subspace. A different choice of subspace interchanges the Bianchi identities with the equations of motion.

6.3 Euler-Lagrange Equations

We shall now proceed to solve the system. Our aim is to make contact with the conventional formulation of a bosonic string propagating on T^n.

We solve the system by solving the constraint first and then using the Euler-Lagrange equations. To solve the constraint, one must divide the coordinates on T^{2n} into n physical coordinates, $X^i \in T^n$, and n dual coordinates, $\tilde{X}_i \in \tilde{T}^n$, that depend on the X^i. This involves a choice and in the first subsection we introduce the notion of a polarization. In the succeeding subsection, we will be able to solve the system and arrive at a set of equations equivalent to a conventional string model.

6.3.1 Solving the constraint

We solve the constraint by splitting the $2n$ coordinates into two complementary and maximally null subspaces with respect to L. Let us assume we have a canonical basis in which L has the form

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{6.14}$$

We can clearly choose the first n coordinates to be the physical ones X^i and the n last ones to be their dual $\tilde{X}_i$. The metric is then $L = 2dX^i dX_i$. However, this is not the most general choice. Any $O(n, n, \mathbb{Z})$ transformation will provide a different split of the coordinates. We thus see that the choice of split is an element $g \in O(n, n, \mathbb{Z})$.

We define a **choice of polarization** $\mathcal{P}$ to be a transformation $\mathcal{P} : T^{2n} \to T^{2n}$ such that $\mathcal{P} L^{-1} \mathcal{P}^T = L^{-1}$. In the basis of (6.14), we write $\mathcal{P}$ as

$$\mathcal{P} = \begin{pmatrix} \Pi^i{}_I \\ \tilde{\Pi}_{iI} \end{pmatrix} \ .$$

Equations (6.2) imply the following relations for $\mathcal{P}$

$$\begin{aligned} \Pi^i{}_I L^{IJ} \Pi^j{}_J &= 0 \\ \tilde{\Pi}_{iI} L^{IJ} \tilde{\Pi}_{jJ} &= 0 \\ \Pi^i{}_I L^{IJ} \tilde{\Pi}_{jJ} &= \delta^i_j \ , \end{aligned} \tag{6.15}$$

while we can also invert them as in (6.3) to get the equivalent

$$\tilde{\Pi}_{iI} \Pi^i{}_J + \tilde{\Pi}_{iJ} \Pi^i{}_I = L_{IJ} \ .$$

Whatever basis we start with, as long as (6.15) holds, $\mathcal{P}$ will transform the coordinates so as to put L in canonical form. It is useful, though, to think that L is already in canonical form and that the transformation $\mathcal{P}$ is a different choice of split into null coordinates.

In group theoretic language, we are decomposing $O(n,n)$ into representations of $GL(n)$. In particular, the $2n$-dimensional representation of $O(n,n)$ decomposes as $2n \to n + n'$, where n and n' are the fundamental and anti-fundamental representations of $GL(n)$. The uppercase indices $i, j, \ldots$ are used for the fundamental representation and the lowercase ones are used for the anti-fundamental. In the *transformed* basis the embedding group $GL(n,\mathbb{Z})$ in $O(n,n,\mathbb{Z})$ is given by matrices of the form

$$h = \begin{pmatrix} M & 0 \\ 0 & (M^{-1})^T \end{pmatrix} \ , \tag{6.16}$$

with $M \in GL(n,\mathbb{Z})$. Polarizations under the equivalence $P \equiv hP$ give the same split, but transform the physical coordinates X^i.

Given a polarization $\mathcal{P}$, we can solve the constraint. We henceforth work in a given canonical basis or, equivalently, having given a projection $\mathcal{P}$. The coordinate momenta are written as

$$P^I = dX^I = \begin{pmatrix} P^i \\ Q_i \end{pmatrix}$$

and the current J is written as

$$J_I = \begin{pmatrix} J_i \\ K^i \end{pmatrix} \ .$$

In this set of coordinates, the metric H is given in canonical form as

$$H = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \ .$$

After rearranging terms, the constraint $P = \star(L^{-1}HP + L^{-1}J)$ gives the solution

$$Q_i = \star G_{ij}dX^j - G_{ij}K^j + B_{ij}dX^j \ , \tag{6.17}$$

while similarly the consistency equation for the current, $J = - \star HL^{-1}J$, yields

$$J_i = - \star G_{ij}K^j + B_{ij}K^j \ . \tag{6.18}$$

It is also useful to decompose the worldsheet one-form K^i as

$$G_{ij}K^j = k_{im} \star dY^m - u_{im}dY^m \ . \tag{6.19}$$

6.3.2 Solving the system

We can now solve the doubled torus system. Using (6.17), the Bianchi identity $dQ_i = 0$ becomes the equation of motion for X^i, which reads

$$d\left(\star G_{ij}dX^j + B_{ij}dX^j - k_{im} \star dY^m + u_{im}dY^m\right) = 0 \ , \tag{6.20}$$

while for Y^m the equation of motion reads

$$d(\star\tilde{G}_{mn}dY^n + \tilde{B}_{mn}dY^n - 2u_{mi}dX^i \\ - 2k_{mi} \star dX^i + u_{mi} \star K^i + k_{mi}K^i) = 0 \ . \tag{6.21}$$

Since the equation of motion for Y^m differs from that given in [19], we shall derive (6.21) here. This also illustrates the dynamics of the doubled torus as a constrained system in the lagrangian picture.

Proof of (6.21). Varying the doubled torus action with respect to Y^m yields

$$d\left(\star G_{mn}dY^n + B_{mn}dY^n + \star A_{Im}dX^I - \tilde{A}_{Im}dX^I\right) = 0 \ .$$

Our focus is on the last two terms proportional to dX^I. Using (6.17), the two last terms after the d-operator are expanded as

$$\begin{aligned}\star A_{Im}dX^I - \tilde{A}_{Im}dX^I =& \star A_{im}dX^i - \tilde{A}_{im}dX^i \\ &+ \star A^i{}_m Q_i - \tilde{A}^i{}_m Q_i \ .\end{aligned}$$

The consistency equation $A = -HL^{-1}\tilde{A}$ gives

$$\begin{aligned}\tilde{A}_{in} =& - G_{ij}A^j{}_n + B_{ij}\tilde{A}^j{}_n \ , \\ A_{in} =& - G_{ij}\tilde{A}^j{}_n + B_{ij}A^j{}_n \ .\end{aligned}$$

With these at hand, the two terms become

$$\begin{aligned}\star A_{Im}dX^I - \tilde{A}_{Im}dX^I =& - \star G_{ij}\tilde{A}^j{}_m dX^i + G_{ij}A^j{}_m dX^i + \star B_{ij}A^j{}_m dX^i \\ & - B_{ij}\tilde{A}^j{}_m dX^i + G_{ij}A^i{}_m dX^j + \star B_{ij}A^i{}_m dX^j \\ & - \star G_{ij}A^i{}_m K^j - \star G_{ij}\tilde{A}^i{}_m dX^j - B_{ij}\tilde{A}^i{}_m dX^j \\ & + G_{ij}\tilde{A}^i{}_m K^j \\ =& + 2G_{ij}A^i{}_m dX^j - \star G_{ij}\tilde{A}^i{}_m dX^j \\ & - \star G_{ij}A^i{}_m K^j + G_{ij}\tilde{A}^i{}_m K^j \ .\end{aligned}$$

Then, using the definition (6.19), we arrive at the equation of motion for Y^m presented in (6.21). □

Both equations of motion can be derived from a sigma model action of $n +$

$\dim \mathscr{N}$ degrees of freedom with lagrangian

$$\mathcal{L}_c = \frac{1}{2}(\mathcal{G}_{\mathcal{MN}} d\mathcal{X}^{\mathcal{M}} \wedge \star d\mathcal{X}^{\mathcal{N}} + \mathcal{B}_{\mathcal{MN}} d\mathcal{X}^{\mathcal{M}} \wedge d\mathcal{X}^{\mathcal{N}}) \ , \tag{6.22}$$

where $\mathcal{X}^{\mathcal{M}} = (Y^m, X^i)$ and the metric and B-field are defined to be

$$\begin{aligned} \mathcal{G}_{ij} =& G_{ij} \ , \quad \mathcal{B}_{ij} = B_{ij} \ , \quad \mathcal{G}_{im} = -k_{im} \ , \quad \mathcal{B}_{im} = u_{im} \ , \\ \mathcal{G}_{mn} =& \frac{1}{2}\tilde{G}_{mn} + \frac{1}{2}G^{ij}(k_{im}k_{jn} - u_{im}u_{jn}) \ , \\ \mathcal{B}_{mn} =& \frac{1}{2}\tilde{B}_{mn} + \frac{1}{2}G^{ij}(u_{mi}k_{nj} - k_{mi}u_{nj}) \ . \end{aligned}$$

At this point it is useful to point out that the system given here is equivalent to the system given in [122], if we rescale $\tilde{G}$ and $\tilde{B}$ by a factor of 1/2 and the doubled torus action by a factor of 2.

6.4 Passive and Active T-duality

Having arrived at conventional string theory, we can now talk about T-duality and how this descends from the doubled torus to the conventional formulation. We shall see that T-duality can be viewed either as active or passive transformations.

An active transformation corresponds to keeping the same polarization (choice of physical coordinates) but making an 'active' transformation on the data (H, J, X). By construction of the theory, both background geometry and dynamics are invariant under $O(n, n, \mathbb{Z})$ transformations. These were given in (6.11). As X transforms like

$$X \longrightarrow gX \ , \quad g \in O(n, n, \mathbb{Z}) \ ,$$

H transforms as

$$H \longrightarrow g^{-T} H g^{-1} \tag{6.23}$$

and reproduces the Buscher rules (6.7). For the transformation (6.23), this is

$$E \longrightarrow E' = (dE + c)(bE + a)^{-1} \ .$$

The current J transforms as

$$J \longrightarrow g^{-T} J \ ,$$

which in the split (J_i, K^i) gives for K^i

$$K^i \longrightarrow K'^i = b^{ij} J_j + a^i{}_j K^j$$

or, by using the solution for J_i given in (6.18),

$$K'^i = (b^{ij} B_{jk} + a^i{}_k) K^k - b^{ij} G_{jk} \star K^k \ .$$

The fields (k_{im}, u_{im}) transform as

$$G^{ij}(k_{jm} \pm u_{jm}) \longrightarrow (b^{ij_1} B_{j_1 j_2} G^{j_2 j} + a^i{}_{j_2} G^{j_2 j} \pm b^{ij})(k_{jm} \pm u_{jm}) \ .$$

A passive transformation consists in keeping the geometric data (H, A, X) fixed, but choosing different physical coordinates. That is, using a different choice of polarization $\mathcal{P}$. To make this clear let us introduce some notation. Having fixed $(\tilde{G}, \tilde{B})$, the conventional theory is said to be defined by the remaining data $(H, A, \mathcal{P})$. As was made clear in the previous discussion, a T-duality transformation from the active viewpoint is a transformation

$$(H, J, \mathcal{P}) \longrightarrow (g^{-t} H g^{-1}, g^{-t} J, \mathcal{P}) \ .$$

The passive transformation is a different choice of polarization

$$(H, J, \mathcal{P}) \longrightarrow (H, J, \mathcal{P} g) \ .$$

This is completely equivalent to the active one through a large diffeomorphism of the doubled torus. In fact, two theories are identical if the data that define them are related by the equivalence relation

$$(H, J, hPg) \equiv (g^{-t} H g^{-1}, g^{-t} J, P) \ ,$$

for some $h \in GL(n, \mathbb{Z})$, where $GL(n, \mathbb{Z})$ is embedded as in (6.16). The distinguishing feature of T-folds, compared to ordinary manifolds, is that in general no global polarization $\mathcal{P}$ can be chosen, even though it is always possible locally.

Another point that has to be made clear is that all of the previous results has been done classically. T-duality is a perturbative duality between quantum theories. Remember that the string momenta are 'quantized' after quantization. While the doubled torus manifests T-duality and equivalence at the classical level, one has no control over whether this will still hold in the quantum realm. The quantum effects have been studied and quantum equivalence has been shown using different methods [118, 122, 123]. Our aim in the following chapter is to analyze the hamiltonian dynamics of the theory, both because the classical dynamics are elucidated, but also because this leads to the canonical quantization of the system.

Chapter 7

Hamiltonian Analysis

In this chapter we analyze the hamiltonian dynamics of the doubled torus system. Since the system is defined with an explicit constraint, the natural framework for an analysis is Dirac's theory of constraints [124, 125]. We find that the constraint is of Dirac second class. This notion will be introduced in section §7.1. In section §7.2 we analyze the constraint and write the hamiltonian. Because of its importance in conformal field theories, we devote a separate section (§7.3) to the energy-momentum tensor.

In section §7.4 we reduce the system to the constraint surface. An important feature of the hamiltonian treatment is that a choice of polarization is not necessary. We illustrate this with a toy model. This chapter serves well our ultimate purpose, which is quantization.

7.1 Dirac Second Class Constraints

In this section we review the theory of Dirac second class constraints. For reasons of simplicity, we work with a point particle mechanical system and assume that Dirac's algorithm for finding and classifying primary and secondary constraints has been completed.

Let us assume that a point particle's motion in phase space $\mathscr{P} = \{(x^i, p_i)\}$ is prescribed by a hamiltonian $\mathcal{H} \in C(\mathscr{P})$. The phase space has a canonical symplectic structure ω

$$\omega = \sum_i dx^i \wedge dp_i \ ,$$

so that, for any quantity $f \in C(\mathscr{P})$, there is an associated vector field called the hamiltonian flow X_f and given by

$$df = -\omega(X_f) \ .$$

The Poisson bracket is defined in terms of the canonical symplectic structure as

$$\{f, g\}_{\mathrm{PB}} = \omega(X_f, X_g) \ . \tag{7.1}$$

If the particle is unconstrained, the dynamic evolution of the quantity f is given by

$$\frac{df}{dt} = \{f, \mathcal{H}\}_{\mathrm{PB}} \ .$$

Often it is necessary to limit the motion of the particle by a constraint in phase space. Constraints fall into two classes.

Dirac *first* class constraints are commonly related to singular lagrangians and gauge degrees of freedom. On the constraint surface, there is a flow generated by some remaining gauge freedom. The physical phase space is then the orbit space of the constraint surface under that flow. We define a Dirac first class constraint to be such that the Poisson bracket of the constraint with any of the system's constraints vanishes on the constraint surface. Equivalently, we define a Dirac first class constraint to be such that it defines a coisotropic submanifold. The process of finding the *physical subspace*, with a non-degenerate symplectic structure, is known as a double symplectic reduction.

On the other hand, a Dirac *second* class constraint is such that on the constraint surface there is no gauge freedom. In fact one can work solely with the canonical Poisson structure: if the initial conditions satisfy the constraint, then time evolution will keep trajectories on the constraint surface. Dirac second class constraints are those constraints that are not first class. Equivalently, we define a Dirac second class constraint to be such that it defines a symplectic submanifold.

For the case of second class constraints, Dirac's motivated definition was that of a Dirac bracket, that would replace the *canonical* Poisson bracket. On the reduced phase space $\mathscr{P}_0 := \{\psi = 0\}$, it is a Poisson structure that satisfies

$$\{f, \psi\}_{\mathrm{D}} = 0 \ , \tag{7.2}$$

for any quantity $f \in C(\mathscr{P})$. Then one reduces the hamiltonian by setting $\psi = 0$. The dynamics for any quantity $f \in C(\mathscr{P}_0)$ are given by the Dirac bracket of f with the reduced hamiltonian.

Assume we have a set of second class constraints $\psi^\alpha = 0$ that define the subspace $\mathscr{P}_0$. The restriction $\omega|_{\mathscr{P}_0}$ is non-degenerate and as such there is a unique decomposition of the tangent space $T\mathscr{P}$, restricted at points of $\mathscr{P}_0$, as

$$T\mathscr{P}|_{\mathscr{P}_0} = T\mathscr{P}_0 \oplus T\mathscr{P}_0^\perp \ ,$$

where $T\mathscr{P}_0^\perp$ the symplectic perpendicular of $T\mathscr{P}_0$. Note that $T\mathscr{P}_0^\perp$ is also a symplectic subspace. We illustrate the decomposition of $T\mathscr{P}|_{\mathscr{P}_0}$ in figure 7.1, where we have added a local coordinate system $\phi^{\bar{\alpha}}$ that complements the total phase space $\{(\psi^\alpha, \phi^{\bar{\alpha}})\}$.

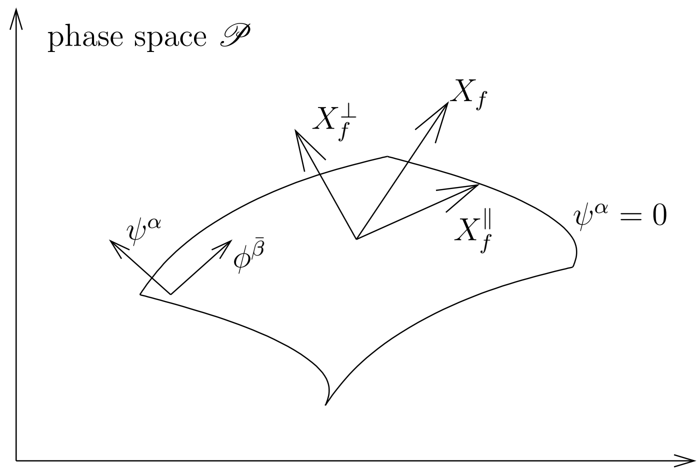

Figure 7.1: The flow X_f decomposes into a flow parallel and perpendicular to the constraint surface ψ^α.

The Dirac bracket defined in (7.2) is given by

$$\{f,g\}_{\mathrm{D}} = \omega(X_f^{\parallel}, X_g^{\parallel}) \ , \tag{7.3}$$

where $X^{\parallel}$ is defined uniquely as the component of X parallel to the constraint surface $\psi^\alpha = 0$ with respect to ω. Equivalently, the Dirac bracket of $f, g \in C(\mathscr{P}_0)$ is given by the formula

$$\{f,g\}_{\mathrm{D}} = \{f,g\}_{\mathrm{PB}} - \sum_{\alpha,\beta} \{f,\psi^\alpha\}_{\mathrm{PB}} \, M_{\alpha\beta} \left\{\psi^\beta, g\right\}_{\mathrm{PB}} \ , \tag{7.4}$$

where the matrix $M_{\alpha\beta}$ is the inverse of

$$M^{\alpha\beta} = \left\{\psi^\alpha, \psi^\beta\right\}_{\mathrm{PB}} \ .$$

Proof of (7.4)*.* Since $T\mathscr{P}_0^{\perp}$ is also symplectic, the inverse of $M^{\alpha\beta}$ exists. The flow parallel to the surface is given by

$$X_f^{\parallel} = X_f - \sum_\alpha \lambda^\alpha X_{\psi^\alpha}$$
$$\omega(X_f^{\parallel}, X_{\psi^\alpha}) = 0 \ .$$

We solve for λ_α by letting the first equation act on $d\psi^\alpha$. Inserting the result into (7.3) we get the Dirac bracket in the form of (7.4). □

7.2 Hamiltonian and Constraints

In this section we shall follow Dirac's method to classify the constraints and write the hamiltonian in a convenient way. It will be shown that the constraint is second class and there are no secondary ones. The most interesting feature of the hamiltonian formalism is the absence of a need to refer to a polarization, in contrast to the lagrangian formalism. Therefore, it is an analysis of the doubled torus with full credit to the system, without additional data.

We begin by writing the lagrangian in the compact form

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu + \dot{q}^\mu j_\mu - V(q) \ , \tag{7.5}$$

where the indices are $\mu = I, n$ and the coordinates are $q^I = X^I$ and $q^n = Y^n$. A dot is a derivative with respect to time, $\dot{q}^\mu \equiv \partial_\tau q^\mu$. The metric in (7.5) is

$$g_{\mu\nu} = \begin{pmatrix} H_{IJ} & A_{In} \\ A_{Jm} & G_{mn} \end{pmatrix} \ .$$

The source terms j_μ are given by

$$\begin{aligned} j_I =& \tilde{A}_{In}Y'^n \\ j_n =& -\tilde{A}_{In}X'^I + B_{nm}Y'^m \ , \end{aligned}$$

and the potential is $V(q) = \frac{1}{2}g_{\mu\nu}q'^\mu q'^\nu$, where a prime is a derivative with respect to the string parameter $q'^\mu \equiv \partial_\sigma q^\mu$.

We find the conjugate momenta $\pi_\mu = g_{\mu\nu}\dot{q}^\nu + j_\mu$. More explicitly, the conjugate momentum of X^I is

$$\pi_I = \frac{\partial L}{\partial \dot{X}^I} = H_{IJ}\dot{X}^J + A_{In}\dot{Y}^n + \tilde{A}_{In}Y'^n \ ,$$

and the conjugate momentum of Y^n is

$$\pi_n = \frac{\partial L}{\partial \dot{Y}^n} = G_{mn}\dot{Y}^m + A_{In}\dot{X}^I - \tilde{A}_{In}X'^I + B_{nm}Y'^m \ .$$

This allows us to calculate the hamiltonian density

$$\begin{aligned} \mathcal{H} =& \pi_\mu\dot{q}^\mu - \mathcal{L} \\ =& \frac{1}{2}g^{\mu\nu}(\pi_\mu - j_\mu)(\pi_\nu - j_\nu) + \frac{1}{2}g_{\mu\nu}q'^\mu q'^\nu \ . \end{aligned} \tag{7.6}$$

Note that $g_{\mu\nu}$ is invertible as a metric on the doubled torus geometry, so its inverse exists.

We now turn towards the constraint and express it in phase space coordinates. The constraint was given in (6.9) as

$$\star P^I = S^I{}_J P^J + L^{IJ} J_J \ .$$

Writing it in its two worldsheet components gives

$$\begin{aligned}\Psi_1 =& P_\tau - S P_\sigma - L J_\sigma \approx 0 \\ \Psi_2 =& P_\sigma - S P_\tau - L J_\tau \approx 0 \ ,\end{aligned}$$

where we omitted the I, J indices for brevity. Using $S^2 = 1$ and $SL \star J = -LJ$, one finds that the two components above are related as

$$\Psi_2 = -S\Psi_1$$

and one can take Ψ_1 as the only Dirac primary constraint. Using $S^2 = 1$ and $SL \star J = -LJ$ again, one can write Ψ_1 as $H^{IJ}(\pi_J - L_{JK} P^K_\sigma)$, where π_J is the conjugate momentum for X^J defined above. Therefore, our primary constraint can be taken to be

$$\Psi_I = \pi_I - L_{IJ} X'^J \approx 0 \ . \tag{7.7}$$

The hamiltonian dynamics are specified by the canonical Poisson structure, which is given by the brackets

$$\begin{aligned}\left\{X^I(\sigma), \pi_J(\sigma')\right\}_{\mathrm{PB}} =& \delta^I_J \delta(\sigma - \sigma') \\ \{Y^n(\sigma), \pi_m(\sigma')\}_{\mathrm{PB}} =& \delta^n_m \delta(\sigma - \sigma')\end{aligned}$$

and all other brackets are zero. The time evolution of any quantity f is given by

$$\dot{f} = \left\{ f, \int \mathcal{H} d\sigma \right\}_{\mathrm{PB}} \ .$$

Our first task is to complete the Dirac method, by classifying the constraint and obtaining any secondary constraints. That is, we have to study the closure and time evolution of the constraint. We find that the time evolution is weakly vanishing

$$\left\{\Psi_I(\sigma), \int_{\sigma'} \mathcal{H}\right\}_{\mathrm{PB}} = \partial_\sigma(-L_{IJ}H^{JK}\Psi_K) \approx 0 \ . \tag{7.8}$$

This means that there are no secondary constraints (provided the constraint also closes). That is, if we impose initial conditions such that $\Psi = 0$, then the constraint is satisfied for all other moments in time. We also have to check the closure of the constraint. We find

$$\{\Psi_I(\sigma_1), \Psi_J(\sigma_2)\}_{\mathrm{PB}} = -2L_{IJ}\delta'(\sigma_1 - \sigma_2) \ . \tag{7.9}$$

The right-hand side is a non-vanishing distribution with non-zero determinant on the constraint surface. This classifies the constraint as Dirac second class.

The Dirac bracket is given by (7.4) in the DeWitt multi-index notation for field theories

$$\{A, B\}_{\mathrm{D}} = \{A, B\}_{\mathrm{PB}} - \int_{\sigma,\sigma'} \{A, \Psi_I(\sigma)\}_{\mathrm{PB}} \, G^{IJ}(\sigma, \sigma') \, \{\Psi_J(\sigma'), B\}_{\mathrm{PB}} \ .$$

The functional inverse of the right-hand side of (7.9) is found to be

$$G^{IJ}(\sigma, \sigma') = \{\Psi_I(\sigma), \Psi_J(\sigma')\}^{-1}_{\mathrm{PB}} = -\frac{1}{4}L^{IJ}\left(\epsilon(\sigma - \sigma') - \epsilon(\sigma' - \sigma)\right) \ .$$

Here $\epsilon(\sigma)$ is the Heaviside step function. We write it in the above convoluted form to remind ourselves that, working with a string, all functions should be periodic. We find the following Dirac brackets

$$\begin{aligned} \left\{X^I(\sigma), X^J(\sigma')\right\}_{\mathrm{D}} &= -\frac{1}{4}L^{IJ}\left(\epsilon(\sigma - \sigma') - \epsilon(\sigma' - \sigma)\right) \\ \left\{X^I(\sigma), \pi_J(\sigma')\right\}_{\mathrm{D}} &= \frac{1}{2}\delta^I_J\delta(\sigma - \sigma') \\ \{\pi_I(\sigma), \pi_J(\sigma')\}_{\mathrm{D}} &= \frac{1}{2}L_{IJ}\delta'(\sigma - \sigma') \ . \end{aligned} \tag{7.10}$$

The Dirac brackets of Y^m and π_m remain canonical

$$\begin{aligned}\left\{X^I(\sigma), Y^m(\sigma')\right\}_{\rm D} &= \{\pi_I(\sigma), Y^m(\sigma')\}_{\rm D} = 0 \\ \left\{X^I(\sigma), \pi_m(\sigma')\right\}_{\rm D} &= \left\{\pi_I(\sigma), \pi_(\sigma')\right\}_{\rm D} = 0 \\ \{Y^m(\sigma), \pi_n(\sigma')\}_{\rm D} &= \delta^m_n \delta(\sigma - \sigma') \ .\end{aligned} \tag{7.11}$$

It is useful to adopt the rotated coordinates $\Phi_I = \pi_I + L_{IJ} X'^J$. The Dirac brackets of the rotated coordinates are given by

$$\begin{aligned}\{\Psi_I(\sigma), A\}_{\rm D} =&0 \\ \{\Phi_I(\sigma), \Phi_J(\sigma')\}_{\rm D} =&2L_{IJ}\delta'(\sigma - \sigma') \\ \{\Phi_I(\sigma), Y^m(\sigma')\}_{\rm D} =&\{\Phi_I(\sigma), \pi_m(\sigma')\}_{\rm D} = 0 \ ,\end{aligned} \tag{7.12}$$

where A is any quantity. The variables Φ_I and Ψ_I, along with Y^m and π_m, are not complete in phase space. This is because the transformation from (X^I, π_I) to (Φ_I, Ψ_I) is not local. Essentially we lose information about the center of mass X^I_0. For the rest of the chapter this will not matter because all the quantities we are interested in depend on $(\Phi_I, \Psi_I, Y^m, \pi_m)$. In particular, we will be able to reduce on the constraint surface $\Psi = 0$. However, we will comment on the importance of the center of mass X^I whenever needed.

We conclude this subsection with the hamiltonian, written in the coordinates $(\Phi_I, \Psi_I, Y^m, \pi_m)$. After some algebra we find

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} Z_\mu Z_\nu - \frac{1}{4} H^{IJ} \Phi_I \Phi_J + \frac{1}{4} H^{IJ} \Psi_I \Psi_J + \frac{1}{2} \tilde{G}_{mn} Y'^m Y'^n \ , \tag{7.13}$$

where we have collected the quantities

$$\begin{aligned}Z_I =&\Phi_I - \tilde{A}_{In} Y'^n \\ Z_m =&\pi_m - \tilde{B}_{mn} Y'^n \ .\end{aligned}$$

We notice that Ψ appears only quadratically.

7.3 Energy Momentum Tensor

We define an energy momentum tensor from the doubled torus lagrangian as

$$T_{ab} = \frac{2}{\sqrt{-h}} \frac{\partial \mathcal{L}}{\partial h^{ab}}\Big|_{h=\eta} \ .$$

h_{ab} is a general worldsheet metric and the above expression is evaluated when h is flat.

By reducing to $\Psi_I = 0$ we impose upon physical quantum states

$$\hat{T}_{ab}\Big|_{\Psi_I=0} |\text{ph.state} >= 0 \ .$$

Due to Weyl invariance $T_{00} = T_{11}$, we need only investigate T_{00} and T_{01}. Written in phase space, for generic $g_{\mu\nu}, j_\mu$, one finds

$$\begin{aligned} T_{00} =& \mathcal{H} \\ T_{01} =& \pi_\mu q^{\mu\prime} = \frac{1}{4} L^{IJ} \Phi_I \Phi_J + \pi_m Y^{m\prime} - \frac{1}{4} L^{IJ} \Psi_I \Psi_J \ , \end{aligned}$$

where $\mathcal{H}$ is the Hamiltonian density (7.6). We first check the closure under the Poisson bracket. It is clear, though, that, since the elements T_{ab} form a closed algebra of constraints, they will also form a closed algebra on the constraint surface $\Psi_I = 0$. That is, by first setting $\Psi_I = 0$ in the above equations, the algebra still closes. This is possible because Ψ_I appears quadratically in both T_{00} and T_{01}. We have already shown that T_{00} closes with Ψ_I, this is essentially equation (7.8). Similarly, T_{01} and Ψ_I close on $\Psi_I = 0$. Again, one can set $\Psi_I = 0$ before calculating the Poisson brackets. The same is true if we switch from Poisson brackets to Dirac brackets.

7.4 Hamiltonian Reduction

We have used the rotated coordinates $(\Psi_I, \Phi_I, Y^m, \pi_m)$, rather than the defining (X^I, π_I, Y^m, π_m). In doing so we neglect the zero modes of X^I, which do not affect our results here. We now reduce the phase space by setting $\Psi = 0$. The reduced hamiltonian density is obtained from the hamiltonian density (7.13) by setting $\Psi_I = 0$. We shall use the same symbol $\mathcal{H}$. It is

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} Z_\mu Z_\nu - \frac{1}{4} H^{IJ} \Phi_I \Phi_J + \frac{1}{2} \tilde{G}_{mn} Y'^m Y'^n \ , \tag{7.14}$$

where the same definitions for Z_μ as in equation (7.13) apply. We also use the same symbol T_{ab} for the energy momentum tensor on the constraint surface. We thus have

$$\begin{aligned} T_{00} =& \mathcal{H} \\ T_{01} =& \frac{1}{4} L^{IJ} \Phi_I \Phi_J + \pi_m Y^{m\prime} \ . \end{aligned} \tag{7.15}$$

The dynamics are given by the Dirac bracket defined on the reduced phase space coordinates, which were given in (7.12). The time evolution of any quantity f is then

$$\dot{f} = \left\{ f, \int \mathcal{H} d\sigma \right\}_{\mathrm{D}} \ .$$

In the lagrangian picture we solved the system using a choice of polarization $\mathcal{P}$. We illustrate the process with the diagram of figure 7.2, where the conventional theories A and B are T-dual. Although T-duality is not manifest in the conventional formalism, theories A and B are derived from the manifestly T-dual doubled torus formalism.

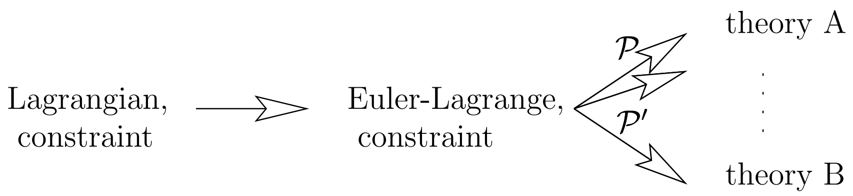

Figure 7.2: Given a polarization $\mathcal{P}$ one arrives at a conventional T-dual theory.

It is interesting to compare the lagrangian method, figure 7.2, with the Dirac method. In the Dirac method, we start with a hamiltonian, constraint and canonical Poisson structure. We then reduce to the constraint surface. One can ask whether the reduced hamiltonian and symplectic structure correspond to a sigma model. The answer is that there are different conventional sigma models, which are in fact T-dual. This is shown in figure 7.3. Although we use the terms "theory

Figure 7.3: First one obtains the reduced hamiltonian on the reduced phase space and the Dirac brackets. Then one can use a polarization P, to show a correspondence with a conventional sigma model.

A" or "theory B" for the conventional picture, it is evident that the theory is in fact one, that of the doubled torus. A polarization is needed only if one wants to interpret the reduced system as a sigma model.

Indeed, recall that the non-canonical Dirac bracket is

$$\{\Phi_I(\sigma), \Phi_J(\sigma')\}_{\rm D} = 2L_{IJ}\delta'(\sigma - \sigma') \ . \tag{7.16}$$

A question one can pose is then "is there a sigma model for which one can write the hamiltonian and Dirac brackets we have?". The T-duality group $O(n, n, \mathbb{Z})$ appears as a subgroup of the group of symplectomorphisms: the group that preserves the right-hand side of the above bracket. A polarization in the hamiltonian formalism is a split of the Φ_I coordinates into canonical (X^i, π_i) coordinates so that they obey

$$\left\{X^i(\sigma), \pi_j(\sigma')\right\}_{\rm D} = \delta^i_j\delta(\sigma - \sigma') \ .$$

However there is a subtlety, since the transformation involved is non-local. In doing so, one should reintroduce the zero modes of the physical field X^i. Furthermore, in the quantum interacting theory the zero modes of both T-dual coordinates are important, as they are conjugate to the left and righ-handed momenta seperately[1].

Coming back to the choice, it corresponds precisely to an element in

$$O(n, n, \mathbb{Z})/GL(n, \mathbb{Z}) \ .$$

[1]I would like to thank Joan Simon and Chris Hull for pointing this out to me.

To elucidate this, remember that we are looking for a lagrangian split of the signature (n, n) metric L. This can be achieved by taking the first n coordinates, canonically conjugate to the last n, when L is in the canonical basis

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

Any $O(n, n, \mathbb{Z})$ transformation preserves the metric L and the periodic conditions of Φ_I, hence, it provides a different split. There is, though, an equivalence between such transformations: transformations under the group $GL(n, \mathbb{Z})$ that is embedded in $O(n, n, \mathbb{Z})$ as

$$\begin{pmatrix} M & 0 \\ 0 & M^{-t} \end{pmatrix} \ , \text{ with } M \in GL(n, \mathbb{Z}) \ .$$

This is equivalent to our definition of a polarization in the lagrangian picture.

7.5 A Toy Model

It is interesting to illustrate the correspondence of the figures 7.2 and 7.3 with a toy model. Let us define the simplest possible doubled torus background. Let it be a trivial T^2 fibration over a flat Minkowski space $\mathscr{E} = M^{1,d-1} \times T^2$ with warped metric

$$g = R(Y)^2 d\theta^2 + R(Y)^{-2} d\tilde{\theta}^2 + \eta_{mn} dY^m dY^n \ .$$

We read off that $J = \tilde{B} = 0$ and that $\tilde{G}_{mn} = \eta_{mn}$ is a flat lorentzian metric. The metric H_{IJ} is indeed an $O(1,1)/(O(1) \times O(1))$ coset metric

$$H = \begin{pmatrix} R^2 & 0 \\ 0 & R^{-2} \end{pmatrix} \ ,$$

and we allow R to be an arbitrary[2] function of Y.

[2]it will be constrained by conformal invariance, but this will not matter here.

By using the lagrangian method we can choose different polarizations, corresponding to the two different elements of $O(1,1,\mathbb{Z})/GL(1,\mathbb{Z})$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

They correspond to the physical coordinates θ or $\tilde{\theta}$ with equations of motion

$$\begin{aligned} d\star(R^2 d\theta) =& 0 \ , \\ d\star(R^{-2} d\tilde{\theta}) =& 0 \ , \end{aligned}$$

while for Y the same equation holds in both polarizations: $d\star(\frac{1}{2}G_{mn}dY^n) = 0$. The equations can be derived from the following lagrangians

$$\begin{aligned} \mathcal{L}_A =& \frac{1}{2}R^2 d\theta \wedge \star d\theta + \frac{1}{4}G_{mn}dY^m \wedge \star dY^n \\ \mathcal{L}_B =& \frac{1}{2}R^{-2} d\tilde{\theta} \wedge \star d\tilde{\theta} + \frac{1}{4}G_{mn}dY^m \wedge \star dY^n \ . \end{aligned}$$

The reduced hamiltonian is

$$\mathcal{H} = \frac{1}{4}(R^{-2}\Phi_1^2 + R^2\Phi_2^2) + \frac{1}{2}(G^{mn}\pi_m\pi_n + G_{mn}Y'^m Y'^n)$$

and the Dirac bracket is

$$\{\Phi_1(\sigma), \Phi_2(\sigma')\}_{\mathrm{D}} = 2\delta'(\sigma - \sigma') \ .$$

We can choose the physical coordinate to be θ or $\tilde{\theta}$, according to whether it satisfies $X' = \frac{1}{2}\Phi_2$ or $X' = \frac{1}{2}\Phi_1$. The canonical conjugate momentum of X is then Φ_1 or Φ_2. By a Legendre transformation,

$$\tilde{\mathcal{L}} = \dot{q}^\mu \pi_\mu - \mathcal{H} \ \text{with} \ \dot{q}^\mu = \frac{\partial \mathcal{H}}{\partial \pi_\mu} \ ,$$

we obtain the lagrangians of the two theories

$$
\begin{aligned}
\tilde{\mathcal{L}}_A =& R^2 d\theta \wedge \star d\theta + \frac{1}{2} G_{mn} dY^m \wedge \star dY^n \\
\tilde{\mathcal{L}}_B =& R^{-2} d\tilde{\theta} \wedge \star d\tilde{\theta} + \frac{1}{2} G_{mn} dY^m \wedge \star dY^n \ .
\end{aligned}
$$

These are scaled by a factor of 2 with respect to what we got from the lagrangian formalism, but the action principle gives identical dynamics

$$
\tilde{\mathcal{L}}_{A/B} = 2\mathcal{L}_{A/B} \ .
$$

This scaling will always occur, but it is irrelevant to the classical dynamics. We could avoid it by a redefinition of the critical lagrangian in equation (6.22).

Note that θ and $\tilde{\theta}$ defined like this have the correct periodicity. Indeed, remember that on the constraint surface we have

$$
\Phi_I = \pi_I + L_{IJ} X'^J \overset{\Psi_I=0}{\approx} 2L_{IJ} X'^J \ .
$$

Since $X^I \cong X^I + 1$, we have $\Phi_I \cong \Phi_I + 2$. It is important to note, though, that quantization might and generically does impose further discretization.

Chapter 8

Quantizing a T-fold

In this chapter we quantize a T-fold using the doubled torus formalism. T-folds are often called monodrofolds. They are conformal field theory (CFT) orbifolds where the monodromy element is a combination of a T-duality transformation and a spacetime symmetry. They can be viewed as Scherck-Schwarz reductions, where the monodromy element is a T-duality transformation. Although they cannot be described by a globally smooth target spacetime, they are in principal of equal importance with smooth backgrounds. There has been considerable interest in these backgrounds [126, 127, 121, 128, 129, 130, 131, 132, 133, 134]. The doubled torus geometry allows us to examine T-folds in the same geometric framework as geometric backgrounds.

In section §8.1 we expand on the notion of a T-fold and set up a model T-fold. In section §8.2 we quantize the model and in section §8.3 we show modular invariance of its partition function.

8.1 Setup

In this section we describe a simple, but nontrivial, bosonic asymmetric orbifold. Our aim is to see the usefulness of the doubled torus formalism. We first describe the asymmetric orbifold in the conventional formulation. The non-smoothness is lifted when we describe the geometry in the doubled torus formulation. We also give the classical equations of motion, as derived from the hamiltonian.

8.1.1 T-fold Geometry

The T-fold we are interested in is constructed, in the non-doubled language, as an S^1 'fiber' over an S^1 base, where a T-duality transformation acts on the fiber as one transverses the base. We can appreciate that this defines a consistent CFT by considering its parent theory. Figure 8.1 shows the endpoints of a segment, which will be glued together by a T-duality transformation. On top of each point of the segment lies a fibered circle. The fiber has radius R and $1/R$ at the two endpoints of the base respectively. As the T-duality transformation is not smooth, one cannot construct a smooth S^1 target space bundle. This is illustrated quite convincingly in figure 8.2, where the two endpoints cannot be patched together. Nevertheless, T-duality is a symmetry of the parent theory and one expects a well-defined CFT.

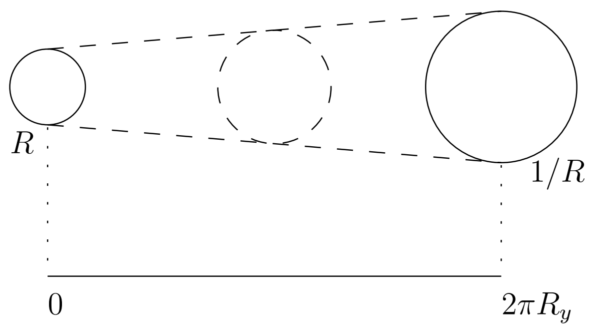

Figure 8.1: The parent theory as an S^1 fiber over a segment

We now describe the same setup using the doubled torus. One has for a parent geometry a two-torus fibration over a segment, figure 8.3. The modulus of the torus is described by the radius R, which is now the size of one of the one-cycles, the other being $1/R$. As one transverses the base, the torus undergoes a transformation which is the only nontrivial element of $O(1,1;\mathbb{Z})$: it exchanges R with $1/R$. The segment can be patched at its ends through a large diffeomorphism. The result is a smooth T^2-fiber bundle (figure 8.4).

Let us describe the doubled torus background in detail. The background is a Minkowski space $\mathscr{N}$ times a nontrivial two-torus T^2 fibered over a circle S^1. We take the coordinates on the doubled fiber to be (x^1, x^2), the coordinate

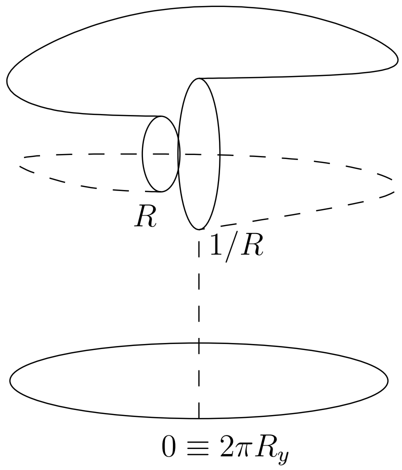

Figure 8.2: One cannot have a smooth target space for such a CFT

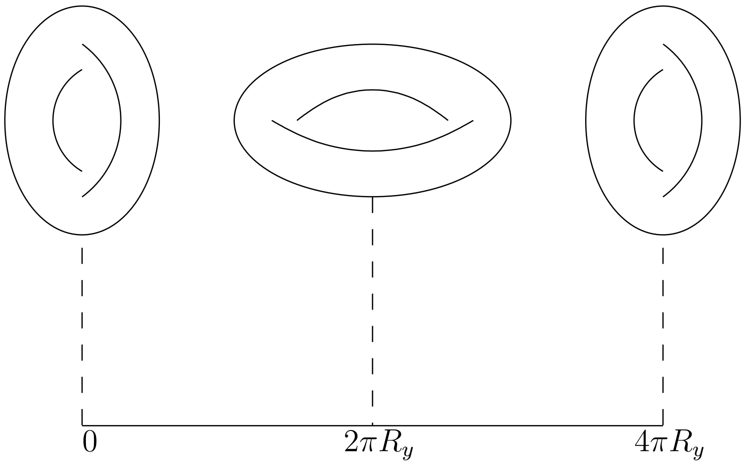

Figure 8.3: The tori at base points 0 and $2\pi R_y$ are diffeomorphic

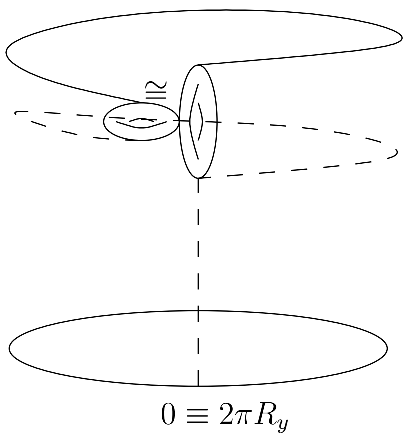

Figure 8.4: The two tori at base points 0 and $2\pi R_y$ can be patched together

on the base S^1 to be y and the coordinates of the Minkowski space y^a. The background is *locally* $\mathscr{N} \times S^1 \times T^2$. When we consider the doubled torus system, we introduce embedding coordinates X^I, Y, Y^a that correspond to x^I, y, y^a. To make the transition easier, we shall, from now on, use the capital letters X^I, Y, Y^a for the coordinates.

For simplicity, we turn all b-fields off. That is, we set

$$A_{mI} = \tilde{A}_{mI} = B_{mn} = 0$$

and we also require no Y dependence

$$\partial_n H_{IJ} = \partial_n G_{mn} = 0 \ ,$$

where m, n are indices for the total base space, $Y^m = (Y^a, Y)$. The manifold $\mathscr{N}$ is flat with Minkowski metric η_{ab}. The coset metric H_{IJ} is given pointwise over the base by

$$H = \begin{pmatrix} R^2 & 0 \\ 0 & R^{-2} \end{pmatrix} \ .$$

We require the metric H to be constant so that the system becomes linear and solvable. Consistency with the monodromy (see e.g., figure 8.1) means that we have $R = 1$. Then the background can be seen as an asymmetric orbifold of a *geometric* background and can be solved using orbifold techniques. We will never the less treat the problem in the doubled formalism: a doubled-torus background with a geometric monodromy around its base. In many of our formulas, when this is possible, we will keep the radius $R = 1$ explicit.

To obtain the doubled torus geometry as the desired T-fold, we identify the base points at 0 and $2\pi R_y$. The fibers are patched with the nontrivial element of $O(1,1;\mathbb{Z})$

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{8.1}$$

This results in the following identifications

$$\begin{aligned} Y &\longrightarrow Y + 2\pi R_y \\ X^I &\longrightarrow M^I{}_J X^J \\ Y^a &\longrightarrow Y^a \ . \end{aligned}$$

It is useful to initially take Y to be the coordinate on a circle with radius $2R_y$: $Y \equiv Y + 4\pi R_y$. The orbifold identifications correspond to a half-shift around the parent circle. We note that the monodromy element is of order 2. The metric H_{IJ} transforms as

$$H \longrightarrow (M^{-1})^T H M^{-1} \ .$$

From this we see that the monodromy corresponds to

$$R \longrightarrow \frac{1}{R} \ ,$$

which is precisely what we want. We now proceed with the string dynamics.

8.1.2 Equations of Motion

Recall that the total phase space is $\mathscr{P} = (X^I, \pi_I, Y^n, \pi_n)$, but on the reduced surface $\Psi = 0$ the physical phase space is $\mathscr{P}_0 = (\Phi_I, Y^n, \pi_n)$. From (7.14) the hamiltonian is

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} Z_\mu Z_\nu - \frac{1}{4} H^{IJ} \Phi_I \Phi_J + \frac{1}{2} G_{mn} Y'^m Y'^m \ ,$$

with $Z_I = \Phi_I$ and $Z_m = \pi_m$. The nontrivial Dirac brackets are

$$\begin{aligned} \{Y^n(\sigma), \pi_m(\sigma')\}_{\mathrm{D}} &= \delta^n_m \delta(\sigma - \sigma') \\ \{\Phi_I(\sigma), \Phi_J(\sigma')\}_{\mathrm{D}} &= 2L_{IJ} \delta'(\sigma - \sigma') \ , \end{aligned}$$

so that the equations of motion, $\dot{f} = \left\{f, \int_{\sigma'} \mathcal{H}(\sigma')\right\}_{\mathrm{D}}$, are

$$\dot{\Phi}_I = L_{IJ} H^{JK} \Phi'_K \tag{8.2}$$

$$d \star dY = d \star dY^a = 0 \ . \tag{8.3}$$

Solving equation (8.3) for the the coordinates Y^a, one obtains

$$Y^a = y_0^a + p^a \tau + \frac{1}{\sqrt{2}} \sum_{k \neq 0} \frac{i b_k^a}{k} e^{-ik\sigma^-} + \frac{1}{\sqrt{2}} \sum_{k \neq 0} \frac{i \tilde{b}_k^a}{k} e^{-ik\sigma^+} \ , \tag{8.4}$$

where the momenta p^a are unconstrained. To solve the d'Alembert equation for the periodic coordinate Y, it is useful to split it, in the usual fashion, into left and right-movers

$$Y = Y_R(\sigma^-) + Y_L(\sigma^+) \ ,$$

the solution of which are

$$\begin{aligned} Y_R(\sigma^-) =& \frac{1}{2} y_0 + p_R \sigma^- + \frac{1}{\sqrt{2}} \sum_{k \neq 0} \frac{i b_k}{k} e^{-ik\sigma^-} \\ Y_L(\sigma^+) =& \frac{1}{2} y_0 + p_L \sigma^+ + \frac{1}{\sqrt{2}} \sum_{k \neq 0} \frac{i \tilde{b}_k}{k} e^{-ik\sigma^+} \ . \end{aligned} \tag{8.5}$$

The boundary conditions for Y will determine a quantization rule for p_L and p_R. We postpone this for section §8.2.3.

We solve equation (8.2) by diagonalizing

$$S^T = L H^{-1} = \begin{pmatrix} 0 & R^2 \\ R^{-2} & 0 \end{pmatrix}$$

into ± 1 eigenspaces, which are spanned by the vectors

$$e_\pm(R) = \begin{pmatrix} R \\ \pm R^{-1} \end{pmatrix} \ .$$

The solution for Φ_I is

$$\Phi_I(\sigma,\tau) = \Phi_{0I} + \Phi_I^{(+1)}(\sigma^+) + \Phi_I^{(-1)}(\sigma^-) \ ,$$

where Φ_{0I} is constant and $\Phi_I^{(\pm 1)}(\sigma^\pm) = f^\pm(\pm)e_\pm(R)$. Note that Φ_I does not have linear terms in $\sigma^\pm$. This is because on the constraint surface we have the (weak) equality $\Phi_I = \Pi_I + L_{IJ}X'^J \approx 2L_{IJ}X'^J$ and X'^J is periodic.

We now determine the periodicity conditions, which fix the solutions of Φ_I and Y completely. We can distinguish two sectors. Sector I is the untwisted sector: as one transverses the base, the closed string is periodic. The condition for sector I is

$$\begin{aligned} \Phi_I(\sigma+2\pi) =& \Phi_I(\sigma) \\ Y(\sigma+2\pi) =& Y(\sigma) + 4\pi R_y m \qquad\qquad m \in \mathbb{Z} \ . \end{aligned}$$

Sector II is the twisted sector: as one transverses the base a period of $2\pi R_y$, the field Φ_I undergoes a monodromy transformation of order 2. The condition for sector II is

$$\begin{aligned} \Phi_I(\sigma+2\pi) =& M_I{}^J\Phi_J(\sigma) \\ Y(\sigma+2\pi) =& Y(\sigma) + 2\pi R_y(2m+1) \qquad\qquad m \in \mathbb{Z} \ . \end{aligned}$$

The periodicity conditions impose constraints on the zero modes Φ_0. Quantization, though, imposes further constraints.

8.2 Quantization

This section is divided into four subsections, dedicated, in turn, to the following. We first describe the quantization of the two sectors: I and II. We then find the quantization condition of the zero modes in the direction of the fiber. Finally, we give the mass formulae and level matching conditions.

8.2.1 Sector I

The boundary conditions are

$$\Phi_I(\sigma+2\pi) = \Phi_I(\sigma)$$
$$Y(\sigma+2\pi) = Y(\sigma) + 4\pi R_y m \ , \qquad\qquad m \in \mathbb{Z} \ .$$

We see that the $\Phi_I^{\pm}$ are periodic functions. Therefore, the solution for Φ_I is

$$\Phi_I(\sigma,\tau) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \sum_{k\neq 0} \tilde{a}_k e^{-ik\sigma^+} e_+ + \sum_{k\neq 0} a_k e^{-ik\sigma^-} e_- \ , \tag{8.6}$$

where the vectors $e_\pm(R)$ are the ± 1 eigenstates of S^T. The constants q_1 and q_2 are related to the winding and momentum numbers that would appear in the conventional non-doubled formalism. In subsection §8.2.3 we will show that q_1 and q_2 obey the quantization condition

$$q_1 q_2 = 2mn, \qquad m,n \in \mathbb{Z} \ . \tag{8.7}$$

We now turn to the boundary conditions for Y. The solution for Y is given in (8.5) and the boundary conditions give

$$p_L + p_R = \frac{1}{2R_y} n_y \ , \qquad\qquad n_y \in \mathbb{Z}$$
$$p_L - p_R = 2R_y w_y \ , \qquad\qquad w_y \in \mathbb{Z} \ .$$

We are ready to quantize the oscillator modes of Φ_I, Y^a, Y. We begin with $Y^m = (Y^a, Y)$, $m = 0,1\dots 24$ and $Y^{24} \equiv Y$. The conjugate momenta are $\pi_n = \eta_{nm}\dot{Y}^m$ and the Dirac bracket

$$\{Y^m(\sigma,\tau), \pi_n(\sigma',\tau)\}_{\mathrm{D}} = \delta^m_n \delta(\sigma-\sigma')$$

is quantized by replacing $\{,\} \to -i[,]$. We obtain the following commutation relations for the modes associated to Y^m:

$$[b_k^m, b_l^n] = k\delta_{k+l}\eta^{mn}; \quad [\tilde{b}_k^m, \tilde{b}_l^n] = k\delta_{k+l}\eta^{mn}; \quad [b_k^m, \tilde{b}_l^n] = 0 \ . \tag{8.8}$$

Next we consider Φ_I, with Dirac bracket

$$\{\Phi_I(\sigma,\tau), \Phi_J(\sigma',\tau)\}_{\mathrm{D}} = 2L_{IJ}\delta'(\sigma-\sigma') \ . \tag{8.9}$$

Replacing $\{,\}_{\mathrm{D}} \to -i[,]$ one arrives at

$$[a_k, a_l] = k\delta_{k+l}; \quad [\tilde{a}_k, \tilde{a}_l] = k\delta_{k+l}; \quad [a_k, \tilde{a}_l] = 0 \ . \tag{8.10}$$

Under M we have $\Phi_I \to {M_I}^J\Phi_J$, so, using the explicit form for M given in (8.1), the eigenvectors which appear in the decomposition (8.6) transform as

$$\Phi^{(\pm)}(\sigma^\pm; R) \ \stackrel{M}{\longrightarrow} \ \pm\Phi^{(\pm)}(\sigma^\pm; R^{-1}) \ ,$$

together with $q_1 \leftrightarrow q_2$. Therefore, the associated action on the modes is

$$\tilde{a}_k \mapsto \tilde{a}_k, \qquad a_k \mapsto -a_k \ .$$

The Hilbert space for the untwisted sector, which we denote by $H_{(+)}$, is built upon a vacuum state $|0\rangle$ that is invariant under the monodromy: $M|0\rangle = |0\rangle$. We now decompose the Hilbert space $H_{(+)}$ into the eigenspaces $H^{\pm}_{(+)}$ under the action of M.

As explained by Hellerman *et al.* [135], the correct action of T-duality on quantum states involves a nontrivial phase. This can be shown by considering the operator product expansion (OPE) of two $+1$ T-eigenstates and requiring that no -1 eigenstates appear on the right-hand side of the equation. In the doubled language, the correct action of T-duality picks up the phase

$$M|q_1, q_2\rangle = (-1)^{\frac{q_1 q_2}{2}} |q_2, q_1\rangle \ . \tag{8.11}$$

This phase is essential for the modular invariance of the partition function. The Hilbert space splits into $H_{(+)} = H^+_{(+)} \oplus H^-_{(+)}$ with

$$H^{\pm}_{(+)} = \left\{ \prod_{i=1}^{N} a_{-n_i} \prod_{j,k,l} \tilde{a}_{-m_j} b_{-r_k} \tilde{b}_{-s_l} \right.$$
$$\left. \Big(|q_1, q_2; n_y, w_y\rangle \pm (-1)^{N+n_y+\frac{q_1 q_2}{2}} |q_2, q_1; n_y, w_y\rangle \Big) \right\} \ . \quad (8.12)$$

Note that the factor of $(-1)^{n_y}$ is due to the Y-shift. The states we construct are in fact "off-shell", since we have not yet imposed physical state conditions. We shall do so in section §8.2.4.

Before we move on to sector II we point out a few interesting features. First, we observe that we have only one set of left-moving modes and one set of right-moving modes. This means that there is no need to make a choice of polarization for the quantum mechanical states. This is in contrast to the lagrangian formulation, where a polarization must be chosen. Here we do not need to do so, because we have moved on the constraint surface in phase space. Secondly, we notice that our orbifold looks very similar to the interpolating orbifolds considered in [135, 129]. This suggests that the doubled torus formalism is equivalent to the conventional non-doubled formulation of these backgrounds. However, there may still be differences in the physical state conditions or the partition function.

8.2.2 Sector II

Sector II has boundary conditions

$$\Phi_I(\sigma + 2\pi) = {M_I}^J \Phi_J(\sigma)$$
$$Y(\sigma + 2\pi) = Y(\sigma) + (2m+1)2\pi R_y \ , \qquad m \in \mathbb{Z} \ .$$

The solutions for Y and Y^a are unchanged from sector I and are given in equations (8.4) and (8.5) respectively. However, the quantization conditions for

p_L, p_R are now changed to

$$p_L + p_R = \frac{n_y}{2R_y} \ , \qquad n_y \in \mathbb{Z}$$
$$p_L - p_R = R_y(2w_y + 1) \ , \qquad w_y \in \mathbb{Z} \ ,$$

because of the different boundary conditions of Y. The oscillator algebras for the modes associated to Y and Y^a are unchanged from those of sector I and are given by equation (8.8).

We now turn our attention to Φ_I, the solution of which can be written as

$$\Phi_I = \Phi_{0I} + e_+(R) f(\sigma^+) + e_-(R) g(\sigma^-) \ .$$

The boundary conditions imply that f is periodic and g is semi-periodic on the closed string coordinate. Therefore, Φ_I can be expanded in modes as

$$\Phi_I(\sigma, \tau) = \begin{pmatrix} q \\ q \end{pmatrix} + e_+(R) \sum_{k \neq 0} \tilde{a}_k e^{-ik\sigma^+} + e_-(R) \sum_{k \in \mathbb{Z} + \frac{1}{2}} a_k e^{-ik\sigma^-} \ ,$$

where the boundary conditions force the constant term to be self dual: $\Phi_{01} = \Phi_{02} = q$. The quantization condition on q is

$$q = \frac{1}{\sqrt{2}} \left(n - \frac{1}{2} \right) , \qquad n \in \mathbb{Z} \ . \tag{8.13}$$

This condition is chosen so that level matching in this sector makes sense [135]. We will prove this in the following section (§8.2.3).

We want to replace the Dirac bracket with an operator commutation relation. *Locally* we naively have

$$[\Phi_I(\sigma, \tau), \Phi_J(\sigma', \tau)] = 2i L_{IJ} \delta'(\sigma - \sigma') \ ,$$

but the distribution $\delta'(\sigma - \sigma')$ cannot be periodic, because this is incompatible with the monodromy transformations that occur when $\sigma \to \sigma + 2\pi$. To get a correct global statement, we replace the right-hand side of the bracket with the monodromy invariant matrix

$$2iL_{IJ}\delta'(\sigma-\sigma') = i\begin{pmatrix} R^2\left[\delta'_{2\pi}(\Delta\sigma)-\delta'_{4\pi}(\Delta\sigma)\right] & \delta'_{2\pi}(\Delta\sigma)+\delta'_{4\pi}(\Delta\sigma) \\ \delta'_{2\pi}(\Delta\sigma)+\delta'_{4\pi}(\Delta\sigma) & R^{-2}\left[\delta'_{2\pi}(\Delta\sigma)-\delta'_{4\pi}(\Delta\sigma)\right] \end{pmatrix} ,$$

where $\Delta\sigma \equiv \sigma-\sigma'$ and $\delta_{2\pi}, \delta_{4\pi}$ are Dirac delta functions with period 2π and 4π respectively. By using this, we calculate the commutation relations for the modes

$$[a_k, a_l] = k\delta_{k+l}; \quad [\tilde{a}_m, \tilde{a}_n] = m\delta_{m+n}; \quad [a_k, \tilde{a}_m] = 0, \tag{8.14}$$

where $k, l \in \mathbb{Z}+\frac{1}{2}$ and $m, n \in \mathbb{Z}$.

The action of the monodromy on the modes is similar to that of sector I

$$\tilde{a}_k \mapsto \tilde{a}_k, \qquad a_k \mapsto -a_k$$

In sector II there is also a nontrivial phase to take into account, namely

$$M\left|q\right\rangle = e^{-\frac{i\pi}{8}}(-1)^{q^2}\left|q\right\rangle \tag{8.15}$$

This phase was obtained, in the non-doubled formulation, in the work of Hellerman and Walcher [135].

We now discuss the off-shell Hilbert space of sector II, which we denote by $H_{(-)}$. First note that we need a twisted vacuum for the right-handed module so that the vacuum flips sign under the monodromy. We only need one twisted vacuum $|0\rangle_-$, such that $M|0\rangle_- = -|0\rangle_-$. The Hilbert space decomposes into the ±1 eigenspaces under the monodromy, $H_{(-)} = H^+_{(-)} \oplus H^-_{(-)}$, with

$$H^{\pm}_{(-)} = \left\{ \frac{1 \pm (-1)^{N+n_y+n(n-1)/2}}{2} \prod_{i=1}^{N} a_{-n_i} \prod_{j,k,l} \tilde{a}_{-m_j} b_{-r_k} \tilde{b}_{-s_l} \left|q, n_y, w_y\right\rangle_- \right\} \tag{8.16}$$

and $n \in \mathbb{Z}$ is related to q by (8.13). The factor of $(-1)^{n_y}$ comes from the Y-shift.

8.2.3 Quantization of Zero Modes

When we wrote the solutions in the two sectors I and II, we gave (equations (8.7) and (8.13)), without proof, the quantization conditions for the zero modes of the field Φ_I. They are

Sector	quantization
I	$q_1 q_2 = 2mn$
II	$q = \frac{1}{\sqrt{2}}\left(n - \frac{1}{2}\right)$

with $m, n \in \mathbb{Z}$. In this section we shall prove them.

First, let us recall the simple case of a quantum point particle on a circle S^1, but considered as an orbifold $\mathbb{R}/\mathbb{Z}$. The parent Hilbert space on $\mathbb{R}$ is made up of momentum states $|p\rangle_{p\in\mathbb{R}}$. If we call the generator of translations $t : x \to x + 2\pi$, then (due to the commutation $[x, p] = i$) we have the action $t\,|p\rangle = \exp(i2\pi p)\,|p\rangle$. The Hilbert space is invariant under t. It consists of the projected states

$$\sum_n t^n\,|p\rangle = \sum_n \exp(in2\pi p)\,|p\rangle = \delta_{2\pi}(2\pi p)\,|p\rangle \;\; ,$$

which implies that $p = 0 \mod 1$. If for some reason the momentum was initially quantized in even integers, the momentum can be further 'fractionated' to take any integer value. Furthermore, we want the operator $\exp(ix)$ to be realized on the Hilbert space and this requires the span of all integer values.

Now let us return to sector I. We will use two different proofs for the quantization condition. We shall first prove it by "cheating" and looking at the equivalence with the conventional model.

The doubled torus constraint (6.9) halves the physical degrees of momentum, winding and oscillator modes. Because it is a differential constraint on X^I, the number of zero modes will not be halved. Therefore, we must put in an extra constraint on X_0^I, so that we have the degrees of freedom of a critical string theory. We impose the physical constraint

$$\Pi^i{}_I X_0^I = X_0^i \; , \qquad \tilde{\Pi}_{\underline{i}I} X_0^I = 0 \; ,$$

where ${\Pi^i}_I$ and $\tilde{\Pi}_{\underline{i}I}$ are the projectors discussed in chapter 6. The indices i correspond to the physical polarization, and $\underline{i}$ to the unphysical polarization. In the above constraint, we have kept the center of mass X_0^i in the direction of the physical space and put to zero the one in the other direction. The physical center of mass should be identified with a coordinate on a circle of radius R. By considering, though, the Dirac bracket from equation (8.9)

$$\left\{X^I(\sigma), \Phi_J(\sigma')\right\}_{\mathrm{D}} = \delta^I_J \delta(\sigma - \sigma') \ ,$$

we can extract the following commutator of the zero mode

$$[X_0^i, \Phi_{0j}] = \delta^i_j \ ,$$

where Φ_{0j} is the "physical" component.

That is, Φ_{0i} is the conjugate momentum to X_0^i and hence $\Phi_{0i} \in \mathbb{Z}$, similarly to the case of a quantum point particle on a circle. For the other direction we can use the fact that $L^{IJ}\Phi_J \approx 2X'^I$ on the constraint surface. Therefore, $\Phi_{0\underline{i}}$ inherits the discretization of the winding modes. That is, $\Phi_{0\underline{i}} \in 2\mathbb{Z}$.

These conditions can be written concisely in matrix form as

$$\Pi\Phi_0 = m, \qquad \tilde{\Pi}\Phi_0 = 2n \ ,$$

where we are now thinking of Φ as a column vector and $m, n \in \mathbb{Z}$. Using the relation

$$(\Pi)^T\tilde{\Pi} + (\tilde{\Pi})^T\Pi = L$$

we arrive at the covariant quantization condition

$$\Phi_0^T L \Phi_0 = 4mn \ , \quad m, n \in \mathbb{Z} \ . \tag{8.17}$$

They are termed covariant, as they are independent of a polarization. This is precisely the condition $q_1 q_2 = 2mn$.

We will now show an alternative derivation of this quantization, which will be the method for sector II. Let us call t the generator of $\mathbb{Z}$ of our orbifold transformation. It acts like M on the fiber and translates by $2\pi R_y$ on the base circle. We write the zero modes as

$$\Phi_0 = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} .$$

We have the following action

$$t\,|q_1, q_2, n_y\rangle = \exp\left(i\pi(n_y + \frac{q_1 q_2}{2})\right) |q_2, q_1, n_y\rangle \ .$$

The factor of $\exp(i\pi n_y)$ is the usual phase coming from the translation on the circle base, whereas the phase $\exp(i\pi q_1 q_2/2)$ is known [135] to be the right T-duality realization for closure of OPEs in sector I. At this stage we do not restrict the quantization of q_1,q_2. After projecting with the formal sum $\sum_n t^n$, the existence of invariant states requires

$$\pi\left(\frac{q_1 q_2}{2} + n_y\right) = 0 \mod 2\pi$$

and for generic $n_y \in \mathbb{Z}$:

$$q_1 q_2 = 2mn, \qquad m, n \in \mathbb{Z}$$

This is precisely the quantization condition (8.17). For specific n_y only a subsector of these modes gives invariant states. But our first aim is the discretization of Φ_0.

We finally turn to sector II. We write the zero mode as

$$\Phi_0 = \begin{pmatrix} q \\ q \end{pmatrix} .$$

The generator t acts [135] like

$$t\,|q,n_y\rangle = \exp\left(i\pi(q^2-\frac{1}{8}+n_y)\right)|q,n_y\rangle \ .$$

Our construction is of an orbifold and we can show modular invariance, level matching and quantization of zero modes by adopting the above phase. The invariant Hilbert space requires, for generic n_y,

$$q^2-\frac{1}{8}=0 \mod 1 \ .$$

The most general choice, with even spacing of the modes q, is

$$q=\frac{1}{\sqrt{2}}\left(n-\frac{1}{2}\right) \ , \tag{8.18}$$

where $n\in\mathbb{Z}$. This is the quantization of the mode cited in equation (8.13).

8.2.4 Physical State Conditions

In this section we consider the physical state conditions that are imposed on the states of the off-shell Hilbert spaces $H_{(\pm)}$. In particular, we will investigate the level matching conditions and mass formulae. Our goal is to show that canonically quantizing the doubled torus is consistent *and* equivalent to quantizing in the non-doubled formulation. This claim is finally proved when we write the partition function.

The energy-momentum tensor of the doubled torus system (7.15) is

$$\begin{aligned}T_{00} =&\mathcal{H}\\ T_{01} =&\frac{1}{4}L^{IJ}\Phi_I\Phi_J+\pi_m Y^{m\prime} \ .\end{aligned}$$

In terms of the coordinates $\sigma^{\pm}$, the only non-zero components of T are $T_{\pm\pm}=\frac{1}{2}(T_{00}\pm T_{01})$, given explicitly by

$$T_{\pm\pm} \ = \ \frac{1}{8}\left(H^{IJ}\pm L^{IJ}\right)\Phi_I\Phi_J+\partial_\pm Y\partial_\pm Y+\eta_{ab}\partial_\pm Y^a\partial_\pm Y^b \ ,$$

where $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$. We now substitute the mode expansions of Φ_I, Y, Y^m to obtain the Virasoro operators $L_m, \tilde{L}_m$. We will do this for both sectors in turn.

We begin with sector I and obtain

$$\begin{aligned} L_m =& \frac{1}{2\pi}\int_0^{2\pi} e^{im\sigma^-} T_{--} d\sigma^- \\ =& \frac{1}{2}\sum_{k=-\infty}^{+\infty} (a_{m-k}a_k + b_{m-k}b_k + \eta_{ab} b^a_{m-k} b^b_k) \ , \end{aligned}$$

where

$$a_0 = \frac{1}{2}\left(\frac{q_1}{R} - q_2 R\right), \qquad b_0 = \sqrt{2} p_R, \qquad b^a_0 = \frac{p^a}{\sqrt{2}} \ . \tag{8.19}$$

Similarly,

$$\begin{aligned} \tilde{L}_m =& \frac{1}{2\pi}\int_0^{2\pi} e^{im\sigma^+} T_{++} d\sigma^+ \\ =& \frac{1}{2}\sum_{k=-\infty}^{+\infty} \left(\tilde{a}_{m-k}\tilde{a}_k + \tilde{b}_{m-k}\tilde{b}_k + \eta_{ab} \tilde{b}^a_{m-k} \tilde{b}^b_k\right) \ , \end{aligned}$$

where

$$\tilde{a}_0 = \frac{1}{2}\left(\frac{q_1}{R} + q_2 R\right), \qquad \tilde{b}_0 = \sqrt{2} p_L, \qquad \tilde{b}^a_0 = \frac{p^a}{\sqrt{2}} \ . \tag{8.20}$$

For the normal ordered zero modes L_0 and $\tilde{L}_0$ we get

$$\begin{aligned} L_0 =& \frac{1}{8}\left(\frac{q_1}{R} - q_2 R\right)^2 + p_R^2 + \frac{1}{4}(p^a)^2 + \sum_{k=1}^{\infty}(a_{-k}a_k + b_{-k}b_k + b^a_{-k}b^a_k) \\ \tilde{L}_0 =& \frac{1}{8}\left(\frac{q_1}{R} + q_2 R\right)^2 + p_L^2 + \frac{1}{4}(p^a)^2 + \sum_{k=1}^{\infty}(\tilde{a}_{-k}\tilde{a}_k + \tilde{b}_{-k}\tilde{b}_k + \tilde{b}^a_{-k}\tilde{b}^a_k) \ . \end{aligned} \tag{8.21}$$

Therefore, the level matching condition is

$$\frac{1}{2} q_1 q_2 + p_L^2 - p_R^2 + \tilde{N} - N = 0 \ , \tag{8.22}$$

where N and $\tilde{N}$ are the usual occupation numbers. Observe that the first term is an integer because of the quantization condition $q_1 q_2 = 2mn$ with $m, n \in \mathbb{Z}$. Hence, level matching *is* satisfied.

The formula for the spectrum of massive states is

$$M^2 = 2\left(p_L^2 + p_R^2 + \frac{q_1^2}{4R^2} + \frac{q_2^2 R^2}{4} + N + \tilde{N} - 2\right) \ , \tag{8.23}$$

where the -2 arises as the zero point energy of 24 left-handed and 24 right-handed integer moded bosonic oscillators, which each contribute $-1/24$.

From the mass formula we see that the state $a_{-1}\tilde{a}_{-1}\,|k^a\rangle$, which corresponds to the metric component along the fiber, is indeed massless as one would expect. However, it belongs to $H^-_{(+)}$, that is, it has eigenvalue -1 under the orbifold action. Therefore, this state will be projected out. This is in agreement with [121] and [129], where it is explained that when there is a nontrivial monodromy, the moduli of the spacetime geometry must take values which are fixed under the action of the monodromy. In our example $R \to R^{-1}$, so the component of the metric with both legs in the fiber has fixed value $R = 1$.

We now consider sector II. First note that

$$T_{--} = \frac{1}{8}\left(\frac{q}{R} - qR\right)^2 + \frac{1}{2}\left(\frac{q}{R} - qR\right)\sum_{k\in\mathbb{Z}+\frac{1}{2}} a_k e^{-ik\sigma^-} + \ldots$$

That is, T_{--} has both integer and half-integer modes. Therefore, it will generically be neither periodic nor anti-periodic. T_{++} is periodic and we require T_{--} to be periodic, but this is only satisfied when $R = 1$. We put $R = 1$ from now on. We then obtain the following modes L_m

$$\begin{aligned} L_m &= \frac{1}{2\pi}\int_0^{2\pi} e^{im\sigma^-} T_{--} d\sigma^- \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}} a_{m-k}a_k + \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(b_{m-k}b_k + \eta_{ab} b^a_{m-k} b^b_k\right) \ , \end{aligned}$$

where b_0 and b_0^a are related to the Y-momenta via (8.19). Similarly $\tilde{L}_m$ is

$$\begin{aligned} \tilde{L}_m &= \frac{1}{2\pi}\int_0^{2\pi} e^{im\sigma^+} T_{++} d\sigma^+ \\ &= \frac{1}{2}\sum_{k=-\infty}^{+\infty}\left(\tilde{a}_{m-k}\tilde{a}_k + \tilde{b}_{m-k}\tilde{b}_k + \eta_{ab}\tilde{b}^a_{m-k}\tilde{b}^b_k\right) \ , \end{aligned}$$

where $\tilde{a}_0 = q$, and $\tilde{b}_0, \tilde{b}^a_0$ are related to the Y-momenta via (8.20). For the normal ordered zero modes, L_0 and $\tilde{L}_0$, we have

$$\begin{aligned} L_0 =& p_R^2 + \frac{1}{4}(p^a)^2 + \sum_{k=\frac{1}{2}}^{\infty} a_{-k}a_k + \sum_{k=1}^{\infty}(b_{-k}b_k + b^a_{-k}b^a_k) \\ \tilde{L}_0 =& \frac{1}{2}q^2 + p_L^2 + \frac{1}{4}(p^a)^2 + \sum_{k=1}^{\infty}(\tilde{a}_{-k}\tilde{a}_k + \tilde{b}_{-k}\tilde{b}_k + \tilde{b}^a_{-k}\tilde{b}^a_k) \end{aligned} \tag{8.24}$$

The zero point energy for the right-movers will be -1, since we have a contribution of $-1/24$ from each of the 24 periodic bosons. For the left-movers the zero point energy is $-45/48$, since we have 23 periodic bosons contributing $-1/24$ each and 1 anti-periodic boson contributing $+1/48$. The condition imposed on physical states is

$$(\tilde{L}_0 - 1)\,|\text{phys}\rangle = (L_0 - \frac{45}{48})\,|\text{phys}\rangle = 0\ .$$

Hence, the level matching condition and mass spectrum formula are given by

$$\frac{1}{2}q^2 + p_L^2 - p_R^2 + \tilde{N} - N - \frac{1}{16} = 0 \tag{8.25}$$

$$M^2 = 2\left(p_L^2 + p_R^2 + \frac{1}{2}q^2 + N + \tilde{N} - (2 - \frac{1}{16})\right)\ . \tag{8.26}$$

The term $-1/16$ in the level matching condition looks problematic when the formula is written in terms of the original zero mode q. Level matching problems are well known to plague asymmetric orbifolds and generally one must make some kind of amendment to make the level matching formula sensible. The solution here is to quantize q appropriately so that the $-1/16$ cancels. This happens if we choose $\sqrt{2}q = n - 1/2$, with $n \in \mathbb{Z}$ [135]. Moreover, in section §8.2.3 we showed that the quantization rule follows directly from the phase adopted in equation (8.15). This is a first success of our model. We now move on to investigate the partition function for this model. We will see that the quantization for q leads to a modular invariant partition function.

8.3 The Partition Function

We now have all the ingredients required to calculate the partition function. By **partition function** we mean the one-loop vacuum energy contribution of the doubled torus. We can imagine string field theory as a theory that allows for the creation and absorption of closed strings, in an analogy to the Feynman diagrams of quantum field theory. The zero-loop energy is by conformal invariance the worldsheet of a sphere. The one-loop diagram will contain one handle, where a string is created at some worldsheet time and reabsorbed at some time later. The partition function we consider is, therefore, a doubled torus system defined on a two-torus worldsheet.

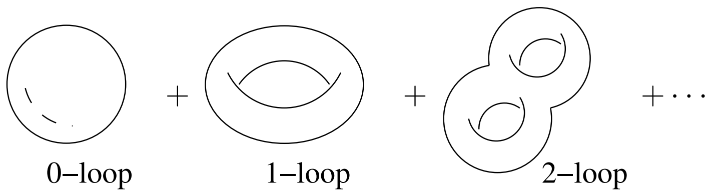

Figure 8.5: World-sheets of the 0, 1 and 2 loop contributions to the vacuum energy.

The zero-loop contribution is a CFT on a sphere and as there are no moduli describing it, it can be reabsorbed into the string coupling constant. The torus worldsheet, on the other hand, has a nontrivial dependence on the complex modulus τ of the torus. For the CFT to be conformally invariant, the partition function should be invariant under the Teichmueller group $SL(2,\mathbb{Z})$, which is generated by the discreet operations

$$
\begin{aligned}
T &: \tau \longrightarrow \tau + 1 \\
S &: \tau \longrightarrow -\frac{1}{\tau}\ .
\end{aligned}
$$

We call this desired property **modular invariance**.

The partition function can also be seen as a one-dimensional generalization of the Feynman-Kac path integral of closed loops. As such, we begin with a closed string, evolve it in euclidean time by $2\pi\tau_2$ and let it come back to itself, translated by $2\pi\tau_1$ in the string direction. This gives the Hilbert trace

$$Z(\tau) = \mathrm{Tr}_H \exp(2\pi\tau_1 \hat{P} - 2\pi\tau_2 \hat{\mathcal{H}}) \ ,$$

where $\hat{P}$ and $\hat{\mathcal{H}}$ are the momentum and hamiltonian respectively. Using the zero order modes L_0, $\tilde{L}_0$, this is rewritten [136, 20] as

$$Z(\tau) = \sum_a \mathrm{Tr}_{H_a^+}(\mathsf{q}^{\hat{L}_0}\mathsf{q}^{\hat{\tilde{L}}_0}) \ ,$$

where the hats on L_0, $\tilde{L}_0$ are included to remind us to include the zero point energy [137]: each periodic boson will contribute a factor of $\mathsf{q}^{-1/24}$ to the energy, while the twisted vacuum zero point energy for each anti-periodic boson is $1/48$. The **nome** is defined as $\mathsf{q} = \exp(2\pi i\tau)$.

It is always useful to use the trace over the Hilbert spaces H_a, rather than the invariant Hilbert spaces H_a^+, so that the partition function is written like

$$Z(\tau) = \sum_{a,b} \frac{1}{2} {Z^a}_b$$

and the partition function in the sector $H_{(a)}$ with b insertions is given by

$${Z^a}_b(\tau) = \mathrm{Tr}_{H_a}(g^b \mathsf{q}^{\hat{L}_0}\mathsf{q}^{\hat{\tilde{L}}_0}) \ .$$

In this formula, the ${Z^a}_b$ can be related to path integrals with appropriate boundary conditions on the σ and τ cycles of the torus. The advantage of this formalism lies in the desired property of **modular covariance**

$$\begin{aligned} {Z(\tau+1)^a}_b =& {Z(\tau)^a}_{b-a} \\ {Z(-1/\tau)^a}_b =& {Z(\tau)^b}_{-a} \ , \end{aligned} \tag{8.27}$$

which implies the modular invariance of the partition function $Z(\tau)$. Of course modular invariance does not necessarily imply modular covariance. Nevertheless the partition functions will be shown to be indeed modular covariant.

To ease the proof of modular invariance we employ one further notion, that of **interpolating orbifolds**. Our orbifold can be thought of as a *nontrivial* product of a flat Minkowski space, a shift orbifold (the base space where we identify $Y \equiv Y + 2\pi R_y$) and the doubled torus fiber with monodromy element M. More precisely the partition function can be factorized as

$$Z(\tau) = \frac{1}{2} Z_{(\mathscr{N})}(\tau) \sum_{a,b=0,1} Z^{a}_{(\Phi)b}(\tau) Z^{a}_{(Y)b}(\tau) \ . \tag{8.28}$$

In the above $Z_{(\mathscr{N})}$ comes from the Minkowski space, $Z_{(Y)}$ comes from the nontrivial base circle of radius R_y and $Z_{(\Phi)}$ comes from the fiber. In fact, each factor is separately modular covariant.

8.3.1 The Interpolating Orbifolds

The use of the term *interpolating orbifold* comes from [128] and references therein. An interpolating orbifold combines a finite order discrete symmetry of a worldsheet theory with a spacetime shift along a compact direction. The term reflects the interpolating behavior as one takes the two extreme limits of the shift parameter. In our case the worldsheet symmetry is the monodromy M on the fiber and the shift is $Y \to Y + 2\pi R_y$ on a circle of original radius, say $2R_y$. These operations are both of order 2.

Equation (8.28) is of course a consequence of the formulae for the modes L_0 and $\tilde{L}_0$, as expressed for sector I in (8.21) and sector II in (8.24), and the direct product structure of the Hilbert space, equations (8.12) and (8.16). In this section we calculate the factors $Z_{(\mathscr{N})}$, $Z_{(Y)}$ and $Z_{(\Phi)}$. After this we will be able to prove modular invariance through modular covariance for each factor separately.

The Trivial Base

We begin with the partition function of the Minkowski space $\mathscr{N}$. The computation is straightforward but let us persist on it, because it will set the example for the rest of the computations. The contribution from the Minkowski space comes from the following parts of the Virasoro zero modes

$$L_0 = \frac{1}{4}(p^a)^2 + \sum_{k=1}^{\infty}(b^a_{-k}b^a_k) + \cdots$$
$$\tilde{L}_0 = \frac{1}{4}(p^a)^2 + \sum_{k=1}^{\infty}(\tilde{b}^a_{-k}\tilde{b}^a_k) + \cdots \ ,$$

with the trace over the Fock space of an oscillator algebra

$$\left\{\prod_{k,l}\left(b^i_{-k}\right)^{n_k}\left(\tilde{b}^j_{-l}\right)^{\tilde{n}_l}|0;p^i\rangle\right\}_{n_k,\tilde{n}_l,p^i} \ .$$

By choosing a light-cone gauge, only 22 space-like coordinates will contribute. Each coordinate gives the same factor of

$$Z_{(\mathbb{R})}(\tau) = (\mathsf{q}\bar{\mathsf{q}})^{-1/24}\int_{\mathbb{R}} dk \exp(-\pi\tau_2 k^2)\sum < \mathsf{q}^{\sum_{k=1}^{\infty}(b_{-k}b_k)}\bar{\mathsf{q}}^{\sum_{k=1}^{\infty}(\tilde{b}_{-k}\tilde{b}_k)} > \ .$$

The $(\mathsf{q}\bar{\mathsf{q}})^{-1/24}$ comes from the zero point energy of the vacuum, the gaussian integral comes from the continuous momenta $(p^a)^2$ and the sum is over the Fock space of a single oscillator algebra $[b_k, b_l] = k\delta_{k+l}$ and its conjugate.

The gaussian integral gives a factor of $(\tau_2)^{-1/2}$, while the trace breaks up into a geometric sum as follows. For each state

$$\cdots(a_{-n})^k\cdots|0\rangle \ ,$$

the operator $\mathsf{q}^{\sum_{k=1}^{\infty}(b_{-k}b_k)}$ will give the value q^{nk} and, thus, summing over all partitions we have

$$\sum_{k=1}^{\infty}\prod_{n=1}^{\infty}\mathsf{q}^{nk} = \prod_{n=1}^{\infty}(1-\mathsf{q}^n)^{-1} \ .$$

The partition function of the Minkowski space is then $Z_{(\mathscr{N})} = (Z_{(\mathbb{R})}(\tau))^{22}$ with

$$Z_{(\mathbb{R})}(\tau) = (\tau_2)^{-1/2}|\eta(\tau)|^{-2}$$

and

$$\eta(\tau) = \mathsf{q}^{1/24} \prod_{i=1}^{\infty} (1 - \mathsf{q}^i) \ .$$

The partition function is modular invariant due to the properties of the Dedekind eta function

$$\begin{aligned} \eta(\tau + 1) =& \eta(\tau) \\ \eta(-1/\tau) =& (-i\tau)^{1/2} \eta(\tau) \end{aligned}$$

The Shift Orbifold

We next turn to the partition function of the shift orbifold. The relevant parts of the Virasoro zero modes are

$$\begin{aligned} L_0 =& p_R^2 + \sum_{k=1}^{\infty} (b_{-k} b_k) + \cdots \\ \tilde{L}_0 =& p_L^2 + \sum_{k=1}^{\infty} (\tilde{b}_{-k} \tilde{b}_k) + \cdots \ , \end{aligned}$$

where, in order to include both sectors, we rewrite the periodicity conditions in a universal way

$$p_{L/R} = \frac{1}{2} (\frac{n_y}{2R_y} \pm R_y (2w_y + a)) \ , \quad n_y, w_y \in \mathbb{Z} \ , a = \begin{cases} 0 & \text{for Sector I} \\ 1 & \text{for Sector II} \end{cases} \ .$$

The oscillator algebra and the zero point energy will give a factor of $|\eta(\tau)|^{-2}$. The gaussian integral over momenta is now replaced by a double sum over p_L and p_R. The result is

$$\begin{aligned} Z_{(Y)}{}^a{}_b(\tau) = |\eta(\tau)|^{-2} \sum_{n_y, w_y} (-1)^{n_y b} \times \\ \exp \left\{ -\pi\tau_2 \frac{n_y}{(2R_y)^2} - \pi\tau_2 (2R_y)^2 (w_y + \frac{a}{2})^2 - 2\pi\tau_1 n_y i (w_y + \frac{a}{2}) \right\} \ . \end{aligned} \quad (8.29)$$

One can recast equation (8.29) into the form

$$Z_{(Y)}{}^a{}_b(\tau) = \sum_{n,w \in \mathbb{Z}} \ \sum_{q=0,1} (-1)^{bq} Z_{2R} \left[2n + q \middle| w + \frac{a}{2} \right] \ ,$$

with

$$
\begin{aligned}
Z_{2R}\left[2n+q\middle|w+\frac{a}{2}\right] = &|\eta(\tau)|^{-2}\times \\
&\exp\left\{-\pi\tau_2\left(\frac{(2n+q)^2}{4R^2}+4R^2(w+\frac{a}{2})^2\right)+2\pi i\tau_1(2n+q)(w+\frac{a}{2})\right\}\ .
\end{aligned}
$$

This is the form presented in [39] and it can easily be generalized to include an "N-shift" on a circle of original radius NR_y as in [129].

We can use the Poisson resummation formula

$$
\sum_n \exp\{-\pi a n^2 + 2\pi i b n\} = a^{-1/2}\sum_n \exp\left\{-\frac{\pi}{a}(n-b)^2\right\} \tag{8.30}
$$

to rewrite equation (8.29) into

$$
\begin{aligned}
Z_{(Y)}{}^a{}_b(\tau) = &|\eta(\tau)|^{-2}(\tau_2)^{-1/2}(2R_y)\times \\
&\sum_{n,w}\exp\left\{-\pi\frac{(2R_y)^2}{\tau_2^2}\left|n+\frac{b}{2}-\left(w+\frac{a}{2}\right)\tau\right|^2\right\}\ .
\end{aligned} \tag{8.31}
$$

This form is manifestly modular covariant. Indeed, under $\tau\to\tau+1$ we replace $n\to n+w$, while under $\tau\to -1/\tau$ we interchange n and w.

The Fiber Orbifold

Now we will obtain the partition function $Z_{(\Phi)}{}^a{}_b$. We shall have to treat the different values of a,b separately.

$a=1$. For sector I the relevant part of the Virasoro zero modes are

$$
\begin{aligned}
L_0 =& \frac{1}{8}\left(\frac{q_1}{R}-q_2R\right)^2+\sum_{k=1}^{\infty}(a_{-k}a_k)+\cdots \\
\tilde{L}_0 =& \frac{1}{8}\left(\frac{q_1}{R}+q_2R\right)^2+\sum_{k=1}^{\infty}(\tilde{a}_{-k}\tilde{a}_k)+\cdots\ .
\end{aligned}
$$

The quantization condition is $q_1q_2 = 2mn$ and we write this as $q_1=\sqrt{2}m\varepsilon$ and $q_2=\sqrt{2}n/\varepsilon$ for some $\varepsilon\in\mathbb{R}$.

The calculation of ${Z_{(\Phi)}}^0{}_0$ is very similar to the calculation of ${Z_{(Y)}}^0{}_0$. After a Poisson re-summation we get

$$
{Z_{(\Phi)}}^0{}_0 = |\eta(\tau)|^{-2}(\tau_2\varepsilon^2)^{-1/2} \times \sum_{m,n} \exp\left\{-\frac{\pi}{\varepsilon^2\tau_2}\left|n\tau - m\right|^2\right\} ,
$$

which indeed satisfies the modular (self-) covariance of equation (8.27).

For the partition function ${Z_{(\Phi)}}^0{}_1$, because of the insertion of the T-duality operator M, there will be some changes. First the trace of the operator

$$
\hat{M}\mathsf{q}^{\sum_{k=1}^{\infty}(a_{-k}a_k)}
$$

will give a minus sign in the geometric series, thus counting a factor of

$$
\prod_{n=1}^{\infty}(1+\mathsf{q}^n)^{-1} ,
$$

while the infinite sum of modes will collapse into a sum where $q_1 = q_2 = \sqrt{2}m$. In the sum we must also include the nontrivial phase of T-duality. The end result is

$$
{Z_{(\Phi)}}^0{}_1 = (\mathsf{q}\bar{\mathsf{q}})^{-1/24}\prod_{n=1}^{\infty}(1+\mathsf{q}^n)^{-1}\prod_{n=1}^{\infty}(1-\bar{\mathsf{q}}^n)^{-1}\sum_{q_1\in\sqrt{2}\mathbb{Z}}(-1)^{q_1^2/2}\bar{\mathsf{q}}^{q_1^2/2} .
$$

Using the definitions of the theta functions given in section §8.3.3, we have the different factors expressed as

$$
\begin{aligned}
\left(\frac{2\eta(\tau)}{\theta_2(\tau)}\right)^{1/2} &= \mathsf{q}^{-1/24}\prod_{n=1}^{\infty}(1+\mathsf{q}^n)^{-1} \\
\overline{\eta(\tau)}^{-1} &= \bar{\mathsf{q}}^{-1/24}\prod_{n=1}^{\infty}(1-\bar{\mathsf{q}}^n)^{-1} \\
\overline{\theta_4(2\tau)} &= \sum_{n\in\mathbb{Z}}(-1)^n\bar{\mathsf{q}}^{n^2} .
\end{aligned}
$$

Therefore,

$$
{Z_{(\Phi)}}^0{}_1 = \left(\frac{2\eta}{\theta_2}\right)^{1/2}\frac{\overline{\theta_4(2\tau)}}{\overline{\eta}} .
$$

$a=2$. For sector II the relevant part of the Virasoro zero modes are

$$L_0 = \sum_{k=\frac{1}{2}}^{\infty} a_{-k} a_k + \cdots$$
$$\tilde{L}_0 = \frac{1}{2} q^2 + \sum_{k=1}^{\infty} \tilde{a}_{-k} \tilde{a}_k + \cdots \ .$$

There are several differences with the untwisted sector. The quantization rule of

$$q = \frac{1}{\sqrt{2}} (n - \frac{1}{2})^2$$

must be used. The geometric sum is now over a half-integer power

$$\sum_{k=1/2}^{\infty} \prod_{n=1}^{\infty} \mathsf{q}^{nk} = \prod_{n=1}^{\infty} (1 - \mathsf{q}^{n-\frac{1}{2}})^{-1} \ .$$

Also the twisted vacuum contributes a factor of $\mathsf{q}^{1/48}$ for the anti-periodic boson. The partition function ${Z_{(\Phi)}}^1{}_0$ is found to be

$$ {Z_{(\Phi)}}^1{}_0 = \mathsf{q}^{1/48} \bar{\mathsf{q}}^{-1/24} \prod_{n=1}^{\infty} (1 - \mathsf{q}^{n-\frac{1}{2}})^{-1} \prod_{n=1}^{\infty} (1 - \bar{\mathsf{q}}^{n})^{-1} \sum_{n \in \mathbb{Z}} \bar{\mathsf{q}}^{\frac{1}{4}(n-\frac{1}{2})^2} \ .$$

Using the definitions of the theta functions given in section §8.3.3, we express the different factors as

$$\left(\frac{\eta(\tau)}{\theta_4(\tau)} \right)^{1/2} = \mathsf{q}^{-1/48} \prod_{n=1}^{\infty} (1 - \mathsf{q}^{n-\frac{1}{2}})^{-1}$$
$$\overline{\eta(\tau)}^{-1} = \bar{\mathsf{q}}^{-1/24} \prod_{n=1}^{\infty} (1 - \bar{\mathsf{q}}^{n})^{-1}$$
$$\overline{\theta_2(\frac{1}{2}\tau)} = \sum_{n \in \mathbb{Z}} \bar{\mathsf{q}}^{\frac{1}{4}(n-\frac{1}{2})^2}$$

and we rewrite the partition function as

$$ {Z_{(\Phi)}}^1{}_0 = \left(\frac{\eta}{\theta_4} \right)^{1/2} \frac{\overline{\theta_2(\frac{1}{2}\tau)}}{\overline{\eta}} \ .$$

For the twisted sector with one insertion we include the nontrivial T-duality phase

$$(-1)^{\frac{n}{2}(n-1)} \ .$$

As we show in section §8.3.3, the zero mode sum can be written as the complex conjugate of the following theta function

$$\sqrt{2}\theta_2(\frac{\tau}{2},-\frac{1}{4}) = \sum_{n\in\mathbb{Z}}(-1)^{\frac{n}{2}(n-1)}\mathsf{q}^{\frac{1}{4}(n-\frac{1}{2})^2} \ . \tag{8.32}$$

The insertion will also send $a_{-k} \to -a_{-k}$, so that the argument of the geometric sum of oscillation modes picks up a sign. The result is

$$Z_{(\Phi)}{}^1{}_1 \ = \ \mathsf{q}^{1/48}\bar{\mathsf{q}}^{-1/24}\prod_{n=1}^{\infty}(1 \ + \ \mathsf{q}^{n-\frac{1}{2}})^{-1}\prod_{n=1}^{\infty}(1 \ - \ \bar{\mathsf{q}}^{n})^{-1}\sum_{n\in\mathbb{Z}}(-1)^{\frac{n}{2}(n-1)}\bar{\mathsf{q}}^{\frac{1}{4}(n-\frac{1}{2})^2} \ .$$

Using

$$\left(\frac{\eta(\tau)}{\theta_3(\tau)}\right)^{1/2} = \mathsf{q}^{1/48}\prod_{n=1}^{\infty}(1+\mathsf{q}^{n-\frac{1}{2}})^{-1} \ ,$$

we finally express the partition function as

$$Z_{(\Phi)}{}^1{}_1 = \left(\frac{2\eta}{\theta_3}\right)^{1/2}\frac{\overline{\theta_2(\frac{1}{2}\tau,-\frac{1}{4})}}{\overline{\eta}} \ .$$

8.3.2 Modular Invariance

In section §8.3.1 we found the partition functions of the interpolated orbifolds. The flat base orbifold

$$Z_{(\mathscr{N})}(\tau) = \left((\tau_2)^{-1/2}|\eta(\tau)|^{-2}\right)^{22}$$

and the shift orbifold

$$Z_{(Y)}{}^a{}_b(\tau) = |\eta(\tau)|^{-2}(\tau_2)^{-1/2}(2R_y)\times \sum_{n,w}\exp\left\{-\pi\frac{(2R_y)^2}{\tau_2^2}\left|n+\frac{b}{2}-\left(w+\frac{a}{2}\right)\tau\right|^2\right\} \tag{8.33}$$

combine with the fiber orbifold

$$\begin{aligned}
{Z_{(\Phi)}}^0{}_0 =& \frac{1}{|\eta|^2\sqrt{\tau_2\epsilon}} \sum_{m,n\in\mathbb{Z}} \exp\left(-\frac{\pi}{\tau_2\epsilon^2}|m+n\tau|^2\right) \\
{Z_{(\Phi)}}^0{}_1 =& \left(\frac{2\eta}{\theta_2}\right)^{1/2} \frac{\overline{\theta_4}(2\tau)}{\overline{\eta}} \\
{Z_{(\Phi)}}^1{}_0 =& \left(\frac{\eta}{\theta_4}\right)^{1/2} \frac{\overline{\theta_2}(\frac{1}{2}\tau)}{\overline{\eta}} \\
{Z_{(\Phi)}}^1{}_1 =& \left(\frac{2\eta}{\theta_3}\right)^{1/2} \frac{\overline{\theta_2}(\frac{\tau}{2};-\frac{1}{4})}{\overline{\eta}}
\end{aligned}$$

to give the partition function of our oribold as

$$Z(\tau) = \frac{1}{2} Z_{(\mathscr{N})}(\tau) \sum_{a,b=0,1} Z^a_{(\Phi)b}(\tau) Z^a_{(Y)b}(\tau) \ .$$

As described in the beginning of section §8.3, modular invariance under the action of $SL(2,\mathbb{Z})$ on the parameter τ is vital for a consistent CFT.

The partition function $Z_{(\mathscr{N})}$ is modular invariant as a consequence of the definition of the Dedekind eta function. Indeed, from equation (8.34) we see that the combination $(\tau_2)^{-1/2}|\eta(\tau)|^{-1}$ is invariant under both S and T. Furthermore, we have noted that ${Z_{(Y)}}^a{}_b(\tau)$ in equation (8.33) is modular covariant. Under $T:\tau\to\tau+1$, we replace $n\to n+w$ in the sum, while under $S:\tau\to-1/\tau$ we interchange n and w. What is left to prove is modular covariance of the fiber partition function ${Z_{(\Phi)}}^a{}_b$. We shall not show the calculation. It is straightforward, but let us note the equations we use. The equations can be found in the next section. Under T we have

$$\begin{aligned}
{Z_{(\Phi)}}^0{}_0(\tau+1) =& {Z_{(\Phi)}}^0{}_0(\tau) && \text{similarly to } {Z_{(Y)}}^a{}_b \\
{Z_{(\Phi)}}^0{}_1(\tau+1) =& {Z_{(\Phi)}}^0{}_1(\tau) && \text{using (8.35a),(8.35b) and (8.35c)} \\
{Z_{(\Phi)}}^1{}_0(\tau+1) =& {Z_{(\Phi)}}^1{}_1(\tau) && \text{using (8.35c) and (8.39)} \\
{Z_{(\Phi)}}^1{}_1(\tau+1) =& {Z_{(\Phi)}}^1{}_0(\tau) && \text{using (8.35b) and (8.39)} \ ,
\end{aligned}$$

while under S we have

$$
\begin{aligned}
{Z_{(\Phi)}}^0{}_0(-1/\tau) =& {Z_{(\Phi)}}^0{}_0(\tau) && \text{similarly to } {Z_{(Y)}}^a{}_b \\
{Z_{(\Phi)}}^0{}_1(-1/\tau) =& {Z_{(\Phi)}}^1{}_0(\tau) && \text{using (8.36a) and (8.36c)} \\
{Z_{(\Phi)}}^1{}_1(-1/\tau) =& {Z_{(\Phi)}}^1{}_1(\tau) && \text{using (8.36b) and (8.38)} .
\end{aligned}
$$

The modular covariance of ${Z_{(\Phi)}}^a{}_b$ relies crucially on the quantization conditions on the zero modes. The quantization conditions were in turn obtained by the phase factors upon quantizing the monodromy. These phases were introduced in [135] in the non-doubled conventional formalism. This improves on earlier work [128, 129, 126], where the same orbifold was found to not be modular invariant, when calculated in the conventional formalism. Although the authors were aware of the importance of the T-duality phases, they did not have the correct ones, with the result of having to look for further solutions of consistency. On the other hand, our results agree with the conventional formalism as presented in [135], that is, we obtain the same partition function.

So we have shown that the doubled S^1 system, considered as a constraint Hamiltonian system, is equivalent quantum mechanically to the conventional non-doubled picture. An important point, though, is that we have not needed to make any choice of 'physical' states. Even though we have not chosen a polarization, it is not surprising that we obtain the same partition function. After all T-dual theories must have the same partition function and our formalism is the T-duality covariant form of the asymmetric orbifold.

8.3.3 Theta functions

In the definition of ${Z_{(\Phi)}}^a{}_b$ we used several Jacobi theta functions [138]. Because definitions of theta functions vary greatly in the literature, we write our definitions here and note some of their properties that are of interest to us.

The theta functions we use are $\theta_i(\tau, z)$, for $i = 2, 3, 4$. They can be defined by the following infinite sums

$$\theta_2(\tau;z) = \sum_{n\in\mathbb{Z}} \mathsf{q}^{\frac{1}{2}(n-\frac{1}{2})^2} e^{i\pi(2n-1)z}$$
$$\theta_3(\tau;z) = \sum_{n\in\mathbb{Z}} \mathsf{q}^{\frac{1}{2}n^2} e^{i2\pi nz}$$
$$\theta_4(\tau;z) = \sum_{n\in\mathbb{Z}} (-1)^n \mathsf{q}^{\frac{1}{2}n^2} e^{i2\pi nz} \ ,$$

or expressed as the infinite products

$$\theta_2(\tau,z) = 2\eta(\tau)\mathsf{q}^{\frac{1}{12}} \cos(\pi z) \prod_{n=1}^{\infty} (1 + 2\mathsf{q}^n \cos(2\pi z) + \mathsf{q}^{2n})$$
$$\theta_3(\tau,z) = \eta(\tau)\mathsf{q}^{-\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 + 2\mathsf{q}^{n-\frac{1}{2}} \cos(2\pi z) + \mathsf{q}^{2n-1}\right)$$
$$\theta_4(\tau,z) = \eta(\tau)\mathsf{q}^{-\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - 2\mathsf{q}^{n-\frac{1}{2}} \cos(2\pi z) + \mathsf{q}^{2n-1}\right) \ .$$

In the definitions we used the nome $\mathsf{q} = \exp(2\pi i\tau)$. The Dedekind eta function can also be defined as an infinite product

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \ .$$

The modular transformation of $\eta(\tau)$ is

$$\eta(\tau+1) = e^{i\frac{\pi}{12}}\eta(\tau) \tag{8.34a}$$
$$\eta(-\frac{1}{\tau}) = (-i\tau)^{1/2}\eta(\tau) \ . \tag{8.34b}$$

The usual modular transformation properties of the theta functions [138] under $T : \tau \to \tau + 1$ are

$$\theta_2(\tau+1,z) = e^{\frac{i\pi}{4}}\theta_2(\tau,z) \tag{8.35a}$$
$$\theta_3(\tau+1,z) = \theta_4(\tau,z) \tag{8.35b}$$
$$\theta_4(\tau+1,z) = \theta_3(\tau;z) \ , \tag{8.35c}$$

while under $S : \tau \to -1/\tau$, they are

$$\theta_2\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z^2}{\tau}} \theta_4(\tau, z) \tag{8.36a}$$

$$\theta_3\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z^2}{\tau}} \theta_3(\tau, z) \tag{8.36b}$$

$$\theta_4\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z^2}{\tau}} \theta_2(\tau, z) \ . \tag{8.36c}$$

As a first application, let us prove equation (8.32). It was stated that

$$\theta_2(\frac{\tau}{2}, -\frac{1}{4}) = \frac{\sqrt{2}}{2} \sum_{n\in\mathbb{Z}} (-1)^{\frac{n}{2}(n-1)} \mathsf{q}^{\frac{1}{4}(n-\frac{1}{2})^2} \ .$$

Proof of (8.32). We have the following consecutive equalities

$$\begin{aligned} \sum_{n\in\mathbb{Z}} (-1)^{\frac{n}{2}(n-1)} \mathsf{q}^{\frac{1}{4}(n-\frac{1}{2})^2} =& 2 \sum_{n\in\mathbb{Z}} (-1)^n \mathsf{q}^{\left(n-\frac{1}{4}\right)^2} \\ =& \sqrt{2} \sum_{n\in\mathbb{Z}} \exp\left(\pi i \left(\frac{1}{4} - \frac{n}{2}\right)\right) \mathsf{q}^{\frac{1}{4}\left(n-\frac{1}{2}\right)^2} \\ =& \sqrt{2}\theta_2(\frac{\tau}{2}, -\frac{1}{4}) \ . \end{aligned} \tag{8.37}$$

The first equality is obtained by splitting the sum into even, $n = 2m$, and odd, $n = -2m + 1$, integers and finding that they give the same contribution. For the second equality, one splits the sum on the right-hand side into even and odd integers and finds that they can be gathered into one sum with a factor of $\cos \pi/4$. The last equality is true by the definition of θ_2. □

Apart from the usual modular transformations (8.35) and (8.36), there are two more transformations that we need. The first one reads

$$\theta_2\left(-\frac{1}{2\tau}, -\frac{1}{4}\right) = (-i\tau)^{1/2} \theta_2\left(\frac{\tau}{2}, -\frac{1}{4}\right) \ . \tag{8.38}$$

Proof of (8.38). We define $\tau' = 2\tau$ and $z' = -\frac{1}{2}\tau$. We have the following equalities

$$\begin{aligned}
\theta_2\left(-\frac{1}{2\tau},-\frac{1}{4}\right) =&\theta_2\left(-\frac{1}{\tau'},-\frac{z'}{\tau'}\right)\\
=&(-i\tau')^{1/2}\exp\left(i\pi\frac{z'^2}{\tau'}\right)\theta_4(\tau',z')\\
=&(-i\tau)^{1/2}\sqrt{2}\mathsf{q}^{1/16}\sum_{n\in\mathbb{Z}}(-1)^n\mathsf{q}^{n^2}\mathsf{q}^{-\frac{n}{2}}\\
=&(-i\tau)^{1/2}\sqrt{2}\sum_{n\in\mathbb{Z}}(-1)^n\mathsf{q}^{\left(n-\frac{1}{4}\right)^2}\\
=&(-i\tau)^{1/2}\theta_2\left(\frac{\tau}{2},-\frac{1}{4}\right)\ ,
\end{aligned}$$

where for the second equality we used the S modular transformation of θ_2 (8.36a) and for the last equality we used equation (8.37). □

The second modular transformation we want to prove reads

$$\sqrt{2}\theta_2\left(\tau+\frac{1}{2},-\frac{1}{4}\right)=e^{i\frac{\pi}{8}}\theta_2\left(\tau,0\right)\ . \tag{8.39}$$

Proof of (8.39). To begin with, we observe that $\theta_2(\tau,0)$ can be written as a sum over positive integers only

$$\theta_2(\tau,0)=\sum_{n\in\mathbb{Z}}\mathsf{q}^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}=2\sum_{n=1}^{\infty}\mathsf{q}^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}\ .$$

On the other hand, we can write

$$\begin{aligned}
\theta_2\left(\tau+\frac{1}{2},-\frac{1}{4}\right) =&\sum_{n\in\mathbb{Z}}\exp\left(i\frac{\pi}{2}\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\right)\mathsf{q}^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}\\
=:&\sum_{n\in\mathbb{Z}}f_n\mathsf{q}^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}\ .
\end{aligned} \tag{8.40}$$

We defined implicitly the series f_n. One can show that it is periodic $f_{n+2}=f_n$ and that

$$f_n+f_{n+1}=\sqrt{2}e^{i\frac{\pi}{8}}\ .$$

Now, due to the periodicity $f_{n+1}=f_{1-n}$, and because the power of $\mathsf{q}^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}$ is symmetric under $n \to 1-n$, we can split the infinite sum of equation (8.40) into $n \geq 1$ and $n < 1$ integers. We arrive at the sum

$$\theta_2\left(\tau + \frac{1}{2}, -\frac{1}{4}\right) = \sqrt{2}e^{i\frac{\pi}{8}} \sum_{n=1}^{\infty} \mathsf{q}^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2} e^{i\frac{\pi}{8}} \theta_2(\tau, 0) \ .$$

$\square$

This concludes the list of properties of theta functions we need to use.

Chapter 9

Supersymmetric Double Torus

So far we have analyzed the bosonic sector of the doubled torus system as a constrained two-dimensional field theory. We showed the equivalence of the classical field theory with the conventional sigma model. We also quantized a particular T-fold, to show its consistency and quantum equivalence with the conventional framework. In this chapter we extend the doubled torus system to include worldsheet supersymmetry.

World-sheet fermions were added to string theory quite soon in its inception [139]. This is required, if one hopes to build realistic models of the world. For the same reasons we are urged to extend the doubled torus to also include fermions on the worldsheet. This will allow for the construction of more complicated orbifolds, hopefully modular invariant and perhaps realistic. Supersymmetric asymmetric orbifolds, corresponding to T-fold backgrounds, have been considered in [128, 129, 135], but not in the doubled formalism.

We will employ the machinery of worldsheet superspace [140, 141]. We set our conventions in section §9.1. In section §9.2 we extend the doubled torus lagrangian into superspace. It is important though to investigate the fate of the constraint. For this, we make an obvious choice that is presented in section §9.3. Our first issue is whether the theory is equivalent to the conventional superstring. We find that it is indeed and we immediately confirm that the nature of the constraint retains its nature of Dirac second class.

9.1 Superspace Formalism

Minkowski spacetime can be seen as the coset space of the Poincaré group modulo the Lorentz group. The induced Poincaré action on the space agrees with the inhomogeneous Lorentz transformations. This led Salam [142] to propose superspace as a generalization of spacetime for supersymmetric theories. Superspace can be defined as the coset space of the Poincaré supergroup modulo the Lorentz group. One of the nice features of superspace field theories is that *often* supersymmetry closes off-shell due to the presence of auxiliary fields [140, 141, 143]. The supersymmetries are realized as translations in superspace.

The worldsheet metric η has signature $(+,-)$ and we define the Clifford algebra

$$\{\rho^a, \rho^b\} = 2\eta^{ab} \ .$$

A representation is

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{9.1a}$$

and

$$\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ . \tag{9.1b}$$

We also introduce the third Pauli matrix $\rho_3 = \rho^0\rho^1 = \sigma^3$, which is the chirality operator. We note the useful property that it acts on vectors as the Hodge dual

$$\rho^3 \cdot V = - \star V \ ,$$

for any $V = V_a\rho^a$.

In $1+1$ dimensions one has the choice of Dirac, Majorana, Weyl or Majorana-Weyl spinors. In the representation of (9.1), Majorana spinors are purely real and the chiral components are diagonal. In order to incorporate $\mathcal{N} = 1$ supersymmetry, we extend the worldsheet with one Majorana spinorial coordinate θ_α.

We take spinorial indices α, β to be Grassmann odd and worldsheet indices a, b to be Grassmann even. Berezin integration is defined by the linear operation

$$\int d\theta_\alpha \theta_\beta = \delta_{\alpha\beta}$$

and 0 otherwise. The normalization of $d^2\theta$ is such that

$$\int d^2\theta(\bar{\theta}\theta) = 1 \ ,$$

where $\bar{\theta}_\alpha = \theta_\beta \rho^0_{\beta\alpha}$ as usual. Also in our conventions, complex conjugation transposes spinors without a sign.

The supercharges Q_α will generate translations in superspace $(\sigma_a, \theta_\alpha)$. We define them as

$$\begin{aligned} Q_\alpha =& \frac{\partial}{\partial \bar{\theta}^\alpha} + i(\rho^a \theta)_\alpha \partial_a \\ \bar{Q}_\alpha =& (Q^* \rho^0)_\alpha = -\frac{\partial}{\partial \theta^\alpha} - i(\bar{\theta}\rho^a)_\alpha \partial_a \end{aligned}$$

and they generate even translations through the anticommutator

$$\{Q_\alpha, Q_\beta\} = -2i(\rho^a \rho^0)_{\alpha\beta}\partial_a \ .$$

The super-derivatives

$$\begin{aligned} \mathcal{D}_\alpha =& \frac{\partial}{\partial \bar{\theta}^\alpha} - i(\rho^a \theta)_\alpha \partial_a \\ \bar{\mathcal{D}}_\alpha =& (D^* \rho^0)_\alpha = -\frac{\partial}{\partial \theta^\alpha} + i(\bar{\theta}\rho^a)_\alpha \partial_a \end{aligned}$$

do not anticommute. They satisfy though the relation

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 2i(\rho^a \rho^0)_{\alpha\beta}\partial_a \ .$$

This follows from charge conjugation

$$\overline{(\rho^\alpha \theta)}_\alpha = (\bar{\theta}\rho^a)_\alpha$$

and

$$\overline{(\frac{\partial}{\partial\theta^\alpha})} = -\frac{\partial}{\partial\bar{\theta}^\alpha} \ .$$

The super-derivatives satisfy the important anti-commutativity property with the supercharges

$$\{Q_\alpha, D_\beta\} = 0 \ .$$

Therefore, for a superfield $\mathbb{F}(\sigma, \theta)$, that transforms under supersymmetry as

$$\delta_\alpha \mathbb{F} = Q_\alpha \mathbb{F} \ ,$$

its super-derivative $\mathcal{D}_\alpha \mathbb{F}$ also transforms as a superfield

$$\delta_\alpha \mathcal{D}_\beta \mathbb{F} = Q_a \mathcal{D}_\beta \mathbb{F} \ .$$

We can expand any superfield $\mathbb{F}$ as

$$\mathbb{F}(\sigma, \theta) = F(\sigma) + \bar{\theta}\psi(\sigma) + \frac{1}{2}\bar{\theta}\theta f(\sigma) \tag{9.2}$$

and its super-covariant derivative is given by

$$\mathcal{D}_\alpha \mathbb{F} = \psi_\alpha + \theta_\alpha f - i(\rho^a\theta)_\alpha \partial_\alpha F + \frac{i}{2}\partial_a(\rho^a\psi)_\alpha \bar{\theta}\theta \ .$$

To show this, we use the Fierz identity

$$\theta_\alpha \bar{\theta}_\beta = -\frac{1}{2}\delta_{\alpha\beta}\bar{\theta}\theta \ ,$$

which implies the useful relation

$$\bar{\theta}\epsilon_1 \bar{\theta}\epsilon_2 = -\frac{1}{2}\bar{\epsilon}_2\epsilon_1\bar{\theta}\theta \ .$$

Given an unconstrained superfield $\mathbb{F}$, a supersymmetric action can be written as the integration over superspace

$$S[\mathbb{F}] = \int d^2\sigma d^2\theta \mathbb{F} \ .$$

The worldsheet lagrangian can be found by replacing the Berezin integration with (the spinorial inner product of) the super-derivatives $\mathcal{D}$ and setting $\theta = 0$, or by expanding and keeping the term proportional to θ^2. In 1+1 dimensions, by defining the component fields in (9.2), it becomes straightforward to use the second method. In the next section we will extend the doubled torus lagrangian into superspace.

9.2 Supersymmetric Lagrangian

The obvious supersymmetric generalization of the doubled torus lagrangian, such that truncating the fermions gives the original lagrangian, is to functionally extend the original lagrangian into superspace. To be more precise, we first define the superfields $\mathbb{X}$ and $\mathbb{Y}$, so that they contain the bosonic embeddings X and Y

$$\begin{aligned} \mathbb{X}^I =& X^I + \bar{\theta}\psi^I + \frac{1}{2}\bar{\theta}\theta f^I \\ \mathbb{Y}^n =& Y^n + \bar{\theta}\chi^n + \frac{1}{2}\bar{\theta}\theta\phi^n \ . \end{aligned}$$

It is useful to consider the collective coordinates $\mathcal{Q}^\mu = (\mathbb{X}^I, \mathbb{Y}^m)$

$$\mathcal{Q}^\mu = q^\mu + \bar{\theta}\psi^\mu + \frac{1}{2}\bar{\theta}\theta f^\mu \ ,$$

with super-derivative expansion

$$\mathcal{D}_\alpha \mathcal{Q}^\mu = \psi^\mu_\alpha + \theta_\alpha f^\mu - i(\rho^a\theta)_\alpha \partial_a q^\mu + \frac{i}{2}\partial_a(\rho^a\psi^\mu)_\alpha(\bar{\theta}\theta) \ .$$

The functional generalization of the doubled torus lagrangian (6.8) is

$$\mathcal{L} = \int d^2\theta \left\{ \frac{1}{2} g_{\mu\nu}(\mathbb{Y}) \bar{D}\mathcal{Q}^\mu D\mathcal{Q}^\nu - \frac{1}{2} b_{\mu\nu}(\mathbb{Y}) \bar{D}\mathcal{Q}^\mu (\rho_3) D\mathcal{Q}^\nu \right\} \ . \tag{9.3}$$

Note that all the spinor indices in the above equations are contracted. We introduced ρ_3, so that the b-field term will appear, due to $\rho^3 \cdot V = - \star V$ for a worldsheet vector V. The fields $g_{\mu\nu}$, $b_{\mu\nu}$ are defined as in the bosonic doubled torus geometry of section §6.1, see also equation (7.5). That is, the metric is

$$g_{\mu\nu} = \begin{pmatrix} H_{IJ} & A_{In} \\ A_{Jm} & G_{mn} \end{pmatrix}$$

and the non-zero components of $b_{\mu\nu}$ are $b_{Im} = -b_{mI} = \tilde{A}_{Im}$ and $b_{mn} = B_{mn}$. By expanding the target spacetime indices we get

$$\begin{aligned} \mathcal{L} = \int d^2\theta \Big\{ & \frac{1}{2} H_{IJ}(\mathbb{Y}) \bar{D}\mathbb{X}^I D\mathbb{X}^J + A_{Im}(\mathbb{Y}) \bar{D}\mathbb{X}^I D\mathbb{Y}^m - \tilde{A}_{Im}(\mathbb{Y}) \bar{D}\mathbb{X}^I (\rho_3) D\mathbb{Y}^m \\ & + \frac{1}{2} G_{mn}(\mathbb{Y}) \bar{D}\mathbb{Y}^m D\mathbb{Y}^n - \frac{1}{2} B_{mn}(\mathbb{Y}) \bar{D}\mathbb{Y}^m (\rho_3) D\mathbb{Y}^n \Big\} \ . \end{aligned} \tag{9.4}$$

We integrate using $\int d^2\theta(\bar{\theta}\theta) = 1$, to obtain a supersymmetric Lagrangian. This gives the bosonic doubled torus lagrangian upon truncation. For example, the first term in (9.4) is expanded as

$$\begin{aligned} H_{IJ}(\mathbb{Y}) \bar{D}_\alpha \mathbb{X}^I D_\alpha \mathbb{X}^J = & H_{IJ}(\mathbb{Y}) \bar{\psi}^I \psi^J + 2 H_{IJ}(\mathbb{Y}) \bar{\psi}^I \theta F^J \\ & - 2i H_{IJ}(\mathbb{Y}) (\bar{\psi}^I \rho^a \theta) \partial_a X^J \\ & + H_{IJ}(\mathbb{Y}) \left(\eta^{ab} \partial_a X^I \partial_b X^J + i \bar{\psi}^I \rho^a \partial_a \psi^J + F^I F^J \right) \bar{\theta}\theta \ , \end{aligned}$$

where

$$\begin{aligned} H_{IJ}(\mathbb{Y}) = H_{IJ}(Y) + \partial_n H_{IJ}(Y) \bar{\theta}\chi^n + & \frac{1}{2} \partial_n H_{IJ}(Y) \bar{\theta}\theta \phi^n \\ & + \frac{1}{2} \partial_m \partial_n H_{IJ}(Y) (\bar{\theta}\chi^m)(\bar{\theta}\chi^n) \ . \end{aligned}$$

Expanding everything in this way and integrating, we arrive at the following supersymmetric lagrangian

$$\begin{aligned}\mathcal{L} =&\frac{1}{2}g_{\mu\nu}\partial_a q^\mu \partial_b q^\nu \eta^{ab} - \frac{1}{2}b_{\mu\nu}\partial_a q^\mu \partial_b q^\nu \epsilon^{ab} \\ &+ \frac{1}{2}g_{\mu\nu} i\bar{\psi}^\mu \not{\partial}\psi^\nu - \frac{i}{2}b_{\mu\nu}\bar{\psi}^\mu \rho^3 \not{\partial}\psi^\nu \\ &+ \frac{1}{2}g_{\rho\sigma,\nu} i\bar{\psi}^\rho \rho^a \psi^\nu \partial_a q^\sigma - \frac{1}{2}b_{\nu\rho,\mu} i\bar{\psi}^\nu \rho^3 \rho^a \psi^\mu \partial_a q^\rho \\ &+ \frac{1}{2}g_{\mu\nu} f^\mu f^\nu + \left(-\frac{1}{2}\Gamma^\mu_{\rho\nu}\bar{\psi}^\rho\psi^\nu - \frac{1}{4}H^\mu{}_{\rho\nu}\bar{\psi}^\rho \rho^3 \psi^\nu\right) g_{\mu\kappa} f^\kappa \\ &- \frac{1}{8}g_{\rho\sigma,\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma + \frac{1}{8}b_{\rho\sigma,\mu\nu}\bar{\psi}^\rho\rho^3\psi^\sigma\bar{\psi}^\mu\psi^\nu\end{aligned} \quad .$$

We can keep this form, or choose to solve for f^μ. This is an auxiliary field with non-propagating dynamics, that is to say its equation of motion is algebraic. By inserting its solution into the lagrangian, worldsheet supersymmetry becomes on-shell. After some work, we obtain the lagrangian

$$\begin{aligned}\mathcal{L} =&\frac{1}{2}g_{\mu\nu}\partial_a q^\mu \partial_b q^\nu \eta^{ab} - \frac{1}{2}b_{\mu\nu}\partial_a q^\mu \partial_b q^\nu \epsilon^{ab} \\ &+ \frac{1}{2}g_{\mu\nu} i\bar{\psi}^\mu \not{\nabla}^+\psi^\nu + \frac{1}{4}R^-_{\mu\nu\rho\sigma}\psi^\mu_+\psi^\nu_+\psi^\rho_-\psi^\sigma_-\end{aligned} \quad , \tag{9.5}$$

where $\not{\nabla}^\pm = \partial_a q^\mu \rho^a \nabla^\pm_\mu$ are the pull-backs of the torsion-full target spacetime connections

$$\nabla^\pm_\mu V^\nu = \nabla_\mu V^\nu \mp \frac{1}{2}H_\mu{}^\nu{}_\rho V^\rho$$

to the worldsheet and R^- is the curvature

$$R^{-\mu}{}_{\nu\rho\sigma} = \left[\nabla^-_\rho, \nabla^-_\sigma\right]^\mu{}_\nu \ .$$

9.3 Constraint Analysis

The supersymmetric lagrangian (9.5) is to be regarded, as in the bosonic case, a pseudo-lagrangian. Although it contains the dynamics of the base coordinates Y^m and their super-partners, most of the dynamics of the compact directions will come from the constraint we impose.

Remember the constraint of the bosonic theory can be written as

$$\dot{X} - L\tilde{A}\dot{Y} = S(X' - L\tilde{A}Y') \ , \tag{9.6}$$

an equation that halves the independent vectors dX^I. Now $\{dX^I\}_I = X^\star T(T^{2n})$, where $X^\star$ is the pull-back of the map from the worldsheet into the target spacetime, $X : \Sigma \to T^{2n}$. The fermions in the supersymmetric sigma model are sections of the bundle

$$X^\star T(T^{2n}) \otimes \sqrt{K} \ ,$$

with $\sqrt{K}$ the, Grassmann odd, Majorana representation of $SO(1,1)$. We want a theory equivalent to the conventional superstring and we should impose a constraint that halves the fermionic degrees of freedom as well. Furthermore, the constraint we impose should be supersymmetric. One can supersymmetrize (9.6) with

$$D_\alpha \mathbb{X}^I - L^{IJ}\tilde{A}_{Jn}(\mathbb{Y})D_\alpha \mathbb{Y}^n = -S^I{}_J(\mathbb{Y})\rho^3_{\alpha\beta}\left(D_\beta \mathbb{X}^J - L^{JK}\tilde{A}_{Kn}(\mathbb{Y})D_\beta \mathbb{Y}^n\right) \ . \tag{9.7}$$

Observe that the constraint is still self-dual, because $S^2 = 1$ remains true on the superfields and

$$A_{In}(\mathbb{Y}) = -H_{IJ}(\mathbb{Y})L^{JK}\tilde{A}_{Kn}(\mathbb{Y}) \ . \tag{9.8}$$

Of course the constraint in (9.7) reduces to (9.6) upon setting the fermions and auxiliary fields to zero. The rest of the chapter presents some important results. First, we show that the constraint is consistent with the lagrangian, in the sense that by imposing it on the Euler-Lagrange equations we get nontrivial dynamics. In the bosonic case an exterior derivative on the constraint gave the Euler-Lagrange equation of X. For the supersymmetric case, acting on the constraint with a $\mathcal{D}$ derivative gives the Euler-Lagrange equation of $\mathbb{X}$.

The second task is to expand the constraint (9.7) in powers of θ. The linear term in θ gives the original bosonic constraint, corrected with some fermionic terms. The constant term gives a constraint on the worldsheet fermions. We show that the constraint on the fermions is such that it halves the fermionic degrees of freedom. This is precisely what we want, a theory that is supersymmetric and that by truncation of the fermions we get a theory equivalent to the conventional bosonic string model. We conclude that the supersymmetric generalization of the doubled torus is indeed equivalent to the conventional superstring theory.

Note though that the last statement is true but for two tests, which we show the theory passes successfully. First, we need to show that to order θ^2, the constraint does not reduce the degrees of freedom furthermore. Indeed, we show that, on-shell, the constraint to order θ^2 is implied by the constraints to order θ^0 and θ^1. The second test is to show that the constraint is Dirac second class.

9.3.1 Consistency with Euler Lagrange equations

Using $S^2 = 1$ and the consistency equation (9.8), we write the constraint (9.7) in the equivalent form

$$C_{I\alpha} := H_{IJ}(\mathbb{Y})D_\alpha \mathbb{X}^J + A_{In}(\mathbb{Y})D_\alpha \mathbb{Y}^n \\ - \tilde{A}_{In}(\mathbb{Y})\rho^3_{\alpha\beta}D_\beta \mathbb{Y}^n + L_{IJ}\rho^3_{\alpha\beta}D_\beta \mathbb{X}^J = 0\ . \quad (9.9)$$

Now the Euler-Lagrange equations for $\mathbb{X}^I$ can be written in a supersymmetric form, by varying the superspace lagrangian (9.3) with respect to $\mathbb{X}$. This gives

$$\bar{D}_\alpha \left(g_{I\mu}(\mathbb{Y})D_\alpha \mathcal{Q}^\mu - b_{I\mu}(\mathbb{Y})\rho^3_{\alpha\beta}D_\beta \mathcal{Q}^\mu \right) = 0 \quad (9.10)$$

or by expanding $g_{I\mu}$ and $b_{I\mu}$

$$\bar{D}_\alpha \Big(H_{IJ}(\mathbb{Y})D_\alpha \mathbb{X}^J + A_{In}(\mathbb{Y})D_\alpha \mathbb{Y}^n - \tilde{A}_{In}(\mathbb{Y})\rho^3_{\alpha\beta}D_\beta \mathbb{Y}^n \Big) = 0\ .$$

Observe that $\bar{D}_\alpha \rho^3_{\alpha\beta} D_\beta = 0$ is true as a consequence of the supersymmetry algebra. Therefore, the constraint $C_{I\alpha} = 0$ implies the equations of motion for $\mathbb{X}$, in complete analogy with the bosonic doubled torus, see for example equations (6.12) and (6.13). Schematically we have

$$C^I_\alpha = 0 \Rightarrow \bar{D}_\alpha C^I_\alpha = 0 \Leftrightarrow \mathrm{eom}(\mathbb{X})\ .$$

9.3.2 Constraint on the components

We consider the constraint of (9.7) to each order in θ. The constant term reads

$$\psi^I - L^{IJ}\tilde{A}_{Jn}(Y)\chi^n = -S^I{}_J(Y)\rho^3\psi^J + H^{IJ}\tilde{A}_{Jn}\rho^3\chi^n \ . \tag{9.11}$$

This halves the independence of the fermions ψ^I, through using the endomorphism of the fiber S. A nice way of writing this is to split the fermions in their chiral parts

$$\begin{aligned}(1+S)\psi_+ =&(1+S)L\tilde{A}\chi_+\\(1-S)\psi_- =&(1-S)L\tilde{A}\chi_- \ .\end{aligned}$$

These constraints seem very natural as $\frac{1}{2}(1\pm S)$ are projectors. Therefore, half of the ψ^I's are constrained in terms of the χ^m's.

The constraint can also be written in the following equivalent ways

$$(1+\rho^3 S)_I{}^J g_{J\mu}\psi^\mu = \left(H^{IJ}+L^{IJ}\rho^3\right) g_{J\mu}\psi^\mu = \left(1+S\rho^3\right)\left(\psi - L\tilde{A}\chi\right) = 0 \ .$$

It is easy to show that the constraint on the fermions is in fact Dirac second class.

From the linear term in θ we obtain the following constraint

$$\dot{X} - SX' - L\tilde{A}\dot{Y} + H^{-1}\tilde{A}Y' = -\frac{i}{2}S\bar{\chi}^n\rho^1\partial_n\left(L\tilde{A}\chi - S\rho^3\psi + H^{-1}\tilde{A}\rho^3\chi\right) \tag{9.12a}$$

$$f - L\tilde{A}\phi = -\frac{1}{2}\bar{\chi}^n\partial_n\left(L\tilde{A}\chi - S\rho^3\psi + H^{-1}\tilde{A}\rho^3\chi\right) \ . \tag{9.12b}$$

The left-hand side of the first equation is the initial bosonic constraint, corrected with additional fermionic terms on the right hand side

$$\star dX - SdX = LAdY + L\tilde{A}\star dY + \text{corrections} \ ,$$

whereas the second equation is automatically satisfied when the auxiliary fields are put on-shell.

In phase space, equation (9.12a) can be written in exactly the same way as the original bosonic constraint, namely

$$\pi_I - L_{IJ}X'^J = 0 \ ,$$

where π_I is the canonical momentum associated to X^I that is derived from the supersymmetric Lagrangian (9.5). The constraint is therefore Dirac second class.

9.3.3 Halving the Degrees of Freedom

We now turn to the term of the constraint that is quadratic in θ. To begin with, let us expand the constraint C^I_α, defined in (9.9), in its components

$$C^I_\alpha = C^{I(0)}_\alpha + \bar{\theta}_\beta C^{I(1)}_{\alpha\beta} + \frac{1}{2}(\bar{\theta}\theta)C^{I(2)}_\alpha \ .$$

But it follows that

$$\begin{aligned} C^{I(0)}_\alpha &= 0 \quad \text{on shell} \\ C^{I(1)}_{\alpha\beta} &= 0 \quad \text{on shell} \qquad \Longrightarrow C^{I(2)}_\alpha = 0 \ . \\ \bar{D}_\alpha C^I_\alpha &= 0 \quad \text{eom for } \mathbb{X}^I \end{aligned}$$

Therefore, we show that to order θ^2 the constraint is automatically satisfied, if the constant and linear terms in θ are conserved on shell[1].

We have seen that the constraint, to zero and first order in θ, are Dirac second class and halve the degrees of freedom. They are

$$\pi_I - L_{IJ}X'^J = 0 \tag{9.13}$$

$$(1+\rho^3 S)_I{}^J g_{J\mu}\psi^\mu = 0 \ . \tag{9.14}$$

On-shell, the constraint to order θ^2 does not provide additional component constraints. Furthermore, the constraint is supersymmetric by construction. It is

[1]that is to say, the time derivatives of these terms are also satisfied.

also obvious that supersymmetrizing minimally a bosonic system, by extending it into superspace, is a unique process. We are thus led to the conclusion that the supersymmetric doubled torus system we presented is classically equivalent to the conventional superstring formulation.

Chapter 10

Conclusion

It is already clear that the concept of spacetime that emerges from string theory has added features to the one used in general relativity or quantum field theory. The consistency of superstring theory necessitates a spacetime described by supergravity, while T-duality allows for backgrounds that are not smooth. In this thesis we explored the supersymmetry of string theory backgrounds and the non-geometric features that arise due to T-duality.

We showed that supergravity solutions possess a superalgebra of symmetries. We presented the construction explicitly for the type IIb and heterotic supergravity theories. We also reproduced a theorem showing that enough supersymmetry implies homogeneity. The Killing superalgbera of a string theory background can be traced back to the supersymmetries of the massless string states. We have thus shown the existence of a superalgebra, which acts on the massless string states. We observe that the construction of a Killing superalgebra does not use the supergravity equations of motion, besides the Bianchi identities. This is evidence for the uniqueness of the supergravity theories we considered and their underlying supergeometry.

Little is known about quantum corrections to supergravity, although the amount of supersymmetry is presumed to persist to all orders [25, 144]. For the heterotic case, which is more tractable, a first order α' correction preserves the construction of the Killing superalgebra [145]. Much less is known about M-theory but there are some results. In the work of [146], it is assumed that both the 11d supergravity equations of motion and supersymmetry variations are corrected to first order by higher derivative terms in such a way that supersymmetry is preserved. In the work of [144], supersymmetric backgrounds are assumed to be deformed, along with the supersymmetry variation, in such a way that supersymmetry is again preserved. If the dimension of the Killing superalgebra remains fixed, it suggests looking at deformations of the Killing superalgebra. The Poincaré superalgebra of flat space is known to be rigid under cohomology deformations, but other backgrounds (such as the M2-brane and M5-brane Killing superalgebras of d=11 supergravity) admit nontrivial deformations [147]. It is natural to assume the deformation parameter as a string theory or (for the case of M-theory) Planck scale quantum correction.

Under certain conditions, one can extract a superalgebra of conserved charges associated to a background. For asymptotically flat spacetimes this is the ADM superalgebra. We computed this for M-theory and found that the odd charges generate, in addition to the ADM momentum, some additional charges. Besides a topological gravitational charge, which can exist in dimensions higher than 4, we found the Page type charges associated to magnetic and electric configurations of M-branes. The ADM formula we produced is universal and can be applied for generic backgrounds, provided the charges converge.

Inspired by the extended ADM superalgebra, we investigated the extension of the Killing superalgebra. From the spacetime perspective, we found a geometric realization of $\mathfrak{osp}(1|32)$ for the $\mathrm{AdS}_4 \times S^7$ background, whereas we argued that the pp-wave does not admit a minimally full extension. This does not exclude the possibility that the pp-wave superalgebra can be extended less than fully. We note that the geometric realization of the Freund-Rubin backgrounds is not universal as the Killing superalgebra is. This can be explained from the topological nature of the extension, whereas the (non-extended) Killing superalgebra is local in nature.

The extended Killing superalgebra is related to the global supersymmetries of a supersymmetric worldvolume theory. There are many results in the literature about what type of charges to expect from this perspective [148, 149, 150, 151, 152]. Nevertheless, the superalgebra is in most cases unknown. $\mathfrak{osp}(1|32)$, and its possible role in M-theory, has featured a lot in the literature from different perspectives [153, 107, 154, 155].

The geometric symmetries we explored are important in string theory. In order to investigate the non-geometric symmetries, we turned to the doubled geometry framework, whereby the dimension of spacetime is increased. We reviewed the framework from the lagrangian perspective and we analyzed the system from the hamiltonian perspective. We found that in the hamiltonian framework, the notion of a polarization, which selects the 'physical' spacetime, is not necessary. This gives credit to the doubled formalism as the right formalism to study T-duality. We found that T-duality in the hamiltonian formalism is a group that preserves the Dirac bracket.

We also used the framework to quantize a model T-fold. We specified a nontrivial doubled geometry and quantized the string canonically. We found that the quantization is equivalent to the results one would find using the conventional approach of asymmetric orbifolds. We found many interesting features of the T-fold, mainly that it was modular invariant and that it stabilizes the modulus of the internal circle. One of the difficulties we came across was the quantization of the zero modes. It would be interesting to investigate the importance of a topological term that was added to the action in [122], in particular its role in our quantization.

The doubled geometry is a framework with many advantages and there is considerable interest in its use [156, 157, 132, 122, 158, 159, 127, 132, 123]. The background field method has been used to find the constraints implied by conformal invariance on the doubed geometry [118, 119]. It would be interesting to further investigate the hamiltonian dynamics and find how various other aspects of T-duality fit in. This would include the study of D-branes and the relation of the doubled geometry with generalized geometry.

We finally extended the doubled geometry to include fermions. We argued that if we do so, we have a system that is at least classically equivalent to the conventional string. Since supersymetry is interesting for theoretical and phenomenological reasons, this was a vital step to make. One of our results was that the supersymmetric constraint is second class and, therefore, quantization can be at least canonically tractable.

Appendix A

Conventions and Notation

Most of our conventions are developed in the text. Nevertheless, we will summarize here some generic conventions that we use.

In 11 dimensions we use a mostly minus signature and in 10 dimensions we use a mostly plus. The Clifford algebra $C\ell(s,t)$ implies the vector space $\mathbb{R}^d$, where $d = s + t$, with metric η of signature (s,t), t being the negative eigenvalues of the metric. It is defined by the ideal $X \cdot X = -\eta(X,X)$. A representation of $C\ell(s,t)$ is given by the gamma matrices, Γ^M in 11 dimensions and γ^m for other signatures. An antisymmetric product $\gamma^{M_1 \cdots M_n}$ has strength factor 1. Spinors are taken to be Grassmann odd when working with field variations and lagrangians (eg. sections §2.1, §2.2, §2.3 and chapter 9) and Grassmann even when discussing Killing spinors or Killing superalgebras of bosonic backgrounds.

An n-form ω is expressed in components

$$\omega = \frac{1}{n!}\omega_{m_1 \dots m_n} dx^{m_1} \wedge \dots dx^{m_n}$$

and the Hodge dual is defined by $\omega \wedge *\omega = |\omega|^2_g \,\mathrm{dvol}$, which is in components

$$\star\omega = \frac{\sqrt{|g|}}{n!(d-n)!}\omega_{m_1 \dots m_n}\varepsilon^{m_1 \dots m_n}{}_{m_{n+1} \dots m_d} dx^{m_{n+1}} \wedge \dots dx^{m_d} \ ,$$

where $\varepsilon_{1\dots d} = 1$. A basis of the n-th exterior vector space is often written as

$$dx^{m_1 \cdots m_n} = dx^{m_1} \wedge \dots dx^{m_n} \ ,$$

or, given an orthonormal basis e^a,

$$e^{a_1 \cdots a_n} = e^{a_1} \wedge \dots e^{a_n} \ .$$

There are some symbols that we use consistently for the same objects. We present them in table A.1.

$\mathscr{M}, \mathscr{N}$	:	manifolds
$\bigwedge^n V$	:	the n-exterior algebra of V
$S^2 V$	:	the symmetric tensor square of V
$\mathscr{L}$	:	the Lie derivative
c_n	:	the subscript indicates the degree of the n-form c
g	:	a metric
dvol	:	the volume element of the metric
∇	:	the Christoffel connection or the spin connection
D	:	the exterior spin derivative
$\mathcal{D}$	:	the gravitino connection of a supergravity theory, or a superspace covariant derivative
$\mathcal{L}$	:	a lagrangian
$\mathcal{H}$	:	a hamiltonian
H	:	a doubled torus fiber metric, or depending on the context a subgroup
$\mathbb{X}, \mathbb{Y} \dots$	:	superspace fields
S, $\mathbb{S}$	:	spinor bundles
ε, $\boldsymbol{\varepsilon}$	:	spinor fields or Killing spinors
$\bar{\varepsilon}\varepsilon'$, $\langle \varepsilon, \varepsilon' \rangle$	:	the spinor inner product of ε and ε'
$\mathfrak{k}, \mathfrak{m} \dots$	:	a Lie (super-)algebra, $\mathfrak{k}$ being reserved for a Killing superalgebra
$\lfloor x \rfloor$	:	the floor of x

Table A.1: Often used symbols

Appendix B

Spinors for Freund-Rubin Backgrounds

In section §5.3 we used spinors in signatures $(1,3)$, $(2,3)$, $(7,0)$, $(8,0)$ and $(1,10)$. In this appendix we elaborate on the respective Clifford modules and spinor inner products.

For a metric η of arbitrary signature (s,t), a representation of the Clifford algebra $\mathrm{C}\ell(s,t)$ is defined by

$$\{\Gamma^A, \Gamma^B\} = -2\eta^{AB} \ .$$

We choose a unitary representation $(\Gamma^A)^\dagger \Gamma^A = \mathbb{1}$. A spin invariant pairing is defined with either of the matrices

$$C = \prod_{a=\text{time}} \Gamma^a$$
$$\tilde{C} = \prod_{a=\text{space}} \Gamma^a$$

The two matrices are related by acting with dvol and so for $d = s + t = \text{odd}$ they coincide up to a sign. Let us denote the two pairings respectively $\langle\,,\rangle_1$ and $\langle\,,\rangle_2$:

$$\langle\varepsilon,\varepsilon'\rangle_1 = \varepsilon^\dagger C\varepsilon'$$
$$\langle\varepsilon,\varepsilon'\rangle_2 = \varepsilon^\dagger \tilde{C}\varepsilon' \ .$$

Using $\Gamma^{A\dagger} = -\eta^{AA}\Gamma^A$, with $X \in \mathbb{R}^{s,t}$, the two pairings have the properties

$$\begin{aligned}\overline{\langle\varepsilon,\varepsilon'\rangle_1} &= (-1)^{\lfloor\frac{t}{2}\rfloor}\langle\varepsilon',\varepsilon\rangle_1 \\ \langle X\cdot\varepsilon,\varepsilon'\rangle_1 &= (-1)^{t+1}\langle\varepsilon, X\cdot\varepsilon\rangle_1\end{aligned} \tag{B.1a}$$

$$\begin{aligned}\overline{\langle\varepsilon,\varepsilon'\rangle_2} &= (-1)^{\lfloor\frac{s+1}{2}\rfloor}\langle\varepsilon',\varepsilon\rangle_2 \\ \langle X\cdot\varepsilon,\varepsilon'\rangle_2 &= (-1)^{s}\langle\varepsilon, X\cdot\varepsilon\rangle_2\end{aligned} \tag{B.1b}$$

In the following we discuss the Clifford algebras for signatures $(1,3)$, $(2,3)$, $(7,0)$, $(8,0)$ and $(1,10)$. More details can be found in [59, 160].

B.0.4 Cl(1,3) and Cl(2,3)

The algebra $\mathrm{C}\ell(1,3)$ is isomorphic to

$$\mathrm{C}\ell(1,3) = \mathrm{Mat}(4,\mathbb{R}) \ .$$

A real Clifford representation γ^a acts on Majorana spinors in $\Delta_A \cong \mathbb{R}^4$.

We use the pairing (B.1b). The symmetric and antisymmetric square of Δ_A into $\bigwedge^*\mathbb{R}^{1,3}$ is

$$S^2\Delta_A = \bigwedge{}^1\mathbb{R}^{1,3} \oplus \bigwedge{}^2\mathbb{R}^{1,3}$$
$$\bigwedge{}^2\Delta_A = \bigwedge{}^0\mathbb{R}^{1,3} \oplus \bigwedge{}^3\mathbb{R}^{1,3} \oplus \bigwedge{}^4\mathbb{R}^{1,3} \ .$$

The algebra $\mathrm{C}\ell(2,3)$ is isomorphic to

$$\mathrm{C}\ell(2,3) = \mathrm{Mat}(4,\mathbb{R}) \oplus \mathrm{Mat}(4,\mathbb{R}) \ .$$

There are two real irreducible Clifford representations, which act on Majorana spinors in $\Delta_A \cong \mathbb{R}^4$ and $\overline{\Delta}_A \cong \mathbb{R}^4$. The two representations can be built from a representation of gamma matrices γ^a of $\mathrm{C}\ell(1,3)$

$$\Gamma^a = \mathrm{dvol}(1,3)\gamma^a$$
$$\Gamma^r = \pm\, \mathrm{dvol}(1,3)\ .$$

The two representations are differantiated by the sign of $\mathrm{dvol}(2,3)$.

For our purposes we use the sign $\mathrm{dvol}(2,3) = +1$, which act on Majorana spinors in Δ_A. The spin generators $S_{AB} \in \mathfrak{spin}(2,3)$ are related to the generators of $\mathrm{C}\ell(1,3)$

$$S_{ab} = -\frac{1}{2}\Gamma_{ab} = -\frac{1}{2}\gamma_{ab} = s_{ab}$$
$$S_{ar} = -\frac{1}{2}\Gamma_{ar} = \mp\frac{1}{2}\gamma_a$$

and an even 2n-form in $\bigwedge^* \mathbb{R}^{2,3}$ gives a 2n-form and a 2n-1 form in $\bigwedge^* \mathbb{R}^{1,3}$. They act in terms of the representation of $\mathrm{C}\ell(1,3)$ like

$$\Gamma_{a_1\cdots a_{2n}} = \gamma_{a_1\cdots a_{2n}}$$
$$\Gamma_{ra_1\cdots a_{2n+1}} = -\gamma_{a_1\cdots a_{2n+1}}\ .$$

We use the pairing (B.1b). Then the spinnor pairing coincides with the one on $\mathrm{C}\ell(1,3)$

$$\langle -, -\rangle_{(2,3)} = \langle -, -\rangle_{(1,3)}\ .$$

The square of Δ_A into $\bigwedge^* \mathbb{R}^{2,3}$ is

$$S^2\Delta_A = \bigwedge{}^2\mathbb{R}^{2,3}$$
$$\bigwedge{}^2 \Delta_A = \bigwedge{}^0 \mathbb{R}^{2,3} \oplus \bigwedge{}^4 \mathbb{R}^{2,3}\ .$$

B.0.5 Cl(7,0) and Cl(8,0)

The algebra $\mathrm{C}\ell(7,0)$ is isomorphic to

$$\mathrm{C}\ell(7,0) = \mathrm{Mat}(8,\mathbb{R}) \oplus \mathrm{Mat}(8,\mathbb{R}) \ .$$

There are two real irreducible representations, which are differentiated by the sign of the volume element. We choose a real representation γ^i, with $\mathrm{dvol}(7,0) = [r]$, acting on real spinors in $\Delta_S \cong \mathbb{R}^8$.

We use the pairing (B.1a), which is equal to the other up to a sign. The symmetric and antisymmetric square of Δ_S into $\bigwedge^* \mathbb{R}^7$ is

$$\begin{aligned} S^2\Delta_S &= \textstyle\bigwedge^0 \mathbb{R}^7 \oplus \bigwedge^3 \mathbb{R}^7 \\ \textstyle\bigwedge^2 \Delta_S &= \textstyle\bigwedge^1 \mathbb{R}^7 \oplus \bigwedge^2 \mathbb{R}^7 \ . \end{aligned}$$

The algebra $\mathrm{C}\ell(8,0)$ is isomorphic to

$$\mathrm{C}\ell(8,0) = \mathrm{Mat}(16,\mathbb{R}) = \mathrm{Mat}(8,\mathbb{R}) \otimes \mathrm{Mat}(2,\mathbb{R}) \ .$$

There is a real Clifford representation, which we can build from a representation γ^i of $\mathrm{C}\ell(7,0)$, where the volume element has sign $[r]$

$$\begin{aligned} \Gamma_r &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1} \\ \Gamma_a &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \gamma_a \ . \end{aligned}$$

The volume element with orientation $dx^r \wedge \prod_{a=1}^{7} dx^a$ is

$$\mathrm{dvol}(8,0) = [r] \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \otimes \mathbb{1} \ .$$

A spinor in signature $(8,0)$ splits into two chiral spinors. We fix a spinor in signature $(7,0)$, which belongs in the module Δ_S, to lift to a chiral spinor of signature $(8,0)$. Its chirality is then $-[r]$. The spin elements in $\mathfrak{spin}(8,0)$ act on Δ_S in terms of the generators of $C\ell(7,0)$

$$\begin{aligned} S_{ij} &= s_{ab} = -\frac{1}{2}\gamma_{ij} \\ S_{ir} &= -\frac{1}{2}\gamma_i \ . \end{aligned}$$

An even $2n$-form in $\bigwedge^* \mathbb{R}^8$ gives a $2n$-form and a $(2n-1)$-form in $\bigwedge^* \mathbb{R}^7$. The latter act in terms of the representation of $C\ell(7,0)$

$$\begin{aligned} \Gamma_{i_1\cdots i_{2n}} &= \gamma_{i_1\cdots i_{2n}} \\ \Gamma_{ri_1\cdots i_{2n+1}} &= -\gamma_{i_1\cdots i_{2n+1}} \ . \end{aligned}$$

We use the pairing (B.1a). Then the spinnor pairing coincides with the one on spinors of $C\ell(7,0)$

$$\langle -,-\rangle_{(8,0)} = \langle -,-\rangle_{(7,0)} \ .$$

For chiral spinors of definite chirality $-[r]$ and an odd form $c \in \bigwedge^{\mathrm{odd}} \mathbb{R}^8$, we have

$$\begin{aligned} \langle \varepsilon, c\,\varepsilon' \rangle &= -[r]\langle \mathrm{dvol}(8,0)\,\varepsilon, c\,\varepsilon' \rangle \\ &= -[r]\langle \varepsilon, \mathrm{dvol}(8,0)\,c\,\varepsilon' \rangle \\ &= -\langle \varepsilon, c\varepsilon \rangle \\ &= 0 \end{aligned}$$

and the square of Δ_S into $\bigwedge^* \mathbb{R}^8$ is

$$\begin{aligned} S^2\Delta_S &= \textstyle\bigwedge^0 \mathbb{R}^8 \oplus \bigwedge^{4+} \mathbb{R}^8 \\ \textstyle\bigwedge^2 \Delta_S &= \textstyle\bigwedge^2 \mathbb{R}^8 \ . \end{aligned}$$

B.0.6 Cl(1,10)

The algebra is isomorphic to

$$C\ell(1,10) = \mathrm{Mat}(32,\mathbb{R}) \oplus \mathrm{Mat}(32,\mathbb{R})$$

and the two real Clifford irreducible representations are differentiated by the sign of the volume element. We use a real representation where the sign is $[r]$. It acts on spinors in the module $\Delta \cong \mathbb{R}^{32}$.

The representation $\Gamma^A = \{\Gamma^a, \Gamma^i\}$ can be built from a representation of gamma matrices for the Clifford algebras $C\ell(1,3)$ and $C\ell(7,0)$

$$\begin{aligned} \Gamma^\mu &= \mathrm{dvol}(1,3)\gamma^\mu \otimes \mathbb{1} \\ \Gamma^i &= \mathrm{dvol}(1,3) \otimes \gamma^i \end{aligned} \tag{B.2}$$

and $\Delta = \Delta_A \otimes \Delta_S$.

We use the pairing (B.1b). Then the spinnor pairing is

$$\langle -, - \rangle|_\Delta = -\langle -, \mathrm{dvol} - \rangle|_{\Delta_S} \times \langle -, - \rangle|_{\Delta_S} \ .$$

The symmetric square of spinors in Δ into $\bigwedge^* \mathbb{R}^{1,10}$ is

$$S^2\Delta = \bigwedge^1 \mathbb{R}^{1,10} \oplus \bigwedge^2 \mathbb{R}^{1,10} \oplus \bigwedge^5 \mathbb{R}^{1,10} \ .$$

Appendix C

The pp-wave Killing Superalgebra

We shall reproduce the Killing superalgebra of the maximally supersymmetric Kowalski-Glikman wave of M-theory [161, 162, 69, 108]. We use the conventions of chapter 2 with $\lambda = 1$. Flat indices $X^{\underset{\sim}{\mu}}$ will be denoted by an under-tilde, while gamma matrices Γ^{μ} will always have flat indices without the under-tilde required. This appendix serves as an example of a Killing superalgebra. We also use the form of the Killing superalgebra in section §5.1, in order to argue that it does not admit a full extension.

The Kowalski-Glikman wave describes a Calan-Wallach space, which is a lorentzian symmetric space, with metric

$$g = dx^{+}dx^{-} + \frac{1}{2}A_{ij}x^{i}x^{j}(dx^{-})^{2} - \delta_{ij}dx^{i}dx^{j} \ , \quad i = 1 \ldots 9$$

and constant flux

$$F = \mu dx^{-} \wedge \Theta \ , \quad \Theta \in \bigwedge\nolimits^{3} \mathbb{R} \ .$$

The curvature two form is found to be $R^{\nabla}_{\underset{\sim}{-i}} = -A_{ij}dx^{j} \wedge dx^{i}$ and 0 otherwise and so the Einstein equation of motion is satisfied provided[1] that

$$\mathrm{Ric}_{\underset{\sim}{-}\underset{\sim}{-}} = \sum_i A_{ii} = -2\mu^2 \Theta^{ijk}\Theta_{ijk} \ .$$

The solution generically admits 16 'basic' Killing spinors. It is maximally supersymmetric, up to SO(9), for $A_{ij} = \delta_{ij} a_i$ and $\Theta = dx^{123}$, where

$$a_i = \begin{cases} \frac{4}{9}\mu^2 & i \leq 3 \\ \frac{4}{36}\mu^2 & i > 3 \ . \end{cases}$$

For later convenience we define $w_i = \sqrt{a_i}$ and

$$\lambda_i = \begin{cases} -\frac{1}{3}\mu & i \leq 3 \\ \frac{1}{6}\mu & i > 3 \ . \end{cases}$$

We immediately compute the nontrivial spin connection $\omega_{\underset{\sim}{-}\underset{\sim}{i}} = A_{ij} x^j dx^-$ and the nontrivial Christoffel symbols

$$\begin{aligned} \nabla_i \partial_i = \nabla_i \partial_- &= A_{ij} x^j \partial_+ \\ \nabla_- \partial_- &= \sum_i A_{ij} x^j \partial_i \ . \end{aligned}$$

The gravitino connection is given by

$$\Omega_\mu = \begin{cases} 0 & \mu = 0 \\ -\frac{\mu}{6}(1 - \Gamma^{-+})I & \mu = - \\ \lambda_i \Gamma^- \Gamma_i I & \mu = i \ , \end{cases}$$

where $I = \Gamma^{123}$ and so $I^2 = -1$.

We will first find the Killing vectors and their algebra, then the Killing spinors and finally find the remaining brackets of the Killing superalgebra, which we denote as usual by $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$. For the bosonic subalgebra, we express the metric space as the coset space G/H, where the Lie algebra of G is an extended

[1] note we use a negative signature for the flat directions $i = 1 \ldots 9$.

Heisenberg algebra $\mathfrak{g} = \text{span}\{e_+, e_-, e_i, e_i^*\}$, with nontrivial brackets

$$[e_, e_i] = e_i^* \quad [e_-, e_i^*] = -a_i e_i \quad [e_i, e_i^*] = -a_i e_+ \tag{C.1}$$

and the Lie algebra of H, denoted by $\mathfrak{h}$, is spanned by e_i^*. Indeed, let us express a coset representative as

$$\sigma(x) = \exp(x^+ e_+) \exp(x^- e_i) \exp(x^i e_i) \ .$$

Then the pull-back of the Maurer-Cartan form is

$$\sigma^- d\sigma = e_+ dx^+ + dx^-(e_+ x^i e_i^* + A_{ij} x^i x^j e_+) + dx^i e_i \ .$$

The Lie algebra split Lie$(G) = \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$, with $\mathfrak{p} = \text{span}\{e_+, e_-, e_i\}$, is symmetric [163] and the $\mathfrak{h}$-invariant metric B, with $B(e_+, e_-) = 1$ and $B(e_i, e_j) = -\delta_{ij}$, gives the metric of the Kowalski-Glikman wave. That is, we identify the tangent space of G/H at the origin with $\mathfrak{p}$ and the group G acts on the left of σ as isometries of the metric. This gives rise to a Lie algebra anti-isomorphism of $\mathfrak{g}$ as the Killing vectors of the isometry flows. If for instance $X \in \mathfrak{g}$, then we can find the Killing vector ξ_X that is isomorphic to X, by solving

$$\sigma^- X \sigma = -\sigma^- d\sigma(\xi_X) + \mathfrak{h} \ .$$

We find the following Killing vectors

$$\begin{aligned}
\xi_{e_-} &= -\partial_- \\
\xi_{e_+} &= -\partial_+ \\
\xi_{e_i} &= -\cos(w_i x^-)\partial_i + x^i w_i \sin(w_i x^-)\partial_+ \\
\xi_{e_+^*} &= -w_i \sin(w_i x^-)\partial_i - a_i x^i cos(w_i x^-)\partial_+ \ .
\end{aligned}$$

In many of these calculations we use the Campbell-Baker-Hausdorff formula in the form

$$\exp(-Y)X\exp(Y) = \exp([X,-])Y \ .$$

These Killing vectors reproduce the Lie algebra of G, equation (C.1). Furthermore, the normalizer of H gives a consistent right action that is an isometry. The remaining isometries are manifest, they are the rotations in the first three and last six coordinates x^i. That is,

$$\xi_{M_{ij}} = x^i\partial_j - x^j\partial_i \ , \quad \text{for } 1 < i,j \leq 3 \text{ or } 4 \leq i,j \geq 9$$

and they rotate canonically the ξ_{e_i} and $\xi_{e_i^*}$. The even part of the Killing algebra is thus $\mathfrak{k}_0 = (\mathfrak{so}(3)\times\mathfrak{so}(6)) \ltimes \mathfrak{g}$.

The Killing spinors satisfy $(\nabla_\mu + \Omega_\mu)\varepsilon = 0$ and this reduces to

$$\begin{aligned}
&\partial_+\varepsilon = 0\\
&\left(\partial_i + \lambda_i\Gamma^-\Gamma_i I\right)\varepsilon = 0\\
&\left(\partial_- - \frac{1}{2}a_i x^i\Gamma^-\Gamma^i - \frac{\mu}{6}\left(1-\Gamma^-\Gamma^+\right)I\right)\varepsilon = 0 \ ,
\end{aligned}$$

from which we find the solution

$$\varepsilon = \exp\left(\frac{\mu}{2}Ix^-\right)\varepsilon_- + \left(1 - x_i\lambda_i\Gamma^-\Gamma_i I\right)\exp\left(\frac{\mu}{6}Ix^-\right)\varepsilon_+$$

where $\Gamma^\pm\varepsilon_\pm = 0$. After some calculations we find the odd-odd bracket. Two Killing spinors $\varepsilon, \varepsilon' \in \mathfrak{k}_1$ square to $V \in \mathfrak{k}_0$ with

$$\begin{aligned}
V = &-\overline{\varepsilon}_+\Gamma^-\varepsilon'_+\xi_{e_-} - \overline{\varepsilon}_-\Gamma^+\varepsilon'_-\xi_{e_+} + \sum_{\substack{i,j\leq 3\\ i,j>3}} \overline{\varepsilon}_+\lambda_j I\Gamma_j{}^i\Gamma^-\varepsilon'_+\xi_{M_{ji}}\\
&- \left(\overline{\varepsilon}_-\Gamma^i\varepsilon'_+ + \overline{\varepsilon}_+\Gamma^i\varepsilon'_-\right)\xi_{e_i} - \frac{1}{w_i}\left(\overline{\varepsilon}_- I\Gamma^i\varepsilon'_+ + \overline{\varepsilon}_-\Gamma^i I\varepsilon'_-\right)\xi_{e_i^*} \ .
\end{aligned} \tag{C.2}$$

We will identify the Killing spinors with their value at the origin and denote the corresponding superalgebra elements by $Q = Q_+ + Q_-$. The even-even bracket is that of (C.1), along with the canonical action of $\mathfrak{so}(3)\times\mathfrak{so}(6)$. The odd-odd bracket is that of (C.2). Many cancelations occur so that the brackets close.

For instance, we frequently use the identity $w_i I\Gamma^i = -2\lambda_i \Gamma^i I$, with no sum over i. The same is true for the even odd part. The Killing vector acts as the Lie derivative on Killing spinors. Using this, we find the remaining brackets

$$\begin{aligned}
[e_+, Q] &= 0 \\
[e_-, Q_-] &= -\frac{\mu}{2} I Q_- \\
[e_-, Q_+] &= -\frac{\mu}{6} I Q_+ \\
[e_i, Q_+] &= \lambda_i \Gamma_i I (\Gamma^- Q_+) \\
[e_i^*, Q_+] &= -\frac{a_i}{2} \Gamma_i (\Gamma^- Q_+) \ ,
\end{aligned} \tag{C.3}$$

while the rotations M_{ij} act canonically as a subgroup of $\mathfrak{spin}(1, 10)$. We note that the element e_+ is central in the superalgebra, whereas no elements of $\mathfrak{k}_1$ are left invariant by the whole of $\mathfrak{k}_0$.

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