The Noetherianity of Idealizers in Skew Extensions

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Ruth Alice Elizabeth Reynolds*)

To my parents

Publications

This thesis is based in part on work of the author contained in a paper published in the Journal of Algebra [32] (Available online with DOI:10.1016/j.jalgebra.2019.06.045). This paper forms the material for Chapter 3 with minimal changes to fit the narrative of the thesis.

Lay Summary

Much as there are two sides to every story, the properties of an object may change depending on the side from which you look at it. Whilst this asymmetrical notion in mathematics is difficult to imagine in the abstract, we are fortunate that we may turn to the real world for an example.

Figure 1: A 'two-faced' painting by Bois-Clair

Someone standing on the left-hand side of the picture in Figure 1 will see a different picture from a viewer on the right-hand side. Hence this painting has a natural asymmetry.

Ring theory is an area of mathematics which has analogues of symmetric objects (commutative rings) where the viewpoint does not matter, and nonsymmetric objects (noncommutative rings) where the viewpoint does matter, like this picture.

Given two whole numbers, we can add these numbers together to get another whole number. We can also multiply these numbers together and we will get a whole number, and we know how to expand brackets:

$$3 + 4 = 7, \quad 3 \times 4 = 12, \quad 3 \times (4 + 4) = 3 \times 4 + 3 \times 4 = 24.$$

Having these notions of addition and multiplication are what makes the set of whole numbers into a *ring* which is called the *ring of integers*, denoted $\mathbb{Z}$. More generally, a ring is a set of elements with two operations, adding and multiplying, which satisfy some rules (known as axioms) such as expanding brackets. Note that the order of multiplication does not matter for the integers: $3 \times 4 = 4 \times 3$. This is a special property which makes $\mathbb{Z}$ a *commutative ring*. On the face of it, it may seem as though every set of elements would give a commutative ring. This is not the case! Consider the set of functions which take one integer and return another. We can define multiplication on this set by composition, and addition by addition of functions so this is a ring. Let F and G be two such functions where

$$F(x) = x + 1 \quad G(x) = x^2.$$

Then the compositions $F \circ G$ and $G \circ F$ are different. We can see this as:

$$F \circ G(2) = F(2^2) = 2^2 + 1 = 5$$
$$G \circ F(2) = G(2 + 1) = (2 + 1)^2 = 9,$$

and hence, as $5 \neq 9$, we have $F \circ G \neq G \circ F$. Thus in this example the order of multiplication matters and this is a *noncommutative ring*.

In noncommutative ring theory we are interested in studying objects with asymmetric properties. Noncommutative rings can have different structures due to the multiplication, so called 'left' and 'right' structures. If a ring has a certain property when viewed from the left, it will not necessarily have the same property when viewed from the right.

Rings contain special subsets called *ideals*. Consider the set of even numbers in $\mathbb{Z}$. Adding two even numbers will give an even result, and if we multiply an even number by any other integer the answer will also be an even number. Being able to add two elements together, or to multiply by elements in the ring, and stay in the same subset is what makes the set of even numbers into an ideal of $\mathbb{Z}$. Idealizers, as the name suggests, are rings which make ideals.

If the ideals of a ring satisfy a certain finiteness property then we call the ring a *noetherian ring*. Essentially, we wish for every ideal to be able to be built out of a finite amount of information (finitely generated) as then they are much easier to handle. Noetherian rings play a vital role in many areas of abstract algebra and were first studied by the emminent mathematician Emmy Noether in the first half of the 20^{th} century. A lot of fundamental results are proved under the assumption that the ring is noetherian and, in some senses, a noetherian ring is thought of as 'well-behaved'. However, it can be incredibly difficult to prove that a ring is noetherian. In mathematics, noetherianity is a property which may be asymmetrical, and in this thesis we use idealizers to create many examples of rings which are noetherian on only one side.

It is important in mathematics to "classify" objects such as rings; which means finding all rings which satisfy certain properties. When mathematicians try to classify noncommutative rings idealizers are often part of the answer! Thus if we want to understand noncommutative rings, understanding idealizers is essential.

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Abstract

Given a right ideal I in a ring R, the idealizer of I in R is the largest subring of R in which I becomes a two-sided ideal. These rings are of interest since they often give examples of rings with asymmetrical properties, and they may also display other pathological behaviour. For example, they provide many examples of noncommutative rings which are right but are not left noetherian. In this thesis we examine the behaviour of idealizers in skew extensions of commutative rings. We first focus on the skew group ring $B = C\#G$, where C is a commutative noetherian domain and G is a finitely generated abelian group. For a prime ideal I of C, we study the idealizer of the right ideal IB in B and we obtain necessary and sufficient conditions for when the idealiser is left and right noetherian and we relate them to geometry, deriving interesting properties. We also give an example of these conditions in practice which translates to a curious number theoretic problem. We next consider idealizers in the second Weyl algebra A_2, which is the ring of differential operators on $\Bbbk[x, y]$ (in characteristic 0). Specifically, let f be a polynomial in x and y which defines an irreducible curve whose singularities are all cusps. We show that the idealizer of the right ideal fA_2 in A_2 is always left and right noetherian, extending the work of McCaffrey.

Contents

Chapter 1

Introduction

Let R be a ring and let I be a right ideal of R. We define the *idealizer subring of I in R* (or simply the *idealizer*) to be

$$\mathbb{I}_R(I) \coloneqq \{r \in R \mid rI \subseteq I\}.$$

That is to say, $\mathbb{I}_R(I)$ is the largest subring of R in which I becomes a two-sided ideal. This ring has no commutative analogue and so it is an intrinsically noncommutative concept. In this thesis we study the behaviour of idealizer subrings in skew extensions. In particular, we are interested in when idealizer subrings are left or right noetherian. Noetherianity is a finiteness property which is desirable for a ring to have, however it can be very complicated to test whether a ring is noetherian. One difficulty is that subrings of noetherian rings are not necessarily noetherian as the following example shows.

Example 1.1.1. Let $\Bbbk[x, y]$ be a polynomial ring in two variables, and let

$$R = \Bbbk + y\Bbbk[x, y] \subsetneqq \Bbbk[x, y].$$

Consider the ascending chain of ideals

$$0 \subseteq (yx) \subseteq (yx, yx^2) \subseteq (yx, yx^2, yx^3) \subseteq \dots$$

We can see that $yx^{i+1} \notin (yx^i)$ as there are no elements in R which will increase the power of x without increasing the power of y too. Hence R is not a noetherian ring, although it is a subring of the noetherian ring $\Bbbk[x, y]$.

This badly behaved ring has a similar construction to the following noncommutative ring:

$$R_J = \Bbbk + y\Bbbk_J[x, y] \subsetneq \Bbbk_J[x, y],$$

where $\Bbbk_J[x, y]$ is the (affine) Jordan plane. The Jordan plane has the following presentation

$$\Bbbk_J[x, y] = \frac{\Bbbk \langle x, y \rangle}{\langle xy - yx - x^2 \rangle}$$

and is a left and right noetherian ring. The ring R_J also has strange behaviour, in particular, this ring is left and right noetherian precisely when the characteristic of the field $\Bbbk$ is 0 [41]. In fact when the characteristic of $\Bbbk$ is 0, R_J is the idealizer of the right ideal $y\Bbbk_J[x, y]$. Part of the reason why idealizers are so interesting to study is that they exhibit such pathological behaviour as we will see in more detail in this thesis.

It is natural in noncommutative ring theory to search for examples of rings which reflect their noncommutative structure. For example, rings which have properties which commutative rings cannot have. In particular, we search for rings which have some kind of asymmetry. In this thesis, specifically in Chapter 3, our study of idealizers allows us to find many examples of rings which are only noetherian on one side.

Robson was the first to study the noetherianity of idealizers. He obtained the following result which shows how the noetherianity of idealizers can completely control the noetherianity of the ambient ring.

Theorem 1.1.2. [33, Theorem 2.3] *Let R be a ring and let I be a maximal right ideal of R. Then R is right noetherian if and only if $\mathbb{I}_R(I)$ is right noetherian.*

We note that Robson does not address left noetherianity in his paper. In this thesis we do address left noetherianity, and we show that an idealizer of a right ideal being left noetherian can be a much more subtle property to prove.

An important classification in noncommutative ring theory is the classification of noncommutative curves by Artin and Stafford.

Theorem 1.1.3. [1] *Let R be a connected graded noetherian domain of GK dimension* 2 *(a noncommutative curve). Then R is one of the following:*

1. *a twisted homogeneous coordinate ring (see* [2] *for a definition);*

2. *an idealizer inside a twisted homogeneous coordinate ring.*

This theorem and related results in GK dimension 3 [36] show it is important to understand the behaviour of idealizers as they should feature in any suitably general class of noncommutative rings.

The first type of skew extension we work with is called the skew group ring. If a group G acts on a ring R, the elements of the skew group ring $R\#G$ are of the form

$$\sum_{g \in G} r_g g$$

where $r_g \in R$ and only finitely many are nonzero. These rings are very well understood. For instance, if R is left and right noetherian, and G is a polycyclic-by-finite group, then $R\#G$ is also a left and right noetherian ring. However, the noetherianity of idealizers in $R\#G$ has not been previously studied. We investigate this in Chapter 3.

Skew group rings are a long established noncommutative construction so it is strange that idealizers had not yet been studied in these foundational rings. However, skew group rings are closely related to twisted homogeneous coordinate rings where idealizers have been studied, hence it is a natural to ask whether the noetherianity of idealizers in skew group rings behaves similarly. In Chapter 3 we answer this question completely for skew group rings of the form $C\#G$ where C is a commutative noetherian domain and G is a finitely generated abelian group.

Theorem 1.1.4. [Theorems 3.1.4 and 3.1.5] *Let C be a commutative noetherian domain, let G be a finitely generated abelian group acting on C and let $B = C\#G$ be the skew group ring. For an prime ideal $I \lhd C$, consider the idealizer subring $\mathbb{I}_B(IB)$. Then the left and right noetherianity of $\mathbb{I}_B(IB)$ depends on the properties of the orbit of $\mathbb{V}(I)$ under action of G on* $\operatorname{Spec} C$.

Theorem 1.1.4 is related to work of Sierra and Rogalski [37, 34] who studied idealizers in twisted homogeneous coordinate rings and they obtained the following result which we paraphrase.

Theorem 1.1.5. [37, Theorem 10.2] *Let $B = B(X, \mathcal{L}, \sigma)$ be a twisted homogeneous coordinate ring, where X is a projective variety and $\sigma \in \operatorname{Aut} X$. Let I be a right ideal in B defined by a closed subvariety Z of X. Then $\mathbb{I}_B(I)$ is left and right noetherian when the orbit of Z under the σ has similar properties to Theorem 1.1.4.*

Although it is not clear from our statements here, studying idealizers in skew group rings and twisted homogeneous coordinate rings also allows for the natural asymmetry of idealizers to arise. Indeed, it becomes evident in Chapter 3 that it is very easy to construct rings which are only noetherian on one side.

Example 1.1.6. Let $C = \Bbbk[x, y]$ and let $\mathbb{Z}^2$ act on C by

$$(m, n) * f(x, y) = f(x + m, y + n).$$

Then, if $B = C\#\mathbb{Z}^2$, the idealizer $\mathbb{I}_B((x, y)B)$ is right, but not left noetherian. In fact, any maximal ideal $M \lhd C$ will give an idealizer $\mathbb{I}_B(MB)$ which is right but not left noetherian. This example leads to an interesting number theoretic problem on the finiteness of integer points on translated curves in $\mathbb{A}^2$. The reasons for this will become clear in Chapter 3.

A related example to this was essential in the proof by Sierra and Walton that the universal enveloping algebra of the Witt algebra is not noetherian [38] which answered a question which had been open for over 20 years.

In Chapter 5 we also study idealizers in a different kind of Ore extension of a commutative ring, the second Weyl algebra A_2. We prove the following result.

Theorem 1.1.7. [Theorem 5.1.6] *Let $f \in \Bbbk[x, y]$ define an irreducible algebraic curve which has only cuspidal singularities. Then the idealizer subring $\mathbb{I}_{A_2}(fA_2)$ is left and right noetherian.*

Theorem 1.1.7 is related to the work of McCaffrey [25] who studied idealizers at nonsingular curves and proved the following result.

Theorem 1.1.8. [25] *Let $f \in \Bbbk[x, y]$ be define a nonsingular curve. Then $\mathbb{I}_{A_2}(fA_2)$ is right noetherian.*

In Chapter 5 we generalise this result.

We are also interested in the ring of differential operators of a commutative ring. If R is the commutative ring in question, denote the ring of differential operators by $\mathscr{D}(R)$. Just as you might relate polynomials to a set of algebraic equations, differential operators provide an algebraic framework to handle differential equations. While the proper definition of a differential operator is too long and technical for an introduction, we can give some sense of how these operators behave. In particular, if $R = \Bbbk[x_1, \ldots, x_n]$ is a polynomial ring, partial derivatives $\partial/\partial x_i$ are examples of differential operators on R. These partial derivatives satisfy the Leibniz (product) rule and so they are a special type of operator called a derivation. Differential operators are a generalisation of derivations. The ring of differential operators on a polynomial ring $\Bbbk[x_1, \ldots, x_n]$ is actually a very well known ring, the n^{th} Weyl algebra A_n. Idealizers are related to rings of differential operators on curves in the following way.

Theorem 1.1.9. [Proposition 5.1.7] *Let $X \subseteq \mathbb{A}^2$ be a curve defined by a polynomial $f \in \Bbbk[x, y]$. Then*

$$\mathscr{D}(\mathcal{O}(X)) \cong \frac{\mathbb{I}_{A_2}(fA_2)}{fA_2}.$$

Smith and Stafford [39] showed that if $X = \operatorname{Spec} R$ is an irreducible singular algebraic curve, then $\mathscr{D}(R)$ shares many properties with $\mathscr{D}(R')$ where R' is the integral closure of R in its field of fractions. In particular, both of these rings are left and right noetherian.

Given the link between idealizers and differential operators described in Theorem 1.1.9, a natural question is whether idealizers at singular curves are left and right noetherian. We answer this question in the affirmative in the case where the singularities are suitably well-behaved.

Example 1.1.10. Consider the curve $X = \mathbb{V}(y^3 - x^2)$. This is a singular curve with a cuspidal singularity at the origin. By [39], $\mathscr{D}(\mathcal{O}(X))$ is closely related to A_1 which is the first Weyl algebra. Hence $\mathscr{D}(\mathcal{O}(X))$ is a left and right noetherian ring.

By Theorem 1.1.7, the idealizer $\mathbb{I}_{A_2}((y^3 - x^2)A_2)$ is also left and right noetherian and the precise details are left till Chapter 5.

Chapter 2

Preliminaries and Motivation

In this chapter we provide the background material for Chapter 3 and we also give some motivation for our interest in idealizers. In particular, we justify our description of idealizers as noncommutative mathematical objects which are important to study for both their interesting and often pathological properties and also how they fit into the wider context of noncommutative ring theory. Throughout this thesis $\Bbbk$ will denote an algebraically closed field of characteristic 0 and all of our rings will be $\Bbbk$-algebras.

2.1 Motivating Examples

We begin this chapter with some examples of subrings with interesting properties. We will subsequently see that these examples are a type of ring called an idealizer, the study of which is the basis of this entire thesis.

Definition 2.1.1. Let R be a ring and let σ be a ring automorphism of R. Let T be the ring which, as a free left R-module is given by

$$\bigoplus_{n \in \mathbb{Z}} Rx^n,$$

and where multiplication is induced from the rule

$$xr = \sigma(r)x.$$

This ring T is referred to as a *skew-Laurent ring over* R, denoted $R[x^{\pm 1}; \sigma]$. This construction may be repeated to form an *iterated skew-Laurent ring over* R if the chosen automorphisms commute.

Example 2.1.2. Let $R = \Bbbk[x, y]$ and let $\sigma, \tau \in \mathrm{Aut}(R)$ be defined by

$$\sigma(x) = x + 1 \quad \sigma(y) = y$$

and

$$\tau(x) = x \quad \tau(y) = y + 1.$$

Then we may form the skew-Laurent polynomial ring $B = R[s^{\pm 1}, t^{\pm 1}; \sigma, \tau]$ where $st = ts$. Subrings of this ring are of particular interest to us in Chapter 3.

Definition 2.1.3. Let R be a ring and let σ be a ring endomorphism of R. An additive map $\delta : R \to R$ is a σ-*derivation* if

$$\delta(rs) = \delta(r)\sigma(s) + r\delta(s)$$

for all $r, s \in R$.

Using this definition we may construct a family of fundamental and well-understood examples of noncommutative rings.

Definition 2.1.4. Let R be a ring, σ a ring endomorphism of R (note that we do not require an automorphism as in Definition 2.1.1), and δ a σ-derivation of R. Let $S = R[x; \sigma, \delta]$ be the ring which, as a free left R-module, is

$$\bigoplus_{n \geq 0} Rx^n$$

where multiplication is induced by the rule

$$xr = \sigma(r)x + \delta(r)$$

for all $r \in R$. We call S an *Ore extension of* R.

A more detailed discussion of these rings may be found in Chapters 1 and 2 of [19].

Example 2.1.5. Let $R = \Bbbk[x]$, $\sigma = \mathrm{id}$ and let $\delta = d/dx$. Then we may form the Ore extension (or formal differential operator ring) $S = \Bbbk[x][y; \mathrm{id}, d/dx] = \Bbbk[x][y; d/dx]$. This is a ring with the following multiplication rule:

$$yx = xy + 1.$$

This ring is more commonly known as the *first Weyl algebra*, and is denoted A_1. By iterating this construction we may form the n^{th} Weyl algebra A_n. These rings, and rings of this form, will be the focus of Chapter 5. For ease of notation, we will often replace the variable y with ∂_x and refer to the Weyl algebra as the ring generated by x and ∂_x with relation

$$\partial_x x = x\partial_x + 1,$$

which we will denote $\mathbb{k}[x, \partial_x]$. We take a similar approach with the second Weyl algebra and denote this as $A_2 = \mathbb{k}[x, y, \partial_x, \partial_y]$.

Let us now consider some subrings of these ring extensions with interesting properties.

Example 2.1.6. Firstly, let us consider the skew-Laurent extension B of $\mathbb{k}[x, y]$ described in Example 2.1.2. Let

$$R' = \mathbb{k}[x, y] + (x, y)B.$$

While this may look like a fairly innocuous ring, it has an interesting asymmetry - this ring is right noetherian but not left noetherian. The results in this thesis show that the reason for this lies in the properties of the orbit of the point $(0, 0) \in \mathbb{A}^2$ under the action by σ and τ. In particular, the set

$$\{(n, m) \in \mathbb{Z}^2 \mid \sigma^n\tau^m((0, 0)) = (0, 0)\}$$

is finite, and hence the ring is right noetherian. However, the set

$$\mathbb{V}(y) \cap \{\sigma^n((0, 0))\}_{n\in\mathbb{Z}}$$

is infinite, and so the ring is not left noetherian. It turns out that this ring is actually the idealizer subring of the right ideal $(x, y)B$ in B and this is a special case of the theory developed in Chapter 3.

Example 2.1.7. Let us now consider the Ore extension from Example 2.1.5 which we recall is the first Weyl algebra A_1. Consider the subring

$$K = \mathbb{k}[x] + xA_1,$$

which has a similar form to the subring in Example 2.1.6. However, in this instance this ring is both left and right noetherian. This turns out to be the idealizer subring of xA_1 in A_1. We will study generalisations of this subring in Chapter 5 and we will provide a more detailed explanation of this phenomenon, and constructions like this, in the background to Chapter 5.

2.2 A Brief History of Idealizers

In this section, we provide a survey of the properties of idealizers found in the literature. Whilst the majority of this does not directly contribute to our research, it does show that idealizers are ubiquitous and provide a wealth of examples of interesting concepts in noncommutative ring theory. Moreover, the properties they have, and their nice relationships with their ambient rings, are highly useful.

Idealizers were first studied in detail by Robson, [33, 24] who used them as a valuable tool to study the structure of a special type of ring called an HNP ring (see Definition 2.2.3). More precisely, when studying an HNP ring R and its representation theory, it is necessary to study all rings S which lie between R and its classical right quotient ring $Q(R)$, which we will define later (Definition 2.3.8). He focused, in particular, on those intermediate rings such that S_R is finitely generated. As Goodearl observed [18], one might view S as some type of localisation of R, whence there is a close relationship between the properties of R, S, and their simple modules.

Robson made particular headway with two types of right ideal, those which are semimaximal or generative. In this section we describe the results he obtained.

2.2.1 Idealizers and Noetherianity

In this section we summarise the work of Robson and Resco on the noetherianity of idealizer subrings. We begin by recalling some basic definitions which we include both for completeness and as a reminder that we are working in a noncommutative setting.

Definition 2.2.1. Let R be a a ring. Then R is *right (left) noetherian* if every ascending chain of right (left) ideals eventually stabilises. That is to say if

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$$

is an ascending chain of right (left) ideals of R, then there exists an $N \in \mathbb{N}$ such that

$I_N = I_{N+k}$ for all $k \geq 1$. We say that R is *noetherian* if it is both left and right noetherian.

We say that a ring R is *simple* if it has no ideals other than 0 and itself. We define a *domain* to be a non-zero ring in which $ab = 0$ implies $a = 0$ or $b = 0$. For a proper ideal P of R, we say that P is a *prime ideal* if whenever $I, J \lhd R$ such that $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. A *prime ring* is a ring in which 0 is a prime ideal.

Definition 2.2.2. We define a ring R to be *right (left) hereditary* if every submodule of a projective right (left) R-module is again projective. We say a ring is *hereditary* if it is both left and right hereditary.

Definition 2.2.3. We say a ring is *HNP* if it is hereditary, noetherian, and prime.

Proposition 2.2.4. *Let R be a $\Bbbk$-algebra which is infinite dimensional as a $\Bbbk$-vector space. If R is simple, then R has no finite dimensional modules.*

Proof. Let M be a finite dimensional left R-module with $\Bbbk$-basis $\{m_1, \ldots, m_n\}$ as a $\Bbbk$-vector space. Then we may construct a map:

$$\begin{aligned} \phi : R &\to M^n \\ r &\mapsto (rm_1, \ldots, rm_n). \end{aligned}$$

Then $\ker \phi = \bigcap_{i=1}^n \mathrm{ann}_R(m_i) = \mathrm{ann}_R M \lhd R$. Further, since R is infinite dimensional and M is finite dimensional, this map must have a non-trivial kernel. Hence, R contains a non-trivial proper ideal and cannot be simple. □

Definition 2.2.5. Let A be a right ideal in a ring S, then A is *semimaximal* if S/A is semisimple. We say that A is *generative* if $SA = S$.

An easy example of a generative right ideal is any non-zero right ideal in a simple ring. Robson showed that idealizers at semimaximal and generative ideals share many properties with the ambient ring. In fact, in HNP rings it is possible to view the ring in which you are idealizing as a localisation of the idealizer, and undo this process by idealizing.

Proposition 2.2.6. [33, Theorem 2.2] *Let A be a semimaximal right ideal in a ring S and let $R = \mathbb{I}_S(A)$. Then R is right noetherian if and only if S is right noetherian.*

Proposition 2.2.7. [33, Proposition 2.3] *Let A be a generative right ideal in a ring S and let $R = \mathbb{I}_S(A)$. Then R is right noetherian if and only if S is right noetherian and S/A is a noetherian right R-module.*

Robson also obtained strong results on the interaction of the dimension theory between these rings, in particular the global dimension.

Definition 2.2.8. Let R be a ring. We define the *right global dimension* as:

$$\operatorname{rgldim} R = \sup\{\operatorname{pdim} M \mid M \text{ is a right } R\text{-module}\},$$

where $\operatorname{pdim} M$ is the minimal length among all finite projective resolutions of M. We may define left global dimension symmetrically.

For an arbitrary ring the right and left global dimensions may be different. However, if R is a noetherian ring then these values coincide. Robson studied idealizers in rings which have a special property related to global dimension. We now provide a useful alternative definition of a hereditary ring.

Lemma 2.2.9. *The following are equivalent for a ring R:*

1. *R is right hereditary;*

2. $\operatorname{rgldim} R = 1$.

The first Weyl algebra is hereditary. In fact, A_1 is HNP and Robson showed that idealizer subrings in HNP rings are particularly well behaved.

Theorem 2.2.10. [33, Theorem 4.3] *Let S be an HNP ring and let A be a semimaximal right ideal in S. Then $\mathbb{I}_S(A)$ is also an HNP ring.*

Theorem 2.2.11. [33, Theorem 7.4] *Let $S = A_1$, the first Weyl algebra. Then the following hold:*

1. *if R is any subring which contains both $\Bbbk$ and a non-zero right ideal A of S, then R is right noetherian;*

2. *if, further, $\mathbb{I}_S(A) \subseteq R$, then R is also left noetherian.*

We note that in all but a few examples considered by Robson, he does not consider the question of left noetherianity for an idealizer at a right ideal.

In [30, 31], Resco also studied idealizers in Weyl algebras. He obtained the following theorem which describes the noetherianity of certain idealizers in higher Weyl algebras.

Theorem 2.2.12. [30, Proposition 2.3] *Let $I \leq_r A_1$ be a right ideal in the first Weyl algebra. Then $\mathbb{I}_{A_n}(IA_n)$ is right and left noetherian.*

In particular he was the first to observe the asymmetric nature of idealizers as displayed in the following theorem.

Theorem 2.2.13. [31, Theorem 2] *Let $A_2 = \Bbbk[x, y, \partial_x, \partial_y]$ be the second Weyl algebra over $\Bbbk$. Then the idealizer $\mathbb{I}_{A_2}((x, y)A_2)$ is a $\Bbbk$-affine domain which is right but not left noetherian.*

In particular, this shows that left noetherianity of an idealizer is not well-understood and warrants investigation. In Chapter 3 we do consider left noetherianity and we shall see shortly that this is a much more subtle property which leads to some very interesting ergodic-type requirements.

2.2.2 Results of Stafford and Rogalski on Noetherianity

In [43], idealizers were a crucial construction which Stafford used to find a number of interesting new examples of rings with peculiar properties. As part of this, he studied the noetherianity of idealizers and obtained the following result.

Lemma 2.2.14. [43, Lemma 1.2] *Let I be a right ideal in a noetherian ring R. Assume that, for every right ideal $J \supseteq I$ of R, $\text{Hom}_R(R/I, R/J)$ is right noetherian as a right $\mathbb{I}_R(I)$-module. Then $\mathbb{I}_R(I)$ is right noetherian.*

We shall use this result a great deal in both Chapters 3 and 5. As with Robson's results on noetherianity, this result is purely a 'right-handed' result. However, Stafford provides the following useful trick which allows us to occasionally completely avoid the problem.

Lemma 2.2.15. [43, Lemma 1.3] *Let x be a regular element in a prime, noetherian ring R such that there exists an antiautomorphism α of R satisfying $\alpha(xR) = Rx$. Then $\mathbb{I}_R(xR)$ is right noetherian if and only if it is left noetherian.*

Rogalski in [35] generalised Lemma 2.2.14 to the following:

Proposition 2.2.16. [35, Proposition 2.1] *Let $T = \mathbb{I}_S(I)$ be the idealizer of the right ideal I in a noetherian ring S, and assume in addition that S_T is finitely generated. The following are equivalent:*

1. *T is right noetherian;*
2. *$\mathrm{Hom}_S(S/I, S/J)$ is a noetherian right T-module (or T/I-module) for all right ideals J of S which contain I.*

Rogalski further provided the first general necessary and sufficient conditions for an idealizer at a right ideal to be left noetherian.

Proposition 2.2.17. [35, Proposition 2.2] *Let S be a noetherian ring with right ideal I, and let $T = \mathbb{I}_S(I)$ be the idealizer subring of I in S. The following are equivalent:*

1. *T is left noetherian;*
2. *T/I is a left noetherian ring, and*

$$\mathrm{Tor}_1^S(S/I, S/K) = (I \cap K)/IK$$

is a noetherian left T-module (or T/I-module) for all left ideals K of S.

As will become clear in later chapters of this thesis, whilst these results do provide necessary and sufficient conditions for the idealizer subring to be left and right noetherian, actually testing whether these conditions hold is highly non-trivial. In particular, the left noetherian condition is very tricky to verify.

2.2.3 Graded and Filtered Rings

Since a lot of the techniques in this thesis rely on the theory of graded and filtered rings, we provide all the necessary definitions here. In particular, we recall the notion of a G-graded ring for more general groups than just $\mathbb{Z}$.

Definition 2.2.18. Let R be a ring and let G be a group. Then R is defined to be a *G-graded ring* if $R = \bigoplus_{g \in G} R_i$ where the R_g are additive abelian subgroups such that

$$R_g R_h \subseteq R_{gh}$$

for all $g, h \in G$. The elements of R_g are referred to as the *homogeneous elements of degree g*.

Remark 2.2.19. We note that we may define an $\mathbb{N}$-graded ring (although $\mathbb{N}$ is not a group) by viewing R as a $\mathbb{Z}$-graded ring where $R_{-i} = 0$ for $i > 0$. An $\mathbb{N}$-graded ring is called *connected graded (cg)* if $R_0 = \Bbbk$.

Definition 2.2.20. Let R be a G-graded ring and let M be a right R-module. We say that M is a *G-graded right R-module* if $M = \bigoplus_{g\in G} M_g$ and

$$M_g R_h \subseteq M_{g*h}$$

for all $g, h \in G$. If N is a submodule of M, we say N is a G-graded submodule if $N = \bigoplus_{g\in G} N_g$ such that $N_g = N \cap M_g$ for all $g \in G$. In this case, the factor module M/N is also G-graded by defining

$$M/N = \bigoplus_{g\in G} M_g/N_g.$$

Definition 2.2.21. Let M and N be two G-graded right R-modules. We define a *graded homomorphism* $\phi : M \to N$ to be a homomorphism of right R-modules which respects the grading. That is to say

$$\phi(M_g) \subseteq N_g$$

for all $g \in G$.

Definition 2.2.22. Let R be a $\mathbb{Z}$-graded ring, we define the n^{th} *Veronese of* R to be the ring

$$R^{(n)} = \bigoplus_{i\in\mathbb{Z}} R_{ni}.$$

Definition 2.2.23. If every ascending chain of graded right ideals stabilises, then we refer to the ring R as *graded right noetherian.*

We have a weaker notion than that of a graded ring which is also useful.

Definition 2.2.24. Let R be a $\Bbbk$-algebra. We say that R is a *filtered ring* if there exists an increasing sequence of $\Bbbk$-vector spaces of R,

$$0 \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$$

such that $R = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$, and the $\mathcal{F}_n$ are compatible with multiplication as follows:

$$\mathcal{F}_i \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$$

for all $i, j \in \mathbb{N}$. We say this is a *finite filtration* if each $\mathcal{F}_i$ is a finite dimensional $\Bbbk$-vector space.

Definition 2.2.25. Let M be a module over a filtered ring $R = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. We say that M is a *filtered module* if there exists an ascending chain of subspaces

$$0 \subseteq M(0) \subseteq M(1) \subseteq M(2) \subseteq \cdots \subseteq M$$

such that $M = \bigcup_{n\in\mathbb{N}} M(n)$ and

$$M(n)\mathcal{F}_i \subseteq M(n+i)$$

for all $n, i \in \mathbb{N}$.

Example 2.2.26. Let R be a finitely generated $\Bbbk$-algebra and let V be a finite-dimensional generating set containing 1. Define $\mathcal{F}_i = V^i$. Then we observe that this forms a finite filtration on R.

Let $S = \bigoplus_{n\in\mathbb{N}} S_n$ be an $\mathbb{N}$-graded ring. Then we may view S as a filtered ring by defining

$$\mathcal{F}_i = S_0 \oplus S_1 \oplus \cdots \oplus S_i.$$

Definition 2.2.27. Let $R = \bigcup_{n\in N} \mathcal{F}_n$ be a filtered ring. Define $A_i = \mathcal{F}_{i+1}/\mathcal{F}_i$. If $a \in \mathcal{F}_n \setminus \mathcal{F}_{n-1}$ then a is said to be an element of *degree* n and $\overline{a} = a + \mathcal{F}_{n-1} \in A_n$ is the *leading term* of a. If c is another element of R of degree m, define

$$\overline{ac} = ac + \mathcal{F}_{m+n-1} \in A_{m+n}.$$

Using this multiplication we form the *associated graded ring* of R with respect to $\mathcal{F}$, $\operatorname{gr} R = \bigoplus_{i\geq 0} A_i$.

Example 2.2.28. Let $A_1 = A_1(\Bbbk) = \Bbbk[x; \partial_x]$ be the first Weyl algebra. We may define a filtration on A_1 by letting

$$\mathcal{F}_n = \Bbbk \left\langle x^i \partial_x^j \mid i + j \leq n \right\rangle;$$

that is to say, $\mathcal{F}_n$ is the $\Bbbk$-space generated by monomials in x and ∂_x of total degree less than or equal to n. Then $\operatorname{gr} A \cong \Bbbk[x, y]$. This filtration is called the *Bernstein filtration of* A_1 and will be used particularly in Chapter 5.

In Chapter 5 we require our rings to have a filtration with a special property.

Definition 2.2.29. Let $R = \bigcup_{n \geq 0} R_n$ be a filtered ring. We say that R is *strongly filtered* if

$$R_i R_j = R_{i+j}$$

for all $i, j \in \mathbb{N}$.

Lemma 2.2.30. *Let A be a finitely generated $\Bbbk$-algebra. Then A has a strong filtration.*

Proof. Let V be a finite generation set for A which contains 1. Define $A(i) = V^i$, then this is a filtration on A. Further

$$A(n) = \underbrace{A(1) \dots A(1)}_{n \text{ times}}$$

and hence the filtration is strong. □

We also briefly mention here one of the measures of dimension that will occur in the thesis.

Definition 2.2.31. Let R be a finitely generated $\Bbbk$-algebra, and let V be a finite-dimensional generating subspace for R that contains 1. The *Gelfand-Kirillov dimension of* R is

$$\begin{aligned} \operatorname{GKdim} R &= \inf\{\alpha \in \mathbb{R} \mid \dim_{\Bbbk}(V^n) \leq n^{\alpha} \text{ for all } n \gg 0\} \\ &= \limsup_{n \to \infty} \log_n \dim_{\Bbbk} V^n. \end{aligned}$$

If R is a finitely generated $\Bbbk$-algebra this definition is independent of the generating space V [22, Lemma 1.1].

Example 2.2.32. Let A_n be the n^{th} Weyl algebra, then $\operatorname{GKdim} A_n = 2n$.

Definition 2.2.33. Let A be a $\Bbbk$-algebra, and let M be a right A-module. The *Gelfand-Kirillov dimension* of M is given by

$$\operatorname{GKdim}(M) = \sup_{V,F} \limsup_{n \to \infty} \log_n \dim_{\Bbbk}(FV^n)$$

where the supremum is taken over all finite dimensional subspaces V of A containing 1 and all finite dimensional subspaces F of M.

If R is a finitely generated commutative $\Bbbk$-algebra, then $\operatorname{GKdim} R = \operatorname{Kdim} R$ where the latter denotes the (classical) Krull dimension. However, for noncommutative rings GK dimension can be very badly behaved. In fact, the following proposition shows that the GK dimension of a ring need not even be an integer.

Proposition 2.2.34. [12] *Let $r \in \mathbb{R}$ with $r \geq 2$. There is an affine algebra R with $\operatorname{GKdim} R = r$.*

The restriction $r \geq 2$ is due to Bergman's gap theorem [5] which states there are no algebras R with $1 < \operatorname{GKdim} R < 2$. However, if we further assume that our algebra is a graded domain then, due to a result of Smoktunowicz [40], there are no such algebras with GK dimension strictly between 2 and 3. For further details on GK dimension see [26, Chapter 8] and [22].

There are many natural connections between a filtered ring and its associated graded ring.

Theorem 2.2.35. [26] *Let R be a filtered ring. If $\operatorname{gr} R$ has any of the following properties, then R also has this property:*

1. *right or left noetherian;*
2. *a domain;*
3. *a prime ring;*
4. *finitely generated;*
5. $\operatorname{gldim} R \leq n$*;*
6. $\operatorname{GKdim} R \leq n$*.*

2.2.4 Twisted Homogeneous Coordinate Rings

We now turn to another important structure in noncommutative ring theory, the twisted homogeneous coordinate ring (THCR). Whilst this may initially seem like a diversion away from the main thrust of this thesis, the theory of idealizers in twisted homogeneous coordinate rings provides a very compelling source of motivation and confirms the importance of studying them. Chapter 3 also works with rings with a similar structure. Since twisted homogeneous coordinate rings are a generalisation of a well-known commutative definition we begin by recalling that commutative picture.

Definition 2.2.36. Let X be a projective variety and let $\mathcal{L}$ be an invertible sheaf on X. We define the *section ring of $\mathcal{L}$ on X* to be the graded ring

$$B(X, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n}).$$

Here the multiplication

$$H^0(X, \mathcal{L}^{\otimes n}) \otimes H^0(X, \mathcal{L}^{\otimes m}) \xrightarrow{\mu} H^0(X, \mathcal{L}^{\otimes (m+n)})$$

is induced from the natural map

$$\mathcal{L}^{\otimes n} \otimes \mathcal{L}^{\otimes m} \xrightarrow{\mu} \mathcal{L}^{\otimes (m+n)}.$$

In general, this ring is very difficult to understand, however if the invertible sheaf $\mathcal{L}$ is *ample* (i.e. for any coherent sheaf $\mathcal{F}$, the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ has nice homological properties when $n \gg 0$) then we have the following result due to Serre.

Theorem 2.2.37. [2] *Let X be a projective variety and $\mathcal{L}$ an ample invertible sheaf. Then the section ring $B = B(X, \mathcal{L})$ is noetherian. Further, we have the following equivalence of categories*

$$\operatorname{coh}(X) \simeq \text{gr-}B / \operatorname{tors-}B$$

where $\operatorname{gr} B$ *denotes the category of noetherian $\mathbb{Z}$-graded right B-modules,* tors-B *the Serre subcategory of finite dimensional modules, and* $\operatorname{coh}(X)$ *the category of coherent sheaves on X.*

Artin and Van den Bergh [2] showed in their landmark paper that this construction has a noncommutative generalisation, in a similar way to how we may form a skew extension from a commutative ring.

Definition 2.2.38. Let X be a projective variety, $\sigma \in \operatorname{Aut}(X)$, and $\mathcal{L}$ an invertible sheaf on X. Let $\mathcal{L}_0 = \mathcal{O}_X$, denote $\mathcal{L}^\sigma := \sigma^*(\mathcal{L})$ the pullback of $\mathcal{L}$ along σ, and

$\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$ for each $n \geq 1$. We have a natural pullback of global sections

$$\sigma^* : H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}^\sigma).$$

Define the *twisted homogeneous coordinate ring* associated to the triple $(X, \mathcal{L}, \sigma)$ to be the graded ring

$$B = B(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}_n),$$

where the multiplication on B is induced by the following maps:

$$H^0(X, \mathcal{L}_m) \otimes H^0(X, \mathcal{L}_n) \xrightarrow{1 \otimes (\sigma^m)^*} H^0(X, \mathcal{L}_m) \otimes H^0(X, \mathcal{L}_n^{\sigma^m}) \xrightarrow{\mu} H^0(X, \mathcal{L}_{m+n})$$

where μ is the natural multiplication of global sections map.

Example 2.2.39. Let k denote an algebraically closed field of arbitrary characteristic. Let us consider the twisted homogeneous coordinate ring $B = B(\mathbb{P}^1, \mathcal{O}(1), \sigma)$ where $\sigma \in \operatorname{Aut}_k(\mathbb{P}^1)$ is defined by

$$\sigma([a : b]) = [a : a + b]$$

and $\mathcal{O}(1)$ is Serre's twisting sheaf. It turns out that this ring is isomorphic to the following ring:

$$k_J[x, y] = \frac{k \langle x, y \rangle}{(xy - yx - x^2)}.$$

The ring $k_J[x, y]$ is referred to as the *Jordan (affine) plane.*

By altering the ample condition in Theorem 2.2.37 to incorporate the action of σ, Artin and Van den Bergh generalised Serre's result to this noncommutative setting.

Theorem 2.2.40. [2] *Let X be a projective variety, $\sigma \in \operatorname{Aut} X$, and $\mathcal{L}$ an appropriately ample invertible sheaf. Then $B = B(X, \mathcal{L}, \sigma)$ is a noetherian ring. Further, we have the following equivalence of categories*

$$\text{gr-}B / \text{tors-}B \simeq \operatorname{coh}(X).$$

Twisted homogeneous coordinate rings are crucial in two useful classification results in noncommutative ring theory: firstly, in the classification of AS regular algebras and secondly, in the following fundamental result of Artin and Stafford.

Theorem 2.2.41. [1] *Let R be a noetherian connected graded domain of GK dimension* 2. *Then there is an integer $k \geq 1$ such that $R^{(k)}$ is one of the following:*

1. *$B(X, \mathcal{L}, \sigma)$ where X is a projective curve, $\sigma \in \operatorname{Aut} X$, and $\mathcal{L}$ is an appropriately ample invertible sheaf on X;*

2. *an idealizer subring in $B(X, \mathcal{L}, \sigma)$.*

The first case of the theorem occurs precisely when some Veronese subring of R is generated in degree 1. Whilst this always holds for commutative rings, it is not always the case for noncommutative rings and idealizers provide a source of examples of this phenomenon.

Example 2.2.42. Recall that k denotes an algebraically closed field of arbitrary characteristic and recall the Jordan plane $k_J[x, y]$ from Example 2.2.39. As may be seen from the example, this ring is in the first case of the classification. However, consider the subring of $k_J[x, y]$:

$$R = k + yk_J[x, y].$$

By a result of Stafford and Zhang [41], this ring is noetherian precisely when the characteristic of the field k is 0. In this case R is actually the idealizer subring of the right ideal $yk_J[x, y]$ inside $k_J[x, y]$ and is an example of a ring in which no Veronese subring may be generated in degree 1; hence, it fits into the second half of the classification.

2.2.5 Idealizers in Twisted Homogeneous Coordinate Rings

We now describe the work of Rogalski and Sierra who studied idealizers in twisted homogeneous coordinate rings. We begin with some definitions of some technical geometrical terms which shall be unmotivated for now.

Definition 2.2.43. Given a subset $\mathcal{C}$ of closed points of a topological space X, we say that $\mathcal{C}$ is *critically dense* if every infinite subset of $\mathcal{C}$ has closure equal to all of X.

Whilst this definition may seem a little difficult to test, Bell [4] showed that in certain situations, critical density is equivalent to classical density.

Proposition 2.2.44. *Let σ be an automorphism of an affine variety over a field of characteristic* 0. *Then*

$$\{\sigma^n(p) \mid n \in \mathbb{Z}\}$$

is dense if and only if it is critically dense.

Example 2.2.45. If k is a field of characteristic zero, and

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$$

is an automorphism on $\mathbb{A}^2$. Consider the orbit of the point $P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ under this automorphism:

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \pi \end{pmatrix}, \begin{pmatrix} 1 \\ \pi^2 \end{pmatrix}, \dots \right\}.$$

This set is dense in $\mathbb{A}^2$ and hence, by Proposition 2.2.44, it is critically dense. For further examples of this phenomenon and of Proposition 2.2.44 failing in characteristic p, see [34, Example 12.9].

We also have the following intersection condition for affine varieties.

Definition 2.2.46. Let $I, J \lhd C$, where C is a commutative ring. Then we say that $V(I)$ and $V(J)$ are *homologically transverse* if

$$\mathrm{Tor}_j^C(C/I, C/J) = 0$$

for all $j \geq 1$. Further, for a group action of G on C, we say $\{V(I^g)\}_{g \in G}$ is *critically transverse* if for all ideals $K \lhd C$, $V(I^g)$ and $V(K)$ are homologically transverse for all but finitely many $g \in G$.

In this example we show that homological transversality is a subtler notion than classical transversality.

Example 2.2.47. Consider the curves defined by ideals the $(y - x^2)$ and (y) in $\Bbbk[x, y]$. Classically, these two curves are not transverse at the origin.

Let us now consider

$$\mathrm{Tor}_j^{\Bbbk[x,y]}\left(\frac{\Bbbk[x, y]}{(y^2 - x)}, \frac{\Bbbk[x, y]}{(y)} \right)$$

for $j \geq 1$. It is easily shown that this is zero when $j \geq 2$ and when $j = 1$,

$$\mathrm{Tor}_1^{\Bbbk[x,y]}\left(\frac{\Bbbk[x, y]}{(y^2 - x)}, \frac{\Bbbk[x, y]}{(y)} \right) \cong \frac{(y^2 - x) \cap (y)}{(y^2 - x)(y)} = \frac{(y^2 - x)y}{(y^2 - x)(y)} = 0.$$

Thus we see that these two curves are homologically transverse. In fact, the coordinate ring of two irreducible plane curves are only *not* homologically transverse when the curves are the same.

In twisted homogeneous coordinate rings we restrict our attention to a specific type of idealizer defined as follows.

Definition 2.2.48. [37] Let X be a projective variety over $\Bbbk$, let σ be an automorphism of X, and let $\mathcal{L}$ be an appropriately ample invertible sheaf on X. Let Z be a closed subscheme of X. Form the twisted homogeneous coordinate ring $B = B(X, \mathcal{L}, \sigma)$, and let I be the right ideal of B generated by sections that vanish on Z. Let

$$R = R(X, \mathcal{L}, \sigma, Z) \coloneqq \mathbb{I}_B(I) = \{x \in B \mid xI \subseteq I\}.$$

Then we call R the *geometric idealizer* of I in B or the *idealizer at Z in B*.

Rogalski studied when idealizer subrings at points in $\mathbb{P}^d$ were right and left noetherian and obtained the following results.

Proposition 2.2.49. [35, Propositions 5.3 and 5.4] *Let $\sigma \in \operatorname{Aut} \mathbb{P}^d$, let $p \in \mathbb{P}^d$, and assume that p is of infinite order under σ. Let $R = R(\mathbb{P}^d, \mathcal{O}(1), \sigma, \{p\})$. Then R is right noetherian. Furthermore, if the set $\{\sigma^n(p) \mid n \in \mathbb{Z}\} \subseteq \mathbb{P}^d$ is critically dense, then R is left noetherian.*

Sierra further generalised this result to the setting where Z is a closed subvariety of a projective variety X.

Proposition 2.2.50. [37] *Let X be a projective variety, let $\sigma \in \operatorname{Aut}(X)$, and let $\mathcal{L}$ be an appropriately ample invertible sheaf on X. Assume that Z is irreducible and of infinite order under σ and form the ring $R(X, \mathcal{L}, \sigma, Z)$.*

1. *If for all $p \in Z$ the set $\{n \geq 0 \mid \sigma^n(p) \in Z\}$ is finite, then R is right noetherian.*

2. *If, further, the set $\{\sigma^n(Z)\}_{n \geq 0}$ is critically transverse, then R is left noetherian.*

This proposition highlights the asymmetry of idealizers and allows us very easily to construct rings which are right, but not left, noetherian. In Chapter 3, we obtain similar results to those of Sierra but we also manage to strengthen them. We also provide an example of these conditions in the wild and show how the properties of the resulting idealizers depend on the answer to an interesting number-theoretic problem.

2.2.6 Idealizers and Universal Enveloping Algebras

We now describe a novel use of the theory of idealizers; namely, Sierra and Walton's proof [38] that the universal enveloping algebra of the Witt algebra is not noetherian. We begin by describing the motivation for this result.

Definition 2.2.51. Let $\mathfrak{g}$ be a Lie algebra and form the free tensor algebra $T(\mathfrak{g})$. We define the *universal enveloping algebra* of $\mathfrak{g}$ to be

$$U(\mathfrak{g}) := \frac{T(\mathfrak{g})}{\langle a \otimes b - b \otimes a - [a, b] \mid a, b \in \mathfrak{g} \rangle}.$$

The representation theory of the Lie algebra and the universal enveloping algebra are intimately linked: the representations of $\mathfrak{g}$ are in one-to-one correspondence with the modules of $U(\mathfrak{g})$. We have the following conjecture to describe the noetherianity of these rings:

Conjecture 2.2.52. [38] *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is finite dimensional if and only if $U(\mathfrak{g})$ is a noetherian ring.*

One direction of this conjecture follows from Theorem 2.2.35, namely, the universal enveloping algebra of a finite dimensional Lie algebra is noetherian. However, proving the converse is an open problem.

The following Lie algebra was suggested by Small and Dean as a potential counterexample.

Definition 2.2.53. We define the *Witt algebra* to be the Lie algebra W with basis $\{e_i\}_{i \in \mathbb{Z}}$ and bracket $[e_n, e_m] = (m - n)e_{n+m}$.

Note, in particular, that this is an infinite dimensional Lie algebra. Small and Dean asked the following question:

Question 2.2.54. [15] *Is $U(W)$ noetherian?*

Sierra and Walton provided an answer to this question with the following theorem.

Theorem 2.2.55. [38, Theorem 0.5] *The ring $U(W)$ is neither left nor right noetherian.*

The theory of idealizers and, in particular, a good understanding of their noetherianity is essential to their proof. In essence, Sierra and Walton constructed a homomorphism ρ from $U(W)$ to a localisation of $\mathbb{I}_{A_2}((x, y)A_2)$ and considered the homomorphic image of W. As in Resco's example 2.2.13, $\mathbb{I}_{A_2}((x,y)A_2)$ is not left noetherian and a similar proof may be used to show that $\rho(U(W))$ is also not left noetherian.

2.3 Further Background Material

Before we move on to Chapter 3, we first provide a few important definitions and examples which are necessary for both this chapter and Chapter 5.

Recall first the definitions of G-graded rings and noetherianity. It turns out that for special types of groups G, we need only consider graded noetherianity.

Definition 2.3.1. Let G be a group. We say that G is *polycyclic-by-finite* if G has a finite chain

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n \lhd G,$$

where G_i is normal in G_{i+1} and each factor $G_{i+1}/G_i \cong \mathbb{Z}$ for $i = 1, \ldots, n-1$, and G/G_n is a finite group.

Theorem 2.3.2. [13] *Let R be a G-graded ring where G is a polycyclic-by-finite group. If R is graded noetherian, then R is noetherian.*

We observe that the question is still open as to whether Theorem 2.3.2 holds for groups G which are not polycyclic-by-finite; for example, group algebras.

In Chapter 3 we are particularly interested in a type of skew extension of a ring by a polycyclic-by-finite group. We define these now.

Definition 2.3.3. Given a ring R and a group G which acts on the left on R, for $g \in G$, $a \in R$ the image of a under action by g will be denoted a^g. We denote by $R\#G$ the *skew group ring* which is a free left R-module with elements of G as a basis and with multiplication determined by

$$(ah)(bg) = (ab^h)(hg)$$

for $g, h \in G$ and $a, b \in R$. Each element of $R\#G$ can be written uniquely as $\sum_{g \in G} r_g g$ with $r_g \in R$ and $r_g = 0$ for all but finitely many $g \in G$.

Example 2.3.4. Recall Example 2.1.2 concerning the ring $B = \Bbbk[x,y][s^{\pm 1}, t^{\pm 1}; \sigma, \tau]$. We may view the actions of the automorphisms σ and τ of $\Bbbk[x,y]$ as an action of $\mathbb{Z}^2$

defined by

$$(n, m) * f(x, y) = \sigma^n \tau^m (f(x, y)).$$

In this way we may rewrite B as a skew group ring $\Bbbk[x, y]\#\mathbb{Z}^2$. This ring, and its idealizers, are of particular interest to us in Chapter 3 where it provides a nice example of the noetherianity conditions we obtain.

Theorem 2.3.5. [26, Theorem 1.5.12] *Let R be a right noetherian ring and let G be a polycyclic-by-finite group which acts on R. Then the skew group ring $R\#G$ is right noetherian.*

This result follows from Theorem 2.3.2 by observing that the category of finitely generated graded right $R\#G$-modules is equivalent to mod-R. We note that, as with Theorem 2.3.2, the converse of this theorem is still open.

For completeness we now include a short section on noncommutative localisation as it is a method that will be used during the thesis without comment. We begin with the commutative picture.

Initially, let R denote a commutative ring and let X denote a multiplicatively closed set of regular elements. Then we may form the quotient ring

$$RX^{-1} = \{rx^{-1} \mid r \in R, x \in X\}$$

where the addition and multiplication operations are the natural ones for fractions. Let R now be a noncommutative ring, let X be a multiplicatively closed subset of regular elements and suppose that $S = RX^{-1}$ is a right ring of fractions, let ax^{-1}, by^{-1} be two fractions. Then the only way for the following equality to hold

$$ax^{-1}by^{-1} = ab(yx)^{-1}$$

would be if x and b were to commute, which we do not wish to assume. However, as we are assuming S exists, we are also assuming that every element of S may be written with a right hand denominator, in particular $x^{-1}b = cz^{-1}$ for some $c \in R$ and $z \in X$. Hence

$$ax^{-1}by^{-1} = a(x^{-1}b)y^{-1} = ac(yz)^{-1}.$$

The existence of c and z is a special condition which we now define.

Definition 2.3.6. Let X be a multiplicative set in a ring R. If, for each $x \in X$ and $r \in R$, there exists a $y \in X$ and $s \in R$ such that $ry = xs$, equivalently $xR \cap rX \neq \emptyset$, we say that X satisfies the *right Ore condition.* A multiplicative set which satisfies this condition is called a *right Ore set.*

Theorem 2.3.7. [19, Exercise 4N] *If R is a right noetherian domain, then the intersection of any two non-zero right ideals is non-empty, so the set $R \setminus \{0\}$ is a right Ore set.*

Definition 2.3.8. Let R be a ring and $X \subseteq R$ a multiplicative set of regular elements in R. A *right ring of fractions* (or *right quotient ring*) *for R with respect to X* is any overring $S \supseteq R$ such that:

1. every element of X is invertible in S;

2. every element of S can be expressed in the form ax^{-1} for some $a \in R$ and $x \in X$.

If $X = R \setminus \{0\}$, then we refer to this quotient ring as the *classical right quotient ring* and denote it $Q(R)$. The case when R has a classical right quotient ring Q is also denoted by saying that R is a *right order in Q.*

Chapter 3

Idealizers in Skew Group Rings

In this chapter we study idealizers in skew group rings. We begin with an introduction to the important results in the chapter.

3.1 Introduction

Introduced by Ore in [29], idealisers are highly noncommutative rings with interesting and sometimes pathological behaviour. For example, Stafford used idealisers to construct a variety of left and right noetherian rings with peculiar ideal structure [43]. Idealisers occur naturally in Artin and Stafford's classification of noncommutative projective curves [1] so we expect idealisers to feature in any sufficiently general noncommutative classification. Hence a better understanding of the behaviour of these subrings is desired. In the Artin-Stafford classification idealisers come from noetherian graded rings which have the property that no Veronese is generated in degree 1. This property never occurs for commutative rings.

Idealisers are often good examples of rings that have different left and right structures. For example, if $B = \mathbb{C}[x, y][t^{\pm}; \sigma]$ is a skew Laurent ring where

$$\sigma(x) = x + 1 \text{ and } \sigma(y) = y,$$

and we consider the right ideal $(x, y)B$, then we show below that $\mathbb{I}_B((x, y)B)$ is right but not left noetherian. The reasons for this will become clear in the chapter, but we mention that this depends on the dynamics of the orbit of (x, y) under σ. A related example was essential in the proof by Sierra and Walton that the universal enveloping algebra of the Witt algebra is not noetherian [38] which answered a question which had been open for over 20 years.

Idealisers were studied in detail by Robson in [33] where he observed that in the case that right ideal I is semimaximal, that is to say, I is the intersection of maximal right ideals, then the properties of the B and $\mathbb{I}_B(I)$ are very closely linked. In particular, the following result characterises the noetherianity of an idealiser of a maximal right ideal.

Theorem 3.1.1. [33, Theorem 2.3] *Let B be a ring, I a maximal right ideal of B. Then B is right noetherian if and only if $\mathbb{I}_B(I)$ is right noetherian.*

We note that Robson does not address left noetherianity at all in his paper and, as a consequence of results in this chapter, a left noetherian version of Theorem 3.1.1 in a graded setting does not hold. Indeed, the idealiser mentioned above is a counterexample.

The objective of this chapter is to investigate the noetherianity of idealisers in skew group rings. In [35] and [37], Rogalski and Sierra considered what it means for idealisers in twisted homogeneous coordinate rings [2] to be left and right noetherian and they obtained the following result which we paraphrase.

Theorem 3.1.2. [37, Theorem 10.2] *Let $B = B(X, \mathcal{L}, \sigma)$ be a twisted homogeneous coordinate ring, where X is a projective variety, $\mathcal{L}$ is an appropriately ample invertible sheaf on X, $\sigma \in \operatorname{Aut} X$, and let I be a right ideal of B corresponding to a closed subscheme $Z \subseteq X$ of infinite order under σ. Then the noetherianity of $\mathbb{I}_B(I)$ is determined by the orbit of Z under σ.*

We state this result precisely in Theorem 3.4.14 but, whilst we are being vague about the technicalities, it is interesting to note that an algebraic condition which is sometimes very difficult to verify, that of noetherianity, in this case can be understood from a geometric perspective.

Let C be a commutative domain and let $\mathbb{Z}$ act on C by powers of $\sigma \in \operatorname{Aut} C$. Based on the behaviour of idealisers in twisted homogeneous coordinate rings, one naturally conjectures that similar geometric conditions to those in Theorem 3.1.2 control noetherianity of idealisers in $C\#\mathbb{Z}$ and indeed we verify that this is the case. However, it is difficult to predict what will control noetherianity of idealisers in $C\#G$ for an arbitrary group G. In this chapter we completely answer the question for G being a finitely generated abelian group.

Definition 3.1.3. Let $I, J \lhd C$, a commutative ring. Then we say that $V(I)$ and $V(J)$ are *homologically transverse* if

$$\mathrm{Tor}_j^C(C/I, C/J) = 0$$

for all $j \geq 1$. Further, for a group action of G on C, we say $\{V(I^g)\}_{g \in G}$ is *critically transverse* if for all ideals $K \lhd C$, $V(I^g)$ and $V(K)$ are homologically transverse for all but finitely many $g \in G$.

Our main results are as follows:

Theorem 3.1.4 (Theorem 3.4.5)**.** *Let C be a commutative noetherian domain, let G be a finitely generated abelian group acting on C and let $B = C\#G$. Let $I \lhd C$ be a prime ideal with trivial stabiliser under the G-action on C. Then the following are equivalent:*

(1) for all points $p \in V(I)$ the set

$$\{h \in G \mid p^h \in V(I)\}$$

is finite;

(2) $\mathbb{I}_B(IB)$ is right noetherian.

Theorem 3.1.5 (Theorem 3.4.13)**.** *With the same setup as Theorem 3.1.4 the following are equivalent:*

(1) $\{V(I^g)\}_{g \in G}$ is critically transverse;

(2) $\mathbb{I}_B(IB)$ is left noetherian.

We give a complete characterisation of noetherianity for idealisers at prime ideals of C with a more general group action in the body of the chapter.

Understanding when the conditions in Theorems 3.1.4 and 3.1.5 hold can be subtle. For example, let $C = \mathbb{C}[x, y]$ and let $\mathbb{Z}^2$ act on $\mathbb{C}[x, y]$ by translation. Then the idealizer of the right ideal $(x - 7y^2 - 1)B$ in B is neither left nor right noetherian. However, if $f \in \mathbb{C}[x, y]$ defines either an irreducible curve of genus ≥ 1 or a line, then $\mathbb{I}_B(IB)$ is left and right noetherian. Full details of this example may be found in Section 5.

In the first section and second section, we focus on group rings of polycyclic-by-finite groups and show that working with graded rings allows us to simplify results from the literature. In the fourth section we prove Theorems 3.1.4 and 3.1.5 and in the final section we give the details of the example mentioned above which raises an interesting question in number theory.

3.2 Preliminaries

The aim of this chapter is to generalise theorems in the literature about idealisers in twisted homogeneous coordinate rings to some other well-chosen noncommutative situations. In this section we give definitions and results which we will need. Our idealisers will be group-graded, and we begin by discussing the noetherianity of group-graded rings.

Chin and Quinn [13] show that if G is a polycyclic-by-finite group and R is a G-graded ring, then all G-graded right ideals of R are finitely generated if and only if R is right noetherian. We note that it is still an open question as to whether this holds for rings graded by arbitrary groups. Indeed, it is even still a question as to whether $R\#G$, the group ring of G, can be noetherian when G is not polycyclic-by-finite. Hence it is clearly not reasonable for us to consider the noetherianity of rings graded by non-polycyclic-by-finite groups.

The proof by Chin and Quinn [13] is rather inexplicit, so we begin with a direct proof in the case that G is a finitely generated abelian group. This generalises a result of Björk [10, Theorem 2.18].

Proposition 3.2.1. *Let G be a finitely generated abelian group and let R be a G-graded ring. If all homogeneous right (left) ideals of R are finitely generated then R is right (left) noetherian.*

Proof. We proceed by induction on $\text{Rank}(G) = n$. Suppose initially that $n = 0$, then $G = \{0, g_1, \dots, g_m\}$ is finite and $R = R_0 \oplus (\bigoplus_{i=1}^m R_{g_i})$. We show that each R_g is finitely generated as a right R_0-module. Indeed, let $I_1 \subsetneq I_2 \subsetneq \dots$ be a strictly ascending chain of right R_0-submodules in R_g, then $I_1R \subseteq I_2R \subseteq \dots$ is an ascending chain of G-graded right ideals in R. Further, since $(I_jR)_g = I_j$ this chain in strictly ascending and so must stabilise, hence R_g is a noetherian right R_0-module. Thus, as G is a finite group, R is finitely generated as a right R_0-module and hence is right noetherian.

So we have proved the case when $n = 0$. Now we suppose that $n > 0$. Then G contains a normal subgroup isomorphic to $\mathbb{Z}$; abusing notation we write $\mathbb{Z} \lhd G$. First we show that R is $G/\mathbb{Z}$-graded. For an element $x \in G$, let $\bar{x}$ denote its image in $G/\mathbb{Z}$ and let $R_{\overline{x}} = \oplus_{a\in\mathbb{Z}} R_{x+a}$. Let $\bar{g}, \bar{h} \in G/\mathbb{Z}$. Then,

$$
\begin{aligned}
R_{\overline{g}} R_{\overline{h}} = \left(\bigoplus_{a\in\mathbb{Z}} R_{g+a}\right)\left(\bigoplus_{b\in\mathbb{Z}} R_{h+b}\right) &= \sum_{a,b\in\mathbb{Z}} R_{g+a}R_{h+b} \\
&\subseteq \sum_{a,b\in\mathbb{Z}} R_{g+a+h+b} \\
&= \sum_{a,b\in\mathbb{Z}} R_{(g+h)+a+b} = R_{\overline{g+h}}.
\end{aligned}
$$

We next prove the claim:

if R has ACC on G-graded right ideals then R has ACC on $G/\mathbb{Z}$-graded ideals. We use the general method of Björk's proof [10, Theorem 2.18]. Let $L \le R_R$ be a $G/\mathbb{Z}$-graded right ideal and let $X \subseteq G$ be a set of coset representatives for $G/\mathbb{Z}$ so then $L = \bigoplus_{x\in X} L_{\bar{x}}$. Let t be an indeterminate. We begin by constructing the *external homogenisation* of a $G/\mathbb{Z}$-homogeneous element $z \in L$. We have $z \in L_{\bar{g}}$ for some $g \in X$, and we write $z = \sum_{h\in\mathbb{Z}} z_{g+h}$. Let $v = v(z) = \max\{h \in \mathbb{Z} \mid z_{g+h} \neq 0\}$ and define $z^* = \sum_{h\in\mathbb{Z}} z_{g+h}t^{v-h} \in R[t]$. Then define $L^* = \langle z^* \mid z \in L\rangle$ as a right ideal of $R[t]$. We note that L^* is G-graded, under the grading on $R[t]$ where R is G-graded and t is given degree $1 \in \mathbb{Z} \le G$. Under this grading, for each $z \in L_{\bar{g}}$, z^* will be G-homogeneous of degree $g + v(z)$ and hence L^* will be G-graded as it is generated by these elements.

There is a positive filtration $F_\bullet$ on $R[t]$ with $F_w = R + Rt + \cdots + Rt^w$. Now consider the image $\sigma(L^*)$ in $\mathrm{gr}_F(R[t])$. Then $\sigma(L^*) = \bigoplus_{w\in\mathbb{N}}(L^* \cap F_w)/(L^* \cap F_{w-1}) \cong \bigoplus_{w\in\mathbb{N}} J(w)t^w$. We observe that the $J(w)$ are right ideals in R such that $J(w) \subseteq J(w+1)$. Further, these right ideals $J(w)$ are G-graded. Indeed, by virtue of coming from L, the $J(w)$ are $G/\mathbb{Z}$-graded. Now suppose $h = \sum_{i=1}^n h_i \in J(w)$ such that $h_i \in R_{g+a_i}$ where the $a_i \in \mathbb{Z}$ are distinct. Then

$$\left(\sum_{i=1}^n h_i\right) t^w + [\text{lower powers of } t] \in L^*.$$

Recall that L^* is G-graded and, letting $N_i = g + a_i + w$, we see

$$\left(\left(\sum_{i=1}^n h_i\right) t^w + [\text{lower powers of } t]\right)_{N_i} = h_i t^w + [\text{lower powers of } t]_{N_i} \in L^*$$

and hence

$$\sigma\left(h_i t^w + ([\text{lower powers of } t])_{N_i}\right) = h_i t^w \in \sigma(L^*).$$

Thus we have an ascending chain of G-graded right ideals of R,

$$J(0) \subseteq J(1) \subseteq J(2) \ldots,$$

which stabilises to J_∞ by assumption.

We claim this is enough to show that $\sigma(L^*)$ is finitely generated. Indeed we reproduce the standard argument from Hilbert's basis theorem. Let $a_1, \ldots, a_n$ be a finite generating set for J_∞ and let $f_i \in \sigma(L^*)$ such that $\mathrm{lt}(f_i) = a_i$ where $\mathrm{lt}(g)$ is the coefficient of the highest power of t in g. Without loss of generality we may assume $\deg_t(f_i) = m$ for all i (else, if m' is the maximum of the t-degrees of the f_i then we may replace any f of lower t-degree with $f_i t^{m'-\deg_t(f_i)}$ Let $L_0 = \left(\sigma(L^*) \cap \sum_{i=0}^{m-1} t^i R[t]\right) + \sum_{i=1}^n f_i R[t]$, then we claim $L_0 = \sigma(L^*)$. Suppose not, obviously $L_0 \subseteq \sigma(L^*)$ so let $f \in \sigma(L^*) \setminus L_0$ be of minimal t-degree $d \geq m$. Since $f \in \sigma(L^*)$, $\mathrm{lt}(f) \in J_\infty$ and as such $\mathrm{lt}(f) = \sum_{i=1}^n a_i r_i$ for some $r_i \in R$. Consider $g = f - \sum_{i=1}^n f_i r_i t^{d-m}$, then this cancels out the leading term of f and since $\deg_t(g) < \deg_t(f)$, by minimality $f = \sum_{i=1}^n f_i r_i t^{d-m} \in L_0$, as required. Hence $\sigma(L^*)$ is finitely generated. Since F is a positive filtration, by [10, Proposition 2.11], L^* is finitely generated.

We now define $\phi : R[t] \to R$ by $\phi(\sum_w t^w x_w) = \sum_w x_w$ (i.e. mapping t to 1). This is a surjective ring homomorphism and $L = \phi(L^*)$, hence L is finitely generated.

So through this we have proved that R being right G-graded-noetherian implies that R is right $G/\mathbb{Z}$-graded noetherian. As $\mathrm{Rank}(G) > \mathrm{Rank}(G/\mathbb{Z})$, by induction on $\mathrm{Rank}(G)$ we conclude that R is right noetherian. □

We have the following useful lemma.

Lemma 3.2.2. *Let $A \subseteq A'$ be subrings of a domain D. Suppose that A is right noetherian and that A contains a non-zero right ideal of D. Then A' is right noetherian.*

Proof. Let $0 \neq J \leq D_D$ such that $J \subseteq A$ and let $0 \neq y \in J$. Then we observe that $yA' \subseteq JD \subseteq A$, so yA' is a right ideal of A, and hence is a finitely generated right A-module. As D is a domain, $A' \cong yA'$ as right A-modules and hence A' is a finitely generated A-module and so is right noetherian. □

Now we give some general results from ring theory which will be used for the proofs of Theorems 3.1.4 and 3.1.5. The starting point for the treatment of both left and right noetherianity is to generalise results of Rogalski and Stafford to the G-graded setting where G is a polycyclic-by-finite group. Note that it is an open question of whether the noetherianity of G-graded rings is completely controlled by homogeneous ideals for only polycyclic-by-finite groups. Hence we only consider rings graded by polycyclic-by-finite groups as those are the only ones whose noetherianity may be confirmed by checking homogeneous ideals.

Lemma 3.2.3 (cf. [43, Lemma 1.1]). *Let G be a polycyclic-by-finite group and let B be a right noetherian G-graded ring. Let $I \leq_{gr} B_B$ and let $R = \mathbb{I}_B(I)$. Suppose further that B/I is a graded-noetherian right R-module. Then R is a right noetherian ring.*

Proof. This is a graded version of [43, Lemma 1.1]. Note that R is G-graded. Let $J \leq_{\text{gr}} R_R$. Then JB and JI are graded right ideals of B, thus are finitely generated, and $JB \geq JR = J \geq JI$. We can write $JB = \sum_{i=1}^{n} s_i B$ and $JI = \sum_{j=1}^{m} r_j I$ where $s_i, r_j \in J$. We note that we may actually assume $\{s_i\} = \{r_j\}$. Indeed, let $\{t_k\} = \{s_i\} \cup \{r_j\}$. Then $\{t_k\}$ generates JB, but we also note that $JI = \sum_j r_j I \subseteq \sum_k t_k I \subseteq JI$ so the $\{t_k\}$ also generate JI. So without loss of generality $m = n$ and $s_i = r_i$ for all $i = 1, \dots, n$.

Then we have a surjection

$$(B/I)^n \xrightarrow{(s_1,\dots,s_n)} JB/JI \supseteq J/JI.$$

Hence, since B/I is graded-noetherian, J/JI is a finitely generated right R-module. Say $J/JI = \sum_{\ell=1}^{p} (u_\ell + JI)R$ for some $u_\ell \in J$. Then observe that $J = \sum_\ell u_\ell R + \sum_i s_i I \subseteq \sum_\ell u_\ell R + \sum_i s_i R \subseteq J$, so J is finitely generated. Hence, by [13], R is right noetherian. □

Definition 3.2.4. Let I, J be right ideals of a ring B. Define

$$(J : I) = \{b \in B \mid bI \subseteq J\}$$

to be the *ideal quotient*.

We alert the reader that this is a symmetric notation for an asymmetric concept. We will not use the corresponding left-handed version. We also observe that if I and J are graded right ideals of B, then $(J : I)$ will be graded as well.

Proposition 3.2.5 (cf. [35, Proposition 2.1])**.** *Let G be a polycyclic-by-finite group and let I be a graded right ideal of a G-graded right noetherian ring B which is a domain. Let $R = \mathbb{I}_B(I)$. Then the following are equivalent:*

(1) R is right noetherian.

(2) For every graded right ideal $J \supseteq I$ of B, $\mathrm{Hom}_B(B/I, B/J)$ is a right graded-noetherian R-module (or R/I-module).

Proof. This is a graded version of [35, Proposition 2.1] noting that the two-sided noetherien hypothesis on the overring B in that result is superfluous. Suppose (2) holds. By Lemma 3.2.3 it suffices to show that B/I is a graded-noetherian right R-module. By hypothesis, $R/I \cong \mathrm{Hom}_B(B/I, B/I)$ is right graded-noetherian and so it suffices to show B/R is right graded-noetherian. To this end, let $K_R \leq_{\mathrm{gr}} B_R$ such that $R_R \leq K_R$. As B is noetherian, let $J = KI = \sum_{i=1}^{n} k_i I$. As $R \leq K$ we have that $I \subseteq J$. Let $C = (J : I) = \{b \in B \mid bI \subseteq J\} \leq_{\mathrm{gr}} B_R$. Since $KI \subseteq J$, $K \subseteq C = C + J$.

We have the following identification: $\mathrm{Hom}_B(B/I, B/J) \cong (J : I)/J = C/J$, thus by hypothesis C/J is a right graded-noetherian R-module. Hence the submodule K/J is finitely generated, say $K/J = \sum_{\ell=1}^{m}(a_\ell + J)R$ where $a_\ell \in K$ and so $K = \sum_{\ell=1}^{m} a_\ell R + \sum_{i=1}^{n} k_i I \subseteq \sum_{\ell=1}^{m} a_\ell R + \sum_{i=1}^{n} k_i R \subseteq K$ is finitely generated as required. Thus, as R is G-graded, R is right noetherian.

Conversely, observing that $R \subseteq B$ which contains a right ideal of B, namely I, we use Lemma 3.2.2 to conclude B_R is finitely generated and the rest of the argument may be found in [35, Proposition 2.1]. □

Proposition 3.2.6 (cf. [35, Proposition 2.2])**.** *Let G be a polycyclic-by-finite group and let R be the idealiser of a graded right ideal I in a left noetherian G-graded ring B . Then the following are equivalent:*

(1) R is left noetherian;

(2) $\frac{BJ \cap R}{J}$ is a left graded-noetherian R-module (or R/I-module) for all finitely generated $J \leq_{\mathrm{gr}\,R} R$;

(3) R/I is left graded-noetherian and $\mathrm{Tor}_1^B(B/I, B/K) = (I \cap K)/IK$ is a graded-noetherian left R-module (R/I-module) for all $K \leq_{\mathrm{gr}\,B} B$.

Proof. The equivalence of statements (1) and (2) may be found in [34, Lemma 5.10], and the argument (1) implies (3) may be found in [35, Proposition 2.2]. We note that these statements are not in the G-graded setting but the proofs follow in the same way.

We now show (3) implies (2). Let J be an arbitrary finitely generated graded left ideal of R and consider the following short exact sequences of left R/I-modules:

$$0 \to \frac{J}{IBJ} \to \frac{BJ \cap R}{IBJ} \to \frac{BJ \cap R}{J} \to 0$$

and

$$0 \to \frac{BJ \cap I}{IBJ} \to \frac{BJ \cap R}{IBJ} \to \frac{BJ \cap R}{BJ \cap I} \to 0.$$

Note $\mathrm{Tor}_1^B(B/I, B/BJ) \cong \frac{I \cap BJ}{IBJ}$ and $\frac{BJ \cap R}{BJ \cap I}$ injects into R/I. Thus, by assumption, the outer terms of the second short exact sequence are graded-noetherian left R/I-modules, so $\frac{BJ \cap R}{IBJ}$ is a graded-noetherian left R/I-module. Hence, from the first short exact sequence $\frac{BJ \cap R}{J}$ is a left graded-noetherian R/I-module as required. □

3.3 Idealisers in Skew Group Rings

We have shown that the noetherianity of idealisers in left and right noetherian domains graded by polycyclic-by-finite groups is completely determined by the properties of Hom and Tor spaces associated to homogeneous ideals. Let us now turn to the situation of interest to us, when B is the skew group ring of a polycyclic-by-finite group, and prove some further useful results. We begin by fixing notation.

Definition 3.3.1. Given a ring A and a group G which acts on A, for $g \in G$, $a \in A$ the image of a under action by g will be denoted a^g. We denote by $A\#G$ the skew group ring which is a free left A-module with elements of G as a basis and with multiplication determined by

$$(ah)(bg) = (ab^h)(hg)$$

for $g, h \in G$ and $a, b \in A$. Each element of $A\#G$ can be written uniquely as $\sum_{g \in G} a_g g$ with $a_g = 0$ for all but finitely many $g \in G$.

Many results in [37] are proved under the assumption that the subscheme Z at which one idealises has infinite order under σ, that is $\mathrm{Stab}_\sigma(Z)$ is trivial. We wish to allow nontrivial stabilisers, which will require some notation, which will be in force for the remainder of the chapter.

From here on $\Bbbk$ will denote an algebraically closed field.

Notation 3.3.2. We let I be a prime ideal in a commutative noetherian affine domain C which is a $\Bbbk$-algebra. Let G be a polycyclic-by-finite group which acts on $\operatorname{Spec} C$. This induces an action on C by pullback and let $B := C\#G$ be the skew group ring which is left and right noetherian by Theorem 3.3.3. Let $K := \operatorname{Stab}_G(I)$ and let $R = \mathbb{I}_B(IB)$ denote the idealiser in B of the right ideal IB. Since IB is graded, so is R.

Note that with our induced action, if

$$M_p = \{c \in C \mid c(p) = 0\}$$

is a maximal ideal of C at a point $p \in \operatorname{Spec} C$. Then for $g \in G$

$$M_p^g = M_{g^{-1}p}$$

and so $V(M_p^g) = g^{-1}V(M_p)$.

We have the following result about the noetherianity of skew group rings from McConnell and Robson.

Theorem 3.3.3. [26, 1.5.12] *Let G be a polycyclic-by-finite group and A a ring. If A is right (left) noetherian then $A\#G$ is right (left) noetherian.*

As noted earlier, it is still an open question as to whether $A\#G$ being noetherian implies that G is a polycyclic-by-finite group. Thus we do not consider groups more general than polycyclic-by-finite as we do not know whether $C\#G$ will be noetherian, and so all of the results from Section 2 no longer hold. Now that we have specified the rings with which we are working, we can be more precise about the structure of $\mathbb{I}_B(IB)$.

Lemma 3.3.4. *Assume Notation 3.3.2 and recall the notation $(J : I)$ from Definition 3.2.4. Let $J \lhd C$ and consider $JB \leq B_B$. Then*

$$(JB : IB) = \bigoplus_{g \in G} (J : I^g)g.$$

Further

$$R = \bigoplus_{g \in G} (I : I^g) g.$$

Proof. We note that as both JB and IB are graded, then $(JB : IB)$ is also graded. Then for $g \in G$ we have the following identifications:

$$\begin{aligned}(JB : IB)_g &= \{c \in C \mid cgIB \subseteq JB\} g \\ &= \{c \in C \mid cI^g B \subseteq JB\} g \\ &= \{c \in C \mid cI^g \subseteq J\} g = (J : I^g) g.\end{aligned}$$

Observing $R = \{b \in B \mid bIB \subseteq IB\} = (IB : IB)$ gives the result.

□

We note that if I has trivial stabiliser under the action by G, then $R = C + IB$ by the primeness of I. We also have the following reductions of Propositions 3.2.5 and 3.2.6 using that B is strongly graded. More generally, we make work out the structure of R/IB for a general prime ideal $I \lhd C$.

Proposition 3.3.5. *Assume Notation 3.3.2. Then $R/IB \cong (C/I)\#K$ is left and right noetherian.*

Proof. We note that for any prime ideal $J \lhd C$,

$$(J : I^g) = \begin{cases} C \text{ if } I^g \subseteq J, \\ J \text{ else.} \end{cases} \tag{1}$$

Then by Lemma 3.3.4 , $R = \bigoplus_{g \in G} (I : I^g) g = \left(\bigoplus_{g \in K} Cg \right) \oplus \left(\bigoplus_{g \in G \setminus K} Ig \right)$. Hence

$$R/IB = \bigoplus_{g \in G} \frac{(I : I^g)}{I} g = \bigoplus_{k \in K} \frac{C}{I} k.$$

This is a free left (C/I)-module which has a basis generated by the elements of K. Thus $R/IB = (C/I)\#K$.

As K is a subgroup of a finitely generated abelian group it is also finitely generated abelian and C/I is clearly noetherian, $(C/I)\#K$ is left and right noetherian by Theorem 3.3.3. □

Theorem 3.3.6. *Assume Notation 3.3.2. Then the following are equivalent:*

(1) $\bigoplus_{g\in G} \operatorname{Tor}_1^C(C/I, C/P^g)g$ *is a finitely generated left* R/IB*-module for all prime ideals* $P \lhd C$*;*

(2) R *is left noetherian.*

Proof. By Proposition 3.2.6, R is left noetherian if and only if

$$\operatorname{Tor}_1^B(B/IB, B/K) = (IB \cap K)/IBK$$

is a noetherian left R/IB-module for all graded left ideals $K \leq B$. Every graded left ideal of B is of the form $K = BJ$ where $J \lhd C$ as B is strongly graded. Also, by Proposition 3.3.5, $R/IB \cong C/I\#K$ is a left and right noetherian ring. Hence R is left noetherian if and only if $\operatorname{Tor}_1^B(B/IB, B/BJ)$ is a finitely generated left R/IB-module for all ideals $J \lhd C$.

We now prove that the following statements are equivalent:

(a) $\operatorname{Tor}_1^B(B/IB, B/BJ)$ is a finitely generated left R/IB-module for all ideals $J \lhd C$;

(b) $\operatorname{Tor}_1^B(B/IB, B/BP)$ is a finitely generated left R/IB-module for all prime ideals $P \lhd C$;

That (a) implies (b) is clear.

We have the following result from commutative algebra [16, Proposition 3.7]:

There exist C-modules M_i and prime ideals $P_i \lhd C$ for $i = 0, \dots, n$ such that

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = C/J$$

and $M_{j+1}/M_j \cong C/P_{j+1}$ for $j = 0, \dots, n-1$.

Assume (b). We show by induction on j that $\operatorname{Tor}_1^B(B/IB, B \otimes_C M_j)$ is a finitely generated left R/IB-module. The statement is trivially true for $j = 0$ as then $B \otimes_C M_0 \cong 0$. Let us now consider the short exact sequence

$$0 \to M_{j-1} \to M_j \to C/P_j \to 0.$$

Applying $B \otimes_C -$ which is exact as B_C is flat, we obtain

$$0 \to B \otimes_C M_{j-1} \to B \otimes_C M_j \to B/BP_j \to 0.$$

Applying $B/IB \otimes_B -$ gives rise to a long exact sequence which contains the following terms

$$\cdots \to \operatorname{Tor}_1^B(B/IB, B \otimes_C M_{j-1}) \xrightarrow{\alpha} \operatorname{Tor}_1^B(B/IB, B \otimes_C M_j) \xrightarrow{\beta} \operatorname{Tor}_1^B(B/IB, B/BP_j) \to \ldots$$

from which we extract the short exact sequence

$$0 \to \ker \beta \to \operatorname{Tor}_1^B(B/IB, B \otimes_C M_j) \xrightarrow{\beta} \operatorname{im} \beta \to 0.$$

Since $\ker \beta \cong \operatorname{im} \alpha$ is a homomorphic image of $\operatorname{Tor}_1^B(B/IB, B \otimes_C M_{j-1})$, which is finitely generated by induction, and $\operatorname{im} \beta$ is a submodule of $\operatorname{Tor}_1^B(B/IB, B/BP_j)$, which is finitely generated by assumption, we conclude $\operatorname{Tor}_1^B(B/IB, B \otimes_C M_j)$ is a finitely generated left R/IB-module as required. Hence (a) holds.

Now that we have shown the equivalence of these statements we have the following identifications:

$$\begin{aligned} \operatorname{Tor}_1^B(B/IB, B/BP) = (IB \cap BP)/IPJ &= \bigoplus_{g \in G} \frac{I \cap P^g}{IP^g} g \\ &= \bigoplus_{g \in G} \operatorname{Tor}_1^C(C/I, C/P^g) g \end{aligned}$$

which completes the proof. □

We also have a similar result for right noetherianity:

Theorem 3.3.7. *Assume Notation 3.3.2. Then the following are equivalent:*

(1) $\bigoplus_{g \in G} \frac{(P:I^g)}{P} g$ *is a finitely generated right* R/IB*-module for all prime ideals* $P \lhd C$ *which contain* I*;*

(2) R *is right noetherian.*

We omit the proof from Proposition 3.2.5 as this follows in exactly the same style as Theorem 3.3.6 follows from Proposition 3.2.6.

3.4 Idealisers in Skew Group Rings of Abelian Groups

3.4.1 Right noetherianity

Let us first consider right noetherianity for the setup of Notation 3.3.2. By Theorem 3.3.7, we must show $\mathrm{Hom}_B(B/IB, B/JB)$ is a noetherian right R/IB-module for all prime ideals $J \lhd C$ which contain I. From here we assume that we are working with group rings over finitely generated abelian groups. As mentioned before, since we are dealing with non-trivial stabilisers, we require some extra notation and a definition.

Definition 3.4.1. For two subgroups H, K of a finitely generated abelian group G, we say H is *complementary* to K if $H \cap K = \{0\}$ and $H \oplus K$ is of finite index in G.

Remark 3.4.2. We note that by the classification theorem for finitely generated abelian groups, that a complement always exists and may be chosen to be a free abelian group with $\mathrm{Rank}(K) = \mathrm{Rank}(G) - \mathrm{Rank}(H)$.

Notation 3.4.3. For a subgroup $H \leq G$ and an ideal $J \lhd C$ we denote specific sets as follows:

$$S_{H,V(J)} \coloneqq \{h \in H \mid h.V(J) \subseteq V(I)\},$$

where $h.p$ is the induced action of G on $\mathrm{Spec}\, C$, and

$$T_{H,V(J)} \coloneqq \{h \in H \mid \mathrm{Tor}_1^C(C/I^{-h}, C/J) \neq 0\}.$$

Lemma 3.4.4. *Assume Notation 3.3.2 and that G is a finitely generated abelian group. The sets $S_{G,V(J)}$ and $T_{G,V(J)}$ are both unions of K cosets in G.*

Proof. We begin with $S_{G,V(J)}$. We must show that if $a \in S_{G,V(J)}$, then $a+K \in S_{G,V(J)}$. Indeed

$$(a + K).V(J) \subseteq K.V(I) \subseteq V(I).$$

Now for $T_{G,V(J)}$ we have

$$\begin{aligned}
\mathrm{Tor}_1^C(C/I^{-a}, C/J) = 0 &\iff \mathrm{Tor}_1^C(C/I, C/J^a) = 0 \\
&\iff \mathrm{Tor}_1^C(C/I^{-k}, C/J^a) = 0 \quad \forall k \in K \text{ as } K = \mathrm{Stab}_G(I) \\
&\iff \mathrm{Tor}_1^C(C/I, C/J^{(a+k)}) = 0 \\
&\iff \mathrm{Tor}_1^C(C/I^{-(a+k)}, C/J) = 0 \quad \forall k \in K.
\end{aligned}$$

□

Theorem 3.4.5. *Assume Notation 3.3.2 and that G is a finitely generated abelian group. Then the following are equivalent:*

(1) there exists a subgroup $H \leq G$, complementary to K, such that for all points $p \in V(I)$ the set $S_{H,p}$ is finite;

(2) $\mathbb{I}_B(IB)$ is right noetherian;

(3) for all subgroups $H \leq G$, complementary to K, and for all points $p \in V(I)$, $S_{H,p}$ is finite.

Proof. We begin by proving (1) implies (2) using the following claims:

(a) For $\pi : G \to G/K$, the canonical map, $\pi(S_{G,p})$ is finite for all $p \in V(I)$;

(b) $\pi(S_{G,V(J)})$ is finite for all prime ideals $J \lhd C$ where $I \subseteq J$;

(c) $\frac{(JB:IB)}{JB}$ is a finitely generated right $(C/I)\#K$-module for all prime ideals $J \lhd C$ where $I \subseteq J$.

We proceed by showing (1) $\implies$ (a) $\implies$ (b) $\implies$ (c) $\implies$ (2).

Let us start with (1) $\implies$ (a). Assume for some complement $H \leq G$ of K that $S_{H,p}$ is finite for all $p \in V(I)$. We note that this implies that $S_{H,p}$ is finite for all $p \in \operatorname{Spec} C$; indeed, if $S_{H,p}$ were infinite for some p, then for any $a \in S_{H,p}$, $S_{H,a.p}$ is infinite and $a.p \in V(I)$.

Now $H + K = H \oplus K$ has finite index, say m, in G and let $a_1, \ldots, a_m \in G$ be coset representatives. Then $G = \bigsqcup_{i=1}^{m}(a_i + H + K)$ and so for $p \in V(I)$,

$$\begin{aligned}
S_{G,p} &= \{\alpha \in G \mid \alpha.p \in V(I)\} \\
&= \bigsqcup_{i=1}^{m} \{\alpha \in a_i + H + K \mid \alpha.p \in V(I)\} \\
&= \bigsqcup_{i=1}^{m} (\{\alpha \in a_i + H \mid \alpha.p \in V(I)\} + K) \qquad \text{as } K = \operatorname{Stab}_G(I) \\
&= \bigsqcup_{i=1}^{m} (a_i + \{h \in H \mid (h + a_i).p \in V(I)\} + K) \\
&= \bigsqcup_{i=1}^{m} (a_i + S_{H,a_i.p} + K).
\end{aligned}$$

Hence, as $S_{H,a_i.p}$ is finite by assumption, $\pi(S_{G,p})$ is finite as required.

For (a) $\implies$ (b),

let $J \lhd C$ be a prime ideal such that $I \subseteq J$. Since $S_{G,V(J)} = \bigcap_{p \in V(J)} S_{G,p}$ the result follows and, as $S_{G,V(J)}$ is a K-set by Lemma 3.4.4, $S_{G,V(J)}$ is a finite union of cosets of K.

Now for (b) $\implies$ (c).

Again, let $J \lhd C$ be prime such that $I \subseteq J$. We have $\frac{(JB:IB)}{JB} = \bigoplus_{g \in G} \frac{(J:I^g)}{J} g$ by Lemma 3.3.4. Then

$$\begin{aligned}\frac{(JB : IB)}{JB} &= \bigoplus_{g \in G} \frac{(J : I^g)}{J} g = \bigoplus_{g \in S_{G,V(J)}} \frac{C}{J} g \quad \text{by Equation (1)} \\ &= \bigoplus_{j=1}^{p} \bigoplus_{k \in K} \frac{C}{J}(b_j + k)\end{aligned}$$

where the $b_j \in G$ are the coset representatives for the finite set $\pi(S_{G,V(J)})$. We note that $(C/J)b_j$ is a right C/I-module. To see this, we must check that I acts trivially on $(C/J)b_j$. Since C acts on $(C/J)b_j$ by $\bar{f} b_j c = \bar{f} c^{b_j} b_j$ and $b_j \in S_{G,V(J)}$, $I^{b_j} \subseteq J$ as required. Hence $(C/J)b_j I = 0$ and

$$\oplus_j \oplus_k (C/J)(b_j + k) \cong (\oplus_j (C/J)(b_j)) \otimes_{C/I} (\oplus_k C/Ik) = \oplus_j \frac{C}{J} b_j \otimes_{C/I} ((C/I)\#K),$$

which is a finitely generated right $(C/I)\#K$-module.

Finally, we must prove that (c) $\implies$ (2). By Theorem 3.3.7, R is right noetherian if and only if $\bigoplus_{g \in G} \frac{(P:I^g)}{P} g$ is a finitely generated right R/IB-module for all prime $J \lhd C$ such that $I \subseteq J$, and so (2) holds.

We move onto (2) implies (3). Suppose that there exists a complement $H \leq G$ to K such that for some $p \in V(I)$,

$$S_{H,p} = \{h \in H \mid h.p \in V(I)\}$$

is infinite. Consider $M = I(p)$, the ideal of C associated to $p \in V(I)$. We note, if $h_1, h_2 \in S_{H,p}$ are distinct, then $h_1 + K \neq h_2 + K$ as $H \cap K = \{0\}$.

Then

$$\begin{aligned}
\bigoplus_{g\in G} \frac{(M:I^g)}{M} g = \bigoplus_{g\in S_{G,p}} \frac{C}{M} g \quad &\geq \bigoplus_{g\in S_{H,p}+K} \frac{C}{M} g \\
&= \bigoplus_{\substack{a\in S_{H,p} \\ k\in K}} \frac{C}{M}(a+k) \\
&= \bigoplus_{a\in S_{H,p}} \frac{C}{M} a \otimes_{C/I} \bigoplus_{k\in K} \frac{C}{I} k \\
&= \bigoplus_{a\in S_{H,p}} \frac{C}{M} a \otimes_{C/I} ((C/I)\#K).
\end{aligned}$$

By assumption, this is not a finitely generated right module over $(C/I)\#K \cong R/IB$, thus R is not right noetherian by Theorem 3.3.7.

The implication (3) implies (1) is trivial. This completes the proof. □

3.4.2 Left noetherianity

Now we turn our attention to left noetherianity of our idealiser R. By Theorem 3.3.6, we must show that $R/IB \cong (C/I)\#K$ is left graded-noetherian and

$$\bigoplus_{g\in G} \operatorname{Tor}_1^C(C/I, C/P^g)g$$

is a noetherian left R-module (equivalently, R/IB-module) for all prime ideals $P \lhd C$. We note that we may not restrict to only prime ideals which contain the ideal I. This is one aspect of the left structure of idealisers which is more complicated and highlights that idealisers are rings which can have different left and right structures.

By Proposition 3.3.5, R/IB is left noetherian. We now characterise left noetherianity of $R = \mathbb{I}_B(IB)$.

Theorem 3.4.6. *Assume Notation 3.3.2 and that G is a finitely generated abelian group. Then the following are equivalent:*

(1) There exists a subgroup $H \leq G$, complementary to K, such that

$$T_{H,V(J)} = \{a \in H \mid \operatorname{Tor}_1^C(C/I^{-a}, C/J) \neq 0\}$$

is finite for all prime ideals $J \lhd C$;

(2) R is left noetherian;

(3) For all complementary subgroups H to K, $T_{H,V(J)}$ is finite for all prime ideals $J \lhd C$.

Proof. We aim to show that (1) implies that $\pi\left(T_{G,V(J)}\right)$ is finite. We observe that $\{a \in H \mid \operatorname{Tor}_1^C(C/I^{-a}, C/J) \neq 0\} = \{a \in H \mid \operatorname{Tor}_1^C(C/I, C/J^a) \neq 0\}$ and we use the latter in the proof for ease of notation. Again, $H \oplus K$ has finite index in G, say m, with coset representatives $a_1, \ldots a_m$. Then we have, for $J \lhd C$,

$$\begin{aligned}
T_{G,V(J)} &= \{\alpha \in G \mid \operatorname{Tor}_1^C(C/I, C/J^\alpha) \neq 0\} \\
&= \bigsqcup_{i=1}^m \{\alpha \in a_i + H + K \mid \operatorname{Tor}_1^C(C/I, C/J^\alpha) \neq 0\} \\
&= \bigsqcup_{i=1}^m \left(a_i + \{\alpha \in H \mid \operatorname{Tor}_1^C(C/I, C/J^{(a_i+\alpha)}) \neq 0\} + K\right) \text{ as in Lemma 3.4.4} \\
&= \bigsqcup_{i=1}^m \left(a_i + T_{H,V(J^{a_i})} + K\right).
\end{aligned}$$

Since $T_{H,V(J^{a_i})}$ is finite for $i = 1, \ldots, m$, we obtain that $\pi\left(T_{G,V(J)}\right)$ is finite. As $\pi\left(T_{G,V(J)}\right)$ is a right K-set by Lemma 3.4.4, $T_{G,V(J)}$ is a finite union of cosets of K.

Then

$$\begin{aligned}
\operatorname{Tor}_1^B(B/IB, B/BJ) &= \frac{IB \cap BJ}{IBJ} = \bigoplus_{g \in G} \frac{Ig \cap gJ}{IgJ} \\
&= \bigoplus_{g \in G} \frac{Ig \cap J^g g}{IJ^g g} = \bigoplus_{g \in G} \frac{I \cap J^g}{IJ^g} g \\
&= \bigoplus_{g \in G} \operatorname{Tor}_1^C(C/I, C/J^g) g \\
&= \bigoplus_{g \in T_{G,V(J)}} \operatorname{Tor}_1^C(C/I, C/J^g) g \\
&= \bigoplus_{j=1}^m \bigoplus_{k \in K} \operatorname{Tor}_1^C(C/I, C/J^{b_j+k})(k + b_j)
\end{aligned}$$

where the $b_j \in G$ are the coset representatives for the finite set $\pi\left(T_{G,V(J)}\right)$.

Observe that $\bigoplus_{k \in K} \operatorname{Tor}_1^C(C/I, C/J^{b_j+k})(b_j+k)$ is a finitely generated left $(C/I)\#K$-module. Indeed, using the K-invariance of I, we see that both $\bigoplus_{k \in K} I \cap J^{b_j+k} k$ and $\bigoplus_{k \in K} IJ^{b_j+k} k$ are graded left ideals of $C\#K$, and hence are finitely generated left $C\#K$-modules. Thus $\bigoplus_{k \in K} \operatorname{Tor}_1^C(C/I, C/J^{b_j+k}) k = \bigoplus_{k \in K} \frac{I \cap J^{b_j+k}}{IJ^{b_j+k}} k$ is a finitely generated left $C\#K$-module. As I acts trivially, $\bigoplus_{k\in K}\operatorname{Tor}_1^C(C/I, C/J^{b_j+k})k$ is also a finitely generated left $(C/I)\#K$-module. Thus $\operatorname{Tor}_1^B(B/IB, B/BJ)$ is a finitely generated left $(C/I)\#K$-module. As J was arbitrary, by Theorem 3.3.6, R is left noetherian.

We now show (2) $\implies$ (3). Suppose there exists a complementary subgroup $H \leq G$ to K such that for some prime ideal $J \lhd C$, $T_{H,V(J)}$ is infinite.

Then

$$\begin{aligned}\operatorname{Tor}_1^B(B/IB, B/BJ) &= \bigoplus_{g\in G}\operatorname{Tor}_1^C(C/I, C/J^g)g = \bigoplus_{g\in T_{G,V(J)}}\operatorname{Tor}_1^C(C/I, C/J^g)g\\ &\geq \bigoplus_{g\in T_{H,V(J)}\oplus K}\operatorname{Tor}_1^C(C/I, C/J^g)g\\ &= \bigoplus_{a\in T_{H,J}}\bigoplus_{k\in K}\operatorname{Tor}_1^C(C/I, C/J^{a+k})(a+k),\end{aligned}$$

which is an infinite direct sum of non-zero $(C/I)\#K$-modules. Hence R is not left noetherian as required. □

We now apply our results in the case that I is a maximal ideal of C. By Theorem 3.4.5, R is right noetherian if and only if there exists a complementary subgroup $H \leq G$ to $\operatorname{Stab}_G(I)$ such that

$$S_{H,p} = \{h \in H \mid h.p = p\}$$

is finite where $p = V(I)$. But $S_{H,p} = H \cap \operatorname{Stab}_G(I) = 0$, so this always holds. Hence R is always right noetherian. We note that this result may be considered a graded version of Theorem 3.1.1.

Let us now consider left noetherianity. We require for some complementary subgroup $H \leq G$

$$\{h \in H \mid \operatorname{Tor}_1^C(C/I^{-h}, C/J) \neq 0\}$$

to be finite for all non-zero prime $J \lhd C$. Then since $\operatorname{Tor}_1^C(C/I^{-h}, C/J)$ is supported on $V((-h).I) \cap V(J)$, $\operatorname{Tor}_1^C(C/I^{-h}, C/J) \neq 0$ if and only if $J^h \subseteq I$ or equivalently $h.p = V(I) \in V(J)$. That is to say, for all proper subvarieties, Z, of $\operatorname{Spec} C$, $\{h.p\}_{h\in H} \cap Z$ is finite. If $|G.p| < \infty$ this automatically holds. Otherwise, this property already exists in the literature and is known as *critical density*.

Definition 3.4.7. Let X be an affine variety and let S be an infinite subset of X. We say S is *critically dense* if $S \cap Y$ is finite for all proper subvarieties Y of X. Equivalently, any infinite subset of S is Zariski dense.

We summarise the preceding discussion in the following theorem.

Theorem 3.4.8. *Assume Notation 3.3.2 and that G is a finitely generated abelian group. Suppose that I is a maximal ideal of C and let $p = V(I)$. Then R is always right noetherian. If $|G.p| < \infty$ then R is left noetherian. Otherwise, R is left noetherian if and only if $G.p$ is critically dense.* □

Finally, to finish this section we observe that the left noetherianity of an idealizer guarantees that it will be right noetherian.

Theorem 3.4.9. *Assume Notation 3.3.2 and that G is a finitely generated abelian group. If the idealizer $\mathbb{I}_B(IB)$ is left noetherian, then it is right noetherian.*

Proof. We prove this by taking the contrapositive. Let us assume that $\mathbb{I}_B(IB)$ is not right noetherian. Then for every subgroup $H \leq G$, complementary to K, there exists a point $p_H \in V(I)$ such that

$$S_{H,p_H} \coloneqq \{h \in H \mid h.p_H \in V(I)\}$$

is infinite. We now show that choosing $J = I(p_H) \lhd C$ also gives

$$T_{H,p_H} = \{h \in H \mid \mathrm{Tor}_1^C(C/I^{-h}, C/J) \neq 0\}$$

is infinite and hence, by Theorem 3.4.6, $\mathbb{I}_B(IB)$ is not left noetherian. Indeed, $S_{H,p_H} \subseteq T_{H,p_H}$, since every $h \in S_{H,p_H}$ will give a non-homologically transverse intersection (the image of the point will lie on $V(I)$), and so T_{H,p_H} is infinite as required. □

3.4.3 Critical Transversality

In the previous two sections we have found conditions on for $\mathbb{I}_B(IB)$ to be right or left noetherian which we note are very different. For the right-hand side we have a condition which is based on the orbit of points in $V(I)$. However, on the other side the condition for left noetherianity is much less clear. In this section we show that this condition has a geometric analogue as Sierra showed for twisted homogeneous coordinate rings in [37] and we prove Theorem 3.1.5.

We now show that the condition of left noetherianity is closely related to the notion of critical transversality as defined in the introduction.

Lemma 3.4.10. *Assume Notation 3.3.2 and that G is a finitely generated abelian group. Let H be the complementary subgroup to $K = \mathrm{Stab}_G(I)$. Then the following are equivalent:*

(1) for all prime ideals $J \lhd C$, the set

$$\{h \in H \mid \mathrm{Tor}_1^C(C/I^{-h}, C/J) \neq 0\}$$

is finite;

(2) for all prime ideals $J \lhd C$, the set

$$A(J) = \{h \in H \mid V(I^{-h}) \text{ is not homologically transverse to } V(J)\}$$

is finite.

Before beginning the proof, we establish some terminology which we will need if $\mathrm{Spec}\, C$ is singular.

Definition 3.4.11. Let $X = \mathrm{Spec}\, C$ be an affine variety. We define the *singular stratification* of X iteratively as follows.

Define $X^{(1)} = \{m \in \mathrm{MaxSpec}(C) \mid \mathrm{gldim}_C\left(C_{(m)}\right) = \infty\}$ to be the singular locus of X, which is closed, and define $X^{(n)}$ to be the singular locus of $X^{(n-1)}$. The singular stratification is preserved under automorphisms.

We have the following Lemma, originally due to Mel Hochster.

Lemma 3.4.12. [37, Lemma 5.3] *Suppose that $V(I)$ is homologically transverse to all parts of the singular stratification of* $\mathrm{Spec}\, C$*. Then*

$$\mathrm{pdim}_C(C/I) < \infty.$$

Proof of Lemma 3.4.10. That (2) implies (1) is trivial.

Assume (1). We may assume that H is infinite and $\mathrm{Stab}_H(I) = \{0\}$ by definition. We note that

$$A(J) = \bigcup_{j \geq 1} \{h \in H \mid \mathrm{Tor}_j^C(C/I^{-h}, C/J) \neq 0\}.$$

We note that whilst standard homological arguments imply that

$$A_j(J) = \{h \in H \mid \operatorname{Tor}_j^C(C/I^{-h}, C/J) \neq 0\}$$

is finite for each $j \geq 1$, this is not enough to conclude (2). However, if the projective dimension of C/I is finite, (2) immediately follows.

We first claim that for any finitely generated C-module M and $j \geq 1$, the set

$$\{h \in H \mid \operatorname{Tor}_j^C(C/I^{-h}, M) \neq 0\}$$

is finite. We induct on $j \geq 1$. Firstly, for $j = 1$. By [16, Proposition 3.7], M has a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with each $M_{i+1}/M_i \cong C/P_i$ for some prime ideal $P_i \lhd C$. A similar argument to Theorem 3.3.6, inducting on n gives that $\{h \in H \mid \operatorname{Tor}_1^C(C/I^{-h}, M) \neq 0\}$ is finite. Now let $j > 1$. We may construct a short exact sequence

$$0 \to K \to C^m \to M \to 0$$

where K is a finitely generated C-module. From the long exact sequence in Tor we obtain

$$\operatorname{Tor}_j^C(C/I^{-h}, M) \cong \operatorname{Tor}_{j-1}^C(C/I^{-h}, K),$$

from which the claim follows by induction as the right-hand side vanishes for all but finitely many $h \in H$.

From the claim, $V(I)$ is homologically transverse to all H-invariant subvarieties of $\operatorname{Spec} C$. This is because, if J is H-invariant, $A_j(J)$ is either all of H or trivial and, as $\{h \in H \mid \operatorname{Tor}_1^C(C/I^{-h}, C/J) \neq 0\}$ is finite, it must be the latter. In particular, $V(I)$ is homologically transverse to the singular stratification of $\operatorname{Spec} C$ as each of the terms in the singular stratification corresponds to a factor C/A where A is an H-invariant ideal of C. Hence C/I has finite projective dimension by Lemma 3.4.12 and $A_j(J) = 0$ for all $j > \operatorname{pdim}_C C/I$. Thus $A(J) = \bigcup_{j \geq 1} A_j(J)$ is finite as required. □

The following Theorem now follows as a corollary of Lemma 3.4.10.

Theorem 3.4.13 (Theorem 3.1.5). *Assume Notation 3.3.2 and that G is a finitely* *generated abelian group. Then R is left noetherian if and only if, for some complement $H \leq G$ of K, $\{I^{-h}\}_{h \in H}$ is critically transverse.* □

As mentioned in the introduction, results obtained by Sierra for idealisers in twisted homogeneous coordinate rings are similar to our results in the case that $G = \mathbb{Z}$.

Theorem 3.4.14. [37, Theorem 10.2] *Let X be a projective variety, let $\sigma \in \operatorname{Aut} X$, let $\mathcal{L}$ be a σ-ample invertible sheaf on X, and let Z be an irreducible, closed subscheme of X of infinite order under σ. Let $B = (X, \mathcal{L}, \sigma)$ be a twisted homogeneous coordinate ring and let I be the right ideal of B corresponding to Z.*

If for all $p \in Z$, the set $\{n \geq 0 \mid \sigma^n(p) \in Z\}$ is finite then $\mathbb{I}_B(I)$ is right noetherian. If the set $\{\sigma^n Z\}_{n \in \mathbb{Z}}$ is critically transverse, then $\mathbb{I}_B(I)$ is left noetherian.

We note that both of the sets involved correspond with those we obtained in Theorems 3.4.5 and 3.4.6 in the case that the stabiliser is trivial and $G = \mathbb{Z}$.

3.5 Idealisers Defined by Subvarieties of the Plane

Now that we have an abstract set of conditions for when idealiser rings are left and right noetherian, let us see how these work in practice. We consider $C = \mathbb{C}[x, y]$ and $\mathbb{Z}^2 \subseteq \mathbb{C}^2$ acting by translation. We will see that the noetherianity of the idealisers we obtain depends on subtle arithmetic results about integer points on varieties.

Example 3.5.1. Let us consider $C = \mathbb{C}[x, y]$, and $\sigma, \tau \in \operatorname{Aut}(\mathbb{C}[x, y])$ defined by

$$\sigma(x) = x + 1, \quad \sigma(y) = y,$$
$$\tau(x) = x, \quad \tau(y) = y + 1.$$

Define $B = C \# \mathbb{Z}^2$ with $\mathbb{Z}^2 = (\sigma, \tau)$ acting on C by translation. By Bezout's theorem, the only irreducible curves with non-trivial stabiliser under this action are lines with rational slope (including slope ∞). We shall treat this case separately.

If $V(I)$ is not such a line then $I \lhd C$ is prime with trivial stabiliser. By Theorem 3.4.5, $\mathbb{I}_B(IB)$ is right noetherian if and only if

$$\{a \in \mathbb{Z}^2 \mid a.p \in V(I)\}$$

is finite for all $p \in V(I)$. This is equivalent to

$$V(I) \cap (b + \mathbb{Z}^2)$$

being finite for all $b \in \mathbb{C}^2$.

Further, $\mathbb{I}_B(IB)$ is left noetherian if and only if

$$\{a \in \mathbb{Z}^2 \mid \operatorname{Tor}_1^C(C/I^{-a}, C/J) \neq 0\}$$

is finite for all prime ideals $J \lhd C$.

Our task is to understand these geometric conditions better. We shall split our work into two cases: when I is and is not maximal. Firstly, when I is maximal, then $\operatorname{Stab}_{\mathbb{Z}^2}(I)$ is trivial and we claim that $\mathbb{I}_B(IB)$ right but it is not left noetherian. Indeed, if I is maximal then $\mathbb{I}_B(IB)$ is easily shown to be right noetherian by Theorem 3.3.7. Also, as $V(I) = p$ for some point $p = (p_1, p_2) \in \mathbb{A}^2$, let us consider the subvariety $Y = V(x - p_1) \subseteq \mathbb{A}^2$. As $\tau^i(p) \in Y$ for all $i \in \mathbb{Z}$, $\mathbb{Z}^2.V(I)$ cannot be critically dense and thus, by Theorem 3.4.8, $\mathbb{I}_B(IB)$ is not left noetherian. We summarise this in the following Proposition.

Proposition 3.5.2. *Assume the setup from Example 3.5.1. Suppose I is a maximal ideal. Then $\mathbb{I}_B(IB)$ is right but not left noetherian.* □

Now we consider what happens when I is prime but not maximal nor corresponds to a line of rational slope. Then $I = (f)$ for some irreducible polynomial $f \in \mathbb{C}[x, y]$ which is not linear in x and y with coefficients in $\mathbb{Q}$ and corresponds to a plane curve in $\mathbb{A}^2$. We now view $\mathbb{I}_B(IB)$ as a subring of the skew field of fractions associated to B, $Q(B)$, in which f is invertible. Then the conjugation action by f:

$$\theta\colon\ Q(B) \to Q(B),$$
$$\theta(a) = f^{-1} a f$$

is an isomorphism of $Q(B)$. We note that $\mathbb{I}_B(IB) = C + fB$, by Lemma 3.3.4 as I has trivial stabiliser, and so $\theta(\mathbb{I}_B(IB)) = C + Bf = \mathbb{I}_B(BI)$ as f commutes with C. Thus θ restricted to $\mathbb{I}_B(IB)$ induces an isomorphism between $\mathbb{I}_B(IB)$ and $\mathbb{I}_B(BI)$. Hence $\mathbb{I}_B(IB)$ is left noetherian if and only if $\mathbb{I}_B(BI)$ is left noetherian which, by the opposite-sided version of Theorem 3.4.5, happens if and only if $V(I) \cap (b + \mathbb{Z}^2)$ is finite for all

$b \in \mathbb{C}^2$. That is to say, $\mathbb{I}_B(IB)$ is left noetherian if and only if it is right noetherian. We note that if $I = (f)$ then $\mathrm{Tor}_1^C(C/I, C/J) \neq 0$ if and only if $J \supseteq I$. So the Tor condition also gives that $\mathbb{I}_B(IB)$ is left noetherian if and only if $V(I) \cap (b + \mathbb{Z}^2)$ is finite. We summarise the above discussion in the following Proposition.

Proposition 3.5.3. *Assume the setup from Example 3.5.1. Suppose I is a non-maximal prime ideal that does not correspond to a line of rational slope. Then $\mathbb{I}_B(IB)$ is left and right noetherian if and only if the set*

$$V(I) \cap (b + \mathbb{Z}^2)$$

is finite for all $b \in \mathbb{C}^2$. □

We now seek to understand which ideals I, or indeed which curves $V(I)$, satisfy this condition. We turn to Siegel's theorem on integral points.

Theorem 3.5.4. [44, Theorem 3.2] *Let C be an affine irreducible curve over a number field k, and suppose it has infinitely many integral points. Then C has genus* 0 *and at most two points at infinity.*

We observe that if we were simply interested in whether $V(I) \cap \mathbb{Z}^2$ was finite where $V(I)$ was defined over some finite field extension of $\mathbb{Q}$ then this would be a straightforward application of this theorem. However, this is not the case and in addition we are interested in (possibly non-rational) translations of $V(I)$ and hence we search for some 'dynamical' Siegel's Theorem. Thus we turn to Lang's generalisation of Siegel's Theorem from [23, Chapter VII pp.121 and Theorem 4]:

Theorem 3.5.5. *If C is an affine curve defined over a ring S finitely generated over $\mathbb{Z}$, and if its genus is $g \geq 1$, then C has only a finite number of points in S.*

We note that if $g \geq 2$ this follows from Falting's Theorem. Applying the results, we obtain:

Theorem 3.5.6. *With the setup from Example 3.5.1, let $X = V(I)$ be an irreducible curve. If X has genus ≥ 1, then $\mathbb{I}_B(IB)$ is left and right noetherian.*

Proof. Let $I = (f(x, y))$ for some irreducible polynomial $f(x, y) \in \mathbb{C}[x, y]$ and suppose that $\mathbb{I}_B(IB)$ is not noetherian. Then, by Proposition 3.5.3 there exists $c, d \in \mathbb{C}$ such that $g \coloneqq f(c + x, d + y)$ has infinitely many integer solutions. Let S be the $\mathbb{Z}$-algebra generated by the coefficients of g. Then g defines a curve over S and g has an infinite number of solutions in S (from its infinite integer solutions). Thus, by Theorem 3.5.5, g must have genus 0. Hence, f also has genus 0 as required. $\square$

For example, if X were a smooth cubic curve then the idealiser associated to this curve will always be left and right noetherian. So any cubic curve with non-trivial j-invariant will give a left and right noetherian idealiser.

However, if we consider a genus 0 curve of the form

$$x^2 - ny^2 = 1$$

where n is a given positive nonsquare integer, then it is a result of Lagrange that this curve has an infinite number of integral points. As an example, consider $x^2 - 7y^2 = 1$, then the integer solutions (x_{k+1}, y_{k+1}) are given by the recurrence formula:

$$x_{k+1} = x_1 x_k + 7y_1 y_k$$
$$y_{k+1} = x_1 y_k + y_1 x_k$$

where $(x_1, y_1) = (8, 3)$. So $\mathbb{I}_B((x^2 - 7y^2 - 1)B)$ is neither right nor left noetherian.

Recall that we did not consider lines with rational slope as they do not have trivial stabiliser. We deal with these lines now.

Proposition 3.5.7. *Assume the setup from Example 3.5.1. Let $I \lhd C$ be a prime ideal in C corresponding to a line in $\mathbb{A}^2$. Then $\mathbb{I}_B(IB)$ is right and left noetherian.*

Proof. If I corresponds to a line of irrational slope, $\mathrm{Stab}_{\mathbb{Z}^2}(I) = 0$ and $V(I) \cap (b + \mathbb{Z}^2) < \infty$ for all $b \in \mathbb{C}^2$, so by Proposition 3.5.3, $\mathbb{I}_B(IB)$ is both left and right noetherian.

Now $I = (f) = (mx - ny - p)$ where $m, n, p \in \mathbb{C}$. Without loss of generality $m, n \in \mathbb{Z}$ as the slope is rational and, by symmetry, we may assume $n \neq 0$. Then, as $\sigma^i \tau^j(f) = f$ if and only if $(i, j) \in (n, m)\mathbb{Z}$, $K = \mathrm{Stab}_{\mathbb{Z}^2}(I) = (n, m)\mathbb{Z}$. Consider $H = (0, 1)\mathbb{Z}$. As $n \neq 0$, $H \cap K = \{0\}$ and, since $H, K \cong \mathbb{Z}$, $H \oplus K = \mathbb{Z}^2$ so H is a complement to K. Further, as $\tau^j(p) \notin V(I)$ for any $p \in V(I)$ and $j \neq 0$, the set $S_{H,p} = \{(0, 0)\}$ and hence, by Theorem 3.4.5, $\mathbb{I}_B(IB)$ is right noetherian. For left noetherianity we must show that the set $\{h \in H \mid I^h$ is not homologically transverse to $J\}$ is finite for all prime $J \lhd C$. Then, as $V(I^h) \subseteq \mathbb{A}^2$ is a plane curve, I^h and J can only have a non homologically transverse intersection if $I^h \subseteq J$. But this can only happen at most once when $H = (0,1)\mathbb{Z}$, hence the set is finite and $\mathbb{I}_B(IB)$ is left noetherian. $\square$

3.6 More General Groups

In this section we briefly consider noetherianity of idealisers in group rings involving polycyclic-by-finite groups. Providing a complement actually exists - something which may not happen in non-abelian groups - the same form of argument as in Theorems 3.4.5 and 3.4.6 goes through. We begin by showing that the sets

$$S_{G,J} = \{g \in G \mid I^g \subseteq J\}$$

and

$$T_{G,V(J)} = \{g \in G \mid \mathrm{Tor}_1^C(C/I^{g^{-1}}, C/J) \neq 0\}$$

are right and left K-sets respectively.

Lemma 3.6.1. *The sets*

$$S_{G,J} = \{g \in G \mid I^g \subseteq J\}$$

and

$$T_{G,V(J)} = \{g \in G \mid \mathrm{Tor}_1^C(C/I^{g^{-1}}, C/J) \neq 0\}$$

are unions of right and left cosets of K in G respectively.

It may be noted that our set $S_{G,J}$ is the same set as $S_{G,V(J)}$ in Notation 3.4.3. Indeed,

$$S_{G,V(J)} = \{g \in G \mid g.V(J) \subseteq V(I)\} = \{g \in G \mid I \subseteq J^{g^{-1}}\} = \{g \in G \mid I^g \subseteq J\}.$$

Proof of Lemma 3.6.1. First for $S_{G,J}$ where $J \supseteq I$. Let $a \in S_{G,J}$ and let $k \in K$. Then

$$I^{ak} \subseteq I^a \subseteq J$$

by the definition of K.

Now for $T_{G,V(J)}$ where $J \lhd C$ is an arbitrary prime ideal. Then

$$\begin{aligned}\mathrm{Tor}_1^C(C/I^{a^{-1}}, C/J) = 0 &\iff \mathrm{Tor}_1^C(C/I, C/J^a) = 0 \\ &\iff \mathrm{Tor}_1^C(C/I^{k^{-1}}, C/J^a) = 0 \\ &\iff \mathrm{Tor}_1^C(C/I, C/J^{ka}) = 0 \\ &\iff \mathrm{Tor}_1^C(C/I^{(ka)^{-1}}, C/J) = 0\end{aligned}$$

as required. We note that neither of these sets are necessarily two-sided K-sets. □

Armed with this result, the generalisation of Theorems 3.4.5 and 3.4.6 follows through exactly the same argument.

Theorem 3.6.2. *Assume Notation 3.3.2 and that* $K = \mathrm{Stab}_G(I)$ *has at least one complementary subgroup. Then the following are equivalent:*

(1) there exists a subgroup $H \leq G$*, complementary to* K*, such that for all points* $p \in \operatorname{Spec} C$ *the set* $S_{H,p}$ *is finite;*

(2) R *is right noetherian;*

(3) for all subgroups $H \leq G$*, complementary to* K*, and for all points* $p \in \operatorname{Spec} C$*,* $S_{H,p}$ *is finite.*

We note that the first condition in Theorem 3.6.2 is for all points $p \in \operatorname{Spec} C$ as opposed to just points in $V(I)$. This is because, if G is no longer abelian, the argument from Theorem 3.4.5 no longer applies and so we cannot reduce to only considering $S_{H,p}$ for $p \in V(I)$.

Theorem 3.6.3. *Assume Notation 3.3.2 and that* K *has at least one complementary subgroup. Then the following are equivalent:*

(1) there exists a subgroup $H \leq G$*, complementary to* K*, such that*

$$T_{H,V(J)} = \{a \in H \mid \mathrm{Tor}_1^C(C/I^{-a}, C/J) \neq 0\}$$

is finite for all prime ideals $J \lhd C$*;*

(2) R *is left noetherian;*

(3) for all complementary subgroups H to K, $T_{H,V(J)}$ is finite for all prime ideals $J \lhd C$.

To close, we consider an example where the stabiliser does not have a complementary subgroup.

Example 3.6.4. Let G be the Heisenberg group with centre $Z(G)$, explicitly:

$$G = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}, \quad Z(G) = \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have that $G/Z(G) \cong \mathbb{Z}^2$, which we let act on $\mathbb{C}[x, y]$ as in Example 3.5.1. Requiring that $Z(G)$ acts trivially gives an induced action of G on $\mathbb{C}[x, y]$ with each maximal ideal having $Z(G)$ as its stabiliser. We claim that the centre does not have a complementary subgroup. Indeed, to obtain $[G : HZ(G)] < \infty$, since $Z(G) = [G, G]$ and the Hirsch length of G is 3, H would have to be free abelian of rank 2. But no subgroup of G of this form intersects the centre trivially. Hence $Z(G)$ is an example of of a subgroup with no complement. Let $B = \mathbb{C}[x, y]\#G$, $M \lhd \mathbb{C}[x, y]$ a maximal ideal, and $R = \mathbb{I}_B(MB)$.

However, we can still determine whether these idealisers are left or right noetherian. Indeed, as M is a maximal ideal, we only have two choices for $J \supseteq M$ in $(JB : MB)/JB$, namely $J = M$ and $J = C$. In either case, $(JB : MB)/JB$ is a finitely generated right $(C/I)\#K$-module and hence, by Theorem 3.3.7, R is right noetherian. For left noetherianity, as M is a maximal ideal of $\mathbb{C}[x, y]$ it is of the form $M = (x - p, y - q)$ for some $p, q \in \mathbb{C}$, consider the ideal $J = (x - p) \lhd \mathbb{C}[x, y]$. Let

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix},$$

then $J^h = J \subseteq M$ for all $h \in H$ and so $\mathrm{Tor}_1^C(C/M, C/J^h) = \mathrm{Tor}_1^C(C/M, C/J) \neq 0$. If $h, h' \in H$ are distinct then $hK \neq h'K$, so $\bigoplus_{h \in H} \mathrm{Tor}_1^C(C/M, C/J^h)h$ cannot be a finitely generated left $(C/I)\#K$-module. Thus, since $\bigoplus_{h \in H} \mathrm{Tor}_1^C(C/M, C/J^h)h \leq \bigoplus_{g \in G} \mathrm{Tor}_1^C(C/M, C/J^g)g$, it follows that $\bigoplus_{g \in G} \mathrm{Tor}_1^C(C/M, C/J^g)g$ cannot be finitely generated as a left $(C/I)\#K$-module and hence, by Theorem 3.3.6, R is not left noetherian.

Chapter 4

Preliminaries on Rings of Differential Operators

4.1 Background on Rings of Differential Operators

In this chapter we describe the background to Chapter 5 and provide context for our results. The rings that interest us are differential operator rings, which are a type of skew extension of a commutative ring that offer us an algebraic approach to working with linear differential equations. Let us start with how to associate a module to a system of differential equations. The interested reader is directed to Coutinho [14] for a more complete introduction. In the remainder of this thesis we consider $\Bbbk$ to be an algebraically closed field of characteristic zero.

Let A_n be the n^{th} Weyl algebra and let $P \in A_n$ be an operator on $\Bbbk[x_1, \dots, x_n]$. We can therefore write $P = \sum_\alpha g_\alpha \partial^\alpha$ where $\alpha \in \mathbb{N}^n$, $g_\alpha \in \Bbbk[x_1, \dots, x_n]$, and if $\alpha = (m_1, \dots, m_n)$ then ∂^α denotes $\frac{\partial^{m_1}}{\partial x_1^{m_1}} \cdots \frac{\partial^{m_n}}{\partial x_n^{m_n}}$. This operator gives rise to a differential equation:

$$P(f) = \sum_\alpha g_\alpha \partial^\alpha (f) = 0,$$

where f is a polynomial (more generally if $\Bbbk = \mathbb{R}$ then f could also be a $\mathcal{C}^\infty$ function on $\mathbb{R}^n$). If $P_1, \dots, P_m$ are operators in A_n we may similarly construct a system of differential equations by setting:

$$P_1(f) = P_2(f) = \cdots = P_m(f) = 0. \tag{4.1}$$

We may associate a finitely generated left A_n-module $A_n/\sum_{i=1}^m A_nP_i$ to this system in the following way. If $g \in \Bbbk[x_1, \ldots, x_n]$ is a solution to (4.1), then we may define an A_n-module homomorphism

$$\begin{aligned} \alpha_g : A_n &\to \Bbbk[x_1, \ldots, x_n] \\ 1 &\mapsto g. \end{aligned}$$

If P' is an operator in the left ideal $\sum_1^m A_nP_i$, then $P'(g) = 0$ by (4.1), and so α_g induces a homomorphism

$$\begin{aligned} \overline{\alpha_g} : \frac{A_n}{\sum_1^m A_nP_i} &\to \Bbbk[x_1, \ldots, x_n] \\ Q &\mapsto \alpha_g(Q). \end{aligned}$$

Hence we may associate a solution to a system of differential equations with an element of $\operatorname{Hom}_{A_n}(A_n/\sum_1^m A_nP_i, \Bbbk[x_1, \ldots, x_n])$. Moreover, it can be shown that any such homomorphism gives rise to a solution of the system of equations. This results in the following theorem:

Theorem 4.1.1. [14, Chapter 6, Theorem 1.2] *Let M be the module associated to a system of differential equations. Then the vector space of polynomial solutions of the system is isomorphic to* $\operatorname{Hom}_{A_n}(M, \Bbbk[x_1, \ldots, x_n])$.

From this theorem we see that the module M associated to a system of differential equations encodes information about the solutions of the system.

4.1.1 The Ring of Differential Operators of a Commutative Ring

As we have just seen, we may bring natural algebraic constructions to the world of differential equations. More generally, we will see that we may define differential operators on commutative rings. We first define the order of a differential operator.

Definition 4.1.2. Let R be a commutative $\Bbbk$-algebra. As with the Weyl algebra, we view an element a of R as an operator in $\operatorname{End}_\Bbbk R$ by the rule $r \mapsto ar$, for every $r \in R$. We define a *differential operator of order n* inductively. We say an operator $P \in \operatorname{End}_\Bbbk R$ is a *differential operator of order zero* if $[P, a] = 0$ for all $a \in R$. Suppose we have defined differential operators of order $< n$. An operator $P \in \operatorname{End}_\Bbbk R$ is a *differential operator* *of order* n if it is not a differential operator of order strictly less than n, and $[a, P]$ has order less than n for all $a \in R$. We denote by $D^n(R)$ the set of all differential operators of order $\leq n$.

We now define a special type of operator, an operator which satisfies the Leibniz rule for differentiation, which we will show is in fact a differential operator of order 1.

Definition 4.1.3. A *derivation* of a commutative $\Bbbk$-algebra R is an operator $D \in \operatorname{End}_{\Bbbk} R$ which satisfies the Leibniz rule:

$$D(ab) = D(a)b + aD(b)$$

for every $a, b \in R$. We denote the $\Bbbk$-vector space of all derivations by $\operatorname{Der}_{\Bbbk} R$.

Note that if $a \in R$ and $D \in \operatorname{Der}_{\Bbbk} R$, then aD is also a derivation and by this action $\operatorname{Der}_{\Bbbk} R$ becomes a left R-module. Further, we observe $D(\Bbbk) = 0$.

We provide a standard proof of the following lemma to give the reader a sense of how to work with the definition of differential operator.

Lemma 4.1.4. *The differential operators of* R *of order* ≤ 1 *are the elements of* $\operatorname{Der}_{\Bbbk} R + R$. *Further,* $D^0(R) = R$.

Proof. Let $P \in D^1(R)$ and set $D = P - P(1)$. Note that D has order ≤ 1 and $D(1) = 0$. By definition, $[D, a]$ has order 0 for all $a \in R$, and so, for all $b \in R$:

$$0 = [[D, a], b](1) = ((Da)b - (aD)b - b(Da) + b(aD))(1) = D(ab) - aD(b) - bD(a).$$

Hence,

$$D(ab) = aD(b) + bD(a)$$

as required, and so $P = D + P(1) \in \operatorname{Der}_{\Bbbk} R + R$. We observe that if $Q \in D^0(R)$ then

$$0 = [Q, a](1) = Q(a) - aQ(1)$$

and hence $Q \in \operatorname{End}_R R = R$. $\square$

Definition 4.1.5. The *ring of differential operators* $\mathscr{D}(R)$ of a commutative $\Bbbk$-algebra R is defined to be the set of all operators of $\operatorname{End}_{\Bbbk} R$ of finite order, with the operations of sum and composition of operators. That is to say

$$\mathscr{D}(R) = \bigcup_{n\geq 0} D^n(R).$$

Remark 4.1.6. We note that the ring of differential operators is naturally a filtered ring with respect to the order of differential operators and, since, if $\alpha \in D^i(R)$ and $\beta \in D^j(R)$, then $[\alpha, \beta] \in D^{i+j-1}(R)$, this implies that we may define a commutative ring structure on

$$\operatorname{gr} \mathscr{D}(R) = \bigoplus_{n\geq 0} D^{n+1}(R)/D^n(R).$$

4.1.2 Rings of Differential Operators on Nonsingular Varieties

In this section we will see that if R is a commutative ring which is the coordinate ring of a nonsingular curve then $\mathscr{D}(R)$ is a particularly pleasant ring. We give the following definition that is not formal, but suffices for our purposes.

Definition 4.1.7. Let R be a commutative $\mathbb{k}$-algebra. We say that R is *regular* if it is the coordinate ring of a nonsingular irreducible affine variety. Equivalently, R is a finitely generated domain and $\operatorname{gldim} R < \infty$.

Theorem 4.1.8. [9] *Let R be a regular ring. Then $\mathscr{D}(R)$ is generated by R and* $\operatorname{Der}_{\mathbb{k}} R$.

The Weyl algebra naturally fits into this context as the following example shows.

Example 4.1.9. We aim to show that $A_1 = \mathscr{D}(\mathbb{k}[x])$. By Theorem 4.1.8, $\mathscr{D}(\mathbb{k}[x])$ is generated by $\mathbb{k}[x]$ and $\operatorname{Der}_{\mathbb{k}} \mathbb{k}[x]$ so we must study the latter.

Indeed, let $D \in \operatorname{Der}_{\mathbb{k}} \mathbb{k}[x]$. Then $D(x^k) = kx^{k-1}D(x)$, and hence

$$(D - D(x)\partial_x)(x^k) = D(x^k) - D(x)kx^{k-1} = 0,$$

where ∂_x denotes $\partial/\partial x$. Thus, as $\{x^i\}_{i\geq 0}$ form a basis for $\mathbb{k}[x]$, we see $D = f\partial_x$ where $f = D(x) \in \mathbb{k}[x]$. Therefore $\mathscr{D}(\mathbb{k}[x])$ is in fact a ring that we have already met, the first Weyl algebra. Using a similar argument, one can show that $\mathscr{D}(\mathbb{k}[x_1 \dots, x_n]) \cong A_n$.

We note that Theorem 4.1.8 does not hold if R is simply a commutative affine domain. The ring $R = \mathbb{k}[x, y]/(y^3 - x^2)$ gives a counterexample; for details see [14, Exercise 3.6]. We list some important properties of $\mathscr{D}(R)$ when $R = \mathcal{O}(X)$ is regular.

Theorem 4.1.10. [9] *Let R be a regular ring and let $X = \operatorname{Spec} R$. Then $\mathscr{D}(R)$ has the following properties:*

1. $\operatorname{gr}\mathscr{D}(R) \cong \mathcal{O}(T^*X)$ *where T^*X denotes the cotangent bundle of X (which is also a nonsingular affine variety);*

2. $\operatorname{gr}\mathscr{D}(R)$ *is a commutative noetherian domain and hence $\mathscr{D}(R)$ is a right and left noetherian domain;*

3. $\operatorname{gldim}(\mathscr{D}(R)) = \dim X$ *and* $\operatorname{GKdim}(\mathscr{D}(R)) = 2\dim X$.

In the case that R is a regular ring, a lot is known about the module structure of $\mathscr{D}(R)$. We state here an important bound on the dimension of $\mathscr{D}(R)$-modules obtained by Bernstein.

Theorem 4.1.11. [7] *If $X = \operatorname{Spec} R$ is a nonsingular variety and M is a nonzero $\mathscr{D}(R)$-module, then* $\operatorname{GKdim}(M) \geq \dim X$.

This strengthens the observation in Proposition 2.2.4 that $\mathscr{D}(R)$ has no nonzero finite dimensional modules.

4.1.3 Holonomic Modules

When $\operatorname{GKdim}(M) = \dim X$, M is a very special type of $\mathscr{D}(R)$-module - a *holonomic module*. These will turn out to be important for our results in Chapter 5 and they are very interesting in their own right. We will now spend a little time going into some detail about their special properties.

Definition 4.1.12. Let $X = \operatorname{Spec} R$ be a nonsingular variety and let M be a finitely generated $\mathscr{D}(R)$-module. We say M is a *holonomic module* if M is zero or $\operatorname{GKdim}(M) = \dim X$.

Example 4.1.13. Let $A_n = \Bbbk[x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}]$ be the n^{th} Weyl algebra, as in Example 2.1.5. Then $\Bbbk[x_1, \ldots, x_n]$ is a left A_n-module where the module structure is induced by the following actions:

$$x_i * f = x_i f \text{ and } \partial_{x_i} * f = \partial f / \partial x_i.$$

The GK dimension of $\Bbbk[x_1, \ldots, x_n]$ and the dimension of $\mathbb{A}^n$ are both n, and hence this is a holonomic module.

Holonomic modules have particularly nice properties.

Theorem 4.1.14. [14, Chapter 10 Theorem 2.2] *Holonomic modules are artinian.*

Corollary 4.1.15. *Any holonomic module has finite length.*

This corollary in particular shows that holonomic modules obtain all their structure from simple holonomic modules. In general, holonomic modules are quite difficult to study, but some progress has been made in the case of the Weyl algebras. The first result is a straightforward observation since A_1 is a Kdim 1 critical module (a notion we will define later; see Subsection 4.1.6).

Lemma 4.1.16. *The simple A_1-modules are holonomic.*

We also observe the following:

Lemma 4.1.17. [3, Corollary 2.3] *Let M, N be two simple A_1-modules. Then $M \otimes_{\Bbbk} N$ is a holonomic module over $A_1 \otimes_{\Bbbk} A_1 \cong A_2$.*

Initially it was thought that all irreducible modules over A_n were holonomic, although the only evidence for this was a lack of counterexamples! However, in 1985, Stafford proved that the following module was a nonholonomic module.

Example 4.1.18. [42, Theorem 1.1] Let $n > 2$ and let $\lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be algebraically independent over $\mathbb{Q}$. By a long calculation the operator

$$r = \partial_{x_1} + \left(\sum_{i=2}^{n} \lambda_i x_1 x_i \partial_{x_i} + x_i \right) + \sum_{i=2}^{n} (x_i - \partial_{x_i})$$

generates a maximal left ideal of A_n. Then $M := A_n/A_n r$ is irreducible. Since $\mathrm{GKdim}(M) = 2n - 1 > n$, M cannot be holonomic.

Bernstein and Lunts [8] subsequently found a different method to construct irreducible A_n-modules of GK dimension $2n-1$ which was much more geometric. In fact, they showed that "almost every" $P \in A_n$ generates a maximal left ideal such that $\mathrm{GKdim}\, A_n/A_n P = 2n-1$, and hence the simple module $A_n/A_n P$ cannot be holonomic.

In Chapter 5 we will be particularly interested in how holonomic modules behave under localisation. We delay discussion of this until the chapter in question.

4.1.4 Rings of Differential Operators over Curves

We now move to describing the work of Smith and Stafford on rings of differential operators over singular curves. We will go into some detail here since we rely heavily both on their constructions and their results for our own work in Chapter 5.

We begin with the setup and a more general definition of the space of $\Bbbk$-linear differential operators between two R-modules for some commutative ring R.

Definition 4.1.19. Let M and N be two modules over a commutative $\Bbbk$-algebra R. We define a $\Bbbk$-linear differential operator from M to N of order n inductively by $\mathscr{D}^{-1}(M,N) = 0$ and for $n \geq 0$,

$$\mathscr{D}^n(M,N) = \{\theta \in \operatorname{Hom}_{\Bbbk}(M,N) \mid [\theta, a] \in \mathscr{D}^{n-1}(M,N) \text{ for all } a \in R\}.$$

We define the set of $\Bbbk$-linear differential operators to be

$$\mathscr{D}_R(M,N) = \bigcup_{n\geq 0} \mathscr{D}^n(M,N).$$

These spaces behave very well under localisation as the following lemma shows.

Lemma 4.1.20. [39, 1.3(d)] *Let S be a multiplicatively closed subset of R, let M be either a finitely generated R-module or an $R[S^{-1}]$-module, and let N be an R-module. Then*

$$R[S^{-1}] \otimes_R \mathscr{D}_R(M,N) \cong \mathscr{D}_{R[S^{-1}]}(MS^{-1}, NS^{-1})$$

We also have another notion of differential operator:

Definition 4.1.21. Let $A \subseteq B$ be commutative $\Bbbk$-algebras. We write

$$\mathscr{D}(B,A) = \{D \in \mathscr{D}(B) \mid D * f \in A \text{ for all } f \in B\},$$

where $D * f$ denotes the action of the operator.

We also have the following proposition which tells us when these are equivalent, which we will show applies in our context.

Proposition 4.1.22. [39, Lemma 2.7] *Let A and B be domains such that $A \subseteq B \subseteq$ $\operatorname{Fract} A$. Then $\mathscr{D}(B,A) = \mathscr{D}_A(B,A)$. In particular, if $S \subseteq A$ is multiplicatively closed, then $\mathscr{D}(B[S^{-1}], A[S^{-1}]) = A[S^{-1}] \otimes_A \mathscr{D}(B,A)$.*

An important technique which Smith and Stafford use is to compare the singular curve X to its normalisation $\widetilde{X}$. We define this now.

Definition 4.1.23. We define the *normalization* of a singular affine curve X to be a variety $\widetilde{X}$ such that $\mathcal{O}(\widetilde{X})$ is the integral closure of $\mathcal{O}(X)$ in its field of fractions.

Note the normalisation is defined for any affine variety. However, by a result of Serre a variety is normal is and only if it is regular in codimension 1 and satisfies the S_2 condition [21, Theorem II.8.22A], and thus a curve is normal if and only if it is nonsingular. The standard way to desingularise any curve is by normalising it. Also, note that there is a canonical surjective map from $\widetilde{X}$ to X.

Example 4.1.24. Consider the cuspidal cubic curve $X = \mathbb{V}(y^3 - x^2)$. This is a singular curve and hence cannot be normal. We normalise the ring

$$R = \frac{\Bbbk[x, y]}{\langle y^3 - x^2 \rangle}.$$

Consider the element $t = x/y \in F(R)$ and notice that it satisfies the equation $t^2 - y = 0$. Hence $t \in \mathcal{O}(\widetilde{X})$.

Hence we have a variety in 3-space (in coordinates x, y, t) which satisfies the following equations

$$y^3 = x^2, \qquad ty = x, \qquad t^2 = y$$

and we have a surjection from this variety to our original curve by restriction to the xy-plane. We observe

$$\frac{\Bbbk[x, y, t]}{\langle y^3 - x^2, ty - x, t^2 - y \rangle} \cong \Bbbk[t]$$

which is integrally closed. As such the normalisation of the cuspidal cubic is the affine line, $\mathbb{A}^1$.

Now we describe the setup.

Notation 4.1.25. Let X be a variety and let $\widetilde{X}$ be the normalisation of X (with normalisation map $\phi : \widetilde{X} \to X$). Then we note that $\mathcal{O}(X) \subseteq \mathcal{O}(\widetilde{X})$ and $\mathcal{O}(\widetilde{X})$ is the integral closure of $\mathcal{O}(X)$ in its ring of fractions $K = \Bbbk(X)$. Denote by $\mathscr{D}(X)$ the ring of differential operators $\mathscr{D}(\mathcal{O}(X))$. Then by Proposition 4.1.22, $K \otimes_{\mathcal{O}(\widetilde{X})} \mathscr{D}(\widetilde{X}) \cong \mathscr{D}(K)$ and hence we may identify $\mathscr{D}(\widetilde{X})$ with its image in $\mathscr{D}(K)$:

$$\mathscr{D}(\widetilde{X}) = \{\theta \in \mathscr{D}(K) \mid \theta * f \in \mathcal{O}(\widetilde{X}) \text{ for all } f \in \mathcal{O}(\widetilde{X})\}.$$

Hence we can view $\mathscr{D}(X)$ and $\mathscr{D}(\widetilde{X})$ as subalgebras of $\mathscr{D}(K)$. We will use this identification without reference throughout Chapter 5.

In our setup, more may be said about the set $\mathscr{D}(B, A)$. We will see that it plays a key role in linking the ring of differential operators over X with that over its normalisation.

Remark 4.1.26. We see from the definition that $\mathscr{D}(B, A)$ is a right ideal of $\mathscr{D}(B)$. Suppose now that $B = \mathcal{O}(\widetilde{X})$ and $A = \mathcal{O}(X)$ and denote $\mathscr{D}(\widetilde{X}, X) \coloneqq \mathscr{D}(\mathcal{O}(\widetilde{X}), \mathcal{O}(X))$. Observe that if $\theta \in \mathscr{D}(\widetilde{X}, X)$,

$$\mathscr{D}(X)\theta(\mathcal{O}(\widetilde{X})) \subseteq \mathscr{D}(X)(\mathcal{O}(X)) \subseteq \mathcal{O}(X),$$

hence $\mathscr{D}(\widetilde{X}, X)$ is also a left ideal of $\mathscr{D}(X)$. Further, by [39, Theorem 2.5], $\mathscr{D}(\widetilde{X}, X)$ is a progenerator for $\mathscr{D}(\widetilde{X})$.

Definition 4.1.27. Let R, R' be two orders in a division ring Q. Then we say that R and R' are *order equivalent* if there exist units $a, b, a', b' \in Q$ such that $aRb \subseteq R'$ and $a'R'b' \subseteq R$. This is denoted $R \sim R'$.

We now summarise the results of Smith and Stafford which link together $\mathscr{D}(X)$ with $\mathscr{D}(\widetilde{X})$.

Theorem 4.1.28. [39, Theorem 2.5] *Let X be a curve with normalisation $\widetilde{X}$. Denote by P the set of $\mathbb{k}$-linear differential operators from $\widetilde{X}$ to X. Then $T \coloneqq \operatorname{End}_{\mathscr{D}(\widetilde{X})} P$ is Morita equivalent to $\mathscr{D}(\widetilde{X})$. Further, $\mathscr{D}(X)$ and T are order equivalent.*

Corollary 4.1.29. [39] *Let X be a curve. Then $\mathscr{D}(X)$:*

1. *is right and left noetherian;*
2. *is a finitely generated $\mathbb{k}$-algebra;*
3. *has (Gabriel-Rentschler) Krull dimension 1;*
4. *has GK dimension 2.*

If our curve is particularly pleasant, that is to say the singularities are cusps, then we may say more about $\mathscr{D}(X)$.

Theorem 4.1.30. [39, Theorem 3.4] *Let X be a curve with normalisation $\widetilde{X}$ such that the normalisation map $\phi : \widetilde{X} \to X$ is bijective. Then $\mathscr{D}(X) \cong \operatorname{End}_{\mathscr{D}(\widetilde{X})} \mathscr{D}(\widetilde{X}, X)$. Hence $\mathscr{D}(X)$ is Morita equivalent to $\mathscr{D}(\widetilde{X})$.*

Corollary 4.1.31. [39] *Let X be a curve with normalisation $\widetilde{X}$ such that the normalisation map $\phi : \widetilde{X} \to X$ is bijective. The ring of differential operators $\mathscr{D}(X)$ is simple and hereditary.*

Example 4.1.32. Recall the cuspidal cubic $C = \mathbb{V}(y^3 - x^2)$ with normalisation $\mathbb{A}^1$. In this case $\mathscr{D}(\mathbb{A}^1, C)$ is contained in $A_1 = \Bbbk[t, \partial]$ (where $\Bbbk[\mathbb{A}^1] = \Bbbk[t]$ contains $\Bbbk[C] = \Bbbk\langle t^2, t^3\rangle$), and

$$\mathscr{D}(\mathbb{A}^1, C) \cong t^2 A_1 + (t\partial - 1)A_1$$

and this gives a Morita equivalence between $\mathscr{D}(C)$ and A_1. As such, $\mathscr{D}(C)$ is simple, hereditary, and noetherian. The isomorphism

$$\mathscr{D}(C) \cong \operatorname{End}_{A_1}\left(t^2 A_1 + (t\partial - 1)A_1\right)$$

was first observed by Musson [28] by explicit calculation.

4.1.5 Simple A_n-modules

In Chapter 5 we utilise various results about simple A_n-modules, in particular a classification in the case $n = 2$. We shall spend a little time now surveying this.

The task of classifying simple modules, and in particular holonomic modules, over rings of differential operators is highly nontrivial. The first progress was made by Block [11]. Bavula and Van Oystaeyen [3] provided a classification of simple modules over the second Weyl algebra.

View A_2 as the tensor product of two copies of A_1: $A_2 \cong \Bbbk[y, \partial_y] \otimes_{\Bbbk} \Bbbk[x, \partial_x]$. Let Q denote the Weyl skew field (the classical quotient ring of A_1) of the first copy of A_1 and let M be a nonzero simple A_2-module. Consider the localisation of M with respect to the denominator set $C = A_1^* = \Bbbk[y, \partial_y]^*$

$$MC^{-1} = Q \otimes_{A_1} M.$$

Either this module is 0, in which case we shall say M is C-torsion, or this module is nonzero, in which case we say M is C-torsionfree. As M is simple, this covers all possibilities and hence the isoclasses of simple A_2-modules may be split into two subisoclasses: C-torsion and C-torsionfree.

Proposition 4.1.33. [3, Proposition 4.1] *Up to isomorphism, the simple C-torsion*

A_2-modules are precisely those simple A_2-modules which are induced from $A_1 \otimes_{\Bbbk} A_1$. That is to say, if M is a simple C-torsion A_2-module then there exist N_1, N_2 which are simple A_1-modules such that

$$M \cong N_1 \otimes_{\Bbbk} N_2.$$

Bavula and Van Oystaeyen go further to classify all simple A_2-modules which we now describe. We shall see that holonomicity plays an important role in this classification.

As above, let Q denote the Weyl skew field. Then we may view $A_2 = A_1 \otimes_{\Bbbk} A_1$ as a subalgebra of $Q \otimes_{A_1} A_1 := A_1(Q)$ (a Weyl algebra over the Weyl skew field).

Theorem 4.1.34. [3, Proposition 5.1] *Let M be a simple A_2-module. Then $\widetilde{M} := A_1(Q) \otimes_{A_2} M$ is an $A_1(Q)$-module and a left Q-vector space. There are three possibilities for $\dim_Q \widetilde{M}$ and the set of simple A_2-modules is accordingly partitioned:*

(1) $\dim_Q \widetilde{M} = 0$ *if and only if $M = N_1 \otimes_{\Bbbk} N_2$ where N_i are simple A_1-modules;*

(2) $1 \leq \dim_Q \widetilde{M} < \infty$ *if and only if M is holonomic but does not fall into Case* 1*;*

(3) $\dim_Q \widetilde{M} = \infty$ *if and only if M is nonholonomic.*

4.1.6 Noncommutative Krull Dimension

For Chapter 5, we work with another measure of dimension not mentioned preiously which is (noncommutative) Krull dimension. In this section we provide a brief introduction to this. We begin by describing the classical Krull dimension which we seek to generalise. The interested reader is directed to [19, Chapter 15] for a more detailed introduction.

Definition 4.1.35. We say that a chain of prime ideals in a ring R

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

has *length n*. We define the *(classical) Krull dimension of R* to be the supremum of all the lengths of chains of prime ideals in R. This is denoted $\operatorname{Cl.Kdim} R$.

As you can see, this definition relies on prime ideals and, as is normally the case in noncommutative ring theory, this means that it does not work well for most noncommutative rings. Hence an alternative definition was required. In order to define this we must first define, by transfinite induction, classes $\mathcal{K}_\alpha$ of R-modules for all ordinals α.

Definition 4.1.36. Let $\mathcal{K}_{-1} = \{0\}$, that is precisely the zero module. Next, consider an ordinal $\alpha \geq 0$; if $\mathcal{K}_\beta$ has been defined for all ordinals $\beta < \alpha$, let $\mathcal{K}_\alpha$ be the class of those R-modules M such that, for every (countable) descending chain

$$M_0 \geq M_1 \geq M_2 \geq \dots$$

of submodules of M, we have $M_i/M_{i+1} \in \bigcup_{\beta<\alpha} \mathcal{K}_\beta$ for all but finitely many indices i. If an R-module M belongs to some $\mathcal{K}_\alpha$ then we define the least such α to be the *Krull dimension of* M denoted $\operatorname{Kdim} M$. If M does not belong to any such $\mathcal{K}_\alpha$, then we say M *does not have Krull dimension.*

Theorem 4.1.37. [19, Lemma 15.3] *If* M *is a noetherian module, then* M *has Krull dimension.*

We have claimed that this is a generalisation of classical Krull dimension, hence we provide some results to justify this to the reader.

Proposition 4.1.38. [19, Lemma 15.13] *Let* R *be a commutative noetherian ring. Then* $\operatorname{Kdim}(R) = \operatorname{Cl.Kdim}(R)$.

Example 4.1.39. Consider first the polynomial ring in n variables, which has classical Krull dimension n, and hence $\operatorname{Kdim}(\Bbbk[x_1, \dots, x_n]) = n$. If A_n is the n^{th} Weyl algebra over a field $\Bbbk$ of characteristic 0, then $\operatorname{Kdim}(A_n) = n$.

Definition 4.1.40. Let M be a module over a ring R. We say that M is α*-critical* for some ordinal α if $\operatorname{Kdim}(M) = \alpha$ and $\operatorname{Kdim}(M/N) < \alpha$ for all nonzero submodules N of M.

Example 4.1.41. We see that the simple modules are precisely the 0-critical modules. Also, the 1-critical modules are those where the module itself is not Artinian, but any proper factor is. For example, $\mathbb{Z}$ viewed as a module over itself is 1-critical.

4.1.7 Idealizers at Smooth Curves

We now turn to the work of McCaffrey which we generalise in Chapter 5. In particular, McCaffrey works with idealizers obtained from smooth curves and proves that they are right noetherian. The aim of Chapter 5 is to generalise this to curves which are singular, but in a relatively well-behaved way.

Theorem 4.1.42. [25] *Let B be a regular $\mathbb{k}$-affine domain and Q a prime ideal in B such that B/Q is regular. Then $\mathbb{I}_{\mathscr{D}(B)}(Q\mathscr{D}(B))$ is right noetherian.*

Since McCaffrey works with irreducible curves which are nonsingular, that is to say prime ideals $Q \lhd B$ such that both B and B/Q are regular, it allows him to localise his ring B at any maximal ideal $\mathfrak{m}$ and obtain a regular local ring with the following property. Suppose B has dimension n, Q prime ideal, and $\mathfrak{m}$ is any maximal ideal containing Q. Then $QB_{\mathfrak{m}}$ has a maximal regular sequence $\{x_1, \ldots, x_i\}$ of length $i = \text{height}(Q)$ which may be extended to a generating set $\{x_1 \ldots, x_i, x_{i+1}, \ldots, x_n\}$ for $\mathfrak{m}B_{\mathfrak{m}}$.

In essence, this construction allows McCaffrey to work in new coordinates in a localised world and replicate the proof that $\mathbb{I}_{A_n}\left(\left(\sum_{j=1}^{i} x_i\right) A_n\right)$ is right and left noetherian (see Theorem 5.1.2). However, we must develop different tools to deal with our rings since the maximal ideals which correspond to the singular points on our curves do not allow the generating set to be extended (since the tangent space is not well-defined). In Chapter 5 now, we describe our new methods to bypass this problem.

Chapter 5

Idealizers in the Second Weyl Algebra

5.1 Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic 0. Throughout this chapter, by 'a variety' we mean 'an irreducible affine algebraic variety over $\Bbbk$' and by 'a curve' we mean an 'irreducible affine algebraic curve over $\Bbbk$'. We denote the ring of regular functions on a variety X by $\mathcal{O}(X)$. In this chapter we are interested in idealizers in rings of differential operators. Recall the second Weyl algebra A_2 defined in Definition 2.1.5. In this chapter we study $\mathbb{I}_{A_2}(fA_2) \subseteq A_2$ where $f \in \Bbbk[x, y]$ defines a curve X such that the normalisation map $\phi : \widetilde{X} \to X$ is injective.

The noetherianity of idealizer subrings in Weyl algebras and more general rings of differential operators has been studied before. Robson was the first to study idealizer subrings in A_1, the first Weyl algebra. Theorem 5.1.1 is a special case of a general result about idealizers in HNP rings, of which A_1 is an example.

Theorem 5.1.1. [33, Theorem 7.4] *Let $I \leq_r A_1$ be nonzero. Then $\mathbb{I}_{A_1}(I)$ is a left and right noetherian ring.*

However, the higher Weyl algebras are no longer hereditary (in fact gldim $A_n = n$) so alternative techniques must be used to study idealizers in these rings. Despite this, it turns out that idealizers at certain types of right ideal in higher Weyl algebras behave very well, as we see in the following theorem.

Theorem 5.1.2. [30, Proposition 2.3] *View $A_1 \subseteq A_n$, the n^{th} Weyl algebra. Let*

$I \leq_r A_1$ be a right ideal in the first Weyl algebra. Then $\mathbb{I}_{A_n}(IA_n)$ is right and left noetherian.

Proof. We simply observe that

$$\begin{aligned}\mathbb{I}_{A_n}(IA_n) &\cong \mathbb{I}_{A_1}(I) \otimes_{\Bbbk} A_{n-1} \\ &\cong \mathbb{I}_{A_1}(I)[x_2, \ldots, x_n][\partial_2; \partial/\partial x_2] \ldots [\partial_n; \partial/\partial x_n]\end{aligned}$$

and hence, by Theorem 5.1.1, the noetherianity of $\mathbb{I}_{A_n}(IA_n)$ is induced from that of $\mathbb{I}_{A_1}(I)$ by viewing it as an Ore extension of $\mathbb{I}_{A_1}(I)$. □

The situation is not so nice for more complicated ideals; for instance we cannot use the same trick on a right ideal of the form $xA_2 + yA_2$ where we view A_2 as $\Bbbk[x, y][\partial_x, \partial_y]$. Indeed, Resco proved that the conclusion of the previous theorem does not hold for the idealizer associated to this ideal .

Theorem 5.1.3. [31, Theorem 2] *The idealizer $\mathbb{I}_{A_2}(xA_2 + yA_2)$ is right but not left noetherian.*

On the other hand, Theorem 5.1.2 shows that $\mathbb{I}_{A_2}(fA_2)$ is right and left noetherian when f is of the form $ax + by + c$ for some $a, b, c \in \Bbbk$. Hence it is natural to ask the following question.

Question 5.1.4. *For which $f \in \Bbbk[x, y]$ is $\mathbb{I}_{A_2}(fA_2)$ right and left noetherian?*

McCaffrey [25] studied these idealizers in the case where f defines a nonsingular curve and obtained the following theorem which we paraphrase.

Theorem 5.1.5. [25] *Let $f \in \Bbbk[x, y]$ define a nonsingular curve. Then the idealizer ring $\mathbb{I}_{A_2}(fA_2)$ is right noetherian.*

In this chapter we strengthen and generalise this result to a class of singular curves.

Theorem 5.1.6. *Let $X = \mathbb{V}(f)$ be a plane curve such that the normalisation map $\phi : \widetilde{X} \to X$ is injective. Then $\mathbb{I}_{A_2}(fA_2)$ is right and left noetherian.*

We reduce this problem to considering the noetherianity of $\mathrm{Hom}_{A_2}(A_2/fA_2, S)$ as a right $\mathbb{I}_{A_2}(fA_2)/fA_2$-module, where S is a simple right A_2-module. We then split this into two cases: when $S \cong A_2/mA_2$ where $m \lhd \Bbbk[x, y]$ is maximal, or when S is not of this form. For the first case, we obtain that the module is noetherian by a careful combinatorial argument. For the second case, we use a localisation argument and the work of [39] to derive the result as a consequence of Bernstein's preservation of holonomicity.

A straightforward application of Theorems 5.1.1 and 5.1.2 shows that $\mathbb{I}_{A_2}(xA_2)$ is right noetherian. The techniques McCaffrey develops for his result are underpinned by the fact that, locally, nonsingular curves look like the affine line. Unfortunately, this is not the case with singular curves and so we must use different techniques.

The reason that we require the normalisation map $\phi : \widetilde{X} \to X$ to be injective is to do with the structure of the ring of differential operators on cuspidal curves in comparison to more complicated curves. The ring of global differential operators over a variety X, $\mathscr{D}(X)$, in the sense of Grothendieck [20, 16.8.1], has many nice properties when X is nonsingular. In particular, $\mathscr{D}(X)$ is a finitely generated, noetherian $\Bbbk$-algebra and, when X is a curve, $\mathscr{D}(X)$ is a simple, hereditary, noetherian, prime (HNP) ring [20, 16.11.2], [9, Chapter 3 Theorem 2.5]. Smith and Stafford further showed that if X is a singular curve then $\mathscr{D}(X)$ is still a finitely generated, noetherian $\Bbbk$-algebra and, if the normalisation map $\phi : \widetilde{X} \to X$ is injective, $\mathscr{D}(X)$ remains a simple hereditary ring, [39].

We also have the following link between rings of differential operators and idealizers.

Proposition 5.1.7. [39, Proposition 1.6] *Let Y be a nonsingular variety and X a subvariety defined by an ideal I of $\mathcal{O}(Y)$. Then*

$$\mathscr{D}(X) \cong \frac{\mathbb{I}_{\mathscr{D}(Y)}(I\mathscr{D}(Y))}{I\mathscr{D}(Y)}.$$

It is natural to ask, given the strong relation between idealizers and rings of differential operators, whether idealizers at singular curves are also left and right noetherian. In this chapter we answer that question in the affirmative, at least when the curve is suitably well-behaved.

5.2 Preliminaries

We shall start with a summary of the definitions and results from the literature concerning idealizers at smooth curves and rings of differential operators of possibly singular curves.

We begin by stating important results from the literature which will be used in the main part of the chapter.

5.2.1 Rings of Differential Operators of Curves

We begin with the case when X is a nonsingular variety. Recall that we write $\mathscr{D}(X)$ for $\mathscr{D}(\mathcal{O}(X))$. Then $\mathscr{D}(X)$ has the following properties, [20, Section 16]:

(a) $\mathscr{D}(X)$ is a finitely generated, simple, noetherian domain.

(b) The global dimension of $\mathscr{D}(X)$ is finite; more precisely $\operatorname{gldim}(\mathscr{D}(X)) = \dim X$. Further, $\operatorname{GKdim}(\mathscr{D}(X)) = 2\dim X$. We see that in the case that X is a curve, $\mathscr{D}(X)$ is an HNP ring with GK dimension 2.

Lemma 5.2.1. [27, Proposition 1.9] *Let S be a multiplicatively closed set in a finitely generated commutative $\Bbbk$-algebra R. Then*

$$\mathscr{D}(RS^{-1}) = \mathscr{D}(R) \otimes_R RS^{-1} = \mathscr{D}(R)S^{-1},$$

that is to say, localising commutes with taking rings of differential operators.

We now move on to the case where X is a curve with singular points. An important construction we require is that of the normalisation of X, which we denote $\widetilde{X}$; that is, $\mathcal{O}(\widetilde{X})$ is the integral closure of $\mathcal{O}(X)$ inside the function field $\Bbbk(X)$. When X is a curve, the normalisation is also a curve. We suppose further that the normalisation map $\phi : \widetilde{X} \to X$ is injective; that is, the singularities of X are all cusps.

Recall from Chapter 4 two sets of differential operators between different modules or $\Bbbk$-algebras.

Definition 5.2.2. Let A be a commutative $\Bbbk$-algebra and let M and N be A-modules. Then we define the space of $\Bbbk$-*linear differential operators from M to N of order at most n* inductively by $\mathscr{D}^{-1}(M,N) = 0$ and for $n \geq 0$:

$$\mathscr{D}^n(M,N) = \{\theta \in \operatorname{Hom}_{\Bbbk}(M,N) \mid [\theta, a] \in \mathscr{D}^{n-1}(M,N) \text{ for all } a \in A\}.$$

We denote the space of all $\Bbbk$-linear differential operators from M to N by $\mathscr{D}_A(M,N)$.

We also have the following definition for differential operators between $\Bbbk$-algebras.

Definition 5.2.3. If $A \subseteq B$ are commutative k-algebras then we write

$$\mathscr{D}(B, A) = \{D \in \mathscr{D}(B) \mid D * f \in A \text{ for all } f \in B\},$$

where $D * f$ denotes the action of the differential operator D on f; that is, the (left) $\mathscr{D}(B)$-module structure on B.

We note that $\mathscr{D}_A(B, A)$ and $\mathscr{D}(B, A)$ are not necessarily equal. However in the case that A and B are both domains such that $A \subseteq B \subseteq$ Fract (A) (the situation in which we are interested), then these two sets are equal [39, Lemma 2.7]. When $B = \mathcal{O}(\widetilde{X})$ and $A = \mathcal{O}(X)$, we shall write $\mathscr{D}(\widetilde{X}, X) := \mathscr{D}(\mathcal{O}(\widetilde{X}), \mathcal{O}(X))$. We note that this is both a right ideal of $\mathscr{D}(\widetilde{X})$ and also a left ideal of $\mathscr{D}(X)$. If $\mathscr{D}(X)$ and $\mathscr{D}(\widetilde{X})$ are Morita equivalent we may use Definition 5.2.2 to define $\mathscr{D}(X, \widetilde{X}) \cong \mathscr{D}_{\mathcal{O}(X))}(\mathcal{O}(X), \mathcal{O}(\widetilde{X}))$, which is $\mathrm{Hom}_{\mathscr{D}(\widetilde{X})}(\mathscr{D}(\widetilde{X}, X), \mathscr{D}(\widetilde{X}))$, the dual of $\mathscr{D}(\widetilde{X}, X)$ [39, Proposition 3.14].

An important technique in this chapter will be to identify $\mathscr{D}(X)$ and $\mathscr{D}(\widetilde{X})$ with subalgebras of $\mathscr{D}(K)$ where K is the fraction field of $\mathcal{O}(X)$. If $K = \Bbbk(X)$ is the field of fractions associated to $\mathcal{O}(X)$, then we may identify $\mathscr{D}(\widetilde{X})$ with its image in $\mathscr{D}(K)$ as follows:

$$\mathscr{D}(\widetilde{X}) = \{D \in \mathscr{D}(K) \mid D * (\mathcal{O}(\widetilde{X})) \subseteq \mathcal{O}(\widetilde{X})\}.$$

We may also identify $\mathscr{D}(\widetilde{X}, X)$ as:

$$\mathscr{D}(\widetilde{X}, X) = \{D \in \mathscr{D}(K) \mid D * (\mathcal{O}(\widetilde{X})) \subseteq \mathcal{O}(X)\}.$$

Then we have the following result due to Smith and Stafford.

Theorem 5.2.4. [39, Theorem 3.4] *Let X be a curve. Then the following are equivalent:*

(1) *The normalisation map $\phi : \widetilde{X} \to X$ is injective;*

(2) *$\mathscr{D}(\widetilde{X})$ is Morita equivalent to $\mathscr{D}(X)$ via the progenerator $\mathscr{D}(\widetilde{X}, X)$.*

From this result we have the following properties of $\mathscr{D}(X)$.

Corollary 5.2.5. [39, Theorems A and B] *Suppose $\phi : \widetilde{X} \to X$ is injective. Then $\mathscr{D}(X)$ is*

(a) right and left noetherian;

(b) a finitely generated $\Bbbk$-algebra;

(c) a hereditary ring with (Gabriel-Rentschler) Krull dimension 1*, and GK dimension* 2*;*

(d) a simple ring.

5.2.2 Holonomic Modules

We are particularly interested in a certain type of module over a ring of differential operators - a holonomic module. Whilst these modules have a simple definition, their properties are surprisingly good. We restate Theorem 4.1.11 here for the reader's convenience.

Theorem 5.2.6. [7] *If $X = \operatorname{Spec} R$ is a nonsingular variety and M is a nonzero $\mathscr{D}(R)$-module, then* $\operatorname{GKdim}(M) \geq \dim X$*.*

Definition 5.2.7. We define a *holonomic* D-module to be a finitely generated module such that $\operatorname{GKdim}(M) = \dim X$.

The following theorem shows that, in the setting in which we are working, holomonic modules behave particularly well.

Theorem 5.2.8 (Bernstein's preservation of holonomicity)**.** [6, Theorem A] *Let $D = \mathscr{D}(X)$ be a ring of differential operators over a smooth algebraic variety X. Let $S \subseteq \mathcal{O}(X)$ be a multiplicatively closed subset and let M be a holonomic right DS^{-1}-module. Then M is holonomic as a right D-module.*

Although Bernstein gives this result as a statement about derived categories of holonomic D-modules, it is well known that open immersions send holonomic D-modules to holonomic D-modules; for a proof, see [17, Theorem 3.23].

5.2.3 Idealizers at curves

In this subsection we summarise the results from the literature which we will use in the main part of the chapter. We will focus on the idealizer at a curve defined by a polynomial $f \in \Bbbk[x, y]$, more precisely, we mean the ring:

$$\mathbb{I}_{A_2}(fA_2) = \{P \in A_2 \mid PfA_2 \subseteq fA_2\}.$$

We recall the notation in Definition 3.2.4 of the colon ideal for two right ideals I, J in a ring R:

$$(J : I) = \{r \in R \mid rI \subseteq J\}.$$

We also recall the following isomorphism.

Proposition 5.2.9. [33, Proposition 1.1] *Let I be a right ideal in a ring R and let J be a right ideal of R which contains I. Then*

$$\mathrm{Hom}_R(R/I, R/J) \cong \frac{(J : I)}{J},$$

considered as right modules over $\mathbb{I}_R(I)$.

Proposition 5.2.10. [26, cf. Prop. 15.5.9] *Let $I, J \lhd R = \Bbbk[x, y]$ and let $\beta \in A_2$. Then*

(1) $\beta * R \subseteq I$ if and only if $\beta \in IA_2$;

(2) $\beta * I \subseteq J$ if and only if $\beta \in (JA_2 : IA_2)$,

where $\beta * R$ denotes the action of A_2 on R as described in Example 4.1.13.

Proof. For (1), this is precisely [26, Proposition 15.5.9 (i)].
For (2) we start with the converse. Let $a \in I$, then $a \in IA_2$. Hence

$$\beta * a = (\beta a) * 1_R \in (JA_2) * 1_R \subseteq JR = J.$$

Now for the forward direction, let $a \in I$ and $r \in R$. Then

$$(\beta a) * r = \beta * ar \in J$$

and hence $(\beta a) * R \subseteq J$. Thus, by (1), $\beta a \in JA_2$, which implies $\beta \in (JA_2 : IA_2)$. □

Corollary 5.2.11. *Let $f, g \in \Bbbk[x, y]$. Then*

$$\mathscr{D}((f), (g)) = \{\theta \in A_2 \mid \theta * (f) \subseteq (g)\} = (gA_2 : fA_2)$$

and setting $f = g$ we obtain

$$\mathbb{I}_{A_2}(fA_2) = \{\theta \in A_2 \mid \theta * p \in (f) \text{ for all } p \in (f)\} = \mathscr{D}((f), (f)).$$

□

We recall the following result about the noetherianity of idealizers from Proposition 2.2.16:

Proposition 5.2.12. [35, Proposition 2.1] *Let I be a right ideal in a noetherian domain B. Let $R = \mathbb{I}_B(I)$. Then the following are equivalent:*

(1) R is right noetherian.

(2) For every right ideal $J \supseteq I$ of B, $\mathrm{Hom}_B(B/I, B/J)$ is a right noetherian R-module (or R/I-module).

We next summarise the work of McCaffrey on idealizers at nonsingular curves [25].

Theorem 5.2.13. [McCaffrey] *Let B be a regular $\Bbbk$-affine domain and Q a prime ideal in B such that B/Q is regular. Then $\mathbb{I}_{\mathscr{D}(B)}(Q\mathscr{D}(B))$ is right noetherian.*

Further:

Proposition 5.2.14. [McCaffrey] *Let B and Q satisfy the assumptions of Theorem 5.2.13. For J any right ideal of $\mathscr{D}(B)$ strictly containing Q, we have*

$$(J : Q\mathscr{D}(B)) = \mathbb{I}_{\mathscr{D}(B)}(Q\mathscr{D}(B)) + J$$

and further, using the identification in Proposition 5.2.9,

$$\frac{(J : Q\mathscr{D}(B))}{J} = \mathrm{Hom}_{\mathscr{D}(B)}\left(\frac{\mathscr{D}(B)}{Q\mathscr{D}(B)}, \frac{\mathscr{D}(B)}{J}\right) \cong \frac{\mathbb{I}_{\mathscr{D}(B)}(Q\mathscr{D}(B)) + J}{J}.$$

Proposition 5.2.15. [McCaffrey] *Let B and Q satisfy the assumptions of Theorem 5.2.13. If J is any right ideal of $\mathscr{D}(B)$ strictly containing Q, then*

$$Q \subsetneqq J \cap \mathbb{I}_{\mathscr{D}(B)}(Q\mathscr{D}(B)).$$

5.2.4 Hereditary Rings

As observed earlier, hereditary rings have particularly nice properties, two of which we detail below.

Lemma 5.2.16. *Let S be a localisation of a hereditary ring. If S is not semisimple, then it is also hereditary.*

Proof. We simply note that the global dimension of the localisation is bounded above by that of the original ring [26, 7.4.3], namely 1. □

We have the following result which shows that simple modules over localisations of HNP rings are very well behaved.

Proposition 5.2.17. *Let R be an HNP ring and let S a right denominator set such that RS^{-1} is not the full Goldie quotient ring of R. Let M be a simple right RS^{-1}-module. Then there exists a simple right R-module, N, such that*

$$M \cong N \otimes_R RS^{-1}.$$

Proof. Let $T := RS^{-1}$. Note that M_T is either Goldie torsion or Goldie torsionfree. If M were Goldie torsionfree then, by [19, Lemma 7.17], T would have nonzero socle and hence, by [19, Theorem 7.15], T is semisimple; this is impossible. Therefore M_T is (Goldie) torsion and thus so is M_R. Consider $mR \leq M$ where $m \in M$ is nonzero. Since m is torsion, $\mathrm{r.ann}_R(mR) \neq 0$, and so $mR \cong R/\mathrm{r.ann}_R(mR)$ has finite length as R is HNP [26, Lemma 6.2.8]. Hence $mR \subseteq M$ contains a simple right R-module N. Then note that

$$N \otimes_R T \cong NT \subseteq M$$

as T is flat over R. By the simplicity of M, $NT = M$. □

5.3 $\mathbb{I}_{A_2}(fA_2)$ is Noetherian

Let $f \in \Bbbk[x, y]$ satisfy the hypotheses of Theorem 5.1.6. In this section, we prove that $\mathbb{I}_{A_2}(fA_2)$ is noetherian on both sides. We start by showing that it is enough to prove that $\mathbb{I}_{A_2}(fA_2)$ is right noetherian, and then we show that we may check two separate cases determined by certain torsion properties.

We begin by setting up notation.

Notation 5.3.1. Let $X = \mathbb{V}(f)$ for $f \in \Bbbk[x, y]$ be a curve such that the normalisation map $\phi : \widetilde{X} \to X$ is injective . Also, let $D := \mathscr{D}(X)$ denote the ring of differential operators on X. Then, by [39, Theorem 3.4] D is Morita equivalent to $A := \mathscr{D}(\widetilde{X})$, that is to say

$$D \cong \mathrm{End}_A(P)$$

where $P = \mathscr{D}(\widetilde{X}, X)$ is a (D, A)-bimodule which is finitely generated and projective on both sides.

We may form the right ideal $fA_2 \leq_r A_2$ and we denote *the idealizer defined by* f *in* A_2 by $\mathbb{I}_{A_2}(fA_2)$.

We begin this study on the noetherianity of $\mathbb{I}_{A_2}(fA_2)$ by proving a short result which allows us to only consider right noetherianity.

Proposition 5.3.2. *To prove Theorem 5.1.6 it suffices to prove the right noetherian statement.*

Proof. Suppose that we have proved that $\mathbb{I}_{A_2}(fA_2)$ is right noetherian for all $f \in \Bbbk[x, y]$ such that $X = \mathbb{V}(f)$ satisfies the hypothesis of Theorem 5.1.6. We show $fA_2f^{-1} \cap A_2 = \mathbb{I}_{A_2}(fA_2)$. Indeed,

$$\begin{aligned}\mathbb{I}_{A_2}(fA_2) &= \{\theta \in A_2 \mid \theta f A_2 \subseteq fA_2\} \\ &= \{\theta \in A_2 \mid f^{-1}\theta f A_2 \subseteq A_2\} \\ &= \{\theta \in A_2 \mid \theta \subseteq fA_2f^{-1}\} \\ &= A_2 \cap fA_2f^{-1}.\end{aligned}$$

Then,

$$\mathbb{I}_{A_2}(fA_2) \cong f^{-1}\mathbb{I}_{A_2}(fA_2)f = A_2 \cap f^{-1}A_2f = \mathbb{I}_{A_2}(A_2f),$$

by symmetry. Now note that the hypotheses on the singularities of X is left-right symmetric. Hence, by using a left-handed version of the proof of right noetherianity, $\mathbb{I}_{A_2}(fA_2)$ is left noetherian as required. □

By Propositon 5.1.7, we have the following identification:

$$D = \mathscr{D}(X) \cong \mathbb{I}_{A_2}(fA_2)/fA_2,$$

and we use this without reference throughout. Proposition 5.2.12 states that $\mathbb{I}_{A_2}(fA_2)$ is right noetherian if and only if $\mathrm{Hom}_{A_2}(A_2/fA_2, A_2/J)$ is a finitely generated right $\mathbb{I}_{A_2}(fA_2)/fA_2 \cong D$-module for all right ideals $J \leq A_2$ which contain fA_2, we will also use this without reference throughout.

We start by showing that A_2/IA_2 is 1-critical (in terms of (Gabriel-Rentschler) Krull dimension), where I denotes any height 1 prime ideal in $\Bbbk[x, y]$.

5.3.1 A_2/IA_2 is 1-critical for all Height 1 Prime Ideals of $\Bbbk[x,y]$

In this section we show that A_2/IA_2 is Kdim 1-critical for all height 1 primes $I \lhd \Bbbk[x,y]$, which we note are principal as $\Bbbk[x,y]$ is a UFD. We do this in order to reduce to considering maximal right ideals $J \leq_r A_2$ which strictly contain fA_2.

Theorem 5.3.3. *Let I be a height 1 prime ideal in $\Bbbk[x,y]$. Then A_2/IA_2 is 1-critical.*

Before we prove this, we provide some auxiliary definitions and results.

Definition 5.3.4. A nonzero module M_R over a ring R is *compressible* if for any submodule $N \leq M$ there exists a monomorphism $M \hookrightarrow N$.

Example 5.3.5. We observe that $\Bbbk[x,y]/I$ is compressible for all prime ideals I since any pair of ideals are subisomorphic.

Lemma 5.3.6. [26, Lemma 6.9.4] *A compressible module with Krull dimension is critical.*

In light of this result, we show that A_2/IA_2 is compressible for any prime ideal $I \lhd \Bbbk[x,y]$. We have the following result which shows how compressible modules behave over Ore extensions.

Proposition 5.3.7. [26, Proposition 6.9.6] *Let R be a ring, δ a derivation of R, and $S = R[x;\delta]$. If M_R is compressible then $(M \otimes_R S)_S$ is compressible.*

Proof of Theorem 5.3.3. We begin by showing $\mathrm{Kdim}(A_2/IA_2) = 1$. For this, we simply observe that A_2 is 2-critical, thus $\mathrm{Kdim}(A_2/IA_2) \leq 1$. Since A_2/IA_2 is not artinian (as an example consider an element $c \in \Bbbk[x,y] \setminus I$ such that $c \notin \Bbbk + I$ and c is not a unit, then the descending chain

$$A_2/IA_2 \supsetneq (cA_2 + IA_2)/IA_2 \supsetneq (c^2A_2 + IA_2)/IA_2 \supsetneq \dots$$

is infinite), we have $\mathrm{Kdim}(A_2/IA_2) = 1$.

We must now show A_2/IA_2 is 1-critical. Recall from Example 5.3.5 that $\Bbbk[x,y]/I$ is compressible for all prime ideals I. Hence, viewing A_2 as an iterated Ore extension, applying Proposition 5.3.7 twice and Lemma 5.3.6 we obtain the result. $\square$

Corollary 5.3.8. *If J is a right ideal of A_2 which strictly contains IA_2, then A_2/J has finite length.* $\square$

5.3.2 Reduction to Simple Modules with a specific Torsion Property

From here on, let $f \in \Bbbk[x, y]$ be such that $X := \mathbb{V}(f)$ satisfies the hypotheses of Notation 5.3.1. In this subsection we show that in our situation, in order to apply Proposition 5.2.12 it is not necessary to consider all right ideals J which contain fA_2, rather just those J which are maximal. We shall then see that considering a certain torsion property splits simple right A_2-modules into two types.

Proposition 5.3.9. *Assume Notation 5.3.1. Then the following are equivalent:*

(1) $\mathrm{Hom}_{A_2}(A_2/fA_2, A_2/J)$ *is a finitely generated right* D*-module for all right ideals* $J \leq_r A_2$ *such that* $fA_2 \subsetneq J$*;*

(2) $\mathrm{Hom}_{A_2}(A_2/fA_2, A_2/J)$ *is a finitely generated right* D*-module for all maximal right ideals* $J \leq_r A_2$ *which contain* fA_2*;*

(3) $\mathrm{Hom}_{A_2}(A_2/fA_2, N)$ *is a finitely generated right* D*-module for all simple right* A_2*-modules* N*.*

(4) $\mathbb{I}_{A_2}(fA_2)$ *is right noetherian.*

Proof. We observe the implications (1) implies (2) and (3) implies (2) are straightforward. We show (2) implies (1) and (3). Assume (2) holds and let N be a simple right A_2-module. Without loss of generality, suppose $\varphi \in \mathrm{Hom}_{A_2}(A_2/fA_2, N)$ is a nonzero homomorphism. Consider $0 \neq m = \varphi(1 + fA_2)$. As N is simple $mA_2 = N$. Hence we have a surjective homomorphism $\psi : A_2 \twoheadrightarrow N$ by $a \mapsto m.a$, and $N \cong A_2/\mathrm{r.ann}_{A_2}(m)$. Clearly $fA_2 \subseteq \mathrm{r.ann}_{A_2}(m) \leq_r A_2$. Now

$$\mathrm{Hom}_{A_2}(A_2/fA_2, N) \cong \mathrm{Hom}_{A_2}(A_2/fA_2, A_2/\mathrm{r.ann}_{A_2}(m))$$

where the right-hand side is a finitely generated right D-module by assumption. Hence (3) holds.

Now we show that (2) implies (1). Let J be a right ideal of A_2 which strictly contains fA_2. By Corollary 5.3.8, A_2/J has finite length, say n, and hence it has a composition series of the form

$$0 \subseteq M_0/J \subseteq M_2/J \subseteq \cdots \subseteq M_n/J = A_2/J$$

where $M_i \leq_r A_2$ are right ideals which contain J and each M_i/M_{i-1} is a simple right A_2-module. We show that $\mathrm{Hom}_{A_2}(A_2/IA_2, M_j/J)$ is a finitely generated right $\mathscr{D}(X)$-module for all $j \geq 0$ by induction on j. If $j = 0$ then M_0/J is simple and the result follows since (3) holds.

Now assume that $j > 0$. Consider the short exact sequence of A_2-modules

$$0 \to M_{j-1}/J \to M_j/J \to M_j/M_{j-1} \to 0.$$

Applying $\mathrm{Hom}_{A_2}(A_2/fA_2, -)$ gives rise to an exact sequence of D-modules

$$0 \to \mathrm{Hom}_{A_2}(A_2/fA_2, M_{j-1}/J) \xrightarrow{\alpha} \mathrm{Hom}_{A_2}(A_2/fA_2, M_j/J) \xrightarrow{\beta} \mathrm{Hom}_{A_2}(A_2/fA_2, M_j/M_{j-1})$$

from which we extract the short exact sequence of D-modules

$$0 \to \mathrm{Hom}_{A_2}(A_2/fA_2, M_{j-1}/J) \to \mathrm{Hom}_{A_2}(A_2/fA_2, M_j/J) \to \operatorname{im}\beta \to 0.$$

As $\mathrm{Hom}_{A_2}(A_2/fA_2, M_{j-1}/J)$ is a finitely generated right D-module by induction, and $\operatorname{im}\beta$ is a submodule of $\mathrm{Hom}_{A_2}(A_2/fA_2, M_j/M_{j-1})$, which is finitely generated since (3) holds and D is right noetherian, $\mathrm{Hom}_{A_2}(A_2/fA_2, A_2/J)$ is finitely generated as required.

Finally we must show the equivalence of (1) and (4). By Proposition 5.2.12, (4) implies (1). For the converse, Proposition 5.2.12 and the fact that D is right noetherian (Corollary 5.2.5) show that we must check that $\mathrm{Hom}_{A_2}(A_2/fA_2, A_2/fA_2)$ is a finitely generated right D-module. But this is clear since

$$\mathrm{Hom}_{A_2}(A_2/fA_2, A_2/fA_2) \cong \frac{(fA_2 : fA_2)}{fA_2} = \frac{\mathbb{I}_{A_2}(fA_2)}{fA_2} \cong D.$$

□

We now split simple A_2-modules into two cases.

Definition 5.3.10. Let M be a right A_2-module and let $\mathcal{C}$ be a right Ore set in A_2. Define

$$t_{\mathcal{C}}(M) := \{m \in M \mid m \cdot a = 0 \text{ for some } a \in \mathcal{C}\}$$

to be the set of $\mathcal{C}$-torsion elements of M. Observe that this is a submodule of M. If $t_{\mathcal{C}}(M) = 0$, we say M is *$\mathcal{C}$-torsionfree*, and if $t_{\mathcal{C}}(M) = M$, M is *$\mathcal{C}$-torsion*.

The following lemma splits simple A_2-modules into two natural cases. When $\mathcal{C} = \Bbbk[y]^*$ we will abuse notation and say M is $\Bbbk[y]$-torsionfree or $\Bbbk[y]$-torsion respectively; similarly if $\mathcal{C} = \Bbbk[x]^*$.

Lemma 5.3.11. *Let M be a simple right A_2-module and consider the right Ore set $\mathcal{C} = \Bbbk[y]^*$ or $\mathcal{C} = \Bbbk[x]^*$. Then either M is $\mathcal{C}$-torsion or $\mathcal{C}$-torsionfree.*

Proof. By [19, Lemma 4.21] $t_{\mathcal{C}}(M)$ is an A_2-submodule of M. Hence, by simplicity of M, either $t_{\mathcal{C}}(M) = 0$ or $t_{\mathcal{C}}(M) = M$. □

We summarise our progress so far. In order to prove that $\mathbb{I}_{A_2}(fA_2)$ is right noetherian, we must show that $\operatorname{Hom}_{A_2}(A_2/fA_2, A_2/J)$ is a finitely generated right D-module for all maximal right ideals J which strictly contain fA_2; these fall into two cases:

(1) A_2/J is $\Bbbk[y]$-torsion and $\Bbbk[x]$-torsion;

(2) A_2/J $\Bbbk[y]$-torsionfree or $\Bbbk[x]$-torsionfree.

We begin with case (2).

5.3.3 The Torsionfree Case

In this section we show that if A_2/J is $\Bbbk[y]$- or $\Bbbk[x]$-torsionfree, then the right D-module $\operatorname{Hom}_{A_2}(A_2/fA_2, A_2/J)$ is finitely generated for all maximal right ideals J which strictly contain fA_2. We will see it is enough to prove this for the $\Bbbk[y]$-torsionfree case.

Lemma 5.3.12. *Let J be a maximal right ideal of A_2 which strictly contains fA_2 and such that A_2/J is $\Bbbk[y]$-torsionfree. Let $g \in \Bbbk[y]$ be the polynomial which defines the y-coordinates of the singular points of X. Note that $S = \{g^n\}_{n \geq 0}$ forms a right Ore set in A_2 (Lemma 5.2.1) and so we may construct the localisation of A_2 by S:*

$$A_2[g^{-1}] = A_2 S^{-1} := \{ab^{-1} \mid a \in A_2,\ b \in S\}.$$

Then the natural map of the following right D-modules

$$\operatorname{Hom}_{A_2}(A_2/fA_2, A_2/J) \to \operatorname{Hom}_{A_2[g^{-1}]}(A_2[g^{-1}]/fA_2[g^{-1}], A_2[g^{-1}]/J[g^{-1}])$$

is injective.

Proof. Recall the definition of the ideal quotient of two right ideals I, J in a ring R

$$(J : I) = \{r \in R \mid rI \subseteq J\}.$$

We have the following natural map of right $\mathbb{I}_{A_2}(fA_2)$-modules:

$$(J : fA_2) \overset{\iota}{\hookrightarrow} (J[g^{-1}] : fA_2[g^{-1}]) \overset{h}{\twoheadrightarrow} \frac{(J[g^{-1}] : fA_2[g^{-1}])}{J[g^{-1}]}$$

and let us consider the map $\pi = h \circ \iota$, so $\pi(a) = a + J[g^{-1}]$. Note that $J \subseteq \ker \pi$ and that the induced map

$$\frac{(J : fA_2)}{J} \to \frac{(J[g^{-1}] : fA_2[g^{-1}])}{J[g^{-1}]}$$

intertwines with the isomorphisms from Proposition 5.2.9 to give the map in the statement of the lemma. If we can show $\ker \pi = J$, then the result will follow by Proposition 5.2.9. To this end, let $m \in \ker \pi$. Then

$$\begin{aligned} m \in J[g^{-1}] &\implies m = jg^{-n} \quad \text{for some } j \in J \\ &\implies mg^n \in J \implies m + J \in A_2/J \text{ is } \mathbb{k}[y]\text{-torsion.} \end{aligned}$$

By our hypothesis, this implies $m \in J$. □

Lemma 5.3.13. *Let X and X' be birational, irreducible, affine curves which contain a common open subset U. Then the set of linear differential operators*

$$\mathscr{D}(X, U) = \mathscr{D}(U) = \mathscr{D}(X', U)$$

are equal.

Further,

$$\mathscr{D}(U) \otimes_{\mathscr{D}(X')} \mathscr{D}(X, X') \subseteq \mathscr{D}(U).$$

Proof. Let the field of fractions associated to $\mathcal{O}(U)$ be denoted $K = \mathbb{k}(U)$. Then we may identify $\mathscr{D}(U)$ with its image in $\mathscr{D}(K)$ as

$$\mathscr{D}(U) = \{\theta \in \mathscr{D}(K) \mid \theta * (\mathcal{O}(U)) \subseteq \mathcal{O}(U)\}.$$

We may similarly identify $\mathscr{D}(X, X')$ as

$$\mathscr{D}(X, X') = \{\theta \in \mathscr{D}(K) \mid \theta * (\mathcal{O}(X)) \subseteq \mathcal{O}(X')\}.$$

Since $\mathscr{D}(X)$ is an HNP ring, $\mathscr{D}(X, X')$ is a rank 1 projective $\mathscr{D}(X)$-module and in this proof we will identify $\mathscr{D}(U) \otimes_{\mathscr{D}(X')} \mathscr{D}(X, X')$ with the subset $\mathscr{D}(U)\mathscr{D}(X, X')$ of the Goldie quotient ring Q. We will do this in future without comment.

As $\Bbbk(X) = \Bbbk(X') = \Bbbk(U)$, we may use a standard argument in projective geometry to find $g \in \mathcal{O}(X)$ (respectively $g' \in \mathcal{O}(X')$) such that $\mathcal{O}(U) = \mathcal{O}(X)[g^{-1}]$ (respectively $\mathcal{O}(U) = \mathcal{O}(X')[g'^{-1}]$).

Then, as $\mathcal{O}(X) \subseteq \mathcal{O}(U)$, $\mathscr{D}(U) \subseteq \mathscr{D}(X, U)$. Now, let $D \in \mathscr{D}(X, U)$ and observe that

$$D * (\mathcal{O}(U)) = D * (\mathcal{O}(X)[g^{-1}]) \subseteq D * (\mathcal{O}(X))[g^{-1}] \subseteq \mathcal{O}(U)[g^{-1}] = \mathcal{O}(U),$$

where the first containment follows by Lemma 5.2.1. So $\mathscr{D}(X, U) = \mathscr{D}(U)$, and similarly, $\mathscr{D}(X', U) = \mathscr{D}(U)$.

Finally,

$$\mathscr{D}(U)\mathscr{D}(X, X') * (\mathcal{O}(X)) = \mathscr{D}(U) * (\mathcal{O}(X')) \subseteq \mathcal{O}(U),$$

and so $\mathscr{D}(U)\mathscr{D}(X, X') \subseteq \mathscr{D}(X, U) = \mathscr{D}(U)$. □

Proposition 5.3.14. *Let J be a maximal right ideal of A_2 which strictly contains fA_2 such that A_2/J is $\Bbbk[y]$-torsionfree and let $g \in \Bbbk[y]$ be the polynomial which defines the y-coordinates of the singular points of X. Then*

$$M := \operatorname{Hom}_{A_2[g^{-1}]}(A_2[g^{-1}]/fA_2[g^{-1}], A_2[g^{-1}]/J[g^{-1}])$$

is a finitely generated right D-module. Consequently, $\operatorname{Hom}_{A_2}(A_2/fA_2, A_2/J)$ *is a finitely generated right D-module.*

Proof. Let the normalisation of X be the curve $\widetilde{X}$ and let $\phi : \widetilde{X} \to X$ be the normalisation map which is bijective. We note that $\mathcal{O}(X)$ and $\mathcal{O}(\widetilde{X})$ may be viewed as subsets of $\Bbbk(X)$, and hence $\mathscr{D}(X)$ and $\mathscr{D}(\widetilde{X})$ can be identified as subsets of $\mathscr{D}(\Bbbk(X))$. Let $A = \mathscr{D}(\widetilde{X})$. We first claim that M is a finite length right module over a localisation of A.

Indeed,

$$M = \frac{(J[g^{-1}] : fA_2[g^{-1}])}{J[g^{-1}]} = \frac{\mathbb{I}_{A_2[g^{-1}]}(fA_2[g^{-1}]) + J[g^{-1}]}{J[g^{-1}]}$$
$$\cong \frac{\mathbb{I}_{A_2[g^{-1}]}(fA_2[g^{-1}])}{\mathbb{I}_{A_2[g^{-1}]}(fA_2[g^{-1}]) \cap J[g^{-1}]},$$

where the second equality holds due to Proposition 5.2.14 as $(f) \lhd \Bbbk[x,y][g^{-1}]$ satisfy the conditions of Theorem 5.2.13. Further, due to Proposition 5.2.15, we note that $fA_2[g^{-1}] \subsetneqq \mathbb{I}_{A_2[g^{-1}]}(fA_2[g^{-1}]) \cap J[g^{-1}]$ and hence

$$\mathscr{D}(\mathcal{O}(X)[\overline{g}^{-1}]) \cong \frac{\mathbb{I}_{A_2[g^{-1}]}(fA_2[g^{-1}])}{fA_2[g^{-1}]} \twoheadrightarrow M,$$

with nontrivial kernel, where $\overline{g}$ is the image of $g \in \Bbbk[x,y]$ in $\mathcal{O}(X)$.

We now show that $C' := \mathcal{O}(X)[\overline{g}^{-1}] \cong \mathcal{O}(\widetilde{X})[G^{-1}]$ where $G \in \mathcal{O}(\widetilde{X})$, whence we may conclude that M is isomorphic to a proper factor of a localisation of A. Indeed, if $\phi : \widetilde{X} \to X$ is our bijective normalisation map then, if Y denotes the set of singular points in X and $\widetilde{Y}$ the corresponding points in $\widetilde{X}$, the restriction of ϕ to $X \setminus Y$ gives an isomorphism $X \setminus Y \cong \widetilde{X} \setminus \widetilde{Y}$. Let $Z = \mathbb{V}(g) \cap X$, which is a finite set of points, and define $\widetilde{Z} := \phi^{-1}(Z)$. Then ϕ also gives an isomorphism

$$X \setminus Z \cong \widetilde{X} \setminus \widetilde{Z}.$$

As in the proof of Lemma 5.3.13, we may find $G \in \mathcal{O}(\widetilde{X})$ which defines $\widetilde{Z}$. Then

$$C' = \mathcal{O}(X \setminus Z) \cong \mathcal{O}(\widetilde{X} \setminus \widetilde{Z}) = \mathcal{O}(\widetilde{X})[G^{-1}].$$

Hence

$$\mathscr{D}(C') = \mathscr{D}(\mathcal{O}(\widetilde{X})[G^{-1}]) = \mathscr{D}(\mathcal{O}(\widetilde{X}))[G^{-1}] = A[G^{-1}],$$

by Lemma 5.2.1.

Let us now look more closely at $A[G^{-1}]$. We claim that this ring is Kdim 1-critical. Indeed, $A[G^{-1}]$ is a localisation of the HNP ring A by 5.2.1(b) , and is thus hereditary by Lemma 5.2.16. Since it is not a division ring, $\operatorname{Kdim}(A[G^{-1}]) = 1$ and, by [26, Corollary 6.2.12], any proper factor has Kdim 0 as required. Hence M has finite length as a right $A[G^{-1}]$-module. Thus, to show that M_D is finitely generated it suffices to show S_D is finitely generated for all simple right modules S of $A[G^{-1}] = \mathscr{D}(C') \supseteq D$. But simple $A[G^{-1}]$-modules are holonomic so, by Theorem 5.2.8, S_A is holonomic and thus is finitely generated. (Note that we are using that $\widetilde{X}$ is nonsingular here.)

Now recall by Theorem 5.2.4 that D and A are Morita equivalent where the progenerator is

$$P := \mathscr{D}(\widetilde{X}, X) = \{\theta \in A \mid \theta * (\mathcal{O}(\widetilde{X})) \subseteq \mathcal{O}(X)\}.$$

We show $A[G^{-1}] \otimes_D P \cong A[G^{-1}]$ as A-modules. We note that $A[G^{-1}] \otimes_D P \subseteq A[G^{-1}]$ by Lemma 5.3.13. We now show the reverse containment. Observe that $P \otimes_A P^*$ (resp. $P^* \otimes_D P$) is a nonzero two sided ideal of D (resp. A), and hence $P \otimes_A P^* = D$ (resp. $P^* \otimes_D P = A$) by simplicity, A by 5.2.1(a) and D by Corollary 5.2.5. Thus

$$A[G^{-1}] = \big(A[G^{-1}] \otimes_D P\big) \otimes_A P^* \subseteq A[G^{-1}] \otimes_A P^* \subseteq A[G^{-1}],$$

where the last containment follows by Lemma 5.3.13 as $P^* = \mathscr{D}(X, \widetilde{X})$. Hence,

$$\big(A[G^{-1}] \otimes_D P\big) \otimes_A P^* = A[G^{-1}] \otimes_D P^*,$$

and as P^* is a progenerator, applying $-\otimes_D P$ to both sides gives $A[G^{-1}] \otimes_A P \cong A[G^{-1}]$.

We claim that $S_D \otimes_D P \cong S_A$. By Proposition 5.2.17, we see that $S \cong S' \otimes_A A[G^{-1}]$ where S' is a finitely generated simple right A-module. Now,

$$\begin{aligned} S \otimes_D P &= S' \otimes_A \big(A[G^{-1}] \otimes_D P\big) \\ &\cong S' \otimes_A A[G^{-1}] = S_A. \end{aligned}$$

As S_A is finitely generated, and finite generation is preserved by Morita equivalence, S_D is finitely generated as required. $\square$

We have shown that if A_2/J is $\Bbbk[y]$-torsionfree where J is a maximal right ideal which contains fA_2, then $\operatorname{Hom}_{A_2}(A_2/fA_2, A_2/J)$ is a finitely generated right D-module. If A_2/J were instead a $\Bbbk[x]$-torsionfree module, an identical method would work.

5.3.4 The Torsion Case

We must show that $\operatorname{Hom}_{A_2}(A_2/fA_2, A_2/J)$ is a finitely generated right D-module for all maximal right ideals J of A_2 which strictly contain fA_2 and such that A_2/J is $\Bbbk[x]-$

and $\Bbbk[y]$-torsion. We shall see that we may effectively reduce to showing that the right D-module:

$$\mathcal{M} = \frac{((x,y)A_2 : fA_2)}{(x,y)A_2}$$

is finitely generated. Recall the definition a strongly filtered ring (Definition 2.2.29).

Definition 5.3.15. Let $R = \bigcup_{n\geq 0} R(n)$ be a filtered ring. We say that R is *strongly filtered* if

$$R(i)R(j) = R(i+j)$$

for all $i, j \in \mathbb{N}$.

Definition 5.3.16. Let T be a module over a ring R with a strong filtration $R = \bigcup R(i)$. We say T is *quasi-filtered* if T may be written as a union of nested finite-dimensional vector spaces $T = \bigcup_{i\in\mathbb{N}} T(i)$ such that there exists an $a \in \mathbb{N}$ such that

$$T(i)R(1) \subseteq T(i+a)$$

for all $i \in \mathbb{N}$. Further, we say T has *generalised linear growth* if there exist $c \in \mathbb{R}_+$, $d \in \mathbb{R}$ such that

$$\dim_{\Bbbk} T(i) \leq ci + d,$$

for all $i \in \mathbb{N}$.

We now prove the following theorem which shows that modules with suitably bounded growth over strongly filtered simple rings will always be finitely generated.

Theorem 5.3.17. *Let A be a strongly filtered infinite dimensional simple ring. Then any quasi-filtered right A-module with generalised linear growth is finitely generated.*

Before we prove this theorem, we first prove some auxiliary lemmata.

Lemma 5.3.18. *Let A be a strongly filtered simple infinite dimensional ring and let U be a nonzero finite dimensional subspace of a right A-module. Then*

$$\dim_{\Bbbk} UA(m) \geq m \quad \forall m \in \mathbb{N}.$$

Proof. We begin by noting that $UA(m) \supsetneqq UA(m-1)$ for all $m \in \mathbb{N}$. Indeed, if this were not the case then $UA = UA(m)$ would be a nonzero finite dimensional A-module since A is strongly filtered, which is impossible by Theorem 2.2.4. Hence

$$\dim_{\Bbbk} UA(m) \geq \dim_{\Bbbk} UA(m-1) + 1 \geq \cdots \geq \dim_{\Bbbk} UA(0) + m \geq m. \qquad \square$$

Lemma 5.3.19. *Let A be a strongly filtered simple infinite dimensional ring and assume that N is not a finitely generated right A-module. Then we may form a infinite ascending chain of finite dimensional subspaces $\{U_i\}_{i\geq 1}$ of N such that $\dim_{\Bbbk} U_iA(m) \geq im$ for all $i, m \geq 1$.*

Proof. We proceed by induction on $i \geq 1$, noting that the preceding lemma gives the base case $i = 1$. Now we assume that there exists a finite dimensional U_i such that $\dim_{\Bbbk} U_iA(m) \geq im$ for all $m \in \mathbb{N}$. Now, since N is not finitely generated, $N \neq U_iA$, so there exists a finite dimensional $U_{i+1} \supsetneq U_i$ with $U_{i+1} \nsubseteq U_iA$. Let $\overline{U_{i+1}}$ denote the image of U_{i+1} in N/U_iA. Consider the factor module

$$\overline{U_{i+1}A} := U_{i+1}A/U_iA = \overline{U_{i+1}}A.$$

Then

$$\begin{aligned} \dim_{\Bbbk} U_{i+1}A(m) &\geq \dim_{\Bbbk} U_iA(m) + \dim_{\Bbbk} \overline{U_{i+1}}A(m) \\ &\geq im + m \end{aligned}$$

where the last inequality follows from Lemma 5.3.18. $\square$

Lemma 5.3.20. *Let A be a strongly filtered simple infinite dimensional ring and let N be a quasi-filtered right A-module with generalised linear growth. Then any strictly ascending chain of finite-dimensional subspaces $\{U_i\}_{i\in\mathbb{N}}$ such that $\dim_{\Bbbk} U_iA(m) \geq im$ for all $i, m \in \mathbb{N}$ is finite.*

Proof. Note that for the U_i as in the statement of the lemma there exists a $u_i \in \mathbb{N}$ such that $U_i \subseteq N(u_i)$, since they are finite dimensional. Further, for all $m \in \mathbb{N}$,

$$U_iA(m) \subseteq N(u_i)A(m) = N(u_i)A(1)^m \subseteq N(u_i + ma)$$

as N is quasi-filtered, which implies that

$$im \leq \dim_{\Bbbk}(U_iA(m)) \leq \dim_{\Bbbk} N(u_i + ma) \leq c(u_i + ma) + d,$$

and hence $i \leq ca$. $\square$

Proof of Theorem 5.3.17. Assume for contradiction that N_A is a quasi-filtered right module with generalised linear growth which is not finitely generated. Then by Lemma 5.3.19 we may form an ascending chain of subspaces $\{U_i\}_{i\in\mathbb{N}}$ such that $\dim_{\Bbbk} U_i A(m) \geq im$ for all $i, m \geq 1$. But this contradicts Lemma 5.3.20. $\square$

Recall Notation 5.3.1. We wish to apply Theorem 5.3.17 to the $D = \mathscr{D}(X)$-module

$$\mathcal{M} = \frac{((x,y)A_2 : fA_2)}{(x,y)A_2}.$$

We now show that in our circumstances we have two filtrations on $D = \mathscr{D}(X)$ which interact well together. For the first filtration, recall the Bernstein filtration $\mathcal{B} = \bigcup_{i\geq 0} \mathcal{B}_i$ on the second Weyl algebra A_2 defined by

$$\mathcal{B}_n = \operatorname{span}_{\Bbbk}\{x^i y^j \partial_x^k \partial_y^\ell \mid i + j + k + \ell \leq n\}.$$

That is to say, $\mathcal{B}_n$ is the $\Bbbk$-subspace of A_2 spanned by monomials in $x, y, \partial_x, \partial_y$ of degree $\leq n$. Recall also, by Proposition 5.1.7, we have that $D \cong \mathbb{I}_{A_2}(fA_2)/fA_2$, and we use this to define a filtration on D. First we view $\mathbb{I}_{A_2}(fA_2)$ as a subring of A_2 with the Bernstein filtration $\mathcal{B}$ and define a filtration on $\mathbb{I}_{A_2}(fA_2)$ by setting $\mathcal{G}_m = \mathbb{I}_{A_2}(fA_2) \cap \mathcal{B}_m$. Then we define a filtration on the ring $\mathbb{I}_{A_2}(fA_2)/fA_2$ by

$$\mathcal{F}_m := (\mathbb{I}_{A_2}(fA_2)/fA_2)_m = \frac{\mathcal{G}_m + fA_2}{fA_2}.$$

By abuse of notation, we will refer to this as the *Bernstein filtration on D*. We use subscripts here to clearly distinguish which filtration on D we are using.

For the second filtration, by Lemma 2.2.30 and Corollary 5.2.5, D has a strong filtration which we denote

$$D(m) = V^m$$

where V is a finite generating set for D containing 1.

We claim these two filtrations interact well together, which we formalise in the following proposition.

Lemma 5.3.21. *Recall Notation 5.3.1. Consider the Bernstein filtration on D and let M be a filtered D-module with respect to the Bernstein filtration. Choose a finite* *generating set V for D containing 1 and define a strong filtration on D as in Proposition 2.2.30 by $D(m) = V^m$. Then M is quasi-filtered in the sense of Definition 5.3.16.*

Proof. Note that each filtered piece of D under the strong filtration is finite dimensional, hence there exists $a \in \mathbb{N}$ such that

$$D(1) \subseteq \mathcal{F}_a.$$

Hence

$$M(i)D(1) \subseteq M(i)\mathcal{F}_a \subseteq M(i+a),$$

as required. □

Proposition 5.3.22. *Let $\gamma, \delta \in \mathbb{k}$. The right D-module*

$$\mathcal{M} = \frac{((x-\gamma, y-\delta)A_2 : IA_2)}{(x-\gamma, y-\delta)A_2}$$

is finitely generated.

Proof. We first observe that for any point $(\gamma, \delta) \in \mathbb{A}^2$, we have the following isomorphism:

$$\frac{((x-\gamma, y-\delta)A_2 : fA_2)}{(x-\gamma, y-\delta)A_2} \cong \frac{((x', y')A_2 : \hat{f}A_2)}{(x', y')A_2},$$

where $\hat{f} \in \mathbb{k}[x, y]$ defines a curve with the same properties as X. Hence without loss of generality it suffices to prove that $\mathcal{M}$ is finitely generated for $\gamma = \delta = 0$.

We use Theorem 5.3.17. We define a filtration on $\mathcal{M}$ by first denoting

$$\Gamma_n := ((x, y)A_2 : fA_2) \cap \mathcal{B}_n$$

and then constructing

$$\mathcal{M}(n) := \frac{\Gamma_n + (x, y)A_2}{(x, y)A_2}.$$

We again abuse notation and refer to this as the *Bernstein filtration on $\mathcal{M}$*. Now we note that $\Gamma_n \cdot \mathcal{G}_m \subseteq \Gamma_{n+m}$ by definition. Then

$$\begin{aligned}\mathcal{M}(n) \cdot \mathcal{F}_m &= \frac{\Gamma_n + (x,y)A_2}{(x,y)A_2} \cdot \frac{\mathcal{G}_m + fA_2}{fA_2} \\ &\subseteq \frac{\Gamma_n \cdot \mathcal{G}_m + (x,y)A_2}{(x,y)A_2} \\ &\subseteq \frac{\Gamma_{n+m} + (x,y)A_2}{(x,y)A_2} = \mathcal{M}(n+m).\end{aligned}$$

So $\mathcal{M}$ is a filtered D-module when D has the induced Bernstein filtration. Now we show $\mathcal{M}$ has generalised linear growth.

We will now use some standard linear algebra techniques related to linear independence of equations to bound the dimension of $\mathcal{M}(n)$ where $n \gg 0$. We begin by describing the important sets we will use. Firstly, given a polynomial $f = \sum_{\alpha,\beta} f_{\alpha\beta} x^\alpha y^\beta \in \Bbbk[x,y]$ we define the set

$$\operatorname{supp}(f) \coloneqq \{(i,j) \mid f_{ij} \neq 0\} \subseteq \mathbb{N}^2.$$

Then we note

$$\operatorname{supp}(fx^a y^b) = \operatorname{supp}(f) + (a,b).$$

Write $A_2/(x,y)A_2 = \Bbbk[\partial_x, \partial_y]$, which we filter by setting $\deg \partial_x = \deg \partial_y = 1$; this induces the Bernstein filtration on $\mathcal{M} \subseteq A_2/(x,y)A_2$. That is to say, we view the filtered pieces of $\mathcal{M}$ as:

$$\mathcal{M}(n) = \{\theta \in \Bbbk[\partial_x, \partial_y]_{\leq n} \mid \theta * fx^i y^j \in (x,y) \quad \forall i,j \in \mathbb{N}\}.$$

We also define

$$\Delta_n = \{(i,j) \in \mathbb{N}^2 \mid i + j \leq n\},$$

Let $N = \max_{\text{totaldeg}}(f)$. Then, if $n > N$, we see

$$\Delta_{n-N} = \{(i,j) \in \mathbb{N}^2 \mid \operatorname{supp}(f) + (i,j) \subseteq \Delta_n\}.$$

Fix $n > N$. We first claim that

$$\dim_{\Bbbk} \mathcal{M}(n) \leq |\Delta_n| - |\Delta_{n-N}|.$$

Put a deg-lex ordering on pairs $(i,j) \in \mathbb{N}^2$, denoted $\prec$, where $(a,b) \prec (c,d)$ if $a+b < c+d$ or, if there is equality of total degree, and (a,b) is lexicographically less than (c,d). Note that

$$(a,b) \prec (c,d) \iff (a+i, b+j) \prec (c+i, d+j). \tag{5.1}$$

Now we show how to give an upper bound for $\dim_{\Bbbk} \mathcal{M}(n)$.

Let $\theta = \sum_{0 \le p+q \le n} a_{pq} \partial_x^p \partial_y^q$ be a general element of $\Bbbk[\partial_x, \partial_y]_{\le n}$, where the a_{pq} are indeterminates. Also let

$$\Phi_{ij} := \left(\theta * f x^i y^j\right)(0,0) = \sum_{(\alpha,\beta) \in \operatorname{supp} f} a_{\alpha+i,\beta+j} (\alpha+i)!(\beta+j)! f_{\alpha\beta}, \tag{5.2}$$

which is a linear combination of the variables a_{pq}. Then $\theta \in \mathcal{M}(n)$ if and only if $\Phi_{ij} = 0$ for all $i, j \in \mathbb{N}$. When $(i,j) \in \Delta_{n-N}$, Φ_{ij} is nonzero as an element of the set $\operatorname{span}_{\Bbbk}\{a_{pq} \mid (p,q) \in \Delta_n\}$.

We extend the monomial ordering on Δ_n to the a_{pq} by defining $a_{\alpha\beta} \prec a_{\alpha'\beta'}$ if and only if $(\alpha,\beta) \prec (\alpha',\beta')$. Further, we denote the element of $\mathbb{N}^2$ associated to the leading term of Φ_{ij} by $\overline{\Phi_{ij}}$, so $\overline{\Phi_{ij}} := (k,\ell)$ where $a_{k\ell} = \operatorname{lt}_{\prec}(\Phi_{ij})$. Then we note that

$$\overline{\Phi_{ij}} = (i,j) + \overline{\Phi_{00}},$$

by (5.2).

Let $\{\beta_{ij}\}_{0 \le i+j \le n} \in \Bbbk$ be not all zero and let

$$M = \max_{\prec}\{(i,j) \mid \beta_{ij} \ne 0\} = (i_M, j_M).$$

If $(i,j) \prec M$, then $\overline{\Phi_{ij}} \prec M + \overline{\Phi_{00}}$ by (5.1). Consider the linear combination of the Φ_{ij},

$$\Phi := \sum_{0 \le i+j \le n} \beta_{ij} \Phi_{ij},$$

viewed as a linear expression in the a_{pq}. Then $\overline{\Phi} = \overline{\Phi_{i_M j_M}} = M + \overline{\Phi_{00}}$ and the leading coefficient of Φ is $\lambda \beta_{i_M j_M}$ where λ is a nonzero coefficient coming from (5.2). Hence, $\Phi = 0$ implies $\beta_{i_M j_M} = 0$, a contradiction.

Thus $\{\Phi_{ij} \mid (i,j) \in \Delta_{n-N}\}$ are linearly independent and the elements of Δ_{n-N} give linearly independent constraints on $\mathcal{M}(n)$. Hence

$$\dim_{\Bbbk} \mathcal{M}(n) \leq |\Delta_n| - |\Delta_{n-N}|,$$

as claimed.

We now observe that

$$\dim_{\Bbbk} \mathcal{M}(n) \leq |\Delta_n| - |\Delta_{n-N}| = \frac{n(n+1)}{2} - \frac{(n-N)(n-N+1)}{2}$$

which is linear in n, and hence $\mathcal{M}$ has generalised linear growth.

Put a strong filtration $D = \bigcup_{n\geq 0} D(n)$ on D as described in Proposition 2.2.30. By Lemma 5.3.21, there exists an $a \in \mathbb{N}$ such that

$$\mathcal{M}(n)D(1) \subseteq \mathcal{M}(n+a)$$

for all $n \geq 0$. Thus, by Theorem 5.3.17, $\mathcal{M}_D$ is finitely generated. □

We now complete the proof of the right noetherianity of $\mathbb{I}_{A_2}(fA_2)$. The following auxiliary lemma aids us.

Lemma 5.3.23. *Let S be a simple $\Bbbk[x]$-torsion right A_1-module, then*

$$S \cong A_1/(x-a)A_1$$

for some $a \in \Bbbk$.

Proof. As S is a simple module with $\Bbbk[x]$-torsion, that implies that $S \cong A_1/K$ where K is a maximal right ideal of A_1 which contains a polynomial p in x (the annihilator of 1). Let us first consider a composition series for the right module A_1/pA_1.

Since $\Bbbk$ is algebraically closed, we may factorise p into linear factors:

$$p = \prod_{i=1}^{n} (x - a_i)$$

and denote

$$p_j = \prod_{i=j}^{n} (x - a_i),$$

for $j = 1, \ldots, n$.

Observe that

$$p_jA_1/p_{j+1}A_1 \cong A_1/(x-a_{j+1})A_1,$$

which is a simple right A_1-module. Hence the following induces a composition series for A_1/pA_1:

$$pA_1 \subseteq p_2A_1 \subseteq \cdots \subseteq p_nA_1 \subseteq A_1.$$

We finish the proof by noting that, as $pA_1 \subseteq K$,

$$A_1/pA_1 \twoheadrightarrow A_1/K,$$

and hence, by the Jordan-Hölder theorem, A_1/K must be isomorphic to a composition factor of A_1/pA_1. That is to say

$$S \cong A_1/K \cong A_1/(x-a)A_1$$

for some $a \in A_1$. □

Proposition 5.3.24. *Assume Notation 5.3.1. Suppose that $J \leq_r A_2$ is maximal and contains fA_2, and that A_2/J is $\Bbbk[y]$- and $\Bbbk[x]$-torsion. Then there exist $a, b \in \Bbbk$ such that*

$$A_2/J \cong A_2/(x-a, y-b)A_2.$$

Proof. Since A_2/J is $\Bbbk[x]$-torsion, this implies that there exists a nonzero polynomial $g \in J \cap \Bbbk[x]$ (the annihilator of the identity). By [3, Proposition 5.1], $A_2/J \cong S_1 \otimes_{\Bbbk} S_2$, where S_1 is a simple $\Bbbk[x, \partial_x]$-module and S_2 is a simple $\Bbbk[y, \partial_y]$-module, and there exists a nonzero $s \in S_1$ such that $sg = 0$. We note, by the same reasoning as Lemma 5.3.11, S_1 is either $\Bbbk[x]$-torsion or $\Bbbk[x]$-torsionfree. As $sg = 0$, S_1 must be $\Bbbk[x]$-torsion. Applying Lemma 5.3.23, we may conclude $S_1 \cong A_1/(x-a)A_1$ for some $a \in \Bbbk$. Similarly, we may conclude that $S_2 \cong A_1/(y-b)A_1$ and thus $A_2/J \cong S_1 \otimes_{\Bbbk} S_2 \cong A_2/(x-a, y-b)A_2$. □

We now prove Theorem 5.1.6.

Proof of Theorem 5.1.6. By Proposition 5.3.9 to show that $\mathbb{I}_{A_2}(fA_2)$ is right noetherian we must show that $\operatorname{Hom}_{A_2}(A_2/fA_2, A_2/J)$ is a finitely generated right D-module for all maximal right ideals $J \leq_r A_2$ which strictly contain fA_2.

Lemma 5.3.11 shows that there are two cases:

1. A_2/J is $\Bbbk[x]$- or $\Bbbk[y]$-torsionfree; or

2. A_2/J is $\Bbbk[x]$- and $\Bbbk[y]$-torsion.

Proposition 5.3.14 covers case (1). For case (2), Proposition 5.3.24 shows that we need only consider right ideals J of the form $(x-a, y-b)A_2$ where $(a,b) \in \Bbbk^2$ and Proposition 5.3.22 shows that

$$\operatorname{Hom}_{A_2}(A_2/fA_2, A_2/(x-a, y-b)A_2) \cong \frac{((x-a, y-b)A_2 : IA_2)}{(x-a, y-b)A_2},$$

is finitely generated as required. Then, by Proposition 5.3.2, $\mathbb{I}_{A_2}(fA_2)$ is left and right noetherian. □

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