SOME ASPECTS OF MAGNETOHYDRODYNAMICS STABILITY

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SOME ASPECTS OF MAGNETOHYDRODYNAMICS/STABILITY

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CHAPTER I

INTRODUCTION

Although essentially a branch of "classical" physics, it is surprising to find that magnetohydrodynamics has been developing only for the past twenty years and had become of great interest only in the last ten years. The only problem which, to the author's knowledge, was discussed before 1940, was the design of an electromagnetic pump. This was considered by Hartman and Lazarus\(^1\) who considered passing an electrically conducting liquid between the pole pieces of a magnet. However, between 1940 and 1950 Alven and his Swedish collaborators developed the theory in order to attempt to describe such phenomena as sunspots, the sun's magnetic field, aurora and the mechanism by which cosmic ray particles are accelerated. The progress of the theory up to 1950 is described in Alven's treatise\(^2\). Interest in the subject increased after Babcock's discovery in 1947 of a star with a variable magnetic field.

In the early 1950's it was first proposed that the earth's magnetic field might be due to the rotation of a core of highly conducting liquid metal. This problem aroused some interest and it became known as the dynamo problem. Recent papers on this subject are, amongst others, by Herzenberg\(^3\), Herzenberg and Lowes\(^4\). The subject really expanded only

\* The attention of the author has been drawn by one of the examiners to earlier work by Larmor, Cowling, Chapman and Ferraro. See references in Elsasser: Amer. J. Physics 23, 590 (1955) and 24, 85 (1956).
when it was realised that it might be able to generate power from a controlled thermonuclear reaction. The method proposed was to heat a gas discharge to a sufficiently large temperature in order that the necessary nuclear processes could take place. In view of the extremely high temperatures involved, any normal means of confining the gas was clearly impossible and the only way in which confinement could be achieved was by using a magnetic field. At the very high temperatures involved, the gas becomes ionised so that ions and electrons move along the magnetic lines of force. Thus we see that if we take a suitable magnetic field configuration we might be able to confine the gas or plasma as it is called when in this state of ionisation, at high temperatures. However it was found that the plasma column is extremely unstable, being particularly unstable towards end effects if the column was linear. The way out of this difficulty was to have the plasma column in the shape of a torus and again it was found that any inhomogeneities in the magnetic field caused instability. There are a lot of experimental details in the Proceedings of the Geneva Conference on the uses of peaceful atomic energy. (6)

There are three methods of tackling the problem of magneto-hydrodynamic stability. The first, as used by Spitzer in his monograph (7), is to approach the topic from Boltzmann's equation. This theory has been developed by, amongst others, Chandrasekhar, Vaufman, Breuckner and Watson (8), and Chew, Low and Goldberger (9). The second is to consider the motion of an individual electron
and ion in the external magnetic field. This method has been considered by Rosenbluth and Longmire.\(^{(10)}\)

The third method and the only one which we shall use in this thesis is to treat the field quantities as continuous functions, so that we can use the usual field equations of hydrodynamics and electromagnetism. Conditions for this assumption to be valid have been given by Bernstein et al.\(^{(11)}\)

They are

a) Quadratic terms in the velocity and current density are negligible.

b) The plasma is locally quasi-neutral.

c) The ratio of the electron mass to the ion mass is negligible compared with unity.

d) Unless otherwise stated, (i.e. in Chapter 2), all heating effects are to be neglected.

We shall also assume a scalar pressure, the condition for this being that there should be many particle collisions during a characteristic time and the displacement current \(\frac{\partial \mathbf{E}}{\partial t}\) in Maxwell's equations, this being justifiable if the magnetohydrodynamic velocity \((c_0^2 M)^{1/2} H\) is small compared with the velocity of light.

The mathematical methods which we use in this thesis are the usual ones of classical mechanics - (a) straightforward perturbation theory, (b) variational methods, and (c) considerations of a system's potential energy. To apply (a) we look for a steady state solution of our field equations. If
this solution is described by a set of functions \( \chi_i(x) \) then we perturb the system so that
\[
\chi_i(x) \rightarrow \chi_i(x) + \chi_i(x, t)
\]
(1.1.1)
it being assumed that the \( \chi_i(x, t) \) are sufficiently small that squares and products of them and their derivatives are negligible. We then try and separate any particular dependence we are interested in. Since we are considering the stability of the system, it is the time dependence of the various \( \chi_i(x, t) \) that we are interested in. Thus we write
\[
\chi_i(x, t) = \chi_i(x) e^{\omega t}
\]
(1.1.2)
say. If \( \omega \) is real then it is obvious that our stability condition is \( \omega < 0 \) and if \( \omega = 0 \) we have what is known as marginal stability. If \( \omega \) is imaginary, then if \( \omega \rightarrow 0 \) we have marginal stability. On the other hand if \( \text{Re } \omega \rightarrow 0 \) but \( \text{Im } \omega > 0 \) then when instability does set in we shall have oscillations of everincreasing amplitude. This point is discussed more fully in section 2.4. Throughout this thesis, unless it is explicitly stated otherwise, we shall consider marginal stability. The variational method, (b), is demonstrated in sections 2.6 and 2.7.

The third method, (c), of discussing stability by energy considerations is described as follows. We state our equilibrium solution and calculate its potential energy and then deform the boundary of the fluid and calculate the change in potential energy. If this change is positive, it means that the system
is stable to that particular boundary deformation, and if the change is negative then the system is unstable. (By potential energy we mean here all energy except kinetic, i.e. gravitational, magnetic and, if the medium is compressible, internal energy.)

The plan of this thesis is as follows. In Chapter 2 we consider the classical problem of a fluid layer heated from below. We have an external magnetic field and also consider the case when there is also an externally applied rotation. Our main conclusion here is that the magnetic field delays the onset of instability, but when there is also an applied rotation the motion of the fluid shows some unexpected behaviour before the effect of the magnetic field predominates. In Chapter 3 we consider another classical problem of hydrodynamics - that of the stability of a fluid confined between two rigid coaxial cylinders, and again we find that the presence of the magnetic field, in the direction of the axes of the cylinder, tends to inhibit the onset of instability. We also note here the analogous features of this problem with that discussed in Chapter 2.

In Chapter 4 we first obtain a limit on the value of the magnetic field that can exist in a mass of self-gravitating fluid without instability setting in. We find that the magnetic energy must be less than the gravitational potential energy of the system. This is rather surprising for in Chapters 2 and 3 we found that the larger the magnitude of the magnetic field, the greater the inhibiting effect it had on the onset of instability. The reason for this limit existing in this particular case may be due to a conflict between the tendency of the fluid to move along
the lines of magnetic force and the self gravitational forces. We go on to use the energy method described previously to consider the stability of fluid spheres, spheroids and cylinders under the action of various internal and external magnetic fields. When the system is axisymmetric then we find that the problems can be reduced to the determination of four scalar functions and we are able to solve for these functions.

These models are of astrophysical interest for they give very rough approximations to stars and galaxies. The roughness of the approximation can be seen from the fact that nuclear sources of energy, which obviously play a large part in the behaviour of a star and heating effects are ignored. For a full discussion of these topics see articles by Ledoux and Walraven\(^{(12)}\), and Ledoux\(^{(12)}\).

In Chapter 5, we consider the problem of the magnetohydrodynamic stability of a compressible fluid. This is the only chapter in this thesis in which compressibility is taken into account. We approach the subject by showing that certain integrals are constants of the motion and deriving equations which the equilibrium quantities must satisfy. In the special case of incompressibility, the equations reduce to those of force free magnetic fields which are of astrophysical interest. We show that these force free fields are stable. Returning to the more general problem, in the case of axial symmetry we are able, as in Chapter 4, to reduce the problem to the determination of six scalar functions. The six resulting equations are integrated and a complete set of integrals of the motion obtained.
In Chapter 6, we consider the stability of twisted magnetic fields. These are magnetic fields which have no radial component (in cylindrical polar coordinates). Although we shall start with axially symmetric configurations we shall see that as soon as we relax the condition that all perturbations should also be axisymmetric then the problem of the stability becomes quite involved even for the simplest of mathematical models. These twisted magnetic fields are of interest in the theory of sunspots and also in the theory of the 'pinch' effect, which we shall not be discussing.

Throughout this thesis, except in Chapter 5, we shall be considering magnetohydrodynamic models. Even with the simplest models, as in Chapter 2, the mathematics required for the determination of the stability of the model is always tedious, even if easy to work out in principle. The only time when we consider a general system is in Chapter 5, and here again, we find that the equations obtained are not soluble in the general case. However, we can solve them in the special case when the system is taken to be axisymmetric and we further assume that it retains this property when disturbed. This is an extremely artificial assumption, for as we have noted above, in Chapter 6 we shall see that if we relax this condition, then even with the simplest models, the stability criteria become quite complicated.
The most general problem which we consider in this chapter is the stability of the motion of an electrically conducting fluid, confined to move between two parallel planes. The fluid is assumed to be viscous and it is subject to a rotation, gravity, and an externally applied magnetic field. It is also heated from below, the bounding planes being maintained at constant temperature. The equations of the problem are the Navier Stokes equation for the motion of a viscous fluid. This equation is coupled to Maxwell's equations by the Lorentz force on a moving conductor. We also use Ohm's Law and the equation of heat conduction. We may formulate the equations as follows (in e.m.u. units)

\[ \begin{align*}
\frac{\partial u}{\partial t} + f (u \cdot \nabla) u - \mu \nabla^2 u &= -\nabla p + \sigma \nabla^2 u + \nabla \cdot \mathbf{J} + 2\mu \nabla \cdot \mathbf{u} \\
\nabla \cdot \mathbf{H} &= 4\pi \mathbf{j} \\
\nabla \cdot \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \\
\nabla \cdot \mathbf{H} &= 0 \\
\mathbf{j} &= \sigma (\mathbf{E} + \mu \nabla \cdot \mathbf{H}) \\
\frac{\partial T}{\partial t} + (u \cdot \nabla) T &= k \nabla^2 T.
\end{align*} \]

where, in writing (2.1.2) we have neglected the displacement
current. In writing these equations we use the following notation: \( \mathbf{u} \) is the fluid velocity, \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic field intensities respectively; \( \mathbf{j} \) the current density, \( \Omega \) the angular velocity of the applied rotation, \( \Phi \) the potential of the external forces acting on the system, \( p \) the pressure, \( \rho \) the mass density, \( T \) the temperature, \( \mu \) the magnetic permeability and \( \sigma, \sigma \) and \( k \) are respectively the coefficients of kinematic viscosity, electrical conductivity and thermometric conductivity.

Eliminating \( \frac{1}{\rho} \) between (2.1.1) and (2.1.2) gives

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \mu \mathbf{E} + \frac{\mu}{\rho} (\mathbf{H} \cdot \nabla) \mathbf{H} + \nabla^2 \mathbf{u} + \frac{1}{\sigma} \nabla \Phi + 2 \mathbf{u} \times \Omega \tag{2.1.7}
\]

We now make our first approximation, which is due to Rayleigh\(^2\). We assume that any variation in the fluid density \( \rho \) is due only to temperature variations and that we consider any variation in \( \rho \) only in so far as it influences the effect of the external potential. To do this we replace the term \( \rho \nabla \Phi \) in (2.1.7) by \( \rho_o (1 - \alpha \Delta T) \nabla \Phi \) where \( \rho_o \) is the density at a mean temperature \( T_o \), \( \Delta T \) is the difference between the local temperature and \( T_o \) and \( \alpha \) is the coefficient of volume expansion of the fluid. Elsewhere in (2.1.7) we replace \( \rho \) by \( \rho_o \) so that (2.1.7) now reads

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \mu \mathbf{E} + \frac{\mu}{\rho_o} (\mathbf{H} \cdot \nabla) \mathbf{H} + \frac{1}{\sigma} \nabla \Phi + 2 \mathbf{u} \times \Omega \tag{2.1.8}
\]
Since we have assumed that $\rho$ is independent of space and time it follows from the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (2.1.9)$$

that

$$\nabla \cdot \mathbf{u} = 0 \quad (2.1.10)$$

Since we are assuming that the external forces on the fluid are gravitational,

$$\nabla \Phi = -g \lambda \quad (2.1.11)$$

where $\lambda$ is a unit vector in the direction of the upward vertical, so that

$$\Phi = -g \lambda \cdot \mathbf{r} \quad (2.1.12)$$

where $\mathbf{r}$ is the radius vector from the origin.

We now turn our attention to the heat conduction equation (2.1.6). Since the fluid is heated from below, we may assume the presence of an adverse temperature gradient $-\beta$ ($\beta > 0$), which is maintained in the direction of $\lambda$. Since we assume that the boundaries are maintained at constant temperatures, we can write

$$T = T_0 - \beta \lambda \cdot \mathbf{r} + \Theta \quad (2.1.13)$$

where $T_0$ is the mean temperature already defined, and $\Theta(r,t)$ is the deviation at $(\mathbf{r}, t)$ of the temperature from its local mean value and $\Theta(r,t)$ vanishes on the boundaries. It follows that

$$\nabla T = T - T_0 = -\beta \lambda \cdot \mathbf{r} + \Theta \quad (2.1.14)$$
Substituting (2.1.12) and (2.1.14) into (2.1.8) gives

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \frac{1}{\mu \eta_0} (u \cdot \nabla) H = \nu \nabla^2 u + \gamma \theta \lambda - \nabla \Pi \tag{2.1.15}
\]
where we have written

\[
\Pi = \frac{1}{\beta_0} \rho + \frac{\mu}{8 \pi \eta_0} H^2 + g A \cdot \mathbf{e} - \frac{1}{2} \beta \sigma (\mathbf{A} \cdot \mathbf{e})^2 \tag{2.1.16}
\]
and

\[
\gamma = \alpha \beta \tag{2.1.17}
\]
Using (2.1.13) to eliminate \( T \) from (2.1.6) gives

\[
\frac{\partial \theta}{\partial t} - \rho A \cdot u + (u \cdot \nabla) \theta = \kappa \nabla^2 \theta \tag{2.1.18}
\]
Combining Maxwell's equations and Ohm's Law gives

\[
\nabla \cdot (\nabla \times \mathbf{H}) = 4 \pi \sigma \left[ -\rho \frac{\partial H}{\partial t} + \gamma \nabla \cdot (u \times \mathbf{H}) \right] \tag{2.1.19}
\]
and expanding the vector products, using (2.1.4) and (2.1.10) gives

\[
\frac{\partial H}{\partial t} + (u \cdot \nabla) H - (H \cdot \nabla) u = \mathbf{Z} \nabla^2 H \tag{2.1.20}
\]
where

\[
\mathbf{Z} = (4 \pi \sigma \rho)^{-1} \tag{2.1.21}
\]

We now linearise the equations by means of perturbation theory. Consider first the system to be in an equilibrium configuration with

\[
u < V, \quad H = H_0, \quad \theta = 0 \quad \text{and} \quad \Pi = \Pi_0 \tag{2.1.22}
\]
Then \( \nu, H_0 \) and \( \Pi_0 \) satisfy

\[
(u \cdot \nabla) \nu - \frac{\mu}{\eta_0} (H_0 \cdot \nabla) H_0 = \nabla^2 \nu + 2 \nu \nabla \lambda - \nabla \Pi_0 \tag{2.1.23}
\]
\begin{align*}
(\mathbf{v} \cdot \nabla) H_0 - (H_0 \cdot \nabla) \mathbf{v} &= \nabla^2 H_0 
\tag{2.1.24}
\int \lambda \cdot \mathbf{v} &= 0 
\tag{2.1.25}
\end{align*}

together with
\begin{align*}
\nabla \cdot \mathbf{v} &= \nabla \cdot H_0 = 0 
\tag{2.1.26}
\end{align*}

Now imagine the system to be perturbed so that
\begin{align*}
\mathbf{\nu} &= \mathbf{v} + \mathbf{\nu} \quad H_0 = H_0 + \lambda, \quad \Pi = \Pi_0 + \Pi', \quad \Theta = \Theta 
\tag{2.1.27}
\end{align*}

where \( \mathbf{\nu}, \lambda, \Pi' \) and \( \Theta \) are assumed to be sufficiently small in order that we may neglect products and powers of them and their derivatives. Substituting (2.1.27) into (2.1.20), (2.1.18), (2.1.14) and (2.1.10) gives, using (2.1.23) - (2.1.26),
\begin{align*}
\frac{\partial \mathbf{\nu}}{\partial t} + (\mathbf{\nu} \cdot \nabla) \mathbf{\nu} + (\mathbf{v} \cdot \nabla) \mathbf{\nu} &= \frac{\mathbf{L}}{\mu \rho_0} \left\{ \left( \mathbf{v} \cdot \nabla \right) H_0 + \left( H_0 \cdot \nabla \right) \mathbf{v} \right\} \\
&= \nabla^2 \mathbf{\nu} + \gamma \Theta \lambda - \nabla \Pi' + 2 \nabla \times \mathbf{\lambda} 
\tag{2.1.28}
\end{align*}

\begin{align*}
\frac{\partial \lambda}{\partial t} + (\mathbf{\nu} \cdot \nabla) H_0 + (\mathbf{v} \cdot \nabla) \lambda &= -\left( \mathbf{H}_0 \cdot \nabla \right) \lambda - \left( H_0 \cdot \nabla \right) \mathbf{v} - \left( \mathbf{v} \cdot \nabla \right) H_0 \\
&= \nabla^2 \lambda 
\tag{2.1.29}
\end{align*}

\begin{align*}
\frac{\partial \Theta}{\partial t} - \beta \lambda \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla) \Theta &= \kappa \nabla^2 \Theta 
\tag{2.1.30}
\end{align*}

\begin{align*}
\nabla \cdot \lambda &= \nabla \cdot \mathbf{v} = 0 
\tag{2.1.31}
\end{align*}
**Section 2**

Chandrasekhar\(^{(3)}\)-(7) in a series of papers has solved equations (2.1.28) - (2.1.31) in the special case when \( V = 0 \) and \( H_0 \) is a constant. In this case (2.1.28) - (2.1.30) reduce to

\[
\frac{\partial \nu}{\partial t} - \frac{\rho}{4 \pi \rho_0} (H_0 \cdot \nabla) \nu = \varsigma \nabla^2 \nu + \gamma \Theta - \nu \nabla \nu + 2 \nu \nabla \Theta \quad (2.2.1)
\]

\[
\frac{\partial }{\partial t} - (H_0 \cdot \nabla) \nu = \zeta \nabla^2 \nu \quad (2.2.2)
\]

\[
\frac{\partial \Delta}{\partial t} - \beta \Delta \cdot \nabla = \kappa \nabla^2 \Theta \quad (2.2.3)
\]

We commence our analysis by taking the divergence of (2.2.1) which gives, using (2.1.31)

\[
\nabla^2 \Pi' = \gamma \Delta \cdot \nabla \Theta + 2 \Delta \cdot \nabla (\nabla \nu) \quad (2.2.4)
\]

Now take the curl of (2.2.1) and (2.2.2) to give

\[
\frac{2}{j} \left( \nabla \nu \right) - \frac{\rho}{4 \pi \rho_0} (H_0 \cdot \nabla) (\nabla \nu) = \gamma \nabla^2 \left( \nabla \nu \right) - \gamma \Delta \cdot \nabla \Theta + 2 \left( \Delta \cdot \nabla \nu \right) \quad (2.2.5)
\]

\[
\frac{2}{j} \left( \nabla \nu \right) - (H_0 \cdot \nabla) (\nabla \nu) = \zeta \nabla^2 \left( \nabla \nu \right) \quad (2.2.6)
\]

and multiplying these last two equations scalarly by \( \Delta \) gives

\[
\frac{2}{j} \left[ (\nabla \nu) \cdot \Delta \right] - \frac{\rho}{4 \pi \rho_0} \left( H_0 \cdot \nabla \right) \left[ (\nabla \nu) \cdot \Delta \right] = \gamma \left[ \nabla^2 \left( \nabla \nu \right) \cdot \Delta \right] + 2 \left( \Delta \cdot \nabla \nu \right) (\nabla \nu) \quad (2.2.7)
\]

\[
\frac{2}{j} \left[ (\nabla \nu) \cdot \Delta \right] - (H_0 \cdot \nabla) \left[ (\nabla \nu) \cdot \Delta \right] = \zeta \left[ \nabla^2 \left( \nabla \nu \right) \cdot \Delta \right] \quad (2.2.8)
\]

Taking the scalar product of \( \Delta \) with (2.2.2) gives

\[
\frac{2}{j} (\nu \cdot \Delta) - (H_0 \cdot \nabla) (\nu \cdot \Delta) = \zeta \nabla^2 (\nu \cdot \Delta) \quad (2.2.9)
\]
Since, by (2.1.31)
\[ \nabla \cdot (\nabla \times \psi) = -\nabla^2 \psi \quad \nabla \cdot (\nabla \times A) = -\nabla^2 A \] (2.2.10)
taking the curl of (2.2.5) gives
\[ -\frac{2}{\gamma} \nabla^2 (\nabla \times \psi) = -\frac{\nu}{4\pi \rho_0} (H_0 \cdot \nabla)(-\nabla^2 \psi) \]
\[ = \nabla^2 (\nabla \times \psi) + 2(\nabla \cdot \nabla)(\nabla \times \psi) - \gamma \phi \nabla^2 - \Delta \nabla \phi \theta \] (2.2.11)
and multiplying this scalarly by \( \nabla \theta \) gives
\[ \frac{2}{\gamma} \nabla^2 (\nabla \times \psi, \theta) = \frac{\nu}{4\pi \rho_0} (H_0 \cdot \nabla) \{ \nabla^2 (\phi, \theta) \} + \nabla^4 (\nabla \times \psi, \theta) \]
\[ + 2(\nabla \cdot \nabla)(\nabla \times \psi) - \gamma \phi \nabla^2 - (\gamma \phi)^2 \theta \] (2.2.12)

If we introduce a new set of variables defined by
\[ \psi = \nabla \cdot \phi \quad \chi = (\nabla \psi) \cdot \phi \quad \phi = (\nabla \times \psi) \cdot \phi \quad \chi = \psi \cdot \chi \] (2.2.13)

then we can write our basic set of equations (2.2.12), (2.2.7), (2.2.8), (2.2.9) and (2.2.3) respectively in the form
\[ \frac{\partial \psi}{\partial t} = \frac{\nu}{4\pi \rho_0} (H_0 \cdot \nabla) \psi + \nabla \psi - \gamma \phi \nabla^2 - (\gamma \phi)^2 \theta + 2 \nabla \cdot \nabla \psi \] (2.2.14)
\[ \frac{\partial \chi}{\partial t} = \frac{\nu}{4\pi \rho_0} (H_0 \cdot \nabla) \phi + \nabla^2 \chi + 2 \nabla \cdot \nabla \psi \] (2.2.15)
\[ \frac{\partial \phi}{\partial t} = (H_0 \cdot \nabla) \phi + 2 \nabla^2 \phi \] (2.2.16)
\[ \frac{\partial \psi}{\partial t} = (H_0 \cdot \nabla) \psi + \nabla^2 \chi \] (2.2.17)
\[ \frac{\partial \theta}{\partial t} = \beta \psi + \kappa \nabla^2 \theta \] (2.2.18)
Section 3

Before we can proceed much further without discussion of the problem we must formulate the boundary conditions. We take a Cartesian system of coordinates in which the positive direction of the $z$-axis is in the upward vertical direction so that we may take the boundary planes to be at $z = 0$ and $z = d$. Since we are assuming that the boundaries are maintained at constant temperatures we must have $\theta = 0$ on a boundary. Also the normal component of the velocity must be zero on a boundary, i.e. $w = 0$ on a boundary. Our next conditions depend on whether the boundary is free or rigid. If it is rigid and we postulate no slip on it then on a rigid boundary $\nu = 0$. Since $\nabla \cdot \nu = 0$ it follows that $\frac{\partial w}{\partial z} = 0$ on a rigid boundary and since $\nu = (\nabla \cdot \nu)$ we must also have $\phi = 0$. On the other hand, if the boundary is free, we require that the viscous stresses must vanish at the boundary. This implies(8) that $\frac{\partial w}{\partial z} = \frac{\partial \phi}{\partial z} = 0$ on a free surface. We now consider the electromagnetic boundary conditions. We can limit the perturbations in the magnetic field to the region between the planes if we consider the boundaries to be perfect conductors. This assumption implies the existence of surface currents and charges which will prevent any electric field in the $xy$ plane and any magnetic field in the $z$ direction from penetrating into the boundaries, i.e. on a perfectly conducting rigid boundary we require $E_x = E_y = 0$ and $\nu = 0$. If the boundary is free and adjoins a vacuum then we require $E_z = 0$ on the boundary in order to prevent any migration of charge across the surface. We may summarise these conditions
as follows. We require
\[ w = \theta = 0 \text{ at } z = 0 \text{ and } z = d \] (2.3.1)

At a perfectly conducting rigid surface
\[ \frac{\partial w}{\partial z} = f = 0 \] (2.3.2)
and
\[ E_x = E_y = \mathcal{E} = 0 \] (2.3.3)

For a free surface we must have
\[ \frac{\partial^2 w}{\partial z^2} = \frac{\partial f}{\partial z} = 0 \] (2.3.4)
and if it adjoins a vacuum
\[ E_z = 0 \] (2.3.5)

Section 4.

We now discuss the mechanism by which instability can set in. There are two ways in which this may happen - by convective instability and by overstability. Consider what happens when the time dependence of each dependent function is given by \( e^{\imath t} \).

If \( \omega \) is imaginary and \( \Re \omega \to 0 \) but \( \Im \omega \to 0 \) then when instability sets in we shall have oscillations of ever increasing amplitude. This is what is meant by overstability. When they were considering the problem of a fluid layer heated from below, Pellew and Southwell(9) showed that if there is maintained convective motion in which the dependent functions are proportional
to $e^{\omega t}$ then $\omega$ must be real. Thus the changeover from stability to instability is characterised by the vanishing of $\omega$. We shall assume that this principle of marginal stabilities can be carried over into the present problem. Chandrasekhar has investigated the validity of this assumption in the case when no rotation is present. He shows that, under terrestrial conditions (e.g. for liquid mercury), it is a valid assumption. However under astrophysical conditions in which

$$\kappa \gg \eta \gg 0 \quad (2.4.1)$$

the assumption holds only if

$$\rho \frac{H_0^2}{\eta} \alpha^2 < 9 \frac{\eta^2}{\kappa} \gamma^3 p_0 \quad (2.4.2)$$

On the other hand, over-stability may occur if

$$\rho \frac{H_0^2}{\eta} \alpha^2 > 9 \frac{\eta^2}{\kappa} \gamma^3 p_0 \quad (2.4.3)$$

We see that marginal stability is characterised by putting $\frac{2}{\lambda t} = 0$ in equations (2.2.14) - (2.2.18) and over-stability by having all dependent functions proportional to $e^{\lambda t}$. Hence we need only develop the theory for the case of over-stability and then put $p = 0$ to give the case of marginal stability.

Section 5.

We shall discuss first the comparatively simple problem which arises when $\Lambda$, $H_0$ and $\Sigma$ act in the same direction. Putting
\[ \Lambda = (0, 0, 1) \quad H_0 = H_0(0, 0, 1) \quad \mathcal{N} = \mathcal{N}(0, 0, 1) \tag{2.5.1} \]

in \((2.2.14) - (2.2.18)\) we obtain

\[ \frac{2}{\tau t} \nabla^2 w = \frac{\mu H_0}{3 \eta J_0} \frac{2}{J_2} \nabla^2 x + \frac{2}{J_2} \nabla^2 w - \frac{J_2}{J_1} \left( \frac{\sum_{k=1}^{n} x_k}{x^2} + \frac{\sum_{k=1}^{n} y_k}{y^2} \right) \theta + 2 \mathcal{N}^2 \frac{J_2}{J_2^2} \tag{2.5.2} \]

\[ \frac{2}{\tau t} f = \frac{\mu H_0}{3 \eta J_0} \frac{2}{J_2} \nabla^2 f + \frac{2}{J_2} \nabla^2 \phi + 2 \mathcal{N}^2 \frac{J_2}{J_2^2} \omega \tag{2.5.3} \]

\[ \frac{2}{\tau t} \phi = H_0 \frac{2}{J_2} f + \eta \nabla^2 \phi \tag{2.5.4} \]

\[ \frac{2}{\tau t} \kappa = H_0 \frac{2}{J_2} w + \eta \nabla^2 \kappa \tag{2.5.5} \]

\[ \frac{2}{\tau t} \theta = \rho \omega + \kappa \nabla^2 \Theta \tag{2.5.6} \]

By analogy with the classical hydrodynamical problem\(^{(9)}\) we look for solutions of \((2.5.2) - (2.5.6)\) which characterise cellular symmetry. Thus we put

\[ F(x, y, z, t) = f(x, y) g(z) e^{\epsilon pt} \tag{2.5.7} \]

where \(f(x, y)\) satisfies

\[ d^2 \left( \frac{2}{\pi k^2} + \frac{2}{\pi l^2} \right) f = -a^2 f \tag{2.5.8} \]

where \(a\) is a constant which arises during the separation of variables. It is related to the size of the cell walls by

\[ a^2 = c^2 \pi^2 \left( \frac{m^2}{L_1^2} + \frac{n^2}{L_2^2} \right) \]

for the case of rectangular cells of sides \(L_1\) and \(L_2\). Here \(m\) and \(n\) are arbitrary even integers. If the cells are hexagonal
with side L then
\[ a = \frac{4}{3} \frac{n \pi}{L} \]
where \( n \) is an integer. It has been shown\(^{(9)}\) that \( a \) is real. Thus we substitute the following expressions into (2.5.2) - (2.5.6)

\[ w(x,y,z,t) = f(x,y) \phi(z,t) e^{ipt} \]
\[ f(x,y) = f(x,y) Z(t) e^{ipt} \]
\[ \phi(x,y,z,t) = \psi(x,y) \Phi(t) e^{ipt} \]
\[ \psi(x,y,z,t) = \psi(x,y) \chi(t) e^{ipt} \]
\[ \Theta(x,y,z,t) = \psi(x,y) \Theta(t) e^{ipt} \]  

(2.5.9)

(Note that the function \( \Phi(t) \) introduced here is not to be confused with the function \( \Phi \) defined by (2.1.12)).

Doing this and changing our scale so that \( z \) is measured in units of \( d \) (i.e. our planes are now at \( z = 0 \) and \( z = 1 \)) we obtain

\[ (D^2 - a^2) (D^2 - a^2 - \frac{i \beta d^2}{\kappa}) W - \frac{2 \Lambda d}{\kappa} DZ + \frac{\mu H_0 d}{4 \pi \rho_0} D(\nabla^2 - a^2) X \]
\[ = \frac{\gamma \alpha d^2}{\kappa} W \]  

(2.5.10)

\[ (D^2 - a^2 - \frac{i \beta d^2}{\kappa}) Z = -\frac{2 \Lambda d}{\kappa} DW - \frac{\mu H_0 d}{4 \pi \rho_0} D \Phi \]  

(2.5.11)

\[ (D^2 - a^2 - \frac{i \beta d^2}{\kappa}) \Phi = -\frac{H_0 d}{\kappa} DZ \]  

(2.5.12)

\[ (D^2 - a^2 - \frac{i \beta d^2}{\kappa}) \chi = -\frac{H_0 d}{\kappa} DW \]  

(2.5.13)

\[ (D^2 - a^2 - \frac{i \beta d^2}{\kappa}) \Theta = -\frac{\beta d^2}{\kappa} W \]  

(2.5.14)

where
\[ D \equiv \frac{d}{dz} \]  

(2.5.15)
Our boundary conditions (2.3.1) - (2.3.5) are now

\[ W = \psi = 0 \quad \text{at} \quad z = 0, 1 \quad (2.5.16) \]

\[ \partial W = \zeta = \chi = 0 \quad \text{on a perfectly conducting rigid surface} \quad (2.5.17) \]

\[ \partial^2 W = \partial^2 z = 0 \quad \text{on a free surface} \quad (2.5.18) \]

\[ \Phi = 0 \quad \text{on a free surface adjoining a vacuum} \quad (2.5.19) \]

After a process of elimination we may obtain the following equation for \( W \)

\[
\left( D^2 - a^2 - \frac{c p d^2}{\kappa} \right) \left[ \left( D^2 - a^2 \right) \left( D^2 - a^2 - \frac{c p d^2}{\kappa} \right) - Q D^2 \right]^2 
+ T D^2 \left( D^2 - a^2 - \frac{c p d^2}{\kappa} \right)^2 W 
= -Ra^2 \left( D^2 - a^2 - \frac{c p d^2}{\kappa} \right) \left( D^2 - a^2 - \frac{c p d^2}{\kappa} \right) - Q D^2 W
\]

where, for brevity, we have introduced new constants

\[
Q = \frac{\rho \Omega^2 d^2}{\kappa n_0^2 \nu} \quad T = \frac{\kappa n_0^2 d^4}{\nu^2} \quad R = \frac{\kappa \rho d^4}{\kappa \nu^2}
\]

Since we are considering the onset of instability either by convection or by overstability, \( p \) is either zero or real. The physical conditions of the problem demand that \( R \) be real. Hence, as it were, we may take the real and imaginary parts of (2.5.20) to give a double eigenvalue problem for \( p \) and \( R \), for given values of \( Q \), \( a^2 \) and \( T \). Since (2.5.20) is a linear
differential equation with constant coefficients, there is no difficulty in principle in solving it but there would be an enormous amount of work involved in its solution. However, it is possible to solve (2.5.20) explicitly in the case when both bounding surfaces are free. The function

\[ W(z) = \sin n\pi z \]  

(2.5.22)

satisfies (2.5.20) and also the requisite boundary conditions, if

\[
\left( n^2 \sigma^2 + a^2 + \frac{c^2 b d^2}{k} \right) \left[ (n^2 \sigma^2 + a^2 + \frac{c^2 b d^2}{k}) \left( n^2 \sigma^2 + a^2 + \frac{c^2 b d^2}{k} \right) + Qn^2 \right]^2
\]

\[ + Tn^2 \left( n^2 \sigma^2 + a^2 + \frac{c^2 b d^2}{k} \right)^2 \right] = R \left( n^2 \sigma^2 + a^2 + \frac{c^2 b d^2}{k} \right) \left[ (n^2 \sigma^2 + a^2 + \frac{c^2 b d^2}{k}) \left( n^2 \sigma^2 + a^2 + \frac{c^2 b d^2}{k} \right) + Qn^2 \right]^2
\]

(2.5.23)

If we put \( p = 0 \) then we can show that for marginal stability \( R \) as a function of \( n \) has a minimum when \( n = 1 \) and we assume that this holds for overstability. Hence, putting \( n = 1 \) in (2.5.23) and equating real and imaginary parts gives

\[
R = \pi^4 \frac{1}{\lambda} \left[ (1+x)^2 - s_1 s_3 \right] p^2 + Q_1 \frac{(1+x)^2 s_3 s_2 b^2}{(1+x)^2 + s_2 p^2} + \frac{T_1}{1+x} \left[ (1+x)^2 s_2 p^2 + Q_1 \right] \left[ (1+x)^2 - s_1 s_3 \right] p^2
\]

(2.5.24)

\[
+ \left[ (1+x)^2 - s_1 s_3 \right] p^2 + Q_1 \right] p^2 + (1+x)^2 (s_1 + s_3) (s_1 + s_2) p^2
\]

\[
O = \frac{T_1}{1+x} \frac{(s_3 - s_1) (1+x)^2 + (s_1 + s_3) Q_1 + s_2 (s_3 - s_1) p^2}{(1+x)^2 - s_1 s_3 p^2 + Q_1 \right]^2 + (1+x)^2 (s_1 + s_2)^2 p^2}
\]

(2.5.25)

\[ + Q_1 \frac{s_3 - s_2}{(1+x)^2 + s_2 p^2} + s_1 + s_3 \]
where
\[ \chi = \frac{a^2}{\pi^3}, \quad Q_1 = \frac{Q}{\pi^2}, \quad T_1 = \frac{T}{\pi^4} \]
\[ S_1 = \frac{d^2}{\pi^4}, \quad S_2 = \frac{d^2}{\pi^6}, \quad S_3 = \frac{d^2}{\pi^8} \]  
(25.26)

For \( p = 0 \), (2.5.24) reduces to equation (60) of Chandrasekhar's paper, i.e., now

\[ R = \pi^4 \frac{1 + x}{x} \frac{\left[ (1 + x)^2 + Q_1 \right] + T_1 (1 + x)}{x \left[ (1 + x)^2 + Q_1 \right]} \]  
(2.5.27)

Numerical solutions of equations (2.5.27) and (2.5.24), (2.5.25) have been calculated by Chandrasekhar (5, 7), and we show his results in graphical form. The calculations are made for liquid mercury at room temperature so that terms in \( S_2 \) arising in (2.5.24), (2.5.25) can be neglected.

Graph of \( \log_{10} \), \( R_c \) against \( \log_{10} Q_1 \) for cellular convection (—) and overstability (⋯) for a given \( T_1 \). The overstability curve is drawn for \( \frac{S}{\kappa} = 0.025 \).
The dependence on $Q_1$ of the wave number $a$ (in units of $a^{-1}$) of the disturbance at which instability sets in convection (---) and overstability (---) for given values of $T_1$.

The figures are taken from Chandrasekhar (7).

We see that the marginal stability graphs exhibit strong discontinuities. For instance for certain ranges of values of the parameters $Q_1$ and $T_1$, the function $R_e(a)$ has two minima, one for when $Q_1$ is less than a certain value, the other when $Q_1$ is greater than this critical value. For instance when $T_1 = 10^5$ and $Q_1 = 80$, the minima are at $a = 18.3$ and $3.38$, so that $R_e = 4.0 	imes 10^5$ and $4.78 	imes 10^5$ while for $Q_1 = 100$ the minima are at $a = 18.2$ and $3.37$ where $R_e = 3.98 	imes 10^5$ and $3.93 	imes 10^5$ respectively. Thus for $Q_1$ slightly less than 100 the cellular wave number for marginal stability decreases suddenly from $18.2$ to $3.4$. This behaviour holds for $T_1 > 2.5 	imes 10^3$. 

---

**Fig. II**

![Graph showing dependence of $a$ on $Q_1$.](image-url)
We see the pattern of behaviour as follows. If $T_i$ is large enough and if we increase the magnetic field from zero then the cells which appear at marginal stability are elongated until the parameter $Q_i$ reaches a critical value when two sets of cells appear simultaneously, one set being much more elongated than the other. As the magnetic field increases further, $R_c$ decreases to a minimum and then increases, indicating that the inhibiting effect of the magnetic field predominates. If, however, $T_i < 2.5 \times 10^3$ there is a rapid variation in $a$ as a function of $Q_i$ and if $T_i < 200$, $a$ is a monotonic increasing function of $Q_i$. Other general features are the initial decrease of $R_c$ with $Q$, for $T_i > 500$, the decrease being very obvious when $Q_i$ passes through its critical value. If $T_i < 500$ then $R_c$ is a monotonic increasing function of $Q_i$.

We see also from Fig. I that in convective stability takes over from overstability approximately where the convection curve takes its minimum value and Fig. II shows that this changeover is accompanied by a sharp discontinuity of the wave number. From the shape of the graphs at the discontinuity we see that at the changeover from overstability to instability the cells become greatly widened. From Fig. I we also see that the overstability curves show no signs of the discontinuities which appear in the marginal stability case.

Section 6

In this section we consider the problem defined in Section 2.2 but with no applied rotation. Our equations are just (2.2.14) - (2.2.18) with $\mu$ put equal to zero. We write them out for convenience.
\[ \frac{\partial^2 w}{\partial t^2} = \frac{\rho}{4 \eta_0} \left( H_0 \cdot \nabla \right)^2 \nabla^2 \chi + \nabla^4 w - \gamma \nabla^2 (\chi \cdot \nabla)^2 \Theta \] (2.6.1)

\[ \frac{\partial \phi}{\partial t} = \frac{\rho}{4 \eta_0} (H_0 \cdot \nabla) \phi + \nabla^2 \phi \] (2.6.2)

\[ \frac{\partial \chi}{\partial t} = (H_0 \cdot \nabla) \chi + \gamma \nabla^2 \phi \] (2.6.3)

\[ \frac{\partial \chi}{\partial t} = (H_0 \cdot \nabla) \chi + \gamma \nabla^2 \chi \] (2.6.4)

\[ \frac{\partial \Theta}{\partial t} = \rho w + \kappa \nabla^2 \Theta \] (2.6.5)

We see that we have two sets of coupled equations - one involving \( w, \chi \) and \( \Theta \) and the other involving \( \phi \) and \( \chi \). We shall consider the first set, i.e., equations (2.6.1), (2.6.4) and (2.6.5). Looking for solutions of the form

\[ f(x, y, z, t) = f(x, y, z) e^{i \rho t} \]

we obtain

\[ i \rho \nabla^4 w = \frac{\rho}{4 \eta_0} \left( H_0 \cdot \nabla \right)^2 \nabla^2 \chi + \nabla^4 w - \gamma \nabla^2 (\chi \cdot \nabla)^2 \Theta \] (2.6.6)

\[ i \rho \chi = (H_0 \cdot \nabla) \chi + \gamma \nabla^2 \chi \] (2.6.7)

\[ i \rho \Theta = \rho w + \kappa \nabla^2 \Theta \] (2.6.8)

where we have put

\[ w(x, y, z, t) = w(x, y, z) e^{i \rho t} \]

\[ \chi(x, y, z, t) = \chi(x, y, z) e^{i \rho t} \]

\[ \Theta(x, y, z, t) = \Theta(x, y, z) e^{i \rho t} \] (2.6.9)
and immediately dropped the primes.

We first consider the case when \( \lambda \) and \( H_0 \) are coplanar, i.e. we put

\[
\lambda = (0, 0, 1) \quad H_0 = H_0(\sin \psi, 0, \cos \psi) \quad (2.6.10)
\]

so that \((2.6.6) - (2.6.8)\) reduce to

\[
\frac{i}{\kappa} \frac{d^2}{dt^2} \kappa^2 w = \frac{P}{4 \pi \rho_0} H_0 \cos \psi d \left( \tan \frac{2}{2} + \frac{2z}{2} \right) (\nabla^2 \psi) + \nabla^4 w
- \frac{\sigma d^2}{2} \left( \frac{2^2}{2} + \frac{2^2}{2} \right) \psi
\]

\[
(2.6.11)
\]

\[
\frac{i}{\kappa} \frac{d^2}{dt^2} \kappa = \frac{H_0}{\kappa} \cos \psi d \left( \tan \frac{2}{2} + \frac{2z}{2} \right) W + \nabla^2 \psi
\]

\[
(2.6.12)
\]

\[
\frac{i}{\kappa} \frac{d^2}{dt^2} \psi = \frac{d^2}{\kappa} W + \nabla^2 \psi
\]

\[
(2.6.13)
\]

where we are again measuring \( z \) in units of \( d \).

Eliminating \( \kappa \) between \((2.6.11)\) and \((2.6.12)\) and of \( \psi \) between \((2.6.13)\) and \((2.6.14)\) gives

\[
\nabla^2 (\nabla^2 - \frac{i}{\kappa} \frac{d^2}{dt^2}) (\nabla^2 - \frac{i}{\kappa} \frac{d^2}{dt^2}) = \frac{\mu H_0^2 d^2 \cos \psi}{4 \pi \rho_0 \kappa} \nabla^2 \left( \tan \frac{2}{2} + \frac{2z}{2} \right) W
- \frac{\sigma d^2}{2} \left( \frac{2^2}{2} + \frac{2^2}{2} \right) (\nabla^2 - \frac{i}{\kappa} \frac{d^2}{dt^2}) \psi
\]

\[
(2.6.14)
\]

and
where
\[ Q' = \frac{\mu H_0 c^2 a + \frac{1}{2} \mu}{\lambda_1 \sigma_1 2} \quad \text{and} \quad R' = \frac{\beta \delta d^4}{k \sigma} \] (2.6.16)

We now look for solutions of (2.6.14) of the form
\[ W(x, y, z) = W(x) \exp \{ i a_1 x + i a_2 y \} \quad \text{and} \quad \Theta(x, y, z) = \Theta(x) \exp \{ i a_1 x + i a_2 y \} \] (2.6.17)

where \( a_1 \) and \( a_2 \) are the wave numbers of the disturbance measured in units of \( d^{-1} \). We note that \( \exp \{ i a_1 x + i a_2 y \} \) satisfies (2.5.8) if
\[ a_1^2 + a_2^2 = a^2 \] (2.6.18)

Also \( \Theta \) and \( W \) may be complex functions. Making the substitution (2.6.17) in (2.6.14), (2.6.15), gives
\[ (D^2 - \alpha^2)(D^2 - \alpha_1^2 - \frac{i \beta d^2}{k})(D^2 - \alpha_2^2 - \frac{i \beta d^2}{k}) W \]
\[ = Q' (D^2 - \alpha^2)(D + i a_1 \tan \phi)^2 W + R' \alpha^2 (D^2 - \alpha_1^2 - \frac{i \beta d^2}{k}) W \] (2.6.19)

\[ (D^2 - \alpha^2)(D^2 - \alpha_1^2 - \frac{i \beta d^2}{k})(D^2 - \alpha_2^2 - \frac{i \beta d^2}{k})(D^2 - \alpha_1^2 - \frac{i \beta d^2}{k}) W \]
\[ = Q' (D^2 - \alpha^2)(D^2 - \alpha_1^2 - \frac{i \beta d^2}{k})(D + i a_1 \tan \phi)^2 W \]
\[ - R' \alpha^2 (D^2 - \alpha_1^2 - \frac{i \beta d^2}{k}) W \] (2.6.20)

where we have written, \( \alpha^2 = a_1^2 + a_2^2 \).
We note that if in (2.5.20) we put $T = 0$ and if in (2.6.20) we put $\psi = 0$ we obtain identical equations. Due to the appearance of the operator $(Di c a, ka n t)^2$ we cannot proceed as we did in section 2.5 and look for solutions for the case when both bounding surfaces are free. However, in the case of marginal stability it is possible to formulate a variational principle.

Putting $p = 0$ in (2.6.19), (2.6.20) we obtain

$$\begin{align*}
\left[ (D^2 - \alpha^2)^2 - Q'(D + i a, ka n t)^2 \right] W &= R^3 \alpha^2 \Theta \\
(D^2 - \alpha^2)^2 \left[ (D^2 - \alpha^2)^2 - Q'(D + i a, ka n t)^2 \right] W &= -R^3 \alpha^2 W
\end{align*}
$$

(2.6.22)

(2.6.23)

The boundary conditions (2.5.16) - (2.5.19) are

$$\begin{align*}
|W| &= |W| = 0 \\
|D^2 W| &= 0 \quad \text{on a rigid surface} \\
|D^2 W| &= 0 \quad \text{on a free surface adjoining a vacuum}
\end{align*}
$$

(2.6.24)

(2.6.25)

(2.6.26)

By (2.6.22), (2.6.24) is equivalent to

$$\begin{align*}
|W| &= \left[ (D^2 - \alpha^2)^2 - Q'(D + i a, ka n t)^2 \right] W = 0 \quad \text{at } z = 0, 1
\end{align*}
$$

(2.6.27)

As before, since (2.6.22), (2.6.23) are linear with constant coefficients, we could, in principle solve the subject to the given boundary conditions. Since $W$ and $\Theta$ may be complex, we have, in effect a system of order twelve and the work involved would be
enormous. However we develop a variational principle as follows. Introduce a function \( F \) by

\[
F = \left( D^2 - x^2 \right)^2 - Q' \left( D + i a_1 \tan \theta \right)^2 W \tag{2.6.28}
\]

so that (2.6.23) is

\[
(D^2 - x^2) F = - R' x^2 W \tag{2.6.29}
\]

and the boundary conditions (2.6.27) are

\[
|W| = |F| = 0 \quad \text{at} \quad z = 0, 1 \tag{2.6.30}
\]

If \( F^* \) is the complex conjugate of \( F \) we have, from (2.6.29)

\[
\int_0^1 F^* (D^2 - x^2) F \, dz = - R' x^2 \int_0^1 F^* W \, dz
\]

\[
\int_0^1 \left[ F^* \frac{DF}{Dz} - \int_0^1 \left[ \frac{DF^*}{Dz} \, F + x^2 F F^* \right] \, dz \right] = - R' x^2 \int_0^1 W \left( D^2 - x^2 \right)^2
\]

\[
- Q' \left( D + i a_1 \tan \theta \right)^2 \int_0^1 W \, dz
\]

\[
- \int_0^1 (D H^2 + x^2 F F^2) \, dz = - R' x^2 \int_0^1 (D^2 - x^2) W^2 \, dz
\]

\[
+ R' x^2 Q' \int_0^1 \left[ D W^2 + a_1^2 \tan \theta + 1 W^2 - 2 x a_1 \tan \theta + W^2 \, dw \right] \, dz
\]

\[
= - R' x^2 \int_0^1 (D^2 - x^2) W^2 \, dz
\]

\[
+ R' x^2 Q' \int_0^1 (D W^2 + a_1^2 \tan \theta + 1 W^2 + x a_1 \tan \theta + W^2 \, dw \, dz
\]

and hence

\[
R' = \frac{\int_0^1 \left[ 1 (D^2 - x^2) W^2 + Q' 1 D W + a_1 \tan \theta + W^2 \, dw \right] \, dz}{\alpha^2 \int_0^1 (D^2 - x^2) W^2 + Q' 1 D W + a_1 \tan \theta + W^2 \, dw \, dz} \frac{I_1}{I_2} \tag{2.6.32}
\]

and hence \( R' \) is the ratio of two positive definite integrals.

In the appendix to this chapter we show that if \( W \) is given an
arbitrary variation \( \delta W \), compatible with the boundary conditions, then the variation in \( R' \), \( \delta R' \), is given by

\[
\alpha^2 I_2 \delta R' = -\int_0^1 \delta F \frac{\partial^2}{(\partial^2 - \alpha^2)} F + \alpha^2 R' W \delta d_2 - \int_0^1 \delta F \frac{\partial^2}{(\partial^2 - \alpha^2)} F + \alpha^2 R' W \delta d_2 \tag{2.6.33}
\]

Hence, to first order, \( \int R' = 0 \) and for all arbitrary variations in \( W \) satisfying the boundary conditions provided that (2.6.29) is satisfied and conversely, if (2.6.29) is satisfied then \( \int R' = 0 \). Then by (2.6.33) the true solution of the problem leads to a minimum value of \( R' \) when \( R' \) is evaluated by (2.6.32). We are thus able to state the following variational principle. Assume an expression for \( F \) such that \( F = 0 \) at \( z = 0, 1 \) and which involves one or more parameters \( A \).

With this function \( F \) solve (2.6.29) for \( W \) satisfying the necessary boundary conditions. Then evaluate \( R' \) by (2.6.32) and minimise it with respect to the \( A \). Then this gives us the best value of \( R' \) for the chosen form of \( F \).
Section 7.

We now apply the variational principle of Section 2.6 to the three possible combinations of boundary conditions:

(a) both bounding surfaces free, (b) both rigid, and (c) one rigid and one free. The procedure is straightforward but there is a lot of algebra involved and so we will sketch the method.

a) When both bounding surfaces are three, it is convenient to consider the boundaries as being at \( z = \pm \frac{1}{2} \). The boundary conditions are then

\[
W|_1 = W|_2 = D|_2 = 0 \quad \text{at} \quad z = \pm \frac{1}{2} \quad (2.7.1)
\]

A suitable trial function is

\[
F = \cos \pi z \quad (2.7.2)
\]

The next step is to solve

\[
\left\{ (D^2 - \alpha^2)^2 - Q' (D + c a_1 \tan \psi) \right\} W = \cos \pi z \quad (2.7.3)
\]

which leads to

\[
W = \gamma_1 e^{\pi \pi z} + \gamma_2 e^{-\pi \pi z} + \sum_{k=1}^{l} A_k e^{i k \pi z} \quad (2.7.4)
\]

where the \( \gamma_i \) and \( \gamma_k \) are constants depending on \( a^2 \), \( Q' \), \( a \), and \( \psi \) and the \( A_k \) are constants of integration. The \( A_k \) are now determined from (2.7.1) and then \( R' \) is evaluated from (2.6.32) for given values of \( Q' \), \( a^2 \) and \( a_1 \tan \psi \).

For the case (b) we again take the boundaries to be at \( z = \pm \frac{1}{2} \) and again take (2.7.2) as our trial function. In the case (c) it is more convenient to keep the boundaries at \( z = 0, 1 \) so that we may use the trial function

\[
F = \sin \pi z \quad (2.7.5)
\]
Section 8.

Before quoting the results of the calculations which Chandrasekhar\(^{(6)}\) performed in the above work, we note what happens when \(\text{H}_0\) and \(\lambda\) act in the same direction. The variational principle is still valid if we put \(\psi = 0\) and \(Q' = Q\) where \(Q\) is as defined in (2.5.21). We also see that \(R'\) is identical with \(R\) as defined in (2.5.21). When \(\psi = 0\) we see that we need no longer consider \(|W|, |DW|\) etc. In this special case we could proceed as in Section 2.5 by finding an explicit solution for \(W\) in the case when both bounding surfaces are free. For the cases (b) and (c) mentioned above we can make a simplifying assumption. Translating our axes so that the boundaries are at \(z = \pm \frac{1}{2}\), we see that for the case (b) when both boundaries are rigid, the boundary conditions require that \(W\) and \(F\) be either even or odd functions and Chandrasekhar states that the even functions give lower values of \(R\). Our procedure is now as before.

In the case (c) when one surface is rigid and the other free, we take the odd solutions. Since they are odd they must vanish at \(z = 0\) so that we may consider (c) as being a special case of (b), the range of the problem now being \(0 \leq z \leq \frac{1}{2}\), i.e. we have the case (b) for a fluid moving between planes a distance \(d' = \frac{1}{2}d\) apart. This means that by definition we shall obtain a value of \(R\) which is sixteen times smaller and our new value of \(Q\) will be a quarter of its original value.

When \(\lambda\) and \(\text{H}_0\) act in the same direction the results obtained are best displayed graphically.\(^{(3)}\)
Curve I is for the case when both bounding surfaces are free.
Curve II is for the case when both bounding surfaces are rigid.
Curve III is for the case when one bounding surface is rigid and the other free.

We see that the general behaviour of the dependence of $R_c$ on $Q$ is the same in the three cases and that the only differences between them are numerical. Thus having solved the problem for one set of boundary conditions, we have effectively solved it for the other two sets. Carrying this result over to the case when there is a rotation acting we may expect that the behaviour of the system in the cases which were not considered will show the same features as in the case when both bounding surfaces are free.

When $\lambda$ and $H_0$ are coplanar, Chandrasekhar\(^{(6)}\) finds that convection first sets in in longitudinal rolls. However we expect that when $\lambda$ and $H_0$ are in the same direction, i.e. $\psi = 0$, that convection will set in with a cellular pattern so that we see that even for small values of $\psi$ we have a discontinuity in the convection pattern. This was explained as follows\(^{(6)}\). When $\lambda$ and $H_0$ are coplanar, different Rayleigh numbers are required for different predetermined patterns of
of convection, the most difficult pattern to excite being transverse rolls. When the Rayleigh number is large enough for these rolls to be excited, the cellular pattern will change. Thus the difference in the Rayleigh numbers required to excite transverse and longitudinal rolls is effectively a measure of the extent of the suppression of a proper cellular convection pattern at marginal stability. Hence when $\psi$ is small, transverse rolls are only slightly suppressed at marginal instability and when $\psi = 0$, this suppression of the cellular pattern disappears and both transverse and longitudinal rolls appear simultaneously at marginal instability.

Section 9.

The problem of the stability of the flow of an electrically conducting fluid between two parallel planes when a magnetic field is acting, has been considered by Stuart$^{(10)}$ and Lock$^{(11)}$. They however neglect gravitational effects and do not allow for any heating of the fluid. The equations of motion are, in our previous notation,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = \frac{1}{\mu_0} \left( \nabla \cdot \nabla^2 u - \nabla \nabla : \nabla u \right) + \nabla \nabla^2 u$$ (2.9.1)

$$\frac{\partial H}{\partial t} + (u \cdot \nabla) H - (H \cdot \nabla) u = \nabla \nabla^2 H$$ (2.9.2)

$$\nabla \cdot H = \nabla \cdot u = 0$$ (2.9.3)

$$\Pi = \frac{1}{\rho_0} \left( \frac{P H^2}{\gamma} + \frac{\nabla \cdot H}{\gamma} \right)$$ (2.9.4)

Stuart$^{(10)}$ considers the problem of parabolic flow under the action of a coplanar magnetic field. He assumes that the bounding
planes are perfect conductors at \( z = \pm d \). The equations (2.9.1) - (2.9.4) admit the steady state solution
\[
\mathbf{u} = \mathbf{V} = \mathbf{U}_0 (1 - \frac{r^2}{a^2}) (1, 0, 0) = U, \ \mathbf{v} \tag{2.9.5}
\]
\[
\mathbf{h} = (h_1, h_2, 0) = \text{constant} = \mathbf{H}_0 \tag{2.9.6}
\]
with
\[
U_0 = -\frac{d^2 \gamma \Pi_0}{2} = \text{constant}. \tag{2.9.7}
\]
Linearising (2.9.1) - (2.9.4) by putting
\[
\mathbf{u} = \mathbf{V} + \mathbf{v}, \quad \mathbf{h} = \mathbf{H}_0 + \mathbf{h}', \quad \Pi = \Pi_0 + \Pi' \tag{2.9.8}
\]
and looking for solutions of the form
\[
\mathbf{u} = \exp \left\{ i (\alpha x + \gamma y - \beta t) \right\} \left( u(x), v(x), w(x) \right) \tag{2.9.9}
\]
where we are measuring \( z \) in units of \( d \) and
\[
(U-c)(D^2-\lambda_1^2) \mathbf{v} = -\mathbf{v} \mathbf{u}'' - \frac{\mu \lambda_1}{4 \pi \rho \chi_1 U_0} (D^2-\lambda_1^2) \Psi \tag{2.9.10}
\]
\[
\left( \frac{U-c}{D^2-\lambda_1^2} \right) \Psi = -\frac{e \Psi}{\alpha_i R} (D^2-\lambda_1^2) \Psi \tag{2.9.11}
\]
where we are measuring \( z \) in units of \( d \) and
\[ c = \frac{P}{\alpha U_0}, \quad \alpha d = \alpha_1, \quad \beta d = \beta_1 \]
\[ \alpha^2 + \lambda^2 = \lambda_1^2, \quad \lambda'd = \lambda_1, \quad U_1 = U_0 U \]
\[ R = \frac{U_0 d}{\beta}, \quad \lambda' = \frac{\alpha h + \beta h_1}{\lambda_1} \]  
\[ (2.9.12) \]

Since we are taking the bounding surfaces to be rigid conductors, we require that the normal component of the magnetic field should vanish on the boundary, i.e. \( \psi = 0 \). The velocity perturbations must also vanish and so must the tangential electric field. Since

\[ \mu \pi \sigma (\xi + \rho \eta n H) = \frac{\partial n H}{\partial z} \]

we must have

\[ \phi' = 0 \quad \text{at} \quad z = \pm 1 \]

The boundary conditions are thus

\[ U = V = W = \psi = \phi' = 0 \quad \text{at} \quad z = \pm 1 \]

The vanishing of \( \phi' \) implies that of \( \xi' \) and \( U = W = 0 \) implies \( \psi' = 0 \). The conditions are thus

\[ U = V' = W = 0 \quad \text{at} \quad z = \pm 1 \]  
\[ (2.9.13) \]

We can solve (2.9.10), (2.9.11) for a two dimensional perturbation i.e. \( \gamma = 0 \) so that \( \lambda_1 = \alpha_1 \) and \( L = \lambda_1 \), if we note that for, say, mercury at room temperature, \( \eta \sigma = 0(5 \times 10^6) \) and so for values of \( R \) in the region of the critical Rayleigh number, i.e.

\[ R = 0(5000) \]

we can neglect the \( \psi \) term on the left hand side of (2.9.11) which then becomes

\[ -\frac{h_1}{M_0} V = -\frac{c \nu}{\alpha_1 \partial R} (D^2 - \alpha_1^2) \psi \]  
\[ (2.9.14) \]
Eliminating $\psi$ between (2.9.14) and (2.9.10) gives an equation for $v$ only,

$$(M-c)(D^2-\alpha^2)u - vU'' + c_qv = \frac{c}{\alpha} (D^2-\alpha^2)v$$  \hspace{1cm} (2.9.15)$$

where

$$q = \frac{\mu_1 l_1^2}{4\eta \alpha_1 U_0 U_0 \alpha_1} = \frac{\mu_1 l_1^2}{4\eta \alpha_1 U_0 U_0 \alpha_1}$$

Since (2.9.15) is a fourth order differential equation for the velocity boundary conditions

$$\frac{U}{U} = \frac{U'}{U'} = 0 \quad \text{at} \quad z = \pm 1$$  \hspace{1cm} (2.9.17)$$

will be sufficient. In general $c$ (=$\frac{\rho}{\alpha} U_0$) is complex, say,

$c = c_r + ic_i$, but $\alpha_1$, $q$ and $R$ are real. Thus we have a complex eigenvalue problem for $c_r$ and $c_i$. If $c_i > 0$ we have instability and stability if $c_i < 0$. If $c_i = 0$ we have neutral stability and this is the case which Stuart considered. If $c_i = 0$ and we eliminate $c_r$ from the eigenfunction relations we obtain a relation between $\alpha_1$ and $R$. We shall call the curves relating $\alpha_1$ and $R$ for given values of $q$ the $(\alpha_1 - R)$ curves. These are curves of neutral stability. Stuart has solved (2.9.15) and finds the curves of neutral stability to be roughly as shown.
The curves are drawn for different values of \( q \). The interiors of the curves are the unstable regions. Stuart estimates that the curves are closed for sufficiently high values of \( R \) and that for \( q \) increasing the curves shrink until for \( q > 0.10 \), they vanish, i.e. there is no neutral stability curve. We see that there are two Rayleigh numbers \( R_{\text{max}} \) and \( R_{\text{crit}} \). For \( R > R_{\text{max}} \) all disturbances are stable, and for all \( R < R_{\text{crit}} \) they are also stable. We note that \( R_{\text{crit}} \) is an increasing function of \( q \). Stuart states that the parameter \( q \) is not the best one with which to describe the motion and that a better one is \( q' = \frac{2}{\alpha} \). He states that the \((\alpha_i, -R)\) curves for constant \( q' \) are much more elongated than shown in Fig. 2.9.1, than the same curves for constant \( q \), but since we have made the approximation \( R = 0(5000) \), we cannot infer the shape of these curves for large values of \( R > 2.5 \times 10^5 \) for mercury at room temperature, i.e. we cannot say whether the curves of neutral stability for constant \( q' \) vanish for sufficiently high values of \( R \). He also shows that
R_{crit} and \( \frac{d}{d\epsilon}(R_{crit}) \) are increasing functions of \( q' \). He concludes that the magnetic field has much the same effect as found previously in this Chapter.

Lock(11) considers the same problem as Stuart, but with two differences - he has a transverse magnetic field and his bounding surfaces are assumed to be non-conducting. In effect he considers the stability of a solution which was found by Hartmann and Lazarus(12). A solution of the steady state equations obtained from (2.9.1) - (2.9.4) is

\[ \mathbf{M} = (U, 0, 0) \quad \mathbf{H} = (H_x, 0, H_z) \]  

(2.9.18)

with

\[ M = M_0 \frac{\cosh M - \cosh \left( \frac{M_0}{a} \right)}{\cosh M - 1} \]  

(2.9.17)

\[ H_x = H \frac{R M}{M(\cosh M - 1)} \left\{ \cosh \left( \frac{M_0}{a} \right) - \frac{2}{a} \cosh M \right\} \]  

(2.9.20)

\[ H_z = \text{constant}, \]

Here \( U_0 \) is the velocity at \( z = 0 \),

\[ M = \mu H d \sqrt{\frac{\sigma}{\rho_o}} \quad R_M = \frac{U d}{\sigma} \quad H = \text{constant} \]  

(2.9.21)

Linearising the general equations (2.9.1) - (2.9.4) in the usual manner and looking for solutions \( \mathbf{v} = (v_x, v_y, v_z) \) and \( \mathbf{h} = (h_x, h_y, h_z) \) where \( \mathbf{v} \) and \( \mathbf{h} \) are functions of \( z \) multiplied by

\[ e^{i \left[ \left( h_x + k_x \right) r + k_y y \right]} \]  

Lock finds that the problem may be reduced to solving the equations

\[ \phi + \frac{i}{\alpha} \frac{\partial}{\partial t} = (\omega - \epsilon) \phi + \frac{\epsilon}{\alpha R_M} \left( \frac{\partial^2 - \alpha^2}{\alpha^2} \right) \phi \]  

(2.9.22)
\[(w - c)(D^2 - \alpha^2) \psi - \psi \nabla^2 \psi + \frac{c}{\alpha R} (D^2 - \alpha^2) \psi = \rho \left( \Phi \left( D^2 - \alpha^2 \right) \phi - \frac{c}{\alpha} D \left( D^2 - \alpha^2 \right) \phi - D^2 \Phi \phi \right) \tag{2.9.23} \]

where
\[L_2 = H \Phi (z) + \rho \left( \nabla^2 \Phi \right) = \rho \left( i \kappa_x \left( x - ct \right) + \kappa_y y \right) \tag{2.9.24} \]
\[V_2 = U_0 \Phi (z) + \rho \left( i \kappa_x \left( x - ct \right) + \kappa_y y \right) \]

and
\[k = \frac{H}{H'} \quad \omega = \frac{U}{U_0} \quad c = \frac{C}{U_0} \quad \alpha = k_x \alpha \quad \alpha^2 = \alpha_x^2 + \alpha_y^2 \quad \rho = \rho \left( \frac{U}{U_0} \right)^2 \quad R = \frac{U_0 d}{\alpha} \tag{2.9.25} \]

and we are measuring \( z \) in units of \( d \).

Lock now uses the simplification provided by liquid mercury at room temperature to simplify (2.9.22) and (2.9.23) by neglecting terms in \( k \) in (2.9.22) and (2.9.23) and the term in \((w - c)\) on the right hand side of (2.9.22). Thus we have an equation for \( \psi \) alone
\[(w - c)(D^2 - \alpha^2) \psi - \psi \nabla^2 \psi + \frac{c}{\alpha R} (D^2 - \alpha^2) \psi = \frac{c M^3}{\alpha R} \nabla^2 \psi \tag{2.9.26} \]

and the boundary conditions are
\[\psi = D \psi = 0 \quad \text{at} \quad z = \pm 1 \tag{2.9.27} \]

He now infers the analogue of Squires' Theorem \(^{(13)}\) to enable him to say that, for a fixed value of \( M \), the motion is more
unstable to a two dimensional disturbance than to a three
dimensional one and so puts $\alpha_1 = \alpha$ in (2.9.26) so that we have

$$(w - c)(D^2 - \alpha^2)\psi - D^2 w \cdot \psi + \frac{c}{\alpha R} (D^2 - \alpha^2)^2 \psi = \frac{i M^2}{\alpha R} D^2 \psi$$

(2.9.28)

Now this is just the Orr-Sommerfeld equation of hydrodynamics
with an extra term on the right hand side and if we have neutral
or amplified oscillations, i.e. $\Im c > 0$ then this term is
negligible. He also neglects two other terms under the same
approximation so that he writes (2.9.28) as

$$(w - c)(D^2 - \alpha^2)\psi - D^2 w \cdot \psi = \frac{-i}{\alpha R} D^4 \psi$$

(2.9.29)

Comparing this last equation with the corresponding one found
by Stuart for the coplanar field (2.9.18) we see that the coplanar
field adds a term to the Orr-Sommerfeld equation, but in the
transverse magnetic field modifies the velocity distribution
(see (2.9.25)). Thus the transverse field is more effective than
the coplanar magnetic field in stabilizing the motion. He then
solves (2.9.29) for the case of neutral stability ($\Im c = 0$)
and obtains for various fixed values of $M$ curves of the
form
These curves are only valid for small $M$. If $M$ is large he obtains curves of a similar shape, plotting $\alpha^* = \alpha/M$ against $R^* = \left(\frac{R}{M}\right)^{\frac{1}{3}}$, $\alpha^*$ being of the order 0.2 to 0.3.

He concludes that if $M$ is large enough the system will be stable to all infinitesimal perturbations but not necessarily stable to finite perturbations. He also concludes that a transverse field is more effective than a parallel field in stabilizing the flow.

**Section 10**

In this section we mention the experiments that have been performed to check the above calculations. When there is no applied rotation, Nakagawa (14) and Girlow (15) have performed experiments to test Chandrasekhar's predictions. They both used mercury as their fluid, confined between planes made of
"Pyrex" and plexiglass respectively. Their magnetic fields were both in the vertical direction, but their methods of heating differed. Nakagawa used an electrical method of heating and Girlow used a circulation of hot water. The temperatures are measured with thermopiles. To determine the critical temperature gradient the following method is used. When the mercury is at rest, the relation between the temperature difference, between the top and bottom of the mercury, is $\Delta \Theta$ and the external temperature difference $\Delta T$ is plotted. $\Delta \Theta$ and $\Delta T$ are also measured when the mercury is in motion and the two curves are 'linearly' extrapolated, the point where they meet giving the critical value of $\Delta \Theta$. There is good experimental agreement with the theory. In further experiments, Nakagawa (16) increased the range of the parameter $Q$ and the agreement remained good.

To test the case when an applied rotation was also present, Nakagawa (17, 18) performed further experiments. His apparatus was essentially as before, the rotation of the mercury being made by rotating the face of the dish containing the mercury. This time the agreement between experiment and theory was not quite so good, since the theoretical case was only calculated for the case when both bounding surfaces were free. However, he did find the discontinuity in the dependence of $R_c$ on $Q$. He also found the good agreement with the values of $a/\lambda$ as predicted by Chandrasekhar.
Appendix

We prove here equation (2.6.33). We have

\[ R' = \frac{\int_0^l \left( (DF)^2 + \alpha^2 |F|^2 \right) dz}{\alpha^2 \int_0^l \left( |Q - \alpha^2 W|^2 + Q_1^2 DW + \tau_1 \tau_2 \alpha^2 W^2 \right) dz} = \frac{I_1}{\alpha^2 I_2} \]

say. Then if we give \( W \) a variation \( \delta W \) compatible with the boundary conditions, then there is a corresponding variation \( \delta F \) in \( F \) and we must have by (2.6.25), (2.6.26) and (2.6.30) that

\[ |\delta W| = |\delta F| = 0 \]

at \( z = 0,1 \) (2.A.2)

\[ |D^2 \delta W| = 0 \]

on a free surface (2.A.3)

\[ |D \delta W| = 0 \]

on a rigid surface (2.A.4)

Then

\[ \delta R' = \frac{1}{\alpha^2} \left( \frac{I_n \delta I_n - I_2 \delta I_2}{I_2} \right) = \frac{1}{\alpha^2} \left( \delta I_1 - \alpha^2 R' \delta I_2 \right) \]

(2.A.5)

Now
\[ \delta I_1 = \delta \int_0^1 \left\{ \delta F \delta F^* + \alpha^2 \delta F \delta F^* \right\} dz \\
= \delta \int_0^1 \left\{ \delta F \delta F^* + \alpha^2 \delta F \delta F^* + \alpha^2 \delta F \delta F^* \right\} dz \\
= \int_0^1 \left\{ \delta F \delta F^* \right\} dz - \int_0^1 \delta F \delta F^* dz + \int_0^1 \delta F \delta F^* dz + \alpha^2 \int_0^1 \delta F \delta F^* dz + \alpha^2 \int_0^1 \delta F \delta F^* dz \\
= - \int_0^1 [\delta F^*(D^2 - \alpha^2)F + DF(D^2 - \alpha^2)F^*] dz \\
\text{(2.A.6)} \]

the terms in square brackets vanishing by (2.A.2).

Also

\[ \delta I_2 = \delta \int_0^1 \left\{ \left[ (D^2 - \alpha^2)W \right] \left[ (D^2 - \alpha^2)W^* \right] \\
+ Q' \left[ DW + i\alpha, \tan + W \right] \left[ DW^* - i\alpha, \tan - W^* \right] \right\} dz \\
= \int_0^1 \left\{ \left[ (D^2 - \alpha^2)SW \right] \left[ (D^2 - \alpha^2)W^* \right] + \left[ D^2 - \alpha^2 \right] W \left[ (D^2 - \alpha^2)SW^* \right] \\
+ Q' \left[ DSW + i\alpha, \tan + SW \right] \left[ DW^* - i\alpha, \tan + W^* \right] + Q' \left[ DSW + i\alpha, \tan + SW \right] \left[ DW^* - i\alpha, \tan + W^* \right] \right\} dz \\
= i_1' + Q'i_2' \] \\
\text{say} \quad \text{(2.A.7)}
Then
\[
\begin{align*}
i_1 &= \int_0^1 \left[ (D^3 - \alpha^2) dW \right] [W, W^{\ast}] + \int_0^1 (D^3 - \alpha^2) W [Q^2, dW^{\ast}] d2 \\
&= \int_0^1 \left[ D^3 W^{\ast} (D^3 - \alpha^2) dW - \alpha^2 W^{\ast} (D^3 - \alpha^2) dW + D^3 W (D^3 - \alpha^2) W^{\ast} d2 \\
&= \left[ D^3 W^{\ast} (D^3 - \alpha^2) dW \right]_0^1 - \int_0^1 D^3 W^{\ast} (D^3 - \alpha^2) dW d2 \\
&= \int_0^1 W^{\ast} (D^3 - \alpha^2) DS d2 - \alpha^2 \int_0^1 W (D^3 - \alpha^2) dW d2 \\
&= \int_0^1 W^{\ast} (D^3 - \alpha^2) dS W d2 - \alpha^2 \int_0^1 W (D^3 - \alpha^2) dW d2 \\
&= \int_0^1 \left[ W^{\ast} (D^3 - \alpha^2) D S W \right] d2 - \int_0^1 W (D^3 - \alpha^2) dW d2 \\
&= \int_0^1 W^{\ast} (D^3 - \alpha^2) dW + W (D^3 - \alpha^2) dW^{\ast} d2 \\
(2.1.8)
\end{align*}
\]

Also
\[
\begin{align*}
i_2 &= \int_0^1 (DS W DW^{\ast} + DW DS W^{\ast}) d2 + \alpha^2 a_1 \int_0^1 (dW W^{\ast} + W W^{\ast}) d2 \\
&= \int_0^1 (DS W DW^{\ast} + DW DS W^{\ast}) d2 + \alpha^2 a_1 \tan a + \int_0^1 \{ WS W^{\ast} - W^{\ast} WS d2 + W D W^{\ast} - W^{\ast} D W d2 \\
&= A_1 + A_2 + A_3 \\
\text{say.} & \quad (2.1.9)
\end{align*}
\]

with
\[
A_1 = \int_0^1 (DS W DW^{\ast} + DW DS W^{\ast}) d2 \\
= [W^{\ast} DS W]_0^1 - \int_0^1 W^{\ast} D^3 DS d2 + [W DS W^{\ast}]_0^1 - \int_0^1 W D S W^{\ast} d2
\]
\[ = - \int_0^1 W^* D^2 SW \, dz - \int_0^1 W^2 SW^* \, dz \]  

and

\[ f_3 = \int_0^1 \left\{ DW DW^* - W^* DS W + W DS W^* - SW^* DW \right\} \, dz \]
\[ = \int_0^1 \left\{ \left[ W^* SW \right]_0 - \int_0^1 W^* DS W \, dz - \int_0^1 W^* DS W \, dz \right\} \]
\[ + \int_0^1 W^0 DS W^* \, dz - \left[ W^0 SW \right]_0 + \int_0^1 W^0 DS W^* \, dz \]
\[ = 2 \int_0^1 \left\{ \int_0^1 W^0 DS W^* \, dz - \int_0^1 W^* DS W \, dz \right\} \]

Thus combining (2.A.9), (2.A.10) and (2.A.11) gives

\[ \varphi_2 = - \int_0^1 \left\{ W \left( D^2 - 2 \int_0^1 \tan \theta + \tan^2 \theta \right) SW^* + \right. \]
\[ + W^* (D^2 - 2 \int_0^1 \tan \theta + \int_0^1 \tan \theta - \theta^2 (\tan \theta)^2) SW \} \, dz \]
\[ = - \int_0^1 \left\{ W \left( D - \varepsilon \right), \tan \theta \right\} \, SW^* + \left[ W^0 \left( D + \varepsilon \right), \tan \theta \right\} \, SW \} \, dz \]

Then, by (2.A.5), (2.A.7)

\[ \alpha^2 I'' R = \delta I' - \alpha^2 R' Q' \]
\[ = - \int_0^1 \left\{ IF^*(D^2 - \theta^2) \, F \right\} \, dz - \int_0^1 \left\{ IF (D^2 - \theta^2) \right\} \, F^* \, dz \]
\[ - \int_0^1 \alpha^2 R' W^* (D^2 - \theta^2) \, SW \, dz - \int_0^1 \alpha^2 R' W (D^2 - \theta^2) \, SW^* \, dz \]
\[ + \int_0^1 \alpha^2 R' Q' W (D + \varepsilon \tan \theta) \, SW \, dz \]
\[ + \int_0^1 \alpha^2 R' Q' W (D - \varepsilon \tan \theta) \, SW^* \, dz \]
\[ \begin{align*}
&= -\int_0^1 dF^* (D^2 - \alpha^2) F d_2 - \int_0^1 dF (D^2 - \alpha^2) F^* d_2 \\
&\quad - \alpha^2 R^1 \int_0^1 W^* \left[ (D^2 - \alpha^2) - Q' (D + \alpha, \tan 4) \right] S W d_2 \\
&\quad - \alpha^2 R^1 \int_0^1 W \left[ (D^2 - \alpha^2) - Q' (D - \alpha, \tan 4) \right] S W^* d_2 \\
&= -\int_0^1 dF^* (D^2 - \alpha^2) F d_2 - \int_0^1 dF (D^2 - \alpha^2) F^* d_2 \\
&\quad - \alpha^2 R^1 \int_0^1 W^* S F d_2 - \alpha^2 R^1 \int_0^1 W S F^* d_2 \\
&= -\int_0^1 dF^* \left\{ (D^2 - \alpha^2) F + \alpha^2 R^1 W \right\} d_2 \\
&\quad - \int_0^1 dF \left\{ (D^2 - \alpha^2) F^* + \alpha^2 R^1 W^* \right\} d_2.
\end{align*} \]
Section 1

In this chapter we consider the stability of the flow of a viscous, incompressible fluid, with finite electrical conductivity, confined between two rigid, coaxial, rotating cylinders. There is an impressed magnetic field in the direction of the axis of the cylinders. The equations of motion of the system are as in Section 2.1 but we neglect gravity and heating effects. We work in cylindrical polar coordinates \((\varphi, \phi, z)\). Then the equations are, in the same notation

\[
\frac{\partial \mathbf{u}}{\partial t} + \left( \mathbf{u} \cdot \nabla \right) \mathbf{u} = \frac{\mu}{\rho} \left( \nabla \times \mathbf{H} \right) - \mathbf{\nabla} p + \eta \nabla^2 \mathbf{u} \quad (3.1.1)
\]

\[
\frac{\partial \mathbf{H}}{\partial t} + \left( \mathbf{u} \cdot \nabla \right) \mathbf{H} = \left( \mathbf{H} \cdot \nabla \right) \mathbf{u} + \eta \nabla^2 \mathbf{H} \quad (3.1.2)
\]

\[
\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{H} = 0 \quad (3.1.3)
\]

\[
\frac{\mathbf{H}}{\nabla} = \frac{1}{\eta} \left( \mathbf{H} + \frac{\mu}{\rho} \nabla^2 \mathbf{H} \right) \quad (3.1.4)
\]

and \(\eta\) is defined by \((2.1.21)\).

The above equations admit the steady state solution

\[
\mathbf{u} = (0, V(\varphi), 0) = (0, A \varphi + B \frac{\varphi}{\omega}, 0) \quad (3.1.5)
\]

\[
\mathbf{H} = (0, 0, H_0) = \text{constant}
\]

\[
\frac{\partial \mathbf{H}}{\partial \omega} = \frac{1}{\omega} \nabla^2 \mathbf{H}
\]

where \(A\) and \(B\) are constants related to the angular velocities of the cylinders. If the cylinders are of radii \(R_1\) and \(R_2\) \((R_2 > R_1)\) and have angular velocities \(\omega_1\) and \(\omega_2\) respectively, then
The procedure used for this problem is identical with that used in Chapter 2. We perturb the system so that

\[ \mathbf{u} = (u_\infty, u_\phi + V, u_z) \quad H = (k_\infty, h_\phi, h_z + h) \]  

(3.1.7)

where \( u_\infty, u_\phi, u_z, k_\infty, h_\phi, h_z \) are sufficiently small so that we may neglect squares and products of them and their derivatives. We shall assume axial symmetry. Putting (3.1.7) into (3.1.1) - (3.1.3) gives, on taking components

\[
\frac{\partial \Pi'}{\partial \alpha} - \frac{V^2}{\alpha^3} = -\frac{2}{\alpha} u_\infty + \Re \left( \nabla^2 u_\infty - \frac{u_\infty}{\alpha^2} \right) + 2 \left( A + \frac{B}{\alpha^2} \right) u_\phi (3.1.8)
\]

\[
\partial = -\frac{2u_\phi}{\alpha} + \Re \left( \nabla^2 u_\phi - \frac{u_\phi}{\alpha^2} \right) - 2Au_\phi + \frac{\mu H_0}{\kappa \eta_0} \frac{\partial h_\phi}{\partial z} (3.1.9)
\]

\[
\frac{\partial \Pi'}{\partial z} = -\frac{2u_2}{\alpha} + \Re \nabla^2 u_2 + \frac{\mu H_0}{\kappa \eta_0} \frac{\partial h_2}{\partial z} (3.1.10)
\]

\[
-h_0 \frac{\partial u_\infty}{\partial z} = -\frac{2h_\infty}{\alpha^2} + \Re \left( \nabla^2 h_\infty - \frac{h_\infty}{\alpha^2} \right) (3.1.11)
\]

\[
-h_0 \frac{\partial u_\phi}{\partial z} = -\frac{2h_\phi}{\alpha^2} + \Re \left( \nabla^2 h_\phi - \frac{h_\phi}{\alpha^2} \right) - \frac{2B}{\kappa^2} h_2 (3.1.12)
\]

\[
-h_0 \frac{\partial u_2}{\partial z} = -\frac{2h_2}{\alpha^2} + \Re \nabla^2 h_2 (3.1.13)
\]
\[
\frac{\partial u}{\partial t} + \frac{1}{c} u \frac{\partial u}{\partial z} + \frac{\partial u^2}{\partial z^2} = 0, \quad \frac{\partial h}{\partial t} + \frac{1}{c} h \frac{\partial h}{\partial z} + \frac{\partial h^2}{\partial z^2} = 0 \quad (3.1.14)
\]

where \[ V^2 \equiv \frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega} \frac{\partial}{\partial \omega} + \frac{\partial^2}{\partial z^2} \]

and \( \Pi' \) is the perturbation in \( \Pi \).

We now look for solutions of these equations which are periodic in both \( t \) and \( z \). Thus putting

\[
\begin{align*}
u_0 &= e^{pt} u(\omega) \cos \lambda z \\
h_0 &= e^{pt} \theta(\omega) \sin \lambda z \\
u_\phi &= e^{pt} v(\omega) \cos \lambda z \\
h_\phi &= e^{pt} \psi(\omega) \\
u_z &= e^{pt} w(\omega) \sin \lambda z \\
h_z &= e^{pt} \xi(\omega) \cos \lambda z
\end{align*}
\]

into (3.1.8) - (3.1.14) gives

\[
\frac{\partial \Pi'}{\partial t} - V^2 \Pi' = \left[ \mathcal{O}(DD^* - \lambda^2 - \frac{p}{\omega}) u + 2(A + \frac{B}{B_\omega}) v + \frac{\mu H_0 \lambda}{4 \pi f_0} \right] \cos \lambda z
\]

\[
0 = \mathcal{O}(DD^* - \lambda^2 - \frac{p}{\omega}) v - 2Au + \frac{\mu H_0 \lambda}{4 \pi f_0} \psi
\]

\[
\frac{\partial \Pi'}{\partial z} = \left[ -\mathcal{O}(DD^* - \lambda^2 - \frac{p}{\omega}) w - \frac{\mu H_0 \lambda}{4 \pi f_0} \chi \right] \sin \lambda z
\]

\[
\lambda H_0 u = \mathcal{O}(DD^* - \lambda^2 - \frac{p}{\omega}) \theta
\]

\[
\lambda H_0 v = \mathcal{O}(DD^* - \lambda^2 - \frac{p}{\omega}) \psi - \frac{2B}{\omega^2} \theta
\]

\[
-\lambda H_0 w = \mathcal{O}(DD^* - \lambda^2 - \frac{p}{\omega}) \psi
\]

\[
D^* u = -\lambda w \\
D^* \theta = \lambda \chi
\]

(3.1.22)
where
\[ D = \frac{d}{d\omega}, \quad D^* = \frac{d}{d\omega} + \frac{1}{\omega} \] (3.1.23)
so that
\[ DD^* = D^*D - \frac{1}{\omega^2} \] (3.1.24)

We note first that equations (1.19), (1.21) and (1.22) are not all independent and then eliminating \( \Pi' \) between (3.1.16) and (3.1.18) by appropriate differentiation and using (3.1.22) we obtain
\[ \frac{2}{\lambda^2} \left( DD^* - \lambda^2 - \frac{p}{\omega} \right) D D^* - \lambda^2 \right) u + \frac{\rho H_0}{\mu_0 \mu_\alpha} \left( DD^* - \lambda^2 \right) \theta = 2(n + B) \frac{\partial u}{\partial x} \] (3.1.25)

Then (3.1.17), (3.1.19), (3.1.20) and (3.1.25) are four equations in four unknowns \( u, v, \theta \) and \( \psi \). We have to solve them subject to the appropriate boundary conditions. The hydrodynamical conditions are that there is no slip on the cylinders, i.e.
\[ u = v = w = 0 \quad \text{at} \quad \omega = R_1 \text{ and } R_2 \] (3.1.26)
which, by (3.1.22) are equivalent to
\[ u = v = D^* \right u = 0 \quad \text{at} \quad \omega = R_1 \text{ and } R_2 \] (3.1.27)

Chandrasekhar\(^{(1)}\) has considered the magnetic conditions and concludes that if the cylinders are perfect conductors then the appropriate conditions are
\[ D^* \psi = \Theta = 0 \quad \text{at} \quad \omega = \ell_1 \text{ and } \ell_2 \] (3.1.28)
On the other hand, if the cylinders are non conductors, Niblett\(^{(3)}\) has remarked that the appropriate equations are
\[ \psi = \Theta = 0 \quad \text{at} \quad \psi = R_1 \text{ and } R_2 \quad (3.1.29) \]

Since our system of equations is of order 10, we now have sufficient conditions to solve the problem.

Section 2

However the four equations which determine the problem have only been solved under a rather stringent approximation - the "small gap approximation". We assume that the radii of the cylinders are very nearly equal so that

\[ d = R_2 - R_1 \ll \frac{1}{2} (R_1 + R_2) = R_0 \quad \text{say.} \quad (3.2.1) \]

Then as Jeffreys\(^{(4)}\) has remarked, we can neglect the term \( \frac{1}{c^2} \) in (3.1.23) and thus write \( D^2 P = P D ) = D^2 \) say. If we assume the principle of the interchange of stabilities then (3.1.17), (3.1.19), (3.1.20), (3.1.25) become

\[ O = \lambda \left( D^2 - \lambda^2 \right) \psi - 2A \eta + \frac{\mu H_0 \lambda}{\eta \sigma j_0} \psi \quad (3.2.2) \]

\[ \lambda H_0 \psi = \eta \left( D^2 - \lambda^2 \right) \Theta \quad (3.2.3) \]

\[ \lambda H_0 \psi = \eta \left( D^2 - \lambda^2 \right) \psi - \frac{2B}{\sigma^2} \Theta \quad (3.2.4) \]

\[ \frac{\partial}{\partial \xi} \left( D^2 - \lambda^2 \right) \eta + \frac{\mu H_0}{\eta \sigma j_0} \left( D^2 - \lambda^2 \right) \Theta = 2( A + \frac{B}{\sigma^2} \psi ) \quad (3.2.5) \]

Chandrasekhar's argument now depends on showing, in effect, that the term \( \frac{2B}{\sigma^2} \Theta \) in (3.2.4) is negligible. The system of equations then reduces to one of order 8 and we may discard two
boundary conditions. Then (3.2.4) gives

\[ u = \frac{2}{\lambda H_0} (D^2 - \lambda^2) \varphi \]  

so that eliminating \( v \) and \( \varphi \) from (3.2.2) and (3.2.5) we have

\[ \left[ (D^2 - \lambda^2)^2 + \frac{\rho H_0^2 \lambda^2}{4 \pi \rho \eta} \right] \varphi = \frac{2H_0}{\lambda \eta} \mu \]  
\[ \left[ (D^2 - \lambda^2)^2 + \frac{\rho H_0^2 \lambda^2}{4 \pi \rho \eta} \right] \mu = \frac{2H_0}{\lambda \eta} \lambda^2 \left( A + \frac{B}{\omega} \right) \times (D^2 - \lambda^2) \varphi \]  

In his two papers on the subject, Chandrasekhar makes two different assumptions. In the first case \(^{(1)}\) when the cylinders are rotating in the same direction, since \( d \ll R_0 \), he writes

\[ A + \frac{B}{\omega^2} \cong A + \frac{B}{R_0^2} = \mathcal{M} \]  

and in the second case \(^{(2)}\), when the cylinders may be rotating in opposite directions, he assumes that the angular velocity distribution in the gap is linear and writes

\[ A + \frac{B}{\omega^2} = \alpha + \beta \omega \]  

where \( \alpha \) and \( \beta \) are constants. Then changing our independent variable from \( \xi \) to \( \xi \) where

\[ \xi = \frac{\tilde{\xi} - R_0}{d} - \frac{1}{2} = \frac{\tilde{\xi} - R_0}{d} \]  

so that we are measuring distances from the mean radius in units of \( d \) and if we also measure \( \lambda \) in units of \( d \), say

\[ \lambda = \frac{a}{d} \]
then (3.2.10) gives

\[ A + \frac{B}{\omega^2} = \lambda_1 \left[ \frac{1}{2} (1+m) - (1-m) \psi \right] \]  

(3.2.12)

where \( m \) is given by (3.1.6).

We now have two sets of equations to solve. If we make the approximation (3.2.9) then we have to solve

\[ \left[ (D^2 - \omega^2) + Qa^2 \right] \psi = 2A d^2 Ho ad \frac{u}{\psi} \]  

(3.2.13)

\[ \left[ (D^2 - \omega^2) + Qa^2 \right] u = \frac{2\eta \lambda_0}{\psi} \frac{1}{Ho ad} \]  

(3.2.14)

whence

\[ \left[ (D^2 - \omega^2) + Qa^2 \right] \psi = -Ta^2 (D^2 - \omega^2) \psi \]  

(3.2.15)

where

\[ T = -\frac{\lambda_0 A d^2}{\psi} \]  

(3.2.16)

On the other hand, if we make the approximation (3.2.10) we have to solve

\[ \left[ (D^2 - \omega^2) + Qa^2 \right] \psi = 2Ad^2 Ho ad \]  

(3.2.13)

\[ \left[ (D^2 - \omega^2) + Qa^2 \right] u = \frac{2\eta d^2}{\psi} \lambda_1 \left\{ \frac{1}{2} (1+m) - (1-m) \psi \right\} \]  

(3.2.17)

whence

\[ \left[ (D^2 - \omega^2) + Qa^2 \right] \psi = -Ta^2 \left\{ \frac{1}{2} (1+m) - (1-m) \psi \right\} \]  

(3.2.18)

where we have put

\[ \psi' = \frac{Ho ad}{\psi} \frac{2Ad^2}{\psi} \]
and immediately dropped the prime and where

\[ T_1 = -4 \frac{\mathcal{L}_1 A}{a^2} d^4 \]  

(3.2.19)

Both sets of equations have to be solved using the boundary conditions already found. They are now

\[ u = D_1 u = (D^2 - a^2) \psi = 0 \quad \text{at} \quad f = \frac{1}{2} \]  

(3.2.20)

and \( \psi = 0 \) if the cylinders are nonconducting \}

or \( D\psi = 0 \) if they are conducting \}

(3.2.21)

**Section 3**

The first set of equations may be solved by a variational principle. To do this we proceed exactly as we did in Chapter 2. First let

\[ P = \left[ (D^2 - a^2)^2 + Qa^2 \right] \psi \quad G = (D^2 - a^2) \psi \]  

(3.3.1)

and then it is possible to show that \( T \) is the ratio of two positive definite integrals

\[ T = \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{ (D^2 - a^2)P \right\}^2 + Qa^2 P^2 \, df}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{ (D\psi)^2 + a^2 \psi^2 \left[ (D\psi)^2 + a^2 \psi^2 \right] \right\} df} = \frac{T_1}{a^2 I_2} \]  

(3.3.2)

and that for any arbitrary variation \( f \psi \) in \( \psi \), compatible with the boundary conditions, that the corresponding variation in \( T \), \( \delta T \), is given by

\[ \delta T = \frac{\partial}{\partial a^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta P \left\{ (D^2 - a^2)^2 + Qa^2 \right\} d^4 \]  

(3.3.3)
so that if (3.2.15) is satisfied then \( \int T = 0 \) and conversely, if \( \int T = 0 \), then (3.2.15) is satisfied. Also we obtain the true minimum value of \( T \) when evaluating \( T \) by (3.3.2). Hence as in Section 2 we have a variational principle as follows: assume for DP an expression involving one or more parameters \( A_i \) and which vanishes at \( \xi = \pm \frac{1}{2} \). Then solve for \( P \) so that \( P(\pm \frac{1}{2}) = 0 \) and then for \( \psi \) using \( \psi = 0 \) or \( D\psi = 0 \) as the case may be and

\[
(D^2 - \alpha^2)\psi = 0 \quad \text{at} \quad \xi = \pm \frac{1}{2}
\]

and then evaluate \( T \) by means of (3.3.2) and minimise it with respect to the \( A_i \).

Both Chandrasekhar\(^{(1)}\) and Niblett\(^{(3)}\) choose the trial function

\[
DP = \frac{1}{2} \sin \eta x - f + \Re \sin 2\eta f
\]

and the only difference in their analyses is that the conditions \( D\psi = 0 \) or \( \psi = 0 \) give different constants of integration. After long and tedious analyses, they both find that \( T \) is of the form

\[
T = \frac{\ell + g\Re + h\Re^2}{\ell + m\Re + n\Re^2}
\]

which has a minimum when

\[
(h m - g n)\Re^2 + 2(\ell l - f n)\Re + (g l - m f) = 0
\]

the constants \( f, g, h, m, n \) and being different in the two cases.
Section 4

The other method used by Chandrasekhar\(^{(2)}\) to solve the set of equations (3.2.13), (3.2.17), (3.2.18) is first to expand \( u \) in terms of orthogonal functions each of which satisfies the boundary conditions as \( u \). Next solve (3.2.13) for \( \psi \) subject to the appropriate boundary conditions and then substitute for \( u \) and \( \psi \) in (3.2.17) and multiply the resultant equation by each of the orthogonal functions. Then integrating with respect to \( \psi \) over \((-\frac{1}{2}, \frac{1}{2})\) gives a set of equations determining the various coefficients in the expansion of \( u \). Then the condition that we should have a non-trivial solution gives us a secular determinantal equation for \( T \).

The orthogonal functions used are

\[
C_m(x) = \frac{\cosh \frac{1}{2} \lambda_m x}{\cosh \frac{1}{2} \lambda_m} - \frac{\cos \lambda_m x}{\cos \frac{1}{2} \lambda_m} \tag{3.4.1}
\]

\[
S_m(x) = \frac{\sinh \frac{1}{2} \mu_m x}{\sinh \frac{1}{2} \mu_m} - \frac{\sin \mu_m x}{\sin \frac{1}{2} \mu_m} \tag{3.4.2}
\]

where \( \lambda_m \) and \( \mu_m \) are respectively the positive roots of

\[
\tanh \frac{1}{2} \lambda + \tan \frac{1}{2} \lambda = 0, \quad \cosh \frac{1}{2} \mu - \cot \frac{1}{2} \mu = 0 \tag{3.4.3}
\]

These functions have the following properties.

a) \( C_m(x) \) is an even function, \( S_m(x) \) is an odd function.

b) They are solutions of

\[
\frac{d^4 y}{dx^4} = x^4 y \tag{3.4.4}
\]

such that

\[
y = y' = 0 \quad \text{at} \quad x = \pm \frac{1}{2} \tag{3.4.5}
\]
c) They satisfy
\[ \int_{\Omega} C_m(x) S_n(x) dx = \int_{\Omega} S_m(x) C_n(x) dx = f_{mn} \]
\[ \int_{\Omega} C_m(x) S_n(x) dx = 0 \]  
(3.4.6)

The work involved in using this method appears to be as long as, if not longer, than that required for the variational method but its advantage is that it allows one to consider the possible counter rotation of the cylinders. In his paper Chandrasekhar considers explicitly the case when \( m = -1 \), i.e. the cylinders are rotating in opposite directions with the same angular velocities.

Section 5

Let \( T_c \) and \( a_c \) be respectively the values of \( T_1 \) and \( a \) for which instability sets in. Chandrasekhar(2) also considered the asymptotic behaviour of the stability of the motion as the magnetic field tended to infinity, i.e. as \( Q \to \infty \). He found that as \( Q \to \infty \), \( a_c^2 \to 0 \) and \( Q a_c^2 \) and \( T_1 a_c^2 \) tend to a finite limit. After some calculation he found that

\[ T_c \to 726 Q \text{ and } a_c \to 25.6 Q^{-\frac{1}{2}} \text{ as } Q \to \infty \text{ (for non-conducting walls)} \]

and \( T_c \to 6203 Q \text{ and } a_c \to 53.4 Q^{-\frac{1}{2}} \text{ as } Q \to \infty \text{ for conducting walls} \).

He also states that when the cylinders rotate in the same direction
\[ \frac{1}{2} T_c (1 + m) \to 107.2 Q \text{ and } a_c \to 15.0 Q^{-\frac{1}{2}} \text{ as } Q \to \infty \text{ for non-conducting walls} \]

\[ \frac{1}{2} T_c (1 + m) \to 651.27 Q \text{ and } a_c \to 52.0 Q^{-\frac{1}{2}} \text{ as } Q \to \infty \text{ for conducting walls} \]
Curve I is the case of two conducting cylinders rotating in opposite directions with $m = -1$.

Curve Ia is the asymptotic direction of curve I.

Curve II is the case of two non-conducting cylinders rotating in opposite directions with $m = -1$. Its asymptotic direction cannot be distinguished from the curve given.

Curve III is for two conducting cylinders rotating in the same direction.

Curve IV is for two non-conducting cylinders rotating in the same direction. (These curves are taken from the papers of Chandrasekhar (1, 2), Niblett (3).

We may interpret these results as follows. As $H$ and/or $\sigma$ increases it becomes much more difficult for the fluid to avoid moving along the line of force of the magnetic field.
field. However one must be careful about this for it is found that in the case when both cylinders rotate in the same direction and when both are conductors then initially the critical wave number of the disturbance increases with $Q$. We might interpret this as being due to "competition" between the angular momentum of the fluid and the magnetic field. We note that each of the four cases shown in Fig. 1 exhibits the same general characteristics - a slow increase in the tendency of the magnetic field to inhibit instability. Then as the magnetic field is increased the rate of inhibition increases and then becomes approximately linear. We also see that if the walls of the cylinders are non-conducting then the inhibiting effect of the magnetic field is less than when they are conductors.

**Section 6**

Donnelly and Ozima\(^5\) have performed experiments to test the validity of the above calculations for the case when the cylinders are conductors and rotate in the same direction. They use a viscometer described in a previous paper and they use mercury as the fluid. The cylinders are made of stainless steel. In effect they measure the viscosity of the fluid which is proportional to the product of the torque and the period of rotation of the inner cylinder against the speed of rotation. They obtain reasonable agreement with the theory but their experimental points lie near to the curve for non-conducting cylinders. They attribute this to the assumption in the theory that the cylinders are perfectly conducting.
Niblett (3) in his paper performed a similar experiment for non-conducting walls. He used perspex cylinders and liquid mercury. The outer cylinder was fixed and he used radioactive isotopes of mercury to determine when instability set in. However he did not get good agreement with the theory.

Section 7

In this section we note the analogous features of the problem considered in this Chapter when the cylinders rotate in the same direction and that considered in Chapter 2, when there is no applied rotation. Jeffreys (4) noted this same analogy in the purely hydrodynamical case. They are:-

a) We can reduce both problems to sets of very similar equations with similar boundary conditions.
b) The equations can be solved by the same variational method.
c) In both cases there appears the parameter $Q$.
d) The magnetic field in both cases exhibits an inhibiting effect on the onset of marginal instability. This effect shows the same general behaviour as $Q$ increases.
CHAPTER 4

Section 1

In this chapter we discuss the problem of the stability of a mass of self-gravitating fluid with infinite electrical conductivity under the action of a magnetic field. Unless otherwise stated we shall assume the fluid to be incompressible and inviscid. We shall derive an energy integral and then use the fact that the system cannot have negative energy to give us an inequality which the gravitational and magnetic energies of the fluid must satisfy. We commence by writing (2.1.1) in the form

\[ \int \frac{du}{dt} - \frac{1}{\mu} (\mathbf{H} \cdot \nabla) \mathbf{H} = -\nabla \left( \rho + \frac{H^2}{2} \right) + \rho \nabla \Phi \]  

(4.1.1)

We have taken the magnetic permeability \( \mu \) to be unity. Multiplying (4.1.1) scalarly by \( \mathbf{r} \) (the radius vector from the origin) and integrating over the volume of the fluid gives

\[ \int \mathbf{r} \cdot \frac{du}{dt} \, d\tau = \frac{1}{\mu} \int \mathbf{r} \cdot (\mathbf{H} \cdot \nabla) \mathbf{H} \, d\tau - \int \mathbf{r} \cdot \nabla \left( \rho + \frac{H^2}{2} \right) \, d\tau + \int \mathbf{r} \cdot \nabla \Phi \, d\tau \]

(4.1.2)

Now

\[ \int \mathbf{r} \cdot \frac{du}{dt} \, dt = \int_{M} \mathbf{r} \cdot \frac{du}{dt} \, d\mathbf{r} = \int_{0}^{M} \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \, d\mathbf{m} = \int_{0}^{M} \mathbf{r} \cdot \frac{d^2 \mathbf{r}}{dt^2} \, d\mathbf{m} = \frac{1}{2} \frac{d^2 \mathbf{r}}{dt^2} \int_{M} \mathbf{r}^2 \, d\mathbf{m} - \int_{0}^{M} \frac{d\mathbf{r}}{dt} \, d\mathbf{m} \]

(4.1.3)

\[ = \frac{1}{2} \frac{d^2 \mathbf{r}}{dt^2} I - 2T \]
say. Now

\[- \int \mathbf{r} \cdot \nabla (p + \frac{H^2}{8\pi}) \, d\tau = -\int_S \left( p + \frac{H^2}{8\pi} \right) \mathbf{n} \cdot d\mathbf{S} + 3 \int_\Omega \left( p + \frac{H^2}{8\pi} \right) d\tau \] (4.1.4)

where we have used the divergence theorem. Now if we assume the boundary of the fluid to be free then the total pressure must vanish there and hence the surface integral in (4.1.4) must vanish. Also

\[\int_\Omega \left( p + \frac{H^2}{8\pi} \right) d\tau = (\gamma - 1) \bar{\mathcal{E}} + M \] (4.1.5)

where \( \gamma \) is the ratio of the specific heats of the fluid and \( \bar{\mathcal{E}} \) is the internal energy and \( M \) the magnetic energy of the fluid. If we consider the fluid to be surrounded by a vacuum then the vanishing of the total pressure implies the vanishing of \( H \) on \( S \). Using this condition we can show that

\[\frac{1}{4\pi} \int_\Omega \mathbf{r} \cdot (\mathbf{H} \cdot \nabla) \mathbf{H} \, d\tau = \frac{1}{4\pi} \int_\Omega \nabla \cdot (\mathbf{H} \times \mathbf{H}) \, d\tau = \frac{1}{4\pi} \int_\Omega \mathbf{H}^2 \, d\tau = -2M \] (4.1.6)

If we write

\[\int_\Omega \mathbf{r} \cdot \nabla \mathcal{E} \, d\tau = \int_0^M \mathbf{r} \cdot \nabla \mathcal{E} \, d\mathcal{M} = \mathcal{E} \] (4.1.7)

then (4.1.2) becomes

\[\frac{1}{2} \frac{d^2 \bar{\mathcal{E}}}{dt^2} = 2T + 3(\gamma - 1) \bar{\mathcal{E}} + M + \mathcal{E} \] (4.1.8)

We can interpret \( I \) as a moment of inertia, \( T \) as the kinetic energy and \( \mathcal{E} \) as the gravitational potential energy of the system. If we have static equilibrium then (4.1.8)
becomes

$$3(\gamma-1)\Phi + M + L = 0$$  \hspace{1cm} (4.1.9)$$

If

$$\ell = \Phi + M + L$$  \hspace{1cm} (4.1.10)$$
is the total energy of the static system, then eliminating $\Phi$ we have

$$\ell = \frac{- (3\gamma-4)\eta - M + 1L}{3(\gamma-1)}$$  \hspace{1cm} (4.1.11)$$

and since $\ell$ cannot be negative then

$$(3\gamma-4)(1L_1 - M) > 0$$

Now, if there were no magnetic field the condition for stability would be $\gamma > \frac{4}{3}$. However, even if $\gamma > \frac{4}{3}$ we see the presence of the magnetic field may induce instability if

$$M > 1L_1$$

We shall be applying this energy principle to gaseous matter for which $\gamma > \frac{4}{3}$. Thus a condition that the configuration be stable is that

$$|L| > M$$  \hspace{1cm} (4.1.12)$$

We shall refer to (4.1.8) as the virial theorem. From (4.1.12) we see that if the magnitude of the magnetic force is great enough then the magnetic field may induce instability.
Section 2

This result suggests a procedure for discussing the stability of a mass of gas under the action of a magnetic field and self-gravitation. The appropriate equations are solved to determine the equilibrium solution and the system is perturbed. Instead of proceeding as we did in Chapters 2 and 3 by solving the equations and determining the perturbations, we calculate directly the total change in energy of the system. If the total change is negative then the system is unstable, since it is not in its state of lowest energy. If the change is positive then the system is stable to the particular perturbation given to it, but is not necessarily stable to another perturbation.

Although this method would appear to be quicker than that used in Chapters 2 and 3, the algebraic calculations are, if anything, longer. We shall use this method to consider the equilibrium of fluid spheres and spheroids, surrounded by a vacuum and under the action of internal and external magnetic fields.

The explicit method used for the consideration of the fluid spheres and spheroids is

1) Define one's model. It must be such that it satisfies all the equilibrium conditions and all the necessary boundary conditions.

2) Give the boundary a deformation. We use the deformation

\[ \varepsilon = R \quad \rightarrow \quad r = R + \varepsilon P_\ell (\eta) \] (4.2.1)

where \( \varepsilon \) is a parameter of smallness (we shall usually neglect terms of \( O(\varepsilon^2) \)) and the \( P_\ell (\eta) \) are Legendre polynomials. Here
\[ \mu = \cos \theta \] in spherical polar coordinates \((r, \theta, \phi)\). The deformation given by (4.2.1) is a special one but a more general one can obviously be built up by summing over various values of \(l\), say

\[ r - R \rightarrow r = R + \sum_{l=1}^{n} l \ell (l) \]  

(4.2.2)

3) Calculate the changes \(\Delta M\) and \(\Delta N\) in the magnetic and gravitational energies respectively.

Section 3

We shall illustrate this procedure by giving rough details of the calculation performed by Chandrasekhar and Fermi\(^{(1)}\) to determine the stability of a fluid sphere of radius \(r = R\) and origin at the centre of the sphere. They assumed an axisymmetric configuration and that the system retained this property when perturbed. The equilibrium configuration is that there are no fluid motions and that it is surrounded by a vacuum. The internal magnetic field \(\vec{H}_i\) is assumed to be uniform and the external field \(\vec{H}_e\) is a dipole field. Thus, in spherical polars \((r, \theta, \phi)\) we may write

\[
\begin{align*}
\vec{H}_i &= H\left( r, -\sqrt{1-r^2}, 0 \right) & & r < R \\
\vec{H}_e &= H\left( (r/R)^3, r, \frac{1}{2}\left(\frac{r^3}{R^3}\right)^{1/2}, 0 \right) & & r > R
\end{align*}
\]  

(4.3.1)

where \(\mu = \cos \theta\). Then the magnetic energy is easily found to be

\[ M = \frac{1}{4} R^3 H^2 \]  

If the deformation is caused by giving a local displacement \(\frac{x}{2}\) to each point of the body, then to first order

\[ \mu = \frac{2x}{5t} \]  

(4.3.2)
and since the fluid is incompressible

\[ \nabla \cdot \mathbf{u} = 0 \tag{4.3.3} \]

which integrates to

\[ \nabla \cdot \mathbf{\xi} = 0 \tag{4.3.4} \]

If we assume that \( \mathbf{\xi} \) is irrotational then we can derive \( \mathbf{\xi} \) from a scalar potential \( \psi \) such that

\[ \nabla^2 \psi = 0 \tag{4.3.5} \]

which has the solution

\[ \psi = \sum_{\ell} A_{\ell} r^\ell P_{\ell}^m(\theta) \tag{4.3.6} \]

The boundary condition is \( \mathbf{\xi}(\mathbf{r}) = \mathbf{\epsilon} P_{\ell}^m(\theta) \) and hence

\[ \mathbf{\xi} = \left\{ \mathbf{\epsilon} \left( \frac{r}{\mathbf{r}} \right)^{\ell-1} P_{\ell}^m(\theta), -\frac{\mathbf{\epsilon}}{r} \left( \frac{r}{\mathbf{r}} \right)^{\ell-1} \mathbf{r} \cdot P_{\ell}^m(\theta), 0 \right\} \tag{4.3.7} \]

where the prime denotes \( \frac{d}{d\theta} \).

To calculate the change in the magnetic field, \( \delta \mathbf{H} \), we know that the change in the electric field is given by

\[ \delta \mathbf{E} = -\mathbf{\nabla} \times \mathbf{H} \]

and that

\[ \mathbf{\nabla} \times \delta \mathbf{E} = -\frac{2}{\mathbf{r}} \delta \mathbf{H} \]

and hence, by (4.3.2)

\[ \mathbf{\nabla} \times (\frac{2}{\mathbf{r}} \times \mathbf{H}) = \frac{2}{\mathbf{r}} \delta \mathbf{H} \]

which integrates to give

\[ \delta \mathbf{H} = \mathbf{\nabla} \times (\mathbf{\xi} \times \mathbf{H}) \tag{4.3.8} \]
Expanding the vector product and using \( \nabla \cdot \mathbf{H}_c = \nabla \cdot \mathbf{J} = 0 \)
this gives, since \( \mathbf{H}_c \) is uniform

\[
\delta \mathbf{H}_c = (\mathbf{H}_c \cdot \nabla) \delta
\]

then (4.3.1), (4.3.7) give

\[
\delta \mathbf{H}_c = \left\{ \epsilon H (l-1) \frac{r^{l-2}}{R^{l-1}} P_{l-1}(\Theta), -m H \left( \frac{r^{l-2}}{R^{l-1}} \right) \right\}_{l=1}^{l^*} = P_l(\Theta), \Omega
\]

and the resultant change in the magnetic energy is

\[
\delta M_i = \frac{1}{4} \int \mathbf{H}_c \cdot \delta \mathbf{H}_c
\]

\[
= \frac{1}{3} \epsilon H^2 R^2 \delta l, 2
\]

after some reduction.

To calculate the change in the external field \( \delta \mathbf{H}_e \), we assume that it is derivable from a scalar potential \( \delta \Phi_e \), satisfying

\[
\nabla^2 (\delta \Phi_e) = 0
\]

of which the appropriate solution is

\[
\delta \Phi_e = \sum_j A_j \frac{P_j(\Theta)}{r^{j+1}}
\]

The boundary condition which determines the \( A_j \) is the continuity of \( \mathbf{n} \cdot \mathbf{H} \). To formulate this condition on the perturbed boundary we note that, given a surface \( \mathbf{f}(r, \Theta, \Phi) = \text{constant} \) then the direction ratios, the normal to it at the point \((r, \Theta, \Phi)\) are given by \((P, Q, S)\) where

\[
P = \frac{\partial f}{\partial r}, \quad Q = \frac{1}{r} \frac{\partial f}{\partial \Theta}, \quad S = \frac{1}{r \sin \Theta} \frac{\partial f}{\partial \Phi}
\]
In this particular problem the surface is

\[ r - \varepsilon \rho_p(r) = R \]

so that

\[ \rho = 1 \quad S = 0 \]

and

\[ Q = -\frac{\varepsilon}{R} \frac{d^l O}{d\theta} \rho_p(r) = -\frac{\varepsilon}{R(1 + \frac{\varepsilon}{R} \rho_p(r))} \frac{d\rho}{d\theta} \frac{d\rho}{d\mu} \rho_p(r) \]

\[ = -\frac{\varepsilon}{R} \left(-1 + \varepsilon \theta \right) \frac{d\rho}{d\mu} \rho_p(r) \]

\[ = \frac{\varepsilon}{R} \left(1 - r^2 \right) \frac{d\rho}{d\mu} \rho_p(r) \]

\[ \rightarrow 0(\varepsilon) \]

Then to \( O(\varepsilon) \) the boundary condition

\[ \frac{n}{(\hat{H} + i \hat{H}^\star)} \bigg|_{r = r + \varepsilon \rho_p} = \frac{n}{(H + i H^\star)} \bigg|_{r = r + \varepsilon \rho_p} \]

\[ \Rightarrow \]

\[ H(r + \varepsilon \rho_p) + H(r) \bigg|_R \frac{\varepsilon}{R} \left(1 - r^2 \right) \rho_p \bigg| (r) = H(r + \varepsilon \rho_p) + H(r) \bigg|_R \frac{\varepsilon}{R} \left(1 - r^2 \right) \rho_p \bigg| (r) \]

\[ (4.3.13) \]

Making the necessary substitutions we find that

\[ A_j = 0 \quad j \neq \ell \pm 1 \]

\[ A_{\ell \pm 1} = \mp \frac{3(\ell + 1)}{2(2\ell + 1)} \]

and hence

\[ \delta H_{er} = \varepsilon \mu \left[ \frac{(\ell - 1)(\ell + 2)}{2(2\ell + 1)} \frac{R^l}{r^{\ell + 1}} \rho_p(r) + \frac{3(\ell + 1)(\ell + 2)}{2(2\ell + 1)} \frac{R^{\ell + 2}}{r^{\ell + 3}} \rho_p(r) \right] \]

\[ \delta H_{e\theta} = \varepsilon \mu \left[ \frac{(\ell - 1)(\ell + 2)}{2(2\ell + 1)} \frac{R^l}{r^{\ell + 1}} \rho_p(r) + \frac{3(\ell + 1)(\ell + 2)}{2(2\ell + 1)} \frac{R^{\ell + 2}}{r^{\ell + 3}} \rho_p(r) \right] \]

\[ (4.3.14) \]

The change in the magnetic energy of the external field is
\[
\delta M_e = \frac{H^2}{8\pi} \int \int \int \left( \frac{R}{r} \right)^\ell \left\{ r^2 \Phi + 2 \left( \frac{1}{4} (1 - \rho^2) \right) r^2 \phi \right\} \phi \theta \phi \\
+ \frac{H^2}{8\pi} \int \int \int \left( \frac{R}{r} \right)^\ell \left\{ 2 \left( \frac{P_i(r)}{P_0(r)} \right) \Phi_r + P_i(r) \Phi_\theta \right\} \phi \theta \phi 
\]

After some reduction we find

\[
\delta M_e = \frac{1}{60} e H^2 R^2 \delta \ell, \quad (4.3.15)
\]

so that the total change in magnetic energy is, by (4.3.11),

\[
\Delta M = \Delta M_c + \Delta M_e = \frac{9}{20} e H^2 R^2 \delta \ell, \quad (4.3.16)
\]

To evaluate the change in the gravitational energy we have to evaluate the integral

\[
\Delta N = \int \int \int \xi \cdot \nabla \Phi_i \phi \\
R + \xi > r > 0
\]

where \( \Phi_i \) is the solution of Poisson's equation

\[
\nabla^2 \Phi_i = -4\pi G \rho 
\]

which is finite at \( r = 0 \) and is such that \( \frac{\Phi_i}{r} \) are continuous with \( \Phi_e \), \( \frac{\Phi_e}{r} \) where \( \Phi_e \) is the solution of

\[
\nabla^2 \Phi_e = 0 
\]

which tends to zero as \( r \to \infty \). Solving (4.3.19) and (4.3.18) and substituting in (4.3.17) gives

\[
\Delta N = \frac{3(e-1)}{(2e+1)^2} (e^2) \frac{GM}{R} 
\]

(4.3.20)
so that the total change in energy of the system is given by

\[ \Delta \mathcal{F} = \frac{g}{20} \mathcal{U}^2 R^2 \mathcal{F}_{1,1} + \frac{3(\ell-1)}{(2\ell+1)^2} \epsilon^2 \mathcal{F}_{1,1} \]  

(4.3.21)

We see that if \( \ell \neq 2 \) then \( \Delta \mathcal{F} = O(\epsilon^3) > 0 \) and we have stability for this deformation. However if \( \ell = 2 \), then \( \Delta \mathcal{F} = O(\epsilon) \) and takes the sign of \( \epsilon \). Thus if \( \epsilon < 0 \) we have instability.

Now the equation

\[ \mathcal{F} - \epsilon \mathcal{P}_\ell(\mu) = \mathcal{F} \]

when \( \epsilon < 0 \) is the equation of an oblate spheroid. We therefore see that the sphere tends to become an oblate spheroid under \( \mathcal{P}_2 \)-deformation.

Section 4

Gjellstedt(2) has considered the problem of the stability of an oblate spheroid under the action of a uniform internal field and an external dipole field. Working in a system of orthogonal coordinates \( (\xi, \psi, \phi) \) defined by

\[
\begin{align*}
\xi &= c \left\{ (1+\xi^2)(1-\mu^2) \right\}^{1/2} \cos \phi \\
\psi &= \xi \left\{ (1+\xi^2)(1-\mu^2) \right\}^{1/2} \sin \phi \\
\phi &= c \xi \mu
\end{align*}
\]

(4.4.1)

so that the spheroid is

\( f = E \)

she finds that for the deformation

\( f = E \rightarrow f = E + \epsilon \mathcal{P}_\ell(\mu) \)

then the changes in the magnetic and potential energies are
given by

\[ \Delta M = 0 \quad \text{if } l \text{ is odd} \]

\[ \Delta M = -\frac{H^2(1+\varepsilon^2)}{\varepsilon} \epsilon \left\{ \frac{\beta}{\varepsilon} + \frac{(H^2/\varepsilon)\varepsilon}{\varepsilon} \right\} \int_0^1 \frac{p_l(\varepsilon)}{\varepsilon} \frac{\varepsilon}{E+1} \frac{\varepsilon}{E+1} \]  

if \( l \) is even, and

\[ \Delta L = -\frac{3}{10} \frac{M^2 \varepsilon}{\varepsilon} \epsilon \left\{ \frac{2}{E+1} \varepsilon + \frac{1}{E+1} \right\} \delta_{d,2} \]

The equilibrium condition is \( \Delta L + \Delta M = 0 \). Using this she concludes that for a \( P_2 \) deformation there is a unique stable configuration for a perfectly conducting fluid spheroid under the action of an internal magnetic field and an external dipole field. For \( P_l \) deformations with \( l \neq 2 \) then the stability depends on the sign of \( \Delta M \).

Gjellstead\(^{(3)}\) also considered similar problems for the case of a fluid sphere under the action of a uniform external magnetic field and zero internal field. She found that

\[ \Delta M = -\frac{1}{4} H^2 R^2 \epsilon \delta_{l,2} \]

and using

\[ \Delta L = \frac{3(l-1)}{(2l+1)^2} \frac{\varepsilon}{R} \frac{M}{R} \]

we see that if \( l=2 \) then, to \( O(\varepsilon) \), \( \Delta \delta \) has the sign of \( -\varepsilon \).

Thus the sphere tends to become a prolate spheroid. If \( l \neq 2 \) then \( \Delta \delta = O(\varepsilon^2) > 0 \) and we have instability.

**Section 5**

We now develop an alternative analysis which may be used for considering the stability of a magnetohydrodynamic system. With the usual assumptions of no viscosity, infinite electrical conductivity and incompressibility, the basic
equations are, from Section 2.1.

\[ \frac{\partial H}{\partial t} = \nabla \times (\mathbf{u} \times H) \]  
(4.5.1)

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{4\pi \rho} (\nabla \times H) \times H - \nabla \left( \frac{b}{\rho} + \Phi + \frac{1}{2} \mathbf{u}^2 \right) \]  
(4.5.2)

\[ \nabla \cdot H = \nabla \cdot \mathbf{u} = 0 \]  
(4.5.3)

We may write (4.5.2) in the form

\[ \frac{\partial \mathbf{u}}{\partial t} = \frac{1}{4\pi \rho} (\nabla \times H) \times H - (\nabla \times \mathbf{u}) \times \mathbf{u} - \nabla \left( \frac{b}{\rho} + \Phi + \frac{1}{2} \mathbf{u}^2 \right) \]  
(4.5.4)

Introducing a new vector \( \mathbf{h} \) by

\[ \mathbf{h} = (4\pi \rho)^{-1/2} H \]  
(4.5.1), (4.5.4) can be written

\[ \frac{\partial \mathbf{h}}{\partial t} = \nabla \times \mathbf{E} \]  
(4.5.5)

\[ \frac{\partial \mathbf{u}}{\partial t} = (\nabla \times \mathbf{h}) \times \mathbf{h} - (\nabla \times \mathbf{u}) \times \mathbf{u} - \nabla \Pi = \mathbf{E} - \nabla \Pi \]  
(4.5.6)

where

\[ \mathbf{E} = \mathbf{u} \times \mathbf{h} \]  
\[ \mathbf{L} = (\nabla \times \mathbf{h}) \times \mathbf{h} - (\nabla \times \mathbf{u}) \times \mathbf{u} \]  
\[ \Pi = \frac{b}{\rho} + \Phi + \frac{1}{2} \mathbf{u}^2 \]  
(4.5.7)

and taking the curl of (4.5.6) gives

\[ \frac{2}{\rho} \nabla \times \mathbf{u} = \nabla \times \mathbf{L} \]  
(4.5.8)

From now on we shall assume that our system has axial symmetry so that we can use a result found by Lust and Schulte\(^{(4)}\). They showed that any solenoidal axisymmetric vector can be written as the sum of a solenoidal vector and a toroidal vector. Now \( \mathbf{h} \) and
\( u \) are solenoidal and since we have assumed axial symmetry we can write, in cylindrical polar coordinates, \((\omega, \phi, z)\).

\[
\dot{u} = l_2 \omega \times T + \nabla \cdot (l_2 \omega \times P) \tag{4.5.9}
\]

\[
u = l_2 \omega \times V + \nabla \cdot (l_2 \omega \times U) \tag{4.5.10}
\]

where \( P, T, V \) and \( U \) are scalar functions independent of \( \phi \), \( r \) is the radius vector from the origin and \( l_2 \) is a unit vector along the axis of symmetry which we take to be in the \( z \)-direction.

Then it is found that

\[
\nabla \cdot \dot{u} = -\varepsilon \frac{A^T}{2 \omega} \frac{d}{dz} \left( -\nabla A_5 \cdot P \frac{d}{dz} \right) + \frac{l^2}{2} \left( \omega^2 T \right) \frac{d}{dz} \frac{d}{dz} \tag{4.5.11}
\]

\[
\nabla \cdot \nu = -\varepsilon \frac{\nabla V}{2 \omega} \frac{d}{dz} \left( -\nabla A_5 \cdot U \frac{d}{dz} \right) + \frac{l^2}{2} \left( \omega^2 U \right) \frac{d}{dz} \frac{d}{dz} \tag{4.5.12}
\]

where \( A_5 \) is an operator defined by

\[
A_5 \equiv \frac{2}{\omega^2} + \frac{2}{\omega^2} + \frac{2}{\omega^2} \tag{4.5.13}
\]

Calculating \( \nabla \) and \( \nabla \cdot \) gives us conditions on \( P, T, U \) and \( V \) and it can be shown that they must satisfy the equations

\[
\frac{\partial^2 P}{\partial \phi^2} = -\frac{1}{\alpha} \left[ \omega^2 T, \omega^2 U \right] \tag{4.5.14}
\]

\[
\frac{\partial V}{\partial \phi} = \frac{1}{\alpha^2} \left\{ \left[ v, \omega^2 P \right] - \left[ T, \omega^2 U \right] \right\} \tag{4.5.15}
\]

\[
\frac{\partial^2 V}{\partial \phi^2} = \frac{1}{2} \left\{ \left[ \omega^2 T, \omega^2 P \right] - \left[ T, \omega^2 U \right] \right\} \tag{4.5.16}
\]

\[
\frac{\partial^2 A_5}{\partial \phi^2} + \frac{2}{\omega^2} \left[ A_5, \omega^2 P \right] - \left[ A_5, \omega^2 U \right] + \omega^2 \left( r^2 - V \right) \tag{4.5.17}
\]

where \( \left[ f, g \right] \) denotes the Jacobian of the functions \( f \) and \( g \) with respect to \( z \) and \( \omega \) i.e.
In the case when the electrical conductivity is finite, more general forms of (4.5.14) and (4.5.15) have been found by Chandrasekhar\(^5\). We see straightaway that \(P\) is independent of \(T\) but that \(T\) depends on \(V\) unless \(V\) is constant, i.e. the toroidal field depends on any non-uniform rotation and that this dependence provides a coupling between \(P\) and \(T\). If we assume the principle of marginal stabilities then the equilibrium equations are obtained by putting \(\frac{2}{2\varepsilon} = 0\) in (4.5.14) - (4.5.17). We denote the resultant equations by (4.5.14a) - (4.5.17a).

Equations (4.5.14a) - (4.5.17a) have been integrated by Woltjer\(^6\) by the same method used to integrate the more general equations obtained in the compressible case. We shall integrate these more general equations in the Appendix to Chapter 5 and thus we need only quote the results of this case. They are

\[
\omega^2 U = F(\omega^2 P) \quad (4.5.19)
\]

\[
V = TF'(\omega^2 P) + G(\omega^2 P) \quad (4.5.20)
\]

\[
\omega^2 T = \omega^2 V F'(\omega^2 P) + H(\omega^2 P) \quad (4.5.21)
\]

\[
\Delta_s P = F'(\omega^2 P) \Delta_s U - \omega^2 TV F''(\omega^2 P) - \omega^2 V G(\omega^2 P) - TH'(\omega^2 P) + U(\omega^2 P) \quad (4.5.22)
\]

where \(F\), \(G\), \(H\) and \(k\) are arbitrary functions of their argument and a prime denotes differentiation with respect to their argument. We have assumed all necessary continuation properties of the functions \(F\), \(G\), \(H\) and \(k\).
Woltjer (6) also showed that the set of equations (4.5.14) - (4.5.17) admit a complete set of integrals of motion. Again since we shall be generalising the results to the compressible medium in Chapter 5, we quote the results. If the boundary is rigid so that

\[ \rho = T = U = 0 \quad \text{on the boundary} \]

then the integrals of motion are

\[ I_1 = \int T \, d^2 r \]

\[ I_2 = \int T H \, d^2 r \]

\[ I_3 = \int (\omega^2 \rho) \, d^2 r \]

\[ I_4 = \int \omega^2 \, d^2 r \]

and the energy integral

\[ 2E = \int \omega^2 \left\{ -p \Delta \rho + T^2 + V^2 - U \Delta \rho \right\} \]

This set of integrals contains those which were obtained by Chandrasekhar (7). Both he and Woltjer remark that the restriction to axial symmetry limits severely the possible motions of an incompressible magnetohydrodynamic system.

Section 6

In this section we first show how equations (4.5.19) - (4.5.22) reduce to results found by other authors. First when there are no poloidal motions, i.e. \( U = 0 \), then we find that

\[ V = \omega (\omega^2 \rho), \quad \vec{\omega} \cdot \nabla T = H(\omega \rho) \] and

\[ \Delta \rho + \omega^2 \frac{\vec{\omega}}{\omega} \cdot \nabla (\omega^2 \rho) \frac{\partial}{\partial \omega} \nabla (\omega^2 \rho) + \frac{1}{\omega^2} H(\omega \rho) H'(\omega \rho) = \kappa (\omega^2 \rho) \tag{4.6.1} \]
which is equivalent to that found by Chandrasekhar\(^{5}\).

When \( T = U = V = 0 \) then we have

\[
\nabla \cdot \mathbf{P} = \kappa (\omega^2 \mathbf{P})
\]

which is Ferraro's\(^8\) result. He showed that in this case the current density is proportional to the distance from the axis, i.e. the current tends to concentrate near the surface of the fluid. He further shows that if \( \kappa (\omega^2 \mathbf{P}) \) is a constant then the magnetic field has the same effect as an angular velocity and that the surface of the sphere becomes oblate.

When \( U = V = 0 \) we have \( \omega^2 \mathbf{T} = \mathbf{H} (\omega^2 \mathbf{P}) \) and

\[
\nabla \cdot \mathbf{P} = -\frac{1}{\omega^2} \mathbf{H} (\omega^2 \mathbf{P}) \mathbf{H}' (\omega^2 \mathbf{P}) + \kappa (\omega^2 \mathbf{P})
\]

(4.6.2)

which is equivalent to that found by Chandrasekhar and Prendergast\(^9\). This characterises the most general magnetic field which can prevail in an axisymmetric configuration of an incompressible medium in gravitational equilibrium.

Further, taking \( \mathbf{H} (\omega^2 \mathbf{P}) = \alpha \omega^2 \mathbf{P} \) and \( \kappa = \text{constant} \), we obtain from (4.6.2)

\[
\nabla \cdot \mathbf{P} = -\alpha^2 \mathbf{P} + \kappa
\]

(4.6.3)

and

\[
\mathbf{T} = \alpha \mathbf{P}
\]

(4.6.4)

These two last equations are the equations which Prendergast used\(^{10}\) to consider the static stability of a fluid sphere under the action of a magnetic field. From (4.6.3) (4.6.4) and (4.5.9) he finds that

\[
\mathbf{L} = -\frac{2}{3} \int \mathbf{P} \mathbf{r} d^3 r - \frac{1}{r} \frac{2}{3} \int \mathbf{P} d^3 r \mathbf{r} \mathbf{r} + \alpha r \int \mathbf{P} d^3 r \mathbf{r} \mathbf{r} \mathbf{r} \mathbf{r} (4.6.5)
\]
The lines of force of the field for $r < R$ are two parameter families of toroids and helices, a typical line of force being a helix wrapped on a toroid.

In a subsequent paper, Prendergast\(^{(11)}\) developed a variational method for determining the stability of the model described above, by the method of small perturbations. He also obtained stability criteria from the virial theorem and from the condition that the minimum value of the internal pressure be greater than or equal to zero. He gives the following results.

a) The virial theorem requires that $M < 1.4$.

b) The pressure criterion requires that $M < 0.5069$.

c) The stability criterion requires that $M < 0.4068$.

In b) and c) it is assumed that $\alpha$ is the lowest root of (4.6.8).

Sykes\(^{(12)}\) has shown that Prendergast's model may be considered to be the limit of a sequence of oblate spheroids rotating about the axis of symmetry. Since we have a rotation...
V must be non zero. In this case we put \( U = 0 \) in (4.5.1) - (4.5.22) so that \( V = \frac{\partial}{\partial t} \) and it is found that

\[
\left[ \frac{\partial}{\partial t} \right]^2 = \beta^2 \omega^2 + V_o^2
\]

and we take \( \kappa(\omega^2 P) = \lambda \) where \( \beta, V_o \) and \( \kappa \) are constants.

When \( U = P \) and \( T = V \), then

\[
F = \omega^2 P
\]

and

\[
G(\omega^2 P) = H(\omega^2 P) = \kappa(\omega^2 P) = 0
\]

This is a solution which was found by Chandrasekhar\(^{(16)}\).

It is equivalent to

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial \omega}
\]

i.e. to

\[
F = \left( \frac{\partial}{\partial \omega} \right)^{\omega^2 P}
\]

and he proved that this solution was stable.

Section 7

In this section we shall discuss the stability of a fluid cylinder of radius \( R \) under the same conditions as before. We first consider the case when all variables are functions of the radial distance \( \bar{\omega} \), only, and the magnetic field is in the \( z \)-direction. Then (4.1.1) reduces to

\[
\int \frac{d\omega}{d\tau} = -\frac{2}{3\omega} \left( H + \frac{H^2}{8\pi} \right) - \frac{2}{\omega} \frac{\partial m(\omega)}{\partial \omega} + P
\]

where \( m(\omega) \) is the mass of fluid contained in a cylinder of radius \( \omega \). Then

\[
\int_0^R 2\pi \frac{d\omega}{d\tau} \omega^2 d\omega = -\int_0^R \frac{2}{\omega} \left( H + \frac{H^2}{8\pi} \right) \omega^2 2\pi d\omega
\]

\[
-\int_0^R 2\pi m(\omega) 2\pi \omega d\omega
\]

(4.7.2)
As before ((4.1.3) - (4.1.5)) the left hand side and the first integral on the right hand side reduce respectively to

\[ \frac{1}{2} \frac{d^2}{dt^2} \int_0^M \omega^2 \, dm - \int_0^M \left( \frac{d\omega}{dt} \right)^2 \, dm \]

and

\[ 2 \{ (r-1) \Phi + M \} \]

where functions, \( M, M, \Phi \) etc. are taken per unit length of the cylinder. The third integral

\[ \int_0^R 2 \Delta m(\delta) 2\pi \omega \, d\delta = \int_0^M 2 \Delta m(\delta) \, dm(\delta) = G M^2 \]

so that (4.7.2) now reads

\[ \frac{1}{2} \frac{d^2}{dt^2} \int_0^M \omega^2 \, dm - \int_0^M \left( \frac{d\omega}{dt} \right)^2 \, dm = 2 (r-1) \Phi + 2M - G M^2 \]  

(4.7.3)

and as in section 4.1 it follows that the stability condition is

\[ M < \frac{1}{2} \frac{\sigma^2}{G} M^2 \]  

(4.7.4)

Chandrasekhar and Fermi\(^{(1)}\) then consider radial pulsations of the cylinder. They use the method of small perturbations, looking for solutions of the perturbed equations of the form \( f(\delta) e^{i\sigma t} \) and find that

\[ \sigma^2 \int_0^R \omega^2 \, d\delta = 2 (r-1) G M^2 + (2 - r) \int_0^R H^2 \omega \, d\delta \]  

(4.7.5)

and thus, by (4.7.3)

\[ \sigma^2 \int_0^M \omega^2 \, dm = 4 \{ (r-1)^2 \Phi + M \} \]  

(4.7.6)

and hence the system is always stable since \( \sigma \) is real.

Simon\(^{(3)}\) has also considered this problem but from a slightly different point of view. He considers the magnetic field \( H_{\perp} \) to fill all space instead of being confined to the cylinder and looks for solutions in which the dependent functions are proportional
to \( \exp \left\{ i (\omega t + m \phi + k z) \right\} \) when \( m \) is integral. If \( k \neq 0 \) he concludes that the presence of the external magnetic field increases the stability even when \( m \neq 0 \), i.e. when the system is dependent on the azimuth angle. On the other hand, if \( k = 0 \), he finds that \( \sigma^2 \) is always positive and is independent of the value of the magnetic field, i.e. the system is always stable.

This result is only for radial pulsations and the question arises - is the cylinder stable under transverse or longitudinal perturbations. The procedure followed is as for the sphere. Let the undisturbed cylinder be \( \bar{\omega} = R_0 \) and let it have a magnetic field \( H_0 \) acting along the axis. Now deform the boundary so that

\[
\bar{\omega} = R_0 \rightarrow \bar{\omega} = R + \alpha \csc k z
\]

where \( R \) is the mean radius of the deformed cylinder. Since the mass per unit length remains constant

\[
R_0^2 = R^2 + \frac{1}{2} \alpha^2
\]

To calculate the change in the gravitational potential energy due to the deformation the external and internal potentials \( V_e \) and \( V_i \) are calculated. They are the solutions of

\[
\begin{align*}
\nabla^2 V_e &= 0 \\
\nabla^2 V_i &= -4 \pi G 
\end{align*}
\]

such that

\[
\begin{align*}
V_e \bigg|_{R + \alpha \csc k z} &= V_i \bigg|_{R + \alpha \csc k z} \\
\frac{\partial V_e}{\partial \bar{\omega}} \bigg|_{R + \alpha \csc k z} &= \frac{\partial V_i}{\partial \bar{\omega}} \bigg|_{R + \alpha \csc k z}
\end{align*}
\]

and it is found that
where $I_n$, $K_n$ are Bessel functions for a purely imaginary argument which have no singularities at the origin and at infinity respectively. We now consider an infinitesimal change in the amplitude of the deformation from $a$ to $a + \delta a$ and consider the change in the potential energy $\delta \mathcal{V}$. To do this we introduce the displacement $\xi$, as for the sphere, assuming that $\nabla \cdot \xi = 0$ and we must have $\nabla \times \xi = 0$ and hence

$$\xi = -\nabla \psi,$$

so that

$$\nabla^2 \psi = 0$$

(4.7.12)

The appropriate solution of which is

$$\psi = A I_0(k\rho) \cosh k \xi$$

and the constant $A$ is found from

$$\xi \psi = a \cosh k \xi$$

so that

$$\xi = a \frac{I_1(k\rho)}{I_0(k\rho)} \cosh k \xi \quad \xi = -a \frac{I_0(k\rho)}{I_1(k\rho)} \sinh k \xi$$

(4.7.13)

Then the displacement $\delta \xi$ required to change $a$ to $a + \delta a$ is

$$\delta \xi = \delta a \frac{I_1(k\rho)}{I_0(k\rho)} \cosh k \xi \quad \delta \xi = -\delta a \frac{I_0(k\rho)}{I_1(k\rho)} \sinh k \xi$$

(4.7.14)

The change in gravitational potential energy per unit length is

$$\delta \mathcal{V} = -2\pi \int \int \psi_k \nabla \cdot \nabla \psi \ d\xi \ d\psi$$

$$= 4 \pi m^2 G R^2 \left\{ \frac{1}{2} - I_0(kR) K_0(kR) \right\} a \delta a$$
so that
\[ \Delta N = \int_0^a \delta A \, dA = 2 \pi^2 \varphi^2 R^2 a^2 \left[ \frac{1}{2} - I_0(kR) K_0(kR) \right]^2 \] (4.7.15)

To calculate the change in the magnetic energy we first write
\[ H = H_0 + h \] (4.7.15)

where \( H \) is the mean field inside the cylinder. To determine \( H \) and \( h \) we note that since lines of force lie on the boundary of the cylinder, the magnetic flux through any cross section must be constant, i.e.,
\[ \int_0^{2\pi} H_0 \, d\omega = \int_0^{2\pi} (H + h) \, d\omega = N_\text{mag} \] (4.7.16)

and that we can write
\[ h = -V\phi \] with \( V^2 \phi = 0 \) (4.7.17)

with the solution
\[ \phi = \sum_{n=1}^{\infty} \frac{a^n A_n}{n k} I_0(2kR) \sin(kz) \] (4.7.18)

Working to order \( a^2 \)
\[ \phi = \frac{a A_1}{k} I_0(kR) \sin(kz) + \frac{a^2 A_2}{2k} I_0(2kR) \sin(2kz) \] (4.7.19)

whence, evaluating \( h \) and then \( N \) to \( O(a^2) \) using
\[ \frac{1}{2} H_0 R_0^2 = \frac{1}{2} H_0 (R^2 + \frac{1}{2} \varphi^2) \]
(from (4.7.8)) and equating coefficients of \( \cos kz \) and \( \cos 2kz \) in (4.7.16) to give three equations, Chandrasekhar and Fermi(1)
found that
\[ H = H_0 \left\{ 1 + \frac{a^2}{R^2} kR \frac{I_0(kR)}{I_1(kR)} \right\} \]

\[ A_1 = -\frac{H}{R} \frac{kR}{I_1(kR)} \quad A_2 = \frac{H}{R^2} \frac{kR}{I_2(kR)} \left\{ \frac{kR I_0(kR)}{I_1(kR)} - \frac{1}{2} \right\} \] (4.7.20)

Then
\[ M = \frac{1}{4} \left\{ \int_0^{R+a \cos kz} \frac{1}{2} \frac{h^2}{\ell^2} \, d\ell \right\} \]
\[ = \frac{1}{8} H_0^2 R_0^2 + \frac{1}{8} a^2 H^2 kR \frac{I_0(kR)}{I_1(kR)} \] (4.7.21)

But the original magnetic energy per unit length is \( \frac{1}{8} H_0^2 R_0^2 \)
and hence
\[ \Delta M = \frac{1}{8} a^2 H^2 kR \frac{I_0(kR)}{I_1(kR)} \] (4.7.22)

From the equilibrium condition
\[ \Delta M + \Delta \mathcal{N} = 0 \]
and the asymptotic behaviour of the various Bessel functions involved, Chandrasekhar and Fermi concluded that the cylinder is stable to this particular deformation if \( k > k_\ast \), where \( k_\ast \) is the single positive root of the equation
\[ \frac{1}{2} - \frac{I_0(kR)}{I_1(kR)} k_0(kR) + kR \frac{I_0(kR)}{I_1(kR)} \left( \frac{H}{H_5} \right)^2 = 0 \] (4.7.23)

where
\[ H_5 = 4\pi \int \overline{J} \] (4.7.24)
and that the cylinder is unstable if \( k < k^* \). Calculation of \( k^* \) demonstrates the stabilising effect of the magnetic field.

It is found that as \( H \) increases the wavelength at which instability sets in increases rapidly, in fact for \( H > H_s \),

\[
k^* R = 0.6811 e^{-2 (H/H_s)^2}
\]

To find the mode of maximum instability we look for the Lagrangian of the motion. It is found that the Lagrangian function per unit length, relative to the equilibrium position is,

\[
\mathcal{L} = \frac{1}{2} \int R^2 \left( \frac{I_0(kR)}{kR I_1(kR)} \left( \frac{d\rho}{dt} \right)^2 + 2 \pi^2 \rho^2 R^2 \mathcal{F}(kR) \right) d\rho
\]

where

\[
\mathcal{F}(kR) = I_0(kR) k_0(kR) - \frac{1}{2} \left( \frac{H}{H_s} \right)^2 kR \frac{I_0(kR)}{I_1(kR)}
\]

Taking the equation of motion for \( \mathcal{L} \) gives rise to and looking for solutions proportional to \( e^{\pm \omega t} \) it is found that \( \omega \) is purely imaginary for \( k > k^* \) and real for \( k < k^* \), which is what one would expect. Now \( \omega = 0 \) when \( k = 0 \) and \( k = k^* \) and thus at an intermediate value of \( k^* \), \( q \) takes a maximum value \( q_m \) say. This value of \( k^* \) gives the mode of maximum instability and it is found that for \( H > H_s \),

\[
k^*_m = 0.4131 e^{-2 (H/H_s)^2}
\]

and

\[
q_m = \frac{1}{2} k^*_m R \left( \frac{4}{17} \rho \right)^{1/2}
\]

Now \( q_m \) is proportional to the time required for instability to set in and again we see the stabilising effect of the magnetic field.

Trehan(14) has also considered this problem for the case when the internal field is force free, i.e. when it satisfies

\[
\nabla \cdot H = \alpha H
\]

(4.7.25)
where \( a \) is a constant. (We shall be discussing force free fields in more detail in section 5.3, but it seems to be more convenient to discuss this problem here). Using equation (4.5.9) it can be shown that (4.7.25) implies that

\[
\nabla^2 \phi + \alpha^2 \phi = 0
\]  

(4.7.27)

where \( \nabla \) is as defined by (4.5.13). Looking for solutions periodic in \( \omega \), we find that

\[
\phi(\omega, z) = A \frac{J_1(\omega \gamma)}{\gamma} \cos \beta z
\]  

(4.7.28)

where \( A \) and \( \beta \) are constants and \( \gamma^2 = \alpha^2 - \beta^2 \). Then the magnetic field is given by

\[
H = A \beta J_1(\omega \gamma) \sin \beta z + A \alpha J_0(\omega \gamma) \cos \beta z \frac{d}{dz} 
\]  

(4.7.29)

Using the same boundary condition as before, i.e. that at the boundary the magnetic field is in the \( z \)-direction we have

\[
J_1(\gamma R_0) = 0
\]  

say \( \gamma R_0 = J_{\gamma,n} \), \( n = 1, 2, 3, \ldots \)  

(4.7.30)

and \( A \) is related to the average value of the mean field per unit length by

\[
\langle H^2 \rangle = A^2 J_0^2(\gamma R_0) \left( \frac{J_{\gamma,n}^2}{R_0^2} + \beta^2 \right)
\]  

(4.7.31)

where \( \langle \rangle \) denotes the average of \( x \) over \( z \) and

\[
\langle H^2 \rangle = \frac{2}{R_0^2} \int_0^{R_0} \langle H^2 \rangle \, dz \, d\theta
\]  

(4.7.32)
The stability is now investigated by the same methods as were used previously in this section. We use (4.7.7), (4.7.8), (4.7.13), (4.7.15) and then a result due to Lundquist (15) that if an element of fluid at \( \mathbf{r} \) is displaced to \( \mathbf{r}' \) so that

\[
\mathbf{r}' = \mathbf{r} + \mathbf{z}
\]  

(4.7.33)

then

\[
\mathcal{H}(\mathbf{r}', t) = \mathcal{H}(\mathbf{r}, 0) \cdot \nabla \mathbf{r}'
\]  

(4.7.34)

The change in magnetic energy per unit length is

\[
\Delta M = \frac{1}{8\pi} \left\langle \int_\mathcal{C} \left\{ |\mathcal{H}(\mathbf{r}', t)|^2 - |\mathcal{H}(\mathbf{r}, 0)|^2 \right\} \, d\tau \right. 
\]  

(4.7.35)

the integration being over the cylinder \( \mathcal{C} = R_0 \). Expanding the integrand in (4.7.35) using (4.7.33) and (4.7.34) it can be shown that

\[
\Delta M = \frac{1}{8\pi} \left\langle \int_\mathcal{C} \left( |\mathcal{H} \cdot \nabla \mathbf{z}|^2 \right) \, d\tau \right. 
\]  

(4.7.36)

and after a long calculation Trehan finds that

\[
\Delta M = \frac{A^2 \alpha^2 \pi^2 \epsilon}{32 I_2^2(\alpha)} \int_0^t (J_1, \mu) \mathcal{G}(\mu, \beta) 
\]  

(4.7.37)

where

\[
\alpha = k R \quad \epsilon = \frac{1}{2} \quad \beta = k
\]

and

\[
\mathcal{G}(\alpha, \beta) = \frac{1}{\alpha^6} \quad \text{x polynomial of degree 9 in } \alpha.
\]

We have assumed that \( k = O(R) \) or less so that the Bessel functions may be expanded in power series. The stability condition
\[ \Delta M + \Delta N = 0 \] gives, using (4.7.15) and (4.7.37),

\[ 0 = \frac{H_s^2}{8} \left\{ \frac{1}{2} - I_0(x) K_0(x) + \left( \frac{H_m}{H_s} \right)^2 \frac{x^2 \varepsilon}{4 I_1(x)} K(x_1, \beta) \right\} \]

(4.7.38)

where

\[ H_m^2 = \frac{A^2}{R^2} \int_0^1 (J_{1,n}(\lambda))^2 \left( J_{1,n} + \beta^2 R \right) \]

\[ H_s^2 = 16 \pi^2 \int_0^2 \xi \, R^2 \]

(4.7.39)

\[ k(x_1, \beta) = (J_{1,n} + \beta^2)^{-1} \xi(x_1, \beta) \]

Consideration of the asymptotic behaviour of the right hand side of (4.7.38) shows that the equation

\[ \frac{1}{2} - I_0(x) K_0(x) + \left( \frac{H_m}{H_s} \right)^2 \frac{x^2 \varepsilon}{4 I_1(x)} K(x_1, \beta) = 0 \]

has only one positive root \( x_+ \). Then we have stability if \( x > x_+ \), instability if \( x < x_+ \). On making the necessary calculations Trehan found that at resonance (\( \beta = k \)) the value of \( (H_m/H_s) \) is less than for any value of \( \beta \) near \( k \). The behaviour of the mode of maximum instability is as in the model discussed by Chandrasekhar and Fermi(1) and the general behaviour of this model is also as discussed previously in this section.
CHAPTER 5

Section 1

In this chapter we consider the problem of magnetohydrodynamic stability from a different standpoint and in more generality. Our discussion is extended to the case of compressible fluids and we use variational methods. The two assumptions made are that the fluid is non-dissipative and that the pressure is a function of density only. The equations of the problem are, as given by Woltjer\(^{(1)}\),

\[
\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{\rho_0} \left( \nabla \times \mathbf{H} \times \mathbf{H} - \rho \left( \nabla \cdot \mathbf{u} \right) \mathbf{u} - \nabla \rho \right) - \rho \nabla \left( \frac{1}{2} \mathbf{u}^2 + \psi \right) \quad (5.1.1)
\]

\[
\frac{\partial \psi}{\partial t} = \nabla \cdot (\mathbf{u} \times \mathbf{H}) \quad (5.1.2)
\]

\[
\frac{\partial \mathbf{u}}{\partial t} = - \nabla \left( \rho \mathbf{u} \right) \quad (5.1.3)
\]

\[
\nabla^2 \psi = 4 \pi \mathcal{E}_p \quad (5.1.4)
\]

\[
\frac{\partial \mathbf{H}}{\partial t} = \frac{\mathbf{J}}{\mu_0} \quad (5.1.5)
\]

\[
\nabla \cdot \mathbf{H} = 0 \quad (5.1.6)
\]

where all symbols are as previously defined. We start by noting that Woltjer has proved the constance of the set of integrals

\[
I_1 - I_6 \quad \text{given by}
\]

\[
I_1 = \int \mathbf{A} \cdot \left( \nabla \times \mathbf{A} \right) \, d\tau
\]

\[
I_2 = \int \mathbf{A} \cdot \mathbf{u} \, d\tau
\]

\[
I_3 = \int \mathbf{p} \cdot \mathbf{v} \cdot \mathbf{u} \, d\tau
\]
\[ I_1 = \oint_T \vec{A} \cdot \vec{r} \cdot \vec{u} \, d\tau \]
\[ I_5 = \oint_T \vec{p} \cdot \vec{r} \cdot \vec{u} \, d\tau \]
\[ I_6 = \oint_T \vec{p} \cdot d\tau \]

Here \( \vec{A} \) is the vector potential of the magnetic field, the gauge being chosen so that there is no magnetic scalar potential, \((\im \im \vec{k})\) is a set of unit vectors in an arbitrary cartesian system of coordinates, and \( \tau \) is the volume of the system under consideration which we assume to be spherical and closed. This assumption gives the boundary conditions

\[ \vec{H} \cdot \vec{n} = \vec{M} \cdot \vec{n} = 0 \quad \text{on the boundary.} \quad (5.1.7) \]

It is obvious that \( I_3, I_4 \) and \( I_5 \) express the conservation of angular momentum and that \( I_6 \) gives the conservation of mass.

To prove that \( I_1 \) is constant, introduce the vector potential into \((5.1.2)\) so that

\[ \frac{\partial \vec{A}}{\partial \tau} = \vec{M} \cdot (\nabla \vec{A}) \quad (5.1.8) \]

whence

\[ (\nabla \vec{A}) \cdot \frac{\partial \vec{A}}{\partial \tau} = 0 \quad (5.1.9) \]

Thus

\[ \frac{2}{\beta} \oint_T \vec{A} \cdot (\nabla \vec{A}) \, d\tau = \oint_T \vec{A} \cdot (\nabla \vec{A}) \cdot \frac{\partial \vec{A}}{\partial \tau} \, d\tau \]
\[ = \oint_T (\nabla \vec{A}) \cdot \frac{\partial \vec{A}}{\partial \tau} \, d\tau + \int_S \vec{A} \cdot \frac{\partial \vec{A}}{\partial \tau} \cdot d\vec{s} \quad (5.1.10) \]

The volume integral vanishes by \((5.1.9)\) and the surface integral is zero if \( \frac{\partial \vec{A}}{\partial \tau} = 0 \) on \( S \). By \((5.1.8)\)

\[ \frac{\partial \vec{A}}{\partial \tau} = \vec{M} \cdot \vec{H} \]

and since \( \vec{H} \cdot \vec{n} = 0 \) on \( S \), \( \frac{\partial \vec{A}}{\partial \tau} \cdot \vec{n} = 0 \) on \( S \), i.e. \( \frac{\partial \vec{A}}{\partial \tau} = 0 \) on \( S \).
is a valid boundary condition. This gives the constancy of $I_1$.

To prove that $I_2$ is constant, multiply (5.1.1) scalarly by $\frac{1}{\gamma} H$ and integrate over $\mathcal{I}$, whence

$$
\int_{\mathcal{I}} H \cdot \frac{\partial U}{\partial t} d\mathcal{I} = -\int_{\mathcal{I}} \{(\nabla \cdot U) \cdot U \cdot H - \frac{1}{\gamma} H \cdot \partial \cdot A \cdot (\frac{1}{2} u^2 + E) \} d\mathcal{I}
$$

$$
= -\int_{\mathcal{I}} \{\nabla \cdot (U \cdot H) + A \cdot \partial \cdot (\frac{1}{2} u^2 + E) \} d\mathcal{I}
$$

$$
= \int_{\mathcal{S}} \{\nabla \cdot (U \cdot H) + (\frac{1}{2} u^2 + E) H + \frac{1}{\gamma} \partial \cdot A \} d\mathcal{S}
$$

all other integrals vanishing since $p = f(\gamma)$ and since $H \cdot n = 0$ on $\mathcal{S}$ we can take $A = 0$ on $\mathcal{S}$. The result follows.

Proving the constancy of $I_3 - I_5$ is equivalent to proving that the magnetic field contributes nothing to the angular momentum of the system. To show this we note that the contribution of the magnetic terms in (5.1.1) to the angular momentum is

$$
\int_{\mathcal{I}} \nabla \times \left\{ (\nabla \cdot H) \nabla H \right\} d\mathcal{I} = \int_{\mathcal{I}} \nabla \times \left\{ (H \cdot \nabla) H - \nabla (\frac{1}{2} H^2) \right\} d\mathcal{I}
$$

$$
= \int_{\mathcal{I}} \left\{ -\nabla \cdot H \nabla H + \frac{1}{2} \nabla \cdot (H^2 \nabla) \right\} d\mathcal{I}
$$

$$
= \int_{\mathcal{S}} H \cdot n \nabla H d\mathcal{S}
$$

and this last surface integral vanishes on a spherical shell. This is the required result. $I_6$ is obtained directly from (5.1.3).

Since the medium is assumed to be non-dissipative, an energy
integral will exist. Multiply (5.1.1) scalarly by \( \mathbf{u} \); this gives using (5.1.3),

\[
\frac{2}{\rho c} \int_T \frac{1}{2} \rho u^2 \, dt = \int_T \left\{ \frac{1}{4 \pi} \left( \nabla^2 \mathbf{H} \right) \cdot \mathbf{H} \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{p} \\
- \mathbf{p} \cdot \nabla \left( \frac{1}{2} \rho u^2 + \mathbf{u} \right) - \frac{1}{2} u^2 \nabla^2 \left( \rho \mathbf{u} \right) \right\} \, dt
\]

\[
= \int_T \left\{ \frac{1}{4 \pi} H \cdot \left[ \nabla \left( \rho \mathbf{u} \right) \right] - \mathbf{p} \cdot \nabla \mathbf{u} + \frac{1}{2} u^2 \nabla (\rho \mathbf{u}) \right\} \, dt
\]

\[
+ \int_S \left\{ \frac{1}{4 \pi} \nabla \mathbf{H} \cdot \left( \rho \mathbf{u} \right) - \mathbf{p} \mathbf{u} - \frac{1}{2} \rho u^2 \mathbf{u} \right\} \, ds
\]

\[
= \int_T \left\{ \frac{-1}{4 \pi} \mathbf{H} - \frac{\rho \mathbf{u}}{\rho c} - \mathbf{p} \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{p} \right\} \, dt
\]

(5.1.2)

the surface integrals vanishing in view of (5.1.7) and, using (5.1.2). Now

\[
\int_T \rho \mathbf{a} \cdot \mathbf{u} \, dt = \int_T \frac{\rho}{S} \frac{ds}{dt} \, dt
\]

\[
= \int_T - \frac{\rho}{S} \frac{ds}{dt} \, dt
\]

\[
= \int_T - \frac{\rho}{S} \frac{2\mathbf{a}}{\mathbf{a}^2 + \mathbf{u} \cdot \nabla A \left( \rho \mathbf{u} \right)} \, dt
\]

\[
= \int_T - \left\{ \frac{\rho}{S} \frac{2 \mathbf{a}}{\mathbf{a}^2 + \mathbf{u} \cdot \nabla A \left( \rho \mathbf{u} \right)} \right\} \, dt
\]

\[
= - \frac{2}{\rho c} \int_T s \frac{\mathbf{a}}{s} \, dt. \quad (5.1.3)
\]
where we have used successively the divergence theorem, (5.1.5) and the total differential given by
\[ \frac{d}{dt} + \mathbf{u} \cdot \nabla = \frac{d}{dt} \]

Also
\[ \int \mathbf{u} \cdot \nabla \Phi \, d\mathbf{r} = -\int \Phi \nabla \cdot (\mathbf{u} \Phi) \, d\mathbf{r} = -\int \Phi \frac{\partial \mathbf{u}}{\partial t} \, d\mathbf{r} \quad (5.1.14) \]

The question now arises as to whether \( \Phi \) is determined from internal or external sources. If the gravitational fields are external, then \( \Phi \) is independent of time and (5.1.14) gives the change in gravitational energy. On the other hand, if the medium is self-gravitating then
\[ \int \Phi \frac{\partial \mathbf{u}}{\partial t} \, d\mathbf{V} = \frac{1}{4\pi G} \int \mathbf{u} \cdot \nabla \Phi \, d\mathbf{V} = \frac{1}{4\pi G} \int \mathbf{u} \cdot \nabla \Phi \, d\mathbf{V} = \int \Phi \frac{\partial \mathbf{u}}{\partial t} \, d\mathbf{V} \quad (5.1.15) \]

where the use of \( \nabla \) indicates that we integrate over all space. Thus
\[ \int \Phi \frac{\partial \mathbf{u}}{\partial t} \, d\mathbf{r} = \frac{1}{2} \frac{d}{d\mathbf{r}} \int \Phi \, d\mathbf{r} \quad (5.1.16) \]

Then substituting (5.1.13) and (5.1.14) or (5.1.16) into (5.1.12) gives
\[ \frac{2}{2\pi} \int \left\{ \frac{1}{2} \mathbf{n}^2 + \frac{1}{2} \mathbf{u}^2 + \mathbf{p} \cdot \mathbf{u} + \frac{3}{2} \mathbf{p} \cdot \mathbf{u} \right\} \, d\mathbf{r} \, d\mathbf{z} = 0 \]

whence
\[ 2E = \int \left\{ \frac{1}{2} \mathbf{u}^2 + \frac{1}{2} \mathbf{p}^2 + \mathbf{p} \cdot \mathbf{u} + \frac{3}{2} \mathbf{p} \cdot \mathbf{u} \right\} \, d\mathbf{r} \, d\mathbf{z} = \text{constant} \quad (5.1.17) \]
Here $q$ is a constant which is unity if the medium is not self-gravitating and a half if it is self-gravitating. This expression is equivalent to the one given by Bernstein et al.\textsuperscript{(13)} except that they consider adiabatic changes of pressure and density.

Section 2

The results of section 5.1 naturally suggest a method for the consideration of the stability of any magneto-hydrodynamic system. The system will be in stable equilibrium when the energy $E$ takes a minimum value compatible with the constancy of the integrals $I_1 - I_6$. We cannot find directly the minima of $E$ but we can find its extrema. Introducing a set of Lagrange multipliers $\frac{x}{8 \pi}, \beta, \lambda, \rho, \sigma$ and $\xi$ the condition that $E$ should take an extreme value subject to the constancy of $I_1 - I_6$ is

$$\delta E - \frac{x}{8 \pi} \delta I_1 - \beta \delta I_2 - \lambda \delta I_3 - \rho \delta I_4 - \sigma \delta I_5 - \xi \delta I_6 = 0 \quad (5.2.1)$$

where $\delta$ denotes a first order variation in the appropriate integral, satisfying the necessary boundary conditions. In order to obtain results from (5.2.1) we first prove the following lemma:

Since $p = f(\Phi)$ then

$$\delta (\Phi \Phi) = \frac{d}{d\Phi} (\Phi \Phi) \delta \Phi$$

Proof. We know that

$$p = -\frac{2 \Phi}{2 x} = \Phi^2 \frac{\delta \Phi}{\delta \Phi}$$

and since $p$ is a function of $\rho$, so must be $\Phi$. Thus

$$\delta \Phi = \frac{d}{d\Phi} \delta \Phi$$

and the result follows.
The result of evaluating (5.2.1) using the lemma is

\[ \nabla \cdot \mathbf{H} = \alpha \mathbf{H} + \lambda \nabla \mathbf{u} \quad (5.2.2) \]

\[ \mathbf{F} (\mathbf{u} - L \mathbf{r}) = \rho \mathbf{H} \quad (5.2.3) \]

\[ \frac{1}{2} \mathbf{u}^2 - \frac{1}{4} \mathbf{c} \cdot \mathbf{u} + \frac{\partial}{\partial s} (\mathbf{F} \cdot \mathbf{u}) + \mathbf{F} = \mathbf{f} \quad (5.2.4) \]

where

\[ \frac{1}{L} = \lambda \mathbf{1} + \mu \mathbf{1} + \delta \mathbf{k} \quad (5.2.5) \]

Equations (5.2.2) - (5.2.5) contain all positions of both stable and unstable equilibrium but further considerations are necessary to separate them. The further calculation necessary is the calculation of the second order change in the energy and we saw in Chapter 4 that even in the incompressible case, this was no easy task. Another point that arises is that this method gives all solutions if and only if the set of integrals \( I_1 - I_6 \) is complete. When the lines of force form nested toroidal surfaces of arbitrary shape the existence of invariants corresponding to the integral of \( \mathbf{A} \cdot \mathbf{H} \) throughout any magnetic surface and the mass contained between any two magnetic surfaces, have been demonstrated by Kruskal and Kulsrud (11). By a magnetic surface we mean a surface of constant pressure. We see that our set of integrals is not complete.

Woltjer (12) has also tried to find invariant integrals when the medium is assumed to consist of a locally neutral mass of electrons and protons. He finds that, to first order in \( \frac{m_e}{e} \) where \( m_e \) is the electron mass and \( e \) the electronic charge
is invariant for a perfectly conducting plasma. If the plasma is neutral and is such that
\[
H \cdot V \frac{P}{n} = H \cdot V \delta(n)
\]
where \( n \) is the number of particles per unit volume, then
\[
\int H \cdot u \, d\tau
\]
is also approximately invariant.

In his discussion of (5.2.2) – (5.2.5) Woltjer(1) makes the following points
a) the only coupling between the magnetic field and the material pressures is through the constant \( \beta \). Then if \( \beta = 0 \) there is no solution in which a "magnetic pressure" balances a hydrostatic pressure.

b) the assumption that the system is enclosed by a rigid wall does not have much effect in the compressible case since, by (5.2.3) if \( \beta \neq 0 \) then as \( \rho \to 0 \), \( H \to 0 \) and if the gravitational forces are sufficiently strong then both \( \rho \) and \( H \) will be comparatively small near the wall. However, in the incompressible case then this assumption is purely artificial.

It is possible to eliminate \( u \) between (5.2.2) – (5.2.5) and to obtain two equations for \( H \) and \( \rho \). However this gives us a non-linear equation to solve.

Kendall(14) has also discussed this problem. Assuming an adiabatic expansion he shows that considering stationary values of the potential energy
\[
V = \int \left( \frac{1}{k_0} H^2 + \frac{k}{\delta - 1} \right) d\tau
\]
where \( \gamma \) is the ratio of specific heats of the gas, that Euler equations corresponding to a stationary value of \( V \) may be written
\[
\frac{1}{4\pi} (\nabla \times \mathbf{H}) \times \mathbf{H} - \nabla \rho = 0
\]

which is just the equation of magnetostatic equilibrium.

**Section 3**

Consider now the incompressible case. Then (5.2.4) is redundant since it was obtained as the coefficient of a variation \( \delta \rho \) in \( \mathbf{F} \) in (5.2.1). Then (5.2.2), (5.2.3) give

\[
\nabla \times \mathbf{H} = \alpha \mathbf{H} + 4\pi \nu \nabla \cdot \left( \frac{\mathbf{F}}{\pi} + \frac{L \wedge \mathbf{r}}{\rho} \right)
\]

so that if \( \rho \neq 4\pi \nu^2 \) then if \( \mathbf{H} = 0 \), i.e. there is no rotation,

\[
\nabla \times \mathbf{H} = \frac{\alpha}{1 - \frac{4\pi \nu^2}{\rho^2}} \mathbf{H}
\]

On the other hand, if \( \rho = 4\pi \nu^2 \) then we require \( \alpha = 0 \). In this case (5.2.2) gives

\[
\nabla \times \mathbf{H} = (4\pi \rho)^{1/2} \nabla \times \mathbf{U}
\]

i.e.

\[
\mathbf{H} = (4\pi \rho)^{1/2} \mathbf{U}
\]

which is a stable solution which we found in section 4.6.

Returning to (5.3.2) we can write it as

\[
\nabla \times \mathbf{H} = \alpha' \mathbf{H} = \alpha \mathbf{H}
\]

A magnetic field \( \mathbf{H} \) which satisfies (5.3.4) is said to be force free. The concept of a force free field is due to Lüst and Schulter. They remarked that in interstellar space where the electrical conductivity of any matter present is extremely high that, allowing for the very low material density, there would be
large currents flowing without any compensating forces. Thus it would be incorrect to postulate that the current density in interstellar space vanishes and that correct assumption is that the Lorentz force vanishes, i.e.

\[ \mathcal{L} = \frac{1}{2} \mathbf{J} \cdot \mathbf{H} = 0 \]  

(5.3.5)

By Maxwell's equations this is equivalent to

\[ (\nabla \times \mathbf{H}) \cdot \mathbf{H} = 0 \]  

(5.3.6)

which has the solution (5.3.4).

A solution of (5.3.4) was found by Chandrasekhar and Kendall(5). If \( \mathbf{a} \) is a fixed unit vector and \( \psi \) a scalar function of position such that

\[ \nabla^2 \psi + \mathbf{a}^2 \psi = 0 \]  

(5.3.7)

then the general solution of (5.3.4) is

\[ \mathbf{H} = \frac{1}{\mathbf{a}} \nabla \times [\nabla \times (\mathbf{a} \psi)] + \nabla \times (\mathbf{a} \psi) \]  

(5.3.8)

The boundary conditions which hold at a discontinuity in \( \mathbf{a} \) have been formulated by Chandrasekhar(6) as follows: at a discontinuity we require that the magnetic field and the normal component of the current density be continuous, i.e. we require the continuity of

\[ \mathbf{H} \cdot \mathbf{n} \]  

and

\[ (\nabla \times \mathbf{H}) \cdot \mathbf{n} \]

i.e. of

\[ \mathbf{H} \cdot \mathbf{n} \]  

and

\[ \mathbf{a} \mathbf{H} \cdot \mathbf{n} \]

Hence the boundary conditions are
\[ \mathbf{H} \cdot \mathbf{H} = 0 \quad \mathbf{H}_\theta \text{ and } \mathbf{H}_\phi \text{ continuous} \quad (5.3.9) \]

Using (5.3.8) and (5.3.9) it can be shown (5) that the energy of the magnetic field is divided equally between the poloidal and toroidal parts of the magnetic field. This is a generalisation of what had been found previously (6).

The first derivation of (5.3.4) was given by Chandrasekhar and Woltjer (7). They obtained it by maximising the magnetic energy subject to a given mean square current distribution. This derivation has two drawbacks:

a) why choose to maximise the magnetic energy subject to a given mean square current distribution, and

b) this method does not lead directly to (5.3.4) but to

\[ \nabla \times (\nabla \times \mathbf{H}) = \alpha^2 \mathbf{H} \quad (5.3.10) \]

which includes (5.3.4) amongst its solutions but further arguments are required to arrive at (5.3.4).

The next variational derivation was given by Woltjer (2). He minimised the magnetic energy subject to the integral \( I_1 \). Working in terms of the vector potential \( \mathbf{A} \) we minimise

\[ \frac{1}{8} \int_\mathcal{Z} \left| \nabla \mathbf{A} \right|^2 d\mathcal{Z} \]

subject to

\[ I_1 = \int_\mathcal{Z} \mathbf{A} \cdot \nabla \mathbf{A} d\mathcal{Z} \]

Using the boundary conditions \( \mathbf{A} = 0 \) which satisfies \( \mathbf{H} = 0 \) and thus (5.1.7) we easily obtain (5.3.4).

We can see that force free fields are stable for since the Lorentz force vanishes there can be no fluid motions and we have found the minimum value of the magnetic energy. Thus any non-
dissipative system will tend to the force free state. This however raises the question - if the medium is non-dissipative what happens to the excess energy? Two proposals that have been made are a) it is used in the process of accelerating cosmic ray particles, and b) it may generate magnetohydrodynamic waves.

So far we have only been considering a closed system but it is obvious that in practical problems the field will have some effect on the external region. It was pointed out that in general there would be surface currents, but that these could be avoided by a suitable choice of the external field.

The stability of force free fields has been considered more analytically in papers by Woltjer and Trehan. The basic assumptions made by Woltjer are a) the initial configuration is static, and b) the fluid is adiabatically compressible. In the usual notation his linearised equations are

\begin{align}
\frac{\partial \Phi}{\partial t} &= - \nabla \cdot \mathbf{p} + \frac{1}{\mu_0} \left\{ (\nabla \times \mathbf{H}) \times \mathbf{b} + (\nabla \times \mathbf{b}) \cdot \mathbf{H} \right\} + \frac{\partial}{\partial t} \Phi \\
\frac{\partial \mathbf{b}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{H}) \\
\mathbf{D} \cdot \mathbf{b} &= \mathbf{D} \cdot \mathbf{H} = 0
\end{align}

Here \( \mathbf{c}_0 \) denotes the velocity of sound in the initial configuration and \( \Phi \) the change in the gravitational energy due to the perturbation. The first step is to introduce the displacement such that, to first order,
\[ \mathcal{M} = \frac{2\alpha}{2t} \]

and if we put
\[ \chi_n(x, t) = \chi(x) e^{\alpha t} \]

(5.3.11) gives
\[ -\sigma^2 \chi = -\nabla \delta \phi + \frac{1}{4\pi} \left\{ (\nabla \phi) \nabla \chi + (\nabla \chi) \nabla \phi \right\} + \int_0^\infty \nabla \Phi \]

Using (5.3.12), (5.3.15) integrates to give
\[ \nabla \Phi = 4\pi G \rho_0 \chi + \nabla F \]

and writing \( \ell \) as the sum of a solenoidal and an irrotational vector we can choose \( F \) so that the solenoidal part of vanishes and we have
\[ \nabla \Phi = 4\pi G \rho_0 \ell \]

We obtain an integral representation for the stability criterion by multiplying (5.3.16) by \( \ell \) and integrating over the volume. Thus
\[ -\sigma^2 \int_0^\infty \nabla \Phi \cdot \ell \, dt = \int_0^\infty \left\{ -\Phi \cos^2 (\nabla \cdot \ell) + \frac{1}{4\pi} \left\{ \left[ \nabla \phi \right] \cdot \left[ \nabla \phi \right] \right\} \right\} \frac{\ell}{\Phi} \]

We only consider the magnetic terms in the right hand side of (5.3.17) to determine the stabilizing effect (if any) of the magnetic field. The rest of the right hand side is Jean's criterion for gravitational stability. In order that the magnetic field may enhance the stability of the system, we require, using (5.3.4)
\[ \int_0^\infty \left\{ \left[ \nabla \phi \right] \cdot \left[ \nabla \phi \right] \right\} dt > 0 \]
This inequality may be reduced as follows - resolving \( \xi \) along any two constant vectors \( \mathbf{a} \) and \( \mathbf{b} \) and also along \( \mathbf{H} \) we can write

\[
\xi \cdot \mathbf{H} = \phi \mathbf{Q} + \psi \mathbf{P}
\]  

(5.3.19)

where \( \phi \) and \( \psi \) are scalar functions and \( \mathbf{Q} = \mathbf{a} \times \mathbf{H} \) and \( \mathbf{P} = \mathbf{b} \times \mathbf{H} \). Without loss of generality we can take \( \mathbf{a} \cdot \mathbf{b} = 0 \), say, in cartesian coordinates that \( \mathbf{a} = \mathbf{i} \) and \( \mathbf{b} = \mathbf{j} \) so that we require

\[
\int \xi (\phi \mathbf{i} \cdot \mathbf{Q} + (\psi \mathbf{j} \cdot \mathbf{P})^2 + \frac{1}{2} (\nabla \mathbf{H}) \cdot \mathbf{Q} + \mathbf{Q} + \mathbf{P}^2 d\mathbf{r} \geq 0
\]  

(5.3.20)

The second term of this integrand may be negative so that the integral has to be reduced still further. Noting that this last condition is independent of whether \( \mathbf{a} \) is a constant or not, we can take the case of axial symmetry and write

\[
\xi = B \mathbf{i} \omega + A \mathbf{j} \phi + C \mathbf{H}
\]  

(5.3.21)

where \( A, B \) and \( C \) are functions of \( \omega \) and \( z \). It can be shown, after some reductions, that the required integral is always positive in this special case, i.e. if the system has axial symmetry then the presence of the force free field enhances its stability.

Trehan\(^{(9)}\) in his paper considers the stability of a force free field embedded in a compressible sphere. His equilibrium fields are specified by

\[
\nabla \cdot \mathbf{H}_e = \alpha \mathbf{H}_e \quad \nabla \cdot \mathbf{H}_c = 0 \quad 0 \leq r \leq R
\]

\[
\nabla \times \mathbf{H}_c = 0 \quad \nabla \cdot \mathbf{H}_e = 0 \quad R < r < \infty
\]  

(5.3.22)
where $R$ is the radius of the sphere. Solving these subject to (5.3.9) gives, if $\mu = \cos \theta$,

$$
H_i = -\frac{\ell}{2\pi} \left[ (1 - \mu^2) \mu \right] - \frac{L_0}{2r} \left[ \frac{1}{r} \right] P_i + \frac{L_0}{2r} \alpha \delta P_i
$$

$$
H_e = \frac{L_0}{2r} \left[ 1 - \left( \frac{R}{r} \right)^2 \right] - \frac{L_0}{2} \left( 1 - \mu^2 \right) \left( 1 + \frac{1}{2} \left( \frac{R}{r} \right)^3 \right)
$$

(5.3.23)

where

$$
P_i = \frac{A_0}{r} \frac{J_{3/2}(\alpha r)}{r^{3/2}} \quad A_0 = \frac{3}{2} B \frac{R^{m_2}}{\alpha} \frac{1}{J_{3/2}(\alpha R)}
$$

(5.3.24)

and $\alpha$ is a root of

$$
J_{3/2}(\alpha R) = 0
$$

(5.3.25)

and $B$ is a constant. Letting $r \to \infty$ in (5.3.23) we see that $B$ gives the strength of the magnetic field at infinity.

After a long calculation he finds that there are three modes of oscillation - two of which are always stable and the third one is stable if

$$
4 C_2 C_3 > C_4^2
$$

where

$$
C_2 = \frac{148492}{13475} \quad C_3 = \frac{408}{55} + \frac{32}{75} \left( \frac{4 \eta_0}{B^2} \right) \left( 3 c_0^2 - 2 \eta_0 \epsilon R^2 \right)
$$

$$
C_4 = -\frac{4518}{1925}
$$

Section 4

In Section 4.5 we obtained a set of equations (4.5.14) - (4.5.17) which gave four equations for the four functions which described the toroidal and poloidal parts of the magnetic and velocity fields, when the system was incompressible and had axial symmetry. In this section we generalise the equations
(4.5.14) - (4.5.17) to the case when the fluid is compressible. We follow Woltjer's (10) method. First the following lemma is required.

If the pressure is a function of the density only then

$$\int \nabla \frac{d}{dp} (\bar{\rho} f) = \nabla b \tag{5.4.1}$$

**Proof**

By (5.1.5),

$$\int \frac{d}{dp} \bar{\rho} f = \frac{\rho}{\bar{s}}$$

so that

$$\nabla \left \{ \frac{d}{dp} (\bar{\rho} f) \right \} = \nabla \left \{ \frac{\bar{s} \rho + b}{\bar{s}} \right \}$$

$$= \nabla \left \{ \frac{1}{\bar{s}} \right \} \bar{s} \rho + b + \frac{1}{\bar{s}} \nabla (\bar{s} \bar{\rho} + b)$$

$$= \nabla \bar{\rho} + \frac{1}{\bar{s}} \nabla b - \frac{\rho}{\bar{s}^2} \nabla \bar{s}$$

Now it is known that

$$b = -\frac{2 \bar{\rho}}{\bar{s}^2}$$

and as

$$p = p(\bar{s})$$

$$\rho = \rho \frac{d}{d \bar{s}}$$

and hence

$$\int \nabla \left \{ \frac{d}{dp} (\bar{\rho} f) \right \} = \int \nabla \bar{\rho} - \int \frac{d}{d \bar{s}} \frac{d}{dp} \bar{s} \rho + \nabla b = \nabla b$$

This result enables us to write (5.1.1) as

$$\int \frac{d}{d \bar{s}} \frac{d}{dp} (\bar{\rho} f) + \frac{1}{2} \bar{s} \frac{d}{dp} \bar{s} \rho = \frac{1}{4 \pi} \int (\nabla \cdot \bar{\rho}) \bar{\rho} - (\nabla \cdot \nabla) \bar{\rho} - \nabla \left \{ \frac{d}{dp} (\bar{\rho} f) \right \} + \frac{1}{2} \bar{s} \frac{d}{dp} \bar{s} \rho \tag{5.4.2}$$
We now proceed as before, assuming axial symmetry. Since \( u \) is not solenoidal we have to introduce the gradient of a scalar into our representation of \( u \). Thus we write

\[
(4\pi)^{-1/2} H = \propto T e^\phi + \nabla (\propto \rho \ e^\phi) \tag{5.4.3}
\]

\[
u = \propto V e^\phi + \nabla (\propto \mathbf{U} e^\phi) + \nabla \mathbf{W} \tag{5.4.4}
\]

Proceeding in the same way as for the incompressible case, we can obtain the following equations

\[
\frac{\partial (\propto U)}{\partial t} = \frac{1}{\propto} \left[ \propto^2 U, \propto^2 \mathbf{P} \right] - \nabla \mathbf{W} \cdot \nabla (\propto^2 \mathbf{P}) \tag{5.4.5}
\]

\[
\frac{\partial \mathbf{T}}{\partial t} = \frac{1}{\propto} \left[ \propto^2 U, \mathbf{T} \right] + \frac{1}{\propto} \left[ \mathbf{V}, \propto^2 \mathbf{P} \right] - \mathbf{V} \nabla \mathbf{W} - \nabla \mathbf{W} \cdot \mathbf{V} \tag{5.4.6}
\]

\[
\frac{\partial (\propto^2 \mathbf{V})}{\partial t} = \frac{1}{\propto} \left[ \propto^2 \mathbf{T}, \propto^2 \mathbf{P} \right] + \frac{1}{\propto} \left[ \propto^2 U, \propto^2 \mathbf{V} \right] - \nabla \mathbf{W} \cdot \nabla (\propto^2 \mathbf{V}) \tag{5.4.7}
\]

\[
\frac{\partial \mathbf{U}}{\partial t} = \frac{1}{\propto^2} \left\{ \left[ \frac{\mathbf{A} \mathbf{P}}{\mathbf{S}}, \propto^2 \mathbf{P} \right] - \left[ \mathbf{A} \mathbf{U}, \propto^2 \mathbf{U} \right] + \left[ \frac{1}{\mathbf{S}}, \propto^2 \mathbf{T} \right] \right\} + \left[ \propto^2 \mathbf{V}, \mathbf{U} \right] \tag{5.4.8}
\]

\[
\frac{2}{\propto^2} \left\{ -\frac{1}{\propto^2} \frac{\partial (\propto^2 \mathbf{U})}{\partial t} + \propto \mathbf{W} \right\}
= \frac{\mathbf{A} \mathbf{P}}{\mathbf{S}} - \frac{\mathbf{A} \mathbf{U}}{\mathbf{S}} \mathbf{U} \mathbf{W} - \mathbf{U} \mathbf{W} \mathbf{U} - \mathbf{V} \frac{\partial (\propto^2 \mathbf{V})}{\partial \mathbf{S}}
+ \frac{2}{\propto^2} \left\{ \frac{\partial (\propto^2 \mathbf{P})}{\partial \mathbf{S}} + \mathbf{W} + \frac{1}{2} \propto^2 \mathbf{V}^2 + \frac{1}{2} \propto^2 \left[ \nabla (\propto^2 \mathbf{U}) \right]^2 \right\}
+ \frac{1}{\propto^2} \left[ \mathbf{W}, \propto^2 \mathbf{U} \right] + \frac{1}{2} \left( \nabla \mathbf{W} \right)^2 \tag{5.4.9}
\]
\[
\frac{2}{\nu t} = \frac{1}{\omega} \left[ \omega^2 U, \beta \right] - \rho \nabla^2 W - \nabla \cdot \mathbf{F}
\]  
(5.4.10)

Here (5.4.5), (5.4.6) follow from (5.1.2), (5.4.7), (5.4.8) from the curl of (5.1.1), (5.4.9) is the \( \tilde{\omega} \)-component of (5.4.2) and (5.4.10) is the continuity equation. If we denote the equilibrium equations by (5.4.5a) - (5.4.10a), i.e. equations (5.4.5) - (5.4.10) with \( \frac{2}{\nu t} = 0 \) then we can integrate (5.4.5a) - (5.4.10a) to some extent. We give the details of the integration in an appendix to this chapter and merely quote the results here. We find that the equations for the six scalars \( P, T, U, V, W \) and \( \rho \) are

\[
\nabla^2 W = -\frac{1}{\omega} \left[ F(\omega^2 \rho), \frac{1}{\beta} \right]
\]  
(5.4.11)

\[
V = \frac{1}{\beta} F'(\omega^2 \rho) + G(\omega^2 \rho)
\]  
(5.4.12)

\[
\omega^2 T = \omega^2 VF'(\omega^2 \rho) + H(\omega^2 \rho)
\]  
(5.4.13)

\[
\Delta_5 U = \frac{1}{\beta} \Delta_5 \left\{ F(\omega^2 \rho) \right\} - \frac{1}{\omega^2} (\nabla \cdot \mathbf{F}(\omega^2 \rho)) \nabla \cdot \mathbf{F}(\omega^2 \rho)
\]  
(5.4.14)

\[
\Delta_5 \rho = \rho \omega^2 V' + F'(\omega^2 \rho) \Delta_5 U - T \omega^2 V F''(\omega^2 \rho)
\]  
(5.4.15)

\[
\frac{d}{dp} \left( \rho \overline{P} \right) + \overline{F} = \frac{1}{2} \omega^2 V^2 + \frac{1}{2} \omega^2 \left[ \nabla \cdot \mathbf{F}(\omega^2 \rho) \right]^2
\]  
(5.4.16)

Here \( F, G, H \) and \( K \) are arbitrary functions of \( \omega^2 \rho \), a prime denotes differentiation with respect to \( \omega^2 \rho \) and all necessary continuity properties have been assumed. Putting \( F(\omega^2 \rho) = 0 \), \( \overline{F} = 0 \) and \( V = 0 \) in (5.4.16) gives
Now in the proof of (5.4.1) we proved the equivalent of the statement that \( \Phi \) was a function of \( \rho \) only. Thus we have

\[
\frac{d}{ds} (\rho \Phi) + \kappa (\omega^2 \rho) = 0
\]

i.e.

\[
\frac{d}{ds} (\omega^2 \rho) \frac{d}{d(\omega^2 \rho)} (\rho \Phi) + \kappa (\omega^2 \rho) = 0
\]

and it follows that

\[
\Phi (\rho) + \kappa (\omega^2 \rho) = 0
\]

which is a result due to Lust and Schulte\(^{(15)}\) that if gravity is neglected then in a magnetostatic configuration the density is a function of \( \tilde{\omega}^2 \rho \). Woltjer\(^{(10)}\) has remarked how in special cases these general results reduce to cases considered by other authors. He has also shown that we can obtain a complete set of integrals of the motion. The integrals

\[
I_1 = \int_T \rho \kappa (\omega^2 \rho) d\tau
\]

\[
I_2 = \int_T H (\omega^2 \rho) d\tau
\]

\[
I_3 = \int_T \left\{ F (\omega^2 \rho) A_5 U - \omega^2 T F' (\omega^2 \rho) \right\} d\tau
\]

\[
I_4 = \int_T \omega^2 U_7 G (\omega^2 \rho) d\tau
\]

are constants of the motion, provided that

\[
\rho = \nabla \cdot \frac{\omega^2 U}{\omega^2} - \frac{1}{\omega^2} \frac{\nabla (\omega^2 U)}{\nabla} \frac{\partial \omega}{\partial s} \cdot n + \frac{1}{\omega^2} \frac{\nabla (\omega^2 U)}{\nabla} \frac{\partial \omega}{\partial s} \cdot l_2 \cdot n = 0
\]

on the boundary, i.e. provided that
\[ \rho = T = \nabla \cdot n = 0 \quad \text{on the boundary} \quad (5.4.17) \]

Our proofs depend on the result that

\[
\int_T \frac{1}{2} \nabla [\phi, \psi] \, dt = - \int_T \frac{1}{2} \phi [\phi, \psi] \, dt - \int_S \frac{1}{2i} \left( \frac{\partial}{\partial n} - \frac{\partial}{\partial \nu} \right) \phi \, ds \quad (5.4.18)
\]

This is best proved by converting the surface integral into a volume integral by means of the divergence theorem. We illustrate the method by proving the invariance of \( I_4 \). We have

\[
\frac{\partial I_4}{\partial t} = \frac{1}{2} \int_T \Delta^2 U \, g(\Delta^2 p) \, dt
\]

\[
= \int_T \Delta^2 V \, g(\Delta^2 p) + \frac{\partial}{\partial t} \frac{\partial^2}{\partial \nu^2} \left( \frac{\partial^2}{\partial \nu^2} g(\Delta^2 p) \right) \, dt
\]

Substituting for the time derivatives from (5.4.5), (5.4.7) and (5.4.10) gives

\[
\frac{\partial I_4}{\partial t} = \int_T \left\{ \frac{1}{2} g(\Delta^2 p) \left[ \Delta^2 T, \Delta^2 \rho \right] + \frac{1}{2} g(\Delta^2 p) \left[ \Delta^2 U, \Delta^2 V \right] 
- g(\Delta^2 p) \nabla \cdot \nabla (\Delta^2 p) + \frac{\partial^2}{\partial \nu^2} \left( \frac{\partial^2}{\partial \nu^2} g(\Delta^2 p) \right) \left[ \Delta^2 U, \rho \right] 
- \Delta^2 V g(\Delta^2 p) \nabla \cdot \nabla (\Delta^2 p) - \Delta^2 V g(\Delta^2 p) \nabla \cdot \nabla (\Delta^2 p) \nabla \cdot \nabla (\Delta^2 p) + \Delta^2 V g(\Delta^2 p) \frac{1}{\nu} \left[ \Delta^2 U, \Delta^2 \rho \right] - \Delta^2 V g(\Delta^2 p) \nabla \cdot \nabla (\Delta^2 p) \nabla \cdot \nabla (\Delta^2 p) \right\} \, dt
\]

Now from (5.4.18), (5.4.17)

\[
\int_T \frac{1}{2} g(\Delta^2 p) \left[ \Delta^2 T, \Delta^2 \rho \right] \, dt = 0
\]

and the remaining terms combine to give
Here we have put $\psi = 1$ in (5.4.18) and then (5.4.17).

These integrals may be interpreted as follows. $I_1$ denotes conservation of mass in each flux tube since $\vec{\omega} \cdot \vec{p} = \text{constant}$ is the boundary of a flux tube. $I_2$, $I_3$ and $I_4$ conserve respectively $A \cdot \vec{H}$, $\vec{u} \cdot \vec{H}$ and the angular momentum in each flux tube.

When we use equations (5.4.3), (5.4.4) the energy integral (5.1.17) may be written

$$E = \frac{1}{2} \int_C \left\{ -\omega^2 \rho \Delta s \cdot \rho + \omega^2 \tau^2 + \rho^2 \nabla^2 \vec{v} + \frac{\rho}{\omega^2} \left[ \nabla (\vec{\omega} \cdot \vec{u}) \right]^2 
+ \frac{2}{\omega \rho} \left[ \nabla \cdot \vec{u} \right]^2 + \rho \left[ \nabla \vec{W} \right]^2 + \epsilon \left[ \vec{\Phi} + \vec{\Phi} \right] \right\} d\tau $$

(5.4.19)

To show that the set of integrals $I_1 - I_4$ is complete we proceed as follows. Since we have four integrals, each of which involves an arbitrary function we have four infinite families of constraints acting on the system. When we minimise the energy subject to these constraints, we must introduce a set of Lagrange multipliers $\alpha_n$ such that

$$\delta E + \sum_{i=1}^{4} \sum_{n=1}^{\infty} \alpha_n \delta I_{1n} = 0 $$

(5.4.20)
where \( I \) is an integral obtained by taking a particular choice of the arbitrary function involved in it, e.g.

\[
I = \int_T \varphi \kappa_n (\omega^2 p) d\zeta
\]  

(5.4.21)

Evaluating (5.4.20), we find that if we introduce four new functions

\[
\kappa = \sum \alpha_{1n} \kappa_n, \quad \lambda = \sum \alpha_{2n} H_n, \quad F = \sum \alpha_{3n} F_n, \quad G = -\sum \alpha_{4n} G_n
\]

then we can obtain equations (5.4.11) - (5.4.16) thus showing that the set, integrals \( I_1 - I_4 \) is complete.

Appendix

We give here the details of the integration of equations (5.4.5a) - (5.4.10a). Integrating

\[
\nabla (\varphi \mu) = 0
\]  

(5.A.1)

gives

\[
\varphi \mu_x \xi_x + \varphi \mu_z \xi_z = -\nabla \left\{ \frac{F(\omega^2 \mu)}{\alpha} \right\}
\]  

(5.A.2)

Using (5.4.4) and taking components gives

\[
\frac{\partial W}{\partial \xi} = -\frac{1}{\omega^2} \frac{\partial (\omega^2 U)}{\partial \xi} - \frac{1}{\omega^5} \frac{\partial F}{\partial \xi}
\]  

(5.A.3)

\[
\frac{\partial W}{\partial \xi} = -\frac{1}{\omega^2} \frac{\partial (\omega^2 U)}{\partial \xi} + \frac{1}{\omega^5} \frac{\partial F}{\partial \xi}
\]  

(5.A.4)

and substituting (5.A.3), (5.A.4) in (5.4.5a) gives

\[
F = F(\omega^2 \mu)
\]  

(5.A.5)

(5.A.3), (5.A.4) also imply
\[ \nabla^2 W = -\frac{1}{\zeta^2} [F(\zeta^2 p), \zeta] \]  
(5.A.6)

and that if \( X(\zeta, z) \) is an arbitrary function then

\[ \frac{1}{\zeta^2} [\zeta^2 U, X] - \zeta^2 W \cdot \nabla X = \frac{1}{\zeta} [F(\zeta^2 p), X] \]  
(5.A.7)

and

\[ \frac{1}{\zeta^2} [\zeta^2 U, X] - \nabla W \cdot \nabla X = \frac{1}{\zeta} [F(\zeta^2 p), X] \]  
(5.A.8)

Using (5.A.7) and (5.A.8), (5.4.6a) - (5.4.8a) become

\[ \left[ F(\zeta^2 p), \frac{I}{\zeta} \right] + \left[ V, \zeta^2 p \right] = 0 \]  
(5.A.9)

\[ \left[ \zeta^2 T, \zeta^2 p \right] + \left[ F(\zeta^2 p), \zeta^2 V \right] = 0 \]  
(5.A.10)

\[ \left[ \frac{A}{\zeta}, \zeta^2 p \right] - \left[ \frac{A}{\zeta^2}, F(\zeta^2 p) \right] + \left[ \frac{T}{\zeta}, \zeta^2 T \right] + \left[ \zeta^2 V, U \right] = 0 \]  
(5.A.11)

Now it follows from the definition of the Jacobian that if \( S \) is an arbitrary function then

\[ \left[ S(\phi), \psi \right] = S'(\phi) \left[ \phi, \psi \right] = \left[ \phi, \psi S'(\phi) \right] \]  
(5.A.12)

Using this, (5.A.9) gives

\[ \left[ \zeta^2 p, \frac{I}{\zeta} F'(\zeta^2 p) \right] - \left[ \zeta^2 p, V \right] = 0 \]

and hence

\[ V = \frac{I}{\zeta} F'(\zeta^2 p) + G(\zeta^2 p) \]  
(5.A.13)

and similarly (5.A.10) gives

\[ \zeta^2 T = \zeta^2 V F'(\zeta^2 p) + H(\zeta^2 p) \]  
(5.A.14)

Here \( H(\zeta^2 p) \) and \( G(\zeta^2 p) \) are arbitrary functions. To integrate (5.A.11) we recall that
\[
\begin{align*}
\{\alpha', \beta'\} &= \{\alpha', \beta\} + \{2, \gamma\} \\
\{\alpha + \beta, \gamma\} &= \{\alpha, \gamma\} + \{\beta, \gamma\}
\end{align*}
\]

The last two terms of (5.A.11) are then given by using (5.A.13), (5.A.14) and (5.A.12)
\[
\left[ \frac{I'}{J} \alpha^2 T \right] + [\alpha^2 V, V] \\
= \left[ \frac{I'}{J} H'(\alpha^2 \rho), \alpha^2 \rho \right] + [\alpha^2 V G'(\alpha^2 \rho), \alpha^2 \rho] + \left[ \frac{I'}{J} \alpha^2 VF'(\alpha^2 \rho), \alpha^2 \rho \right]
\]

and also
\[
\left[ \frac{1}{J} \Delta_r U, F(\alpha^2 \rho) \right] = \left[ \frac{1}{J} \Delta_r U, F'(\alpha^2 \rho), \alpha^2 \rho \right]
\]

Substituting back into (5.A.11) we find that
\[
A^2_\rho = \rho K'(\alpha^2 \rho) + F'(\alpha^2 \rho) \Delta_r U - T \alpha^2 VF'(\alpha^2 \rho) \\
- T H'(\alpha^2 \rho) = \alpha^2 \rho V G'(\alpha^2 \rho)
\]

(5.A.15)

whence \(K'(\alpha^2 \rho)\) is the derivative with respect to \(\alpha^2 \rho\) of an arbitrary function \(K(\alpha^2 \rho)\). To integrate (5.4.9a) we substitute for \(\frac{1}{J} \Delta_r P, \alpha^2 T\) and \(\frac{2W}{\alpha^2}\) from (5.A.15), (5.A.14) and (5.A.4) and after a short calculation we find
\[
\begin{align*}
\frac{1}{2} \{ \frac{d}{d\rho} (\alpha^2 P) + \alpha^2 V^2 + \frac{1}{2} \alpha^2 \left[ \nabla (\alpha^2 U) \right]^2 + \frac{1}{2} \left[ W, \alpha^2 U \right] \\
+ \frac{1}{2} (\nabla W)^2 + U (\alpha^2 P) - \alpha^2 V G(\alpha^2 P) \} &= 0
\end{align*}
\]

where
\[
\begin{align*}
\frac{d}{d\rho} (\alpha^2 P) + \alpha^2 V^2 + \frac{1}{2} \alpha^2 \left[ \nabla (\alpha^2 U) \right]^2 + \frac{1}{2} \left[ W, \alpha^2 U \right] \\
+ \frac{1}{2} \nabla W^2 + K(\alpha^2 P) - \alpha^2 V G(\alpha^2 P) &= 0
\end{align*}
\]

(5.A.16)

where we have chosen the arbitrary function of integration to be zero. Equations (5.A.15), (5.A.16) can be simplified further if we note that taking the \(z\) derivative of (5.A.3) and the \(\rho\) derivative of (5.A.4) gives
\[ \Omega^5 \mathcal{U} = \frac{1}{\mathcal{F}} \Delta_x \left\{ \frac{F(\omega^2 \mathcal{P})}{\omega^2} \right\} - \frac{1}{\omega^2 \rho^2} \nabla \cdot \nabla \left\{ \Delta \frac{F(\omega^2 \mathcal{P})}{\omega^2} \right\} \]  
(5.17)

Eliminating \( W \) from (5.16) using (5.13) and (5.14) gives, after a short calculation,

\[ \frac{d}{d\rho} \left( \frac{\omega^2}{\rho} \right) + \bar{c} + \frac{1}{2} \alpha^2 \mathcal{V}^2 + \frac{1}{2 \alpha^2 \rho^2} \left\{ \nabla^2 F(\omega^2 \mathcal{P}) \right\}^2 - \alpha^2 \mathcal{V} \mathcal{C} (\omega^2 \mathcal{P}) + k(\omega^2 \mathcal{P}) = 0 \]  
(5.18)
Section 1

By a twisted magnetic field we mean one which, working in cylindrical polar coordinates, has no radial component. We shall consider first the simplest problem, from the mathematical point of view. We have a twisted magnetic field embedded in an infinitely long cylinder of radius $R$, filled with incompressible inviscid, infinitely conducting fluid and there are fluid motions along the magnetic lines of force. The equations of motion are, in the form given by Trehan\(^1\), are, in the usual notation,

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P + \frac{1}{4\pi} (\mathbf{H} \cdot \nabla) \mathbf{H} \quad (6.1.1)
\]

\[
\frac{\partial \mathbf{H}}{\partial t} = (\mathbf{H} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{H} \quad (6.1.2)
\]

\[
\nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{u} = 0 \quad (6.1.3)
\]

\[
\Pi = P + \frac{H^2}{8\pi} \quad (6.1.4)
\]

We shall assume that there is no magnetic field outside the cylinder which is taken to be surrounded by the same fluid. Then on the surface $\omega = R$ there will be a current sheet. Another boundary condition is provided by assuming that the pressure at $\omega = R$ remains constant. Now (6.1.1) - (6.1.4) admit the steady state solution

\[
\mathbf{H} = \mathbf{H}(0, \frac{\omega}{P}, 1) \quad (6.1.5)
\]
\[
\mathbf{u} = \frac{\alpha H}{(4\pi p)^2} (0, \frac{\alpha}{p}, 1) = \alpha \mathbf{v}_A \quad \text{say} \tag{6.1.6}
\]

\[
\Pi = \Pi_0 - (1 - \alpha^2) \frac{H^2 \omega^2}{8\pi p} \tag{6.1.7}
\]

where \( p \) and \( \alpha \) are constants, \( 2\pi p \) being the pitch of the helices formed by the magnetic lines of force. Perturbing and linearising the equations in the usual manner, we find that if

\[
H \rightarrow H + \epsilon, \quad \mathbf{v}_A \rightarrow \mathbf{v}_A + \mathbf{u}, \quad \Pi_0 \rightarrow \Pi_0 + \epsilon \Pi \tag{6.1.8}
\]

then, if we let

\[
Y = (4\pi p)^{-2} \mathbf{X} - \mathbf{u} \tag{6.1.9}
\]

\[
\mathbf{y} = (4\pi p)^{-2} \mathbf{y} + \mathbf{u} \tag{6.1.10}
\]

then \( \mathbf{X} \) and \( \mathbf{y} \) must satisfy

\[
\frac{\partial \mathbf{X}}{\partial t} = \nabla \Pi - (1 + \alpha)(\mathbf{v}_A \cdot \nabla) \mathbf{X} - (1 - \alpha)(\mathbf{y} \cdot \nabla) \mathbf{v}_A \tag{6.1.11}
\]

\[
\frac{\partial \mathbf{y}}{\partial t} = -\nabla \Pi + (1 + \alpha)(\mathbf{X} \cdot \nabla) \mathbf{v}_A + (1 - \alpha)(\mathbf{v}_A \cdot \nabla) \mathbf{y} \tag{6.1.12}
\]

Looking for solutions of (6.1.10) and (6.1.11) of the form

\[
\mathbf{X} = Y(\omega) e^{ip_2} \{ e^{i(ot + k z + m \phi)} \} \tag{6.1.13}
\]

\[
\mathbf{y} = Y(\omega) e^{ip_2} \{ e^{i(ot + k z + m \phi)} \} \tag{6.1.14}
\]

\[
\Pi = \Pi(\omega) e^{ip_2} \{ e^{i(ot + k z + m \phi)} \} \tag{6.1.15}
\]

where \( m \) is any integer, and taking components, we can show that
\[ \Pi(\tilde{\omega}) \] must satisfy
\[ \frac{d^2 \Pi}{d\tilde{\omega}^2} + \frac{i}{\tilde{\omega}} \frac{d \Pi}{d\tilde{\omega}} + \left( k^2 \chi^2 - \frac{m^2}{\tilde{\omega}^2} \right) \Pi = 0 \] (6.1.16)

where
\[ \chi^2 = \left( \frac{2 V_A^2}{p} \right)^2 \left\{ \frac{\alpha \sigma - (1-\alpha^2) k V_A}{(\sigma + \gamma k V_A)(\sigma - \beta k V_A)} \right\} \] (6.1.17)
\[ k = k + \frac{m}{p} \quad R = 1 + \alpha \quad \beta = 1 - \alpha \] (6.1.18)

Since we are assuming the pressure to be constant at \( \tilde{\omega} = R \) we must have \( \Pi = 0 \) at \( \tilde{\omega} = R \). Thus the appropriate solution of (6.1.16) is
\[ \Pi(\tilde{\omega}) = J_m(k \chi R) \] (6.1.19)

where \( J_m(x) \) is the Bessel function of order \( m \) and satisfies
\[ J_m(k \chi R) = 0 \] (6.1.20)

If \( J_{m,n} \) is the \( n \)-th root of (6.1.20), then by (6.1.17) and (6.1.20) we have
\[ \left( \frac{2 V_A}{p} \right)^2 \left\{ \frac{\alpha \sigma - (1-\alpha^2) k V_A}{(\sigma + \gamma k V_A)(\sigma - \beta k V_A)} \right\} = 1 + \frac{J_{m,n}^2}{(k R)^2} = \eta^2 \] (6.1.21)
say. Expanding (6.1.21) gives a quadratic equation for \( \sigma \) and since by (6.1.13) - (6.1.15) our stability condition is that \( \sigma \) be real, the discriminant of this quadratic equation must be positive semi-definite. Hence the stability criterion is
\[ (1 - \alpha^2)\left( \pm \frac{2}{p^2} - k \right) k \leq \alpha^2 \left( k + \frac{1}{p^2} \right)^2 \] (6.1.22)
i.e.
\[ 1 \leq \alpha^2 + \left\{ 1 - (\pm \eta k p + m) \right\} \] (6.1.22)
If $a \geq 1$ it is obvious that this inequality holds and we have stability. Now the constant $a$ is a measure of the relative amounts of energy contained in the magnetic and velocity fields and $a \geq 1$ is equivalent to saying that the energy in the velocity field be greater than the energy in the magnetic field.

That the system be stable when $a = 1$ is what we would expect since this corresponds to $V = V_A$ and we have already seen in section 4.6 that this is stable. This result was proved for the magnetic and velocity fields given by (6.1.6) and $V = V_A$ by Trehan.$^2$

If $a < 1$ then instability may set in for certain values of $k$ and $m$. A case of particular interest is when $a = 0$ and then (6.1.22) reduces to

$$-\mu \leq kR + m \leq \mu$$

(6.1.23)

The inequalities (6.1.23) have been studied in detail by Roberts.$^3$ He finds that if $m > 2$ then we always have stability. If $m = 1$ we have stability if $kR < \frac{1}{0.5} \ldots$ and if $kR > \frac{1}{0.5} \ldots$, then we have instability if $R > (2^{\frac{1}{2} - 1})^{\frac{1}{2} \ldots} > \ldots$. The mode $m = 0$ is unstable if $R > 1.202 \ldots$ and the $m = -1$ mode is always unstable for some sufficiently small $k$. If $m < -1$, then the system is unstable for some values of $k$. He also shows that the modes of maximum instability are those of greatest negative $m$. He also calculates the stability of the system if the cylinder is of finite length, neglecting end effects. This problem has also been considered by Dungey and Loughhead.$^4$ and their results are in agreement with the above. In their paper they state that the
stability of the model is independent of the compressibility and density distribution of the fluid, in the case of marginal stability, and Loughhead\textsuperscript{(5)} claims to prove this. However what they actually prove is that if $\sigma = 0$ then the possible values of the wave numbers $k$ are independent of the compressibility and the density distribution. However, Taylor\textsuperscript{(6)} has pointed out that all conditions on the set of possible values of $k$ may also be satisfied by some non zero values of $\sigma$, some of which may give rise to instabilities.

**Section 2**

The model considered above is mathematically the simplest but is extremely idealised. Roberts\textsuperscript{(3)} has also considered the same model with an external field given by

$$
\Pi = \Pi(0, \frac{k^2}{\rho}, 1), \quad \sigma > R
$$

(6.2.1)

$\Pi_{\infty}$ being the (constant) pressure at infinity and finds that all modes with $m = 0$ are unstable. If $m \geq 2$ then we have stability and if $m = 0$ and $1$ then instabilities are present if $R \geq 1.2024 \rho$ and $R \geq 3.0147 \rho$ respectively. We thus see that this model behaves in much the same way as that described in section 6.1.

When we have fluid motions along the lines of force and assume that the system retains its axial symmetry when perturbed then we can approach the problem using the methods of section 4.5. This has been done by Trehan and Reid\textsuperscript{(7)}. We see that equations (4.5.14) - (4.5.17) admit the stationary state solution
\[ \rho = \mathcal{M} = x \quad T = \mathcal{V} = y \] (6.2.2)

say. Letting
\[ \chi = \frac{1}{2} \nu \nu_A \quad \gamma = \nu_A + g(\alpha) \] (6.2.3)

with \[ \nu_A = H(4 \pi f)^{-1/2} \] being the Alfvén velocity corresponding to a magnetic field in the z-direction. If we let
\[ g(\alpha) = \frac{1}{\rho} \quad \text{for} \quad 0 \leq \alpha \leq R \] (6.2.4)
\[ g(\alpha) = \frac{1}{\rho} \frac{R^2}{\alpha^2} \quad R \leq \alpha \leq \infty \] (6.2.5)

then we have
\[ H = H(0, \omega, g(\alpha), 1) \quad \mathcal{V} = \nu_A(0, \omega, g(\alpha), 1) \] (6.2.6)

We see that we are considering the same model as before, except that we have fluid motions along the lines of force outside the cylinder. If we now perturb the system so that
\[ \rho = x \rightarrow \rho = x + \rho_1 \quad \mathcal{M} = x \rightarrow \mathcal{M} = x + \mathcal{U} \]
\[ T = y \rightarrow T = y + T_1 \quad \mathcal{V} = y \rightarrow \mathcal{V} = y + \mathcal{V}_1 \] (6.2.7)

where \( \rho_1, \mathcal{U}_1, T_1 \) and \( \mathcal{V}_1 \) are such that squares and products of them and their derivatives may be neglected. Chandrasekhar has shown that if we introduce two new functions \( \delta \) and \( \psi \) by
\[ \delta = \rho_1 - \mathcal{M}_1, \quad \psi = T_1 - \mathcal{V}_1 \] (6.2.8)

then our perturbed equations are
\[ \omega \frac{d^2}{dt^2} \delta = -2 \mathcal{V} + \omega \mathcal{X} + 2 \mathcal{V} \frac{\partial^2 \psi}{\partial z^2} + 2 \mathcal{V} \psi \frac{\partial \delta}{\partial z} \] (6.2.9)
Substituting for \( X \) and \( Y \) from (6.2.6) we can obtain an equation for \( f \) alone, i.e.

\[
\left( \frac{\partial^2}{\partial t^2} + V_A \frac{\partial}{\partial x} \right)^2 \Delta_5 f = -4 V_A^2 \tilde{g}''(\tilde{\omega}) \frac{\partial^2 \tilde{f}}{\partial t^2}
\]

(6.2.11)

and the boundary conditions on \( f \) are that it be continuous at \( \tilde{\omega} = R \) and \( \delta \to 0 \) as \( \omega \to \infty \). Looking for a solution of (6.2.11) of the form

\[
f(\tilde{\omega}, r, t) = f(\tilde{\omega}) e^{i(r \tilde{t} + k \tilde{x})}
\]

(6.2.12)

then \( f(\tilde{\omega}) \) must satisfy

\[
b^2 f = \alpha^2 \tilde{g}''(\tilde{\omega}) f + \frac{1}{\omega^3} \frac{d}{d\tilde{\omega}} \tilde{\omega}^3 \frac{df}{d\tilde{\omega}}
\]

(6.2.13)

where

\[
\alpha^2 = \left( \frac{2 V_A k}{\sigma + 2 V_A k} \right)^2
\]

(6.2.14)

Trehan and Reid showed that it is possible to formulate a variational method for the determination of \( k \). They showed that

\[
b^2 = \frac{\alpha^2 \int_0^\infty \tilde{\omega}^3 \tilde{g}''(\tilde{\omega}) \tilde{s}^2 \, d\tilde{\omega} - \int_0^\infty \tilde{\omega}^3 \left( \frac{ds}{d\tilde{\omega}} \right)^2 \, d\tilde{\omega}}{\int_0^\infty \tilde{s}^2 \tilde{w}^3 \, d\tilde{w}}
\]

(6.2.15)

and that

\[
\int_0^\infty \delta \cdot \delta \tilde{w}^3 \, d\tilde{w} = 0 \quad \forall i + j
\]

(6.2.16)
\( \delta_e \) being the eigensolution of the equation for \( \delta(\hat{\omega}) \), obtained by substitution of (6.2.12) in (6.2.11), corresponding to the eigenvalue \( k \). They also state that any variation in \( k^2 \) resulting from any variation in \( \delta \) compatible with the boundary conditions, provided that (6.2.13) is satisfied. We thus have a variational principle for determining \( k^2 \). Writing (6.2.15) in the form

\[
\dot{k}^2 = \frac{\alpha^2 I_i - I_2}{I_3} \tag{6.2.17}
\]

and using (6.2.14) it follows that

\[
\frac{(\sigma + 2 \nu \lambda k)^2}{(2 \nu \lambda k)^2} = \frac{I_i}{k^2 I_3 + I_2} \tag{6.2.18}
\]

and since the integrals \( I_i \) are positive definite we see that \( \sigma \) is always real and that the model under consideration is stable.

The consideration of this last model by this method, brings out the point we made in Chapter 1 that the assumption that an axisymmetric system retains its axial symmetry when perturbed is an artificial assumption. We saw that for a mathematically simpler model than this last one, that if we allowed asymmetry into the system then the behaviour was quite complicated.

Lundquist\(^{9}\) has considered the stability of a twisted magnetic field given by, in cartesian coordinates,

\[
H = \left[ -H_\phi(\omega) \frac{\partial}{\partial x}, H_\phi(\omega) \frac{x}{\partial \phi}, H_2(\omega) \frac{\partial}{\partial z} \right] \tag{6.2.19}
\]
where $\omega^2 = x^2 + y^2$ and $\omega \phi = \frac{\pi}{2}$. The field is confined to a cylinder of radius $R$. He finds an integral expression for the change in magnetic energy when an element of fluid is given a displacement $\frac{\pi}{2}$. The change in energy is

$$\Delta M = \frac{1}{i} \int_{\Gamma} \left( \frac{d^2 x}{2\pi} \frac{d^2 y}{2\pi} + H_x H_y \frac{\partial^2 \phi}{\partial x \partial y} \right) d\Gamma$$

(6.2.20)

where

$$\Gamma = |p + \frac{H^2}{8\pi} + \text{const}$$

(6.2.21)

With the field given by (6.2.19), the condition (6.2.20) gives

$$\int_0^R H_x^2 \omega d\omega > 2 \int_0^R H_x^2 \omega d\omega$$

(6.2.22)

which may be interpreted by saying that the field is unstable when the magnetic energy due to the twisting is greater than twice the magnetic energy of the untwisted field.

He also uses (6.2.20) to consider the stability of a force free magnetic field. The equation

$$(\nabla \times H) \times H = 0$$

has the solution

$$H = H(0, J_1(\alpha r), J_0(\alpha r))$$

(6.2.23)

and using (6.2.20) it can be shown that this field is stable, which is what we expect.
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REFERENCES

Chapter 1.

1) Hartmann, J. & Lazarus, F. Math.-fys. Medd. 15, Nos. 6, 7, 1937.
REFERENCES (Contd.)

Chapter 2.

2) Rayleigh, Lord. 1916, Phil. Mag. (6), 32, 529.

Chapter 3.

REFERENCES (CONTD.)

Chapter 4.


Chapter 5.

REFERENCES (CONT'D.)


Chapter 6
