Diagrammatic Coaction of Two-Loop Feynman Integrals

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Abstract

When evaluating Feynman integrals as Laurent series in the dimensional regulator epsilon one encounters families of iterated integrals, the simplest of which are the multiple polylogarithms. These functions are known to possess a structure called the coaction which captures their analytic properties and the set of functional relations they obey. It has been found that this coaction, when applied to a one-loop Feynman integral, may be expressed using integrals corresponding to subgraphs, as well as cut integrals. In the present work we will explore how this diagrammatic coaction generalises to two-loop Feynman integrals and related questions.

Expressing Feynman integrals using generalised hypergeometric functions is a useful alternative to considering them in Laurent series form. The properties of these functions have been well studied and can be invoked in the study of Feynman integrals. Importantly, we will see that hypergeometric functions also possess a coaction which may be used in computing coactions of Feynman integrals.

We will compute the coactions for a range of two-loop graphs and establish how they differ from one-loop cases. Specifically, the correspondence between subgraphs and cuts observed at one loop will be preserved while multiple master integrals for a given graph can appear at two loops, as can multiple cuts associated with a particular subset of propagators. The appropriate generalisation of deformation terms in the diagrammatic coaction will also be considered.

Given the important role cut integrals play in this picture, we will also examine their calculation. There are also many subtle features involved in specifying how these cuts are defined, and in creating elegant dual bases of master integrals and cuts, which will be explored.

Lay Summary

Field theories can be used to describe the probabilities of certain scattering events between particles. A common approach to computing these probabilities employs mathematical objects known as Feynman integrals, which are often challenging to compute. It is therefore of great interest to determine if they possess mathematical structures which can provide insight into these objects and potentially simplify their calculation.

One such structure is called the coaction, which, loosely speaking, breaks down certain functions into simpler pieces that are more easily manipulated. This structure can be defined on mathematical functions that are used to express some Feynman integrals, and has been extended to a subset of these Feynman integrals themselves. In this thesis, we examine further extensions of this structure.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Parts of this work have been published in [1], and in the proceedings [2–4].

(James C. Matthew, May 2020)

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Contents

List of Figures

Chapter 1

Introduction

Perturbative expansions of scattering amplitudes can be expressed using Feynman integrals. Each such integral can be thought of as corresponding to a Feynman diagram, with the edges of the diagrams representing propagators from the integral, and the diagram possessing a loop for every internal momentum integrated over. To determine higher-order terms in the perturbative expansion, one must then compute Feynman integrals with greater numbers of loops.

When computing Feynman integrals, one finds many cases that exhibit divergences if computed without a suitable regulator. One method of introducing such a regulator is to compute integrals in a non-integer number of dimensions such as $D = 4 - 2\epsilon$. It is these integrals computed in dimensional regularisation which we shall be interested in throughout the present work.

When such a regularised integral is expanded in ϵ, the functions which appear as coefficients of ϵ are certain classes of iterated integrals. It is well established that, for one-loop integrals, the multiple polylogarithms [5] are sufficient to express any such function appearing in the Laurent expansion. At two loops there are a number of examples where this continues to be the case but one soon finds that there are cases where the more general elliptic polylogarithms are required (see, for instance, [6, 7]). There is a systematic description of this class of functions analogous to that for the multiple polylogarithms which has been developed in, for example, [8–10].

These classes of iterated integral have the property in common that they are periods: integrals of rational functions with rational coefficients over domains specified by inequalities on polynomials with rational coefficients [11]. A closely related notion is that of a motivic period, an object which carries information about the integrand and contour used to compute the period, and this space supports an algebraic structure called the coaction [12]. This coaction on the motivic periods refines the earlier notion of a coaction on multiple polylogarithms described in [5], and has been applied to gain insight into the structure of, for instance, ϕ^4 periods [13, 14] and the electron anomalous magnetic moment [15].

The coaction, and the related notion of the symbol which was introduced to the physics literature in [16], have been widely employed to assist in simplifying expressions for Feynman integrals and amplitudes. Similar methods have also begun to be developed for the elliptic polylogarithms [17], though we will not further consider these cases. The symbol is also vital for the so-called bootstrap method of determining amplitudes where the symbol is deduced by applying various physical constraints (see [18] for an early example). The further constraints provided by the coaction itself have now begun to be applied in this same programme [19]. Lastly, we mention that the coaction has been applied to gain insight into the structure of certain string amplitudes [20, 21].

When evaluated in integer numbers of dimensions, those Feynman integrals which converge are periods and so their coaction can be computed [22]. Alternatively, when a Feynman integral is evaluated in dimensional regularisation, the coaction can be found order by order in ϵ as the coefficients of the Laurent expansion are periods. This latter problem was examined for the one-loop case in [23, 24], where it was found that the coaction computed order by order admits a remarkable closed-form expression with a diagrammatic interpretation. Determining how this coaction generalises to polylogarithmic two-loop integrals will be the ultimate motivation for the work of this thesis. These two-loop integrals can be used to obtain higher precision predictions for experimental results. The related uncertainties in the theoretical results can be greater than the experimental uncertainties, and some consider this an important reason to study two-loop integrals.

A key insight of [24] was the role played by cut Feynman integrals in the coaction.

These cut integrals may be thought of as regular Feynman integrals with some subset of the propagators placed on shell. Traditionally, this constraint has been implemented by replacing these propagators with delta functions which impose the on-shell condition. The resulting cuts have been studied extensively in connection with the discontinuities of amplitudes and Feynman integrals [25–28]. Another way to define these cuts is to take residues at the poles where the propagators vanish. This notion has been employed in, for instance, the unitarity method (see [29–32] for examples). This alternative definition of cuts was explored in [33], where cuts on any subset of propagators of a one-loop Feynman integral were defined. A useful technique in the evaluation of cuts is the Baikov representation [34–37], where a change of variables is implemented so that the propagators themselves are the new parameters to be integrated over.

When evaluating Feynman integrals and their cuts in dimensional regularisation, it is found that the results can be expressed using functions belonging to a class known as the hypergeometric functions which includes the well-known Gauss ${}_2F_1$ function. There are many examples in the literature of such expressions: see, for instance, [24] for many one-loop examples, and also [38–42]. It then follows that any relation on Feynman integrals or their cuts can be explained by some corresponding functional identity on hypergeometric functions. Given the ubiquity of hypergeometric functions in mathematical physics, the identities which they satisfy have been well studied and we will make use of many of these throughout this thesis. We will be interested in computing the coaction of those hypergeometric functions which have Laurent series expansions expressible using multiple polylogarithms [1]. This problem has also recently been studied from the motivic perspective in [43].

Let us now describe the content of this thesis. We begin in chapter 2 by outlining various background material on polylogarithms, cuts and the coaction at one loop, and the properties of hypergeometric functions. In chapter 3 we discuss how a closed form of the coaction of various hypergeometric functions can be found. Then in chapters 4 and 5 we consider master integrals and cuts of various two-loop graphs, before using these in chapter 6 to explore the coaction at two loops.

Chapter 2

Background

The cuts and coactions of one-loop Feynman integrals have been studied extensively in [24, 33] and will be reviewed in this section along with various other prerequisites for the two-loop case.

We begin with multiple polylogarithms, a class of iterated integrals that occur in the Laurent expansion of dimensionally regulated Feynman integrals and, in particular, are the only such class of functions to appear at one loop. We will review their coaction and explain how it can be cast in a form which anticipates the structure of the diagrammatic coaction.

We then examine the one-loop Feynman integrals themselves, explaining how a convenient basis of these integrals is selected and how the cuts of one-loop graphs may be computed. We then write down the diagrammatic coaction for the one-loop case and comment on important features of its structure that we will later generalise to two loops.

Lastly, we review the definition and useful properties of the class of hypergeometric functions which will play an important role in this thesis.

2.1 Multiple Polylogarithms

The multiple polylogarithms are a class of functions defined by iterated integrals which can be used to express certain Feynman integrals. They generalise the well-known families of classical polylogs and the multiple zeta values, and are described by Goncharov in [5, 44] along with their Hopf algebra structure, which is essential for their use in many practical calculations.

We will summarise the discussion of the coaction presented in [45], which describes the results of Goncharov with some modifications for the case of zeta values originating with [46].

2.1.1 Definitions

We begin by recalling the definition of the multiple polylogs as the family of iterated integrals:

$$\begin{aligned} I(a_0; a_1, \ldots, a_n; a_{n+1}) &= \int_{a_0}^{a_{n+1}} \frac{dt}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t) \\ I(a_0; ; a_1) &= 1. \end{aligned} \tag{2.1}$$

This class of functions incorporates the classical polylogarithms $\text{Li}_n(z)$ via

$$I(0; 1, \underline{0}_{n-1}; z) = -\text{Li}_n(z), \tag{2.2}$$

and hence also includes the zeta values

$$\zeta_n = \sum_{m=1}^{\infty} \frac{1}{m^n} = \text{Li}_n(1). \tag{2.3}$$

There is a commonly used alternative notation for the polylogs:

$$\begin{aligned} G(a_1, \ldots, a_n; z) &= \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \\ G(; z) &= 1, \end{aligned} \tag{2.4}$$

which obeys the relation

$$G(a_1, \ldots, a_n; z) = I(0; a_n, \ldots, a_1; z). \tag{2.5}$$

We note that the functions in equation (2.1) are not more general than those in (2.4), because each integral with lower limit a_0 in (2.1) may be written as a difference of two integrals with lower limit 0, a procedure which can be iterated to express any function from (2.1) using those of (2.4).

The weight of a polylogarithm is defined to be the number of integrations required to obtain it, so that $I(a_0; a_1, \ldots, a_n; a_{n+1})$ has weight n. Likewise, the zeta value ζ_n has weight n. If we denote the vector space over the rational numbers of all such weight n objects by $\mathcal{H}_n$, and set $\mathcal{H}_0 = \mathbb{Q}$, then the space $\mathcal{H}$ of all polylogarithms is given by

$$\mathcal{H} = \bigoplus_{n=0} \mathcal{H}_n. \tag{2.6}$$

Now consider an algebra $\mathcal{A}$ defined to be a vector space with associative and distributive multiplication and possessing a unit element. Then a coalgebra is an algebra with an additional map, a coproduct, which is a linear mapping $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ obeying the following:

1. Coassociativity: $(\mathbb{I} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathbb{I}) \circ \Delta$, where $\mathbb{I}$ is the identity map.
2. Compatibility with multiplication: $\Delta(ab) = \Delta(a)\Delta(b)$.
3. Compatibility with grading: $\Delta(a)$ must contain only terms with total weight equal to the weight of a.

We distinguish from the above the notion of a coaction, which is a similar mapping but with an image in the tensor product of two spaces which need not coincide with the domain $\mathcal{A}$.

In the specific case of the multiple polylogarithms, there is an algebra known as the shuffle algebra which may be imposed according to the relation

$$I(a_p; a_1, \ldots, a_n; a_q) I(a_p; a_{n+1}, \ldots, a_{n+m}; a_q) = \sum_\sigma I(a_p; \sigma(a_1, \ldots, a_{n+m}); a_q), \tag{2.7}$$

where σ is any shuffle on $a_1, \ldots, a_{n+m}$, i.e. a permutation of these $n+m$ elements that preserves the ordering of $a_1, \ldots, a_n$ and of $a_{n+1}, \ldots, a_{n+m}$.

Following [5], a coproduct can be defined on these functions with generic arguments as follows:

$$\Delta I(a_0; a_1, \ldots, a_n; a_{n+1}) \tag{2.8}$$
$$= \sum_{0=i_0<i_1<\ldots<i_k<i_{k+1}=n+1} \left[I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right].$$

In the above, we are summing over all subsequences of the $a_1, \ldots, a_n$. The first entries then contain the given subsequence as arguments of the polylogarithms, while the corresponding second entries are products of polylogarithms with the arguments that sit between the elements of the subsequence.

It is easily demonstrated from this definition that the coproducts of $\log^n z$ and the classical polylogarithms $\mathrm{Li}_n(z)$ are

$$\Delta \log^n z = \sum_{k=0}^{n} \binom{n}{k} \log^{n-k} z \otimes \log^k z \tag{2.9}$$

$$\Delta \mathrm{Li}_n(z) = 1 \otimes \mathrm{Li}_n(z) + \sum_{k=0}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\log^k z}{k!}. \tag{2.10}$$

Following [45] we then consider a coaction $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}/(i\pi\mathcal{A})$ which acts on $I(a_0; a_1, \ldots, a_n; a_{n+1})$ according to (2.8), but with the coaction of $i\pi$ now given by $\Delta(i\pi) = i\pi \otimes 1$. This avoids the inconsistency [47] which would otherwise exist that, from (2.9) with $z = 1$, we have

$$\Delta\zeta_n = \zeta_n \otimes 1 + 1 \otimes \zeta_n, \tag{2.11}$$

which then implies, as $\zeta_4 = \frac{2}{5}\zeta_2^2$, that

$$\Delta\zeta_4 = \frac{2}{5}(\Delta\zeta_2)^2 = \zeta_4 \otimes 1 + \frac{4}{5}\zeta_2 \otimes \zeta_2 + 1 \otimes \zeta_4. \tag{2.12}$$

This inconsistency between (2.11) and (2.12) is cured by working modulo $i\pi$ in the second entry, as then (2.11) becomes $\Delta\zeta_n = \zeta_n \otimes 1$ for n even and so (2.12) now reads $\Delta\zeta_4 = \zeta_4 \otimes 1$.

Given a weight n polylogarithmic function, we can denote by $\Delta_{i,n-i}$ the terms in its coaction where the first entry has weight i and the second has weight $n-i$. This notation can be extended to describe components of the map $(\Delta \otimes \mathbb{I}) \circ \Delta$. By coassociativity, these are identical to the components of $(\mathbb{I} \otimes \Delta) \circ \Delta$. Here we denote by $\Delta_{i,j,n-i-j}$ the terms where the first, second and third entries of the tensor have, respectively, weights i, j and $n-i-j$. This notation generalises to any number of subsequent applications of Δ.

We conclude by giving a pair of useful identities for the coactions of x^ϵ and $e^{\gamma_E \epsilon}\Gamma(1+\epsilon)$, where γ_E is the Euler-Mascheroni constant:

$$\Delta x^\epsilon = x^\epsilon \otimes x^\epsilon \tag{2.13}$$

$$\Delta e^{\gamma_E \epsilon}\Gamma(1+\epsilon) = e^{\gamma_E \epsilon}\Gamma(1+\epsilon) \otimes e^{\gamma_E \epsilon}\Gamma(1+\epsilon). \tag{2.14}$$

The former of these can be proven using (2.9):

$$\begin{aligned}\Delta x^\epsilon &= \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \Delta \log^i x \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{\epsilon^{i-j}}{(i-j)!} \log^{i-j} x \otimes \frac{\epsilon^j}{j!} \log^j x \\ &= \left(\sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \log^i x \right) \otimes \left(\sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} \log^j x \right) \\ &= x^\epsilon \otimes x^\epsilon. \end{aligned} \tag{2.15}$$

The latter follows from the identity $e^{\gamma_E \epsilon}\Gamma(1+\epsilon) = e^{\sum_{i=2}^{\infty} \frac{(-\epsilon)^i}{i} \zeta_i}$. We note that the factor of $e^{\gamma_E \epsilon}$ is included in the above to obtain a quantity which depends only on the zeta values ζ_i, for which the coaction is well defined. The number γ_E is not known to be expressible as a period [45] and so it is unclear if a coaction can exist on functions of this number. In what follows, we will always normalise integrals in such a way that γ_E is removed.

The identities (2.13) and (2.14) are the first examples of a phenomenon we will see repeated throughout this thesis: coactions on certain functions expanded to Laurent series can be expressed in a closed form. It is far from obvious in advance that this should be possible.

2.1.2 An Alternative Form of the Coaction

We now give another way to state this coaction which was discussed in, for instance, [24]. Let us write

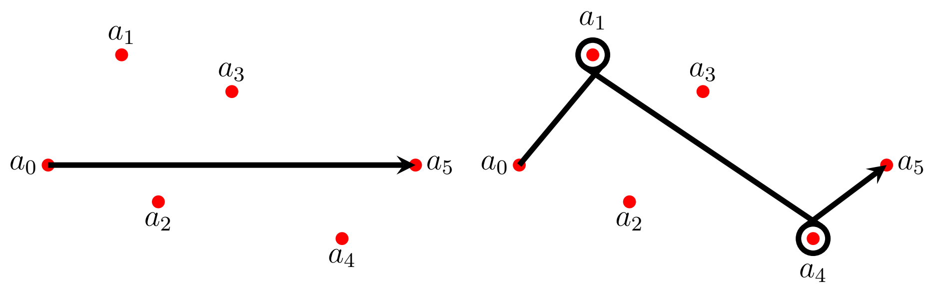
$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_\gamma \omega, \tag{2.16}$$

where

$$\begin{aligned} \omega =& \frac{dt_1}{t_1 - a_n} \wedge \frac{dt_2}{t_2 - a_{n-1}} \wedge \ldots \wedge \frac{dt_n}{t_n - a_1} && (2.17) \\ \gamma =& \left\{ a_0 \leq t_1 \leq a_{n+1}, a_0 \leq t_2 \leq t_1, \ldots, a_0 \leq t_n \leq t_{n-1} \right\}. && (2.18) \end{aligned}$$

Now we consider what happens when we modify the integration contour γ such that it now encircles some subset $\{a_{i_1}, \ldots, a_{i_m}\}$ of the poles $\{a_1, \ldots, a_n\}$ of the integrand as depicted in figure 2.1 for the specific case $n = 4$. Denote this new contour by $\gamma_{a_{i_1},\ldots,a_{i_m}}$. We will compute in the simplest non-trivial case, γ_{a_i}, the integral $\int_{\gamma_{a_i}} \omega$:

$$\begin{aligned} &\int_{\gamma_{a_i}} \omega && (2.19) \\ &= \int_{a_0}^{a_{n+1}} \frac{dt_1}{t_1 - a_n} \int_{a_0}^{t_1} \frac{dt_2}{t_2 - a_{n-1}} \cdots \int_{a_0}^{t_{n-i-1}} \frac{dt_{n-i}}{t_{n-i} - a_{i+1}} \mathrm{Res}_{t_{n-i+1}=a_i} \int_{a_0}^{t_{n-i}} \frac{dt_{n-i+1}}{t_{n-i+1} - a_i} \\ &\int_{a_0}^{t_{n-i+1}} \frac{dt_{n-i+2}}{t_{n-i+2} - a_{i-1}} \cdots \int_{a_0}^{t_{n-1}} \frac{dt_n}{t_n - a_1} \\ &= \int_{a_0}^{a_{n+1}} \frac{dt_1}{t_1 - a_n} \int_{a_0}^{t_1} \frac{dt_2}{t_2 - a_{n-1}} \cdots \int_{a_0}^{t_{n-i-1}} \frac{dt_{n-i}}{t_{n-i} - a_{i+1}} \theta(a_0 \leq a_i \leq t_{n-i}) \end{aligned}$$



(a) The contour $\gamma_\emptyset$ (b) The contour $\gamma_{(a_1,a_4)}$

Figure 2.1: Contours which specify the coaction of the multiple polylogarithms illustrated for the function $I(a_0; a_1, a_2, a_3, a_4; a_5)$

$$
\begin{aligned}
&\times \int_{a_0}^{a_i} \frac{dt_{n-i+2}}{t_{n-i+2}-a_{i-1}} \cdots \int_{a_0}^{t_{n-1}} \frac{dt_n}{t_n - a_1} \\
=&\int_{a_i}^{a_{n+1}} \frac{dt_1}{t_1 - a_n} \int_{a_i}^{t_1} \frac{dt_2}{t_2 - a_{n-1}} \cdots \int_{a_i}^{t_{n-i-1}} \frac{dt_{n-i}}{t_2 - a_{i+1}} \\
&\times \int_{a_0}^{a_i} \frac{dt_{n-i+2}}{t_{n-i+2}-a_{i-1}} \cdots \int_{a_0}^{t_{n-1}} \frac{dt_n}{t_n - a_1} \\
=&I(a_0; a_1, \ldots, a_{i-1}; a_i) I(a_i; a_{i+1}, \ldots, a_n; a_{n+1}),
\end{aligned}
$$

where $\text{Res}_{x=x_0}$ denotes the operation of replacing the integral over x by the residue of the integrand at x_0 and we have used the equality

$$
\begin{aligned}
&\{(t_1, \ldots, t_{n-i}) | a_0 \le t_1 \le a_{n+1}, \ldots, a_0 \le t_{n-i} \le t_{n-i-1}, a_0 \le a_i \le t_{n-i}\} \\
=&\{(t_1, \ldots, t_{n-i}) | a_i \le t_1 \le a_{n+1}, \ldots, a_i \le t_{n-i} \le t_{n-i-1}\}
\end{aligned} \tag{2.20}
$$

to obtain the correct integration domain for the first $n-i$ integrals. Note that the result $I(a_0; a_1, \ldots, a_{i-1}; a_i) I(a_i; a_{i+1}, \ldots, a_n; a_{n+1})$ is the second entry that is paired with $I(a_0; a_i; a_{n+1})$ in the coaction formula (2.8). One can similarly verify that the integral $\int_{\gamma_{a_{i_1}, \ldots, a_{i_m}}} \omega$ is the second entry paired with $I(a_0; a_{i_1}, \ldots, a_{i_m}; a_{n+1})$. Thus there is a pairing between the forms which retain only certain factors and the contours that encircle the poles associated to these factors. An analogous result can also be shown to hold in the alternative notation of (2.4). This is a structure that will reoccur in the coactions of Feynman integrals and hypergeometric functions.

We note that the operation defined above is consistent with rewriting the integral so that the integrations are performed in a different order or, equivalently, applying the shuffle product of (2.7), as we will demonstrate in the specific example of the function $G(a, b; z)$. Following the technique of (2.19), it is found that

$$\begin{aligned} G_a(a,b;z) =& \text{Res}_{u=a} \int_0^z \frac{du}{u-a} \int_0^u \frac{dv}{v-b} = \int_0^a \frac{dv}{v-b} = G(b;a) \\ G_b(a,b;z) =& \text{Res}_{v=b} \int_0^z \frac{du}{u-a} \int_0^u \frac{dv}{v-b} = \int_0^z \frac{du}{u-a} \theta(u-b) = G(a;z) - G(a;b). \end{aligned} \tag{2.21}$$

But we can equally well reverse the order of integration, in which case we find

$$\begin{aligned} G_b(a,b;z) =& \text{Res}_{v=b} \int_0^z \frac{du}{u-a} \int_0^u \frac{dv}{v-b} = \text{Res}_{v=b} \int_0^z \frac{dv}{v-b} \int_v^z \frac{du}{u-a} \\ =& \int_b^z \frac{du}{u-a} = G(a;z) - G(a;b). \end{aligned} \tag{2.22}$$

Using these results along with the trivially computed terms $G_\emptyset(a, b; z) = G(a, b; z)$ and $G_{(a,b)}(a, b; z) = 1$, the coaction

$$\begin{aligned} &\Delta G(a,b;z) \\ =& 1 \otimes G(a,b;z) + G(a;z) \otimes G(b;a) + G(b;z) \otimes [G(a;z) - G(a;b)] + G(a,b;z) \otimes 1 \end{aligned} \tag{2.23}$$

is correctly reproduced.

2.1.3 Discontinuities and Derivatives

It can be shown (see, for example, [45]) that the coaction defined above interacts with the operations of taking discontinuities and derivatives in the following manner:

$$\Delta \circ \text{Disc} = (\text{Disc} \otimes \mathbb{I}) \circ \Delta \tag{2.24}$$

$$\Delta \circ \frac{\partial}{\partial z} = \left(\mathbb{I} \otimes \frac{\partial}{\partial z} \right) \circ \Delta. \tag{2.25}$$

The weight of a polylogarithm is lowered by one when applying the discontinuity operator. For instance at weight one we have $\mathrm{Disc}\log(s_i) = 2\pi i\theta(-s_i)$. Thus (2.24) implies that $\Delta_{0,n-1} \circ \mathrm{Disc} f = (\mathrm{Disc} \otimes \mathbb{I}) \circ \Delta_{1,n-1} f$. Suppose that a weight n polylogarithmic function f obeys $\Delta_{1,n-1} f = \sum_i \log(s_i) \otimes g_i$, for a collection of weight $n-1$ functions $\{g_i\}$. Then there is the equality

$$\mathrm{Disc} f = \sum_i 2\pi i\theta(-s_i) g_i \tag{2.26}$$

which only holds modulo $i\pi$ as the g_i sit in the second entry. Similarly, if $\Delta_{n-1,1} f = \sum_i h_i \otimes \log(s_i)$ then

$$\frac{\partial f}{\partial z} = \sum_i h_i \frac{\partial \log(s_i)}{\partial z}. \tag{2.27}$$

We may extend these results to Laurent series of hypergeometric functions or Feynman integrals which are expressible using polylogarithms. It follows immediately from the above that if we now set $\sum_{n=1}^{\infty} \Delta_{1,n-1} f = \sum_i \log(s_i) \otimes g_i$, then $\mathrm{Disc} f$ continues to be given by $\sum_i 2\pi i\theta(-s_i) g_i$, where now the g_i contain a sum of polylogarithms of different weights. This generalisation also applies to (2.27). By this technique we are able to write the discontinuities of a Feynman integral using its cuts, and write the derivatives using the basis of master integrals.

2.1.4 The Symbol

The symbol map of a polylogarithmic function f can be defined recursively over the weight by writing the differential of the function as

$$df = \sum_i f_i \; d\log(g_i), \tag{2.28}$$

then as in [16] the symbol is defined to be

$$\mathcal{S}(f) = \sum_i \mathcal{S}(f_i) \otimes g_i. \tag{2.29}$$

This mapping reduces a polylogarithmic function to a sum of tensor products of weight one objects. It is equivalent, modulo $i\pi$, to performing the maximal iteration of the coaction with the map $\Delta_{1,1,\ldots,1}$.

2.2 Master Integrals at One Loop

In the following sections we will review the cuts and diagrammatic coactions of one-loop Feynman integrals. In order to simplify this discussion, we follow [24] and define a basis of integrals. Consider the set of integrals

$$\tilde{J}(\{p_i.p_j\}; m_1^2, \ldots, m_n^2) = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \prod_{i=1}^{n} \frac{1}{(k + \sum_{j=1}^{i} p_j)^2 - {m_i}^2} \tag{2.30}$$

with $D = 2\lceil \frac{n}{2} \rceil - 2\epsilon$, where $\lceil x \rceil$ denotes the ceiling of x. This family is a basis for the space of one-loop Feynman integrals in dimensions $D = 2n - 2\epsilon$, with propagators having positive integer exponents and with numerator insertions of the form $(\alpha k + \sum_{i=1}^{n} \beta_i p_i)^2$ also raised to positive integer powers [24]. One can verify this by writing the numerators as linear combinations of the propagators, applying dimension shift relations [48, 49] to convert every integral to the dimension $D = 2\lceil \frac{n}{2} \rceil - 2\epsilon$ and then using integration by parts relations [50, 51] to eliminate integrals with non-unit exponents for any of the propagators.

This basis proves to be particularly simple as the choice of dimension produces integrals that are uniform weight at each order in the Laurent expansion in ϵ. The integrals are further simplified after normalising by the leading singularity, i.e. the lowest order in the ϵ expansion of the maximal cut. Following this normalisation the polylogarithms in the Laurent series expansion have no coefficients which depend on the scales $p_i.p_j$ and m_i^2, and so the functions are said to be pure. After normalising the integrals $\tilde{J}$ of (2.30) in this manner, we denote the result by J.

2.3 Cuts of Feynman Integrals

To write down the diagrammatic coaction in the following section we will require the notion of a cut integral. These cuts of a Feynman integral can be regarded as objects with the contour of integration modified to encircle some subset of the poles where propagators vanish. This can be made more precise with the theory of multivariate residues and the notion of the Leray coboundary [52], but we omit these details and provide a definition of the cuts where they are computed by taking residues at these poles [33]. We will also relate this definition to the less general case of unitarity cuts, which are restricted to real kinematics and certain configurations of cut propagators.

2.3.1 One-Loop Cuts from Residues

We follow the treatment of [33] and restrict our attention to the integrals of the form (2.30) and, for a cut on m of the n propagators, form a list of cut propagators $\{k^2 - m_{i_1}^2, (k+q_1)^2 - m_{i_2}^2, \ldots, (k+q_{m-1})^2 - m_{i_m}^2\}$ and a list of those which are uncut $\{(k+q_m)^2 - m_{i_{m+1}}^2, \ldots, (k+q_{n-1})^2 - m_{i_n}^2\}$. It will also be required that $q_1^2 > 0$, a condition which can always be fulfilled for non-vanishing cuts with some ordering of the cut propagators. Now adopt the parametrisation

$$
\begin{aligned}
q_1 &= (q_1^0, \underline{0}_{D-1}) \\
q_2 &= (q_2^0, q_2^1, \underline{0}_{D-2}) \\
\vdots & \qquad \vdots \\
q_{n-1} &= (q_{n-1}^0, \ldots, q_{n-1}^{n-2}, \underline{0}_{D-n+1}) \\
k &= k_0 \left(1, \beta\cos\theta_1, \beta\cos\theta_2\sin\theta_1, \ldots, \beta\cos\theta_{n-2}\prod_{i=1}^{n-3}\sin\theta_i, \beta\left(\prod_{i=1}^{n-2}\sin\theta_i\right)\underline{1}_{D-n+1} \right).
\end{aligned}
\tag{2.31}
$$

As we demonstrate in appendix A, this choice of parametrisation for k produces the integration measure

$$\int d^D k = \frac{2\pi^{\frac{D+1-n}{2}}}{\Gamma\left(\frac{D-n+1}{2}\right)} \int_{-\infty}^{+\infty} dk_0 k_0^{D-1} \int_0^{\infty} d\beta \beta^{D-2} \prod_{j=1}^{n-2} \int_0^{\pi} d\theta_j \sin^{D-2-j}\theta_j. \tag{2.32}$$

Before using this measure to perform the cut calculation we make the further changes of variable $\cos\theta_i = 2x_i - 1$. Continuing to denote by $\text{Res}_{x=x_0}$ the operator which replaces the integration on x with the residue of the integrand at the pole $x = x_0$, the cut on the m propagators is defined to be

$$\begin{aligned} &\mathcal{C}_{i_1,\ldots,i_m} \int d^D k \frac{1}{k^2 - m_{i_1}^2} \prod_{j=1}^{n-1} \frac{1}{(k+q_j)^2 - m_{i_{j+1}}^2} \qquad (2.33)\\ =&\frac{2\pi^{\frac{D+1-n}{2}}}{\Gamma\left(\frac{D-n+1}{2}\right)} \text{Res}_{k_0=\tilde{k}_0} \text{Res}_{\beta=\tilde{\beta}} \text{Res}_{x_1=\tilde{x}_1} \ldots \text{Res}_{x_{m-2}=\tilde{x}_{m-2}} \int_{-\infty}^{+\infty} dk_0 k_0^{D-1} \int_0^{\infty} d\beta \beta^{D-2} \\ &\times \left\{ \prod_{j=1}^{n-2} \int_0^1 2dx_j \left[2\sqrt{x(1-x)}\right]^{D-3-j} \right\} \frac{1}{k^2 - m_{i_1}^2} \prod_{j=1}^{n-1} \frac{1}{(k+q_j)^2 - m_{i_{j+1}}^2}, \end{aligned}$$

where the poles $\tilde{k}_0$, $\tilde{\beta}$, $\tilde{x}_1$,..., $\tilde{x}_{m-2}$ are defined by

$$\begin{aligned} &(k+q_{j+1})^2 - m_{i_{j+2}}^2\big|_{x_j=\tilde{x}_j} = 0 \\ &k^2 - m_{i_1}^2\big|_{\beta=\tilde{\beta}} = 0 \\ &(k+q_1)^2 - m_{i_2}^2\big|_{\beta=\tilde{\beta},k_0=\tilde{k}_0} = 0. \end{aligned} \tag{2.34}$$

When n is even, we will transform the Γ function from the integration measure using the well-known Legendre duplication formula

$$\Gamma(z) = 2^{1-2z}\sqrt{\pi}\frac{\Gamma(2z)}{\Gamma\left(z+\frac{1}{2}\right)} \tag{2.35}$$

in order that all the Γ functions in our expression take the form $\Gamma(m + a\epsilon)$ where $m, a \in \mathbb{Z}$.

For the computation of single cuts it is easier to rotate into Euclidean space and adopt a similar parametrisation of the momenta. Specifically, each of the momenta in (2.31) now has a corresponding Euclidean momentum such that if $l = (l_0, l_1, \ldots, l_{D-1})$ then $l^E = (-il_0, l_1, \ldots, l_{D-1})$. There is now a different parametrisation for the loop momentum k^E:

$$k^E = |k^E| \left(\cos\theta_0, \cos\theta_1 \sin\theta_0, \ldots, \cos\theta_{n-2} \prod_{j=0}^{n-3} \sin\theta_j, \left(\prod_{j=0}^{n-2} \sin\theta_j \right) \underline{1}_{D-n+1} \right), \tag{2.36}$$

with a corresponding integration measure

$$\int d^D k^E = \frac{\pi^{\frac{D+1-n}{2}}}{\Gamma\left(\frac{D-n+1}{2}\right)} \int_0^\infty d|k^E|^2 \left(|k^E|^2\right)^{\frac{D-2}{2}} \prod_{j=0}^{n-2} \int_0^\pi d\theta_j \sin^{D-2-j}\theta_j. \tag{2.37}$$

In this parametrisation, we can compute a single cut by taking the cut propagator to be $|k^E|^2 + m^2$, which produces a pole at $|k^E|^2 = -m^2$.

Given this framework for computing cuts, there are a variety of general results for near to maximal cuts which may be written in a compact form, as well as conditions under which certain cuts vanish [33]. For instance, single cuts on massless propagators vanish, as do two-propagator cuts for which the corresponding invariant $(p_i + p_{i+1} + \ldots + p_{j-1} + p_j)^2$ is null and three-propagator cuts which isolate an entirely massless vertex.

We finish by mentioning the relations governing cut contours associated with singularities at infinite momentum, which we have disregarded above. These singularities at infinite momentum are known as singularities of the second type [53]. They can be made explicit by adopting an integral representation in complex projective space $\mathbb{CP}^{D+1}$ using the method of [54]. We will be particularly interested in the result of [55] which demonstrates that cut contours associated with this singularity are not independent from those which encircle only poles where propagators vanish.

Indeed if C is some subset of the propagators then there is the relation

$$\Gamma_{C,\infty} = -2a_C\Gamma_C - \sum_{C\subset X\subseteq[n]} (-1)^{\lceil|C|/2\rceil+\lceil|X|/2\rceil}\Gamma_X, \tag{2.38}$$

where Γ_C is the contour which defines the cut on the propagators C and the coefficient a_C is

$$a_C = \begin{cases} 1 & |C| \text{ odd} \\ 0 & |C| \text{ even} \end{cases} . \tag{2.39}$$

The different coefficients for odd and even cases will prove important in explaining an asymmetry which exists between terms in the diagrammatic coaction where the number of propagators is odd and those where it is even.

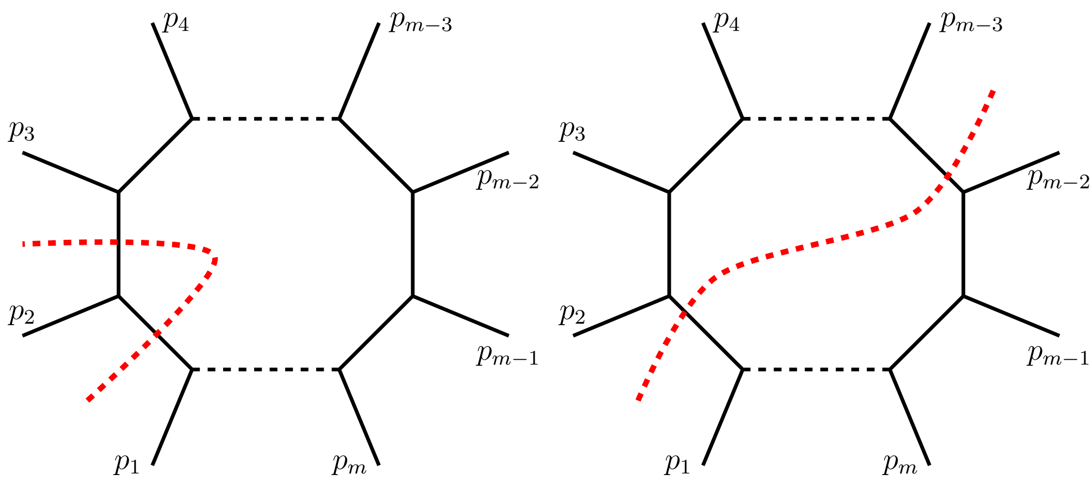
2.3.2 Unitarity Cuts

The notion of a unitarity cut is less general than that of performing a cut by taking residues. We will however find the concept useful in thinking about cuts and discontinuities of two-loop integrals, and so we discuss it briefly here for the one-loop case. In the following, we will need to use the function δ^+, defined by $\delta^+(p^2 - m^2) = \theta(p_0)\delta(p^2 - m^2)$. The δ^+ function thus amounts to a regular δ function with an energy flow condition overlayed.

A unitarity cut can then be defined by drawing a line through the diagram which partitions the vertices into two sets depending on which side of the line they fall. The propagators which are intersected by the line are placed on shell by replacing them in the integral representation with a δ^+ function according to the rule $(k+q)^2 - m^2 \to \delta^+((k+q)^2 - m^2)$ and so that energy flows in only one direction through the cut.

At one loop such a cut must place two propagators on shell. When these are adjacent, the cut has some external momentum p_i flowing through it, and when they are not adjacent this momentum is a sum $p_i + p_{i+1} + \ldots + p_j$. These cuts are said to be on the p^2 channel and $(p_i + p_{i+1} + \ldots + p_j)^2$ channel respectively, and give the discontinuities of the Feynman integral with respect to the variables p^2 and

$(p_i + p_{i+1} + \ldots + p_j)^2$ [27, 28].



(a) Unitarity cut on the p_2^2 channel

(b) Unitarity cut on the $(p_2 + p_3 + \ldots + p_{m-4} + p_{m-3})^2$ channel

Figure 2.2: Single unitarity cuts of a one-loop graph

An iterated unitarity cut is defined by drawing multiple lines through the diagram, with all of the cut propagators being replaced with δ^+ functions as above. Such a cut corresponds to the iterated discontinuity on each of the cut channels [56].

These definitions have two important differences from the residue definition that we have given above. Firstly, the integrals which contain delta functions are only non-vanishing when there is a solution in real kinematics for the propagators being placed on shell. There are cases, such as the maximal cut of the box integral with no internal masses and null external momenta, where the propagators can be placed on shell only in complex kinematics and so a non-vanishing result can only be obtained by using the definition (2.33). Secondly, the θ functions from the energy flow conditions carry information about the kinematic region in which the cut is non-vanishing. Single cuts in the p^2 channel, for instance, vanish when $p^2 < 0$ as we can go to a frame where $p = (0, p_1, \underline{0}_{D-2})$ and then the product of theta functions $\theta(k_0)\theta(-k_0)$ in the cut cause the integral to vanish. When performing two-loop unitarity cuts with a loop by loop method this information indicates a domain of integration in the outer loop, a feature that we will return to in chapter 5.

2.4 The Diagrammatic Coaction of One-Loop Feynman Integrals

Given a Feynman integral which is expressible in terms of multiple polylogarithms, we may compute its coaction. The result possesses a remarkable closed form which we will describe in this section. For what follows, we retain the convention for the $\tilde{J}$ given in (2.30), with dimensionality $D = 2\lceil \frac{n}{2} \rceil - 2\epsilon$. We will review the coactions of the integrals J normalised by leading singularity.

With this convention it is observed [24] that

$$\Delta J_E = \sum_{\emptyset \subset X \subseteq E} \left(J_X + a_X \sum_{e \in X} J_{X \setminus e} \right) \otimes \mathcal{C}_X J_E, \tag{2.40}$$

where E is the set of propagators of the graph, J_X denotes the integral corresponding to a graph with propagators not belonging to X contracted, and the coefficients a_X are given by:

$$a_X = \begin{cases} 1/2 & |X| \text{ even} \\ 0 & |X| \text{ odd} \end{cases} . \tag{2.41}$$

This equation is to be interpreted as holding order by order in ϵ when the Laurent series of the functions involved are calculated and the coaction of polylogarithms is applied to each function in the expansion of J_E. It applies irrespective of the internal masses and external kinematics. The terms $J_{X\setminus e}$ are called deformation terms [24] as they do not fit the simple form $J_X \otimes \mathcal{C}_X J_E$ of the other expressions in the coaction. As we will mention in section 2.5, the deformation terms can be accounted for with reference to the relation (2.38). The presence of such terms, and their coefficients, depend on the precise way in which we define our master integrals and cuts.

In order to check the validity of the diagrammatic coaction formula for a given Feynman integral, it can be expanded in ϵ along with all the other relevant Feynman integrals and cuts from some parameter integral representations as we will outline in section 2.6.3. One must then apply the coaction formula (2.8) to each term and use various functional relations on the polylogarithms to match ΔJ_E to the diagrammatic expression in (2.40). These manipulations can be performed using the package PolyLogTools [57].

There is a substantial body of evidence presented in [24] that this result holds in general:

1. It has been checked up to weight four for a range of graphs whose explicit expressions are known.

2. The divergences introduced by, for instance, the $1/\epsilon$ order term in the expansion of a tadpole graph always cancel where required, while in the divergent cases they reproduce the singularities. Thus the ϵ pole structure of the integral J_E is always reproduced by its diagrammatic coaction.

3. It is understood how the graphical coaction reproduces the terms $1 \otimes J_E$ and $J_E \otimes 1$ which must feature in the coaction. Specifically, the maximal cut of an integral expands to $1 + \mathcal{O}(\epsilon)$ to give the $J_E \otimes 1$ term, while writing J_E as a sum of certain cuts allows the $1 \otimes J_E$ term to be reconstructed.

4. The coaction has been used to correctly obtain the differential equations obeyed by a number of one-loop integrals. One can also recover the fact that the discontinuities of these integrals are given by their cuts, as we describe in section 2.4.1.

5. It takes a similar form to the coaction of the polylogarithms described in 2.1.2, which we discuss in section 2.5.

This diagrammatic coaction can also be extended [24] to cut integrals in the following manner:

$$\Delta \mathcal{C}_C J_E = \sum_{C \subseteq X \subseteq E} \left(\mathcal{C}_C J_X + a_X \sum_{e \in X \setminus C} C_C J_{X \setminus e} \right) \otimes \mathcal{C}_X J_E. \tag{2.42}$$

2.4.1 Discontinuities of One-Loop Feynman Integrals

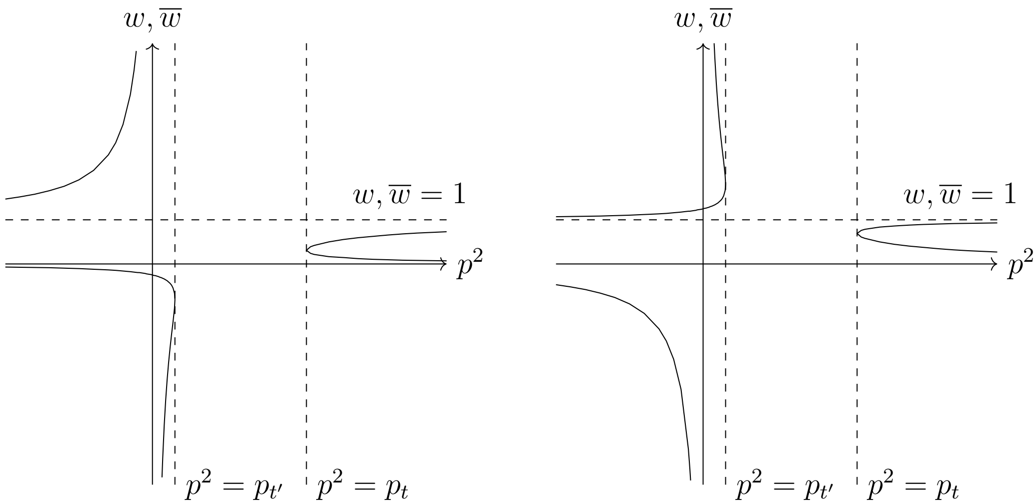
As the diagrammatic coaction agrees with that for multiple polylogarithms, it must obey the relations (2.24), permitting a derivation of the discontinuities and differential equations of a one-loop integral of the form specified in (2.30). In this section we illustrate this approach by proving that the discontinuity with respect to the invariant $s_{i,j} = (p_i + \ldots + p_j)^2$ is given by the cut in the corresponding channel. We will use the result (2.26), and so we must find those terms in the coaction where the first entry has weight one.

It can be shown that the only non-cancelling weight one contributions come from bubble and tadpole terms [24]. Given a one-loop integral J with n propagators, let us denote by $B_{i,j}$ the bubble integrals that correspond to the graphs where all but propagators i and j are contracted, and similarly let T_i be the analogous tadpole integrals. Then it follows that if $\mathcal{P}_i$ denotes projection onto the space of weight i polylogarithms, we have

$$\sum_{i=0}^{\infty} \Delta_{1,i-1} J = \sum_{1 \leq i < j \leq n} \mathcal{P}_1 \left(B_{i,j} + \frac{1}{2} T_i + \frac{1}{2} T_j \right) \otimes \mathcal{C}_{i,j} J + \sum_{1 \leq i \leq n} \mathcal{P}_1 T_i \otimes \mathcal{C}_i J. \tag{2.43}$$

We will now demonstrate that the function $\mathcal{P}_1 \left(B_{i,j} + \frac{1}{2} T_i + \frac{1}{2} T_j \right)$ has a discontinuity only in the region $p^2 > (m_i + m_j)^2$, where p is the momentum flowing into the bubble $B_{i,j}$.

The discontinuities of the logarithms appearing in the first entry can be determined by making use of the fact that they obey $f(z^*) = f^*(z)$, from which it follows that their discontinuities across the branch cut are equal to twice their imaginary parts. For instance, the logarithm $\log(z - i0)$ has discontinuity 0 for $z > 0$ and $2\pi i$ for $z < 0$.



(a) The region $0 < m_i^2 < m_j^2$ (b) The region $0 < m_j^2 < m_i^2$

Figure 2.3: The variables $w, \overline{w}$ for different relative sizes of the masses m_i and m_j

Finding this imaginary part is straightforward in cases where either of the masses vanish in the corresponding bubble. If both vanish then the weight one term in the expansion of $B_{i,j}$ is given by $\log(-p^2 - i0)$, and so the discontinuity is $2\pi i\, \mathcal{C}_{i,j} J$ when $p^2 > 0$. When only one vanishes, say the j^{th}, the relevant term in the expansion of $B_{i,j} + \frac{1}{2} T_i$ is $\log(-p^2 + m_i^2 - i0)$ and so there is a discontinuity when $p^2 > m_i^2$, which is again given by the corresponding channel cut. When neither mass vanishes, we find at weight one a combination of logarithms with arguments related to the variables w and $\overline{w}$ defined by $w\overline{w} = \frac{m_i^2}{p^2}$ and $(1-w)(1-\overline{w}) = \frac{m_j^2}{p^2}$, and so it becomes necessary to know the behaviour of these variables and their imaginary parts.

Solving the defining relations for w and $\overline{w}$ we find they are given by

$$\begin{aligned} w =& \frac{1}{2}\left[1 + \frac{m_i^2}{p^2} - \frac{m_j^2}{p^2} + \sqrt{\lambda\left(1, \frac{m_i^2}{p^2}, \frac{m_j^2}{p^2}\right)}\right] \\ \overline{w} =& \frac{1}{2}\left[1 + \frac{m_i^2}{p^2} - \frac{m_j^2}{p^2} - \sqrt{\lambda\left(1, \frac{m_i^2}{p^2}, \frac{m_j^2}{p^2}\right)}\right], \end{aligned} \tag{2.44}$$

where $\lambda(a,b,c) = a^2+b^2+c^2-2ab-2ac-2bc$. In figure 2.3 we show these variables for different relative sizes of the masses. Between the physical threshold $p_t = (m_i+m_j)^2$ and the anomalous threshold $p_{t'} = (m_i-m_j)^2$, it follows from the factorisation $\lambda\left(1, \frac{m_i^2}{p^2}, \frac{m_j^2}{p^2}\right) = \frac{1}{p^4}(p^2-p_t)(p^2-p_{t'})$ that the variables are not real.

We also require the imaginary parts of the variables, which can be deduced by making the replacements $m_i^2 \to m_i^2 - i\epsilon_1$, $m_j^2 \to m_j^2 - i\epsilon_2$ in (2.44):

$$\begin{aligned} w &\to w + \frac{i}{p^2\sqrt{\lambda\left(1, \frac{m_i^2}{p^2}, \frac{m_j^2}{p^2}\right)}}[\epsilon_1(1-w)+\epsilon_2 w] \\ \overline{w} &\to \overline{w} - \frac{i}{p^2\sqrt{\lambda\left(1, \frac{m_i^2}{p^2}, \frac{m_j^2}{p^2}\right)}}[\epsilon_1(1-\overline{w})+\epsilon_2 \overline{w}]. \end{aligned} \tag{2.45}$$

The weight one term is given by

$$\begin{aligned} &\int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2-m_i^2+i0} \frac{1}{(k+p)^2-m_j^2+i0} \\ =&\frac{1}{p^2(w-\overline{w})}[\log(1-w)-\log(-w)-\log(1-\overline{w})+\log(-\overline{w})] + \mathcal{O}(\epsilon). \end{aligned} \tag{2.46}$$

In the region where $p_{t'} < p^2 < p_t$, we observe that $\overline{w} = w^*$, and so it follows that both $\log(1-w)-\log(-w)-\log(1-\overline{w})+\log(-\overline{w})$ and $w-\overline{w}$ are imaginary and thus their ratio is real and there is no discontinuity. Above the threshold p_t, it follows from figure 2.3 that the only terms with imaginary parts are $\log(-w)$ and $\log(-\overline{w})$, and due to (2.45) they are of opposite sign. The expression (2.46) then has an overall imaginary part. Below the anomalous threshold $p_{t'}$ a similar analysis shows that any imaginary parts of the logarithms cancel, irrespective of the relative sizes of the masses. Thus it follows that there is a discontinuity only in the region above the threshold p_t and, from (2.26), we see that this discontinuity is given by $2\pi i\, \mathcal{C}_{i,j} J$.

We can similarly evaluate the discontinuities of any integral J in the one-loop basis with respect to m_i^2. The term $\mathcal{P}_1 T_i \otimes \mathcal{C}_i J$ in (2.43) gives a contribution $2\pi i\, \mathcal{C}_i J$

to this discontinuity in the region $m_i^2 < 0$. It can also be shown that the other terms in (2.43) do not give any contribution to the discontinuity in this region. This property relies on a cancellation of the imaginary parts of $\mathcal{P}_1 B_{i,j}$ and $\mathcal{P}_1\left(\frac{1}{2}T_i\right)$. Thus the single cuts of an integral J describe its discontinuity with respect to the internal masses.

2.5 General Form of the Coaction

We have now seen that the coactions of the multiple polylogarithms and of the one-loop Feynman integrals and their cuts can both be cast in the form

$$\Delta \int_\gamma \omega = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega \tag{2.47}$$

with bases of forms and contours ω_i and γ_i, and ω and γ lying in the spaces spanned by these objects. This property was first conjectured in [23]. The ω_i can be understood to generate a cohomology group related to the integral $\int_\gamma \omega$, while the γ_i generate the homology group. Of course there are then equivalence classes of forms and contours which produce identical integrals, but throughout the remainder of this thesis we will work with particular representatives of these classes.

For a given set of such forms and contours we can define a period matrix containing all the integrals of each form over all the contours. This matrix is given by $P_{i,j} = \int_{\gamma_i} \omega_j$. In both cases where (2.47) holds, the bases satisfy a certain duality condition which was stated in [23] as

$$\mathcal{P}_{ss} P_{i,j} = \delta_{i,j} \mod i\pi, \tag{2.48}$$

where $\mathcal{P}_{ss}$ is a projector onto the space of semi-simple objects which obey the relation $\Delta x = x \otimes 1$. This condition is easily verified for the bases described in section 2.1.2 for the case of the polylogarithms [24]. A similar analysis can be applied to the case of Feynman integrals to demonstrate that (2.40) can be cast in this form. Using a relation on the cut integrals which follows from (2.38), it can be verified [24] that the condition (2.48) is obeyed when the bases of forms and contours are chosen so that (2.47) reproduces (2.40). Crucially, these forms and contours are not simply those associated with the master integrals J_X and cuts $\mathcal{C}_X$, but we must also include deformations to reproduce the terms $a_X \sum_{e\in X} J_{X\setminus e}$.

As we will see in section 3.3, there is a condition equivalent to (2.48) derivable from intersection theory. Throughout the remainder of this thesis we will often appeal to the notion of dual bases of forms and contours, which should be understood to refer to either (2.48) or its analogue from intersection theory.

We also remark that the form (2.47) of the coaction admits a change to either the basis of forms or of contours, so long as a corresponding change is made to the other basis to preserve the duality condition. Suppose that we have a new basis of forms $\{\omega_i'\}$ related to the old basis by a rotation matrix $\mathcal{M}$ such that the equality $\int_\gamma \omega_i' = \sum_{j=1}^n \mathcal{M}_{i,j} \int_\gamma \omega_j$ holds for any contour γ expressible as a linear combination of the $\{\gamma_j\}$, then applying a corresponding rotation $(\mathcal{M}^{-1})^T$ to the contours preserves the duality:

$$\begin{aligned}\mathcal{P}_{ss} P'_{i,j} =& \mathcal{P}_{ss} \int_{\gamma_i'} \omega_j' \\ =& \sum_{k,l=1}^n \mathcal{M}_{j,l} \mathcal{M}^{-1}_{k,i} \mathcal{P}_{ss} \int_{\gamma_k} \omega_l \\ =& \sum_{k,l=1}^n \mathcal{M}_{j,l} \mathcal{M}^{-1}_{k,i} \delta_{k,l} \mod i\pi \\ =& \delta_{i,j},\end{aligned} \tag{2.49}$$

where we assume that $\mathcal{M}$ is defined so that $\mathcal{P}_{ss}\mathcal{M}_{i,j} = \mathcal{M}_{i,j}$.

The full period matrix is not left unchanged, and transforms to

$$P' = (\mathcal{M}^{-1})^T P \mathcal{M}^T, \tag{2.50}$$

but the form of the coaction is preserved:

$$\begin{aligned}\sum_i \int_\gamma \omega'_i \otimes \int_{\gamma'_i} \omega &= \sum_{i,j,k} \mathcal{M}^{-1}_{j,i} \mathcal{M}_{i,k} \int_\gamma \omega_j \otimes \int_{\gamma_k} \omega \\ &= \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega.\end{aligned} \tag{2.51}$$

2.6 Hypergeometric Functions

Hypergeometric functions are encountered in many branches of mathematical physics, including in the evaluation of Feynman integrals and their cuts. Indeed, with certain assumptions, it can be shown by Mellin Barnes integration [58] that the general form of a Feynman integral can be expressed as a hypergeometric function [59]. An alternative argument to this effect can be derived from the differential equation approach [60]. It is also found in every example that we deal with that the cuts are also of this form.

The simplest definition of the family of generalised hypergeometric functions, due to Horn [61], defines the functions as series

$$\sum_{m_1,\ldots,m_n=0}^{\infty} C(m_1,\ldots,m_n;a_1,\ldots,a_N) x_1^{m_1} \ldots x_n^{m_n} \tag{2.52}$$

with the coefficients obeying the constraint that, for each i, $\frac{C(m_1,\ldots,m_i+1,\ldots,m_n;a_1,\ldots,a_N)}{C(m_1,\ldots,m_i,\ldots,m_n;a_1,\ldots,a_N)}$ is a rational function of the $\{m_i\}$, for whatever values of the $\{x_i\}$ that the series converges. A more modern approach is to define these functions from a system of differential equations known as a GKZ system [62].

The classical theory of hypergeometric functions, including their differential equations, integral representations and certain transformations, is detailed in [63, 64] and we shall outline some relevant aspects below. We will also discuss more recent results concerning contiguous relations and reduction formulae, which will be important for the manipulation of expressions for Feynman integrals and their cuts that appear in the coaction.

2.6.1 The ${}_2F_1$ Function

First, consider the Gauss ${}_2F_1$ function defined for $|x| < 1$ by

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{x^n}{n!}, \tag{2.53} $$

where the Pochhammer symbols are given by $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$. It is clear that this is a hypergeometric function in the sense of (2.52), as we have $C(n; a, b, c) = \frac{(a)_n (b)_n}{(c)_n n!}$ and thus $\frac{C(n+1;a,b,c)}{C(n;a,b,c)} = \frac{(a+n)(b+n)}{(c+n)(1+n)}$.

Outside of the region $|x| < 1$, the function can be defined by analytic continuation using the Kummer connection formulae. For instance, when α, β and γ are generic we have

$$ \begin{aligned} {}_2F_1(\alpha, \beta; \gamma; x) = & \frac{\Gamma(\beta - \alpha)\Gamma(\gamma)}{\Gamma(\gamma - \alpha)\Gamma(\beta)} (-x)^{-\alpha} {}_2F_1 \left(\alpha, \alpha - \gamma + 1; \alpha - \beta + 1; \frac{1}{x} \right) \\ & + \frac{\Gamma(\alpha - \beta)\Gamma(\gamma)}{\Gamma(\gamma - \beta)\Gamma(\alpha)} (-x)^{-\beta} {}_2F_1 \left(\beta - \gamma + 1, \beta; \beta - \alpha + 1; \frac{1}{x} \right), \end{aligned} \tag{2.54} $$

which follows from the Mellin-Barnes integral representation of the ${}_2F_1$ [65]:

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(\alpha + s)\Gamma(\beta + s)\Gamma(-s)}{\Gamma(\gamma + s)} (-x)^s. \tag{2.55} $$

Other cases, as well as the results for non-generic parameters, are fully detailed in [66].

It is easily verified that the ${}_2F_1$ function has the integral representation

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)\Gamma(\alpha)} \int_0^1 \mathrm{d}u \, u^{\alpha - 1} (1 - u)^{\gamma - \alpha - 1} (1 - ux)^{-\beta}, \tag{2.56} $$

from which a simple change of variables proves the relations

$$ {}_2F_1(\alpha, \beta; \gamma; x) = (1 - x)^{-\beta} {}_2F_1 \left(\gamma - \alpha, \beta; \gamma; \frac{x}{x - 1} \right) \tag{2.57} $$

$$
\begin{aligned}
=&(1-x)^{-\alpha}{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{x}{x-1}\right)\\
=&(1-x)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;x).
\end{aligned}
$$

A number of ${}_2F_1$ functions with parameters α, β and γ each differing by integers are said to be contiguous. It may be shown that there are linear relations between any three independent contiguous ${}_2F_1$ functions. One method to derive these relations is to use partial fraction identities along with integration by parts relations generated from the expression

$$\int_0^1 du\frac{\partial}{\partial u}u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}=0. \tag{2.58}$$

We will prefer to use a technique where the integer shifts are generated by differential operators. This method was originally proposed for some simple hypergeometric functions in [67] and has since been extended to an algorithmic method by [68]. Firstly, define the differential operator $\theta=x\frac{d}{dx}$ and then note from the series representation of the ${}_2F_1$ that

$$
\begin{aligned}
(\theta+\alpha){}_2F_1(\alpha,\beta;\gamma;x)=&\alpha\,{}_2F_1(\alpha+1,\beta;\gamma;x) \qquad (2.59)\\
(\theta+\beta){}_2F_1(\alpha,\beta;\gamma;x)=&\beta\,{}_2F_1(\alpha,\beta+1;\gamma;x)\\
(\theta+\gamma-1){}_2F_1(\alpha,\beta;\gamma;x)=&(\gamma-1){}_2F_1(\alpha,\beta;\gamma-1;x)\\
\theta\,{}_2F_1(\alpha,\beta;\gamma;x)=&\frac{\alpha\beta}{\gamma}x\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;x).
\end{aligned}
$$

These relations imply that the ${}_2F_1$ function obeys the second-order differential equation

$$\left[\theta(\theta+\gamma-1)-x(\theta+\alpha)(\theta+\beta)\right]{}_2F_1(\alpha,\beta;\gamma;x)=0 \tag{2.60}$$

and we may then use this to complete the set of raising and lowering operators. The lowering operator for α, for instance, is derived by noting $\theta(\theta+\gamma-1)=(\theta+\alpha)^2+(\gamma-2\alpha-1)(\theta+\alpha)-\alpha(\gamma-\alpha-1)$. Using (2.60) with $\theta(\theta+\gamma-1)$ written in this way, we observe that every term except $\alpha(\gamma-\alpha-1){}_2F_1(\alpha,\beta;\gamma;x)$ contains a factor $\theta+\alpha$, and so we write

$$\begin{aligned}
&\alpha(\gamma-\alpha-1)\,{}_2F_1(\alpha,\beta;\gamma;x)\\
&=[(\theta+\alpha)+(\gamma-2\alpha-1)-x(\theta+\beta)]\,(\theta+\alpha)\,{}_2F_1(\alpha,\beta;\gamma;x)\\
&=[(\theta+\alpha)+(\gamma-2\alpha-1)-x(\theta+\beta)]\,\alpha\,{}_2F_1(\alpha+1,\beta;\gamma;x).
\end{aligned}$$

Replacing α by $\alpha-1$ we find that

$${}_2F_1(\alpha-1,\beta;\gamma;x)=\frac{1}{\gamma-\alpha}\,[(1-x)\theta+(\gamma-\alpha-x\beta)]\,{}_2F_1(\alpha,\beta;\gamma;x). \tag{2.61}$$

The remaining two operators can be found similarly and are:

$$\begin{aligned}
{}_2F_1(\alpha,\beta-1;\gamma;x)&=\frac{1}{\gamma-\beta}\,[(1-x)\theta+(\gamma-\beta-x\alpha)]\,{}_2F_1(\alpha,\beta;\gamma;x) \qquad (2.62)\\
{}_2F_1(\alpha,\beta;\gamma+1;x)&=\frac{\gamma}{x(\gamma-\alpha)(\gamma-\beta)}\,[(1-x)\theta-x(\alpha+\beta-\gamma)]\,{}_2F_1(\alpha,\beta;\gamma;x).
\end{aligned}$$

Now any differential operator acting on a ${}_2F_1$ may be written as a linear combination of θ and the identity operator $\mathbb{I}$ due to (2.60) and thus given three independent contiguous ${}_2F_1$ functions we may find a linear relation among them. This is achieved by selecting one of these functions, which we will write as ${}_2F_1(\alpha,\beta;\gamma;x)$ and using it to express the other two via relations of the form

$$\begin{aligned}
{}_2F_1(\alpha+m_1,\beta+m_2;\gamma+m_3;x)&=(a\theta+b)\,{}_2F_1(\alpha,\beta;\gamma;x) \qquad (2.63)\\
{}_2F_1(\alpha+n_1,\beta+n_2;\gamma+n_3;x)&=(c\theta+d)\,{}_2F_1(\alpha,\beta;\gamma;x).
\end{aligned}$$

By eliminating $\theta\,{}_2F_1(\alpha,\beta;\gamma;x)$ we are left with a linear relation among the three functions.

Lastly, we note that for non-generic values of the parameters α, β and γ the ${}_2F_1$ function admits certain non-linear transformations of argument such as:

$${}_2F_1(\alpha,\beta;\alpha-\beta+1;x) \tag{2.64}$$

$$=(1+\sqrt{x})^{-2\alpha}{}_2F_1\left(\alpha, \alpha-\beta+\frac{1}{2}; 2\alpha-2\beta+1; \frac{4\sqrt{x}}{(1+\sqrt{x})^2}\right),$$

which is valid in the region $|x| < 1$. These quadratic relations can be derived by finding transformations of variable which leave the form of the differential equation (2.60) unchanged as detailed in [69].

2.6.2 Generalised Hypergeometric Functions

We now introduce several classes of generalised hypergeometric function that will occur throughout this thesis: the ${}_{p+1}F_p$ and Appell functions, and briefly list some of their important properties.

We begin with the function ${}_{p+1}F_p(\alpha_1, \ldots, \alpha_{p+1}; \beta_1, \ldots, \beta_p; x)$, an immediate generalisation of the ${}_2F_1$ with more parameters which is defined by

$${}_{p+1}F_p(\alpha_1, \ldots, \alpha_{p+1}; \beta_1, \ldots, \beta_p; x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_{p+1})_n}{(\beta_1)_n \ldots (\beta_p)_n} \frac{x^n}{n!}, \tag{2.65}$$

and also note the definitions of the Appell functions:

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{2.66}$$

$$F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{2.67}$$

$$F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{2.68}$$

$$F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{2.69}$$

All of the above are defined within a suitable radius of convergence, with analytic continuation defining them in other regions.

These functions have the integral representations [66]

$$
\begin{aligned}
&{}_{p+1}F_p(\alpha_1,\ldots,\alpha_{p+1};\beta_1,\ldots,\beta_p;x) \\
=&\frac{\Gamma(\beta_p)}{\Gamma(\alpha_{p+1})\Gamma(\beta_p-\alpha_{p+1})} \\
&\times\int_0^1 du\, u^{\alpha_{p+1}-1}(1-u)^{\beta_p-\alpha_{p+1}-1}{}_pF_{p-1}(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_{p-1};x)
\end{aligned} \tag{2.70}
$$

$$
\begin{aligned}
&F_1(\alpha;\beta,\beta';\gamma;x,y) \\
=&\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \\
&\times\int_0^1 du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'} \\
=&\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \\
&\times\int_0^1 dv\int_0^{1-v} du\, u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux-vy)^{-\alpha}
\end{aligned} \tag{2.71}
$$

$$
\begin{aligned}
&F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y) \\
=&\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')} \\
&\times\int_0^1 dv\int_0^1 du\, u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-ux-vy)^{-\alpha}
\end{aligned} \tag{2.72}
$$

$$
\begin{aligned}
&F_3(\alpha,\alpha';\beta,\beta';\gamma;x,y) \\
=&\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \\
&\times\int_0^1 dv\int_0^{1-v} du\, u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}(1-vy)^{-\alpha'}
\end{aligned} \tag{2.73}
$$

$$
F_4(\alpha,\beta;\gamma,\gamma';x(1-y),y(1-x)) \tag{2.74}
$$

$$
\begin{aligned}
=&\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma'-\beta)} \\
&\times \int_0^1 dv \int_0^1 du\, u^{\alpha-1} v^{\beta-1} (1-u)^{\gamma-\alpha-1} (1-v)^{\gamma'-\beta-1} (1-ux)^{\alpha-\gamma-\gamma'+1} (1-vy)^{\beta-\gamma-\gamma'+1} \\
&\times (1-ux-vy)^{\gamma+\gamma'-\alpha-\beta-1},
\end{aligned}
$$

where ${}_1F_0(\alpha;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n = (1-x)^{-\alpha}$.

The F_4 integral representation is of particular interest due to the difficulty of its derivation, which was first given in [70] (see also [71]), and the difference in the variables used in the series and integral forms. It is often a challenging problem to find the simplest variables to be used in an expansion of a Feynman integral. For instance, experience shows that for integrals with three scales, such as a bubble with two internal masses, variables of the type w and $\bar{w}$ defined by $w\bar{w} = \frac{m_1^2}{p^2}$ and $(1-w)(1-\bar{w}) = \frac{m_2^2}{p^2}$ as in section 2.4.1 are an optimal choice. Given an easily derived expression for this bubble in terms of the F_4 function with arguments $\frac{m_1^2}{p^2}$ and $\frac{m_2^2}{p^2}$, the integral representation immediately suggests variables closely related to w and $\bar{w}$.

We note that each of these classes of function have their own analytic continuation formulae: the ${}_{p+1}F_p$ has a relation similar to (2.54) derived from the Mellin-Barnes integral, while the Appell series have various analytic continuation formulae derivable from their series representations by writing one of the sums as a ${}_2F_1$ and applying the formulae known for this class [72]. The F_1, F_2 and F_3 functions also have transformations of the form (2.57) which follow from their integral forms.

We note that the raising and lowering operators for generating contiguous relations generalise in a straightforward way to the ${}_{p+1}F_p$, but now the differential equation has degree $p+1$ and so these relations generally involve $p+1$ terms. For the Appell functions the method requires two differential operators: $\theta = x\frac{\partial}{\partial x}$ and $\phi = y\frac{\partial}{\partial y}$. It is again simple to raise parameters appearing in the numerator of the series representation and lower those in the denominator, but the remaining cases involve more complicated higher degree operators derived in [73]. These raising and lowering operators depend on θ, ϕ, $\theta\phi$ and the identity operator $\mathbb{I}$. The only exception is the F_1 function [73], where the operator $\theta\phi$ is always linearly dependent on θ and ϕ due to the relation

$$\left(\theta\phi - \frac{\beta' y}{x-y}\theta + \frac{\beta x}{x-y}\phi\right) F_1(\alpha;\beta,\beta';\gamma;x,y) = 0. \tag{2.75}$$

From this we see that a general F_2, F_3 or F_4 function can be written as a linear combination of four contiguous functions. This is achieved in the same way as the ${}_2F_1$ case by writing relations analogous to (2.63) and eliminating those terms which depend on θ, ϕ and $\theta\phi$. For the F_1, only θ and ϕ need to be eliminated and so we only require three contiguous functions. Of course, for each of the Appell functions there is the possibility of non-trivial contiguous relations involving fewer terms when the system of equations in the differential operators is not generic.

Given the collection of raising and lowering operators, it is then trivial to write down the differential equations obeyed by these classes of function, similar to how we wrote down (2.60). For the F_4 function, for instance, we have

$$\begin{aligned}
[\theta(\theta+\gamma-1) - x(\theta+\phi+\alpha)(\theta+\phi+\beta)]F_4(\alpha,\beta;\gamma,\gamma';x,y) &= 0 \\
[\phi(\phi+\gamma'-1) - y(\theta+\phi+\alpha)(\theta+\phi+\beta)]F_4(\alpha,\beta;\gamma,\gamma';x,y) &= 0.
\end{aligned} \tag{2.76}$$

We will find it more convenient to express these in the form

$$\begin{aligned}
0 = &\big\{(1-x-y)\theta^2 - 2x\theta\phi - [(\alpha+\beta)x - (\gamma-1)(1-y)]\theta \\
&-(\alpha+\beta+1-\gamma')x\phi - \alpha\beta x\big\}\, F_4(\alpha,\beta;\gamma,\gamma';x,y) \\
0 = &\big\{(1-x-y)\phi^2 - 2y\theta\phi - [(\alpha+\beta)y - (\gamma'-1)(1-x)]\phi \\
&-(\alpha+\beta+1-\gamma)y\theta - \alpha\beta y\big\}\, F_4(\alpha,\beta;\gamma,\gamma';x,y),
\end{aligned} \tag{2.77}$$

by solving for the terms which involve the operators θ^2 and ϕ^2.

We conclude by mentioning some transformation formulae derived in the sequence of papers [74–76]. The key insight of [74] is that many integral representations of hypergeometric functions can be written in the form of a single average or multiple average. The single average is

$$\int d\mu_b(u) f\left(\sum_{i=1}^{n} u_i Z_i\right), \tag{2.78}$$

where $u, Z \in \mathbb{R}^n$ with $0 < u_i < 1$ and $\sum_{i=1}^{n} u_i = 1$, and $d\mu_b$ is an integration measure defined in [74]. The multiple averages are immediate generalisations of this where the vector Z_i is replaced by a multi-index object and the integration is performed over a number of sets of variables u_i, v_i etc. Specialising to the case where the function f is given by $f(z) = z^n$ with $n \in \mathbb{Z}$, there is a generating function for the multiple averages which provides many useful properties, including their reduction to simpler forms when certain combinations of parameters vanish. By writing the Appell F_4 as a multiple average [75] a number of useful transformations and reduction formulae can be derived [76]. For instance, there are the formulae

$$\begin{aligned} &F_4(\alpha, \beta; \gamma, \beta; x(1-y), y(1-x)) \\ =&(1-y)^{-\alpha} F_1\left(\alpha; 1+\alpha-\gamma, \beta-1-\alpha+\gamma; \gamma; \frac{x}{1-y}, x\right) \end{aligned} \tag{2.79}$$

$$\begin{aligned} &F_4(\alpha, \beta; \gamma, \gamma; x^2, y^2) \\ =&\sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}(\gamma-1/2)_m(\gamma-1/2)_n}{(\gamma)_{m+n}(2\gamma-1)_{m+n}} \frac{(x+y)^{2m}}{m!} \frac{(x-y)^{2n}}{n!} \end{aligned} \tag{2.80}$$

$$\begin{aligned} &F_4(\alpha, \beta; \gamma, 1+\alpha-\gamma; x(1-y), y(1-x)) \\ =&F_2(\alpha, \beta, \beta; \gamma, 1+\alpha-\gamma; x, y) \end{aligned} \tag{2.81}$$

$$\begin{aligned} &F_4(\alpha, \beta; \gamma, 1+\alpha+\beta-\gamma; x(1-y), y(1-x)) \\ =&{}_2F_1(\alpha, \beta; \gamma; x)\, {}_2F_1(\alpha, \beta; 1+\alpha+\beta-\gamma; y), \end{aligned} \tag{2.82}$$

which we will use in our discussions of certain two-loop Feynman integrals.

2.6.3 Laurent Series of Hypergeometric Functions

As we have seen, the hypergeometric functions that arise in the evaluation of Feynman integrals have parameters that depend on a dimension $D = 2N - 2\epsilon$ for some $N \in \mathbb{Z}^+$, and when expanded around $\epsilon = 0$ the coefficients of ϵ will be some iterated integrals. In this section we detail how to perform these expansions in the case where the coefficients are multiple polylogarithms.

The expansion of such functions is a well-studied problem. One can use contiguous relations to limit the number of cases which must be studied and then expand these particular functions. This is done either using the integral representation or by writing the series form, collecting the coefficients of each power of ϵ as series in the variables x_i and matching these to the series definitions of polylogarithms [77–82]. In what follows we will prefer to use the integration method.

We shall be limited to the case of hypergeometric functions where the parameters take the form $m+n\epsilon$ for $m, n \in \mathbb{Z}$ as these are the commonly encountered cases that produce polylogs. In certain cases where a number of parameters are instead of the form $\frac{m}{2} + n\epsilon$ with $m, n \in \mathbb{Z}$ it is possible to remove the half integers by non-linear transformations such as (2.64), but we will not further consider this.

Let us illustrate this method for the case of the ${}_2F_1$, assuming our parameters are such that the integral representation converges. Take a simple case, such as

$$\begin{aligned} &\frac{\Gamma(1+a\epsilon)\Gamma(1+b\epsilon)}{\Gamma(2+(a+b)\epsilon)}{}_2F_1(1+a\epsilon, 1-c\epsilon; 2+(a+b)\epsilon; x) \\ &= \int_0^1 du u^{a\epsilon}(1-u)^{b\epsilon}(1-ux)^{-1+c\epsilon}. \end{aligned}$$

We may perform the ϵ expansion within the integral:

$$\begin{aligned} &\int_0^1 du u^{a\epsilon}(1-u)^{b\epsilon}(1-ux)^{-1+c\epsilon} \\ &= -\frac{1}{x}\sum_{l,m,n=0}^{\infty} a^l b^m c^n \frac{1}{l!m!n!}\epsilon^{l+m+n} \int_0^1 \frac{du}{u-\frac{1}{x}} G^l(0;u)G^m(1;u)G^n\left(\frac{1}{x};u\right), \end{aligned}$$

and it is then clear that the result consists of polylogs and is uniform weight at each order in ϵ as $G^l(0;u)G^m(1;u)G^n\left(\frac{1}{x};u\right)$ may be written as a weight $l+m+n$ polylog using the shuffle product.

Generally though there is an extra step before we may expand in ϵ, which is to account for endpoint singularities in the integral. We do this using a well-known method which is outlined, for instance, in [83]. As an example, take the integral

$$\begin{aligned}&\frac{\Gamma(a\epsilon)\Gamma(1+b\epsilon)}{\Gamma(1+(a+b)\epsilon)}{}_2F_1(a\epsilon,-c\epsilon;1+(a+b)\epsilon;x)\\ &=\int_0^1 du\, u^{-1+a\epsilon}(1-u)^{b\epsilon}(1-ux)^{c\epsilon}\end{aligned}$$

and define $f(u)=(1-ux)^{c\epsilon}$. Then we rewrite the above as

$$\int_0^1 du\, u^{-1+a\epsilon}(1-u)^{b\epsilon}\left[f(u)-f(0)\right]+\int_0^1 du\, u^{-1+a\epsilon}(1-u)^{b\epsilon}f(0). \tag{2.83}$$

The second term can be recognised as $\frac{f(0)}{a\epsilon}\frac{\Gamma(1+a\epsilon)\Gamma(1+b\epsilon)}{\Gamma(1+(a+b)\epsilon)}$ and is easily expanded in ϵ while for the first term integration now commutes with ϵ expansion of the integrand and so we may proceed as before. There are similar prescriptions to eliminate endpoint singularities for other cases, such as:

$$\begin{aligned}&\int_0^1 du\, u^{-1+a\epsilon}(1-u)^{-1+b\epsilon}f(u) &(2.84)\\ =&\int_0^1 du\, u^{-1+a\epsilon}(1-u)^{-1+b\epsilon}[f(u)-uf(1)-(1-u)f(0)]\\ &+\int_0^1 du\, u^{-1+a\epsilon}(1-u)^{-1+b\epsilon}f(0)+\int_0^1 du\, u^{a\epsilon}(1-u)^{-1+b\epsilon}[f(1)-f(0)],\end{aligned}$$

where we note that $u^{-1+a\epsilon}(1-u)^{-1+b\epsilon}[f(u)-uf(1)-(1-u)f(0)]$ is now finite around both $u=0$ and $u=1$. For multiple integrals, one can apply these prescriptions to each integral iteratively, for example:

$$\int_0^1\int_0^1 dvdu\, u^{-1+a\epsilon}v^{-1+b\epsilon}f(u,v) \tag{2.85}$$

$$
\begin{aligned}
&= \int_0^1 du\, u^{-1+a\epsilon} \int_0^1 dv\, v^{-1+b\epsilon} f(0,v) + \int_0^1 du\, u^{-1+a\epsilon} \int_0^1 dv\, v^{-1+b\epsilon} [f(u,v) - f(0,v)] \\
&= \int_0^1 du\, u^{-1+a\epsilon} \left[\int_0^1 dv\, v^{-1+b\epsilon} f(0,0) + \int_0^1 dv\, v^{-1+b\epsilon} [f(0,v) - f(0,0)] \right] \\
&\quad + \int_0^1 du\, u^{-1+a\epsilon} \left[\int_0^1 dv\, v^{-1+b\epsilon} [f(u,0) - f(0,0)] \right. \\
&\quad \left. + \int_0^1 dv\, v^{-1+b\epsilon} [f(u,v) - f(0,v) - f(u,0) + f(0,0)] \right].
\end{aligned}
$$

Thus we can obtain prescriptions for singularities at any combination of $u = 0$, $u = 1$, $v = 0$ and $v = 1$.

Chapter 3

Coaction of Hypergeometric Functions

As we have seen in section 2.6, the class of hypergeometric functions is believed to be sufficient to describe any Feynman integral, and given that the functions and their identities are well studied they provide a useful language to describe Feynman integrals.

It is therefore of considerable interest to determine whether the coaction of these functions can be stated in the compact form of section 2.5 by finding suitable bases of forms and contours. In fact we can construct the coaction in exactly this way, as we now describe and illustrate with a number of examples. In each example we take the integral representation of the hypergeometric function in question, determine the relevant bases, and then verify that the coaction is given by formula (2.47).

We begin with the example of the ${}_2F_1$ to illustrate the method, then outline a number of other examples and conclude with some observations about the theory in general.

This chapter is based on the contents of the paper [1].

3.1 The Gauss ${}_2F_1$ Function

Recall that the ${}_2F_1$ function has integral representation

$$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}{}_2F_1(\alpha,\beta;\gamma;x)=\int_0^1 du u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}. \tag{3.1}$$

As we have seen, this integral is expressible using multiple polylogarithms when the exponents $\alpha-1$, $\gamma-\alpha-1$ and $-\beta$ are each of the form $m+n\epsilon$ with $m,n\in\mathbb{Z}$ and we will limit ourselves to this case in what follows. Of course, this also captures the cases where one or two of the $\{\alpha,\beta,\gamma\}$ are of the form $\frac{m}{2}+n\epsilon$ with $m,n\in\mathbb{Z}$ and the function is transformable so as to eliminate the half integer via the transformations such as (2.64).

We will find the coaction of this function with generic parameters in the following subsection, and then examine the various properties of this result in the later subsections.

3.1.1 Integrands and Contours in the Generic Case

In the following we will construct the coaction of the ${}_2F_1$ regularised by the Γ functions from its integral representation, i.e. the object on the right hand side of (3.1). The coaction of the ${}_2F_1$ itself can then be found easily by using the identity (2.14). For concreteness, let us choose α, β and γ such that the integral is

$$\int_0^1 du\, u^{m+a\epsilon}(1-u)^{n+b\epsilon}(1-ux)^{p+c\epsilon}. \tag{3.2}$$

Let us start by selecting a basis of forms. It is known from the contiguous relations for the ${}_2F_1$ function we described in section 2.6.1 that there is a basis of two master integrals for the set of functions

$$\{{}_2F_1(1+m+a\epsilon,-p-c\epsilon;2+m+n+(a+b)\epsilon;x)|m,n,p\in\mathbb{Z}\}. \tag{3.3}$$

As integer shifts to the argument of the function $\Gamma(a)$ are equivalent to multiplication by a rational function of the argument a, the set

$$\left\{ \int_0^1 du\, u^{m+a\epsilon}(1-u)^{n+b\epsilon}(1-ux)^{p+c\epsilon} \middle| m,n,p \in \mathbb{Z} \right\} \tag{3.4}$$

also has two master integrals. This corresponds to the statement that any of the family of objects $\{du\, u^{m+n\epsilon}(1-u)^{n+b\epsilon}(1-ux)^{p+c\epsilon}|m,n,p \in \mathbb{Z}\}$ can be written, modulo forms with vanishing integral over the domain $[0,1]$, using only two independent elements of this family where specific values of m, n and p are selected. We can choose

$$\begin{aligned} \omega_1 =& b\epsilon\, du\, u^{a\epsilon}(1-u)^{-1+b\epsilon}(1-ux)^{c\epsilon} \\ \omega_2 =& c\epsilon x\, du\, u^{a\epsilon}(1-u)^{b\epsilon}(1-ux)^{-1+c\epsilon}, \end{aligned} \tag{3.5}$$

which both lead to uniform-weight integrals. The normalisation coefficients $b\epsilon$ and $c\epsilon x$ are chosen to produce unit elements in the period matrix, as we will see below.

In the examples we have seen previously, the modified integration contours encircled poles of the integrand. Generically, the integrand (3.2) does not have poles and so we need another definition of the contours. If we cannot encircle the points where u, $1-u$ and $1-ux$ vanish, the natural alternative is to have them as endpoints of the integration domain, with the endpoint singularities of the integrals regulated by the ϵ part of the exponent.

We also mention that we expect there to be only two independent choices of contour due to the fact that the space of forms is two-dimensional and these spaces must have the same dimension. It is trivial that when we have three points $u=0$, $u=1$ and $u=\frac{1}{x}$ there are only two independent choices of contour with each endpoint being one of these points. Less obviously, when a ${}_2F_1$ integrand is integrated between one of these points and $u=\infty$ we do not get an independent result, as we will demonstrate below. With all of the above in mind let us then make the selection of contours:

$$\gamma_1 =\{u|0 \leq u \leq 1\} \tag{3.6}$$

$$\gamma_2 = \left\{ u | 0 \leq u \leq \frac{1}{x} \right\}.$$

We note that the integrals $\int_0^1 \omega_1$ and $\int_0^{\frac{1}{x}} \omega_2$ both have a non-vanishing coefficient of ϵ^0 in their Laurent series due to their regulated endpoint singularities, while the expansions of $\int_0^1 \omega_2$ and $\int_0^{\frac{1}{x}} \omega_1$ begin at order ϵ. In fact, with the normalisation of (3.5), the Laurent series of the former integrals each begin as $1 + \mathcal{O}(\epsilon)$ due to the arguments given in 2.6.3. Specifically, for the integral over γ_1 we can make the change of variables $u \to 1-u$ and split up the integrand as in (2.83), while for γ_2 we must first perform a trivial change of variable $u \to \frac{u}{x}$. Thus with the choices (3.5) and (3.6) we find that

$$P_{i,j} = \delta_{i,j} + \mathcal{O}(\epsilon), \tag{3.7}$$

and so our bases are dual to each other in the sense described in section 2.5. Hence we expect to have

$$\Delta \int_\gamma \omega = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega. \tag{3.8}$$

In order to check this formula it is sufficient to verify the result for each element of the period matrix i.e. by making any selection of ω and γ for which $\omega \in \{\omega_1, \omega_2\}$ and $\gamma \in \{\gamma_1, \gamma_2\}$. If it holds for the elements of these bases then it must also hold for any integrands and contours that are in the span of these bases:

$$\begin{aligned} \Delta \int_\gamma \omega =& \Delta \int_{\sum_j b_j \gamma_j} \sum_i a_i \omega_i \\ =& \sum_{i,j} a_i b_j \Delta \int_{\gamma_j} \omega_i \\ =& \sum_{i,j,k} a_i b_j \int_{\gamma_j} \omega_k \otimes \int_{\gamma_k} \omega_i \\ =& \sum_k \int_{\sum_j b_j \gamma_j} \omega_k \otimes \int_{\gamma_k} \sum_i a_i \omega_i \end{aligned} \tag{3.9}$$

$$= \sum_k \int_\gamma \omega_k \otimes \int_{\gamma_k} \omega.$$

For the case of the ${}_2F_1$, it has been checked that, up to weight four in the expansions of each integral, the coaction does indeed take this form for each of the four elements of the period matrix with the choices (3.5) and (3.6).

Having found the relation (3.8), one can then evaluate the integrals which appear in each entry. The first entries are each of the form (3.1). Meanwhile the second entry integrals are also able to be expressed using the ${}_2F_1$ function, a property which we discuss in more detail in section 3.1.3. If we set $\omega = du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}$ then these integrals are given by

$$\begin{aligned} \int_{\gamma_1} \omega =& \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\, {}_2F_1(\alpha,\beta;\gamma;x) \\ \int_{\gamma_2} \omega =& x^{-\alpha}\frac{\Gamma(\alpha)\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\, {}_2F_1\left(\alpha, 1+\alpha-\gamma; 1+\alpha-\beta; \frac{1}{x}\right). \end{aligned} \tag{3.10}$$

Let us also return to the question of contours with an endpoint at infinity. Choosing the second endpoint to be $u = 1$ for simplicity we find, by making the change of variable $u \to \frac{1}{u}$ in the integral, that if $\gamma_\infty = \{u|1 < u < \infty\}$ then the integral of the form ω defined above is

$$\begin{aligned} &\int_{\gamma_\infty} \omega \\ =&(-1)^{\gamma-\alpha-\beta-1}x^{-\beta}\frac{\Gamma(\beta-\gamma+1)\Gamma(\gamma-\alpha)}{\Gamma(\beta-\alpha+1)}{}_2F_1\left(\beta-\gamma+1, \beta; \beta-\alpha+1; \frac{1}{x}\right). \end{aligned} \tag{3.11}$$

Now if $z = m+a\epsilon$, then we can observe that $\Gamma(z)\Gamma(1-z) = (-1)^m \frac{1}{a\epsilon}\Gamma(1+a\epsilon)\Gamma(1-a\epsilon)$. Using this, along with the results (3.10) and (3.11), we can see that (2.54) implies a linear relation which holds modulo $i\pi$ among integrals of ω over γ_1, γ_2 and γ_∞. This fact can also be derived from homology theory [84]. We will also find in the following section 3.2 when constructing contours for other hypergeometric functions that we need not consider contours extending to infinity.

Now we can take the formula (3.8) and use it to write down the coaction for the integral (3.2) which is of special interest as it is closely related to the ${}_2F_1$ function itself. Using the identity (2.14) we can then obtain

$$
\begin{aligned}
&\Delta {}_2F_1\left(\alpha,\beta;\gamma;x\right) \qquad (3.12)\\
=&{}_2F_1\left(1+\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon;x\right)\otimes {}_2F_1\left(\alpha,\beta;\gamma;x\right)\\
&-\frac{\beta_\epsilon\epsilon}{1+\gamma_\epsilon\epsilon}{}_2F_1\left(1+\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;2+\gamma_\epsilon\epsilon;x\right)\otimes\left\{x^{1-\alpha}\frac{\Gamma(1-\beta)\Gamma(\gamma)}{\Gamma(1+\alpha-\beta)\Gamma(\gamma-\alpha)}\right.\\
&\left.\times{}_2F_1\left(\alpha,1+\alpha-\gamma;1+\alpha-\beta;\frac{1}{x}\right)\right\},
\end{aligned}
$$

where α, β and γ are assumed to take the forms $[\alpha]+\alpha_\epsilon\epsilon$, $[\beta]+\beta_\epsilon\epsilon$ and $[\gamma]+\gamma_\epsilon\epsilon$ respectively, with $[\alpha]$, $[\beta]$ and $[\gamma]$ denoting the integer parts of the parameters.

3.1.2 Degenerate Limits

The result (3.12) accommodates the limits $b\to 0$ and $c\to 0$ of the parameters a and b from (3.2), even though the underlying integrals used to arrive at this formula are no longer well defined in these limits. We can verify by expansion that the coaction formula continues to hold in such limits, but if we wish to consider the entries of the coaction as integrals we must reconsider the meaning of the contours. For the limit $p+c\epsilon\to -1$, we must reinterpret γ_2 when integrating over ω_2. Given that the endpoint singularity at $u=\frac{1}{x}$ is no longer regulated by the $c\epsilon$ exponent, we are now back in the regime where we should interpret the contour as encircling the point.

Performing the contour integral with this new definition gives

$$
\begin{aligned}
&\mathrm{Res}_{u=1/x}\, c\epsilon x\int_0^{1/x} du\, u^{m+a\epsilon}(1-u)^{n+b\epsilon}(1-ux)^{-1} \qquad (3.13)\\
=&-c\epsilon x^{-m-a\epsilon}\left(1-\frac{1}{x}\right)^{n+b\epsilon}.
\end{aligned}
$$

This agrees, up to normalisation, with the limit of the corresponding term in (3.12):

$$\begin{aligned}&\lim_{p+c\epsilon\to-1} c\epsilon x\int_0^{1/x} du\, u^{m+a\epsilon}(1-u)^{n+b\epsilon}(1-ux)^{p+c\epsilon} \\ =&\lim_{p+c\epsilon\to-1}\left\{c\epsilon x^{-m-a\epsilon}\frac{\Gamma(1+m+a\epsilon)\Gamma(1+p+c\epsilon)}{\Gamma(2+m+p+(a+c)\epsilon)}\right. \\ &\left.{}_2F_1\left(1+m+a\epsilon,-n-b\epsilon;2+m+p+(a+c)\epsilon;\frac{1}{x}\right)\right\} \\ =&x^{-m-a\epsilon}\left(1-\frac{1}{x}\right)^{n+b\epsilon}.\end{aligned} \tag{3.14}$$

Thus we can still write the coaction in the form (2.47), subject to having a new definition of the contours. The formula (3.12) has also been checked in the case with $x=1$, where the ${}_2F_1$ function can be summed in a closed form. In this case, it is found that

$$\begin{aligned}{}_2F_1(\alpha,\beta;\gamma;1)=&\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-\beta-1} \\ =&\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}.\end{aligned} \tag{3.15}$$

The coaction can then be obtained by using (2.14) and is consistent with setting $x=1$ in (3.12).

3.1.3 The Second Entries

In the coaction (3.12) there is the interesting property that the function

$$x^{-\alpha}\,{}_2F_1\left(\alpha,1+\alpha-\gamma;1+\alpha-\beta;\frac{1}{x}\right)$$

appearing in the second entry is the second solution to the differential equation (2.60) obeyed by ${}_2F_1(\alpha,\beta;\gamma;x)$. This results from the construction of the second entries by modifying the integration contour.

Computing the derivative of this contour integral, we find

$$\begin{aligned}
&\frac{d}{dx}\int_0^{\frac{1}{x}} du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} \\
&= \int_0^{\frac{1}{x}} du\, \frac{d}{dx} u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} - \left\{\frac{1}{x^2}u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}\right\}\Bigg|_{u=\frac{1}{x}}.
\end{aligned} \tag{3.16}$$

We can perform this calculation for values of the parameters such that the second term vanishes, and so then evaluating the derivative of second degree we find similarly that

$$\begin{aligned}
&\frac{d^2}{dx^2}\int_0^{\frac{1}{x}} du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} \\
&= \int_0^{\frac{1}{x}} du\, \frac{d^2}{dx^2} u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}.
\end{aligned} \tag{3.17}$$

Now the differential equation (2.60) obeyed by ${}_2F_1(\alpha,\beta;\gamma;x)$ is a linear relation among this function and its first and second-order derivatives, which evaluate respectively to

$$\int_0^1 du\, \frac{d}{dx} u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} \tag{3.18}$$

and

$$\int_0^1 du\, \frac{d^2}{dx^2} u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}. \tag{3.19}$$

Such a linear relation can be found by integration by parts and partial fractions, and we are free to change the integration contour provided that the change does not generate any boundary terms. As both 0 and $\frac{1}{x}$ are vanishing points of the integrand we can change the contour $[0,1]$ to $[0,\frac{1}{x}]$ and thus obtain the same linear relation among the objects

$$\left\{\int_0^{\frac{1}{x}} du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta},\right. \tag{3.20}$$
$$\int_0^{\frac{1}{x}} du\, \frac{d}{dx} u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta},$$
$$\left.\int_0^{\frac{1}{x}} du\, \frac{d^2}{dx^2} u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}\right\},$$

which, as we have seen, are the derivatives of $\int_0^{\frac{1}{x}} du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}$. Thus this function obeys the same differential equation as ${}_2F_1(\alpha,\beta;\gamma;x)$.

3.1.4 Projection onto a Basis

We conclude by mentioning that the formula (3.12) suggests a method for deriving contiguous relations on the ${}_2F_1$ functions. If we write down the coaction of some ${}_2F_1$ which is contiguous to ${}_2F_1(1+2\epsilon,\epsilon;1-\epsilon;x)$ and ${}_2F_1(1+2\epsilon,1+\epsilon;2-\epsilon;x)$ then it can be expressed as a linear combination of these functions by expanding the second entries at weight zero and comparing the terms in the coaction given by the map $\sum_{n=0}^{\infty}\Delta_{n,0}$. These second entries are given by the integrals of some form ω over the contours γ_1 and γ_2, and so this approach is analogous to the unitarity method for finding linear relations among Feynman integrals. Of course, the diagrammatic coaction at one loop also encoded this property, as will the two-loop coaction that we will find in chapter 6.

Similarly, we can find relations that hold modulo $i\pi$ expressing the ${}_2F_1$ function in terms of the second entries of the coaction by examining the $\sum_{n=0}^{\infty}\Delta_{0,n}$ components. With the choice of contour basis used in (3.12) this will only yield a trivial result as one of our contours coincides with that in the integral representation of the ${}_2F_1$, but the method may also be applied after writing the coaction in some different basis.

3.2 Further Examples

We now apply the technique above to the calculation of coactions for a number of other hypergeometric functions. In each case, we will know from contiguous relations how many independent forms make up our basis for the first entries of the coaction and thus also the number of contours for the second-entry basis. We will find convenient choices for these bases and write down the coaction with each entry expressed using the same class of hypergeometric function. This will be possible in each case considered for generic arguments on the same grounds as the ${}_2F_1$ case described in section 3.1.3.

In each case we will emphasize the coactions of the hypergeometric functions themselves but, as for the ${}_2F_1$, the formula (2.47) will hold under expansion of the relevant integrals in ϵ. Thus we determine the coactions of each element of the period matrix and functions which are linear combinations of these elements. As before, it is these period matrix elements which are checked by expansion to verify the coaction.

We will examine the cases of ${}_{p+1}F_p$ and Appell functions, which we have introduced in section 2.6.2, as these will be required later for our examples of the diagrammatic coaction.

3.2.1 The ${}_{p+1}F_p$ Functions

Recall that the function defined by

$${}_{p+1}F_p(\alpha_1,\ldots,\alpha_{p+1};\beta_1,\ldots,\beta_p;x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_{p+1})_n}{(\beta_1)_n \ldots (\beta_p)_n} \frac{x^n}{n!} \tag{3.21}$$

has the integral representation

$$\begin{aligned}&{}_{p+1}F_p(\alpha_1,\ldots,\alpha_{p+1};\beta_1,\ldots,\beta_p;x) \\ =&\frac{\Gamma(\beta_p)}{\Gamma(\alpha_{p+1})\Gamma(\beta_p-\alpha_{p+1})}\int_0^1 du\, u^{\alpha_{p+1}-1}(1-u)^{\beta_p-\alpha_{p+1}-1}{}_pF_{p-1}(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_{p-1};xu)\end{aligned} \tag{3.22}$$

$$= \left[\prod_{i=1}^{p} \frac{\Gamma(\beta_i)}{\Gamma(\alpha_{i+1})\Gamma(\beta_i - \alpha_{i+1})}\right] \int_0^1 \dots \int_0^1 \left[\prod_{i=1}^{p} du_i\, u_i^{\alpha_{i+1}-1}(1-u_i)^{\beta_i-\alpha_{i+1}-1}\right] \\ \times \left(1 - x\prod_{i=1}^{p} u_i\right)^{-\alpha_1}.$$

Proceeding in the same way as for the case of the ${}_2F_1$, we will construct the coaction of the integral

$$\int_0^1 \dots \int_0^1 \left[\prod_{i=1}^{p} du_i\, u_i^{m_i+a_i\epsilon}(1-u_i)^{n_i+b_i\epsilon}\right] \left(1 - x\prod_{i=1}^{p} u_i\right)^{p+c\epsilon}. \tag{3.23}$$

By analogy to the ${}_2F_1$ case we choose to define the bases:

$$\begin{aligned} \omega_0 =& \epsilon^p \left[\prod_{i=1}^{p} b_i\right] du_1 \wedge \ldots \wedge du_p \left[\prod_{i=1}^{p} u_i^{a_i\epsilon}(1-u_i)^{-1+b_i\epsilon}\right] \left(1 - x\prod_{i=1}^{p} u_i\right)^{c\epsilon} \qquad (3.24) \\ \omega_{j\geq 1} =& xc\epsilon^p \left[\prod_{\substack{i=1\\ i\neq j}}^{p} \frac{b_i(a_i-a_j-b_j)}{a_i-a_j+b_i-b_j}\right] du_1 \wedge \ldots \wedge du_p \left[\prod_{i=1}^{p} u_i^{a_i\epsilon}\right] \left[\prod_{\substack{i=1\\ i\neq j}}^{p} (1-u_i)^{-1+b_i\epsilon}\right] \\ & \times (1-u_j)^{b_j\epsilon} \left(1 - x\prod_{i=1}^{p} u_i\right)^{-1+c\epsilon} \end{aligned}$$

and

$$\begin{aligned} \gamma_0 =& \{(u_1, \ldots, u_p) | 0 \leq u_i \leq 1 \text{ for all } i\} \qquad (3.25) \\ \gamma_{j\geq 1} =& \left\{(u_1, \ldots, u_p) \middle| 0 \leq u_j \leq \frac{1}{x\prod_{i\neq j} u_i}, 0 \leq u_i \leq 1 \text{ for } i \neq j\right\}. \end{aligned}$$

There must be $p+1$ elements of each basis as we know this is the number of master integrals of the system from the arguments of section 2.6.2.

With these choices of p factors in the integrand having exponents of the form $-1+n\epsilon$ we will obtain uniform-weight integrals. Placing the integration endpoints where these factors vanish we expect to obtain the dual contours, provided that we have normalised our objects as in (3.24). We will prove that these are the correct normalisation factors below.

Before proceeding, we prove an integration identity that will be required to evaluate the second-entry integrals:

$$
\begin{aligned}
&\int_0^1 \cdots \int_0^1 \left[\prod_{i=1}^{p} du_i u_i^{A_i}(1-u_i)^{B_i}\right]\left(1-\frac{u_j}{x\prod_{i\neq j}u_i}\right)^C \qquad (3.26)\\
=&\int_0^1 \cdots \int_0^1 \left[\prod_{i\neq j} du_i u_i^{A_i}(1-u_i)^{B_i}\right]\frac{\Gamma(1+A_j)\Gamma(1+B_j)}{\Gamma(2+A_j+B_j)}\\
&\times {}_2F_1\left(1+A_j,-C;2+A_j+B_j;\frac{1}{x\prod_{i\neq j}u_i}\right)\\
=&(-x)^{1+A_j}\frac{\Gamma(-1-A_j-C)\Gamma(1+A_j)}{\Gamma(-C)}\int_0^1 \cdots \int_0^1 \left[\prod_{i\neq j} du_i u_i^{1+A_i+A_j}(1-u_i)^{B_i}\right]\\
&\times {}_2F_1\left(1+A_j,-B_j;2+A_j+C;x\prod_{i\neq j}u_i\right)\\
&+(-x)^{-C}\frac{\Gamma(1+A_j+C)\Gamma(1+B_j)}{\Gamma(2+A_j+B_j+C)}\int_0^1 \cdots \int_0^1 \left[\prod_{i\neq j} du_i u_i^{A_i-C}(1-u_i)^{B_i}\right]\\
&\times {}_2F_1\left(-1-A_j-B_j-C,-C;-A_j-C;x\prod_{i\neq j}u_i\right)\\
=&(-x)^{1+A_j}\frac{\Gamma(-1-A_j-C)\Gamma(1+A_j)}{\Gamma(-C)}\prod_{i\neq j}\frac{\Gamma(2+A_i+A_j)\Gamma(1+B_i)}{\Gamma(3+A_i+A_j+B_i)}\\
&\times {}_{p+1}F_p\left(\{2+A_i+A_j\}_{i\neq j},1+A_j,-B_j;\{3+A_i+B_i+A_j\}_{i\neq j},2+A_j+C;x\right)\\
&+(-x)^{-C}\frac{\Gamma(1+A_j+C)\Gamma(1+B_j)}{\Gamma(2+A_j+B_j+C)}\prod_{i\neq j}\frac{\Gamma(1+A_i-C)\Gamma(1+B_i)}{\Gamma(2+A_i+B_i-C)}\\
&\times {}_{p+1}F_p\left(\{1+A_i-C\}_{i\neq j},-1-A_j-B_j-C,-C;\{2+A_i+B_i-C\}_{i\neq j},-A_j-C;x\right).
\end{aligned}
$$

With the generic object $\omega = du_1\wedge\ldots\wedge du_p\left[\prod_{i=1}^p u_i^{\alpha_{i+1}-1}(1-u_i)^{\beta_i-\alpha_{i+1}-1}\right]\left(1-x\prod_{i=1}^p u_i\right)^{-\alpha_1}$

we have the trivial integral

$$\int_{\gamma_1} \omega = \left[\prod_{i=1}^{p} \frac{\Gamma(\alpha_{i+1})\Gamma(\beta_i - \alpha_{i+1})}{\Gamma(\beta_i)}\right] {}_{p+1}F_p(\alpha_1, \ldots, \alpha_{p+1}; \beta_1, \ldots, \beta_p; x). \tag{3.27}$$

Meanwhile, for $j \geq 1$, (3.26) implies that

$$\begin{aligned}
&\int_{\gamma_j} \omega \qquad (3.28)\\
&= \int_0^1 \cdots \int_0^1 \left[\prod_{\substack{i=1\\ i\neq j}}^{p} du_i\, u_i^{\alpha_{i+1}-1}(1-u_i)^{\beta_i-\alpha_{i+1}-1}\right]\\
&\quad \times \int_0^{\frac{1}{x\prod_{i\neq j} u_i}} du_j u_j^{\alpha_{j+1}-1}(1-u_j)^{\beta_j-\alpha_{j+1}-1}\left(1 - x\prod_{i=1}^{p} u_i\right)^{-\alpha_1}\\
&= x^{-\alpha_{j+1}} \int_0^1 \cdots \int_0^1 \left[\prod_{i=1}^{p} du_i\right]\left[\prod_{\substack{i=1\\ i\neq j}}^{p} u_i^{\alpha_{i+1}-\alpha_{j+1}-1}(1-u_i)^{\beta_i-\alpha_{i+1}-1}\right] u_j^{\alpha_{j+1}-1}\,(1-u_j)^{-\alpha_1}\\
&\quad \times \left(1 - \frac{u_j}{x\prod_{i\neq j} u_i}\right)^{\beta_j-\alpha_{j+1}-1}\\
&= x^{-\alpha_j+1}\left\{(-x)^{\alpha_{j+1}} \frac{\Gamma(1-\beta_j)\Gamma(\alpha_{j+1})}{\Gamma(\alpha_{j+1}-\beta_j+1)}\left\{\prod_{i\neq j}\frac{\Gamma(\alpha_{i+1})\Gamma(\beta_i-\alpha_{i+1})}{\Gamma(\beta_i)}\right\} {}_{p+1}F_p\left(\{\alpha_i\};\{\beta_i\};x\right)\right.\\
&\quad + (-x)^{-\beta_j+\alpha_{j+1}+1}\frac{\Gamma(\beta_j-1)\Gamma(1-\alpha_1)}{\Gamma(-\alpha_1+\beta_j)}\prod_{i\neq j}\frac{\Gamma(1+\alpha_{i+1}-\beta_j)\Gamma(\beta_i-\alpha_{i+1})}{\Gamma(1+\beta_i-\beta_j)}\\
&\quad \left.\times {}_{p+1}F_p\left(\{1+\alpha_i-\beta_j\};\{1+\beta_i-\beta_j\}_{i\neq j}, 2-\beta_j; x\right)\right\}.
\end{aligned}$$

The elements of the period matrix follow from specialising the values of the m_i in the above formula. By using the results

$$\begin{aligned}
{}_{p+1}F_p(\{1+A_i\epsilon\}, B\epsilon; \{1+C_i\epsilon\}; x) =& 1 + \mathcal{O}(\epsilon) \\
{}_{p+1}F_p(\{A_i\epsilon\}; \{B_i\epsilon\}; x) =& 1 + \mathcal{O}(\epsilon)
\end{aligned} \tag{3.29}$$

$$ {}_{p+1}F_p(\{1+A_i\epsilon\};\{1+B_i\epsilon\},2+C\epsilon;x)=\sum_{n=0}^{\infty}\frac{x^n}{n+1}+\mathcal{O}(\epsilon)=-\frac{1}{x}\log(1-x)+\mathcal{O}(\epsilon), $$

we can establish that $P_{i,j}=\delta_{i,j}+\mathcal{O}(\epsilon)$.

The coaction of a generic ${}_{p+1}F_p$ function may then be written down using (2.47), though we omit the full expression for brevity. The first entries of this coaction are simple to determine from the integral representation (3.22), while the second entries follow from the results (3.27) and (3.28).

3.2.2 The Appell F_1

Another direct generalisation of the ${}_2F_1$ is the integral

$$ \begin{aligned} &F_1(\alpha;\beta,\beta';\gamma;x,y) \\ =&\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 du\,u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}. \end{aligned} \tag{3.30} $$

The coaction can be constructed for parameters which are of the form $m+n\epsilon$ with $m,n\in\mathbb{Z}$ on the regularised integral:

$$ \int_0^1 du\,u^{m+a\epsilon}(1-u)^{n+b\epsilon}(1-ux)^{p+c\epsilon}(1-uy)^{q+d\epsilon}. \tag{3.31} $$

We know from the contiguous relations that there are three master integrals for this family, and so select forms and contours which are an immediate generalisation of the ${}_2F_1$ case:

$$ \begin{aligned} \omega_0 =&b\epsilon\,du\,u^{a\epsilon}(1-u)^{-1+b\epsilon}(1-ux)^{c\epsilon}(1-uy)^{d\epsilon} \\ \omega_1 =&c\epsilon x\,du\,u^{a\epsilon}(1-u)^{b\epsilon}(1-ux)^{-1+c\epsilon}(1-uy)^{d\epsilon} \\ \omega_2 =&d\epsilon y\,du\,u^{a\epsilon}(1-u)^{b\epsilon}(1-ux)^{c\epsilon}(1-uy)^{-1+d\epsilon} \\ \gamma_0 =&\{u|0\leq u\leq 1\} \\ \gamma_1 =&\left\{u\middle|0\leq u\leq\frac{1}{x}\right\} \end{aligned} \tag{3.32} $$

$$\gamma_2 = \left\{ u \middle| 0 \leq u \leq \frac{1}{y} \right\}.$$

These bases are easily verified to be dual to each other, and we find second-entry integrals for $\omega = u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}$ given by

$$\begin{aligned}\int_{\gamma_0} \omega =& \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F_1\left(\alpha;\beta,\beta';\gamma;x,y\right) \\ \int_{\gamma_1} \omega =& x^{-\alpha}\frac{\Gamma(\alpha)\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)} F_1\left(\alpha;1+\alpha-\gamma,\beta';1+\alpha-\beta;\frac{1}{x},\frac{y}{x}\right) \\ \int_{\gamma_2} \omega =& y^{-\alpha}\frac{\Gamma(\alpha)\Gamma(1-\beta')}{\Gamma(1+\alpha-\beta')} F_1\left(\alpha;1+\alpha-\gamma,\beta;1+\alpha-\beta';\frac{1}{y},\frac{x}{y}\right).\end{aligned} \tag{3.33}$$

The coaction of the F_1 function itself, following the notation of (3.12), can be found to be

$$\begin{aligned}&\Delta F_1\left(\alpha;\beta,\beta';\gamma;x,y\right) \\ =& F_1\left(1+\alpha_\epsilon\epsilon;\beta_\epsilon\epsilon,\beta'_\epsilon\epsilon;1+\gamma_\epsilon\epsilon;x,y\right) \otimes F_1\left(\alpha;\beta,\beta';\gamma;x,y\right) \\ &- \frac{\beta_\epsilon\epsilon x}{1+\gamma_\epsilon\epsilon} F_1\left(1+\alpha_\epsilon\epsilon;1+\beta_\epsilon\epsilon,\beta'_\epsilon\epsilon;2+\gamma_\epsilon\epsilon;x,y\right) \\ &\otimes x^{-\alpha}\frac{\Gamma(\gamma)\Gamma(1-\beta)}{\Gamma(\gamma-\alpha)\Gamma(1+\alpha-\beta)} F_1\left(\alpha;1+\alpha-\gamma,\beta';1+\alpha-\beta;\frac{1}{x},\frac{y}{x}\right) \\ &- \frac{\beta'_\epsilon\epsilon y}{1+\gamma_\epsilon\epsilon} F_1\left(1+\alpha_\epsilon\epsilon;\beta_\epsilon\epsilon,1+\beta'_\epsilon\epsilon;2+\gamma_\epsilon\epsilon;x,y\right) \\ &\otimes y^{-\alpha}\frac{\Gamma(\gamma)\Gamma(1-\beta')}{\Gamma(\gamma-\alpha)\Gamma(1+\alpha-\beta')} F_1\left(\alpha;1+\alpha-\gamma,\beta;1+\alpha-\beta';\frac{1}{y},\frac{x}{y}\right).\end{aligned} \tag{3.34}$$

3.2.3 The Appell F_1 as a Double Integral

The Appell F_1 also admits a double integral representation similar to the other Appell functions, which provides an alternative method of constructing the coaction. This integral representation is given by

$$F_1(\alpha;\beta,\beta';\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \tag{3.35}$$

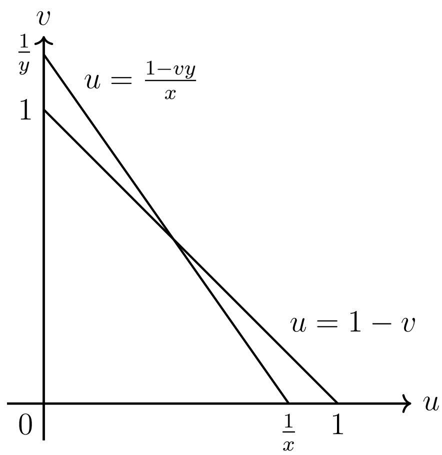


Figure 3.1: Vanishing hyperplanes for the integrand of the F_1 two-variable integral representation

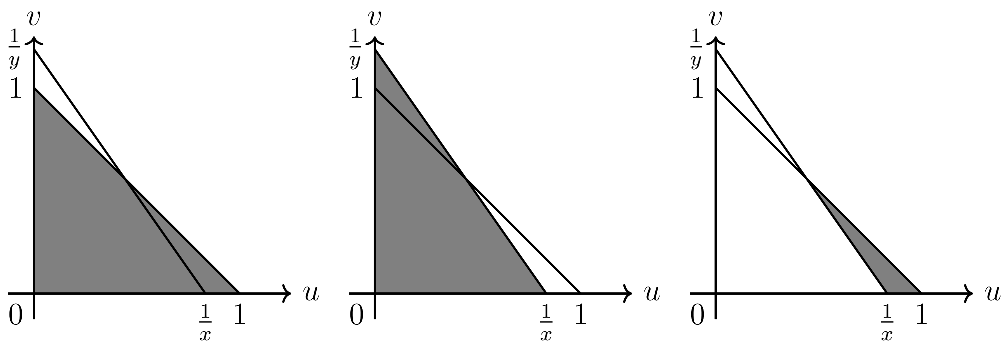
$$\times \int_0^1 dv \int_0^{1-v} du\, u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-xu-yv)^{-\alpha}.$$

As usual, we begin by constructing the coaction on a regularised integral

$$\int_0^1 dv \int_0^{1-v} du\, u^{m+a\epsilon} v^{n+b\epsilon} (1-u-v)^{p+c\epsilon} (1-xu-yv)^{q+d\epsilon} \tag{3.36}$$

where we can achieve duality of the forms and contours by the familiar method of placing the integration domain endpoints at the vanishing points of factors in the integrand which have an integer part of -1 in their exponents. Such a basis is

$$\begin{aligned}
\omega_1 =& ac\epsilon^2\, du \wedge dv\, u^{-1+a\epsilon} v^{b\epsilon} (1-u-v)^{-1+c\epsilon} (1-xu-yv)^{d\epsilon} \qquad (3.37)\\
\omega_2 =& ad\epsilon^2 y\, du \wedge dv\, u^{-1+a\epsilon} v^{b\epsilon} (1-u-v)^{c\epsilon} (1-xu-yv)^{-1+d\epsilon}\\
\omega_3 =& cd\epsilon^2 (y-x) du \wedge dv\, u^{a\epsilon} v^{b\epsilon} (1-u-v)^{-1+c\epsilon} (1-xu-yv)^{-1+d\epsilon}\\
\gamma_1 =& \{(u,v) | 0 \le u \le 1-v, 0 \le v \le 1\}\\
\gamma_2 =& \left\{ (u,v) \middle| 0 \le u \le \frac{1-yv}{x}, 0 \le v \le \frac{1}{y} \right\}
\end{aligned}$$



(a) γ_1 (b) γ_2 (c) γ_3

Figure 3.2: The contours γ_i for the F_1 system

$$\gamma_3 = \left\{ (u,v) \middle| \frac{1-yv}{x} \leq u \leq 1-v, 0 \leq v \leq \frac{x-1}{x-y} \right\}.$$

With reference to figures 3.1 and 3.2, we see that γ_1 and γ_2 are the triangular regions lying under the lines $1-u-v=0$ and $1-ux-vy=0$, while γ_3 is the region bounded by these lines and $v=0$.

The choice $\omega = du \wedge dv\, u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-xu-yv)^{-\alpha}$ gives the integrals

$$\begin{aligned}
\int_{\gamma_1} \omega =& \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)} F_1(\alpha;\beta,\beta';\gamma;x,y) \qquad (3.38)\\
\int_{\gamma_2} \omega =& x^{-\beta} y^{-\beta'} \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(1-\alpha)}{\Gamma(1-\alpha+\beta+\beta')} F_1\left(1+\beta+\beta'-\gamma;\beta,\beta';1-\alpha+\beta+\beta';\frac{1}{x},\frac{1}{y}\right)\\
\int_{\gamma_3} \omega =& x^{1+\beta'-\gamma}(x-1)^{\gamma-\alpha-\beta}(x-y)^{-\beta'} e^{i\pi\alpha} \frac{\Gamma(1-\alpha)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(1-\alpha-\beta+\gamma)}\\
& \times F_1\left(1-\beta;1-\alpha,\beta';1-\alpha-\beta+\gamma;1-x,\frac{y(x-1)}{x-y}\right),
\end{aligned}$$

the first of which follows immediately from (3.35), while the remaining two can be expanded into the series from (2.66).

This change to the new bases can be interpreted as being of the form described in (2.49), (2.50) and (2.51). The matrix $\mathcal{M}$ describing the transformation is most easily derived by looking at how the first entries of the coaction are rotated, since the first entries from both (3.32) and (3.37) can be directly written in F_1 form using (3.30) and (3.35). The relations between the two bases may then be found by using the contiguous relation method. This results in a matrix

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & \frac{\beta'_\epsilon - \alpha_\epsilon}{\beta'_\epsilon} \\ 0 & 0 & \frac{\alpha_\epsilon}{\beta'_\epsilon} \\ 0 & -\frac{\alpha_\epsilon}{\beta_\epsilon} & \frac{\alpha_\epsilon}{\beta'_\epsilon} \end{pmatrix} \tag{3.39}$$

which transforms the basis of forms in (3.32) to that of (3.37). Having established this transformation from the first entries, we may use it to derive non-trivial relations among the two sets of integration contours.

3.2.4 The Appell F_4

The construction of the Appell F_4 coaction raises certain issues that are not present in our previous examples if we work from the usual integral representation:

$$\begin{aligned} &F_4(\alpha,\beta;\gamma,\gamma';x(1-y),y(1-x)) \\ =&\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma'-\beta)} \\ &\times \int_0^1 dv \int_0^1 du u^{\alpha-1} v^{\beta-1} (1-u)^{\gamma-\alpha-1}(1-v)^{\gamma'-\beta-1}(1-ux)^{\alpha-\gamma-\gamma'+1}(1-vy)^{\beta-\gamma-\gamma'+1} \\ &\times (1-ux-vy)^{\gamma+\gamma'-\alpha-\beta-1}. \end{aligned} \tag{3.40}$$

We may try a construction of the four basis elements by fixing the endpoints of a collection of contours by analogy to the previous functions considered, but this fails to produce a basis of contours. We will return to this method below, but first demonstrate the use of an alternative integral representation to obtain the coaction. It is known that the F_4 function can also be expressed in integral form using the result [85]:

$$\frac{\Gamma(1-\gamma)\Gamma(1-\gamma')\Gamma(\gamma'+\gamma'-\alpha-1)}{\Gamma(1-\alpha)}F_4(\alpha,\beta;\gamma,\gamma';x(1-y),y(1-x)) \tag{3.41}$$
$$=\int_{\Gamma_1} t_1^{\beta-\gamma}t_2^{\beta-\gamma'}(1-t_1-t_2)^{\gamma+\gamma'-\alpha-2}(t_1t_2-x(1-y)t_2-y(1-x)t_1)^{-\beta}dt_1dt_2,$$

where the contour Γ_1, along with contours Γ_2, Γ_3 and Γ_4, is defined in [85] such that the integrals of $\omega = dt_1 \wedge dt_2\, t_1^{\beta-\gamma}t_2^{\beta-\gamma'}(1-t_1-t_2)^{\gamma+\gamma'-\alpha-2}[t_1t_2-x(1-y)t_2-y(1-x)t_1]^{-\beta}$ are given by

$$\begin{aligned}
\int_{\Gamma_1}\omega =&\frac{\Gamma(1-\gamma)\Gamma(1-\gamma')\Gamma(\gamma+\gamma'-\alpha-1)}{\Gamma(1-\alpha)} \\
&\times F_4(\alpha,\beta,\gamma,\gamma';x(1-y),y(1-x)) \\
\int_{\Gamma_2}\omega =&\frac{\Gamma(\alpha+1-\gamma)\Gamma(\beta+1-\gamma)\Gamma(1-\beta)\Gamma(\gamma+\gamma'-\alpha-1)}{\Gamma(2-\gamma)\Gamma(\gamma')}e^{i\pi(\alpha+\beta-\gamma-\gamma')}[x(1-y)]^{1-\gamma} \\
&\times F_4(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,\gamma';x(1-y),y(1-x)) \\
\int_{\Gamma_3}\omega =&\frac{\Gamma(\alpha+1-\gamma')\Gamma(\beta+1-\gamma')\Gamma(1-\beta)\Gamma(\gamma+\gamma'-\alpha-1)}{\Gamma(\gamma)\Gamma(2-\gamma')}e^{i\pi(\alpha+\beta-\gamma-\gamma')}[y(1-x)]^{1-\gamma'} \\
&\times F_4(\alpha-\gamma'+1,\beta-\gamma'+1,\gamma,2-\gamma';x(1-y),y(1-x)) \\
\int_{\Gamma_4}\omega =&\frac{\Gamma(\gamma-1)\Gamma(\gamma'-1)\Gamma(1-\beta)}{\Gamma(\gamma+\gamma'-\beta-1)}[x(1-y)]^{1-\gamma}[y(1-x)]^{1-\gamma'} \\
&\times F_4(\alpha-\gamma-\gamma'+2,\beta-\gamma-\gamma'+2,2-\gamma,2-\gamma';x(1-y),y(1-x)).
\end{aligned} \tag{3.42}$$

Continuing to follow the conventions of [85], there is a basis of forms

$$\begin{aligned}
\omega_1 =&dt_1\wedge dt_2\, t_1^{-1+a\epsilon}t_2^{-1+b\epsilon}(1-t_1-t_2)^{-1+c\epsilon}[t_1t_2-x(1-y)t_2-y(1-x)t_1]^{d\epsilon} \\
\omega_2 =&dt_1\wedge dt_2\, t_1^{a\epsilon}t_2^{-1+b\epsilon}(1-t_1-t_2)^{-1+c\epsilon}[t_1t_2-x(1-y)t_2-y(1-x)t_1]^{d\epsilon} \\
\omega_3 =&dt_1\wedge dt_2\, t_1^{-1+a\epsilon}t_2^{b\epsilon}(1-t_1-t_2)^{-1+c\epsilon}[t_1t_2-x(1-y)t_2-y(1-x)t_1]^{d\epsilon} \\
\omega_4 =&(1-x-y)dt_1\wedge dt_2\, t_1^{a\epsilon}t_2^{b\epsilon}(1-t_1-t_2)^{-1+c\epsilon}[t_1t_2-x(1-y)t_2-y(1-x)t_1]^{-1+d\epsilon}.
\end{aligned} \tag{3.43}$$

While these are not dual to the contours of (3.42), it is found that the period

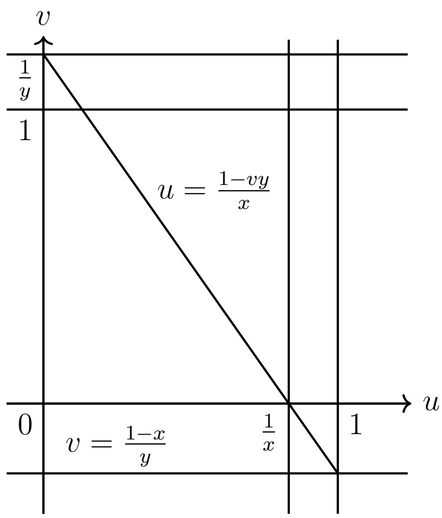


Figure 3.3: Vanishing hyperplanes for the integrand of the extended F_4 integral representation

matrix obeys

$$P = \frac{1}{\epsilon^2}\begin{pmatrix} \frac{a+b+c+2d}{c(a+d)(b+d)} & \frac{1}{c(b+d)} & \frac{1}{c(a+d)} & \frac{a+b+c+2d}{c(a+d)(b+d)} \\ -\frac{1}{ac} & 0 & -\frac{b+d}{ac(c+b+d)} & \frac{1}{dc} \\ \frac{1}{bc} & -\frac{a+d}{bc(d+a+c)} & 0 & \frac{1}{dc} \\ \frac{1}{(a+d)(b+d)} & 0 & 0 & -\frac{d+a+b}{d(a+d)(b+d)} \end{pmatrix} + \mathcal{O}(\epsilon^{-1}) \qquad (3.44)$$

and so we can apply a suitable rotation to (3.43) or (3.42) to find dual bases such that the coaction can be written down in the form (2.47). Given its presence in many relevant examples, we give the full form of the F_4 coaction in appendix B, along with a form that uses another first-entry basis which is required for a certain degenerate limit that one encounters in some Feynman integrals.

We will also demonstrate the construction of the coaction using the approach of the previous sections for a specific example that will be needed in chapter 6. The integral representation (3.40) can be generalised by adding more terms created by performing partial fractions on two of the factors in the integrand. Specifically, note that

$$\frac{1}{(1-u)(1-xu-yv)} = \frac{1}{1-x-yv}\left[\frac{1}{1-u} - \frac{x}{1-xu-yv}\right].$$

Thus we are prompted to consider the integral

$$\int_0^1 du \int_0^1 dv\, u^{m+a\epsilon} v^{n+b\epsilon} (1-u)^{p+c\epsilon} (1-v)^{q+d\epsilon} (1-ux-vy)^{r+g\epsilon} \times (1-ux)^{s+h\epsilon} (1-vy)^{t+j\epsilon} (1-x-vy)^{w+k\epsilon}. \tag{3.45}$$

We can write down the basis of forms

$$\begin{aligned}
\omega_1' =& u^{a\epsilon} v^{b\epsilon} (1-u)^{-1+c\epsilon} (1-v)^{-1+d\epsilon} (1-ux-vy)^{g\epsilon} (1-ux)^{h\epsilon} (1-vy)^{j\epsilon} \\
& \times (1-x-vy)^{k\epsilon} \\
\omega_2' =& u^{a\epsilon} v^{b\epsilon} (1-u)^{-1+c\epsilon} (1-v)^{d\epsilon} (1-ux-vy)^{g\epsilon} (1-ux)^{h\epsilon} (1-vy)^{-1+j\epsilon} \\
& \times (1-x-vy)^{k\epsilon} \\
\omega_3' =& u^{a\epsilon} v^{b\epsilon} (1-u)^{-1+c\epsilon} (1-v)^{d\epsilon} (1-ux-vy)^{g\epsilon} (1-ux)^{h\epsilon} (1-vy)^{j\epsilon} \\
& \times (1-x-vy)^{-1+k\epsilon} \\
\omega_4' =& u^{a\epsilon} v^{b\epsilon} (1-u)^{c\epsilon} (1-v)^{-1+d\epsilon} (1-ux-vy)^{-1+g\epsilon} (1-ux)^{h\epsilon} (1-vy)^{j\epsilon} \\
& \times (1-x-vy)^{k\epsilon} \\
\omega_5' =& u^{a\epsilon} v^{b\epsilon} (1-u)^{c\epsilon} (1-v)^{d\epsilon} (1-ux-vy)^{-1+g\epsilon} (1-ux)^{h\epsilon} (1-vy)^{-1+j\epsilon} \\
& \times (1-x-vy)^{k\epsilon} \\
\omega_6' =& u^{a\epsilon} v^{b\epsilon} (1-u)^{c\epsilon} (1-v)^{d\epsilon} (1-ux-vy)^{-1+g\epsilon} (1-ux)^{h\epsilon} (1-vy)^{j\epsilon} \\
& \times (1-x-vy)^{-1+k\epsilon} \\
\omega_7' =& u^{a\epsilon} v^{b\epsilon} (1-u)^{c\epsilon} (1-v)^{-1+d\epsilon} (1-ux-vy)^{g\epsilon} (1-ux)^{-1+h\epsilon} (1-vy)^{j\epsilon} \\
& \times (1-x-vy)^{k\epsilon} \\
\omega_8' =& u^{a\epsilon} v^{b\epsilon} (1-u)^{c\epsilon} (1-v)^{d\epsilon} (1-ux-vy)^{g\epsilon} (1-ux)^{-1+h\epsilon} (1-vy)^{-1+j\epsilon} \\
& \times (1-x-vy)^{k\epsilon} \\
\omega_9' =& u^{a\epsilon} v^{b\epsilon} (1-u)^{c\epsilon} (1-v)^{d\epsilon} (1-ux-vy)^{g\epsilon} (1-ux)^{-1+h\epsilon} (1-vy)^{j\epsilon}
\end{aligned} \tag{3.46}$$

$$\times (1 - x - vy)^{-1+k\epsilon},$$

along with a basis of contours

$$\begin{aligned}
\gamma_1' &= \{(u,v)|0 \le u \le 1, 0 \le v \le 1\} \\
\gamma_2' &= \left\{(u,v)|0 \le u \le 1, 0 \le v \le \frac{1}{y}\right\} \\
\gamma_3' &= \left\{(u,v)|0 \le u \le 1, \frac{1-x}{y} \le v \le 0\right\} \\
\gamma_4' &= \left\{(u,v)|0 \le u \le \frac{1-yv}{x}, 0 \le v \le 1\right\} \\
\gamma_5' &= \left\{(u,v)|0 \le u \le \frac{1-yv}{x}, 0 \le v \le \frac{1}{y}\right\} \\
\gamma_6' &= \left\{(u,v)|0 \le u \le \frac{1-yv}{x}, \frac{1-x}{y} \le v \le 0\right\} \\
\gamma_7' &= \left\{(u,v)|0 \le u \le \frac{1}{x}, 0 \le v \le 1\right\} \\
\gamma_8' &= \left\{(u,v)|0 \le u \le \frac{1}{x}, 0 \le v \le \frac{1}{y}\right\} \\
\gamma_9' &= \left\{(u,v)|0 \le u \le \frac{1}{x}, \frac{1-x}{y} \le v \le 0\right\}.
\end{aligned} \tag{3.47}$$

These bases are not dual to each other, but we can compute the period matrix and, as above, use it to find a transformation which produces a dual basis. One such transformation sets

$$\begin{aligned}
\omega_1 =& cd\epsilon^2\omega_1' \\
\omega_2 =& cjy\epsilon^2\omega_2' \\
\omega_3 =& c(g+k)y\epsilon^2\left(\omega_3' - \frac{gx}{g+k}\omega_6'\right) \\
\omega_4 =& dgx\epsilon^2\omega_4' \\
\omega_5 =& g(a+g+j)xy\epsilon^2\left(\omega_5' + \frac{c}{(a+g+j)x}\omega_2' + \frac{h}{a+g+j}\omega_8'\right)
\end{aligned} \tag{3.48}$$

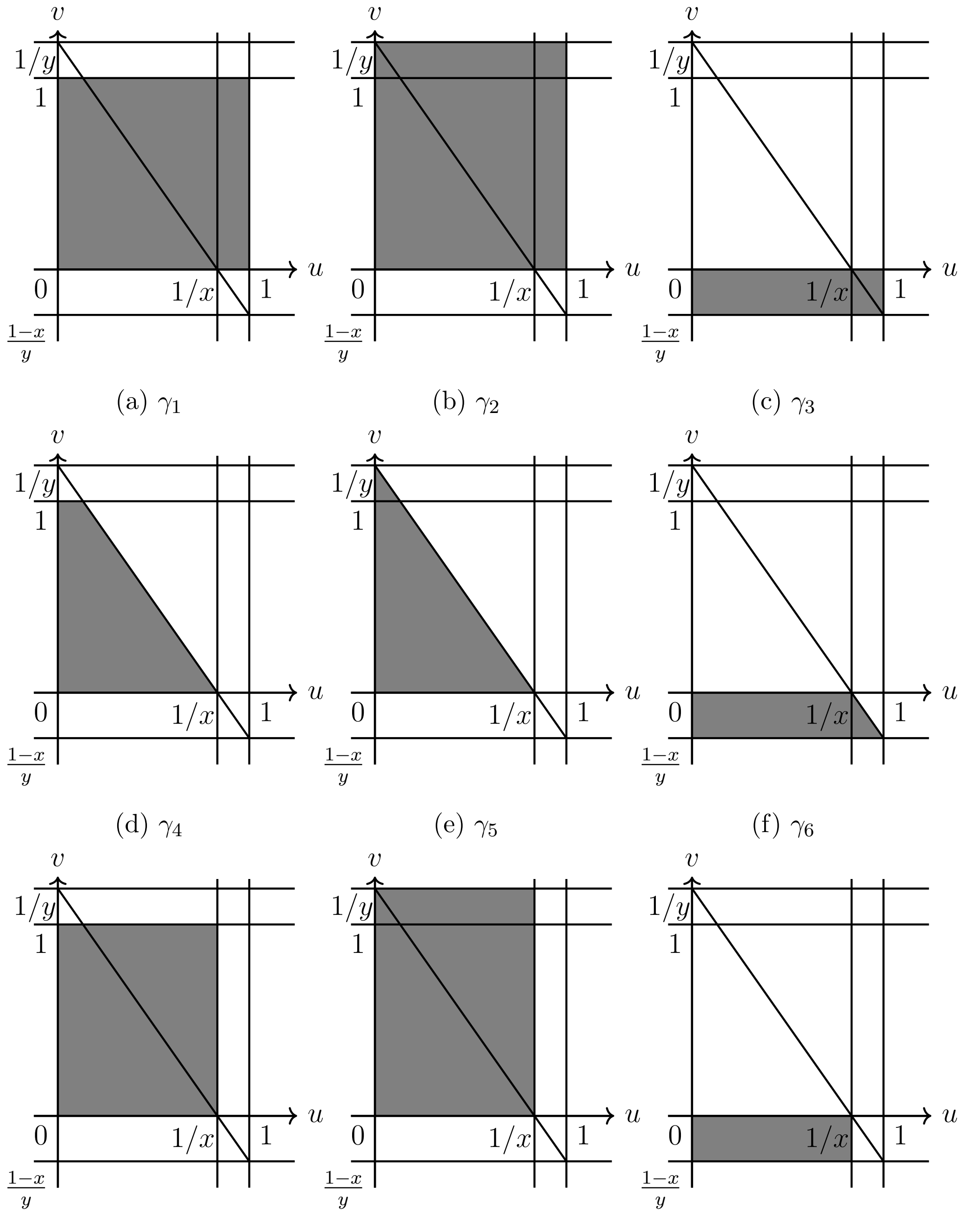
$$
\begin{aligned}
\omega_6 =& g(c+k)xy\epsilon^2\left(\omega_6' - \frac{c}{(c+k)x}\omega_3'\right)\\
\omega_7 =& dhx\epsilon^2\omega_7'\\
\omega_8 =& jhxy\epsilon^2\omega_8'\\
\omega_9 =& khxy\epsilon^2\omega_9'
\end{aligned}
$$

while leaving the contours unchanged as $\gamma_i = \gamma_i'$. The lines bounding these contours are shown in figure 3.3 while the contours themselves are depicted in figure 3.4.

The integral representation of the Appell F_4 differs from this extended family in two respects: firstly the factor $1 - x - yv$ is absent, and secondly the exponents of the remaining seven factors are related as they depend on only four independent parameters α, β, γ and γ'. After these constraints are implemented we expect that the nine-dimensional system we have above will reduce to the four-dimensional F_4 system. While the exact mechanism by which this occurs remains elusive for generic arguments, we can demonstrate it in the physically relevant case of the function $F_4(1+\epsilon, 1; 1-\epsilon, 1-\epsilon; x(1-y), y(1-x))$, which appears in the expression for the sunset integral with two masses.

We can determine $\Delta F_4(1+\epsilon, 1; 1-\epsilon, 1-\epsilon; x(1-y), y(1-x))$ by using (2.14) along with the coaction of (3.45) specialised to the relevant values of the parameters. In this limit, the bases $\{\omega_i\}$ and $\{\gamma_j\}$ acquire linear relations that did not hold for generic values of the parameters. The number of independent elements does not reduce to four though, and so our final coaction is expressed using a larger number of basis elements from the larger space. Despite this, it is found that the coaction collects into a four-term expression which takes the same form as (B.4). The F_4 functions appearing in the second entries of this coaction can be described using the contours (3.47) by the relations

$$
\begin{aligned}
&2\epsilon^2\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}\int_{\gamma_1}\omega \\
=&F_4(1+\epsilon, 1; 1-\epsilon, 1-\epsilon; x(1-y), y(1-x))
\end{aligned}
\tag{3.49}
$$



(a) γ_1 (b) γ_2 (c) γ_3

(d) γ_4 (e) γ_5 (f) γ_6

(g) γ_7 (h) γ_8 (i) γ_9

Figure 3.4: The contours γ_i for the extended F_4 system

$$\epsilon^2 \left(2\int_{\gamma_1} -4\int_{\gamma_2} +2\int_{\gamma_3} - \int_{\gamma_4} -3\int_{\gamma_5} -2\int_{\gamma_6} +6\int_{\gamma_8} \right) \omega$$
$$=[x(1-y)]^\epsilon F_4(1+\epsilon, 1+2\epsilon; 1+\epsilon, 1-\epsilon; x(1-y), y(1-x))$$

$$\epsilon^2 \left(\int_{\gamma_1} +2\int_{\gamma_2} -3\int_{\gamma_3} -3\int_{\gamma_5} +3\int_{\gamma_6} \right) \omega$$
$$=[y(1-x)]^\epsilon F_4(1+\epsilon, 1+2\epsilon; 1-\epsilon, 1+\epsilon; x(1-y), y(1-x))$$

$$\frac{\epsilon^2}{3} \left(4\int_{\gamma_1} -2\int_{\gamma_2} -2\int_{\gamma_3} -2\int_{\gamma_4} -6\int_{\gamma_5} +2\int_{\gamma_6} +6\int_{\gamma_8} \right) \omega$$
$$=\frac{\Gamma(1-\epsilon)^3}{\Gamma(1-3\epsilon)}[x(1-y)]^\epsilon[y(1-x)]^\epsilon F_4(1+2\epsilon, 1+3\epsilon; 1+\epsilon, 1+\epsilon; x(1-y), y(1-x)),$$

where $\omega = du \wedge dv\, u^\epsilon(1-u)^{-1-2\epsilon}(1-v)^{-1-\epsilon}(1-ux)^{3\epsilon}(1-vy)^{2\epsilon}(1-ux-vy)^{-1-3\epsilon}$. The first relation above follows from the usual integral form of the F_4, while the others are derived by expanding all the relevant objects in ϵ and discovering the relations at the level of polylogarithms.

We speculate that there might be a generalisation of these relations to arbitrary arguments α, β, γ and γ' of the F_4 which would allow the coaction to be deduced for a general F_4 with near to integer parameters. Such relations are not derivable by expansion to polylogarithms as we require them to hold for arbitrary parameters of the F_4, and it remains unknown how they might be found. It is still possible however to write down the coaction of the F_4 function using the larger space of forms and contours.

3.2.5 The Appell F_2 and F_3

For completeness, we also include the remaining two Appell functions. We start with the F_3 function, which has integral representation

$$\int_0^1 dv \int_0^{1-v} du\, u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-xu)^{-\alpha}(1-yv)^{-\alpha'} \tag{3.50}$$

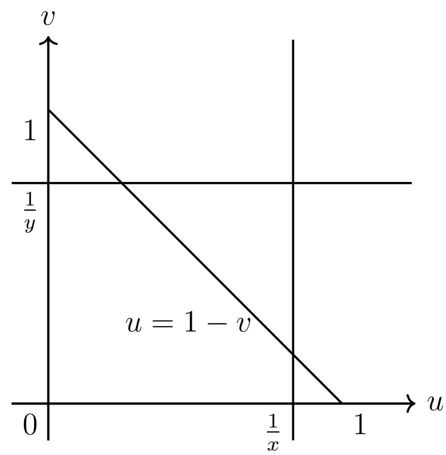


Figure 3.5: Vanishing hyperplanes for the integrand of the F_3 integral representation

$$=\frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)}F_3(\alpha,\alpha';\beta,\beta';\gamma;x,y).$$

The coaction can be straightforwardly found in the form (2.47) for the family of integrals

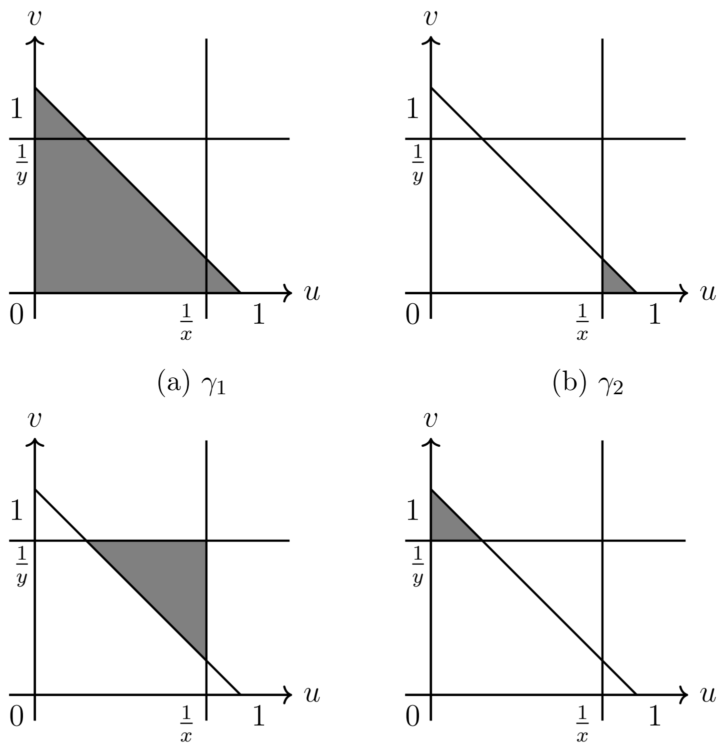
$$\int_0^1 dv \int_0^{1-v} du\, u^{m+a\epsilon} v^{n+b\epsilon}(1-u-v)^{p+c\epsilon}(1-xu)^{q+d\epsilon}(1-yv)^{r+g\epsilon}, \tag{3.51}$$

where we can write down the dual bases

$$\begin{aligned}
\omega_1 =& ab\epsilon^2\, du\wedge dv\, u^{-1+a\epsilon}v^{-1+b\epsilon}(1-u-v)^{c\epsilon}(1-xu)^{d\epsilon}(1-yv)^{g\epsilon} \\
\omega_2 =& -bc\epsilon^2 x\, du\wedge dv\, u^{a\epsilon}v^{-1+b\epsilon}(1-u-v)^{c\epsilon}(1-xu)^{-1+d\epsilon}(1-yv)^{g\epsilon} \\
\omega_3 =& cd\epsilon^2 xy\, du\wedge dv\, u^{a\epsilon}v^{b\epsilon}(1-u-v)^{c\epsilon}(1-xu)^{-1+d\epsilon}(1-yv)^{-1+g\epsilon} \\
\omega_4 =& -ad\epsilon^2 y\, du\wedge dv\, u^{-1+a\epsilon}v^{b\epsilon}(1-u-v)^{c\epsilon}(1-xu)^{d\epsilon}(1-yv)^{-1+g\epsilon}
\end{aligned} \tag{3.52}$$

and

$$\gamma_1 = \{(u,v)|0\le u\le 1-v, 0\le v\le 1\} \tag{3.53}$$



(a) γ_1 (b) γ_2 (c) γ_3 (d) γ_4

Figure 3.6: The contours γ_i for the F_3 system

$$\gamma_2 = \left\{ (u,v) \middle| \frac{1}{x} \le u \le 1 - v, 0 \le v \le 1 - \frac{1}{x} \right\}$$
$$\gamma_3 = \left\{ (u,v) \middle| 1 - v \le u \le \frac{1}{x}, 1 - \frac{1}{x} \le v \le \frac{1}{y} \right\}$$
$$\gamma_4 = \left\{ (u,v) \middle| 0 \le u \le 1 - v, \frac{1}{y} \le v \le 1 \right\}.$$

The integrals of $\omega = du \wedge dv\, u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-xu)^{-\alpha} (1-yv)^{-\alpha'}$ over these contours can be found by expanding certain factors in the integrand and comparing the result to the series form of the F_3:

$$\int_{\gamma_1} \omega = \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)} F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y) \tag{3.54}$$

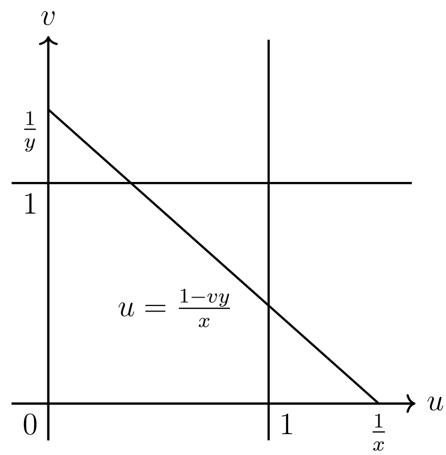


Figure 3.7: Vanishing hyperplanes for the integrand of the F_2 integral representation

$$
\begin{aligned}
\int_{\gamma_2} \omega =& (x-1)^{\gamma-\beta-\alpha} x^{1-\gamma} e^{i\pi\alpha} \frac{\Gamma(1-\alpha)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(1-\alpha-\beta+\gamma)} \\
&\times F_3\left(1-\beta, \alpha'; 1-\alpha, \beta'; 1-\alpha-\beta+\gamma; 1-x, \frac{x-1}{x} y\right) \\
\int_{\gamma_3} \omega =& -\frac{x^{\alpha'+\beta'-\gamma} y^{\alpha+\beta-\gamma'} e^{i\pi(\gamma-\beta-\beta')}}{(x+y-xy)^{\alpha+\alpha'+\beta+\beta'-\gamma-1}} \frac{\Gamma(1-\alpha)\Gamma(1-\alpha')\Gamma(\gamma-\beta-\beta')}{\Gamma(2+\gamma-\alpha-\alpha'-\beta-\beta')} \\
&\times F_3\left(1-\beta, 1-\beta'; 1-\alpha, 1-\alpha'; 2+\gamma-\alpha-\alpha'-\beta-\beta'; \frac{x+y-xy}{y}, \frac{x+y-xy}{x}\right) \\
\int_{\gamma_4} \omega =& (y-1)^{\gamma-\beta'-\alpha'} y^{1-\gamma} e^{i\pi\alpha'} \frac{\Gamma(\beta)\Gamma(1-\alpha')\Gamma(\gamma-\beta-\beta')}{\Gamma(1-\alpha'-\beta'+\gamma)} \\
&\times F_3\left(\alpha, 1-\beta'; \beta, 1-\alpha'; 1-\alpha'-\beta'+\gamma; \frac{y-1}{y} x, 1-y\right).
\end{aligned}
$$

For the case of the F_2 function, with integral representation

$$
\begin{aligned}
&\int_0^1 dv \int_0^1 du\, u^{\beta-1} v^{\beta'-1} (1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-xu-vy)^{-\alpha} \\
=& \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}{\Gamma(\gamma)\Gamma(\gamma')} F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y),
\end{aligned}
\tag{3.55}
$$

the evaluation of the second-entry integrals in the form of F_2 functions is non-trivial

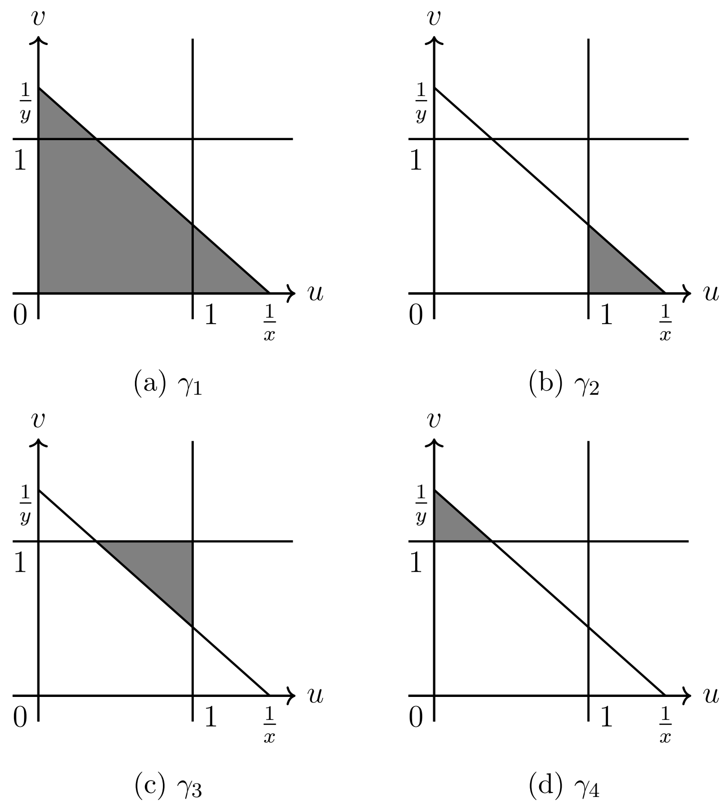


Figure 3.8: The contours γ_i for the F_2 system

and requires a choice of contours described in [86]. Even without these contours we can still write down a correct coaction on the integrals

$$\int_0^1 dv \int_0^1 du\, u^{m+a\epsilon} v^{n+b\epsilon} (1-u)^{p+c\epsilon} (1-v)^{q+d\epsilon} (1-xu-vy)^{r+g\epsilon} \tag{3.56}$$

in the form (2.47), using the dual bases

$$\begin{aligned} \omega_1 =& ab\epsilon^2\, du \wedge dv\, u^{-1+a\epsilon} v^{-1+b\epsilon} (1-u)^{c\epsilon} (1-v)^{d\epsilon} (1-xu-vy)^{g\epsilon} \\ \omega_2 =& -bc\epsilon^2\, du \wedge dv\, u^{a\epsilon} v^{-1+b\epsilon} (1-u)^{-1+c\epsilon} (1-v)^{d\epsilon} (1-xu-vy)^{g\epsilon} \\ \omega_3 =& cd\epsilon^2\, du \wedge dv\, u^{a\epsilon} v^{b\epsilon} (1-u)^{-1+c\epsilon} (1-v)^{-1+d\epsilon} (1-xu-vy)^{g\epsilon} \\ \omega_4 =& -ad\epsilon^2\, du \wedge dv\, u^{-1+a\epsilon} v^{b\epsilon} (1-u)^{c\epsilon} (1-v)^{-1+d\epsilon} (1-xu-vy)^{g\epsilon} \end{aligned} \tag{3.57}$$

and

$$\begin{aligned}\gamma_1 &= \left\{(u,v)\middle|0 \le u \le \frac{1-vy}{x}, 0 \le v \le \frac{1}{y}\right\} \\ \gamma_2 &= \left\{(u,v)\middle|1 \le u \le \frac{1-vy}{x}, 0 \le v \le \frac{1-x}{y}\right\} \\ \gamma_3 &= \left\{(u,v)\middle|\frac{1-vy}{x} \le u \le 1, \frac{1-x}{y} \le v \le 1\right\} \\ \gamma_4 &= \left\{(u,v)\middle|0 \le u \le \frac{1-vy}{x}, 1 \le v \le \frac{1}{y}\right\}.\end{aligned} \tag{3.58}$$

While these bases specify a coaction in the form (2.47), we will prefer to use the contours of [86], which we will denote here as Γ_1, Γ_2, Γ_3 and Γ_4. With these contours the integrals of the form

$$\omega = du \wedge dv\, u^{\beta-1} v^{\beta'-1} (1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-xu-yv)^{-\alpha} \tag{3.59}$$

are known to evaluate to

$$\begin{aligned}\int_{\Gamma_1} \omega =& \frac{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\beta')\Gamma(\gamma'-\beta')}{\Gamma(\gamma)\Gamma(\gamma')} F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y) \\ \int_{\Gamma_2} \omega =& -e^{i\pi(\beta-\gamma)} x^{1-\gamma} \frac{\Gamma(1-\alpha)\Gamma(\gamma-1)\Gamma(\beta')\Gamma(\gamma'-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma')} \\ & \times F_2(\alpha-\gamma+1;\beta-\gamma+1,\beta';2-\gamma,\gamma';x,y) \\ \int_{\Gamma_3} \omega =& -e^{i\pi(\beta'-\gamma')} y^{1-\gamma'} \frac{\Gamma(1-\alpha)\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\gamma'-1)}{\Gamma(\gamma)\Gamma(\gamma'-\alpha)} \\ & \times F_2(\alpha-\gamma'+1;\beta,\beta'-\gamma'+1;\gamma,2-\gamma';x,y) \\ \int_{\Gamma_4} \omega =& e^{i\pi(\beta'+\beta-\gamma'-\gamma)} x^{1-\gamma} y^{1-\gamma'} \frac{\Gamma(1-\alpha)\Gamma(\gamma-1)\Gamma(\gamma'-1)}{\Gamma(\gamma+\gamma'-\alpha-1)} \\ & \times F_2(\alpha-\gamma'-\gamma+2;\beta-\gamma+1,\beta'-\gamma'+1;2-\gamma,2-\gamma';x,y).\end{aligned} \tag{3.60}$$

It may be shown that using this basis but leaving the forms of (3.57) unchanged produces a period matrix P' related to the matrix P computed with (3.57) and

(3.58) by the relation $P' = K^{-1}P$, where the rotation K is

$$K = \begin{pmatrix} \frac{cd}{(a+c)(b+d)} & \frac{d}{b+d} & \frac{c}{a+c} & 1 \\ -\frac{ad}{(a+c)(b+d)} & \frac{d}{b+d} & -\frac{a}{a+c} & 1 \\ \frac{ab}{(a+c)(b+d)} & -\frac{b}{b+d} & -\frac{a}{a+c} & 1 \\ -\frac{bc}{(a+c)(b+d)} & -\frac{b}{b+d} & \frac{c}{a+c} & 1 \end{pmatrix}. \tag{3.61}$$

From this, we can infer that the new contours Γ_i are related to the γ_i by a rotation K^{-1}. We may use this fact, along with the coaction written down using contours (3.58), to obtain a form of the coaction where the second entries are expressed using the Appell F_2 function.

3.3 Positive Geometries and Intersection Theory

In this section we describe some more formal developments outlined in [1] which expand our understanding of the hypergeometric integrals and their coaction that we have discussed in the preceding sections. Specifically we will look at positive geometries, a class of geometric spaces to which we can always associate canonical forms, and intersection theory, which provides a method for computing the lowest order of the period matrix that we have used in establishing the duality of our bases of forms and contours.

3.3.1 Integrals and Positive Geometries

In what follows we will restrict our attention to integrals $\int_\gamma \omega$ such that ω takes the form

$$\omega = du_1 \wedge \ldots \wedge du_n \prod_{i \in I} P_i(u_1, \ldots, u_n, x_1, \ldots, x_n)^{\alpha_i}, \tag{3.62}$$

with the P_i each being a polynomial function of the integration parameters $\{u_i\}$ and scales $\{x_i\}$, and with each $\alpha_i = m_i + a_i\epsilon$ for some $m_i, a_i \in \mathbb{Z}$. The integration contour

γ must have a boundary which is a subset of $\cup_{i\in I}\{(u_1,\ldots,u_n)|P_i(u_1,\ldots,u_n,x_1,\ldots,x_n) = 0\}$. We further impose that the polynomials P_i are chosen so that when the integrals are expanded in ϵ, the coefficients are able to be expressed using only multiple polylogarithms. It can be verified that each of the examples we have considered in the preceding sections of this chapter obeys all of these constraints.

The form ω may be further split into parts Φ and φ given by

$$\begin{aligned}\Phi =& \prod_{i\in I} P_i(u_1,\ldots,u_n,x_1,\ldots,x_n)^{a_i\epsilon} \\ \varphi =& du_1\wedge\ldots\wedge du_n \prod_{i\in I} P_i(u_1,\ldots,u_n,x_1,\ldots,x_n)^{m_i}.\end{aligned} \tag{3.63}$$

Now let us turn to positive geometries. We will not provide the general definition of these objects, which is supplied in [87], and instead consider only a restricted case relevant to the above integrals. We will define $\{\Gamma_j\}$ to be the set of cells bounded by the hypersurfaces $\{(u_1,\ldots,u_n)|P_i(u_1,\ldots,u_n,x_1,\ldots,x_n) = 0\}$. Then the positive geometries we consider will be the pairs $(\mathbb{P}^n(\mathbb{C}),\Gamma_j)$ and their associated canonical forms $\Omega(\mathbb{P}^n(\mathbb{C}),\Gamma_j)$.

For instance, if a domain Γ is bounded by the region defined according to $P_1(u_1,\ldots,u_n) = \ldots = P_m(u_1,\ldots,u_n) = 0$ with polynomial functions P_i which are each linear in the u_i then its canonical form is given by

$$d\log\frac{P_2(u_1,\ldots,u_n)}{P_1(u_1,\ldots,u_n)}\wedge\ldots\wedge d\log\frac{P_m(u_1,\ldots,u_n)}{P_{m-1}(u_1,\ldots,u_n)}. \tag{3.64}$$

The canonical forms are also known when all but one of the polynomial functions are linear [87]. Taken together with the case where all are linear, this covers each of the hypergeometric integrals that we have worked with in the preceding sections. The set of canonical forms supplies a basis for the objects φ of (3.63) which carry the integer parts of the exponents. Multiplying each of these basis elements by Φ we obtain a basis of forms ω_i for our original object ω from (3.62).

We conclude by mentioning two other results which are relevant to the construction of bases for the forms and contours. Firstly, we note the dimensionality of these bases is less than or equal to the number of solutions to the equation

$$d\text{log}\Phi = 0, \tag{3.65}$$

with the equality being realised under certain known conditions [88–91]. This result provides a method to determine the number of basis elements solely from the integral representation and without reference to the counting of master integrals that was described in section 2.6. Secondly, it has been established [88] that a basis of contours can always be found using only those which do not extend to infinity, in agreement with our experience that this is possible for the examples considered in the previous sections.

3.3.2 Intersection Theory

We have previously found dual bases $\{\omega_i\}$ and $\{\gamma_j\}$ of forms and contours by taking candidate bases for each category of objects, integrating the forms over the contours and rotating the bases as required to fulfil the condition (2.48).

There is a more indirect way we can address this problem. Given some form $\Phi\varphi$, one can write down the covariant differential $\nabla_\Phi = d + d\text{log}\Phi\wedge$ and so the cohomology groups that are associated with forms modulo total covariant derivatives can also be determined. These are known as twisted cohomology groups [88]. We can pair a form φ to another form ψ by considering the dual twisted cohomology group with covariant differential $\nabla_{\Phi^{-1}}$, and then define [88, 92] the intersection number

$$\langle \varphi, \psi \rangle_\Phi = \frac{1}{(2\pi i)^2} \int_{\mathbb{P}^n(\mathbb{C})\setminus \bigcup_{i\in I}\{(u_1,\ldots,u_n)|P_i(u_1,\ldots,u_n,x_1,\ldots,x_m)=0\}} \iota_\Phi(\varphi) \wedge \psi \tag{3.66}$$

with a map ι which produces a form $\iota(\varphi)$ that has compact support. There are a number of alternative ways to evaluate this integral under certain special circumstances outlined in [92–94].

For the contours described above in section 3.3.1, we have seen that a contour γ can be associated with a canonical form $\Omega(\gamma)$. It may be shown [93] that if $\omega = \Phi\varphi$

as before, then

$$\int_\gamma \omega = (1 + \mathcal{O}(\epsilon))\langle \Omega(\gamma), \varphi \rangle_\Phi. \tag{3.67}$$

Specialising this statement to particular choices of ω and γ we can write the lowest order of the period matrix using intersection numbers. The calculation of intersection numbers thus provides an alternative method to obtain dual bases of forms and contours for use in the coaction.

Chapter 4

Master Integrals and Cuts of Two-Loop Feynman Integrals

Before we address the diagrammatic coaction at two loops, we will need to find convenient bases of master integrals and cuts. We have already seen in chapters 2 and 3 that, for a wide range of integrals, the coaction can be written down in a simple form when bases that are dual to each other can be found. We will replicate this construction for a range of two-loop Feynman integrals.

We discuss the choices of master integrals that will be used for a number of relevant examples. Finding these master integrals will be substantially harder than the one-loop case, where there was a simple definition that produced pure, uniform-weight functions for any one-loop graph and configuration of masses and external kinematics, and we will be forced to treat each two-loop case individually.

4.1 Master Integrals at Two Loops

We have seen that at one loop there is a convenient basis of pure, uniform-weight integrals. Specifically, these basis integrals took the form [24]

$$\tilde{J}(\{p_i.p_j\}; m_1^2, ..., m_n^2) = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \prod_{i=1}^{n} \frac{1}{(k + \sum_{j=1}^{i} p_j)^2 - m_i{}^2} \tag{4.1}$$

with dimensionality $D = 2\lceil \frac{n}{2} \rceil - 2\epsilon$, and with the possibility that the external momenta p_j^2 and internal masses m_i^2 vanish when we are not in the generic kinematic configuration. For the two-loop case we will not arrive at such a simple basis for a generic integral, but will nevertheless obtain a convenient basis for each of the graphs whose coactions we examine.

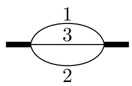
In the one-loop case there was a single master integral corresponding to each subgraph. It is well known that this property no longer holds at two loops, and so in the following sections we will write down as many master integrals as are necessary to form a basis. This number can be found, for instance, by using the package FIRE [95].

We also note that in the one-loop case, each integral $\tilde{J}$ was normalised by its leading singularity. At two loops, there is not generally a notion of a unique maximal cut and so we will be forced to reconsider how our graphs are normalised.

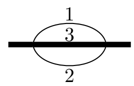
Throughout this section we will make use of numerators in Feynman integrals to define bases of master integrals. We will allow such numerators which are of the form $(\alpha k + \alpha' l + \sum_i \beta_i p_i)^2$ raised to positive integer powers, where the internal loop momenta are k and l, and the $\{p_i\}$ are the external momenta. At two loops, it is generally not possible to reduce a Feynman integral with such a numerator, or with propagator exponents ν_i such that $\nu_i \in \mathbb{Z}$ and $\nu_i \geq 2$, to a combination of integrals where there are no numerators and each propagator has unit power.

We will also encounter cases where there is not an obvious choice for the dimensionality of the integral. For instance, the double-edged triangle topology, shown in figures 4.1e, 4.1f, 4.1g, 4.1h and 4.1i for various mass and momenta configurations, has a bubble loop and a triangle loop. From this, with the logic of the one-loop basis (4.1), we might wish to assign it both dimensionality $D = 2 - 2\epsilon$ and $D = 4 - 2\epsilon$.

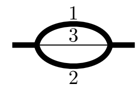
As we will see in the following chapter, with our usual method of adopting a parametrisation of the momenta and taking residues the computation of cuts for integrals with numerators is not substantially harder than for those without. The complexity of these calculations is reduced when the numerator is able to be placed in the loop which is integrated second, but in any case the cut calculation is simpler than for integrals with higher than unit powers on the propagators. Throughout this



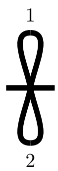
(a) Zero-mass sunset



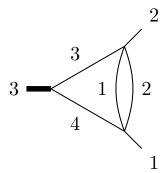
(b) One-mass sunset



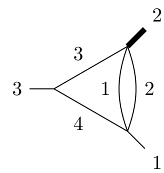
(c) Two-mass sunset



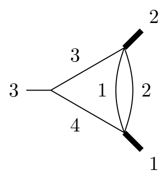
(d) Two-mass tadpoles



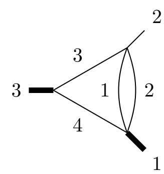
(e) Symmetric one-scale double-edged triangle



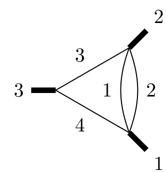
(f) Asymmetric one-scale double-edged triangle



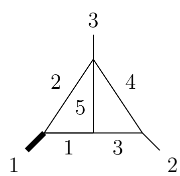
(g) Symmetric two-scale double-edged triangle



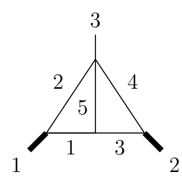
(h) Asymmetric two-scale double-edged triangle



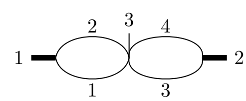
(i) Three-scale double-edged triangle



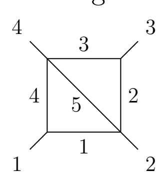
(j) One-scale adjacent triangles



(k) Symmetric two-scale adjacent triangles



(l) Zero-mass adjacent bubbles



(m) Two-scale diagonal box

Figure 4.1: Examples of two-loop graphs with various mass and momentum configurations

section we will therefore introduce bases of master integrals that have numerators and unit powers on each propagator. We will consider all the cases shown in figure 4.1. These examples possess varying numbers of internal masses, external scales, propagators and top-sector master integrals, and so computing their coactions will give us a good idea of the features of the two-loop diagrammatic coaction.

4.2 Sunset Basis

For the sunset integral, each loop possesses two propagators, suggesting that the dimensionality which should be selected to produce a pure, uniform-weight integral is $D = 2 - 2\epsilon$. We do not consider the case with three masses, which cannot be evaluated using only multiple polylogarithms, but will examine each of the graphs shown in figures 4.1a, 4.1b and 4.1c. Let us start with the most generic of these three: the two-mass case, to which there is associated the family of Feynman integrals

$$\begin{aligned}&S(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5;D;p^2,m_1^2,m_2^2) \\ =&e^{2\gamma_E\epsilon}\int\frac{d^Dk}{i\pi^{D/2}}\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{[k^2-m_1^2]^{\nu_1}[l^2-m_2^2]^{\nu_2}[(k+l+p)^2]^{\nu_3}[(k+p)^2]^{\nu_4}[(l+p)^2]^{\nu_5}}.\end{aligned} \tag{4.2}$$

The simplest candidate for a master integral associated with this graph is the unit power case with no numerators: $S(1,1,1,0,0;2-2\epsilon;p^2,m_1^2,m_2^2)$, which can be evaluated using the more general result

$$\begin{aligned}&S(\nu_1,\nu_2,\nu_3,0,0;D;p^2,m_1^2,m_2^2) \\ =&(-1)^{\nu_1+\nu_2+\nu_3}(m_1^2)^{D-\nu_1-\nu_2-\nu_3}e^{2\gamma_E\epsilon}\frac{\Gamma(D/2-\nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(D/2)} \\ &\times\Big\{\Gamma(\nu_1+\nu_2+\nu_3-D)\Gamma(-D/2+\nu_2+\nu_3)\Gamma(D/2-\nu_2) \\ &\times F_4\left(\nu_1+\nu_2+\nu_3-D,-D/2+\nu_2+\nu_3;D/2,1+\nu_2-D/2;\frac{p^2}{m_1^2},\frac{m_2^2}{m_1^2}\right) \\ &+(m_2^2/m_1^2)^{D/2-\nu_2}\Gamma(\nu_1+\nu_3-D/2)\Gamma(\nu_2-D/2)\Gamma(\nu_3)\end{aligned} \tag{4.3}$$

$$\times F_4\left(\nu_1+\nu_3-D/2,\nu_3;D/2,1-\nu_2+D/2,\frac{p^2}{m_1^2},\frac{m_2^2}{m_1^2}\right)\Bigg\}.$$

This result follows from evaluating a sunset with no internal masses (which in turn is computed by using a result for a massless one-loop bubble integral with arbitrary powers on the propagators), and using Mellin-Barnes integration to obtain the case with two masses.

Using the integral representation for the Appell F_4 function, the terms in the above relation can be expanded in ϵ for each choice of the parameters ν_1, ν_2 and ν_3. It is found that for $\nu_1 = \nu_2 = \nu_3 = 1$, the result is uniform weight and proportional to a pure function with some coefficient $\frac{1}{\sqrt{\lambda(p^2,m_1^2,m_2^2)}}$, where $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$. For convenience, we will also want to define an integral that is a function only of dimensionless ratios of the variables, and which has a weight n polylogarithm as the coefficient of ϵ^n in its Laurent expansion. We therefore define a first master integral

$$\begin{aligned}&S^{(1)}(p^2,m_1^2,m_2^2)\\&=\epsilon^2\sqrt{\lambda\left(1,\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right)}(p^2)^{1+2\epsilon}\,S(1,1,1,0,0;2-2\epsilon;p^2,m_1^2,m_2^2).\end{aligned} \tag{4.4}$$

Another possibility is to insert numerators that cancel factors arising from integration over the first loop momentum. Taking the integral representation of $S(1,1,1,0,0;D;p^2,m_1^2,m_2^2)$ and examining the integral over l only, we see that

$$\begin{aligned}&\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{(l^2-m_2^2)(k+l+p)^2}\\&=-\Gamma\left(1-D/2\right)\left[m_2^2-(k+p)^2\right]^{D-3}{}_2F_1\left(D/2-1,2-D/2;D/2;\frac{(k+p)^2}{(k+p)^2-m_2^2}\right).\end{aligned} \tag{4.5}$$

In $D = 2-2\epsilon$, multiplication of this result by a factor of $m_2^2-(l+p)^2$ produces a pure function. Performing the integration over the other loop momentum k leads to another pure function, and so we define the second master integral $S^{(2)}$ by including this numerator $m_2^2-(l+p)^2$. We can also swap the order of loop integration and include an analogous factor $m_1^2 - (k+p)^2$, providing a third master integral $S^{(3)}$. We then have the integrals

$$\begin{aligned} S^{(2)}(p^2, m_1^2, m_2^2) =& \epsilon^2 (p^2)^{2\epsilon} \left[m_2^2\, S(1,1,1,0,0; 2-2\epsilon; p^2, m_1^2, m_2^2) \right. \\ & \left. - S(1,1,1,-1,0; 2-2\epsilon; p^2, m_1^2, m_2^2) \right] \\ S^{(3)}(p^2, m_1^2, m_2^2) =& \epsilon^2 (p^2)^{2\epsilon} \left[m_1^2\, S(1,1,1,0,0; 2-2\epsilon; p^2, m_1^2, m_2^2) \right. \\ & \left. - S(1,1,1,0,-1; 2-2\epsilon; p^2, m_1^2, m_2^2) \right] . \end{aligned} \tag{4.6}$$

Simple expressions for the integral $S^{(1)}$ are easily obtained from (4.3):

$$\begin{aligned} & S^{(1)}(p^2, m_1^2, m_2^2) \\ =& \frac{1-x-y}{x(1-y)} e^{2\gamma_E \epsilon} \Gamma(1+\epsilon) \left(\frac{p^2}{m_1^2} \right)^{1+2\epsilon} \\ & \times \left\{ -\Gamma(1+2\epsilon)\Gamma(1-\epsilon) F_4 \left(1+2\epsilon, 1+\epsilon; 1-\epsilon, 1+\epsilon; \frac{p^2}{m_1^2}, \frac{m_2^2}{m_1^2} \right) \right. \\ & \left. + \Gamma(1+\epsilon) \left(\frac{m_2^2}{m_1^2} \right)^{-\epsilon} F_4 \left(1+\epsilon, 1; 1-\epsilon, 1-\epsilon, \frac{p^2}{m_1^2}, \frac{m_2^2}{m_1^2} \right) \right\} \\ =& e^{2\gamma_E \epsilon} \frac{1-x-y}{x(1-y)} \left\{ \Gamma^2(1+\epsilon) \left(\frac{m_1^2}{p^2} \right)^{-\epsilon} \left(\frac{m_2^2}{p^2} \right)^{-\epsilon} F_4 \left(1, 1+\epsilon; 1-\epsilon, 1-\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2} \right) \right. \\ & - 2 \frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{m_1^2}{p^2} \right)^{-2\epsilon} \left(-\frac{m_1^2}{p^2} \right)^{\epsilon} F_4 \left(1+\epsilon, 1+2\epsilon; 1-\epsilon, 1+\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2} \right) \\ & - 2 \frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{m_1^2}{p^2} \right)^{-\epsilon} \left(\frac{m_2^2}{p^2} \right)^{-\epsilon} \left(-\frac{m_1^2}{p^2} \right)^{\epsilon} \\ & \times F_4 \left(1+\epsilon, 1+2\epsilon; 1+\epsilon, 1-\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2} \right) \\ & + 3 \frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)} \left(\frac{m_1^2}{p^2} \right)^{-2\epsilon} \left(-\frac{m_1^2}{p^2} \right)^{2\epsilon} \\ & \left. \times F_4 \left(1+2\epsilon, 1+3\epsilon; 1+\epsilon, 1+\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2} \right) \right\} , \end{aligned} \tag{4.7}$$

where we have used the analytic continuation formula [72]

$$
\begin{aligned}
&F_4(\alpha,\beta;\gamma,\gamma';x,y) \qquad (4.8)\\
=&\frac{\Gamma(\gamma')\Gamma(\beta-\alpha)}{\Gamma(\gamma'-\alpha)\Gamma(\beta)}(-y)^{-\alpha}F_4\left(\alpha,\alpha+1-\gamma';\gamma,\alpha+1-\beta;\frac{x}{y},\frac{1}{y}\right)\\
&+\frac{\Gamma(\gamma')\Gamma(\alpha-\beta)}{\Gamma(\gamma'-\beta)\Gamma(\alpha)}(-y)^{-\beta}F_4\left(\beta,\beta+1-\gamma';\gamma,\beta+1-\alpha;\frac{x}{y},\frac{1}{y}\right)
\end{aligned}
$$

to express the result using F_4 functions with arguments $\frac{m_1^2}{p^2}$ and $\frac{m_2^2}{p^2}$.

The other integrals $S^{(2)}$ and $S^{(3)}$ can still be evaluated using (4.3), but only indirectly by using integration by parts relations to express them in another basis of master integrals with $\nu_4=\nu_5=0$, such as $S(1,1,1,0,0;2-2\epsilon;p^2,m_1^2,m_2^2)$, $S(2,1,1,0,0;2-2\epsilon;p^2,m_1^2,m_2^2)$ and $S(1,2,1,0,0;2-2\epsilon;p^2,m_1^2,m_2^2)$. This leads to a result with sums of several contiguous F_4 functions. We might suspect that these functions can be related to an expression with a smaller number of terms by contiguous relations as described in 2.6.2. As we will see in the following chapter, computing the cuts of $S^{(2)}$ and $S^{(3)}$ yields much more compact expressions. The simpler expressions appearing in the cuts can be shown to equal the linear combinations resulting from (4.3), allowing us to find the expressions

$$
\begin{aligned}
&S^{(2)}(p^2,m_1^2,m_2^2) \qquad (4.9)\\
=&e^{2\gamma_E\epsilon}\Gamma(1+\epsilon)^2\left(\frac{p^2}{m_1^2}\right)^{2\epsilon}\left(\frac{m_2^2}{m_1^2}\right)^{-\epsilon}\Bigg\{-1-\frac{2\epsilon}{1-\epsilon}\frac{m_2^2}{m_1^2}F_4\left(1+\epsilon,1;1-\epsilon,2-\epsilon;\frac{p^2}{m_1^2},\frac{m_2^2}{m_1^2}\right)\\
&+\left(\frac{m_2^2}{m_1^2}\right)^{\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)}F_4\left(2\epsilon,\epsilon;1-\epsilon,\epsilon;\frac{p^2}{m_1^2},\frac{m_2^2}{m_1^2}\right)\Bigg\}\\
=&e^{2\gamma_E\epsilon}\Bigg\{-\left(\frac{m_1^2}{p^2}\right)^{-\epsilon}\left(\frac{m_2^2}{p^2}\right)^{-\epsilon}\Gamma^2(1+\epsilon)\\
&-\frac{2\epsilon}{1-\epsilon}\Gamma^2(1+\epsilon)\left(\frac{m_1^2}{p^2}\right)^{-\epsilon}\left(\frac{m_2^2}{p^2}\right)^{1-\epsilon}F_4\left(1,1+\epsilon;1-\epsilon,2-\epsilon;\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right)\\
&+2\frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{m_1^2}{p^2}\right)^{-2\epsilon}\left(-\frac{m_1^2}{p^2}\right)^{\epsilon}F_4\left(\epsilon,2\epsilon;1-\epsilon,\epsilon;\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4\epsilon}{1-\epsilon}\frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{m_1^2}{p^2}\right)^{-\epsilon}\left(\frac{m_2^2}{p^2}\right)^{1-\epsilon}\left(-\frac{m_1^2}{p^2}\right)^{\epsilon}\\
&\times F_4\left(1+\epsilon,1+2\epsilon;1+\epsilon,2-\epsilon;\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right)\\
&-\frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}\left(\frac{m_1^2}{p^2}\right)^{-2\epsilon}\left(-\frac{m_1^2}{p^2}\right)^{2\epsilon}F_4\left(2\epsilon,3\epsilon;1+\epsilon,\epsilon;\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right)\Bigg\}
\end{aligned}
$$

and

$$
\begin{aligned}
&S^{(3)}(p^2,m_1^2,m_2^2) \qquad (4.10)\\
=&e^{2\gamma_E\epsilon}\Gamma(1+\epsilon)^2\left(\frac{p^2}{m_1^2}\right)^{2\epsilon}\left(\frac{m_2^2}{m_1^2}\right)^{-\epsilon}\Bigg\{-1+2F_4\left(\epsilon,1;1-\epsilon,1-\epsilon;\frac{p^2}{m_1^2},\frac{m_2^2}{m_1^2}\right)\\
&-\left(\frac{m_2^2}{m_1^2}\right)^{\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)}F_4\left(2\epsilon,1+\epsilon;1-\epsilon,1+\epsilon;\frac{p^2}{m_1^2},\frac{m_2^2}{m_1^2}\right)\Bigg\}\\
=&e^{2\gamma_E\epsilon}\Bigg\{-\left(\frac{m_1^2}{p^2}\right)^{-\epsilon}\left(\frac{m_2^2}{p^2}\right)^{-\epsilon}\Gamma^2(1+\epsilon)\\
&-\frac{2\epsilon}{1-\epsilon}\Gamma^2(1+\epsilon)\left(\frac{m_1^2}{p^2}\right)^{1-\epsilon}\left(\frac{m_2^2}{p^2}\right)^{-\epsilon}F_4\left(1,1+\epsilon;2-\epsilon,1-\epsilon;\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right)\\
&+2\frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{m_1^2}{p^2}\right)^{-\epsilon}\left(\frac{m_2^2}{p^2}\right)^{-\epsilon}\left(-\frac{m_1^2}{p^2}\right)^{\epsilon}F_4\left(\epsilon,2\epsilon;\epsilon,1-\epsilon;\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right)\\
&+\frac{4\epsilon}{1-\epsilon}\frac{\Gamma^2(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{m_1^2}{p^2}\right)^{1-2\epsilon}\left(-\frac{m_1^2}{p^2}\right)^{\epsilon}\\
&\times F_4\left(1+\epsilon,1+2\epsilon;2-\epsilon,1+\epsilon;\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right)\\
&-\frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}\left(\frac{m_1^2}{p^2}\right)^{-2\epsilon}\left(-\frac{m_1^2}{p^2}\right)^{2\epsilon}F_4\left(2\epsilon,3\epsilon;\epsilon,1+\epsilon;\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right)\Bigg\}.
\end{aligned}
$$

We also mention that this two-mass sunset contains the subgraph which is a product of two tadpoles as shown in figure 4.1d. We will assign to this graph a master integral

$$
J(p^2,m_1^2,m_2^2)=\epsilon^2(p^2)^{2\epsilon}S(1,1,0,0,0;2-2\epsilon;p^2,m_1^2,m_2^2) \qquad (4.11)
$$

$$=e^{2\gamma_E\epsilon}\Gamma^2(1+\epsilon)\left(\frac{m_1^2}{p^2}\right)^{-\epsilon}\left(\frac{m_2^2}{p^2}\right)^{-\epsilon}.$$

For the one-mass sunset case, we can obtain a basis of two pure uniform-weight master integrals by making the replacements $m_1^2 \to m^2$ and $m_2^2 \to 0$ in our two-mass results. We only require two master integrals in this limit and for convenience we select the limits of the first and second integrals from the two-mass case, as these immediately reduce to single-term expressions. We will change the loop momentum in (4.2) by sending $k \to -k-l-p$ to obtain the family of integrals:

$$\begin{aligned}&S(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5;D;p^2,m^2) \\ =&e^{2\gamma_E\epsilon}\int\frac{d^Dk}{i\pi^{D/2}}\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{[k^2]^{\nu_1}[l^2]^{\nu_2}[(k+l+p)^2-m^2]^{\nu_3}[(k+l)^2]^{\nu_4}[(l+p)^2]^{\nu_5}}.\end{aligned} \tag{4.12}$$

Then we can write the master integrals for the one-mass sunset as

$$\begin{aligned}S^{(1)}(p^2,m^2) =&\epsilon^2\left(\frac{p^2}{m^2}-1\right)(m^2)^{1+2\epsilon}S(1,1,1,0,0;2-2\epsilon;p^2,m^2) \\ S^{(2)}(p^2,m^2) =&-\epsilon^2(m^2)^{2\epsilon}S(1,1,1,-1,0;2-2\epsilon;p^2,m^2).\end{aligned} \tag{4.13}$$

These can be evaluated by taking suitable limits in the above expressions for the two-mass case, producing the results

$$\begin{aligned}&S^{(1)}(p^2,m^2) \\ =&\left(1-\frac{p^2}{m^2}\right)e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)\,{}_2F_1\left(1+2\epsilon,1+\epsilon;1-\epsilon;\frac{p^2}{m^2}\right) \\ &S^{(2)}(p^2,m^2) \\ =&e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)\,{}_2F_1\left(2\epsilon,\epsilon;1-\epsilon;\frac{p^2}{m^2}\right).\end{aligned} \tag{4.14}$$

When both masses vanish, there is a single master integral, which we can take to be

$$\begin{aligned} S(p^2) =&\frac{1}{3}\epsilon^2(-p^2)^{1+2\epsilon}e^{2\gamma_E\epsilon}\int\frac{d^Dk}{i\pi^{D/2}}\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{k^2l^2(k+l+p)^2} \\ =&e^{2\gamma_E\epsilon}\frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}. \end{aligned} \tag{4.15}$$

We will encounter this massless sunset as a master integral when considering several different graphs. Indeed, the massless sunset is a subgraph of the double-edged triangles, adjacent triangles and diagonal box shown in figure 4.1. For each of these graphs, we will multiply by a different factor to make their corresponding master integral dimensionless, for instance we will include $(-p_3^2)^{1+2\epsilon}$ in the definition of the master integrals associated with the graph of figure 4.1i. It will then be understood in the following chapters that the master integrals belonging to the subgraphs will also be multiplied by the same factor without any other changes to the normalisation. For example, the sunset subgraph of figure 4.1i with propagators 1, 2 and 4 will then have master integral

$$\begin{aligned} S(p_1^2,p_3^2) =&\frac{1}{3}\epsilon^2(-p_3^2)^{1+2\epsilon}e^{2\gamma_E\epsilon}\int\frac{d^Dk}{i\pi^{D/2}}\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{k^2l^2(k+l+p_1)^2} \\ =&e^{2\gamma_E\epsilon}\frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}\left(\frac{p_1^2}{p_3^2}\right)^{-1-2\epsilon} \end{aligned} \tag{4.16}$$

instead of that given in (4.15). These replacements ensure consistency between master integrals for a graph and its subgraphs.

4.3 Bases for Other Feynman Integrals

We will now assign master integrals to the remaining graphs shown in figure 4.1, starting with the family of graphs 4.1e, 4.1f, 4.1g, 4.1h and 4.1i. We then look at the graphs 4.1j and 4.1k, and its subtopology 4.1l, before finishing with 4.1m.

4.3.1 Double-Edged Triangle Basis

Consider first the graph 4.1i as it is the most generic of the double-edged triangle graphs in figure 4.1. Allowing each loop momentum to have a different number of dimensions, the family of Feynman integrals associated with this graph is given by

$$\begin{aligned}&P(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7;D_1,D_2;p_1^2,p_2^2,p_3^2)\\&=e^{2\gamma_E\epsilon}\int\frac{d^{D_2}l}{i\pi^{D_2/2}}\int\frac{d^{D_1}k}{i\pi^{D_1/2}}\frac{1}{(k^2)^{\nu_1}[(k+l+p_2)^2]^{\nu_2}(l^2)^{\nu_3}[(l-p_3)^2]^{\nu_4}}\\&\quad\times\frac{1}{[(k+p_3)^2]^{\nu_5}[(k+p_2)^2]^{\nu_6}[(l+p_2)^2]^{\nu_7}}.\end{aligned}\tag{4.17}$$

Integrating first over the momentum k associated with the bubble loop of the graph with a dimension D_1 and then evaluating the remaining integral over l in D_2 dimensions using the results of, for instance, [83] or [96], we find that when only the first four propagators are retained

$$\begin{aligned}&P(\nu_1,\nu_2,\nu_3,\nu_4,0,0,0;D_1,D_2;p_1^2,p_2^2,p_3^2)\\&=e^{2\gamma_E\epsilon}\int\frac{d^{D_2}l}{i\pi^{D_2/2}}\frac{1}{(l^2)^{\nu_3}[(l-p_3)^2]^{\nu_4}}\int\frac{d^{D_1}k}{i\pi^{D_1/2}}\frac{1}{(k^2)^{\nu_1}[(k+l+p_2)^2]^{\nu_2}}\\&=e^{2\gamma_E\epsilon}(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}(-p_3^2)^{D_1/2+D_2/2-\nu_1-\nu_2-\nu_3-\nu_4}\\&\quad\times\frac{\Gamma(D_1/2-\nu_1)\Gamma(D_1/2-\nu_2)\Gamma(\nu_1+\nu_2-D_1/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(D_1-\nu_1-\nu_2)}\\&\quad\times\left\{\left(\frac{p_1^2}{p_3^2}\right)^{\frac{D_1+D_2}{2}-\nu_1-\nu_2-\nu_4}\frac{\Gamma(D_1/2+D_2/2-\nu_1-\nu_2-\nu_3)}{\Gamma(\nu_1+\nu_2-D_1/2)\Gamma(\nu_4)}\right.\\&\quad\times\frac{\Gamma(\nu_1+\nu_2+\nu_4-D_1/2-D_2/2)\Gamma(D_2/2-\nu_4)}{\Gamma(D_2+D_1/2-\nu_1-\nu_2-\nu_3-\nu_4)}\\&\quad\times F_4\left(\begin{matrix}\nu_3,D_2/2-\nu_4\\1+D_1/2+D_2/2-\nu_1-\nu_2-\nu_4,1+\nu_1+\nu_2+\nu_3-D_1/2-D_2/2\end{matrix};\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right)\\&\quad+\frac{\Gamma(D_1/2+D_2/2-\nu_1-\nu_2-\nu_3)}{\Gamma(\nu_3)\Gamma(\nu_4)}\\&\quad\times\frac{\Gamma(D_1/2+D_2/2-\nu_1-\nu_2-\nu_4)\Gamma(\nu_1+\nu_2+\nu_3+\nu_4-D_1/2-D_2/2)}{\Gamma(D_2+D_1/2-\nu_1-\nu_2-\nu_3-\nu_4)}\end{aligned}\tag{4.18}$$

$$\times F_4\left(\begin{matrix}\nu_1+\nu_2-D_1/2,\nu_1+\nu_2+\nu_3+\nu_4-D_1/2-D_2/2\\ 1+\nu_1+\nu_2+\nu_4-D_1/2-D_2/2,1+\nu_1+\nu_2+\nu_3-D_1/2-D_2/2\end{matrix};\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right)$$
$$+\left(\frac{p_1^2}{p_3^2}\right)^{D_1/2+D_2/2-\nu_1-\nu_2-\nu_4}\left(\frac{p_2^2}{p_3^2}\right)^{D_1/2+D_2/2-\nu_1-\nu_2-\nu_3}\frac{\Gamma(D_1/2+D_2/2-\nu_1-\nu_2)}{\Gamma(\nu_3)\Gamma(\nu_4)}$$
$$\times\frac{\Gamma(\nu_1+\nu_2+\nu_3-D_1/2-D_2/2)\Gamma(\nu_1+\nu_2+\nu_4-D_1/2-D_2/2)}{\Gamma(\nu_1+\nu_2-D_1/2)}$$
$$\times F_4\left(\begin{matrix}D_2+D_1/2-\nu_1-\nu_2-\nu_3-\nu_4,D_1/2+D_2/2-\nu_1-\nu_2\\ 1+D_1/2+D_2/2-\nu_1-\nu_2-\nu_4,1+D_1/2+D_2/2-\nu_1-\nu_2-\nu_3\end{matrix};\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right)$$
$$+\left(\frac{p_2^2}{p_3^2}\right)^{\frac{D_1+D_2}{2}-\nu_1-\nu_2-\nu_3}\frac{\Gamma(D_1/2+D_2/2-\nu_1-\nu_2-\nu_4)}{\Gamma(\nu_1+\nu_2-D_1/2)\Gamma(\nu_3)}$$
$$\times\frac{\Gamma(\nu_1+\nu_2+\nu_3-D_1/2-D_2/2)\Gamma(D_2/2-\nu_3)}{\Gamma(D_2+D_1/2-\nu_1-\nu_2-\nu_3-\nu_4)}$$
$$\times F_4\left(\begin{matrix}\nu_4,D_2/2-\nu_3\\ 1+\nu_1+\nu_2+\nu_4-D_1/2-D_2/2,1+D_1/2+D_2/2-\nu_1-\nu_2-\nu_3\end{matrix};\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right)\Bigg\},$$

where, for compactness, we have adopted the notation

$$F_4(\alpha,\beta;\gamma,\gamma';x,y)=F_4\left(\begin{matrix}\alpha,\beta\\ \gamma,\gamma'\end{matrix};x,y\right).$$

We may also slightly modify this calculation to obtain the result for the integral with numerator $P(\nu_1,\nu_2,\nu_3,\nu_4,0,0,-a;D_1,D_2;p_1^2,p_2^2,p_3^2)$. The factor $(l+p_2)^2$ in the numerator is also produced by integrating the first loop, and so we need only shift the exponent of this factor in the above calculation to compute this integral.

For ease of calculation we will adopt the basis of integrals consisting of the mixed dimension object $P(1,1,1,1,0,0,0;2-2\epsilon,4-2\epsilon;p_1^2,p_2^2,p_3^2)$ and the integral with numerator $(l+p_2)^2$ raised to unit power $P(1,1,1,1,0,0,-1;2-2\epsilon,2-2\epsilon;p_1^2,p_2^2,p_3^2)$, both of which are uniform weight. We can then make them pure and dimensionless by including certain normalisation factors:

$$P^{(1)}(p_1^2,p_2^2,p_3^2)=\epsilon^2(-p_3^2)^{1+2\epsilon}P(1,1,1,1,0,0,-1;2-2\epsilon,2-2\epsilon;p_1^2,p_2^2,p_3^2) \tag{4.19}$$

$$P^{(2)}(p_1^2, p_2^2, p_3^2) = \epsilon^3 \sqrt{\lambda\left(1, \frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right)} (-p_3^2)^{1+2\epsilon} P(1,1,1,1,0,0,0; 2-2\epsilon, 4-2\epsilon; p_1^2, p_2^2, p_3^2).$$

Now consider the other cases of double-edged triangle graphs shown in figure 4.1. The Feynman integrals corresponding to the symmetric two-scale and asymmetric one-scale cases are reducible to sunset integrals and so we need not define any master integrals for these graphs. For the symmetric one-scale case, we can take as our master integral either $P^{(1)}$ or $P^{(2)}$ from (4.19) in the limit $p_1^2 \to 0$, $p_2^2 \to 0$ as these integrals are proportional to each other in this limit. Lastly, for the two-scale asymmetric case we may choose either $P^{(1)}$ or $P^{(2)}$ with $p_2^2 \to 0$, which we denote by $\tilde{P}$ and P, respectively. These integrals are not proportional to each other, but instead we can express either as a linear combination of the other and a sunset integral with external momentum p_1.

4.3.2 Adjacent Triangles and Diagonal Box Bases

We finish by mentioning two further Feynman integrals whose coactions we will study: the adjacent triangles with two scales and the box with a diagonal, shown in figures 4.1k and 4.1m respectively. The integrals associated with the graph of figure 4.1j are reducible and so we need not assign a master integral to this graph. Let us write down the families of integrals corresponding to each of the remaining graphs:

$$\begin{aligned} &T(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5; D; p_1^2, p_2^2) \\ =& e^{2\gamma_E\epsilon} \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{[k^2]^{\nu_1}[(k-p_1)^2]^{\nu_2}[l^2]^{\nu_3}[(l-p_2)^2]^{\nu_4}[(k+l)^2]^{\nu_5}} \end{aligned} \tag{4.20}$$

$$\begin{aligned} &B(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5; D; s, t) \\ =& e^{2\gamma_E\epsilon} \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{[k^2]^{\nu_1}[(k+p_2+l)^2]^{\nu_2}[(k+p_2+p_3+l)^2]^{\nu_3}[(k-p_1)^2]^{\nu_4}[l^2]^{\nu_5}}, \end{aligned} \tag{4.21}$$

where $s = (p_1+p_2)^2$ and $t = (p_2+p_3)^2$.

For each graph we require only a single corresponding master integral and in fact the simplest choice - with no numerators, unit powers on each propagator and $D =$

$4-2\epsilon$ dimensions - will produce a uniform-weight integral in each case. Including normalisation factors these integrals can be made pure and dimensionless, with weight n polylogarithms as coefficients of ϵ^n in their expansions, and so we define

$$\begin{aligned}
&T(p_1^2,p_2^2) \\
=&\epsilon^4(p_1^2-p_2^2)(-p_1^2)^\epsilon(-p_2^2)^\epsilon\, T(1,1,1,1,1;4-2\epsilon;p_1^2,p_2^2) \\
=&e^{2\gamma_E\epsilon}\epsilon(p_1^2-p_2^2)\left\{\frac{1}{(1-2\epsilon)}\frac{\Gamma^2(1+\epsilon)\Gamma^4(1-\epsilon)}{\Gamma^2(1-2\epsilon)}\frac{1}{p_2^2}\left(\frac{p_1^2}{p_2^2}\right)^{-\epsilon}{}_2F_1\left(1-\epsilon,1-2\epsilon;2-2\epsilon;1-\frac{p_1^2}{p_2^2}\right)\right. \\
&-\frac{1}{2(1-2\epsilon)}\frac{\Gamma(1+2\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)}\left[\frac{1}{p_1^2}\left(\frac{p_2^2}{p_1^2}\right)^{-\epsilon}{}_3F_2\left(1-\epsilon,1,1-2\epsilon;1+\epsilon,2-2\epsilon;1-\frac{p_2^2}{p_1^2}\right)\right. \\
&\left.\left.+\frac{1}{p_2^2}\left(\frac{p_1^2}{p_2^2}\right)^{-\epsilon}{}_3F_2\left(1-\epsilon,1,1-2\epsilon;1+\epsilon,2-2\epsilon;1-\frac{p_1^2}{p_2^2}\right)\right]\right\}
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&B(s,t) \\
=&\epsilon^4(s+t)s^\epsilon t^\epsilon B(1,1,1,1,1;4-2\epsilon;s,t) \\
=&-\epsilon(s+t)e^{2\gamma_E\epsilon}\frac{1}{2(1-2\epsilon)}\frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}\left[\frac{1}{s}\left(\frac{t}{s}\right)^{-\epsilon}{}_2F_1\left(1-2\epsilon,1-2\epsilon;2-2\epsilon;1+\frac{t}{s}\right)\right. \\
&\left.+\frac{1}{t}\left(\frac{s}{t}\right)^{-\epsilon}{}_2F_1\left(1-2\epsilon,1-2\epsilon;2-2\epsilon;1+\frac{s}{t}\right)\right].
\end{aligned} \tag{4.23}$$

These integrals are computed using the differential equation method [97–99], with the particular cases above given in [100]. For the adjacent triangle graph 4.1j, we replace the normalisation above with one which depends only on p_1^2 such that the integral is dimensionless.

Finally, we mention that the graph of figure 4.1l will have master integral

$$\begin{aligned}
I(p_1^2,p_2^2)=&\frac{1}{4}\epsilon^2(-p_1^2)^{1+\epsilon}(-p_2^2)^{1+\epsilon}T(1,1,1,1,0;2-2\epsilon,p_1^2,p_2^2) \\
=&e^{2\gamma_E\epsilon}\frac{\Gamma^2(1+\epsilon)\Gamma^4(1-\epsilon)}{\Gamma^2(1-2\epsilon)},
\end{aligned} \tag{4.24}$$

which is computed trivially as a product of two massless bubble graphs.

Chapter 5

Cuts of Two-Loop Feynman Integrals

Recall that in 2.3.1, the cut on a collection of propagators of a Feynman integral was defined by parametrising the momenta and replacing some of the integrations by residues. We will begin here by providing another perspective: that the cuts are solutions to certain differential equations. We will discuss how this is related to the definition by contour integration for one-loop graphs and its generalisation to the two-loop case. We then provide a number of further examples of cuts which will be used in the following chapter.

Throughout our discussion of two-loop graphs we will be primarily interested only in those cuts where each loop of the graph contains some cut propagator. From a coaction perspective, these are the cut contours which should be dual to differential forms of two-loop integrals. It will be found in the next chapter that the coaction of a two-loop graph can be expressed solely in terms of this subset of the cuts and the dual master integrals, and so our discussion is mostly limited to these cases.

In this chapter, it will be understood that since cuts appear in the second entry of the coaction, where we work modulo $i\pi$, we can compute our cuts modulo $i\pi$ also. Hence we will freely use the fact that, modulo $i\pi$, there are the equalities $(-x)^\epsilon = x^\epsilon$ and $\Gamma(1+\epsilon)\Gamma(1-\epsilon) = 1$.

5.1 Cuts From Differential Equations and Contours: a One-Loop Example

Suppose that we have a Feynman integral J which obeys some collection of first-order differential equations taking the form

$$\partial_{\underline{x}} J = \mathcal{A}\underline{J}, \tag{5.1}$$

where components of $\underline{x}$ are the scales m_i^2 and $s_{i,j}$ of the integral J, $\underline{J}$ is the vector of master integrals for the graph and its subgraphs, and $\mathcal{A}$ is some matrix containing rational functions of the scales. We can define a related system by replacing J and each entry of $\underline{J}$ with a new object, the cut on some collection of propagators. When the cut in question is the maximal cut of the graph, this new system will be specified by the same differential operators as the homogeneous part of the system obeyed by the uncut integral, a fact which has been investigated in [36, 101–103]. We will see evidence in the following calculations that the differential equations for non-maximal cuts can also be found by this method, and will assume this to be a general property of cuts. This behaviour is analogous to that described for the second entries of the coaction of hypergeometric functions in section 3.1.3.

Let us demonstrate this with a simple one-loop case: the one-mass bubble integral $B(p^2, m^2)$ defined by (2.30). This integral obeys the equations

$$\begin{pmatrix} \partial_{p^2} \\ \partial_{m^2} \end{pmatrix} B = \begin{pmatrix} \frac{p^2+\epsilon(m^2+p^2)}{(m^2-p^2)p^2} & -\frac{\epsilon}{(m^2-p^2)p^2} \\ -\frac{1+2\epsilon}{m^2-p^2} & \frac{\epsilon}{(m^2-p^2)m^2} \end{pmatrix} \begin{pmatrix} B \\ T \end{pmatrix}, \tag{5.2}$$

where $T(m^2)$ is the integral corresponding to the tadpole subgraph. We can thus write down equations defining cuts $\mathcal{C}_1 B$ and $\mathcal{C}_{1,2} B$:

$$\begin{pmatrix} \partial_{p^2} \\ \partial_{m^2} \end{pmatrix} \mathcal{C}_1 B = \begin{pmatrix} \frac{p^2+\epsilon(m^2+p^2)}{(m^2-p^2)p^2} & -\frac{\epsilon}{(m^2-p^2)p^2} \\ -\frac{1+2\epsilon}{m^2-p^2} & \frac{\epsilon}{(m^2-p^2)m^2} \end{pmatrix} \begin{pmatrix} \mathcal{C}_1 B \\ \mathcal{C}_1 T \end{pmatrix} \tag{5.3}$$

$$\begin{pmatrix} \partial_{p^2} \\ \partial_{m^2} \end{pmatrix} \mathcal{C}_{1,2}B = \begin{pmatrix} \frac{p^2+\epsilon(m^2+p^2)}{(m^2-p^2)p^2} & -\frac{\epsilon}{(m^2-p^2)p^2} \\ -\frac{1+2\epsilon}{m^2-p^2} & \frac{\epsilon}{(m^2-p^2)m^2} \end{pmatrix} \begin{pmatrix} \mathcal{C}_{1,2}B \\ 0 \end{pmatrix},$$

where the tadpole cut $\mathcal{C}_{1,2}T$ vanishes as there is no second propagator and thus no residue at this pole, and we presume that the cut $\mathcal{C}_1 T = e^{\gamma_E \epsilon} \frac{1}{\Gamma(1-\epsilon)}(-m^2)^{-\epsilon}$ has already been computed using (2.33) or some other method. It is clear from the latter of these equations that the maximal cut takes the form

$$\mathcal{C}_{1,2}B = C(\epsilon)(p^2)^{\epsilon}(p^2 - m^2)^{-1-2\epsilon} \tag{5.4}$$

for some unknown function $C(\epsilon)$. Meanwhile, the solution for the single cut is given by

$$\begin{aligned} &\mathcal{C}_1 B(p^2, m^2) \\ =&C(\epsilon)(p^2)^{\epsilon}(p^2 - m^2)^{-1-2\epsilon} + e^{\gamma_E \epsilon} \frac{1}{\Gamma(1-\epsilon)} \frac{(-m^2)^{-\epsilon}}{p^2} {}_2F_1\left(1, 1+\epsilon; 1-\epsilon; \frac{m^2}{p^2}\right). \end{aligned} \tag{5.5}$$

Of course, our differential equation carries only information about the dependence of the integral on the scales m^2 and p^2, so we expect that it will not allow us to determine the function C. The coefficient of the other term is fixed only because we assumed a certain coefficient of the inhomogeneous term $\mathcal{C}_1 T$ in our differential equation (5.3). The remaining coefficient must be found by performing contour integrals of the type described in section 2.3.1.

We may also take each of the above differential equations in (5.3) and derive second-order operators which annihilate each of B, $\mathcal{C}_1 B$ and $\mathcal{C}_{1,2}B$. In fact, the operators are the same for each of these functions. This is not surprising as we can take the second-order differential equations for the uncut integral and replace this integral with its cuts in the same manner that we did for the first-order equations above. Thus if we have found the general solution to this equation for the uncut integral, its cuts are linear combinations of these same functions.

The result (5.5) for the single cut provides a two-dimensional solution space but, as we have seen, cuts at one loop can be defined by a unique contour integral. The single cut as defined in (2.33) evaluates to the integral

$$\begin{aligned} &e^{\gamma_E\epsilon}\frac{1}{\Gamma(-\epsilon)}\frac{(-m^2)^{-\epsilon}}{p^2}\int_0^1 du\,(1-u)^{-1-\epsilon}\left(1-\frac{m^2}{p^2}u\right)^{-1-\epsilon} \\ =&e^{\gamma_E\epsilon}\frac{1}{\Gamma(1-\epsilon)}\frac{(-m^2)^{-\epsilon}}{p^2}{}_2F_1\left(1,1+\epsilon;1-\epsilon;\frac{m^2}{p^2}\right). \end{aligned} \tag{5.6}$$

The second solution can be obtained by modifying the integral in the above by changing its integration limits so the integral is now given by

$$e^{\gamma_E\epsilon}\frac{1}{\Gamma(-\epsilon)}\frac{(-m^2)^{-\epsilon}}{p^2}\int_0^{\frac{p^2}{m^2}} du\,(1-u)^{-1-\epsilon}\left(1-\frac{m^2}{p^2}u\right)^{-1-\epsilon}, \tag{5.7}$$

which is a linear combination of the single and double cuts. This is analogous to the case of the ${}_2F_1$, where we saw that the different integrals $\int_{\gamma_i}\omega$ appearing in the second entry of the coaction were also the solutions to the differential equations obeyed by the original function.

The second solution to the differential equations for the single cut in (5.3) can be interpreted as originating from the cut contour that also encircles the pole at infinity (2.38). In writing down (5.3) we did not exclude the possibility that our cut contour encircled this pole along with those from the two massive propagators. Hence we see that the procedure of changing the integration limits in integral representations such as (5.7) can have the effect of producing a cut which encircles the pole at infinity along with the collection of propagators where the residues were originally taken. The definition (2.33) is known to not encircle this pole at infinity, and once we have specified this property of the contour, we obtain (5.6).

Throughout the remainder of this chapter, we will use this differential equation approach to define the space of functions in which our cuts should fall, before identifying choices of integration contour which produce these cuts. This will enable us to interpret our cuts as contour integrals similar to the approach taken at one loop, and also resolve ambiguities such as that seen in (5.5) over which solutions to our differential equations should be defined to be the cuts on a particular subset of the propagators.

5.2 One-Mass Sunset

We have defined the basis of master integrals $S^{(1)}$ and $S^{(2)}$ for the sunset with one internal mass (4.13). Given the formula (2.47), and its success in describing the coaction at one loop, we can try to apply the same formula to the two-loop case. With two differential forms, corresponding to $S^{(1)}$ and $S^{(2)}$, there should be two dual contours. If we are to generalise the one-loop case, then these contours should be able to be interpreted as maximal cuts. As we will see, there are indeed two contours that we can define to be maximal cuts.

Let us begin with the system of differential equations for $S^{(1)}$ and $S^{(2)}$. As these integrals are dimensionless, we regard them as functions of a single variable $z = \frac{p^2}{m^2}$ and find differential equations

$$\frac{d}{dz}\begin{pmatrix} S^{(1)} \\ S^{(2)} \end{pmatrix} = \frac{\epsilon}{2z}\begin{pmatrix} \frac{(3+5z)}{1-z} & -3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} S^{(1)} \\ S^{(2)} \end{pmatrix}. \tag{5.8}$$

Unlike the system of equations for the bubble, replacing each integral with its maximal cut does not cause any terms to vanish, and so the system for the maximal cuts takes an identical form to that for the uncut integrals:

$$\frac{d}{dz}\begin{pmatrix} \mathcal{C}_{1,2,3}S^{(1)} \\ \mathcal{C}_{1,2,3}S^{(2)} \end{pmatrix} = \frac{\epsilon}{2z}\begin{pmatrix} \frac{(3+5z)}{1-z} & -3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \mathcal{C}_{1,2,3}S^{(1)} \\ \mathcal{C}_{1,2,3}S^{(2)} \end{pmatrix}. \tag{5.9}$$

This is due to the fact that each of these master integrals is associated with the sunset graph itself, unlike the case of the bubble where the second integral belonged to the tadpole subgraph.

By differentiating each of (5.8) and (5.9) a second time and expressing the result using the operator $\theta = z\frac{d}{dz}$ we are able to obtain the results

$$\left[z(\theta+1+2\epsilon)(\theta+1+\epsilon) - \theta(\theta-\epsilon)\right]\frac{1}{1-z}\begin{pmatrix} S^{(1)} \\ \mathcal{C}_{1,2,3}S^{(1)} \end{pmatrix} = 0 \tag{5.10}$$

$$[z(\theta+2\epsilon)(\theta+\epsilon)-\theta(\theta-\epsilon)]\begin{pmatrix} S^{(2)} \\ \mathcal{C}_{1,2,3}S^{(2)} \end{pmatrix}=0.$$

Comparing this to (2.60) and using the well-known general solution of these equations, we are able to write down the space of solutions for the maximal cuts:

$$\begin{aligned}\mathcal{C}_{1,2,3}S^{(1)} =& A(\epsilon)(1-z)\,{}_2F_1\left(1+2\epsilon,1+\epsilon;1-\epsilon;z\right)\\ &+B(\epsilon)(1-z)z^{-1-2\epsilon}\,{}_2F_1\left(1+2\epsilon,1+3\epsilon;1+\epsilon;\frac{1}{z}\right)\\ \mathcal{C}_{1,2,3}S^{(2)} =& C(\epsilon)\,{}_2F_1(2\epsilon,\epsilon;1-\epsilon;z)+D(\epsilon)z^{-2\epsilon}\,{}_2F_1\left(2\epsilon,3\epsilon;1+\epsilon;\frac{1}{z}\right).\end{aligned} \tag{5.11}$$

The solution spaces for the uncut integrals are identical. We will determine the coefficients for the maximal cuts in the following sections by evaluating them as integrals.

5.2.1 Maximal Cuts of $S^{(1)}$

We can also compute these same objects by adopting the parametrisation that was used at one loop (2.31). In order to take the residues we will have to perform this procedure one loop at a time. We will demonstrate this for the case of $S^{(1)}$ first, and $S^{(2)}$ will follow with only a slight modification of the calculation.

To perform the calculation this way we will have to choose which loop to take first in the calculation. There are two such choices available to us: we may either place the massive propagator in the first loop or in the second. We will perform both of these calculations and compare the results, taking first the case where the mass is in the first loop. We number the propagators as in figure 4.1b, and so to determine the cut of $S^{(1)}$ we must evaluate

$$\mathcal{C}_1\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2}\mathcal{C}_{2,3}\int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{k^2[(k+l+p)^2-m^2]}, \tag{5.12}$$

where $\mathcal{C}_{i,j,\ldots,k}$ denotes the operation of replacing integrals over the coordinates by residues at the poles where propagators i, j, $\ldots$ and k vanish. We initially leave the domains of integration over the remaining coordinates unspecified, and will define particular cut contours by selecting domains afterwards. The cut of the inner loop is a one-loop calculation and can be performed using the method described in 2.3.1 to find the result

$$\begin{aligned}
&\mathcal{C}_{1,2}\int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{k^2[(k+p)^2-m^2]} \\
=&\frac{1}{i\pi^{D/2}}\frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}\text{Res}_{k_0=\frac{m^2-p^2}{2\sqrt{p^2}}}\text{Res}_{\beta=1}\int dk_0k_0^{D-1}\int d\beta\beta^{D-2} \\
&\times\frac{1}{k_0^2(1-\beta^2)\left[p^2-m^2+2k_0\sqrt{p^2}\right]} \\
=&\frac{2}{2\pi i}\frac{\Gamma(D/2)}{\Gamma(D-1)}(p^2-m^2)^{D-3}(p^2)^{1-D/2}.
\end{aligned}\tag{5.13}$$

Using this one-loop expression, we can find the cut

$$\mathcal{C}_{2,3}\int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{k^2[(k+l+p)^2-m^2]}\tag{5.14}$$

of the sunset inner loop by making the replacement $p\to l+p$ in (5.13). The cut (5.12) then becomes

$$\begin{aligned}
&\frac{2}{2\pi i}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\mathcal{C}_1\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2}[(l+p)^2]^{\epsilon}[(l+p)^2-m^2]^{-1-2\epsilon} \\
=&\frac{2}{2\pi i}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{2\pi^{\frac{1-2\epsilon}{2}}}{\Gamma\left(\frac{1-2\epsilon}{2}\right)}\text{Res}_{\beta=1}\frac{1}{i\pi^{D/2}}\int dl_0l_0^{1-2\epsilon}\int d\beta\beta^{-2\epsilon}\frac{1}{l_0^2(1-\beta^2)} \\
&\times\left[l_0^2(1-\beta^2)+p^2+2\sqrt{p^2}l_0\right]^{\epsilon}\left[l_0^2(1-\beta^2)+p^2+2\sqrt{p^2}l_0-m^2\right]^{-1-2\epsilon} \\
=&-\frac{\Gamma^2(1-\epsilon)}{\Gamma^2(1-2\epsilon)}\frac{2^{2-2\epsilon}}{(2\pi i)^2}\int dl_0l_0^{-1-2\epsilon}\left(p^2+2\sqrt{p^2}l_0\right)^{\epsilon}\left(p^2-m^2+2\sqrt{p^2}l_0\right)^{-1-2\epsilon}.
\end{aligned}\tag{5.15}$$

In order to reproduce the result of (5.11) we will select a pair of integration contours: in the first case we define the l_0 integration to take place over the domain $-\frac{\sqrt{p^2}}{2}<l_0<0$, while in the second we choose $-\frac{p^2-m^2}{2\sqrt{p^2}}<l_0<0$. Any other suitable domains of integration must then result in cuts which are linearly dependent on those produced by this selection. With these choices, we get the results

$$
\begin{aligned}
&\mathcal{C}_1 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{l^2} \mathcal{C}_{2,3} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2[(k+l+p)^2 - m^2]} \qquad (5.16)\\
=& -\frac{\Gamma^2(1-\epsilon)}{\Gamma^2(1-2\epsilon)} \frac{2^{2-2\epsilon}}{(2\pi i)^2} \left\{ \begin{array}{l} \int_{-\frac{\sqrt{p^2}}{2}}^{0} dl_0 l_0^{-1-2\epsilon} \left(p^2 + 2\sqrt{p^2} l_0\right)^{\epsilon} \left(p^2 - m^2 + 2\sqrt{p^2} l_0\right)^{-1-2\epsilon} \\ \int_{-\frac{p^2-m^2}{2\sqrt{p^2}}}^{0} dl_0 l_0^{-1-2\epsilon} \left(p^2 + 2\sqrt{p^2} l_0\right)^{\epsilon} \left(p^2 - m^2 + 2\sqrt{p^2} l_0\right)^{-1-2\epsilon} \end{array} \right. \\
=& \frac{\Gamma^2(1-\epsilon)}{\Gamma^2(1-2\epsilon)} \frac{4}{(2\pi i)^2} \left\{ \begin{array}{l} (p^2 - m^2)^{-1-2\epsilon} \int_0^1 dl_0 l_0^{-1-2\epsilon} (1-l_0)^{\epsilon} \left(1 - \frac{p^2}{p^2-m^2} l_0\right)^{-1-2\epsilon} \\ (p^2)^{2\epsilon} (p^2 - m^2)^{-1-4\epsilon} \int_0^1 dl_0 l_0^{-1-2\epsilon} (1-l_0)^{-1-2\epsilon} \left[1 - \left(1 - \frac{m^2}{p^2}\right) l_0\right]^{\epsilon} \end{array} \right. \\
=& -\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{2}{\epsilon(2\pi i)^2} \left\{ \begin{array}{l} (p^2 - m^2)^{-1-2\epsilon} \frac{\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)} {}_2F_1\left(-2\epsilon, 1+2\epsilon; 1-\epsilon; \frac{p^2}{p^2-m^2}\right) \\ 2(p^2)^{2\epsilon} (p^2 - m^2)^{-1-4\epsilon} \frac{\Gamma(1-2\epsilon)}{\Gamma(1-4\epsilon)} {}_2F_1\left(-2\epsilon, -\epsilon; -4\epsilon; 1 - \frac{m^2}{p^2}\right) \end{array} \right. .
\end{aligned}
$$

Including the prefactors $\epsilon^2 \left(\frac{p^2}{m^2} - 1\right) (m^2)^{1+2\epsilon} e^{2\gamma_E \epsilon}$ which appear in the definition (4.13) and normalising by $(2\pi i)^2$ we now have a pair of maximal cuts of $S^{(1)}$:

$$
\begin{aligned}
&\mathcal{C}'^{(1)}_{1,2,3} S^{(1)} \qquad (5.17)\\
=& -2\epsilon e^{2\gamma_E \epsilon} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} (z-1)^{-2\epsilon} {}_2F_1\left(-2\epsilon, 1+2\epsilon; 1-\epsilon; \frac{z}{z-1}\right) \\
=& -2\epsilon e^{2\gamma_E \epsilon} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} (1-z) {}_2F_1\left(1+\epsilon, 1+2\epsilon; 1-\epsilon; z\right) \\
&\mathcal{C}'^{(2)}_{1,2,3} S^{(1)} \\
=& -4\epsilon e^{2\gamma_E \epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-4\epsilon)} z^{2\epsilon} (z-1)^{-4\epsilon} {}_2F_1\left(-2\epsilon, -\epsilon; -4\epsilon; 1 - \frac{1}{z}\right) \\
=& -\epsilon e^{2\gamma_E \epsilon} (1-z) \left\{ \frac{1}{\Gamma(1-2\epsilon)} {}_2F_1\left(1+2\epsilon, 1+\epsilon; 1-\epsilon; z\right) \right. \\
& \left. -3z^{-1-2\epsilon} \frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)} {}_2F_1\left(1+2\epsilon, 1+3\epsilon; 1+\epsilon; \frac{1}{z}\right) \right\},
\end{aligned}
$$

where we have related the results to the functions appearing in (5.11) by using (2.57) and analytic continuation formulae such as (2.54).

We will regard these calculations as defining contours $\Gamma'^{(1)}_{1,2,3}$ and $\Gamma'^{(2)}_{1,2,3}$ so that our cuts are given by

$$\mathcal{C}'^{(1)}_{1,2,3}S^{(1)} = \int_{\Gamma'^{(1)}_{1,2,3}} \omega_{S^{(1)}} \qquad (5.18)$$
$$\mathcal{C}'^{(2)}_{1,2,3}S^{(1)} = \int_{\Gamma'^{(2)}_{1,2,3}} \omega_{S^{(1)}},$$

where the form $\omega_{S^{(1)}}$ is

$$\begin{aligned}&\omega_{S^{(1)}} \\ =&\epsilon^2\left(\frac{p^2}{m^2}-1\right)(m^2)^{1+2\epsilon}e^{2\gamma_E\epsilon}\frac{1}{(i\pi^{1-\epsilon})^2}d^Dk\wedge d^Dl\frac{1}{k^2l^2[(k+l+p)^2-m^2]}.\end{aligned} \qquad (5.19)$$

Comparing these cuts to (5.11) we see that the selection of two independent contours of integration in (5.16) has allowed us to recover the maximal cuts of the sunset. An alternative method to compute these cuts is to make a different choice of loop ordering such that the integration over the two massless propagators is performed first. With this choice we find that one of the cuts is given by

$$\begin{aligned}&\mathcal{C}_3\int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{k^2-m^2}\mathcal{C}_{1,2}\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2[(k+l+p)^2]} \\ =&\frac{2}{2\pi i}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\mathcal{C}_3\int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{k^2-m^2}[(k+p)^2]^{-1-\epsilon} \\ =&-\frac{2}{2\pi i}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{\sqrt{\pi}\Gamma(\frac{1}{2}-\epsilon)}\text{Res}_{|k^E|^2=-m^2}\int d|k^E|^2\left(|k^E|^2\right)^{-\epsilon} \\ &\times\int_0^\pi d\theta\,\sin^{-2\epsilon}\theta\frac{(-|k^E|^2-|p^E|^2-2|k^E||p^E|\cos\theta)^{-1-\epsilon}}{|k^E|^2+m^2} \\ =&-\frac{1}{2\pi i}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{2^{1-2\epsilon}\left(m^2\right)^{-\epsilon}}{\sqrt{\pi}\Gamma(\frac{1}{2}-\epsilon)}\int_0^1 dx[x(1-x)]^{-\frac{1}{2}-\epsilon}[m^2+p^2-2\sqrt{m^2p^2}(2x-1)]^{-1-\epsilon}\end{aligned} \qquad (5.20)$$

$$
\begin{aligned}
&= -\frac{1}{2\pi i}\frac{2\,(m^2)^{-\epsilon}}{\Gamma(1-2\epsilon)}[(\sqrt{m^2}+\sqrt{p^2})^2]^{-1-\epsilon}{}_2F_1\left(\frac{1}{2}-\epsilon, 1+\epsilon; 1-2\epsilon; \frac{4\sqrt{m^2p^2}}{(\sqrt{m^2}+\sqrt{p^2})^2}\right)\\
&= -\frac{1}{2\pi i}\frac{2\,(m^2)^{-1-2\epsilon}}{\Gamma(1-2\epsilon)}{}_2F_1\left(1+\epsilon, 1+2\epsilon; 1-\epsilon; \frac{p^2}{m^2}\right).
\end{aligned}
$$

We normalise this result by $2\pi i$ and write down a cut of $S^{(1)}$:

$$
\mathcal{C}''^{(1)}S^{(1)} = -\,2\epsilon^2 e^{2\gamma_E\epsilon}\,(z-1)\,\frac{1}{\Gamma(1-2\epsilon)}{}_2F_1\left(1+\epsilon, 1+2\epsilon; 1-\epsilon; z\right). \tag{5.21}
$$

This cut agrees modulo $i\pi$ with that found by the other loop ordering, up to a normalisation factor of ϵ. The second cut $\mathcal{C}''^{(2)}S^{(1)}$ can then be written down to match $\mathcal{C}'^{(2)}S^{(1)}$ up to this same normalising factor by a suitable change to the integration domain in the definition of the ${}_2F_1$ which appears in (5.21). This second cut can be found by replacing the domain $0 < x < 1$ of the x integration in (5.20), but it is computationally simpler to take the final result and use it to infer the other cut. We conclude that, in this example, the choice of the loop ordering does not affect the space spanned by the two cuts.

5.2.2 Maximal Cuts of $S^{(2)}$

We will also require the cuts of the second master integral $S^{(2)}$. Starting with the integral $S(1,1,1,-1,0;2-2\epsilon;p^2,m^2)$ we need to compute the cut

$$
\mathcal{C}_1\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2}\mathcal{C}_{2,3}\int\frac{d^Dk}{i\pi^{D/2}}\frac{(k+l)^2}{k^2[(k+l+p)^2-m^2]}. \tag{5.22}
$$

We may take l to be null and adopt the parametrisation

$$
\begin{aligned}
l+p =& \sqrt{(l+p)^2}(1,\underline{0}_{D-1}) \\
l =& \frac{l.p}{\sqrt{(l+p)^2}}(1,1,\underline{0}_{D-2}),
\end{aligned} \tag{5.23}
$$

from which it follows that

$$
\begin{aligned}
&\mathcal{C}_{2,3}\int\frac{d^Dk}{i\pi^{D/2}}\frac{(k+l)^2}{k^2[(k+l+p)^2-m^2]} \\
=&\frac{1}{i\pi^{1-\epsilon}}\frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)}\mathrm{Res}_{k_0=\frac{m^2-(l+p)^2}{2\sqrt{(l+p)^2}}}\mathrm{Res}_{\beta=1}\int dk_0k_0^{1-2\epsilon}\int d\beta\beta^{-2\epsilon}\int_0^\pi d\theta\sin^{-1-2\epsilon}\theta \\
&\times\frac{k_0^2(1-\beta^2)+2k_0\frac{l.p}{\sqrt{(l+p)^2}}(1-\beta\cos\theta)}{k_0^2(1-\beta^2)\left[k_0^2(1-\beta^2)+(l+p)^2-m^2+2k_0\sqrt{(l+p)^2}\right]} \\
=&-2l.p\frac{1}{2\pi i}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}[m^2-(l+p)^2]^{-2\epsilon}[(l+p)^2]^{-1+\epsilon}.
\end{aligned}
\tag{5.24}
$$

Then we may evaluate (5.22):

$$
\begin{aligned}
&\mathcal{C}_1\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2}\mathcal{C}_{2,3}\int\frac{d^Dk}{i\pi^{D/2}}\frac{(k+l)^2}{k^2[(k+l+p)^2-m^2]} \\
=&-\frac{2}{2\pi i}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{i\pi^{1-\epsilon}}\frac{2\pi^{\frac{1}{2}-\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)}\mathrm{Res}_{\beta=1}\int dl_0l_0^{1-2\epsilon}\int d\beta\beta^{-2\epsilon} \\
&\times\frac{l_0\sqrt{p^2}}{l_0^2(1-\beta^2)}[m^2-(l+p)^2]^{-2\epsilon}[(l+p)^2]^{-1+\epsilon} \\
=&\frac{1}{(2\pi i)^2}\frac{\Gamma^2(1-\epsilon)}{\Gamma^2(1-2\epsilon)}2^{2-2\epsilon}\sqrt{p^2}\int dl_0l_0^{-2\epsilon}\left(p^2+2l_0\sqrt{p^2}\right)^{-1+\epsilon}\left(m^2-p^2-2l_0\sqrt{p^2}\right)^{-2\epsilon}.
\end{aligned}
\tag{5.25}
$$

We will follow the prescription given for the cuts of $S^{(1)}$ and perform this cut with domains of integration $-\frac{\sqrt{p^2}}{2}<l_0<0$ and $\frac{m^2-p^2}{2\sqrt{p^2}}<l_0<0$. This produces cuts

$$
\begin{aligned}
&\mathcal{C}_1\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2}\mathcal{C}_{2,3}\int\frac{d^Dk}{i\pi^{D/2}}\frac{(k+l)^2}{k^2[(k+l+p)^2-m^2]} \\
=&\frac{1}{(2\pi i)^2}\frac{\Gamma^2(1-\epsilon)}{\Gamma^2(1-2\epsilon)}2^{2-2\epsilon}\sqrt{p^2}\left\{\begin{array}{l}\displaystyle\int_{-\frac{\sqrt{p^2}}{2}}^0 dl_0l_0^{-2\epsilon}\left(p^2+2l_0\sqrt{p^2}\right)^{-1+\epsilon}\left(m^2-p^2-2l_0\sqrt{p^2}\right)^{-2\epsilon}\\ \displaystyle\int_{\frac{m^2-p^2}{2\sqrt{p^2}}}^0 dl_0l_0^{-2\epsilon}\left(p^2+2l_0\sqrt{p^2}\right)^{-1+\epsilon}\left(m^2-p^2-2l_0\sqrt{p^2}\right)^{-2\epsilon}\end{array}\right.
\end{aligned}
\tag{5.26}
$$

$$=\frac{1}{(2\pi i)^2}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-4\epsilon)}\frac{2}{\epsilon}\begin{cases} (p^2-m^2)^{-2\epsilon}{}_2F_1\left(1-2\epsilon,2\epsilon;1-\epsilon;\frac{p^2}{p^2-m^2}\right)\frac{\Gamma(1-4\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)} \\ (p^2)^{-1+2\epsilon}(p^2-m^2)^{1-4\epsilon}\frac{\epsilon}{1-4\epsilon}{}_2F_1\left(1-2\epsilon,1-\epsilon;2-4\epsilon;1-\frac{m^2}{p^2}\right) \end{cases}.$$

Finally, we normalise these cuts by $(2\pi i)^2$ and include the prefactors which are present in the definition of $S^{(2)}$ to obtain its cuts:

$$\begin{aligned}
&\mathcal{C}'^{(1)}_{1,2,3}S^{(2)} \\
=&-2\epsilon e^{2\gamma_E\epsilon}\frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}(z-1)^{-2\epsilon}{}_2F_1\left(1-2\epsilon,2\epsilon;1-\epsilon;\frac{z}{z-1}\right) \\
=&-2\epsilon e^{2\gamma_E\epsilon}\frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}{}_2F_1\left(\epsilon,2\epsilon;1-\epsilon;z\right) \\
&\mathcal{C}'^{(2)}_{1,2,3}S^{(2)} \\
=&-2\epsilon e^{2\gamma_E\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-4\epsilon)}z^{-1+2\epsilon}(z-1)^{1-4\epsilon}\frac{\epsilon}{1-4\epsilon}{}_2F_1\left(1-2\epsilon,1-\epsilon;2-4\epsilon;1-\frac{1}{z}\right) \\
=&\epsilon e^{2\gamma_E\epsilon}\left[-\Gamma(1+2\epsilon){}_2F_1\left(2\epsilon,\epsilon;1-\epsilon;z\right)+\frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}z^{-2\epsilon}{}_2F_1\left(2\epsilon,3\epsilon;1+\epsilon;\frac{1}{z}\right)\right].
\end{aligned} \tag{5.27}$$

We note that these cuts of $S^{(2)}$ have been computed by taking the residues in an identical manner to $\mathcal{C}'^{(1)}_{1,2,3}S^{(1)}$ and $\mathcal{C}'^{(2)}_{1,2,3}S^{(1)}$, and then integrating over the same domains in l_0 as these cuts, and so we can write

$$\begin{aligned}
\mathcal{C}'^{(1)}_{1,2,3}S^{(2)} &= \int_{\Gamma'^{(1)}_{1,2,3}}\omega_{S^{(2)}} \\
\mathcal{C}'^{(2)}_{1,2,3}S^{(2)} &= \int_{\Gamma'^{(2)}_{1,2,3}}\omega_{S^{(2)}},
\end{aligned} \tag{5.28}$$

with the contours from (5.18) and the form

$$\omega_{S^{(2)}} = -\epsilon^2(m^2)^{2\epsilon}e^{2\gamma_E\epsilon}\frac{1}{(i\pi^{1-\epsilon})^2}d^Dk\wedge d^Dl\frac{(k+l)^2}{k^2l^2[(k+l+p)^2-m^2]}. \tag{5.29}$$

5.2.3 Maximal Cuts Dual to $S^{(1)}$ and $S^{(2)}$

We have already established in section 2.5 that the formula (2.47) along with duality condition (2.48) describes the coaction of one-loop Feynman integrals and so we would like to express our two-loop examples in the same formalism. In the present example, we have a two-dimensional space of forms spanned by $\omega_{S^{(1)}}$ and $\omega_{S^{(2)}}$ which are associated with our chosen master integrals $S^{(1)}$ and $S^{(2)}$. The contours $\Gamma'^{(1)}_{1,2,3}$ and $\Gamma'^{(2)}_{1,2,3}$ that we have found in this section are not dual to $\omega_{S^{(1)}}$ and $\omega_{S^{(2)}}$, as we can find by expanding (5.17) and (5.27) that the period matrix of our system is

$$\begin{pmatrix} \int_{\Gamma'^{(1)}_{1,2,3}} \omega_{S^{(1)}} & \int_{\Gamma'^{(1)}_{1,2,3}} \omega_{S^{(2)}} \\ \int_{\Gamma'^{(2)}_{1,2,3}} \omega_{S^{(1)}} & \int_{\Gamma'^{(2)}_{1,2,3}} \omega_{S^{(2)}} \end{pmatrix} = -2\epsilon \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} + \mathcal{O}(\epsilon^2). \tag{5.30}$$

It is then clear that we can define contours

$$\begin{aligned} \Gamma^{(1)}_{1,2,3} &= -\frac{1}{4\epsilon}\Gamma'^{(2)}_{1,2,3} \\ \Gamma^{(2)}_{1,2,3} &= -\frac{1}{2\epsilon}\left(\Gamma'^{(1)}_{1,2,3} - \frac{1}{2}\Gamma'^{(2)}_{1,2,3}\right) \end{aligned} \tag{5.31}$$

which are dual to $\omega_{S^{(1)}}$ and $\omega_{S^{(2)}}$. We can explain this diagonalisation by considering the exponents of the various factors in the integrand in (5.16) and (5.26). Integrating between the two limits $l_0 = 0$ and $l_0 = \frac{m^2-p^2}{2\sqrt{p^2}}$ we touch an endpoint singularity regularised by ϵ for $S^{(1)}$ but not for $S^{(2)}$ and so the contour $\Gamma'^{(2)}_{1,2,3}$ is proportional to $\Gamma^{(1)}_{1,2,3}$. Meanwhile, when the integration occurs between $l_0 = 0$ and $l_0 = -\frac{\sqrt{p^2}}{2}$ we obtain non-zero terms at lowest order for both master integrals and so must take a linear combination of the two contours to find $\Gamma^{(2)}_{1,2,3}$. This procedure is similar to the construction of the contours for the ${}_2F_1$ coaction given in section 3.1.1, but now one of our integrands no longer takes such a simple form.

Finally, we will also trivially define, in the notation of (2.47), $\omega_1 = \omega_{S^{(1)}}$, $\omega_2 = \omega_{S^{(2)}}$, $\gamma_1 = \Gamma^{(1)}_{1,2,3}$ and $\gamma_2 = \Gamma^{(2)}_{1,2,3}$. Generally, we will find that the contours dual to our forms $\{\omega_i\}$ are not simply individual cut contours as in this case, but linear combinations of these contours. This was already the case at one loop, where the deformation terms could be interpreted in this manner.

5.2.4 Single Cut and Discontinuities of the One-Mass Sunset

We conclude by discussing the cut on the single massive propagator, which we will illustrate for $S^{(1)}$. Although these cuts will not prove necessary for the diagrammatic coaction, they express the discontinuities of the sunset integrals with respect to m^2 and we will wish to examine how this is encoded in the coaction. The most direct way to compute the single cut $\mathcal{C}_3 S^{(1)}$ which expresses the mass discontinuity will be to place the massive propagator in the outer loop:

$$\begin{aligned}&\mathcal{C}_3 \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2 - m^2} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{l^2 (k+l+p)^2} \\ =&\frac{2}{\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\mathcal{C}_3 \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2 - m^2} \frac{1}{[(k+p)^2]^{1+\epsilon}}.\end{aligned} \tag{5.32}$$

Using the one-loop result for the single cut of a one-mass bubble where a residue is taken at $k^2 = m^2$ and the remainder of the space is integrated over:

$$\begin{aligned}&\mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2-m^2)[(k+p)^2]^b} \\ =&-i\frac{\pi^{\frac{1}{2}-\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)} \frac{1}{i\pi^{1-\epsilon}} \mathrm{Res}_{|k^E|^2=-m^2} \int d|k^E|^2 \left(|k^E|^2\right)^{-\epsilon} \int_0^\pi d\theta \sin^{-2\epsilon}\theta \\ &\times \frac{1}{(|k^E|^2+m^2)\left(-|k^E|^2-|p^E|^2-2|k^E||p^E|\cos\theta\right)^b} \\ =&-\Gamma(1+\epsilon)\frac{\left(m^2\right)^{-\epsilon}}{(p^2)^b} {}_2F_1\left(b, b+\epsilon; 1-\epsilon; \frac{m^2}{p^2}\right),\end{aligned} \tag{5.33}$$

we then find that

$$\mathcal{C}_3 \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2 - m^2} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{l^2 (k+l+p)^2} \tag{5.34}$$

$$= -\frac{2}{\epsilon}\Gamma(1+2\epsilon)\left(m^2\right)^{-\epsilon}(p^2)^{-1-\epsilon}{}_2F_1\left(1+\epsilon,1+2\epsilon;1-\epsilon;\frac{m^2}{p^2}\right),$$

from which we can immediately write down

$$\begin{aligned}\mathcal{C}_3 S^{(1)} =& -2\epsilon(z-1)e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)z^{-1-\epsilon}{}_2F_1\left(1+\epsilon,1+2\epsilon;1-\epsilon;\frac{1}{z}\right) \\ =& -\epsilon(z-1)e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Bigg[{}_2F_1\left(1+\epsilon,1+2\epsilon;1-\epsilon;z\right) \\ & +3\frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)}z^{-1-2\epsilon}{}_2F_1\left(1+3\epsilon,1+2\epsilon;1+\epsilon;\frac{1}{z}\right)\Bigg].\end{aligned} \tag{5.35}$$

We can also use this example to explore the conjecture that the outcome of a cut calculation does not depend on the chosen ordering of the loops by evaluating the cut

$$\begin{aligned}&\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2}\mathcal{C}_3\int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{(k^2-m_1^2)(k+l+p)^2} \\ =& -\Gamma(1+\epsilon)(m^2)^{-\epsilon}\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2}\frac{1}{(l+p)^2}{}_2F_1\left(1,1+\epsilon;1-\epsilon;\frac{m_1^2}{(l+p)^2}\right) \\ =&2\epsilon\Gamma(1-\epsilon)\Gamma(1+2\epsilon)(m^2)^{-\epsilon}\int_0^1 du\,u^\epsilon(1-u)^{-1-2\epsilon}\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2[(l+p)^2-um^2]} \\ =&2\Gamma(1+2\epsilon)(m^2)^{-1-2\epsilon}\int_0^1 du\,u^{-1}(1-u)^{-1-2\epsilon}{}_2F_1\left(1,1+\epsilon;1-\epsilon;\frac{p^2}{um^2}\right) \\ =& -2\epsilon\Gamma(1+2\epsilon)(m^2)^{-1-2\epsilon}\int_0^1 du\,u^{-1}(1-u)^{-1-2\epsilon}\int_0^1 dv(1-v)^{-1-\epsilon}\left(1-v\frac{p^2}{um^2}\right)^{-1-\epsilon}.\end{aligned} \tag{5.36}$$

In order to obtain the same cut as we computed in (5.34), we must replace the domain of integration over the parameter v by $[\frac{um^2}{p^2},1]$. With this change we find that

$$\begin{aligned}&\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2}\mathcal{C}_3\int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{(k^2-m_1^2)(k+l+p)^2} \\ =& -2\epsilon\Gamma(1+2\epsilon)(m^2)^{-1-2\epsilon}\int_0^1 du\,u^{-1}(1-u)^{-1-2\epsilon}\int_0^{1-\frac{um^2}{p^2}} dv\,v^{-1-\epsilon}\left(1-(1-v)\frac{p^2}{um^2}\right)^{-1-\epsilon}\end{aligned} \tag{5.37}$$

$$=4\Gamma^2(1+2\epsilon)\Gamma^2(1-\epsilon)(m^2)^{-\epsilon}(p^2)^{-1-\epsilon}\int_0^1 du\, u^{\epsilon}(1-u)^{-1-2\epsilon}\left(1-\frac{um^2}{p^2}\right)^{-1-2\epsilon}$$
$$=-\frac{2}{\epsilon}\Gamma(1+2\epsilon)(m^2)^{-\epsilon}(p^2)^{-1-\epsilon}{}_2F_1\left(1+\epsilon,1+2\epsilon;1-\epsilon;\frac{m^2}{p^2}\right).$$

This example demonstrates that performing the cut with a different loop ordering need not always return the same result, although with a suitable modification of the integration contour we can once again obtain the cut (5.34).

Of course, the single cut is not linearly independent of the two maximal cuts and the uncut integral. Each of these four objects obeys the same second-order differential equation which we stated for the uncut integral and maximal cuts in (5.10). This equation has only two independent solutions and so given any three of the four integrals there must exist a linear relation among them.

One such relation can be derived by computing the discontinuities of

$$\int \frac{d^D k}{i\pi^{D/2}}\int \frac{d^D l}{i\pi^{D/2}}\frac{1}{k^2 l^2[(k+l+p)^2-m^2]} \tag{5.38}$$

from its parameter integral form:

$$-\frac{1}{\epsilon^2}\frac{\Gamma(1+2\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(-2\epsilon)}\int_0^1 du\, u^{\epsilon}(1-u)^{-1-2\epsilon}(m^2-up^2)^{-1-2\epsilon}. \tag{5.39}$$

We will use the fact that

$$\text{Disc}(z^{a\epsilon}) = 2\pi i\frac{a\epsilon}{\Gamma(1-a\epsilon)\Gamma(1+a\epsilon)}|z|^{a\epsilon}\theta(-z). \tag{5.40}$$

It follows that in the kinematic region $p^2 < m^2 < 0$, $(m^2-up^2)^{-2\epsilon}$ has a discontinuity when $u < \frac{m^2}{p^2}$, while in the region $0 < m^2 < p^2$ this same factor has a discontinuity when $u > \frac{m^2}{p^2}$. The integral (5.39) then has discontinuities in these same regions given by restricting the domain of integration to that specified by the above inequalities on u, as one can demonstrate from the coaction of the ${}_2F_1$ function. When $p^2 < m^2 < 0$

we find the result

$$\begin{aligned} &2\pi i \frac{2}{\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(-2\epsilon)}\int_0^{\frac{m^2}{p^2}} du\, u^\epsilon (1-u)^{-1-2\epsilon}(m^2-up^2)^{-1-2\epsilon} \\ =&2\pi i \frac{2}{\epsilon}\Gamma(1+2\epsilon)(m^2)^{-\epsilon}(p^2)^{-1-\epsilon}{}_2F_1\left(1+\epsilon, 1+2\epsilon; 1-\epsilon; \frac{m^2}{p^2}\right) \end{aligned} \tag{5.41}$$

while for $0 < m^2 < p^2$ we obtain

$$\begin{aligned} &2\pi i \frac{2}{\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(-2\epsilon)}\int_{\frac{m^2}{p^2}}^{1} du\, u^\epsilon (1-u)^{-1-2\epsilon}(m^2-up^2)^{-1-2\epsilon} \\ =&2\pi i \frac{2}{\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma^2(1-\epsilon)\Gamma(-2\epsilon)}{\Gamma(-4\epsilon)}(p^2)^{2\epsilon}(m^2-p^2)^{-1-4\epsilon}{}_2F_1\left(-2\epsilon, -\epsilon; -4\epsilon; 1-\frac{m^2}{p^2}\right). \end{aligned} \tag{5.42}$$

It is clear that the momentum discontinuity should be given by the cut with integration domain $-\frac{p^2-m^2}{2\sqrt{p^2}} < l_0 < 0$ in (5.16) as this domain corresponds to the conditions $(l+p)^2 > m^2$ and $l_0 < 0$ which one imposes in evaluating the discontinuity as a unitarity cut. Comparing the above to (5.16) we find that this is indeed the case, while we see from (5.34) that the mass discontinuity is given by the single cut up to a sign.

We also notice that the sum of the integrals (5.41) and (5.42) is equal to the product of -2ϵ and the integral (5.39). Stated in terms of $S^{(1)}$ and its cuts this result is

$$2\epsilon\, S^{(1)} = \mathcal{C}_3 S^{(1)} + 4\epsilon\, \mathcal{C}^{(1)}_{1,2,3} S^{(1)}, \tag{5.43}$$

where the factor of 4ϵ originates from our definition of the cut $\mathcal{C}^{(1)}_{1,2,3}$. A similar analysis for $S^{(2)}$ produces the same relation:

$$2\epsilon\, S^{(2)} = \mathcal{C}_3 S^{(2)} + 4\epsilon\, \mathcal{C}^{(1)}_{1,2,3} S^{(2)}, \tag{5.44}$$

where the cut $\mathcal{C}_3 S^{(2)}$ is given by

$$\mathcal{C}_3 S^{(2)} = -2\epsilon e^{2\gamma_E \epsilon} \frac{\Gamma(1+2\epsilon)\Gamma(1+3\epsilon)}{\Gamma^3(1+\epsilon)} z^{-2\epsilon} {}_2F_1\left(2\epsilon, 3\epsilon; 1+\epsilon; \frac{1}{z}\right). \tag{5.45}$$

For completeness we also include the relations

$$\begin{aligned} S^{(1)} =& \mathcal{C}^{(1)}_{1,2,3} S^{(1)} + \mathcal{C}^{(2)}_{1,2,3} S^{(1)} \\ S^{(2)} =& \mathcal{C}^{(1)}_{1,2,3} S^{(2)} + \mathcal{C}^{(2)}_{1,2,3} S^{(2)} \end{aligned} \tag{5.46}$$

which will be explained by the diagrammatic coaction in the following chapter.

5.3 The Two-Mass Sunset

The sunset integral with two internal masses combines the features of the one-mass case of the previous section with our bubble example in 5.1: there are multiple maximal cuts, but we also have a master integral from the product of tadpoles subgraph and so will want to find a non-maximal cut where only the two massive propagators are placed on shell.

We again start by using the differential equations for these objects to find the spaces occupied by their cuts. Let us regard the three master integrals $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ defined in (4.4) and (4.6), and the master integral J of (4.11) as functions of the variables $z_1 = \frac{m_1^2}{p^2}$ and $z_2 = \frac{m_2^2}{p^2}$. Then the differential equations with respect to these variables obeyed by each integral are

$$\theta \begin{pmatrix} S^{(1)} \\ S^{(2)} \\ S^{(3)} \\ J \end{pmatrix} = \mathcal{A} \begin{pmatrix} S^{(1)} \\ S^{(2)} \\ S^{(3)} \\ J \end{pmatrix} \tag{5.47}$$

$$\phi \begin{pmatrix} S^{(1)} \\ S^{(2)} \\ S^{(3)} \\ J \end{pmatrix} = \mathcal{B} \begin{pmatrix} S^{(1)} \\ S^{(2)} \\ S^{(3)} \\ J \end{pmatrix},$$

where $\theta = z_1 \frac{\partial}{\partial z_1}$, $\phi = z_2 \frac{\partial}{\partial z_2}$ and the matrices $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{aligned} \mathcal{A} =& \epsilon \begin{pmatrix} \frac{1-5z_1^2+z_2^2+4z_1-2z_2+4z_1z_2}{\lambda(1,z_1,z_2)} & 2\frac{1-z_2}{\sqrt{\lambda(1,z_1,z_2)}} & \frac{1+3z_1-z_2}{\sqrt{\lambda(1,z_1,z_2)}} & \frac{1+z_1-z_2}{\sqrt{\lambda(1,z_1,z_2)}} \\ -1+z_1+z_2 & -2 & -1 & -1 \\ -2z_1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ \mathcal{B} =& \epsilon \begin{pmatrix} \frac{1+z_1^2-5z_2^2-2z_1+4z_2+4z_1z_2}{\lambda(1,z_1,z_2)} & \frac{1-z_1+3z_2}{\sqrt{\lambda(1,z_1,z_2)}} & 2\frac{1-z_1}{\sqrt{\lambda(1,z_1,z_2)}} & \frac{1-z_1+z_2}{\sqrt{\lambda(1,z_1,z_2)}} \\ -2z_2 & 0 & 0 & 1 \\ -1+z_1+z_2 & -1 & -2 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned} \tag{5.48}$$

We can immediately write down the systems of equations obeyed by the cuts $\mathcal{C}_{1,2}$ of each of these four integrals:

$$\begin{aligned} \theta \begin{pmatrix} \mathcal{C}_{1,2}S^{(1)} \\ \mathcal{C}_{1,2}S^{(2)} \\ \mathcal{C}_{1,2}S^{(3)} \\ \mathcal{C}_{1,2}J \end{pmatrix} =& \mathcal{A} \begin{pmatrix} \mathcal{C}_{1,2}S^{(1)} \\ \mathcal{C}_{1,2}S^{(2)} \\ \mathcal{C}_{1,2}S^{(3)} \\ \mathcal{C}_{1,2}J \end{pmatrix} \\ \phi \begin{pmatrix} \mathcal{C}_{1,2}S^{(1)} \\ \mathcal{C}_{1,2}S^{(2)} \\ \mathcal{C}_{1,2}S^{(3)} \\ \mathcal{C}_{1,2}J \end{pmatrix} =& \mathcal{B} \begin{pmatrix} \mathcal{C}_{1,2}S^{(1)} \\ \mathcal{C}_{1,2}S^{(2)} \\ \mathcal{C}_{1,2}S^{(3)} \\ \mathcal{C}_{1,2}J \end{pmatrix}. \end{aligned} \tag{5.49}$$

Meanwhile, the maximal cuts of the integrals $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ are described by the simpler systems of equations

$$\begin{aligned}
\theta \begin{pmatrix} \mathcal{C}_{1,2,3}S^{(1)} \\ \mathcal{C}_{1,2,3}S^{(2)} \\ \mathcal{C}_{1,2,3}S^{(3)} \end{pmatrix} &= \tilde{\mathcal{A}} \begin{pmatrix} \mathcal{C}_{1,2,3}S^{(1)} \\ \mathcal{C}_{1,2,3}S^{(2)} \\ \mathcal{C}_{1,2,3}S^{(3)} \end{pmatrix} \\
\phi \begin{pmatrix} \mathcal{C}_{1,2,3}S^{(1)} \\ \mathcal{C}_{1,2,3}S^{(2)} \\ \mathcal{C}_{1,2,3}S^{(3)} \end{pmatrix} &= \tilde{\mathcal{B}} \begin{pmatrix} \mathcal{C}_{1,2,3}S^{(1)} \\ \mathcal{C}_{1,2,3}S^{(2)} \\ \mathcal{C}_{1,2,3}S^{(3)} \end{pmatrix},
\end{aligned} \tag{5.50}$$

with

$$\begin{aligned}
\tilde{\mathcal{A}} &= \epsilon \begin{pmatrix} \frac{1-5z_1^2+z_2^2+4z_1-2z_2+4z_1z_2}{\lambda(1,z_1,z_2)} & 2\frac{1-z_2}{\sqrt{\lambda(1,z_1,z_2)}} & \frac{1+3z_1-z_2}{\sqrt{\lambda(1,z_1,z_2)}} \\ -1+z_1+z_2 & -2 & -1 \\ -2z_1 & 0 & 0 \end{pmatrix} \\
\tilde{\mathcal{B}} &= \epsilon \begin{pmatrix} \frac{1+z_1^2-5z_2^2-2z_1+4z_2+4z_1z_2}{\lambda(1,z_1,z_2)} & \frac{1-z_1+3z_2}{\sqrt{\lambda(1,z_1,z_2)}} & 2\frac{1-z_1}{\sqrt{\lambda(1,z_1,z_2)}} \\ -2z_2 & 0 & 0 \\ -1+z_1+z_2 & -1 & -2 \end{pmatrix}.
\end{aligned} \tag{5.51}$$

Let us address the equations for $S^{(1)}$ and its cuts first. It can be shown that we have the second-order system of equations:

$$\begin{aligned}
\mathcal{D}_1^{(1)} \frac{1}{\sqrt{\lambda(1,z_1,z_2)}} \begin{pmatrix} S^{(1)} \\ \mathcal{C}_{1,2}S^{(1)} \\ \mathcal{C}_{1,2,3}S^{(1)} \end{pmatrix} &= 0 \\
\mathcal{D}_2^{(1)} \frac{1}{\sqrt{\lambda(1,z_1,z_2)}} \begin{pmatrix} S^{(1)} \\ \mathcal{C}_{1,2}S^{(1)} \\ \mathcal{C}_{1,2,3}S^{(1)} \end{pmatrix} &= 0,
\end{aligned} \tag{5.52}$$

with the differential operators

$$\mathcal{D}_1^{(1)} = (1-z_1-z_2)\theta^2 - 2z_1\theta\phi - [(2+5\epsilon)z_1 - \epsilon(1-z_2)]\theta \tag{5.53}$$

$$\begin{aligned} &- 2(1+2\epsilon)z_1\phi - (1+2\epsilon)(1+3\epsilon)z_1 \\ \mathcal{D}_2^{(1)} =&(1-z_1-z_2)\phi^2 - 2z_2\theta\phi - [(2+5\epsilon)z_2 - \epsilon(1-z_1)]\phi \\ &- 2(1+2\epsilon)z_2\theta - (1+2\epsilon)(1+3\epsilon)z_2. \end{aligned}$$

When deriving the second-order equations for $S^{(1)}$, we take the expressions for $\theta S^{(1)}$ and $\phi S^{(1)}$ from (5.47), along with the expression for $\theta\phi S^{(1)}$, and solve them for $S^{(2)}$, $S^{(3)}$ and J. This allows the objects $\theta^2 S^{(1)}$ and $\phi^2 S^{(1)}$ to be written solely in terms of $S^{(1)}$, $\theta S^{(1)}$, $\phi S^{(1)}$ and $\theta\phi S^{(1)}$. The procedure for the two propagator cut is identical. For the maximal cut however, we are only required to eliminate the cuts of $S^{(2)}$ and $S^{(3)}$, but not of J, and so there is an extra independent relation:

$$\mathcal{D}_3^{(1)} \frac{1}{\sqrt{\lambda(1,z_1,z_2)}} \mathcal{C}_{1,2,3} S^{(1)} = 0, \tag{5.54}$$

with

$$\begin{aligned} &\mathcal{D}_3^{(1)} \\ =&\theta\phi - \frac{1+2\epsilon}{\lambda(1,z_1,z_2)} \left[z_2(1+3z_1-z_2)\theta + z_1(1-z_1+3z_2)\phi + 2(1+3\epsilon)z_1z_2\right]. \end{aligned} \tag{5.55}$$

Solving the equations $\mathcal{D}_1^{(1)} f(z_1,z_2) = \mathcal{D}_2^{(1)} f(z_1,z_2) = 0$, we find the general solution

$$\begin{aligned} &f(z_1,z_2) \\ =&A\, F_4(1+2\epsilon,1+3\epsilon;1+\epsilon,1+\epsilon;z_1,z_2) + B\, z_1^{-\epsilon} F_4(1+2\epsilon,1+\epsilon;1-\epsilon,1+\epsilon;z_1,z_2) \\ &+ C\, z_2^{-\epsilon} F_4(1+2\epsilon,1+\epsilon;1+\epsilon,1-\epsilon;z_1,z_2) + D\, z_1^{-\epsilon} z_2^{-\epsilon} F_4(1+\epsilon,1;1-\epsilon,1-\epsilon;z_1,z_2), \end{aligned} \tag{5.56}$$

which follows from the results of the contour integrals (3.42).

Of the terms in (5.56), only the functions $F_4(1+2\epsilon,1+3\epsilon;1+\epsilon,1+\epsilon;z_1,z_2)$, $z_1^{-\epsilon} F_4(1+2\epsilon,1+\epsilon;1-\epsilon,1+\epsilon;z_1,z_2)$ and $z_2^{-\epsilon} F_4(1+2\epsilon,1+\epsilon;1+\epsilon,1-\epsilon;z_1,z_2)$ obey the relation $\mathcal{D}_3^{(1)} f(z_1,z_2) = 0$. This may be demonstrated by employing (2.79) and (2.80) to re-express these solutions using F_1 functions, and then applying the relation (2.75). It follows that our maximal cuts should take the form

$$\begin{aligned}&\mathcal{C}_{1,2,3}S^{(1)} \\ =&A\,F_4(1+2\epsilon,1+3\epsilon;1+\epsilon,1+\epsilon;z_1,z_2)+B\,z_1^{-\epsilon}F_4(1+2\epsilon,1+\epsilon;1-\epsilon,1+\epsilon;z_1,z_2)\\ &+C\,z_2^{-\epsilon}F_4(1+2\epsilon,1+\epsilon;1+\epsilon,1-\epsilon;z_1,z_2),\end{aligned} \tag{5.57}$$

while the two propagator cut can be a combination of all four terms from (5.56).

The differential equations for $S^{(2)}$ and $S^{(3)}$ have a more complex form than for the case of $S^{(1)}$. We examine only $S^{(2)}$, as the case of $S^{(3)}$ will follow from swapping the masses m_1 and m_2. We define the three operators:

$$\begin{aligned}\mathcal{D}_1^{(2)}=&(1-z_1-z_2)\theta^2-2z_1\theta\phi-[5\epsilon z_1-\epsilon(1-z_2)]\theta-(1+4\epsilon)z_1\phi-6\epsilon^2 z_1\\ \mathcal{D}_2^{(2)}=&(1-z_1-z_2)\phi^2-2z_2\theta\phi-[5\epsilon z_2-(\epsilon-1)(1-z_1)]\phi-4\epsilon z_2\theta-6\epsilon^2 z_2\\ \mathcal{D}_3^{(2)}=&\theta\phi-\frac{1}{\lambda(1,z_1,z_2)}\left\{2\epsilon z_2(1+3z_1-z_2)\theta+[1-z_1+z_2+\epsilon(2-2z_1+6z_2)]z_1\phi+12\epsilon^2 z_1 z_2\right\}.\end{aligned} \tag{5.58}$$

It follows from (5.47), (5.49) and (5.50) that there are the second-order differential equations

$$\begin{aligned}&\left[\mathcal{D}_1^{(2)}+\frac{z_1\lambda(1,z_1,z_2)}{-z_1(1-z_1+z_2)+\epsilon(-1+z_1^2-z_2^2+2z_2)}\mathcal{D}_3^{(2)}\right]\begin{pmatrix}S^{(2)}\\ \mathcal{C}_{1,2}S^{(2)}\\ \mathcal{C}_{1,2,3}S^{(2)}\end{pmatrix}=0\\ &\left[\mathcal{D}_2^{(2)}+\frac{(1-z_1)\lambda(1,z_1,z_2)}{-z_1(1-z_1+z_2)+\epsilon(-1+z_1^2-z_2^2+2z_2)}\mathcal{D}_3^{(2)}\right]\begin{pmatrix}S^{(2)}\\ \mathcal{C}_{1,2}S^{(2)}\\ \mathcal{C}_{1,2,3}S^{(2)}\end{pmatrix}=0,\end{aligned} \tag{5.59}$$

with the maximal cut obeying the additional constraint $\mathcal{D}_3^{(2)}\mathcal{C}_{1,2,3}S^{(2)}=0$. Similar to the case of $S^{(1)}$ we infer that there are three possible solutions for the maximal cut, giving the general solution

$$\begin{aligned}&\mathcal{C}_{1,2,3}S^{(2)} \\ =&A\,F_4(2\epsilon,3\epsilon;1+\epsilon,\epsilon;z_1,z_2)+B\,z_1^{-\epsilon}F_4(\epsilon,2\epsilon;1-\epsilon,\epsilon;z_1,z_2)\end{aligned} \tag{5.60}$$

$$+ C\, z_2^{1-\epsilon} F_4(1+\epsilon, 1+2\epsilon; 1+\epsilon, 2-\epsilon; z_1, z_2).$$

There is a fourth solution to (5.59): $z_1^{-\epsilon} z_2^{-\epsilon} + \frac{2\epsilon}{1-\epsilon} z_1^{-\epsilon} z_2^{1-\epsilon} F_4(1, 1+\epsilon; 1-\epsilon, 2-\epsilon; z_1, z_2)$, and so, similar to the case of $S^{(1)}$, the cut $\mathcal{C}_{1,2}S^{(2)}$ can be a combination of the three maximal cuts plus this fourth term.

5.3.1 Maximal Cuts

Now consider the calculation of the cuts by taking residues of a parametrised integral form, beginning with the maximal cuts, which we will demonstrate for the first and second master integrals $S^{(1)}$ and $S^{(2)}$. The cuts of the third master integral follow from a similar calculation and we omit the details for brevity. For the first two master integrals, it will be sufficient to consider

$$\mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{[(k+p)^2]^a}{k^2 - m_1^2} \mathcal{C}_{2,3} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 - m_2^2)(k+l+p)^2}, \tag{5.61}$$

where we have chosen for convenience to compute these cuts by integrating first over a loop containing one massive and one massless propagator. The inner-loop cut is easily performed using (5.13), which leads to

$$\begin{aligned}
&\mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{[(k+p)^2]^a}{k^2 - m_1^2} \mathcal{C}_{2,3} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 - m_2^2)(k+l+p)^2} \qquad (5.62)\\
=&\frac{2}{2\pi i} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{1}{i\pi^{1-\epsilon}} \mathcal{C}_1 \int d^D k \frac{1}{k^2 - m_1^2} [(k+p)^2]^{a+\epsilon} [(k+p)^2 - m_2^2]^{-1-2\epsilon}\\
=&-\frac{2i}{2\pi i} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{\pi^{\frac{1}{2}-\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)} \frac{1}{i\pi^{1-\epsilon}} \mathrm{Res}_{|k^E|^2 = -m_1^2} \int d|k^E|^2 \left(|k^E|^2\right)^{-\epsilon} \int d\theta \sin^{-2\epsilon}\theta \frac{1}{|k^E|^2 + m_1^2}\\
&\times \left(-|k^E|^2 - |p^E|^2 - 2|k^E||p^E| \cos\theta\right)^{a+\epsilon} \left(-|k^E|^2 - |p^E|^2 - 2|k^E||p^E| \cos\theta - m_2^2\right)^{-1-2\epsilon}\\
=&-\frac{2i}{2\pi i} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{\pi^{\frac{1}{2}-\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)} \frac{1}{i\pi^{1-\epsilon}} \left(m_1^2\right)^{-\epsilon} \int d\theta \sin^{-2\epsilon}\theta \left(m_1^2 + p^2 - 2\sqrt{m_1^2 p^2} \cos\theta\right)^{a+\epsilon}\\
&\times \left(-m_2^2 + m_1^2 + p^2 - 2\sqrt{m_1^2 p^2} \cos\theta\right)^{-1-2\epsilon}
\end{aligned}$$

$$= -\frac{2i}{2\pi i}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\pi^{\frac{1}{2}-\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)}\frac{1}{i\pi^{1-\epsilon}}\left(m_1^2\right)^{-\epsilon}2^{-2\epsilon}\int dx[x(1-x)]^{-\frac{1}{2}-\epsilon}$$
$$\times \frac{[m_1^2+p^2-2\sqrt{m_1^2}\sqrt{p^2}(2x-1)]^{a+\epsilon}}{[-m_2^2+m_1^2+p^2-2\sqrt{m_1^2}\sqrt{p^2}(2x-1)]^{1+2\epsilon}}.$$

The above expression can be rewritten using an F_1 function but, as described above in the discussion of the differential equations, we will prefer a result that uses the F_4 function. This can be obtained by splitting the factor $-m_2^2+m_1^2+p^2-2\sqrt{m_1^2}\sqrt{p^2}(2x-1)$ using Mellin-Barnes integration. We will also select a particular maximal cut by specifying the integration domain so that $0<x<1$, and the other two cut contours will then follow from the final result. With these manipulations, the cut becomes

$$\mathcal{C}_1\int\frac{d^Dk}{i\pi^{D/2}}\frac{[(k+p)^2]^a}{k^2-m_1^2}\mathcal{C}_{2,3}\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{(l^2-m_2^2)(k+l+p)^2} \tag{5.63}$$
$$= -\frac{2i}{2\pi i}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\pi^{\frac{1}{2}-\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)}\frac{1}{i\pi^{1-\epsilon}}\left(m_1^2\right)^{-\epsilon}2^{-2\epsilon}\frac{1}{2\pi i}\frac{1}{\Gamma(1+2\epsilon)}\int_{-i\infty}^{+i\infty}dz\Gamma(1+2\epsilon+z)\Gamma(-z)$$
$$\times(-m_2^2)^{-1-2\epsilon-z}\int_0^1 dx[x(1-x)]^{-\frac{1}{2}-\epsilon}\left[m_1^2+p^2-2\sqrt{m_1^2}\sqrt{p^2}(2x-1)\right]^{a+\epsilon+z}$$
$$= -\frac{2}{(2\pi i)^2}\left(m_1^2\right)^{-\epsilon}\int_{-i\infty}^{+i\infty}dz\Gamma(1+2\epsilon+z)\Gamma(-z)(-m_2^2)^{-1-2\epsilon-z}\left[\left(\sqrt{m_1^2}+\sqrt{p^2}\right)^2\right]^{a+\epsilon+z}$$
$$\times{}_2F_1\left(\frac{1}{2}-\epsilon,-a-\epsilon-z;1-2\epsilon;\frac{4\sqrt{m_1^2}\sqrt{p^2}}{(\sqrt{m_1^2}+\sqrt{p^2})^2}\right)$$
$$= -\frac{2}{(2\pi i)^2}\left(m_1^2\right)^{-\epsilon}\int_{-i\infty}^{+i\infty}dz\Gamma(1+2\epsilon+z)\Gamma(-z)(-m_2^2)^{-1-2\epsilon-z}(p^2)^{a+\epsilon+z}$$
$$\times{}_2F_1\left(-a-\epsilon-z,-a-z;1-\epsilon;\frac{m_1^2}{p^2}\right)$$
$$= -\frac{2}{(2\pi i)^2}\left(m_1^2\right)^{-\epsilon}\sum_{m=0}^{\infty}\int_{-i\infty}^{+i\infty}dz\Gamma(1+2\epsilon+z)\Gamma(-z)(-m_2^2)^{-1-2\epsilon-z}(p^2)^{a+\epsilon+z}$$
$$\times\frac{(-a-\epsilon-z)_m(-a-z)_m}{(1-\epsilon)_m}\frac{(m_1^2/p^2)^m}{m!}$$

$$
\begin{aligned}
&= -\frac{2}{2\pi i}\left(m_1^2\right)^{-\epsilon}\left(p^2\right)^{-1+a-\epsilon}\sum_{m,n=0}^{\infty}\Gamma(1+2\epsilon+n)\\
&\quad\times\frac{(-a+1+\epsilon+n)_m(-a+1+2\epsilon+n)_m}{(1-\epsilon)_m}\frac{(m_1^2/p^2)^m}{m!}\frac{(m_2^2/p^2)^n}{n!}\\
&= -\frac{2}{2\pi i}\left(m_1^2\right)^{-\epsilon}\left(p^2\right)^{-1+a-\epsilon}\Gamma(1+2\epsilon)\sum_{m,n=0}^{\infty}\frac{(-a+1+\epsilon)_{m+n}(-a+1+2\epsilon)_{m+n}(1+2\epsilon)_n}{(1-\epsilon)_m(-a+1+\epsilon)_n(-a+1+2\epsilon)_n}\\
&\quad\times\frac{(m_1^2/p^2)^m}{m!}\frac{(m_2^2/p^2)^n}{n!}.
\end{aligned}
$$

Now it is clear that in the case $a = 0$ the above sum can be written as an F_4 function, and so after normalisation by $2\pi i$ we obtain the result

$$
-2\left(m_1^2\right)^{-\epsilon}\left(p^2\right)^{-1-\epsilon}\Gamma(1+2\epsilon)F_4\left(1+\epsilon,1+2\epsilon;1-\epsilon,1+\epsilon;\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right). \tag{5.64}
$$

For the case $a = 1$ we must use the identity

$$
\begin{aligned}
&\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{m+n}(\delta+1)_n}{(\gamma)_m(\gamma')_n(\delta)_n}\frac{x^m}{m!}\frac{y^n}{n!}\\
=&F_4(\alpha,\beta;\gamma,\gamma';x,y)+\frac{1}{\delta}\frac{\alpha\beta y}{\gamma'}F_4(\alpha+1,\beta+1;\gamma,\gamma'+1;x,y)
\end{aligned} \tag{5.65}
$$

to obtain the cut

$$
2\left(m_1^2\right)^{-\epsilon}\left(p^2\right)^{-\epsilon}\Gamma(1+2\epsilon)F_4\left(\epsilon,2\epsilon;1-\epsilon,\epsilon;\frac{m_1^2}{p^2},\frac{m_2^2}{p^2}\right) \tag{5.66}
$$

of the integral

$$
\int\frac{d^Dk}{i\pi^{D/2}}\frac{m_2^2-(k+p)^2}{k^2-m_1^2}\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{(l^2-m_2^2)(k+l+p)^2}. \tag{5.67}
$$

Now in (3.42) we noted a basis of four contours for the F_4 function. We select the three of these contours which produce (5.64) and the other two maximal cuts from (5.57). This gives the results

$$\mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2 - m_1^2} \mathcal{C}_{2,3} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 - m_2^2)(k+l+p)^2} \tag{5.68}$$
$$= \begin{cases} -2\left(m_1^2\right)^{-\epsilon}(p^2)^{-1-\epsilon}\Gamma(1+2\epsilon)F_4\left(1+\epsilon, 1+2\epsilon; 1-\epsilon, 1+\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right) \\ -2(m_2^2)^{-\epsilon}(p^2)^{-1-\epsilon}\Gamma(1+2\epsilon)F_4\left(1+\epsilon, 1+2\epsilon; 1+\epsilon, 1-\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right) \\ (p^2)^{-1-2\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma(1+3\epsilon)}{\Gamma^3(1+\epsilon)}F_4\left(1+2\epsilon, 1+3\epsilon; 1+\epsilon, 1+\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right) \end{cases}.$$

For $S^{(2)}$, using these same contours produces the cuts

$$\mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{m_2^2 - (k+p)^2}{k^2 - m_1^2} \mathcal{C}_{2,3} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 - m_2^2)(k+l+p)^2} \tag{5.69}$$
$$= \begin{cases} 2\left(m_1^2\right)^{-\epsilon}(p^2)^{-\epsilon}\Gamma(1+2\epsilon)F_4\left(\epsilon, 2\epsilon; 1-\epsilon, \epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right) \\ \frac{4\epsilon}{1-\epsilon}(m_2^2)^{1-\epsilon}(p^2)^{-1-\epsilon}\Gamma(1+2\epsilon)F_4\left(1+\epsilon, 1+2\epsilon; 1+\epsilon, 2-\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right) \\ -\frac{1}{3}(p^2)^{-2\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma(1+3\epsilon)}{\Gamma^3(1+\epsilon)}F_4\left(2\epsilon, 3\epsilon; 1+\epsilon, \epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right) \end{cases}.$$

The corresponding cuts of $S^{(1)}$ and $S^{(2)}$ then follow immediately by including the prefactors from the definitions (4.4) and (4.6), and we denote them by $\mathcal{C}'^{(i)}_{1,2,3}S^{(j)}$:

$$\begin{aligned} &\mathcal{C}'^{(1)}_{1,2,3}S^{(1)} \\ =& -2\epsilon^2 e^{2\gamma_E\epsilon}\sqrt{\lambda(1,z_1,z_2)}z_1^{-\epsilon}\Gamma(1+2\epsilon)F_4(1+\epsilon, 1+2\epsilon; 1-\epsilon, 1+\epsilon; z_1, z_2) \\ &\mathcal{C}'^{(2)}_{1,2,3}S^{(1)} \\ =& -2\epsilon^2 e^{2\gamma_E\epsilon}\sqrt{\lambda(1,z_1,z_2)}z_2^{-\epsilon}\Gamma(1+2\epsilon)F_4(1+\epsilon, 1+2\epsilon; 1+\epsilon, 1-\epsilon; z_1, z_2) \\ &\mathcal{C}'^{(3)}_{1,2,3}S^{(1)} \\ =&\epsilon^2 e^{2\gamma_E\epsilon}\sqrt{\lambda(1,z_1,z_2)}\frac{\Gamma(1+2\epsilon)\Gamma(1+3\epsilon)}{\Gamma^3(1+\epsilon)}F_4(1+2\epsilon, 1+3\epsilon; 1+\epsilon, 1+\epsilon; z_1, z_2) \\ &\mathcal{C}'^{(1)}_{1,2,3}S^{(2)} \\ =&2\epsilon^2 e^{2\gamma_E\epsilon}z_1^{-\epsilon}\Gamma(1+2\epsilon)F_4(\epsilon, 2\epsilon; 1-\epsilon, \epsilon; z_1, z_2) \end{aligned} \tag{5.70}$$

$$\begin{aligned}
&\mathcal{C}'^{(2)}_{1,2,3}S^{(2)}\\
=&\frac{4\epsilon^3}{1-\epsilon}e^{2\gamma_E\epsilon}z_2^{1-\epsilon}\Gamma(1+2\epsilon)F_4\left(1+\epsilon,1+2\epsilon;1+\epsilon,2-\epsilon;z_1,z_2\right)\\
&\mathcal{C}'^{(3)}_{1,2,3}S^{(2)}\\
=&-\frac{1}{3}\epsilon^2e^{2\gamma_E\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma(1+3\epsilon)}{\Gamma^3(1+\epsilon)}F_4\left(2\epsilon,3\epsilon;1+\epsilon,\epsilon;z_1,z_2\right).
\end{aligned}$$

It can also be shown that the cuts of $S^{(3)}$ are given by

$$\begin{aligned}
\mathcal{C}'^{(1)}_{1,2,3}S^{(3)}=&\frac{4\epsilon^3}{1-\epsilon}e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)z_1^{1-\epsilon}F_4(1+\epsilon,1+2\epsilon;2-\epsilon,1+\epsilon;z_1,z_2)\\
\mathcal{C}'^{(2)}_{1,2,3}S^{(3)}=&2\epsilon^2e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)z_2^{-\epsilon}F_4(\epsilon,2\epsilon;\epsilon,1-\epsilon;z_1,z_2)\\
\mathcal{C}'^{(3)}_{1,2,3}S^{(3)}=&-\frac{1}{3}\epsilon^2e^{2\gamma_E\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma(1+3\epsilon)}{\Gamma^3(1+\epsilon)}F_4(2\epsilon,3\epsilon;\epsilon,1+\epsilon;z_1,z_2).
\end{aligned} \tag{5.71}$$

Similar to the one-mass case, the cut contours $\Gamma'^{(1)}_{1,2,3}$, $\Gamma'^{(2)}_{1,2,3}$ and $\Gamma'^{(3)}_{1,2,3}$ which produce these cuts are not dual to the forms $\omega_{S^{(1)}}$, $\omega_{S^{(2)}}$ and $\omega_{S^{(3)}}$ of our master integrals, but we can rotate them to a set of new contours

$$\begin{pmatrix}\Gamma^{(1)}_{1,2,3}\\ \Gamma^{(2)}_{1,2,3}\\ \Gamma^{(3)}_{1,2,3}\end{pmatrix}=\frac{1}{\epsilon^2}\begin{pmatrix}-\frac{1}{2}&-\frac{1}{2}&-3\\ 1&\frac{1}{2}&3\\ \frac{1}{2}&1&3\end{pmatrix}\begin{pmatrix}\Gamma'^{(1)}_{1,2,3}\\ \Gamma'^{(2)}_{1,2,3}\\ \Gamma'^{(3)}_{1,2,3}\end{pmatrix} \tag{5.72}$$

which do obey the duality condition $\int_{\Gamma^{(i)}_{1,2,3}}\omega_{S^{(j)}}=\delta_{i,j}+\mathcal{O}(\epsilon)$ for $i,j\in\{1,2,3\}$.

5.3.2 Two-Propagator Cut and Dual Cut Contours

We turn now to the cut on the two massive propagators. We have seen that this cut may be a linear combination of the four terms in (5.56), and we interpret this freedom in the same way as for the one-loop bubble integral we studied in section 5.1: depending on how we define our contour of integration we may introduce the maximal cuts via two-loop analogues of the relation (2.38). This freedom also existed at one loop, but it was known that the definition of cuts given in section 2.3.1 did not encircle the poles at infinity and so we will compute the two-propagator cut by iterating the one-loop cutting procedure.

We again include a numerator $[(k+p)^2]^a$ and demonstrate the cut for the first two master integrals. We choose to integrate first over a loop containing only one of the masses and so can write the cut as

$$\mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{[(k+p)^2]^a}{k^2 - m_1^2} \mathcal{C}_2 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 - m_2^2)(k+l+p)^2}. \tag{5.73}$$

We can then compute the cut of the sunset by using the result (5.33):

$$\begin{aligned}
&\mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{[(k+p)^2]^a}{k^2 - m_1^2} \mathcal{C}_2 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 - m_2^2)(k+l+p)^2} \\
=& -\Gamma(1+\epsilon)(m_2^2)^{-\epsilon} \mathcal{C}_1 \int d^D k \frac{1}{(k^2 - m_1^2)[(k+p)^2]^{1-a}} {}_2F_1\left(1, 1+\epsilon; 1-\epsilon; \frac{m_2^2}{(k+p)^2}\right) \\
=& -\Gamma(1+\epsilon)(m_2^2)^{-\epsilon} \sum_{n=0}^{\infty} (m_2^2)^n \frac{(1+\epsilon)_n}{(1-\epsilon)_n} \mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 - m_1^2)[(k+p)^2]^{1-a+n}} \\
=& \Gamma^2(1+\epsilon) \frac{(m_1^2)^{-\epsilon} (m_2^2)^{-\epsilon}}{(p^2)^{1-a}} \sum_{n=0}^{\infty} \frac{(1+\epsilon)_n}{(1-\epsilon)_n} \frac{(m_2^2)^n}{(p^2)^n} {}_2F_1\left(1-a+n, 1-a+n+\epsilon; 1-\epsilon; \frac{m_1^2}{p^2}\right) \\
=& \Gamma^2(1+\epsilon) \frac{(m_1^2)^{-\epsilon} (m_2^2)^{-\epsilon}}{(p^2)^{1-a}} \sum_{m,n=0}^{\infty} \frac{(1-a+n)_m (1-a+n+\epsilon)_m (1+\epsilon)_n}{(1-\epsilon)_m m! (1-\epsilon)_n} \left(\frac{m_1^2}{p^2}\right)^m \left(\frac{m_2^2}{p^2}\right)^n.
\end{aligned} \tag{5.74}$$

So taking $a = 0$, we obtain

$$\begin{aligned}
&\mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2 - m_1^2} \mathcal{C}_2 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 - m_2^2)(k+l+p)^2} \\
=& \Gamma^2(1+\epsilon) \frac{(m_1^2)^{-\epsilon} (m_2^2)^{-\epsilon}}{p^2} \sum_{m,n=0}^{\infty} \frac{(1)_{m+n} (1+\epsilon)_{m+n}}{(1-\epsilon)_m (1-\epsilon)_n m! n!} \left(\frac{m_1^2}{p^2}\right)^m \left(\frac{m_2^2}{p^2}\right)^n \\
=& \Gamma^2(1+\epsilon) \frac{(m_1^2)^{-\epsilon} (m_2^2)^{-\epsilon}}{p^2} F_4\left(1, 1+\epsilon; 1-\epsilon, 1-\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right),
\end{aligned} \tag{5.75}$$

while for $a = 1$ we find

$$\begin{aligned}
&\mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{(k+p)^2}{k^2 - m_1^2} \mathcal{C}_2 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 - m_2^2)(k+l+p)^2} \qquad (5.76)\\
=&\Gamma^2(1+\epsilon) \left(m_1^2\right)^{-\epsilon} (m_2^2)^{-\epsilon} \sum_{m,n=0}^{\infty} \frac{(n)_m (n+\epsilon)_m (1+\epsilon)_n}{(1-\epsilon)_m (1-\epsilon)_n m!} \left(\frac{m_1^2}{p^2}\right)^m \left(\frac{m_2^2}{p^2}\right)^n \\
=&\Gamma^2(1+\epsilon) \left(m_1^2\right)^{-\epsilon} (m_2^2)^{-\epsilon} \left[{}_2F_1 \left(0, \epsilon; 1-\epsilon; \frac{m_1^2}{p^2}\right)\right. \\
&\left. + \frac{1+\epsilon}{1-\epsilon} \frac{m_2^2}{p^2} \sum_{m,n=0}^{\infty} \frac{(1)_{m+n} (1+\epsilon)_{m+n} (2+\epsilon)_n}{(1-\epsilon)_m (2-\epsilon)_n (1+\epsilon)_n m! n!} \left(\frac{m_1^2}{p^2}\right)^m \left(\frac{m_2^2}{p^2}\right)^n \right] \\
=&\Gamma^2(1+\epsilon) \left(m_1^2\right)^{-\epsilon} (m_2^2)^{-\epsilon} \left[1 + \frac{1+\epsilon}{1-\epsilon} \frac{m_2^2}{p^2} \left\{ F_4 \left(1, 1+\epsilon; 1-\epsilon, 2-\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right)\right.\right. \\
&\left.\left. + \frac{1}{2-\epsilon} \frac{m_2^2}{p^2} F_4 \left(2, 2+\epsilon; 1-\epsilon, 3-\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right) \right\} \right].
\end{aligned}$$

Using the differential operator method for deriving contiguous relations as described in section 2.6.2, it follows that

$$\begin{aligned}
F_4\left(2, 2+\epsilon; 1-\epsilon, 3-\epsilon; z_1, z_2\right) =& \frac{2-\epsilon}{(1+\epsilon) z_2} \phi\, F_4(1, 1+\epsilon; 1-\epsilon, 2-\epsilon; z_1, z_2) \qquad (5.77)\\
F_4(1, 1+\epsilon; 1-\epsilon, 1-\epsilon; z_1, z_2) =& \left[\frac{1}{1-\epsilon} \phi + 1\right] F_4(1, 1+\epsilon; 1-\epsilon, 2-\epsilon; z_1, z_2),
\end{aligned}$$

where $\phi = z_2 \frac{\partial}{\partial z_2}$. We can then eliminate $\phi\, F_4(1, 1+\epsilon; 1-\epsilon, 2-\epsilon; z_1, z_2)$ to obtain a linear relation between the functions $F_4\left(2, 2+\epsilon; 1-\epsilon, 3-\epsilon; z_1, z_2\right)$, $F_4(1, 1+\epsilon; 1-\epsilon, 1-\epsilon; z_1, z_2)$ and $F_4(1, 1+\epsilon; 1-\epsilon, 2-\epsilon; z_1, z_2)$. Using this relation along with (5.75) and (5.76), we obtain the simplified expression

$$\begin{aligned}
&\mathcal{C}_1 \int \frac{d^D k}{i\pi^{D/2}} \frac{m_2^2 - (k+p)^2}{k^2 - m_1^2} \mathcal{C}_2 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 - m_2^2)(k+l+p)^2} \qquad (5.78)\\
=& -\Gamma^2(1+\epsilon) \left(m_1^2\right)^{-\epsilon} (m_2^2)^{-\epsilon} \left[1 + \frac{2\epsilon}{1-\epsilon} \frac{m_2^2}{p^2} F_4 \left(1, 1+\epsilon; 1-\epsilon, 2-\epsilon; \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right)\right].
\end{aligned}$$

Thus the cuts of the master integrals $S^{(1)}$ and $S^{(2)}$ can be found using (5.75), (5.78) and the definitions (4.4) and (4.6):

$$\begin{aligned}\mathcal{C}'_{1,2}S^{(1)} =&\epsilon^2 e^{2\gamma_E\epsilon}\sqrt{\lambda(1,z_1,z_2)}\Gamma^2(1+\epsilon)z_1^{-\epsilon}z_2^{-\epsilon}F_4\left(1,1+\epsilon;1-\epsilon,1-\epsilon;z_1,z_2\right) \quad (5.79)\\ \mathcal{C}'_{1,2}S^{(2)} =&-\epsilon^2 e^{2\gamma_E\epsilon}\Gamma^2(1+\epsilon)z_1^{-\epsilon}z_2^{-\epsilon}\left[1+\frac{2\epsilon}{1-\epsilon}z_2F_4\left(1,1+\epsilon;1-\epsilon,2-\epsilon;z_1,z_2\right)\right],\end{aligned}$$

with the cut of $S^{(3)}$ following from a similar calculation and given by the expression

$$\begin{aligned}&\mathcal{C}'_{1,2}S^{(3)} \quad (5.80)\\ &=-\epsilon^2 e^{2\gamma_E\epsilon}\Gamma^2(1+\epsilon)z_1^{-\epsilon}z_2^{-\epsilon}\left[1+\frac{2\epsilon}{1-\epsilon}z_1F_4(1,1+\epsilon;2-\epsilon,1-\epsilon;z_1,z_2)\right].\end{aligned}$$

It is clear from the above expressions that these cuts vanish when we take either of the limits $z_1 \to 0$ or $z_2 \to 0$. This must be the case as the maximal cut of the double tadpole integral J vanishes in these limits. We note that these cuts can be given an alternative definition such that they are non-zero by integrating first over a bubble containing both cut propagators. The result of this alternative calculation is found to be linearly dependent on the other cuts we have computed above and will play no role in the diagrammatic interpretation of the coaction so we will not further explore it here.

Let us label the contour used to produce these cuts on the two massive propagators as $\Gamma'_{1,2}$ so that $\mathcal{C}'_{1,2}S^{(i)} = \int_{\Gamma'_{1,2}}\omega_{S^{(i)}}$. When performing integrals over this contour we parametrise the momenta of each loop in Euclidean space, take the residues at $|l^E|^2 = -m_2^2$ and $|k^E|^2 = -m_1^2$, and then integrate over the remaining space in k and l. In (5.74), this integration over the remaining space was performed within the earlier result (5.33) and consisted of integrating over an angular variable θ from 0 to π for each loop. If we evaluate the same cut of the master integral J of the product of tadpoles graph, the integrand has no dependence on these angular variables and so they are trivially integrated out of the integration measure, from which it follows that the cut is

$$\mathcal{C}'_{1,2}J \quad (5.81)$$

$$
\begin{aligned}
&= \int_{\Gamma'_{1,2}} \omega_J \\
&= \frac{\pi^{D/2}}{\Gamma(D/2)} \frac{i^2}{(i\pi^{D/2})^2} \mathrm{Res}_{|l^E|^2=-m_2^2} \mathrm{Res}_{|k^E|^2=-m_1^2} \int d|k^E|^2 |k^E|^{D-2} \int d|l^E|^2 |l^E|^{D-2} \\
&\quad \times \frac{1}{(|k^E|^2+m_1^2)(|l^E|^2+m_2^2)} \\
&= \epsilon^2 \Gamma^2(1+\epsilon) z_1^{-\epsilon} z_2^{-\epsilon}.
\end{aligned}
$$

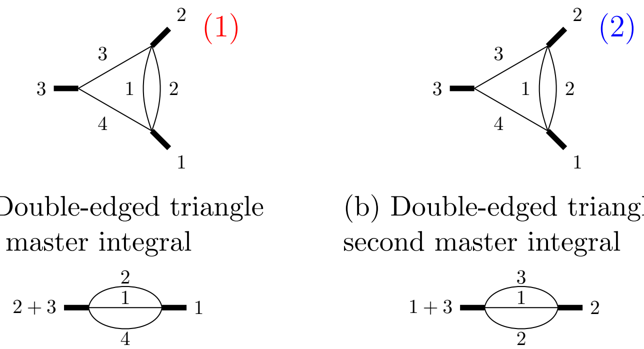
Now we want to find dual bases of forms and contours for the full system of four master integrals and cuts, so we start by setting $\omega_1 = \omega_{S^{(1)}}$, $\omega_2 = \omega_{S^{(2)}}$, $\omega_3 = \omega_{S^{(3)}}$ and $\omega_4 = \omega_J$. It is then clear from the comment below (5.72) and the observation $\int_{\Gamma^{(i)}_{1,2,3}} \omega_J = 0$ for $i \in \{1,2,3\}$ that the contours $\gamma_1 = \Gamma^{(1)}_{1,2,3}$, $\gamma_2 = \Gamma^{(2)}_{1,2,3}$ and $\gamma_3 = \Gamma^{(3)}_{1,2,3}$ are dual to the first three forms. We then notice from (5.81) that setting $\Gamma_{1,2} = \frac{1}{\epsilon^2}\Gamma'_{1,2}$ leads to $\int_{\Gamma_{1,2}} \omega_J = 1 + \mathcal{O}(\epsilon)$. However, the $\Gamma_{1,2}$ is not dual to ω_J as integrating ω_1, ω_2 and ω_3 over this contour produces a non-vanishing result at weight zero. In fact the dual contour is $\gamma_4 = \Gamma_{1,2} + \Gamma^{(1)}_{1,2,3} + \Gamma^{(2)}_{1,2,3} + \Gamma^{(3)}_{1,2,3}$.

5.4 Further Two-Loop Examples

5.4.1 Double-Edged Triangle Graph

For the double-edged triangle graph, let us begin by computing the cuts for the most generic kinematic configuration that we will consider: that with three non-null external momenta, but no internal masses, as shown in figure 5.1. The cuts for cases where some combination of the external momenta are null then follow by taking suitable limits of the generic results. We have already given definitions of two master integrals for this graph in (4.19), and start by determining maximal cuts dual to each of these objects. We expect that there will be two independent maximal cut contours, and that selecting suitable linear combinations of them will yield the cuts dual to our master integrals.

As we have seen in the sunset examples, we must make a selection of which



(a) Double-edged triangle first master integral

(b) Double-edged triangle second master integral

(c) First sunset subgraph

(d) Second sunset subgraph

Figure 5.1: Master integrals for the three-scale double-edged triangle system

loop momentum to integrate over first. In the present example, we observe that the bubble loop is the far simpler choice as the loop and its channel cut each evaluate to trivial functions. Integrating first over this loop, we write down the maximal cuts

$$\begin{aligned}\mathcal{C}_{3,4}\int\frac{d^{D_2}l}{i\pi^{D_2/2}}\frac{[(l+p_2)^2]^a}{l^2(l-p_3)^2}\mathcal{C}_{1,2}\int\frac{d^{D_1}k}{i\pi^{D_1/2}}\frac{1}{k^2(k+l+p_2)^2}\\ =\frac{2}{2\pi i}\frac{\Gamma(D_1/2)}{\Gamma(D_1-1)}\mathcal{C}_{3,4}\int\frac{d^{D_2}l}{i\pi^{D_2/2}}\frac{1}{l^2(l-p_3)^2}[(l+p_2)^2]^{D_1/2-2+a}.\end{aligned} \tag{5.82}$$

We proceed with the usual momentum parametrisation, which now takes the form

$$\begin{aligned}p_3 =&\sqrt{p_3^2}\left(1,\underline{0}_{D_2-1}\right)\\ p_2 =&\frac{1}{2\sqrt{p_3^2}}\left(p_1^2-p_2^2-p_3^2,\sqrt{\lambda(p_1^2,p_2^2,p_3^2)},\underline{0}_{D_2-2}\right)\\ l =&l_0\left(1,\beta\cos\theta,\beta\sin\theta\underline{1}_{D_2-2}\right),\end{aligned} \tag{5.83}$$

from which it follows that the cut is

$$\mathcal{C}_{3,4}\int\frac{d^{D_2}l}{i\pi^{D_2/2}}\frac{[(l+p_2)^2]^a}{l^2(l-p_3)^2}\mathcal{C}_{1,2}\int\frac{d^{D_1}k}{i\pi^{D_1/2}}\frac{1}{k^2(k+l+p_2)^2} \tag{5.84}$$

$$
\begin{aligned}
=&\frac{1}{i\pi^{D_2/2}}\frac{2}{2\pi i}\frac{\Gamma(D_1/2)}{\Gamma(D_1-1)}\frac{2\pi^{D_2/2-1}}{\Gamma(D_2/2-1)}\text{Res}_{l_0=\frac{\sqrt{p_3^2}}{2}}\text{Res}_{\beta=1}\int dl_0 l_0^{D_2-1}\int d\beta\beta^{D_2-2}\\
&\times\int d\theta\sin^{D_2-3}\theta\frac{1}{l_0^2(1-\beta^2)(l^2+p_3^2-2l_0\sqrt{p_3^2})}\\
&\times\left(l^2+p_2^2+\frac{l_0}{\sqrt{p_3^2}}\left[p_1^2-p_2^2-p_3^2-\beta\cos\theta\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}\right]\right)^{D_1/2-2+a}\\
=&\frac{2}{(2\pi i)^2}\frac{\Gamma(D_1/2)}{\Gamma(D_1-1)\Gamma(D_2/2-1)}(p_3^2)^{D_1/2+D_2/2-4+a}\int dx[x(1-x)]^{D_2/2-2}\\
&\times\left(\frac{1}{2}\left[z_1+z_2-1-(2x-1)\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{D_1/2-2+a},
\end{aligned}
$$

where we have left the dimensions D_1 and D_2 generic as they differ for our two master integrals.

Given this result we can select two independent contours of integration for our maximal cuts. It is clear that the maximal cuts evaluate to ${}_2F_1$ functions, and we can choose the integration domains of $[0,1]$ and $\left[0,\frac{p_1^2+p_2^2-p_3^2+\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}}{2\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}}\right]$. Then the cuts of $P^{(1)}$ and $P^{(2)}$, after including a factor of $(2\pi i)^2$, are given by

$$
\begin{aligned}
&\mathcal{C}'^{(1)}_{1,2,3,4}P^{(1)} \qquad (5.85)\\
=&2\epsilon^3\Gamma(1+2\epsilon)\int_0^1 dx[x(1-x)]^{-1-\epsilon}\left(\frac{1}{2}\left[z_1+z_2-1-(2x-1)\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{-\epsilon}\\
=&-4\epsilon^2\Gamma^2(1+2\epsilon)\Gamma^2(1-\epsilon)\left(\frac{1}{2}\left[z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{-\epsilon}\\
&\times{}_2F_1\left(-\epsilon,\epsilon;-2\epsilon;\frac{2\sqrt{\lambda(1,z_1,z_2)}}{z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{C}'^{(1)}_{1,2,3,4}P^{(2)} \qquad (5.86)\\
=&-2\epsilon^3\sqrt{\lambda(1,z_1,z_2)}\Gamma(1+2\epsilon)\int_0^1 dx[x(1-x)]^{-\epsilon}\\
&\times\left(\frac{1}{2}\left[z_1+z_2-1-(2x-1)\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{-1-\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
=&-2\epsilon^3\sqrt{\lambda(1,z_1,z_2)}\frac{\Gamma(1+2\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\left(\frac{1}{2}\left[z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{-1-\epsilon}\\
&\times {}_2F_1\left(1-\epsilon,1+\epsilon;2-2\epsilon;\frac{2\sqrt{\lambda(1,z_1,z_2)}}{z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{C}'^{(2)}_{1,2,3,4}P^{(1)} \qquad (5.87)\\
=&2\epsilon^3\Gamma(1+2\epsilon)\int_0^{\frac{z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}}{2\sqrt{\lambda(1,z_1,z_2)}}}dx[x(1-x)]^{-1-\epsilon}\\
&\times\left(\frac{1}{2}\left[z_1+z_2-1-(2x-1)\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{-\epsilon}\\
=&2\epsilon^3\Gamma(1+2\epsilon)\left[\sqrt{\lambda(1,z_1,z_2)}\right]^{\epsilon}\left(\frac{1}{2}\left[z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{-2\epsilon}\\
&\times\int_0^1 dx x^{-1-\epsilon}(1-x)^{-\epsilon}\left(1-\frac{z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}}{2\sqrt{\lambda(1,z_1,z_2)}}x\right)^{-1-\epsilon}\\
=&-2\epsilon^2\frac{\Gamma(1+2\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\sqrt{\lambda(1,z_1,z_2)}\right]^{\epsilon}\left(\frac{1}{2}\left[z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{-2\epsilon}\\
&\times {}_2F_1\left(-\epsilon,1+\epsilon;1-2\epsilon;\frac{z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}}{2\sqrt{\lambda(1,z_1,z_2)}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{C}'^{(2)}_{1,2,3,4}P^{(2)} \qquad (5.88)\\
=&-2\epsilon^3\sqrt{\lambda(1,z_1,z_2)}\Gamma(1+2\epsilon)\int_0^{\frac{z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}}{2\sqrt{\lambda(1,z_1,z_2)}}}dx[x(1-x)]^{-\epsilon}\\
&\times\left(\frac{1}{2}\left[z_1+z_2-1-(2x-1)\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{-1-\epsilon}\\
=&-2\epsilon^3\Gamma(1+2\epsilon)\left[\sqrt{\lambda(1,z_1,z_2)}\right]^{\epsilon}\left(\frac{1}{2}\left[z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{-2\epsilon}\\
&\times\int_0^1 dx x^{-\epsilon}(1-x)^{-1-\epsilon}\left[1-\frac{z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}}{2\sqrt{\lambda(1,z_1,z_2)}}x\right]^{-\epsilon}
\end{aligned}
$$

$$
=2\epsilon^2 \frac{\Gamma(1+2\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\sqrt{\lambda(1,z_1,z_2)}\right]^{\epsilon} \left(\frac{1}{2}\left[z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}\right]\right)^{-2\epsilon}
\times {}_2F_1\left(1-\epsilon,\epsilon;1-2\epsilon;\frac{z_1+z_2-1+\sqrt{\lambda(1,z_1,z_2)}}{2\sqrt{\lambda(1,z_1,z_2)}}\right).
$$

We then take the contours $\Gamma'^{(1)}_{1,2,3,4}$ and $\Gamma'^{(2)}_{1,2,3,4}$ which define these cuts and rotate them to find the contours dual to $\omega_{P^{(1)}}$ and $\omega_{P^{(2)}}$ in the same way as was done for the sunset examples in the previous sections. Expanding the integrals (5.85), (5.86), (5.87) and (5.88) up to weight one terms, we find the period matrix

$$
\begin{pmatrix} \int_{\Gamma'^{(1)}_{1,2,3,4}} \omega_{P^{(1)}} & \int_{\Gamma'^{(1)}_{1,2,3,4}} \omega_{P^{(2)}} \\ \int_{\Gamma'^{(2)}_{1,2,3,4}} \omega_{P^{(1)}} & \int_{\Gamma'^{(2)}_{1,2,3,4}} \omega_{P^{(2)}} \end{pmatrix} = 2\epsilon^2 \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix} + \mathcal{O}(\epsilon^3) \tag{5.89}
$$

and so the correct transformation is

$$
\begin{aligned}
\Gamma^{(1)}_{1,2,3,4} &= -\frac{1}{4\epsilon^2}\Gamma'^{(1)}_{1,2,3,4} \\
\Gamma^{(2)}_{1,2,3,4} &= \frac{1}{2\epsilon^2}\left(-\frac{1}{2}\Gamma'^{(1)}_{1,2,3,4} + \Gamma'^{(2)}_{1,2,3,4}\right).
\end{aligned} \tag{5.90}
$$

The double-edged triangles have as a subgraph the sunset with no internal masses, and so we expect that to write down the coaction we will also require cuts dual to these graphs. These objects are the cuts $\mathcal{C}_{1,2,3}$ and $\mathcal{C}_{1,2,4}$, where we continue to use the numbering shown in figures 4.1 and 5.1. To define this contour we briefly consider the maximal cuts of one of these sunset integrals, say the sunset with incident momentum p_2.

Taking the massless limit of (5.15) we find that the cut

$$
\mathcal{C}_3 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{l^2} \mathcal{C}_{1,2} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2(k+l+p_2)^2} \tag{5.91}
$$

is given by

$$-\frac{\Gamma^2(1-\epsilon)}{\Gamma^2(1-2\epsilon)}\frac{2^{2-2\epsilon}}{(2\pi i)^2}\int dl_0 l_0^{-1-2\epsilon}\left(p_2^2+2\sqrt{p_2^2}l_0\right)^{-1-\epsilon}. \tag{5.92}$$

We follow the prescription adopted in our previous examples of integrating between the points at which each factor of the integrand vanishes to select the integration domain $\left[0,-\frac{\sqrt{p^2}}{2}\right]$. The parameter integral in the above is then equal to a beta function. Note that this domain of integration is also that which we would arrive at when computing the triple cut of this graph as a unitarity cut, as we described in section 5.2.4 for the one-mass case. One can check that the cut computed with these integration limits obeys the relevant differential equation as we found in our treatment of sunsets with internal masses.

Let the cut contour where these are the limits of l_0 integration be called $\Gamma'_{1,2,3}$, then the integral over this contour is

$$\frac{1}{(2\pi i)^2}\frac{6}{\epsilon}\frac{\Gamma^3(1-\epsilon)\Gamma(1+3\epsilon)}{\Gamma(1-2\epsilon)}(p_2^2)^{-1-2\epsilon}. \tag{5.93}$$

Thus the cut of the master integral $S(p_2^2,p_3^2)$ is given by $\frac{1}{(2\pi i)^2}2\epsilon\frac{\Gamma^3(1-\epsilon)\Gamma(1+3\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{p_2^2}{p_3^2}\right)^{-1-2\epsilon}$ and so to find the cut contour such that $\int_{\Gamma_{1,2,3}}\omega_S = 1+\mathcal{O}(\epsilon)$ we must normalise by the factor $(2\pi i)^2\frac{1}{2\epsilon}$. We use this contour to define the cut $\mathcal{C}_{1,2,3}$ of $S(p_2^2,p_3^2)$ and also the same cut of the other integrals in the system.

To evaluate this cut of $P^{(1)}$ and $P^{(2)}$ we must now parametrise the momenta using

$$\begin{aligned}
p_2 =& \sqrt{p_2^2}\left(1,\underline{0}_{D_2-1}\right) \\
p_3 =& \frac{1}{2\sqrt{p_2^2}}\left(p_1^2-p_2^2-p_3^2,\sqrt{\lambda(p_1^2,p_2^2,p_3^2)},\underline{0}_{D_2-2}\right) \\
l =& l_0\left(1,\beta\cos\theta,\beta\sin\theta\underline{1}_{D_2-2}\right),
\end{aligned} \tag{5.94}$$

so that the momenta in propagators 1, 2 and 3 are identical to those used in the sunset cut (5.91). The integral $\int_{\Gamma_{1,2,3}} \omega$ of

$$\omega = \frac{1}{i\pi^{D_1/2}} \frac{1}{i\pi^{D_2/2}} d^{D_1}k \wedge d^{D_2}l \frac{[(l+p_2)^2]^a}{l^2(l-p_3)^2} \frac{1}{k^2(k+l+p_2)^2} \tag{5.95}$$

over this contour is then given by

$$\begin{aligned}
&\int_{\Gamma_{1,2,3}} \omega \qquad (5.96)\\
=&\frac{(2\pi i)^2}{2\epsilon} \frac{1}{i\pi^{D_2/2}} \frac{2}{2\pi i} \frac{\Gamma(D_1/2)}{\Gamma(D_1-1)} \frac{2\pi^{D_2/2-1}}{\Gamma(D_2/2-1)} \mathrm{Res}_{\beta=1} \int_0^{-\frac{\sqrt{p_2^2}}{2}} dl_0 l_0^{D_2-1}\\
&\times \int d\beta \beta^{D_2-2} \int_0^\pi d\theta \sin^{D_2-3}\theta \frac{1}{l_0^2(1-\beta^2)} \left(l^2 + p_2^2 + 2l_0\sqrt{p_2^2}\right)^{D_1/2-2+a}\\
&\times \left(l^2 + p_3^2 - \frac{l_0}{\sqrt{p_2^2}}\left[p_1^2 - p_2^2 - p_3^2 - \beta\cos\theta\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}\right]\right)^{-1}\\
=&-\frac{2}{\epsilon} \frac{\Gamma(D_1/2)}{\Gamma(D_1-1)\Gamma(D_2/2-1)} 2^{D_2-3} \int_0^{-\frac{\sqrt{p_2^2}}{2}} dl_0 l_0^{D_2-3} \left(p_2^2 + 2l_0\sqrt{p_2^2}\right)^{D_1/2-2+a}\\
&\times \int_0^1 dx [x(1-x)]^{D_2/2-2} \left(p_3^2 - \frac{l_0}{\sqrt{p_2^2}}\left[p_1^2 - p_2^2 - p_3^2 - (2x-1)\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}\right]\right)^{-1}\\
=&-\frac{1}{\epsilon} \frac{\Gamma(D_1/2)}{\Gamma(D_1-1)\Gamma(D_2/2-1)} \frac{1}{p_3^2} (p_2^2)^{D_2/2+D_1/2-3+a} \int_0^1 dl_0 l_0^{D_2-3}(1-l_0)^{D_1/2-2+a}\\
&\times \int_0^1 dx [x(1-x)]^{D_2/2-2}\\
&\times \left(1 + \frac{l_0}{2p_3^2}\left[p_1^2 - p_2^2 - p_3^2 + \sqrt{\lambda(p_1^2,p_2^2,p_3^2)}\right] - \frac{\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}}{p_3^2} x l_0\right)^{-1},
\end{aligned}$$

where we use the same domain of integration over the variable l_0 as in the sunset cut, but integrate over the full range of the new variable x.

In the above cut calculation, our integrand depends on one more variable, the angle θ which is later replaced by x, compared to the sunset maximal cut. Thus when integrating over the remaining space we must explicitly perform an integral over θ, rather than using an integration measure where the θ integral has already been performed.

We expect from writing down the differential equations for this cut as described in section 5.1 that there will be multiple possible contours of integration which could be taken to define a cut on the three propagators 1, 2 and 3. Selecting this contour $\Gamma_{1,2,3}$ provides an unambiguous definition of the cut on these propagators of the double-edged triangle, or indeed of any graph which contains this sunset as a subgraph. Since we will ultimately want to express the coaction of two-loop graphs in the form (2.47), it is necessary that we define each of our cuts as integrals over a certain contour as we have done here.

The parameter integral in the above cut calculation can be recognised as an example of the S_1 generalised Kampé de Fériet function [83]:

$$\begin{aligned} &S_1(\alpha,\alpha';\beta;\gamma;\delta;x,y) \\ =&\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\alpha')_{m+n}(\beta)_m}{(\gamma)_{m+n}(\delta)_m}\frac{x^m}{m!}\frac{y^n}{n!} \\ =&\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)\Gamma(\beta)\Gamma(\delta-\beta)}\int_0^1 du\int_0^1 dv\, u^{\alpha-1}v^{\beta-1}(1-u)^{\gamma-\alpha-1}(1-v)^{\delta-\beta-1} \\ &\times(1-uvx-uy)^{-\alpha'}. \end{aligned} \tag{5.97}$$

Then it follows from this definition and the analytic continuation formula [83]

$$\begin{aligned} &S_1(\alpha,\alpha';\beta;\gamma;\delta;x,y) \\ =&\frac{\Gamma(\alpha'-\alpha)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha')}(-y)^{-\alpha}F_2\left(\alpha,\beta;\alpha+1-\gamma;\delta;\alpha+1-\alpha';-\frac{x}{y},\frac{1}{y}\right) \\ &+\frac{\Gamma(\alpha-\alpha')\Gamma(\gamma)}{\Gamma(\gamma-\alpha')\Gamma(\alpha)}(-y)^{-\alpha'}F_2\left(\alpha',\beta;\alpha'+1-\gamma;\delta;\alpha'+1-\alpha;-\frac{x}{y},\frac{1}{y}\right) \end{aligned} \tag{5.98}$$

that we can write the cuts as

$$\int_{\Gamma_{1,2,3}}\omega \tag{5.99}$$

$$
\begin{aligned}
&= -\frac{1}{\epsilon}\frac{\Gamma(D_1/2)\Gamma(D_1/2-1+a)\Gamma(D_2/2-1)}{\Gamma(D_1-1)\Gamma(D_1/2+D_2-3+a)}\frac{1}{p_3^2}(p_2^2)^{D_2/2+D_1/2-3+a} \\
&\quad\times S_1\Bigg(D_2-2,1;D_2/2-1;D_1/2+D_2-3+a;D_2-2; \\
&\qquad\qquad \frac{\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}}{p_3^2}, -\frac{1}{2p_3^2}\left[p_1^2-p_2^2-p_3^2+\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}\right]\Bigg) \\
&= -\frac{1}{\epsilon}\frac{\Gamma(D_1/2)\Gamma(D_1/2-1+a)\Gamma(D_2/2-1)}{\Gamma(D_1-1)}\frac{1}{p_3^2}(p_2^2)^{D_2/2+D_1/2-3+a} \\
&\quad\times\left[\frac{\Gamma(3-D_2)}{\Gamma(D_1/2-1+a)}\left(\frac{1}{2p_3^2}\left[p_1^2-p_2^2-p_3^2+\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}\right]\right)^{2-D_2}\right. \\
&\quad\times F_2\Bigg(D_2-2,D_2/2-1;2-a-D_1/2;D_2-2;D_2-2; \\
&\qquad\qquad \frac{2\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}}{p_1^2-p_2^2-p_3^2+\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}}, -\frac{2p_3^2}{p_1^2-p_2^2-p_3^2+\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}}\Bigg) \\
&\quad+\frac{\Gamma(D_2-3)}{\Gamma(D_1/2+D_2-4+a)\Gamma(D_2-2)}\frac{2p_3^2}{p_1^2-p_2^2-p_3^2+\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}} \\
&\quad\times F_2\Bigg(1,D_2/2-1;5-a-D_1/2-D_2;D_2-2;4-D_2; \\
&\qquad\qquad \left.\frac{2\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}}{p_1^2-p_2^2-p_3^2+\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}}, -\frac{2p_3^2}{p_1^2-p_2^2-p_3^2+\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}}\Bigg)\right],
\end{aligned} \tag{5.100}
$$

from which $\mathcal{C}_{1,2,3}P^{(1)}$ and $\mathcal{C}_{1,2,3}P^{(2)}$ follow by specialising the values of a, D_1 and D_2, and including the relevant prefactors from the definition (4.19). The cuts $\mathcal{C}_{1,2,4}$ are found by swapping the momenta p_1 and p_2.

Similar to the case of the two-mass sunset, we will ultimately want expressions for all the cuts of this graph written using the Appell F_4 function, as it is this function which we used to express the uncut integrals in section 4.3.1. When computing the coactions of these integrals, each entry of the coaction will then be expressed using an F_4 function according to the results stated in appendix B. Such relations can be found by expansion in ϵ of the cuts and of the coaction second entries. We speculate that they may also be derivable using the relations (2.81) and (2.82) along with other transformations described in section 2.6, although the exact derivation remains unknown.

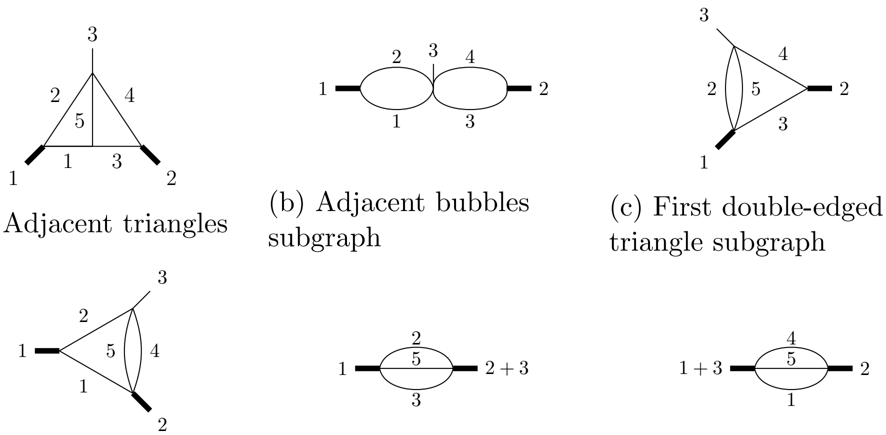
We now want to find a full set of dual forms and contours for the whole system. We have already obtained the maximal cut contours $\Gamma^{(1)}_{1,2,3,4}$ and $\Gamma^{(2)}_{1,2,3,4}$ dual to $\omega^{(1)}_P$ and $\omega_{P^{(2)}}$, and specified contours $\Gamma_{1,2,3}$ and $\Gamma_{1,2,4}$ such that the corresponding maximal cuts of the sunset subgraphs expand to $1 + \mathcal{O}(\epsilon)$. It remains only to find the expansions of the cuts $\mathcal{C}_{1,2,3}P^{(i)}$ and $\mathcal{C}_{1,2,4}P^{(i)}$ and establish if any deformation terms are required.

We see from the exponents on the factors l_0, $1 - l_0$, x and $1 - x$ in the integral (5.96) that there will be a deformation term for the integral $P^{(1)}$ but not for $P^{(2)}$. In fact, it is found that $\int_{\Gamma_{1,2,3}} \omega_{P^{(1)}} = -1+\mathcal{O}(\epsilon)$ and $\int_{\Gamma_{1,2,4}} \omega_{P^{(1)}} = -1+\mathcal{O}(\epsilon)$, from which we conclude that we need to add a deformation term of $\int_{\Gamma^{(1)}_{1,2,3,4}}$ to each of these triple cuts to obtain the dual contours. Thus we have the forms and contours $\omega_1 = \omega_{P^{(1)}}$, $\omega_2 = \omega_{P^{(2)}}$, $\omega_3 = \omega_{S_{1,2,3}}$, $\omega_4 = \omega_{S_{1,2,4}}$, $\gamma_1 = \Gamma^{(1)}_{1,2,3,4}$, $\gamma_2 = \Gamma^{(2)}_{1,2,3,4}$, $\gamma_3 = \Gamma_{1,2,3} + \Gamma^{(1)}_{1,2,3,4}$ and $\gamma_4 = \Gamma_{1,2,4} + \Gamma^{(1)}_{1,2,3,4}$.

For the various graphs where one or two of the variables p_1^2, p_2^2 and p_3^2 vanish, our basis does not contain elements corresponding to Feynman integrals which vanish or are reducible. We also no longer require two master integrals for the double-edged triangle graph itself in any of these limits, and are free to choose either the limit of $P^{(1)}$, or $P^{(2)}$, as we mentioned in the previous chapter. We expect from this that there should be only one maximal cut contour in this limit, and it is clear from (5.84) that this is indeed the case as the factor $z_1 + z_2 - 1 - (2x - 1)\sqrt{\lambda(1, z_1, z_2)}$ degenerates when either z_1 or z_2 vanish.

5.4.2 Adjacent Triangles

Let us now turn to the case of the adjacent triangles with two external scales. All the relevant master integrals are depicted in figure 5.2, and as in the preceding examples we will have to take the maximal cut contours of each graph and use them to construct a set of contours dual to our master integrals. We only include in the

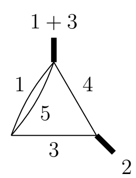


(a) Adjacent triangles (b) Adjacent bubbles subgraph (c) First double-edged triangle subgraph

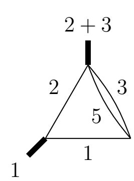
(d) Second double-edged triangle subgraph (e) First sunset subgraph (f) Second sunset subgraph

Figure 5.2: Master integrals for the adjacent triangles system

figure those graphs to which a non-vanishing and non-reducible master integral is assigned. We note in particular the absence of the diagrams



and



as the corresponding Feynman integrals are reducible.

Now consider computing the maximal cut of this graph. Suppose we wish to compute this integral by integrating first over one of the two triangular loops, say the triangle with edges 1, 2 and 5 as shown in figure 5.2. As we are computing a maximal cut of this graph, the propagator 3 will also be placed on shell, and if we already assume this to be the case when integrating over the first loop, the result will be zero. This follows from the reducibility of the one-loop triangle integral with no internal masses and only two non-null external momenta.

We note that the first-loop cut may also be evaluated without placing propagator 3 on shell. There may then be some suitable contour which can be used to define a maximal cut for which, instead of taking a conventional residue at the point where propagator 3 vanishes, some integration is performed with this point as a boundary of the integration domain. This would be analogous to the construction of contours for hypergeometric integrals described in the sections 3.1.1 and 3.1.2. For simplicity however, we will instead choose the other possible ordering of the loops, and integrate first over the box loop consisting of propagators 1, 2, 3 and 4.

Performing the maximal cut of this box cut using the one-loop cut definition produces the result

$$\begin{aligned}&\mathcal{C}_5 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{l^2} \mathcal{C}_{1,2,3,4} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2 (k-p_1)^2 (k+l)^2 (k+l+p_2)^2} \\ =&\frac{1}{(2\pi i)^2} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \mathcal{C}_5 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{l^2} \frac{\left[(p_1^2\ p_2.l + p_2^2\ p_1.l + 2 p_1.l\ p_2.l)\right]^{-1-2\epsilon}}{\left[(p_1.l + p_2.l)(p_1^2\ p_2.l + p_2^2\ p_1.l)\right]^{-\epsilon}}.\end{aligned} \tag{5.101}$$

For compactness, let us write the momentum parametrisation using the variables p, q and r defined by

$$\begin{aligned} p_1 =& \sqrt{p_1^2}(1, \underline{0}_{D-1}) = (r, \underline{0}_{D-1}) \\ p_2 =& \left(-\frac{p_1^2 + p_2^2}{2\sqrt{p_1^2}}, \frac{p_1^2 - p_2^2}{2\sqrt{p_1^2}}, \underline{0}_{D-2} \right) = (p, q, \underline{0}_{D-2}) \\ l =& l_0 (1, \beta \cos\theta, \beta \sin\theta \underline{1}_{D-1}). \end{aligned} \tag{5.102}$$

We then find that after taking the β residue, it remains to integrate over l_0 and θ:

$$\mathcal{C}_5 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{l^2} \mathcal{C}_{1,2,3,4} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2 (k-p_1)^2 (k+l)^2 (k+l+p_2)^2} \tag{5.103}$$

$$
\begin{aligned}
=&\frac{1}{(2\pi i)^2}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\mathcal{C}_5\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2}\frac{[(p_1^2\;p_2.l+p_2^2\;p_1.l+2p_1.l\;p_2.l)]^{-1-2\epsilon}}{[(p_1.l+p_2.l)(p_1^2\;p_2.l+p_2^2\;p_1.l)]^{-\epsilon}}\\
=&\frac{1}{(2\pi i)^2}\frac{1}{\Gamma(1-2\epsilon)}2\pi^{1-\epsilon}\frac{1}{i\pi^{2-\epsilon}}\mathrm{Res}_{\beta=1}\int dl_0 l_0^{3-2\epsilon}\int d\beta\beta^{2-2\epsilon}\int d\theta\sin^{1-2\epsilon}\theta\\
&\times\frac{1}{l_0^2(1-\beta^2)}\frac{[(p_1^2+2l_0r)l_0(p-\beta q\cos\theta)+p_2^2\,l_0r]^{-1-2\epsilon}}{\{[l_0r+l_0(p-q\beta\cos\theta)][p_1^2\,l_0(p-q\beta\cos\theta)+p_2^2\,l_0r]\}^{-\epsilon}}\\
=&-\frac{1}{(2\pi i)^3}\frac{1}{\Gamma(1-2\epsilon)}2^{2-2\epsilon}\int dl_0 l_0^{-2\epsilon}\int dx[x(1-x)]^{-\epsilon}\\
&\times\frac{[(p_1^2+2l_0r)(p+q-2xq)+p_2^2r]^{-1-2\epsilon}}{\{[r+(p+q-2xq)][p_1^2(p+q-2xq)+p_2^2r]\}^{-\epsilon}}\\
=&\frac{1}{(2\pi i)^3}\frac{1}{\Gamma(1-2\epsilon)}2^{2-2\epsilon}(p_1^2-p_2^2)^{2\epsilon}\int dl_0 l_0^{-2\epsilon}\int dx\\
&\times\left\{\sqrt{p_1^2}x(p_1^2-p_2^2)+2l_0[p_2^2+x(p_1^2-p_2^2)]\right\}^{-1-2\epsilon}.
\end{aligned}
$$

Now, unlike certain cases we have studied, we cannot appeal to the notion of a unitarity cut to define the integration domain for this cut. Instead, by analogy with the previous cases, let us perform the integration between the points $l_0=0$ and $l_0=-\frac{\sqrt{p_1^2}x(p_1^2-p_2^2)}{2[p_2^2+x(p_1^2-p_2^2)]}$ where the integrand vanishes. Evaluating this integral gives the result

$$
\begin{aligned}
&\mathcal{C}_5\int\frac{d^Dl}{i\pi^{D/2}}\frac{1}{l^2}\mathcal{C}_{1,2,3,4}\int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{k^2(k-p_1)^2(k+l)^2(k+l+p_2)^2} \qquad (5.104)\\
=&\frac{1}{(2\pi i)^3}\frac{1}{\Gamma(1-2\epsilon)}2^{2-2\epsilon}(p_1^2-p_2^2)^{2\epsilon}\int dx\int_0^{-\frac{\sqrt{p_1^2}x(p_1^2-p_2^2)}{2[p_2^2+x(p_1^2-p_2^2)]}}dl_0 l_0^{-2\epsilon}\\
&\times\left[\sqrt{p_1^2}x(p_1^2-p_2^2)+2l_0(p_2^2+x(p_1^2-p_2^2))\right]^{-1-2\epsilon}\\
=&-\frac{2}{(2\pi i)^3}\frac{\Gamma(-2\epsilon)}{\Gamma(1-4\epsilon)}(p_1^2-p_2^2)^{-2\epsilon}(p_1^2)^{-2\epsilon}\int dx x^{-4\epsilon}[p_2^2+x(p_1^2-p_2^2)]^{-1+2\epsilon}.
\end{aligned}
$$

We then perform the x integration on a similarly restricted domain between $x=0$

and $x = -\frac{p_2^2}{p_1^2 - p_2^2}$:

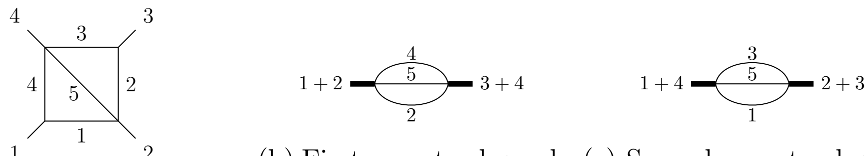
$$\begin{aligned}
&\mathcal{C}_5 \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{l^2} \mathcal{C}_{1,2,3,4} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2(k-p_1)^2(k+l)^2(k+l+p_2)^2} \\
&= -\frac{2}{(2\pi i)^3} \frac{\Gamma(-2\epsilon)}{\Gamma(1-4\epsilon)} (p_1^2 - p_2^2)^{-2\epsilon} (p_1^2)^{-2\epsilon} \int_0^{-\frac{p_2^2}{p_1^2-p_2^2}} dx x^{-4\epsilon} [p_2^2 + x(p_1^2 - p_2^2)]^{-1+2\epsilon} \\
&= -\frac{1}{2\epsilon^2} \frac{1}{(2\pi i)^3} \Gamma(1+2\epsilon)(p_1^2 - p_2^2)^{-1+2\epsilon} (p_1^2)^{-2\epsilon} (p_2^2)^{-2\epsilon}.
\end{aligned} \tag{5.105}$$

The other cuts of this graph will also be required to compute the diagrammatic coaction. We will again rely on our definition of the cut contours as those which give the maximal cuts of the corresponding subgraphs. For instance, we will need the contour used to compute the maximal cut of the double-edged triangle of figure 5.2c in order to find the cut $\mathcal{C}_{2,3,4,5}T$ of the adjacent triangle graph. This maximal cut contour was previously constructed in section 5.4.1. The maximal cuts of the graphs 5.2b, 5.2d, 5.2e and 5.2f are used similarly to compute the cuts $\mathcal{C}_{1,2,3,4}T$, $\mathcal{C}_{1,2,4,5}T$, $\mathcal{C}_{2,3,5}T$ and $\mathcal{C}_{1,4,5}T$ of T.

It can be shown that the ϵ expansions of each of these cuts begin at weight one, and so, provided we use the double-edged triangle master integral P with mixed dimensionality from section 4.3.1 instead of the master integral $\tilde{P}$ with a numerator, there are no deformation terms present in this system. A basis of dual forms and contours is then given by $\omega_1 = \omega_T$, $\omega_2 = \omega_I$, $\omega_3 = \omega_{P_{2,3,4,5}}$, $\omega_4 = \omega_{P_{1,2,4,5}}$, $\omega_5 = \omega_{S_{2,3,5}}$, $\omega_6 = \omega_{S_{1,4,5}}$, $\gamma_1 = \Gamma_{1,2,3,4,5}$, $\gamma_2 = \Gamma_{1,2,3,4}$, $\gamma_3 = \Gamma_{2,3,4,5}$, $\gamma_4 = \Gamma_{1,2,4,5}$, $\gamma_5 = \Gamma_{2,3,5}$ and $\gamma_6 = \Gamma_{1,4,5}$.

Let us also comment briefly on the cut of the one-scale adjacent triangle graph of figure 4.1j. As this graph is reducible, we expect that its maximal cut must vanish. This is confirmed by the result (5.105), but we can see this at the level of the parameter integral (5.103). Taking $p_2^2 \to 0$ in (5.103), the final result now reads

$$\frac{1}{(2\pi i)^3} \frac{1}{\Gamma(1-2\epsilon)} 2^{2-2\epsilon} \frac{1}{p_1^2} \int dl_0 l_0^{-2\epsilon} \left(\sqrt{p_1^2} + 2l_0\right)^{-1-2\epsilon} \int dx x^{-1-2\epsilon}. \tag{5.106}$$



(a) Diagonal box (b) First sunset subgraph (c) Second sunset subgraph

Figure 5.3: Master integrals for the diagonal box system

From our usual prescription of integrating between the points where the integrand vanishes, we see that the result must be zero as the two parameter integrals decouple and there is only one value of x for which the integrand vanishes. There is then no domain in x over which to perform the integral and we declare the result to be zero.

5.4.3 Diagonal Box

Finally, we compute the cuts of the box with diagonal, shown in figure 5.3. We note that the only cuts required in this example are the maximal cut $\mathcal{C}_{1,2,3,4,5}$ and the two cuts $\mathcal{C}_{1,3,5}$ and $\mathcal{C}_{2,4,5}$. These latter cuts are found by computing the discontinuities of the integral with respect to s and t. Meanwhile, we will argue from the diagrammatic results of the following chapter that the maximal cut of this graph should appear in the coaction paired with the uncut integral. As we are not continuing further to compute the cuts of graphs for which the diagonal box is a subgraph, we will not require this maximal cut contour to be known explicitly and can instead find the maximal cut from the coaction.

We may take the discontinuities of the diagonal box integral by computing its coaction, expanding the first entries to weight one, and then using the results of section 2.1.3. The cuts $\mathcal{C}_{1,3,5}$ and $\mathcal{C}_{2,4,5}$ are then found by including the various normalisation factors present in the definitions of the master integral $B(s,t)$ and the cut contours. These cuts are given by the expressions

$$\mathcal{C}_{1,3,5}B \tag{5.107}$$

$$= -\frac{1}{6\epsilon^3}\frac{1}{1-2\epsilon}e^{2\gamma_E\epsilon}\frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}\frac{1}{s}\left(\frac{t}{s}\right)^{-\epsilon}{}_2F_1\left(1-2\epsilon,1-2\epsilon;2-2\epsilon;1+\frac{t}{s}\right)$$

$\mathcal{C}_{2,4,5}B$

$$= -\frac{1}{6\epsilon^3}\frac{1}{1-2\epsilon}e^{2\gamma_E\epsilon}\frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}\frac{1}{t}\left(\frac{s}{t}\right)^{-\epsilon}{}_2F_1\left(1-2\epsilon,1-2\epsilon;2-2\epsilon;1+\frac{s}{t}\right).$$

The coaction can then be expressed in a form where each object in the first entry is identified as a graph. The second entry paired with the diagonal box is identified as the maximal cut:

$$\mathcal{C}_{1,2,3,4,5}B = e^{2\gamma_E\epsilon}\frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}\frac{(s+t)^{2\epsilon}}{s^\epsilon t^\epsilon}. \tag{5.108}$$

We can check that this object does indeed satisfy the relevant differential equation.

In principle, we can compute this cut by defining an appropriate contour of integration similar to that constructed in (5.104) and (5.105). Although the procedure for choosing the exact integration contour in this case is unclear, we will derive the integrand below in order to illustrate the complexity which is involved in choosing such contours for diagrams with higher numbers of external legs.

To compute the cut with our usual method, we cannot integrate over one of the maximally cut triangles first as these cuts vanish due to the fact that $p_1^2 = p_3^2 = 0$. Instead, we integrate first over the box loop with propagators 1, 2, 3 and 4, which has maximal cut

$$\frac{1}{(2\pi i)^2}e^{\gamma_E\epsilon}\frac{\Gamma(D/2-1)}{\Gamma(D-3)}\left(\frac{s+t}{2}\right)^{2-D/2}\left[-(s+t)l.p_2-2\left(l.p_3+\frac{t}{2}\right)\left(l.p_1+\frac{s}{2}\right)\right]^{D/2-3}$$

when it is assumed that $l^2=0$. Adopting the parametrisation

$$\begin{aligned} p_1+p_2 =& \sqrt{s}(1,\underline{0}_{D-1}) \\ p_2 =& \frac{\sqrt{s}}{2}(1,1,\underline{0}_{D-2}) \\ p_3 =& \left(-\frac{\sqrt{s}}{2},-\frac{1}{\sqrt{s}}\left(t+\frac{s}{2}\right),\sqrt{-t\left(1+\frac{t}{s}\right)}\right) \end{aligned} \tag{5.109}$$

$$l = l_0(1, \beta\cos\theta_1, \beta\cos\theta_2\sin\theta_1, \beta\sin\theta_1\sin\theta_2\, \underline{1}_{D-3})$$

and taking the residue at $\beta = 1$ as usual, we find that the maximal cut is

$$\begin{aligned}
&\mathcal{C}'_{1,2,3,4,5}B \qquad (5.110)\\
=& -\frac{1}{(2\pi i)^2} e^{2\gamma_E\epsilon} \frac{\Gamma(D/2-1)}{\Gamma(D-3)} \left(\frac{s+t}{2}\right)^{2-D/2} \frac{\pi^{\frac{D-3}{2}}}{\Gamma\left(\frac{D-3}{2}\right)} \int dl_0 l_0^{D-3} \int dx\, 2^{D-3}[x(1-x)]^{D/2-2}\\
&\times \int dy\, 2^{D-4}[y(1-y)]^{\frac{D-5}{2}} \left\{ -\sqrt{s}(s+t)l_0(1-x) - 2\left(\sqrt{s}l_0 x + \frac{s}{2}\right)\right.\\
&\left.\times \left[l_0 \left(-\frac{\sqrt{s}}{2} + \frac{1}{\sqrt{s}}(t+s/2)(2x-1) - \sqrt{-t\left(1+\frac{t}{s}\right)} 2\sqrt{x(1-x)}(2y-1)\right) + t/2 \right]\right\}^{D/2-3}.
\end{aligned}$$

It then remains to select a normalisation and domain of integration over the variables l_0, x and y such that the result is (5.108). We see that no matter which integration variable we select first, the result of the first integral is a ${}_2F_1$ function. It must then be the case that performing the subsequent integrations produces a result which is completely reducible to the trivial function appearing in (5.108). Quite generally, one can imagine performing maximal cuts of graphs with many external legs and having to find one or more maximal cut contours from parameter integral forms such as (5.110). Although we do not offer a procedure for finding such contours, all the examples we have considered previously suggest that it is possible.

5.5 General Properties

Now that we have illustrated the calculation of cuts for a range of two-loop graphs, let us describe in general terms the procedure for defining and computing a basis of cut contours. We begin with maximal cuts. Suppose we have some two-loop graph and a number of top-sector master integrals corresponding to it. Then we expect to have the same number of independent maximal cut contours for the graph in question.

The maximal cuts evaluated using these contours must ultimately satisfy the homogeneous parts of the system of differential equations obeyed by the graph. To determine suitable contours, we adopt the same momentum parametrisation as at one loop and take residues. Writing down the integral over the first-loop momentum and replacing integrals by residues at each pole we find a unique result where every integration is replaced by a residue. Performing the same operation for the outer-loop integral we no longer find that every integration is eliminated except in the most trivial cases, and it becomes necessary to define a domain of integration for the remaining variables. The construction of these integration domains follows a similar procedure to that detailed for hypergeometric functions in chapter 3. Specifically, we try integrating between the points at which the various factors of the integrand vanish. One can then check that the resulting contours obey the expected differential equations. Having found these maximal cut contours $\Gamma'^{(i)}_{1,\ldots,n}$, one finds in each of our examples that they can also be rotated to new contours $\Gamma^{(i)}_{1,\ldots,n}$ such that $\int_{\Gamma^{(i)}_{1,\ldots,n}} \omega_{J^{(j)}} = \delta_{i,j} + \mathcal{O}(\epsilon)$, where $\omega_{J^{(j)}}$ is the form associated with the j^{th} top-sector master integral of the graph.

As a practical matter, it can sometimes be convenient to define these cuts using a more complicated class of hypergeometric function for consistency with the non-maximal cuts, as we did for the two-mass sunset. It was found in that case that instead of using an F_1 integral representation to define three maximal cuts we could instead transform the result into an Appell F_4 function and use three of the four domains of integration that were described in chapter 3 for this family of integrals. These three domains were distinguished from the fourth by the fact that they obeyed the expected differential equations of the maximal cut.

When all the Feynman integrals associated with a particular graph are reducible to linear combinations of integrals from subgraphs, we define the maximal cut to be zero. This follows from our requirement that the maximal cuts must satisfy the homogeneous parts of the differential equations for the uncut integral. For a reducible integral, these homogeneous parts can be written without the integral itself and so the corresponding equations for the maximal cut simply state that its derivatives vanish. Also, when finding the coaction of two-loop integrals, we ultimately want to mimic the construction of dual bases of forms and contours which was used for polylogarithms, one-loop graphs and hypergeometric functions. As a reducible graph does not contribute to the basis of forms, we expect there will be no cut contour from the maximal cut of such a graph either.

If our graph has some collection of propagators $\{D_1, \ldots, D_n\}$ and we wish to find its cuts on $\{D_{i_1}, \ldots, D_{i_m}\}$, then we use the maximal cut contours of the graph with only propagators $\{D_{i_1}, \ldots, D_{i_m}\}$ to define these cuts. For every extra variable used to parametrise the momenta of the larger graph we must add an extra parameter integral and use the corresponding integration measure formula, instead of applying the integration measure from the maximal cut calculation where these extra variables had already been integrated out. The integrations over the extra variables are performed over their full ranges so that the domain of integration is identical to the calculation of the maximal cut. We can then see that the cut may be written as an integral over the maximal cut of the subgraph and the remaining uncut propagators. This implies that if the graph with propagators $\{D_{i_1}, \ldots, D_{i_m}\}$ has vanishing maximal cut, then the cuts of all larger graphs on exactly this collection of propagators must also vanish. Lastly, we mention that the non-maximal cuts defined in this way should satisfy the corresponding differential equations written down for the cut as described in section 5.1.

Given master integrals for a two-loop graph and all of its two-loop subgraphs along with the maximal cut contours for each of these integrals, we can then compute the full period matrix. It is possible for non-maximal cuts of our graphs to have weight zero terms in their ϵ expansions, in which case one must then perform a rotation on either the contours or the forms to make the bases dual. This procedure at one loop resulted in the presence of the deformation terms, and we have seen that this can also happen at two loops.

Let us comment on the choice of loop order during the calculations. We have demonstrated in the case of the one-mass sunset maximal cut that the cuts computed are the same regardless of whether the internal mass appears in the first loop to be integrated over or in the second. One often finds, however, that the cut integrals are prohibitively difficult to compute with one choice of loop ordering. The cuts of the sunset with two masses, for instance, were computed by splitting the masses between the two loops. Had they both been placed in the inner loop then the result of this first loop integration would have been a function of the non-trivial variables w and $\overline{w}$ defined by $w\overline{w} = \frac{m_1^2}{p^2}$ and $(1-w)(1-\overline{w}) = \frac{m_2^2}{p^2}$. It is unclear in general how to perform cut integrals where such variables appear and recover results obtained with the simpler choice of loop ordering. Sometimes we also find that a certain loop ordering results in the inner-loop maximal cut vanishing, and so we are compelled to order the loops in the opposite way. We have also seen in the case of the one-mass sunset single cut that placing the mass in the inner loop requires a very specific choice of integration domain to recover the correct result. We will not definitively resolve the question of how loop ordering affects the result of the cut calculation, but we will find in every example considered that there is some loop ordering that allows the calculation of the cuts which appear in the coaction.

Finally, we comment on how these cuts interact with the second-type Landau singularities at infinite momentum. This was explored in detail at one loop [33], where it was known that the cut contours did not encircle this pole at infinite momentum. The homology relation (2.38) was even used to explain the coefficients of the deformation terms in the coaction. These results were derived by writing the integrals in projective space, as we mentioned in section 2.3.1. We have not attempted in this work to explore how cut integrals at two loops can be written in this space, and it remains unclear how these properties might generalise to a generic two-loop graph. The only cases that we can comment on are those where our cuts can be thought of as iterated one-loop cuts, and so it is clear that the cut contour does not encircle poles where either loop momentum is infinite, such as the two-mass sunset cut on both massive propagators. For cases where our cuts are not iterated one-loop cuts, however, we can make no such statements.

Chapter 6

Diagrammatic Coaction of Two-Loop Feynman Integrals

Throughout this section we will give results for the coactions of a number of two-loop Feynman integrals, with the objective of gathering evidence of a general structure of the coaction at two loops. As we will illustrate explicitly for the one-mass sunset, this can be done by applying the results of chapter 3 to expressions for the integrals written using hypergeometric functions. In these calculations we will generally have to use a variety of functional relations of the kinds described in section 2.6 to match the terms of the hypergeometric coaction to objects which have a diagrammatic interpretation. It is common, for instance, that the functions in the first entries of the hypergeometric coaction that we selected in chapter 3 only match expressions for Feynman integrals up to integer shifts of the parameters, and so we have to use contiguous relations to arrive at the diagrammatic form. For the second entries, analytic continuation identities such as (2.54) are often used to relate the terms of the coaction to cut integrals.

Of course, we could also simply expand all the relevant objects in ϵ up to some finite order and compare the coaction of the expansion to some assumed form of the diagrammatic coaction. Indeed, this method was used to find the form of the diagrammatic coaction at one loop [24]. We will prefer not to use this technique though, as it leads to significantly less compact calculations that make no use of the underlying structure of the expressions which depend on hypergeometric functions. One must also make a conjecture about the form of the coaction in advance, unlike when using the hypergeometric method where the relevant integrals and cuts are generated from the form of the coaction and the application of functional relations. Observing the hypergeometric functions that emerge from the coaction also provides helpful insights into how the cuts should be defined in the first place, as we can use the bases of integration contours studied in chapter 3 to help motivate the domains of integration chosen in the cut calculations. This information is not readily apparent from the ϵ expanded form.

6.1 One-Mass Sunset

6.1.1 Computing the Coaction

We have already obtained expressions for a pair of master integrals of the one-mass sunset graph, and computed their cuts. Here we compute the coactions of each of these master integrals and express the results in a form analogous to the one-loop coaction of (2.40) where the first entries are the master integrals and the second entries are cuts.

Recall that in section 4.2 we wrote down the master integrals

$$\begin{aligned} S^{(1)} =& \epsilon^2 \left(\frac{p^2}{m^2} - 1 \right) (m^2)^{1+2\epsilon} J(1,1,1,0,0;2-2\epsilon;p^2,m^2) \\ S^{(2)} =& -\epsilon^2 (m^2)^{2\epsilon} J(1,1,1,-1,0;2-2\epsilon;p^2,m^2), \end{aligned} \tag{6.1}$$

which evaluated to

$$\begin{aligned} S^{(1)} =& (1-z)\, e^{2\gamma_E \epsilon} \Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon) {}_2F_1\left(1+2\epsilon, 1+\epsilon; 1-\epsilon; z\right) \\ S^{(2)} =& e^{2\gamma_E \epsilon} \Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon) {}_2F_1\left(2\epsilon, \epsilon; 1-\epsilon; z\right). \end{aligned} \tag{6.2}$$

We have expressions for the coactions of the ${}_2F_1$ functions appearing above, given by (3.12) specialised to the particular values of parameters appearing in our master integrals:

$$\begin{aligned}
&\Delta {}_2F_1\left(1+2\epsilon,1+\epsilon;1-\epsilon;z\right) \\
=&{}_2F_1\left(1+2\epsilon,\epsilon;1-\epsilon;z\right)\otimes {}_2F_1\left(1+2\epsilon,1+\epsilon;1-\epsilon;z\right) \\
&-\frac{3\epsilon}{1-\epsilon}{}_2F_1\left(1+2\epsilon,1+\epsilon;2-\epsilon;z\right)\otimes\left\{z^{-2\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)}\right. \\
&\left.\times {}_2F_1\left(1+2\epsilon,1+3\epsilon;1+\epsilon;\frac{1}{z}\right)\right\}
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
&\Delta {}_2F_1\left(2\epsilon,\epsilon;1-\epsilon;z\right) \\
=&{}_2F_1\left(1+2\epsilon,\epsilon;1-\epsilon;z\right)\otimes {}_2F_1\left(2\epsilon,\epsilon;1-\epsilon;z\right) \\
&-\frac{\epsilon}{1-\epsilon}{}_2F_1\left(1+2\epsilon,1+\epsilon;2-\epsilon;z\right)\otimes\left\{z^{1-2\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)}\right. \\
&\left.\times {}_2F_1\left(2\epsilon,3\epsilon;1+\epsilon;\frac{1}{z}\right)\right\}.
\end{aligned} \tag{6.4}$$

We also observe that (2.14) implies

$$\begin{aligned}
&\Delta[e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)] \\
=&\Delta[e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)]\Delta[e^{-\gamma_E\epsilon}\Gamma(1-\epsilon)]\Delta[e^{\gamma_E\epsilon}\Gamma(1+\epsilon)] \\
=&[e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\otimes e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)][e^{-\gamma_E\epsilon}\Gamma(1-\epsilon)\otimes e^{-\gamma_E\epsilon}\Gamma(1-\epsilon)] \\
&\times[e^{\gamma_E\epsilon}\Gamma(1+\epsilon)\otimes e^{\gamma_E\epsilon}\Gamma(1+\epsilon)] \\
=&[e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)]\otimes[e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)],
\end{aligned} \tag{6.5}$$

where we have used the compatibility of the coaction with multiplication mentioned in section 2.1.1.

Now we must relate the functions ${}_2F_1(1+2\epsilon,\epsilon;1-\epsilon;z)$ and ${}_2F_1(1+2\epsilon,1+\epsilon;2-\epsilon;z)$ appearing in the first entries of the coaction to ${}_2F_1(1+2\epsilon,1+\epsilon;1-\epsilon;z)$ and ${}_2F_1(2\epsilon,\epsilon;1-\epsilon;z)$, which appear in (6.2). This is easily done with the method described in section 2.6.1, and we find

$$\begin{aligned}
&{}_2F_1(1+2\epsilon,\epsilon;1-\epsilon;z) \\
=&\frac{1}{4}\left[(1-z){}_2F_1(1+2\epsilon,1+\epsilon;1-\epsilon;z)+3{}_2F_1(2\epsilon,\epsilon;1-\epsilon;z)\right] \\
&{}_2F_1(1+2\epsilon,1+\epsilon;2-\epsilon;z) \\
=&\frac{1-\epsilon}{4\epsilon z}\left[(1-z){}_2F_1(1+2\epsilon,1+\epsilon;1-\epsilon;z)-{}_2F_1(2\epsilon,\epsilon;1-\epsilon;z)\right].
\end{aligned} \tag{6.6}$$

These contiguous relations imply that

$$\begin{aligned}
&\Delta{}_2F_1\left(1+2\epsilon,1+\epsilon;1-\epsilon;z\right) \\
=&\frac{1}{4}\left[(1-z){}_2F_1(1+2\epsilon,1+\epsilon;1-\epsilon;z)+3{}_2F_1(2\epsilon,\epsilon;1-\epsilon;z)\right]\otimes{}_2F_1\left(1+2\epsilon,1+\epsilon;1-\epsilon;z\right) \\
&-\frac{3}{4z}\left[(1-z){}_2F_1(1+2\epsilon,1+\epsilon;1-\epsilon;z)-{}_2F_1(2\epsilon,\epsilon;1-\epsilon;z)\right]\otimes\left\{z^{-2\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)}\right. \\
&\left.\times{}_2F_1\left(1+2\epsilon,1+3\epsilon;1+\epsilon;\frac{1}{z}\right)\right\} \\
=&(1-z){}_2F_1(1+2\epsilon,1+\epsilon;1-\epsilon;z)\otimes\left[\frac{1}{4}{}_2F_1\left(1+2\epsilon,1+\epsilon;1-\epsilon;z\right)\right. \\
&\left.-\frac{3}{4}z^{-1-2\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)}{}_2F_1\left(1+2\epsilon,1+3\epsilon;1+\epsilon;\frac{1}{z}\right)\right] \\
&+{}_2F_1(2\epsilon,\epsilon;1-\epsilon;z)\otimes\left[\frac{3}{4}{}_2F_1\left(1+2\epsilon,1+\epsilon;1-\epsilon;z\right)\right. \\
&\left.+\frac{3}{4}z^{-1-2\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)}{}_2F_1\left(1+2\epsilon,1+3\epsilon;1+\epsilon;\frac{1}{z}\right)\right].
\end{aligned} \tag{6.7}$$

Using this result, along with (6.5), we find

$$\begin{aligned}
&\Delta S^{(1)} \\
=&\Delta(1-z)e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon){}_2F_1\left(1+2\epsilon,1+\epsilon;1-\epsilon;z\right) \\
=&(1-z)e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon){}_2F_1(1+2\epsilon,1+\epsilon;1-\epsilon;z)\otimes\Big\{(1-z)e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)
\end{aligned} \tag{6.8}$$

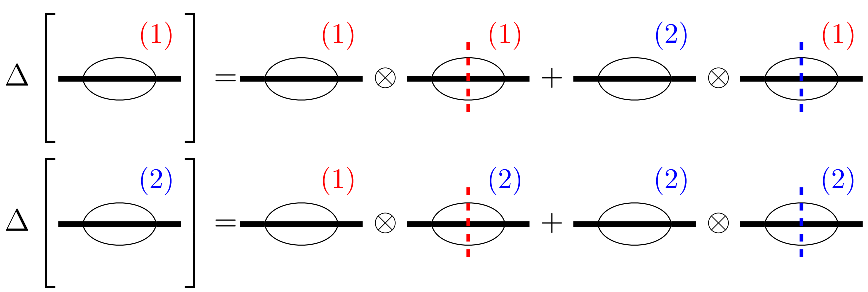
$$\times \left[\frac{1}{4}{}_2F_1\left(1+2\epsilon, 1+\epsilon; 1-\epsilon; z\right) - \frac{3}{4}z^{-1-2\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)}{}_2F_1\left(1+2\epsilon, 1+3\epsilon; 1+\epsilon; \frac{1}{z}\right)\right]\Bigg\}$$
$$+ e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon){}_2F_1(2\epsilon, \epsilon; 1-\epsilon; z) \otimes \Bigg\{(1-z)e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)$$
$$\times \left[\frac{3}{4}{}_2F_1\left(1+2\epsilon, 1+\epsilon; 1-\epsilon; z\right) + \frac{3}{4}z^{-1-2\epsilon}\frac{\Gamma^3(1-\epsilon)}{\Gamma(1-3\epsilon)}{}_2F_1\left(1+2\epsilon, 1+3\epsilon; 1+\epsilon; \frac{1}{z}\right)\right]\Bigg\}.$$

We can then immediately recognise the objects appearing in this coaction as the master integrals $S^{(1)}$ and $S^{(2)}$, and the cuts $\mathcal{C}^{(1)}_{1,2,3}S^{(1)}$ and $\mathcal{C}^{(2)}_{1,2,3}S^{(1)}$ of $S^{(1)}$ defined by (5.17) and (5.31). A similar calculation may be performed to find $\Delta S^{(2)}$, starting from (6.4). The coactions can then be written in a diagrammatic form:

$$\begin{aligned}\Delta S^{(1)} =& S^{(1)} \otimes \mathcal{C}^{(1)}_{1,2,3}S^{(1)} + S^{(2)} \otimes \mathcal{C}^{(2)}_{1,2,3}S^{(1)} \\ \Delta S^{(2)} =& S^{(1)} \otimes \mathcal{C}^{(1)}_{1,2,3}S^{(2)} + S^{(2)} \otimes \mathcal{C}^{(2)}_{1,2,3}S^{(2)}.\end{aligned} \tag{6.9}$$

We observe that these formulae each have the form (2.47), with the forms and contours now interpreted as we have described in chapters 4 and 5.

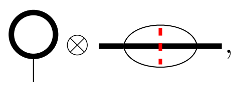
In order to represent this result graphically, we will write each master integral as a sunset graph with a label (1) or (2) at the top right. The cuts will be written with dotted lines through the propagators placed on shell as at one loop, but now with these lines in different colours to indicate which cut we are referring to. The cuts $\mathcal{C}^{(1)}_{1,2,3}$ will appear in red, while $\mathcal{C}^{(2)}_{1,2,3}$ will be blue. With this convention, we can write the results (6.9) as



(6.10)

6.1.2 Properties of the One-Mass Sunset Coaction

It is observed that the coactions (6.9) take a form which is analogous to the one-loop case: there is a pairing between integrals and cuts where the propagators left uncontracted in the first entry are cut in the second. We can observe that the coactions only involves two-loop integrals in the first entry and so, given the pairing of integrals and cuts, only cuts where there is a propagator in each loop placed on shell are present in the second entry. Specifically, there is no term



even though it obeys the pairing between integrals and cuts. As the tadpole integral is not linearly dependent on the two sunset master integrals it can never be present in the coaction. In contrast, the single cut could appear in the second entry if we so choose, as we will explain below. We also observe that, with the presence of only one mass, there are no non-vanishing two-loop integrals that can be formed by contracting propagators of the sunset. The feature new to two loops is that the coaction can contain a sum over different terms where the first-entry integrals correspond to the same graph, and the cuts have the same collection of propagators placed on shell.

In writing down the coactions (6.10) we have used a very specific choice of master integrals $S^{(1)}$ and $S^{(2)}$, and maximal cuts $\mathcal{C}^{(1)}_{1,2,3}$ and $\mathcal{C}^{(2)}_{1,2,3}$ which were chosen by imposing a duality condition. We may ask if the form of these coactions is specific to only these particular choices of integrals and cuts, or is preserved if some other choice is made instead.

We have seen in (2.51) that when rotations $\mathcal{M}$ and $(\mathcal{M}^{-1})^T$ are applied to a basis of forms $\{\omega_i\}$ and contours $\{\gamma_i\}$, the form of a coaction (2.47) is preserved. When considering the coaction of some Feynman integral, however, we would like to have a diagrammatic interpretation for each entry in the coaction, and so we must consider how applying such rotations affects this interpretation.

In this example, the coaction is expressed using only master integrals of the sunset graph itself and the maximal cuts. We are free to regard the forms ω_1 and ω_2 rotated by $\mathcal{M}$ as defining a new basis of master integrals for the sunset graph, while the contours found by applying $(\mathcal{M}^{-1})^T$ likewise define new cuts. Thus we may transform to any dual basis of master integrals and cuts, and still preserve the diagrammatic interpretation of the coaction shown in (6.10).

We conclude by discussing how the discontinuities of the sunset integrals are captured by the coaction. As we have seen in section 2.4.1, the diagrammatic coaction at one loop encodes the fact that mass and momentum discontinuities of Feynman integrals are given by single propagator and channel cuts, respectively. We shall now explain how this same property can be recovered for the one-mass sunset.

We demonstrated in section 5.2.4 that the single cuts of $S^{(1)}$ and $S^{(2)}$ are each linearly dependent on the maximal cuts of these respective graphs. It follows that the coactions (6.9) can then be re-expressed using the cuts which correspond to discontinuities in the second entry instead of using both maximal cuts. It is then simple to demonstrate using the method of section 2.1.3 that the coaction correctly encodes these discontinuities, as we find by expansion of the first entries that

$$\sum_{n=0}^{\infty} \Delta_{1,n} S^{(i)} = \log(m^2) \otimes \mathcal{C}_3 S^{(i)} + \log(m^2 - p^2) \otimes 4\epsilon\, \mathcal{C}_{1,2,3}^{(1)} S^{(i)}. \tag{6.11}$$

We will not wish to write the coaction using the single cut as, unlike (6.9), there is no clear diagrammatic way to interpret the resulting structure.

6.2 Coaction of the Two-Mass Sunset

The coaction of the two-mass sunset integrals can be computed in a similar way to the one-mass case by using the formulae of appendix B, and we find that it takes the form (2.47), with the forms $\{\omega_i\}_{i=1,2,3,4}$ corresponding to the master integrals $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ and J of the sunset and tadpoles defined in section 4.2, and the contours $\{\gamma_i\}_{i=1,2,3,4}$ of section 5.3. We recall that the contours γ_1, γ_2 and γ_3 give the three maximal cuts, while γ_4 is a sum of the cut on two massive propagators and the maximal cuts. For $S^{(1)}$, the coaction can then be written

$$\begin{aligned}\Delta S^{(1)} =&\Delta \int_\gamma \omega_1 \\ =&\sum_{i=1}^{4} \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega_1 \\ =&S^{(1)} \otimes \mathcal{C}^{(1)}_{1,2,3} S^{(1)} + S^{(2)} \otimes \mathcal{C}^{(2)}_{1,2,3} S^{(1)} + S^{(3)} \otimes \mathcal{C}^{(3)}_{1,2,3} S^{(1)} \\ &+ J \otimes (\mathcal{C}_{1,2} S^{(1)} + \mathcal{C}^{(1)}_{1,2,3} S^{(1)} + \mathcal{C}^{(2)}_{1,2,3} S^{(1)} + \mathcal{C}^{(3)}_{1,2,3} S^{(1)}) \\ =&(S^{(1)} + J) \otimes \mathcal{C}^{(1)}_{1,2,3} S^{(1)} + (S^{(2)} + J) \otimes \mathcal{C}^{(2)}_{1,2,3} S^{(1)} + (S^{(3)} + J) \otimes \mathcal{C}^{(3)}_{1,2,3} S^{(1)} \\ &+ J \otimes \mathcal{C}_{1,2} S^{(1)},\end{aligned} \tag{6.12}$$

where we are free to collect the terms of the coaction according to either common first entries or second entries.

The coactions of the other master integrals can also be computed and are found to have the same structure:

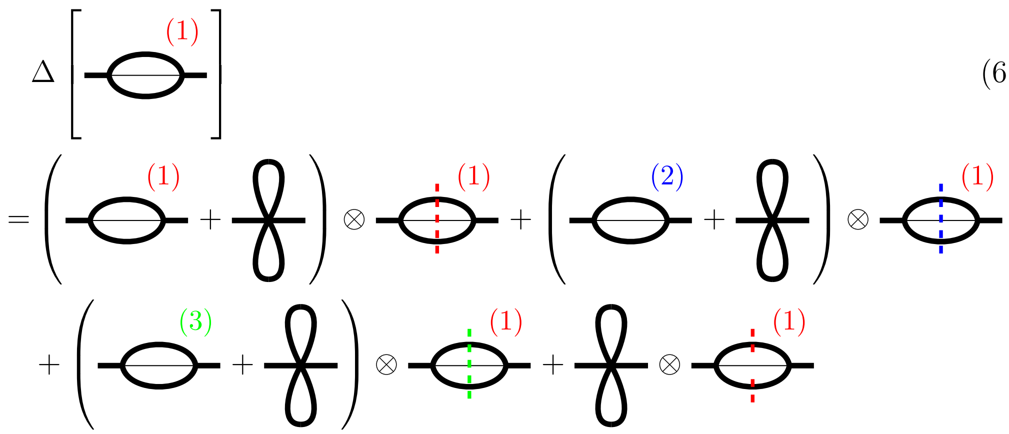
$$\begin{aligned}\Delta S^{(2)} =&S^{(1)} \otimes \mathcal{C}^{(1)}_{1,2,3} S^{(2)} + S^{(2)} \otimes \mathcal{C}^{(2)}_{1,2,3} S^{(2)} + S^{(3)} \otimes \mathcal{C}^{(3)}_{1,2,3} S^{(2)} \\ &+ J \otimes (\mathcal{C}_{1,2} S^{(2)} + \mathcal{C}^{(1)}_{1,2,3} S^{(2)} + \mathcal{C}^{(2)}_{1,2,3} S^{(2)} + \mathcal{C}^{(3)}_{1,2,3} S^{(2)})\end{aligned} \tag{6.13}$$

$$\begin{aligned}\Delta S^{(3)} =&S^{(1)} \otimes \mathcal{C}^{(1)}_{1,2,3} S^{(3)} + S^{(2)} \otimes \mathcal{C}^{(2)}_{1,2,3} S^{(3)} + S^{(3)} \otimes \mathcal{C}^{(3)}_{1,2,3} S^{(3)} \\ &+ J \otimes (\mathcal{C}_{1,2} S^{(3)} + \mathcal{C}^{(1)}_{1,2,3} S^{(3)} + \mathcal{C}^{(2)}_{1,2,3} S^{(3)} + \mathcal{C}^{(3)}_{1,2,3} S^{(3)}).\end{aligned} \tag{6.14}$$

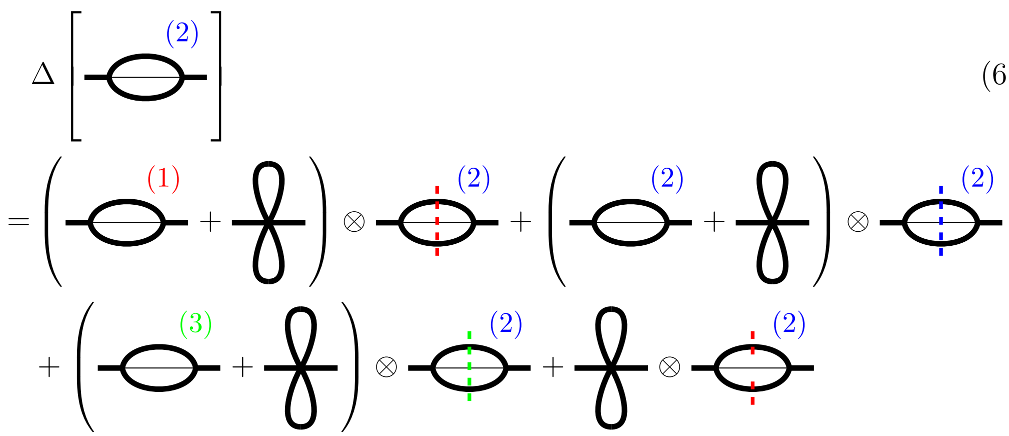
We also note the trivial result

$$\Delta J =J \otimes \mathcal{C}_{1,2} J. \tag{6.15}$$

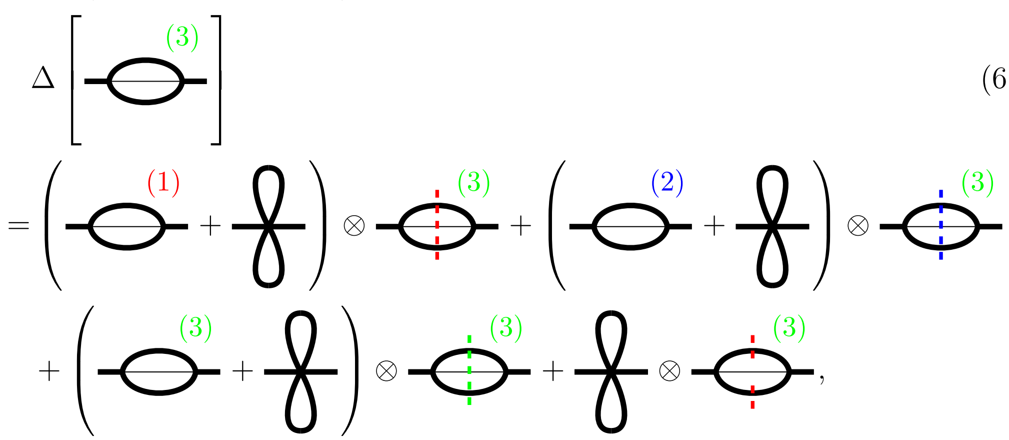
We will represent this in a similar way to the coaction of the one-mass sunset, but now with green labels for the third sunset master integral and the third maximal cut. The two-propagator cut is coloured red as it is the first (and only) cut corresponding to this given collection of propagators being placed on shell. With this convention, the coactions of the sunset integrals are:



(6.16)

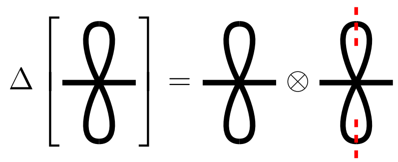


(6.17)



(6.18)

and the coaction of the tadpoles is



(6.19)

6.2.1 Properties of the Two-Mass Sunset Coaction

These coactions have more complicated transformations under changes of basis than in the one-mass case. Given the complexity of a fully generic rotation of the bases ω_i and γ_j, we will instead consider a number of simpler transformations which are more easily interpretable.

Let us start by considering a transformation $\mathcal{M}$ which acts only on the three sunset integrals, rotating the forms according to

$$\begin{pmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{pmatrix} = \mathcal{M} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{6.20}$$

so that the new maximal cut contours dual to these forms are

$$\begin{pmatrix} \gamma_1' \\ \gamma_2' \\ \gamma_3' \end{pmatrix} = (\mathcal{M}^{-1})^T \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}. \tag{6.21}$$

As we have demonstrated in (2.51), this leaves unchanged the form of part of the coaction:

$$\sum_{i=1}^{3} \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega = \sum_{i=1}^{3} \int_\gamma \omega_i' \otimes \int_{\gamma_i'} \omega, \tag{6.22}$$

but when expressed using the new contours, the term $\int_\gamma \omega_4 \otimes \int_{\gamma_4} \omega$ no longer takes the same form. In section 5.3, we wrote the contour γ_4 as a sum of the cut contours for the four cuts $\mathcal{C}_{1,2}$, $\mathcal{C}^{(1)}_{1,2,3}$, $\mathcal{C}^{(2)}_{1,2,3}$ and $\mathcal{C}^{(3)}_{1,2,3}$. The two-propagator cut is not changed under the transformation (6.21), but if we set

$$\mathcal{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{pmatrix} \tag{6.23}$$

then the remaining terms are now written

$$\begin{aligned} &\int_{\gamma_1} \omega + \int_{\gamma_2} \omega + \int_{\gamma_3} \omega \\ =&\int_{m_{1,1}\gamma_1'+m_{2,1}\gamma_2'+m_{3,1}\gamma_3'} \omega + \int_{m_{1,2}\gamma_1'+m_{2,2}\gamma_2'+m_{3,2}\gamma_3'} \omega \\ &+ \int_{m_{1,3}\gamma_1'+m_{2,3}\gamma_2'+m_{3,3}\gamma_3'} \omega \\ =&(m_{1,1}+m_{1,2}+m_{1,3}) \int_{\gamma_1'} \omega + (m_{2,1}+m_{2,2}+m_{2,3}) \int_{\gamma_2'} \omega \\ &+ (m_{3,1}+m_{3,2}+m_{3,3}) \int_{\gamma_3'} \omega. \end{aligned} \tag{6.24}$$

We may then write down the coactions of our new sunset master integrals as

$$\begin{aligned} \Delta S'^{(i)} =& S'^{(1)} \otimes \mathcal{C}'^{(1)}_{1,2,3} S'^{(i)} + S'^{(2)} \otimes \mathcal{C}'^{(2)}_{1,2,3} S'^{(i)} + S'^{(3)} \otimes \mathcal{C}'^{(3)}_{1,2,3} S'^{(i)} \\ &+ J \otimes [\mathcal{C}_{1,2} S'^{(i)} + (m_{1,1}+m_{1,2}+m_{1,3}) \mathcal{C}'^{(1)}_{1,2,3} S'^{(i)} \\ &+ (m_{2,1}+m_{2,2}+m_{2,3}) \mathcal{C}'^{(2)}_{1,2,3} S'^{(i)} + (m_{3,1}+m_{3,2}+m_{3,3}) \mathcal{C}'^{(3)}_{1,2,3} S'^{(i)}], \end{aligned} \tag{6.25}$$

which are identical to the forms (6.12), (6.13) and (6.14), save for the new coefficients of the deformation terms. In our original basis, we found that $\mathcal{C}_{1,2}S^{(1)}$, $\mathcal{C}_{1,2}S^{(2)}$ and $\mathcal{C}_{1,2}S^{(3)}$ each expand to -1 at the lowest order in ϵ, and so, because $\int_{\gamma_i} \omega_j = \delta_{i,j} + \mathcal{O}(\epsilon)$, the deformation terms $\Gamma^{(1)}_{1,2,3} + \Gamma^{(2)}_{1,2,3} + \Gamma^{(3)}_{1,2,3}$ were included in the contour γ_4 to ensure that $\int_{\gamma_4} \omega_i = 0 + \mathcal{O}(\epsilon)$ for $i = 1, 2, 3$. Now in our new basis, we have

$$\mathcal{C}_{1,2} S'^{(1)} = -(m_{1,1}+m_{1,2}+m_{1,3}) + \mathcal{O}(\epsilon) \tag{6.26}$$

$$\mathcal{C}_{1,2}S'^{(2)} = -\left(m_{2,1} + m_{2,2} + m_{2,3}\right) + \mathcal{O}(\epsilon)$$
$$\mathcal{C}_{1,2}S'^{(3)} = -\left(m_{3,1} + m_{3,2} + m_{3,3}\right) + \mathcal{O}(\epsilon)$$

while $\int_{\gamma_i'} \omega_j' = \delta_{i,j} + \mathcal{O}(\epsilon)$, and so the deformation term coefficients change accordingly.

As well as rotating the forms and contours within the three-dimensional spaces of sunset integrals and maximal cuts we may also change the forms ω_1, ω_2 and ω_3 by the transformation

$$\begin{pmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \omega_4, \tag{6.27}$$

which may be restated as

$$\begin{pmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \\ \omega_4 \end{pmatrix} = \mathcal{M} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} \tag{6.28}$$

with

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.29}$$

The corresponding transformation on the contours is then given by

$$(\mathcal{M}^{-1})^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\alpha & -\beta & -\gamma & 1 \end{pmatrix}, \tag{6.30}$$

from which we find the coactions of the new master integrals are

$$\begin{aligned}\Delta S'^{(i)} =& S'^{(1)} \otimes \mathcal{C}^{(1)}_{1,2,3} S'^{(i)} + S'^{(2)} \otimes \mathcal{C}^{(2)}_{1,2,3} S'^{(i)} + S'^{(3)} \otimes \mathcal{C}^{(3)}_{1,2,3} S'^{(i)} \\ &+ J \otimes [\mathcal{C}_{1,2} S'^{(i)} + (1-\alpha)\mathcal{C}^{(1)}_{1,2,3} S'^{(i)} \\ &+ (1-\beta)\mathcal{C}^{(2)}_{1,2,3} S'^{(i)} + (1-\gamma)\mathcal{C}^{(3)}_{1,2,3} S'^{(i)}],\end{aligned} \tag{6.31}$$

and so we have once again preserved the form of the coaction except for possibly changing the coefficients of the deformation terms.

Finally, suppose we give a different definition to the cut $\mathcal{C}_{1,2}$ such that the new cut $\mathcal{C}'_{1,2}$ is some linear combination of $\mathcal{C}_{1,2}$ and the maximal cuts $\mathcal{C}^{(1)}_{1,2,3}$, $\mathcal{C}^{(2)}_{1,2,3}$ and $\mathcal{C}^{(3)}_{1,2,3}$. As we described for one loop in chapter 5 with reference to the differential equations for cut integrals, there is an ambiguity in defining the cuts. Indeed, for the two-mass sunset, a cut $\mathcal{C}'_{1,2}$ on the two massive propagators can be chosen to be any linear combination

$$\mathcal{C}'_{1,2} = \mathcal{C}_{1,2} + \alpha\mathcal{C}^{(1)}_{1,2,3} + \beta\mathcal{C}^{(2)}_{1,2,3} + \gamma\mathcal{C}^{(3)}_{1,2,3} \tag{6.32}$$

where $\mathcal{C}_{1,2}$ must have unit coefficient so that $\mathcal{C}'_{1,2} J = 1 + \mathcal{O}(\epsilon)$. With this new definition we then immediately have

$$\begin{aligned}\Delta S^{(i)} =& S^{(1)} \otimes \mathcal{C}^{(1)}_{1,2,3} S^{(i)} + S^{(2)} \otimes \mathcal{C}^{(2)}_{1,2,3} S^{(i)} + S^{(3)} \otimes \mathcal{C}^{(3)}_{1,2,3} S^{(i)} \\ &+ J \otimes \Big[\mathcal{C}'_{1,2} S^{(i)} + (1-\alpha)\,\mathcal{C}^{(1)}_{1,2,3} S^{(i)} \\ &+ (1-\beta)\,\mathcal{C}^{(2)}_{1,2,3} S^{(i)} + (1-\gamma)\,\mathcal{C}^{(3)}_{1,2,3} S^{(i)}\Big].\end{aligned} \tag{6.33}$$

So we conclude that with changes to the master integrals and cuts we can alter the coefficients of the deformation terms that appear in the coactions of the sunset integrals $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$, while the form of the remaining terms $S^{(1)} \otimes \mathcal{C}^{(1)}_{1,2,3} S^{(i)} + S^{(2)} \otimes \mathcal{C}^{(2)}_{1,2,3} S^{(i)} + S^{(3)} \otimes \mathcal{C}^{(3)}_{1,2,3} S^{(i)} + J \otimes \mathcal{C}_{1,2} S^{(i)}$ is unchanged. Hence the deformation term structure that we observe in the coaction is contingent upon the definitions of our master integrals and cuts. We must then ensure that the definitions of these objects that we adopt at two loops are analogous to those used at one loop in order that our diagrammatic coaction is a direct generalisation of the one-loop case.

In the two-mass sunset example, our coaction is indeed such a generalisation. The first master integral is analogous to the master integrals used at one loop: we give the propagators unit power, do not include any numerators, and evaluate the integral in $2-2\epsilon$ dimensions as this is the natural number of dimensions for each loop of the graph. With this choice for the first master integral we obtain a deformation term given by $J \otimes \mathcal{C}^{(1)}_{1,2,3} S^{(i)}$ in the coaction of $S^{(i)}$. Our other two choices of master integrals also produce analogous deformation terms and so we conclude that they are an appropriate choice to complete the set of three integrals. For the cut on two massive propagators we define the particular cut contour as an iterated one-loop cut, as mentioned in section 5.3, and so expect that this cut has the same property of not encircling the poles at infinite momentum as the cuts at one loop.

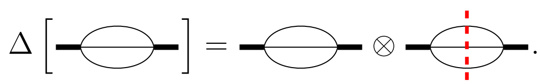
6.3 Coactions of Further Two-Loop Feynman Integrals

Throughout this section, we will state the coactions of the remaining graphs whose master integrals and cuts were considered in the previous chapters. We will not supply the details of how the coactions are computed, as this is performed in every case using the hypergeometric coactions of chapter 3 along with various hypergeometric identities. We will find in each case that the form (2.47) continues to hold, with the $\{\omega_i\}$ and $\{\gamma_j\}$ selected to be dual to each other as described throughout chapter 5. We will not find any structure in these examples which is more general than that which was present in the two-mass sunset, and so we need not repeat our analysis from the previous section of the structure of the coaction and its behaviour under changes of basis.

We begin with the zero-mass sunset graph, with master integral defined in (4.15). We defined the maximal cut of this graph in (5.92) and the discussion which followed it. The coaction takes the trivial form

$$\Delta S = S \otimes \mathcal{C}_{1,2,3} S, \tag{6.34}$$

which can be written diagrammatically as



(6.35)

Now we turn to the family of double-edged triangle graphs. We will illustrate the coactions of each of the graphs with three or fewer non-null external momenta. For clarity, we will begin to write the diagrams with labels on each propagator and label subgraphs with the numbers of the propagators that they contain. This will allow us to distinguish between different sunsets or double-edged triangles which appear as subgraphs.

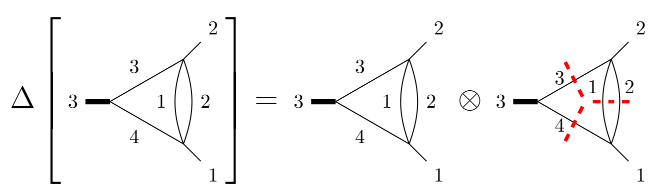
We start with the one-scale cases. The symmetric graph of figure 4.1e has no subgraphs with non-vanishing Feynman integrals and its coaction takes the form

$$\Delta P = P \otimes \mathcal{C}_{1,2,3,4} P. \tag{6.36}$$

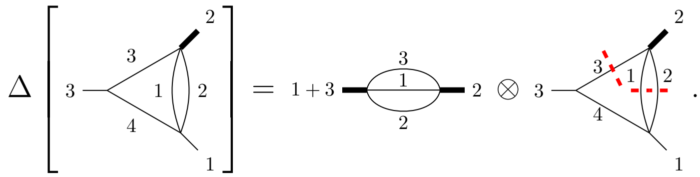
The asymmetric one-scale case shown in figure 4.1f is a reducible graph and has a single subgraph with a non-vanishing Feynman integral: the zero-mass sunset. The coaction takes the form

$$\Delta P = S_{1,2,3} \otimes \mathcal{C}_{1,2,3} P. \tag{6.37}$$

These coactions are represented diagrammatically as



(6.38)

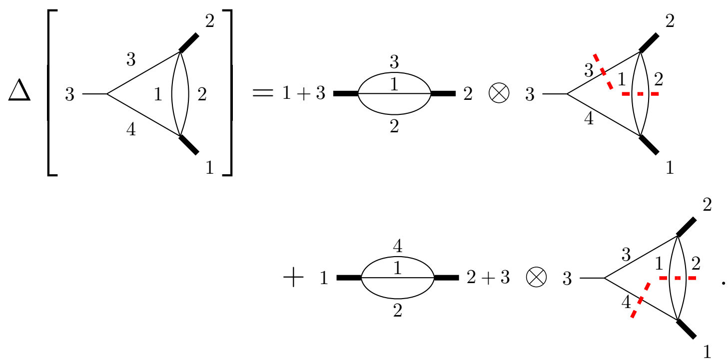


(6.39)

Now take the two-scale cases. The symmetric graph 4.1g is reducible and has two sunset subgraphs, each of which contribute to the coaction:

$$\Delta P = S_{1,2,3} \otimes \mathcal{C}_{1,2,3} P + S_{1,2,4} \otimes \mathcal{C}_{1,2,4} P. \tag{6.40}$$

This is written diagrammatically as



(6.41)

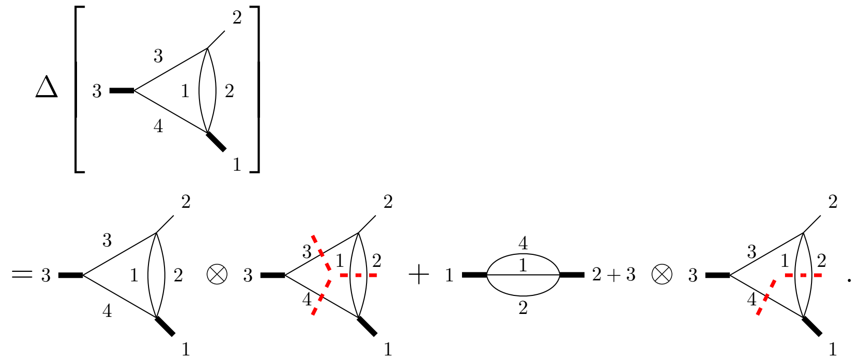
The asymmetric case 4.1h, meanwhile, is non-reducible and has a single sunset subgraph. Recall that in the three-scale case, one of our choices for the master integrals had an associated deformation term, while the other did not. This property continues to hold after taking the limit $p_2^2 \to 0$, and so the form of the coaction will depend on which master integral we select. Specifically, we can have the coaction

$$\Delta P = P \otimes \mathcal{C}_{1,2,3,4} P + S_{1,2,4} \otimes \mathcal{C}_{1,2,4} P, \tag{6.42}$$

or using the other master integral we obtain

$$\Delta \tilde{P} = (\tilde{P} + S_{1,2,4}) \otimes \mathcal{C}_{1,2,3,4} \tilde{P} + S_{1,2,4} \otimes \mathcal{C}_{1,2,4} \tilde{P}, \tag{6.43}$$

where the master integrals P and $\tilde{P}$ are defined in the text beneath equation (4.19). The first of these results can be written as



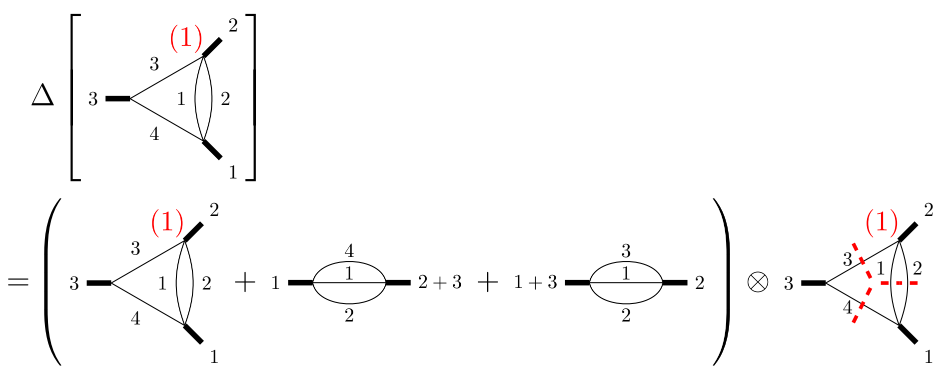
(6.44)

Lastly, take the three-scale case with two master integrals corresponding to the double-edged triangle and a pair of sunset subgraphs. The coactions of the master integrals $P^{(1)}$ and $P^{(2)}$ defined in (4.19) are

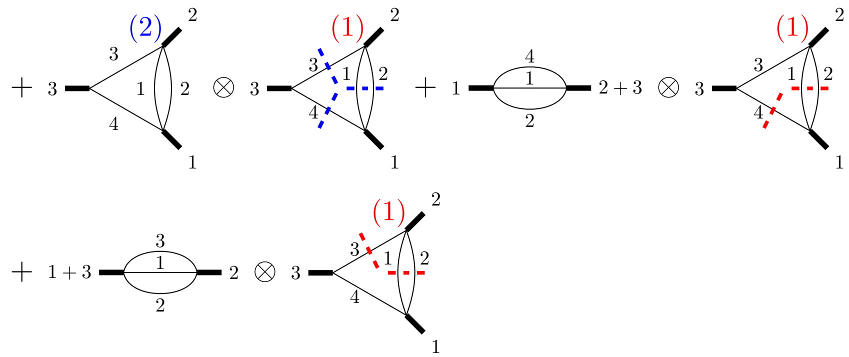
$$\begin{aligned}\Delta P^{(1)} =&(P^{(1)} + S_{1,2,3} + S_{1,2,4}) \otimes \mathcal{C}^{(1)}_{1,2,3,4} P^{(1)} + P^{(2)} \otimes \mathcal{C}^{(2)}_{1,2,3,4} P^{(1)} \\ &+ S_{1,2,3} \otimes \mathcal{C}_{1,2,3} P^{(1)} + S_{1,2,4} \otimes \mathcal{C}_{1,2,4} P^{(1)}\end{aligned} \tag{6.45}$$

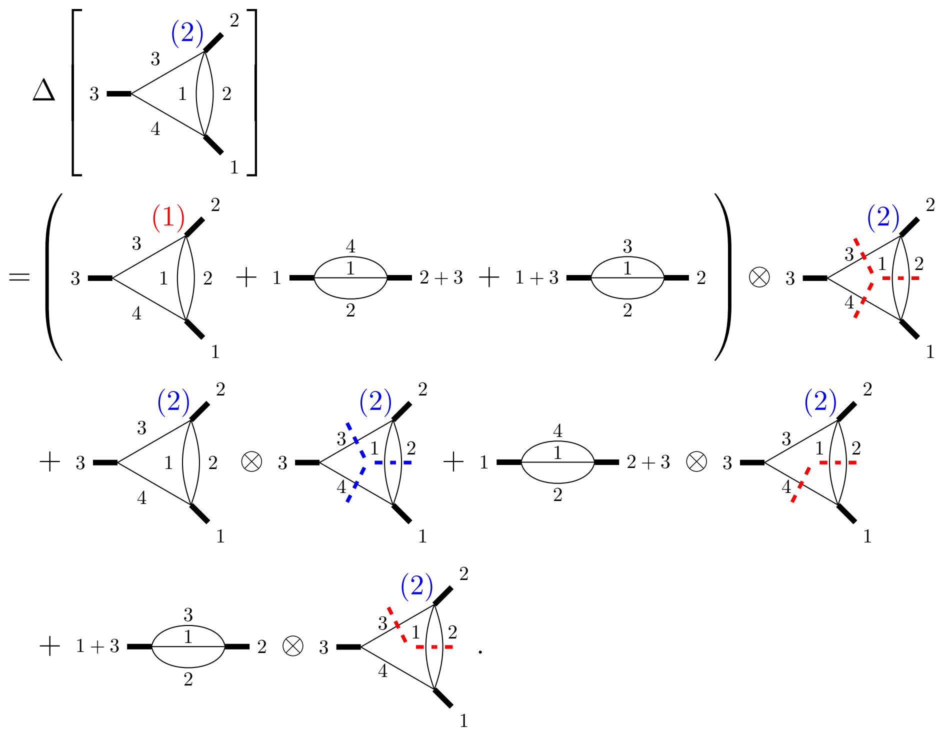
$$\begin{aligned}\Delta P^{(2)} =&(P^{(1)} + S_{1,2,3} + S_{1,2,4}) \otimes \mathcal{C}^{(1)}_{1,2,3,4} P^{(2)} + P^{(2)} \otimes \mathcal{C}^{(2)}_{1,2,3,4} P^{(2)} \\ &+ S_{1,2,3} \otimes \mathcal{C}_{1,2,3} P^{(2)} + S_{1,2,4} \otimes \mathcal{C}_{1,2,4} P^{(2)},\end{aligned} \tag{6.46}$$

which can be restated as



(6.47)





(6.48)

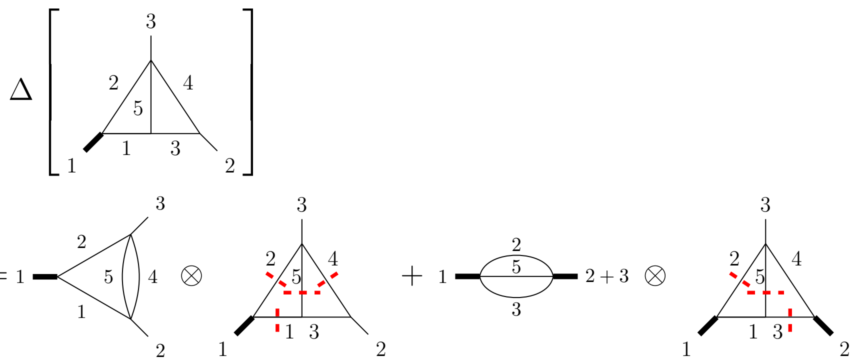
Now consider the adjacent triangle graphs of figures 4.1j and 4.1k. The former of these is reducible to a linear combination of a double-edged triangle and a sunset integral, each of which contribute to its coaction:

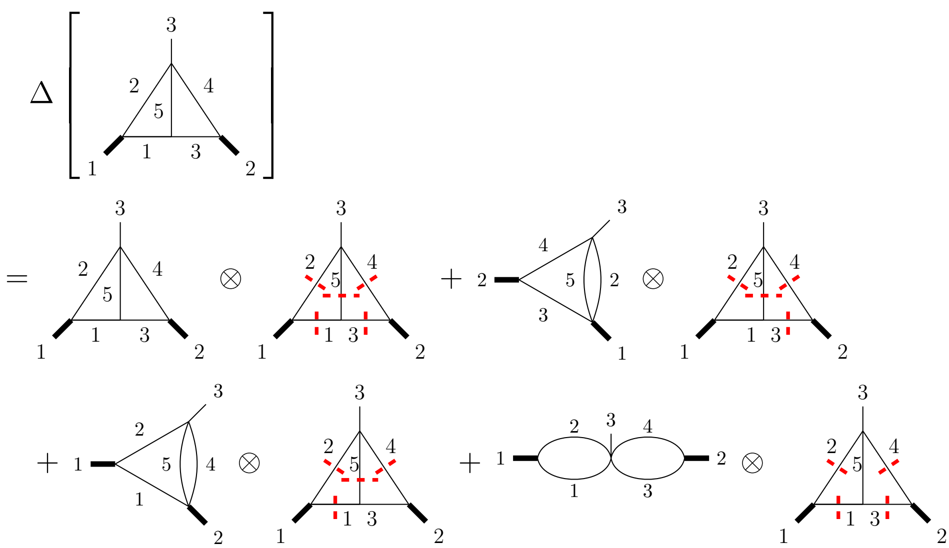
$$\Delta T = S_{2,3,5} \otimes \mathcal{C}_{2,3,5} T + P_{1,2,4,5} \otimes \mathcal{C}_{1,2,4,5} T. \tag{6.49}$$

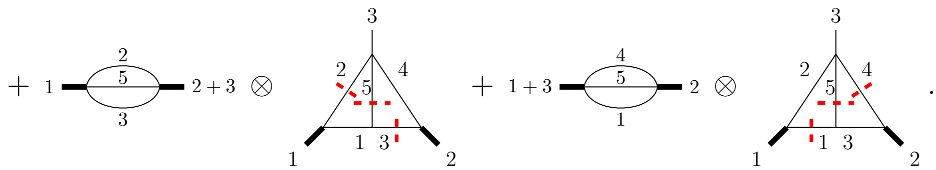
The two-scale case, in contrast, is not reducible and all its non-vanishing two-loop subgraphs appear in the coaction:

$$\begin{aligned}\Delta T =& T \otimes \mathcal{C}_{1,2,3,4,5} T + I_{1,2,3,4} \otimes \mathcal{C}_{1,2,3,4} T + P_{1,2,4,5} \otimes \mathcal{C}_{1,2,4,5} T \\ &+ P_{2,3,4,5} \otimes \mathcal{C}_{2,3,4,5} T + S_{2,3,5} \otimes \mathcal{C}_{2,3,5} T + S_{1,4,5} \otimes \mathcal{C}_{1,4,5} T.\end{aligned} \tag{6.50}$$

We can write each of these results diagrammatically:


(6.51)

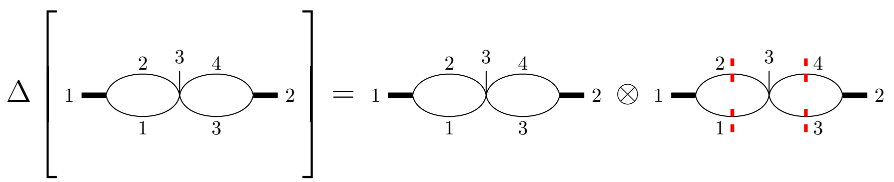

(6.52)



For completeness, we also mention the graph of figure 4.1l, which has the trivial coaction

$$\Delta I = I \otimes \mathcal{C}_{1,2,3,4} I, \tag{6.53}$$

represented diagrammatically as

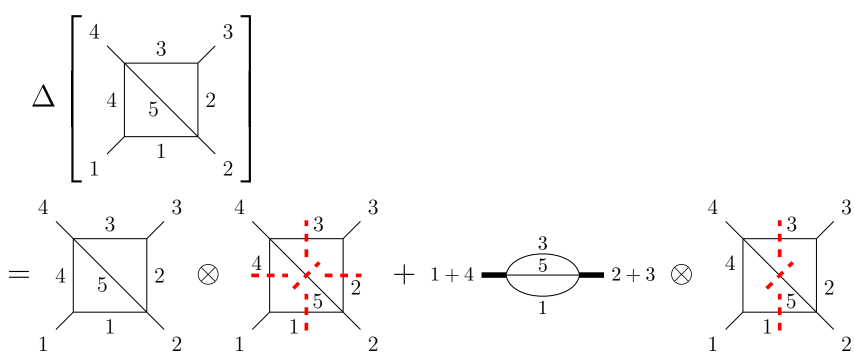


(6.54)

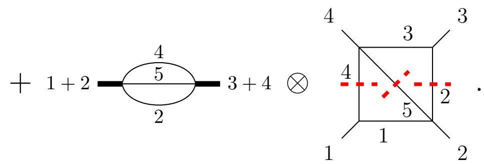
Lastly, there is the diagonal box graph. This graph contains double-edged triangle subgraphs, but they have only a single scale and so are reducible. They do not contribute to the coaction of the integral (4.23):

$$\Delta B = B \otimes \mathcal{C}_{1,2,3,4,5} B + S_{1,3,5} \otimes \mathcal{C}_{1,3,5} B + S_{2,4,5} \otimes \mathcal{C}_{2,4,5} B, \tag{6.55}$$

which can be restated as



(6.56)



6.4 General Structure

In each of the examples we have considered here, the coaction takes the form

$$\Delta \int_\gamma \omega = \sum_{i=1}^{n} \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega \tag{6.57}$$

with the bases of forms $\{\omega_i\}$ from the master integrals of chapter 4 and cut contours $\{\gamma_i\}$ of chapter 5. These coactions generalise the one-loop feature that there is a pairing between graphs and cuts: every propagator which is not placed on shell in the second-entry cut is contracted in the corresponding first-entry subgraph. At two loops though, we must now allow sums of terms where the same graph appears multiple times in the first entry and the same collection of propagators is placed on shell in the second entry. Although we have only encountered here examples for which these terms occur at the top sector and maximal cut, we speculate that the coactions of more complicated two-loop graphs can also involve these terms for subgraphs and non-maximal cuts.

The coactions we have seen contain sums over all such pairings of subgraphs and cuts for which neither the graph or cut vanishes, with the constraint that the graph must also have two loops (and so the cut must place at least one propagator in each loop on shell). The requirement that only two-loop integrals are present in the first entry might be expected from the fact that the set of master integrals for a two-loop graph contains exclusively other two-loop integrals. It is also anticipated by the fact that our integrals are normalised by factors of $e^{L\gamma_E\epsilon}$ at L loops and so each entry of the coaction is expected to contain an L loop integral or cut carrying this factor. We also note that any uncut integral can be written as a sum of its cuts and so an integral with a loop containing no cut propagators can be replaced by a linear combination of objects where every loop contains at least one propagator which has been cut.

The other main feature present at one loop was the deformation terms, and we also see evidence of these in our two-loop examples. As discussed in section 6.2.1, the presence of these terms and their coefficients are contingent upon the choice of bases for the forms and cut contours. In order to understand if the terms in our sunset and double-edged triangle examples can be viewed as generalisations of the one-loop deformation terms, we must address whether our master integrals and cuts are themselves generalisations of those used at one loop.

For the sunset, we explained that our first master integral was the obvious choice to make based on the rule for finding uniform-weight pure master integrals at one loop. The other two master integrals gave rise to the same deformation term coefficients. Similarly, our two-propagator cut contour had the property of the one-loop cuts that it did not encircle the poles at infinity. Based on these observations we can argue that the coaction of the two-mass sunset is indeed analogous to the coactions at one loop described by (2.40). The situation is far less clear for the double-edged triangle, where it is not obvious if either our master integrals or cuts are appropriate generalisations of the one-loop objects.

One may then ask if there is a diagrammatic rule which explains these terms and their coefficients. For instance, we may guess that there should be deformation terms with coefficient one half for each closed loop of the graph with an even number of propagators. These deformation terms could be formed by contracting a single propagator in the loop with an even number of edges. Such a rule would imply that the double-edged triangle coaction should be restated using different bases to eliminate the deformation terms. Meanwhile, in the sunset case, there would be a deformation term consisting of a product of tadpoles and it would have a unit coefficient as the massless propagator contracted to form the tadpoles sits in both of the two loops of the graph. Of course, it is impossible to ascertain the accuracy of such a rule from only a few examples and so more two-loop cases will have to be considered before we can predict with any confidence where deformation terms should occur. We also have available the family of graphs which are products of one-loop objects, as their coaction can be trivially deduced and contains deformation terms whenever these are present for either of the one-loop graphs in the product.

Let us also describe briefly the way in which discontinuities are encoded in these coactions. We mentioned in the case of the one-mass sunset that the single cut was expressed as a linear combination of the two maximal cuts but did not appear directly in the coaction as we have chosen to express it. We see a similar behaviour in (6.52), where the discontinuity of the adjacent triangles with respect to p_1^2 is given as a sum of two cuts in this channel, only one of which is present in the coaction. This other channel cut is in fact a linear combination of other cuts appearing in the coaction, as we expect from the fact that our cut contours form a basis of all the possible cuts. We do not however know in general which terms of the coaction can contribute to the discontinuities and which relations among cuts are then required to recover the discontinuities themselves. This is in contrast to one loop where we described in section 2.4.1 how to find the discontinuities of any one-loop Feynman integral.

One final matter we have not addressed is the coaction of cut Feynman integrals. It was known at one loop that these cut integrals had a coaction which was also of the same form (6.57) as that of the uncut integrals. In this work we have not attempted to compute the coactions of cuts, instead only focusing on the uncut integrals. However, given the broad applicability of the formula (6.57) that we have demonstrated, it would be very surprising if the coactions of cuts did not also take this form.

Chapter 7

Conclusions and Outlook

In this thesis we have detailed recent work on finding closed-form expressions for the coactions of classes of integrals which can be expanded using multiple polylogarithms. Specifically, we have looked at various commonly encountered hypergeometric functions in chapter 3, and then at a number of polylogarithmic two-loop Feynman integrals in chapter 6. The key result linking these cases is that the coactions of every integral we have considered can be expressed in the simple form

$$\Delta \int_\gamma \omega = \sum_{i=1}^{n} \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega \tag{7.1}$$

for dual bases $\{\omega_i\}$ and $\{\gamma_i\}$.

In chapter 3 we began by considering the well-known ${}_2F_1$ function. From the parameter integral representation of this function we could determine the necessary bases to express the coaction by pairing integrands with integration domains such that there were regulated endpoint singularities in certain cases but not others. This procedure generalised straightforwardly to other common hypergeometric functions and we concluded the chapter by describing a quite general class of integrals for which it is, in principal, possible to obtain these bases.

Of course, one soon encounters examples where there are difficult obstacles to completing this construction. We saw for the Appell F_4 function that the usual integral representation of this function could be thought of as being embedded in a larger space of integrals, and we determined the coaction of a specific example of an F_4 by taking a suitable limit in this larger space. It was unclear how this procedure applied for generic parameters, although suitable contours could still be found from an alternative integral representation which was fortuitously available in the literature.

For Feynman integrals, the construction of the dual bases was more involved. We began in chapter 4 by providing uniform-weight pure master integrals for various two-loop graphs. We then detailed the construction of cut contours in chapter 5 by examining the specific case of maximal cuts. These contours then provided an unambiguous way to define non-maximal cuts. The cut contours we constructed carried with them a particular choice for the ordering of the loops in the calculation, and it remains to be considered whether these results can also be found from another method that does not require such a constraint.

Upon computing the coactions of these Feynman integrals it was found that they took the form (7.1), with the first and second entries of the coaction being identified, respectively, as the master integrals of the system and the cuts. The pairing between subgraphs and cuts from one loop was preserved. We also discovered examples of deformation terms appearing at two loops, although we demonstrated that the presence of these terms depends on the way in which we define our master integrals and cuts. It is unclear if there are canonical choices of these objects such that the deformation terms can be interpreted according to a diagrammatic rule similar to that found at one loop.

Finally, we mention that no examples falling outside the class of multiple polylogarithms have been explored in this work. Whether or not a formula similar to (7.1) can describe coactions outside this class remains to be seen.

Appendix A

Integration Measure

Suppose that we wish to change the integration variables in N dimensional Minkowski space from k^i to a new set of variables according to

$$\begin{aligned}&(k^0, k^1, \ldots, k^{N-1}) \\ =&\Bigg(k_0, k_0\beta\cos\theta_1, k_0\beta\cos\theta_2\sin\theta_1, \ldots, k_0\beta\cos\theta_n\prod_{j=1}^{n-1}\sin\theta_j, \ldots, \\ &k_0\beta\cos\theta_{N-2}\prod_{j=1}^{N-3}\sin\theta_j, k_0\beta\prod_{j=1}^{N-2}\sin\theta_j\Bigg),\end{aligned} \tag{A.1}$$

where the integration range is $-\infty < k_0 < +\infty$, $\beta \geq 0$, $0 \leq \theta_i \leq \pi$ for all but the last angle and $0 \leq \theta_{N-2} \leq 2\pi$ by analogy with the familiar circular and spherical coordinates. Then, defining I_N to be the determinant of the corresponding Jacobian matrix $\frac{\partial(k^0,\ldots,k^{N-1})}{\partial(k_0,\beta,\theta_1,\ldots,\theta_{N-2})}$, we find

$$I_N = \det\begin{pmatrix} J_0 & J_1 & J_2 & \ldots & J_n & \ldots & J_{N-2} & J_{N-1}\end{pmatrix} \tag{A.2}$$

with the columns given by

$$J_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad J_{N-1} = \begin{pmatrix} \beta \prod_{j=1}^{N-2} \sin\theta_j \\ k_0 \prod_{j=1}^{N-2} \sin\theta_j \\ k_0\beta\cos\theta_1 \prod_{j=2}^{N-2} \sin\theta_j \\ \vdots \\ k_0\beta\cos\theta_{N-2} \prod_{j=1}^{N-3} \sin\theta_j \end{pmatrix} \tag{A.3}$$

and, for $1 \leq n \leq N-1$,

$$J_n = \begin{pmatrix} \beta\cos\theta_n \prod_{j=1}^{n-1} \sin\theta_j \\ k_0\cos\theta_n \prod_{j=1}^{n-1} \sin\theta_j \\ k_0\beta\cos\theta_n\cos\theta_1 \prod_{j=2}^{n-1} \sin\theta_j \\ \vdots \\ k_0\beta\cos\theta_n\cos\theta_{n-1} \prod_{j=1}^{n-2} \sin\theta_j \\ -k_0\beta \prod_{j=1}^{n} \sin\theta_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{A.4}$$

This determinant can be evaluated by expanding in minors along the bottom row to show that:

$$\begin{aligned} I_N =& \left(k_0\beta\sin\theta_{N-2}\prod_{j=1}^{N-3}\sin\theta_j\right)\sin\theta_{N-2}\, I_{N-1} + \left(k_0\beta\cos\theta_{N-2}\prod_{j=1}^{N-3}\sin\theta_j\right)\cos\theta_{N-2}\, I_{N-1} \\ =& k_0\beta\left(\prod_{j=1}^{N-3}\sin\theta_j\right) I_{N-1} \end{aligned}$$

where we have noted that the minors are equal to $\sin\theta_{N-2}\, I_{N-1}$ and $\cos\theta_{N-2}\, I_{N-1}$. It then follows that for $N \geq 3$ we have $I_N = k_0^{N-1}\beta^{N-2}\sin^{N-3}\theta_1 \ldots \sin\theta_{N-3}$. Now suppose our external momenta depend only on n variables: k_0, β and the first $n-2$

angles. Then we can perform the integrations over the remaining angles:

$$
\begin{aligned}
&2\prod_{j=n-1}^{N-2}\int_0^{\pi} d\theta_j \sin^{N-2-j}\theta_j \\
=&2\prod_{j=n-1}^{N-2} 2^{N-2-j}\frac{\Gamma^2\left(\frac{N-1-j}{2}\right)}{\Gamma(N-1-j)} \\
=&2\prod_{j=n-1}^{N-2}\sqrt{\pi}\frac{\Gamma\left(\frac{N-1-j}{2}\right)}{\Gamma\left(\frac{N-j}{2}\right)} \\
=&2\sqrt{\pi}^{N-n}\frac{\Gamma(1/2)}{\Gamma\left(\frac{N-n+1}{2}\right)} \\
=&\frac{2\pi^{\frac{N-n+1}{2}}}{\Gamma\left(\frac{N-n+1}{2}\right)},
\end{aligned}
$$

where we replace the last integration over the range $[0, 2\pi]$ with $[0, \pi]$ and multiply by a factor of 2 outside the product. Given this we can write down the corresponding expression when N is replaced with a number of dimensions D:

$$
\int d^D k = \frac{2\pi^{\frac{D+1-n}{2}}}{\Gamma\left(\frac{D-n+1}{2}\right)}\int_{-\infty}^{+\infty} dk_0 k_0^{D-1}\int_0^{\infty} d\beta\beta^{D-2}\prod_{j=1}^{n-2}\int_0^{\pi} d\theta_j \sin^{D-2-j}\theta_j. \tag{A.5}
$$

The proof of the Euclidean measure proceeds in a similar manner except here we do not have a k_0 component so our matrix is lacking one row and column from this but the parametrisation is otherwise unchanged. Note that the final result is stated as

$$
\int d^D k^E = \frac{\pi^{\frac{D+1-n}{2}}}{\Gamma\left(\frac{D-n+1}{2}\right)}\int_0^{\infty} d|k^E|^2\left(|k^E|^2\right)^{D/2-1}\prod_{j=0}^{n-2}\int_0^{\pi} d\theta_j \sin^{D-2-j}\theta_j, \tag{A.6}
$$

where we have chosen to write $2d|k^E||k^E|^{D-1} = d|k^E|^2\left(|k^E|^2\right)^{D/2-1}$.

Finally, we note that in order to evaluate parametrised cut integrals it will be useful to perform the change of variables $\cos\theta_i = 2x_i - 1$, under which we find

$$\int_0^\pi d\theta \sin^\alpha\theta = 2^\alpha \int_0^1 dx[x(1-x)]^{\frac{\alpha-1}{2}}. \tag{A.7}$$

Appendix B

Full Form of the Appell F_4 Coaction

Here we will write two forms of the coaction of the Appell F_4 function. We will first provide the coaction which results from using the forms selected in (3.43), before writing it in an alternative form which is required for certain calculations involving Feynman integrals.

B.1 First Form

Let $\alpha = [\alpha] + \alpha_\epsilon \epsilon$, $\beta = [\beta] + \beta_\epsilon \epsilon$, $\gamma = [\gamma] + \gamma_\epsilon \epsilon$ and $\gamma' = [\gamma'] + \gamma'_\epsilon \epsilon$, where $[x]$ denotes the integer part of x, then:

$$\begin{aligned}
&\Delta F_4(\alpha, \beta; \gamma, \gamma'; x(1-y), y(1-x)) \qquad \text{(B.1)}\\
&= \Bigg[\frac{\alpha_\epsilon(-\beta_\epsilon + \gamma_\epsilon + \gamma'_\epsilon)}{2\gamma_\epsilon \gamma'_\epsilon} F_4(1 + \alpha_\epsilon \epsilon, \beta_\epsilon \epsilon; 1 + \gamma_\epsilon \epsilon, 1 + \gamma'_\epsilon \epsilon; x(1-y), y(1-x)) \\
&\quad + \frac{-\alpha_\epsilon + \gamma'_\epsilon}{2\gamma'_\epsilon} F_4(\alpha_\epsilon \epsilon, \beta_\epsilon \epsilon; \gamma_\epsilon \epsilon, 1 + \gamma'_\epsilon \epsilon; x(1-y), y(1-x)) \\
&\quad + \frac{-\alpha_\epsilon + \gamma_\epsilon}{2\gamma_\epsilon} F_4(\alpha_\epsilon \epsilon, \beta_\epsilon \epsilon; 1 + \gamma_\epsilon \epsilon, \gamma'_\epsilon \epsilon; x(1-y), y(1-x))
\end{aligned}$$

$$+\frac{\alpha_\epsilon\beta_\epsilon(1-x-y)}{2\gamma_\epsilon\gamma'_\epsilon}F_4(1+\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))\Bigg]$$

$$\otimes F_4(\alpha,\beta;\gamma,\gamma';x(1-y),y(1-x))$$

$$+\Bigg[\frac{\alpha_\epsilon(\alpha_\epsilon-\gamma_\epsilon)(-\beta_\epsilon+\gamma_\epsilon)(-\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}{2\gamma_\epsilon^2(\gamma'_\epsilon)^2}$$

$$\times F_4(1+\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$

$$+\frac{(\alpha_\epsilon-\gamma_\epsilon)(\beta_\epsilon-\gamma_\epsilon)(\alpha_\epsilon-\gamma'_\epsilon)}{2\gamma_\epsilon(\gamma'_\epsilon)^2}F_4(\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$

$$+\frac{(\alpha_\epsilon-\gamma_\epsilon)(\beta_\epsilon-\gamma_\epsilon)(\alpha_\epsilon+\gamma_\epsilon)}{2\gamma_\epsilon^2\gamma'_\epsilon}F_4(\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$

$$+\frac{\alpha_\epsilon\beta_\epsilon(\beta_\epsilon-\gamma_\epsilon)(-\alpha_\epsilon+\gamma_\epsilon)(1-x-y)}{2\gamma_\epsilon^2(\gamma'_\epsilon)^2}$$

$$\times F_4(1+\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))\Bigg]$$

$$\otimes\Bigg[e^{i\pi(\alpha+\beta-\gamma-\gamma')}\frac{\sin(\pi\gamma')}{\pi}\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\gamma)}{\Gamma(1-\gamma)\Gamma(2-\gamma)}$$

$$[x(1-y)]^{1-\gamma}F_4(1+\alpha-\gamma,1+\beta-\gamma;2-\gamma,\gamma';x(1-y),y(1-x))\Bigg]$$

$$+\Bigg[\frac{\alpha_\epsilon(\alpha_\epsilon-\gamma'_\epsilon)(-\beta_\epsilon+\gamma'_\epsilon)(-\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}{2\gamma_\epsilon^2(\gamma'_\epsilon)^2}$$

$$\times F_4(1+\alpha_\epsilon\epsilon,\beta-\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$

$$+\frac{(\alpha_\epsilon-\gamma'_\epsilon)(\beta_\epsilon-\gamma'_\epsilon)(\alpha_\epsilon+\gamma'_\epsilon)}{2\gamma_\epsilon(\gamma'_\epsilon)^2}F_4(\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$

$$+\frac{(\alpha_\epsilon-\gamma_\epsilon)(\alpha_\epsilon-\gamma'_\epsilon)(\beta_\epsilon-\gamma'_\epsilon)}{2\gamma_\epsilon^2\gamma'_\epsilon}F_4(\alpha_\epsilon\epsilon,\beta-\epsilon\epsilon;1+\gamma_\epsilon\epsilon,\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$

$$+\frac{\alpha_\epsilon\beta_\epsilon(\beta_\epsilon-\gamma'_\epsilon)(-\alpha_\epsilon+\gamma'_\epsilon)(1-x-y)}{2\gamma_\epsilon^2(\gamma'_\epsilon)^2}$$

$$\times F_4(1+\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))\Bigg]$$

$$\otimes\Bigg[e^{i\pi(\alpha+\beta-\gamma-\gamma')}\frac{\sin(\pi\gamma)}{\pi}\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1+\alpha-\gamma')\Gamma(1+\beta-\gamma')}{\Gamma(1-\gamma')\Gamma(2-\gamma')}$$

$$[y(1-x)]^{1-\gamma'}F_4(1+\alpha-\gamma',1+\beta-\gamma';\gamma,2-\gamma';x(1-y),y(1-x))\Bigg]$$

$$
\begin{aligned}
&+\Bigg[\frac{\alpha_\epsilon((\alpha_\epsilon-\gamma_\epsilon)(\beta_\epsilon-\gamma_\epsilon)-(\alpha_\epsilon+\beta_\epsilon)\gamma'_\epsilon+(\gamma'_\epsilon)^2)}{2\gamma_\epsilon\gamma'_\epsilon(-\alpha_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}\\
&\times F_4(1+\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))\\
&\frac{(\alpha_\epsilon-\gamma'_\epsilon)(\alpha_\epsilon-\gamma_\epsilon+\gamma'_\epsilon)}{2\gamma'_\epsilon(-\alpha_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}F_4(\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))\\
&+\frac{(\alpha_\epsilon-\gamma_\epsilon)(\alpha_\epsilon+\gamma_\epsilon-\gamma'_\epsilon)}{2\gamma_\epsilon(-\alpha_{i+1}\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}F_4(\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,\gamma'_\epsilon\epsilon;x(1-y),y(1-x))\\
&+\frac{\alpha_\epsilon\beta_\epsilon(1-x-y)}{2\gamma_\epsilon\gamma'_\epsilon}F_4(1+\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))\Bigg]\\
&\otimes\Bigg[\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(-1+\gamma)\Gamma(-1+\gamma')}{\Gamma(1-\gamma)\Gamma(1-\gamma')\Gamma(-1-\alpha+\gamma+\gamma')\Gamma(-1-\beta+\gamma+\gamma')}[x(1-y)]^{1-\gamma}[y(1-x)]^{1-\gamma'}\\
&F_4(2+\alpha-\gamma-\gamma',2+\beta-\gamma-\gamma';2-\gamma,2-\gamma';x(1-y),y(1-x))\Bigg].
\end{aligned}
$$

B.2 Second Form

With the choice of forms made in the previous section, we see that the functions appearing in the first entry are

$$
\begin{aligned}
&F_4(1+\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))\\
&F_4(\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))\\
&F_4(\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,\gamma'_\epsilon\epsilon;x(1-y),y(1-x))\\
&F_4(1+\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x)).
\end{aligned}
\tag{B.2}
$$

Observe that the coaction given in the previous section is ill-defined for cases where either α_ϵ or β_ϵ is zero. We can understand this by noticing that the basis of functions used in the first entries of the coaction degenerates in these cases as $F_4(\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$ and $F_4(\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$ both become equal to 1.

For such cases we must express the first entries using alternative functions, such as

$$\begin{aligned}
&F_4(1+\alpha_\epsilon\epsilon, \beta_\epsilon\epsilon; 1+\gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x)) \\
&F_4(\alpha_\epsilon\epsilon, 1+\beta_\epsilon\epsilon; 1+\gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x)) \\
&F_4(\alpha_\epsilon\epsilon, 1+\beta_\epsilon\epsilon; \gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x)) \\
&F_4(1+\alpha_\epsilon\epsilon, 1+\beta_\epsilon\epsilon; 1+\gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x)).
\end{aligned} \tag{B.3}$$

This yields an alternative formula for the coaction

$$\begin{aligned}
&\Delta F_4(\alpha, \beta; \gamma, \gamma'; x(1-y), y(1-x)) \\
&= \Bigg[-\frac{\beta_\epsilon(-\gamma_\epsilon+\gamma'_\epsilon)}{\gamma'_\epsilon(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)} F_4(\alpha_\epsilon\epsilon, 1+\beta_\epsilon\epsilon; \gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x)) \\
&\quad - \frac{\beta_\epsilon(\alpha_\epsilon-\gamma_\epsilon)(\alpha_\epsilon-\gamma_\epsilon-\gamma'_\epsilon)(-\beta_\epsilon+\gamma'_\epsilon)}{(\alpha_\epsilon-\beta_\epsilon)\gamma_\epsilon\gamma'_\epsilon(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)} \\
&\quad \times F_4(\alpha_\epsilon\epsilon, 1+\beta_\epsilon\epsilon; 1+\gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x)) \\
&\quad - \frac{\alpha_\epsilon(\beta_\epsilon-\gamma_\epsilon)(\beta_\epsilon-\gamma'_\epsilon)(\beta_\epsilon-\gamma_\epsilon-\gamma'_\epsilon)}{(\alpha_\epsilon-\beta_\epsilon)\gamma_\epsilon\gamma'_\epsilon(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)} \\
&\quad \times F_4(1+\alpha_\epsilon\epsilon, \beta_\epsilon\epsilon; 1+\gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x)) \\
&\quad + \frac{\alpha_\epsilon\beta_\epsilon(\gamma'_\epsilon-\gamma_\epsilon x-\gamma'_\epsilon y+\beta_\epsilon(-1+x+y))}{\gamma_\epsilon\gamma'_\epsilon(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)} \\
&\quad \times F_4(1+\alpha_\epsilon\epsilon, 1+\beta_\epsilon\epsilon; 1+\gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x))\Bigg] \\
&\quad \otimes F_4(\alpha, \beta; \gamma, \gamma'; x(1-y), y(1-x)) \\
&\quad + \Bigg[\frac{\beta_\epsilon(\alpha_\epsilon-\gamma_\epsilon)(\beta_\epsilon-\gamma_\epsilon)(\gamma_\epsilon+\gamma'_\epsilon)}{\gamma_\epsilon(\gamma'_\epsilon)^2(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)} F_4(\alpha_\epsilon\epsilon, 1+\beta_\epsilon\epsilon; \gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x)) \\
&\quad - \frac{\beta_\epsilon(\beta_\epsilon-\gamma_\epsilon)(-\alpha_\epsilon+\gamma_\epsilon)(\beta_\epsilon\gamma_\epsilon+\alpha_\epsilon(\beta_\epsilon-\gamma_\epsilon-\gamma'_\epsilon))(-\alpha_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}{(\alpha_\epsilon-\beta_\epsilon)(\gamma_\epsilon)^2(\gamma'_\epsilon)^2(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)} \\
&\quad \times F_4(\alpha_\epsilon\epsilon, 1+\beta_\epsilon\epsilon; 1+\gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x)) \\
&\quad + \frac{\alpha_\epsilon\beta_\epsilon(\alpha_\epsilon-\gamma_\epsilon)(-\beta_\epsilon+\gamma_\epsilon)(\beta_\epsilon-\gamma'_\epsilon)(-\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}{(\alpha_\epsilon-\beta_\epsilon)(\gamma_\epsilon)^2(\gamma'_\epsilon)^2(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)} \\
&\quad \times F_4(1+\alpha_\epsilon\epsilon, \beta_\epsilon\epsilon; 1+\gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x))
\end{aligned} \tag{B.4}$$

$$- \frac{\alpha_\epsilon\beta_\epsilon(\alpha_\epsilon-\gamma_\epsilon)(\beta_\epsilon-\gamma_\epsilon)(-(\gamma_\epsilon+\gamma'_\epsilon)(-1+y)+\beta_\epsilon(-1+x+y))}{(\gamma_\epsilon)^2(\gamma'_\epsilon)^2(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}$$
$$\times F_4(1+\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))]$$
$$\otimes\Bigg[e^{i\pi(\alpha+\beta-\gamma-\gamma')}\frac{\sin(\pi\gamma')}{\pi}\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\gamma)}{\Gamma(1-\gamma)\Gamma(2-\gamma)}$$
$$[x(1-y)]^{1-\gamma}F_4(1+\alpha-\gamma,1+\beta-\gamma;2-\gamma,\gamma';x(1-y),y(1-x))\Bigg]$$
$$+\Bigg[\frac{\beta_\epsilon(\alpha_\epsilon-\gamma'_\epsilon)(-\beta_\epsilon+\gamma'_\epsilon)(\gamma_\epsilon+\gamma'_\epsilon)}{\gamma_\epsilon(\gamma'_\epsilon)^2(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}F_4(\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$
$$-\frac{\beta_\epsilon^2(\alpha_\epsilon-\gamma_\epsilon)(\alpha_\epsilon-\gamma'_\epsilon)(\beta_\epsilon-\gamma'_\epsilon)(\alpha_\epsilon-\gamma_\epsilon-\gamma'_\epsilon)}{(\alpha_\epsilon-\beta_\epsilon)(\gamma_\epsilon)^2(\gamma'_\epsilon)^2(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}$$
$$\times F_4(\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$
$$-\frac{\alpha_\epsilon\beta_\epsilon(\beta_\epsilon-\gamma_\epsilon)(\beta_\epsilon-\gamma'_\epsilon)(\beta_\epsilon-\gamma_\epsilon-\gamma'_\epsilon)(-\alpha_\epsilon+\gamma'_\epsilon)}{(\alpha_\epsilon-\beta_\epsilon)(\gamma_\epsilon)^2(\gamma'_\epsilon)^2(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}$$
$$\times F_4(1+\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$
$$-\frac{\alpha_\epsilon\beta_\epsilon(\alpha_\epsilon-\gamma'_\epsilon)(-\beta_\epsilon+\gamma'_\epsilon)((\gamma_\epsilon+\gamma'_\epsilon)x-\beta_\epsilon(-1+x+y))}{(\gamma_\epsilon)^2(\gamma'_\epsilon)^2(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}$$
$$\times F_4(1+\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))]$$
$$\otimes\Bigg[e^{i\pi(\alpha+\beta-\gamma-\gamma')}\frac{\sin(\pi\gamma)}{\pi}\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1+\alpha-\gamma')\Gamma(1+\beta-\gamma')}{\Gamma(1-\gamma')\Gamma(2-\gamma')}$$
$$[y(1-x)]^{1-\gamma'}F_4(1+\alpha-\gamma',1+\beta-\gamma';\gamma,2-\gamma';x(1-y),y(1-x))\Bigg]$$
$$+\Bigg[-\frac{\beta_\epsilon(\gamma_\epsilon-\gamma'_\epsilon)}{\gamma'_\epsilon(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}F_4(\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$
$$-\frac{\beta_\epsilon(\alpha_\epsilon-\gamma_\epsilon)(\alpha_\epsilon(-\beta_\epsilon+\gamma_\epsilon)+\beta_\epsilon(-\gamma_\epsilon+\gamma'_\epsilon))}{(\alpha_\epsilon-\beta_\epsilon)\gamma_\epsilon\gamma'_\epsilon(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}$$
$$\times F_4(\alpha_\epsilon\epsilon,1+\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$
$$-\frac{\alpha_\epsilon\beta_\epsilon(\beta_\epsilon-\gamma_\epsilon)(\beta_\epsilon-\gamma'_\epsilon)}{(\alpha_\epsilon-\beta_\epsilon)(\gamma_\epsilon\gamma'_\epsilon(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon))}$$
$$\times F_4(1+\alpha_\epsilon\epsilon,\beta_\epsilon\epsilon;1+\gamma_\epsilon\epsilon,1+\gamma'_\epsilon\epsilon;x(1-y),y(1-x))$$
$$+\frac{\alpha_\epsilon\beta_\epsilon(\gamma_\epsilon-\gamma'_\epsilon x-\gamma_\epsilon y+\beta_\epsilon(-1+x+y))}{\gamma_\epsilon\gamma'_\epsilon(-2\beta_\epsilon+\gamma_\epsilon+\gamma'_\epsilon)}$$

$$
\times F_4(1+\alpha_\epsilon\epsilon, 1+\beta_\epsilon\epsilon; 1+\gamma_\epsilon\epsilon, 1+\gamma'_\epsilon\epsilon; x(1-y), y(1-x))]
$$

$$
\otimes \left[\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(-1+\gamma)\Gamma(-1+\gamma')}{\Gamma(1-\gamma)\Gamma(1-\gamma')\Gamma(-1-\alpha+\gamma+\gamma')\Gamma(-1-\beta+\gamma+\gamma')}[x(1-y)]^{1-\gamma}[y(1-x)]^{1-\gamma'}\right.
$$

$$
\left. F_4(2+\alpha-\gamma-\gamma', 2+\beta-\gamma-\gamma'; 2-\gamma, 2-\gamma'; x(1-y), y(1-x))\right].
$$

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