Fano threefolds and algebraic families of surfaces of Kodaira dimension zero

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Contents

Chapter 1

Introduction

1.1 Historical background

At the end of the 19-th century a considerable progress in the classification of algebraic varieties was made. The breaking through idea in the study of the geometry of a given (normal projective) algebraic variety X was to consider (complete) *linear systems* of (Weil) divisors on X. Namely, to any divisor D on X, which is just the formal finite integral sum of irreducible hypersurfaces on X, $D := \sum d_i D_i$, one can associate a finite-dimensional linear space $|D|$, which consists of all rational functions on X with poles in the support of D. If $|D| \neq \emptyset$ and the common zero locus $\mathrm{Bs}(|D|)$ of functions $f_1, \ldots, f_N$, which form a basis of $|D|$, is of codimension $\geqslant 2$ on X, then one can define a rational map $\Phi_{|D|} : X \dashrightarrow \mathbb{P}^N$ into projective space $\mathbb{P}^N$, mapping any point x in X to the point $[f_1(x) : \ldots : f_N(x)]$ in $\mathbb{P}^N$. The map $\Phi_{|D|}$ is not defined exactly at the locus $\mathrm{Bs}(|D|)$, which does not depend on the choice of f_i's and got the name of the *base locus* of the linear system $|D|$. Conversely, any rational map of X into any projective space leads to a linear system $|D|$ for some divisor D

on X, with $\mathrm{Bs}(|D|) \subset X$ of codimension $\geqslant 2$.

The above simple and convenient way of looking on rational maps of algebraic varieties into projective spaces had been developed in the framework of Italian school of Algebraic Geometry. The application was made towards the (partial) solution of one of the fundamental problems in Algebraic Geometry, namely, the *problem of biregular classification of smooth projective algebraic varieties* over the field $\mathbb{C}$ of complex numbers. Roughly, the main idea was that to any (smooth projective) algebraic variety X one can associate the *canonical divisor* K_X on X, which is just the divisor of zeroes of the top exterior power $\bigwedge^{\dim X} \Omega^1_X$ of the cotangent bundle Ω^1_X on X. The corresponding linear system $|K_X|$, or, more generally, the multiple linear systems $|nK_X|$, $n \in \mathbb{N}$, provide a source of *birational invariants* of X, such as the dimension $h^0(X, K_X) := \dim |K_X|$ of $|K_X|$ (or, more generally, $h^0(X, nK_X) := \dim |nK_X|$), the maximal dimension $\kappa(X) := \max_{n\in\mathbb{N}} \dim \Phi_{|nK_X|}(X)$ of images $\Phi_{|nK_X|}(X)$ of X under the rational maps $\Phi_{|nK_X|}$ (the *Kodaira dimension* of X), and etc. Namely, for any birational isomorphism (or *birational map*) $X \dashrightarrow Y$, i.e., the invertible rational map between (smooth projective) algebraic varieties X and Y (cf. the above construction of $X \dashrightarrow \mathbb{P}^N$), the corresponding numbers $h^0(X, K_X)$ and $h^0(Y, K_Y)$, $\kappa(X)$ and $\kappa(Y)$, etc., are the same for both X and Y. Further, one divides the class of all (smooth projective) algebraic varieties of a given dimension into the subclasses, via separating these varieties by values of their birational invariants, and then tries to establish the classification in each of the subclasses.

The above approach provided a satisfactory classification of smooth projective algebraic curves and surfaces over $\mathbb{C}$ (see [36], [75], [8], [25], [86], [88]). On this way, among the others, another fundamental problem appeared, namely, the so called *rationality problem* for algebraic varieties. More precisely, given such a variety X, one asks for criteria of existence of a birational map, say $\Psi : \mathbb{P}^{\dim X} \dashrightarrow X$, or, in the other words, for criteria of *rationality* of X.

Remark 1.1.1. From the point of view of the classification of algebraic varieties it is more natural to consider another, more up to date problem, namely, the *problem of characterization of the projective space*, when one asks for criteria of existence of a biregular isomorphism $X \simeq \mathbb{P}^{\dim X}$. The solution to this problem can be extracted from the papers [33], [48], [54], [65].

In the case of smooth projective curves, it is not difficult to see that X is rational iff $h^0(X, nK_X) = 0$ for all n (then one says that the Kodaira dimension of X is negative). On the other hand, the situation in dimensions $\geqslant 2$ is much more complicated. However, in the case of surfaces there is a *Castelnuovo criterion*, stating that given smooth projective surface X is rational whenever the Kodaira dimension of X is negative and the *irregularity* $q(X) := \dim H^0(X, \Omega^1_X)$ of X is zero (these two conditions are, of course, necessary for X to be rational). Since the latter criterion had been established, it was tempting to find similar criteria in terms of vanishing of some (discrete) birational invariants to characterize rational varieties in dimensions $\geqslant 3$. This led to the fruitful part of the so called *Lüroth Problem.* Namely, one can generalize the rationality problem to its *unirational* counterpart, replacing "birational map Ψ" with "dominant map Ψ" in the above considerations. Note that unirationality of X is easily seen to be equivalent to the inclusion $\mathbb{C}(X) \subset \mathbb{C}(\mathbb{P}^{\dim \mathbb{X}})$ of fields of rational functions. The latter implies that $\kappa(X) < 0$ and $q(X) = 0$ for unirational X. In particular, in dimensions 1 and 2, as we have seen, the class of unirational (smooth projective) algebraic varieties coincides with the class of rational ones. But in higher dimensions we arrive to

Question 1.1.2 (Lüroth Problem). *Let X be a (smooth projective) algebraic variety (over $\mathbb{C}$) with $\dim X \geqslant 3$. Suppose that X is unirational. Is X rational?*

The approach to answer Question 1.1.2 was developed by G. Fano, one of the brightest representatives of the Italian school of Algebraic Geometry. In the series of papers [26], [27], [28], G. Fano studied normal projective varieties over $\mathbb{C}$, which are close to rational, namely, those with the ample anticanonical divisor. Such varieties were called later *Fano varieties* (cf. Definition 2.3.14). Again, as follows from the above considerations, Fano varieties of dimensions 1 and 2 are rational, since, obviously, for a given such X we have $\kappa(X) < 0$ and $q(X) = 0$. However, as G. Fano had shown, already in dimension 3 there are non-rational Fano varieties. More precisely, G. Fano considered a smooth projective quartic threefold $X \subset \mathbb{P}^4$, which is a Fano variety, since K_X is the class of the minus hyperplane section of X due to the adjunction. In [27], [28], G. Fano proved that every such X is non-rational. On the other hand, unirationality of some particular X was shown in the paper [87]:

Example 1.1.3 (Segre's Example). The smooth projective quartic threefold

$$\big(x_0^4 + x_0x_4^3 + x_1^4 - 6x_1^2x_2^2 + x_2^4 + x_3^4 + x_3^3x_4 = 0\big) \subset \mathbb{P}^4 = \mathrm{Proj}\big(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]\big)$$

is unirational.

Thus, Question 1.1.2 gets the negative answer. Yet, papers [27], [28] contained some inaccurate and even incorrect statements (see also [85]), the main ideas of G. Fano were later recovered by V. A. Iskovskikh and Yu. I. Manin in the paper [44], where the precise proof of the non-rationality of every smooth projective quartic threefold was obtained:

Theorem 1.1.4. *For every smooth projective quartic threefold $X \subset \mathbb{P}^4$, the group* $\mathrm{Bir}(X)$ *of birational automorphisms of X coincides with the group* $\mathrm{Aut}(X)$ *of regular (projective) automorphisms of X. In particular, X is non-rational.*

The crucial observation was that there exists a (not necessary complete) linear system of divisors, say $\mathcal{D}$ on X, with high multiplicity at some loci, provided there is a non-regular birational automorphism $\sigma \in \mathrm{Bir}(X)$. More precisely, for any divisor D on X one has $D \in |-nK_X|$ for some $n = n(D)$ by the Lefschetz theorem. Then the stated condition for $\mathcal{D}$, called the *Nöether–Fano inequality*, is that the *pair* $(X, \frac{1}{n}D)$ *is not canonical* for a general $D \in \mathcal{D}$ (see Definition 2.2.7). After that, using the intersection theory on X, one gets a contradiction, hence showing that in fact $\sigma \in \mathrm{Aut}(X)$.

Remark 1.1.5. The Nöether–Fano inequality method of proving non-rationality of Fano varieties got further development. For example, non-rationality of each of the 95 general quasi-smooth *terminal* (see Definition 2.2.7) hypersurfaces in weighted projective spaces $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ was proved in [18], and the group of birational automorphisms of every such hypersurface was described. Further, non-rationality of some other smooth Fano threefolds was proved by V. A. Iskovskikh and A. V. Pukhlikov in [46], and non-rationality of a general smooth projective hypersurface of degree N in $\mathbb{P}^N$, $N \geqslant 5$, was proved by A. V. Pukhlikov in [80].

It turns out, however, that the class of Fano varieties is not only the source of counterexamples to the Lüroth Problem. Namely, in the second half of the 20th century, after the powerful machinery of the *Theory of Minimal Models* (or *Minimal Model Program, or MMP*) due to S. Mori, Y. Kawamata, J. Kollár, M. Reid, V. V. Shokurov and others has been developed (see [58], [64], [66], [90] and Section 2.3), it was realized that Fano varieties serve as the natural building blocks for algebraic varieties of negative Kodaira dimension, one of the two possible outcomes produced by the MMP. On this way, let us mention the following generalization of Theorem 1.1.4, made by A. Corti:

Theorem 1.1.6 (see [17]). *A smooth projective quartic threefold in* $\mathbb{P}^4$ *is not birationally isomorphic to any fibration by surfaces of negative Kodaira dimension.*

In general, however, a given algebraic variety of negative Kodaira dimension may have non-trivial birational structure of fibration by Fano varieties. Thus, from the point of view of classification of algebraic varieties, it makes it very important to have a biregular classification of Fano varieties. The smooth case of this problem was already considered by Italian algebraic geometers. The two-dimensional case, which is the first non-trivial case by the above considerations, was completely settled down (see [20]), leading to the well-known description of *del Pezzo surfaces.* Three-dimensional case was treated by G. Fano in [27], [28], [29], yet the complete classification of smooth Fano threefolds was carried out only a quoter-century passed in papers of V. A. Iskovskikh, S. Mori and S. Mukai (see [42], [43], [67]), where the original ideas of G. Fano were improved and the powerful methods of the MMP applied.

On the other hand, varieties occurring in the MMP possess some mild singularities, say, *terminal* $\mathbb{Q}$*-factorial* (see Section 2.2), which leads one to the problem of classification of singular Fano varieties. Even more, the latter problem is naturally generalized to the classification problem of Fano varieties with, in some sense, the worst possible class of singularities, namely, *canonical Gorenstein* singularities (see Section 2.2). For instance, every $\mathbb{Q}$-*Fano* threefold, i.e., a threefold with only terminal $\mathbb{Q}$-factorial singularities and Picard number 1 (see Section 2.3), for which the anticanonical image is three-dimensional, is birational to some Fano threefold with canonical Gorenstein singularities (see [1]).

On the other hand, a natural complement to the class of smooth Fano threefolds as varieties which contain a smooth K3 surface as an ample divisor, is the class of three-dimensional normal algebraic varieties which contain a smooth Enriques surface as an ample divisor. If one requires further that the latter threefolds are not the cones, then we arrive at *Fano–Enriques threefolds*. As G. Fano showed, these varieties are always singular (see [30] and also [16]). At the same time, every Fano–Enriques threefold possesses a double cover by a Fano threefold with canonical Gorenstein singularities.

Smooth Fano threefolds and Fano–Enriques threefolds are, in some sense, the general representatives of the class of three-dimensional algebraic varieties with ample anticanonical divisor and canonical singularities (see [50, Remark 1.10]). However, as the case of smooth Fano threefolds shows, the classification of Fano threefolds with canonical Gorenstein singularities is a hard and interesting problem (see [50], [51], [79], [78]).

In the present thesis, we apply what is known about the geometry of the above mentioned Fano threefolds to study algebraic families of surfaces of Kodaira dimension zero which correspond to a given Fano threefold X in the natural way, such as linear subsystems in $|-K_X|$ and etc. (see Sections 1.2 and 1.3 for exposition).

1.2 Formulation of the main results

The main results of the thesis are contained in the papers [10] and [49]. we formulate the main results obtained.

The first problem deals with the smooth projective quartic threefold X over $\mathbb{C}$ in $\mathbb{P}^4$ and is related to Theorem 1.1.6. Namely, the natural step further from the result of Theorem 1.1.6 is the following

Problem 1.2.1. Does X possess any birational structure of fibration by surfaces of Kodaira dimension zero? Classify all such structures up to birational equivalence.

The first part of Problem 1.2.1 asks if there exists a rational map $f : X \dashrightarrow Z$ with Z being a curve and the general fibre of f being a surface birationally isomorphic to a smooth surface of Kodaira dimension zero. Equivalently, stated in terms of linear systems (see Section 1.1), the first part of Problem 1.2.1 asks of there exists a one-dimensional linear system (a *pencil*) $\mathcal{H}$ on X whose general element is an irreducible surface birationally isomorphic to a smooth surface of Kodaira dimension zero. Such $\mathcal{H}$, if it exists, is called a *Halphen pencil* (see Definition 3.1.1).

Example 1.2.1. Any one-dimensional linear system in the anticanonical linear system $|-K_X|$ is a Halphen pencil by Bertini theorem and the adjunction formula.

Example 1.2.1 gives a positive answer to the first part of the Problem 1.2.1.

Further, given two Halphen pencils $\mathcal{H}_1$ and $\mathcal{H}_2$ on X, we say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are *equivalent* if there exists a birational map $\chi : X \dashrightarrow X$ such that, say $\mathcal{H}_1$ is the proper transform of $\mathcal{H}_2$, i.e., $\mathcal{H}_1 = \chi_*^{-1}(\mathcal{H}_2)$ (see Section 2.1 for the notation). Then, in connection with the second part of Problem 1.2.1, one may ask if every Halphen pencil on X is equivalent to that from Example 1.2.1.

Remark 1.2.2. Similarly, in view of Theorem 1.1.6 and Problem 1.2.1, one may study birational structures on X of fibrations by elliptic curves. This has been completely settled down in [12].

The description of Halphen pencils on X under the above equivalence relation turns out to be quite natural from the view point of the proof of Theorem 1.1.4. More precisely, once there exists a Halphen pencil $\mathcal{H}$ on X, one can prove (see Section 3.1) that the pair $(X, \frac{1}{n}H)$ is canonical but not terminal for a general $H \in \mathcal{H} \subset |-nK_X|$ (cf. the arguments after Theorem 1.1.4). This is a "degenerate" form of the Nöether–Fano inequality (cf. Section 1.1). The following example illustrates this phenomenon:

Example 1.2.3 (Bertini, Dolgachev). Let C be a smooth elliptic curve over $\mathbb{C}$ in $\mathbb{P}^2 = \mathrm{Proj}(\mathbb{C}[x, y, z])$ defined by the equation $f(x, y, z) = 0$ for a cubic homogeneous polynomial $f(x, y, z)$. Let $P_1, \ldots, P_9$ be nine distinct points on C such that the divisor

$$\sum_{i=1}^{9} P_i - \mathcal{O}_{\mathbb{P}^2}(3)\big|_C$$

is m-torsion for some $m \in \mathbb{N}$. Then there exists a curve $Z \subset \mathbb{P}^2$ of degree $3m$ and multiplicity m at each point P_i. Consider the pencil $\mathcal{P}^m$ of curves on $\mathbb{P}^2$ with a general element $P^m \in \mathcal{P}^m$ given by the equation

$$\lambda f^m + \mu g = 0,$$

where $g = 0$ is the equation of Z, $\lambda, \mu \in \mathbb{C}$. Then P^m is birationally isomorphic to an elliptic curve. $\mathcal{P}^m$ is called the *m-standard Halphen pencil*. One may pose the problem of description of Halphen pencils for $\mathbb{P}^2$, similar to Problem 1.2.1 for X, running the same arguments as above. It turns out that every Halphen pencil on $\mathbb{P}^2$ is equivalent to the m-standard one for some m (see [7], [21]).

The proof of the result mentioned in Example 1.2.3 heavily relied on the description of the structure of the group $\mathrm{Bir}(\mathbb{P}^2)$ (see [21]). In the case of the quartic X, we have even more restrictions, namely, we have $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ (see Theorem 1.1.4), which moves Problem 1.2.1, roughly, to the description of hypersurfaces in $\mathbb{P}^4$ whose restriction to X has high multiplicity along some locus. On this way, one gets the following

Theorem 1.2.4 ([14], [9]). *Suppose that the quartic X is general. Then every Halphen pencil on X is cut out by hyperplanes*

$$\lambda l_1\big(x, y, z, t, w\big) + \mu l_2\big(x, y, z, t, w\big) = 0 \subset \mathbb{P}^4 = \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t, w]\Big)$$

for some linear forms l_1, l_2 and $\lambda, \mu \in \mathbb{C}$.

Remark 1.2.5. In connection with Theorem 1.2.4, a similar result was obtained in [14] for each of the 95 general quasi-smooth terminal hypersurfaces in weighted projective spaces $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ (cf. Remark 1.1.5).

The assertion of Theorem 1.2.4 is erroneously proved in [9] without the assumption that the threefold X is general. The following example was constructed in [41]:

Example 1.2.6 ([41]). Suppose that X is given by the equation

$$w^3x + w^2q_2\big(x, y, z, t\big) + wxp_2\big(x, y, z, t\big) + q_4\big(x, y, z, t\big) = 0$$

in $\mathbb{P}^4 = \mathrm{Proj}\Big(\mathbb{C}[x,y,z,t,w]\Big)$ for some forms q_i, p_i of degree i. Let $\mathcal{H}_2$ be the pencil on X which is cut out by hypersurfaces

$$\lambda x^2 + \mu\Big(wx + q_2\big(x,y,z,t\big)\Big) = 0,$$

where λ, $\mu \in \mathbb{C}$. Then $\mathcal{H}_2$ is a Halphen pencil if $q_2(0,y,z,t) \neq 0$ by [14, Theorem 2.3]. More precisely, a general $H \in \mathcal{H}_2$ is birationally isomorphic to a smooth K3 surface.

Let us now state our first main result:

Theorem 1.2.7. *Let $\mathcal{H}$ be a Halphen pencil on X. Then one of the following holds:*

- *either $\mathcal{H}$ is cut out on X by hyperplanes*

  $$\lambda l_1\big(x,y,z,t,w\big) + \mu l_2\big(x,y,z,t,w\big) = 0 \subset \mathbb{P}^4 = \mathrm{Proj}\Big(\mathbb{C}[x,y,z,t,w]\Big)$$

  for some linear forms l_1, l_2 and $\lambda, \mu \in \mathbb{C}$,

- *or the threefold X can be given by the equation*

  $$w^3x + w^2q_2\big(x,y,z,t\big) + wxp_2\big(x,y,z,t\big) + q_4\big(x,y,z,t\big) = 0$$

  in $\mathbb{P}^4 = \mathrm{Proj}\Big(\mathbb{C}[x,y,z,t,w]\Big)$ such that $q_2(0,y,z,t) \neq 0$, and $\mathcal{H}$ is cut out on the threefold X by hypersurfaces

  $$\lambda x^2 + \mu\Big(wx + q_2\big(x,y,z,t\big)\Big) = 0$$

  for some forms q_i, p_i of degree i and $\lambda, \mu \in \mathbb{C}$.

Theorem 1.2.7 provides the complete solution to Problem 1.2.1 (see also Section 3.7 for discussion of the related problems).

The second problem, treated in the present thesis, deals with smooth *primitively polarized* K3 surfaces. Recall that every such a surface S possesses an ample line bundle L such that the class of L in the Picard group $\mathrm{Pic}(S)$ of S is not divisible. The number $g := (L^2)/2 + 1$ is called the *genus* of S (see Section 2.1 for the notation).

Let $\mathcal{K}_g$ be the moduli space of smooth primitively polarized K3 surfaces of genus g. $\mathcal{K}_g$ is known to be a quasi-projective algebraic variety (see [5], [92]). This makes it possible to study the questions about rationality, unirationality, rational connectedness, value of the Kodaira dimension, and etc., for $\mathcal{K}_g$ (see [31], [37], [38], [91] for the related questions about the moduli spaces of curves).

S. Mukai's vector bundle method, developed to classify the higher dimensional Fano manifolds of Picard number 1 and coindex 3 (see [68], [71]), allowed to prove the following

Theorem 1.2.8 (see [70], [73], [69], [74]). *The moduli space $\mathcal{K}_g$ is unirational for $g \in \{2, \ldots, 10, 12, 13, 18, 20\}$.*

Remark 1.2.9. At the same time, on the opposite to Theorem 1.2.8, $\mathcal{K}_g$ has non-negative Kodaira dimension for general $g \geqslant 43$ (see [35], [61], [62]).

In principle, the proof of unirationality of $\mathcal{K}_g$ is based on the observation that a general K3 surface S_g with primitive polarization L_g and "not very big" g is an anticanonical section of a smooth Fano 3-fold X_g of genus g so that $L_g = -K_{X_g}\big|_{S_g}$ (see [70], [69], [72]). The latter gives a rational dominant map from the moduli space $\mathcal{F}_g$ of pairs (X_g, S_g), where $S_g \in |-K_{X_g}|$ is smooth, to $\mathcal{K}_g$ by sending (X_g, S_g) to S_g, with $\mathcal{F}_g$ typically being a rational algebraic variety. However, this construction has the restriction that X_g must have Picard number 1, which does not hold for most g (see [67], [42], [43], [45]).

In order to generalize the above arguments for every possible g, to a given smooth Fano 3-fold V of genus g one associates the Picard lattice $R_V :=$ Pic(V), equipped with the pairing $(D_1, D_2) := D_1 \cdot D_2 \cdot (-K_V)$ for D_1, $D_2 \in$ Pic(V), and considers the moduli space $\mathcal{K}_g^{R_V}$ of all smooth K3 surfaces S_g, equipped with a primitive embedding $R_V \hookrightarrow \mathrm{Pic}(S_g)$ which maps $-K_V$ to an ample class on S_g of square g (let us call such S_g a K3 *surface of type* R_V). A beautiful result due to A. Beauville states that a general K3 surface of type R_V is the anticanonical section of a smooth Fano 3-fold X_g of genus g such that $R_{X_g} \simeq R_V$ (see [6]). The proof employs the same idea as above, but instead of $\mathcal{F}_g$ the moduli space $\mathcal{F}_g^{R_V}$ of pairs (X_g, S_g), where $S_g \in |-K_{X_g}|$ is smooth and X_g is equipped with the lattice isomorphism $R_{X_g} \simeq R_V$, is considered. Again the forgetful map $(X_g, S_g) \mapsto S_g$ from $\mathcal{F}_g^{R_V}$ to $\mathcal{K}_g^{R_V}$ turns out to be generically surjective. However, these arguments can be applied only to some $g \leqslant 33$ (see [67], [42], [43], [45]).

The above arguments lead to the following

Question 1.2.10. *Is* $\mathcal{K}_{33}$ *(respectively,* $\mathcal{K}_{36}$*) unirational (cf. Theorem 1.2.8 and Remark 1.2.9)?*

To develop an approach to answer Question 1.2.10, let us take, say $\mathcal{K}_{36}$ and try to prove that it is unirational. For the latter, we employ the above ideas to realize a general smooth primitively polarized K3 surface of genus 36 as an anticanonical section of some Fano 3-fold, which must be singular in this case (see [67], [42], [43], [45]). The natural candidate for the latter is the Fano 3-fold X of *genus* 36 (see Definition 2.3.14 and Remark 2.3.15 for the notation), constructed and studied in [50], [51]. This X has only one singular point (see Corollary 4.2.8) and the anticanonical linear system $|-K_X|$ gives an embedding $X \hookrightarrow \mathbb{P}^{37}$ (see Remark 4.2.10), which implies that a general surface $S \in |-K_X|$ is smooth. Also the Picard group of X is generated by K_X (see Corollary 4.2.9).

Unfortunately, the divisor class group of X has two generators, K_X and some surface E (see Corollary 4.2.9), so that the restrictions $K_X\big|_S$ and $E\big|_S$ generate a primite sublattice R_S in $\mathrm{Pic}(S)$. In particular, the Picard number of S must be at least 2, and hence S can not be general. However, all lattices R_S, $S \in |-K_X|$, are isomorphic to the lattice $\mathrm{R} \simeq \mathbb{Z}^2$ with the associated quadratic form $70x^2 + 4xy - 2y^2$ (see the end of Section 4.2), and, as above, we can consider the moduli space $\mathcal{K}_{36}^{\mathrm{R}}$ of K3 surfaces of type R. On the other hand, we may also consider the moduli space $\mathcal{F}$ of pairs $(X^\sharp, S^\sharp)$, where $X^\sharp$ is a Fano 3-fold of genus 36 with canonical Gorenstein singularities and $S^\sharp \in |-K_{X^\sharp}|$ is smooth (see Remark 1.2.12 for the precise description of $\mathcal{F}$). On this way, we prove the following

Theorem 1.2.11. *The forgetful map $s : \mathcal{F} \longrightarrow \mathcal{K}_{36}^{\mathrm{R}}$ is generically surjective.*

Remark 1.2.12. In the proof of Theorem 1.2.11, we do not appeal to Akizuki–Nakano Vanishing Theorem, used in [6] to show that $\mathcal{F}_g$ (or $\mathcal{F}_g^{R_V}$) is a smooth stack, since it is not clear how to apply this theorem in the singular case. Instead, we note that X is unique up to an isomorphism (see Proposition 4.2.5), and, moreover, it admits a crepant resolution $f : Y \longrightarrow X$, with Y being also unique up to an isomorphism (see Proposition 4.2.6). Then one can prove (see Proposition 4.3.1) that $\mathcal{F}$ carries the structure of a normal scheme, being the geometric quotient $U/\mathrm{Aut}(Y)$ of an open subset U in $\mathbb{P}^{37}$ by the group $\mathrm{Aut}(Y)$ of regular automorphisms of Y. The proof of Theorem 1.2.11 then goes along the same lines as in [6] (see Lemma 4.3.7 below).

Remark 1.2.13. Taking $X = \mathbb{P}(1,1,1,3)$ in the above considerations, one might apply the arguments from [6] directly (cf. Remark 1.2.12) to prove that the moduli space $\mathcal{K}_{10}$ is unirational (see [10], [49] for geometric properties of $\mathbb{P}(1,1,1,3)$).

Theorem 1.2.11 (cf. Remark 1.2.12) gives only unirational hypersurface in $\mathcal{K}_{36}$ (see Corollary 4.4.1) but not the whole $\mathcal{K}_{36}$, and hence the answer to Question 1.2.10 is still to go (see, however, Section 4.4, where we provide some properties of a general smooth K3 surface of type R, which allows one to embed a general smooth K3 surface of genus 33 (respectively, of genus 36) into the Grassmanian $G(3,14)$ (respectively, $G(4,13)$) and makes one hope to proof unirationality of $\mathcal{K}_{33}$ (respectively, of $\mathcal{K}_{36}$), applying the methods from [70], [69], [72]).

1.3 Description of the thesis

The thesis consists of four chapters. First chapter is introductory. In Chapter 2, we recall some basic facts from the singularity theory of algebraic varieties (see Section 2.2) and the theory of minimal models (see Section 2.3), which will be used throughout the rest of the thesis. We also make some conventions on the notions and notation used in the thesis (see Section 2.1).

Each Chapter 3 and 4 starts with some preliminary results (see Sections 3.1 and 4.1, respectively). Each Chapter 3 and 4 ends with some corollaries and conclusive remarks (see Sections 3.7 and 4.4, respectively).

In Chapter 3, we prove Theorem 1.2.7, providing the complete description of Halphen pencils on a smooth projective quartic threefold X in $\mathbb{P}^4$. Let $\mathcal{M}$ be such a pencil. Firstly, we show that $\mathcal{M} \subset |-nK_X|$ for some $n \in \mathbb{N}$, and the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical but not terminal. Further, if the *set of non-terminal centers* $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ (see Remark 2.2.8) does not contain points, we show that $n = 1$ (see Section 3.2). Finally, if there is a point $P \in \mathbb{CS}(X, \frac{1}{n}\mathcal{M})$, in Section 3.1 we show first that a general $M \in \mathcal{M}$ has multiplicity $2n$ at P (cf. Example 1.2.3). After that, analyzing the shape of the Hessian of the equation of X at the point P, we prove that $n = 2$ and $\mathcal{M}$ coincides with the exceptional Halphen pencil from Example 1.2.6 (see Sections 3.3–3.6).

In Chapter 4, we prove Theorem 1.2.11, which shows, in particular, that a general smooth K3 surfaces of type R is an anticanonical section of the Fano threefold X with canonical Gorenstein singularities and genus 36. In Section 4.2, we prove that X is unique up to an isomorphism and has a unique singular point, providing the geometric quotient construction of the moduli space $\mathcal{F}$ in Section 4.3 (cf. Remark 1.2.12). Finally, in Section 4.3 we prove that the forgetful map $\mathcal{F} \longrightarrow \mathcal{K}_{36}^{\mathrm{R}}$ is generically surjective.

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Chapter 2

Preliminary part

2.1 Basic notions and notation

We use standard notions and facts from the theory of algebraic varieties and schemes (see [39], [34]). We also use standard notions and facts from the theory of minimal models and singularities of pairs (see [59], [53], [57]). All algebraic varieties are assumed to be defined over $\mathbb{C}$. Throughout the thesis we use standard notions and notation from [57], [59], [53], [34], [39]. However, let us introduce some:

- We denote by μ_n the cyclic group of order n.
- $\mathbb{Z}$ is the ring of integers, $\mathbb{N}$ is the set of natural numbers, $\mathbb{Q}$ and $\mathbb{R}$ are the fields of rational and real numbers, respectively. For $\mathbb{K} = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, $\mathbb{K}_{>0}$ (respectively, $\mathbb{K}_{\geqslant 0}$) is the set of all positive (respectively, nonnegative) elements in $\mathbb{K}$.
- $\alpha \gg 0$ stands for a sufficiently big number $\alpha \in \mathbb{R}$.
- We denote by $\llcorner \alpha \lrcorner$ the greatest integer not bigger than $\alpha \in \mathbb{R}$.

- We denote by $|\Sigma|$ the cardinality of the set Σ.

- For a $\mathbb{Q}$-Cartier divisor M (respectively, a linear system $\mathcal{M}$) and an algebraic cycle Z on a projective normal variety V, we denote by $M\big|_Z$ (respectively, $\mathcal{M}\big|_Z$) the restriction of M to Z. We denote by $Z_1 \cdot \ldots \cdot Z_k$ the intersection of algebraic cycles $Z_1, \ldots, Z_k$, $k \in \mathbb{N}$, in the Chow ring of V. We denote by $(Z_1, \ldots, Z_k)_P$ the local intersection of Z_i at a point $P \in V$.

- $M_1 \equiv M_2$ (respectively, $Z_1 \equiv Z_2$) stands for the numerical equivalence of two $\mathbb{Q}$-Cartier divisors M_1, M_2 (respectively, two algebraic 1-cycles Z_1, Z_2) on a projective normal algebraic variety V. We denote by $N_1(V)$ the group of classes of algebraic cycles on V modulo numerical equivalence. We denote by $\rho(V)$ the Picard number of V. We denote by $[Z]$ the class in $N_1(V)$ of a 1-cycle Z on V. A *Curve* means an *effective* 1*-cycle*.

- We denote by $\mathrm{mult}_Z(\Gamma)$ the multiplicity of an algebraic cycle Γ on an algebraic variety V along reduced cycle $Z \subset V$.

- $D_1 \sim D_2$ stands for the the linear equivalence of two Weil divisors D_1, D_2 on a projective normal algebraic variety V. We denote by $\mathrm{Pic}(V)$ (respectively, by $\mathrm{Cl}(V)$) the group of Cartier (respectively, Weil) divisors on V modulo linear equivalence. The elements from $\mathrm{Cl}(V) \otimes \mathbb{Q}$ are called $\mathbb{Q}$*-divisors*. We denote by $D_1 \sim_{\mathbb{Q}} D_2$ the $\mathbb{Q}$-linear equivalence of any two $\mathbb{Q}$-divisors D_1, D_2 on V. We set $\llcorner D \lrcorner := \sum \llcorner d_i \lrcorner D_i$ for any $\mathbb{Q}$-divisor $D := \sum d_i D_i$. We also denote by $\mathrm{Supp}(D)$ the support of D.

- For a Weil divisor D on a normal algebraic variety V, we denote by $\mathcal{O}_V(D)$ the corresponding divisorial sheaf on V (sometimes we denote both by $\mathcal{O}_V(D)$ (or by D)).

- For a coherent sheaf F on a projective normal variety V, we denote by $H^i(V, F)$ the i-th cohomology group of F. We set $h^i(V, F) := \dim H^i(V, F)$ and $\chi(V, F) := \sum_{i=1}^{\dim V}(-1)^i h^i(V, F)$. We also denote by $c_i(F)$ the i-th Chern class of F.

- For a vector bundle E on a smooth algebraic variety V, we denote by $\mathbb{P}_V(E)$ (or simply by $\mathbb{P}(E)$ if no confusion is likely) the associated projective bundle. We also set $\det E := c_1(E)$.

- We denote by $T_P(V)$ the Zariski tangent space to an algebraic variety V at a point $P \in V$. For V smooth and a smooth hypersurface $D \subset V$, we denote by $T_V\langle D\rangle$ the subsheaf of the tangent sheaf on V which consists of all vector fields tangent to D.

- For a Cartier divisor M on a projective normal variety V, we denote by $|M|$ the corresponding complete linear system on V. For an algebraic cycle Z on V, we denote by $|M - Z|$ the linear subsystem in $|M|$ consisting of all the divisors passing through Z. For a linear system $\mathcal{M}$ on V, we denote by $\mathrm{Bs}(\mathcal{M})$ the base locus of $\mathcal{M}$. If $\mathcal{M}$ does not have base components, we denote by $\Phi_{\mathcal{M}}$ the corresponding rational map.

- For a birational map $\psi : V' \dashrightarrow V$ between projective normal varieties and an algebraic cycle Z (respectively, a linear system $\mathcal{M}$) on V, we denote by $\psi_*^{-1}(Z)$ (respectively, by $\psi_*^{-1}(\mathcal{M})$) the proper transform of Z (respectively, of $\mathcal{M}$) on V'.

- We denote by $\kappa(V)$ the Kodaira dimension of a projective normal variety V.

- We denote by $\mathrm{Sing}(V)$ the singular locus of an algebraic variety V. We denote by $(O \in V)$ the analytic germ of a point $O \in \mathrm{Sing}(V)$. A birational morphism $f : W \longrightarrow V$ with smooth W is called a *resolution of singularities of* V. A birational morphism $f : W \longrightarrow V$ with smooth W is called a *log resolution of singularities of the pair* (V, D) for some $\mathbb{Q}$-divisor D on V if the components of the divisor $f_*^{-1}(D)$ and of the f-exceptional locus all have simple normal crossings.

- We denote by $\mathbb{F}_n$ the Hirzebruch surface with the class of a fiber l and the minimal section h of the natural projection $\mathbb{F}_n \to \mathbb{P}^1$ such that $(h^2) = -n$, $n \in \mathbb{Z}_{\geqslant 0}$.

2.2 Singularities of algebraic varieties

In the present section, we formulate several notions from singularity theory of algebraic varieties, which will be then used throughout the thesis. We also introduce several known facts about singularities of algebraic varieties (see [81], [84], [53], [57], [59], [58] for the proofs).

Definition 2.2.1. A normal algebraic variety X is called $\mathbb{Q}$-*factorial* (or X *has* $\mathbb{Q}$-*factorial singularities*) if every $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-*Cartier*, i.e., for any $D \in \mathrm{Cl}(X) \otimes \mathbb{Q}$, some multiple rD is Cartier, where $r = r(D) \in \mathbb{N}$. A normal algebraic variety X is called *Gorenstein* (respectively, $\mathbb{Q}$-*Gorenstein*) (or X *has Gorenstein (*respectively*,* $\mathbb{Q}$-*Gorenstein) singularities*) if the canonical divisor K_X is Cartier and X is a Cohen–Macaulay scheme (respectively, the canonical divisor K_X is $\mathbb{Q}$-Cartier). A singularity $(O \in X)$ (or a point O on X) is called $\mathbb{Q}$-*factorial* (respectively, *Gorenstein,* $\mathbb{Q}$-*Gorenstein*) if X is that near O.

Let us consider the pair (X, D), where $D := \sum d_i D_i$ is a $\mathbb{Q}$-divisor on a normal algebraic variety X, such that the divisor $K_X + D$ is $\mathbb{Q}$-Cartier.

Definition 2.2.2. Let $f : Y \longrightarrow X$ be a birational morphism with the collection of irreducible exceptional divisors E_i (E_i are called *exceptional divisors over* X). Then the equality

$$K_Y \equiv f^*(K_X + D) + \sum a(f_*^{-1}(D_i), X, D) f_*^{-1}(D_i) + \sum a(E_i, X, D) E_i$$

holds. Here

- $a(f_*^{-1}(D_i), X, D) := -d_i$;
- $a(E_i, X, D) \in \mathbb{Q}$ is called the *discrepancy of the exceptional divisor* E_i *with respect to* (X, D).

The number $a(E_i, X, D)$ depends only on the discrete valuation corresponding to E_i of the field $\mathbb{C}(X)$. Hence the notion of the discrepancy $a(E, X, D)$ of the exceptional divisor E over X with respect to (X, D) makes sense, since it does not depend on the birational morphism $f : Y \longrightarrow X$, for which E is exceptional.

Definition 2.2.3. *The discrepancy of the pair* (X, D) is the number

$$\operatorname{discrep}(X, D) := \inf_E \{a(E, X, D)\},$$

where the infimum is taken over all exceptional divisors E over X.

Directly from Definitions 2.2.2 and 2.2.3 one gets the following statements:

Lemma 2.2.4. *Let* D' *be an effective* $\mathbb{Q}$*-Cartier divisor on* X. *Then* $\operatorname{discrep}(X, D) \geqslant \operatorname{discrep}(X, D + D')$.

Lemma 2.2.5. *Let $|H|$ be the basepoint-free linear system of Cartier divisors on X. Then for a general element $H \in |H|$ the inequality* $\mathrm{discrep}(X, D) \leqslant \mathrm{discrep}(H, D|_H)$ *holds.*

Lemma 2.2.6. *Let $|H|$ be a basepoint-free linear system of Cartier divisors on X. Then for a general element $H \in |H|$ the equality* $\mathrm{discrep}(X, D + H) = \min\{0, \mathrm{discrep}(X, D)\}$ *holds.*

Definition 2.2.7. The pair (X, D) has

- *terminal singularities* (or the pair (X, D) is *terminal*) if $\mathrm{discrep}(X, D) > 0$;
- *canonical singularities* (or the pair (X, D) is *canonical*) if $\mathrm{discrep}(X, D) \geqslant 0$;
- *purely log terminal singularities* (or the pair (X, D) is *purely log terminal*) if $\mathrm{discrep}(X, D) > -1$;
- *Kawamata log terminal singularities* (or the pair (X, D) is *Kawamata log terminal*) if $\mathrm{discrep}(X, D) > -1$ and $\llcorner D \lrcorner \leqslant 0$;
- *divisorially log terminal singularities* (or the pair (X, D) is *divisorially log terminal*) if the exists a log resolution $f : Y \longrightarrow X$ with the exceptional locus $\bigcup E_i$ of pure codimension 1 on Y, such that $a(E_i, X, D) > -1$ for all i;
- *log canonical singularities* (or the pair (X, D) is *log canonical*) if $\mathrm{discrep}(X, D) \geqslant -1$.

In the case of the pair $(X, 0)$, we will say that X has the corresponding type of singularities.

Remark 2.2.8. One can also formulate the local versions of Definitions 2.2.2–2.2.7 for singularities of the pair (X, D) near a given cycle Z on X by considering exceptional divisors E over X with centers at Z, i.e., $f(E) = Z$ for any birational morphism $f : Y \longrightarrow X$. This suggests to introduce the *set of non-terminal centers* $\mathbb{CS}(X, D)$ of (X, D) as the set of all cycles on X near which (X, D) fails to be terminal. In the other words, given any $Z \in \mathbb{CS}(X, D)$, one has $a(E, X, D) \leqslant 0$ for some exceptional divisor E over X with the center at Z. We note also that all the above notions can be generalized to the case of pairs $(X, \mathcal{D})$, where $\mathcal{D}$ is a linear system on X without fixed components, by considering general $D \in \mathcal{D}$.

The next statement allows one to calculate the discrepancy of the pair (X, D) on a particular log resolution of singularities of (X, D):

Lemma 2.2.9 (see [59, Corollary 2.32]). *In the notation of Definition 2.2.7, we have*:

- $d_i \leqslant 1$ *for all* i;
- *for every log resolution* $f : Y \longrightarrow X$ *with the collection of exceptional divisors* E_i*, the equality*

$$\mathrm{discrep}(X, D) = \min\Big\{\min_i\{a(E_i, X, D)\}, \min_j\{1 - d_j\}, 1\Big\}$$

  holds, provided that $a(E_i, X, D) \geqslant -1$ *for all* i.

Remark 2.2.10. According to [59, Proposition 2.41], every divisorially log terminal pair (X, D) with $\llcorner D \lrcorner = 0$ is Kawamata log terminal. In particular, (X, D) is purely log terminal. On the other hand, it is easy to construct an example of a divisorially log terminal pair (X, D) with $\llcorner D \lrcorner \neq 0$, which is not purely log terminal. However, by [59, Proposition 5.51] any divisorially log terminal pair (X, D) is purely log terminal iff the reduced part $\llcorner D \lrcorner$ is a disjoint union of its components.

Remark 2.2.11. It follows from Lemma 2.2.9 that $\mathrm{discrep}(X, 0) = 1$ for smooth X. In particular, X is contained in the class of terminal varieties. Note also that the pairs with log canonical singularities have the minimal possible finite discrepancy, since for every pair (X, D) either $-1 \leqslant \mathrm{discrep}(X, D) \leqslant 1$, or $\mathrm{discrep}(X, D) = -\infty$ (see [59, Corollary 2.31]).

Example 2.2.12. Let $(O \in X)$ be a $\mathbb{Q}$-Gorenstein singularity with $\dim X = 2$. Then the notion of the terminal singularity does not give anything new, since it follows from the Hodge index theorem that on every resolution of the singularity $(O \in X)$ all exceptional divisors over X with centers at O are (-1)-curves, which implies that the surface X is smooth at the point O. Furthermore, in the case of the canonical singularities a complete classification is known (see [23], [59, Theorem 4.20]). More precisely, every two-dimensional canonical singularity $(O \in X)$ is analytically isomorphic to one of the following hypersurface singularities in $\mathbb{C}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z])$:

- $\mathbf{A_n}$: $x^2 + y^2 + z^{n+1} = 0$, $n \geqslant 1$;
- $\mathbf{D_n}$: $x^2 + y^2 z + z^{n-1} = 0$, $n \geqslant 4$;
- $\mathbf{E_6}$: $x^2 + y^3 + z^4 = 0$;
- $\mathbf{E_7}$: $x^2 + y^3 + yz^3 = 0$;
- $\mathbf{E_8}$: $x^2 + y^3 + z^5 = 0$.

The above singularities are called *Du Val singularities* (see [3], [4] for other descriptions of Du Val singularities).

Example 2.2.12 shows that two-dimensional canonical singularities are always Gorenstein. Then Lemma 2.2.5 implies the following

Proposition 2.2.13. *Let X be an algebraic variety with only canonical singularities. Then the divisor K_X is Cartier in codimension 2 on X.*

But not every singular algebraic variety X has K_X being a Cartier divisor:

Example 2.2.14. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2. Let X be the cone over C with the vertex O. Then the divisor K_X can not be even $\mathbb{Q}$-Cartier. Indeed, let $Y \longrightarrow X$ be the blow up of the point O. Then Y is a smooth ruled surface over C, which implies that $\rho(X) = 1$. Hence, if K_X were a $\mathbb{Q}$-Cartier divisor, then it must be $\mathbb{Q}$-proportional to the hyperplane section of the cone X. Lifting the corresponding equality to Y, one easily gets a contradiction.

Example 2.2.15. Let $X \subset \mathbb{P}^6$ be the cone with the vertex O over the Veronese surface. Then the singularity $(O \in X)$ is isomorphic to the quotient $\mathbb{C}^3/\tau$, where τ is the involution on $\mathbb{C}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z])$ acting via $(x, y, z) \mapsto (-x, -y, -z)$. Hence K_X is not Cartier, but $2K_X$ is. Furthermore, let $f : Y \longrightarrow X$ be the blow up of the vertex O with the exceptional divisor E. Then Y is smooth and $K_Y \equiv f^*(K_X) + \dfrac{1}{2}E$. The latter implies that X has only terminal singularities (see Lemma 2.2.9). Hence the terminal singularities in dimensions $\geqslant 3$ have more complicated structure (cf. Example 2.2.12).

Further, for every $\mathbb{Q}$-Gorenstein singularity there exists a natural construction which allows one to switch to the case of singularity with canonical divisor being Cartier:

Example 2.2.16 (see [84, 3.6]). Consider a $\mathbb{Q}$-Gorenstein singularity $(O \in X)$ and a $\mathbb{Q}$-Cartier divisor D on X near O. Let $n \in \mathbb{Z}$ be the smallest number such that $nD \sim \mathcal{O}_X$. Let s be a generator of the sheaf $\mathcal{O}_X(-nD)$. Then multiplication by s gives an isomorphism $s : \mathcal{O}_X(nD) \simeq \mathcal{O}_X$. Let us now construct the sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ in the following way:

$$\mathcal{A} := \mathcal{O}_X \oplus \mathcal{O}_X(D) \oplus \ldots \oplus \mathcal{O}_X((n-1)D),$$

where multiplication is defined as follows:

$$\mathcal{O}_X(aD) \otimes \mathcal{O}_X(bD) \to \mathcal{O}_X((a+b)D), \qquad \text{if } a+b < n$$

and

$$\mathcal{O}_X(aD) \otimes \mathcal{O}_X(bD) \xrightarrow{s} \mathcal{O}_X((a+b-n)D), \qquad \text{if } a+b \geqslant n.$$

Set $Y := \mathrm{Spec}_X(\mathcal{A})$. Then Y is a normal algebraic variety with the natural morphism $\pi : Y \longrightarrow X$. Furthermore, π is a Galois cover with the group μ_n and ramification exactly at those points on X, where D is not Cartier. Moreover, $\pi^{-1}(O)$ consists of a unique point and $\pi^*(D)$ is a Cartier divisor. In particular, if we take for D the canonical divisor K_X on the $\mathbb{Q}$-Gorenstein algebraic variety X, then we arrive at the normal algebraic variety Y, with K_Y being a Cartier divisor, for which there is a cyclic finite cover $\pi : Y \longrightarrow X$.

On the other hand, we have the following

Proposition 2.2.17 (see [57, Prpoposition 3.16]). *Let $f : Y \longrightarrow X$ be a finite morphism and D_X a $\mathbb{Q}$-divisor on X. Let D_Y be the $\mathbb{Q}$-divisor on Y for which the equality*

$$K_Y + D_Y \equiv f^*(K_X + D_Y)$$

holds. Then the pair (X, D_X) *is log canonical* (*respectively, Kawamata log terminal*) *iff the pair* (Y, D_Y) *is log canonical* (*respectively, Kawamata log terminal*)*.*

Proposition 2.2.17 and the construction of the cyclic cover in Example 2.2.16 allow one to reduce the study of arbitrary canonical $\mathbb{Q}$-Gorenstein singularities to the case of canonical singularities with canonical divisor being Cartier. In the three-dimensional case, this leads to classification of all terminal and canonical singularities. Let us introduce few more notions:

Definition 2.2.18. A singularity $(O \in X)$ (or a point O on X), where $\dim X = 3$, is called cDV if either O is smooth, or for the zero locus H of a general generator of the maximal ideal $\mathfrak{m}_O$ of the point O in the local ring $\mathcal{O}_{O,X}$ the singularity $(O \in H)$ is Du Val. The latter condition is equivalent to that the singularity $(O \in X)$ is analytically isomorphic to a hypersurface singularity of the form

$$f(x, y, z) + tg(x, y, z, t) = 0$$

in $\mathbb{C}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t])$, where $f(x, y, z) = 0$ is the equation of a Du Val singularity (see Example 2.2.12) and $g(x, y, z, t)$ is an arbitrary polynomial.

Definition 2.2.19. A singularity $(O \in X)$ with $\dim X = n$ is called *rational* (respectively, *elliptic*) if for some resolution $f : Y \longrightarrow X$ of the singularity of X at O the equality $R^i f_* \mathcal{O}_X = 0$ holds for all $i > 0$ (respectively, $R^i f_* \mathcal{O}_X = 0$ for all $0 < i < n - 1$ and $R^{n-1} f_* \mathcal{O}_X = \mathbb{C}$).

Example 2.2.20. By resolving the singularity $(O \in X)$ from Example 2.2.12 by a sequence of blow ups, it can be easily seen that the exceptional divisors over X with centers at O are rational (-2)-curves. This implies that the

Du Val singularities are all rational. The latter also follows from the more general fact that all divisorially log terminal singularities are rational (see [59, Theorem 5.22]).

Now let $(O \in X)$ be a canonical singularity such that $\dim X = 3$ and K_X is Cartier near O. Then near O the threefold X is a Cohen–Macaulay scheme (see [89]), i.e., $(O \in X)$ is a Gorenstein singularity. On the other hand, we have the following results:

Theorem 2.2.21 (see [57, Theorem 11.1]). *A singularity $(O \in X)$, with K_X being a Cartier divisor near O, is rational iff it is canonical.*

Theorem 2.2.22 (see [81, Theorem 2.6] and Lemma 2.2.5). *Let $(O \in X)$ be a rational Gorenstein singularity. Then for the zero locus H of a general generator of the maximal ideal $\mathfrak{m}_O$ of the point O in the local ring $\mathcal{O}_{O,X}$ the singularity $(O \in H)$ is Gorenstein and is either rational or elliptic. Conversely, if for the zero locus H of a general generator of the maximal ideal $\mathfrak{m}_O$ the singularity $(O \in H)$ is rational Gorenstein, then $(O \in X)$ is also a rational Gorenstein singularity.*

Theorems 2.2.21, 2.2.22 and the description of the Du Val singularities in Examples 2.2.12, 2.2.20 imply that all cDV singularities are canonical. Moreover, we have the following

Theorem 2.2.23 (see [81, Theorem 2.11]). *Let $(O \in X)$ be a three-dimensional canonical Gorenstein singularity. Then there exists a birational morphism $f : Y \longrightarrow X$ such that*

- *the algebraic variety Y has only* cDV *singularities;*
- *the equality $K_Y \equiv f^*(K_X)$ holds.*

Example 2.2.24. Consider a hypersurface singularity $(O \in X)$ in $\mathbb{C}^4 = \operatorname{Spec}(\mathbb{C}[x, y, z, t])$ given by the equation

$$x^2 + h(y, z, t) = 0,$$

where $\deg h \geqslant 4$. It follows from Theorem 2.2.22 and description of the two-dimensional elliptic Gorenstein singularities (see [63], [59, 4.4], [82, 4.23]) that the singularity $(O \in X)$ is canonical. As a candidate for the morphism $f : Y \longrightarrow X$ from Theorem 2.2.23 one can take the restriction to X of the *weighted blow up* of $\mathbb{C}^4$ with weights $(2, 1, 1, 1)$ (see Example 2.2.28 below).

Proposition 2.2.25 (see [81, Corollary 2.14]). *In the notation of Theorem 2.2.23, the f-exceptional locus has pure codimension* 1 *on* Y *and is a union of rational and ruled surfaces.*

Example 2.2.26. Lemma 2.2.9 and Proposition 2.2.25 imply that the singularity of the cone the vertex over a smooth K3 surface is worse than canonical.

Theorem 2.2.27 (see [84], [60], [59], [81]). *Let* $(O \in X)$ *be a* cDV *singularity. Then the discrepancy of every exceptional divisor over* X *with the center at* O *is strictly positive. In particular, every isolated* cDV *singularity is terminal. Conversely, every three-dimensional terminal Gorenstein singularity is* cDV*. In particular, all three-dimensional terminal singularities are isolated.*

The above results provide a satisfactory description of terminal and canonical singularities in dimension 3. In the next example, we consider a particular case of the purely log terminal singularities:

Example 2.2.28. Consider the affine space $\mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$ and the action of the group μ_m via $x_i \mapsto \varepsilon^{a_i} x_i$, where ε is a primitive m-root of unity and $a_i \in \mathbb{Z}_{\geqslant 0}$. The quotient $X := \mathbb{C}^n/\mu_m(a_1, \ldots, a_n)$ is called *cyclic quotient singularity* (or just *singularity*) *of type* $\frac{1}{m}(a_1, \ldots, a_n)$. Remark 2.2.10 and Theorem 2.2.17 imply that X has purely log terminal singularities. Cyclic quotient singularities can be resolved by a sequence of *weighted blow ups*. More precisely, for $X = \mathbb{C}^n/\mu_m(a_1, \ldots, a_n)$ with $a_i \in \mathbb{N}$, the weighted blow up with weights $(a_1, \ldots, a_n)$ is the birational morphism $f : Y \longrightarrow X$ such that the algebraic variety Y is covered by the affine charts $U_1, \ldots, U_n$,

$$U_i \simeq \mathbb{C}^n_{y_1,\ldots,y_n}/\mu_{a_i}(-a_1, \ldots, m, \ldots, -a_n),$$

where m stands on the i-th place. The coordinates on X and U_i are related in the following way:

$$x_i := y_i^{a_i/m}, \qquad x_j := y_j y_i^{a_j/m} \qquad \text{for } j \neq i.$$

In each chart U_i, the f-exceptional divisor E is given by the equation $y_i = 0$. The latter gives the equality

$$K_Y = f^*(K_X) + (-1 + \sum a_i/m)E.$$

In particular, any two-dimensional cyclic quotient singularity can be resolved by a sequence of weighted blow ups, which is actually the minimal resolution (see [32, 2.3]). Furthermore, there are nice criteria for a cyclic quotient singularity to be canonical and Gorenstein, respectively (see [81]). Namely, the singularity of type $\frac{1}{m}(a_1, \ldots, a_n)$ is canonical iff $\sum a_i \geqslant m$, and is Gorenstein iff $\sum a_i$ is divisible by m.

Further, let us recall some results about the inversion of adjunction. Consider a pair (X, D) such that $\dim X = 3$ and $D := \sum d_i D_i$ is a *boundary*, i.e., $0 < d_i \leqslant 1$ for all i. Set $D' := \llcorner D \lrcorner \neq 0$. Then the following results take place:

Theorem 2.2.29 (see [90, Corollary 3.8]). *If the pair* (X, D) *is divisorially log terminal and all irreducible components of the divisor* D' *are* $\mathbb{Q}$*-Cartier, then all these components are normal surfaces and intersect normally.*

Theorem 2.2.30 (see [90, Proposition 3.9, Corollary 3.10]). *Under the assumptions of Theorem 2.2.29, let* $S \subset D'$ *be an irreducible component. Then there exists an effective* $\mathbb{Q}$*-divisor* $\mathrm{Diff}_S(D - S)$ *on* S *such that*

$$K_S + \mathrm{Diff}_S(D - S) \sim_{\mathbb{Q}} \big(K_X + D\big)\big|_S$$

and $\mathrm{Supp}(\mathrm{Diff}_S(D - S)) \supseteq D_i \cap S$ *for all* i*. Furthermore, for every prime Weil divisor* W *on* S *there is an analytic isomorphism*

$$(X, S, W) \simeq \big(\mathbb{C}^3 = \mathrm{Spec}\big(\mathbb{C}[x_1, x_2, x_3]\big), \big(x_1 = 0\big), \big(x_1 = x_2 = 0\big)\big)/\mu_n(1, q, 0)$$

near the general point of the cycle W*, where* q*,* $n \in \mathbb{N}$*,* $q \leqslant n$ *and* $(q, n) = 1$*. In particular, if* X *is smooth in codimension* 2 *on* S*, then* $\mathrm{Diff}_S(D - S) = 0$*.*

Example 2.2.31. Let X be the cone with the vertex O over a rational normal curve $C_n \subset \mathbb{P}^n$. The blow up of X at O gives a birational morphism $\mathbb{F}_n \longrightarrow X$, which implies that the group $\mathrm{Cl}(X)$ is generated by the generatrix L of the cone X. Furthermore, we have $K_X \sim -(n+2)L$, $\mathcal{O}_X(nL) \simeq \mathcal{O}_X(1)$, and hence

$$\big(K_X + L\big)\big|_L \sim_{\mathbb{Q}} K_L + \big(1 - \frac{1}{n}\big)O.$$

Thus, $\mathrm{Diff}_L(0) = \big(1 - \frac{1}{n}\big)O$.

Theorem 2.2.32 (see [90], [57], [58] (cf. Lema 2.2.4)). *Under the assumptions of Theorem 2.2.30, we have:*

- *if the divisor* $D - S$ *is* $\mathbb{Q}$*-Cartier, then the pair* (X, D) *is purely log terminal near* S *iff the pair* $(S, \mathrm{Diff}_S(D - S))$ *is Kawamata log terminal;*

- *if the pair (X, S) is purely log terminal and the divisor $D-S$ is $\mathbb{Q}$-Cartier, then the pair (X, D) is log canonical near S iff the pair $(S, \mathrm{Diff}_S(D-S))$ is log canonical.*

Finally, let us formulate a result about the factoriality of three-dimensional algebraic varieties:

Theorem 2.2.33 (see [52, Lemma 5.1]). *Let X be a three-dimensional algebraic variety with terminal Gorenstein $\mathbb{Q}$-factorial singularities. Then X is factorial.*

Remark 2.2.34. As Example 2.2.15 shows, being $\mathbb{Q}$-factorial only is not enough for an algebraic variety to be factorial.

2.3 Results from minimal model theory

In the present section, we formulate several notions from the theory of minimal models of algebraic varieties, which will be then used throughout the thesis. We also introduce several known facts from the theory of minimal models of algebraic varieties (see [53], [59] for the proofs).

Definition 2.3.1. A $\mathbb{Q}$-Cartier divisor L on a normal algebraic variety X is called *big* if $\dim H^0(X, \mathcal{O}_X(mL)) > cm^{\dim X}$ for some constant $c > 0$ and $m \gg 0$. A $\mathbb{Q}$-Cartier divisor L on X is called *nef* if $L \cdot Z \geqslant 0$ for every curve Z on X.

Definition 2.3.2. For a normal algebraic variety X, the closure $\overline{NE}(X)$ in $\mathbb{R}^{\rho(X)}$ of the minimal convex cone, which contains the classes of all curves on X, is called *Mori cone of X*. For a $\mathbb{Q}$-Cartier divisor L on X, we set:

$$\overline{NE}(X)_{L\geqslant 0} := \{Z \in N_1(X) \otimes \mathbb{R} \text{ with } [Z] \in \overline{NE}(X) \bigm| L \cdot Z \geqslant 0\}$$

and

$$\overline{NE}(X)_{L<0} := \overline{NE}(X) \setminus \overline{NE}(X)_{L\geqslant 0}.$$

Definition 2.3.3. For a normal algebraic variety X, a halfline $R \subseteq \overline{NE}(X)$ of the form

$$R := \overline{NE}(X) \cap \big(\{Z \in N_1(X) \otimes \mathbb{R} \text{ with } [Z] \in \overline{NE}(X) \bigm| L \cdot Z = 0\}\big) \otimes \mathbb{R}_{\geqslant 0},$$

where $L \neq 0$ is some nef divisor on X, is called *extremal ray on* X.

Extremal rays play a very important role in the description of the structure of Mori cones of algebraic varieties:

Theorem 2.3.4 (see [59], [53]). *Let X be a $\mathbb{Q}$-factorial algebraic variety with a boundary D such that the pair (X, D) is purely log terminal. Then the following decomposition holds:*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+D\geqslant 0} + \sum R_i.$$

Here $R_i \subseteq \overline{NE}(X)_{K_X+D<0}$ are extremal rays such that

- *R_i are discrete in the halfspace $N_1(X)_{K_X+D<0} \otimes \mathbb{R}$;*
- *$R_i = \mathbb{R}_{\geqslant 0}[C_i]$ for all i, where C_i is a rational curve on X.*

One of the constituents in the proof of Theorem 2.3.4 is the following

Theorem 2.3.5 (see [59, Theorem 2.70], [56]). *Let (X, D) be a Kawamata log terminal pair and L a nef and big $\mathbb{Q}$-divisor on X. Then for the $\mathbb{Q}$-Cartier Weil divisor $N \equiv K_X + D + L$ the equality*

$$H^i(X, \mathcal{O}_X(N)) = 0$$

holds for all $i > 0$.

The following is the relative version of Theorem 2.3.5:

Theorem 2.3.6 (see [53]). *Under the assumptions of Theorem 2.3.5, let* $f : X \longrightarrow X'$ *be a morphism on a normal algebraic variety* X' *and* M *be a* $\mathbb{Q}$*-Cartier Weil divisor on* X *such that the divisor* $M - (K_X + D)$ *is* f*-ample (over* X'*). Then the equality*

$$R^i f_*(\mathcal{O}_X(M)) = 0$$

holds for all $i > 0$.

Theorem 2.3.4 provides some useful information on the geometry of the algebraic variety X via the following

Theorem 2.3.7. *Under the assumptions of Theorem 2.3.4, let* L *be a Cartier divisor on* X *such that* L *is nef and*

$$F_L := \{Z \in N_1(X) \otimes \mathbb{R} \text{ with } [Z] \subset \overline{NE}(X) \bigm| L \cdot Z = 0\} \subseteq \overline{NE}(X)_{K_X+D<0}.$$

Then there exists a morphism with connected fibers (the $(K_X + D)$-extremal contraction of the face F_L*),* $\text{cont}_{F_L} : X \longrightarrow X'$*, such that for every curve* Z *on* X*,* $\text{cont}_{F_L}(Z)$ *is a point iff* $[Z] \in F_L$*. Furthermore, the exact sequence*

$$0 \to N_1(X/X') \to N_1(X) \to N_1(X') \to 0$$

takes place.

Definition 2.3.8. Under the assumptions of Theorem 2.3.7, if F_L is an extremal ray and $\dim X' < \dim X$, then $\text{cont}_{F_L} : X \longrightarrow X'$ is called the *log Mori fibration*. In the particular case, when $D = 0$, X has terminal singularities and X' is a point, algebraic variety X is called $\mathbb{Q}$-*Fano variety*.

Together with the above notions and facts, which we need for the future reference, we will also use the existence of the log MMP in dimensions $\leqslant 3$ (see [58], [90]). Namely, we will use the fact, under the assumptions of Theorem 2.3.4, that after one applies the log minimal model to the pair (X, D) with $\dim X \leqslant 3$, one gets a pair (X', D') with singularities not worse than (X, D) has and such that either the $\mathbb{Q}$-divisor $K_{X'} + D'$ is nef, or X' possesses a structure of the log Mori fibration. Moreover, there is a birational map $\chi : X \dashrightarrow X'$ such that D' is the proper transform of D with respect to χ.

Theorem 2.3.9 (see [58], [90]). *Let (X, D) be a log canonical pair and $\dim X = 3$. Then there exist an algebraic variety $\widetilde{X}$ and a birational morphism $f : \widetilde{X} \longrightarrow X$ such that*

- *$\widetilde{X}$ is $\mathbb{Q}$-factorial;*
- *the equality $K_{\widetilde{X}} + \widetilde{D} \equiv f^*(K_X + D)$ holds for some boundary $\widetilde{D}$ on $\widetilde{X}$;*
- *the pair $(\widetilde{X}, \widetilde{D})$ is divisorially log terminal.*

Proof. Let $h : Y \longrightarrow X$ be a log resolution of singularities of the pair (X, D). Write

$$K_Y + D_Y \equiv h^*(K_X + D) + A - B,$$

where $D_Y := g_*^{-1}(D)$, A, B are effective exceptional divisors without common components such that B is a boundary. After running the log MMP over X with respect to the pair $(Y, D_Y + B)$, we get an algebraic variety $\widetilde{X}$ and a birational morphism $f : \widetilde{X} \longrightarrow X$ such that

- $\widetilde{X}$ is $\mathbb{Q}$-factorial;
- the equality $K_{\widetilde{X}} + \widetilde{D} \equiv f^*(K_X + D)$ holds for some boundary $\widetilde{D}$ on $\widetilde{X}$;

- the pair $(\widetilde{X}, \widetilde{D})$ is divisorially log terminal.

□

More precise version of Theorem 2.3.9, which is proved by the similar arguments, is the following

Proposition 2.3.10. *Let X be a three-dimensional algebraic variety with canonical singularities. Then there exist an algebraic variety Y with terminal $\mathbb{Q}$-factorial singularities and a birational morphism $f : Y \longrightarrow X$ such that the equality $K_Y \equiv f^*(K_X)$ holds.*

Definition 2.3.11. In the notation of Proposition 2.3.10, Y (or f) is called *terminal $\mathbb{Q}$-factorial modification of X*.

Example 2.3.12. Let $X \subset \mathbb{P}^4 = \mathrm{Proj}(\mathbb{C}[x, y, z, t, w])$ be the cone with the vertex O over the quadric surface, given by the equation $xy = zw$. Then X is not $\mathbb{Q}$-factorial, since the divisors $(x = w = 0)$ and $(y = z = 0)$ intersect just at the point O. However, there exists a birational morphism $f : Y \longrightarrow X$ such that Y is smooth and the f-exceptional locus is isomorphic to $\mathbb{P}^1$ (see [32, 2.6]). In particular, we have $K_Y = f^*(K_X)$, and hence Y is a terminal $\mathbb{Q}$-factorial modification of X.

Remark 2.3.13. In the notation of Proposition 2.3.10, let Y and Y' be two terminal $\mathbb{Q}$-factorial modifications of X. Then, since Y and Y' are minimal models over X (cf. the proof of Theorem 2.3.9), [55, Theorem 4.3] implies that the induced birational map $Y \dashrightarrow Y'$ is either an isomorphism or a sequence of K_Y-flops over X.

Further, let us give the following

Definition 2.3.14. A normal algebraic variety X is called *Fano variety* (respectively, *weak Fano variety*) if it has canonical Gorenstein singularities and the anticanonical divisor $-K_X$ is ample (respectively, nef and big).

Remark 2.3.15. It follows from the Riemann–Roch formula (see [83]) and Theorem 2.3.5 that $\dim|-K_X| = -\frac{1}{2}K_X^3+2$ for any (weak) Fano threefold X. The number $(-K_X)^3$ (respectively, $\dfrac{1}{2}(-K_X)^3+1$) is called *(anticanonical) degree* (respectively, *genus*) of X.

Remark 2.3.16. Passing to a "good" resolution of X, from Theorem 2.3.6 and the Leray spectral sequence one can deduce that $\mathrm{Pic}(X) \simeq H^2(X,\mathbb{Z})$ for any (weak) Fano threefold X (see [45, Proposition 2.1.2]).

Let us now consider several examples:

Example 2.3.17. In the notation of Proposition 2.3.10, if X is Gorenstein, then Y is also Gorenstein. Moreover, Theorem 2.2.33 implies that Y is factorial in this case. Further, if X is a Fano threefold, then Y is a weak Fano threefold. Conversely, for every weak Fano threefold Y with terminal factorial singularities, its image $X := \Phi_{|-nK_Y|}(Y)$ for some $n \in \mathbb{N}$ is a Fano threefold such that $K_Y = \Phi^*_{|-nK_Y|}(K_X)$ (see [52]).

Example 2.3.18. Let $X \subset \mathbb{P}^{10}$ be the cone over the anticanonically embedded surface $\mathbb{P}^2$. Let $f : Y \longrightarrow X$ be the blow up of the vertex on X. Then $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(3))$ and f is the birational contraction of the negative section of the $\mathbb{P}^1$-bundle Y. From the relative Euler exact sequence we obtain $-K_Y \sim 2M$, where $\mathcal{O}_Y(M) \simeq \mathcal{O}_Y(1)$ is the tautological sheaf on Y (see [67, Proposition 4.26]). On the other hand, f is given by the linear system $|M|$, which implies that Y is a weak Fano threefold such that $K_Y = f^*(K_X)$. In particular, X

is a Fano threefold and $f : Y \longrightarrow X$ is its terminal $\mathbb{Q}$-factorial modification. Furthermore, one can show that X is the weighted projective space $\mathbb{P}(1,1,1,3)$ (see for example [79]). It also follows easily from [22] that $(-K_X)^3 = 72$.

Example 2.3.19. Consider the weighted projective space $X := \mathbb{P}(6,4,1,1)$. The singular locus of X is a curve $L \simeq \mathbb{P}^1$ such that for some two points P and Q on L the germs $(P \in X)$ and $(Q \in X)$ are the singularities of type $\frac{1}{6}(4,1,1)$ and $\frac{1}{4}(2,1,1)$, respectively, and for every point $O \in L \setminus \{P, Q\}$ the singularity $(O \in X)$ is analytically isomorphic to $((0,o) \in \mathbb{C} \times U)$, where $(o \in U)$ is the singularity of type $\frac{1}{2}(1,1)$ (see [40, 5.15]). Hence the singularities of X are canonical and Gorenstein (see Example 2.2.28). On the other hand, we have $\mathcal{O}_X(-K_X) \simeq \mathcal{O}_X(12)$ (see [22, Theorem 3.3.4] and Theorem 4.1.1 below), which implies that the divisor $-K_X$ is ample. Thus, X is a Fano threefold.

Let $f_1 : Y_1 \longrightarrow X := Y_0$ be the weighted blow up of the point P with weights $\frac{1}{6}(4,1,1)$ (see Example 2.2.28). Then the singular locus of the threefold Y_1 is a curve L_1 such that for some two points P_1 and Q_1 on L_1 the germs $(P_1 \in Y_1)$ and $(Q_1 \in Y_1)$ are the singularities of type $\frac{1}{4}(2,1,1)$ and for every point $O \in L_1 \setminus \{P_1, Q_1\}$ the singularity $(O \in Y_1)$ is analytically isomorphic to $((0,o) \in \mathbb{C} \times U)$, where $(o \in U)$ is the singularity of type $\frac{1}{2}(1,1)$.

Let $f_2 : Y_2 \longrightarrow Y_1$ be the weighted blow up of the points P_1 and Q_1 with weights $\frac{1}{4}(2,1,1)$. Then the singular locus of the threefold Y_2 is a curve $L_2 \simeq \mathbb{P}^1$ such that for every point O on L_2 the singularity $(O \in Y_2)$ is analytically isomorphic to $((0,o) \in \mathbb{C} \times U)$, where $(o \in U)$ is the singularity of type $\frac{1}{2}(1,1)$.

Finally, the blow up $f_3 : Y_3 \longrightarrow Y_2$ of the curve L_2 on Y_2 leads to a smooth threefold $Y := Y_3$ and a birational morphism $f : Y \longrightarrow X$. By construction, we have $K_{Y_i} = f_i^*(K_{Y_{i-1}})$ (cf. Example 2.2.28) and f is the composition of f_i, $i \in \{1, 2, 3\}$. Hence $K_Y = f^*(K_X)$ and Y is a terminal $\mathbb{Q}$-factorial modification of X.

Remark 2.3.20. In the notation of Examples 2.3.18 and 2.3.19, the morphism f is a composition of extremal birational contractions and the exceptional locus of f has pure codimension 1 on Y. This implies that there are no small K_Y-trivial extremal contractions on Y. In particular, every terminal $\mathbb{Q}$-factorial modification of X is isomorphic to Y (see Remark 2.3.13).

Let us now state several facts about the extremal rays and extremal contractions on weak Fano threefolds with terminal factorial singularities. Let Y be such a threefold and X the corresponding Fano threefold (see Remark 2.3.17). Let also $\mathrm{ext} : Y \longrightarrow Y'$ be a K_Y-negative extremal contraction. We have:

Proposition 2.3.21 (see [79]). *The cone $\overline{NE}(Y)$ is polyhedral and generated by contractible extremal rays.*

Proof. Consider the pair (W, D) such that

- the threefold W is $\mathbb{Q}$-factorial and the pair (W, D) is Kawamata log terminal;
- $K_W + D \equiv 0$;
- the irreducible components of the divisor D generate the linear space $N^1(W)$ over $\mathbb{R}$ dual to $N_1(W)$.

Then [79, Lemma 4.2] implies that the statement of proposition is true for W. The existence of such (W, D) follows from [79, Lemma 4.3]. □

Theorem 2.3.22 (see [79, Proposition 4.11]). *If* $\dim Y' = 1$, *then* $(-K_X)^3 \leqslant 54$.

Theorem 2.3.23 (see [79, Proposition 5.2]). *If* $\dim Y' = 2$, *then the following holds:*

- *if* $\mathrm{ext}: Y \longrightarrow Y'$ *is not a* $\mathbb{P}^1$*-bundle, then* $(-K_X)^3 \leqslant 54$*;*
- *if* $\mathrm{ext}: Y \longrightarrow Y'$ *is a* $\mathbb{P}^1$*-bundle, then either* $(-K_X)^3 \leqslant 64$ *or* $(-K_X)^3 = 72$ *and* $X = \mathbb{P}(1,1,1,3)$ *(cf. Example 2.3.18).*

Theorem 2.3.24. *If* ext *is a birational morphism, then the following holds:*

- ext *is divisorial with the exceptional divisor* E*;*
- *if* $O := \mathrm{ext}(E)$ *is a point, then* ext *is the blow up of* Y' *at* O*. Furthermore,* Y' *is a weak Fano threefold with terminal singularities and* $(-K_{Y'})^3 > (-K_Y)^3$*. Moreover,* Y' *is factorial except for the case when* $E \simeq \mathbb{P}^2$, $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ *and* $(O \in Y')$ *is the singularity of type* $\frac{1}{2}(1,1,1)$ *(cf. Example 2.2.28);*
- *if* $C := \mathrm{ext}(E)$ *is a curve, then* Y' *is smooth near* C*,* C *is reduced and irreducible, and* ext *is the blow up of* Y' *at* C*.*

Proof. The statement follows from [19], [64] and the equality

$$K_Y \equiv \mathrm{ext}^*(K_{Y'}) + \alpha E$$

for some $\alpha \in \mathbb{Q}_{>0}$. □

Corollary 2.3.25. *Under the assumptions of Theorem 2.3.24, if* $\mathrm{ext}(E) \in Y'$ *is the singularity of type* $\frac{1}{2}(1,1,1)$*, then* $f(E)$ *is a plane on* X *(i.e., the surface* $\Pi \simeq \mathbb{P}^2$ *on* X *such that* $K_X^2 \cdot \Pi = 1$*).*

Proof. Since $E \simeq \mathbb{P}^2$ and $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$, we get

$$K_Y^2 \cdot E = \big(\mathrm{ext}^*\big(K_{Y'}\big) + \frac{1}{2}E\big)^2 \cdot E = \frac{1}{4}E^3 = 1,$$

which implies that $f(E)$ is a surface $\Pi \simeq \mathbb{P}^2$ on X such that $K_X^2 \cdot \Pi = K_Y^2 \cdot E = 1$. □

Proposition 2.3.26 (see [79, Proposition 4.5]). *If $C := \mathrm{ext}(E)$ is a curve, then Y' has terminal factorial singularities and one of the following holds:*

- *Y' is a weak Fano threefold with $(-K_{Y'})^3 \geqslant (-K_Y)^3$;*
- *$K_{Y'} \cdot C > 0$ and C is the only curve on Y' which intersects $K_{Y'}$ positively. Furthermore, in this case $C \simeq \mathbb{P}^1$ and either $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or $E \simeq \mathbb{F}_1$.*

Proof. Let us use the arguments from the proof of Proposition–definition 4.5 in [79]. By Theorem 2.3.24, the curve C is reduced and irreducible, the threefold Y' is smooth near C and ext is the blow up of Y' at C. In particular, Y' has terminal factorial singularities and $K_Y = \mathrm{ext}^*(K_{Y'}) + E$. Further, we have

$$K_Y^3 = K_{Y'}^3 + 3\mathrm{ext}^*\big(K_{Y'}\big) \cdot E^2 + E^3, \tag{2.3.1}$$

and if Y' is a weak Fano threefold, then we obtain

$$0 \leqslant \big(-K_Y\big)^2 \cdot E = 2\mathrm{ext}^*\big(K_{Y'}\big) \cdot E^2 + E^3$$

and

$$0 \leqslant \big(-K_Y\big) \cdot \big(-\mathrm{ext}^*\big(K_{Y'}\big)\big) \cdot E = \mathrm{ext}^*\big(K_{Y'}\big) \cdot E^2.$$

This together with (2.3.1) gives the inequality $(-K_{Y'})^3 \geqslant (-K_Y)^3$.

Now, if Y' is not a weak Fano threefold, then for some irreducible curve Z on Y' we get $K_{Y'} \cdot Z > 0$. It is easy to see that in fact $Z = C$. Then

Proposition 2.3.21 and [64, Corollary 1.3] imply that $C \simeq \mathbb{P}^1$. In particular, $E \simeq \mathbb{F}_n$ for some $\mathbb{Z}_{\geqslant 0}$. Further, since $-K_Y\big|_E$ is a section of the $\mathbb{P}^1$-bundle $E \longrightarrow C$, we have

$$-K_Y\big|_E \sim h + (n+a)l$$

on E for some $a \in \mathbb{Z}$. Moreover, we have $a \geqslant 0$, since the divisor $-K_Y$ is nef. Then we get

$$0 > -K_{Y'} \cdot C = K_Y^2 \cdot E - 2 + 2p_a(C) = \big(-K_Y\big|_E\big)^2 - 2 = n + 2a - 2,$$

which implies that $a = 0$ and $n \leqslant 1$. Proposition 2.3.26 is completely proved.

□

Corollary 2.3.27. *In the assumptions of Proposition 2.3.26, the following holds*

- *if $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then $f(E)$ is a line on X (i.e., the curve $\Gamma \simeq \mathbb{P}^1$ on X such that $-K_X \cdot \Gamma = 1$) and X is singular along $f(E)$;*
- *if $E \simeq \mathbb{F}_1$, then $f(E)$ is a plane on X (cf. Corollary 2.3.25).*

Proof. In the notation from the proof of Proposition 2.3.26, if $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then $-K_Y \cdot h = 0$ and $-K_Y \cdot l = 1$. This implies that $f(E)$ is a curve $\Gamma \simeq \mathbb{P}^1$ on X such that $-K_X \cdot \Gamma = 1$.

If $E \simeq \mathbb{F}_1$, then $-K_Y \cdot h = 0$ and h is the only curve on E which intersects K_Y by zero. This implies that $f(E)$ is a surface $\Pi \simeq \mathbb{P}^2$ on X such that $K_X^2 \cdot \Pi = K_Y^2 \cdot E = 1$. □

Chapter 3

Smooth quartic threefold

3.1 Auxiliary results

Let X be a smooth quartic threefold in $\mathbb{P}^4$.

Definition 3.1.1. A *Halphen pencil* $\mathcal{M}$ on X is a one-dimensional linear system whose general element is an irreducible subvariety birational to a smooth variety of Kodaira dimension zero.

We fix for the rest of the chapter X and $\mathcal{M}$ from Definition 3.1.1. Let us obtain several properties of the pair $(X, \mathcal{M})$. Firstly, we have

$$\mathcal{M} \subset |-nK_X|$$

for some $n \in \mathbb{N}$, since $-K_X = \mathcal{O}_X(1)$ by the adjunction formula and $\mathrm{Pic}(X) = \mathbb{Z}K_X$ by Lefschetz theorem.

Further, put $\mu := 1/n$. Then

- the pair $(X, \mu\mathcal{M})$ is canonical by [41, Theorem A],
- the pair $(X, \mu\mathcal{M})$ is not terminal by [14, Theorem 2.1].

Let $\mathbb{CS}(X, \mu\mathcal{M})$ be the set of non-terminal centers of $(X, \mu\mathcal{M})$ (see Remark 2.2.10). Then

$$\mathbb{CS}\Big(X, \mu\mathcal{M}\Big) \neq \varnothing,$$

because $(X, \mu\mathcal{M})$ is not terminal. Let M_1 and M_2 be two general surfaces in $\mathcal{M}$.

Lemma 3.1.2. *Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a point $P \in X$. Then*

$$\mathrm{mult}_P\big(M\big) = n\mathrm{mult}_P\big(T\big) = 2n,$$

where M is any surface in $\mathcal{M}$, and T is the surface in $|-K_X|$ that is singular at P.[1)]

Proof. It follows from [80, Proposition 1] that the inequality

$$\mathrm{mult}_P\Big(M_1 \cdot M_2\Big) \geqslant 4n^2$$

holds. Let H be a general surface in $|-K_X|$ such that $P \in H$. Then

$$4n^2 = H \cdot M_1 \cdot M_2 \geqslant \mathrm{mult}_P\Big(M_1 \cdot M_2\Big) \geqslant 4n^2,$$

which gives $(M_1 \cdot M_2)_P = 4n^2$. Arguing as in the proof of [80, Proposition 1], we see that

$$\mathrm{mult}_P\big(M_1\big) = \mathrm{mult}_P\big(M_2\big) = 2n,$$

because $(M_1 \cdot M_2)_P = 4n^2$. Similarly, we see that

$$4n = H \cdot T \cdot M_1 \geqslant \mathrm{mult}_P\big(T\big)\mathrm{mult}_P\big(M_1\big) = 2n\mathrm{mult}_P\big(T\big) \geqslant 4n,$$

which implies that $\mathrm{mult}_P(T) = 2$. Finally, we also have

$$4n^2 = H \cdot M \cdot M_1 \geqslant \mathrm{mult}_P\big(M\big)\mathrm{mult}_P\big(M_1\big) = 2n\mathrm{mult}_P\big(M\big) \geqslant 4n^2,$$

where M is any surface in $\mathcal{M}$, which completes the proof. □

[1)]Note that, since X is smooth, T is a normal surface by the Serre's Criterion of normality.

Lemma 3.1.3. *Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a point $P \in X$. Then*

$$M_1 \cap M_2 = \bigcup_{i=1}^{r} L_i,$$

where $L_1, \ldots, L_r$ are lines on the threefold X that pass through the point P, $r \in \mathbb{N}$.

Proof. Let H be a general surface in $|-K_X|$ such that $P \in H$. Then

$$4n^2 = H \cdot M_1 \cdot M_2 = \mathrm{mult}_P\Big(M_1 \cdot M_2\Big) = 4n^2$$

by Lemma 3.1.2. Then $\mathrm{Supp}(M_1 \cdot M_2)$ consists of lines on X that pass through P. □

Lemma 3.1.4. *Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a point $P \in X$. Then*

$$n/3 \leqslant \mathrm{mult}_L\big(\mathcal{M}\big) \leqslant n/2$$

for every line $L \subset X$ that passes through the point P.

Proof. Let D be a general hyperplane section of X through L. Then we have

$$M\Big|_D = \mathrm{mult}_L\big(\mathcal{M}\big)L + \Delta,$$

where M is a general surface in $\mathcal{M}$ and Δ is an effective divisor such that

$$\mathrm{mult}_P\big(\Delta\big) \geqslant 2n - \mathrm{mult}_L\big(\mathcal{M}\big).$$

On the surface D we have $L \cdot L = -2$. Then

$$n = \Big(\mathrm{mult}_L\big(\mathcal{M}\big)L + \Delta\Big) \cdot L = -2\mathrm{mult}_L\big(\mathcal{M}\big) + \Delta \cdot L$$

on D. But $\Delta \cdot L \geqslant \mathrm{mult}_P(\Delta) \geqslant 2n - \mathrm{mult}_L(\mathcal{M})$. Thus, we get

$$n \geqslant -2\mathrm{mult}_L\big(\mathcal{M}\big) + \mathrm{mult}_P\big(\Delta\big) \geqslant 2n - 3\mathrm{mult}_L\big(\mathcal{M}\big),$$

which implies that $n/3 \leqslant \mathrm{mult}_L\big(\mathcal{M}\big)$.

Further, let T be the surface in $|-K_X|$ that is singular at P. Then $T\cdot D$ is reduced and

$$T\cdot D = L+Z,$$

where Z is an irreducible plane cubic curve such that $P\in Z$. Then

$$3n=\Big(\mathrm{mult}_L\big(\mathcal{M}\big)L+\Delta\Big)\cdot Z = 3\mathrm{mult}_L\big(\mathcal{M}\big)+\Delta\ \cdot Z$$

on the surface D. Note that the set $\Delta\cap Z$ is finite by Lemma 3.1.3. In particular, we have

$$\Delta\ \cdot Z\geqslant \mathrm{mult}_P\big(\Delta\big)\geqslant 2n-\mathrm{mult}_L\big(\mathcal{M}\big),$$

because $\mathrm{Supp}(\Delta)$ does not contain the curve Z. Thus, we get

$$3n\geqslant 3\mathrm{mult}_L\big(\mathcal{M}\big)+\mathrm{mult}_P\big(\Delta\big)\geqslant 2n+2\mathrm{mult}_L\big(\mathcal{M}\big),$$

which implies that $\mathrm{mult}_L\big(\mathcal{M}\big)\leqslant n/2$. □

Finally, we will need the following two simple results, which turn out to be the cornerstone of the proof of Theorem 1.2.7:

Lemma 3.1.5. *Let S be a surface, O a smooth point on S and R an effective Weil divisor on S. Then for every linear system $\mathcal{D}$ on S without fixed components and general curves D_1, $D_2\in\mathcal{D}$, we have*

$$\mathrm{mult}_O\Big(D_1\cdot R\Big)=\mathrm{mult}_O\Big(D_2\cdot R\Big)\leqslant \mathrm{mult}_O\big(R\big)\mathrm{mult}_O\Big(D_1\cdot D_1\Big).$$

Proof. Put $S_0=S$ and $O_0=O$. Let us consider a sequence of blow ups

$$S_n\xrightarrow{\ \pi_n\ }S_{n-1}\xrightarrow{\ \pi_{n-1}\ }\cdots\xrightarrow{\ \pi_2\ }S_1\xrightarrow{\ \pi_1\ }S_0$$

such that π_1 is the blow up of the point O_0, and π_i is the blow up of the point O_{i-1} that is contained in the curve E_{i-1}, where E_{i-1} is the exceptional curve of π_{i-1}, and $i = 2, \ldots, n$.

Let D_j^i be the proper transform of D_j on S_i for $i = 0, \ldots, n$ and $j = 1, 2$. Then

$$D_1^i \equiv D_2^i \equiv \pi_i^*\Big(D_1^{i-1}\Big) - \mathrm{mult}_{O_{i-1}}\Big(D_1^{i-1}\Big)E_i \equiv \pi_i^*\Big(D_2^{i-1}\Big) - \mathrm{mult}_{O_{i-1}}\Big(D_2^{i-1}\Big)E_i$$

for $i = 1, \ldots, n$. Set $d_i := \mathrm{mult}_{O_{i-1}}(D_1^{i-1}) = \mathrm{mult}_{O_{i-1}}(D_2^{i-1})$ for $i = 1, \ldots, n$.

Let R^i be the proper transform of R on the surface S_i for $i = 0, \ldots, n$. Then

$$R^i \equiv \pi_i^*\Big(R^{i-1}\Big) - \mathrm{mult}_{O_{i-1}}\Big(R^{i-1}\Big)E_i$$

for $i = 1, \ldots, n$. Set $r_i := \mathrm{mult}_{O_{i-1}}(R^{i-1})$ for $i = 1, \ldots, n$. In particular, $r_1 = \mathrm{mult}_O(R)$.

We may choose the blow ups $\pi_1, \ldots, \pi_n$ in a way such that $D_1^n \cap D_2^n$ is empty in the neighborhood of the exceptional locus of $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$. Then we get

$$\mathrm{mult}_O\Big(D_1 \cdot D_2\Big) = \sum_{i=1}^{n} d_i^2$$

by definition of the local multiplicity.

We may also choose the blow ups $\pi_1, \ldots, \pi_n$ in a way such that $D_1^n \cap R^n$ and $D_2^n \cap R^n$ are empty in the neighborhood of the exceptional locus of $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$. Then, as above, we get

$$\mathrm{mult}_O\Big(D_1 \cdot R\Big) = \mathrm{mult}_O\Big(D_2 \cdot R\Big) = \sum_{i=1}^{n} d_i r_i,$$

where some numbers among $r_1, \ldots, r_n$ may be zero. Then

$$\mathrm{mult}_O\Big(D_1 \cdot R\Big) = \mathrm{mult}_O\Big(D_2 \cdot R\Big) = \sum_{i=1}^{n} d_i r_i \leqslant \sum_{i=1}^{n} d_i r_1 \leqslant$$
$$\leqslant \sum_{i=1}^{n} d_i^2 r_1 = \mathrm{mult}_O\Big(R\Big)\mathrm{mult}_O\Big(D_1 \cdot D_2\Big),$$

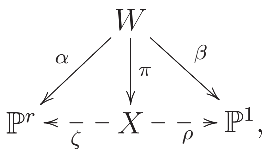
because $d_i \leqslant d_i^2$ and $r_i \leqslant r_1 = \mathrm{mult}_O(R)$ for every $i = 1, \ldots, n$. $\square$

Theorem 3.1.6 (see [14, Theorem 2.2]). *Let $\mathcal{B}$ be a linear system on a threefold V such that a general surface in $\mathcal{B}$ is irreducible. Let $\mathcal{P}$ be a pencil on V such that*

$$\mathrm{Supp}(P) \cap \mathrm{Supp}(B) \subseteq \Sigma,$$

where $P \in \mathcal{P}$, $B \in \mathcal{B}$ are general, and $\Sigma \subset V$ is a Zariski closed proper subset which does not depend on P and B. Then $\mathcal{B}$ and $\mathcal{P}$ coincide.

Proof. Let $\rho : V \dashrightarrow \mathbb{P}^1$ be the rational map induced by the pencil $\mathcal{P}$ and $\zeta : V \dashrightarrow \mathbb{P}^r$ be the rational map induced by the linear system $\mathcal{B}$. We then consider a simultaneous resolution of both of the rational maps as follows:



where W is a smooth variety, π is a birational morphism, α and β are morphisms. Let Λ be a Zariski closed subset of the variety W such that the morphism

$$\pi\big|_{W \setminus \Lambda} : W \setminus \Lambda \longrightarrow V \setminus \pi\big(\Lambda\big)$$

is an isomorphism. Let also Δ be the union of the set Λ and the closure of the proper transform of the set $\Sigma \setminus \pi(\Lambda)$ on the variety W. Then the set Δ is a proper subvariety of W.

Suppose that the pencil $\mathcal{P}$ is different from the linear system $\mathcal{B}$. Let B_W be the pull-back of a general hyperplane on $\mathbb{P}^r$ via the morphism α and P_W be a general fiber of the morphism β. Then the intersection $B_W \cap P_W$ is non-empty and the support $\mathrm{Supp}(B_W \cap P_W)$ is not contained in Δ. Hence we have

$$\mathrm{Supp}\big(\pi\big(B_W\big)\big) \cap \mathrm{Supp}\big(\pi\big(P_W\big)\big) \not\subset \Sigma,$$

where $\pi(B_W)$ and $\pi(P_W)$ are general divisors in the linear systems $\mathcal{B}$ and $\mathcal{P}$, respectively. The obtained contradiction proves Theorem 3.1.6. □

3.2 Curves

We use notation and conventions of Section 3.1. In the present section, we prove the following result:

Proposition 3.2.1. *Suppose that* $\mathbb{CS}(X, \mu\mathcal{M})$ *contains a curve. Then* $n = 1$.

Suppose that the set $\mathbb{CS}(X, \mu\mathcal{M})$ contains a curve Z. Then it follows from Lemmas 3.1.3 and 3.1.4 that $\mathbb{CS}(X, \mu\mathcal{M})$ does not contain points of the threefold X and

$$\mathrm{mult}_Z\big(\mathcal{M}\big) = n, \tag{3.2.1}$$

since the pair $(X, \mu\mathcal{M})$ is canonical but not terminal (see Lemmas 2.2.5, 2.2.6 and Theorem 2.2.32). Then $\deg(Z) \leqslant 4$ by [14, Lemma 2.1].

Lemma 3.2.2. *Suppose that* $\deg(Z) = 1$. *Then* $n = 1$.

Proof. Let $\pi : V \to X$ be the blow up of X along the line Z. Let $\mathcal{B}$ be the proper transform of the pencil $\mathcal{M}$ on the threefold V, and let B be a general surface in $\mathcal{B}$. Then

$$B \sim -nK_V \tag{3.2.2}$$

by (3.2.1). There is a commutative diagram

$$\begin{array}{ccc} & V & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\eta} \\ X & \overset{}{\underset{\psi}{\dashrightarrow}} & \mathbb{P}^2, \end{array}$$

where ψ is the linear projection from the line Z and η is the morphism induced by the complete linear system $|-K_V|$. Thus, it follows from (3.2.2) that $\mathcal{B}$ is the pull-back of a pencil $\mathcal{P}$ on $\mathbb{P}^2$ by η. In particular, the base locus of $\mathcal{B}$ is contained in the union of fibers of η.

The set $\mathbb{CS}(V, \mu\mathcal{B})$ is not empty by [14, Theorem 2.1]. It easily follows from (3.2.1) that the set $\mathbb{CS}(V, \mu\mathcal{B})$ does not contain points because $\mathbb{CS}(X, \mu\mathcal{M})$ contains no points. In particular, there is an irreducible curve $L \subset V$ such that

$$\mathrm{mult}_L\big(\mathcal{B}\big) = n$$

and $\eta(L)$ is a point $Q \in \mathbb{P}^2$. Let C be a general curve in $\mathcal{P}$. Then $\mathrm{mult}_Q(C) = n$. But

$$C \sim \mathcal{O}_{\mathbb{P}^2}\big(n\big)$$

by (3.2.2). Thus, we see that $n = 1$, since a general surface in $\mathcal{M}$ is irreducible. $\square$

Thus, we may assume that the set $\mathbb{CS}(X, \mu\mathcal{M})$ does not contain lines.

Lemma 3.2.3. *The curve $Z \subset \mathbb{P}^4$ is contained in a plane.*

Proof. Suppose that Z is not contained in any plane in $\mathbb{P}^4$. Let us show that this assumption leads to a contradiction. Since $\deg(Z) \leqslant 4$, we have

$$\deg\big(Z\big) \in \big\{3, 4\big\},$$

and Z is smooth if $\deg(Z) = 3$. If $\deg(Z) = 4$, then Z may have at most one double point.

Suppose that Z is smooth. Let $\alpha\colon U \to X$ be the blow up of X at Z, and let F be the exceptional divisor of the morphism α. Then the base locus of the linear system

$$\Big|\alpha^*\Big(- \deg\big(Z\big) K_X\Big) - F\Big|$$

does not contain any curve. Let D_1 and D_2 be the proper transforms on U of two sufficiently general surfaces in the linear system $\mathcal{M}$. Then it follows from (3.2.1) that

$$\Bigg(\alpha^*\Big(- \deg\big(Z\big) K_X\Big) - F\Bigg) \cdot D_1 \cdot D_2 =$$

$$= n^2 \Bigg(\alpha^*\Big(- \deg\big(Z\big) K_X\Big) - F\Bigg) \cdot \Bigg(\alpha^*\Big(- K_X\Big) - F\Bigg)^2 \geqslant 0,$$

since the cycle $D_1 \cdot D_2$ is effective. On the other hand, we have

$$\Big(\alpha^*\Big(-\deg\big(Z\big)K_X\Big)-F\Big)\cdot\Big(\alpha^*\Big(-K_X\Big)-F\Big)^2 = \Bigg(3\deg\big(Z\big)-\Big(\deg\big(Z\big)\Big)^2-2\Bigg) < 0,$$

which is a contradiction. Thus, the curve Z is not smooth.

Thus, we see that Z is a quartic curve with a double point O.

Let $\beta\colon W \to X$ be the composition of the blow up of the point O with the blow up of the proper transform of the curve Z. Let G and E be the exceptional surfaces of the birational morphism β such that $\beta(E) = Z$ and $\beta(G) = O$. Then the base locus of the linear system

$$\Big|\beta^*\big(- 4K_X\big) - E - 2G\Big|$$

does not contain any curve.

Let R_1 and R_2 be the proper transforms on W of two sufficiently general surfaces in $\mathcal{M}$. Put $m = \mathrm{mult}_O(\mathcal{M})$. Then it follows from (3.2.1) that

$$\left(\beta^*\Big(-4K_X\Big) - E - 2G\right) \cdot R_1 \cdot R_2 =$$

$$= \left(\beta^*\Big(-4K_X\Big) - E - 2G\right) \cdot \left(\beta^*\Big(-nK_X\Big) - nE - mG\right)^2 \geqslant 0,$$

and $m < 2n$, since the set $\mathbb{CS}(X, \mu\mathcal{M})$ does not contain points. Then we get

$$\left(\beta^*\Big(-4K_X\Big)-E-2G\right)\cdot\left(\beta^*\Big(-nK_X\Big)-nE-mG\right)^2 = -8n^2+6mn-m^2 < 0,$$

which is a contradiction. □

If $\deg(Z) = 4$, then $n = 1$ by Lemma 3.2.3 and [14, Theorem 2.2].

Lemma 3.2.4. *Suppose that* $\deg(Z) = 3$. *Then* $n = 1$.

Proof. Let $\mathcal{P}$ be the pencil in $|-K_X|$ that contains all hyperplane sections of X that pass through the curve Z. Then the base locus of $\mathcal{P}$ consists of the curve Z and a line $L \subset X$.

Let D be a sufficiently general surface in the pencil $\mathcal{P}$, and let M be a sufficiently general surface in the pencil $\mathcal{M}$. Then D is a smooth surface, and

$$M\Big|_D = nZ + \mathrm{mult}_L\big(\mathcal{M}\big)L + B \equiv nZ + nL, \qquad (3.2.3)$$

where B is a curve whose support does not contain neither Z nor L.

On the surface D, we have $Z \cdot L = 3$ and $L \cdot L = -2$. Intersecting (3.2.3) with L, we get

$$n = \big(nZ + nL\big) \cdot L = 3n - 2\mathrm{mult}_L\big(\mathcal{M}\big) + B \cdot L \geqslant 3n - 2\mathrm{mult}_L\big(\mathcal{M}\big),$$

which easily implies that $\mathrm{mult}_L(\mathcal{M}) \geqslant n$. But the inequality $\mathrm{mult}_L(\mathcal{M}) \geqslant n$ is impossible because we assumed that $\mathbb{CS}(X, \mu\mathcal{M})$ contains no lines. □

Lemma 3.2.5. *Suppose that* $\deg(Z)=2$. *Then* $n=1$.

Proof. Let $\alpha\colon U\to X$ be the blow up of the curve Z. Then $|-K_U|$ is a pencil, whose base locus consists of a smooth irreducible curve $L\subset U$ (cf. the proof of Lemma 3.2.4).

Let D be a general surface in $|-K_U|$. Then D is a smooth surface.

Let $\mathcal{B}$ be the proper transform of the pencil $\mathcal{M}$ on the threefold U. Then

$$-nK_U\Big|_D \equiv B\Big|_D \equiv nL,$$

where B is a general surface in $\mathcal{B}$. But $L^2=-2$ on the surface D. Then

$$L\in \mathbb{CS}\Big(U,\mu\mathcal{B}\Big)$$

which implies that $\mathcal{B}=\Big|-K_U\Big|$ by [14, Theorem 2.2]. Then $n=1$. □

The assertion of Proposition 3.2.1 is proved.

3.3 Points

We use notation and conventions of Section 3.1.

Remark 3.3.1. To prove Theorem 1.2.7, it is enough to show that X can be given by

$$w^3x+w^2q_2\big(x,y,z,t\big)+wxp_2\big(x,y,z,t\big)+q_4\big(x,y,z,t\big)=0$$

in $\mathbb{P}^4=\mathrm{Proj}\Big(\mathbb{C}[x,y,z,t,w]\Big)$, where q_i and p_i are homogeneous polynomials of degree $i\geqslant 2$ such that $q_2(0,y,z,t)\neq 0$.

Suppose that $n\neq 1$. Then Proposition 3.2.1 implies that there is a point $P\in X$ such that

$$P\in \mathbb{CS}\Big(X,\mu\mathcal{M}\Big).$$

It follows from Lemmas 3.1.2–3.1.4 that

- there are finitely many distinct lines $L_1, \ldots, L_r \subset X$ containing $P \in X$,
- the equality $\mathrm{mult}_P(M) = 2n$ holds, and
$$n/3 \leqslant \mathrm{mult}_{L_i}\big(M\big) \leqslant n/2,$$
where M is a general surface in the pencil $\mathcal{M}$,
- the equality $\mathrm{mult}_P(T) = 2$ holds, where $T \in |-K_X|$ is the surface such that $\mathrm{mult}_P(T) \geqslant 2$,
- the base locus of the pencil $\mathcal{M}$ consists of the lines $L_1, \ldots, L_r$, and
$$\mathrm{mult}_P\Big(M_1 \cdot M_2\Big) = 4n^2,$$
where M_1 and M_2 are sufficiently general surfaces in $\mathcal{M}$.

Lemma 3.3.2. *The equality* $\mathbb{CS}(X, \mu\mathcal{M}) = \{P\}$ *holds.*

Proof. The set $\mathbb{CS}(X, \mu\mathcal{M})$ does not contain curves by Proposition 3.2.1.

Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a point $Q \in X$ such that $Q \neq P$. Then $r = 1$.

Let D be a general hyperplane section of X that passes through L_1. Then
$$M\Big|_D = \mathrm{mult}_{L_1}\big(\mathcal{M}\big)L_1 + \Delta,$$
where M is a general surface in $\mathcal{M}$ and Δ is an effective divisor such that
$$\mathrm{mult}_P\big(\Delta\big) \geqslant 2n - \mathrm{mult}_{L_1}\big(\mathcal{M}\big) \leqslant \mathrm{mult}_Q\big(\Delta\big).$$

On the surface D, we have $L_1^2 = -2$. Then we get
$$n = \Big(\mathrm{mult}_{L_1}\big(\mathcal{M}\big)L_1 + \Delta\Big) \cdot L_1 =$$
$$-2\mathrm{mult}_{L_1}\big(\mathcal{M}\big) + \Delta \cdot L \geqslant -2\mathrm{mult}_{L_1}\big(\mathcal{M}\big) + 2\Big(2n - \mathrm{mult}_{L_1}\big(\mathcal{M}\big)\Big),$$

which gives $\mathrm{mult}_{L_1}(\mathcal{M}) \geqslant 3n/4$. On the other hand, $\mathrm{mult}_{L_1}(\mathcal{M}) \leqslant n/2$ by Lemma 3.1.4, a contradiction. □

The quartic threefold X can be given by the equation

$$w^3x + w^2q_2\big(x, y, z, t\big) + wq_3\big(x, y, z, t\big) + q_4\big(x, y, z, t\big) = 0$$

in $\mathbb{P}^4 = \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t, w]\Big)$, where q_i is a homogeneous polynomial of degree $i \geqslant 2$.

Remark 3.3.3. The lines $L_1, \ldots, L_r \subset \mathbb{P}^4$ are given by the equations

$$x = q_2\big(x, y, z, t\big) = q_3\big(x, y, z, t\big) = q_4\big(x, y, z, t\big) = 0,$$

for the surface T is cut out on X by $x = 0$, and $\mathrm{mult}_P(T) = 2$ iff $q_2(0, y, z, t) \neq 0$.

Let $\pi \colon V \to X$ be the blow up of the point P with the exceptional divisor E. Then we have

$$\mathcal{B} \equiv \pi^*\big(-nK_X\big) - 2nE \equiv -nK_V,$$

where $\mathcal{B} := \pi_*^{-1}(\mathcal{M})$.

Remark 3.3.4. The pencil $\mathcal{B}$ has no base curves in E because

$$\mathrm{mult}_P\big(M_1 \cdot M_2\big) = \mathrm{mult}_P\big(M_1\big)\mathrm{mult}_P\big(M_2\big).$$

Let $\bar{L}_i$ be the proper transform of the line L_i on the threefold V for $i = 1, \ldots, r$. Then

$$B_1 \cdot B_2 = \sum_{i=1}^{r} \mathrm{mult}_{\bar{L}_i}\Big(B_1 \cdot B_2\Big)\bar{L}_i,$$

where B_1 and B_2 are proper transforms of M_1 and M_2 on the threefold V, respectively.

Lemma 3.3.5. *Let Z be an irreducible curve on X such that $Z \notin \{L_1, \ldots, L_r\}$. Then*

$$\deg\big(Z\big) \geqslant 2\mathrm{mult}_P\big(Z\big),$$

and the equality $\deg(Z) = 2\mathrm{mult}_P(Z)$ implies that

$$\bar{Z} \cap \left(\bigcup_{i=1}^{r} \bar{L}_i\right) = \varnothing,$$

where $\bar{Z}$ is a proper transform of the curve Z on the threefold V.

Proof. The curve $\bar{Z}$ is not contained in the base locus of the pencil $\mathcal{B}$. Then

$$0 \leqslant B_i \cdot \bar{Z} \leqslant n\Big(\deg\big(Z\big) - 2\mathrm{mult}_P\big(Z\big)\Big),$$

which implies the required assertions. □

To conclude the proof of Theorem 1.2.7, it is enough to show that

$$q_3\big(x, y, z, t\big) = xp_2\big(x, y, z, t\big) + q_2\big(x, y, z, t\big)p_1\big(x, y, z, t\big),$$

where p_1 and p_2 are some homogeneous polynomials of degree 1 and 2, respectively.

3.4 Good points

We use notation and conventions of Sections 3.1 and 3.3. Suppose that the conic

$$q_2(0, y, z, t) = 0 \subset \mathbb{P}^2 = \mathrm{Proj}\Big(\mathbb{C}[y, z, t]\Big)$$

is reduced and irreducible. In the present Section, we prove the following result:

Proposition 3.4.1. *The polynomial $q_3(0, y, z, t)$ is divisible by $q_2(0, y, z, t)$.*

Let us prove Proposition 3.4.1. Suppose that $q_3(0, y, z, t)$ is not divisible by $q_2(0, y, z, t)$.

Let $\mathcal{R}$ be the linear system on the threefold X that is cut out by quadrics

$$xh_1(x, y, z, t) + \lambda\Big(wx + q_2(x, y, z, t)\Big) = 0,$$

where h_1 is an arbitrary linear form and $\lambda \in \mathbb{C}$. Then $\mathcal{R}$ does not have fixed components.

Lemma 3.4.2. *Let R_1 and R_2 be general surfaces in the linear system $\mathcal{R}$. Then*

$$\sum_{i=1}^{r} \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big) \leqslant 6.$$

Proof. We may assume that R_1 is cut out by the equation

$$wx + q_2(x, y, z, t) = 0,$$

and R_2 is cut out by $xh_1(x, y, z, t) = 0$, where h_1 is sufficiently general. Then

$$\mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big) = \mathrm{mult}_{L_i}\Big(R_1 \cdot T\Big).$$

Put $m_i = \mathrm{mult}_{L_i}(R_1 \cdot T)$. Then

$$R_1 \cdot T = \sum_{i=1}^{r} m_i L_i + \Delta,$$

where $m_i \in \mathbb{N}$, and Δ is a cycle, whose support contains no lines passing through P.

Let $\bar{R}_1$ and $\bar{T}$ be the proper transforms of R_1 and T on V, respectively. Then

$$\bar{R}_1 \cdot \bar{T} = \sum_{i=1}^{r} m_i \bar{L}_i + \Omega,$$

where Ω is an effective cycle, whose support contains no lines passing through P.

The support of the cycle Ω does not contain curves that are contained in the exceptional divisor E, because $q_3(0,y,z,t)$ is not divisible by $q_2(0,y,z,t)$ by our assumption. Then

$$6 = E \cdot \bar{R}_1 \cdot \bar{T} = \sum_{i=1}^{r} m_i \big(E \cdot \bar{L}_i\big) + E \cdot \Omega \geqslant \sum_{i=1}^{r} m_i \big(E \cdot \bar{L}_i\big) = \sum_{i=1}^{r} m_i,$$

which is exactly what we want. □

Let M and R be general surfaces in $\mathcal{M}$ and $\mathcal{R}$, respectively. Put

$$M \cdot R = \sum_{i=1}^{r} m_i L_i + \Delta,$$

where $m_i \in \mathbb{N}$, and Δ is a cycle, whose support contains no lines passing through P.

Lemma 3.4.3. *The cycle Δ is not trivial.*

Proof. Suppose that $\Delta = 0$. Then $\mathcal{M} = \mathcal{R}$ (see Theorem 3.1.6). But $\mathcal{R}$ is not a pencil. □

We have $\deg(\Delta) = 8n - \sum_{i=1}^{r} m_i$. On the other hand, the inequality

$$\mathrm{mult}_P\big(\Delta\big) \geqslant 6n - \sum_{i=1}^{r} m_i$$

holds, because $\mathrm{mult}_P(M) = 2n$ and $\mathrm{mult}_P(R) \geqslant 3$. It follows from Lemma 3.3.5 that

$$\deg\big(\Delta\big) = 8n - \sum_{i=1}^{r} m_i \geqslant 2\mathrm{mult}_P\big(\Delta\big) \geqslant 2\left(6n - \sum_{i=1}^{r} m_i\right),$$

which implies that $\sum_{i=1}^{r} m_i \geqslant 4n$. But it follows from Lemmas 3.1.5 and 3.1.4 that

$$m_i \leqslant \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)\mathrm{mult}_{L_i}\big(M\big) \leqslant \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)n/2$$

for every $i = 1, \ldots, r$, where R_1 and R_2 are general surfaces in $\mathcal{R}$. Then

$$\sum_{i=1}^{r} m_i \leqslant \sum_{i=1}^{r} \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big) n/2 \leqslant 3n$$

by Lemma 3.4.2, which is a contradiction.

The assertion of Proposition 3.4.1 is proved.

3.5 Bad points

We use notation and conventions of Sections 3.1 and 3.3. Suppose that the conic

$$q_2(0, y, z, t) = 0 \subset \mathbb{P}^2 = \mathrm{Proj}\Big(\mathbb{C}[y, z, t]\Big)$$

is reduced and reducible. Therefore, we have

$$q_2(x, y, z, t) = \big(\alpha_1 y + \beta_1 z + \gamma_1 t\big)\big(\alpha_2 y + \beta_2 z + \gamma_2 t\big) + x p_1\big(x, y, z, t\big)$$

where $p_1(x, y, z, t)$ is a linear form, and $(\alpha_1 : \beta_1 : \gamma_1) \in \mathbb{P}^2 \ni (\alpha_2 : \beta_2 : \gamma_2)$. In the present Section, we prove the following result:

Proposition 3.5.1. *The polynomial $q_3(0, y, z, t)$ is divisible by $q_2(0, y, z, t)$.*

Suppose that $q_3(0, y, z, t)$ is not divisible by $q_2(0, y, z, t)$. Then without loss of generality, we may assume that $q_3(0, y, z, t)$ is not divisible by $\alpha_1 y + \beta_1 z + \gamma_1 t$.

Let Z be the curve in X that is cut out by the equations

$$x = \alpha_1 y + \beta_1 z + \gamma_1 t = 0.$$

Remark 3.5.2. The equality $\mathrm{mult}_P(Z) = 3$ holds, but Z is not necessary reduced.

Hence, it follows from Lemma 3.3.5 that $\mathrm{Supp}(Z)$ contains a line among $L_1, \ldots, L_r$.

Lemma 3.5.3. *The support of the curve Z does not contain an irreducible conic.*

Proof. Suppose that $\mathrm{Supp}(Z)$ contains an irreducible conic C. Then

$$Z = C + L_i + L_j$$

for some $i \in \{1, \ldots, r\} \ni j$. Then $i = j$, because otherwise the set

$$\Big(C \cap L_i\Big) \bigcup \Big(C \cap L_j\Big)$$

contains a point that is different from P, which is impossible by Lemma 3.3.5. We see that

$$Z = C + 2L_i,$$

and it follows from Lemma 3.3.5 that $C \cap L_i = P$. Then C is tangent to L_i at the point P.

Let $\bar{C}$ be the proper transform of the curve C on the threefold V. Then

$$\bar{C} \cap \bar{L}_i \neq \varnothing,$$

which is impossible by Lemma 3.3.5. The assertion is proved. □

Lemma 3.5.4. *The support of the curve Z consists of lines.*

Proof. Suppose that $\mathrm{Supp}(Z)$ does not consist of lines. It follows from Lemma 3.5.3 that

$$Z = L_i + C,$$

where C is an irreducible cubic curve. But $\mathrm{mult}_P(Z) = 3$. Then

$$\mathrm{mult}_P\big(C\big) = 2,$$

which is impossible by Lemma 3.3.5 □

We may assume that there is a line $L \subset X$ such that $P \notin L$ and

$$Z = a_1 L_1 + \cdots + a_k L_k + L,$$

where $a_1, a_2, a_3 \in \mathbb{N}$ such that $a_1 \geqslant a_2 \geqslant a_3$ and $\sum_{i=1}^{k} a_i = 3$.

Remark 3.5.5. We have $L_i \neq L_j$ whenever $i \neq j$.

Let H be a sufficiently general surface of X that is cut out by the equation

$$\lambda x + \mu\Big(\alpha_1 y + \beta_1 z + \gamma_1 t\Big) = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Then H has at most isolated singularities.

Remark 3.5.6. The surface H is smooth at the points P and $L \cap L_i$, where $i = 1, \ldots, k$.

Let $\bar{H}$ and $\bar{L}$ be the proper transforms of H and L on the threefold V, respectively.

Lemma 3.5.7. *The inequality $k \neq 3$ holds.*

Proof. Suppose that the equality $k = 3$ holds. Then H is smooth. Put

$$B\Big|_{\bar{H}} = m_1\bar{L}_1 + m_2\bar{L}_2 + m_3\bar{L}_3 + \Omega,$$

where B is a general surface in $\mathcal{B}$, and Ω is an effective divisor on $\bar{H}$ whose support does not contain any of the curves $\bar{L}_1$, $\bar{L}_2$ and $\bar{L}_3$. Then

$$\bar{L} \not\subseteq \mathrm{Supp}\big(\Omega\big) \not\supseteq \bar{H} \cap E,$$

because the base locus of the pencil $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Then

$$n = \bar{L} \cdot \Big(m_1\bar{L}_1 + m_2\bar{L}_2 + m_3\bar{L}_3 + \Omega\Big) = \sum_{i=1}^{3} m_i + \bar{L} \cdot \Omega \geqslant \sum_{i=1}^{3} m_i,$$

which implies that $\sum_{i=1}^{3} m_i \leqslant n$. On the other hand, we have

$$-n = \bar{L}_i \cdot \Big(m_1\bar{L}_1 + m_2\bar{L}_2 + m_3\bar{L}_3 + \Omega\Big) = -3m_i + L_i \cdot \Omega \geqslant -3m_i,$$

which implies that $m_i \geqslant n/3$. Thus, we have $m_1 = m_2 = m_3 = n/3$ and

$$\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = \Omega \cdot \bar{L}_2 = \Omega \cdot \bar{L}_3 = 0,$$

which implies that $\mathrm{Supp}(\Omega) \cap \bar{L}_1 = \mathrm{Supp}(\Omega) \cap \bar{L}_2 = \mathrm{Supp}(\Omega) \cap \bar{L}_3 = \varnothing$.

Let B' be another general surface in $\mathcal{B}$. Arguing as above, we see that

$$B'\Big|_{\bar{H}} = \frac{n}{3}\Big(\bar{L}_1 + \bar{L}_2 + \bar{L}_3\Big) + \Omega',$$

where Ω' is an effective divisor on the surface $\bar{H}$ such that

$$\mathrm{Supp}\big(\Omega'\big) \cap \bar{L}_1 = \mathrm{Supp}\big(\Omega'\big) \cap \bar{L}_2 = \mathrm{Supp}\big(\Omega'\big) \cap \bar{L}_3 = \varnothing.$$

One can easily check that $\Omega \cdot \Omega' = n^2 \neq 0$. Then

$$\mathrm{Supp}\big(\Omega\big) \cap \mathrm{Supp}\big(\Omega'\big) \neq \varnothing,$$

because $|\mathrm{Supp}(\Omega) \cap \mathrm{Supp}(\Omega')| < +\infty$ due to generality of the surfaces B and B'.

The base locus of the pencil $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Hence, we have

$$\mathrm{Supp}\big(B\big) \cap \mathrm{Supp}\big(B'\big) = \bigcup_{i=1}^{r} \bar{L}_i,$$

but $\bar{L}_i \cap \bar{H} = \varnothing$ whenever $i \notin \{1, 2, 3\}$. Hence, we have

$$\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \cup \big(\mathrm{Supp}\big(\Omega\big) \cap \mathrm{Supp}\big(\Omega'\big)\big) = \mathrm{Supp}\big(B\big) \cap \mathrm{Supp}\big(B'\big) \cap \bar{H} = \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3,$$

which implies that $\mathrm{Supp}(\Omega) \cap \mathrm{Supp}(\Omega') \subset \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3$. In particular, we see that

$$\mathrm{Supp}(\Omega) \cap \Big(\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3\Big) \neq \varnothing,$$

because $\mathrm{Supp}(\Omega) \cap \mathrm{Supp}(\Omega') \neq \varnothing$. But $\mathrm{Supp}(\Omega) \cap \bar{L}_i = \varnothing$ for $i = 1, 2, 3$. □

Lemma 3.5.8. *The inequality $k \neq 2$ holds.*

Proof. Suppose that the equality $k=2$ holds. Then $Z=2L_1+L_2+L$. Put

$$B\Big|_{\bar{H}}=m_1\bar{L}_1+m_2\bar{L}_2+\Omega,$$

where B is a general surface in $\mathcal{B}$, and Ω is an effective divisor on $\bar{H}$ whose support does not contain the curves $\bar{L}_1$ and $\bar{L}_2$. Then $\bar{L}\not\subseteq\mathrm{Supp}(\Omega)\not\supseteq\bar{H}\cap E$ and

$$n=\bar{L}\cdot\Big(m_1\bar{L}_1+m_2\bar{L}_2+\Omega\Big)=m_1+m_2+\bar{L}\cdot\Omega\geqslant m_1+m_2,$$

which implies that $m_1+m_2\leqslant n$. On the other hand, we have

$$\bar{T}|_{\bar{H}}=2\bar{L}_1+\bar{L}_2+\bar{L}+E\Big|_{\bar{H}}\equiv\Big(\pi^*\Big(-K_X\Big)-2E\Big)\Big|_{\bar{H}},$$

where $\bar{T}$ is the proper transform of the surface T on the threefold V. Then

$$-1=\bar{L}_1\cdot\Big(2\bar{L}_1+\bar{L}_2+\bar{L}+E\Big|_{\bar{H}}\Big)=2\Big(\bar{L}_1\cdot\bar{L}_1\Big)+2,$$

which implies that $\bar{L}_1\cdot\bar{L}_1=-3/2$ on the surface $\bar{H}$. Then

$$-n=\bar{L}_1\cdot\Big(m_1\bar{L}_1+m_2\bar{L}_2+\Omega\Big)=-3m_1/2+L_1\cdot\Omega\geqslant-3m_1/2,$$

which gives $m_1\geqslant 2n/3$. Similarly, we see that $\bar{L}_2\cdot\bar{L}_2=-3$ on the surface $\bar{H}$. Then

$$-n=\bar{L}_2\cdot\Big(m_1\bar{L}_1+m_2\bar{L}_2+\Omega\Big)=-3m_2+L_2\cdot\Omega\geqslant-3m_2,$$

which implies that $m_2\leqslant n/3$. Thus, we have $m_1=2m_2=2n/3$ and

$$\Omega\cdot\bar{L}=\Omega\cdot\bar{L}_1=\Omega\cdot\bar{L}_2=0,$$

which implies that $\mathrm{Supp}(\Omega)\cap\bar{L}_1=\mathrm{Supp}(\Omega)\cap\bar{L}_2=\varnothing$.

Let B' be another general surface in $\mathcal{B}$. Arguing as above, we see that

$$B'\Big|_{\bar{H}}=\frac{2n}{3}\bar{L}_1+\frac{n}{3}\bar{L}_2+\Omega',$$

where Ω' is an effective divisor on $\bar{H}$ whose support does not contain $\bar{L}_1$ and $\bar{L}_2$ such that

$$\mathrm{Supp}\big(\Omega'\big)\cap\bar{L}_1=\mathrm{Supp}\big(\Omega'\big)\cap\bar{L}_2=\varnothing,$$

which implies that $\Omega\cdot\Omega'=n^2$. In particular, we see that

$$\mathrm{Supp}\big(\Omega\big)\cap\mathrm{Supp}\big(\Omega'\big)\neq\varnothing,$$

and arguing as in the proof of Lemma 3.5.7 we obtain a contradiction. □

It follows from Lemmas 3.5.7 and 3.5.8 that $Z=3L_1+L$. Put

$$B\Big|_{\bar{H}}=m_1\bar{L}_1+\Omega,$$

where B is a general surface in $\mathcal{B}$, and Ω is a curve such that $\bar{L}_1\not\subseteq\mathrm{Supp}(\Omega)$. Then

$$\bar{L}\not\subseteq\mathrm{Supp}\big(\Omega\big)\not\supseteq\bar{H}\cap E,$$

because the base locus of $\mathcal{B}$ consists of the curves $\bar{L}_1,\ldots,\bar{L}_r$. Then

$$n=\bar{L}\cdot\Big(m_1\bar{L}_1+\Omega\Big)=m_1+\bar{L}\cdot\Omega\geqslant m_1,$$

which implies that $m_1\leqslant n$. On the other hand, we have

$$\bar{T}|_{\bar{H}}=3\bar{L}_1+\bar{L}+E\Big|_{\bar{H}}\equiv\Big(\pi^*\Big(-K_X\Big)-2E\Big)\Big|_{\bar{H}},$$

where $\bar{T}$ is the proper transform of the surface T on the threefold V. Then

$$-1=\bar{L}_1\cdot\Big(3\bar{L}_1+\bar{L}+E\Big|_{\bar{H}}\Big)=3\bar{L}_1\cdot\bar{L}_1+2,$$

which implies that $\bar{L}_1\cdot\bar{L}_1=-1$ on the surface $\bar{H}$. Then

$$-n=\bar{L}_1\cdot\Big(m_1\bar{L}_1+\Omega\Big)=-m_1+L_1\cdot\Omega\geqslant -m_1,$$

which gives $m_1\geqslant n$. Thus, we have $m_1=n$ and $\Omega\cdot\bar{L}=\Omega\cdot\bar{L}_1=0$. Then $\mathrm{Supp}(\Omega)\cap\bar{L}_1=\varnothing$.

Let B' be another general surface in $\mathcal{B}$. Arguing as above, we see that

$$B'\Big|_{\bar{H}} = n\bar{L}_1 + \Omega',$$

where Ω' is an effective divisor on $\bar{H}$ whose support does not contain $\bar{L}_1$ such that

$$\mathrm{Supp}\big(\Omega'\big) \cap \bar{L}_1 = \varnothing,$$

which implies that $\Omega \cdot \Omega' = n^2$. In particular, we see that $\mathrm{Supp}(\Omega) \cap \mathrm{Supp}(\Omega') \neq \varnothing$.

The base locus of the pencil $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, L_r$. Hence, we have

$$\mathrm{Supp}\big(B\big) \cap \mathrm{Supp}\big(B'\big) = \bigcup_{i=1}^{r} \bar{L}_i,$$

but $\bar{L}_i \cap \bar{H} = \varnothing$ whenever $\bar{L}_i \neq \bar{L}_1$. Then $\mathrm{Supp}(\Omega) \cap \bar{L}_1 \neq \varnothing$, because

$$\bar{L}_1 \cup \Big(\mathrm{Supp}\big(\Omega\big) \cap \mathrm{Supp}\big(\Omega'\big)\Big) = \mathrm{Supp}\big(B\big) \cap \mathrm{Supp}\big(B'\big) \cap \bar{H} = \bar{L}_1,$$

which is a contradiction. The assertion of Proposition 3.5.1 is proved.

3.6 Very bad points

We use notation and conventions of Sections 3.1 and 3.3. Suppose that $q_2 = y^2$. The proof of Proposition 6.1 implies that $q_3(0, y, z, t)$ is divisible by y. Then

$$q_3 = yf_2\big(z, t\big) + xh_2\big(z, t\big) + x^2 a_1\big(x, y, z, t\big) + xyb_1\big(x, y, z, t\big) + y^2 c_1\big(y, z, t\big)$$

where a_1, b_1, c_1 are linear forms, f_2 and h_2 is are homogeneous polynomials of degree two. In the present Section, we prove the following result:

Proposition 3.6.1. *The equality $f_2(z, t) = 0$ holds.*

Let us prove Proposition 3.6.1 by reductio ad absurdum. Suppose that $f_2(z,t) \neq 0$.

Remark 3.6.2. By choosing suitable coordinates, we may assume that $f_2 = zt$ or $f_2 = z^2$.

We must use smoothness of the threefold X by analyzing the shape of q_4. We have

$$q_4 = f_4\big(z,t\big)+xu_3\big(z,t\big)+yv_3\big(z,t\big)+x^2a_2(x,y,z,t)+xyb_2(x,y,z,t)+y^2c_2(y,z,t),$$

where a_2, b_2, c_2 are homogeneous polynomials of degree two, u_3 and v_3 are homogeneous polynomials of degree three, and f_4 is a homogeneous polynomial of degree four.

Lemma 3.6.3. *Suppose that $f_2(z,t) = zt$ and*

$$f_4\big(z,t\big) = t^2 g_2\big(z,t\big)$$

for some $g_2(z,t) \in \mathbb{C}[z,t]$. Then $v_3(z,0) \neq 0$.

Proof. Suppose that $v_3(z,0) = 0$. The surface T is given by the equation

$$w^2y^2 + yzt + y^2c_1\big(x,y,z,t\big) + t^2g_2\big(z,t\big) + yv_3\big(z,t\big) + y^2c_2\big(x,y,z,t\big) = 0$$

in $\mathbb{P}^3 = \mathrm{Proj}\Big(\mathbb{C}[y,z,t,w]\Big)$, since T is cut out on X by the equation $x = 0$. Then T has non-isolated singularity along the line $x = y = t = 0$, which is impossible because X is smooth. □

Arguing as in the proof of Lemma 3.6.3, we obtain the following corollary.

Corollary 3.6.4. *Suppose that $f_2(z,t) = zt$ and*

$$f_4\big(z,t\big) = z^2 g_2\big(z,t\big)$$

for some $g_2(z,t) \in \mathbb{C}[z,t]$. Then $v_3(0,t) \neq 0$.

Lemma 3.6.5. *Suppose that $f_2(z,t)=zt$. Then $f_4(0,t)=f_4(z,0)=0$.*

Proof. We may assume that $f_4(z,0)\neq 0$. Let $\mathcal{H}$ be the linear system on X that is cut out by

$$\lambda x+\mu y+\nu t=0,$$

where $(\lambda:\mu:\nu)\in\mathbb{P}^2$. Then the base locus of $\mathcal{H}$ consists of the point P.

Let $\mathcal{R}$ be a proper transform of $\mathcal{H}$ on the threefold V. Note that the base locus of $\mathcal{R}$ consists of a single point that is not contained in any of the curves $\bar{L}_1,\ldots,\bar{L}_r$.

The linear system $\mathcal{R}|_B$ has no base points, where B is a general surface in $\mathcal{B}$. But

$$R\cdot R\cdot B=2n>0,$$

where R is a general surface in $\mathcal{R}$. Then $\mathcal{R}|_B$ is not composed from a pencil, which implies that the curve $R\cdot B$ is irreducible and reduced by the Bertini theorem.

Let H and M be general surfaces in $\mathcal{H}$ and $\mathcal{M}$, respectively. Then $M\cdot H$ is irreducible and reduced. Thus, the linear system $\mathcal{M}|_H$ is a pencil.

The surface H contains no lines passing through P, and H can be given by the equation

$$w^3x+w^2y^2+w\Big(y^2l_1\big(x,y,z\big)+xl_2\big(x,y,z\big)\Big)+l_4\big(x,y,z\big)=0$$

in $\mathbb{P}^3=\mathrm{Proj}\Big(\mathbb{C}[x,y,z,w]\Big)$, where $l_i(x,y,z)$ is a homogeneous polynomials of degree i.

Arguing as in Example 1.2.6, we see that there is a pencil $\mathcal{Q}$ on the surface H such that

$$\mathcal{Q}\sim\mathcal{O}_{\mathbb{P}^3}\big(2\big)\Big|_H,$$

general curve in $\mathcal{Q}$ is irreducible, and $\mathrm{mult}_P(\mathcal{Q}) = 4$. Arguing as in the proof of Lemma 3.1.2, we see that $\mathcal{M}|_H = \mathcal{Q}$ by [14, Theorem 2.2]. Let M be a general surface in $\mathcal{M}$. Then

$$M \equiv -2K_X,$$

and $\mathrm{mult}_P(M) = 4$. The surface M is cut out on X by an equation

$$\lambda x^2 + x\Big(A_0 + A_1\big(y,z,t\big)\Big) + B_2\big(y,z,t\big) + B_1\big(y,z,t\big) + B_0 = 0,$$

where A_i and B_i are homogeneous polynomials of degree i, and $\lambda \in \mathbb{C}$.

It follows from $\mathrm{mult}_P(M) = 4$ that $B_1(y,z,t) = B_0 = 0$.

The coordinated (y,z,t) are also local coordinates on X near the point P. Then

$$x = -y^2 - y\Big(zt + yp_1\big(y,z,t\big)\Big) + \text{higher order terms},$$

which is a Taylor power series for $x = x(y,z,t)$, where $p_1(y,z,t)$ is a linear form.

The surface M is locally given by the analytic equation

$$\lambda y^4 + \Big(-y^2 - yzt - y^2 p_1\big(y,z,t\big)\Big)\Big(A_0 + A_1\big(y,z,t\big)\Big) + B_2\big(y,z,t\big) + \text{higher order terms} = 0,$$

and $\mathrm{mult}_P(M) = 4$. Hence, we see that $B_2(y,z,t) = A_0 y^2$ and

$$A_1\big(y,z,t\big)y^2 + A_0 y\Big(zt + yp_1\big(y,z,t\big)\Big) = 0,$$

which implies that $A_0 = A_1(y,z,t) = B_2(y,z,t) = 0$. Hence, we see that a general surface in the pencil $\mathcal{M}$ is cut out on X by the equation $x^2 = 0$, which is a absurd. $\square$

Arguing as in the proof of Lemma 3.6.5, we obtain the following corollary.

Corollary 3.6.6. *Suppose that* $f_2(z,t) = z^2$. *Then* $f_4(0,t) = 0$.

Let $\mathcal{R}$ be the linear system on the threefold X that is cut out by cubics

$$xh_2\big(x,y,z,t\big) + \lambda\Big(w^2x + wy^2 + q_3\big(x,y,z,t\big)\Big) = 0,$$

where h_2 is a form of degree 2, and $\lambda \in \mathbb{C}$. Then $\mathcal{R}$ has no fixed components.

Let M and R be general surfaces in $\mathcal{M}$ and $\mathcal{R}$, respectively. Put

$$M \cdot R = \sum_{i=1}^{r} m_i L_i + \Delta,$$

where $m_i \in \mathbb{N}$, and Δ is a cycle, whose support contains no lines among $L_1, \ldots, L_r$.

Lemma 3.6.7. *The cycle* Δ *is not trivial.*

Proof. Suppose that $\Delta = 0$. Then $\mathcal{M} = \mathcal{R}$ (see Theorem 3.1.6). But $\mathcal{R}$ is not a pencil. □

We have $\mathrm{mult}_P(\Delta) \geqslant 8n - \sum_{i=1}^{r} m_i$, because $\mathrm{mult}_P(\mathcal{M}) = 2n$ and $\mathrm{mult}_P(\mathcal{R}) \geqslant 4$. Then

$$\deg\big(\Delta\big) = 12n - \sum_{i=1}^{r} m_i \geqslant 2\mathrm{mult}_P\big(\Delta\big) \geqslant 2\Big(8n - \sum_{i=1}^{r} m_i\Big)$$

by Lemma 3.3.5, because $\mathrm{Supp}(\Delta)$ does not contain any of the lines $L_1, \ldots, L_r$.

Corollary 3.6.8. *The inequality* $\sum_{i=1}^{r} m_i \geqslant 4n$ *holds.*

Let R_1 and R_2 be general surfaces in the linear system $\mathcal{R}$. Then

$$m_i \leqslant \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)\mathrm{mult}_{L_i}(M) \leqslant \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)n/2$$

for every $1 \leqslant i \leqslant 4$ by Lemmas 3.1.5 and 3.1.4. Then

$$4n \leqslant \sum_{i=1}^{r} m_i \leqslant \sum_{i=1}^{r} \mathrm{mult}_{L_i}\Big(R_1 \cdot R_2\Big)n/2.$$

Corollary 3.6.9. *The inequality* $\sum_{i=1}^{r}\mathrm{mult}_{L_i}(R_1\cdot R_2)\geqslant 8$ *holds.*

Now we suppose that R_1 is cut out on the quartic X by the equation

$$w^2x+wy^2+q_3\big(x,y,z,t\big)=0,$$

and R_2 is cut out by $xh_2\big(x,y,z,t\big)=0$, where h_2 is sufficiently general. Then

$$\sum_{i=1}^{r}\mathrm{mult}_{L_i}\Big(R_1\cdot T\Big)=\sum_{i=1}^{r}\mathrm{mult}_{L_i}\Big(R_1\cdot R_2\Big)\geqslant 8,$$

where T is the hyperplane section of the hypersurface X that is cut out by $x=0$. But

$$R_1\cdot T=Z_1+Z_2,$$

where Z_1 and Z_2 are cycles on X such that Z_1 is cut out by $x=y=0$, and Z_2 is cut out by

$$x=wy+f_2\big(z,t\big)+yc_1\big(x,y,z,t\big)=0.$$

Lemma 3.6.10. *The equality* $\sum_{i=1}^{r}\mathrm{mult}_{L_i}(Z_1)=4$ *holds.*

Proof. The lines $L_1,\ldots,L_r\subset\mathbb{P}^4$ are given by the equations

$$x=y=q_4\big(x,y,z,t\big)=0,$$

which implies that $\sum_{i=1}^{r}\mathrm{mult}_{L_i}(Z_1)=4$. □

Hence, we see that $\sum_{i=1}^{r}\mathrm{mult}_{L_i}(Z_2)\geqslant 4$. But Z_2 can be considered as a cycle

$$wy+f_2\big(z,t\big)+yc_1\big(y,z,t\big)=f_4\big(z,t\big)+yv_3\big(z,t\big)+y^2c_2(y,z,t)=0$$

in $\mathbb{P}^3=\mathrm{Proj}\Big(\mathbb{C}[y,z,t,w]\Big)$, and by setting $u=w+c_1(y,z,t)$, we see that Z_2 can be considered as a cycle

$$uy+f_2\big(z,t\big)=f_4\big(z,t\big)+yv_3\big(z,t\big)+y^2c_2(y,z,t)=0$$

in $\mathbb{P}^3 = \mathrm{Proj}\Big(\mathbb{C}[y,z,t,u]\Big)$, and we can consider the set of lines $L_1, \ldots, L_r$ as the set of curves in $\mathbb{P}^3$ given by $y = f_4(z,t) = 0$.

Lemma 3.6.11. *The inequality $f_2(z,t) \neq zt$ holds.*

Proof. Suppose that $f_2(z,t) = zt$. Then it follows from Lemma 3.6.5 that

$$f_4\big(z,t\big) = zt\big(\alpha_1 z + \beta_1 t\big)\big(\alpha_2 z + \beta_2 t\big)$$

for some $(\alpha_1 : \beta_1) \in \mathbb{P}^1 \ni (\alpha_2 : \beta_2)$. Then Z_2 can be given by

$$uy + zt = yv_3\big(z,t\big) + y^2 c_2\big(y,z,t\big) - uy\big(\alpha_1 z + \beta_1 t\big)\big(\alpha_2 z + \beta_2 t\big) = 0$$

in $\mathbb{P}^3 = \mathrm{Proj}\Big(\mathbb{C}[y,z,t,u]\Big)$, which implies $Z_2 = Z_2^1 + Z_2^2$, where Z_2^1 and Z_2^2 are cycles in $\mathbb{P}^3$ such that Z_2^1 is given by

$$y = uy + zt = 0,$$

and Z_2^2 is given by $uy + zt = v_3(z,t) + yc_2(y,z,t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0$.

We may assume that L_1 is given by $y = z = 0$, and L_2 is given by $y = t = 0$. Then

$$Z_2^1 = L_1 + L_2,$$

which implies that $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) \geqslant 2$.

Suppose that $r = 4$. Then $\alpha_1 \neq 0$, $\beta_1 \neq 0$, $\alpha_2 \neq 0$, $\beta_2 \neq 0$. Hence, we see that

$$L_1 \not\subseteq \mathrm{Supp}\big(Z_2^2\big) \not\supseteq L_2,$$

because $v_3(z,t) + yc_2(y,z,t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$ does not vanish on L_1 and L_2. But

$$L_3 \not\subseteq \mathrm{Supp}\big(Z_2^2\big) \not\supseteq L_4,$$

because zt does not vanish on L_3 and L_4. Then $\sum_{i=1}^{r}\mathrm{mult}_{L_i}(Z_2^2)=0$, which is impossible.

Suppose that $r=3$. We may assume that $(\alpha_1,\beta_1)=(1,0)$, but $\alpha_2\neq 0\neq\beta_2$. Then

$$L_2\not\subseteq\mathrm{Supp}\big(Z_2^2\big),$$

because $v_3(z,t)+yc_2(y,z,t)-uz(\alpha_2 z+\beta_2 t)$ does not vanish on L_2. We have

$$f_4\big(z,t\big)=z^2t(\alpha_2 z+\beta_2 t),$$

which implies that $v_3(0,t)\neq 0$ by Corollary 3.6.4. Hence, wee see that

$$L_1\not\subseteq\mathrm{Supp}\big(Z_2^2\big)\not\supseteq L_3,$$

because $v_3(z,t)+yc_2(y,z,t)-uz(\alpha_2 z+\beta_2 t)$ and zt do not vanish on L_1 and L_3, respectively, which implies that $\sum_{i=1}^{r}\mathrm{mult}_{L_i}(Z_2^2)=0$. The latter is a contradiction.

We see that $r=2$. We may assume that $(\alpha_1,\beta_1)=(1,0)$, and either $\alpha_2=0$ or $\beta_2=0$.

Suppose that $\alpha_2=0$. Then $f_4(z,t)=\beta_2 z^2t^2$. By Lemma 3.6.3 and Corollary 3.6.4, we get

$$v_3\big(0,t\big)\neq 0\neq v_3\big(z,0\big),$$

which implies that $v_3(z,t)+yc_2(y,z,t)-\beta_2 zt$ does not vanish on neither L_1 nor L_2. Then

$$L_1\not\subseteq\mathrm{Supp}\big(Z_2^2\big)\not\supseteq L_2,$$

which implies that $\sum_{i=1}^{r}\mathrm{mult}_{L_i}\big(Z_2^2\big)=0$, which is a contradiction.

We see that $\alpha_2\neq 0$ and $\beta_2=0$. We have $f_4(z,t)=\alpha_2 z^3t$. Then

$$v_3\big(0,t\big)\neq 0$$

by Corollary 3.6.4. Then $L_1 \not\subseteq \mathrm{Supp}(Z_2^2)$ because the polynomial

$$v_3(z,t) + yc_2(y,z,t) - \alpha_2 z^2$$

does not vanish on L_1.

The line L_2 is given by the equations $y = t = 0$. But Z_2 is given by the equations

$$uy + zt = v_3\big(z,t\big) + yc_2\big(y,z,t\big) - \alpha_2 uz^2 = 0,$$

which implies that $L_2 \not\subseteq \mathrm{Supp}(Z_2^2)$. Then $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction. □

Therefore, we see that $f_2(z,t) = z^2$. It follows from Corollary 3.6.6 that

$$f_4\big(z,t\big) = zg_3\big(z,t\big)$$

for some $g_3(z,t) \in \mathbb{C}[z,t]$. We may assume that L_1 is given by $y = z = 0$.

Lemma 3.6.12. *The equality $g_3(0,t) = 0$ holds.*

Proof. Suppose that $g_3(0,t) \neq 0$. Then $\mathrm{Supp}(Z_2) = L_1$, because Z_2 is given by

$$uy + z^2 = zg_3(z,t) + yv_3\big(z,t\big) + y^2c_2(y,z,t) = 0,$$

and the lines $L_2, \ldots, L_r$ are given by the equations $y = g_3(z,t) = 0$.

The cycle $Z_2 + L_1$ is given by the equations

$$uy + z^2 = z^2g_3\big(z,t\big) + zyv_3\big(z,t\big) + zy^2c_2\big(y,z,t\big) = 0,$$

which implies that the cycle $Z_2 + L_1$ can be given by the equations

$$uy + z^2 = zyv_3\big(z,t\big) + zy^2c_2\big(y,z,t\big) - uyg_3\big(z,t\big) = 0.$$

We have $Z_2 + L_1 = C_1 + C_2$, where C_1 and C_2 are cycles in $\mathbb{P}^3$ such that C_1 is given by

$$y = uy + z^2 = 0,$$

and the cycle C_2 is given by the equations

$$uy + z^2 = zv_3\big(z,t\big) + zyc_2\big(y,z,t\big) - ug_3\big(z,t\big) = 0.$$

We have $C_1 = 2L_2$. But $L_1 \not\subseteq \mathrm{Supp}(C_2)$ because the polynomial

$$zv_3\big(z,t\big) + zyc_2\big(y,z,t\big) - ug_3\big(z,t\big)$$

does not vanish on L_1, because $g_3(0,t) \neq 0$. Then

$$Z_2 + L_1 = 2L_2,$$

which implies that $Z_2 = L_1$. Then $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2) = 1$, which is a contradiction. $\square$

Thus, we see that $r \leqslant 3$ and

$$f_4\big(z,t\big) = z^2\big(\alpha_1 z + \beta_1 t\big)\big(\alpha_2 z + \beta_2 t\big)$$

for some $(\alpha_1 : \beta_1) \in \mathbb{P}^1 \ni (\alpha_2 : \beta_2)$. Then

$$v_3\big(0,t\big) \neq 0$$

by Corollary 3.6.4. But Z_2 can be given by the equations

$$uy + z^2 = yv_3\big(z,t\big) + y^2c_2\big(y,z,t\big) - uy\big(\alpha_1 z + \beta_1 t\big)\big(\alpha_2 z + \beta_2 t\big) = 0$$

in $\mathbb{P}^3 = \mathrm{Proj}\Big(\mathbb{C}[y,z,t,u]\Big)$, which implies $Z_2 = Z_2^1 + Z_2^2$, where Z_2^1 and Z_2^2 are cycles on $\mathbb{P}^3$ such that Z_2^1 is given by

$$y = uy + z^2 = 0,$$

and the cycle Z_2^2 is given by the equations

$$uy + z^2 = v_3\big(z,t\big) + yc_2\big(y,z,t\big) - u\big(\alpha_1 z + \beta_1 t\big)\big(\alpha_2 z + \beta_2 t\big) = 0,$$

which implies that $Z_2^1 = 2L_1$. Thus, we see that $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) \geqslant 2$.

Lemma 3.6.13. *The inequality $r \neq 3$ holds.*

Proof. Suppose that $r = 3$. Then $\beta_1 \neq 0 \neq \beta_2$, which implies that

$$L_1 \not\subseteq \mathrm{Supp}\big(Z_2^2\big),$$

because $v_3(z,t) + yc_2(y,z,t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$ does not vanish on L_1. But

$$L_2 \not\subseteq \mathrm{Supp}\big(Z_2^2\big) \not\supseteq L_3,$$

because $\beta_1 \neq 0 \neq \beta_2$. Then $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction. □

Thus, we see that either $r = 1$ or $r = 2$.

Lemma 3.6.14. *The inequality $r \neq 2$ holds.*

Proof. Suppose that $r = 2$. We may assume that

- either $\beta_1 \neq 0 = \beta_2$,
- or $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 \neq 0$.

Suppose that $\beta_2 = 0$. Then $f_4(z,t) = \alpha_2 z^3(\alpha_1 z + \beta_1 t)$ and

$$L_1 \not\subseteq \mathrm{Supp}\big(Z_2^2\big),$$

because $v_3(z,t) + yc_2(y,z,t) - \alpha_2 uz(\alpha_1 z + \beta_2 t)$ does not vanish on L_1. But L_2 is given by

$$y = \alpha_1 z + \beta_1 t = 0,$$

which implies that z^2 does not vanish on L_2, because $\beta_1 \neq 0$. Then

$$L_2 \not\subseteq \mathrm{Supp}\big(Z_2^2\big),$$

which implies that $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

Hence, we see that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 \neq 0$. Then $L_1 \not\subseteq \mathrm{Supp}(Z_2^2)$, because

$$v_3(z,t) + yc_2(y,z,t) - u(\alpha_1 z + \beta_1 t)^2$$

does not vanish on L_1. But $L_2 \not\subseteq \mathrm{Supp}(Z_2^2)$, because z^2 does not vanish on L_2. Then

$$\sum_{i=1}^{r} \mathrm{mult}_{L_i}\big(Z_2^2\big) = 0,$$

which is a contradiction. □

We see that $f_4(z,t) = z^2$ and $f_4(z,t) = \mu z^4$ for some $0 \neq \mu \in \mathbb{C}$. Then Z_2^2 is given by

$$uy + z^2 = v_3\big(z,t\big) + yc_2\big(y,z,t\big) - \mu z^2 = 0,$$

where $v_3(0,t) \neq 0$ by Corollary 3.6.4. Thus, we see that $L_1 \not\subseteq \mathrm{Supp}(Z_2^2)$, because

$$v_3\big(z,t\big) + yc_2\big(y,z,t\big) - \mu z^2$$

does not vanish on L_1. Then $\sum_{i=1}^{r} \mathrm{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

The assertion of Proposition 3.6.1 is proved.

The assertion of Theorem 1.2.7 follows from Propositions 3.2.1, 3.4.1, 3.5.1, 3.6.1.

3.7 Corollaries and conclusive remarks

From Theorem 1.2.7 and Example 1.2.6 we get the following

Corollary 3.7.1. *A general surface in any Halphen pencil on a smooth quartic threefold in* $\mathbb{P}^4$ *is birationally isomorphic to a* K3 *surface.*

Let us also point out the relation between the result of Theorem 1.2.7 and the following local invariant of a point on the smooth quartic threefold $X \subset \mathbb{P}^4$. Namely, set

$$m(O) := \sup\{\lambda \in \mathbb{Q} \mid \text{the linear system } \big|n(\pi^*(-K_X) - \lambda E)\big| \\ \text{has no fixed components for } n \gg 0\},$$

where $\pi\colon Y \longrightarrow X$ is the blow up of the point O with $E := \pi^{-1}(O)$. We call the number $m(O)$ *mobility threshold* of X at O. It can be easily seen that $1 \leqslant m(O) \leqslant 2$. Furthermore, Example 1.2.6 and Lemma 3.1.2 show that the upper bound $m(O) = 2$ can be achieved, and hence Theorem 1.2.7 characterizes those points on X for which the equality $m(O) = 2$ holds. On the other hand, the lower bound $m(O) = 1$ is also achieved and holds precisely when X contains a cone with the vertex at O. Thus, it is reasonable to pose the following

Problem 3.7.1. Given X as above, calculate $m(O)$ for every point O on X and describe those O with the given value $m(O)$.

There is also a "simplified" version of the mobility threshold:

$$s(O) := \sup\{\lambda \in \mathbb{Q} \mid \text{the linear system } \big|n(\pi^*(-K_X) - \lambda E)\big| \\ \text{has no fixed components for } n \gg 0, \text{ and } \mathrm{Bs}\big|n(\pi^*(-K_X) - \lambda E)\big| \subset E\}.$$

According to [24, Theorem 1.2], $s(O)$ is nothing but the *Seshadri constant* of $-K_X$ at the point O, i.e., $s(O) := \sup\big\{\lambda \in \mathbb{Q} \mid \pi^*(-K_X) - \lambda E \text{ is nef}\big\}$. One can consider a problem, similar to Problem 3.7.1, for $s(O)$.

Chapter 4

Polarized K3 surfaces

4.1 Auxiliary results

Before starting the proof of Theorem 1.2.11, let us state some auxiliary results. Firstly, we introduce few more properties of Fano threefolds (see Definition 2.3.14):

Theorem 4.1.1 (see [22], [40]). *Let* $\mathbb{P} := \mathbb{P}(a_0, \ldots, a_n)$ *be a* well-formed *weighted projective space (i.e.,* $\gcd(a_0, \ldots, \hat{a}_i, \ldots, a_n) = 1$ *for all* i*). Then*

- $(\mathcal{O}_{\mathbb{P}}(1))^n = \dfrac{1}{a_0 \ldots a_n}$;
- $\mathrm{Pic}(\mathbb{P}) \simeq \mathbb{Z}$;
- $\omega_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}}) = \mathcal{O}_{\mathbb{P}}(-\sum_{i=0}^{n} a_i)$.

Theorem 4.1.2 (see [79, Theorem 1.5]). *Let* V *be a Fano threefold. Then* $(-K_V)^3 \leqslant 72$ *with equality iff either* $V = \mathbb{P}(3,1,1,1)$ *or* $V = \mathbb{P}(6,4,1,1)$ *(cf. Examples 2.3.18, 2.3.19).*

Theorem 4.1.3 (see [47]). *Let V be a Fano threefold. If* $\mathrm{Bs}|-K_V|\neq\emptyset$*, then* $(-K_V)^3\leqslant 22$.

Theorem 4.1.4 (see [15, Theorem 1.5]). *In the notation of Theorem 4.1.3, let* $\mathrm{Bs}|-K_V|=\emptyset$ *and* $\Phi_{|-K_V|}: V\longrightarrow\mathbb{P}^{g+1}$ *be the corresponding morphism. If* $\Phi_{|-K_V|}$ *is not an embedding, then* $(-K_V)^3\leqslant 40$.

Theorem 4.1.5 (see [15, Theorem 1.6]). *In the notation of Theorem 4.1.4, if* $\Phi_{|-K_V|}$ *is an embedding and* $\Phi_{|-K_V|}(V)$ *is not an intersection of quadrics, then* $(-K_V)^3\leqslant 54$.

From Theorems 4.1.3, 4.1.4 and 4.1.5 we get the following

Corollary 4.1.6. *Let V be a Fano threefold such that* $(-K_V)^3\geqslant 54$*. Then the linear system* $|-K_V|$ *gives an embedding of V into* $\mathbb{P}^{g+1}$ *such that the image* $\Phi_{|-K_V|}(V)$ *is an intersection of quadrics.*

Lemma 4.1.7. *In the notation of Corollary 4.1.6, let* $\pi: V\dashrightarrow V'$ *be the linear projection from a linear space* $\Lambda\subset V$ *such that* $\dim V'=3$*. Then the map π is birational.*

Proof. This is obvious because V is an intersection of quadrics. □

Lemma 4.1.8. *In the notation of Corollary 4.1.6, let* $\pi: V\dashrightarrow V'$ *be the linear projection from a singular* cA_1 *point* $O\in V$*. Then V' is an anticanonically embedded Fano threefold such that* $(-K_{V'})^3=(-K_V)^3-2$.

Proof. Consider the blow up $\sigma: W\longrightarrow X'$ of the X' at O:

$$\begin{array}{ccc} & W & \\ {}^{\sigma}\swarrow & & \searrow^{\tau} \\ V & \overset{\pi}{\dashrightarrow} & V'. \end{array}$$

The linear projection π is given by the linear system $\mathcal{H} \subset |-K_V|$ of all hyperplane sections of V passing through O. Further, since $O \in V$ is a singular cA_1 point, the singularity $(O \in V)$ is analytically isomorphic to

$$\big((0,0,0,0) \in \big(x^2+y^2+z^2+\varepsilon t^n=0\big)\big) \subset \mathbb{C}^4 = \mathrm{Spec}\big(\mathbb{C}[x,y,z,t]\big),$$

where $n \geqslant 2$, for some $\varepsilon \in \mathbb{C}$ (see [2]). The latter implies that W has at most canonical Gorenstein singularities and for a general surface $H \in \mathcal{H}$ we have

$$\sigma_*^{-1}(H) = \sigma^*(H) - E_\sigma,$$

where E_σ is the σ-exceptional divisor. On the other hand, from the adjunction formula we get the equality

$$K_W = \sigma^*(K_V) + E_\sigma.$$

Thus, the morphism $\tau : W \longrightarrow V'$ is given by the linear system $\sigma_*^{-1}(\mathcal{H}) \subseteq |-K_W|$. In particular, W is a weak Fano threefold, since $\sigma_*^{-1}(\mathcal{H})$ is basepoint-free on W and $(-K_W)^3 = (-K_V)^3-2 > 0$. Then $\dim V' = 3$ and π is birational (see Lemma 4.1.7). Thus, since τ is a crepant morphism, the threefold X has only canonical Gorenstein singularities (cf. Example 2.3.17). Moreover, we have $\mathcal{O}_X(-K_X) \simeq \mathcal{O}_X(1)$ by construction, which implies that V' is an anticanonically embedded Fano threefold with $(-K_{V'})^3 = (-K_V)^3 - 2$. $\square$

Further, let us state some facts about deformations of algebraic varieties and their quotients by algebraic groups:

Proposition 4.1.9 (see [6]). *Let V be a smooth algebraic variety and $W \subseteq V$ a smooth closed subvariety. Then the first order deformations of the pair (V, W) are parameterized by the group $H^1(V, T_V\langle W\rangle)$. Moreover, the map* *which associates to a first order deformation of* (V, W) *the corresponding deformation of* W, *is the induced map* $H^1(r) : H^1(V, T_V\langle W\rangle) \longrightarrow H^1(W, T_W)$.

Theorem 4.1.10 (see [6]). *Let* Σ *be a* K3 *surface*[1)] of type L and genus g *for an integral lattice* L *(i.e., there exists a primitive embedding* $\mathrm{L} \hookrightarrow \mathrm{Pic}(\mathbb{Z})$ *of lattices, which maps some element in* L *to an ample class* $L \in \mathrm{Pic}(\mathbb{Z})$ *with* $(L^2) = 2g - 2$*). Then*

- *the first order deformations of* (Σ, L) *are parameterized by the orthogonal of* $c_1(\mathrm{L}) \subset H^1(\Sigma, \Omega^1_\Sigma)$ *in* $H^1(\Sigma, T_\Sigma)$*;*
- *the moduli stack* $\mathcal{K}_g^{\mathrm{L}}$ *of all* K3 *surfaces of type* L *is smooth, irreducible, and of dimension* $20 - \mathrm{rank}(\mathrm{L})$.

Theorem 4.1.11 (see [76]). *Let* V *be a normal scheme on which an algebraic group scheme* G *acts properly. Then the geometric quotient* V/G *exists as a normal algebraic scheme.*

4.2 Beginning of the proof of Theorem 1.2.11

Let X be the Fano threefold of genus 36 (or degree 70).[2)] Let us present the construction and some properties of X.

Consider the weighted projective space $\mathbb{P} := \mathbb{P}(1, 1, 4, 6)$ with weighted homogeneous coordinates x_0, x_1, x_2, x_3 of weights 1, 1, 4, 6, respectively. $\mathbb{P}$ is a Fano 3-fold of degree 72 (see Example 2.3.19 and Theorem 4.1.1). Furthermore, the linear system $|-K_{\mathbb{P}}|$ gives an embedding of $\mathbb{P}$ in $\mathbb{P}^{38}$ such that the image $\Phi_{|-K_{\mathbb{P}}|}(\mathbb{P})$ is an intersection of quadrics (see Corollary 4.1.6). In what follows, we assume that $\mathbb{P} \subset \mathbb{P}^{38}$ is anticanonically embedded.

[1)] Throughout this chapter all K3 surfaces are assumed to be smooth.
[2)] This X was found by I. Cheltsov

Lemma 4.2.1. $L := \mathrm{Sing}(\mathbb{P})$ *is a line on* $\mathbb{P}$ *(i.e.,* $L \simeq \mathbb{P}^1$ *and* $-K_{\mathbb{P}} \cdot L = 1$*).*

Proof. The curve L is given by equations $x_0 = x_1 = 0$ on $\mathbb{P}$ (see [40, 5.15]). This implies that $L \simeq \mathbb{P}^1$. It remains to show that $-K_{\mathbb{P}} \cdot L = 1$.

Let S be a surface on $\mathbb{P}$ with equation $x_0 = 0$. Then $L \subset S$ and

$$-K_{\mathbb{P}} \cdot L = -K_{\mathbb{P}}\big|_S \cdot L,$$

where the last intersection is taken on $S = \mathbb{P}(6, 4, 1) \simeq \mathbb{P}(3, 2, 1)$ (see [40, 5.7]).

Since $\mathcal{O}_{\mathbb{P}}(-K_{\mathbb{P}}) \simeq \mathcal{O}_{\mathbb{P}}(12)$ (see Theorem 4.1.1), a general element $D \in |-K_{\mathbb{P}}|$ is given by equation

$$a_0x_3^2+x_2^3+a_6(x_0,x_1)x_3+a_2(x_0,x_1)x_2x_3+a_4(x_0,x_1)x_2^2+a_8(x_0,x_1)x_2+a_{12}(x_0,x_1) = 0$$

on $\mathbb{P}$, where $a_i(x_0, x_1)$ are homogeneous polynomials of degree i in the variables x_0, x_1. Furthermore, in weighted projective coordinates y_0, y_1, y_2 on $S = \mathbb{P}(3, 2, 1)$ of weights 3, 2, 1, respectively, the cycle $D\big|_S$ is given by the equation

$$b_0y_0^2 + y_1^3 + b_1y_0y_3^3 + b_2y_0y_1y_2 + b_3y_1^3 + b_4y_1^2y_2^2 + b_5y_1y_2^4 + b_6y_2^6 = 0,$$

where $b_i \in \mathbb{C}$ (see [40, 5.10]). This implies that $\mathcal{O}_S(D\big|_S) \simeq \mathcal{O}_S(6)$. On the other hand, we have $\mathcal{O}_S(L) \simeq \mathcal{O}_S(1)$. Thus, we get

$$-K_{\mathbb{P}} \cdot L = -K_{\mathbb{P}}\big|_S \cdot L = 1,$$

where the last intersection is taken on $S = \mathbb{P}(3, 2, 1)$ (see Theorem 4.1.1). $\square$

Further, there are two points P and Q on L such that the germs $(P \in \mathbb{P})$ and $(Q \in \mathbb{P})$ are the singularities of types $\frac{1}{6}(4, 1, 1)$ and $\frac{1}{4}(2, 1, 1)$, respectively, and for every point O in $L \setminus \{P,\ Q\}$ the germ $(O \in \mathbb{P})$ is locally analytically isomorphic to the singularity $((0, o) \in \mathbb{C} \times W)$, where $(o \in W)$ is the singularity of type $\frac{1}{2}(1, 1)$ (see Example 2.3.19).

Proposition 4.2.2. *L is the unique line on $\mathbb{P}$.*

Proof. Let $L_0 \neq L$ be another line on $\mathbb{P}$. Since $-K_{\mathbb{P}} \sim \mathcal{O}_{\mathbb{P}}(12)$, we have

$$\mathcal{O}_{\mathbb{P}}(1) \cdot L_0 = \frac{1}{12}, \tag{4.2.1}$$

which implies that $L \cap L_0 \neq \emptyset$. Consider the crepant resolution $\phi : T \longrightarrow \mathbb{P}$ of $\mathbb{P}$ from Example 2.3.19. Set $L_0' := \phi_*^{-1}(L_0)$, $E_Q := \phi^{-1}(Q)$, $E_P := \phi^{-1}(P)$ and $E_L := \overline{\phi^{-1}(L \setminus \{P, Q\})}$, the Zariski closure in T of $\phi^{-1}(L \setminus \{P, Q\})$. These are all the components of the ϕ-exceptional locus. Furthermore, we have $E_P = E_P^{(1)} \cup E_P^{(2)}$, where $E_P^{(i)}$ are irreducible components of the divisor E_P such that $E_P^{(1)} \cap E_L = \emptyset$ and $E_P^{(2)} \cap E_L \neq \emptyset$.

Since $\rho(\mathbb{P}) = 1$, the group $N_1(T)$ is generated by the classes of ϕ-exceptional curves and some curve Z on T such that $R := \mathbb{R}_+[Z]$ is the K_T-negative extremal ray (see Proposition 2.3.21). In particular, since $-K_T \cdot L_0' = 1$, Proposition 2.3.21 implies that

$$L_0' \equiv Z + E^*, \tag{4.2.2}$$

where E^* is a linear combination with nonnegative coefficients of irreducible ϕ-exceptional curves. Further, the linear projection π_L of $\mathbb{P}$ from L is given by the linear system $\mathcal{H} \subset |-K_{\mathbb{P}}|$ of all hyperplane sections of $\mathbb{P}$ containing L. In addition, π_L maps L_0 to the point because $L \cap L_0 \neq \emptyset$ and $\mathbb{P}$ is the intersection of quadrics. On the other hand, ϕ factors through the blow up of $\mathbb{P}$ at L (see Example 2.3.19). Hence the linear system $\phi_*^{-1}\mathcal{H}$ is basepoint-free on T and $H \cdot L_0' = 0$ for $H \in \phi_*^{-1}\mathcal{H}$. In particular, $H \in |-K_T - E_L|$.

Lemma 4.2.3. *In* (4.2.2)*, the support* Supp(E^*) *of E^* is either $\emptyset$ or e_P, where $e_P \subset E_P^{(1)}$.*

Proof. As we saw, the face of the Mori cone $\overline{NE}(T)$, which corresponds to the nef divisor H, contains the class of the curve L'_0. Then from (4.2.2) we get

$$H \cdot Z = H \cdot E^* = 0.$$

In particular, H intersects trivially every curve in $\mathrm{Supp}(E^*)$. On the other hand, we have $\mathrm{Supp}(E^*) \subseteq \{e_P, e_Q, e_L\}$, where e_P, e_Q, e_L are the curves in E_P, E_Q, E_L, respectively. But for $e_P \subset E_P^{(2)}$ intersections $H \cdot e_P$, $H \cdot e_Q$, $H \cdot e_L$ are all non-zero. Thus, $\mathrm{Supp}(E^*)$ is either $\emptyset$ or e_P, where $e_P \subset E_P^{(1)}$. □

Consider the extremal contraction $f_R : T \longrightarrow T'$ of R. The morphism f_R is birational with exceptional divisor E_R (see Theorems 2.3.22, 2.3.23 and 2.3.24).

Lemma 4.2.4. *The divisor $-K_{T'}$ is not nef.*

Proof. Suppose that $-K_{T'}$ is nef, i.e., T' is a weak Fano 3-fold (with, possibly, non-Gorenstein singularities). If T' has only terminal factorial singularities, then, since $(-K_{T'})^3 \geqslant (-K_T)^3 = 72$ (see Theorem 2.3.24 and Proposition 2.3.26), T' is a terminal $\mathbb{Q}$-factorial modification of either $\mathbb{P}(1,1,1,3)$ or $\mathbb{P}(1,1,4,6)$ (see Theorem 4.1.2 and Remark 2.3.20). In particular, we have either $\rho(T') = 5$ or 2 (see Examples 2.3.18 and 2.3.19). On the other hand, $\rho(T') = \rho(T) - 1 = 4$, a contradiction.

Thus, the singularities of T' are worse than factorial. In this case, $f_R(E_R)$ is a point (see Theorem 2.3.24 and Proposition 2.3.26), and we get

$$E_P \cap E_R = E_Q \cap E_R = \emptyset. \tag{4.2.3}$$

On the other hand, it follows from (4.2.2) that $-K_{\mathbb{P}} \cdot \phi_*(Z) = 1$, i.e., $\phi(Z)$ is a line on $\mathbb{P}$. In particular, as for L_0 above, we have $\phi_*(Z) \cap L \neq \emptyset$. But then (4.2.3) implies that $0 = K_{T'} \cdot Z = -1$, a contradiction. □

It follows from Lemma 4.2.4 that $-K_{T'}$ is not nef and $E_R = \mathbb{F}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$ (see Corollary 2.3.27). But if $E_R = \mathbb{F}_1$, then $\phi(E_R)$ is a plane on $\mathbb{P}$ such that $L \not\subset \phi(E_R)$ (see Corollary 2.3.27). This implies that there is a line on $\mathbb{P}$ not intersecting L, a contradiction (see (4.2.1)). Finally, in the case when $E_R = \mathbb{P}^1 \times \mathbb{P}^1$, we have $Z \subset E_R = E_L$ (see Corollary 2.3.27), and if $\mathrm{Supp}(E^*) = \emptyset$ in (4.2.2), then $L_0 = L$, a contradiction. Hence, by Lemma 4.2.3, we get $\mathrm{Supp}(E^*) = e_P$, where $e_P \subset E_P^{(1)}$ is an irreducible curve. Further, on E_R we have

$$Z \sim l, \qquad E_P\big|_{E_R} = E_P^{(2)}\big|_{E_R} \sim h \sim E_Q\big|_{E_R},$$

which implies that $E_P^{(2)} \cdot Z = E_Q \cdot Z = 1$. On the other hand, since $L_0 \neq L$, we have either $E_P^{(2)} \cdot L_0' = 0$ or $E_Q \cdot L_0' = 0$. Then, intersecting (4.2.2) with $E_P^{(2)}$ and E_Q, we get a contradiction because $E_P^{(2)} \cdot e_P,\ E_Q \cdot e_P \geqslant 0$.

Thus, $L_0 = L$. Proposition 4.2.2 is completely proved. $\square$

Coming back to the construction of X, take any point O in $L \setminus \{P,\ Q\}$ and consider the linear projection $\pi : \mathbb{P} \dashrightarrow \mathbb{P}^{37}$ from O. Then the image of π is a Fano 3-fold X_O of degree 70 (see Lemma 4.1.8).

Proposition 4.2.5. *For any point O' in $L \setminus \{P,\ Q,\ O\}$, the image of the linear projection $\mathbb{P} \dashrightarrow \mathbb{P}^{37}$ from O' is a Fano 3-fold $X_{O'}$ isomorphic to X_O.*

Proof. In the above notation, L is given by equations $x_0 = x_1 = 0$ on $\mathbb{P}$, with equations of P and Q being $x_0 = x_1 = x_2 = 0$ and $x_0 = x_1 = x_3 = 0$, respectively (see [40, 5.15]). Then the torus $(\mathbb{C}^*)^3$, acting on $\mathbb{P}$, acts transitively on the set $L \setminus \{P,\ Q\}$, which induces an isomorphism $X_{O'} \simeq X_O$. $\square$

In what follows, because of Proposition 4.2.5, we fix the point $O \in L \setminus \{P,\ Q\}$, the linear projection $\pi : \mathbb{P} \dashrightarrow \mathbb{P}^{37}$ from O, and denote the image of

π by X. Let us construct a terminal $\mathbb{Q}$-factorial modification of X. Consider the blow up $\sigma : W \longrightarrow \mathbb{P}$ of $\mathbb{P}$ at O, and the following commutative diagram:

$$\begin{array}{ccc} & W & \\ {}^{\sigma}\swarrow & & \searrow^{\mu} \\ \mathbb{P} & \underset{\pi}{\dashrightarrow} & X. \end{array}$$

The type of the singularity $(O \in \mathbb{P})$ implies that W has at most canonical Gorenstein singularities (cf. the proof of Lemma 4.1.8). Moreover, we have $\mathrm{Sing}(W) = \sigma_*^{-1}(L)$ and the singularities of W are exactly of the same kind as of $\mathbb{P}$, i.e., locally near every point in $\mathrm{Sing}(W)$, W is isomorphic to $\mathbb{P}$. Then, resolving the singularities of W in the same way as for $\mathbb{P}$ (see Example 2.3.19), we arrive at the birational morphism $\tau : Y \longrightarrow W$, with Y being smooth and $K_Y = \tau^*(K_W)$. Set $f := \tau \circ \mu : Y \longrightarrow X$.

Proposition 4.2.6. *$f : Y \longrightarrow X$ is a terminal $\mathbb{Q}$-factorial modification of X. Moreover, Y is unique up to isomorphism, i.e., every smooth weak Fano 3-fold of degree 70 is isomorphic to Y.*

Proof. The linear projection π is given by the linear system $\mathcal{H} \subset |-K_{\mathbb{P}}|$ of all hyperplane sections of $\mathbb{P}$ passing through O. For a general $H \in \mathcal{H}$, we have

$$\sigma_*^{-1}(H) = \sigma^*(H) - E_\sigma,$$

where E_σ is the σ-exceptional divisor. On the other hand, from the adjunction formula we get

$$K_W = \sigma^*(K_{\mathbb{P}}) + E_\sigma.$$

Thus, the morphism $\mu : W \longrightarrow X$ is given by the linear system $\sigma_*^{-1}(\mathcal{H}) \subseteq |-K_W|$. Furthermore, since $\mathbb{P}$ is an intersection of quadrics, π is a birational map, which implies that μ and f are also birational with $K_Y = f^*(K_X)$. In particular, $(-K_Y)^3 = (-K_X)^3 = 70$ (cf. the proof of Lemma 4.1.8).

Thus, it remains to prove that every smooth weak Fano 3-fold of degree 70 is isomorphic to Y. Let Y' be another smooth weak Fano 3-fold of degree 70. Then its image under the morphism $f' := \Phi_{|-nK_{Y'}|}$, $n \in \mathbb{N}$, is a Fano threefold X' such that $K_{Y'} \equiv f'^*(K_{X'})$ (see Example 2.3.17). Since $(-K_{Y'})^3 = (-K_{X'})^3 = 70$, we get $X' \simeq X$ and Y' is a terminal $\mathbb{Q}$-factorial modification of X. Then, since Y and Y' are relative minimal models over X, the induced birational map $Y \dashrightarrow Y'$ is either an isomorphism or a sequence of K_Y-flops over X (see Remark 2.3.13). Then, since Y and Y' are relative minimal models over X, the induced birational map $Y \dashrightarrow Y'$ is either an isomorphism or a sequence of K_Y-flops over X .

Lemma 4.2.7. *Every K_Y-trivial extremal birational contraction $f_1 : Y \longrightarrow Y_1$ is divisorial.*

Proof. Suppose that f_1 is small. In the notation from the proof of Proposition 4.2.2, denote by $E_{Y,L}$, $E_{P,L}^{(i)}$, $E_{Q,L}$ the proper transforms on Y of surfaces E_L, $E_P^{(i)}$, E_Q, respectively. The resolution $\tau : Y \longrightarrow W$ (or $\phi : T \longrightarrow \mathbb{P}$) is locally toric near Sing(W). In particular, we have $E_{P,L}^{(1)} \simeq \mathbb{F}_4$, $E_{P,L}^{(2)} \simeq \mathbb{F}_2$, $E_{Q,L} \simeq \mathbb{F}_2$, $E_{Y,L} \simeq \mathbb{F}_m$ for some $m \in \mathbb{N}$ (see Example 2.3.19), and hence the only possibility for f_1 is to contract the curve $Z = h$ on $E_{Y,L}$ such that $\tau(Z) = \sigma_*^{-1}(L)$.

On the other hand, using Theorem 2.3.24 and Proposition 2.3.26, it is not difficult to see that Y is obtained by the blow up of the 3-fold T at the curve $\phi^{-1}(O) \simeq \mathbb{P}^1$ (see [10], [49]). Furthermore, since $\mathbb{P}$ is singular along the line, we have $E_L \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (see Corollary 2.3.27), and hence $E_{Y,L} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, a contradiction. □

It follows from Lemma 4.2.7 that $Y' \simeq Y$. Proposition 4.2.6 is completely proved. □

Corollary 4.2.8. Sing(X) *consists of a unique point.*

Proof. Since the morphism $\mu : W \longrightarrow X$ is given by the linear system $\sigma_*^{-1}(\mathcal{H}) \subseteq |-K_W| = |\sigma^*(-K_{\mathbb{P}}) - E_\sigma|$ (see the proof of Proposition 4.2.6), it contracts only $\sigma_*^{-1}(L) = \mathrm{Sing}(W)$ to the unique singular point on X (see Proposition 4.2.2). □

Corollary 4.2.9. *We have* $\mathrm{Pic}(X) = \mathbb{Z} \cdot K_X$ *and* $\mathrm{Cl}(X) = \mathbb{Z} \cdot K_X \oplus \mathbb{Z} \cdot E$, *where* $E := \mu_*(E_\sigma)$.

Proof. This follows from the construction of X and equalities $\rho(\mathbb{P}) = 1$, $(-K_X)^3 = 70$. □

Remark 4.2.10. It follows from the construction of X that $f = \Phi_{|-K_Y|}$ and $X \subseteq \mathbb{P}^{37}$ is anticanonically embedded (cf. Corollary 4.1.6).

Remark 4.2.11. Since Y is a smooth weak Fano 3-fold, we have $\mathrm{Pic}(Y) \simeq H^2(Y, \mathbb{Z})$ (see Remark 2.3.16) and $H^2(Y, \mathcal{O}_Y) = 0$ by Theorem 2.3.5.

It follows from Corollary 4.2.8 that a general surface $S \in |-K_X|$ is smooth. Furthermore, Corollary 4.2.9 implies that the cycles $K_X\big|_S$ and $E\big|_S$ are not divisible in $\mathrm{Pic}(S)$, linearly independent in $H^2(S, \mathbb{Q})$, and hence they generate a primite sublattice R_S in $\mathrm{Pic}(S)$. It follows from the construction of X that all lattices R_S, $S \in |-K_X|$, are isomorphic to the lattice $\mathrm{R} \simeq \mathbb{Z}^2$ with the associated quadratic form $70x^2 + 4xy - 2y^2$, and we can consider the moduli stack $\mathcal{K} := \mathcal{K}_{36}^{\mathrm{R}}$ of K3 surfaces of type R (see Theorem 4.1.10). $\mathcal{K}$ is actually an algebraic space because the forgetful map $\mathcal{K} \longrightarrow \mathcal{K}_{36}$ is representable and 1-to-1 in our case (see [6, (2.5)]).[3)]

4.3 End of the proof of Theorem 1.2.11

We use notation and conventions of Section 4.2. Since $f : Y \longrightarrow X$ is the crepant resolution (see Proposition 4.2.6), it follows from Corollary 4.2.8 that we can assume a general $S \in |-K_X|$ to be a surface in $|-K_Y|$ on Y. We can also assume that $S \cap \mathrm{Exc}(f) = \emptyset$ for the f-exceptional locus $\mathrm{Exc}(f)$. Further, it follows from Remark 4.2.10 that the points in $(\mathbb{P}^{37})^*$, corresponding to such S, form an open subset $U \subset (\mathbb{P}^{37})^*$. Consider the natural (faithful) action of the group $G := \mathrm{Aut}(Y)$ on U. Shrinking U if necessary, we obtain the following

Proposition 4.3.1. *The geometric quotient U/G exists as a smooth scheme.*

Proof. Let us calculate the group G first. Take $g \in \mathrm{Aut}(\mathbb{P})$ to be an automorphism of $\mathbb{P}$ which fixes the point O. Then g lifts to the automorphism of Y (see the construction of X and Y in Section 4.2). Conversely, take any $g \in G$.

Lemma 4.3.2. *The morphism $\tau : Y \longrightarrow W$ is g-equivariant.*

Proof. Since the morphism $f = \Phi_{|-K_Y|} : Y \longrightarrow X$ is g-equivariant (see Remark 4.2.10), it follows from the construction of Y in Section 4.2 that the irreducible components of $\mathrm{Exc}(f)$ are all g-invariant. Thus, since $\mathrm{Pic}(Y)$ is generated by K_Y, the irreducible components of E_f and $E_{Y,\sigma} := \tau_*^{-1}(E_\sigma)$, it is enough to prove that $g(E_{Y,\sigma}) = E_{Y,\sigma}$. Suppose that $g(E_{Y,\sigma}) \neq E_{Y,\sigma}$. Then,

[3)] It can be also easily seen that the class of a (-2)-curve in $\mathrm{Pic}(S)$ is unique and generated by the conic $E|_S$.

since all the curves in E_σ (respectively, in $\tau_*(g(E_{Y,\sigma}))$) are numerically proportional and τ is divisorial, we must have $E_\sigma \cap \tau_*(g(E_{Y,\sigma})) = \emptyset$. The latter implies that there exists a curve $C \equiv \sigma_*(-K_W \cdot \tau_*(g(E_{Y,\sigma})))$ on $\mathbb{P}$ with $-K_{\mathbb{P}} \cdot C = 4$ and $C \cap L = \emptyset$. On the other hand, since $-K_{\mathbb{P}} \sim \mathcal{O}_{\mathbb{P}}(12)$, we get $\mathcal{O}_{\mathbb{P}}(1) \cdot C = \dfrac{1}{3}$, a contradiction. $\square$

It follows from Lemma 4.3.2 that g acts on W. Further, considering the induced g-action on the cone $\overline{NE}(W)$, we obtain, since $\mathrm{Pic}(W) = \mathbb{Z} \cdot K_W \oplus \mathbb{Z} \cdot E_\sigma$, that $\sigma : W \longrightarrow \mathbb{P}$ is g-equivariant. The latter gives a g-action on $\mathbb{P}$ with the fixed point O.

Thus, G is isomorphic to the stabilizer in $\mathrm{Aut}(\mathbb{P})$ of the point O, and to describe the G-action on U we may consider the action of the corresponding subgroup in $\mathrm{Aut}(\mathbb{P})$ on the linear system $|-K_{\mathbb{P}} - O|$. Note that, since $(P \in \mathbb{P})$, $(Q \in \mathbb{P})$ and $(O \in \mathbb{P})$ are the pairwise non-isomorphic singularities, every $g \in G$ fixes every point on L. Finally, since $\mathcal{O}_{\mathbb{P}}(1)$, $\mathcal{O}_{\mathbb{P}}(4)$, $\mathcal{O}_{\mathbb{P}}(6)$ are G-invariant, the g-action on $\mathbb{P}$ can be described as follows:

$$\begin{aligned} x_0 &\mapsto ax_0 + bx_1, \\ x_1 &\mapsto cx_0 + dx_1, \\ x_2 &\mapsto \lambda^4 x_2 + f_4(x_0, x_1), \\ x_3 &\mapsto \lambda^6 x_3 + x_2 f_2(x_0, x_1) + f_6(x_0, x_1), \end{aligned} \tag{4.3.1}$$

where $\lambda \in \mathbb{C}^*$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{C})/\{\pm 1\}$, $f_i := f_i(x_0, x_1)$ are homogeneous polynomials of degree i in x_0, x_1. On the other hand, since $-K_{\mathbb{P}} \sim \mathcal{O}_{\mathbb{P}}(12)$, a general element in $|-K_{\mathbb{P}} - O|$ can be given by the equation

$$\begin{aligned} \alpha x_3^2 + x_2^3 + a_6(x_0, x_1)x_3 + a_2(x_0, x_1)x_2x_3 + a_4(x_0, x_1)x_2^2 + \\ + a_8(x_0, x_1)x_2 + a_{12}(x_0, x_1) = 0 \end{aligned} \tag{4.3.2}$$

on $\mathbb{P}$, where $a_i := a_i(x_0, x_1)$ are homogeneous polynomials of degree i in x_0, x_1, and $\alpha \in \mathbb{C}^*$ is fixed.

Take a general surface S_0 on $\mathbb{P}$ with the equation (4.3.2) such that $a_2 = a_4 = a_6 = 0$.

Lemma 4.3.3. *If S_0 is g-invariant for some $g \neq \mathrm{id}$ from* (4.3.1)*, then $f_2 = f_4 = f_6 = 0$, $c = b = 0$, $a = d = \sqrt{-1}$, $\lambda^4 = 1$.*

Proof. g-invariance of S_0 implies that $f_2 = f_4 = f_6 = 0$ and

$$\begin{aligned} a_8(x_0, x_1) &= a_8(ax_0 + bx_1, cx_0 + dx_1), \\ a_{12}(x_0, x_1) &= a_{12}(ax_0 + bx_1, cx_0 + dx_1). \end{aligned} \tag{4.3.3}$$

Without loss of generality we may assume that $a_8 = x_0 x_1 b_6$ for some $b_6 := b_6(x_0, x_1)$ coprime to x_0 and x_1. Then (4.3.3) and generality of S_0 imply that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$, and we get:

$$a^{12} = 1, \qquad a^{i+1}d^{7-i} = 1, \qquad a^i d^{6-i} = a^j d^{6-j}$$

for all $0 \leqslant i,\ j \leqslant 6$. In particular, $a = d$, $a^8 = a^{12} = 1$, i.e., $a = d = \sqrt{-1}$. Finally, since $x_2 \mapsto \lambda^4 x_2$ (see (4.3.1)) and hence $a_8(x_0, x_1) = \lambda^4 a_8(x_0, x_1)$ (see (4.3.2)), we get $\lambda^4 = 1$. □

Lemma 4.3.4. *Let $g \in G$, given by* (4.3.1)*, be such that $f_2 = f_4 = f_6 = 0$, $c = b = 0$, $a = d = \sqrt{-1}$, $\lambda = \pm\sqrt{-1}$. Then $g = \mathrm{id}$.*

Proof. We have

$$\begin{aligned} g([x_0 : x_1 : x_2 : x_3]) = [\sqrt{-1}x_0 : \sqrt{-1}x_1 : (\sqrt{-1})^4 x_2 : (\sqrt{-1})^6 x_3] = \\ = [x_0 : x_1 : x_2 : x_3] \end{aligned} \tag{4.3.4}$$

on $\mathbb{P}$. Hence $g = \mathrm{id}$. □

It follows from Lemmas 4.3.3 and 4.3.4, since $\lambda^4 = 1$ implies $\lambda^2 = \pm 1$, that the stabilizer of S_0 in G is a group of order 2, generated by some $g_0 \in G$ with $\lambda^2 = 1$ (see (4.3.1)). Consider the normal algebraic subgroup $G' \subset G$ generated by $g^{-1}g_0g$ for all $g \in G$, i.e., generators of G' are all the elements in G for which $f_4 = 0$, $c = b = 0$, $a = d = \sqrt{-1}$ and $\lambda = 1$ in (4.3.1). Then the G'-action on U is proper, and we can consider the geometric quotient $U' := U/G'$, which exists as a normal scheme (see Theorem 4.1.11). Further, take the $G'' := G/G'$-equivariant factorization map $\pi_G : U \longrightarrow U'$ and consider the induced G''-action on U'. Shrinking U if necessary, we obtain

Lemma 4.3.5. *The G''-action on U' is free and proper.*

Proof. Let S_0' be the image on U' of S_0 under π_G. Then we have $G'' \cdot S_0' \simeq G''$ for the G''-orbit of S_0', and, by the dimension count, there exists a Zariski open subset in U' with a free G''-action. □

Lemma 4.3.5 and Theorem 4.1.11 imply that the geometric quotient $U/G \simeq U'/G''$ exists as a smooth scheme. Proposition 4.3.1 is completely proved. □

Set $\mathcal{F} := U/G$ to be the scheme from Proposition 4.3.1. It follows from Proposition 4.2.6 and Remark 4.2.10 that $\mathcal{F}$ is a (coarse) moduli space which parameterizes the pairs $(Y^\sharp, S^\sharp)$ consisting of smooth weak Fano 3-fold $Y^\sharp$ of degree 70 and smooth surface $S^\sharp \in |-K_{Y^\sharp}|$ (cf. [6, (2.2)]). These give the following

Lemma 4.3.6. *For $o := (Y, S) \in \mathcal{F}$, we have $H^1(Y, T_Y\langle S\rangle) = T_o\mathcal{F}$.*

Proof. This follows from the fact that $\mathcal{F}$ is smooth and $H^1(Y, T_Y\langle S\rangle)$ parameterizes the first order deformations of (Y, S) (see Proposition 4.1.9). □

Consider the forgetful morphism $s : \mathcal{F} \longrightarrow \mathcal{K}$, which sends (Y, S) to S.

Lemma 4.3.7. *s is generically surjective.*

Proof. Consider the restriction map $r : T_Y\langle S\rangle \longrightarrow T_S$. It fits into the exact sequence

$$0 \longrightarrow \Omega^2_Y \longrightarrow T_Y\langle S\rangle \xrightarrow{r} T_S \longrightarrow 0, \tag{4.3.5}$$

since $\mathrm{Ker}(r) = T_Y(-S)$ is a subsheaf of $T_Y\langle S\rangle$ consisting of the vector fields vanishing along S, for which we have $T_Y(-S) \simeq \Omega^2_Y$. From (4.3.5) we get the exact sequence

$$H^1(Y, T_Y\langle S\rangle) \xrightarrow{H^1(r)} H^1(S, T_S) \xrightarrow{\partial} H^2(Y, \Omega^2_Y).$$

The map ∂ is dual to the restriction map $i : H^1(Y, \Omega^1_Y) \longrightarrow H^1(S, \Omega^1_S)$ (see [6]). In particular, $\mathrm{Ker}(\partial)$ is the orthogonal of $\mathrm{Im}(i)$. On the other hand, we have $\mathrm{Im}(i) = \mathbb{Z} \cdot c_1(K_Y\big|_S) \oplus \mathbb{Z} \cdot c_1(\tau_*^{-1}(E_\sigma)\big|_S) \simeq \mathbb{Z} \cdot K_X\big|_S \oplus \mathbb{Z} \cdot E\big|_S$ (see Corollary 4.2.9 and Remark 4.2.11), and hence $H^1(r)$ coincides with the tangent map to s at (Y, S), with $\mathrm{Im}(H^1(r)) = \mathrm{Ker}(\partial)$ being the tangent space to $\mathcal{K}$ at S (see Lemma 4.3.6 and Theorem 4.1.10). Thus, since $\mathcal{K}$ is irreducible (see Theorem 4.1.10), we get that s is generically surjective. $\square$

Theorem 1.2.11 is completely proved.

4.4 Corollaries and conclusive remarks

Since the forgetful map $\mathcal{K}^{\mathrm{R}}_{36} \longrightarrow \mathcal{K}_{36}$ is representable and 1-to-1 (see the end of Section 4.2), from Theorem 1.2.11, the construction of $\mathcal{F}$ and quasi-projectivity of $\mathcal{K}_{36}$ we deduce the following

Corollary 4.4.1. *There exists a* 18*-dimensional unirational algebraic variety which parameterizes up to isomorphism all smooth* K3 *surfaces of type* R*. For general such surface S, $S \in |-K_X|$ and the Picard lattice of S is isomorphic to* R.

Remark 4.4.2. On the opposite, it follows from the proof of Theorem 1.2.11 and [11], [13], [77] that no general smooth primitively polarized K3 surface S of genus 36 can be an ample anticanonical section of a normal algebraic 3-fold, except for the cone over S.

Remark 4.4.3. It would be interesting to know whether the map s from Theorem 1.2.11 is 1-to-1 and $\mathcal{K}_{36}^{\mathrm{R}}$ is rational (it follows from the proof of Theorem 1.2.11 that s is generically étale).

Take S as in Corollary 4.4.1. The generators of the group $\mathrm{Pic}(S)$ are $H := -K_X\big|_S$ and $C := E\big|_S$ (see the end of Section 4.2).

Lemma 4.4.4. *$H - 4C$ is an ample divisor on S.*

Proof. Let Z be an irreducible curve on S such that $(H - 4C) \cdot Z \leqslant 0$. Then we have $Z \neq C$. Write

$$Z = aH + bC$$

in $\mathrm{Pic}(S)$ for some $a, b \in \mathbb{Z}$. Note that $a > 0$ because the linear system $|m(H + C)|$ is basepoint-free for $m \gg 0$ (it provides the contraction of the (-2)-curve C) and $(H + C) \cdot Z = 72a$. On the other hand, we have

$$0 \geqslant (H - 4C) \cdot Z = 62a + 10b,$$

which implies that $b < -6a$. But then we get

$$(Z^2) = 70a^2 + 4ab - 2b^2 \leqslant -26a^2 < -2,$$

which is impossible. Thus, $(H-4C)\cdot Z>0$ for every curve $Z\subset S$, and hence $H-4C$ is ample by the Nakai–Moishezon criterion, since $(H-4C)^2=22$. $\square$

Remark 4.4.5. Using exactly the same arguments as in the proof of Lemma 4.4.4, one can prove that $H-rC$ are the ample divisors on S for $r=1,\ 2,\ 3$, which provide primitive polarizations on S of genus 33, 28, 21, respectively. This can also be seen via the geometric argument. Indeed, let $p_1:\mathbb{P}^{37}\dashrightarrow\mathbb{P}^{34}$ be the linear projection from the plane Π passing through the conic C. The blow up $f_1:Y_1\longrightarrow X$ of C resolves the indeterminacy of p_1 on X and gives a morphism $g_1:Y_1\longrightarrow X_1:=p_1(X)$. It can be easily seen that Y_1 is a weak Fano threefold and $X_1\subset\mathbb{P}^{34}$ is an anticanonically embedded Fano threefold of genus 33 (see the proof of Proposition 6.15 in [49]). Moreover, we get $\mathrm{Pic}(Y_1)=\mathbb{Z}\cdot K_{Y_1}\oplus\mathbb{Z}\cdot E_{f_1}$, where $E_{f_1}\simeq\mathbb{F}_4$ is the f_1-exceptional divisor, and the morphism g_1 contracts the surface $f_{1*}^{-1}(E)$ to the point. In particular, the locus $\mathrm{Sing}(X_1)$ consists of a unique point, $\mathrm{Pic}(X_1)=\mathbb{Z}\cdot K_{X_1}$ and $\mathrm{Cl}(X_1)=\mathbb{Z}\cdot K_{X_1}\oplus\mathbb{Z}\cdot E_1$, where $E_1:=g_{1*}(E_{f_1})$. One can prove that E_1 is a cone over a rational normal curve of degree 4 such that $E_1=X_1\cap\mathbb{P}^5$. In particular, there exists a rational normal curve $C_1\subset X_1\setminus\mathrm{Sing}(X_1)$ of degree 4 with $C_1=X_1\cap\Pi_1$ for some $\Pi_1\simeq\mathbb{P}^4$. Proceeding with X_1, Π_1, etc. in the same way as with X, Π, etc. above, we obtain three other anticanonically embedded Fano threefolds $X_2\subset\mathbb{P}^{29}$, $X_3\subset\mathbb{P}^{22}$, $X_4\subset\mathbb{P}^{13}$ of genus 28, 21, 12, respectively, such that $\mathrm{Sing}(X_r)$ consists of a unique point, $\mathrm{Pic}(X_r)=\mathbb{Z}\cdot K_{X_r}$ and $\mathrm{Cl}(X_r)=\mathbb{Z}\cdot K_{X_r}\oplus\mathbb{Z}\cdot E_r$ for all r, where E_r is a cone over a rational normal curve of degree $2+2r$. By construction, a general surface $S_r\in|-K_{X_r}|$ is isomorphic to S, $1\leqslant r\leqslant 4$. Furthermore, identifying S with S_r, we get that $-K_{X_r}\big|_{S_r}\sim H-rC$ is an ample divisor on S, which provides a primitive polarization on S of genus $36 - 2r - r^2$, $1 \leqslant r \leqslant 4$.

Let us recall the following

Definition 4.4.6 (see [72, Definition 3.8]). A K3 surface Σ with a primitive polarization L and genus $g := (L^2)/2 + 1$ is called *BN general* if $h^0(\Sigma, L_1)h^0(\Sigma, L_2) < g + 1$ for every pair of non-zero divisors $L_1, L_2 \in \mathrm{Pic}(\Sigma)$ such that $L_1 + L_2 = L$.

Lemma 4.4.7. *The surface S is BN general with respect to polarization $H - 4C$.*

Proof. Let

$$H - 4C = L_1 + L_2$$

for some $L_1,\ L_2 \in \mathrm{Pic}(S)$. We may assume that both $h^0(S, L_1), h^0(S, L_2) > 0$. Write

$$L_i = a_i H + b_i C$$

in $\mathrm{Pic}(S)$ for some $a_i,\ b_i \in \mathbb{Z}$, $i \in \{1, 2\}$. Note that $a_i \geqslant 0$ for $i \in \{1, 2\}$ (see the proof of Lemma 4.4.4). Thus, we get that, say $a_1 = 1$ and $a_2 = 0$. Then, in particular, $b_2 \neq 0$.

Now, if $b_2 < 0$, then $h^0(S, L_2) = 0$ and we are done. Further, if $b_2 > 0$, then $b_1 \leqslant -5$ and hence we get

$$h^0(S, L_1)h^0(S, L_2) = h^0(S, H + b_1 C) < h^0(S, H - 4C) = 13,$$

since $h^0(S, L_2) = h^0(S, b_2 C) = 1$. □

Remark 4.4.8. In the same way, as in the proof of Lemma 4.4.7, one can prove that the surface S is also BN general with respect to polarizations $H - iC$ for $i = 1, 2, 3$.

It follows from Lemma 4.4.7 and [72] that S possesses a *rigid* vector bundle E_3 of rank 3 with respect to polarization $H - 4C$, i.e., E_3 is stable, $\det E_3 = H - 4C$, $H^0(S, E_3) = 7$, $H^i(S, E_3) = 0$ for all $i > 0$, and E_3 is unique for these properties (see [72, Theorem 4.5]). Moreover, Remark 4.4.8, [72, Theorem 4.5] and simple properties of vector bundles on a smooth surface imply the following

Lemma 4.4.9. *$\hat{E}_3 := E_3 \otimes \mathcal{O}_S(C)$ is a rigid vector bundle on S of rank 3 with respect to polarization $H - C$, i.e., $\hat{E}_3$ is stable, $\det \hat{E}_3 = H - C$, $H^0(S, \hat{E}_3) = 14$, $H^i(S, \hat{E}_3) = 0$ for all $i > 0$, and $\hat{E}_3$ is unique for these properties.*

The morphism $\Phi_{E_3} : S \longrightarrow G(3, 7)$ given by E_3 is an embedding of S, where $G(3, 7) \subset \mathbb{P}(\bigwedge^3 \mathbb{C}^7)$ is considered with respect to the Plücker embedding so that $\mathcal{O}_{G(3,7)}(1)\big|_S \simeq \mathcal{O}_S(H - 4C)$ (see [68]). Moreover, for the universal vector bundle $\mathcal{E}_7$ on $G(3, 7)$ we have

$$S = G\big(3, 7\big) \cap \big(\sigma_1 = \sigma_2 = \sigma_3 = 0\big) \cap \big(\lambda = 0\big), \qquad (4.4.1)$$

where $\sigma_i \in H^0(G(3, 7), \bigwedge^2 \mathcal{E}_7) \simeq \bigwedge^2 \mathbb{C}^7,\ \lambda \in H^0(G(3, 7), \bigwedge^3 \mathcal{E}_7) \simeq \bigwedge^3 \mathbb{C}^7$ (see [72], [68]). Furthermore, the arguments in [72], [68] imply that every general primitively polarized K3 surface of genus 12 is given in $G(3, 7)$ by equations of the form (4.4.1).

Remark 4.4.10. In the same way, applying Lemma 4.4.9 and the arguments from [72], [68], we obtain the embedding $\Phi_{\hat{E}_3} : S \hookrightarrow G(3, 14) \subset \mathbb{P}(\bigwedge^3 \mathbb{C}^{14})$ such that $\mathcal{O}_{G(3,14)}(1)\big|_S \simeq \mathcal{O}_S(H - C)$. In order to determine the equations of $S \subset G(3, 14)$, we fix a basis $\{z_1,\ \ldots,\ z_7,\ z_1^*,\ \ldots,\ z_7^*\}$ of $H^0(G(3, 14),\ \mathcal{E}_{14}) \simeq \mathbb{C}^{14}$, where $\mathcal{E}_{14}$ is the universal vector bundle $G(3,14)$, so that

$$p_{i_1,i_2,i_3} := z_{i_1} \wedge z_{i_2} \wedge z_{i_3}, \qquad q_{i_1,i_2,i_3} := z^*_{i_1} \wedge z_{i_2} \wedge z_{i_3}, \tag{4.4.2}$$
$$q^*_{i_1,i_2,i_3} := z^*_{i_1} \wedge z^*_{i_2} \wedge z_{i_3}, \qquad p^*_{i_1,i_2,i_3} := z^*_{i_1} \wedge z^*_{i_2} \wedge z^*_{i_3}$$

are the coordinates for the Plücker embedding $G(3,14) \hookrightarrow \mathbb{P}(\wedge^3\mathbb{C}^{14})$ for various i_1, i_2, i_3.[4)] Take also the global sections $\sigma_1, \sigma_2, \sigma_3 \in H^0(G(3,14), \bigwedge^2 \mathcal{E}_{14}) \simeq \bigwedge^2 \mathbb{C}^{14}$, $\lambda \in H^0(G(3,14), \bigwedge^3 \mathcal{E}_{14}) \simeq \bigwedge^3 \mathbb{C}^{14}$ as above and set

$$\sigma_r := \sum_{1 \leqslant i,j \leqslant 7} \alpha^{(r)}_{i,j} z_i \wedge z_j, \qquad \lambda := \sum_{1 \leqslant j_1,j_2,j_3 \leqslant 7} \alpha_{j_1,j_2,j_3}\, p_{j_1,j_2,j_3},$$

$r = 1,2,3$, for some $\alpha^{(r)}_{i,j}$, $\alpha_{j_1,j_2,j_3} \in \mathbb{C}$. Then one can prove that

$$S = G\big(3,14\big) \cap \bigcap_{1 \leqslant i_1,i_2,i_3 \leqslant 7} \big(p_{i_1,i_2,i_3} = q_{i_1,i_2,i_3} = p^*_{i_1,i_2,i_3} = q^*_{i_1,i_2,i_3}\big) \cap \tag{4.4.3}$$
$$\cap \big(\sigma_1 = \sigma_2 = \sigma_3 = 0\big) \cap \big(\lambda = 0\big) \cap \mathbb{P}^{33}$$

for various $i_1,\ i_2,\ j_1,\ \ldots,\ j_4$ (the details of the proof of this fact will appear elsewhere). Again, using the arguments from [72], [68], one can prove that every general primitively polarized K3 surface of genus 33 is given in $G(3,14)$ by equations of the form (4.4.3).

Remark 4.4.11. For the vector bundle E_3, we have a surjective morphism $w : \mathbb{C}^7 \otimes \mathcal{O}_S \longrightarrow E_3$, since E_3 is generated by its global sections (see [72, Definition 4.1]). Then the dual $E_3^\vee$ to $\mathrm{Ker}(w)$ is a vector bundle of rank 4 with $\det E_3^\vee = H - 4C$. We expect, by considering the morphisms $\Phi_{E_3^\vee}$ and $\Phi_{E_3^\vee \otimes \mathcal{O}_S(C)}$ from S to Grassmanians $G(4,7)$ and $G(4,13)$, respectively, that S can be embedded into $G(4,7)$ (respectively, into $G(4,13)$) as a primitively polarized K3 surface of genus 12 (respectively, 36) and is given in $G(4,7)$ (respectively, in $G(4,13)$)

[4)] For not to put the extra notation, in (4.4.2) we allow also equal i_1, i_2, i_3, thus identifying $\mathbb{P}(\bigwedge^3 \mathbb{C}^{14})$ with a linear subspace in $\mathbb{P}^{419}$ in the obvious way.

by equations dual to (4.4.1) (respectively, to (4.4.3)). Again, applying the arguments from [72], [68], we expect that the same holds for every general primitively polarized K3 surface of genus 12 (respectively, 36).

In the view of (4.4.1) and Remarks 4.4.10, 4.4.11 let us finish with the following

Conjecture 4.4.12. *The moduli spaces $\mathcal{K}_{33}$ and $\mathcal{K}_{36}$ are birationally isomorphic to the moduli space $\mathcal{K}_{12}$.*

The positive solution to Conjecture 4.4.12 would imply that both $\mathcal{K}_{33}$ and $\mathcal{K}_{36}$ are unirational (see [74], [68]).

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