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CHAPTER 1.
INTRODUCTION.

According to quantum mechanics, the stationary states of a (free) system of \( n \) nucleons are characterized by the (simultaneous) eigenvalues of a complete, commuting set (including the Hamiltonian) of linear, hermitian operators corresponding to the observables of the system. These stationary states are determined from the time-independent Schrödinger (eigenvalue) equation

\[
H(1,2,\ldots,n) \, \Psi_k(1,2,\ldots,n) = E_k \, \Psi_k(1,2,\ldots,n), \quad (1.1)
\]

together with suitable continuity, boundary and symmetry conditions imposed on the \( \Psi_k(1,\ldots,n) \). In eq. (1.1), \( H(1,\ldots,n) \) is the operator corresponding to the total energy of the system, i.e. its Hamiltonian; \( k \) the set of (discretely or continuously varying) simultaneous eigenvalues distinguishing the different states; and \( E_k \) is a constant, the energy of the system in the state \( \Psi_k(1,\ldots,n) \).

The coordinates \( x_i, y_i, z_i, s_i, \tau_i \) of the \( i \) th nucleon are here collectively denoted by \( i \cdot \) (\( i = 1,2,\ldots,n \)).

Now the Hamiltonian of a system may be of the form

\[
H = H^{(0)} + H^{(1)} \quad (1.2)
\]

where \( H^{(1)} \), assumed independent of time, is a small (transient; say, \( t = t_o \) to \( t = t_1 \)) perturbation. Since the time-development of the system is given by the time-dependent Schrödinger equation,

\[
H(1,2,\ldots,n) \psi(1,\ldots,n;t) = \frac{-i}{\hbar} \frac{\partial}{\partial t} \psi(1,\ldots,n;t) \quad (1.3)
\]

then, if the system starts off (at \( t = t_o \)) by being in a stationary state, say \( \psi^{(0)}_k(1,\ldots,n) \), of \( H^{(0)} \), it will, at a later time (\( t = t \)) be in a state \( \psi(1,\ldots,n;t) \) which satisfies (1.3) and for which

\[
\psi(1,\ldots,n;t) = \psi^{(0)}_k(1,\ldots,n). \]
In the usual time-dependent perturbation theory, in first-order approximation, the probability, per unit time, of finding the system in state $\psi_{k'}^{(o)}(1,\ldots, n)$ is given as proportional to

$$
| H_{k,k'}^{(1)} |^2 = | \langle \psi_{k'}^{(o)}(1,\ldots, n) | H^1 \psi_k^{(o)}(1,\ldots, n) \rangle |^2
$$

(1.4)

where the matrix element, $T_{k',k}^{(1)}$, of an operator, $T$, between the states $\psi_{k'}^{(o)}(1,\ldots, n)$ and $\psi_k^{(o)}(1,\ldots, n)$ is defined by

$$
T_{k',k} = \left( \psi_{k'}^{(o)}(1,\ldots, n), T \psi_k^{(o)}(1,\ldots, n) \right) \equiv \int \psi_{k'}^{(o)}(1,\ldots, n)^* T \psi_k^{(o)}(1,\ldots, n) \, d\tau
$$

(1.5)

In eq. (1.5) the symbol $\int \ldots d\tau$ means integration over position coordinates and summations over spin $(s)$ and isobaric spin $(\varepsilon)$ coordinates of all the nucleons; $^*$ denotes the complex conjugate.

Usually, $H^{(1)}$ depends on some operators, $T, S, U, \ldots$, (which can act on the state vectors $\psi_k^{(o)}(1,\ldots, n)$), and on the experimental conditions. For example, $L_2$ in atomic $L$-coupling, or the magnetic dipole interaction $\hat{L}_1 \cdot \vec{H}$ of a system with magnetic moment $\hat{L}_1$ in an external field $\vec{H}$. Thus, generally, the stationary state properties of a (free) system are measured by introducing a small (transient) perturbation, depending on a number of operators (which need not commute with $H^{(o)}$) and the fixed external conditions, and measuring the transition probabilities (related to relative intensity of the transition 'lines') of the perturbed system.

Writing the eigenvalues of the observables, $A, B, \ldots$, of $H^{(o)}$ in the form $\langle \psi_{k}^{(o)}(1,\ldots, n), A \psi_{k}^{(o)}(1,\ldots, n) \rangle, \ldots$, we see that of the experimentally measurable properties (and these are the quantities of primary interest) of a system, such as the total angular momentum of a stationary state (of $H^{(o)}$), its magnetic moment, transition probabilities to other states under the influence of various perturbations, and so on, some are specified by the "matrix elements" of linear, hermitian operators which commute with $H^{(o)}$,
and some are connected, through the matrix elements of $H^{(1)}$, to the matrix elements of operators which may or may not commute with $H^{(0)}$. Thus an important theoretical task is the calculation of matrix elements (1.5) for operators of interest. But the exact solution of the many-body eigenvalue problem for $H^{(0)}$, assuming that we were, which we are not, in possession of the real interaction forces of an n-nucleon system; generally, $H^{(0)}$ specifies some hypothetical model, and thence the explicit evaluation of the matrix elements (1.5) presents such formidable mathematical difficulties that we are forced to exploit the symmetries of the physical system under consideration to simplify the calculations, and in order to make predictions based only on considerations of these basic symmetries.

Now consider a group, $G$, (for definitions see chapter 2) of operations $R, S, T, \ldots$, which can be thought of as mappings of the space of coordinates $\vec{x}, x_1, y_1, \ldots, r_n$ onto itself: $\vec{x} \rightarrow R\vec{x}$

For example, $R$ could be a rotation of the position coordinates, affecting only $x_i, y_i, z_i$ ($i = 1, 2, \ldots, n$); or it could be a permutation of the coordinates of the nucleons. Then we can define a group, $P_G$, of operators $R, S, T, \ldots$, defined by

$$P_R f(\vec{x}) = f(R^T\vec{x})$$

which map a function $f$ of $\vec{x}$ onto another such function: $f \rightarrow f' = P_R f$. It is easy to see that $P_G$ is isomorphic to $G$; for

$$P_S P_R f(\vec{x}) = P_R f(S^{-1}\vec{x})$$

$$= f(R^{-1}S^{-1}\vec{x})$$

$$= P_S P_R f$$

Hence $P_R P_S = P_T$ whenever $RS = T$. 

An operator, $A$, is said to be invariant under the group $P_G$ if
\[ QAQ^{-1} = A, \quad \text{for all } Q \in P_G, \]
i.e., applied to a function, $f(\xi)$, $QAQ^{-1}$ and $A$ yield the same
resultant function. Henceforth we shall take $P_G$ to be the group
of operators, $P_R$, under which the Hamiltonian, $H^{(0)}$, is invariant.
This group is sometimes called the symmetry group of $H^{(0)}$. For
example, for the system of $n$ nucleons, $H^{(0)}$ will be 'invariant under
permutations' of the nucleons: $(S_n)$; and, provided the system is
not in an external field picking out a direction in space, also
under rotations: $(R_3)$. The parity operator, $\mathcal{P}$,
\[ \mathcal{P} f (x, y, z, s, \tau) = f(-x, -y, -z, s, \tau), \]
is a further example of an operator under which $H^{(0)}$ will be
invariant. ($G$ in this case consisting of the identity, $I$, and $\mathcal{P}$).

Then the eigenfunctions of $H^{(0)}$ could be carriers of irreducible
representations (for details see chapters 2 and 3) of $P_G$, and could
be labelled by the indices characterizing the columns of these
irreducible representations. This enables us to use the elegant
and powerful tool of the theory of group representations. In
particular, the carriers and properties of the permutation groups,
$S_n$, and the rotation group, $R_3$, become of special interest. Hence
chapter 2 is concerned with general results of the theory of groups
and their representations, while chapter 3 is devoted to a study
of the detailed properties of the irreducible representations of
$S_n$ and $R_3$.

Furthermore the linear, hermitian operators (which may or
may not commute with $H^{(0)}$) whose matrix elements are connected
with the experimentally measurable properties of the system have
one or more components such that under the group $P_G$ they transform
according to a representation of $G$. When this representation is
an irreducible one, they will be called irreducible tensor operators.
An important example of such an operator is the angular momentum
operator, defined as any vector operator $\mathbf{J}$ satisfying the com-
mutation relations

$$[J_k, J_l] = i\hbar \epsilon_{klm} J_m.$$  \hfill (1.7)

Thus, the operators

$$i\hbar^{-1} l_x = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \quad i\hbar^{-1} l_y = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \quad i\hbar^{-1} l_z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x},$$

do satisfy the above commutation relations, and transform under the
group of operators $P_R$ (where $R \in \mathfrak{g}_3$), according to a 3x3 irreducible representa-
tion of $\mathfrak{g}_3$. (Cf. chapter 4, eq. (4.3) and the discussion follow-
ing). Now consider any function $f(x, y, z)$, and evaluate $P_R f(x, y, z)$
where $R$ is an infinitesimal rotation $\omega x$, about the $x$-axis, $\omega y$ about
$y$-axis and $\omega z$ about $z$-axis. The matrix corresponding to $R$, up
to the first order in $\omega x, \omega y, \omega z$, will be:

$$R = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & -\omega_y \\
0 & 1 & \omega_x & 0 & 1 & -\omega_z \\
0 & -\omega_x & 1 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & \omega_z & -\omega_y \\
-\omega_x & 1 & \omega_z \\
\omega_y & -\omega_x & 1
\end{pmatrix}
$$

Then

$$P_R f(x, y, z) = f(R^{-1} x, R^{-1} y, R^{-1} z)
= f(x + \omega y z - \omega z y, y - \omega z z + \omega_z x, z - \omega y x + \omega x y)
= f(x, y, z) + (\omega y z - \omega_z y) \frac{\partial f}{\partial x} + (\omega z z - \omega y z) \frac{\partial f}{\partial y} + (\omega y x - \omega x y) \frac{\partial f}{\partial z}.
$$

Hence the angular momentum operators, $l_x, l_y, l_z$, are the
infinitesimal generators of the group $P_\mathfrak{g}$ which is isomorphic
to $\mathfrak{g}_3$. Thus the carriers of the irreducible representations of
$\mathfrak{g}_3$ will have definite angular momentum properties: so that there
exists an intimate connection between the transformation property
of the state vector of an n-nucleon system under the group of
operators \( R \in R_3 \) and the angular momentum of the system in the
state described by that state vector.

This exploitation of the symmetries of the physical system,
and the introduction of tensor operators was originally due to
H. Weyl, E. P. Wigner and others; but their systematic and detailed
development, and application to atomic and nuclear spectroscopy is
mainly the work of G. Racah (Racah, G., A, 1942a, 1942b, 1943,
1949, 1951); H.A. Jahn and others.

Chapter 4, then, is devoted to a detailed study of irreducible
tensor operators and the calculation of their matrix elements in
the space of state vectors which are carriers of irreducible representa-
tions of \( R_3 \). The coupling of the angular momenta of two or more
subsystems to a resultant angular momentum of the total system,
connected as it is with the reduction of direct products representa-
tions of \( R_3 \), is presented in chapter 3 for the state vectors,
and extended to irreducible tensor operators in chapter 4, where
it provides an elegant and powerful tool in the evaluation of their
matrix elements.

While the emphasis throughout is on a clear and thorough
(though, certainly not exhaustive) presentation of the methods
and techniques — essentially, the wholesale employment of group-
representation-theoretical concepts — which have been found useful
in theoretical studies of nuclear structure, (reference may be
made to the series of papers on the subject by Jahn et al,
Flowers et al, and others) simple and immediate applications of
the formalism have been worked out whenever possible, especially
in chapter 4,
Moreover, Chapter 5 in its entirety is concerned with the applications of the concepts developed in the earlier chapters to theoretical nuclear spectroscopy, and to the theory of angular correlation of successive nuclear radiations. None of the material — except, perhaps, one or two formulae of the appendix — is claimed as original.
CHAPTER 2

GROUPS AND REPRESENTATION OF GROUPS

A set or collection, G, of elements R, S, T, ..., which satisfies the following four requirements is called a group:

(a) To each ordered pair, R, S of arbitrary elements of G there corresponds a unique element, RS of G. This correspondence is termed multiplication, and the resulting element called the product of the two.

In symbols: $R \in G, S \in G \implies RS \in G$.

(b) There exists in G at least one element, I, called the identity, such that $IR = R \quad \forall R \in G$.

(c) For each element $S \in G$ there exists at least one element, $S^{-1} \in G$, called the inverse of S, such that $S^{-1}S = I$.

(d) The operation of "multiplication" is associative:

\[ R(ST) = (RS)T \quad \forall R, S, T \in G. \]

From these follow: $SS^{-1} = I$, $RI = R$, and the uniqueness of I and $S^{-1}$. Elaborate discussions of abstract group theory will be found in standard books on modern algebra, or group theory. (e.g. Wigner 1931, Ledermann 1953, Birkhoff 1954.) Here we give a summary of the main ideas which will be of use later on.

Homomorphism, Isomorphism.

Two groups, G and $G'$, are said to be homomorphic if there exists a mapping, $\phi$, of one onto the other: $R \to R' = \phi R$, $R \in G, R' \in G'$, such that $RS = T \implies \phi R \phi S = \phi T$.

Thus two groups are homomorphic if multiplication is preserved under
a mapping of one onto the other. If, moreover, the multiplication-preserving mapping is one-one—i.e. to each element of G corresponds only one element of G', and each element of G' is associated with only one element of G—then the two groups are said to be isomorphic.

Isomorphic groups have the same abstract properties, i.e. properties which do not depend on the specific nature of the group elements (operators, matrices, numbers, etc.), but which depend on the abstract structure of the group. They are sometimes said to be different realizations (by operators, matrices, etc.) of the same abstract group.

Classes.

Two elements, A, B, of G are called equivalent if there exists an element T ∈ G such that A = TBT⁻¹. The transformation S → R = TST⁻¹ is called an equivalence transformation by T. The subset, C, of element of G equivalent to a given element constitute a class of G.

It is evident that all the elements of C are equivalent to each other.

Subgroups, Cosets, Invariant subgroups.

A subset, G', of elements of G itself satisfying the group postulates is called a subgroup. Those other than I, and G itself, are named proper subgroups.

The set of elements \{AP\}, R ∈ G, P ranging over G', form a left coset of G'. Evidently G' itself is a coset, say by R ≡ I. The number of distinct cosets of a subgroup will be called its index. Right cosets are defined similarly by \{G'R\}, \{G'R\}, ...

If G' consists entirely of one or more classes, then it is termed an invariant subgroup. It then follows that left and right cosets are identical:

\[ \{sa'\} = \{a's\} \] except for the order of terms.

The cosets of an invariant subgroup, G', form the factor group,
$G'$, whose elements are the cosets of $G'$, and where the product of two cosets is the set of distinct products of elements one taken from each coset, with regard to order of multiplication. All subgroups of index 2 are invariant.

**Examples of groups.**

1. The numbers $G: \{1, i, -1, -i\}$, under ordinary multiplication, form a commutative ($RS = SR$, all $R, S \in G$) finite group of order 4. The set $\{1, -1\}$ is an invariant subgroup. The mapping $f: 1 \rightarrow 1, i \rightarrow i, A^2 \rightarrow -1, A^3 \rightarrow -i, (A^4 = I)$ establishes an isomorphism between $G$ and the (cyclic) group $C_4 = \{1, A, A^2, A^3\} (A^4 = I)$.

2. Denoting the operation of replacing 1 by $\alpha_1$, 2 by $\alpha_2$, ..., $n$ by $\alpha_n$ by $f_\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_i \neq \alpha_j$ for $i \neq j$ and all $\alpha_i$ taken from the set $1, 2, \ldots, n$, and defining the product $p_{\alpha \beta}$ of $p_\alpha$ and $p_\beta$ by sequential performance, first of $p_\alpha$, then of $p_\beta$, it is easy to see that the $n!$ such operations form a finite group of order $n!$. It is called the symmetric group of order $n!$, (or degree $n$) and often denoted by $S_n$.

3. The totality of $2 \times 2$ non-singular matrices with rational elements forms an (infinite, discrete) group under matrix multiplication.

4. The set of real, linear substitutions $x_\mu = a_{\mu \nu} x_\nu$, $\mu, \nu = 1, 2, 3$, (summation convention) which leave $ds^2 = \delta_{\mu \nu} dx_\mu dx_\nu$ invariant, forms a group under matrix multiplication: $e_{\mu \nu} = a_{\mu \tau} b_{\tau \nu}$. The matrix $a_{\mu \nu}$ is orthogonal -- $a_{\mu \tau} a_{\nu \tau} = \delta_{\mu \nu}$ -- and of determinant $\pm 1$. We will call it the 3-dimensional orthogonal group, $O_3$.

The invariant subgroup, consisting of all $3 \times 3$ real, orthogonal matrices with determinant $+1$ is called the 3-dimensional rotation
These examples show that the group requirements are wholly independent of the nature of the elements; they also illustrate some of the widely different types of groups possible. Thus in ex. (1) there are only four elements, abstractly denoted by \( G_1, G_2, G_3, G_4 \), with the multiplication table

\[
\begin{array}{cccc}
G_1 & G_2 & G_3 & G_4 \\
G_1 & G_1 & G_3 & G_4 \\
G_2 & G_4 & G_1 & G_2 \\
G_3 & G_2 & G_4 & G_1 \\
G_4 & G_3 & G_2 & G_1 \\
\end{array}
\]

Examples (3) and (4) illustrate groups containing an infinite (enumerateable and non-enumerateable, respectively) number of elements. An element of either group is completely specified by giving the values of a finite number of independent parameters. In the case of ex. (3), there are four independent matrix elements, and each can be taken from the set of all real, rational numbers, subject only to the non-vanishing of the determinant of the matrix, whereas the nine quantities \( a_{\mu \nu} \) of ex. (4) are subject to the six relations \( a_{\mu \mu} a_{\nu \tau} = \delta_{\mu \nu} \). Thus there are only 3 independent parameters uniquely characterizing \( a_{\mu \nu} \). To each triplet of values of these parameters, lying within a certain region of the "parameter space" there corresponds a group element, and vice-versa.

A group is said to be finite-continuous, or parametric, if each group element can be uniquely specified by the values of a finite number of independent, essential parameters, \( \alpha^1, \ldots, \alpha^r \), varying in a certain region. \( r \) is sometimes called the order of
the group. Thus $O_3$ and $R_3$ are examples of 3-parametric groups. In general, if we have the point-transformations of an n-dimensional space:

$$\tilde{x}^i = f^i(x'_1, x'_2, \ldots, x'_n; \alpha'_1, \ldots, \alpha'_n), \quad i = 1, 2, \ldots, n.$$  \hfill (2.1)

depending on $r$ independent, essential parameters, such that

$$\left| \frac{\partial f^i}{\partial x^j} \right| = \frac{\partial (f^1, \ldots, f^n)}{\partial (x'_1, \ldots, x'_n)} \neq 0, \quad \hfill (2.2a)$$

$$f^i(f(x; \alpha); \alpha_3) = f^i(x'; \alpha'), \quad \text{where} \quad \alpha_3^3 = \varphi(x'_2; \alpha'). \quad \hfill (2.2b)$$

and

$$x^i = f^i(x; \alpha) \quad \text{where} \quad \alpha^i = \psi^i(x). \quad \hfill (2.2c)$$

then the transformations (2.1) form a $G_r$. For the details of the theory of continuous groups, as well as for various topological considerations of parametric groups, we refer to (Lie 1924, 1927; Eisenhart 1933; Rosen A; Pontrjagin 1939; Montgomery 1955). When all the functions $f^i(x; \alpha)$ are linear in the $x'$s, so that (2.1) can be written in the matrix notation

$$\begin{pmatrix} \tilde{x}^1 \\ \vdots \\ \tilde{x}^n \end{pmatrix} = \begin{pmatrix} \alpha_1(x') \\ \vdots \\ \alpha_n(x') \end{pmatrix} \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix},$$

where each element of the matrix depends on the $r$ essential, independent parameters, then the group is called linear, or an $r$-parametric linear group. The group multiplication (2.2b) then becomes ordinary matrix multiplication, and the group could have been defined as the totality of certain $n \times n$ matrices whose elements depend in a particular way on the parameters.

The region of variability of the $\alpha'$s in (2.1) may be (simply or multiply) connected, or it may consist of a number of separate (simply or multiply connected) regions. One then speaks of simply-continuous (einfachkontinuierliche) and mixed-continuous (gemischtkontinuierliche) groups respectively.
Besides the groups considered so far, there are other kinds of groups. For example, in General Relativity one considers the totality of coordinate transformations

\[ \hat{x}^i = g^i(x), \quad i = 1, 2, \ldots, n. \]

where \( \left| \frac{\partial g^i}{\partial x^j} \right| \to 0 \) and the \( g^i(x) \) satisfy certain continuity conditions. Here again, the group multiplication is defined by

\[ \hat{x}^i = g^i(f(x)) \]

as the product transformation of first \( \hat{x}^i = f^i(x) \) and then \( \hat{x}^i = g^i(x) \).

The group element is the functional form of the \( g^i(x) \). Or, one can impose certain restrictions — besides \( \left| \frac{\partial g^i}{\partial x^j} \right| \to 0 \) — on the functional form of \( g^i(x) \), provided the group postulates are satisfied. These types of groups could be called infinite-continuous. The group element would be specified completely by the continuously infinite set of values of \( g^i(x) \), or each \( g^i(x) \) could be expanded in a 'Fourier integral expansion' and the continuously infinite set of values of the coefficients taken as 'parameters' of the group element.

For further details, see (Liet 1924, 1927).

**Direct product groups.**

Lastly, we introduce one more concept of abstract group theory. Given two groups, \( G \) and \( G' \), we can form a third group, \( G \times G' \), called their direct product (group), whose elements are ordered pairs of elements of \( G \) and \( G' \): \( \{R, S\} \), \( R \in G \), \( S \in G' \). With multiplication defined by

\[ \{R, S\} \{T, U\} = \{RT, SU\}, \]

all the group requirements can be readily verified. The subgroups \( \{I, S'\} \), \( S' \) ranging over \( G' \), and \( \{R, I\} \), \( R \) ranging over \( G \), are invariants; for, e.g., \( \{R, T\} \{I, S\} \{R, T\}^{-1} = \{R, T\}I, S' \{R^{-1}, T^{-1}\} = \{I, T', S', T'^{-1}\}; \)
i.e. with an element \( \{ I, g' \} \) the subgroup contains all the elements equivalent to it. If \( G \) and \( G' \) are both finite, of orders \( g \) and \( g' \) respectively, then \( G \times G' \) will be a finite group of order \( gg' \).

If \( G \) is finite, of order \( g \), but \( G' \) is simply-continuous, \( r \)-parametric, then \( G \times G' \) will be mixed-continuous, \( r \)-parametric. That is, if \( G \{ G_1, G_2, \ldots, G_g \} \) and \( G : \{ \alpha', \ldots, \alpha' \} \) then the elements of \( G \times G' \) would be given by the \( r \) parameters \( \alpha, \ldots, \alpha' \), and the index \( i = 1, 2, \ldots, g \). It is not difficult to see that all mixed-continuous groups could be regarded as the direct product of a number of finite and simply-continuous ones. For example, \( O_3 \) is isomorphic to \( G_2 \times R_3 \).

Group representation.

If we can associate an \( m \times m \) non-singular matrix, \( D(R) \), with the element \( R \) of a group \( G \), such that the collection of matrices \( D(R) \), \( R \in G \), forms a homomorphic group, i.e. such that

\[
D(R) D(S) = D(T), \quad \text{whenever } RS = T \text{ in } G,
\]

then the matrices \( D(R) \) are said to form a representation of \( G \).

Since \( IR = R \), all \( R \in G \), then \( D(I) D(R) = D(R) \). As all the matrices are assumed non-singular, it follows that \( D(I) = I_m \), where

\[
(I_m)_{\alpha\beta} = \delta_{\alpha\beta}, \quad \beta, \alpha = 1, 2, \ldots, m.
\]

Again, from \( D(R) D(R^{-1}) = D(I) = I_m \) it follows that

\[
D(R^{-1}) = (D(R))^{-1}
\]

If the representation matrices are all unitary, i.e. if

\[
D(R) D(R) = I_m
\]

where

\[
(D(R))_{\alpha\beta} = \overline{(D(R)_{\beta\alpha})^*},
\]

then the representation is said to be unitary.

For finite groups, the \( m^2 \) elements of the representation matrices, \( D_{\alpha\beta} (G_i) \), \((i = 1, 2, \ldots, g; \alpha, \beta = 1, \ldots, m)\) can be thought of as functions of the
discrete index $i$. For $r$-parametric groups, we require that $D(R)$, $R \in G$, be functions of the $r$ parameters, $\alpha'_1, \ldots, \alpha'_r$, such that

$$D(\alpha'_1) D(\alpha'_2) = D(\alpha'_3)$$

whenever

$$\alpha'_3 = \alpha'(\alpha'_2; \alpha'_1), \quad \rho = 1, 2, \ldots, r.$$  

completely specifies the abstract group being represented.

It is evident that assigning in $1 \times 1$ matrix (1) to every element of any group fulfil the requirements of a representation. It will be called the identity representation.

**Direct sum representation.**

Suppose we are given two (possibly identical) representations, $D^{(1)}(R)$ and $D^{(2)}(R)$ of $G$. Then the matrices $D^{(1)}(R) + D^{(2)}(R)$, where the direct sum, $A \oplus B$, of an $m \times m$ matrix $A$, with an $n \times n$ matrix $B$ is defined by

$$(A + B)_{i,j} = A_{i,j}, \quad 1 \leq i,j \leq m$$

$$= B_{i,j}, \quad m+1 \leq i,j \leq m+n$$

$$= 0, \quad \text{otherwise}.$$  

i.e.

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$  

will also form a representation of $G$.

Indeed

$$\begin{pmatrix} D^{(1)}(R) & 0 \\ 0 & D^{(2)}(R) \end{pmatrix} \begin{pmatrix} D^{(1)}(S) & 0 \\ 0 & D^{(2)}(S) \end{pmatrix} = \begin{pmatrix} D^{(1)}(T) & 0 \\ 0 & D^{(2)}(T) \end{pmatrix}.$$  

The set $D^{(1)}(R) \oplus D^{(2)}(R)$ is called the direct sum representation of $G$.

**Direct product representation.**

Another way of forming a representation of $G$ from two given
ones is to take the direct product of the corresponding matrices. The

direct product, \( A \times B \), of an \( m \times n \) matrix \( A \) by an \( m' \times n' \) matrix \( B \)

is defined by

\[
(A \times B)_{i,k,j,l} = A_{i,j} B_{k,l}, \quad (i=1,2,\ldots,m; \quad j=1,\ldots,n; \quad k=1,2,\ldots,m'; \quad l=1,\ldots,n')
\]

where each row and each column of \( A \times B \) is labelled by two indices

such that the row \( i,k \) lies above the row \( i',k' \) if \( i'>i \), or \( i'=i \)

and \( k'>k \); and likewise, the column \( j,l \) stands to the left of the column \( j',l' \) if \( j'>j \), or \( j'=j \), \( l'>l \).

Thus \( A \times B \) is an \( m \times m' \times n \times n' \) matrix which can be pictured as:

\[
A \times B = \left( \begin{array}{cccc}
A_{1,1} B & A_{1,2} B & \cdots & A_{1,n} B \\
A_{2,1} B & A_{2,2} B & \cdots & A_{2,n} B \\
\vdots & \vdots & \ddots & \vdots \\
A_{m,1} B & A_{m,2} B & \cdots & A_{m,n} B \\
\end{array} \right)
\]

It is easy to verify that the set of matrices \( D(1) (A) \times D(2) (B) \) do,
in fact, form a representation, called the direct product representation of \( G \).

If \( G \) is the direct product of two groups, \( G' \) and \( G'' \), then

the direct product of a representation of \( G' \) and one of \( G'' \) will

be a representation of \( G \).

**Equivalent representations.**

Let \( D(R) \) be an \( m \)-dimensional representation of \( G \). If \( Q \) is

an \( m \times m \) non-singular matrix independent of any group element, then

the collection of matrices \( Q D(R) Q^{-1} \) will also form a representation

of \( G \). Equivalent representations of \( G \) are not considered as

essentially different; by different representations we shall imply

non-equivalent ones.
Irreducible representations.

If a representation \( D(R) \) is such that

\[
Q^{-1} D(R) Q = \begin{pmatrix} D^{(1)}(R) & M \\ 0 & D^{(2)}(R) \end{pmatrix}
\]

\((M \text{ may or may not be zero.})\)

then it is said to be a reducible representation. It is clear that

\( D^{(1)}(R) \) and \( D^{(2)}(R) \) themselves will be (possibly reducible) representations of \( G \). In the special case of \( M = 0 \), the representation is called completely reducible. Thus complete reducibility implies reducibility, but not always conversely. We now state, without proofs, a few basic results of representation theory.

It can be shown that for finite groups, and for parametric groups whose region of variability of the parameters is bounded and closed — called compact parametric groups — reducibility implies complete reducibility. Thus there are only two kinds of representations of such groups: reducible ones and irreducible ones.

When \( D(R) \) has been brought to the form

\[
Q^{-1} D(R) Q = D^{(1)}(R) + D^{(2)}(R)
\]

then either \( D^{(1)}(R) \) and \( D^{(2)}(R) \) are reducible, or irreducible. If they are both irreducible, then \( D(R) \) is equivalent to the direct sum of two irreducible representations of \( G \). If at least one of \( D^{(1)}(R) \), \( D^{(2)}(R) \) is reducible, say by \( \xi^{(1)} \) and \( \xi^{(2)} \) respectively, then \( D(R) \) can be reduced by \( \xi^{(1)} + \xi^{(2)} \) to the direct sum of a number of representations of \( G \) each of lower dimension than \( D(R) \). Evidently this process must eventually stop; for, at worst, we arrive at a completely diagonal matrix, each element of which will be a 1x1 irreducible representation of \( G \). Thus the representations of these groups are either irreducible, or equivalent to the direct sum of a number of irreducible representations, some of which may be repeated a number
of times. Weyl (Weyl 1931, chapter III, §6) has shown that the irreducible constituents of a reducible representation, \( D(\mathbf{R}) \), are uniquely determined by \( D(\mathbf{R}) \); and, except for order of occurrence, do not depend on the process of the reduction of \( D(\mathbf{R}) \).

For finite groups, the number of inequivalent irreducible representations is equal to the number, \( c \), of classes in the group. Thus we can label the different irreducible representations of a finite group \( G \) by an index \( k \), taking the values \( 1, 2, \ldots, c \). We shall denote the dimension of the irreducible representation \( D^k(\mathbf{R}) \) by \( d_k \).

Compact parametric groups have, in general, an infinite number of different irreducible representations which can be labelled by a set of (discretely or continuously varying) labels \( k; \ D^k(\mathbf{R}) \). We omit a discussion of their labelling, but merely refer to (Kacah 1950) for parametric groups which do not possess commutative invariant subgroups (called semi-simple Lie groups).

Restriction to subgroups.

A representation, \( D(\mathbf{R}) \), of \( G \) furnishes a representation of any subgroup, \( G' \), of \( G \); for this we merely need consider the subset, \( D(\mathbf{R}), \mathbf{R} \in G' \), as the representation matrices of \( G' \). This process will be called restriction of a representation to a subgroup.

Now an irreducible representation of \( G \) need not remain irreducible under restriction to \( G' \). We can now transform the collection \( D(\mathbf{R}), \mathbf{R} \in G \), with the matrix \( Q \) which reduces the representation \( D(\mathbf{R}), \mathbf{R} \in G' \) to the direct sum of a number of irreducible representations of \( G' \). Then the collection \( Q \cdot D(\mathbf{R}) \cdot Q^{-1}, \mathbf{R} \in G \), under restriction to \( G' \), will already appear in reduced form; it will be said to be in standard form with respect to \( G' \). It should be pointed out,
though, that this does not fix the representation $D(R)$, since a reordering of the constituent irreducible representations of $G'$ along the 'diagonal' — which can be achieved by an equivalence transformation with a permutation matrix — will still leave $D(R), R \in G$, in standard form with respect to $G'$.

**Reduction of direct product representation.**

If $D^{k'}(R')$ and $D^{k''}(R'')$ are irreducible representations of $G'$ and $G''$ respectively, then $D^{k'}(R') \times D^{k''}(R'')$ will be an irreducible one of $G' \times G''$. In particular, $G'$ and $G''$ may be the same group $G$.

Now $G' \times G$ contains the subgroup $\{R_1, R\}$, $R \in G$, which is isomorphic to $G$. Thus, under restriction to this subgroup, $D^{k'}(R) \times D^{k''}(R)$ will provide a (in general) reducible representation of $G$. i.e. in general, the direct product representation of two irreducible ones is reducible. Hence there will exist a constant, non-singular matrix, $Q(k', k'')$, which will reduce $D^{k'} \times D^{k''}$:

$$Q(k', k'') \cdot D^{k'} \times D^{k''} \left( Q(k', k'') \right)^{-1} = \sum_{k'''} a_{k'''}^{k', k''} D^{k'''}$$

where $\sum$ means direct summation, and

$$a_{k} D^{k} = \frac{D^{k} + D^{k+1} + \ldots + D^{k}}{a_{k} \text{ times}}$$

If a group $G$ is such that:

(a) the elements $R$ and $R^{-1}$ are in the same class, and
(b) $a_{k}$ is either zero or one, i.e. in the reduction of $D^{k'} \times D^{k''}$ each irreducible representation of $G$ occurs at most once,

then $G$ is called a simply-reducible (S.R.) group (Wigner A). For example $S_n$ for $n = 1, 2, 3, 4$, and $R_3$ are S.R. groups.

**Group summation and group integration.**

Let $G : \{G_1, \ldots, G_i, \ldots, G_g\}$ be a finite group of order $g$; and associate a number $J(G_i)$ with each element $G_i$ of $G$. ($i = 1, 2, \ldots, g$).

We can then form

$$J = \sum_{i=1}^{g} J(G_i) ; \quad (2.3)$$

and it is evident that

$$J = \sum_{i=1}^{g} J(G_i) = \frac{g}{2} \sum_{i=1}^{g} J(G_i, G_k) , \quad (2.4)$$
where \( G_j \) and \( G_k \) are any two fixed elements of \( G \), because in both sums the same finite \((s)\) number of finite numbers \( J(G_j)\) are added, only in different orders.

For parametric groups we can likewise associate a number with each group element. In this case, \( J(R), R \in G \), becomes a function, \( J(\alpha_1, \ldots, \alpha_r) \) of the \( r \) independent, essential parameters uniquely specifying \( R \). We omit the details of the generalization of the concept of summation over \( G \), (2.3), to parametric groups, and merely note that the summation is replaced by a multiple integration:

\[
\sum_{R \in G} J(R) \rightarrow \int_{\alpha} J(\alpha_1, \ldots, \alpha_r) \left( \frac{\partial}{\partial \alpha_1} \cdots \frac{\partial}{\partial \alpha_r} \right) q^m_{\alpha} \, d\alpha_1 \cdots d\alpha_r,
\]

where the integration is over the multi-dimensional region of variability of the \( \alpha \)'s; \( p^s(q^{-1}R) \) is the \( s \) th parameter of \( q^{-1}R \), \( s = 1, 2, \ldots, r \), and the \( q^1 \cdots q^r; \alpha_1 \cdots \alpha_r \) are the parameters of \( q \) and \( R \) respectively.

Thus

\[
\int_{R \in G} J(R) \overset{\text{def}}{=} g^{-1} \int_{\alpha} J(\alpha) \left( \frac{\partial}{\partial \alpha_1} \cdots \frac{\partial}{\partial \alpha_r} \right) q^m_{\alpha} \, d\alpha_1 \cdots d\alpha_r
\]

\[
= \int_{R \in G} J(SR)
\]

(2.5)

where \( S \) is any fixed element of \( G \), and \( g^{-1} \) is a "normalization" constant. The verification is straightforward:

\[
\int_{R \in G} J(SR) = g^{-1} \int_{\alpha} J(\alpha) \left( \frac{\partial}{\partial \alpha_1} \cdots \frac{\partial}{\partial \alpha_r} \right) q^m_{\alpha} \, d\alpha_1 \cdots d\alpha_r
\]

\[
= g^{-1} \int_{X} J(x) \left( \frac{\partial}{\partial x^1} \cdots \frac{\partial}{\partial x^r} \right) p^s(S^{-1}x) \, dx^1 \cdots dx^r,
\]

where \( SR = X \), i.e., \( p^s(SR) = x^s \), and a change of variables, \( \alpha^s = p^s(S^{-1}x) \) has been carried out.
Thus
\[
\int_{\mathcal{R} \in G} J(SR) = \frac{g}{d} \int_{x} \cdots \int_{x} J(x) \cdot \left( \frac{\partial (-p(\eta, x))}{\partial (-x^i \cdots)} \right) d^x \cdot d^x
\]
\[
= \int_{\chi \in G} J(\chi)
\]

use being made of multiplication of Jacobians.

**Orthogonality relations.**

One of the most important and useful results of the theory of group representations is the orthogonality relations of matrix elements of irreducible representations of a group.

Let \( D^k(\mathcal{R}) \) and \( D^{k'}(\mathcal{R}) \) be two unitary, irreducible representations of a finite group, \( G \), of order \( g \). Then the relations
\[
\sum_{i=1}^{g} \delta_{\alpha, \alpha'} \delta_{\beta, \beta'} D^k_{\alpha', \beta'}(G_i) D^{k'}_{\alpha, \beta}(G_i) = \frac{g}{d} \delta_{kk'} \delta_{\alpha \alpha'} \delta_{\beta \beta'}
\]
are called the orthogonality relations. For proof see (Wigner 1931, chapter IX; Murnaghan 1938, chapter III).

In the case of compact parametric groups — for which a convergent integration (2.5) over the group manifold can be defined — the summation in (2.6) should be replaced by integration, and \( g \) by \( \int_{\mathcal{R} \in G} \).

Then, in general,
\[
\int_{\mathcal{R} \in G} D^k_{\alpha', \beta'}(R) D^{k'}_{\alpha, \beta}(R) = \frac{\delta_{kk'}}{d} \delta_{\alpha \alpha'} \delta_{\beta \beta'}
\]
where
\[
\int_{R} J(R) \sum_{i=1}^{g} J(G_i)
\]
for finite groups of order \( g \),
\[
= \int_{\cdots \int_{x}} J(x) \left( \frac{\partial (-p(\eta, x))}{\partial (-x^i \cdots)} \right) d^x \cdots d^x
\]
for compact groups.

**Group characters.**

The quantities \( \chi(R) = \delta_{\rho} D(R) = \sum_{\alpha} D_{\alpha}(R) \) are called the characters of the representation \( D(R) \). Since
\[
\delta_{\rho} D(S RS^{-1}) = \delta_{\rho} D(S) D(R) (D(S))^{-1} = \delta_{\rho} D(R)
\]
we see that to all the elements of a class corresponds the same character in a given representation. Thus \( \chi(R) \) is a class function. Moreover, since \( \text{Sp} \subseteq D(R) \subseteq \text{Sp} \cdot D(R) \), equivalent representations have the same characters. The converse of this holds for the irreducible representations of groups for which reducibility implies complete reducibility. Thus, for such groups, the necessary and sufficient condition for the equivalence of two irreducible representations is the equality of their characters.

Setting \( \beta = \alpha, \beta' = \alpha' \) in eq. (2.7), and summing over \( \alpha \) from 1 to \( d_k \), and over \( \alpha' \) from 1 to \( d_{k'} \), we obtain

\[
\int_{R \in G} \chi^{k'}(R)^{*} \chi^k(R) = q \delta_{kk'}.
\]  

(2.8)

A knowledge of all the \( \chi^{k}(R), (R \in G, \text{all } k) \) makes it, in principle, possible to determine the multiplicity of the constituent irreducible representations of a given one. For this, let \( D(R) \) be a given representation of \( G \), whose \( \chi^{k}(R) \) are known.

Then

\[
\chi(R) = D \cdot D(R) = \sum_{k} a^k R^{k}(R).
\]  

(2.9)

Multiplying both sides of (2.9) by \( \chi^{k'}(R)^{*} \), and making use of (2.8) we obtain:

\[
q a^k_{k'} = \int_{R \in G} \chi^{k'}(R)^{*} \chi^k(R).
\]  

(2.10)

Two-valued representations.

If there corresponds a non-singular, mm matrix, \( D(R) \), to the element \( R \in G \), such that

\[
D(R) D(S) = \mathbf{1} D(T), \text{ whenever } RS = T \text{ in } G,
\]

then the matrices \( D(R), R \in G, \) are said to form a two-valued (or spin) representation of \( G \). It will be shown later that \( H \) admet of (an infinite number of different, irreducible) two-valued representation
whereas — and this will not be proved — \( \tau \) possess only proper representations.

**Carriers.**

Suppose we have a set of \( d^{(m)} \) functions, \( f^{(m)}_i(\xi) \) \((i = 1, 2, \ldots, d^{(m)})\). \( \xi \) may denote the coordinates \( x, y, z \) of a single nucleon, or \( x, y, z, s, \tau \) of a nucleon, or the totality \( x, y, \ldots, \tau \) of an \( n \)-nucleon system. And suppose further that we have a group \( G \) of operations \( \mathcal{R} \) for which \( \xi = \mathcal{R} \xi \), all \( \mathcal{R} \in G \), define a quantity \( \xi \) of the same type as \( \xi \). (Cf. page 134.) We can further define

\[
P^{(m)}_\mathcal{R} f^{(m)}_i(\xi) = \mathcal{R}_i^{(m)} f^{(m)}_i(\xi); \quad P^{(m)}_\mathcal{R} f^{(m)}_i(\xi) \text{ is the symbol of a function, and the}
\]

\( d^{(m)} \) functions \( P^{(m)}_\mathcal{R} f^{(m)}_i(\xi) \) are of quite different functional form from the \( d^{(m)} \) functions \( f^{(m)}_i(\xi) \).

Now given a \( d \)-dimensional representation, \( D^{(m)}(\mathcal{R}) \), of \( G \), we say that \( f^{(m)}_i(\xi) \), \( i = 1, 2, \ldots, d^{(m)} \), are carriers of \( D^{(m)}(\mathcal{R}) \) if

\[
P^{(m)}_\mathcal{R} f^{(m)}_i(\xi) = \sum_{j=1}^{d^{(m)}} f^{(m)}_j(\xi) D_{i j}(\mathcal{R})
\]

and the function \( f^{(m)}_i(\xi) \) is said to transform, under \( P^{(m)}_\mathcal{R} \), according to the \( i \) \text{th} column of \( D^{(m)}(\mathcal{R}) \). The representation \( D^{(m)}(\mathcal{R}) \) may or may not be irreducible. If it is reducible, then there will be a matrix \( \mathcal{Q} \) such that

\[
\mathcal{Q} D^{(m)}(\mathcal{R}) \mathcal{Q}^{-1} = \sum_\mathcal{K} a_\mathcal{K} D^{(m)}(\mathcal{K}),
\]

the \( D^{(m)}(\mathcal{K}) \) being the irreducible representations of \( G \).

Writing (2.14) in the matrix notation,

\[
\begin{pmatrix}
P^{(m)}_\mathcal{R} f^{(m)}_1(\xi), & \cdots & P^{(m)}_\mathcal{R} f^{(m)}_{d^{(m)}}(\xi)
\end{pmatrix} =
\begin{pmatrix}
f^{(m)}_1(\xi), & \cdots & f^{(m)}_{d^{(m)}}(\xi)
\end{pmatrix}
\begin{pmatrix}
D^{(m)}(\mathcal{R})
\end{pmatrix}
\]

makes it easy to see that taking certain linear combinations of the \( f^{(m)}_i(\xi) \), \( i = 1, \ldots, d^{(m)} \), in the form

\[
\begin{pmatrix}
f^{(m)}_1(\xi), & \cdots & f^{(m)}_{d^{(m)}}(\xi)
\end{pmatrix} =
\begin{pmatrix}
f^{(m)}_1(\xi), & \cdots & f^{(m)}_{d^{(m)}}(\xi)
\end{pmatrix}
\begin{pmatrix}
\mathcal{Q}^{-1}
\end{pmatrix}
\]
'partitions' the set of the \( d \) functions into subsets such that the members of a set transform among themselves according to an irreducible constituent of \( D^{(m)}(R) \). We could then label each subset by the labels of the irreducible representation according to which it transforms, the members of the subset being distinguished by the different columns of that representation. In particular, \( D^{(m)}(R) \) may be the direct product of two irreducible representations of \( G \).

In the next chapter we illustrate and amplify the results so far developed for the particular case of \( S_n \) and \( \mathbb{R} \), which are of importance in the applications.
CHAPTER 3.

SPECIALIZATION TO $S_n$ AND $R_3$.

The symmetric group, $S_n$.

The symmetric group, $S$, consists of the totality of the operations $p_\alpha = (\alpha_1 \alpha_2 \ldots \alpha_n)$ as defined in chapter 2, ex. (2). The product, $p_\alpha p_\beta$, of $p_\alpha$ and $p_\beta$ is

$$p_\alpha p_\beta = (\alpha_1 \alpha_2 \ldots \alpha_n)(\beta_1 \beta_2 \ldots \beta_n) = (\alpha_1 \alpha_2 \ldots \alpha_n),$$

which is again a permutation of $1, 2, \ldots, n$. The identity is

and the reciprocal of $p_\alpha$ is $(\alpha_1 \alpha_2 \ldots \alpha_n)$ which can be brought to the form $(1, 2, \ldots, n)$ by a rearrangement of the columns. The associative law for (3.1) is easy to verify.

Interchanges $p_{m-1,m}$.

There are different ways of writing down a given permutation. Thus, it is possible to write any permutation as a product of interchanges of the form $p_{(m-1,m)}$ ($m = 2, 3, \ldots, n$). For cycle notation see next paragraph. E.g.


or

$$(142)(3) = (14)(42)(3)$$


It is easy to see that this can be done for any permutation whatever. Hence $S$ can be generated from the $n-1$ elements $(12), (23), \ldots, (m-1,m), \ldots, (n-1,n)$. Moreover, it is known that when an arbitrary permutation is written in the many possible different ways as a product of general interchanges, then the number of the interchanges is either always even, or always odd; the permutation is
accordingly said to be even or odd. In the above examples, 
(134)(25) is odd, while (142)(3) is even. Thus the elements of $S_n$ fall into two categories: even ones, odd ones. On multiplying every even permutation by, say, (12), we obtain an odd permutation; and likewise multiplication of an odd one by (12) yields an even one. Thus there are $\frac{n!}{2}$ permutations of each kind. A little reflection will convince one that the even permutations form an invariant subgroup of $S_n$.

**Cycle notation.**

One well-known way of writing a permutation is that of writing it as a product of cycles:

$$
\left( \begin{array}{cccc}
1 & 2 & \ldots & n \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\end{array} \right) = (1 \alpha_1 \alpha_2 \ldots)(\ldots) = \ldots
$$

where the cycle $(1 \alpha_1 \alpha_2 \ldots)$ means $1$ is replaced by $\alpha_1$, $\alpha_1$ by $\alpha_2$, until we reach $\alpha_n$ which is replaced by $1$; and then the cycle is closed. If this does not exhaust the permutation, a second cycle can be started with an integer not already in the first, and so on.

E.g., $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 12345 & 54321 \end{pmatrix} = (134)(25)$, or $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 34125 & 12345 \end{pmatrix} = (13)(24)(5)$.

Now the cycle structure of a permutation $A$ and $S A S^{-1}$, where $S$ ranges over $S_n$, is the same. (See, e.g. Murnaghan 1938, page 9.) Hence the elements of a class of $S_n$ have a common cycle structure.

Since the total number of integers in the various cycles must equal $n$, a cycle structure is specified by a partition of $n$. For example, the class of $S_5$ containing the element $(134)(25)$ can be characterized by the partition $5 = 3 + 2 \equiv \{ 3, 2 \}$; whereas the class containing $(13)(25)(4)$ is labelled by the partition $5 = 2 + 2 + 1 \equiv \{ 2^2, 1 \}$. Hence the number of distinct classes of $S_n$ equals $p(n)$, the number of
partitions of \( n \). E.G.:

\[
p(4) = 5 \quad \text{for} \quad 4 = 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1
\]

\[
p(5) = 7 \quad \text{etc.}
\]

Representations of \( S_n \):

As the number of inequivalent, irreducible representations of a group equals the number of its distinct classes, we can label the former as well as the latter, for \( S_n \), by partitions of \( n \).

Each partition \( n = n_1+n_2+\ldots+n_k \), \( n_i > n_{i+1} \), of \( n \) can be drawn as a Young shape, where we put \( n_1 \) boxes in the first row, \( n_2 \) in the second, \( \ldots \), \( n_k \) in the \( k \)th, the first box in each row being directly underneath each other. Thus \( (134)(25) \) is in the class labelled by \( \{3,2\} \rightarrow \begin{array}{c} \text{3} \\ \text{2} \end{array} \), and \( (13)(25)(4) \) in that labelled by \( \{2,1\} \rightarrow \begin{array}{c} \text{2} \\ \text{1} \end{array} \), etc., which we can also use to label inequivalent, irreducible representations of \( S_5 \).

Later on we shall give rules for associating definite inequivalent (standard) irreducible representations of \( S_n \) with different shapes of \( n \). Since \( S_{n-1} \triangleleft S_{n-2} \ldots \) are successive subgroups of \( S_n \) an irreducible representation of \( S_n \) will, in general, be reducible under restriction to \( S_{n-1} \triangleleft S_{n-2} \ldots \). The question arises as to which irreducible representations of \( S_{n-1} \) (labelled, according to our rules, by partitions of \( n-1 \)) appear in a given (standard) irreducible representation of \( S_n \) under restriction to \( S_{n-1} \). Omitting the proof, we give the following result (Jahn A, lecture 2) which is of importance. In the irreducible representation \( \{n_1,\ldots,n_k\} \rightarrow \begin{array}{c} \text{n} \\ \text{k} \end{array} \) of \( S_n \), under restriction to \( S_{n-1} \), only (and every one of) those irreducible representations of \( S_{n-1} \) occur which (according to our
rules) are labelled by (permissible) Young shapes of \(n-1\) obtainable from the original shape.

i.e. \[\{n_1, \ldots, n_k\} \rightarrow \{n_{-1}, \ldots, n_k\} + \ldots + \{n_1, \ldots, n_{k-1}\},\] \((3.2)\)

where \(\rightarrow\) means restriction to \(S_{n-1}\), provided that, in the \(i\)th curly bracket on the right-hand side of \((3.2)\), the condition \(n_i \geq n_{i+1}\) holds. \((i=1,2,\ldots,k).\)

E.g. \(\{4,3,2\} \rightarrow \{2,2,2\} + \{4,2,3\} + \{4,3,2,1\}\).

\begin{align*}
\[
\begin{array}{ccc}
\text{Dimension of } D^{[n_1, \ldots, n_k]} \\
\end{array}
\end{align*}

Let us denote the carriers of \(D^{[n_1, \ldots, n_k]}(R), R \in S_n\), by \(\mathcal{D}_{i}^{[n_1, \ldots, n_k]}\), \(i=1,2,\ldots,d\). Since \(D^{[n_1, \ldots, n_k]}\) is assumed to be in standard form with respect to \(S_{n-1}, S_{n-2}, \ldots, S_{n-k}\), \(\mathcal{D}_{i}^{[n_1, \ldots, n_k]}\) will consist of sets of functions, the members of a set transforming, under \(P_{R}, R \in S_{n-1}\), according to a (standard) irreducible representation of \(S_{n-1}\) contained in \(D^{[n_1, \ldots, n_k]}\). The number of such sets, from \((3.2)\), will equal the number of ways the integer \(n\) can be written in a box of the shape \(\{n_1, \ldots, n_k\}\) such that it occurs at the end of a row and at the foot of a column; i.e. such that deleting it from the original shape yields a (permissible) shape of \(n-1\). The set of functions corresponding to any one of the shapes of \(n-1\) thus obtained will itself consist of sets of functions, one belonging to each shape of \(n-2\) obtainable from the chosen shape of \(n-1\) by deleting the box numbered \(n-1\) in a permissible manner. And so on until only one box is left. This can be labelled in only one way. It then follows that \(d_{\{n_1, \ldots, n_k\}}\) equals the number of ways the integers \(1,2,\ldots,n\) can be written in the boxes of the shape \(\{n_1, \ldots, n_k\}\) of \(n\)
in such a way that the successive deletion of \(n, n-1, \ldots, 2\) leaves a (permissible) shape of \(n-1, n-2, \ldots, 1\). A shape of \(n\) with the integers \(1, 2, \ldots, n\) written in its boxes such that the ascending order is preserved both along the rows, and down the columns will be called a standard tableau of that shape.

Thus \(d\{n, \ldots, n_k\}\) is the number of standard tableau of shape \(\{n, \ldots, n_k\}\). It can be shown (Rutherford 1948, § 10; Boerner 1955, page 101) that this yields

\[
d\{n, \ldots, n_k\} = \frac{n! \prod_{i=1}^{k} (n_i - n_j + j - i)}{\prod_{i=1}^{k} (n_i + k - i)!}.
\] (3.3)

For the case of \(k = 2\), (3.3) reduces to

\[
d\{n, n-n_1\} = \binom{n}{n_1} - \binom{n}{n_1+1}, \quad n_1 \geq \frac{n}{2}.
\] (3.4)

**Yamanouchi symbols.**

From the preceding discussion it is apparent that a member of the carriers of \(D\{n, \ldots, n_k\}\) as well as the column of \(D\{n, \ldots, n_k\}\) to which it belongs, can be labelled by the standard tableau which specifies (by success removal of \(n, n-1, \ldots\)) its transformation property, under \(P_R, R \in S_{n-1}, R \in S_{n-2}, \ldots\), according to particular columns of particular irreducible representations of \(S_{n-1}, S_{n-2}, \ldots\). E.g., a function transforming according to the standard irreducible representation \[
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\] of \(S_5\) and \[
\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}
\] of \(S_4\) of \(S_3\), \[
\begin{array}{c}
\begin{array}{c}
C
\end{array}
\end{array}
\] of \(S_2\) and \[
\begin{array}{c}
\begin{array}{c}
D
\end{array}
\end{array}
\] of \(S_1\) could be labelled by \[
\begin{array}{c}
\begin{array}{c}
E
\end{array}
\end{array}
\] .

Now each standard tableau can be specified by a string of
integers \( \sigma_n, \sigma_{n-1}, \ldots, \sigma_i \), such that in the given standard tableau, \( q \) appears in the \( q \) th row, \( (q = 1, 2, \ldots, n) \). The string of integers \( \sigma_n, \sigma_{n-1}, \ldots, \sigma_1 \) is called a Yamanouchi symbol; it can now be used to label the carrier functions, and the columns, of \( B^{(n, \ldots, n)} \).

\[ \begin{align*}
\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} & \equiv (111) ; \\
\begin{pmatrix} 1 & 2 \end{pmatrix} & \equiv (12) ; \\
\begin{pmatrix} 1 & 3 \end{pmatrix} & \equiv (13) ; \\
\begin{pmatrix} 2 & 3 \end{pmatrix} & \equiv (23) ; \\
\end{align*} \]

and the carrier functions:

\[ \begin{align*}
\varphi^{[3]}_{(111)} ; & \quad \varphi^{[2]}_{(121)} ; \\
\varphi^{[3]}_{(211)} ; & \quad \varphi^{[2]}_{(131)} ; \\
\varphi^{[3]}_{(231)} ; & \quad \varphi^{[3]}_{(321)} .
\end{align*} \]

Young-Yamanouchi representation of \( S_n \).

As already stated, \( S_n \) can be generated from the \( n-1 \) elements \( (1, 2), (2, 3), \ldots, (n-1, n) \); so it is necessary to give only the matrices corresponding to these elements in any representation of \( S_n \). We now give the matrix elements of the matrix corresponding to \((n-1, n)\) in the orthogonal, standard, irreducible representation of \( S_n \), labelling the rows and columns of the matrices by the standard tableaux of the given shape: (Jahn A, chapter 2.)

\[ \begin{align*}
(a) & \left< \begin{array}{c|c} m-1 & m \\ \hline m & m \end{array} \right| \left( \begin{array}{c|c} m-1 & m \\ \hline m & m \end{array} \right) > = +1 , & (3.5a) \\
(b) & \left< \begin{array}{c|c} m-1 & m \\ \hline m & m \end{array} \right| \left( \begin{array}{c|c} m-1 & m \\ \hline m & m \end{array} \right) > = -1 , & (3.5b) \\
(c) & \left< b \begin{array}{c|c} m & m \\ \hline m-1 \end{array} \right| \left( \begin{array}{c|c} m-1 & m \\ \hline m-1 & m \end{array} \right) > = \frac{+1}{a + b - 2} , & (3.5c) \\
(d) & \left< \begin{array}{c|c} m & m \\ \hline m & m \end{array} \right| \left( \begin{array}{c|c} m & m \\ \hline m & m \end{array} \right) > = \frac{-1}{a + b - 2} , & (3.5d)
\end{align*} \]
(e) \[
\left< \begin{array}{c|c}
(m-1, n) & m \\
\hline
m & m
\end{array} \right> = \sqrt{1 - \frac{1}{(a+b-2)^2}}
\]

(3.5e)

(f) All other matrix elements are zero;

where in (3.5a) to (3.5e) the integers \(1, 2, \ldots, m-2, m+1, \ldots, n\) occur in the same places in the shape.

As an example, the orthogonal, standard irreducible representation of \(S_3\) is given below:

\[
S_3 : \quad \begin{array}{cccccc}
& I & (12) & (23) & (31) & (123) & (132) \\
123 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
213 & 1 & 1 & 1 & 1 & 1 & 1 \\
132 & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} \\
321 & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} \\
\end{array}
\]

It will be seen that the two matrices corresponding to \(I\) and \((12)\) - i.e. \(S_2\) - in, say, the representation \(\Box\), already appear in reduced form, being the direct sum of the identity and the anti-symmetric representations of \(S_3\):

\[
\Box \rightarrow \square + \emptyset
\]
The rotation group, \( R_3 \).

A convenient way of parametrizing \( R_3 \) (see chapter 2, ex. (4) and the discussion following) is by means of the Euler angles, \( \alpha, \beta, \gamma \). For this we make use of the homomorphism between \( R_3 \) and \( U_2 \), the group of all \( 2 \times 2 \) unitary, unimodular (i.e. of determinant \( +1 \)) matrices.

Let

\[
U_2(\alpha, \beta, \gamma) = \begin{pmatrix}
\cos \frac{\alpha}{2} e^{\frac{i(\beta - \gamma)}{2}} & \frac{i}{2} e^{\frac{i(\beta + \gamma)}{2}} \\
-\sin \frac{\alpha}{2} e^{\frac{i(\beta - \gamma)}{2}} & \cos \frac{\alpha}{2} e^{\frac{i(\beta + \gamma)}{2}}
\end{pmatrix}
\]

(3.6)

where the range of variability of \( \alpha, \beta, \gamma \) is \( 0 \leq \alpha \leq 4\pi \), \( 0 \leq \beta \leq 2\pi \), \( 0 \leq \gamma \leq 4\pi \).

Then

\[
S = U_2(\alpha, \beta, \gamma) \Gamma \cdot \mathcal{S} \left( U_2(\alpha, \beta, \gamma)^{-1} \right)
\]

(3.7)

where \( \Gamma \equiv (x, y, z) \) are the cartesian coordinates of a point and

\[
\mathcal{S} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

(3.8)

are the standard Pauli matrices, (cf. chapter 4.) defines a transformation:

\[
\Gamma \cdot \mathcal{S} \rightarrow S = U_2(\alpha, \beta, \gamma) \Gamma \cdot \mathcal{S} U_2(\alpha, \beta, \gamma)^{-1}
\]

of the traceless, hermitian matrix \( \Gamma \cdot \mathcal{S} \),

\[
\Gamma \cdot \mathcal{S} = \begin{pmatrix}
x & y - iz \\
y + iz & -x
\end{pmatrix}
\]

by \( U_2(\alpha, \beta, \gamma) \). Hence \( S \) will also be traceless and hermitian, and can be written as \( S = \Gamma' \cdot \mathcal{S} \), with \( \Gamma' = (x', y', z') \) real, since \( \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \) form a complete base for all \( 2 \times 2 \) traceless matrices; the reality of \( \Gamma' \) depending on the hermitian characters of \( S \).

Hence (3.7) becomes:

\[
\Gamma' \cdot \mathcal{S} = U_2(\alpha, \beta, \gamma) \Gamma \cdot \mathcal{S} \left( U_2(\alpha, \beta, \gamma)^{-1} \right)
\]

(3.9)

defining a real, linear transformation \( \Gamma \rightarrow \Gamma' \) depending on \( \alpha, \beta, \gamma \).
Taking the determinant of both sides of (3.9), we obtain:

\[-(x'^2 + y'^2 + z'^2) = -(x^2 + y^2 + z^2),\]

showing that \( r \rightarrow r' = R_3(\alpha \beta \gamma) r \). Eq. (3.9) can be written in the form:

\[
\begin{pmatrix}
  z' & x'-iy' \\
  x'+iy' & z'
\end{pmatrix} =
\begin{pmatrix}
  \cos \beta e^{i\alpha} & \sin \beta e^{i\alpha} \\
  -\sin \beta e^{-i\alpha} & \cos \beta e^{-i\alpha}
\end{pmatrix}
\begin{pmatrix}
  z & x-iw' \\
  x+iw' & z
\end{pmatrix}
\begin{pmatrix}
  \cos \beta e^{i\alpha} & -\sin \beta e^{i\alpha} \\
  \sin \beta e^{-i\alpha} & \cos \beta e^{-i\alpha}
\end{pmatrix}
\]

which, after lengthy algebra, and taking real and complex parts, yields:

\[
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix} =
\begin{pmatrix}
  \cos \beta \cos \gamma + \sin \beta \sin \gamma & \sin \beta \cos \gamma - \cos \beta \sin \gamma & 0 \\
  -\sin \beta \cos \gamma + \cos \beta \sin \gamma & \cos \beta \cos \gamma + \sin \beta \sin \gamma & 0 \\
  \sin \gamma & \sin \gamma & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\]

(3.10)

Now, after performing a rotation \( \gamma \) about the \( z \)-axis, followed by one of \( \beta \) about \( y \)-axis and finally \( \alpha \) about the \( z \)-axis, the coordinates of a point \( P \) in the two frames of reference will be related by:

\[
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix} =
\begin{pmatrix}
  \cos \alpha \cos \beta & \sin \alpha \sin \beta & 0 \\
  \sin \alpha \cos \beta & \cos \alpha \sin \beta & 0 \\
  \sin \beta & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix}
\]

(3.11)

which, again after some tedious matrix multiplication reduces to (3.10). The sense of the rotations in (3.11) is always that of a right handed screw in relation to the cyclic order \( x \) to \( y \), \( y \) to \( z \), \( z \) to \( x \). Thus the rotation specified by (3.11) coincides with that given by \( \frac{1}{2} (\alpha \beta \gamma) \), eq. (3.9).

As \( +\frac{1}{2} (\alpha \beta \gamma) \) and \( -\frac{1}{2} (\alpha \beta \gamma) \) will generate the same \( R_3(\alpha \beta \gamma) \), it is easy to see that (disregarding the \( \tau \) sign) there is a one-one
correspondence between \( u_2 \) and \( R_3 \)

One must, of course, still show that under this correspondence multiplication is preserved. Since

\[
\begin{align*}
    u_2(\alpha \beta \gamma) &= u_2(\alpha \beta \gamma) u_2(0 \beta 0) u_2(0 \gamma),
    \quad \text{for } \alpha \in U(1), \beta \in U(2), \gamma \in U(3),
\end{align*}
\]

we merely need to show this for

\[
\begin{align*}
    u_2(0 \beta 0) u_2(0 \gamma) &\leftrightarrow R_3(0 \beta 0) R_3(0 \gamma),
\end{align*}
\]

and

\[
\begin{align*}
    R_3(0 \beta 0) R_3(0 \gamma) &\leftrightarrow R_3(0 \beta 0) R_3(0 \gamma).
\end{align*}
\]

Both these can easily be verified by direct matrix multiplication. But we also verify this directly from the "defining" eq. (3.9):

thus

\[
\begin{align*}
    (a) \quad \tau^t \tau &= (R(1) \tau)^t \tau = u(1) \tau \cdot \tau (u(1))^{-1},
    \quad \text{from (b)}
\end{align*}
\]

where \( R(1) \) means \( R_3(0 \beta \gamma) \), and likewise for \( R(2) \), \( u(1) \), \( u(2) \).

Now let \( R(3) \) correspond to \( u(1)u(2) \): then

\[
\begin{align*}
    (R(3) \tau)^t \tau &= u(1) u(2) \tau \cdot \tau (u(1) u(2))^{-1}
    \quad \text{from (b)}
\end{align*}
\]

Hence

\[
\begin{align*}
    \pm u(1) u(2) \leftrightarrow R(1) R(2).
\end{align*}
\]

Representations of \( R_3 \):

It can be shown (e.g. Wigner 1931, chapter XV) that there is an enumerable infinity of unitary, irreducible representations of \( u_2 \), which can be labelled by \( j \) where \( j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \); and that those labelled by integer \( j \) are one-valued, those labelled by half-integer \( j \) are two-valued representations of \( R_3 \).

The unitary, irreducible (one or two valued) representation,

\[
\begin{align*}
    D_j(\alpha, \beta, \gamma), \quad \text{of } R \text{ is } 2j + 1 \text{ dimensional, and its matrix elements are given by: (Wigner: loc. cit)}
\end{align*}
\]

\[
\begin{align*}
    D_{m,n}^j(\alpha, \beta, \gamma) &= e^{im \alpha} \sum_{r} \frac{(-1)^{r} \sqrt{(l+m)!(l-m)! (l+n)!(l-n)!}}{(l+n-r)! (l-m-r)! r! (r+m-n)!} c_{s} \frac{2^{j+n-m-2r} \beta^{2j+n-m-2r}}{2j+n-m} \cdot e^{in \gamma}
    \quad \text{for } m, n = 0, 1, 2, \ldots, j,
\end{align*}
\]

(3.12)
where the rows and columns of \( D(\alpha, \beta, \gamma) \) are labelled by \( m \) and \( n \) respectively, \((m, n = \pm j, j = 1, 2, \ldots, -j)\)

From (3.12), the matrix representing a rotation \( \gamma \) about the \( z \)-axis will be: (only the term \( r = 0 \) survives in the summation.)

\[
D_{m,n}^j(\alpha, \beta, \gamma) = \hat{e}^{in\gamma} \delta_{m,n} = \left( \begin{array}{ccc}
\hat{e}^{ij\gamma} & \hat{e}^{i(j-1)\gamma} & 0 \\
\hat{e}^{i(j-1)\gamma} & \hat{e}^{ij\gamma} & 0 \\
0 & \hat{e}^{i(j-1)\gamma} & \hat{e}^{ij\gamma}
\end{array} \right)
\]

(3.13)

It will be seen that \( D_{m,n}^j(\alpha, \beta, \gamma + 2\pi) = \hat{e}^{2\pi i} D_{m,n}^j(\alpha, \beta, \gamma) \). Since \( u_2(\alpha, \beta, \gamma + 2\pi) = -u_2(\alpha, \beta, \gamma) \) the single-valued representations of \( R_3 \) are those (which all the same matrix to \( I u_2(\alpha, \beta, \gamma) \)) for which \( \hat{e}^{2\pi i} = 1 \), i.e. \( n \) (and hence \( j \)) integer. The half-integer representations satisfy

\[
D^j(R(\phi)) D^j(R(\psi)) = \hat{e}^{j(\phi - \psi)} D^j(R(\phi R(\psi)), \quad j = \frac{1}{2}, \frac{3}{2}, \ldots
\]

The characters of \( D^j(\alpha, \beta, \gamma) \).

Since all rotations of equal angle about any axis, (say \( \phi \) about the \( z \)-axis) — and every element of \( R_3 \) is effectively a rotation about some axis — constitute a class of \( R_3 \), and since character is a class function, we have: (using (3.13).)

\[
\chi^j(\phi) = \chi^j(0,0,\phi) = \sum_{m=-j}^{j} D_{m,m}^j(0,0,\phi) = \sum_{m=-j}^{j} \hat{e}^{im\phi}
\]

\[
= \frac{\sin((j+\frac{1}{2})\phi)}{\sin\frac{\phi}{2}}
\]

(3.14)
where we have denoted a class of $R_3$ by the angle of rotation, $\phi$.

\[ D^{\frac{1}{2}}(\alpha, \beta, \gamma) \text{ and } D^{\frac{1}{2}}(\alpha, \beta, \gamma). \]

From eq. (3.13) we write down the matrix elements of $D^{\frac{1}{2}}(\alpha, \beta, \gamma)$:

\[
\begin{align*}
D^{\frac{1}{2}}_{\frac{1}{2}, \frac{1}{2}}(\alpha, \beta, \gamma) &= e^{i\frac{\alpha}{2}} \cos \frac{\beta}{2} e^{i\frac{\gamma}{2}}, \\
D^{\frac{1}{2}}_{\frac{1}{2}, -\frac{1}{2}}(\alpha, \beta, \gamma) &= e^{i\frac{\alpha}{2}} \sin \frac{\beta}{2} e^{i\frac{\gamma}{2}}, \\
D^{\frac{1}{2}}_{-\frac{1}{2}, \frac{1}{2}}(\alpha, \beta, \gamma) &= -e^{-i\frac{\alpha}{2}} \sin \frac{\beta}{2} e^{i\frac{\gamma}{2}}, \\
D^{\frac{1}{2}}_{-\frac{1}{2}, -\frac{1}{2}}(\alpha, \beta, \gamma) &= e^{-i\frac{\alpha}{2}} \cos \frac{\beta}{2} e^{-i\frac{\gamma}{2}};
\end{align*}
\]

which is exactly the matrix $u_2(\alpha, \beta, \gamma)$, eq. (5.6), i.e. the matrices $u_2(\alpha, \beta, \gamma)$ form a unitary (unimodular) irreducible representation of themselves.

But

\[
D^{\frac{1}{2}}(\alpha, \beta, \gamma) = e^{i\frac{\alpha}{2}} \begin{pmatrix}
\frac{\alpha}{2} & \frac{\beta}{2} & \frac{\gamma}{2} \\
\frac{\cos \beta}{2} & 1 - \cos \beta & -i \sin \beta \\
\frac{i \sin \beta}{2} & -i \sin \beta & 1 + \cos \beta
\end{pmatrix}
\]

is not identical with $R_3(\alpha, \beta, \gamma)$, though they are equivalent.

It should be mentioned that all the representation matrices agree with the standard phase convention of Condon and Shortley (Condon 1955, chapter III), (Rose 1955, §8), Racah, Jahn and others.

**Carriers of $D^{\frac{1}{2}}(\alpha, \beta, \gamma)$**.

Any set of $2^{j+1}$ functions, $f^{j}_{m}(\xi)$, $m = j, j - 1, \ldots, -j$, for which

\[
P_{R} f^{j}_{m}(\xi) = \sum_{m = -j}^{j} f^{j}_{m}(\xi) D^{j}_{m, m}(R), \quad R \in R_3.
\]

holds constitute carriers of $D^{\frac{1}{2}}(\alpha, \beta, \gamma)$. 
Let \( f_{m_1}^J(\xi) \) be a set of carriers of \( D_{m_1}^J \), and \( g_{m_2}^{J_2}(\xi) \) that of \( D_{m_2}^{J_2} \). Then we have:

\[
\left( f_{m_1}^J(\xi), g_{m_2}^{J_2}(\xi) \right) = \left( P_R f_{m_1}^J(\xi), P_R g_{m_2}^{J_2}(\xi) \right)
\]

\[
= \sum_{m_1, m_2} \left( f_{m_1}^J(\xi), g_{m_2}^{J_2}(\xi) \right) D_{m_1}^J(R) D_{m_2}^{J_2}(R)
\]

Integrating over \( R \):

\[
g \left( f_{m_1}^J, g_{m_2}^{J_2} \right) = \sum_{m_1, m_2} \left( f_{m_1}^J, g_{m_2}^{J_2} \right) \delta_{J_1, J_2} \delta_{m_1, m_2} \frac{g}{2J_1 + 1}
\]

i.e.,

\[
\left( f_{m_1}^J(\xi), g_{m_2}^{J_2}(\xi) \right) = \delta_{J_1, J_2} \delta_{m_1, m_2} C(J_1),
\]

where \( C(J_1) \) is independent of \( m_1 \) and \( m_2 \). Thus functions belonging to different \( D^{J_1}(R) \), or to different columns of the same \( D^{J_1}(R) \) are orthogonal; and the scalar product of two (the same, or different) functions belonging to the same column of the same \( D^{J_1}(R) \) is independent of \( m \).

The orthonormalized surface harmonics, \( Y_{m}^{J}(\theta, \varphi) \):

\[
Y_{m}^{J}(\theta, \varphi) = \frac{(-1)^{J+m}}{2^J J!} \sqrt{\frac{(2J+1)(J-m)!}{2\pi (J+m)!}} \sin^m \theta e^{im\varphi} \left( \frac{d}{d \cos \theta} \right)^m \sin \theta,
\]

are examples of a carrier set for integral \( J \). For, firstly, since

\[
(R(\theta_0, \gamma))^{-1}(\theta, \varphi) \rightarrow \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix} = \begin{pmatrix} \sin \theta \cos (\varphi + \gamma) \\ \sin \theta \sin (\varphi + \gamma) \\ \cos \theta \end{pmatrix}
\]
then, from (3.18),
$$P \{ 0, o, \gamma \} \ Y^j_m (0, \varphi) = Y^j_m (0, \varphi) = Y^j_m (0, \varphi) \cdot e^{im\gamma},$$
and from (3.13)
$$= \sum_{m'=-j}^j Y^j_{m'} (0, \varphi) \ D^{ij}_{m'm} (0, 0, \gamma).$$
And secondly,
$$P \{ 0, \beta, 0 \} \ Y^j_m (0, \varphi) = \sum_{m'=-j}^j Y^j_{m'} (0, \varphi) \ D^{ij}_{m'm} (\beta),$$
where, due to (3.19), only \( Y \)'s of the same \( j \) appear in the expansion
of \( P \{ 0, \beta, 0 \} \ Y^j_m (0, \varphi) \) in terms of surface harmonics. Now a matrix
commuting with \( D^{ij} (0, 0, \gamma) \) must be a diagonal matrix; and, on con-
sidering the elements of the row labelled by \( m = 0 \) of the products,
either way, of that diagonal matrix and \( D^{ij} (\beta) \), it will be seen
that the diagonal matrix must be a multiple of the identity.
Hence, by Schur's lemma, the representation carried by \( Y^j_m \),
m = \( j, \ldots, -j \), must be irreducible. Eq. (3.19) shows that it
must also be unitary; hence the result.

\underline{Reduction of \( D^j (\alpha \beta \gamma) \times D^j (\alpha \beta \gamma) \).}

Unless \( j_1 \) or \( j_2 \) (or both) is zero, \( D^j (\alpha \beta \gamma) \times D^j (\alpha \beta \gamma) \) will be
reducible.

There must exist a matrix, \( S (j_1 j_2) \), which reduces \( D^j (\alpha j_2) \times D^j (\alpha j_2) \):
$$S (j_1 j_2) \ D^j (\alpha j_2) \times D^j (\alpha j_2) (S (j_1 j_2))^{-1} = \sum_{J=0, \pm 1/2, \pm 3/2, \ldots} a (j_1 j_2) \ D^J.$$  \( \{2.20\} \)

Setting \( \alpha = \beta = \varphi \), \( \gamma = \varphi \), and taking the spur of both sides of (3.20):
$$\sum_J a (j_1 j_2) \chi^J (\varphi) = \chi^j_1 (\varphi) \chi^j_2 (\varphi) \sum_J a (j_1 j_2) \ D^J \Rightarrow \sum_J a (j_1 j_2) \chi^J (\varphi) \sum_J \sum_{\lambda=J} e^{i\lambda \varphi} \chi^J (\varphi) = \sum_J \sum_{\lambda=J} e^{i\lambda \varphi} \chi^J (\varphi).$$
Hence, due to the orthogonality relations, (2.8), for the characters,

\[ a_{J-M} = \begin{cases} 1 & \text{for } J = J_1, J_1 + 1, \ldots, |J_1 - J_2|, \\ 0 & \text{otherwise}. \end{cases} \]

As the inverse of any element of \( R_3 \) (say a rotation \( \phi \) about an axis \( \mathbf{1} \)) is in the same class as that element, this shows that \( R_3 \) is a simply reducible group.

The rows and columns of \( S^{(J_1, J_2)} \) can be labelled by \( J, M \) and \( m_1, m_2 \), respectively, such that the \( J, M \) row lies above \( J', M' \) one if \( J > J' \), or \( J = J', M > M' \); and the \( m_1, m_2 \) column lies to the left of the \( m_1', m_2' \) one if \( m_1 + m_2 = m_1' + m_2' \), or \( m_1 + m_2 = m_1' + m_2' \), \( m > m' \).

**Wigner coefficients.**

An alternative way of writing \( S^{(J_1, J_2)} \) is the standard Condon and Shortley (Direct) notation:

\[ S^{(J_1, J_2)}_{J, M; m_1, m_2} = (J_1, J_2, m_1, m_2 | J, M). \]

As Wigner (Wigner 1931, chapter XVII) was the first to give the general expression

\[
(J_1, J_2, m_1, m_2 | J, J) = \frac{\delta_{M, M}}{\sqrt{(J_1 + J_2)! (J_1 - J_2)!}} \Delta(J_1, J_2, J) \left( \frac{(J_1 + J_2)! (J_1 - J_2)!}{(J_1 + m_1)!(J_1 - m_1)!(J_2 + m_2)!(J_2 - m_2)!} \right) \times \frac{\sum_{r} (-1)^{J_1 + J_2 + r} (J_1 + J_2 - r)! (J_1 - m_1 + r)! (J_2 - m_2 + r)!}{(J_1 - J_2 - r)! (J_1 - r)! (J_2 + r)!}, \tag{3.21}
\]

where

\[ \Delta(a b c) = \Delta(b a c) = \Delta(c a b) = \left( \frac{(a + b + c)!}{(a - b + c)! (a + b - c)! (a - b + c)!} \right)^{1/2} \]

and

\[ [J] = 2J + 1. \tag{3.23} \]

we shall call them Wigner coefficients.

Racah (Racah 1942) has given an equivalent, more convenient form,
\[
\langle \ell_2 m_1 m_2 | J_2 J_2 J_M \rangle = \delta_{M, m_1 m_2} A(J_2 J_2) \sqrt{\frac{2J_2}{(J_2 + M)!(J_2 - M)!}} (J_2 m_1) (J_2 m_2) \times \\
\sum_r (-1)^r \frac{r!}{(J_2 + r - J_M)! (J_2 - r - J_M)!} (J_2 m_1 + r) (J_2 m_2 - r) \quad (3.24)
\]

As $D(J_2)$ has unitary, and (3.21) or (3.24) show that the phase has been so chosen that $S(J_2 J_2)$ is real, it follows that $S(J_2 J_2)$ is a real, $(2J_2 + 1)(2J_2 + 1)$ dimensional orthogonal matrix:

\[
S(J_2 J_2) S(J_2 J_2) = \tilde{S}(J_2 J_2) S(J_2 J_2) = I_{(2J_2 + 1)(2J_2 + 1)} \quad (3.25)
\]

From eqs. (3.21) — or (3.24) — and (3.25) follow various useful symmetry and orthogonality relations for the Wigner coefficients, which we summarise below:

\[
\langle \ell_2 m_1 m_2 | J_2 J_2 J_M \rangle = (-1)^{J_2 + J - J'} \langle J_2 J_2 m_1 m_2 | J_2 J_2 J_M \rangle \quad (3.26a)
\]

\[
= (-1)^{J_2 - J'} \langle J_2 m_1 m_2 | J_2 m_1 m_2 \rangle \quad (3.26b)
\]

\[
= (-1)^{J_2 - J} \sqrt{\frac{(J_2 + J - M)!}{(J_2 - J - M)!}} \langle J_2 m_1 m_2 | J_2 m_1 m_2 \rangle \quad (3.26c)
\]

and

\[
\sum_{m_1} \langle J_2 m_1 M - M | J_2 J_2 J_M \rangle (J_2 m_1 M' - M | J_2 J_2 J_M' \rangle = \delta_{M, M'} \quad (3.27a)
\]

\[
\sum_{J_2} \langle J_2 m_1 m_2 | J_2 J_2 J_M m_2 \rangle (J_2 m_1 m_2' | J_2 J_2 J M' \rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'} \quad (3.27b)
\]

Again, from (3.24) we can immediately calculate:

\[
\langle 0 0 m_1 0 | J_2 J_2 J_M \rangle = \delta_{J_2 J_2} \delta_{m_1 m_1} \quad (3.28)
\]

\[
\langle J_2 J_2 J_2 | J_2 J_2 J_2 J_2 J_2 \rangle = 1 \quad (3.29)
\]
Eq. (3.28), together with (3.26a), yields:

\[
\begin{align*}
(j_1 j_1 m_1 m_1 | j_1 j_1 0 0) &= \frac{(-1)^{j_1 - m_1}}{\sqrt{|j_1|}}. \\
\end{align*}
\]  

(3.30)

From (3.24) and (3.26a) we obtain:

\[
\begin{align*}
(j_1 j_2 0 0 | j_1 j_2 l 0) &= \frac{(-1)^{j_2 + l} \Delta(j_1 j_2) g! \sqrt{|j_1|}}{(g - j_1)! (g - j_2)! (g - l)!}, \\
&= 0 \quad j_{1+2} = \text{odd}. \\
\end{align*}
\]  

(3.31)

Finally, for completeness, we give the relation

\[
\sum_{j_{12} j'_{12} m_{12} m'_{12}} (j_1 j_2 m_1 m_2 | j_{12} j'_{12} m'_{12}) D_{j_1 m_{11}}^{j_1} D_{j_2 m_{21}}^{j_2} = \delta_{jj'} D_{m_1 m_2}^{j'},
\]  

(3.32)

which follows from the definition of \( S(j_1 j_2) \).

All other symmetry and orthogonality relations of the Wigner coefficients can be obtained from successive combination of some or all of the above ones.

**Coupling of angular momenta.**

The Wigner coefficients are of immense use in applications. Thus, when we wish to couple two (kinematically separate, but not necessarily non-interacting) subsystems, each with sharp angular momentum, to obtain a whole system with sharp total angular momentum, we proceed as follows:

Let \( \psi_{m_1}^{j_1}(\xi_1) \), \( \psi_{m_2}^{j_2}(\xi_2) \) be the \( m_1 \), \( m_2 \) members, respectively, of two sets of \( 2_{j_1 + 1}, 2_{j_2 + 1} \) functions which describe the subsystems (collective coordinates \( \xi_1, \xi_2 \)) with angular momenta \( j_1, j_2 \); transforming, (see chapter 1.) under \( R \), \( R \in \mathbb{R} \), according to \( D_{j_1}^{j_1}(R), D_{j_2}^{j_2}(R) \).
Then the \((2j_1+1)(2j_2+1)\) functions 
\[ \psi_{m_1}^{j_1}(\xi_1) \psi_{m_2}^{j_2}(\xi_2), \quad (m_1 = j_1 \cdots -j_1; m_2 = j_2 \cdots -j_2), \]
will be carriers of the (reducible) representation \(D_1 \times D_2^1\),
i.e.,
\[ P_R \psi_{m_1}^{j_1}(\xi_1) \psi_{m_2}^{j_2}(\xi_2) = P_R \psi_{m_1}^{j_1}(\xi_1), P_R \psi_{m_2}^{j_2}(\xi_2), \]

\[ = \sum_{m_1',m_2'} \psi_{m_1'}^{j_1}(\xi_1) \psi_{m_2'}^{j_2}(\xi_2) D_{m_1'}^{j_1}(R) D_{m_2'}^{j_2}(R), \quad (3.33) \]

\[ = \sum_{m_1',m_2'} \psi_{m_1'}^{j_1} \psi_{m_2'}^{j_2} (D_1 \times D_2^1)_{m_1',m_2'}^{m_1,m_2}, \quad (3.34) \]

so that the product functions do not describe a total (coupled) system with unique angular momentum.

By either writing \((3.34)\) as a matrix equation and remembering that \(S(j_1,j_2)\) reduces \(D_1 \times D_2^1\), or multiplying both sides of \((3.33)\) by \((j_1 j_2 M_1 M_2 | j_1 j_2 J M)\), summing over \(m_1, m_2\), and using \((3.32)\) we see that the linear combinations
\[ \psi_{j_1 j_2}^{(j_1 j_2)}(\xi_1, \xi_2) = \sum_{m_1} \psi_{m_1}^{j_1}(\xi_1) \psi_{m_2}^{j_2}(\xi_2) (j_1 j_2 m_1 M_1 m_2 M_2 | j_1 j_2 J M), \quad (3.35) \]

transform as:
\[ P_R \psi_{j_1 j_2}^{(j_1 j_2)}(\xi_1, \xi_2) = \sum_{M_1'} \psi_{j_1 j_2}^{(j_1 j_2)}(\xi_1, \xi_2) D_{M_1'}^{j_1}(R) D_{M_2'}^{j_2}(R), \quad (3.36) \]

To solve \((3.35)\) for \( \psi_{m_1}^{j_1}(\xi_1) \psi_{m_2}^{j_2}(\xi_2) \), we multiply both sides by
\[
\psi_{j_1j_2}(s_1) \psi_{j_3}(s_3) = \sum_{j} \langle j | j_1j_2 | j_j_1 \rangle \langle j_3 | j_1 | j \rangle \psi_{j}(s_3).
\]

It should be noted that the total number of 'coupled' functions, \( \psi_{j}(s_3) \), is, of course, \((2j+1)(2j'+1)\):

\[
\sum_{j=j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1).
\]

Eqs. (5.35) and (5.37) effectively summarize the problem of coupling of two angular momenta.

**Racah coefficients.**

When there are more than two subsystems, (such as, for example, the orbital angular momenta of more than two electrons in atomic \( L-S \) coupling) then we are faced with a new problem: the complete description of the angular momentum of the total system requires a specification of the order of the couplings of the angular momenta of the several constituents to a given total angular momentum of the whole system.

Thus, in order to arrive at total angular momentum \( J \), of a system composed of three subsystems, of angular momenta \( j_1, j_2 \), and \( j_3 \), one could either couple \( j_1 \) and \( j_2 \) to obtain \( j_{12} \), then couple \( j_{12} \) to \( j_3 \) (to a total \( J \)), or one could couple \( j_1 \) to \( j_{23} \), the result of coupling \( j_2 \) to \( j_3 \).

On the first scheme of coupling we have:

\[
\psi_{j_1j_2j_3}(L_{12}, L_{23}) = \sum_{j_{12}} \langle j_{12} | j_1 \rangle \langle j_2 \rangle \psi_{j_{12}}(L_{12}) \psi_{j_3}(L_{23})
\]
whereas, on the second scheme of coupling:

\[
\psi_{1/2}(l_{13}; l_{23}) = \sum_{m_2} \left( \begin{array}{c} J_1 \ J_2 \ J_3 \ J_4 \\ m_1 \ m_2 \ m_3 \ m_4 \end{array} \right) \psi_{1/2}^{J_1} \psi_{1/2}^{J_2} \psi_{1/2}^{J_3} \psi_{1/2}^{J_4}
\]

the total number of functions, \((2J_1+1)(2J_2+1)(2J_3+1)\), being preserved in both cases.

By a double application of (3.37) we have, from (3.38):

\[
\psi_{m_1}^{J_1} \psi_{m_2}^{J_2} \psi_{m_3}^{J_3} = \sum_{J_4, \ J_4'} \left( \begin{array}{c} J_1 \ J_2 \ J_3 \ J_4 \\ m_1 \ m_2 \ m_3 \ m_4 \end{array} \right) \psi_{J_4}^{J_1} \psi_{J_4'}^{J_2} \psi_{J_4'}^{J_3} \psi_{m_4}^{J_4'}
\]

Inserting this in (3.39), yields:

\[
\psi_{1/2}(l_{13}; l_{23}) = \sum_{m_2, m_3} \left( \begin{array}{c} J_1 \ J_2 \ J_3 \ J_4 \\ m_1 \ m_2 \ m_3 \ m_4 \end{array} \right) \psi_{J_4'}^{J_1} \psi_{J_4'}^{J_2} \psi_{m_4}^{J_4'}
\]

The orthogonality of carriers, eq. (3.17), shows that the coefficient of \(\psi_{E(l_{13}; l_{23})J'}^{M'}\) involves a \(\delta_{JJ'} \delta_{MM'}\). Hence we can carry out the summation over \(J'\) and \(M'\) in (3.40), and write the result in the standard Dirac notation (cf. chapter 4.)

\[
\left| \frac{l_{13}}{l_{12}} \right| = \sum_{J, M} \left| \left[ \frac{l_{13}}{l_{12}} \right] \right| \left< \frac{l_{13}}{l_{12}} \right| \left| \frac{l_{13}}{l_{12}} \right>
\]

where the transformation coefficient, depending on the six \(J's\), is:

\[
\left< \frac{l_{13}}{l_{12}} \right| \left| \frac{l_{13}}{l_{12}} \right> = \sum_{m_1, m_2, m_3} \left( \begin{array}{c} J_1 \ J_2 \ J_3 \ J_4 \\ m_1 \ m_2 \ m_3 \ m_4 \end{array} \right) \psi_{m_1}^{J_1} \psi_{m_2}^{J_2} \psi_{m_3}^{J_3} \psi_{m_4}^{J_4}
\]

This complicated sum over products of four Wigner coefficients has
been carried out by Racah who gives (Racah 1942 b)

$$\left< [J_{1} \ell_{3} \ell_{1}] \left| J_{1}(J_{1} \ell_{3} \ell_{1}) J_{23} \right> \right> = \sqrt{\frac{[J_{1} \ell_{3} \ell_{1}] - 1}{2 \ell_{3} \ell_{1}}} \ W(J_{1} \ell_{3} \ell_{1} ; J_{23} \ell_{23} J_{23} J_{23} \ell_{23}) \quad (3.42)$$

where the explicit expression for $W(J_{1} \ell_{3} \ell_{1} ; J_{23} \ell_{23})$, called a Racah coefficient, is

$$W(J_{1} \ell_{3} \ell_{1} ; J_{23} \ell_{23}) = A(J_{1} \ell_{3} \ell_{1}) A(J_{23} \ell_{23}) A(J_{1} \ell_{3} \ell_{1}) X \times \sum_{\ell} \frac{(-1)^{r} (J_{1} \ell_{3} \ell_{1} + J_{23} \ell_{23} + J - r)! \left\{ (J_{1} \ell_{3} \ell_{1} + J_{23} \ell_{23} - J + r)! \right\}^{-1}}{J_{1} \ell_{3} \ell_{1} + J_{23} \ell_{23} + J - r)! \left\{ (J_{1} \ell_{3} \ell_{1} + J_{23} \ell_{23} - J + r)! \right\}^{-1}} \quad (3.43)$$

An equivalent expression, given by (Biedenharn 1952) is:

$$W(J_{1} \ell_{3} \ell_{1} ; J_{23} \ell_{23}) = A(J_{1} \ell_{3} \ell_{1}) A(J_{23} \ell_{23}) A(J_{1} \ell_{3} \ell_{1}) X \times \sum_{\ell} \frac{(-1)^{r} (J_{1} \ell_{3} \ell_{1} + J_{23} \ell_{23} + J - r)! \left\{ (J_{1} \ell_{3} \ell_{1} + J_{23} \ell_{23} - J + r)! \right\}^{-1}}{J_{1} \ell_{3} \ell_{1} + J_{23} \ell_{23} + J - r)! \left\{ (J_{1} \ell_{3} \ell_{1} + J_{23} \ell_{23} - J + r)! \right\}^{-1}} \quad (3.44)$$

The Racah coefficients have a number of symmetry and orthogonality relations which can be obtained from either of the eqs. (3.41), (3.43) or (3.44). We summarize them below:

(Racay 1942 b, Biedenharn 1952; see also Jahn 1951).

$$W(J_{1} \ell_{3} \ell_{1} ; J_{23} \ell_{23}) = W(J_{1} \ell_{3} \ell_{1} ; J_{23} \ell_{23}) \quad (3.45a)$$

$$= W(J_{1} \ell_{3} \ell_{1} ; J_{23} \ell_{23}) \quad (3.45b)$$

$$= W(J_{1} \ell_{3} \ell_{1} ; J_{23} \ell_{23}) \quad (3.45c)$$

$$= (-1)^{J_{1} \ell_{3} \ell_{1} - J_{23} \ell_{23}} W(J_{23} \ell_{23} ; J_{1} \ell_{1}) \quad (3.45d)$$
and
\[ \sum_{J_{n}} \left[ J_{n} \right] W(J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}) W(J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}) = \frac{\delta_{J_{n}J_{n}^{\prime}}} \left[ J_{n} \right], \] (3.46a)

\[ \sum_{J_{n}} (-1)^{-J_{n}+J_{2}+J_{3}+J_{4}+J_{5}+J_{6}} \left[ J_{n} \right] W(J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}) W(J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}) = W(J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}). \] (3.46b)

Another useful relation which can be obtained from (3.41) is:
\[ \sum_{J_{n}} (J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}) (J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}) (J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}) (J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}) = \sqrt{[J_{1}][J_{2}]} W(J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}) W(J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}) W(J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}). \] (3.47)

It is more convenient, though, following (Wigner A), and (Jahn 1954), to define the Wigner 6j-symbol:
\[ \{ J_{1} J_{2} J_{3} \} \{ J_{4} J_{5} J_{6} \} = (-1)^{J_{1}+J_{2}+J_{3}} W(J_{1} J_{2} J_{3} J_{4} J_{5} J_{6}), \] (3.48)
in terms of which the basic symmetry relations (3.45) take the extremely simple form:

(a) Any permutation of the columns leaves the Wigner 6j-symbol invariant.

(b) Simultaneous inversion of any two columns leaves it invariant.

Thus
\[ \{ J_{1} J_{2} J_{3} \} = \{ J_{1} J_{2} J_{3} \} = \{ J_{1} J_{2} J_{3} \} = \{ J_{1} J_{2} J_{3} \} = \{ J_{1} J_{2} J_{3} \}. \] (3.49)

It is evident that by repeated use of (3.49) any one of the six 6j's can be moved to any position. Hence any Wigner 6j-symbol with one zero can be obtained from: (see Jahn 1951, table 20.)
\[ \{ J_{1} J_{2} J_{3} \} = \delta_{J_{1}J_{2}} \delta_{J_{3}J_{2}} (-1)^{J_{1}+J_{2}+J_{3}} \frac{\sqrt{[J_{1}][J_{2}][J_{3}][J_{4}][J_{5}][J_{6}]}}. \] (3.60)
The Racah coefficients have been tabulated by (Hiedenharn 1952) for the values \( \frac{1}{2}, 1, \frac{3}{2}, 2 \) of \( J_2 \) — and, due to (3.49), for any other of the \( J \)'s. Jahn (Jahn 1951) has tabulated the function

\[
U(J_2, J_3; J_{13}) = \sqrt{[2 J_2 + 1][2 J_3 + 1]} W(J_2, J_3; J_{13}) = \langle \Omega_{J_2} \Omega_{J_3} \Omega_J | \Omega_{J_{13}} \rangle
\]

(3.51)

for the ranges \( \frac{1}{2}, 1, \frac{3}{2}, 2 \) of \( J_3 \). For \( J_3 = \frac{5}{2} \) see Table 9 of Flowers (1952d).

The orthogonality relation (3.46a) and the sum rule (3.46b) can also be obtained directly from (3.42) by 'coupling, recoupling', as follows:

\[
\sum_{J_{12}} \begin{bmatrix} J_1 & (J_2, J_3) & J_{13} \end{bmatrix} J_{12} J_{13} J_{123} = \sum_{J_{12}} \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix} J_{12} J_{13} J_{123} U(J_1, J_2, J_3; J_{12}, J_{13})
\]

(3.52)

on the other hand,

\[
\sum_{J_{12}} \begin{bmatrix} J_1 & (J_2, J_3) & J_{13} \end{bmatrix} J_{12} J_{13} J_{123} = (-1)^{J_2 + J_3 - J_{13}} \sum_{J_{13}} \begin{bmatrix} J_1 & (J_2, J_3) & J_{13} \end{bmatrix} J_{12} J_{13} J_{123} U(J_1, J_2, J_3; J_{13}, J_{23})
\]

(3.53)

Comparison of (3.53) with (3.52) yields:

\[
\sum_{J_{13}} (-1)^{J_2 + J_3 - J_{13}} U(J_2, J_3; J_{13}) U(J_1, J_2, J_3; J_{12}, J_{13}) = U(J_1, J_2, J_3; J_{12}, J_{23})
\]

which is equivalent to (3.46b).
Again, we have:

\[
\begin{align*}
\left| \sum_{J_{12}} \left( (J_{12})_{J_{12} J_{23}} J_{12} \right) \right| &= \sum_{J_{12}} \left( \left| (J_{12})_{J_{12} J_{23}} J_{12} \right| \right) U(J_{12} J_{23} ; J_{12} J_{23}) \\
&\equiv \sum_{J_{12}} \left( (-1)^{J_{12} + J_{23} - J_{12} - J_{23}} \left| \left( J_{12} \right)_{J_{12} J_{23}} J_{12} \right| \right) U(J_{12} J_{23} ; J_{12} J_{23}) \\
&\equiv \sum_{J_{12}} \left( (-1)^{J_{12} + J_{23} - J_{12} - J_{23}} \left| \left( J_{12} \right)_{J_{12} J_{23}} J_{12} \right| \right) U(J_{12} J_{23} ; J_{12} J_{23}) U(J_{12} J_{23} ; J_{12} J_{23}) \\
&\equiv \sum_{J_{12}} \left( \left| \left( J_{12} \right)_{J_{12} J_{23}} J_{12} \right| \right) U(J_{12} J_{23} ; J_{12} J_{23}) U(J_{12} J_{23} ; J_{12} J_{23}) \tag{3.54}
\end{align*}
\]

where we have used (3.45) and the fact that \( J_{12} + J_{23} - J_{12} - J_{23} \) is always an integer. By (3.17), (3.54) yields

\[
\sum_{J_{12}} U(J_{12} J_{23} ; J_{12} J_{23}) U(J_{12} J_{23} ; J_{12} J_{23}) = \delta_{J_{12} J_{12}} \tag{3.55a}
\]

Similarly,

\[
\sum_{J_{23}} U(J_{12} J_{23} ; J_{12} J_{23}) U(J_{12} J_{23} ; J_{12} J_{23}) = \delta_{J_{23} J_{23}} \tag{3.55b}
\]

In terms of Wigner \( 6j \)-symbols, (3.55) become:

\[
\sum_{\chi} [\chi] \left\{ \begin{array}{ccc} J_{1} & J_{2} & \chi \\ J_{4} & J_{5} & J_{6} \end{array} \right\} \left\{ \begin{array}{ccc} J_{1} & J_{2} & \chi \\ J_{4} & J_{5} & J_{6} \end{array} \right\} = \frac{\delta_{J_{6} J_{5}^{'}}}{[J_{6}]} \tag{3.56a}
\]

and

\[
\sum_{\chi} [\chi] \left\{ \begin{array}{ccc} J_{1} & J_{2} & J_{3} \\ J_{4} & J_{5} & \chi \end{array} \right\} \left\{ \begin{array}{ccc} J_{1} & J_{2} & J_{3} \\ J_{4} & J_{5} & \chi \end{array} \right\} = \frac{\delta_{J_{3} J_{3}^{'}}}{[J_{3}]} \tag{3.56b}
\]

Eqs. (3.55) show that, for given \( J_{1}, J_{2}, J_{3} \) and fixed \( J \), the \( U \) functions
(or the Wigner $6j$-symbols multiplied by $\sqrt{\frac{1}{[j_{12}][j_{13}][j_{23}]}}$) are the elements of an orthogonal matrix, connecting the different schemes of coupling, whose rows and columns are labelled by $j_{12}$ and $j_{23}$ respectively.

As a final illustration of the 'coupling-recoupling' technique, we give the derivation of a sum rule for Racah coefficients quoted in (Biedenharn 1952, eq. (17)).

First:

$$\sum_{j_{12}} \left\{ \begin{array}{c} j_{1} \end{array} \begin{array}{c} j_{2} \end{array} \begin{array}{c} (j_{3} J_{4}) \end{array} \begin{array}{c} j_{3} \end{array} \begin{array}{c} j_{4} \end{array} \right\}^{J_{M}} = \sum_{j_{23}} \left\{ \begin{array}{c} (j_{1} J_{2}) \end{array} \begin{array}{c} j_{1} \end{array} \begin{array}{c} (j_{3} J_{4}) \end{array} \begin{array}{c} j_{3} \end{array} \begin{array}{c} j_{4} \end{array} \right\}^{J_{M}} U(j_{1} J_{2} j_{3} J_{4} j_{12} J_{23} J_{34})$$

and secondly, when the recoupling is done in three stages:

$$\sum_{j_{123}} \left\{ \begin{array}{c} j_{1} \end{array} \begin{array}{c} j_{2} \end{array} \begin{array}{c} (j_{3} J_{4}) \end{array} \begin{array}{c} j_{3} \end{array} \begin{array}{c} j_{4} \end{array} \right\}^{J_{M}} = \sum_{j_{23}} \left\{ \begin{array}{c} j_{1} \end{array} \begin{array}{c} j_{23} \end{array} \begin{array}{c} j_{3} \end{array} \begin{array}{c} j_{4} \end{array} \right\}^{J_{M}} U(j_{1} J_{23} j_{34} j_{4} j_{3} j_{34})$$

Comparison of (3.57) and (3.58) gives, in terms of $W$’s and with a change of notation:

$$\sum_{x} \left[ x \right] W(j_{3} J_{4} j_{12} J_{13} ; x_{1}) W(j_{1} J_{4} j_{5} J_{4} ; j_{12} J_{12}) W(j_{1} J_{5} j_{1} J_{3} ; J_{12})$$

$$= W(j_{1} J_{2} J_{3} ; J_{4} J_{4} ; x_{1}) W(j_{5} J_{4} J_{5} ; J_{12} J_{13})$$

which can be brought to the form of (Biedenharn 1952, eq. (17)) by a rearrangement of $x$, and renaming of the variables.

Eq. (3.59) can be written in terms of Wigner $6j$-symbols:

$$\sum_{x} \left[ x \right] \left[ x \right]^{\frac{1}{2}} \left[ x \right]^{\frac{1}{2}} \left[ x \right]^{\frac{1}{2}} = (-1)^{J_{1}+J_{2}+J_{3}+J_{4}+J_{5}+J_{6}}$$

(3.60)
Racah (quoted by Siedenharn 1952) has shown that the $W'$s, or the $U'$s, are completely determined (apart from a phase factor which can be chosen by setting $U(0 \mid J_L; J_{P 1} J_{P 2}) = \delta_{J_L J_{P 1}} \delta_{J_{P 2} J_{L 3}}$) by eqs. (3.45), (3.46) and (3.59); so that no further independent relations exist between $W'$s. It should be pointed out that, due to this, the Racah coefficients are completely independent of the choice of $D^J$ and of the Wigner coefficients, (which latter do depend on the explicit choice of $D^J(\alpha \beta \gamma)$).

All the concepts and results developed for Wigner and Racah coefficients, and their uses in 'coupling-recoupling' technique, will be used in the next chapter in the evaluation of matrix elements of irreducible tensor operators.
CHAPTER 4.

IRREDUCIBLE TENSOR OPERATORS.

Operators.

When a rule, $T$, is given which associates with every function, $f(\xi)$, of a certain class, $c(T)$, another function, $g(\xi)$, we write:

$$g(\xi) = T f(\xi);$$

$T$ is the symbolic expression of this rule, and we say that $g(\xi)$ is the result of operating with $T$ on $f(\xi)$. Examples of such operators are $\frac{1}{\delta \xi}$, multiplication by $c(\xi)$, taking logarithm, raising to the $\xi^k$ power, and so on. The only requirement on the rule is that the association $f(\xi) \rightarrow g(\xi) = T f(\xi)$ should be unique. If, moreover, $T$ has the property that

$$T \left( a_1 f_1(\xi) + a_2 f_2(\xi) \right) = a_1 T f_1(\xi) + a_2 T f_2(\xi)$$

where $a_1$ and $a_2$ are constants, then it is said to be a linear operator; of the example quoted above, only $\frac{1}{\delta \xi}$ and multiplication by $c(\xi)$ are linear.

Subject to the convergence of the integral, the scalar product of $\phi_1(\xi)$ and $\psi_1(\xi)$ is defined by

$$(\phi_1(\xi), \psi_1(\xi)) = \int \phi_1^*(\xi) \psi_1(\xi) d\xi;$$

when $\psi_1(\xi) = T \phi_2(\xi)$, the scalar product becomes

$$(\phi_1(\xi), T \phi_2(\xi)) = \int \phi_1^*(\xi) T \phi_2(\xi) d\xi.$$

An operator is called hermitian if

$$(\phi_1(\xi), T \phi_2(\xi)) = (T \phi_1(\xi), \phi_2(\xi)).$$
Irreducible tensor operators.

A set of $2k + 1$ operators, $T_k^{q}(\xi)$, (k integer, $q = k, k-1, \ldots, -k$), which transform under $P_R$, $R \in R_3$, according to $D^k(R)$:

$$P_R T_k^{q}(\xi) = T_k^{q}(R^{-1} \xi) = \sum_{q' = k}^{q} T_k^{q}(\xi) D_k^{q'}(R),$$

(4.1)

are said to be the components of an irreducible tensor operator of order $k$. Irreducible tensor operators of order 0 and 1 are generally called scalar and vector operators, respectively. For example, the operators

$$L_i^1 = \frac{i}{\sqrt{2}}(x^i_2 - z^i_2 + i z^i_2) + z^i_2 - 2i z^i_2 - i z^i_2 + i x^i_2),$$

$$L_i^2 = \delta^i_2,$$

are, in fact, form the components of a vector operator. For illustration we carry through the details of the transformation property of $L_i^1$ for $R = \{0 \beta 0\}$.

Thus

$$P_{0 \beta 0} L_i^1(x, y, z) = L_i^1(\xi, \eta, \xi)$$

where

$$\left( \begin{array}{c} \xi \\ \eta \\ \xi \end{array} \right) = \left( \begin{array}{ccc} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{array} \right) \left( \begin{array}{c} x \\ y \\ z \end{array} \right),$$

(4.3)

since

$$P_{0 \beta 0}^{-1} = \left( \begin{array}{ccc} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{array} \right).$$

Hence

$$P_{0 \beta 0} L_i^1 = \frac{e^{i\beta}}{\sqrt{2}}(x^i_2 - z^i_2 + i z^i_2 - i x^i_2).$$

(4.4)

Now

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial z},$$

and the partial derivatives of $x, y, z$ w.r.t. $\xi, \eta, \xi$ can be calculated from the solution of eq. (4.3):

$$\left( \begin{array}{c} x \\ y \\ z \end{array} \right) = \left( \begin{array}{ccc} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{array} \right) \left( \begin{array}{c} \xi \\ \eta \\ \xi \end{array} \right),$$

(4.5)
Substituting $\xi, \eta, \zeta$ as functions of $x, y, z$ from (4.3), and their various partial derivatives evaluated with the help of (4.5), into (4.4) we finally get:

$$P_{\{0,0\}} L^I (x,y,z) = \frac{k}{i\sqrt{2}} \left[ \sin \beta \left( \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} \right) + \cos \beta \left( \frac{y^2}{\sqrt{2}} - \frac{z^2}{\sqrt{2}} \right) + i \left( \frac{z}{\sqrt{2}} - \frac{x}{\sqrt{2}} \right) \right]. \quad (4.6)$$

On the other hand,

$$\sum_{q' = 1}^{\infty} L^I \left( x', y', z \right) D^I_{q', q} (0,0,0) = L^I (x,y,z) \frac{1 + \cos \beta}{2} + L^I_{-1} (x,y,z) \frac{-\sin \beta}{\sqrt{2}} + L^I_{1} (x,y,z) \frac{1 - \cos \beta}{2},$$

where the $D^I_{q', q} (0,0,0)$ are taken from eq. (3,15).

Thus

$$\sum_{q' = 1}^{\infty} L^I \left( x', y', z \right) D^I_{q', q} (0,0,0) = \frac{k}{i\sqrt{2}} \left[ \frac{1 + \cos \beta}{2} \left( \frac{y^2}{\sqrt{2}} - \frac{z^2}{\sqrt{2}} \right) + i \left( \frac{z}{\sqrt{2}} - \frac{x}{\sqrt{2}} \right) \right] - \frac{\sin \beta}{\sqrt{2}} \frac{k}{i} \left( \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} \right).$$

which, on collecting terms, reduces exactly to (4.6) for $P_{\{0,0\}} L^I (x,y,z)$.

For completeness, we should also verify the transformation property of $L^I_0$ and $L^I_{-1}$; and repeat the whole for $\{0,0,\gamma\}$. But we omit the tedious algebra.

As a general example of (multiplicative) irreducible tensor operators of any order $k$, we mention the (orthonormal) surface harmonics $Y^k_l (\theta, \varphi)$ eq. (3.18). (see discussion following (3.18)).

Matrix representation of operators.

Suppose we are given an irreducible tensor operator of order $k$, $T^k_l$, and a complete orthonormal set of functions, $\{\alpha\}$. Operating by $T^k_l$ on $|\alpha\rangle$ we obtain another set, $|\beta\rangle$:

$$|\beta\rangle = T^k_l |\alpha\rangle,$$

which can be expanded in terms of $|\alpha\rangle$:

$$T^k_l |\alpha\rangle = \sum_{\alpha'} |\alpha'\rangle <\alpha' | T^k_l |\alpha\rangle. \quad (4.7)$$
(For a treatment of Dirac notation, etc., see, e.g., Dirac 1947, chapters I, II, III; or Condon 1953, chapters II, III.)

Eq. (4.7) can be written in the matrix form:

$$(\langle T_{\frac{1}{2}}^k | \alpha' \rangle, T_{\frac{1}{2}}^k | \alpha'' \rangle, \ldots ) = (|\alpha'\rangle, |\alpha''\rangle, \ldots ) \langle \alpha'| T_{\frac{1}{2}}^k | \alpha'\rangle, \langle \alpha''| T_{\frac{1}{2}}^k | \alpha''\rangle, \ldots$$

The effect of operating by $T_{\frac{1}{2}}^k$ on any other set, $|\gamma\rangle$ say, is completely known if we know the matrix $\langle \alpha'| T_{\frac{1}{2}}^k | \alpha\rangle$; for:

$$T_{\frac{1}{2}}^k | \gamma\rangle = T_{\frac{1}{2}}^k \sum_{\alpha} | \alpha\rangle \langle \alpha | \gamma\rangle$$

$$= \sum_{\alpha} T_{\frac{1}{2}}^k | \alpha\rangle \langle \alpha | \gamma\rangle$$

$$= \sum_{\alpha} | \alpha'\rangle \langle \alpha' | T_{\frac{1}{2}}^k | \alpha\rangle \langle \alpha | \gamma\rangle.$$

The matrix $\langle \alpha'| T_{\frac{1}{2}}^k | \alpha\rangle$ thus completely characterizes the operator $T_{\frac{1}{2}}^k$.

The function space $|\alpha j m\rangle$

The set $|\alpha\rangle$ could be the eigenfunctions of commuting linear, hermitian operators. Thus $J^2 = J_1^2 + J_2^2 + J_3^2$, see eq. (1.7), and $J_3$ could be part of such a set of operators, and their eigenfunctions will be labelled $|\alpha j m\rangle$:

$$J^2 |\alpha j m\rangle = (j + l) |\alpha j m\rangle,$$

$$J_3 |\alpha j m\rangle = m |\alpha j m\rangle.$$

When the Hamiltonian, $H$, of a system is invariant under $P_R, R \in R_3$, then $H$ will commute with $J^2$ and $J_3$, and $|\alpha j m\rangle$ will be eigenfunctions of $H$ also. (cf. discussion in chapter 1.)

We are thus interested in the matrix elements of $T_{\frac{1}{2}}^k$ in the space of functions $|\alpha j m\rangle$.

Reduced matrix elements.

The $(2k+1)(2j+1)$ products $T_{\frac{1}{2}}^k |\alpha j m\rangle$ transform under $P_R, R \in R_3$. 

according to $D^k_j$. The $(2J+1)$ linear combinations
\[ \sum_{q} (k j q m | kj JM) T^k_q | \alpha J M q \rangle = | (T^k_j) \alpha J M \rangle \]
(4.8)
on the other hand, transform according to $D^J_j$. With the help of
\[(3.27b), we can solve (4.8) for $T^k_q | \alpha j m \rangle$.
\[ T^k_q | \alpha j m \rangle = \sum_{j} (k j q m | kj \bar{J} q+m \rangle | (T^k_j) \alpha J q+m \rangle \]
The notation $| (T^k_j) \alpha J M \rangle$ is simply a short hand for the linear
combinations $\sum_{q} (k j q M q | kj JM) T^k_q | \alpha J M q \rangle$, eq. (4.8).

Using (4.9), we have:
\[ \langle \alpha' j' m' | T^k_q | \alpha j m \rangle = \sum_{j} \langle \alpha' j' m' | (T^k_j) \alpha J q+m \rangle (k j q m | kj \bar{J} q+m \rangle \]
(4.10)

Eq. (5.17) now tells us that in the summation in (4.10) there will be
a factor $\delta_{q+m, q+m'}$, so that (4.10) becomes:
\[ \langle \alpha' j' m' | T^k_q | \alpha j m \rangle = (k j q m | kj j' m') \langle \alpha' j' m' | (T^k_j) \alpha j m' \rangle . \]
Writing
\[ \langle \alpha' j' m' | (T^k_j) \alpha j m' \rangle = \frac{(\alpha' j' \parallel T^k \parallel \alpha j)}{\sqrt{\epsilon[j']}} \]
(4.12)
eq (4.11) finally becomes:
\[ \langle \alpha' j' m' | T^k_q | \alpha j m \rangle = \frac{(k j q m | kj j' m')(\alpha' j' \parallel T^k \parallel \alpha j)}{\sqrt{\epsilon[j']}} . \]
(4.13)

Eq. (3.17) again tells us that $(\alpha' j' \parallel T^k \parallel \alpha j)$, called a reduced
matrix element of $T^k_q$, will be independent of $m'$ and $m$. And the
factor $\sqrt{\epsilon[j']}$ is not incorporated in the reduced matrix element in
order to agree with the notation of Racah, Jahn and others.

As an example, we evaluate the reduced matrix elements of the
angular momentum (vector) operators, $L^l_q$, eq. (4.2).

Now
\[ \langle \alpha' j' m' | L^l_q | \alpha j m \rangle = (i j q m | i j' m') (\alpha' j' \parallel L^l \parallel \alpha j) \left\{ \epsilon[j'] \right\}^{-1/2} , \]
(4.14)

\[ \text{11. This derivation is from Jahn (A), lecture 3.} \]
Considering \( q = 0 \), and using \( L_0 |\alpha j m\rangle = m k |\alpha j m\rangle \), (4.14) becomes

\[
\delta_{jj'} \delta_{mm'} m k = \delta_{mm'} (1 j o m | i j j' m' \rangle \langle \alpha j' \parallel L' \parallel \alpha j) \sqrt{[j]}
\]
from which we obtain:

\[
(a' j' \parallel L' \parallel \alpha j) = \frac{\delta_{jj'} \sqrt{[j']}}{(1 j o m | i j j' m \rangle)} m k.
\]

Now from (3.24) we easily find that

\[
(1 j o m | i j j' m \rangle) = \frac{-m}{\sqrt{j (j + 1)}};
\]

hence

\[
(a' j' \parallel L' \parallel \alpha j) = \delta_{jj'} \sqrt{j (j + 1)} \langle j j' \parallel \rangle k.
\] (4.15)

Finally

\[
\langle a' j' m' | L' \parallel \alpha j m \rangle = \delta_{jj'} \langle i j q m | i j j' m' \rangle \sqrt{j (j + 1)} k.
\] (4.16)

Eq. (4.16) embodies all the matrix elements of the angular momentum operators. (see, e.g., Peenberg 1953, table 2.)

Selection rules.

It will be seen from (4.13) that the matrix elements of \( T^k_q \)
have been written as the product of two factors: one,

\[
(k j q m | k j j' m' \rangle (2 j' + 1)^{-1/2},
\]

independent of the specific nature of \( T^k_q \), (being the same for the \( q^{th} \) component of any irreducible tensor operator of order \( k \)), the second, \( (a' j' \parallel T^k_q \parallel \alpha j) \), depends on the actual nature of \( T^k_q \) and is independent of \( m \) and \( q \).

Finally the selection rules

\[
\Delta m = m' - m = 0, t 1, \ldots, tk; \quad |\Delta j| = 0, 1, \ldots, k; \quad (4.17)
\]
for the non-vanishing of the matrix elements of \( T^k_q \) follow from the occurrence of the Wigner coefficient in (4.13).

Tensor products of tensor operators.

Following Racah (Racah 1942b; A) we define the irreducible
tensor product of two such operators, \( T_{Q_1} \) and \( U_{Q_2} \) by
\[
(T_{Q_1} U_{Q_2})_Q^K = \sum_{Q_1} T_{Q_1} U_{Q_2} (k_1 k_2 q_1 - q_1 | k_1 k_2 K Q), \quad K \neq 0.
\] (4.18)
and
\[
(T_{Q_1} U_{Q_2})_Q^K = \sum_{Q_2} (-i)^{k} \sqrt{[k]} (k k q_2 - q_2) T_{Q_1} U_{Q_2} = \sum_{Q_2} (-i)^{k} T_{Q_1} U_{Q_2}.
\] (4.19)

We now consider the matrix elements of \( (T_{Q_1} U_{Q_2})_Q^K \) where \( T_{Q_1} \) and \( U_{Q_2} \) operate on the whole system \( | \alpha j m \rangle \).

Now
\[
\langle \alpha' j' m' | (T_{Q_1} U_{Q_2})_Q^K | \alpha j m \rangle = (K_j Q m | K_j j' m') \langle \alpha' j' m' | [T_{Q_1} U_{Q_2}]_Q^K | \alpha j m \rangle.
\] (4.20)

By means of a \( U \) function we write (4.20) as:
\[
\langle \alpha' j' m' | (T_{Q_1} U_{Q_2})_Q^K | \alpha j m \rangle = (K_j Q m | K_j j' m') \sum_{\alpha'' x} \langle \alpha' j' m' | [T_{Q_1} U_{Q_2}]_Q | \alpha'' x \rangle \frac{(\alpha'' x || U_{Q_2} || \alpha j)}{\sqrt{x}} \times U(k k, j' j, K x).
\] (4.21)

Since, from the definition of reduced matrix elements,
\[
| (U_{Q_2})_{\alpha x} \rangle = \sum_{\alpha''} | \alpha'' x \rangle \frac{(\alpha'' x || U_{Q_2} || \alpha j)}{\sqrt{x}},
\]
(4.21) can be written:
\[
\langle \alpha' j' m' | (T_{Q_1} U_{Q_2})_Q^K | \alpha j m \rangle = (K_j Q m | K_j j' m') \sum_{\alpha'' x} \langle \alpha' j' m' | (T_{Q_1} U_{Q_2})_Q | \alpha'' x \rangle \frac{(\alpha'' x || U_{Q_2} || \alpha j)}{\sqrt{x}} \times U(k k, j' j, K x)
\]
\[
= (K_j Q m | K_j j' m') \sum_{\alpha'' x} \frac{(\alpha' j' || T_{Q_1} || \alpha'' x) (\alpha'' x || U_{Q_2} || \alpha j)}{\sqrt{x}} \times U(k k, j' j, K x) \frac{[j']}{[x]} [x].
\] (4.22)

This is the general expression for the matrix elements of \( (T_{Q_1} U_{Q_2})_Q^K \).

In particular, when \( K = Q = 0 \) we obtain:
\[
\frac{(-1)^{k}}{\sqrt{[k]}} \langle \alpha' j' m' | T_{Q_1} U_{Q_2} | \alpha j m \rangle = (0 j 0 m | 0 j j' m') \sum_{\alpha'' x} \frac{(\alpha' j' || T_{Q_1} || \alpha'' x) (\alpha'' x || U_{Q_2} || \alpha j)}{\sqrt{x}} \times \frac{[x]}{[j']} \frac{[j]}{[k]} \frac{[j']}{[x]}.
\] (4.23)
where, from (3.50), \( U(k; j; 0, x) = \delta_{jj'} (-1)^{k} \sqrt{\frac{[\kappa]}{[k][j]}} \). (also see Jahn 1951, table 20).

As \( \langle o j m| o j'm' \rangle = \delta_{jj'} \delta_{mm'} \), (4.23) becomes

\[
\langle \alpha' j' m' | T^{k} U^{k} | \alpha j m \rangle = \delta_{jj'} \delta_{mm'} \sum_{j} \frac{(-1)^{j}}{[j]} \sum_{\alpha''} (-1)^{j''} \langle \alpha'' j'' | T^{k} | \kappa' x \alpha'' | U^{k} | \kappa x \rangle (4.24)
\]

which agrees with the expression given by Racah (Racah A, eq. (165); 1942b, eq. (33).)

Wigner 9j-symbols.

More often, though, we are interested in the matrix elements of \( (T^{k_{1}} U^{k_{2}})^{K} \) in the function space \( |J_{1} j_{1} J_{2} j_{2} J M \rangle \), where \( T^{k_{1}} U^{k_{2}} \) act only on the subsystems (1), (2) of the total system. Hence-forth the suffixes \( 1, 2, \ldots \) will always denote quantities referring to subsystems \( 1, 2, \ldots \); and the different values of a quantity referring to the same subsystem distinguished by dashes.

By 'coupling-recoupling' technique we have:

\[
\langle (J_{1}', J_{2}') J_{3}' m' | (T^{k_{1}} U^{k_{2}})^{K} | (J_{1}, J_{2}) J m \rangle = \langle K J_{1} J_{2} J_{3}' | K J_{1} J_{2} J_{3}' \rangle \langle (J_{1}', J_{2}') J_{3}' m' | [(T^{k_{1}} U^{k_{2}})^{K}] (J_{1}, J_{2}) J m \rangle \]

\[
= \langle K J_{1} J_{2} J_{3}' | K J_{1} J_{2} J_{3}' \rangle \sum \langle (J_{1}', J_{2}') J_{3}' m' | [(T^{k_{1}} J_{1})]_{j''} (U^{k_{2}} J_{2}) J_{3}' \rangle \times \langle (k_{1}, J_{1}) J_{1}'' (k_{2}, J_{2}) J_{2}'' J_{3}' \rangle \times \langle (k_{1}, J_{1}) J_{1}'' (k_{2}, J_{2}) J_{2}'' J_{3}' \rangle (4.25)
\]

Following Jahn (Jahn 1954; also see Wigner A) we define the Wigner 9j-symbol by:

\[
\sqrt{[J_{1}'][J_{2}'][K][J]} \begin{cases} 
    k_{1} & J_{1} & J_{1}' \\
    k_{2} & J_{2} & J_{2}' \\
    K & J & J'
\end{cases} = \langle [(k_{1}, J_{1})]_{1}' (k_{2}, J_{2}) J_{2}' J_{3}' | [k_{3}, k_{2}] K | J_{1} J_{2} J \rangle (4.26)
\]
In terms of the Wigner 9j-symbol, (also using (3.17), (4.26) can be written

\[
\langle (J_1, J_2, J_3) | (T_{k_1} U_{k_2})^K Q | (J'_1, J'_2, J'_3) \rangle = (K J Q M | K J J' M') \sqrt{[K][J][J'][K][J]} \left\{ \begin{array}{ccc} k_1 & J_1 & J'_1 \\ k_2 & J_2 & J'_2 \\ K & J & J' \end{array} \right\} \times \\
\frac{(J'_1 \| T_{k_1} \| J_1) (J'_2 \| U_{k_2} \| J_2)}{\sqrt{[J'_1][J'_2]}}
\]

\[
= (K J Q M | K J J' M') \sqrt{[K][J][J']} \left\{ \begin{array}{ccc} k_1 & J_1 & J'_1 \\ k_2 & J_2 & J'_2 \\ K & J & J' \end{array} \right\} (J'_1 \| T_{k_1} \| J_1) (J'_2 \| U_{k_2} \| J_2). \quad (4.27)
\]

From the general definition of reduced matrix elements, though,

\[
\langle (J_1, J'_2, J_3) | (T_{k_1} U_{k_2})^K Q | (J'_1, J_2, J'_3) \rangle = \frac{(K J Q M | K J J' M') (J'_1 \| T_{k_1} U_{k_2}^K) Q | (J'_1, J_2, J'_3)}{\sqrt{[J'_1][J'_2]}} \quad (4.28)
\]

Comparison of (4.27) and (4.28) yields

\[
(J'_1 \| (T_{k_1} U_{k_2})^K Q | (J'_1, J_2, J'_3)) = \sqrt{[K][J][J']} \left\{ \begin{array}{ccc} k_1 & J_1 & J'_1 \\ k_2 & J_2 & J'_2 \\ K & J & J' \end{array} \right\} (J'_1 \| T_{k_1} \| J_1) (J'_2 \| U_{k_2} \| J_2), \quad (4.29)
\]

which agrees with (Jahn 1954, eq. (4).)

It may be remarked that Wigner (Wigner A, eq. (78)) actually defines the Wigner 9j-symbol by eq. (4.29). From the above derivation, his definition is equivalent to our one.

Eq. (4.27) is a general result; in particular, for the scalar
product we obtain:

\[
\langle (j_1' j_2') j'_m' | T_k U^k | (j_1 j_2) j m \rangle = \delta_{j j'} \delta_{m m'} (-i)^{j_1 + j_2 + j} \{ j'_m' \}_{j' k} \{ j_1 j_2 j \}_{j k} (j'_m' \parallel U^k \parallel j) (j_1 \parallel T_k \parallel j_2) (j \parallel U^k \parallel j_2),
\]

where we have used: (Jahn 1954)

\[
\left\{ \begin{array}{ccc}
  k & j_1 & j_1' \\
  k' & j_2 & j_2' \\
  0 & j & j'
\end{array} \right\} = \delta_{kk'} \delta_{j j'} \frac{(-1)^{j_1 + j_2 + j}}{\sqrt{[k][k']}} \left\{ \begin{array}{ccc}
  j_1 & j_2 & j \\
  j'_1 & j'_2 & j'
\end{array} \right\},
\]

which can be obtained from (4.34). (see later).

Eq. (4.30) agrees with (Racah 1942b, eq. (38)).

Properties of Wigner 9j-symbols.

The Wigner 9j-symbols satisfy a great many symmetry and orthogonality relations and various sum rules, all of which can be derived by extensive use of the 'coupling-recoupling' technique.

Thus, from the defining eq. (4.26):

\[
\left[ \begin{array}{ccc}
  (j_1 j_4) & (j_2 j_5) & (j_6 j_7) \\
  (j_8 j_9)
\end{array} \right] = \sum_{j} \sqrt{\frac{\left\{ j_1 j_2 j_3 \right\}}{\left\{ j_4 j_5 j_6 \right\}}} \left\{ j_1 j_2 j_3 \right\} \left\{ j_4 j_5 j_6 \right\} \left\{ j_7 j_8 j_9 \right\},
\]

But this could also be achieved as follows:

\[
\left[ \begin{array}{ccc}
  (j_1 j_4) & (j_2 j_5) & (j_6 j_7) \\
  (j_8 j_9)
\end{array} \right] = \sum_{x} (-1)^{j_2 + j_5 - j_7} \left\{ j_1 j_4 (j_1 j_2) j_8 \right\} \langle j_9 \rangle U(j_8 j_5 j_7; j_9 j_7)
\]

\[
= \sum_{x} (-1)^{j_2 + j_5 - j_8 + j_8 - x} \left\{ j_1 j_2 (j_4 j_5) j_8 \right\} \langle j_9 \rangle U(j_8 j_5 j_7; j_9 j_7) \times U(j_8 j_5 j_7; j_8 j_7 x)
\]

\[
= \sum_{x} (-1)^{j_2 + j_5 + y - j_7 - x} \left\{ j_1 j_2 (j_4 j_5) j_8 \right\} \langle j_9 \rangle U(j_8 j_5 j_7; j_9 j_7) U(j_8 j_5 j_7; j_8 j_7 x)
\]
The comparison of (4.33) with (4.32), on setting \( z = \frac{1}{3}, y = \frac{1}{6} \) and changing from \( U \) functions to Wigner \( 6j \)-symbols, yields:

\[
\begin{align*}
\{ J_1, J_2, J_3 \} & = \sum_{x}^{2x} (-1)^{x} \left\{ J_1, J_2, J_3 \right\} \left\{ J_4, J_5, J_6 \right\} \left\{ J_1, J_4, J_7 \right\} .
\end{align*}
\]

which is — due to the symmetry relations (3.49) — identical with (Jahn 1954, eq. (2); Wigner A, eq. (78a)). By the same method we can obtain a number of other relations like (4.34); e.g.

\[
\begin{align*}
\{ J_1, J_2, J_3 \} & = \sum_{x}^{2x} (-1)^{x} \left\{ J_1, J_2, J_3 \right\} \left\{ J_4, J_5, J_6 \right\} \left\{ J_1, J_4, J_7 \right\} .
\end{align*}
\]

The symmetry relations follow from (4.34) or (4.35) or from the expression

\[
\begin{align*}
\sqrt{\frac{\pi}{8}} \{ J_1, J_2, J_3 \} \{ J_4, J_5, J_6 \} & = \sum_{m} \left( J_1, J_2, m \right) \left( J_4, J_5, m \right) \left( J_6, J_7, m \right) \left( J_1, J_4, m \right) \left( J_5, J_6, m \right) \left( J_3, J_7, m \right) .
\end{align*}
\]

obtained by taking the three Wigner coefficients on the L.H.S. of (4.36) over to the R.H.S. Here we merely list these symmetries; for details, reference may be made to (Jahn 1954).

(a) A reflection of the Wigner \( 9j \)-symbol in either of its two diagonals \( \left( J_1, J_5, J_9 \right) \) or \( \left( \frac{1}{3}, J_5, J_7 \right) \) leaves it unchanged:

\[
\begin{align*}
\left\{ J_1, J_2, J_3 \right\} & = \frac{1}{2} \left\{ J_1, J_4, J_7 \right\} = \left\{ J_9, J_6, J_3 \right\} .
\end{align*}
\]
(b) An odd (even) permutation of the rows or the columns multiplies it by \((-1)\), \((+1)\), where \(J = \sum \frac{J}{J_1} \).

\[ \begin{pmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \\ J_7 & J_8 & J_9 \end{pmatrix} = \begin{pmatrix} \frac{J}{J_1} & \frac{J}{J_2} & \frac{J}{J_3} \\ \frac{J}{J_4} & \frac{J}{J_5} & \frac{J}{J_6} \\ \frac{J}{J_7} & \frac{J}{J_8} & \frac{J}{J_9} \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \\ J_7 & J_8 & J_9 \end{pmatrix} = \begin{pmatrix} \frac{J}{J_4} & \frac{J}{J_5} & \frac{J}{J_6} \\ \frac{J}{J_7} & \frac{J}{J_8} & \frac{J}{J_9} \\ \frac{J}{J_1} & \frac{J}{J_2} & \frac{J}{J_3} \end{pmatrix} \]

(4.36b)

It should be noted that \(J\) is always an integer. Combinations of (4.36) yield the complete group of 72 symmetries of the Wigner 9j-symbols. And from (4.36) we see that any one of the 9 \(j\)'s can be moved to any other position. Thus when one \(j\) is zero, we can obtain the Wigner 9j symbol from

\[ \begin{pmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \\ J_7 & J_8 & 0 \end{pmatrix} = \begin{pmatrix} \delta_{J_3 J_5} & \delta_{J_5 J_7} & \delta_{J_7 J_9} \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \]

(4.37)

which is derivable from (4.54) or (4.55).

In order to get the orthogonality relations, we note that:

\[ \sum_{J_9} \left[ \begin{pmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \\ J_7 & J_8 & J_9 \end{pmatrix} \right] = \sum_{J_9} \sqrt{[J_3][J_5][J_7]} \left[ \begin{pmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \\ J_7 & J_8 & J_9 \end{pmatrix} \right] \]

\[ = \sum_{J_9} \sum \sqrt{\left[ \begin{pmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \\ J_7 & J_8 & J_9 \end{pmatrix} \right]} \]

from which the orthogonality relations:

(4.38a)
and
\[
\sum_{j_7, j_8} [j_3, j_6] [j_7, j_8] \begin{pmatrix}
   j_1 & j_2 & j_3 \\
   j_4 & j_5 & j_6
\end{pmatrix}
= \frac{\delta_{j_3, j_7} \delta_{j_6, j_8}}{[j_3, j_6] [j_7, j_8]},
\quad (4.38b)
\]

follow easily. (For (4.38a) we use (4.36a), and for (4.38b) we rename the variables, and use (4.36a).)

Eqs. (4.38) show that, for given \( j_1, j_2, j_4, j_5 \) and fixed \( j_9 \), the Wigner 9j-symbols, multiplied by \( \sqrt{[j_3, j_6] [j_7, j_8]} \), are the elements of an orthogonal matrix, connecting the different coupling schemes, whose rows and columns are labelled by \( j_3, j_6 \) and \( j_7, j_8 \) respectively.

(\textit{cf.} the similar statement following (3.56).)

Finally, we derive just one sum rule for the Wigner 9j-symbols:
\[
\left[ (j_1, j_4, j_7) (j_2, j_5, j_8) \right]_{j_9} = \sum_{j_3, j_6} \left[ (j_3, j_6) (j_4, j_7, j_8) \right]_{j_9} \sqrt{[j_3, j_6] [j_7, j_8]}
= \sum_{x, y} (-1)^{j_2 + j_5 - j_8}
\left[ (j_1, j_5) \times (j_4, j_7, j_8) \right]_{j_9} \sqrt{[x][y][j_7][j_8]}
\left[ j_1, j_2, j_3 \right] \left[ j_4, j_5, j_6 \right] \left[ j_7, j_8 \right]
\quad (4.39)
\]

But on the other hand:
\[
\left[ (j_1, j_4, j_7) (j_2, j_5, j_8) \right]_{j_9} = (-1)^{j_2 + j_5 - j_8}
\left[ (j_1, j_4, j_7) (j_2, j_5, j_8) \right]_{j_9}
= \sum_{x, y} (-1)^{j_2 + j_5 - j_8}
\left[ (j_1, j_5) \times (j_4, j_7, j_8) \right]_{j_9} \sqrt{[x][y][j_7][j_8]}
\left[ j_1, j_2, j_3 \right] \left[ j_4, j_5, j_6 \right] \left[ j_7, j_8 \right]
\quad (4.40)
\]

Comparison of (4.39) and (4.40) yields
\[
\sum_{j_3, j_6} (-1)^j [j_3, j_6] [j_1, j_2, j_3] \left[ j_4, j_5, j_6 \right] = (-1)^j [j_4, j_5, j_6] [j_7, j_8]
\quad (4.41)
\]
Applications of the Wigner 6j-symbols.

An obvious application of the results so far developed is the calculation of matrix elements of $L \cdot S$ in atomic LS coupling; see (Condon 1953, section 37). From (4.30) we have: ($k = 1$)

$$\langle L'S' \mid L \cdot S \mid L'S \rangle \propto \delta_{LM} W(L \cdot S L'S'; J J \sum_{\alpha} \langle \alpha \mid \alpha' \mid \alpha' \rangle \langle \alpha' \mid L \cdot S \rangle \langle \alpha' \rangle$$

where $L = (\frac{L}{2}, \frac{L}{2})$, $S' = (\frac{S}{2}, \frac{S}{2})$ act only on parts of the system characterized by $L$, $S$ respectively. But now (4.15) comes to our aid; so:

$$\langle L'S' \mid L \cdot S' \mid L'S \rangle \propto \delta_{LU} \delta_{SS'} \delta_{JM} \sqrt{L(L+1)S(S+1)[L][S]} W(LLSS', JJ).$$

We now look up the expression for the $W$ in (4.42) in (Diedenhorn 1953, table II); with this, (4.42) becomes: (apart from $1$)

$$\langle L'S' \mid L \cdot S' \mid L'S \rangle \propto \delta_{LU} \delta_{SS'} \delta_{JM} \sqrt{L(L+1)S(S+1)[L][S]} \frac{J(J+1) - L(L+1) - S(S+1)}{2\sqrt{L(L+1)S(S+1)[L][S]}}.$$

The Landé interval rule follows from (4.43) exactly as in (Condon 1953).

Again, we may be interested in the matrix elements of an operator which acts only on subsystem (1). These can be calculated from (4.27) by setting $k = 0$. ($k = k'$)

Thus:

$$\langle J_1' J_2' \mid T^{k_1}_{k_2} \mid J_1 J_2 \rangle = (k_1 J q_1 M \mid k_1 J J' M') \sqrt{[k_1][J][J']} \left\{ \begin{array}{c} k_1 J J' M' \\ k_1 J J' M' \end{array} \right\} \left\{ \begin{array}{c} J_1' J_2' \\ J_1 J_2 \end{array} \right\} \sqrt{[J][J']}$$(4.44)

where we have used (4.37), and $(J' \mid I \mid J) = \delta_{JJ}, \sqrt{[J'][J]}.$
Comparing (4.44) with

$$\langle (J', J'_z) J'M' \parallel T'_{k_1} \parallel (J, J_z) JM \rangle = \langle (J', J'_z) J'M' \parallel T_{k_1} || (J, J_z) JM \rangle \frac{\langle (J', J'_z) J'M' \parallel T'_{k_1} \parallel (J, J_z) JM \rangle}{\sqrt{|J'|}}$$

we obtain; (in agreement with eq. (44a) of Racah 1943b.)

$$\langle (J', J'_z) J' \parallel T'_{k_1} \parallel (J, J_z) J \rangle = \delta_{J'J} \delta_{J'_zJ_z} (-1)^{J+J'+k_1} \sqrt{|J|} \frac{|J'|}{|J|} W(J', J'_z, J, J_z, k_1) \langle (J, J_z) J'M' \parallel T'_{k_1} \parallel (J, J_z) JM \rangle \tag{4.45}$$

Likewise:

$$\langle (J', J'_z) J'M' \parallel U'_{k_2} \parallel (J, J_z) JM \rangle = \delta_{J'J} \delta_{J'_zJ_z} (-1)^{J+J'+k_1} \langle (J', J'_z) J'M' \parallel U_{k_2} \parallel (J, J_z) JM \rangle \frac{|J'|}{|J|} \frac{|J'_z|}{|J_z|} W(J', J'_z, J, J_z, k_1) \langle (J, J_z) J'M' \parallel U'_{k_2} \parallel (J, J_z) JM \rangle \tag{4.46}$$

Thus in calculating the magnetic moment of a Russell-Saunders level characterized by LSJ we make use of (4.44) and (4.46):

$$\mu_{LSJ} = \hbar_b \langle (LS)JM \parallel L'_0 + 2S'_0 \parallel (LS)JM \rangle \frac{1}{\hbar}$$

where $\hbar_b = \frac{\hbar}{2m_o c}$

$$= \langle (J, J_z) J'M' \parallel \frac{|J'|}{|J|} \frac{|J'_z|}{|J_z|} \frac{|J'|}{|J|} \frac{|J'_z|}{|J_z|} W(J', J'_z, J, J_z, k_1) \langle (J, J_z) J'M' \parallel U'_{k_2} \parallel (J, J_z) JM \rangle \tag{4.47}$$

Using (4.15), the fact that $W(J, J_z, J'_z, J'_z, J, J_z, J_z) = W(J, J_z, J'_z, J'_z, J, J_z, J_z)$ and looking up (Biedenharn 1952, table II; or Jahn 1951, table 22) for the relevant Racah coefficients, (4.47) yields:

$$\mu_{LSJ} = \frac{\hbar_b}{\hbar} (-1)^{J} \sqrt{(J+1) J} \left\{ \begin{array}{l}
L(L+1) + J(J+1) - S(S+1) \\
2 \sqrt{L(L+1)(L+1)(S+1)(S+1)}
\end{array} \right\} \frac{\sqrt{L(L+1)[L]} \hbar}{J(J+1) J}$$

$$+ 2 \frac{S(S+1) + J(J+1) - L(L+1)}{2 \sqrt{S(S+1)(S+1)} \sqrt{L(L+1)[L]}} \frac{\sqrt{S(S+1)[S]} \hbar}{J(J+1) J}$$

$$= \frac{3 J(J+1) - L(L+1) + S(S+1)}{2 J(J+1)} M \frac{\hbar}{\hbar} \tag{4.48}$$
Writing (4.48) in the form
\[ \mathcal{H}_{LSJ} = \frac{g}{\mathcal{H}_{LSJ}} M \mathcal{H}_{J} \]
where
\[ \frac{g}{\mathcal{H}_{LSJ}} = \frac{3 J(J+1) - L(L+1) + S(S+1)}{2 J(J+1)} \]
we obtain the familiar expression for \( g_{LSJ} \), the Landé g-factor.

From the various sum rules for Wigner 6j and 9j-symbols follow sum rules for intensities of spectral lines; but we omit their discussion.

A further application of Wigner 9j-symbols occurs in transforming from one coupling scheme to another. For example, the transformation from \( jj \) to \( ls \) coupling can be affected as:
\[ [(l_1, l_2) L \left( \frac{1}{2}, \frac{1}{2} \right) S] J = \sum_{J_1 J_2} \left[ (l_1, \frac{1}{2}) J_1 \left( l_2, \frac{1}{2} \right) J_2 \right] J \sqrt{\frac{4J_1 J_2 L S (L+S+1)}{L_1 L_2 S_1 S_2}} \left\{ \begin{array}{ccc} l_1 & \frac{1}{2} & J_1 \\ l_2 & \frac{1}{2} & J_2 \\ L & S & J \end{array} \right\} \] (4.49)

where \( J_i \) can take the two values \( l_i + \frac{1}{2} \) and \( S \) is either 0 or 1. When \( S = 0 \) (and then \( J = L \)), the Wigner 9j-symbol in (4.49) reduces to a Racah coefficient. A more complicated transformation coefficient, namely:
\[ \langle [l_0 \left( l_1 \frac{1}{2} \right)] J \{ J_0 \left( J_1 \frac{1}{2} \right) J \} \rangle = \sqrt{\frac{4J_0 J_1 J_2 L_0 L_1 L_2 S_0 S_1 S_2}{L_0 L_1 L_2 S_0 S_1 S_2}} \left\{ \begin{array}{ccc} J_0 & J_1 & J_2 \\ J_3 & J_4 & J_5 \\ l_0 & l_1 & l_2 \end{array} \right\} \] (4.50)

where (4.50) is taken as the definition of the Jahn 12j-symbol. (Jahn 1954, addendum) can be evaluated with the help of Wigner 9j-symbol.

Thus:
\[
\sum _{x} \left\{ J_{0} \left[ (J_{1} J_{2}) \frac{1}{3} (J_{3} J_{4}) \frac{1}{6} \right] x \right\}_{\frac{1}{9}} J_{9} \right\} \sum _{x} \left\{ J_{0} \left[ (J_{1} J_{2}) \frac{1}{3} (J_{3} J_{4}) \frac{1}{6} \right] x \right\}_{\frac{1}{9}} U(J_{0} J_{1} J_{2} J_{3}; x)
\]

\[
\sum _{x} \left\{ J_{0} \left[ (J_{1} J_{2}) \frac{1}{3} (J_{3} J_{4}) \frac{1}{6} \right] x \right\}_{\frac{1}{9}} \sum _{x} \left\{ J_{0} \left[ (J_{1} J_{2}) \frac{1}{3} (J_{3} J_{4}) \frac{1}{6} \right] x \right\}_{\frac{1}{9}} U(J_{0} J_{1} J_{2} J_{3}; x) \times U(J_{0} J_{1} J_{2} J_{3}; x)
\]

Hence:

\[
\begin{pmatrix}
J_{1} J_{2} J_{3} J_{4} \\
J_{4} J_{5} J_{6} J_{7}
\end{pmatrix}
= (-1)^{J_{3}+J_{6}+J_{7}+J_{9}} \sum _{x} \left\{ J_{1} J_{2} J_{3} J_{4} J_{5} J_{6} J_{7} J_{8} \right\}_{x} \times \sum _{x} \left\{ J_{1} J_{2} J_{3} J_{4} J_{5} J_{6} J_{7} J_{8} \right\}_{x}
\]

(4.51)

where we have used the fact that \( J_{1}+J_{2} - 2x \) is always even.

Putting (4.34) into (4.51) gives:

\[
\begin{pmatrix}
J_{1} J_{2} J_{3} J_{4} \\
J_{4} J_{5} J_{6} J_{7}
\end{pmatrix}
= (-1)^{J_{3}+J_{6}+J_{7}+J_{9}} \sum _{x} \left\{ J_{1} J_{2} J_{3} J_{4} J_{5} J_{6} J_{7} J_{8} \right\}_{x} \times \sum _{x} \left\{ J_{1} J_{2} J_{3} J_{4} J_{5} J_{6} J_{7} J_{8} \right\}_{x}
\]

(4.52)
but unfortunately we can not carry out the summation over \( x \) — which occurs in 4 of the Wigner 6j-symbols. But remembering (3.60) we try to write (4.52) so as to involve \( x \) in only 3 of the Wigner 6j-symbols! Now on looking carefully at (4.34) we notice that \( \frac{1}{3} \), \( \frac{1}{5} \) and \( \frac{1}{7} \) occur only in one Wigner 6j-symbol; the rest occurring in only two. (The same is true, for \( \frac{1}{1}, \frac{1}{6}, \frac{1}{9} \), in (4.35)). Thus with the help of the symmetry relations (4.35) we first bring \( x \) to either of the positions 3, 5 or 7 (1, 6 or 3), and then use (4.34). With this trick, (4.51) becomes:

\[
\begin{vmatrix}
\frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{9} \\
\frac{1}{7} & \frac{1}{8} & \frac{1}{9}
\end{vmatrix} = (-1)^{R} \sum_{x y} (-1)^{\left[ \frac{x}{y} \right]} \left\{ \frac{1}{x} \frac{1}{y} \right\} \left\{ \frac{1}{z} \frac{1}{w} \right\} \left\{ \frac{1}{u} \frac{1}{v} \right\} \left\{ \frac{1}{t} \frac{1}{s} \right\} \left\{ \frac{1}{r} \frac{1}{q} \right\} \left\{ \frac{1}{p} \frac{1}{o} \right\} \left\{ \frac{1}{n} \frac{1}{m} \right\} (4.53)
\]

where \( R \) is the sum of the twelve \( j \)'s of the symbol. Eq. (4.53) agrees exactly with the unpublished result of J.P. Elliott quoted by (Ord-Smith 1954). The Jahn 12j-symbols have a number of symmetry, orthogonality and sum rules, some of which we discuss in the appendix.

Finally we may mention that the Nj-symbols \( (N = 6, 9, 12, \ldots) \) are of use in calculations of fractional parentage coefficients (see Chapter 5) both in the atomic and nuclear case. These, as well as other applications of the preceding methods, will be the subject matter of the next chapter.
A: **Nuclear spectroscopy.**

An important problem in the theoretical elucidation of nuclear structure is the solution of the many-body equation

\[ H(1,2,\ldots,A)\Psi_k(1,\ldots,A) = E_k \Psi_k(1,\ldots,A), \]

which yields the energy levels and the corresponding eigenfunctions of the system. (cf. Chapter 1). Taking account of two-body central (Wigner, Majorana, Heisenberg and Bartlett exchange) and tensor forces, the Hamiltonian can be written as:

\[ H = \sum_{i=1}^{A} \left( T_i + Z_i(r_i) \frac{\hbar^2}{2M} \cdot \mathbf{s}_i \right) + \sum_{i<j=1}^{A} V_{ij} \]

in the usual notation. We assume the spin-orbit coupling term to be large enough to validate the 'jj-coupling' model (see later); then it is convenient to write (5.1) in the form

\[ H = \left[ \sum_{i=1}^{A} \left( T_i + Z_i(r_i) \frac{\hbar^2}{2M} \cdot \mathbf{s}_i + V_i(r_i) \right) \right] + \left[ \sum_{i<j=1}^{A} V_{ij} - \sum_{i=1}^{A} V_i(r) \right], \]

\[ = \sum_{i=1}^{A} H^{(0)}(i) + H^{(I)}, \]

\[ (5.3) \]
In (5.2), \( V_i (r_i) \) is a 'suitably' chosen static, spherically symmetric potential — generally a square-well, or a harmonic oscillator potential; and we assume that \( \sum_{\langle j \neq i \rangle}^{A} v_{ij} - \sum_{\langle \epsilon \rangle}^{A} v_i \) can be treated as a small perturbation. For a detailed discussion of these matters reference may be made to (Flowers 52a; Pryce 54; Jensen 55; see also Blin-Stoyle 56). A discussion of the perturbation calculations will be omitted; but it may be pointed out that the approach adopted (which we could call the independent particle, jj-coupling model) would be completely rigorous, within the assumptions of (5.1), were we to take those linear combinations of the complete set of antisymmetrized eigenfunctions of \( H^{(0)} \) which diagonalize \( H = H^{(0)} + H^{(1)} \). As is well known, this would be a complete 'configuration mixing' solution of the problem; but due to the extremely high order of the secular determinants to be solved, the computations would be 'impossible'. Here we merely consider pure configurations, and illustrate the application of some of the concepts developed so far.

A configuration of \( n \) particles outside closed shells is completely specified by giving \( n \, \nu \, \lambda \) values. Here \( \nu \) is the order of occurrence of the level — of a single particle in the chosen potential field — with orbital angular momentum specified by \( \lambda \). We treat the case of equivalent particles, i.e., the configuration

\[
(\nu \lambda) = \nu^1 \lambda^1 \nu^2 \lambda^2 \cdots \nu^n \lambda^n.
\]

Now an eigenfunction of \( \sum_{\langle \epsilon \rangle}^{A} H^{(1)} \) will be a linear combination of expressions of the form

\[
\phi_{\nu^1 \lambda^1 \mu^1 \tau^1}^{(1)} \phi_{\nu^2 \lambda^2 \mu^2 \tau^2}^{(2)} \ldots \phi_{\nu^n \lambda^n \mu^n \tau^n}^{(n)} \omega^{(1)} \omega^{(2)} \cdots \omega^{(n)}.
\]

(5.4)
where \( \phi_{m_i}^{l j} (i) \) is the result of coupling \( f_{m_i}^l (i) \) to \( \chi_{m_j}^{l 2} (i) \):

\[
\phi_{m_i}^{l j} (i) = \sum f_{m_i}^l (i) \chi_{m_j}^{l 2} \left( \frac{\ell}{2} m_i \frac{m_i - m_j}{2} \right) ;
\]

and the \( f \)'s are products of the form

\[
f_{m_i}^l (i) = R_{\nu l} (r_i) Y_{\mu l} (\theta_i, \phi_i) .
\]

The \( R_{\nu l} (r_i) \) satisfy the radial part of \( H^{(0)} (i) \).

For nucleons (which are 'half-integral spin' particles) the group-theoretical statement of Pauli's principle is that the eigenfunction of a state occurring in nature must transform, under \( S_h \), according to the one-dimensional antisymmetric irreducible representation of \( S_h \). We have (as can be justified) neglected the \( A-n \) core nucleons. In choosing the appropriate linear combinations of (5.4) to satisfy this requirement, we consider the space-spin and the isobaric spin parts separately.

The \( 2^n \) products \( \omega_{m_1}^{l 1} (i) \omega_{m_2}^{l 2} (i) \ldots \omega_{m_n}^{l n} (i) \) spread out a \( 2^n \)-dimensional, reducible representation of \( S_h \). In the reduction of this representation, due to the two-valued nature of the isobaric spin variable, only those standard orthogonal irreducible representations of \( S_h \) will occur which are labelled by Young shapes, of \( n \), of at most two rows:

\[
\begin{array}{c}
\sqrt{\frac{n_1}{n_2}} \\
\end{array}
, 
\quad n_1 + n_2 = n, \quad n_1 \gg n_2 .
\]

This can be characterized by one number,

\[
T = \frac{n_1 - n_2}{2} = n_1 - \frac{n}{2} .
\]

(Compare the similar situation, with respect to \( S_A \), in atomic LS-coupling). Moreover, each such representation will occur \( 2T + 1 \) times. We have the dimensional check:
\[ \sum_{T=0}^{n} (2T+1) d_{\{\frac{n}{2}+T, \frac{n}{2}-T\}} = 2^n \]

For a lucid account of the details, in atomic LS-coupling, we refer to (Wigner 31, Chapter XXII, §§6, §7, §8).

The \(2T+1\) sets of \(d_{\{\frac{n}{2}+T, \frac{n}{2}-T\}}\) functions which are carriers of \(D_{\frac{n}{2}+T, \frac{n}{2}-T}\), and which have arisen from the 'reduction' of the product functions \(\omega_{\lambda_1}^{(1)} \ldots \omega_{\lambda_n}^{(n)}\), can be put in a box with \(d_{\{\frac{n}{2}+T, \frac{n}{2}-T\}}\) rows and \(2T+1\) columns:

\[ \begin{array}{cccc}
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linear combinations of the product functions into a box with
\[ d_{n'}(U_{2j+1}) \text{ rows and } d_{n'}(U_{2j+1}) \text{ columns,} \]
where the rows and columns of the box can be labelled by \((r')\),
the different Young symbols of \(\{n_1', \ldots, n_{2j+1}'\}\), and
\(\alpha_1, (\alpha = 1, 2, \ldots, d_{n'}(U_{2j+1}))\), respectively. Thus a typical function
will look like \(\prod_{(r')}^N \sum_{T} \Omega T^{(r')} (r, \alpha, T, M_T)\).

The \(d_{n'}(U_{2j+1}) \times d_{n'}(U_{2j+1})\) functions will transform, under \(S^n\), according to \(D_{n'}(U_{2j+1}) \times D_{n'}(U_{2j+1})\).

Now, Wigner shows (loc. cit. above) that, in the reduction of
\(D_{n'}(U_{2j+1}) \times D_{n'}(U_{2j+1})\), the antisymmetric representation will
occur (only once) if and only if the two irreducible representations
are dual. That is, if the Young shape corresponding to \(\{n_1', \ldots, n_{2j+1}'\}\)
is the 'transposed' of that corresponding to \(\{\frac{\alpha}{2} - T, \frac{\alpha}{2} - T\}\), and
provided we multiply all the representation matrices of
by \(\varepsilon_{P}\), where
\[ \varepsilon_{P} = +1 \text{ if } P \text{ is an even permutation,} \]
\[ = -1 \text{ if } P \text{ is an odd permutation.} \]

It is clear that the dimensions of \(D_{n'}(U_{2j+1})\) and \(D_{n'}(U_{2j+1})\) are equal. And then
\[ \psi_{MT}^{(1 \ldots n)} = \frac{1}{\sqrt{d}} \sum_{(r)} \Phi_{(r)}^{\alpha} (1 \ldots n) \Omega_{(r)MT}^{T} (1 \ldots n), \quad (5.5) \]

will be a normalized antisymmetric function of all the coordinates of the \( n \) nucleons.

Now it is known in group-representation theory that a partition of \( n \) into at most \( 2j+1 \) integers (such as, e.g., \( \left\{ \frac{n}{2}+T, \frac{n}{2}-T \right\} \)), provided \( n \leq \lambda(2j+1) \) labels an irreducible representation of \( U_{2j+1} \) — the group of all \( (2j+1) \)-dimensional unitary matrices — contains \( D^j(R) \) — 'isomorphic' to \( R_3^\prime \) as a subgroup; hence \( D^{J}(R), R \in U_{2j+1} \), under restriction to \( R_3^\prime \), will break up into the direct sum of a number of \( D^J(R) \) of \( R_3^\prime \). The possible values, and multiplicities, of for a given partition of \( n \) have been considered in detail by Jahn (50) and Flowers (52b) for nuclear LST-coupling, and by Flowers (52c) for nuclear jj-coupling. Thus \( \alpha \) in (5.5) is replaced by an 'almost complete' set of quantum numbers, including \( J \) and \( M_J \). Hence the specification of \( T \) (for given \( n \)) already fixes the possible values of \( J_3 \); and the arrangement into boxes of the isobaric spin and space-spin functions of definite symmetry properties looks like:
And

\[ \Phi_{\lambda}^{T J M_T M_J} (1, \ldots, n) = \frac{1}{\sqrt{d_{\lambda}}} \sum_{\Omega_T} \phi_{\lambda}^{T M_T} \Phi_{\Omega_T M_J}^{T (1, \ldots, n)} \]  

(5.6)

the quantum numbers \( \beta \) (see the papers of Jahn and Flowers; also Racah A., 54) being necessary to distinguish between repeated \( J \)-values.

In the absence of \( H^{(I)} \), all the states of the configuration \( (\nu_1 l_1 j_1)^n (\nu_2 l_2 j_2)^n \ldots \) will be degenerate. Thus the eigenfunction of a 'level' of given \( J \) (and parity, which in this case is \( (-1)^n l_1 + \ldots \) ) would be a linear combination of all the eigenfunctions of all configurations which can lead to the given \( J \) (and parity). In the presence of \( H^{(I)} \) this 'level' will split into a number of states, all with the given \( J \) and parity. To find the separations of these states from the 'unperturbed' level we have to diagonalize the matrix of \( H^{(I)} \) in the chosen eigenfunctions of the suitable configurations. Of course, in order to find the total angular momentum and parity of, say, the ground state one has to diagonalize the matrix of \( H^{(I)} \) in the eigenfunctions of all configurations, and take the lowest energy value.

Since the coupling and antisymmetrization of antisymmetric functions of groups of \( \nu \), \( n \), \ldots \), equivalent nucleons is well known, (e.g. Condon 53, Chapter VIII) we shall consider equivalent nucleons, \( (\nu l j)^n \); and, thus, we are interested in the matrix elements of various operators between the different states of the configuration \( (\nu l j)^n \).
**Fractional parentage coefficients.**

In the calculation of such matrix elements the concept of 'parentage' of a state, reintroduced by Racah (43), is very useful. We write

\[ \psi_{\beta T J}^{M_T M_J} (l_1, \ldots, n) = \sum_{\beta', T', J'} \psi_{\beta' T' J'}^{M_T' M_J'} (l_1, \ldots, n-1) \psi_{M_T M_J}^{M_T' M_J'} (l_n) \langle (\mathbf{J}_n) \beta J \mid \frac{1}{2} T \rangle \frac{1}{2} M_T \frac{1}{2} M_J \mid (\mathbf{J}_n) \beta J \rangle, \tag{5.7} \]

where \( \psi_{\beta' T' J'}^{M_T' M_J'} (l_1, \ldots, n-1) \) is antisymmetric with respect to \( S_{n-1} \).

The Wigner coefficients take care of angular momentum properties, while the \( \langle (\mathbf{J}_n) \beta J \mid \frac{1}{2} T \rangle \frac{1}{2} M_T \frac{1}{2} M_J \rangle \), called fractional parentage coefficients, (f.p.c.), are independent of \( M_T \) and \( M_J \), and are chosen so that \( \psi_{\beta T J}^{M_T M_J} (1, \ldots, n) \) is antisymmetric under \( S_n \).

To see how this helps the calculation of matrix elements, consider

\[ \langle (\mathbf{J}_l) \beta' T' J' M_T' M_J' \mid \sum_{i=1}^{n} f_i \mid (\mathbf{J}_l) \beta J M_T M_J \rangle \tag{5.8} \]

\[ = n \langle (\mathbf{J}_l) \beta' T' J' M_T' M_J' \mid f_1 \mid (\mathbf{J}_l) \beta J M_T M_J \rangle, \]

where \( f_i \) is a single particle irreducible tensor operator acting on the \( i \)th nucleon. With the help of (5.7) we can write (5.8) as:
\[
\left\langle (\nu \lambda j)^{n} \beta^{' \cdot} T^{' \cdot} J^{' \cdot} M^{' \cdot} M^{' j} \right| \sum_{i=1}^{n} f_{i} \left| (\nu \lambda j)^{n} \beta T J M M^{' j} \right\rangle
\]

\[
= n \sum_{\beta^{' \cdot} T^{' \cdot} J^{' \cdot}} \left\langle (\nu \lambda j)^{n} \beta^{' \cdot} T^{' \cdot} J^{' \cdot} M^{' \cdot} M^{' j} \right| f_{n} \left| (\nu \lambda j)^{n} \beta T J M M^{' j} \right\rangle \times \\
\times \left( (\nu \lambda j)^{n-1} \beta^{' \cdot} T^{' \cdot} J^{' \cdot} \frac{1}{2} j \right) \left( (\nu \lambda j)^{n} \beta^{' \cdot} T^{' \cdot} J^{' \cdot} \frac{1}{2} j \right) \left( (\nu \lambda j)^{n} \beta^{' \cdot} T^{' \cdot} J^{' \cdot} \frac{1}{2} j \right)
\]

We now remind ourselves of (4.46), with whose help (5.9) becomes:

\[
\left\langle (\nu \lambda j)^{n} \beta^{' \cdot} T^{' \cdot} J^{' \cdot} M^{' \cdot} M^{' j} \right| \sum_{i=1}^{n} f_{i}^{k, k'} \left| (\nu \lambda j)^{n} \beta T J M M^{' j} \right\rangle
\]

\[
= n \sum_{\beta^{' \cdot} T^{' \cdot} J^{' \cdot}} \left( (\nu \lambda j)^{n-1} \beta^{' \cdot} T^{' \cdot} J^{' \cdot} \frac{1}{2} j \right) \left( (\nu \lambda j)^{n} \beta^{' \cdot} T^{' \cdot} J^{' \cdot} \frac{1}{2} j \right) \left( (\nu \lambda j)^{n} \beta^{' \cdot} T^{' \cdot} J^{' \cdot} \frac{1}{2} j \right) \times \\
\times \delta_{\beta^{' \cdot} \beta \cdot} \delta_{T^{' \cdot} T} \delta_{J^{' \cdot} J} \delta_{J^{' \cdot} J^{' \cdot}} \left( k^{' \cdot} T^{' \cdot} M^{' j} \mid k^{' \cdot} J^{' \cdot} M^{' j} \right) \left( k^{' \cdot} T^{' \cdot} M^{' j} \right) \sqrt{|J||T|} W(j j j J J k k') \\
W(\frac{1}{2}, T^{' \cdot} j T ; T^{' \cdot} k) \left( i \parallel f_{n}^{k', k} \mid j \right) \left( i \parallel f_{n}^{k', k} \mid \frac{1}{2} \right) \left( i \parallel f_{n}^{k', k} \mid \frac{1}{2} \right) (-1)^{\cdot}
\]

\[
(5.10)
\]

where we assume that \( f_{i} \) is an irreducible tensor operator of order \( k' \) in space-spin 'space', and of order \( k'' \) in isobaric spin 'space'.

Eq. (5.10) is a general result; in particular, when \( k'' = 0 \), i.e., when \( f_{i} \) commutes with rotations in isobaric spin space, we obtain
\[
\left< (v_\ell j)^n \beta_{T'J'M'M'_j} \right| \sum_{i} \left. f_{i}^{k} \right| (v_\ell j)^n \beta_{TJM'M_j} \right>
\]

\[
= n \delta_{T'T} \delta_{M'M} \sum_{T} \left( k' J' q' M'_{j} | k' J' J'M'_j \right) \sqrt{\frac{\mathcal{L}[J][\tau]}{W(J'j'j;j'k')}} \times
\]

\[
\times \left( j || f^{k'} || j \right) \left( (v_\ell j)^{-1} \beta_{T'J',j} \right) \left( (v_\ell j)^n \beta_{TJ} \right) \left( (v_\ell j)^{-1} \beta_{T'J',j} \right) \left( (v_\ell j)^n \beta_{TJ} \right)
\]

\[\text{(5.11)}\]

Compare a similar relation given by Schwartz and De Shalit (54, eq. (19)).

**Calculation of the f.p.c.**

Here we give a (rather inelegant) method of calculating the fractional parentage coefficients (following Racah 45): Since

\[
\left< (v_\ell j)^n \beta_{TJM'M_j} \right> = \sum_{\beta_{TJ}} \left< (v_\ell j)^{-1} \beta_{\tau J, j} \right| \beta_{TJM'M_j} \right> \left( (v_\ell j)^{-1} \beta_{\tau J, j} \right)
\]

\[\text{and}\]

\[
= \sum_{\beta_{TJ}} \left< (v_\ell j)^{-2} \beta_{\tau J, j} \left( j || f \right) \beta_{TJM'M_j} \right> U\left( \frac{T}{2} \frac{T}{2} ; \frac{T}{2} \frac{T}{2} \right) \times
\]

\[
\times \left( \frac{\tau^J}{\beta_{TJ}} \right) \left( (v_\ell j)^{-2} \beta_{\tau J, j} \right) \left( (v_\ell j)^{-1} \beta_{\tau J} \right) \times
\]

\[\text{(5.12)}\]

it follows that the f.p.c. satisfy the system of simultaneous equations:
\[
\sum_{\beta \tau J} \left[ \left( \langle \psi_{\ell}^{\beta} \tilde{f} \rangle \right)^{n-2} \frac{1}{\beta} \bar{J} \left( \psi_{\ell}^{\beta} \tilde{f} \right) \left\langle \psi_{\ell}^{\beta} \tilde{f} \right\rangle \right]_{\frac{1}{\beta} \bar{J} \beta \tau J} \times \left( \psi_{\ell}^{\beta} \tilde{f} \right)_{\frac{1}{\beta} \bar{J} \beta \tau J} = 0 \quad \text{for } \tau'' + \tau = \text{even.} \quad (5.13)
\]

Hence once the \( \left( \langle \psi_{\ell}^{\beta} \tilde{f} \rangle \right)^{n-2} \frac{1}{\beta} \bar{J} \left( \psi_{\ell}^{\beta} \tilde{f} \right) \left\langle \psi_{\ell}^{\beta} \tilde{f} \right\rangle \) are known, we can use (5.13) to calculate \( \left( \langle \psi_{\ell}^{\beta} \tilde{f} \rangle \right)^{n-2} \frac{1}{\beta} \bar{J} \left( \psi_{\ell}^{\beta} \tilde{f} \right) \left\langle \psi_{\ell}^{\beta} \tilde{f} \right\rangle \).

For clear details of the properties, factorization, calculation, generalization and uses of the f.p.c., we refer to Racah (A), Jahn (51a, 51b, 54b), Flowers (52d) and others.

Finally we should point out that the concepts of tensor operators, and f.p.c., are not only in the different coupling schemes of the independent particle model (for LST-coupling see the papers of Jahn; for intermediate coupling Lenes, 53, 54, 55a, 55b and Blin-Stoyle 56) but also in the theory of \( \beta \)- and \( \gamma \)-decay. (See the relevant chapters of the book, '\( \beta \) and \( \gamma \) Spectroscopy' Ed. K. Siegbahn). In the next section we briefly treat the derivation of a basic formula of angular correlation theory.

B: Angular correlation theory.

In the experimental measurement of the properties of nuclear energy levels, the angular correlation of successive nuclear radiations provides a valuable and elegant tool.

The theory was originally considered by Hamilton (40), and has been extended by many workers; (Racah 51). It has been reviewed
extensively, (Biedenharn 53; Frauenfelder 55), where more details and further references to the literature may be found.

Here, following Racah (51) closely, we merely give a few applications of irreducible tensor operator algebra to the theory.

Consider a nucleus in an energy level A of sharp angular momentum \( J_A \), and parity \( \pi_A \), decaying into an intermediate level B (of the same, or 'daughter' nucleus) with \( J_B, \pi_B \), by the emission of radiation \( R_1 \) in direction \( \mathbf{k}_1 \) (and polarization \( \xi_1 \)) which then decays into the final level C, \( (J_C, \pi_C) \), by emitting \( R_2 \) in direction \( \mathbf{k}_2 \) (and polarization \( \xi_2 \)). The interaction Hamiltonian for the emission of \( R \) along \( \mathbf{k} \), which we take as the quantisation \((x-y)\) axis, can be written as

\[
H = \sum_{LM} \alpha_{LM}(A_i) \, T^L_M(X_i), \quad (5.14)
\]

where \( \alpha_{LM}(A_i) \) depending on the set of vector arguments \( A_i \), specifying the properties of \( R \), characterize the emitted 'particles', \( R \); and \( T^L_M(X_i) \) are irreducible tensor operators of order \( L \), depending on the nuclear coordinates \( X_i \). When the 'particle' carries away unique angular momentum (e.g. pure \( M\) or \( M \) \( \gamma\text{-transitions} \) then the summation over \( L \) will reduce to one value of \( L \) only.

Having chosen \( \mathbf{k}_1, \mathbf{k}_2 \) as the quantization axes for \( R_1, R_2 \) respectively, we now use the transformation property of \( T^L_M(X_i) \) under the group \( R_3 \) to obtain:
\[ H_1 = \sum_{L_1 M_1} \alpha_{L_1 M_1} (A_{i,1}) T_{M_1}^{L_1} (X_{i,1}) , \]  
(5.15)

\[ H_2 = \sum_{L_2 M_2} \alpha_{L_2 M_2} (A_{i,2}) T_{M_2}^{L_2} (X_{i,2}) D_{k,M_2}^{L_2} (\alpha \beta \gamma) . \]  
(5.16)

Now, according to Hamilton, the probability of the cascade emission is given by

\[ W(k, \gamma) = \sum_{M_A M_C} \left| \sum_{M_B} \left( J_{M_A} | H_1 | J_{M_B} \right) \left( J_{M_B} | H_2 | J_{M_C} \right) \right|^2 , \]  
(5.17)

\[ = \sum_{M_A M_C} \left( J_{M_A} | H_1 | J_{M_B} \right) \left( J_{M_B} | H_2 | J_{M_C} \right)^* \left( J_{M_B} | H_2 | J_{M_C} \right)^* . \]  
(5.18)

Putting (5.15) and (5.16) into (5.18) we obtain:

\[ W(\alpha \beta \gamma) = \sum_{L M_k} \alpha_{L M_k} (A_{i,1}) \alpha_{L' M_k'}^{*} (A_{i,1}) \alpha_{L_2 M_2} (A_{i,2}) \alpha_{L_2' M_2'}^{*} (A_{i,2}) X \]

\[ \times \left( J_{M_A} | T_{M_1}^{L_1} | J_{M_B} \right) \left( J_{M_B} | T_{M_2}^{L_2} | J_{M_C} \right)^* X \]

\[ \times \left( J_{M_A} | T_{M_1'}^{L_1'} | J_{M_B'} \right)^* \left( J_{M_B'} | T_{M_2'}^{L_2'} | J_{M_C} \right)^* D_{k,M_2}^{L_2} (\alpha \beta \gamma) D_{k',M_2'}^{L_2} (\alpha \beta \gamma) , \]

(5.19)

where the summation over \( L M_k \) means summation over all such quantities.

As pointed out by Racah (footnote 9, Racah 51), we have:
\[
L' \frac{D}{\mu', \Lambda_2^t} (-\alpha \beta) = (-1) \frac{D}{\Lambda_2^t} \frac{D}{\mu', \Lambda_2^t} (-\alpha \beta), \tag{5.20}
\]

and also

\[
(-1) \frac{D}{\Lambda_2^t} \frac{D}{\Lambda_2^t} (-\alpha \beta) = (-1) \sum_k \left( \frac{L_2 L_2'}{2} \Lambda_2 \frac{L_2'}{2} \Lambda_2 \right) \frac{D}{\mu', \Lambda_2^t} (-\alpha \beta). \tag{5.21}
\]

Eq. (5.20) follows from (3.12) on taking the complex conjugate of both sides, changing the summation index to \( r' = r + m - n \) and comparing the result with (3.12). And (5.21) can be obtained from the definition of \( S_{(j_1, j_2)} \); or from (3.32) on taking the two Wigner coefficients to the L.H.S.

With (5.20), (5.21), inserted into (5.19) we have:

\[
W(\alpha \beta) = \sum_{LM, L'M', k} \alpha_{LM} \alpha_{L'M'}^* \left( \frac{L_1 J_6 M_6}{2} \frac{L_1 J_6 M_6}{2} \right) \left( J_A \parallel T^{-1} \parallel J_8 \right) \left( [J_A] \right)^{-1/2} \left( \frac{L_1 J_6 M_6}{2} \frac{L_1 J_6 M_6}{2} \right) x
\]

\[
x \left( J_B \parallel T^{-1} \parallel J_c \right) \left( [J_B] \right)^{-1/2} \left( \frac{L_2 J_6 M_6}{2} \frac{L_2 J_6 M_6}{2} \right) \left( J_B \parallel T^{-1} \parallel J_c \right) \left( [J_B] \right)^{-1/2} \tag{5.22}
\]

\[
x \left( \frac{L_2 L_2'}{2} \frac{L_2' L_2}{2} \frac{L_2' L_2}{2} \frac{L_2 L_2'}{2} \right) \frac{D}{\mu', \Lambda_2^t} (-\alpha \beta).
\]

We now use (3.47) to express the sum of products of three Wigner coefficients as a multiple of one Wigner and one Racah coefficient:

\[
e.g. \sum \left( \frac{L_2 J_6 M_6}{2} \frac{L_2 J_6 M_6}{2} \right) \left( \frac{L_2 J_6 M_6}{2} \frac{L_2 J_6 M_6}{2} \right) x
\]

\[
\times \left( \frac{L_2 L_2'}{2} \frac{L_2' L_2}{2} \frac{L_2' L_2}{2} \frac{L_2 L_2'}{2} \right) (-1)^{L_2 + J_8 - \frac{L_2}{2} - \frac{J_8}{2}} \sqrt{[J_8] [k]} W(\frac{L_2}{2}, \frac{L_2}{2}, \frac{L_2}{2}, \frac{L_2}{2}) x \left( \frac{J_8 M_8}{2} \frac{J_8 M_8}{2} \right) \left( J_B \parallel J_B \parallel J_B \parallel J_B \right).
\]
Thus \( W(\alpha \beta \gamma) = \sum L \alpha_{L1 M_1} \alpha_{L'1 M'_1} \star (J_A \parallel J'_A \parallel J_B)(J_A \parallel J'_A \parallel J_B) \star \times (J_B \parallel J'_B \parallel J_C)(J_B \parallel J'_B \parallel J_C) \star W(J_B L_2 J_B L'_2 ; J_A k) W(J_B L'_2 J_B L_2 ; J_A k) \times \)
\( \times (L, L'_1 M_1 - M'_1 | L L'_1 k)(L, L'_1 M_1 - M'_1 | L L'_1 k)(L, L'_1 M_1 - M'_1 | L L'_1 k)(L, L'_1 M_1 - M'_1 | L L'_1 k)(L, L'_1 M_1 - M'_1 | L L'_1 k)(L, L'_1 M_1 - M'_1 | L L'_1 k)(L, L'_1 M_1 - M'_1 | L L'_1 k) \times (-1) \times D^k \quad (5.23) \)

Further, following Racah, define:

\[ C_{k \tau} (L L') = \sum_{M M'} \alpha_{L M} \alpha_{L' M'} (L, L' M - M' | L L k \tau) \quad (5.24) \]

At last, in terms of \( C_{k \tau} (L L') \), (5.23) becomes

\[ W(\alpha \beta \gamma) = \sum_{L k \tau} W(J_B L_2 J_B L'_2 ; J_A k) W(J_A L'_2 J_A L_2 ; J_A k) (J_A \parallel J'_A \parallel J_B) \times \]
\( \times (J_B \parallel J'_B \parallel J_C)(J_B \parallel J'_B \parallel J_C) \star (J_B \parallel J'_B \parallel J_C)(J_B \parallel J'_B \parallel J_C) \star \)
\( \times C_{k \tau_1} (L L'_1) C_{k \tau_2} (L L'_2) \frac{D^k}{\tau_1, \tau_2, \gamma} (\alpha \beta \gamma) \quad (5.25) \]

This is the general expression for the angular correlation function of two successive radiations, of arbitrarily mixed multipolarity and polarization. In particular, when both radiations have unique angular momentum, and we do not observe their polarization, (i.e. 'directional' correlation) then (5.25) simplifies greatly:

\[ W_1(\alpha \beta) = \sum_k A_{k \gamma} P_{k \gamma}(\cos \theta) \quad (5.26) \]
where

\[ A_{jk} = W(J_2 L_2 L'_2; J_k)W(J_1 L_1 L'_1; J_k) \times \]
\[ \times c_{k0}(L_1 L'_1) c_{k0}(L_2 L'_2) \]

since the reduced matrix elements are angle independent, and

\[ D_{0,0}^j (\alpha \beta \gamma) = P_j (\cos \beta) . \]

Finally we may point out that all these powerful concepts have also
been applied (e.g. see Blatt, 52; and Lane 55c.) to the theory of
nuclear reactions.
APPENDIX

PROPERTIES OF THE 12-t-SYMBOL.

For convenience, we restate (with a change of notation) the definition (4.50) and eqs. (4.51), (4.53):

\[
\left\{ \begin{array}{cccc}
    J_1 & J_2 & J_3 & p \\
    J_4 & J_5 & J_6 & q \\
    J_7 & J_8 & J_9 & r
  \end{array} \right\} = \left( \prod_{i<j} [j_i, j_j] \right)^{-\frac{1}{2}} \left\{ \begin{array}{cccc}
    r_1 & r_2 & r_3 & p \\
    r_4 & r_5 & r_6 & q \\
    r_7 & r_8 & r_9 & r
  \end{array} \right\} \quad (A.1)
\]

\[
= (-1)^{J_1+J_5+J_7+J_9} \sum_{x} (-1)^x [x] \left\{ \begin{array}{cccc}
    J_1 & J_2 & J_3 & p \\
    J_4 & J_5 & J_6 & q \\
    J_7 & J_8 & J_9 & x
  \end{array} \right\}, \quad (A.2)
\]

\[
= (-1)^R \sum_{x} (-1)^x [x] \left\{ \begin{array}{cccc}
    J_3 & J_2 & J_1 & p \\
    J_4 & J_5 & J_6 & q \\
    J_7 & J_8 & J_9 & x
  \end{array} \right\} \quad (A.3)
\]

**Symmetries.**

It can be readily verified that the permutations

\[
\begin{pmatrix}
  J_1 & J_2 & J_3 & p \\
  J_4 & J_5 & J_6 & q \\
  J_7 & J_8 & J_9 & r
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
  J_1 & J_2 & J_3 & p \\
  J_4 & J_5 & J_6 & q \\
  J_7 & J_8 & J_9 & r
\end{pmatrix}
\]

are the generators \((\alpha^2 = \beta^2 = I, \quad \alpha \beta = \beta \alpha)\) of a group (isomorphic to
the 'vierergruppe') of 4 symmetries of the 12j-symbols given by
(John 1954, eq. (A6).)
Again, the R.H.S. of (A.3) is invariant under
\[
\begin{pmatrix}
1_1 & 1_2 & 1_3 \\
1_5 & 1_7 & 1_2 & 1_3 & 1_6 & 1_9 & 1_8 & 1_9
\end{pmatrix}
\]
Hence
\[
\lambda : \begin{cases}
\{1_1, 1_2, 1_3, 1_4, 1_5, 1_6, 1_7, 1_9\} \\
\{1_4, 1_5, 1_6, 1_9\} \\
\{1_6, 1_7, 1_9\} \\
\{1_4, 1_7, 1_9\}
\end{cases}
\]
(A.46)
together with \(\alpha\) and \(\beta\) of (A.4), \(\gamma^4 = \alpha^4 = \gamma^4 = \gamma^8 = \beta^4\) generate
a group of 16 symmetries of the 12j-symbols:
\[
I, \alpha, \beta, \gamma, \alpha \beta, \alpha \gamma, \alpha \gamma \beta, \alpha \gamma \beta, \alpha \gamma \beta \alpha \gamma \beta \alpha \gamma \beta \alpha \gamma \beta \alpha \gamma \beta \alpha \gamma \beta \alpha \gamma \beta.
\]
This group of symmetries has been tabulated by (Ord-Smith 1954).

**Orthogonality.**

Using the 'coupling-recoupling' method, we get:
\[
\left| \left\{ \begin{smallmatrix}
j_1 j_2 j_3 j_4 j_5 & j_6 j_7 j_8 j_9
\end{smallmatrix} \right\} \right|^2 = \sum_{j_{12}} \left| \left\{ \begin{smallmatrix}
j_1 + j_2 & j_3 & j_4 & j_5 & j_6 & j_7 & j_8 & j_9
\end{smallmatrix} \right\} \right|^2
\]
\[
= \sum_{j_{12}} \left| \left\{ \begin{smallmatrix}
j_1 \pm j_2 & j_3 & j_4 & j_5 & j_6 & j_7 & j_8 & j_9
\end{smallmatrix} \right\} \right|^2
\]
from which follows (using symmetries of Wigner 12j-symbols):
\[
\sum_{j_{12}} \left| \left\{ \begin{smallmatrix}
j_1 & j_2 & j_3 & j_4 & j_5 & j_6 & j_7 & j_8 & j_9
\end{smallmatrix} \right\} \right|^2
\]
\[
= \frac{\delta_{j_1 j_2} \delta_{j_3 j_4} \delta_{j_5 j_6} \delta_{j_7 j_8} \delta_{j_9 j_{12}}}{\left| \begin{smallmatrix}
j_1 & j_2 & j_3 & j_4 & j_5 & j_6 & j_7 & j_8 & j_9
\end{smallmatrix} \right|^2}
\]
(A.78)
Likewise, starting from \( \left\{ \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right\}_3 \), we could obtain:

\[
\sum_{j} \frac{\delta_{j_3 j} \delta_{j_3} \delta_{j_6 j_6}}{[j_3][j_4][j_5][j_6]} = \begin{pmatrix} j_3 & j_4 \\ j_5 & j_6 \end{pmatrix} .
\tag{A.7b}
\]

Eqs. (A.7) show that, for fixed \( r \), given \( j_3, j_4, j_5 \), and fixed \( j_3 \), the Jahn 12j-symbols, multiplied by \( \sqrt{\frac{[j_3][j_4][j_5][j_6][j_7][j_8][j_9]}{[j_3][j_4][j_5][j_6][j_7][j_8][j_9]}} \) form the elements of an orthogonal transformation matrix whose rows and columns are labelled by \( j_3, j_4, j_5 \) and \( j_7, j_8, j_9 \) respectively.

**Sum rules.**

One has:

\[
\left\{ \left[ r \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right]_3 \right\}_{j_3} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{j_6} \right\} = (-1)^{j_3 + j_4 + j_5 - 1 + j_7 - 1 - j_9} \sum_{x y z} \left\{ \left[ r \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right]_3 \right\}_{x} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{y} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{z} \right\} = (-1)^{j_3 + j_4 + j_5 - 1 + j_7 - 1 - j_9} \sum_{x y z} \left\{ \left[ r \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right]_3 \right\}_{x} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{y} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{z} \right\}
\]

Comparison of (A.6) with (A.6) gives:

\[
\sum_{x} (-1)^{x} \left\{ \left[ x \right] \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right\}_{x} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{y} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{z} \right\} = (-1)^{R + r + j_7 - j_9} \left\{ \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right\}_{R + r + j_7 - j_9} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{j_3} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{j_4} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{j_5} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{j_6} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{j_7} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{j_8} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)_{j_9} \right\},
\tag{A.9}
\]

where \( R = j_3 + j_4 + j_5 + j_6 + j_7 + j_8 + j_9 + p + q + r \).
Again, from

\[
\left\{ \begin{array}{c}
\{ r \left( \frac{1}{2}, \frac{1}{2} \right) \} \quad \{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right\} \left( \begin{array}{c}
\gamma \\
\end{array} \right) = (-1)^{J_6 + J_7 - J_8} \sum_{x, y} (-1)^{-Z} \left\{ \begin{array}{c}
\{ r \left( \frac{1}{2}, \frac{1}{2} \right) \} \quad \{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right\} \sqrt{[x][y][z][\gamma][\gamma][\gamma][\gamma][\gamma]} \left( \begin{array}{c}
\{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right) 
\]

\[
= (-1)^{J_6 + J_7 - J_8} \sum_{x, y} (-1)^{-Z} \left\{ \begin{array}{c}
\{ r \left( \frac{1}{2}, \frac{1}{2} \right) \} \quad \{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right\} \times \left( \begin{array}{c}
\{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right) \sqrt{[x][y][z][\gamma][\gamma][\gamma][\gamma][\gamma]} \left( \begin{array}{c}
\{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right) 
\]

we obtain:

\[
\sum_{x, y, z} (-1)^{-Z} \{ x \} \{ y \} \{ z \} \left( \begin{array}{c}
\{ J_6, J_7, J_8 \} \\
\end{array} \right) = (-1)^{J_6 + J_7 - J_8} \left( \begin{array}{c}
\{ J_6, J_7, J_8 \} \\
\end{array} \right). 
\]

(A.20)

We derive just one more interesting sum-rule:

\[
\left\{ \begin{array}{c}
\{ r \left( \frac{1}{2}, \frac{1}{2} \right) \} \quad \{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right\} = (-1)^{J_6 + J_7 - J_8} \sum_{x, y, z} (-1)^{-Z} \left\{ \begin{array}{c}
\{ r \left( \frac{1}{2}, \frac{1}{2} \right) \} \quad \{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right\} \times \left( \begin{array}{c}
\{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right) \sqrt{[x][y][z][\gamma][\gamma][\gamma][\gamma][\gamma]} \left( \begin{array}{c}
\{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right) 
\]

\[
= (-1)^{J_6 + J_7 - J_8} \sum_{x, y, z} (-1)^{-Z} \left\{ \begin{array}{c}
\{ r \left( \frac{1}{2}, \frac{1}{2} \right) \} \quad \{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right\} \times \left( \begin{array}{c}
\{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right) \sqrt{[x][y][z][\gamma][\gamma][\gamma][\gamma][\gamma]} \left( \begin{array}{c}
\{ \frac{1}{2}, \frac{1}{2} \} \\
\end{array} \right) 
\]

(A.21)
On the other hand:

\[
\left\{ (r_{13}, r_{14}, r_{17}, r_{15}, r_{18}) \right\}_q = (-1)^{j_y + j_z + \frac{1}{2}} \left\{ (r_{15}, r_{17}) \right\}_q
\]

\[
= (-1)^{j_y + j_z + \frac{1}{2} - \frac{1}{2}} \sum_{\lambda} (-1)^{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}} \left\{ (r_{15}, r_{17}) \right\}_q \left\{ (r_{15}, r_{17}) \right\}_q U_{\nu \nu} \left( j_y, j_z, j \right) \tag{A.12}
\]

Equating (A.12) to (A.11) we get, (using (A.4a)):

\[
\sum_{xyz} (-1)^{x-y} \left\{ (j_{13}, j_{14}, j_{15}, j_{16}, j_{18}) \right\}_q \left\{ (j_{15}, j_{17}) \right\}_q = \delta_{j_y} \delta_{j_z} \left( \frac{1}{2} \right) \left\{ (j_{15}, j_{17}) \right\}_q \left\{ (j_{15}, j_{17}) \right\}_q \tag{A.13}
\]

Special values.

Finally, due to the symmetries (A.5), any \( j \) can be brought to the position either of \( j_q \) or of \( q \).

Now, since

\[
\left\{ j_{13}, j_{14}, j_{15}, j_{16}, j_{18} \right\}_q = \left\{ j_{15}, j_{17} \right\}_q = \frac{\delta_{j_y} \delta_{j_z}}{\sqrt{[p][x]}},
\]

the summation in

\[
\left\{ (j_{13}, j_{14}, j_{15}, j_{16}, j_{18}) \right\}_q = (-1)^{x_1} \sum_{x} (-1)^{x} \left\{ (j_{13}, j_{14}, j_{15}, j_{16}, j_{18}) \right\}_q \left\{ (j_{15}, j_{17}) \right\}_q \left\{ (j_{15}, j_{17}) \right\}_q,
\]

obtained from (A.2), reduces to one term. so:

\[
\left\{ (j_{13}, j_{14}, j_{15}, j_{16}, j_{18}) \right\}_q = \frac{\delta_{p j_{16}} \delta_{j_{18}}}{\sqrt{[j_{16}][j_{18}]}} \left\{ (j_{13}, j_{14}, j_{15}, j_{16} \right\}_q \left\{ (j_{15}, j_{17}) \right\}_q \left\{ (j_{15}, j_{17}) \right\}_q \tag{A.14}
\]
And from (A.3) it (likewise) follows:

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 & j_4 & j_5 & j_6 & j_7 & j_8 & j_9
\end{pmatrix}
\begin{pmatrix}
  R + r + j_3 + j_6 + j_9
\end{pmatrix}
\delta_{r_1, j_3} \delta_{j_6, j_9} (-1)^{j_3 + j_6 + j_9}
\frac{\sqrt{\left| r \right| \left| j_9 \right|}}{R + r + j_3 + j_6 + j_9}
\begin{pmatrix}
  j_3 & j_4 & j_5 & j_6 & j_7 & j_8 & j_9
\end{pmatrix}
\begin{pmatrix}
  j_4 \quad j_1 \quad j_2 \quad j_3 \quad j_4 \quad j_5
\end{pmatrix}
\tag{A.15}
\]

Eqs. (A.14) and (A.15) give all the necessary information for evaluating any Jahn 12j-symbol with one zero.
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