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Effect of Spatial Dimensionality on the Chaotic Properties of Turbulent Flow

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Doctor of Philosophy
The University of Edinburgh
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The phenomenon of fluid turbulence is found almost universally in the world around us, however, there is still much that is not understood about the underlying physics. What is known about turbulent fluid flows is that their dynamics can be vastly different depending on the spatial dimension; these differences are particularly stark between two and three dimensions. Additionally, it is known that turbulent flows exhibit deterministic chaos, which manifests itself as an extreme sensitivity to initial conditions. This has important consequences for real world predictability, where finite measurement precision eventually leads to a total loss of predictability. This work is focussed on the effect of the spatial dimension on the chaotic properties of turbulent flows.

The computational cost of performing fully resolved simulations, known as direct numerical simulation (DNS), of turbulent flows at even modest Reynolds numbers can be enormous. This cost increases rapidly with both the Reynolds number and the spatial dimension. Compounding this issue, in order to measure chaotic properties, for example Lyapunov exponents, in such simulations requires the concurrent evolution of many velocity fields. As a result, only now is it beginning to become feasible to perform systematic measurements of these chaotic properties of turbulent flows, albeit only in the idealised case of homogenous and isotropic turbulence (HIT). It should be noted that these fully resolved studies are entirely unfeasible beyond three spatial dimensions at present, and will remain so for the foreseeable future. As such, it is necessary to turn to models where the degrees of freedom are reduced, in our case a popular two point closure: the eddy damped quasi normal Markovian (EDQNM) approximation. To this end, a parallel $d$-dimensional EDQNM code has been developed for this thesis in order to study predictability in higher spatial dimensions. Included here is some discussion of the details needed to write such a code.
This thesis presents the results of such studies in both two and three spatial dimensions, with a focus on the scaling properties of the Kolmogorov-Sinai entropy and the attractor dimension. In three dimensions simple dimensional analysis and the Kolmogorov 1941 (K41) theory predicts that this scaling will be determined entirely by the Reynolds number of the flow. In our results it is seen that this is true, but the rate of scaling is not entirely consistent with K41, nor popular intermittency models. However, in two dimensions it is found that these quantities have a dependence on not only the Reynolds number, but also on the system size and the length scale at which energy is injected. This was not predicted by simple dimensional arguments and provides further evidence of non-universal behaviour in two dimensional HIT.

It has long been observed that features of both two and three dimensional turbulence co-exist in the Earth's atmosphere, largely as a result of the geometry of the system. This geometry is best described as a thin layer, and in previous experiments and simulations a transition between two and three dimensional phenomenology has been observed as the layer height is varied. By performing DNS of thin layer turbulence, we find the Lyapunov exponents can be used as an indicator of this transition. The predictability times either side of the transition are different, which may have consequences for atmospheric forecasting. This co-existence of two and three dimensional dynamics is also found in stratified systems, those undergoing rotation, and those under the influence of strong magnetic fields. Hence, these results may be applicable to a wider range of situations. Additionally, we also present results of non-integer dimensional turbulence using the EDQNM approximation to allow us to disentangle the role of the cascade in the predictability transition found in the thin-layer case.

Anomalous scaling in the structure functions of HIT has drawn numerous comparisons with critical phenomena in the literature; in particular with the idea of an upper critical dimension for turbulence. Using the EDQNM model, we have performed a numerical study of HIT in higher dimensions. Here we find an enhanced forward energy cascade with increasing dimension evidenced by greater velocity derivative skewness and dimensionless dissipation rate. Despite these changes, in general the statistical picture seems to be very similar as the spatial dimension increases from three. However, between five and six dimensions the chaotic properties show a dramatic phase transition to a non-chaotic regime which we relate to the energy cascade as a function of spatial dimension.
Turbulence is a state of fluid flow characterised by rapid variations in both space and time of quantities such as velocity and pressure. These rapid variations lead to turbulent fluids exhibiting extremely complex structure, to the point of appearing random. However, turbulent flows are in fact entirely deterministic, yet paradoxically they are not entirely predictable. Take, for example, weather forecasts of the atmosphere which is dominated by turbulent flow.

From the point of view of physics, turbulence is an unsolved problem, however, the solution of the turbulence problem will not render turbulent flows predictable. The lack of predictability in turbulent flows stems from their complexity and our inability to measure their properties with infinite precision. Any small error in our measurement is quickly amplified until our predicted flow is completely different than the true flow.

This Thesis is focused on gaining a deeper understanding of the predictability of turbulent flows. In particular, the influence of the spatial dimension of the flow is considered in detail. Real flows are of course three dimensional, however, in certain situations they can become effectively two-dimensional. This is important, as two-dimensional turbulent flows have drastically different dynamical behaviour than in three dimensions. Understanding predictability in both these cases will aid not only theoretical understanding, but also help to build more accurate models for atmospheric predictability and thus weather forecasting.

It is also possible, in a mathematical sense, to allow the spatial dimension to take on a more abstract character. In this work, we will consider the spatial dimension as a parameter than can vary freely. Doing so allows us to study self-consistent variations of the turbulence problem, with the aim of relating our findings back to the real world three-dimensional case.
In particular, approaching this abstract case from the viewpoint of predictability provides an illuminating link with the real world turbulence problem.
DECLARATION

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Parts of this work have been published in [18], [49], [51], [50] and [52].

(Daniel Clark, April 2022)
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It has often been said that in life only two things are certain: death and taxes. However, looking at the world around us it becomes apparent that this statement has a glaring omission. Indeed, in life there are in fact three certainties: death, taxes and turbulence. Of course, this is a somewhat facetious remark, but there is some truth to it. Almost all fluid flows, natural or man-made, exhibit turbulence. This is true across an incredible range of length scales: from the flow of blood in an artery all the way to the generation of galactic-scale magnetic fields.

The omnipresent nature of fluid turbulence justifies it as a problem of great importance, however, at the most fundamental level, it remains unsolved. One of the root causes of difficulty in developing a fundamental theory of turbulence is the non-linear nature of the Navier-Stokes equations which govern turbulent flow. This non-linearity couples together different length scales within a turbulent flow and allows for inter-scale energy transfer. As a result, turbulent flows exhibit complex spatial and temporal structure. The combination of fluctuations and excitation across a wide range of length scales and spatial and temporal structure provides us with a useful working definition of what constitutes a turbulent flow. An additional defining feature of turbulence is that it is highly dissipative and thus, without external injection of energy, a turbulent flow will simply decay.
Simply referring to turbulence as an unsolved problem, whilst correct, does a disservice to the substantial progress that has been made. Beginning with Reynolds’ work on pipe flow \cite{186} in the late 1800s, through the work of Kolmogorov \cite{120} and on to the quantum field theory-inspired work initiated by Kraichnan \cite{122}, a substantial body of knowledge has accumulated. Perhaps most influential for the work in this Thesis, was the introduction by Taylor \cite{212} of the idealised case of statistically homogeneous and isotropic turbulence (HIT), upon which the majority of study into the fundamental problem of turbulence has been built.

Of course, real world turbulent flows are, in general, not homogeneous and isotropic due to the presence of mean boundaries and mean flows. However, away from these boundaries and in coordinates moving with the mean flow, there are regions of the flow in which the small scales can be considered homogeneous and locally isotropic. As such, the results from the more analytically tractable case of HIT may be of use in more general and realistic cases involving, amongst many other phenomena, rotation, stratification and external magnetic fields.

Even in the idealised case of HIT, things are not simple. In fact, the removal of boundaries and mean flows leaves us only with the full non-linearity of the Navier-Stokes equations. This fact allows for HIT to display a diverse range of interesting phenomena. In this Thesis we are focused on two main ideas within HIT. The first is the influence of the spatial dimension on the dynamics of the flow where, for example, it is observed that two- and three-dimensional turbulence exhibit dramatically different properties. The second is the idea of deterministic chaos and finite predictability in turbulence. Both of these concepts collide when it comes to atmospheric predictability where two- and three-dimensional effects can coexist. However, we will also go further and consider spatial dimensions greater than three, as well as non-integer dimensions as a method of exploring the mechanisms behind turbulence.

Before delving further into the influence of the spatial dimension, deterministic chaos and their intersection, we will first provide a short overview of the basics of turbulence theory. Our focus will be on topics that are used extensively throughout the rest of this Thesis and thus will be non-exhaustive. For further details the following textbooks are recommended: Davidson \cite{59}, Landau and Lifshitz \cite{129}, \cite{134}, \cite{148,149} Monin and Yaglom \cite{156} and Tennekes and Lumley \cite{214}. We do not follow any of these works exactly, instead drawing from each where appropriate.
1.1 The Navier-Stokes Equations

We begin by considering a fluid which exists in an infinite boundary-free domain in any spatial dimensions $d \geq 2$. Our fluid has a density $\rho(x, t)$ which, for the time being, varies in space and time. We will work throughout this Thesis in Eulerian coordinates where we consider all quantities to be defined as fields. As is standard, we consider a small sub-domain $V$ within our infinite domain and find the mass, $m$, contained within this domain to be

$$m = \int_V \rho(x, t) \, dx.$$  \hfill (I.1)

By considering mass conservation and the flow of mass into and out of $V$ due to the velocity field $u(x, t)$ of our fluid, it can be shown that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0,$$ \hfill (I.2)

where for brevity we have omitted both space and time dependence. Through the product rule we can re-express this as

$$\frac{D \rho}{Dt} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0,$$ \hfill (I.3)

where we have introduced the material derivative $D/Dt$. The material derivative allows us to make a connection with what are known as Lagrangian coordinates, in which the trajectories of individual fluid elements are followed.

For the rest of this Thesis we will only be concerned with incompressible fluids. For such fluids the material derivative of the density is zero, which from the above then implies

$$\nabla \cdot \mathbf{u} = 0,$$ \hfill (I.4)

which we will refer to as the incompressibility condition. Focussing solely on solenoidal velocity fields may seem like quite a severe restriction, nevertheless so long as $|\mathbf{u}| \ll c$ where $c$ is the speed of sound in the fluid, the usage of the incompressibility condition is valid. A great number of flows of practical significance fall into this category, however, this is not the only reason we restrict ourselves to incompressible flows. Perhaps most importantly, incompressible flows are much more tractable from an analytic perspective.

By making use of the incompressibility condition and now considering the conservation of momentum within our sub-domain $V$, the Navier-Stokes equations can be derived. We omit the details here but they can be found in full in [129]. For our incompressible case the
Navier-Stokes equations are then
\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u},
\]
\[
\nabla \cdot \mathbf{u} = 0
\] (I.5)
in which \( P(x, t) \) is the pressure field and \( \nu \) is the kinematic viscosity of the fluid. We note here that, as we have seen above, the incompressibility condition restricts us to constant density flows and for simplicity we take \( \rho = 1 \). Additionally, we mention here that incompressibility implies a direct relationship between the velocity and pressure fields which can be seen by taking the divergence of the Navier-Stokes equations. Doing so we find
\[
\nabla^2 P = -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})
\] (I.6)
In our infinite domain we can solve this using a Green's function to find an expression for \( P \). This process results in
\[
P(x, t) = \begin{cases} 
-\frac{1}{2\pi} \int d\mathbf{x} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \ln |\mathbf{x} - \mathbf{x}_0| & \text{if } d = 2, \\
\frac{1}{(d-2)A_d} \int d\mathbf{x} \frac{\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})}{|\mathbf{x} - \mathbf{x}_0|^{d-1}} & \text{if } d > 2,
\end{cases}
\] (I.7)
where \( A_d = 2 \{(\pi)^{d/2}/\Gamma(d/2)\} \) is the surface area of a \( d \)-dimensional unit sphere. In this expression we find that \( P \) is non-local and thus allows the velocity at one point in space to influence all others via the pressure field.

Anticipating that we will be working solely with HIT in this Thesis, we will not discuss in any detail here the standard Reynolds decomposition of the velocity field into mean and fluctuating parts. In HIT there can be no mean flow and therefore our velocity field \( \mathbf{u} \) is composed entirely of fluctuations and also satisfies
\[
\langle \mathbf{u} \rangle = 0,
\] (I.8)
for some suitable averaging process.

## I.1.1 Fourier Transform of the Navier-Stokes Equations
The use of Fourier analysis in the study of HIT has become standard since the work of Batchelor [12]. We have already discussed that turbulence is a problem involving energy exchange over many length scales. By transforming the turbulence problem into Fourier space we can gain a clearer image of these interactions across length scales.
We define the $d$-dimensional Fourier transform of the velocity field as

$$u(k, t) = \frac{1}{(2\pi)^d} \int dx \, u(x, t) e^{-ik \cdot x}, \quad (I.9)$$

and similarly for any other fields of interest. The inverse transform is then given by

$$u(x, t) = \int dk \, u(k, t) e^{ik \cdot x}. \quad (I.10)$$

To Fourier transform the Navier-Stokes equations we make use of two properties of the Fourier transform. The first is the conversion of derivatives into algebraic expressions, i.e.

$$\nabla \cdot P(x, t) = \nabla \cdot \int dk \, P(k, t) e^{ik \cdot x} = \int dk \, i k \cdot N_{\alpha \beta}(k, t) e^{ik \cdot x}. \quad (I.11)$$

Second is the fact that the Fourier transform of a product is a convolution. This allows us to write the non-linear term, suppressing again the time dependence for brevity, as

$$u_\beta(x) \partial_\beta u_\alpha(x) = \partial_\beta \left( u_\alpha(x) u_\beta(x) \right) = \int dk \, i k \beta N_{\alpha \beta}(k, t) e^{ik \cdot x}, \quad (I.12)$$

where we have used index notation and the Einstein summation convention. Additionally, in the above equation the first equality makes use of the incompressibility condition and $N_{\alpha \beta}(k)$ is the Fourier transform of the non-linear term defined as

$$N_{\alpha \beta}(k) = \int dp \, u_\beta(p) u_\alpha(k - p). \quad (I.13)$$

Now, replacing all the terms in the Navier-Stokes equation with their Fourier representations we find

$$\frac{\partial}{\partial t} \int dk \, u_\alpha(k) e^{ik \cdot x} + \int dk \, i k \beta N_{\alpha \beta}(k, t) e^{ik \cdot x} = -i \int dk \, k \alpha P(k, t) e^{ik \cdot x} - \nu \int dk \, k^2 u_\alpha(k, t) e^{ik \cdot x}. \quad (I.14)$$

Of course, this equation must hold for arbitrary $k$ and thus we have

$$\frac{\partial u_\alpha(k)}{\partial t} + i k \beta \int dp \, u_\beta(p) u_\alpha(k - p) = -i k \alpha P(k, t) - \nu k^2 u_\alpha(k, t), \quad (I.15)$$

which is the Navier-Stokes equation in Fourier space.

We can simplify this equation somewhat and write it in a more symmetric form which we will use throughout this Thesis by considering the incompressibility condition in Fourier space where we can write it as

$$k_\alpha u_\alpha(k) = 0. \quad (I.16)$$
Taking the scalar product of equation I.15 with $k$ we find

$$P(k, t) = -i \frac{k_\alpha k_\beta}{k^2} \int dp \, u_\beta(p) u_\alpha(k - p). \tag{I.17}$$

Now, we can substitute this expression for the pressure and after some rearrangement obtain

$$\left( \frac{\partial}{\partial t} + \nu k^2 \right) u_\alpha(k) = -i k_\beta \left( \delta_{\alpha \gamma} - \frac{k_\alpha k_\gamma}{k^2} \right) \int dp \, u_\beta(p) u_\gamma(k - p),$$

$$= -i k_\beta P_{\alpha \gamma}(k) \int dp \, u_\beta(p) u_\gamma(k - p). \tag{I.18}$$

In the second line of the above equation we have introduced the projection operation $P_{\alpha \beta}(k)$ which acts, as the name suggests, to project the velocity field onto a divergence free space, and thus enforces the incompressibility condition. We can go one step further and exploit the symmetry of the right hand side of the above equation to symmetrise with respect to $\beta$ and $\gamma$, giving us

$$\left( \frac{\partial}{\partial t} + \nu k^2 \right) u_\alpha(k) = -i \frac{1}{2} k_\beta P_{\alpha \gamma}(k) \int dp \, u_\beta(p) u_\gamma(k - p) - i \frac{1}{2} k_\gamma P_{\alpha \beta}(k) \int dp \, u_\beta(p) u_\alpha(k - p),$$

$$= \frac{1}{2i} P_{\alpha \beta \gamma}(k) \int dp \, u_\beta(p) u_\gamma(k - p). \tag{I.19}$$

Here we have defined the inertial transfer operator $P_{\alpha \beta \gamma}(k) = k_\beta P_{\alpha \gamma}(k) + k_\gamma P_{\alpha \beta}(k)$. This symmetrised form of the Navier-Stokes equation in Fourier space will serve as the starting point for the analytical turbulence approximation derived in Chapter II.

### I.1.2 The Reynolds Number

Returning for now to real space, we will consider the physical meaning of the remaining terms in the Navier-Stokes equation, having already considered the pressure field. First, we consider the non-linear term $u \cdot \nabla u$, where for brevity we have dropped the explicit time dependence, which is responsible for mixing and energy transfer across different length scales in the flow. This can be seen clearly in the Fourier space expression we have just derived, in which the non-linear term becomes a convolution and can be seen to couple together different wave-numbers, and thus length-scales, in the flow through what are known as triadic interactions.

Secondly, we have the dissipative term $\nu \nabla^2 u$ which is responsible for the conversion of kinetic energy into heat in the flow through friction between fluid layers as a result of the viscosity. If we look at the Fourier space representation of this term we find that it is...
proportional to $k^2$ and thus acts most strongly on large $k$, which corresponds to the smallest scales of the flow. Turbulence, at least in three dimensions, is known to be highly dissipative, hence this form, alongside the role of the non-linear term in transferring energy across length scales, suggests a symmetry breaking transfer of energy from large to small scales in turbulence. We will return to this idea later in this Chapter.

Depending on which of these two terms is dominant we expect different flow behaviour to be observed. Indeed, in general turbulent flows are characterised by a dimensionless parameter known as the Reynolds number, defined as the ratio of these terms. That is,

$$\text{Re} = \frac{|\mathbf{u} \cdot \nabla \mathbf{u}|}{\nu|\nabla^2 \mathbf{u}|} \sim \frac{UL}{\nu},$$

in which $U$ and $L$ are characteristic velocity and length scales, respectively. These characteristic scales are defined depending on the flow under consideration. For example, in pipe flow $U$ would be the mean velocity and $L$ the pipe diameter. The choice of such characteristic scales for HIT is more difficult and we will defer their introduction until later in this Chapter.

The Reynolds number then measures the importance of inertial and viscous effects. The dynamics of large Reynolds number flows are dominated by the inertial transfer of energy. Whilst for low Reynolds numbers the flow dynamics are dominated by the dissipation of energy. Osbourne Reynolds first established the importance of the parameter which came to be known as the Reynolds number through his experiments on pipe flow [186]. It was observed that as the Reynolds number was increased by increasing the flow velocity, the flow transitioned from what is known as laminar flow to fully developed turbulence. Laminar flow is characterised by a smooth velocity field with the fluid flowing in layers with little to no mixing, in stark contrast to the turbulent case. In between these two states there is a complicated transition to turbulence. In this Thesis we focus solely on the case of fully developed turbulence.

I.1.3 Homogeneity and Isotropy

We stress here that when we speak of HIT we are referring to a statistical state. That is, if we were to look at the velocity field at any given instant, it would not be homogeneous or isotropic. Indeed, if it was then there could be no structure within the flow and thus HIT would not be a particularly interesting problem to study. Furthermore, when we look at the instantaneous velocity field, it's properties undergo large fluctuations in space and time and appear almost random.
Consequently, we require some method by which to compute average values of our flow in order to quantify its properties. If we consider turbulence not sustained by external forces which will decay in time, then typically this averaging process would be an average over an ensemble of flows starting from slightly different but comparable initial conditions. For example, a collection of flows all starting with the same initial energy but with different spatial velocity fields. If, instead, we consider a flow which is sustained through some external energy injection, known as stationary turbulence, then a suitable average may involve multiple snapshots in time.

In each of the above situations, when we make measurements of the velocity field the values we measure and the frequency at which we measure them are determined by the probability density function (PDF) of the velocity field. Full knowledge of the PDF for arbitrary flow states would constitute a solution to the turbulence problem. Of course, at least currently, the PDF is not known and this Thesis is not directly concerned with attempts at its determination. For our purposes, the PDF allows us to give a mathematical definition of homogeneity, isotropy as well as stationarity.

**Homogeneity**  
We say the velocity field is homogeneous if its PDF is invariant under spatial translations.

**Isotropy**  
A velocity field is said to be isotropic if its PDF is invariant to spatial rotations.

**Stationarity**  
A stationary velocity field exhibits time translational symmetry of the underlying PDF.

Thus, we see that each of these cases introduces a certain symmetry to the velocity field PDF. These symmetries impose restrictions on the flow. Homogeneous flows cannot have boundaries, indeed, for viscous flows the boundary layers formed would have different statistical properties when compared to the bulk flow, thus breaking homogeneity. Furthermore, isotropic flows cannot contain a mean flow direction. It then appears that restricting ourselves to HIT leaves us with little relevance to real world turbulence. However, far from boundaries and in suitably defined coordinates, moving with any mean flow, HIT is seen as a good approximation in numerous real world flows, including atmospheric and oceanic turbulence. In Section 1.2 we will return to this point and discuss the concepts of local homogeneity and isotropy.
### 1.2 Turbulence as a Function of Spatial Dimension

At the very beginning of this Chapter we alluded to the changing nature of turbulence in different spatial dimensions. Having introduced the Navier-Stokes equations in a general sense, we are now ready to discuss the phenomenological changes seen when the spatial dimension, \( d \), is varied. We begin with the real world case of three dimensional turbulence. Although as we will see throughout this Chapter, and indeed this Thesis as a whole, the two dimensionalization of real world flows is a common and important occurrence. Additionally, we also consider the non-physical cases of \( d > 3 \) and non-integer dimensions as a different method of approaching the real world turbulence problem.

#### 1.2.1 The Energy and Transfer Spectra

Before focusing on specific dimensions, we will first introduce one of the most studied objects in fluid turbulence, the energy spectrum, \( E(k) \). This gives a measure of the distribution of energy across the numerous length scales in the flow. The energy spectrum satisfies the following equation, which we will derive fully in Chapter II, known as the Lin equation

\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) E(k, t) = T(k, t). \tag{I.21}
\]

On the right hand side of the above equation we have the transfer spectrum, \( T(k) \). This term is responsible for the transfer of energy between different length scales in the flow. Note, it only moves energy around the system and does not create or remove energy, therefore

\[
\int_{0}^{\infty} T(k, t) = 0. \tag{I.22}
\]

Now since we are working with isotropic turbulence, these spectra depend only on the magnitude of the wave-vector \( k \). Additionally, both spectra are defined in terms of averages of the velocity field and hence are statistical in nature.

Hidden in the simple looking equation (I.21) is the turbulent closure problem. This is the fact that \( E(k, t) \) is defined in terms of a second order moment of the velocity field, whilst \( T(k, t) \) is a third order quantity. As such, there is no analytical method which will allow us to determine \( E(k, t) \) for a general flow. We return to the closure problem and its implications in more detail in Chapter II.

Using the conservative property of \( T(k, t) \) expressed above, we can write a useful expression for the decay of energy in a turbulent flow. First we note that the total energy, \( E(t) \), is given
by
\[ E(t) = \frac{1}{2} \langle u_a u_a \rangle = \int_0^\infty dk E(k, t), \quad \text{(I.23)} \]
then by integrating the Lin equation over all \( k \) we find
\[ \frac{\partial}{\partial t} E(t) = -2\nu \int_0^\infty dk k^2 E(k, t) = -\varepsilon, \quad \text{(I.24)} \]
where \( \varepsilon \) is the energy dissipation rate. A further result of the transfer spectrum’s energy conservation is that, in the absence of viscosity, the total energy is conserved. This inviscid situation is known as ideal flow and as such the total energy is an ideal invariant for three dimensional flows.

I.2.2 Three Dimensional Turbulence

The study of turbulence in three dimensions has been dominated by the work of Kolmogorov and his 1941 theory (K41) [117, 118, 120]. However, the inspiration for this theory comes from the work of Richardson who described a process by which structures in the flow known as eddies progressively breakdown into smaller and smaller structures until their energy is dissipated. This process describes a situation where energy is transferred from large to small scales in a three dimensional flow and is referred to as a (direct) energy cascade.

If we consider the Lin equation we can see that the transfer spectrum must be responsible for this cascade, as it expresses the transfer of energy into modes with magnitude \( k \) from all other modes. Due to the conservation properties of \( T(k) \), a system without viscosity would tend towards an equilibrium state with equipartition of energy amongst the modes. However, when viscosity is present that \( k^2 \) factor is symmetry breaking, in the sense that high wave-numbers have their energy dissipated faster than low wave-numbers. This then leads to the flow of energy from large to small scales and the cascade. An important consequence of this cascade is the generation of smaller and smaller scales as the Reynolds number of a flow increases. We can see this by noticing that the Reynolds number is proportional to the inverse of the viscosity. Hence, a larger Reynolds number can be thought of as a smaller viscosity. Then, as the viscosity is lowered, larger values of \( k \) are required to obtain the same amount of dissipation.

Using the transfer spectrum, we can obtain an expression for the flux of energy through wave-number \( k \). We denote this flux by \( \Pi(k) \) and it is defined as
\[ \Pi(k) = \int_k^\infty dp \ T(p), \quad \text{(I.25)} \]
which expresses the flow of energy through wave-number $k$ from wave-numbers less than $k$ to those greater than $k$.

### I.2.2.1 Kolmogorov’s 1941 Theory

Kolmogorov further developed the concept of the energy cascade in what has become known as the K41 theory, which is arguably the most important, and certainly the most influential, piece of work relating to HIT. In this theory Kolmogorov introduces two similarity hypotheses for statistically stationary turbulence that can be considered locally isotropic and is of sufficiently high Reynolds number:

1. The velocity field statistics depend solely on the viscosity, $\nu$, and the mean dissipation rate, $\langle \varepsilon \rangle$.

2. There exists a range of intermediate sized length scales in the flow where the velocity field statistics depend solely on $\varepsilon$.

These are two very simple ideas, but the consequences of these hypotheses are numerous and far reaching.

First, we will give a definition to locally isotropic turbulence. In Kolmogorov’s original presentation of his theory he worked solely in real space considering velocity differences at varying separations, rather than the velocity itself. A flow is then locally isotropic over a region of radius $r$ if the velocity differences are isotropic up to this separation. This is a less restrictive condition than full isotropy of the velocity field and opens up the possibility of regions of locally isotropic flow existing at smaller scales even when the large scale flow which supplies energy is anisotropic.

The first of these hypothesis posits that the small scales of high Reynolds number flows are independent of the large scales that generated them. That is, the cascade process prevents the large scales of the flow from directly influencing the small scales and these small scales are universal in nature. The second hypothesis suggests a subrange of scales within the universal range that have statistics independent of viscosity. By dimensional analysis we can construct a length scale from $\nu$ and $\langle \varepsilon \rangle$ that gives us a lower bound on the subrange described in the second hypothesis. In doing so, we obtain the Kolmogorov microscale length

$$\eta = \left( \frac{\nu^3}{\langle \varepsilon \rangle} \right)^{\frac{1}{4}}. \quad (I.26)$$

Using the relationship $\langle \varepsilon \rangle \sim U^3/L$ where $U$ and $L$ are once more characteristic velocity and
length scales, we find that $\eta$ corresponds to

$$Re = \frac{U \eta}{\nu} = 1,$$  \hspace{1cm} (I.27)

that is the length scale beyond which viscosity begins to become dominant. Thus for scales larger than this but still small compared with the large scales $L$, the inertial transfer of energy is dominant. We refer to this as the inertial subrange, often shortened to the inertial range. In the same manner we can also define a Kolmogorov microscale time, $\tau$ associated with this length as

$$\tau = \left(\frac{\nu}{\varepsilon}\right)^{\frac{1}{2}}.$$  \hspace{1cm} (I.28)

**Structure Functions**  Due to working with velocity differences in real space, Kolmogorov presented his K41 in terms of what are known as longitudinal structure functions. These are the moments of the velocity differences defined as

$$S_n(r) = \left\langle \delta u^n_r \right\rangle,$$  \hspace{1cm} (I.29)

where

$$\delta u_r = (u(x + r) - u(x)) \cdot \frac{r}{|r|}.$$  \hspace{1cm} (I.30)

We will go into more detail regarding these functions later in this Chapter as well as in Chapter [IV]. However, for our current discussion we only need this simple definition.

Now, again using dimensional analysis, Kolmogorov conjectured that the form of the second order longitudinal structure function $S_2(r)$ in the universal region should be

$$S_2(r) = \sqrt{\nu \left\langle \varepsilon \right\rangle} f(r/\eta),$$  \hspace{1cm} (I.31)

where $f$ is an undetermined universal function. It is possible to go further than this if we consider the inertial range only. Here, by dimensional analysis, coupled with an assumption that the velocity gradient skewness is constant, Kolmogorov derived the form

$$S_2(r) = A\left\langle \varepsilon \right\rangle^{-\frac{2}{3}} r^2 \eta \ll r \ll L,$$  \hspace{1cm} (I.32)

where $A$ is a universal constant.

Kolmogorov additionally made a prediction concerning the third order structure function. To do so he demonstrated that the second and third order structure functions satisfied the Karman-Howarth equation [61], which is the real space analogue of the Lin equation. Then considering the infinite Reynolds number limit, which corresponds to the limit $\nu \to 0$, he
Figure I.1  Illustration of the Kolmogorov energy spectrum in three dimensional turbulence. Dashed line shows a $k^{-5/3}$ power-law.

derived

$$S_3(r) = -\frac{4}{5} \langle \varepsilon \rangle r, \quad (I.33)$$

which is commonly known as the four-fifth law and is often referred to as one of the only exact results in turbulence.

**Spectral Form** It has become popular to consider the Kolmogorov theory in spectral space. Here the concept of the energy cascade is far clearer. The Fourier transform of Kolmogorov’s prediction for the second order structure function yields the following expression for the energy spectrum in the inertial range

$$E(k) = C_3 \langle \varepsilon \rangle \frac{2}{5} k^{-\frac{5}{3}} \frac{1}{L} \ll k \ll \frac{1}{\eta}, \quad (I.34)$$

where $C_3$ is a universal constant. This prediction has been extensively evaluated in both experimental and numerical studies. Although good agreement has been found, this theory is not confirmed. Furthermore, Kolmogorov’s hypotheses can be used to make predictions
about higher order statistical quantities, however, these predictions have been found to be far less accurate when compared with experiment. This has come to be known as anomalous scaling; we will return to this topic later in this Chapter in Section 1.3. We show an illustration of the Kolmogorov spectrum in Figure 1.1; the bump at low $k$ corresponds to a region of energy injection.

The use of the mean energy dissipation in both the real and spectral forms of the Kolmogorov inertial range laws is somewhat paradoxical. Indeed, the inertial range is characterised as a range of scales where inertial energy transfer, without the influence of dissipation, dominates. To understand its appearance, we note that, as we are considering stationary turbulence, the mean dissipation rate must be numerically equal to the mean energy injection rate. If the energy injection happens at the large scales and the dissipation at the small scales, then the amount of energy flowing between these scales due to the cascade must also be numerically equal to this. Hence, the flux in the inertial range must be numerically equal to the mean energy dissipation rate. That is, $\langle \varepsilon \rangle = \langle \Pi(k) \rangle = \langle \Pi \rangle$ for $k$ in the inertial range. It would perhaps be more appropriate to use $\langle \Pi \rangle$ in the Kolmogorov inertial range predictions as it is a true inertial range quantity. This allows us to define the inertial range as the region of the flow with a constant mean flux of energy in spectral space.

### I.2.2.2 The Role of Vorticity

To close our discussion on the dynamics of three dimensional turbulence, we now consider the vorticity of the flow. Defined as the curl of the velocity field, the vorticity describes rotational motion in a fluid. We denote the vorticity by

$$\mathbf{\omega}(\mathbf{x}, t) = \nabla \times \mathbf{u}(\mathbf{x}, t) \quad (I.35)$$

By taking the curl of the Navier-Stokes equation we can derive an expression for the vorticity

$$\frac{\partial \mathbf{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{\omega} = \mathbf{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{\omega}. \quad (I.36)$$

For our purposes, the most important term is, $\mathbf{\omega} \cdot \nabla \mathbf{u}$, which is known as the vortex stretching term.

In three dimensional turbulence, the vortex stretching term leads to the production of the vorticity analogue to energy, the enstrophy, which is defined

$$Z(t) = \frac{1}{2} \langle \omega_a \omega_a \rangle = \int_0^\infty dk \, Z(k) = \int_0^\infty dk \, k^2 E(k), \quad (I.37)$$
with $Z(k)$ being the enstrophy spectrum. As a result of this production term, the enstrophy is not an ideal invariant for three-dimensional flows. To understand how this term produces enstrophy, we observe that $\omega \cdot \nabla u$ is only positive when the vorticity and velocity gradient are parallel. As such, the tubes of vorticity found in three-dimensional turbulence are stretched along their axis. Then due to the conservation of angular momentum the vorticity must increase during this process.

An important consequence of vortex stretching and the production of enstrophy can be seen if we look at the energy dissipation as defined in equation (I.24). Looking at the definition of the enstrophy spectrum above we can re-express the energy dissipation equation as

$$\varepsilon = 2\nu \int_0^\infty dk Z(k) = 2\nu Z(t).$$

Hence we see that the energy dissipation rate is proportional to the enstrophy. As $\nu \to 0$, corresponding to the infinite Reynolds number limit, the vortex stretching term is able to stretch vortex tubes to progressively smaller radii, leading to a greater production of enstrophy. If we hold the energy injection constant then the flux through the inertial range must also remain constant, and therefore the same is true of the energy dissipation as we take this limit. That is, even in the limit of zero viscosity, turbulent flows continue to dissipate energy due to a balancing act between dissipation and increased enstrophy production.

It is often suggested that vortex stretching is the physical mechanism behind the energy cascade. This feels intuitive, since as the vortex tubes are stretched, the transfer energy to smaller and smaller scales until small enough for viscosity to act. We will not dwell on this point, however we will say the true picture of the cascade appears to be more complicated than the simple picture presented by vortex stretching. We return to this point in Chapter IV.

### 1.2.3 Two Dimensional Turbulence

The case of two-dimensional turbulence is not a simplified version of the three-dimensional case we have just described, in fact that could not be further from the truth. Two-dimensional turbulence displays an extremely rich array of phenomena not seen in the three-dimensional case. We will not be able to do justice to the field of two-dimensional turbulence in the short introduction, instead we refer interested readers to the textbooks by Davidson [59] and Lesieur [134].

The study of two-dimensional turbulence appears at first glance to be a purely academic
exercise, given we live in a three dimensional world. However, there are a number of
situations in real world turbulent flows where two-dimensional behaviour is observed. For
example, in the Earth’s atmosphere the vertical direction is severely constrained compared
to the horizontal directions. This then leads to two-dimensionalization of the flow above a
certain length scale.

I.2.3.1 Vorticity Dynamics in Two Dimensions

We begin our discussion where we ended things for the three-dimensional case, the
vorticity. Arguably, in two-dimensions the vorticity plays an even more important role and
it is responsible for many of the uniquely two-dimensional turbulent phenomena. Now,
considering the definition of the vorticity in terms of the curl of the velocity field, which
is only defined in three dimensions, it is clear vorticity in two-dimensions needs careful
definition. If our flow is two-dimensional with only $x$ and $y$ components, then using the
definition of curl, the vorticity will point in the non-existent $z$ direction. However, flows in
two dimensions can still show swirling rotational behaviour, however this rotation is always
in the plane. As such, we find the vorticity is a scalar quantity in two dimensions, defined as

$$\omega = \partial_x u_y - \partial_y u_x. \quad (I.39)$$

Using the above definition of two-dimensional vorticity, we form an evolution equation
from the Navier-Stokes equation

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = +\nu \nabla^2 \omega. \quad (I.40)$$

Most notable in this equation is the absence of a vortex stretching term. This is a result of
the vorticity being effectively perpendicular to the velocity gradients in two dimensions.
As a consequence of there being no vortex stretching, the enstrophy in two-dimensions
provides an additional ideal invariant. Moreover, the vorticity itself in conserved along fluid
trajectories.

This is a drastically different situation when compared to the three-dimensional case. Our
concern now is how this affects the energy transfer processes in two dimensions and what
happens to the energy cascade. To begin answering this question we look at the energy
dissipation rate given by

$$\varepsilon = 2\nu Z(t), \quad (I.41)$$

now since in two dimensions the enstrophy is conserved in the zero viscosity, infinite
Reynolds number limit, there is no dissipation of energy. This then suggests that there is no longer a cascade of energy from large to small scales in two dimensions.

A similar equation for the dissipation of enstrophy, \( \eta \), can be derived

\[
\eta = 2\nu \int_0^\infty dk k^2 Z(k) = 2\nu \int_0^\infty dk k^4 E(k) = 2\nu P(t),
\]  

(I.42)

where \( P(t) \) is the palinstrophy. The palinstrophy is not conserved in two dimensions, hence it is possible there is finite enstrophy dissipation in the limit of zero viscosity, however, the balancing of palinstrophy production and dissipation is not guaranteed to behave in the same way as enstrophy production and dissipation in three dimensions. In any case, this suggests that there may be a flow of enstrophy from large to small scales.

### I.2.3.2 Kraichnan’s Contribution

The cascading behaviour in two-dimensional turbulence was studied in detail by Kraichnan [123] who developed a theory analogous to the K41 theory. We will not present the full details of Kraichnan’s work on this subject here but merely provide an overview.

By considering the dual conservation of energy and enstrophy in the flow, Kraichnan deduced that energy must be transferred from small to large scales, whilst enstrophy is transferred from large to small scales. Then, making the same assumptions as Kolmogorov about local isotropy and the existence of inertial subranges for each quantity, leads to a dual cascade scenario. In the direct cascade of enstrophy the assumption is that the energy spectrum should only depend on \( k \) and the enstrophy dissipation.

Hence, by dimensional analysis we have

\[
E(k) = C'_2 \left( \frac{\eta}{\nu} \right)^{\frac{2}{3}} k^{-3}
\]  

(I.43)

in the enstrophy inertial range. As we did for the three dimensional case, we can also use dimensional analysis to obtain the lower bound length scale for this inertial range which we denote as \( \chi \). Just as in three dimensions the full universal range is dependant only on \( \eta \) and \( \nu \) thus we have

\[
\chi = \left( \frac{\nu^{\frac{3}{2}}}{\eta} \right)^\frac{1}{3}
\]  

(I.44)

We can also define a timescale, \( \tau \), in the same manner

\[
\tau = \frac{1}{\eta^{\frac{1}{2}}}
\]  

(I.45)
Following the same procedure and assumptions leads to the energy spectrum taking the form

$$E(k) = C_2 \langle \varepsilon \rangle^{\frac{3}{2}} k^{-\frac{5}{3}} ,$$

in the energy inertial range. We note that this is the same form as for the three-dimensional case, however, the cascade is in the opposite direction. Due to the conservation of enstrophy, the inverse cascade must be the result of vortex mergers in real space. That is, regions of vorticity combining, thus producing larger vortices and passing energy to larger scales in the process.

The dynamical picture in two dimensions is then as follows. If we inject energy, and thus enstrophy, at some length scale $l_f$, then the enstrophy will flow to smaller scales and be removed by viscosity. However, the energy will flow upscale to larger and larger scales until it reaches the scales on the size of the system. There is no mechanism for efficient removal of energy at the large scales. Hence, in a truly two dimensional system, if there is a constant energy injection, the attempts of the system to reach a stationary state will lead to
the formation of a condensate at the scale of the system. For quasi two-dimensional systems in the real world, which are embedded in three dimensional space, friction with the top and bottom of the domain will act to remove large scale energy. We show an illustration of the dual cascade spectrum of two-dimensional turbulence in Figure 1.2, here we can see that either side of a small spike due to energy injection we have two distinct cascade regions.

1.2.4 \( d \)-dimensional Turbulence

We now consider the dynamical behaviour of turbulence when the spatial dimension is allowed to vary freely, which we refer to as \( d \)-dimensional turbulence. Far less is known about \( d \)-dimensional turbulence, indeed this Thesis is largely devoted to its study. As such, we choose here to simply mention some of the historical work and motivations for the study of \( d \)-dimensional turbulence here, before providing far greater detail in Chapters IV and V.

1.2.4.1 Crossover dimensions

In a series of works in the early 1970’s by Nelkin [161, 162], an analogy between fluctuations in turbulence at high Re and critical-point fluctuations in critical phenomena was made. These gave a prediction of the scaling exponents of turbulence and studied the existence of a crossover dimension, below which the Kolmogorov scaling relations in K41 become exact [120], in analogy with mean field theory becoming exact above the crossover dimension \( d=4 \) for the Ising model [225]. The existence of such crossover dimensions was criticised later in [76]. Based on Nelkin’s ideas, Kraichnan looked at the statistical properties of a passive scalar field advected by an incompressible \( d \)-dimensional turbulent flow, finding a strong dependence on dimensionality and a suppression of temporal fluctuations in the stretching rates at \( d \to \infty \) [121]. Subsequent numerical work in this direction was done for incompressible flows in [81, 144] and further generalizations to compressible flows in [46, 82]. The idea of crossover dimensions in fully developed turbulence also led to the use of dynamic renormalization group methods [62, 70, 71].

The existence of a critical dimension between two and three dimensions was further investigated in [72, 76] by considering non-integer dimensions, such that the conservation laws are weakly broken. In these works, an EDQNM approximation is used to show that for \( d \geq 2 \) the direction of the cascade reverses. A different approach is to modify the aspect ratio of the lattice from cubic \( d=3 \) to \( d=2 \) observing the same cascade reverse [15]. The second of these is a more realistic representation of atmospheric turbulence, however, it is less suited for analytic investigation.
1.2.4.2 The large $d$ limit

These dramatic differences in dynamical behaviour between two- and three-dimensional turbulence led to further investigations into higher dimensions. These studies were typically motivated by the anomalous scaling observed in three-dimensional experimental studies, with the hope that the problem may simplify in the limit of large dimension. To this end, Fournier et al. [73] considered the infinite dimensional limit. No obvious simplifications were found; however, they were unable to rule out the vanishing of intermittency in infinite dimensions as was found by Kraichnan for a passive scalar advected by a random velocity field. The differences between turbulence in three dimensions and in higher dimensions is far more subtle than between two and three dimensions. This is likely related to a lack of additional positive-definite ideal invariants for higher dimensions.

Additionally, Fournier et al. [73] considered the role of pressure, and thus the incompressibility condition, in infinite dimensions. It had been suggested that the incompressibility condition should weaken as the dimension increases, leading to the possibility of turbulence tending towards Burgers equation statistics in this limit. We can see this more explicitly by writing the incompressibility condition in $d$-dimensions as

$$\partial_1 u_1 + \partial_2 u_2 + \cdots + \partial_d u_d = 0. \quad (I.47)$$

Now it is clear that as $d$ becomes large it is possible for large fluctuations in individual longitudinal velocity gradients to occur meaning the incompressibility condition becomes less restrictive. This suggests the possibility of the flow exhibiting shock like behaviour in the high $d$ limit. If this is the case then we might expect to observe Burgers scaling, i.e. $E(k) \sim k^{-2}$, in the inertial range.

However, in [73] the pressure was found to continue to play an important role even in infinite dimensions. This result has been questioned by Falkovich et al. [69], who found that in the infinite dimensional limit incompressible turbulence may still have Burgers scaling, with the discrepancy in findings attributed to the role of Gaussian initial conditions in [73]. Finally, recent work by Rozali studied relativistic turbulence in a large number of spatial dimensions, where it is found that equations of motions are simplified in the limit $d \to \infty$ [189].

As a last point on the role of pressure in the large $d$ limit we reconsider equation [1.6]. We can write this equation as

$$\partial^2 P = (\partial_j u_i)(\partial_i u_j), \quad (I.48)$$

where we see that the right hand side is a sum over $d^2$ terms. Therefore as $d \to \infty$ it should
be expected that the pressure has less and less influence on individual velocity components, even if this influence never completely vanishes. This may have implications for the ability of the pressure field to restore isotropy to a flow in high dimension.

1.3 Anomalous scaling

1.3.1 Intermittency

The K41 theory of turbulence is appealing in its simplicity, however, its validity remains an open, and somewhat controversial, question. In 1962, motivated by a criticism from Landau [129], Kolmogorov updated his theory to account for what has become known as internal intermittency in his K62 theory [119]. It can be observed that as the flow Re increases, the bulk of energy dissipation becomes concentrated into a smaller and smaller fraction of the overall flow volume. This led Kolmogorov to replace \( \langle \varepsilon \rangle \) in equation I.34 with a local average

\[
\varepsilon_l = \int_{|r|<l} \varepsilon(x+r)\,dx.
\]  

(I.49)

Then through the assumption that \( \varepsilon_l \) is log-normally distributed, the inertial range energy spectrum should be amended to

\[
E(k) = C \langle \varepsilon \rangle^{\frac{5}{3}} k^{-\frac{5}{3}} (kL)^{-\mu},
\]  

(I.50)

in which \( L \) is the energy injection length-scale and \( \mu \) is the anomalous exponent. Experimental measurement of the energy spectrum [88] from around the time of the publication of the K62 theory was unable to distinguish between the K41 and K62 forms, suggesting that the value of \( \mu \) must be small.

Whilst for the energy spectrum things looked unclear, measurements of real space quantities presented what appeared to be a less ambiguous picture. So far, we have considered the spectral space representations of K41 and K62, however, Kolmogorov worked in real space, considering the statistical properties of velocity differences across a distance \( r \) denoted by \( \delta v_r \). The moments of these differences are known as structure functions, and we can express their form in the inertial range as

\[
S_n(r) = \langle \delta v_r^n \rangle = A_n \langle \varepsilon \rangle^{\frac{n}{2}} \left( \frac{L}{r} \right)^{\frac{\mu(n-3)}{2}} \sim r^{\xi_n},
\]  

(I.51)

where for K41 \( \mu = 0 \) and in K62 \( \mu \neq 0 \). Note, that the energy spectrum results presented above can be obtained via Fourier transform of the second order structure function. Initial
experimental measurements of these structure functions, for example in [216], found that
at fourth order the exponent $\zeta_4$ did not agree with the K41 prediction and was in better
agreement with the K62 value. However, things were further complicated by the more
detailed experimental study of Anselmet et al. [4] showing that at higher orders of the
structure functions neither K41 nor K62 values were seen for the exponent. Attempts to
reconcile theory with the experimental results led to the development of numerous fractal
models, including the influential multi-fractal model [75]. We will not attempt to give
a comprehensive overview of the developments in the study of internal intermittency in
turbulence here. For a more detailed overview of developments in intermittency models
of turbulence and evidence for the existence of corrections to K41 due to intermittency see
[74, 206].

### I.3.2 Finite Reynolds number effects

In the development of K62 and the intermittency models discussed above, there is an ex-
plicit assumption that anomalous scaling in turbulence is a result of internal intermittency,
and thus as K41 does not account for this, it must be incorrect. However, this assumption
should be critically appraised. As mentioned in the introduction, even though for stationary
turbulence there is the kinematic relationship $\langle \Pi \rangle = \langle \varepsilon \rangle$ the two processes are different and
as such fluctuations in one quantity do not imply the same fluctuations in the other. Indeed,
Kraichnan [125, 126] highlighted that K41 cannot be ruled out solely due to spatial variation
in the energy dissipation rate. Kraichnan additionally raises the point that even for $l$ in the
inertial range, $\bar{\varepsilon}_l$ is not an inertial range quantity and that instead the spatial fluctuations
of $\Pi(x)$ in the inertial range should be considered. Then, Kraichnan further observed that if
there is sufficient mixing of energy in space to prevent the fluctuations in $\Pi(x)$ growing as
the cascade proceeds to smaller scales then it is possible, although not guaranteed, that K41
is correct.

Supposing for now that K41 is indeed correct, how can the experimental measurements
of anomalous exponents when considering higher order structure functions be explained?
The main argument used in answering this question is centred around the stipulation in
the K41 theory that the Reynolds number of the flow should be sufficiently large for the
theory to hold. The claim is then that deviations from the K41 scaling exponents are a
result of finite Reynolds number rather than intermittency. The importance of this finite
Reynolds effect was considered and quantified by Qian [180, 181, 182, 183]. These studies
suggest a Reynolds number limit, below which a true inertial range in the spirit of K41
cannot exist, whilst also claiming the influence of a finite Reynolds number on the small-
scale statistics of the flow decays slowly with increasing Re. This finite Reynolds number effect was also explored by Lundgren \[141\], who through matched asymptotic expansions found corrections to K41 that decayed with Re, suggesting it is correct in the limit of infinite Reynolds number. In recent years, a substantial body of work in this area has formed, both experimental and numerical \[6,7,63,146,150,208,210\]. In these works, it is observed that as the Reynolds number is increased the small-scale statistics appear to tend towards the expected Kolmogorov values. These works also stress that care must be taken when using data obtained from experimental flows, which are typically anisotropic at the largest scales. The stipulation that the flow is locally isotropic is central to K41, as such, it must be ensured that large scale anisotropy does not influence the small-scale statistics. Indeed, this may have been the case in atmospheric boundary layer measurements as performed in \[216\]. Furthermore, it is shown that the higher order statistics require higher values of the Reynolds number before the finite Reynolds number effect becomes negligible, casting doubt on the exponents measured for high order structure functions. For a more comprehensive overview of the finite Reynolds number effect see \[149\].

I.3.3 A critical dimension for turbulence

While corrections to K41 due to either intermittency or finite Reynolds numbers have received substantial and sustained interest, we consider now a third possibility that has been explored periodically over the last half century. The K41 inertial range energy spectrum exhibits scale invariance which has drawn many to make comparisons between fluid turbulence and critical phenomena \[1,36,60,85,161,162,188,221,230\]. These analogies are interesting when considering the concept of a critical spatial dimension found in critical phenomena \[84,225\]. Above this dimension the predictions for the critical exponents of the system are given by mean field theory. With its use of the mean energy transfer rate, it has been suggested that K41 is in fact a mean field theory, exact only above a critical dimension \[14,200\].

I.4 Chaos Theory Approach to Turbulence

Turbulent fluid flows exhibit complex and, at first glance, apparently random motions. Consequently, our ability to predict their behaviour is limited. Given that such fluids are governed by deterministic equations of motion, for example the Navier-Stokes equations introduced earlier in this Chapter for non-conducting fluids, their lack of exact predictability appears paradoxical. This aspect of turbulent flows can be understood as a consequence
of deterministic chaos and an extreme sensitivity to initial conditions. As a result, any error in measuring the state of the system, no matter how small, will be amplified as the system evolves, resulting in a finite predictability time. Turbulent flows are ubiquitous in the universe and as such, quantifying their predictability may have wide reaching applications. Furthermore, fluid turbulence is in many ways representative of extended dynamical systems in general, and therefore such results may also be of more broad interest.

In order to quantify the predictability of turbulent flows, we consider an approach based on the mathematical field of Chaos Theory. In particular, as mentioned above, what we are concerned with is deterministic chaos. This approach differs from the more traditional statistical one used earlier in this Chapter when discussing the phenomenology of turbulence in different dimensions. Instead of considering an ensemble average we are now interested in the individual trajectories of the system within a suitably defined state space. In this state space each possible configuration of the system exists as a single point. As we treat our fluid as a continuum, this space will be effectively infinite dimensional. As the system evolves in time it will trace out a trajectory in this infinite dimensional space. By studying these trajectories, it is then possible to begin to understand to what extent turbulent flows are predictable. Furthermore, as we will see throughout this Thesis, we can also relate such predictability results to the problem of scaling in turbulence. In the remainder of this Section we introduce the mathematical machinery needed throughout this Thesis and discuss the real world application of such ideas. We focus here on the introduction of these concepts, leaving their application to HIT until Chapter III.

I.4.1 Lyapunov Exponents

For our purposes we can define a dynamical system as an \( N \) dimensional system of first order differential equations specifying the time evolution, given by the trajectory \( x(t) \), of an initial condition \( x(0) \). For an explicit example consider

\[
\frac{dx(t)}{dt} = F[x(t)].
\]  

(I.52)

Note that in Fourier space the Navier-Stokes equations can be written in this manner.

We now consider an infinitesimally small perturbation to \( x(0) \) given by \( \delta(0) \) such that at \( t = 0 \)

\[
x'(0) = x(0) + \delta(0).
\]  

(I.53)
Then at all future times \( t \) we have

\[
\mathbf{x}'(t) = \mathbf{x}(t) + \mathbf{\delta}(t),
\]

(I.54)

where \( \mathbf{\delta}(t) \) gives the separation between the two trajectories. As long as \( \mathbf{\delta}(t) = \mathbf{x}'(t) - \mathbf{x}(t) \) is small it can be considered a tangent vector. We can then find the time evolution of \( \mathbf{\delta}(t) \) from the linearised equation

\[
\frac{d\mathbf{\delta}(t)}{dt} = \sum_{j=1}^{d} \frac{\partial F_j}{\partial x_j} \mathbf{x}(t) \mathbf{\delta}(t).
\]

(I.55)

Then making use of a theorem by Oseledec [173] there exists a basis \( \{ \mathbf{e}_i \} \) in this tangent space such that for large \( t \)

\[
\mathbf{\delta}(t) = \sum_{i=1}^{N} c_i e^{\lambda_i t} \mathbf{e}_i.
\]

(I.56)

The \( \{ \lambda_i \} \) are known as the Lyapunov exponents of the system and are ordered by convention as \( \lambda_1 > \lambda_2 > \cdots > \lambda_N \). Note, we will usually refer to this collection of exponents as the Lyapunov spectrum of the system. This then gives us a relation for the separation between trajectories \( \mathbf{\delta}(t) \)

\[
|\mathbf{\delta}(t)| \approx |\mathbf{\delta}(0)| e^{\lambda_1 t}.
\]

(I.57)

Therefore, if \( \lambda_1 \) is positive, we have an exponential divergence between \( \mathbf{x}(t) \) and \( \mathbf{x}'(t) \) and the system is said to exhibit deterministic chaos. This rapid divergence between two initially close states is known as an extreme dependence on initial conditions.

### I.4.2 Kolmogorov-Sinai Entropy

There exist deep mathematical links between chaos theory and information theory. Indeed, it is said that deterministically chaotic systems produce information. To understand this we introduce now the concept of an entropy for a dynamical system known as the Kolmogorov-Sinai entropy which is said to give the rate of information production in deterministically chaotic systems.

The Kolmogorov Sinai entropy is a generalisation of the concept of entropy to dynamical systems. To extend the concept we first perform a partitioning of our \( d \)-dimensional state space into \( n \) distinct regions \( P_i \) with \( i = 1, \ldots, n \) each of volume \( V_i \). We assign an invariant probability density function, \( \rho(\mathbf{x}) \), to our phase space so the probability \( p_i \) of being found in the \( i^{th} \) partition of phase space is

\[
p_i = \int_{P_i} \rho(\mathbf{x}) dV_i.
\]

(I.58)
We can then use these $p_i$ to calculate the Shannon entropy for the system. However, this is not yet the Kolmogorov-Sinai entropy, this quantity will be dependant on how we partition our state space and the Kolmogorov-Sinai entropy is an invariant of the system. Also we have made the claim the Kolmogorov-Sinai entropy gives the rate of information production, hence, it will need to be based on the time evolution of the system. Consider now $p_{i_1 i_2 ... i_n}$ as being the probability of being in partition $i_1$ at $t = \tau$, then $i_2$ at $t = \tau + \delta t$ and so on until at $t = \tau + n\delta t$ we are in partition $i_n$. We can then define the entropy $H_n$ as

$$H_n = - \sum_{i_1, ..., i_n} p_{i_1 i_2 ... i_n} \log_2 p_{i_1 i_2 ... i_n}. \quad (I.59)$$

The information required to predict the trajectory of the system at $t = \tau + (n + 1)\delta t$ will be given by $H_{n+1} - H_n$, provided we know the trajectory up to $t = \tau + n\delta t$. We can then define the Kologorov-Sinai entropy, $h_{KS}$

$$h_{KS} = \lim_{\delta t \to 0} \lim_{l \to 0} \lim_{N \to \infty} \frac{1}{N\tau} \sum_{n=0}^{N-1} (H_{n+1} - H_n). \quad (I.60)$$

The limit $l \to 0$ gives independence from how we partition phase space, making this an invariant quantity of the dynamical system. Hence, $h_{KS}$ quantifies this information production rate. If a finite positive value is measured for $h_{KS}$ the system is said to be deterministically chaotic. Purely stochastic systems have $h_{KS} = \infty$ whereas periodic and purely deterministic systems have $h_{KS} = 0$.

We can also make a connection with the Lyapunov spectrum defined in the previous Section and the Kolmogorov-Sinai entropy. Indeed it can be shown that

$$h_{KS} = \sum_{\lambda_j > 0} \lambda_j. \quad (I.61)$$

This definition then makes the connection between chaos and information production explicit. Indeed, if we have two different initial conditions that both lie within the same cell of our partitioned state space they are indistinguishable at the level of partitions. However, if the system is deterministically chaotic, then as the two trajectories evolve they will eventually be found in different cells due to the exponential divergence. As such, the information required to predict such trajectories is determined by the Lyapunov exponents.

### I.4.3 Strange Attractors

So far we have introduced quantifiers for the rate of divergence of trajectories in state space. However, such divergence cannot continue indefinitely and there are constraints placed
upon trajectories due to the equations of motion of the system. This leads to the concept of an attractor in state space, which we can loosely define as a finite subset of the state space where all possible evolutions of the equations of motion take place eventually. That is, for any initial condition, the trajectory through phase space will end up on the attractor given enough time. There are numerous types of attractor, perhaps easiest to understand is the so-called limit cycle attractor characterised by periodic trajectories. Consider for example a driven single pendulum with some form of frictional damping. At a steady state this pendulum will follow a periodic circular trajectory in state space.

Of particular interest for fluid turbulence, which, as it is governed by a partial differential equation will have an infinite dimensional state space, is the notion of an attractor dimension. Given that the attractor is a finite subset, this suggests that turbulent flow can be represented by a finite number of degrees of freedom, perhaps even a small enough number to allow for analytical approaches. However, as we will see in Chapter III the attractor for fluid turbulence, while finite, grows rapidly with the Reynolds number of the flow.

As we have already mentioned, initially close trajectories of systems which exhibit deterministic chaos will diverge exponentially in time. This behaviour leads to a particular kind of attractor in such systems known as a strange attractor. Such attractors are defined by their fractal structure, manifesting itself as a typically non-integer attractor dimension. All chaotic systems, i.e. those with at least one positive Lyapunov exponent, posses strange attractors in their state space, however, it is also possible for non-chaotic systems to exhibit strange attractors although their discussion is beyond the scope of this work. The key difference between these strange attractors and standard attractors is the non-periodic trajectories seen on a strange attractor. In particular, for both normal and strange attractors trajectories must remain on the attractor, however, trajectories on strange attractors follow complex paths due to the fractal structure of the attractor.

Much like with the Kolmogorov-Sinai entropy the attractor dimension can also be determined via the Lyapunov exponents. We can give an intuitive explanation for this as follows. If we consider the evolution of a turbulent flow without any driving forcing to add energy, then over time the flow will become less and less turbulent as all its energy is dissipated. In our state space this corresponds to a shrinking of volume elements, as all trajectories evolve towards the same point. In terms of Lyapunov exponents, as the system decays, more of the spectrum becomes negative and trajectories converge in more and more directions until all exponents become negative.

If, however, the system is driven to a stationary state, we will have a Lyapunov spectrum composed of both positive and negative exponents. The competition between expansion
and contraction in different state space directions will result in a certain volume in state space being preserved. This idea was exploited by Kaplan and Yorke [111] to estimate the dimension of the strange attractor, \( \dim(A) \), as

\[
\dim(A) = j + \frac{\sum_{i=1}^{j} \lambda_i}{|A_{j+1}|},
\]

(I.62)

where \( j \) is determined via

\[
\sum_{i=0}^{j} \lambda_i > 0 \quad \text{and} \quad \sum_{i=0}^{j+1} \lambda_i < 0.
\]

(I.63)

We will make use of this expression at many points in this Thesis to allow us to calculate the attractor dimension from our numerical work.

### I.4.4 Chaos and Real World Predictability

Given the sophisticated mathematical definitions of the Lyapunov exponent, Kolmorogov-Sinai entropy and attractor dimension, it is not immediately clear the relevance they have for real world systems and their predictability. Indeed, all three are related in some way to the behaviour of nearby trajectories in state space. However, when it comes to a system like the Earth's atmosphere, we only see a single realisation.

Continuing with example of the Earth's atmosphere, we can begin to understand the application of the machinery of chaos theory and develop a physical interpretation of the Kolmogorov-Sinai entropy. To do so, we consider how the state of the atmosphere can be measured in practice. Experimental measurements of quantities such as the temperature, humidity and wind velocity, amongst many others, are made at a finite number of weather monitoring stations located at points of interest or convenience. As a result, we will have a limited spatial resolution when it comes to constructing the state space of the atmosphere.

This situation is exactly the same as described in our introduction of the Komogorov-Sinai entropy through state space partitioning. All we can say about the true state of the system is which partition it is in when we make our measurement. In effect, the finite spatial resolution introduces a perturbation between the measured and true states of the system. In order to provide a weather forecast this measured data is then used as the initial conditions in a simulation of the atmosphere. Now as the atmosphere is chaotic the small error in the simulation initial conditions compared to the true initial conditions will become amplified over time at a rate given by the Kolmogorov-Sinai entropy.

As such, if we wish to keep the error on our atmospheric forecasts within a given error
tolerance, the Kolmogorov-Sinai entropy tell us how often we must remeasure the state of the system. Both have units of inverse time and can thus be used to define a predictability time for the system. This also further illustrates the idea that chaotic systems produce information. As the atmosphere evolves and diverges from our approximate state, we need to remeasure to account for this new information, and the Kolmogorov-Sinai entropy gives the rate at which this information is produced.

The dimension of the strange attractor is harder to give a physical meaning to. However, as mentioned in the previous Section, given that the motion of the system must be confined to the attractor this suggests it should be possible to describe the dynamics of the system using dim(A) degrees of freedom. Such ideas are becoming increasingly popular in computational fluid dynamics through the use of proper orthogonal decomposition in which the dynamical system describing fluid flow is decomposed into a large, but finite, number of modes ordered in their importance to the overall flow.

We finish by noting the potential difficulties in applying such ideas to real world systems. The definition of the Lyapunov exponents requires that the separation between state space trajectories be infinitesimal at all times. Of course, in a system like the atmosphere the distance between trajectories is not bounded to be infinitesimal. As a result, the concepts from chaos theory we have discussed are best applied to localised weather phenomena rather than large scale behaviour. A number of extensions to the Lyapunov exponent have been introduced to handle finite separations. Most notably the finite size Lyapunov exponent \[9\], however, we will not discuss this further here.

### 1.5 Thesis Outline

The remainder of this Thesis is arranged as follows: Chapter II is an introduction to a particular statistical closure for turbulence and the description of a numerical code developed to study this closure. The main aim of this Chapter is to provide context for the results of later Chapters. However, it is also intended, along with the appendices, to provide the details required to produce such a numerical code.

In Chapter III the results of studies into the chaotic properties of two and three dimensional turbulence using DNS are presented. Here, the focus is on the Reynolds number scaling of these quantities. It is found that in two dimensions, the chaotic properties show non-universal behaviour, in direct contrast with the three dimensional results.

We then turn our attention to higher dimensional turbulence in Chapter IV. Results from the
numerical code developed in Chapter II are utilised to study turbulence in dimensions from four to twenty including non-integer dimensions. It is found that enstrophy production is maximised between five and six dimensions. This is interesting when compared with the chaotic properties of higher dimensional turbulence in this closure. A remarkable transition to a non-chaotic regime is found at $d \approx 5.8$. These two results are discussed in the context of a critical dimension for turbulence.

The properties of turbulence between two and three dimensions are studied in Chapter V. This work is motivated by the observations of mixed two- and three-dimensional turbulence in the Earth’s atmosphere and the implications of such dynamics for predictability. We utilise both DNS of turbulence in a thin-layer as well as closure calculations of non-integer dimensional turbulence. In the former case, we observe a discontinuous transition in predictability which may have implications for the next generation of numerical weather forecasting models.

Finally, in Chapter VI we present the overall conclusions of this work. Note that each Chapter also contains its own conclusion for the work therein.
In Chapter I we briefly touched upon the closure problem of fluid turbulence, glossing over the technical details. To understand this problem in more detail we introduce schematic equations for the second and third order velocity field moments

\[ \frac{\partial}{\partial t} \langle u_\alpha u_\beta \rangle = L_\gamma \left[ \langle u_\alpha u_\beta u_\gamma \rangle \right], \]

\[ \frac{\partial}{\partial t} \langle u_\alpha u_\beta u_\gamma \rangle = L_\delta \left[ \langle u_\alpha u_\beta u_\gamma u_\delta \rangle \right], \]  

(II.1)

where \( L_\alpha \) is an unspecified differential operator. In equation (II.1) we find the evolution of the second order moment depends on the third order moment and similarly for the third order moment on the fourth. In fact, at all orders the same issue presents itself and we are faced with an infinite hierarchy of equations describing the statistical evolution of a turbulent flow. No clever algebraic manipulations can alleviate this issue, in fact, it is entirely justified to say that this is the problem of fluid turbulence.

Over the past century a diverse range of approaches to tackling the closure problem have been developed. These range from the, at times exceedingly, complex analytical theories of turbulence, which are free of arbitrary parameters, to the simpler closure models in which empirical constants are set to give agreement with real world experiment. It is outside the
This Thesis aims to give a full account of all developments in this area of turbulence, however, to provide background for the work presented in Chapters IV and V, we will fully describe one of the most well studied closure models, the Eddy Damped Quasi-Normal Markovian approximation, here.

This Chapter is arranged as follows, first, starting from the Navier-Stokes equations in Fourier space we will derive an equation for the evolution of the third order velocity field moment. Subsequently, we will introduce the quasi-normal approximation and show how it can be used to close the infinite moment hierarchy. We will then discuss the shortcomings of this approximation and how it can be adapted, resulting in the eddy damped quasi-normal Markovian (EDQNM) approximation. From here, we then derive an equation for the correlation spectrum between two different velocity fields under this closure which can then be used to measure the maximal Lyapunov exponent within the EDQNM closure. Finally, we will show how the EDQNM equations can be discretised and solved numerically. The literature on many of these topics is overly terse in the present author’s opinion, as such, we will endeavour to give a full account here. Indeed, it is hoped that the details in this Chapter will provide a useful reference for the writing of codes to numerical study this closure.

II.1 Moment Equations in Fourier Space

We begin with the incompressible Navier-Stokes equation expressed in Fourier space

\[
\left( \frac{\partial}{\partial t} + \nu k^2 \right) u_\alpha(k) = \frac{1}{2i} P_{\alpha\beta\gamma}(k) \int d\mathbf{p} u_\beta(p) u_\gamma(k-p),
\]

where, as mentioned in Chapter I, the incompressibility condition is imposed by the inertial transfer operator \( P_{\alpha\beta\gamma}(k) \). Throughout this Section we will work in the full \( d \)-dimensional case, therefore we have \( \alpha, \beta, \cdots = 1, \ldots, d \) and, since we consider moments at equal times, we drop explicit time dependence. We are now interested in forming an equation for the second order velocity field moment \( C_{\alpha\sigma}(k) \). Now, homogeneity enforces that the following relationship holds

\[
\langle u_\alpha(k) u_\sigma(k') \rangle = C_{\alpha\sigma}(k) \delta(k + k').
\]

In light of this condition, we are interested in an equation for the evolution of \( \langle u_\alpha(\mathbf{k}) u_\sigma(-\mathbf{k}) \rangle \).

To proceed we first write the equation for \( u_\sigma(-\mathbf{k}) \)

\[
\left( \frac{\partial}{\partial t} + \nu k^2 \right) u_\sigma(-\mathbf{k}) = -\frac{1}{2i} P_{\sigma\beta\gamma}(k) \int d\mathbf{p} u_\beta(p) u_\gamma(-\mathbf{k}-\mathbf{p}),
\]
where we have made use of the property $P_{a\beta\gamma}(-k) = -P_{a\beta\gamma}(k)$. We then multiply equation [II.2] by $u_\alpha(-k)$ and equation [II.3] by $u_\alpha(k)$ before then adding the resultant equations together and averaging. At the end of this process we obtain

$$
\left(\frac{\partial}{\partial t} + 2vk^2\right) \langle u_\alpha(k)u_\alpha(-k) \rangle = \frac{1}{2i}P_{a\beta\gamma}(k) \int dp \langle u_\alpha(-k)u_\beta(p)u_\gamma(k-p) \rangle - \frac{1}{2i}P_{a\beta\gamma}(k) \int dp \langle u_\alpha(k)u_\beta(p)u_\gamma(-k-p) \rangle.
$$

(II.5)

It is possible to express this equation in a slightly different form by considering the real-valued nature of the velocity field. If we consider the complex conjugate of $u_\alpha(k)$, we find it must satisfy

$$
u_\alpha^*(k) = u_\alpha(-k),
$$

(II.6)

hence if we take the complex conjugate of equation [II.2] we find the equation for $u_\alpha(-k)$ can also be expressed as

$$
\left(\frac{\partial}{\partial t} + vk^2\right) u_\alpha(-k) = -\frac{1}{2i}P_{a\beta\gamma}(k) \int dp \ u_\beta(-p)u_\gamma(-k+p),
$$

(II.7)

leading to equation [II.5] being rewritten as

$$
\left(\frac{\partial}{\partial t} + 2vk^2\right) \langle u_\alpha(k)u_\alpha(-k) \rangle = \frac{1}{2i}P_{a\beta\gamma}(k) \int dp \langle u_\alpha(-k)u_\beta(p)u_\gamma(k-p) \rangle - \frac{1}{2i}P_{a\beta\gamma}(k) \int dp \langle u_\alpha(k)u_\beta(p)u_\gamma(-k+p) \rangle.
$$

(II.8)

From here we form an equation for the energy spectrum $E(k)$ by first taking the trace of equation [II.8]

$$
\left(\frac{\partial}{\partial t} + 2vk^2\right) \langle u_\alpha(k)u_\alpha(-k) \rangle = \text{Im} \left[ P_{a\beta\gamma}(k) \int dp \langle u_\alpha(-k)u_\beta(p)u_\gamma(k-p) \rangle \right],
$$

(II.9)

where we have used the fact that

$$
\langle u_\alpha(-k)u_\beta(p)u_\gamma(k-p) \rangle = \langle u_\alpha(k)u_\beta(-p)u_\gamma(-k+p) \rangle^*.
$$

(II.10)

We then recall the expression for the energy spectrum for $d$-dimensional HIT

$$
\langle u_\alpha(k)u_\beta(k') \rangle \delta(k+k') = \frac{2P_{a\beta}(k)E(k)}{(d-1)A_d k^{d-1}} \delta(k+k'),
$$

(II.11)

and write the equation for the evolution of $E(k)$ as

$$
\left(\frac{\partial}{\partial t} + 2vk^2\right) E(k) = \frac{A_d k^{d-1}}{2} \text{Im} \left[ P_{a\beta\gamma}(k) \int dp \ C_{a\beta\gamma}(-k,p,k-p) \right]
$$

(II.12)

\[33\]
where $T(k)$ is the energy transfer spectrum, $C_{\alpha\beta\gamma}(-k, -p, k + p)$ is the third order velocity field moment

$$C_{\alpha\beta\gamma}(k, k', k'')\delta(k + k' + k'') = \langle u_\alpha(k) u_\beta(k') u_\gamma(k'') \rangle,$$

and the delta functions have been cancelled on each side. We also note that homogeneity has forced this integration to only involve triads of wave vectors $\{k, k', k''\}$ that form the legs of a triangle with $k + k' + k'' = 0$.

The closure problem is again elucidated in Equation (II.12). However, we will proceed now as if we are unaware of the issue and derive an equation for the triple moment in an attempt to find a closed equation. We begin, as we did for the second order moment, by forming an evolution equation for the third order moment. To do so, we first multiply the equation for $u_\alpha(-k)$ by $u_\beta(p)u_\gamma(k - p)$

$$u_\beta(p)u_\gamma(k - p)\left(\frac{\partial}{\partial t} + \nu k^2\right)u_\alpha(-k) = -\frac{1}{2i} P_{\alpha\sigma\delta}(k)$$

$$\times \int dm u_\beta(p)u_\gamma(k - p)u_\sigma(m)u_\delta(-k - m),$$

we then multiply the equation for $u_\beta(p)$ by $u_\alpha(-k)u_\gamma(k - p)$

$$u_\alpha(-k)u_\gamma(k - p)\left(\frac{\partial}{\partial t} + \nu p^2\right)u_\beta(p) = \frac{1}{2i} P_{\beta\sigma\delta}(p)$$

$$\times \int dm u_\alpha(-k)u_\gamma(k - p)u_\sigma(m)u_\delta(p - m),$$

and then finally perform the same procedure for $u_\gamma(k - p)$, multiplying by $u_\alpha(-k)u_\beta(p)$

$$u_\alpha(-k)u_\beta(p)\left(\frac{\partial}{\partial t} + \nu|k - p|^2\right)u_\gamma(k - p) = \frac{1}{2i} P_{\gamma\sigma\delta}(k - p)$$

$$\times \int dm u_\alpha(-k)u_\beta(p)u_\sigma(m)u_\delta(k - p - m).$$

We now take these three equations, add them together and average to obtain and equation for $C_{\alpha\beta\gamma}(-k, p, q)$, where $q = k - p$

$$\left(\frac{\partial}{\partial t} + \nu(k^2 + p^2 + q^2)\right)C_{\alpha\beta\gamma}(-k, p, q)$$

$$= -\frac{1}{2i} P_{\alpha\sigma\delta}(k) \int dm C_{\beta\gamma\delta}(p, q, m, -k - m)$$

$$+ \frac{1}{2i} P_{\beta\sigma\delta}(p) \int dm C_{\alpha\gamma\delta}(-k, q, m, p - m)$$

$$+ \frac{1}{2i} P_{\gamma\sigma\delta}(q) \int dm C_{\alpha\beta\delta}(-k, p, m, q - m).$$

At this stage we are ready to introduce the concept of quasi-normality and demonstrate how it can be used to close the moment hierarchy.
II.2 Quasi-Normality and Realisability

II.2.1 Quasi-Normality

The idea of assuming quasi-normality in order to close the infinite moment hierarchy of the Navier-Stokes equation can be traced back to Millionschikov [153], and was popularised in the West by Proudman and Reid [179], as well as Tatsumi [211]. In short, in this method of closure it is assumed that the fourth order moment can be factored into products of second order moments, as can be done for the moments of a normal distribution. In doing so, as the third order moment remains non-zero, this method does not make the velocity field itself normal. This defining feature has led to this approximation being referred to as Quasi-Normal.

Using this idea we can factor the first fourth order moment in Equation II.17 as

\[
\langle u_\beta(p)u_\gamma(q)u_\sigma(m)u_\delta(-k-m) \rangle = \langle u_\beta(p)u_\gamma(q) \rangle \langle u_\sigma(m)u_\delta(-k-m) \rangle \\
+ \langle u_\beta(p)u_\sigma(m) \rangle \langle u_\gamma(q)u_\delta(-k-m) \rangle \\
+ \langle u_\beta(p)u_\delta(-k-m) \rangle \langle u_\gamma(q)u_\sigma(m) \rangle,
\]

allowing the hierarchy to be closed. To make this expression easier to work with, we use the fact that \( q = k - p \) so that we can re-express the above as

\[
\langle u_\beta(p)u_\gamma(q)u_\sigma(m)u_\delta(-k-m) \rangle = C_{\beta\gamma}(p)C_{\sigma\delta}(m)\delta(k) \\
+ C_{\beta\sigma}(m)C_{\gamma\delta}(q)\delta(m+p) \\
+ C_{\beta\delta}(p)C_{\gamma\sigma}(m)\delta(m+q),
\]

then exploiting the isotropy of our problem we have

\[
\langle u_\beta(p)u_\gamma(q)u_\sigma(m)u_\delta(-k-m) \rangle = P_{\beta\gamma}(p)P_{\sigma\delta}(m)C(p)C(m)\delta(k) \\
+ P_{\beta\sigma}(m)P_{\gamma\delta}(q)C(m)C(q)\delta(m+p) \\
+ P_{\beta\delta}(p)P_{\gamma\sigma}(m)C(p)C(m)\delta(m+q).
\]

Notably, the delta function in the first term will force \( k = 0 \) and thus, since \( P_{a\beta\gamma}(0) = 0 \), this term cannot contribute. The analogous terms for the other fourth order moments in
Equation [II.17] are

\[
\left\langle u_a(-k)u_\gamma(q)u_\sigma(m)u_\delta(p-m) \right\rangle = P_{\alpha\gamma}(k)P_{\sigma\delta}(m)C(k)C(m)\delta(p)
\]
\[
+ P_{\sigma\gamma}(m)P_{\gamma\delta}(q)C(m)C(q)\delta(m-k),
\]
\[
+ P_{\alpha\delta}(k)P_{\gamma\sigma}(m)C(k)C(m)\delta(m-q),
\]

and

\[
\left\langle u_a(-k)u_\beta(p)u_\sigma(m)u_\delta(q-m) \right\rangle = P_{\alpha\beta}(k)P_{\sigma\delta}(m)C(k)C(m)\delta(q)
\]
\[
+ P_{\sigma\beta}(m)P_{\beta\delta}(p)C(m)C(p)\delta(m-k),
\]
\[
+ P_{\alpha\delta}(k)P_{\beta\sigma}(m)C(k)C(m)\delta(m-p).
\]

Substituting the expressions we have just derived into Equation [II.17] we can use the delta functions in the second and third terms of each quasi-normal expansion to perform the integration over \(m\). Upon doing so we find

\[
\left( \frac{\partial}{\partial t} + \nu \left( k^2 + p^2 + q^2 \right) \right) C_{\alpha\beta\gamma}(\mathbf{k}, \mathbf{p}, \mathbf{q})
\]
\[
= i P_{\alpha\sigma\gamma}(k)P_{\beta\delta}(p)P_{\gamma\delta}(q)C(p)C(q)
\]
\[
- i P_{\beta\sigma\gamma}(p)P_{\alpha\gamma}(k)P_{\gamma\delta}(q)C(k)C(q)
\]
\[
- i P_{\gamma\sigma\delta}(q)P_{\alpha\beta}(k)P_{\beta\delta}(p)C(k)C(p).
\]

(II.23)

If we now recall the definition of the energy transfer spectrum in \(d\)-dimensional HIT from equation [II.12] we can write

\[
\left( \frac{\partial}{\partial t} + \nu \left( k^2 + p^2 + q^2 \right) \right) T(k)
\]
\[
= \frac{A_d k^{d-1}}{2} P_{\alpha\beta\gamma}(k) \int d\mathbf{p} \int d\mathbf{q} \left[ P_{\alpha\sigma\delta}(k)P_{\beta\delta}(p)P_{\gamma\delta}(q)C(p)C(q)
\right.
\]
\[
- P_{\beta\sigma\gamma}(p)P_{\alpha\gamma}(k)P_{\gamma\delta}(q)C(k)C(q)
\]
\[
- P_{\gamma\sigma\delta}(q)P_{\alpha\beta}(k)P_{\beta\delta}(p)C(k)C(p) \right] \delta(k-p-q).
\]

(II.24)

Note that here we have introduced an integral over \(q\) and a delta function to make the relationship between vectors explicit. Although we have now closed the moment hierarchy, the equation we have derived is still formidable. To proceed, we need to evaluate the contractions of the various projection and inertial transfer operators which are present. This is a fairly straightforward procedure, however it is very tedious so is left to Appendix A. The
main results are three geometric factors \(a_{kpqd}, b_{kpqd}\) and \(b_{kpqd}\) defined as

\[
a_{kpqd} = \frac{P_{\alpha\beta\gamma}(k)P_{\alpha\beta\gamma}(p)P_{\gamma\delta}(\phi)}{4k^2},
\]
\[
b_{kpqd} = \frac{P_{\alpha\beta\gamma}(k)P_{\alpha\beta\gamma}(p)P_{\gamma\delta}(\phi)}{2k^2},
\]
\[
b_{kpqd} = \frac{P_{\alpha\beta\gamma}(k)P_{\gamma\delta}(\phi)P_{\alpha\beta\gamma}(p)}{2k^2}.
\] (II.25)

In Appendix A, it is also shown that these expressions are related such that

\[
2a_{kpqd} = b_{kpqd} + b_{kpqd}.
\] (II.26)

Using this fact, and performing a relabelling of the integration variables \(p \rightarrow q\), Equation II.24 becomes

\[
\left(\frac{\partial}{\partial t} + \nu(k^2 + p^2 + q^2)\right)T(k)
= 2A_d k^{d+1} \int dp \int dq \ b_{kpqd} \left[ C(p)C(q) - C(q)C(k) \right] \delta(k-p-q),
\] (II.27)

which is clearly a far simpler expression. The complete set of manipulations to move from Equation II.24 to Equation II.27 can be also be found in Appendix A.

We are now at a position in our analysis where we can solve for \(T(k)\) in terms of the isotropic second order moment. To do so, we use an integrating factor and the assumption that at \(t = 0\) the velocity field is Gaussian so that \(T(k, 0) = 0\) for all \(k\). The result of this is an equation for the transfer spectrum

\[
T(k) = 2A_d k^{d+1} \int_0^t dt' \int dp \int dq \ e^{-\nu(k^2+p^2+q^2)(t-t')} \ b_{kpqd} \left[ C(p)C(q) - C(q)C(k) \right] \delta(k-p-q).
\] (II.28)

We now substitute this expression into Equation II.12, resulting in

\[
\left(\frac{\partial}{\partial t} + 2\nu k^2\right)E(k) = 2A_d k^{d+1} \int_0^t dt' \int dp \int dq \ e^{-\nu(k^2+p^2+q^2)(t-t')} \times \ b_{kpqd} \left[ C(p)C(q) - C(q)C(k) \right] \delta(k-p-q).
\] (II.29)

Finally, we express the right hand side in terms of the energy spectrum to find

\[
\left(\frac{\partial}{\partial t} + 2\nu k^2\right)E(k) = \frac{8k^2}{A_d(d-1)} \int_0^t dt' \int dp \int dq \ e^{-\nu(k^2+p^2+q^2)(t-t')} \times \ b_{kpqd} \left[ \frac{k^{d-1}E(p)(q) - p^{d-1}E(q)(k)}{p^{d-1}q^{d-1}} \right] \delta(k-p-q),
\] (II.30)
which we will refer to as the Quasi-Normal equation.

II.2.2 Realisability

If we are given an initial condition for the energy spectrum at time \( t = 0 \), that is \( E(k, 0) \), then by evolving Equation II.30 forward in time, it is possible to study the behaviour of \( E(k) \). Even though this equation is a massive simplification compared to the infinite moment hierarchy of the pure Navier-Stokes equations, it can still only be solved numerically in general.

In the 1960s, computers reached the stage that meaningful numerical studies of the Quasi-Normal equation could be carried out. The results of these studies were quite unexpected, showing a serious flaw in the equations. Numerous authors [164–166] found that, after a short time period, the energy spectrum became negative. As this is unphysical behaviour, it is said that the Quasi-Normal equations are non-realisable. As the Quasi-Normal equation is simply an approximation, if the energy spectrum was only negative in the deep dissipative region whilst being reasonable elsewhere, the approximation may have been useful. However, it is found that the spectrum can become negative in the inertial range even at low Re, thus the approximation is not helpful. It is quite surprising that what seems like a reasonable approximation leads to such unphysical behaviour. We now turn to additional modifications that can be made to the Quasi Normal equations such that they are rendered realisable.

II.3 The EDQNM Approximation

The non-realisability of the Quasi Normal equations was explored at length by Orszag [171]. The explanation given for the unphysical behaviour of the Quasi Normal equation is related to the time integral in Equation II.30. Consider the exponential in this expression, at low viscosity and/or time differences that are not too large, this expression will be essentially unity. As such, under these conditions, in this integral all points in time are weighted approximately equally and the system has a long memory of the past. If the viscosity is large, then this situation is avoided and a memory cut off is imposed.

In the Navier-Stokes equations the non-linear term has a scrambling effect on correlations between Fourier modes in addition to the simple viscous damping. Orszag suggested that the absence of this scrambling effect in the Quasi Normal equation was the source of the unphysical behaviour. That is, the memory time of the system is too long in the approximation, allowing correlations over large lengths of time that are not present in the
true system.

II.3.1 Eddy Damping

It is possible to reintroduce the effects of scrambling due to the non-linearity of the Navier-Stokes equations. To do so, the exponential factor in the time integral should be modified to include not only damping due to molecular viscosity, but also from eddy-viscosity. This is a form of effective viscosity acting at scales far larger than where molecular viscosity is dominant.

This idea of an effective viscosity was first introduced by Boussinesq [34]. We can consider the shear stress, $\tau$, in a turbulent flow as being composed of two parts, one due to molecular viscosity and the other due the Reynolds stress tensor. In Newtonian fluids, the part due to molecular viscosity, $\tau_\mu$, is proportional to the velocity gradient, i.e. for a shear in the $x$-direction

$$\tau_\mu = \mu \frac{\partial U}{\partial x}, \quad (\text{II.31})$$

in which $\mu$ is the dynamic viscosity. Boussinesq made the assumption that the Reynolds stress tensor was also proportional to the mean shear, with constant of proportionality given by the eddy-viscosity, $\mu_t$. Therefore, we can write

$$\tau = (\mu + \mu_t) \frac{\partial U}{\partial x}, \quad (\text{II.32})$$

and we observe that the non-linear term from which the Reynolds stress tensor arises leads to an additional transfer of momentum in the flow beyond that of viscosity alone. Importantly, unlike the molecular viscosity, which is a property of the fluid itself, the eddy-viscosity is dependent on the flow structure and thus has spatial dependence.

Applying these ideas to the Quasi-Normal equations suggests the exponential term on the right hand side of Equation (II.30) should be modified to include the effects of eddy-viscosity, such that

$$e^{-\nu(k^2 + p^2 + q^2)(t - t')} \rightarrow e^{-\left((\nu(k^2 + p^2 + q^2) + \mu_{kpq})(t - t')\right)}, \quad (\text{II.33})$$

where $\mu_{kpq}$ has dimensions of inverse time and gives the rate of damping of the third order correlations by eddy-viscosity. Note that since the eddy-viscosity has spatial dependence, $\mu_{kpq}$ will vary with $k$, $p$ and $q$.

In isotropic turbulence we have

$$\mu_{kpq} = \mu_k + \mu_p + \mu_q, \quad (\text{II.34})$$
however the choice for $\mu_k$ is arbitrary and a number of suggestions have been made in the literature. Here we only consider the simplest of these suggestions, which are not self-consistent, but do give the Kolmogorov energy spectrum in their inertial ranges. For examples of self-consistent closure approximations, see the Direct Interaction Approximation of Kraichnan [122] and the Local Energy Transfer Theory of McComb [145] amongst numerous others. Orszag proposed

$$\mu_k = A \left( k^3 E(k) \right)^{\frac{1}{3}}, \quad (II.35)$$

where $A$ is a free parameter which can be tuned to give agreement with experimental measurements of the Kolmogorov constant. This choice does give the $-5/3$ inertial range exponent of the Kolmogorov 1941 theory, however, it is not guaranteed to strictly increase as $k$ increases. This is not entirely realistic given that with units of inverse time $\mu_k$ acts as a characteristic frequency for the system which we would expect to increase with $k$. This was rectified in [178] using the form

$$\mu_k = A \sqrt{\int_0^k dr \, r^2 E(r, t)}. \quad (II.36)$$

It is this form which is used throughout this Thesis. Although this form is standard in the literature it is in a sense arbitrary and not guaranteed to reproduce fully physical behaviour at all scales. For example for $k = 0$ the above expression provides no eddy-damping to the largest scales. It may be then be more appropriate to consider an expression of the form

$$\mu_k = B \int_0^k dr \, f(r) + C \int_k^\infty dr \, g(r), \quad (II.37)$$

however such a study is beyond the scope of this Thesis. In any case, this limitation must be appreciated when interpreting later results.

### II.3.2 Markovianisation

The introduction of eddy-damping to the Quasi-Normal equation on its own is not enough to ensure realizability, for this a final modification is needed. This step is known as Markovianisation, stemming from the idea of a Markov chain. A Markovian process is one in which the time evolution depends only on the current state of the system. That is, the state of the system before the immediate present has no influence on the next evolution of the system.

For our purposes, the end result of the Markovianisation process is the replacement of the
time integral in the Eddy Damped Quasi-Normal equation with the factor $\theta_{kpq}$, resulting in
\[
\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k) = \frac{8k^2}{A_d(d-1)^2} \int dp \int dq \theta_{kpq} b_{kpq,d} \left(\frac{E(p)E(q) - E(q)E(k)}{p^{d-1}q^{d-1}}\right) \delta(k - p - q).
\]
(II.38)

The justification for this modification can be understood by considering that the timescale given by the inverse of $\nu(k^2 + p^2 + q^2) + \mu_{kpq}$ will be far smaller than that of the energy spectrum products. Thus, the energy spectrum terms can be considered as slowly varying and we can set $t' = t$ for these terms. Hence, our time integral now only contains the eddy-damped exponential term which can be integrated giving
\[
\theta_{kpq}(t) = \int_0^t dt' e^{-((\nu(k^2 + p^2 + q^2) + \mu_{kpq})(t - t'))} = \frac{1 - e^{-(\nu(k^2 + p^2 + q^2) + \mu_{kpq})t}}{\nu(k^2 + p^2 + q^2) + \mu_{kpq}}.
\]
(II.39)

This is a drastic simplification and the equations now only depend on the current state of the system. This is a particularly important simplification for numerical calculations of these equations, since without Markovianisation we would be required to store all the previous states of the system in order to compute the time integral. For long and/or very high max $k$ simulations, this would quickly cause efficiency issues due to memory usage.

II.3.3 The EDQNM Equation

Equation (II.39) could already be considered as the Eddy Damped Quasi-Normal Markovian (EDQNM) equation. However, we can perform some additional algebraic manipulations to further reduce the complexity and render the equation in the form that we will use for our numerical calculations.

In the form shown in Equation (II.39) we have two $d$-dimensional integrals over all space, although due to the delta function one is trivial. We can greatly simplify the integral by noting that it is made up of one radial integral and $(d-1)$-angular integrals. The integrand depends only on the magnitude $p$ and, through $q$, on the angle between $k$ and $p$. Hence, by following the steps outlined in Appendix B we can transform to bipolar coordinates and the evolution equation for $E(k)$ becomes the EDQNM equation
\[
(\partial_t + 2\nu k^2) E(k) = 8K_d \int_{\Omega(k)} dp dq \frac{k}{pq} b^{(d)}_{kpq} \theta_{kpq}(t) \left(\frac{\sin \alpha}{k}\right)^{d-3} \left[k^{d-1}E(p)E(q) - p^{d-1}E(q)E(k)\right],
\]
(II.40)
in which $\Omega(k)$ is the region in which $k, p$ and $q$ can form a triangle, $\alpha$ is the angle between $p$ and $q$ as shown in Figure A.1 and $K_d = A_{d-1}/(d-1)^2A_d$. A representation of this region for an arbitrary value of $k$ is shown in Figure II.1. Hence, we have reduced our initial integral
over $d$-dimensional space to a two-dimensional integral over a restricted region. As a result, as the spatial dimension is increased this integral does not become any more expensive to compute numerically, a massive advantage when compared to direct numerical simulations of the Navier-Stokes equations.

The appearance of the exponent $d - 3$ for the term $(\sin \alpha)/k$ in the above equation is at first glance somewhat intriguing. In particular, it seems to single out $d = 3$ as a special case in which this term vanishes. Through the angle $\alpha$ this term is related to all three wave-vectors in the triad integral, hence it’s disappearance for $d = 3$ feels notable. However, given what is known about two dimensional turbulence it is much more deserving of the distinction as a special dimension for turbulence, so why is this term not $d - 2$? The answer to this is contained in the geometric term $b_{kpq}^{(d)}$ which is made up of a number of trigonometric terms depending on all three angles in the wave-vector triangle. Hence, the vanishing of the $\sin \alpha$ term is merely a coincidence.

II.3.4 Conservation Properties

In Chapter I we saw that the transfer spectrum $T(k)$ has the property

$$\int_0^\infty dk \ T(k) = 0,$$

which is an expression of the conservation of energy by the non-linear term. We saw that this was a result of the antisymmetry $T(k)$ under the exchange of $k$ and $p$. Looking at the EDQNM equation, it can be shown that this property is still satisfied even after the many modifications that have been made to obtain this approximation. To see this, we will work term-by-term, beginning with the energy spectrum products

$$k^{d-1}E(p)E(q) - p^{d-1}E(q)E(k) = k^{d-1}E(k)E(q) - k^{d-1}E(q)E(p),$$

which we see are antisymmetric. The eddy-damping term $\theta_{kpq}(t)$ is by design fully symmetric in $k$, $p$ and $q$. Additionally, by the law of sines, the $\sin \alpha/k$ term is also fully symmetric in $k$, $p$ and $q$. In Appendix A we show that

$$k^2 b_{kpq}^{(d)} = p^2 b_{pkq},$$

and therefore we have

$$\frac{k}{p} b_{kpq}^{(d)} = \frac{p}{k} b_{pkq}^{(d)}.$$
Hence, $\frac{k}{pq}b^{(d)}_k$ is symmetric under the exchange of $k$ and $p$. Thus, the integral on the right hand side of the EDQNM equation is antisymmetric under this exchange, and as a result also displays energy conservation.

Finally, it can be shown that the EDQNM equation also satisfies the detailed triad-by-triad energy conservation that the Navier-Stokes equations satisfy. This feature places important restrictions on the numerical approaches that can be utilised for the EDQNM equation. In order to preserve the energy conservation of the non-linear term, the numerical scheme must preserve detailed triad-by-triad conservation and thus must be fully symmetric in $k$, $p$ and $q$. We will return to this point in Section II.5.

II.4 EDQNM equation for the Correlation Spectrum in d Dimensions

Before considering how the EDQNM equations can be solved numerically, we consider here how error growth and chaos can be studied using this approximation. As discussed in Chapter I, our inability to predict the state of a real-world turbulent flow arbitrarily far into the future is due to our inability to measure the initial state with infinite precision.

Making this more concrete, we can think of the true state of our flow as being represented by the velocity field $u^{(1)}(k)$ and our measured state as being given by $u^{(2)}(k)$. We can then define the correlation between these two fields as

$$W_{ij}(k) = \left\langle u_i^{(1)}(k)u_j^{(2)}(-k) \right\rangle,$$  \hspace{1cm} (II.45)

then as we are focussing on isotropic turbulence we have

$$W_{ij}(k) = P_{ij}(k)W(k).$$  \hspace{1cm} (II.46)

If the two fields are completely decorrelated at a given $k$ we have $W(k) = 0$ and if they are completely correlated we find $W(k) = C(k)$.

Considering again a real-world measurement with finite precision, what we would expect to find is that for small $k$, $W(k) \approx C(k)$, then as $k$ increases we find $W(k)$ drops and reaches zero at the limit of our experimental precision. From this initial state of correlation at large scales and decorrelation at small scales, the system will then evolve and over time the decorrelation, or error, at the smallest scales will progressively contaminate larger and larger scales. To study this process we need to obtain an equation for the time evolution of
$W(k)$, which can be found by following the same process as we used to derive the EDQNM equations for a single velocity field.

We begin by writing the Navier-Stokes equations in Fourier space for both fields, for $u^{(1)}(k)$

$$\left( \frac{\partial}{\partial t} + \nu k^2 \right) u^{(1)}_a(k) = \frac{1}{2i} P_{\alpha\beta\gamma}(k) \int dp \, u^{(1)}_\beta(p) u^{(1)}_\gamma(k-p),$$  \hspace{1cm} (II.47)

and for $u^{(2)}(-k)$

$$\left( \frac{\partial}{\partial t} + \nu k^2 \right) u^{(2)}_a(-k) = -\frac{1}{2i} P_{\alpha\beta\gamma}(k) \int dp \, u^{(2)}_\beta(-p) u^{(2)}_\gamma(-k+p).$$  \hspace{1cm} (II.48)

From here, as we did for the single field, we multiply the equation for $u^{(1)}(k)$ by $u^{(2)}(-k)$ and vice versa before adding and averaging to obtain

$$\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) \langle u^{(1)}_a(k) u^{(2)}_a(-k) \rangle = \frac{1}{2i} P_{\alpha\beta\gamma}(k) \int dp \, \langle u^{(2)}_\beta(-k) u^{(1)}_\gamma(k-p) \rangle$$

$$- \frac{1}{2i} P_{\alpha\beta\gamma}(k) \int dp \, \langle u^{(1)}_\gamma(k) u^{(2)}_\beta(p) u^{(2)}_\gamma(-k+p) \rangle.$$  \hspace{1cm} (II.49)

The process from here is essentially identical to the single field case, however, the presence of two different fields prevents us from simplifying the above by taking the imaginary part. Thus, equations for both the triple moments above need to be found and the quasi-normal hypothesis applied in both cases.

Due to the similarity to the derivation of the single field EDQNM equation, we will not go through the full derivation of the EDQNM correlation spectrum equation. Instead we simply present it here

$$\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) E_W(k) = T_{W}(k) - T_{X}(k)$$  \hspace{1cm} (II.50)

in which $E_W(k) = A_d k^{d-1} W(k)$, $T_{W}(k)$ is the transfer of correlated energy and $T_{X}(k)$ is the transfer of correlated energy into uncorrelated energy. The full form for $T_{W}(k)$ is given by

$$T_{W}(k) = 8K_d \int_{\Omega(k)} dp dq \frac{k}{pq} \left( \frac{\sin \alpha}{k} \right)^{d-3} \theta_{kpq} b_{kpq}^{(d)} (E_W(p)E(q)k^{d-1} - E(q)E_W(k)p^{d-1})$$  \hspace{1cm} (II.51)

where we observe the same $k \leftrightarrow p$ antisymmetry as seen in the energy spectrum evolution equation, meaning this transfer term conserves correlated energy, that is

$$\int_{0}^{\infty} dk T_{W}(k) = 0.$$  \hspace{1cm} (II.52)

Of course, we know that from the lack of predictability observed in turbulent flows, something must destroy this conservation of correlated energy. The additional term $T_{X}(k)$
is responsible for the breaking of this conservation, looking at its full expression we have

\[
T_X(k) = 8K_d \int_{\Omega(k)} dp dq \frac{k}{pq} \left( \frac{\sin \alpha}{k} \right)^{d-3} \theta_{kpq} \delta^{(d)} k p q E_W(p) E_{\Delta}(q) k^{d-1}
\] (II.53)

and

\[
\int_0^\infty dk T_X(k) \neq 0.
\] (II.54)

Thus \( T_X(k) \) acts to convert correlated energy into uncorrelated energy. To quantify this we consider the spectrum of uncorrelated energy given by

\[
E_{\Delta}(k) = E(k) - E_W(k).
\] (II.55)

We will consider these transfer terms in greater detail in Chapter IV where we will also discuss how these equations may be used to find the maximal Lyapunov exponent.

### II.5 Numerical Considerations

So far in this Chapter, we have derived numerous integro-differential equations for the time-evolution of the energy spectrum in the EDQNM approximation. In general, it is not possible to find solutions to these equations analytically, thus typically a numerical approach is taken. As discussed in the introduction to this Chapter, the details of how to perform numerical calculations of these equations are scattered throughout decades of the literature, and in many cases far too little information is given to reproduce results. In the course of the work carried out for this Thesis, a \( d \)-dimensional EDQNM code has been produced. This Section compiles all the details needed to create such a code in one place, and aims to fill in many of the gaps seen in the literature.

#### II.5.1 Discretising the EDQNM equation

The restriction of the integrals to the domain \( \Omega(k) \) in Equation II.40 introduces some complication when it comes to numerical calculations of the EDQNM equation. As is standard in numerical integration, we need to discretise this domain in some form. A process for doing so was outlined in [131], here we present the details of this process which is the same as used in our EDQNM code.

We first perform a truncation in \( k \)-space and restrict ourselves to work with \( k_{\text{bot}} < k < k_{\text{top}} \) with similar restrictions then applied to both \( p \) and \( q \). We then look to evaluate the EDQNM
expression on a finite set of points in this domain, i.e. we find \( E(k_i) \) with \( i = 0, \ldots, N - 1 \) with the \( k_i \) spaced logarithmically such that \( k_i = 2^{i/F} \), where \( F > 2 \) sets the resolution. In either case, each point \( k_i \) is taken as being representative of a surrounding region, \( k_{i-1/2} < k_i < k_{i+1/2} \), of length \( \Delta k_i \). For linear spacing \( \Delta k_i = 1 \) for all \( i \), however, for logarithmic spacing \( \Delta k_i = k_{i+1/2} - k_{i-1/2} = \Lambda k_i \) with \( \Lambda = 2^{F/2} - 2^{-F/2} \). The use of logarithmic spacing is important in allowing for very high Reynolds numbers to be studied under the EDQNM approximation. Such spacing is justified due to the smooth nature of \( E(k) \). If linear spacing is used, the numerical cost of the EDQNM approximation is almost that of full three-dimensional direct numerical simulation.

Now, rather than considering the domain for a single value of \( k \), we look to the full three-dimensional domain \( \Omega \) in \((k, p, q)\) space. We find by considering the cosine rule that \( \Omega \) is the region defined by the interior of the planes \( k + p - q = 0 \), \( k - p + q = 0 \) and \( k - p - q = 0 \) for \( k, p, q > 0 \). For both linear and logarithmic spacing, around each point \((k_i, p_m, q_n)\) there is an associated volume \( V_{imn} = \Delta k_i \Delta p_m \Delta q_n \). However, it is not always the case that this volume will lie entirely within the three planes which define \( \Omega \), and thus only a fraction \( v_{imn} \) of the total volume associated with these points \((k_i, p_m, q_n)\) should be counted as contributing to the integral. Additionally, there will be certain points \((k_i, p_m, q_n)\) which lie outside \( \Omega \), but a portion of their associated volume lies within. These pose an issue because, if the point for evaluation lies outside the region in which a triangle can be formed, the \( \sin \alpha \) term in Equation II.40 will be undefined here.

Obtaining values for the volume fraction \( v_{imn} \) and determining these outlying points
with contributing associated volume is the major challenge in discretising the EDQNM equations. Frustratingly, in [131] although values to check \( v_{imn} \) are given, the method for obtaining them is described as “... a straightforward but complex exercise in solid geometry and computer logic.” with no additional information. As such, we describe how these volume fractions may be obtained in detail in Appendix C.

Once we have these volume fractions, we are ready to write a discretised version of the EDQNM equation. Going from continuous to discrete space our double integral becomes a double sum with

\[
\int_{\Omega(k)} dp dq - \sum_m \sum_n v_{imn} \Delta p_m \Delta q_n. \tag{II.56}
\]

We can then write the EDQNM equation as

\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) E(k_i) = 8Kd \sum_m \sum_n v_{imn} \frac{k_i}{p_m q_n} b_{k_ip_m q_n}^{(d)} \theta_{k_ip_m q_n}(t) \left( \frac{\sin \alpha}{k_i} \right)^{d-3} \times \left[ k_i^{d-1} E(p_m) E(q_n) - p_m^{d-1} E(q_n) E(k_i) \right] \Delta p_m \Delta q_n. \tag{II.57}
\]

Due to the symmetric nature of \( v_{imn} \), this discretisation scheme satisfies the necessary symmetries for detailed energy conservation. Hence, if a further sum over \( k \) is carried out, then the right hand side gives as close to zero as is possible due to finite numerical precision.

The above method is not the only approach that we could have taken. Indeed, our use of the central point in a volume as being representative of the whole volume requires some justification. For the variables \( k_i, p_m \) and \( q_n \) there is no issue as these vary smoothly across each volume, as do the energy spectrum terms. Where we may encounter problems is in the geometric terms \( b_{k_ip_m q_n}^{(d)} \) and \((\sin \alpha/k_i)^{d-3}\) which can vary rapidly across the volumes.

Concerned about this point, Leith and Kraichnan [132] devised a modification to the above scheme, whereby they average the \( \sin \alpha \) term over the associated volume lying within \( \Omega \) and use this value in the summation. Doing this and writing \( b_{k_ip_m q_n}^{(d)} \) in a different way, they are able to maintain the symmetry properties of integrals and thus detailed energy conservation. This problem was further investigated by Bowman [35], who provided an elegant algebraic method to perform such averaging. Indeed, Bowman notes that increased accuracy can be gained by averaging common symmetric factors within the integrand, even if the non-symmetric terms are still evaluated at the central point.

The accuracy gains from these averaging processes are not substantial and their implementation further complicates the numerical scheme. Comparable accuracy can be achieved by using a finer numerical grid spacing, i.e. larger \( F \), in the computations. We return to this point in Section II.5.4 where we discuss a straightforward method of testing if our resolution
II.5.2 Time Evolution

So far, we have performed a spatial discretisation of the EDQNM equation. In order to study solutions to this equation, we require now to perform a temporal discretisation and a method of advancing the equation in time. As is standard, we split the time domain into sections of length $\Delta t$, known as the time-step, such that we can write $t_i = t\Delta t$.

Numerical time integration of a differential equation is a mature field, with a range of methods for approximating time derivatives. These vary from the simplest finite difference approaches to complicated multi-step approaches and from fixed values of $\Delta t$ to adaptive time-steps. Our approach is on the simpler side, being a two-step predictor-corrector method as used by Leith and Kraichnan [132].

As the EDQNM equation is a linear differential equation in time, the accuracy of our scheme is improved by handling the viscous term exactly through the use of an integrating factor. That is we have

$$e^{2\nu k^2 t} \left( \frac{\partial}{\partial t} + 2\nu k^2 \right) E(k) = \frac{\partial}{\partial t} \left( e^{2\nu k^2 t} E(k) \right). \quad (II.58)$$

We can then write the EDQNM equation as

$$e^{2\nu k^2 t} E(k, t) = 8K_d \int_0^t ds e^{2\nu k^2 s} \sum_m \sum_n v_{imn} \frac{k_i}{p_m q_n} b_{k_i p_m q_n}^{(d)} \theta_{k_i p_m q_n} (s) \left( \frac{\sin \alpha}{k_i} \right)^{d-3} \Delta p_m \Delta q_n,$$

in which we have restored explicit time dependence. Let us now consider the solution after an amount of time given by $\Delta t$, we then have

$$e^{2\nu k^2 t + \Delta t} E(k, t + \Delta t) = 8K_d \int_0^{t + \Delta t} ds e^{2\nu k^2 s} \sum_m \sum_n v_{imn} \frac{k_i}{p_m q_n} b_{k_i p_m q_n}^{(d)} \theta_{k_i p_m q_n} (s) \left( \frac{\sin \alpha}{k_i} \right)^{d-3} \Delta p_m \Delta q_n$$

$$= e^{2\nu k^2 t} E(k, t) + 8K_d \int_t^{t + \Delta t} ds e^{2\nu k^2 s} \sum_m \sum_n v_{imn} \frac{k_i}{p_m q_n} b_{k_i p_m q_n}^{(d)} \theta_{k_i p_m q_n} (s) \left( \frac{\sin \alpha}{k_i} \right)^{d-3} \Delta p_m \Delta q_n.$$

Hence, to find the energy spectrum at the next time step requires knowledge of the current time and the discretisation of a time integral, and we will denote the integral as $I$ for brevity.
with

\[ I = 8 k_d \int_{t}^{t + \Delta t} ds e^{2vK^2 s} \sum_{m} \sum_{n} v_{imn} \frac{k_i}{p_m q_n} \theta_{k_i, p_m q_n} \left( \frac{\sin \alpha}{k_i} \right)^{d-3} \]

\times \left[ k_i^{d-1} E(p_m, t) E(q_n, t) - p_m^{d-1} E(q_n, t) E(k_i, t) \right] \Delta p_m \Delta q_n. \tag{II.61} \]

To approximate this integral we consider the values at its endpoints with

\[ I(t) = e^{2vK^2 t} \sum_{m} \sum_{n} v_{imn} \frac{k_i}{p_m q_n} b^{(d)} \theta_{k_i, p_m q_n} \left( \frac{\sin \alpha}{k_i} \right)^{d-3} \]

\times \left[ k_i^{d-1} E(p_m, t) E(q_n, t) - p_m^{d-1} E(q_n, t) E(k_i, t) \right] \Delta p_m \Delta q_n \Delta t, \tag{II.62} \]

and

\[ I(t + \Delta t) = e^{2vK^2 t + \Delta t} \sum_{m} \sum_{n} v_{imn} \frac{k_i}{p_m q_n} b^{(d)} \theta_{k_i, p_m q_n} \left( \frac{\sin \alpha}{k_i} \right)^{d-3} \]

\times \left[ k_i^{d-1} E(p_m, t + \Delta t) E(q_n, t + \Delta t) - p_m^{d-1} E(q_n, t + \Delta t) E(k_i, t + \Delta t) \right] \Delta p_m \Delta q_n \Delta t. \tag{II.63} \]

Clearly, evaluating \( I(t + \Delta t) \) requires knowledge of the energy spectrum at the next time step.

To obtain an approximation of these values, we define the predicted solution \( E_p(k, t) \) as

\[ e^{2vK^2 t + \Delta t} E_p(k, t + \Delta t) = e^{2vK^2 t} E(k, t) + I(t). \tag{II.64} \]

Now, taking this predicted solution, we evaluate \( I(t + \Delta t) \) to give us \( I_p(t + \Delta t) \), which is then given by the above with the replacement of \( E(k_i) \rightarrow E_p(k_i) \) and likewise for \( p_m \) and \( q_n \). We then take an average of the values from the integral endpoints to approximate \( I \) as

\[ I = \frac{1}{2} \left( I(t) + I_p(t + \Delta t) \right). \tag{II.65} \]

Therefore, we have our predictor step

\[ E_p(k_i, t + \Delta t) = e^{-2vK^2 \Delta t} \left( E(k_i, t) + I(t) \right) \tag{II.66} \]

and then finally our corrector step

\[ E(k_i, t + \Delta t) = \frac{1}{2} \left[ E_p(k_i, t + \Delta t) + e^{-2vK^2 \Delta t} \left( E(k_i, t) + I_p(t + \Delta t) \right) \right]. \tag{II.67} \]

To utilise this method, we need to compute the triad integral on the right hand side of the EDQNM equation twice per time-step and store both actual and predicted energy spectra. It is possible to go through a similar process to derive higher order versions.
of the approximation, e.g. the fourth order Runge-Kutta method, however, higher order approaches require additional evaluations of the triad integral. It is then a balance between the expense of computing the integral, which is the most expensive operation in our numerical calculation, and being able to take larger values of $\Delta t$. We will discuss this further in Section II.5.4.

### II.5.3 Setting the Kolmogorov Constant

In order to make use of the EDQNM model, it is necessary to specify the value of the free parameter $A$ seen in Equation [II.36]. The value of this constant can be shown to fix the value of the Kolmogorov constant, $C_d$, which is important for a number of numerical measurements. Here, we extend the method used by [148] to derive of the relationship between $A$ and $C_d$ to the $d$-dimensional case. An alternative derivation of the relationship for the three-dimensional case can also be found in [3].

To begin, we consider the eddy-damping rate defined in Equation [II.36] in the limit $\nu \to 0$, such that the energy dissipation rate remains constant and, taking the energy spectrum to be a Kolmogorov spectrum to infinity, find

$$
\mu_k = \frac{\sqrt{3}}{2} A \sqrt{C_d} \varepsilon^\frac{1}{2} k^\frac{2}{3}.
$$

(II.68)

On dimensional grounds, in the inertial range we can take $\mu_k$ to have the form

$$
\mu_k = \beta_d \varepsilon^\frac{1}{2} k^\frac{2}{3},
$$

(II.69)

and thus we have

$$
C_d = \frac{4\beta_d^2}{3A^2}.
$$

(II.70)

To make further progress we now need to relate $C_d$ and $\beta$ in the EDQNM model. We begin by considering the forced Lin equation at stationary state

$$
-T(k) = -2\nu k^2 E(k) + F(k),
$$

(II.71)

where $F(k)$ is the forcing spectrum. Now, in taking the limit of zero viscosity, the dissipation range will move to the smallest possible scales, hence we have

$$
2\nu k^2 E(k) \to \varepsilon \delta(k - \infty),
$$

(II.72)
and energy conservation then implies that
\[ F(k) \rightarrow \varepsilon \delta(k) . \quad (II.73) \]

Upon integrating both sides of the Lin equation up to a value \( \kappa \), which by the symmetry properties of the integral is arbitrary so long as it is neither 0 nor \( \infty \), we find
\[ - \int_{0}^{\kappa} dk T(k) = \varepsilon . \quad (II.74) \]

In the EDQNM model from Equation [II.40] we have a closed expression for \( T(k) \), using which we can perform this integration and relate \( C_d \) and \( A \). Hence, we have
\[ 8K_d \int_{0}^{\kappa} dk \int \frac{k}{p} b_{kpq}^{(d)} \theta_{kpq} \]
\[ \times \left[ \sin^{d-3}(\beta) p^2 E(q) E(k) - \sin^{d-3}(\alpha) k^2 E(p) E(q) \right] = \varepsilon . \quad (II.75) \]

We can re-express the above integral as
\[ -8K_d C_d^2 \int_{0}^{\kappa} dk \int_{[k-p]}^{k+p} dp \int \frac{k}{pq} b_{kpq}^{(d)} \theta_{kpq} \]
\[ \times \left[ \sin^{d-3}(\beta) p^2 q^{-\frac{2}{d}} k^{-\frac{2}{d}} - \sin^{d-3}(\alpha) k^2 p^{-\frac{2}{d}} q^{-\frac{2}{d}} \right] \frac{1}{\beta_d \left( k^\frac{2}{d} + p^\frac{2}{d} + q^\frac{2}{d} \right)} = 1 . \quad (II.76) \]

The two sine terms can also be expressed in terms of \( k, p \) and \( q \) as
\[ \sin(\alpha) = \sqrt{1 - \left( \frac{p^2 + q^2 - k^2}{2pq} \right)^2} \quad \text{and} \quad \sin(\beta) = \sqrt{1 - \left( \frac{k^2 + q^2 - p^2}{2kq} \right)^2} . \quad (II.77) \]

The resulting integral must be evaluated numerically and, if we denote it by \( I_d \), then we have
\[ I_d = \int_{0}^{\kappa} dk \int_{[k-p]}^{k+p} dp \int \frac{k}{pq} b_{kpq}^{(d)} \theta_{kpq} \]
\[ \times \left[ \sin^{d-3}(\beta) p^2 q^{-\frac{2}{d}} k^{-\frac{2}{d}} - \sin^{d-3}(\alpha) k^2 p^{-\frac{2}{d}} q^{-\frac{2}{d}} \right] \frac{1}{\beta_d \left( k^\frac{2}{d} + p^\frac{2}{d} + q^\frac{2}{d} \right)} , \quad (II.78) \]

and
\[ \frac{C_d^2 I_d}{\beta_d} = \frac{1}{8K_d} . \quad (II.79) \]

Then, using Equation [II.70] we can eliminate \( \beta_d \), after which we find
\[ C_d = \left( \frac{\sqrt{3}}{16K_d I_d} \right)^{\frac{2}{3}} A^\frac{2}{3} . \quad (II.80) \]
This is our desired result and after numerical evaluation of $I_d$ this can be used to fix $C_d$ in simulation.

To make this more concrete, we present the results of this procedure for the cases $d = 3$ and $4$ where we take $\kappa = 1$. For $d = 3$ the dimensional factor $K_d$, which results from performing spherical integration in $d$-dimensions, is $K_3 = 1/8$, hence, we look to solve

$$C_3 = \left(\frac{\sqrt{3}}{2I_d}\right)^{\frac{2}{3}} \lambda_1^\frac{2}{3}.$$

(II.81)

It is found that $I_3 \approx 0.19038$ and therefore for $d = 3$ we have

$$C_3 \approx 2.75 \lambda_1^{\frac{2}{3}}.$$

(II.82)

This pre-factor differs very slightly compared with [3], however this is likely a result of different methods of numerical integration. Following a similar procedure for $d = 4$ it is found that

$$C_4 \approx 2.6 \lambda_1^{\frac{2}{3}}.$$

(II.83)

### II.5.4 Resolution Requirements

The relationship between the free parameter of the EDQNM eddy-damping term and the Kolmogorov constant derived in the previous Section provides a useful method for checking the accuracy of numerical calculations. This relationship was found using a higher order numerical approximation to the triad integral. If the value for $F$ is too low, the integrals will not converge to the values above and we will obtain a different Kolmogorov constant than expected for a given value of $A$. Conversely, we can say if the value of the Kolmogorov constant we measure is consistent with the above relationship given the value of $A$ we are using, then the calculation is adequately resolved.

This is an important check to make given the nature of our approximation. Indeed, we have opted not to average rapidly varying terms and additionally we have discarded contributing volumes whose central points lie outside $\Omega$. Hence it is imperative that we ensure our approximation to the integral is still accurate. Looking at the sin $\alpha$ term in the EDQNM equation, we see that as the spatial dimension is increased, the rapid variation of this term is amplified. As such, obtaining the desired Kolmogorov constant requires increasingly high values of $F$ as $d$ increases. As we will see in Chapter IV, this places a limit on the spatial dimensions that can be feasibly studied.

Another facet of the spatial resolution in these calculations is the choice of $k_{\text{hot}}$ and $k_{\text{top}}$. For
$d \geq 3$ we have a direct cascade of energy, and as we will see in Chapter [IV] and Kolmogorov scaling in the EDQNM approximation. Hence, as we are interested in the small scales of the flow, we typically take $k_{\text{bot}} = 1$ to allow us to focus our computational effort at high $k$. We then choose $k_{\text{top}}$ large enough to capture the dissipative region, \textit{i.e.} we take $k_{\text{top}} \eta > 1$, where $\eta$ is the Kolmogorov micro-scale as defined in Chapter [I]. If we are considering two-dimensional turbulence where there is an inverse cascade of energy to large scales we will typically take $k_{\text{bot}} < 1$ to resolve the large scales, and the inverse cascade. A criterion similar to the $d \geq 3$ case can be determined for $k_{\text{top}}$, however, instead of $\eta$ we consider $\chi$, which as defined in Chapter [I] provides a measure of the dissipative scales in the direct enstrophy cascade.

We must also consider temporal resolution in this discussion. Too large a time-step and the system can become unstable. On the other hand, too small a time-step leads to excessive computational effort. For calculations of the energy spectrum evolution, we choose our time-step such that we can resolve the important timescales of the system. As turbulence is a problem of many length-scales, and thus also time-scales, we need to make a choice of which time-scale is most important. One possible choice is the time-scale associated with convective motions

$$t_c = (u k_{\text{top}})^{-1}.$$  \hfill (II.84)

Additionally, we could consider the timescale associated with the process of energy dissipation given by

$$t_d = \frac{1}{\nu k_{\text{top}}^2}.$$  \hfill (II.85)

The second of these can be determined \textit{a priori}, hence in general we choose $\Delta t < t_d$. The choice of $\Delta t$ is then coupled to the Reynolds number of the flow, with higher Reynolds numbers requiring smaller time-steps.

For studies into the predictability of turbulence, using the evolution equation for the correlation energy spectrum, we find additional constraints are placed on $\Delta t$. In these calculations it is observed that the spectrum of decorrelated energy undergoes rapid evolution in the early stages of the calculation with a timescale proportional to the inverse of the maximal Lyapunov exponent. Thus, the time-step must be chosen small enough to capture this evolution correctly. We touch upon this further in Chapter [IV], where we will see this further restricts the spatial dimensions we can study. It is likely that the implementation of a higher order time-stepping algorithm would improve the stability of the code during the early stages of predictability calculations and allow for larger time-steps to be taken. Whether this would lead to an overall performance increase is unclear as higher order schemes require additional evaluations of the triad integrals which may outweigh any speed
up from a larger time-step.

II.5.5 Parallelisation

When compared with direct numerical simulation, numerical calculations of the EDQNM equations require a low amount of computational effort. However, that is not to say it is a trivial calculation. Indeed, calculations high Reynolds numbers require a large $k_{\text{top}}$ and a small value for $\Delta t$. Furthermore, as the spatial dimension is increased, $F$ must also be increased, further increasing the computational load.

To increase the speed at which calculations can be performed, and thus increase the Reynolds numbers that can be explored, we can exploit parallel computing in our EDQNM code. The parallelisation of the EDQNM equations turns out to be a fairly straightforward process. Nevertheless, we provide details of the choices made in the code here.

Our approach makes use of the Message Passing Interface (MPI), and is thus built upon the idea of distributed memory parallelism. A shared memory approach, utilising either something like OpenMP or even general purpose graphics processing units, could be a worthwhile development, although it has not been pursued here.

We denote by $P$ the number of processors used in our calculations. We first make the restriction that the number of points, $N$, that we discretise the $k$ axis into must be a whole numbered multiple of $P$, i.e. $N = nP$ for some natural number $n$. This allows each processor to compute a subset, of size $n$, of the $E(k_i)$. However, we reach an immediate difficulty in that in the summations over $p_m$ and $q_n$ it is possible for contributions from $E(p_m)$ and/or $E(q_n)$ that are not located within the subset allocated to a given processor.

As we compute the $v_{imn}$ prior to beginning time evolution, it is simple to determine which values of $k_i, p_m$ and $q_n$ will interact, as these will have $v_{imn} \neq 0$. From this, it would in principle be possible to perform a complex set of communications between processors to store the necessary values of the energy spectrum beyond the initially allocated subset on each processor. However, it is simpler, and in fact necessary for the computation of $\theta_{k_i p_m q_n}(t_i)$, to distribute the entire energy spectrum to each processor. However, each core still only computes $E(k_i)$ for a subset of the $k_i$, thus there is still an expected performance increase following this approach.

The steps relevant to the parallelisation of the code are as follows:

1. Based on $P$ and $N$ each processor is allocated a subset of the computation of the
2. Compute the initial energy spectrum \( E(k_i, 0) \) on all processors.

3. Each processor computes the relevant subset of \( \nu_{lmn} \) factors.

4. Each processor computes the initial eddy-damping terms \( u_{kpq} \) for all \( k, p, q \).

5. The predictor step is carried out on each core for the subset of \( E(k_i) \) that has been allocated and \( E_p(k_i) \) determined.

6. The values of \( E_p(k_i, t + \Delta t) \) on each processor are sent to the other processors with the result that each processor has the entire predicted spectrum.

7. The eddy-damping factors are recomputed on each processor.

8. Each processor computes its subset of \( E(k_i, t + \Delta t) \)

9. The values of \( E(k_i, t + \Delta t) \) on each processor are sent to the other processors with the result that each processor has the entire spectrum for the next time-step.

Steps 4 through 9 are then repeated until the desired end time is reached.
The study of the chaotic properties of dynamical systems began with the pioneering work of Lorenz \[139\], exploring what is effectively a low dimensional model of the Navier-Stokes equations. These ideas were then employed by Ruelle and Takens \[193\] to describe a mechanism by which turbulence can be generated in a fluid flow. Approaching turbulence through chaos theory differs from the more standard statistical approach \[12\] in the sense that, instead of considering averaged properties of flows, we consider the properties of individual trajectories in a suitably defined state space of the system. Through such methods, a diverse range of problems in fluid dynamics have been studied including in weather and atmospheric predictability \[131, 132, 140\], as well as for the solar wind and other magneto-hydrodynamic systems \[95, 101, 128, 226\].

Starting with the pioneering studies of Leith and Kraichnan \[131, 132\], a large body of work dedicated to the study of predictability in turbulent fluid flows has formed. These initial studies made use of closure approximations in order to render the problem computationally feasible. Unfortunately, these closure models have a number of shortcomings that may reduce their ability to correctly quantify predictability in turbulence. Perhaps most notably, as
they are typically defined in terms of ensemble averaged quantities, they do not provide any information about the spatial structure of the flow which is likely to influence predictability. Additionally, while these models are well-known to give excellent agreement with the K41 theory of turbulence\cite{120}, the validity, or lack thereof, of K41 itself is still unsettled\cite{74}. As such, the applicability of the results of these models to true fluid turbulence may be limited.

As computing power increased, it became possible to perform predictability measurements in direct numerical simulations of turbulence. Since such simulations fully resolve all the relevant scales of the system in both space and time, they provide far more reliable results when compared to closures. However, such simulations come with a large computational expense, so progress has been made at a moderate rate. This is especially true for predictability studies where the computational cost is at least twice that of a standard simulation.

Historically, the majority of work regarding the predictability of fluid turbulence has centred around the measurement of the maximal Lyapunov exponent of the system. This gives a measure of the rate at which nearby trajectories in the state space diverge, and thus also provides a measure of the predictability time of the system. Due to the aforementioned computational expense, these studies began by focusing on the less demanding case of two-dimensional turbulence\cite{23,118} as well as moderate Reynolds number three-dimensional turbulence\cite{115}. More recently, a number of studies at higher Reynolds number in three dimensions have been performed\cite{19,26,155}, as well as a study into predictability in magnetohydrodynamic turbulence\cite{95}. Each of these hydrodynamical studies independently found that the maximal Lyapunov exponent scaled faster with the Reynolds number than predicted by Ruelle\cite{190} using the K41 theory. This is of particular interest, as by using the multi-fractal model\cite{17}, developed in an attempt to capture the effects of internal intermittency in turbulence, it can be shown that the maximal exponent should scale slower than predicted by Ruelle\cite{56}, opposite to what was found in DNS.

Features of two-dimensional turbulence, although not truly realizable itself, can be seen across a wide range of fluid systems. For example, in systems where one dimension is constrained compared to the others, such that the fluid exists in a thin layer, two-dimensional effects have been observed\cite{228}. Moreover, in atmospheric measurements here on Earth\cite{160} and elsewhere in the solar system\cite{237} evidence of two-dimensional phenomenology has been found. As such, understanding the predictability of two-dimensional HIT may be of more relevance to atmospheric predictability than the three-dimensional case in many situations. Consequently, in this Chapter we will study the predictability of turbulence in both dimensions.
It is also possible to study more than simply the behaviour of the maximal Lyapunov exponent. There exist as many exponents as there are degrees of freedom in the system and these are said to form a Lyapunov spectrum. However, since the calculation of each additional exponent desired comes with further computational cost, the study of the Lyapunov spectrum in fluid turbulence is still at a comparatively early stage when compared to that of the maximal exponent, although seems to be following the same path of development. Initially, studies were restricted to shell models [90, 231, 232] but soon progressed to DNS studies in two [89, 90] and three-dimensional Poiseuille flow [113], as well as a highly symmetric homogeneous and isotropic turbulence (HIT) system [218].

By measuring all of the positive Lyapunov exponents of a system, the Kolmogorov-Sinai (KS) entropy, which quantifies the rate of information production of the system and gives a more accurate quantification of predictability, can be estimated. Even at the low Reynolds numbers achieved in the studies mentioned above, the dimension of the attractor, and by extension the number of positive Lyapunov exponents, is in the order of hundreds. Hence, HIT is a distinctly high dimensional chaotic system and methods used for low dimensional systems are likely to give poor estimates of both the KS entropy and the attractor dimension [66].

Such is the computational expense in measuring the KS entropy, that only very recently, and at moderate Reynolds number, has it become possible to measure the scaling of the KS entropy in three-dimensional HIT. Owing to the reduced computational effort required to study two-dimensional turbulence, investigation into the attractor dimension scaling was first performed in the 1980s [90]. Computing power is now such that a systematic study of the chaotic properties of both two- and three-dimensional turbulence is finally possible.

In this Chapter we present a systematic study of the KS entropy and attractor dimension for homogeneous isotropic turbulence in two and three dimensions. Using DNS to study the exact evolution of the Navier-Stokes equations, we test a number of Reynolds number scaling relations derived by Ruelle [191]. As such, our results are only limited by the resolution of our simulations and the enormous computational cost of these measurements. Unfortunately, this restricts us to a limited range of Reynolds numbers, particularly in three dimensions, and thus the applicability to fully turbulent flows is not yet conclusive. These results offer one of the first numerically rigorous tests of the various mathematical relations and conjectures in the literature associated with the KS entropy and attractor dimension for the Navier-Stokes equation.

This Chapter is arranged as follows. We begin with a discussion of the numerical approach taken to obtain the results in the Chapter. This includes both the DNS algorithm as well
as the method for obtaining Lyapunov exponents. We then present theoretical predictions for the scaling of both the KS entropy and the attractor dimension for two- and three-dimensional turbulence. Next, we show results from each dimension. Lastly, we discuss the implications of these results and possible links between turbulence and other systems through information theory.

### III.1 Numerical method

In order to complete a model-independent study of the chaotic properties of incompressible two- and three-dimensional HIT, we perform DNS of the Navier-Stokes equations in two and three spatial dimensions with a large scale hypo-viscous dissipation term in the two-dimensional case

\[
\partial_t u_i + u_j \partial_j u_i = -\partial_i P + \nu \nabla^2 u_i + \mu \nabla^{-2} u_i + f_i, \\
\partial_i u_i = 0, \quad i, j = 1, 2, 3.
\] (III.1)

Here, \( u(x, t) \) is the velocity field, \( P(x, t) \) is the pressure field, \( \mu \) is the hypo-viscosity, which is set to zero in the three-dimensional case, \( f(x, t) \) is an external force that we will specify and discuss later in this Section and \( \nu \) is the viscosity. To obtain our results we have made use of the EddyBurgh code [97], a modification of that described in [236], and as such we make use of the pseudo-spectral method with full de-aliasing using the two-thirds rule. In order to ensure the flow is well resolved, we ensure that \( k_{\text{max}} / k_d \gtrsim 1.25 \) for all our simulations, where \( k_d = 1/l_d \). In [113] it was found that insufficient resolution led to an underestimation of the attractor dimension, though, by following the criteria above in this work we found such issues were avoided.

For the three-dimensional case, all our simulations are carried out in a periodic cube of side length \( 2\pi \). However, for the two-dimensional case our domain is a periodic square with side length \( x = \pi/2, \pi \) or \( 2\pi \). The motivation for varying the domain size in the two-dimensional case will be discussed in Section III.3.2. For the specific details of how the pseudo-spectral code needs to be modified to allow for variable domain sizes see the discussion in Chapter V Section V.2.

To obtain a stationary state in the two-dimensional case some form of large-scale dissipation is necessary, as otherwise the inverse cascade will eventually lead to the formation of a condensate on the scale of the system size [123]. In real world flows which show two-dimensional behavior, the large scale dissipation is given by friction between the two-dimensional flow and the three-dimensional system it is contained within. As such, friction...
terms which depend on the fluid velocity either linearly or quadratically are often used [24]. However, such terms have an effect on all scales of the flow, and given that our predictions in Equations III.12-III.17 depend on the small-scale enstrophy dissipation rate, we opt for a large-scale dissipation that effectively does not act on the small-scales. As a consequence of the large computational cost of our simulations, we have not tested how our results would be affected by the use of friction as opposed to the inverse Laplacian used here. However, any difference should be small. This inverse Laplacian term makes more sense in Fourier space where it becomes $k^{-2}$, which highlights that it most strongly influences the large length scales of the flow. To quantify the effects of this term, we use the hypo-viscous Reynolds number, $Re_{\mu}$, where

$$Re_{\mu} = \frac{u}{\mu L^3}. \quad (III.2)$$

This term is derived from the ratio of inertial to hypo-viscous forces. As such, when it is small, the hypo-viscous term is dominant at the large scales. This allows us to ensure no large scale condensate has formed, indeed, in all our two-dimensional runs we ensure $Re_{\mu}$ remains small.

Notably, we do not employ any form of hyper-viscosity in either our two or three-dimensional simulations. This is very often used in two-dimensional HIT simulations to increase the enstrophy inertial range. It is arguable that the use of hyper-viscosity is akin to that of an effective viscosity employed in methods such as large-eddy simulation, and as such acts as a form of closure. Hence, we choose to avoid any ambiguity stemming from the use of hyper-viscosity in our study.

Throughout this work, when we refer to the Reynolds number we are considering the integral-scale Reynolds number, see Chapter IV. To define this we need to first define the integral length scale, $L$, which gives the rough size of the largest eddys in the flow. The appropriate defintion of $L$ in either two or three dimensions is given in equation IV.16. The integral scale Reynolds number is then given by

$$Re = \frac{UL}{\nu}, \quad (III.3)$$

where $U$ is the RMS velocity.

### III.1.1 Forcing

Due to the presence of dissipative terms in the Navier-Stokes equations, energy must be injected into the system in order for a statistically stationary state to be achieved. In two dimensions the computational expense was low enough to perform simulations using two
different forcing functions to ensure our results are independent of the way this energy is injected. Indeed, we find the choice of forcing does not affect our results. We expect the same to hold for three-dimensional turbulence but did not test for this. The first of these functions is defined as

$$f(k, t) = \begin{cases} \frac{\epsilon}{2E_f}u(k, t) & \text{if } |k| \approx k_f, \\ 0 & \text{else}, \end{cases}$$  \hspace{1cm} (III.4)$$

where $E_f = E(k_f)$ is the energy in the forcing band and $\epsilon$ is the energy injection rate. More explicitly, the forcing acts on the ring of modes satisfying $k_f - 1/2 < |k| \leq k_f + 1/2$. This forcing has been widely used for studies of three-dimensional turbulence and allows the rate of energy injection to be held constant in time, as such this is the function used in all our three-dimensional simulations.

The second forcing employed is a delta-correlated in time stochastic force with amplitude

$$f_{amp} = \begin{cases} \sqrt{\Delta t} & \text{if } |k| \approx k_f, \\ 0 & \text{else}, \end{cases}$$  \hspace{1cm} (III.5)$$

where $\Delta t$ is the simulation time step. This choice then ensures that $\langle u \cdot f \rangle = \epsilon$, i.e. on average the energy injection is given by $\epsilon$. This method of forcing is used commonly in simulations of two-dimensional turbulence. Once again the forcing function is active only on modes which satisfy $k_f - 1/2 < |k| \leq k_f + 1/2$.

### III.1.2 Computation of the Lyapunov spectrum

In chaotic systems there exists a Lyapunov exponent for every degree of freedom. As such, it is possible to define a set of such exponents, arranged in descending order, known as the Lyapunov spectrum. This concept can be defined in more formal terms and a good account of this for the case of fluid turbulence is given in [191]. As in Chapter [ ] we take the KS entropy to be given by the sum of positive Lyapunov exponents

$$h_{KS} = \sum_{\lambda_i > 0} \lambda_i.$$  \hspace{1cm} (III.6)$$

Therefore, in each case we need to measure a number of exponents. Unfortunately, the method to compute these exponents comes with a number of computational challenges. Firstly, \textit{a priori} we do not know in advance the number of positive exponents. Secondly, each exponent requires the simultaneous integration of another velocity field, and finally,
many iterations are required to obtain averaged values for the exponents. Consequently, the
computational cost involved in computing the KS entropy for fully resolved turbulent flows
is high, and indeed we will show that even at moderate Re the number of positive exponents
is in the thousands for the three-dimensional case.

Here we summarise the algorithm put forward by Benettin [16] for measuring multiple
Lyapunov exponents. We begin by evolving a reference velocity field, \( u_0 \), until it reaches
a statistically steady state. We then make \( M \) copies of this field, labelled \( u_i \), \( i = 1 \ldots M \).
A unique small perturbation field is then applied to each copy. This perturbation field
has a Gaussian distribution with zero mean and a variance of size \( \delta_0 \), chosen such that
the perturbation may be considered infinitesimal. For each field we use the finite time
Lyapunov exponent (FTLE) method [174] and measure the growth of the difference fields
\( \delta_i(t) = u_i - u_0 \), rescaling the difference to its original size at time intervals of \( \Delta t \)

\[
\mathbf{u}_i(k, \Delta t) = \mathbf{u}_0(k, \Delta t) + \frac{\mathbf{u}_i(k, \Delta t) - \mathbf{u}_0(k, \Delta t)}{\delta_0},
\]

such that each perturbation continues to grow in the correct direction. The FTLEs are then
given by

\[
\gamma_i(\Delta t) = \frac{1}{\Delta t} \ln \left( \frac{||\delta_i(\Delta t)||}{\delta_0} \right),
\]

and the Lyapunov exponents \( \lambda_i \) are found by averaging over many iterations. Currently,
this algorithm simply measures the largest Lyapunov exponent \( M \) times, as this growth
in this direction of the phase space will dominate all others. To circumvent this issue, we
orthogonalize the \( \delta_i \) after each measurement of the \( \gamma_i \) using the modified Gram-Schmidt
algorithm. For details of how this algorithm is defined for DNS of HIT, and in particular
how the inner product is defined, see [89, 201]. If an infinite number of iterations were
performed, this algorithm would return exponents ordered such that \( \lambda_1 > \lambda_2 > \cdots > \lambda_M \).
As a result of a finite number of iterations, our spectra are not monotonically decreasing,
however, the ordering achieved is reasonable as it is in [18, 113]. This ordering property
allows us to be confident we have found all positive exponents by choosing \( M \) large enough
that a tail of negative exponents persist after averaging. This orthogonalization step scales
with \( M^2 \) and thus becomes a major bottleneck in these calculations. Additionally, we note
that, as in [18, 113], we perform this procedure in the state space of the system, as opposed
to in the tangent space as utilized in [89]. By ensuring our perturbation field is small enough,
these two methods should give consistent results.

When using the stochastic forcing function, extra care must be taken in the implementation
of this algorithm. If a new random force is generated for each of the \( M \) copy fields, then
the forcing acts as an effective perturbation every time-step and destroys the exponential

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divergence of the fields. Therefore, if a stochastic force is being used, the random force should be generated only once each time-step and then this force is applied to all fields.

### III.1.3 Sampling errors

In the computation of the Lyapunov exponents using the algorithm described in the previous Section, an average must be performed to find the value of each exponent. The mathematical definition of the Lyapunov exponent calls for an average over an infinite number of iterations of the FTLE algorithm. Of course, in practice this cannot be done and only a finite number of iterations are performed. As a consequence, sampling errors are introduced into the computed mean value of each exponent.

Such errors are further complicated in the case of turbulent fluid flow by the fact that, depending on the sampling frequency, the values obtained may be highly correlated. Typically, to avoid the complications these correlations cause, samples are taken at larger time intervals. However, given the massive numerical cost involved in the computation of many Lyapunov exponents, this is only viable for very low resolution cases.

A number of methods for computing the sampling error of correlated data have been
developed for use in DNS [102,169]. They share a common feature in that they both make use of an extension of the central limit theorem to weakly dependant variables, see [23] for details. We focus on the method detailed in [169] and use it to find the standard deviation in our measurements of the exponents, and thus the Kolmogorov-Sinai entropy. Importantly, this method lets us make use of all the possibly correlated samples we have and as such is an efficient use of our computational effort.

In Figure III.1 we show a partial Lyapunov spectrum from a $512^2$ simulation. Here, it is clear that the largest exponents take the longest time to converge, as was reported in [113]. For the lesser exponents, their error bars are smaller than the points themselves. Given that the Kolmogorov-Sinai entropy is determined by the sum of a large number of exponents, the influence of the largest exponents reduced convergence is damped by the quick convergence of the remaining exponents. As such, the largest relative errors are found in cases with very few positive exponents.

### III.2 Scaling Predictions

#### III.2.1 Three Dimensions

Theoretical predictions for the chaotic properties of HIT are typically mathematically complex. However, they can often be simplified if we ignore the effects of intermittency. This assumes the energy dissipation takes its average value everywhere in the flow, and works in the framework of Kolmogorov’s 1941 theory (K41) [120]. For the maximal Lyapunov exponent, Ruelle predicted [190] it would be determined by the shortest time-scale in the flow, which will be the Kolmogorov microscale time. In K41 this leads to a prediction of

$$\lambda_1 \sim Re^{1/2}. \quad (\text{III.9})$$

For the attractor dimension the standard scaling for K41, based on purely dimensional grounds, was given by Landau and Lifshitz [129] and gives

$$\dim(A) \sim Re^{\frac{2}{3}}. \quad (\text{III.10})$$

This result was also predicted by Ruelle assuming K41 in [191]. A number of additional mathematically derived estimates for the attractor dimension scaling have also been proposed [54, 83], which predict scalings of $Re^3$ and $Re^{\frac{11}{2}}$ respectively. There also exists a prediction which makes use of the multi-fractal model for intermittency, where the scaling
exponent is found to be slightly less than the K41 value [151].

Turning now to the KS entropy the most notable prediction was also made by Ruelle [191]. Which assuming again the K41 theory gives the scaling of the KS entropy, $h_{KS}$, in HIT

$$h_{KS} \sim \frac{1}{\tau_\eta} \left( \frac{L}{\eta} \right)^3 \sim \frac{1}{T \text{Re}^{11/4}}. \quad (\text{III.11})$$

Here, the Kolmogorov micro-scale is given by $\eta = (\nu^3/\epsilon)^{1/4}$ and the corresponding timescale by $\tau_\eta = (\nu/\epsilon)^{1/2}$ and the large eddy turnover time is defined as $T = L/U$. We note here that the KS entropy scaling is equal to the scaling of the fastest time-scale multiplied by that of the attractor dimension. For a description of the argument justifying this relationship see [199, 223].

Deviations from K41 have been well studied in the literature [4, 86, 103, 110], with both intermittency and finite Reynolds number effects claimed to be possible causes [17, 77, 119, 149]. In studies of the maximal Lyapunov exponent [19, 26] for HIT, different scaling behaviour than that predicted by both K41 [190] and the multi-fractal model [56] was observed. As such, exact agreement with this simplified prediction for the KS entropy and/or the attractor dimension would be very unexpected, and indeed our results also display a conflicting scaling behaviour in the three-dimensional case.

### III.2.2 Two Dimensions

We have just seen that for three-dimensional HIT, there exist a number of theoretical predictions for the scaling behaviour of the maximal Lyapunov exponent, attractor dimension and KS entropy. The simplest of such predictions were based on dimensional arguments and the K41 theory. These ideas can be applied in an analogous way to two-dimensional HIT where the dual cascade picture [13, 123, 133], caused by conservation of both energy and enstrophy, modifies the expected scaling behaviour.

The scaling behavior of the maximal Lyapunov exponent, $\lambda_1$, in two-dimensional HIT can be found by following the Ruelle argument [190] that on dimensional grounds it should be proportional to the inverse of the fastest timescale in the flow. In three-dimensional HIT this is the Kolmogorov timescale, corresponding to the small scales of the flow, which then suggests a scaling with the Reynolds number. On the other hand, for two-dimensional HIT, assuming the energy spectrum is $E(k) \sim k^{-3}$ in the direct cascade and therefore neglecting, for now, any logarithmic corrections, there is only one timescale throughout the entire direct enstrophy cascade. This timescale, $\tau$, is determined solely by the enstrophy dissipation rate,
\[ \eta, \text{ and is given by } \tau \sim \eta^{-1/3}. \] As such, we then have

\[ \lambda_1 \sim \frac{1}{\tau} \sim \eta^{\frac{1}{3}}. \tag{III.12} \]

Therefore, at odds with the three-dimensional case, \( \lambda_1 \) scales independently of the Reynolds number. Furthermore, this suggests that in some sense the small scales of the flow retain information about the larger scales.

Turning to the KS entropy, once again on dimensional grounds alone we can estimate the scaling behavior. In this instance, as with the three-dimensional case, the entropy should scale with the fastest timescale in the flow multiplied by the total number of excited modes. To apply this method to two-dimensional flows, we need to determine the scaling of the total number of excited modes, which is also the scaling of the attractor dimension, \( \text{dim}(A) \). We do so by considering the ratio of the largest scales in the flow, \( L \), to the smallest given by the dissipation length scale, \( \chi = (\nu^3/\eta)^{1/6} \) where \( \nu \) is the viscosity. This gives us

\[ \text{dim}(A) \sim \left( \frac{L}{\chi} \right)^2 \sim \text{Re}, \tag{III.13} \]

which in turn implies that the KS entropy, \( h_{KS} \), will scale as

\[ h_{KS} \sim \frac{1}{\tau} \text{Re} = \eta^{\frac{1}{3}} \text{Re}. \tag{III.14} \]

It has been shown by Kraichnan that in order for a constant enstrophy flux in the direct cascade of two-dimensional turbulence to exist there must be a logarithmic correction to the energy spectrum [124]. This correction will then affect the predicted scaling behavior of the various chaotic quantities we are interested in. This was considered by Ohkitani [167] and introduces an additional logarithmic dependence on the Reynolds number \( \text{Re} \) to each quantity as follows

\[ \lambda_1 \sim (\eta \log \text{Re})^{\frac{1}{3}}, \tag{III.15} \]

\[ \text{dim}(A) \sim \text{Re} (\log \text{Re})^{\frac{1}{3}}, \tag{III.16} \]

and

\[ h_{KS} \sim \eta^{\frac{1}{3}} \text{Re} (\log \text{Re})^{\frac{2}{3}}. \tag{III.17} \]

These differ from the previous predictions only by logarithmic factors and thus may be hard to distinguish in practice, as such we will not pursue these strongly here.

Finally, in the preceding Section we discussed the scaling predictions for the KS-entropy
and attractor dimension were derived for three-dimensional turbulence made by Ruelle [191]. These predictions were extended to the two-dimensional case by Lieb [138]. If the energy dissipation rate, $\varepsilon$, is taken to be constant throughout the fluid for the KS entropy, we have

$$h_{KS} \sim \frac{\varepsilon}{\nu^2} V,$$  \hspace{1cm} (III.18)

where $V$ is the volume of the system. For the attractor dimension it is found that

$$\text{dim}(A) \sim \sqrt{\frac{\varepsilon}{\nu^3}} V.$$  \hspace{1cm} (III.19)

We note here that there also exist a number of more mathematically rigorous scaling laws for some of these quantities, see for example [55]. These are typically expressed in terms of a generalized Grashof number which can be related to the Reynolds number. The dimensional scaling laws in Equations III.12-III.19 are consistent with the rigorous upper bounds in [55], thus we will focus on these simpler dimensional predictions here.

### III.3 Results

#### III.3.1 Three Dimensions

Our results span a range from $\text{Re} \approx 50$ to $\text{Re} \approx 212$. Unfortunately, the number of positive exponents scales so quickly it would take excessive computational effort to explore higher Re flows. In Section III.1 we outlined a spatial resolution criteria to ensure our simulations were adequately resolved. As such, our three-dimensional runs were performed on grids ranging from $48^3$ to $150^3$ collocation points. In doing so, we can be confident that our grid is fine enough to resolve the small scales of our flow.

##### III.3.1.1 Kolmogorov-Sinai Entropy

After non-dimensionalizing the KS entropy by multiplying by the large-eddy turnover time, we find our data is well fit by a power-law of the form $h_{KS} T = C \text{Re}^\alpha$. A plot of this scaling behaviour is shown in Figure III.2. The exponent is found to take the value $\alpha = 2.65 \pm 0.06$ with the constant given by $C = 0.0008 \pm 0.0002$.

Previous studies using shell models [90, 232] found the Re scaling behaviour for both the maximal Lyapunov exponent and the KS entropy to be the same and to follow the K41 prediction of $\text{Re}^{0.5}$ at odds with what is observed in our DNS. Meanwhile, using a multi-
Figure III.2 The main plot shows $Re$ vs $h_{KS}$ with the fit $0.0008 Re^{2.65}$ as a dashed black line. To improve clarity, errors are not shown on this log-log scale.

Affine field [27], it is shown that the space-time entropy of the entire velocity field should scale according to the simplified prediction we are considering here. The reason for the large discrepancy between these results is well elucidated in [223], where on dimensional grounds they find the shell model result corresponds to measurement of the velocity at a single point, whilst the Ruelle result is for the KS entropy of the entire field. Since in our DNS we study the Lyapunov exponents for the entire velocity field, the Ruelle entropy result is the appropriate prediction for this work.

Our data shows the Re scaling exponent is less than expected when following the simplified prediction, although only slightly. Intermittency is often invoked to explain such differences, however, the multi-fractal model predicted deviations [8, 57] in the opposite direction than that observed in DNS [19, 26] for the related largest Lyapunov exponent. As such, it is hard to make the claim that this is an adequate explanation for our results, although we cannot rule it out definitively. Another possible interpretation is that the discrepancy is due to finite Re effects. This is an attractive hypothesis, considering the modest Re values achieved, and indeed the K41 scaling is predicated on the existence of an inertial range, which will be limited in these simulations.
Lyapunov spectra for $\text{Re} = 90(\triangle), 123(\circ), 153(\square)$ and $212(\times)$ rescaled by $h_{KS}$ and the number of positive exponents, $N$. Using this normalization, Lyapunov spectra measured at different $\text{Re}$ are shown to collapse onto the same curve, indicating a scaling property.
III.3.1.2 Lyapunov Spectrum and Attractor Dimension

Beyond the Re scaling of the KS entropy there are also conjectures related to the shape of the Lyapunov spectrum for fluid turbulence. The main question raised is related to the distribution of exponents about $\lambda \approx 0$, and whether or not it diverges around this point [192]. In the $\beta$-model of turbulence, which attempts to extend the K41 theory to account for intermittency, the spectrum of exponents can become singular at $\lambda \approx 0$ [191]. This simple fractal model has since been superseded by the multi-fractal model, but to our knowledge there does not exist a prediction for the distribution using this theory. Measurements of the spectrum in shell models seemed to confirm this divergence [232], however, this was later suspected to be the result of the discretization in wave-number space [233], which is also present in the $\beta$-model.

For the range of Re tested in our DNS results there is no singularity in the Lyapunov spectra at $\lambda \approx 0$. To demonstrate this, we will need to obtain an estimate for the dimension of the attractor underlying the flow at each Re. Firstly, in order to make a sufficient estimate, we relate the attractor dimension to the Lyapunov exponents via the Kaplan-Yorke conjecture...
To do so, we require the integer \( j \) such that

\[
\sum_{i=0}^{j} \lambda_i \geq 0, \quad \text{and} \quad \sum_{i=0}^{j+1} \lambda_i < 0,
\]

(III.20)

and the attractor dimension is then

\[
\dim(A) = j + \frac{\sum_{i=0}^{j} \lambda_i}{|\lambda_{j+1}|}.
\]

(III.21)

Clearly, this requires far more exponents to be obtained than for the KS entropy calculation, meaning the computational cost becomes unfeasible at relatively low Re. Secondly, following [113], we can identify an Re scaling behaviour for the shape of the Lyapunov spectra in the region of positive exponents, which allows us to estimate the dimension of the higher Re cases. This can be seen in Figure [III.3] and if we consider the Re as a control parameter for the system, then this scaling is also seen in other systems of differential equations [112, 143]. Due to the spectra exhibiting a universal shape upon rescaling, we can take the attractor dimension to be proportional to the number of positive exponents. Explicitly, if we have one case where we know the attractor dimension, \( D_1 \), for a given Re value which has \( N_1 \) positive exponents, then for any other Re we can estimate the attractor dimension as \( D_i \approx (N_i / N_1) D_1 \), even if we only know \( N_i \).

For a small subset of cases at lower Re, it is computationally feasible to measure enough exponents to make use of the Kaplan-Yorke dimension formula. Using these directly measured dimensions, we then estimate the attractor dimension for the higher Re cases. We find that in our \( \text{Re} \approx 212 \) simulation the attractor dimension is \( D \approx 5704 \). This result highlights that HIT is in a class of extremely high dimensional systems, even when compared to other chaotic fluid systems, such as Rayleigh-Bernard convection [229].

Finally, in Figure [III.4] we present a different method of normalizing the spectra, this time using the estimated dimensions rather than the total number of positive exponents. This normalization was used in [232] to provide evidence of a divergence in the Lyapunov spectra around \( \lambda \approx 0 \) for a shell model. Therefore, as we set out to demonstrate for the case of the full evolution of the Navier-Stokes equations, as studied in our DNS, Figure [III.4] shows no such divergence at this point across a range of different Re values.

The collapse of the Lyapunov spectra for different values of Re shown in Figure [III.4] suggests it may be possible to relate the shape of Lyapunov spectrum to more traditional measures of scaling such as the energy spectrum. Indeed, this was considered in [234] in a shell model of fully developed turbulence. The results presented in that work rely on the localisation of the Lyapunov exponents in wave-number space. If this is shown for direct numerical
simulation results then it their results suggest that the positive Lyapunov exponents should correspond to inverse timescales in the inertial range and a power-law should be obtained for the density distribution function of the Lyapunov spectrum. To test this prediction requires far more exponents than can be obtained with current computing power, however it should be the subject of future work.

We now turn to the scaling behaviour for the number of positive exponents, $N$, with $Re$. Due to the shape similarity of the Lyapunov spectra, this scaling is also indicative of that for the attractor dimension, and hence the number of active degrees of freedom in the flow. Remarkably, the estimate made on dimensional grounds is closest to what is observed in our DNS data, where we find the best fit to be of the form $N = bRe^\gamma$ with $\gamma = 2.35 \pm 0.05$ and $b = 0.008 \pm 0.002$, as can be seen in Figure III.5. Thus, our data suggests that the number of positive exponents, and therefore the attractor dimension, grows with $Re$ at a rate slightly faster than the Landau estimate. This result is in line with the findings in [19, 26] concerning the largest Lyapunov exponent, whereby the correction to the K41 result is in the opposite direction when compared to the multi-fractal model value. As such, this suggests that if intermittency corrections are responsible for the deviations for K41 scaling seen here, then a different intermittency model, which captures the behaviour seen in our results, is needed.
If we consider that the number of operations in a DNS grows as \( \text{Re}^3 \) then the scaling of \( N \) suggests that, as a lower bound, upon doubling \( \text{Re} \), \( 2^{5.35} \approx 40 \) times more operations are needed to measure the entropy. Thus, at present, the results in this study are on the limit of what is computationally possible in three-dimensions. The rapid growth of \( N \) is not surprising if we think of it as being proportional to the active degrees of freedom in the flow. As the Reynolds number is increased more and more small scale modes become active.

We have seen that the relative sizes of the smallest structures in the flow when compared to the largest then gives the Landau estimate for the scaling of \( N \). From a computational viewpoint we may be concerned that our numerical resolution may influence the value of \( N \) we measure. Indeed, this can be the case if a simulation is under-resolved. If this is the case the dissipative dynamics of the system will not be fully captured leading to incorrect values for the number of positive and negative exponents. The resolution requirements used in this work appear to be sufficient to correctly estimate the value of \( N \). This was tested by performing a number of simulations at a given value of \( \text{Re} \) whilst varying the resolution. Here it was found that the values of \( N \) and for \( h_{KS} \) converged as the resolution was increased.

### III.3.2 Two Dimensions

#### III.3.2.1 Maximal Lyapunov Exponent

As discussed in Section III.2.2, depending on the form of the energy spectrum in the direct enstrophy cascade region, there exist two possible scaling predictions for the maximal Lyapunov exponent, \( \lambda_1 \). The first of these predictions is valid if the energy spectrum takes the form \( E(k) \sim k^{-3} \) and is given by Equation III.12; notably, this prediction has no \( \text{Re} \) dependence and is determined solely by the enstrophy dissipation rate, \( \eta \). We note here that in our simulations, given the high computational demands imposed by the computation of a large number of Lyapunov exponents, we only achieve modest resolution. To illustrate this, in Figure III.6 we show the energy spectra from the highest resolution simulation in our data-set. It is clear that the spectrum in the enstrophy cascade region is steeper than \( k^{-3} \). This is not surprising given the resolution achieved, and has been observed in previous studies [130, 168]. In Figure III.7 we show \( \lambda_1 \) against \( \eta \) and we find our data is well fit by a power-law of the form \( \lambda_1 = a\eta^{1/3} \), with \( a = 0.42 \pm 0.01 \). The calculation of the maximal exponent only requires the simultaneous integration of two velocity fields and is much less resource intensive when compared to the calculation of the entropy and attractor dimension. As such, by computing the maximal exponent in separate simulations, the number of samples is far larger, \( N_s \approx 5,000 \) in all cases, leading to small error.
Figure III.6  *Energy spectrum from a 512² simulation with* $k_f = 7$ *and* $k_{\text{min}} = 1$. *Dashed line shows* $k^{-3}$ *scaling.*

Figure III.7 *Plot of the enstrophy dissipation rate,* $\eta$, *against the largest Lyapunov exponent,* $\lambda_1$. *Dashed line shows the fit* $0.42 \eta^{1/3}$. 
It was suggested by Kraichnan [124] that in order for the enstrophy flux to be constant in the direct cascade inertial range, then there should be a logarithmic correction to the energy spectrum. This alters the scaling prediction of Equation III.12 to that of Equation III.15 and introduces a dependence on Re. To test this we plot in Figure III.8 the product $\lambda_1 \tau$, which, if Equation III.12 is correct, should be constant for all Re, against the Reynolds number. In doing so, we find $\lambda_1 \tau$ slowly increases with Re, with the data being well fit by a power-law of the form $\lambda_1 \tau = \beta Re^\gamma$, where $\beta = 0.16 \pm 0.02$ and $\gamma = 0.16 \pm 0.02$.

To properly test the logarithmic correction suggested by Kraichnan requires a massive range of Reynolds numbers to be considered. The cost of such computations makes testing this relationship prohibitively expensive. To illustrate the issue we show in Figure III.9 a fit of the product $\lambda_1 \tau$ against log(Re). Here we find the exponent to be nearly unity, three times bigger than predicted. However, it seems likely that the result is being heavily skewed by the low Re cases. Given the numerous measurements of the logarithmic correction in the literature it seems most likely that this is the explanation for this discrepancy.
We turn now to the KS entropy for forced two-dimensional turbulence. Once again, in Section III.2.2 we presented two scaling predictions derived via dimensional arguments. The first Equation III.13 is derived for the case where there are no logarithmic corrections to the energy spectrum, whilst Equation III.16 is valid with these corrections. Both cases have a dependence on the enstrophy dissipation time and the Reynolds number, with the corrected prediction introducing an additional logarithmic dependence on Re. Due to the cost of computing the KS entropy scaling quickly with Re (see [18] for the three-dimensional case which is more severe but illustrative), our results likely will not be able to quantify this logarithmic dependence, so we will focus on the prediction of Equation III.12.

Upon testing the prediction of Equation III.13 against our data we find there is no scaling behaviour and the value of $h_{\text{KS}} \tau$ varies by orders of magnitude for the same value of Re. Within this data-set there are a range of different values used for the forcing length scale $k_f$, as well as three different physical box side lengths. If we fix the box side length at $2\pi$, we find what has the appearance of three separate close to parallel lines, one corresponding to each value of $k_f$. This shows there is some form of scaling with the integral scale Reynolds number, but that this is not the full picture. In [90] the dimension of the attractor in two-
dimensional HIT was found to be dependent on \( k_f \), however the exact dependence was not investigated. As such, the fact that our results for the entropy also show a \( k_f \) dependence is not overly surprising, despite being at odds with Equation III.13.

In order to correct the prediction in Equation III.13 to account for this forcing scale dependence, we will consider what was found in [89], where it was observed that the attractor dimension grew at the same rate as the number of modes in the inverse energy cascade inertial range. Using this a reasonable ansatz for the correction factor, \( C \), is

\[
C \sim \left( \frac{k_f}{k_{\text{min}}} \right)^2, \tag{III.22}
\]

where \( k_{\text{min}} \) is determined by the side length, \( x \), of the box our fluid resides within using

\[
k_{\text{min}} = \frac{2\pi}{x}. \tag{III.23}
\]

The typical choice in simulations is \( x = 2\pi \) restricting the allowed wavenumbers to integer values. By choosing \( x = \pi \) and \( x = \pi/2 \) we have \( k_{\text{min}} = 2 \) and \( k_{\text{min}} = 4 \) respectively. Since energy is injected at \( k_f \), then a natural lower bound for the inverse cascade is \( k_{\text{min}} \), and thus \( C \) has the desired scaling behavior. We then consider a corrected scaling prediction for the KS entropy of the form

\[
h_{\text{KS}} \tau \left( \frac{k_{\text{min}}}{k_f} \right)^2 \sim \text{Re}. \tag{III.24}
\]

Using this new scaling prediction, our data is shown in Figure III.10. It can be seen that all points fall on a straight line, thus indicating a power-law scaling. We find the data is well fit by a power-law of the form \( h_{\text{KS}} \tau \left( k_{\text{min}}/k_f \right)^2 = a\text{Re}^b \), with \( a = 0.0018 \pm 0.0005 \) and \( b = 0.9 \pm 0.03 \). There is a notable spread in this data which we do not believe to be solely the result of sampling errors, which are reasonably small in general. Instead, this spread is likely explained by the use of the correction factor \( C \) defined in Equation III.22. This factor does not contain any information regarding the structure of the underlying flow and is merely a ratio of length scales. However, without the use of such a term, no scaling at all is found for simulations with differing \( k_f \) and \( k_{\text{min}} \) values. It is likely a more flow specific correction can be found, but we do not pursue that here.

We now turn our attention to the scaling prediction given in Equation III.18. It is interesting that this prediction must also be corrected by \( C \), or else the same issue of different scaling behaviours for each value of \( k_f \) and \( k_{\text{min}} \) appears once more. We show this in Figure III.11 in which we have non-dimensionalized the entropy using \( \sqrt{\nu/\varepsilon} \). The spread in the data here is more pronounced than for the simpler scaling of Equation III.1 once more this is a combination of small sampling errors and the use of the correction factor \( C \), the effect may
Figure III.10  Plot of Re against the Kolmogorov-Sinai entropy $h_{KS}$ scaled by the enstrophy dissipation time-scale and the ratio of $k_f$ to $k_{min}$, fit corresponds to $0.0018Re^{0.9}$. The values of $(k_{min},k_f)$ are varied and displayed as $(1,3)$ red ($\times$), $(1,5)$ red ($\Box$), $(1,7)$ red ($\circ$), $(1,9)$ red ($\triangle$), $(1,11)$ red ($\triangledown$), $(2,4)$ blue ($\bigcirc$), $(2,6)$ blue ($\bigtriangleup$), $(4,8)$ black ($\Box$) and $(4,12)$ black ($\circ$).
be exacerbated in this case by the logarithmic scale and small values on the y-axis.

Our data thus shows that the scaling of the Kolmogorov-Sinai entropy for two-dimensional turbulence exhibits a dependence on both the forcing length scale and the system size. This is very much at odds with the picture in three-dimensional turbulence, where we found the the scaling of the entropy is very close to that predicted in the K41 theory, depending only on small scale features of the flow. This is perhaps best explained by considering the nature of the triadic interactions in two-dimensional turbulence. In [142], it was shown that non-local triad interactions have an important effect on both the energy and enstrophy inertial ranges. When viewed in physical space this manifests itself in the appearance of long-lived coherent vortices, which then influence the small scales. As such, the fact that in two dimensions the large scales have a direct effect on the chaotic properties of the flow should not come as a surprise.

Figure III.11  Plot of the scaling prediction for the Kolmogorov-Sinai entropy given by Ruelle and Lieb. The values of \((k_{\text{min}}, k_f)\) are varied and displayed as \((1, 3)\) red \((\times)\), \((1, 5)\) red \((\square)\), \((1, 7)\) red \((\odot)\), \((1, 9)\) red \((\vartriangle)\), \((1, 11)\) red \((\triangledown)\), \((2, 4)\) blue \((\odot)\), \((2, 6)\) blue \((\odot)\), \((4, 8)\) black \((\square)\) and \((4, 12)\) black \((\odot)\).
III.3.2.3 Attractor Dimension

By computing a large enough subset of the Lyapunov spectrum, it is also possible to make an estimate of the dimension of the attractor for forced two-dimensional HIT. To do so, we again make use of the Kaplan-Yorke conjecture seen in Equation III.21. This definition makes quantifying the effect of the error in the Lyapunov exponents on the attractor dimension difficult. Any fluctuation in values of the exponents will affect the value of $j$ in a complex manner. As such, we do not attempt to compute the error here.

Naturally, the numerical computation of the attractor dimension comes at a severe computational cost. However, as with the entropy, when compared to the three-dimensional case, the calculation for the attractor dimension is more favourable and a reasonable measurement of the scaling behaviour can be made. The results of this measurement are displayed in Figure III.12 in which we have plotted against Re the scaling prediction of Equation III.13 corrected by the scaling factor $C$ described previously. Upon doing so, we find the data is well fit by a power-law of the form $\dim(A) \left( \frac{k_{\text{min}}}{k_f} \right)^2 = c Re^d$ with $c = 0.055 \pm 0.02$ and $d = 0.78 \pm 0.04$. As with the entropy, we also find that when considering the Ruelle-Lieb prediction of Equation III.19, the correction factor is once again necessary and this is demonstrated in Figure III.13. Although the scatter is less in these Figures, it is still present. This is again a result of a combination of sampling error and the use of the corrective term $C$. Notably, it is clear that data points with either low Re or higher $k_{\text{min}}$, which have the fewest positive exponents, show the largest spread.

The attractor dimension gives a measure of the total number of active degrees of freedom in the flow. In [89], the attractor dimension was found to grow with the width of the energy inertial range. Our results are in agreement with this finding, although we also find a contribution from the enstrophy inertial range due to the dependence on the ratio of large to small scales in the flow measured by Re.

III.3.2.4 Lyapunov spectrum

It is also of interest to investigate the shape of the Lyapunov spectrum, in particular, the distribution of exponents about $\lambda \approx 0$. It was suggested by Ruelle [191, 192] that it may be possible for the distribution of exponents to become singular about this point. Using the GOY shell model [231, 232] it was found that in both two and three dimensions the distribution of exponents did indeed become singular. However, it was later suggested that this divergence was caused by the numerical discretization employed in these works [233].
Figure III.12  Plot of $Re$ against the attractor dimension, $\dim(A)$ scaled by the ratio of $k_f$ to $k_{min}$. The fit corresponds to $0.055Re^{0.78}$. The values of $(k_{min}, k_f)$ are varied and displayed as (1, 3) red ($\times$), (1, 5) red (□), (1, 7) red (○), (1, 9) red (Δ), (1, 11) red (□), (2, 4) blue (○), (2, 6) blue (◇), (4, 8) black (□) and (4, 12) black (○).
Figure III.13  Plot of the attractor dimension scaling prediction given by Ruelle and Lieb. The values of $(k_{\text{min}}, k_f)$ are varied and displayed as $(1, 3)$ red ($\times$), $(1, 5)$ red (□), $(1, 7)$ red (○), $(1, 9)$ red (△), $(1, 11)$ red (□), $(2, 4)$ blue (○), $(2, 6)$ blue (□), $(4, 8)$ black (□) and $(4, 12)$ black (○).
Figure III.14  Lyapunov spectra normalized by $h_{KS}$ and $\text{dim}(A)$. The results of a number of simulations are shown here to highlight the similarity property of the spectra.

In Figure III.14 we show the Lyapunov spectra from a number of our simulations scaled by both their Kolmogorov-Sinai entropies and the attractor dimensions, such that the spectra collapse onto a single curve. From this Figure, it is clear there is no divergence around $\lambda \approx 0$ in our simulations. This is consistent with what has been found for three-dimensional turbulence earlier in this Chapter as well as in [92, 113], although it should be noted that in [92] a ‘knee’-like structure was found around $\lambda \approx 0$, which is also seen in simulations of Rayleigh-Bénard convection [229]. This structure does not, however, appear to be a true divergence.

III.4 Conclusions and Final Remarks

This Chapter has been focused on the calculation of a number of standard measures of chaos in two- and three- dimensional forced incompressible HIT using pseudo-spectral DNS. A number of scaling predictions for these quantities have been made in the literature and, using our numerical results, we have tested a subset of them.

In three dimensions, we have shown that in HIT the KS entropy exhibits a scaling law of the form $h_{KS} T \sim \text{Re}^{2.65}$ and this exponent is less than that predicted using the K41
theory. Further to this, Ruelle raised a question concerning the distribution of Lyapunov exponents in fluid turbulence and a possible divergence around $\lambda \approx 0$. Our results demonstrate that the Lyapunov spectrum for the incompressible periodic Navier-Stokes equations at the Re studied do not exhibit a divergence at this point. Moreover, we identify a Reynolds number independent shape of the spectra, allowing us to conjecture that there is no divergence at any Re. Additionally, we have also investigated the scaling of the number of positive Lyapunov exponents with Re. This, coupled with the Re independent spectra shape, allows us to also estimate the scaling behaviour of the attractor dimension. We found that this quantity scales faster than is predicted using K41 physical arguments, and the opposite of what is predicted if intermittency is accounted for via the multi-fractal model. Caution should however be applied given the low Re values that could be studied via this method using current computing power. The nature of these results may exhibit differences to flows at higher Re which will be of interest to study when the requisite computational resources become available.

Looking now at the two-dimensional case, it was found that the maximal exponent displays a weak dependence on the Reynolds number of the flow, consistent with the logarithmic correction to the energy spectrum suggested by Kraichnan. However, it was seen that for the Kolmogorov-Sinai entropy and attractor dimension, the predictions made on dimensional arguments were not sufficient. Corrections relating to the forcing length scale and system size were then found to be required. It is suggested that these corrections are required due to non-local effects in two-dimensional turbulence as a result of coherent vortices. It should be noted that we have focused on this work on the enstrophy cascade region due to all of our scaling predictions being related to the enstrophy dissipation time scale. However given the form of the corrections needed for the entropy and attractor dimension scaling, it is possible that allowing for an inverse cascade to develop will change the predictability picture. This should be addressed in future work.

When comparing results from each dimension, it is found that these chaotic properties scale with Re far more slowly in two-dimensional turbulence. Furthermore, since these chaotic properties depend on large scale details of the flow in two dimensions, as opposed to only on small scale features in three dimensions, they provide further evidence of non-universality in two-dimensional turbulence. Given the two-dimensional phenomenology seen in the atmospheres of the Earth and Jupiter, this may have important implications for atmospheric predictability. In reality, this two-dimensional phenomenology is not the entire story, as such systems are likely more accurately described by thin layer turbulence. In thin layer turbulence, it is found that there are a number of critical points where the system transitions from purely three-dimensional behaviour to coexisting two-
and three-dimensional phenomenology, and then from this state to purely two-dimensional turbulence [15]. It is, in fact, not just thin layer systems in which this kind of behaviour is seen. Indeed, in systems undergoing rotation, as well as stratification and influence from an external magnetic field, a similar transition from three-dimensional to two-dimensional behaviour is seen [197, 203, 204]. The predictability of such systems is studied in detail in Chapter V.

Such is the complexity of atmospheric systems, that even all of the variants discussed above only begin to scratch the surface. As such, simplified models which approximate the true dynamics of the atmosphere are often used. These models also exhibit sensitivity to initial conditions and, in some cases, Lyapunov spectra have been measured [219]. In one such case in a coupled atmosphere-ocean model [220], a large number of near zero Lyapunov exponents are found, suggesting a possible divergence in the spectra. Whether this divergence is simply a feature of the simplified model or of the true dynamics is an interesting question, however, given the computation cost of even the simple case studied in this work, its answer is likely some way off.

It may be of additional interest to compare the entropy scaling results presented in this Chapter to that seen in condensed matter [159] and quantum field theoretic systems [21, 91]. Indeed, if a similar scaling with a suitably defined control parameter is observed, then through Ornstein's isomorphism theorem [170] there may exist connections at the level of information production between these seemingly disparate systems. This approach strips away the qualitative features that may distinguish different systems of complexity and reduces them all to a common set of quantitative measures that can be systematically cross compared. Such an approach could be beneficial in utilising theoretical knowledge for one type of system to understand another with similar information theoretic content, and likewise for transfer of applications between such related systems. To develop such a program, a key step is to have accurate measurements of the Lyapunov spectra and KS entropy for a diverse set of complex systems.

Through the use of numerical experiment, the results of this Chapter extend the understanding of the links between Eulerian turbulence and deterministic chaos. By utilizing the KS entropy as a measure of information production in HIT, it may also be possible to make a connection from HIT to information theory and related areas such as algorithmic complexity [37]. Further to this, through the work of Ornstein [170], all systems satisfying certain technical conditions with the same KS entropy are isomorphic to each other. Following this line of argument, it may then be possible that other physical systems and HIT are in fact related at the level of information production. The results presented in this paper may then be relevant to a wide range of chaotic systems. In particular, the idea that other
strongly coupled systems may be connected to HIT is already being explored through the Anti-de Sitter space conformal field theory (AdS/CFT) fluid-gravity correspondence [185], which may have a counterpart in information theory.
CHAPTER IV

TURBULENCE BEYOND THREE DIMENSIONS

IV.1 Background and Motivation

As discussed in previous Chapters, much of the progress in developing our understanding of turbulence can be traced to the work of Kolmogorov and his three 1941 papers [117, 118, 120], in which the K41 theory was first described. The results in these papers were derived for the idealised case of HIT, however they are also remarkably applicable to real world flows under certain conditions. One of the most important predictions of the K41 theory, valid at sufficiently high Reynolds number, is the existence of a range of intermediate sized eddies in the flow referred to as the inertial range, characterised by scale invariance and a constant energy flux. This scale invariance manifests itself clearly in the power-law form of the K41 energy spectrum in the inertial range

\[ E(k) = C \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}} , \]  

where \( \varepsilon \) is the constant energy flux, which will be equal to the rate of viscous energy dissipation, and \( C \) is a universal constant.

The scale invariance of the inertial range is reminiscent of that seen in critical phenomena.
close to the critical point. Following this line of argument, there have been numerous analogies comparing turbulence to critical phenomena [1,36,60,78,85,161,230]. A salient feature of many such critical systems is the existence of an upper critical dimension, above which fluctuations are suppressed and the mean field theory values for critical exponents become exact. These ideas have their roots in the work of Ginzburg [84] as well as that of Wilson in the application of renormalisation group methods to critical phenomena [224,225]. For turbulence, a case can be made that the K41 theory, since it uses the mean energy flux in the form of the inertial range energy spectrum, is in fact a kind of mean field theory itself [14,200].

This interpretation of K41 is interesting in light of the measurement of deviations from the exponents predicted by K41 for both the energy spectrum and the structure functions. Both intermittency and finite Reynolds number effects have been theorised as being responsible for such deviations [17,77,119,149], which leads to Equation IV.1 being re-expressed in the form

\[ E(k) \propto k^{-5/3-\mu}. \]  

From here an analogy can be drawn once more to critical phenomena, in which a similar anomalous exponent, which vanishes for mean field theory, is seen when looking at two point correlation functions. Naturally, this has led to speculation about whether an upper critical dimension for turbulence exists and what its associated value would be [136,137,163,188]. Results from a recent study [20] which performed direct numerical simulation of four spatial dimensional HIT found, amongst other results, in particular a suppression of energy fluctuations in going from three to four dimensions, which thus have raised further interesting questions relating to a critical dimension in turbulence. There have also been claims related to, and a small number of studies investigating, the possibility of simplification in infinite dimensions [72,73,121].

Closure approximations have their roots in quantum field theory (QFT). Initially work in employing QFT methods to the turbulence problem was pioneered by Kraichnan [122], Wyld [227] and Edwards [67] to develop a perturbation theory for the Navier-Stokes equations, and this subsequently led to various approximation schemes. It is interesting to note that QFT is also a subject in which the behaviour of systems in different dimensions has been an area of sustained interest in systems including string theory, gauge theory, and ADS-CFT. Thus aside from the tools for calculation this area has helped developed for the field of fluid turbulence, there is also relevance in appreciating this conceptual point and thus in placing more focus on understanding the dimensional behaviour of fluid turbulence.

The EDQNM closure scheme has seen widespread use in both two and three-dimensional
turbulence (see Chapter II for an introduction to the EDQNM approximation and see lesieur1987turbulence for a more in-depth review), where it has produced numerous qualitative results. The EDQNM approximation allows for the investigation of very high Reynolds number flows at relatively low computational cost and has the added benefit that extension to any dimension incurs little additional computational expense when compared with DNS. The EDQNM closure is compatible with the Kolmogorov energy spectrum and is well suited for the study of energy transfer in isotropic turbulence. Additionally, it was noted by Orszag [172] that the quasi-normal approximation is analogous to the random phase approximation of many-body physics, which is also closely linked to the Gaussian approximation. Therefore, if higher dimensional turbulence shows a systematic improvement in agreement with the EDQNM approximation, it may in its own right be an indicator towards a simplification in the turbulent dynamics.

If forward energy transfer does indeed become stronger with higher spatial dimension this may result in an increased bottleneck effect [68]. This effect manifests itself as a pile up of energy in the near dissipative range of the flow and has been observed both experimentally [152, 194] and numerically [114]. It has been suggested by Herring et al. [93] that this effect is a result of viscosity suppressing the non-linear transfer of energy to the smallest scales. Hence, by varying the spatial dimension of the system it is possible to investigate these claims of viscous energy transfer suppression.

In this Chapter our focus is on higher dimensions, studying turbulence between three and twenty spatial dimensions. Before tackling predictability in higher dimensional turbulence we will present some more standard statistical results which will help to guide our interpretation of our predictability study. We have seen already in this Thesis that the chaotic properties of turbulence have been shown to be highly dependent on the spatial dimension for the two- and three-dimensional cases. As such, we may expect to see some influence on the chaotic properties based on the effects seen as the dimension is increased. Additionally, any changes to the chaotic properties may help shed light on the nature of turbulence in higher dimensions.

The Chapter is organised as follows: we begin with an overview of the state of numerical studies into turbulence in higher spatial dimensions. Following this, we recall some details of the EDQNM approximation and expand upon how it may be used to quantify the predictability of turbulence. Next, we present a large number of results from our EDQNM calculation from both statistical and predictability measurements. Finally, we discuss the implications of our results, focusing mainly on the energy cascade and vortex stretching.
IV.2 Numerical Studies Beyond Three Dimensions

Given the analytical challenges turbulence poses, the use of numerical methods, in particular DNS, has been invaluable. However, only recently has it become possible to perform DNS of turbulence in greater than three dimensions. These studies began with the work of \[87, 207, 235\] looking at DNS of decaying turbulence in four and five spatial dimensions. Due to the extreme computational cost of such simulations, the highest resolution achieved in four dimensions was \(256^4\) collocation points. Before discussing the main results of these works, we wish to mention the need for caution in the interpretation of results from studies of free decay at modest resolution. In \[181\] it was suggested that a true inertial range will not be observed until the Taylor Reynolds number, \(Re_\lambda\), is greater than approximately two thousand. We will present a definition of the Reynolds number in Section IV.3. This is an important point because, even in the highest resolution DNS of three-dimensional turbulence \[105\] performed to date, with \(16,384^3\) collocation points, this value has not been achieved. Furthermore, it can be seen \[31\] that free decay of turbulence requires even higher values for an appreciable inertial range. As such, there will have been no true inertial range in these higher dimensional DNS studies, making any definitive conclusion about K41 impossible at this point.

In any case, in \[87\] the intermittency of the dissipation rate is studied in several ways. Most notably, they looked at the total dissipation and a surrogate dissipation often used in experiment. They found reduced intermittency for the total dissipation alongside increased intermittency of the surrogate. However, the reduction of intermittency in the total dissipation was a larger effect than the increase in the surrogate. However, as discussed in Chapter I, it is in fact the fluctuations of the energy transfer rate that are relevant for the validity of K41, and this quantity was not measured in \[87\]. Measurements were also made of the longitudinal structure functions in four dimensions, where it was found that deviations were larger when compared with three dimensions. However, as the authors note, there is no inertial range present in their simulations, hence this finding cannot be considered conclusive. Indeed, it is also conceivable that the finite Reynolds number effect in four dimensions differs in strength from that in three dimensions, further complicating the interpretation of the data found in \[87\].

A more recent study \[20\] performed DNS of stationary four-dimensional turbulence at an increased resolution of up to \(512^4\) collocation points. Of course, even at this level of resolution, no inertial range can be found, so no attempt to measure anomalous scaling was made in this work. Instead, this work focused on the role of fluctuations by considering the variance of the total energy at stationary state alongside measurements.
of the velocity derivative skewness and dimensionless dissipation rate. When considering these fluctuations there is a dramatic reduction when going from three to four dimensions of approximately an order of magnitude at \( \text{Re}_L \approx 1000 \), suggesting a possible simplification of the dynamics. The increase from three to four dimensions is not enough for arguments based on the central limit theorem to explain the reduction, although of course this will become a factor at higher dimensions. Once more, at the resolution available, caution should be taken in the interpretation of this result, however, how these fluctuations behave with increasing dimension may be of interest considering the analogies between turbulence and critical phenomena.

IV.3 Theory

In the following, for brevity we will drop the explicit time dependence of the velocity field. For our purposes, we are primarily interested in the second and third order two point velocity correlations, which are given by

\[
C_{\alpha\beta}(r) = \langle u_\alpha(x) u_\beta(x + r) \rangle, \\
C_{\alpha\beta\gamma}(r) = \langle u_\alpha(x) u_\beta(x) u_\gamma(x + r) \rangle,
\] (IV.3)

with \( \alpha, \beta, \gamma = 1, \ldots, d \), higher order correlations are given by analogous expressions. More specifically, we will be focussed on the second and third order longitudinal correlations defined as

\[
C_{LL} = \frac{r_\alpha r_\beta}{r^2} C_{\alpha\beta}(r) = u^2 f(r), \\
C_{LLL} = \frac{r_\alpha r_\beta r_\gamma}{r^3} C_{\alpha\beta\gamma}(r) = u^3 K(r),
\] (IV.4)

where \( f(r) \) and \( K(r) \) are scalar correlation functions and \( u \) is the RMS velocity. These functions are intimately related to the longitudinal structure functions of the same order. We recall here the Fourier transform of the Navier-Stokes equation derived in Chapter[1]

\[
(\partial_t + v k^2) u_\alpha(k) = \frac{1}{2i} P_{\alpha\beta\gamma}(k) \int dp \ u_\beta(p) u_\gamma(k - p),
\] (IV.5)

where \( P_{\alpha\beta\gamma}(k) = \delta_{\alpha\beta} P_{\alpha\gamma}(k) + \delta_{\gamma\beta} P_{\alpha\beta}(k) \) is the inertial transfer operator and \( P_{\alpha\beta}(k) = \delta_{\alpha\beta} - k_i k_j / k^2 \) is the projection operator which imposes the incompressibility condition. Homogeneity requires that the corresponding second order velocity correlation in Fourier space takes the form

\[
C_{\alpha\beta}(k) = \langle u_\alpha(k) u_\beta(-k) \rangle.
\] (IV.6)
It can be shown that this correlator is related to the energy spectrum, $E(k)$, through

$$\langle u_a(k) u_\beta(-k) \rangle = \frac{2P_{a\beta}(k)E(k)}{(d-1)A_dk^{d-1}}. \quad \text{(IV.7)}$$

We can then form an equation for $E(k)$

$$\left(\partial_t + 2\nu k^2\right)E(k) = \frac{iA_d k^{d-1}}{2} P_{a\beta\gamma}(k) A_{a\beta\gamma}(k) = T(k). \quad \text{(IV.8)}$$

Here, $T(k)$ is the energy transfer spectrum and $A_{a\beta\gamma}(k)$ is defined as

$$A_{a\beta\gamma}(k) = \int d\mathbf{p} \, C_{a\beta\gamma}(k, \mathbf{p}, -\mathbf{k} - \mathbf{p}), \quad \text{(IV.9)}$$

where $C_{a\beta\gamma}(k, \mathbf{p}, -\mathbf{k} - \mathbf{p}) = \langle u_a(k) u_\beta(\mathbf{p}) u_\gamma(-\mathbf{k} - \mathbf{p}) \rangle$ is the spectral third order velocity field moment. More detailed derivations and interpretation of these quantities can be found in [148].

### IV.3.1 Second Order Structure Function

The dimensional dependence of HIT is well elucidated in the form of the velocity correlation functions, and thus in turn the structure functions. Here, we derive the relationship between the second and third order longitudinal structure functions and the energy and transfer spectra respectively. As a result, we are then able to evaluate these structure functions in our numerical EDQNM results. The method used here is the $d$-dimensional extension of that used in Bos et al. [31]. We begin by considering the second order longitudinal structure function

$$S_2^{(d)}(r) = 2(u^2 - C_{LL}) = 2\left(u^2 - \frac{r_a r_\beta}{r^2} C_{a\beta}(r)\right). \quad \text{(IV.10)}$$

These correlations are related to their spectral analogues via a $d$-dimensional inverse Fourier transform

$$C_{a\beta}(r) = \int d\mathbf{k} \, C_{a\beta}(k)e^{i\mathbf{k} \cdot \mathbf{r}} = \int d\mathbf{k} \, \frac{2P_{a\beta}(k)E(k)}{(d-1)A_d k^{d-1}} e^{i\mathbf{k} \cdot \mathbf{r}}. \quad \text{(IV.11)}$$
In HIT, these transforms are simplified by the fact that the correlators must be spherically symmetric, thus we have

\[
C_{LL} = \int d^d k \frac{2(1 - \cos^2 \theta) E(k)}{(d - 1)S_d k^{d-1}} e^{ik \cdot r} \\
= 2 \Gamma \left( \frac{d}{2} \right) \int_0^\infty dk E(k) \frac{J_{d/2}(kr)}{(kr)^{d/2}}.
\]

(IV.12)

Here, we have defined \( \theta \) as the angle between \( \mathbf{k} \) and \( \mathbf{r} \), such that \( \mathbf{k} \cdot \mathbf{r} = kr \cos \theta \), and \( J_n(x) \) is the \( n \)-th order Bessel function of the first kind. Additionally, for \( u^2 \) we find

\[
u^2 = 2 \int_0^\infty dk E(k).
\]

(IV.13)

Inserting these two results into Equation IV.10 yields

\[
S_2^{(d)}(r) = 2 \int_0^\infty dk E(k) \left[ 2 - 2 \frac{d}{2} \Gamma \left( \frac{d}{2} \right) \frac{J_{d/2}(kr)}{(kr)^{d/2}} \right],
\]

(IV.14)

thus we have related \( S_2^{(d)}(r) \) to \( E(k) \). It can be verified that this gives the standard results for two and three dimensions as found in Davidson [59]. Additionally, this result allows us to determine the integral length scale in \( d \)-dimensional HIT. The integral length scale is defined as

\[
L_d = \int_0^\infty dr f(r),
\]

(IV.15)

where \( f(r) \) is the scalar correlation function defined in Equation IV.4, such that \( f(r) = C_{LL}/u^2 \). Therefore, for the \( d \)-dimensional case we have

\[
L_d = \frac{2 \Gamma \left( \frac{d}{2} \right)}{u^2} \int_0^\infty dk E(k) \int_0^\infty dr \frac{J_{d/2}(kr)}{(kr)^{d/2}} = \frac{\Gamma \left( \frac{d}{2} \right) \sqrt{\pi}}{\Gamma \left( \frac{d+1}{2} \right) u^2} \int_0^\infty dk E(k) k^{-1}.
\]

(IV.16)

This expression will be needed when defining our integral scale Reynolds number in \( d \)-dimensional turbulence. It is also possible to generalise the Taylor microscale \( \lambda_d \), which gives the average size of the dissipative eddies, to \( d \)-dimensions. This length scale is defined through fitting a parabola to the small \( r \) expansion of the scalar longitudinal correlation function \( f(r) \), i.e.

\[
f(r) = 1 - \frac{r^2}{2 \lambda_d^2} + \mathcal{O}(r^4).
\]

(IV.17)
From Equations [IV.4] and [IV.12] we find through expansion for small $r$

$$f(r) = \frac{C_{LL}}{u^2} = 1 - \frac{r^2}{d(d+2)u^2} \int_0^\infty dk \, k^2 E(k) + o(r^4), \quad \text{(IV.18)}$$

where if we recall that

$$\varepsilon = 2\nu \int_0^\infty dk \, k^2 E(k), \quad \text{(IV.19)}$$

we then have

$$f(r) = 1 - \frac{\varepsilon r^2}{2d(d+2)\nu u^2} + o(r^4). \quad \text{(IV.20)}$$

Hence, the Taylor microscale in $d$-dimensions is given by

$$\lambda_d = \sqrt{\frac{d(d+2)\nu}{\varepsilon} u}. \quad \text{(IV.21)}$$

### IV.3.2 Third Order Structure Function

A similar analysis can be performed for the third order structure function, $S_3(r)$, whereby it is related to the transfer spectrum. We begin with the relation

$$S_3^{(d)}(r) = 6C_{L,L} = 6\frac{r_\alpha r_\beta r_\gamma}{r^3} C_{\alpha\beta\gamma}(r) \quad \text{(IV.22)}$$

and find that upon Fourier transform we have

$$\mathcal{F}[C_{\alpha\beta\gamma}(r)] = \mathcal{A}_{\alpha\beta\gamma}(k), \quad \text{(IV.23)}$$

with $A_{a\beta\gamma}(k)$ defined as in Equation [IV.9] From here, we can observe that

$$S_3^{(d)}(r) = 6\frac{r_\alpha r_\beta r_\gamma}{r^3} \int dk \mathcal{A}_{\alpha\beta\gamma}(k) e^{ik \cdot r}. \quad \text{(IV.24)}$$

Now all that remains is to express this in terms of the transfer function and perform the Fourier integrals. From Equation [IV.8] we have

$$\frac{i A_d k^{d-1}}{2} P_{a\beta\gamma}(k) \mathcal{A}_{a\beta\gamma}(k) = T(k), \quad \text{(IV.25)}$$

and, as $\mathcal{A}_{a\beta\gamma}(k)$ is a third rank solenoidal tensor symmetric in the indices $\beta$ and $\gamma$, we must have $\mathcal{A}_{a\beta\gamma}(k) = P_{a\beta\gamma}(k) \mathcal{A}(k)$. Thus by evaluating the product $P_{a\beta\gamma}(k)P_{a\beta\gamma}(k) = 2(d-1)k^2$ we have

$$\mathcal{A}_{a\beta\gamma}(k) = \frac{P_{a\beta\gamma}(k) T(k)}{i(d-1)A_d k^{d+1}}. \quad \text{(IV.26)}$$
Hence, for $S_3^{(d)}(r)$ we have the following

$$S_3^{(d)}(r) = 6 \frac{r a r \beta r \gamma}{r^3} \int dk \frac{P_{a \beta \gamma}(k) T(k)}{i(d-1)A_d k^{d+1}} e^{i kr} . \quad \text{(IV.27)}$$

In the same way as we did for $S_2^{(d)}(r)$ we can evaluate this integral by taking $\theta$ to be the angle between $k$ and $r$, which upon doing so we find

$$\frac{r a r \beta r \gamma}{r^3} P_{a \beta \gamma}(k) = 2k \cos \theta \left(1 - \cos^2 \theta\right) . \quad \text{(IV.28)}$$

We then once more evaluate all but two of the $d$ Fourier integrals to obtain

$$S_3^{(d)}(r) = \frac{12 A_{d-1}}{i(d-1)A_d} \int_0^\infty dk \frac{T(k)}{k} \int_0^\pi d\theta \cos \theta \left(1 - \cos^2 \theta\right) \sin^{d-2} \theta e^{i kr \cos \theta} . \quad \text{(IV.29)}$$

The inner integral is formidable, however it can be evaluated using computer algebra software. The result when restricted to integer dimensions is then found to be

$$S_3^{(d)}(r) = 3 \Gamma \left(\frac{d}{2}\right) r \int_0^\infty dk 2^{1+\frac{d}{2}} T(k) \frac{J_{1+\frac{d}{2}}(kr)}{(kr)^{1+\frac{d}{2}}} . \quad \text{(IV.30)}$$

As a check we compare this result with the case for $d = 2$ derived in [47]

$$S_3^{(2)}(r) = 12r \int_0^\infty dk T(k) \frac{J_2(kr)}{(kr)^2} . \quad \text{(IV.31)}$$

Now, clearly upon inserting $d = 2$ into equation [IV.30] we recover the result above. Furthermore using properties of Bessel functions it can also easily be shown that for $d = 3$ the above reduces to the expected expression as seen in [31].

Finally, we consider a small $r$ expansion of the third order structure function in $d$ dimensions

$$S_3^{(d)} = 12r \frac{d(d+2)}{d(d+2)(d+4)} \int_0^\infty dk T(k) - \frac{6r^3}{d(d+2)(d+4)} \int_0^\infty dk k^2 T(k) + \mathcal{O}(r^5) . \quad \text{(IV.32)}$$

From this expansion and the conservation properties of $T(k)$, we can see $S_3^{(d)}(r) \sim r^3$ for small $r$ and $d \geq 3$. In two dimensions the second term also vanishes as a result of enstrophy conservation. This expansion is also of practical use for evaluation of $S_3^{(d)}(r)$ for very small $r$ numerically, where floating point arithmetic errors can arise.
IV.3.3 Enstrophy Production and Skewness

Vorticity and enstrophy play an important role in the behaviour of two- and three-dimensional turbulence. With the absence of vortex stretching in the two-dimensional case being fundamental to the dynamical differences between dimensions. Enstrophy production is also known to be linked to the velocity derivative skewness, hereafter referred to simply as the skewness, of the flow. To generalise these concepts to arbitrary spatial dimension, we first introduce the vorticity 2-form

$$ \Omega_{\alpha\beta}(x) = \partial_{\alpha} u_{\beta}(x) - \partial_{\beta} u_{\alpha}(x). $$  \hspace{1cm} (IV.33)

We can re-express the Navier-Stokes equation using $\Omega_{\alpha\beta}$ as

$$ \partial_t u_{\alpha} + \Omega_{\beta\alpha} u_{\beta} = -\partial_{\alpha} \left( P + \frac{u^2}{2} \right) + \nu \nabla^2 u_{\alpha}, $$  \hspace{1cm} (IV.34)

which is valid in any dimension, and is equivalent to the rotational form of the NSE in three dimensions. Using Equation [IV.34] we are then able to derive an evolution equation for $\Omega_{\alpha\beta}(x)$

$$ \partial_t \Omega_{\alpha\beta} + u_{\gamma} \partial_{\gamma} \Omega_{\alpha\beta} + \Omega_{\alpha\gamma} S_{\gamma\beta} + \Omega_{\gamma\beta} S_{\alpha\gamma} = \nu \nabla^2 \Omega_{\alpha\beta}, $$  \hspace{1cm} (IV.35)

where $S_{\alpha\beta}(x) = \left( \partial_{\alpha} u_{\beta}(x) + \partial_{\beta} u_{\alpha}(x) \right) / 2$ is the strain tensor. Enstrophy in three dimensions is defined in terms of the vorticity, $\omega(x)$, as

$$ Z(t) = \frac{1}{2} \langle \omega_{\alpha} \omega_{\alpha} \rangle = \int_{0}^{\infty} \text{d}k \ k^2 E(k), $$  \hspace{1cm} (IV.36)

where for this case we also have $\omega_{\alpha} = \epsilon_{\alpha\beta\gamma} \Omega_{\beta\gamma} / 2$, which suggests the correct form of enstrophy in terms of the two form is

$$ Z(t) = \frac{1}{4} \langle \Omega^2_{\alpha\beta} \rangle = \int_{0}^{\infty} \text{d}k \ k^2 E(k). $$  \hspace{1cm} (IV.37)

To be confident this second equality holds, we will first form an equation for the evolution of $u^2$ using the vorticity 2-form

$$ \frac{1}{2} \partial_t u_{\alpha} u_{\alpha} + u_{\alpha} \Omega_{\beta\alpha} u_{\beta} = -\partial_{\alpha} \left( P + \frac{u^2}{2} \right) u_{\alpha} + \nu \partial_{\beta} u_{\alpha} \Omega_{\beta\alpha} - \frac{\nu}{2} \Omega^2_{\alpha\beta}. $$  \hspace{1cm} (IV.38)

Upon averaging and invoking homogeneity we find

$$ \partial_t E(t) = -\frac{\nu}{2} \langle \Omega^2_{\alpha\beta} \rangle, $$  \hspace{1cm} (IV.39)
which, when compared to the standard result, is
\[ \langle \Omega_{\alpha\beta}^2 \rangle = 2 \langle (\partial_\alpha u_\beta)^2 \rangle. \] (IV.40)

From here, it can be shown that indeed the second equality in Equation [IV.37] holds as
\[ \langle (\partial_\alpha u_\beta)^2 \rangle = -\langle u_\alpha \nabla^2 u_\alpha \rangle = 2 \int_0^\infty dk k^2 E(k), \] (IV.41)
and the equality is proved. Thus, we are confident Equation [IV.37] is a consistent generalisation of enstrophy to all dimensions.

In order to relate the production of enstrophy to skewness, we require an equation for the generalised enstrophy, which we can obtain from the \( d \)-dimensional vorticity equation above
\[ \partial_t Z(t) = -\frac{1}{2} \langle \Omega_{ij} \Omega_{ik} S_{kj} + \Omega_{ij} \Omega_{kj} S_{ik} \rangle - \frac{\nu}{2} \langle \Omega_{ij} \nabla^2 \Omega_{ij} \rangle. \] (IV.42)

Following steps similar to those for the enstrophy, we can show that the palinstrophy generalises as
\[ P(t) = \frac{1}{4} \langle \Omega_{ij} \nabla^2 \Omega_{ij} \rangle = \int_0^\infty dk k^4 E(k). \] (IV.43)

Now, in order to express Equation [IV.42] in terms of the skewness, we consider the von Kármán-Howarth equation [61] in \( d \)-dimensions expressed in terms of the second and third order two point longitudinal correlations
\[ \partial_t C_{LL} = \frac{1}{r^{d+1}} \partial_r \left[ r^{d+1} C_{LL,L} \right] + \frac{2\nu}{r^{d+1}} \partial_r \left[ r^{d+1} \partial_r C_{LL} \right]. \] (IV.44)

From the preceding discussion we now recognise the integrals in Equation [IV.18] as being the total energy, enstrophy and palinstrophy, hence we have
\[ C_{LL} = \frac{2}{d} E(t) - \frac{r^2}{d(d+2)} Z(t) + \frac{r^4}{4d(d+2)(d+4)} P(t) + O(r^5). \] (IV.45)

Using this expansion in Equation [IV.44] and the fact that \( C_{LL,L} \sim r^3 \) for small \( r \), we find to zeroth order in \( r \)
\[ \partial_t E(t) = -2\nu Z(t). \] (IV.46)

This is entirely equivalent to Equation [IV.39] and represents the decay of energy in turbulent flows without external forcing. Continuing now to \( O(r^2) \)
\[ \partial_t Z(t) = -d(d+2)(d+4) \frac{C_{LL,L}}{r} \bigg|_{r=0} - 2\nu P(t). \] (IV.47)

Reassuringly, both these expressions are consistent with what is derived directly from the
NSE, see Davidson \cite{59} for the two and three dimensional cases. Also, we note here that, since this derivation required $C_{LLL} \sim r^3$ for small $r$, in two dimensions the first term on the right hand side vanishes. Recalling that $S_3^{(d)}(r) = 6C_{LLL}$ and the skewness, $S_0$, can be expressed

$$S_0 = \left. \frac{S_3^{(d)}(r)}{\left[ S_2^{(d)}(r) \right]^{\frac{3}{2}}} \right|_{r \to 0},$$

we then write the enstrophy equation as

$$\partial_t Z(t) = -S_0 \Lambda(d) Z(t)^{\frac{3}{2}} - 2\nu P(t),$$

where the $O(r^2)$ term of the small $r$ expansion of $S_2^{(d)}(r)$ has been used and the dimensional factor is

$$\Lambda(d) = \frac{(d + 4)}{3} \sqrt{\frac{2}{d(d + 2)}}.$$  \hspace{1cm} (IV.50)

The dimensional factor, $\Lambda(d)$, in this equation is a decreasing function of $d$ with an asymptotic limit of $\sqrt{2}/3$. As such, with increasing dimension a larger skewness is required to generate the same amount of enstrophy.

In the above, we have demonstrated that, in any dimension, enstrophy production is governed by the action of the strain field on the generalised vorticity. This action can be thought of as the stretching and folding of structures analogous to vortices in all dimensions. From another viewpoint, this stretching is seen to be caused by a non-zero skewness. However, a larger skewness value is required with increasing dimension to produce the same level of vortex stretching.

We can also consider the skewness as a function of dimension using Equation (IV.48) and the small $r$ expansions for the structure functions. This gives

$$S_0(d) = -\frac{1}{\Lambda(d)} \int_0^\infty dk k^2 T(k) \left[ \int_0^\infty dk k^2 E(k) \right]^{-\frac{2}{3}}.$$ \hspace{1cm} (IV.51)

### IV.4 Numerical Approach

Due to the tremendous computational expense of DNS in higher dimensions, in this Chapter we utilise the EDQNM closure described in detail in Chapter II. As we noted in Chapter II, this approximation has a number of useful properties that make it suitable for numerical studies of higher dimensional turbulence. Additionally, it allows for a straightforward analytic continuation into non-integer dimensions, allowing us to truly
consider the spatial dimension as a parameter.

In obtaining the results obtained in this Chapter, we have made use of the numerical approach to the EDQNM equation described in Chapter II. For the standard statistical measurements we study the normal EDQNM equation given in Equation II.40. Whereas for our investigation into the predictability of higher dimensional turbulence we focus on the coupled system of equations described in Section II.4 of Chapter II. For convenience we will reproduce this system of equations here

\[
[\partial_t + 2\nu k^2] E(k) = T(k) + F(k), \quad (IV.52a)
\]

\[
[\partial_t + 2\nu k^2] E_W(k) = T_W(k) - T_X(k) + F(k), \quad (IV.52b)
\]

\[
[\partial_t + 2\nu k^2] E_\Delta(k) = T_\Delta(k) + T_X(k), \quad (IV.52c)
\]

in which

\[
T(k) = 8K_d \int \int_{\Omega(k)} dp dq \frac{k}{pq} \left( \sin \frac{\alpha}{k} \right)^{d-3} \theta_{kpq} b_{kpq,d} \left[ E(p)E(q)k^{d-1} - E(q)E(k)p^{d-1} \right],
\]

\[
T_W(k) = 8K_d \int \int_{\Omega(k)} dp dq \frac{k}{pq} \left( \sin \frac{\alpha}{k} \right)^{d-3} \theta_{kpq} b_{kpq,d} \left[ E_W(p)E(q)k^{d-1} - E(q)E_W(k)p^{d-1} \right],
\]

\[
T_\Delta(k) = 8K_d \int \int_{\Omega(k)} dp dq \frac{k}{pq} \left( \sin \frac{\alpha}{k} \right)^{d-3} \theta_{kpq} b_{kpq,d} \left[ E_\Delta(p)E(q)k^{d-1} - E(q)E_\Delta(k)p^{d-1} \right],
\]

\[
T_X(k) = 8K_d \int \int_{\Omega(k)} dp dq \frac{k}{pq} \left( \sin \frac{\alpha}{k} \right)^{d-3} \theta_{kpq} b_{kpq,d} E_W(p)E_\Delta(q)k^{d-1}.
\]

In the above we have introduced the forcing spectrum \( F(k) \) which allows us to study the EDQNM equations at steady state. We note here that \( F(K) \) has to be the same in both (IV.52a) and (IV.52b), as otherwise the growth of error in the system will be driven by this term and not the underlying turbulent dynamics. Additionally, we note here that as in previous studies, we make the arbitrary choice to use the same eddy damping factor for both \( E(k) \) and \( E_W(k) \) equations.

We have also introduced an additional equation when compared to Chapter II, that is the equation for the decorrelated energy spectrum \( E_\Delta(k) = E(k) - E_W(k) \). In general, we will find
this to be a more useful measure of error growth in the EDQNM equations.

An important detail to note for our numerical study of the EDQNM equations, is that the values used for the free parameter $A$ in the eddy-damping term for arbitrary dimension $d$ are obtained from previous numerical works. The values of the Kolmogorov constants $C_3$ and $C_4$ in three and four dimensions, respectively, are obtained from DNS simulations in [20]. For higher dimensions, there is no DNS data available, instead the Kolmogorov constants, $C_d$, are obtained from simulations using the Lagrangian renormalised approximation (LRA) in [87], that is another closure approximation that does not depend on the choice of any parameter. Additionally, the main results in this work depend only on the growth rate of $E_\Delta(t)$ and therefore they are unaffected by the choice of $A$. This was tested by performing large variations of $A$ across a range of EDQNM calculations.

As we outlined in Chapter II, the parameters $F$ and $\Delta t$ are set such that for every run, we make sure that the energy spectrum is consistent with the specified Kolmogorov constant for that spatial dimension. Furthermore, in our predictability measurements we ensure that the Lyapunov exponents are invariant under changes in either temporal and spatial resolution. Due to the rapid temporal changes the $E_W(k, t)$ spectrum undergoes as the system evolves, the time-step must be substantially lower in predictability measurements than for more standard statistical measurements.

Throughout this work we make use of a simple form for the forcing spectrum

$$F(k) = \begin{cases} \frac{\varepsilon}{\Delta_k} & \text{if } k \leq \Delta_k \\ 0 & \text{if } k > \Delta_k \end{cases},$$

(IV.54)

where $\Delta_k$ is the forcing band, set to $\Delta_k = 2$ in all runs. This forcing allows us to set the dissipation rate $\varepsilon$ a priori, making it a trivial matter to ensure $k_{\text{top}}$ is sufficiently large to capture the dissipative region of the system.

For both statistical and predictability measurements we are interested in the properties of higher dimensional turbulence at steady state. As such, for each calculation we begin from an energy spectrum that is initially zero for all $k$. As the system is advanced forwards in time the forcing spectrum injects energy into the system, which over time allows a Kolmogorov energy spectrum to form. At this point, we are now able to make our statistical measurements, for example of the structure functions. However, for our predictability measurements further work is needed. Using our steady state energy spectrum we initialise
the correlated energy spectrum such that we have

\[ E_\Delta(k, t_0) = \frac{B}{1 + \exp\left(-\frac{4(k-k_p)}{k_p}\right)}, \quad (IV.55) \]

whereby we concentrate the error to the smallest scales by setting \( k_p \approx 0.9k_{\text{top}} \). This approach mimics the effect of finite measurement precision in real world turbulence prediction. We set \( B \) such that the total decorrelated energy

\[ E_\Delta(t_0) = \int_0^\infty dk E_\Delta(k, t_0) = 10^{-7}. \quad (IV.56) \]

To find the maximal Lyapunov exponent in our EDQNM calculations we note that from the definition of \( E_\Delta(t) \) we have \( |\delta u(t)| = (2E_\Delta(t))^{1/2} \) and thus we have

\[ E_\Delta(t) \propto \exp(2\lambda t). \quad (IV.57) \]

Hence, by simultaneously evolving the system of equations in Equation [IV.52] we can compute \( E_\Delta(t) \) and find \( \lambda \).

In this Chapter, we also present comparisons between EDQNM calculations and full DNS calculations for both three and four spatial dimensions. For these DNS calculations we make use of the same pseudo-spectral DNS code used in previous chapters, for details refer to [236] and [94]. We have also obtained Lyapunov exponents for these DNS runs, for details on the method to obtain these exponents refer to [96] and the discussion in Chapter III. The three dimensional DNS results range from \( N = 64^3 \rightarrow 2048^3 \) collocation points and from \( N = 64^4 \rightarrow 512^4 \) collocations points in our four dimensional DNS. However, note that due to the computational cost, Lyapunov exponents were only obtained in the four dimensional DNS up \( N = 256^4 \).

IV.5 Results

IV.5.1 Energy and Transfer Spectra

In the DNS study of Berera [20], the observed scaling of the energy spectrum was consistent with that predicted by K41 in both the three and four dimensional cases insofar as within a given dimension the energy spectra were found to collapse upon scaling by Kolmogorov variables. However, the spatial dimension was found to have an influence on the shape of the energy spectrum in the near dissipative region. It was observed that, when comparing
four to three dimensions, dissipative effects did not become dominant until smaller scales, evidenced by the presence of a seemingly extended inertial range. Before we investigate this behaviour in higher dimensions using the EDQNM closure, we need to understand to what extent the model is capable of reproducing the effects seen in three and four-dimensional DNS. To this end, in Figure IV.1, we show energy spectra from both DNS and EDQNM in three and four dimensions scaled by the Kolmogorov constant. In both dimensions we have the same $\nu$ and $\varepsilon$ across DNS and EDQNM. A good collapse of the data can be seen in the inertial range and, approaching the dissipative region, the EDQNM closure captures the extended inertial range reasonably well. However, once the dissipative region is reached, the EDQNM results begin to diverge from those of the DNS in both dimensions, and it does not appear that this divergence is worse in one dimension over the other. The discrepancies at the large scales, small $k\eta$, are due to the forcing differences between DNS and EDQNM simulations.

Having verified that the EDQNM approximation can satisfactorily reproduce properties of the energy spectra seen in three and four dimensional DNS, we now turn to even higher spatial dimensions. In Figure IV.2a, we plot the energy spectra from EDQNM simulations for three, six, seven, ten and twenty dimensions, scaled by the appropriate Kolmogorov constant for each dimension, once more we keep $\nu$ and $\varepsilon$ constant across dimensions. Here, it can be seen that increasing spatial dimension is accompanied by a growing accumulation of energy on the edge of the inertial range. Such behaviour may be indicative of an enhanced forward transfer of energy as the spatial dimension increases. This view is consistent with the theoretical arguments which conjecture that as the spatial dimension tends to infinity the nature of the triadic interactions leads to all energy being transferred in the forward direction to the small scales [73]. Further arguments have suggested that the appearance of such an energy bottleneck is the result of triad interactions being damped by viscosity at the smallest scales [93]. In particular, non-local triad interactions are damped by viscosity leading to the transfer of energy becoming less effective. Hence, if the forward energy transfer is enhanced and, as appears to be the case, the aforementioned viscous damping of certain triads is not influenced by the dimension, then this pile up should be expected to increase with dimension. It is also possible that as the dimension changes the balance between local and non-local interactions changes leading to further increase of the bottleneck effect. It is possible to separate the local and non-local interactions analytically in the EDQNM equations as can be seen in [134]. The extension of this method to the $d$-dimensional case has not yet been carried out. However, it may be of interest to investigate the balance between local and non-local triads as a function of the spatial dimension.

In Figure IV.2b, we show the compensated spectra which give an even clearer demonstration that the bottleneck effect becomes more pronounced with increasing dimension. In DNS
Figure IV.1  (a) Comparison between DNS (solid lines) and EDQNM (dashed lines) of energy spectra scaled by the Kolmogorov micro-scale, $\eta$, for three (blue) and four (black) dimensions. (b) Compensated energy spectra for the same data.
Figure IV.2  (a) EDQNM energy spectra scaled by the Kolmogorov micro-scale, $\eta$, for three, six, seven, ten and twenty dimensions, the darker the shade of the line the higher the dimension with twenty being black. Dashed line shows $k^{-5/3}$ scaling. All dimensions have the same viscosity and energy dissipation.  (b) Compensated spectra for the same data, colours the same.
results [20], this view is further supported by an increased skewness in four dimensions. We will return to skewness in Section IV.5.2 where it is evaluated for EDQNM results. It is also clear that in all cases we observe a inertial range with a $k^{-5/3}$ power-law scaling which persists over a number of decades in wave-number space. As we go to higher dimensions we find this scaling region appears to become progressively shortened by the increased bottleneck effect. Although not shown, in all dimensions we find a collapse within a given dimension of energy spectra across a range of Re values when rescaled by $\nu$ and $\varepsilon$. That is, the energy spectrum is found to take on a universal shape in each dimension.

The energy transfer spectrum, $T(k)$, provides a natural measure of the exchange of energy between different scales in turbulent flows. However, in closing the infinite moment hierarchy using the quasi-normal hypothesis, the transfer spectrum is directly effected by the closure. Moreover, $T(k)$ is also directly influenced by the eddy-damping assumption. As such, if the quasi-normal and/or the eddy-damping assumptions are not sound we should expect the transfer spectra produced in the EDQNM simulations to show differences when compared to DNS transfer spectra. Indeed, in Figure IV.4 we see far larger discrepancies between EDQNM and DNS for the non-linear energy transfer than we did in the corresponding energy spectra. However, the qualitative behaviour going from three to four dimensions is the same in both DNS and EDQNM results. The peak non-linear transfer is greater and found at smaller scales in four dimensions compared to three dimensions. However, in contrast with results for the energy spectra, the agreement between the non-linear transfer in DNS and EDQNM appears to be better in four dimensions than in three dimensions, insofar as the peak transfer occurs at similar scales in both DNS and EDQNM. Without higher dimensional DNS results we cannot say whether this better agreement between DNS and EDQNM is purely coincidental or in fact evidence that four dimensional turbulence is in effect more mean-field-like. In Figure IV.3, the energy flux is displayed. For both DNS and EDQNM the energy flux remains roughly constant until smaller scales in four dimensions relative to three dimensions. Once more, due to the forcing differences at larger scales there is a greater disagreement between DNS and EDQNM results for both the non-linear transfer and the energy flux. In light of these comparisons, we should be more cautious in our interpretation of results derived from the transfer spectrum in EDQNM.

Turning once more to purely EDQNM results, we look at the dependence on the spatial dimension of the non-linear energy transfer. In Figure IV.4 we plot the non-linear energy transfer for a number of dimensions. Here, the trend of the peak non-linear energy transfer moving to smaller scales as the dimension increases is observed to continue to around six dimensions, at which point it begins to move to larger scales again. This potential crossover at six dimensions is interesting given the work of Liao [136, 137], where a possible
Figure IV.3  (a) Comparison between DNS (solid lines) and EDQNM (dashed lines) of non-linear transfer in three (blue) and four (black) dimensions. (b) Normalised energy flux for DNS (solid lines) and EDQNM (dashed lines) in three (blue) and four (black) dimensions.
Figure IV.4  (a) EDQNM Non-linear energy transfer for three, six, seven, ten and twenty dimensions, the darker the shade of the line the higher the dimension with twenty being black. (b) Normalised energy flux for same data coloured in the same manner.
upper critical dimension for turbulence at six dimensions is conjectured. Of course, since our results are obtained via a closure approximation they should not be over-interpreted. Furthermore the conclusions in the work by Liao are based upon a renormalisation group approach applied to freely decaying nearly incompressible turbulence. Hence, it is not immediately clear how applicable they are to the truly incompressible steady state results presented here. We merely intend to point out a possible connection with one of the only theoretical predictions of a critical dimension for turbulence. In Figure IV.4b we show the spectral energy flux for a range of spatial dimensions. In line with what was found for the compensated energy spectra, we observe a scaling range of several decades in all dimensions. In this Figure it is clear that above three dimensions there is an increased energy transfer to smaller scales, as evidenced by the flux dropping slower as we enter the dissipative region.

IV.5.2 Skewness

In Chapter I we derived Equation IV.49 relating the production of the generalised enstrophy in \(d\)-dimensions to the skewness, \(S_0\). Higher values of \(S_0\) are then associated with greater vortex-stretching, which would then provide the mechanism for the increased forward energy transfer in higher dimensions. Using Equation IV.51 we can measure the effect of spatial dimension on \(S_0\) in our EDQNM simulations. These results are presented in Figure IV.5a. It is observed that in the EDQNM equations the skewness reaches a maximum value of around \(-0.72\) at seven dimensions, before remaining roughly constant until ten dimensions, beyond which \(S_0\) decreases. If the trend seen in Figure IV.5a continues then the skewness may vanish for infinite spatial dimension. When compared to the results for skewness in the DNS of Berera et al. [20], both three and four-dimensional EDQNM results exhibit a lower value of \(S_0\). This is not a surprising result and has been observed in EDQNM simulations in three dimensions [31] and is likely a result of the assumptions made in the EDQNM model e.g. the quasi-normal or eddy-damping assumptions. The EDQNM approximation is also known to exhibit a constant asymptotic value for \(S_0\) at sufficiently high Reynolds number and the results presented in Figures IV.5a and IV.5b are of this asymptotic value in all cases. The existence of this asymptotic value is predicated on the exponent in the inertial range being \(-5/3\), hence, given our energy spectra results in all dimensions we can be confident of these asymptotic values.

In Equation IV.49 due to the dimensional pre-factor of the skewness term, at larger spatial dimensions, higher skewness values are required to generate the same level of enstrophy production. In Figure IV.5 we show the pre-factor, \(-S_0\Lambda(d)\), of the enstrophy
Figure IV.5 (a) EDQNM Velocity derivative skewness vs dimension. (b) Enstrophy production term $-S_0 \Lambda(d)$ vs dimension.
production term in Equation [IV.49] for a range of dimensions. These results suggest that until five dimensions there is an increase in enstrophy production, then starting at six dimensions there is a reduction in enstrophy production and thus likely also in vortex stretching, although it is possible that the balance between vortex stretching and strain self-amplification changes such that the enstrophy production reduces without a reduction in vortex stretching. If we consider that the action of vortex stretching produces smaller scales in the flow, then this is consistent with the accumulation of energy at the end of the inertial range. Once more, if this trend continues, then for infinite dimension there will be no enstrophy production and hence no vortex stretching. The vanishing of vortex stretching, and thus of velocity derivative skewness, at infinite dimension is quite an extreme scenario, and it may be that a non-zero but finite asymptotic value is reached instead. This is consistent with what was seen in [87] where a finite but non-zero asymptotic skewness value at infinite dimension was predicted using the LRA. This finite value is reached after a consistent increase with dimension. This is at odds with what is seen in our simulations, where a maximum skewness is reached at a finite dimension. However, if the skewness does vanish at infinite dimension a more drastic statement may then be that in infinite dimensions there is no energy flux in either direction and thus no turbulence. We stress here that since these are closure results it is not possible to make any definitive claims. The significance of this maximum skewness dimension is unknown, and may be a result of the closure assumptions. As such, without higher dimensional DNS results, we cannot make a definitive statement. Furthermore, to ensure that the skewness reaching a maximum at finite dimension is not a result of the values of free parameter, $A$. To understand the influence of $A$ on our results we have performed another set of simulations where $A$ is chosen such that $C_d = 1$ in all dimensions. The results of these simulations are presented in Figure [IV.6]. In Figure [IV.6], it is clear that even when the influence of $A$ on the free parameter is removed the skewness still looks to decrease at high dimension after a peak somewhere below 10 dimensions. Without data at far higher dimension we cannot speculate on the asymptotic behaviour of the velocity derivative skewness. What we can say is that, at least for the EDQNM closure, a peak value is observed which is solely determined by the triadic interactions of the system. The significance, if there is any, of this peak skewness dimension is unknown.

As a result of these $C_d = 1$ simulations, we are confident that the appearance of a maximum skewness dimension is a real effect in the EDQNM equations, with the caveat that the Kolmogorov constant does not increase with dimension. Given the observed Kolmogorov constants in the four dimensional DNS of both [87] and [20] and the predictions of the LRA, there is evidence that this caveat holds.
Figure IV.6  (a) EDQNM Velocity derivative skewness vs dimension for $C_d = 1$ runs. (b) Enstrophy production term $-S_0 \Lambda(d)$ vs dimension for $C_d = 1$ runs.
An important further point to consider for the infinite dimensional problem is that of the energy spectrum normalisation in this limit. In [73] it was found that to each order in perturbation theory the energy spectrum had a finite limit if a rescaled time variable is used. However, this time rescaling can be shown to be equivalent to a rescaling of the energy spectrum. The latter rescaling is equivalent to a finite energy per velocity component as opposed to a finite total energy. If the total energy is to remain finite, then as we tend to infinite dimension, each individual velocity component will tend to zero, suggesting zero skewness. However if the components remain non-zero then the skewness will be finite but non-zero.

Once more, these results are interesting considering the work of Liao [136, 137] suggesting the possibility of a critical dimension of six for turbulence. Further, studying this behaviour numerically using the EDQNM approximation becomes progressively more difficult with increasing dimension. For example, at $d = 20$ with a numerical resolution of $F = 50$ we find an error of about 2% in the expected value for the Kolmogorov constant using the free parameter in Appendix D. Compared with $d = 3$ where a resolution of $F = 16$ gives an error much less than a percent it is clear that the resolution costs of higher dimensions quickly becomes an issue.

IV.5.3 Third Order Structure Functions

As discussed in Chapter I the longitudinal structure functions have frequently been measured in experimental [4] and numerical studies of turbulence [86, 103]. In such studies, it is found that, particularly at higher orders, these structure functions show deviations from the scaling predicted by the K41 theory. Such deviations are typically attributed to intermittency corrections, although there are also arguments suggesting these are simply finite Reynolds number corrections due to K41 being an asymptotic theory [5, 7, 11, 119, 131, 133, 208, 213].

The effect of the spatial dimension on these corrections is an interesting question, and comparisons to critical phenomena, in particular anomalous exponents, have been made [161, 188]. Of course, the EDQNM model does not account for intermittency and therefore measurements of structure functions in such simulations cannot answer questions regarding these deviations. Furthermore, we are not aware of a method by which to calculate beyond third order structure functions from spectral quantities. However, if such a method were available it is likely it could, potentially with some difficulty, be applied to the EDQNM model. Indeed, it was shown by Kraichnan [122] that in the Direct Interaction Approximation (DIA) it is possible to study quantities beyond third order. This was then
demonstrated by Chen et. al. [48] for the DIA and by Bos et. al. [29] for the EDQNM model. As such, even if intermittency was present in the EDQNM model we would be unlikely to find substantial corrections at these low orders, even in three dimensions.

Putting questions regarding intermittency and anomalous exponents aside, we wish to test Equation IV.27 using our DNS and EDQNM results. In Figures IV.7a and IV.7b, the third order structure functions computed using Equation IV.27 in EDQNM are compared to those in DNS. Good agreement is seen in the inertial range with both DNS and EDQNM exhibiting the expected scaling. Looking to the dissipative region, we find that all our data follows \( r^3 \) scaling, however, this scaling begins at a different point in DNS compared to EDQNM. Given our energy and transfer spectra results, these deviations in small scale behaviour are expected. The most interesting feature of these Figures can be found in Figure IV.7b where the agreement between DNS and EDQNM appears to be better in four dimensions when compared with three. This is consistent with what was seen in the non-linear transfer, perhaps not surprisingly given that the transfer spectrum is used in determining \( S_3^{(d)}(r) \). Without higher dimensional DNS data it is impossible to know if this better agreement is due to the EDQNM approximation becoming more accurate with increasing dimension or simply a coincidence.

As already stated, the EDQNM model does not exhibit intermittency. However, it does capture well the finite Reynolds number effect. Indeed, in [31] the EDQNM model was compared with the multi-fractal model for three dimensional turbulence giving comparable results for low \( Re \) suggesting it is difficult to distinguish between intermittency effects and finite Reynolds effects in this region. We have not studied the multi-fractal model in four dimensions to make this comparison, however, given the better agreement between EDQNM and DNS in four dimensions for \( S_3^{(d)}(r) \) it may be the case that the finite Reynolds number effect becomes dominant over intermittency in higher dimensions.

Looking now at Figure IV.8a, we see the third order longitudinal structure function for our EDQNM data in four, five and six dimensions scaled by appropriate Kolmogorov quantities. It can be seen that in all dimensions we have \( r^3 \) scaling in the small \( r \) limit, as is seen in three dimensional turbulence and is predicted from the small \( r \) expansion of \( S_3^{(d)}(r) \). Turning to the \( d \)-dimensional von Kármán-Howarth equation, it can be shown that in the inertial range for the third order structure function, we should find

\[
S_3^{(d)}(r) \approx -\frac{12}{d(d + 2)} \varepsilon r , \tag{IV.58}
\]

which reduces to the standard four-fifths law of three dimensional turbulence. Indeed, in Figure IV.8a we can see that each dimension follows its own \(-12/d(d + 2)\) law in the
Figure IV.7  Comparison of third order structure functions in DNS (solid lines) and EDQNM (short dashed lines) for three (blue) and four (black) dimensions: (a) scaled by appropriate Kolmogorov quantities (b) scaled by energy dissipation and $r$. In (a) the dotted lines correspond to the expected $\frac{12r}{d(d+2)}$ inertial range scaling behaviours whilst in (b) they are for the values $\frac{12}{d(d+2)}$. 
Figure IV.8  EDQNM Third order structure functions scaled by appropriate Kolmogorov quantities for three, six, seven, ten and twenty dimensions, the darker the shade of the line the higher the dimension with twenty being black. Dashed lines represent appropriate power-law scaling for each dimension, i.e. $12r/d(d+2)$ for the inertial range. (b) Third order structure function scaled by energy dissipation and $r$ for the same data with the same colouring.
inertial range. Once more in all dimensions we observe a long scaling region. For a clearer comparison in Figure IV.8b we normalise each dimension by the expect inertial range value such that all cases show an scaling range at 1. In doing so we find differences across dimensions, in particular by \( d = 20 \) the scaling region begins at higher \( r/\eta \) than in lower dimensions.

IV.5.4 Dissipative Anomaly

In both experimental [39, 205] and numerical studies [104, 110, 222] of three dimensional turbulence, including in EDQNM [30], there is a large body of evidence which indicates the existence of a non-zero energy dissipation rate, even in the limit of zero viscosity. This is known as the dissipative anomaly. In [20] an increased value for this asymptotic dissipation rate was observed in four dimensions when compared with three. This result once again suggests an enhancement of the forward energy cascade in four dimensions compared with three. We should then expect that beyond four dimensions this asymptotic dissipation rate should increase further given our spectral and skewness results.

The dimensionless dissipation rate is defined as

\[
C_\varepsilon = \frac{\varepsilon L}{u^3}, \quad (IV.59)
\]

and its Reynolds number dependence can be shown to be approximately described by the relationship [64, 147]

\[
C_\varepsilon(\text{Re}) = C_{\varepsilon,\infty} + \frac{C}{\text{Re}}, \quad (IV.60)
\]

where \( \text{Re} = uL/\nu \) is the integral scale Reynolds number in which \( L \) is defined by Equation IV.16.

In Figure IV.9a we show the dimensionless dissipation rate against \( \text{Re} \) for a wide range of \( \text{Re} \) values for three, four and five dimensions. In all cases we observe \( C_\varepsilon \) tending to a constant asymptotic value. This asymptotic value is seen to grow with dimension as would be expected from increased forward energy transfer. We find that in three dimensions this asymptotic value in EDQNM is \( C_{\varepsilon,\infty}^{3d} = 0.38 \) lower than what is seen in DNS where \( C_{\varepsilon,\infty}^{3d} \approx 0.5 \) in the forced case. A higher value was found in the EDQNM of Bos et al. [30] however they take a different choice of \( A \) which will directly influence the value found. This raises an important point for our results given our values of \( A \) for \( d \) greater than four are estimated from another closure approximation in order to set the Kolmogorov constant. However, so long as the Kolmogorov constants decrease to a constant value as \( d \) increases the qualitative trends should hold.
Figure IV.9  

(a) $C_\varepsilon$ vs $Re$ for three, four and five dimensions, darker colour indicates higher dimension with five dimensions in black. Dashed lines indicate the value for $C_{\varepsilon,\infty}$. 

(b) $C_{\varepsilon,\infty}$ against spatial dimension, $d$.  

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In four dimensions we find $C_{\varepsilon,\infty}^{4d} = 0.96$ lower than the value of 1.26 found in DNS \[20\]. Since $C_{\varepsilon}$ is defined in terms of large scale quantities, where the forcing is active, these discrepancies between DNS and EDQNM are not surprising given the differences at small $k$ seen in Figure IV.3. In Figure IV.9b we show the asymptotic dimensionless dissipation rate against the spatial dimension. Here we find the asymptotic value grows with dimension until at least 20 dimensions. Beyond here the resolution requirements render further calculation increasingly difficult. As $C_{\varepsilon,\infty}$ is related to the Kolmogorov constant such measurements would become exceptionally sensitive to resolution errors.

### IV.5.5 Reynolds number scaling of Lyapunov exponents

Turning now to the chaotic properties of higher dimensional turbulence, we first consider the dependence of the maximal Lyapunov exponent, $\lambda$, on the Reynolds number. Figure IV.10 shows the data obtained from approximately twenty-five simulations for each dimension $d = 3, 4$ and 5 using the EDQNM approximation. A scaling of the form of $\lambda \sim \text{Re}^\alpha$ is found, with $\alpha = 0.5$ in all cases when a least-squares fit is computed. Clearly, for all three cases Ruelle’s prediction seems to hold. For $d = 3$ the scaling agrees with the Ruelle dimensional prediction with the value of 0.5 being within the error bounds.

Next, in Figure IV.11 we present a comparison of the Re scaling of the maximal Lyapunov exponent found in both DNS and EDQNM. This comparison is made using results over a comparable range of Re in both cases. The DNS results were obtained in previous works \[19, 96\] whilst the EDQNM and four dimensional DNS results were collected as part of the work in this Thesis. In three dimensions the Ruelle scaling is observed in both DNS and EDQNM. In four dimensions the Reynolds number is far too low for an inertial range to exist hence we would not expect the Ruelle relation to hold. This is also true at low Reynolds number in three dimensions where a divergence from the Ruelle prediction is also seen. The last few points from the four dimensional DNS are tending towards Ruelle scaling so this further supports the argument that the Reynolds number simply needs to be increased. Unfortunately, as we discussed earlier in this Chapter, computing power is not sufficient at this time for such measurements to be made.

There are a number of important observations we can make from the EDQNM-DNS comparison. First, whilst the scaling of the Lyapunov exponents seems to be well captured by the EDQNM approximation, the absolute values may not be. Notably, the EDQNM exponents in three dimensions are lower in value than those in DNS at the same Re value, whilst in four dimensions they are of the same order. Second, we find that the absolute value of the exponents decreases going from three to four dimensions in both EDQNM and DNS.
Figure IV.10  Scaling of Lyapunov exponents $\lambda$ with $Re$ for dimensions $d = 3(\square)$, $4(\circ)$ and $5(\triangle)$ using the EDQNM closure.

Figure IV.11  Scaling of Lyapunov exponents for both EDQNM (□) and DNS (■) in $d = 3$ (blue) and $d = 4$ (black) in the region $Re < 5000$. Ruelle scaling fit for EDQNM data shown as dashed lines.
This lends some credibility to the EDQNM results and suggests that the idea of a critical dimension for error growth may exist in DNS as well as EDQNM, and thus be a property of the true Navier-Stokes equations. Finally, as the Reynolds number is lowered the Lyapunov exponents begin to diverge from the Ruelle scaling and tend towards zero. This also occurs for both DNS and EDQNM data.

**IV.5.6 Error decay and critical dimension in \( d > 5 \)**

We now report what is arguably the most important result of this Chapter, the existence of a critical dimension, \( d_c \), between five and six dimensions, for error growth in the EDQNM equations. Above this dimension we find the error, \( E_\Delta(t) \), stops growing and instead decays. We note here that we are not necessarily suggesting that this is a critical dimension in terms of anomalous scaling in the spirit we described in the first chapter of this Thesis. However, we will discuss possible connections between these two ideas, however, as we are working with a closure approximation we are not in a position to make definitive statements.

When we look at dimensions near \( d_c \) we find there is an initial period of fast error growth, however, this then changes to a decay of error. This initial rapid growth may be the short time super exponential error growth discussed in Li et al. [135]. We show this transition from error growth to decay in Figure IV.12. In this Figure we see that for lower values of Re the fractional error growth \( E_\Delta(t)/E \) the initial growth is similar across dimensions. Following this cases below \( d_c \) the exhibit exponential growth and those above the critical dimension begin to decay exponentially. Now, when we look at the higher Reynolds number cases we find that after the initial super exponential period the cases below \( d_c \) show a short period of exponential growth before saturation. For those above \( d_c \) the growth is never exponential and there is a transient period where the growth becomes decay. Notably, the closer we get to the critical dimension from below the lower the saturation of the fractional growth becomes. We will explore the reasons for this in more detail later in this Chapter.

Given the Re dependence of the transition between error growth and decay which is seen in Figure IV.12 it is possible that we simply require higher and higher Reynolds numbers for error growth as the dimension is increased. To probe this a number of calculations at low Re in three dimensions were performed and it was found that for Re < Re_c \( \approx 12 \) the error always decays, while above this value it would always grow exponentially. Repeating this process while increasing \( d \) leads to Figure IV.13. Here we find that as the spatial dimension increases this critical Reynolds number increases drastically. We find the increase is well fit by the functional form

\[
Re_c(d) \sim \frac{1}{(d - d_c)^2},
\]

(IV.61)
and we find $d_c \approx 5.88$. We do not know a priori what $Re_c$ is for a given dimension, and finding it quickly becomes unfeasible due to the rapid divergence as a function of $d$. Given the divergence from Ruelle scaling towards zero chaos seen in both EDQNM and DNS seen in Figure IV.11, we also checked for the vanishing of chaos using DNS. Although not shown here, these DNS results showed a similar behaviour and we found an increase in the critical Reynolds number for chaos going from three to four dimensions. As such, it is possible a critical dimension for error growth also exists for true Navier-Stokes turbulence.

For completeness, we show the behaviour of the Lyapunov exponents against $Re$ for $d = 6, 7$ and $8$ in Figure IV.14. Note, we show here the absolute value of $\lambda$ and use a linear scale for the y-axis. Contrary to the case for $d < d_c$ in which error grows and the growth rate depends on the Reynolds number, the error decay rate does not have a dependence on $Re$ for $d > d_c$, as expected given the divergence of $Re_c$ shown in Figure IV.13. Physically we can understand this by considering the timescales of the flow, for example, the Re dependence in the error growth rate for $d < d_c$ is related to the fact that the maximum rate is associated to the smallest timescales $\tau$. In the cases where the error decays, it is not clear which timescale is appropriate, however it appears this timescale must, in contrast to the growth cases, be Re independent.
Figure IV.13  Critical Reynolds number vs dimension. The dotted line corresponds to the curve fit $Re_c \sim (d - d_c)^{-2}$.

Figure IV.14  Absolute value of the error decay rate and its dependence on $Re$ for dimensions $d = 6$ (□), $d = 7$ (○) and $d = 8$ (△).
Figure IV.15  Final spectrum of $E_{\Delta}(k)$ where the labelled (solid) lines correspond to different dimensions with $d = 3, 4, 5, 5.5, 5.7$ and $5.85$, together with the steady state spectrum of $E(k)$ represented by the red (dotted) line. All runs have the same viscosity $\nu = 2 \times 10^{-6}$, this viscosity value corresponds to the largest Reynolds number explored in each dimension.

IV.5.7  Error spectrum

The transition from error growth to decay as a function of dimension can be further investigated by considering the error spectrum $E_{\Delta}(k, t)$. The properties of this spectrum have received attention in previous works [19, 26, 131, 132]. Here, it was found that the exponential growth phase manifests itself in a particular shape for the error spectrum. This spectrum initially peaks in the dissipation range before growing in a self similar fashion. The most important feature is that the decorrelation proceeds from small to large scales in the form of an inverse error cascade. When the two fields become fully decorrelated a K41 spectrum is found for the error spectrum. By studying the error spectrum we can further study the lower saturation of the fractional error growth as $d$ approaches $d_c$ as seen in Figure IV.12.

By studying the error spectrum across dimensions we find that there is a critical wavenumber $k_c$ below which the evolution of the error spectrum halts, preventing the full decorrelation of the two solutions. The same is observed for $d > 3$ for all values of Re. This critical wave-number $k_c$ increases with $d$, as shown in Figure IV.15. For each dimension, the value $k_c$ is found to be approximately constant across all Re. Furthermore, we see that for $d \approx d_c$, there is a strong correlation in the inertial range, whereas only the dissipative scales become decorrelated. For values $d > d_c$, the error decays so there is not a stable final
spectrum to do such an analysis. This is suggestive of a connection with the critical Re value needed for error growth. If $k_c$ is outside of the dissipative region, error growth can occur, however, as $d$ approaches $d_c$ it appears that $k_c \to \infty$.

In three dimensions the location of $k_c$ is at the edge of the forced wave-number. To understand the role of large scale forcing in such an effect, we study the evolution of $r(k, t) = E_\Delta(k, t) / E(k, t)$ when the forcing spectrum is set to 0. This is done just after $E(k, t)$ reaches steady state. For $d = 3$, there is still some correlation at low wave-numbers, but it is much less than in the forcing case, suggesting forcing may play a role here. This may be consequence of using the same forcing spectrum for both $E(k)$ and $W(k)$. In doing so we are making use of a correlated force which will likely act to prevent the decorrelation of the two fields penetrating the forcing scales. The gap between $E$ and $E_\Delta$ for $k < k_c$ is still present for $d = 4$ and 5. We also observed that for $d = 6$, $r(t)$ also decays without forcing. Given the divergence of $Re_c$ it is unlikely the forcing at small wave-numbers influences $k_c$ other than near $d = 3$. We note here that in the DNS of Boffetta and Musacchio [26], $r(t)$ does not saturate at unity and is influenced by the forcing at late times in line with what we see here.

An additional caveat to the location of $k_c$ in our work may be in the choice of eddy damping factor used in the $E_W(k)$ equation. As discussed in Section IV.4 this choice is arbitrary but does ensure the correlation spectrum is realisable. This choice may also introduce some correlation into the system, however, it is not clear what a more appropriate eddy damping factor would be. It is possible the ideas used in [10] to introduce different damping factors could be employed for the correlation spectrum but we have not pursued this here.

For dimensions above the critical dimension we find a different behaviour for the evolution of the energy spectrum. Here we find there is an initial transient behaviour before the spectrum peaks at approximately the Kolmogorov micro-scale. This transient phase corresponds to the initial super exponential growth seen for the fractional growth. After this point the spectrum begins to decay whilst maintaining its shape, that is it continues to peak at the Kolmogorov micro-scale length. This is consistent with what we see in Figure IV.15 where as we approached $d_c$ the value of $k_c$ moved closer and closer to the Kolmogorov micro-scale.

### IV.5.8 Error Cascades

Following Leith and Kraichnan [132] we consider now the various cascade rates of Equation [IV.52] In the above, the terms $T(k)$, $T_W(k)$ and $T_\Delta(k)$ represent the transfer of energy, correlated energy and decorrelated energy into a wave-number $k$ respectively. Whilst $T_X(k)$
gives the transfer of correlated energy into uncorrelated energy at a given $k$. Focusing on the error equations we note that both $T_W(k, t)$ and $T_\Delta(k)$ are conservative due the antisymmetry of the integrands in $k \leftrightarrow p$. This means that the total rates $\int_0^\infty dk T_W(k) = 0$, hence, these transfer terms simply transport correlated and uncorrelated energy, respectively, across the different scales, however no net energy is created or removed in these processes. On the other hand, $T_X(k, t)$ is non-conservative and we define $\epsilon_X = \int_0^\infty dk T_X(k)$ which is the rate of total transfer between correlated and uncorrelated energy. Finally, the correlated and uncorrelated energy dissipation rates are given by $\epsilon_{W,\Delta} = 2\nu \int_0^\infty k^2 E_{W,\Delta}$. Thus, the net error growth (or decay) is given by

$$\partial_t \epsilon_{\Delta}(t) = \epsilon_{\text{net}} \equiv \epsilon_X - \epsilon_\Delta.$$

We will consider the evolution of these cascade rates in time. In Figure IV.16 we show the time evolution of the net error growth given by the difference between production and dissipation of uncorrelated energy for three, four and five spatial dimensions. In all cases we find the growth rises rapidly, this stage corresponds to the exponential growth of error and is the period in which we measure the maximal Lyapunov exponent. The net growth rate becomes smaller as the spatial dimension is increased, which is consistent with the results we have shown previously. After this initial period the net growth rate exhibits a period of linear growth before entering a final period of saturation. For $d = 3$ we see that the value of the net growth rate in the linear phase agrees with Leith and Kraichnan's prediction,
Figure IV.17 $\epsilon_{\text{net}}/\epsilon$ with a symmetrical logarithmic scale in the vertical axis, for dimensions $d = 5, 5.3, 5.5, 5.7, 5.85, 5.9, 5.95$ and $6$ and $\nu = 2 \cdot 10^{-6}$. The line colour becomes darker as we move from 5 to 6 dimensions.

$\epsilon_{\text{net}} = 0.23\epsilon$ which was computed as an approximation to the infinite inertial range case by fixing $E(k)$ in time as a Kolmogorov spectrum. This prediction is shown as a dashed line in Figure IV.16. We also note here that at all times the net error growth is non-negative in these dimensions.

In Figure IV.17 we now turn to the region between five and six dimensions, where we have identified a critical error growth dimension. Below the critical dimension the situation is very similar to what was seen in the three, four and five dimensions. However, once the critical dimension is reached we find that the net error growth rate is no longer strictly positive. What we find is that after an initial period of error growth, corresponding to the super exponential growth discussed earlier, the net error growth rate becomes negative. We note that it takes longer for the dissipation rate of uncorrelated energy $\epsilon_\Delta$ to reach saturation as $d$ increases. Additionally, we see that the sharp peak that is typically observed for $d \leq 5$ and that corresponds to the initial exponential growth stage, starts to soften gradually as $d$ approaches $d_c$.

From the above analysis it is clear there is a competition between the generation of uncorrelated energy and its sweeping out by the cascade. To show more explicitly the interplay between uncorrelated and correlated energy we look at the evolution of the flux of correlated energy

$$
\Pi_W(k) = -\int_0^k dp \ T_W(p) = \int_{k}^{\infty} dp \ T_W(p) ,
$$

(IV.63)
Figure IV.18 (Top) Evolution of the of the correlated energy flux $\Pi_W(k)$ corresponding to the correlated energy cascade (solid lines), along with the energy flux $\Pi_u(k)$ corresponding to the energy cascade (dotted), and (Bottom) compensated transfer spectrum $k T_X(k)/\epsilon$. Time evolution goes from light to dark lines. The viscosity for all runs is $\nu = 2 \cdot 10^{-6}$. (a) $d = 3$ and $t/T = 0.5, 5, 7$ and 10, (b) $d = 4$ and $t/T = 0.5, 2.5, 7, 9$ and 11, (c) $d = 5$ and $t/T = 0.3, 2, 4, 5, 7$ and 9

that represents the forward cascade of correlated energy compared to the energy flux

$$\Pi_u(k) = -\int_0^k dp T(k) = \int_k^{\infty} dp T(k),$$

(IV.64)

together with the evolution of the transfer from correlated to uncorrelated energy $T_X(k)$.

We observe in Figure IV.18 that the cascade of correlated energy towards higher wave-numbers becomes more effective as the dimension grows. In Figure IV.18a it is clear that the correlated energy is transferred forward from the forcing range towards a wave-number where it is then transferred (as observed in the $T_X$ plot) from correlated to uncorrelated energy, which is then swept out by the uncorrelated energy flux $\Pi_\Delta = \Pi_u - \Pi_W$, which is not shown in the plot. Initially, the transfer from correlated to uncorrelated energy takes place at the dissipation wave-number, but as the system evolves, this transfer decreases towards $k_c$. For $d = 3$, the value of $k_c$ is at the very forcing range, whereas for $d = 5$ the forward cascade is more effective and the transfer to uncorrelated energy occurs at the inertial range. Finally, in Figure IV.19 we show the flux transition from $d = 5.3$ to $d = 6$. There we observe that as dimension increases, the transfer from correlated to uncorrelated occurs at the dissipation range, and the cascade of energy and correlated energy are equal for the whole inertial range. As such, by considering the correlated energy flux we find the the value $k_c$ is determined by this flux.

It is clear that below the critical dimension and at $\text{Re} > \text{Re}_c$, the production of uncorrelated energy wins this competition and the error between the two velocity fields grows. However, at the critical dimension the situation is altered and the removal of uncorrelated energy by the cascade begins to dominate. This is interesting in light of the results we found earlier in this Chapter, most notably the finding of a maximum enstrophy production near five dimensions. The roles played by vortex stretching and strain self-amplification, and
thus also enstrophy production, in the three-dimensional turbulent energy cascade have been the subject of numerous recent investigations [32, 38, 44, 107]. Indeed, Bos [32] demonstrated that removing both vortex stretching and strain self-amplification from the EDQNM closure in three dimensions results in enstrophy conservation and an inertial range with exponent -3. However, removing this term also acts to suppress strain self-amplification so a distinction between the importance of vortex stretching versus strain self-amplification could not be made. In Carbone and Bragg [44] the two effects were distinguished, and strain self-amplification was identified as having the dominant effect on the energy cascade. Nevertheless, the dimension of maximum enstrophy production will correspond to the case where the combined influence of both effects on the production enstrophy are maximised.

We note that the maximal enstrophy production dimension relies on measurements of the velocity derivative skewness, which is influenced by the choice of the parameter $\beta$ in the EDQNM closure. As discussed earlier in this work, the choice of $\beta$ in higher dimensions relies on calculations from the Lagrangian renormalised approximation, hence the exact dimension of maximal enstrophy production cannot be ascertained exactly. However, given what is known about the Kolmogorov constant from DNS of two-, three- and four-dimensional turbulence, the values for $\beta$ used here give conceivable Kolmogorov constants for the higher dimensional cases. As such, it is plausible that the dimension of maximal
enstrophy production lies between five and six. The determination of $d_c$ for error growth in this work is not affected by the choice of $\beta$ as it is found by considering energy ratios where it is cancelled out. Hence, given this relationship between strain self-amplification, vortex stretching, enstrophy production and the energy cascade, it is plausible that the maximum enstrophy production dimension seen earlier and critical dimension for the error in this work are related.

From the findings in [44] it would appear that the sweeping out of uncorrelated energy by the cascade is dominated by strain self amplification. In the same work the vortex stretching term was found to be associated with fluctuations of the cascade around its average value. It is tempting to associate these fluctuations, and thus vortex stretching, with error growth at the smallest scales, however this requires further investigation. Indeed, it is also possible that strain self-amplification plays a role in the generation of error. The picture, then, is that as we approach the critical dimension for error growth, we also approach the maximum enstrophy production dimension. During this approach we find the Lyapunov exponents become smaller at a given Re. At $d \geq d_c$ the cascade is able to fully sweep out the uncorrelated energy and thus prevent any error growth. Understanding the interplay between strain self-amplification and vortex stretching as both $d_c$ and the maximum enstrophy production dimension are approached may thus provide valuable information about the energy cascade itself.

The proposed role of vortex stretching and strain self-amplification in error growth and the production of uncorrelated energy is also considered in Chapter V of this Thesis. Here the transition between two- and three-dimensional dynamics in a thin-layer was studied via the maximal Lyapunov exponent. A discontinuous transition in the scaling of the exponent was found as the flow moved from two- to three-dimensional dynamics. On the two-dimensional side of the transition there is neither vortex stretching nor strain self-amplification and it is found that the Lyapunov exponent is determined by the injection of enstrophy. On the three-dimensional side, as in this work, the exponents are found to depend on the Reynolds number. The discontinuous nature of the transition suggests that these two processes are intimately related to error growth. Notably, two dimensional turbulence is still chaotic without either process, hence disentangling the individual roles of each process in the growth of error may be challenging.
IV.6 Conclusions and Final Remarks

Motivated by the four dimensional DNS results presented in Berera et al. [20], in this Chapter we have performed a thorough investigation into the effect of the spatial dimension in the EDQNM model of turbulence. While this is only an approximation to true fluid turbulence, we find that it is able to satisfactorily reproduce many of the dimensional effects seen in DNS. To facilitate this study, a number of standard results from three dimensional turbulence have been extended to $d$-dimensional turbulence. Some of these quantities have been discussed in the literature with theoretical ideas as to how they will behave in higher dimensions. This Chapter has presented for the first time both numerical results using EDQNM in a range of dimensions above three and expressions for the second and third order structure function in terms of spectral quantities for any dimension $d$. Furthermore, an equation relating the production of enstrophy to skewness in $d$-dimensional turbulence was derived from the von Kármán-Howarth equation.

In measurements of the energy and transfer spectra and the skewness as function of spatial dimension, we have found a consistent picture suggesting the forward energy cascade becomes enhanced as spatial dimension increases. In terms of spectra, this can be seen as an increase in the bottleneck effect in the near dissipation region of the energy spectra, which grows with dimension. Further to this, in the transfer spectra, we see a larger peak in the non-linear transfer. We also observe that the position of this peak first moves to smaller scales before then reversing and moving back to larger scales, which we posit is a result of suppression of transfer to the dissipative modes as suggested by Herring et al. [93]. We find the skewness reaches a maximum value around seven or eight dimensions and appears to be tending towards zero as the dimension becomes very large. In light of the enstrophy-skewness equation we have derived, this corresponds to a reduction in small scale vortex stretching which is again consistent with the enhanced forward transfer bottleneck effect. The possibility of a zero skewness value in the limit of infinite dimensions poses interesting questions for the fate of turbulence in this limit.

Additionally, we have measured the third order structure functions in higher dimensions using the spectral relations we have derived. We find here that each dimension has its own analogue to the four-fifths law of three dimensional turbulence. Finally, we studied the effect of the spatial dimension on the asymptotic dissipation rate. Due to being defined in terms of large-scale quantities this proves more difficult to accurately measure in the EDQNM model. However, we do find an increase in this asymptotic dissipation rate with dimension, which is a continuation of the trend seen in the DNS performed in Berera et al. [20]. This is also consistent with the existence of an enhanced forward transfer of energy in
higher dimensions.

These results are interesting for a number of reasons. Importantly, they confirm many of the results found in four spatial dimensions from DNS Berera et al. [20]. The fact the EDQNM results show consistency with DNS in three and four dimensions, then suggests the trends found by this method at even higher dimensions, which at present are computationally too demanding for DNS, should have some reliability. Thus, this Chapter has helped to examine some of the theoretical ideas that have been in the literature for decades on the behaviour of turbulence in dimensions greater than three. Furthermore, the appearance of an increased bottleneck effect in higher dimensions may help to shed light on the standard three dimensional bottleneck effect.

In Berera et al. [20] the scaling behaviour of fluctuations with Reynolds number was measured and found to decrease in four dimensions compared with three dimensions. In this work temporal fluctuations in the total energy were measured and found to scale slower with Re in four dimensions. It would be interesting to understand how such fluctuations would scale in even higher dimensions, especially in light of the results in this Chapter concerning the bottleneck effect and velocity derivative skewness. However, measurement of these fluctuations is unfortunately outside the scope of EDQNM calculations and would require future DNS study.

Motivated by the idea of critical dimension for fluid turbulence above which K41 becomes exact we have performed a numerical study of the evolution of the error $E(\Delta, t)$ for homogeneous and isotropic $d$-dimensional turbulence. We consider spatial dimensions ranging from 3 to 8, including non-integer values, using an EDQNM closure approximation. Our work has focused on analysing growth of $\Delta(t)$ through the maximal Lyapunov exponent and the production and removal of uncorrelated energy.

A critical dimension for error growth is $d_c \approx 5.88$ is found, above which the error $\Delta(t)$ decays instead of growing for large times. This value is close to the dimension of maximal enstrophy production found earlier in this Chapter. It is proposed that these dimensions are the related and thus that enstrophy production and error growth in turbulence are linked. As the critical dimension is approached the competition between the generation of uncorrelated energy, and thus error, and the sweeping out of error by the energy cascade begins to move in favour of the cascade. We suggest this is related to a change in the relative strengths of strain self-amplification and vortex stretching. These two processes are directly related to the energy cascade [32, 38, 44, 107] and to the production term in the enstrophy equation.

In [44], the energy cascade is found to be primarily driven by strain self-amplification, whilst vortex stretching is more closely related to fluctuation in the cascade. The results presented
in Chapter V suggest the presence of both these processes leads to very different chaotic properties when comparing two- and three-dimensional turbulence. However, given that the removal of error by the cascade becomes dominant at $d_c$ and the relationship between vortex stretching and cascade fluctuations, it is possible that error production is primarily driven by vortex stretching, although strain self-amplification may play some role and there is the possibility of dimensional dependence on which process drives the chaotic dynamics. This scenario is plausible at least in terms of the picture presented by Equation IV.62. Distinguishing between these two processes is difficult in the EDQNM closure, hence future work in DNS to understand the role they each play in error growth would valuable.

Considering the discussion of cascade fluctuations and their role in determining the validity of K41 made by Kraichnan [125, 126] the possibility of a reduction in the importance of vortex stretching beyond $d_c$ is interesting. If a connection between vortex stretching and error growth can be made, our work may suggest a suppression of fluctuations beyond $d_c$. This, of course, can only be entirely speculative due to the use of a closure approximation in this work. Still, it is a property that appears consistently in the numerical calculations of the EDQNM equations we have used. The EDQNM closure approximation is widely used in many theoretical and numerical works so even if this property is not present in DNS or real flows, it is still an interesting property to look at in the future using other closure approximations, or different numerical realizations of the same equations. The trend observed of decreasing $\lambda$ when going from $d = 3$ to $d = 4$, certainly sets the question for higher dimensions, but that remains for future analysis. Studying the error growth in $d$-dimensions using other closure models could discard or support this finding. Additionally, an interesting direction for further study, is to look at DNS at small box sizes in $d = 6$ to see if the error growth is present or not. Although the range of Re that can be explored in DNS is very small, these simulations might shed some light on the existence of the non-chaotic regime.

Another interesting angle to view these results from is the possibility of Burgers equation dynamics at high spatial dimension. As discussed earlier in this work, this idea has been considered in previous works [69, 73, 87]. Due to the incompressibility condition being the sum of $d$ terms as $d$ is increased the effect of the pressure term on each individual velocity component will be diminished. The Burgers equation is given by the Navier-Stokes equation with the pressure gradient term removed, hence the suggestions that the dynamics of high dimensional turbulence may resemble those of the Burgers equation. Our results provide another possible connection in this direction as it is known that the Burgers equation is integrable and thus does not exhibit chaos [53, 100]. It may be that the transition to a non-chaotic regime is related to the role of pressure, however this will require further
investigation. This is the opposite picture to what was described in the previous paragraph. If the Navier-Stokes equations in high dimension exhibit Burgers equations statistics this will involve an extreme anomalous scaling.

In summary, using the EDQNM approximation for isotropic turbulence we have found a critical dimension for error growth at $d_c \approx 5.88$. By considering our statistical results for the $d$-dimensional EDQNM equations, we propose this dimension is related the dimension of maximal enstrophy production found in Section IV.5.2. By considering the competition between the energy cascade and error production we relate our findings to fluctuations in the energy cascade and the idea of a critical dimension above which K41 is exact. Additionally, we consider the possibility of a Burgers equation statistics limit for the Navier-Stokes equation in high spatial dimension where an extreme anomalous scaling contrary to K41 could be expected. Determining which of these scenarios is correct will require further study of turbulence in spatial dimensions above three.
As touched upon in Chapters I and III, the dynamical behaviour of turbulent fluid flows is known to be vastly different in two and three dimensions. The three dimensional case is characterised by a forward cascade of energy from large to small scales separated by an inertial range where the energy spectrum takes the form $E(k) \sim k^{-5/3}$. In the two-dimensional case, the existence of a second quadratic invariant, the enstrophy, leads to a dual cascade scenario: an inverse cascade of energy from small to large scales and a direct enstrophy cascade from large to small scales. These cascades exhibit scaling regions of $E(k) \sim k^{-5/3}$ and $E(k) \sim k^{-3}$, respectively. Much of our understanding of turbulence in three dimensions can be attributed to Kolmogorov [119, 120], whilst in two dimensions the groundwork was laid by Kraichnan [123].

Despite these differences, there is a growing body of evidence that two- and three-dimensional turbulent dynamics can co-exist under certain circumstances. Perhaps the first demonstration of this was in measurements of the energy spectrum in the Earth’s atmosphere [160], in which the data was interpreted as showing both forward enstrophy and energy cascades. One possible explanation is that the geometry of the atmosphere is such that the vertical
direction is constrained compared to the other two, with the result that above a certain length-scale the system is effectively two-dimensional. This situation is often referred to as thin-layer turbulence. The co-existence of two- and three-dimensional phenomenology, that is both forward and inverse energy cascades, has been observed in both experimental and numerical studies of thin-layers \[15, 22, 42, 45, 106, 158, 198, 217, 228, 237\]. In the numerical investigations, it was found that by reducing the thickness of the fluid layer the system transitions from fully three dimensional behaviour to mixed two- and three-dimensional dynamics and then onto purely two-dimensional. Such transitions are not restricted to thin-layer turbulent systems; they have also been seen in turbulence undergoing rotation, exhibiting stratification, those under the influence of strong magnetic fields and in axis-symmetric flows \[41, 176, 184, 197, 202\]. For a more comprehensive review of such systems and cascade behaviour see \[2\].

These prior studies into transitions between two- and three-dimensional turbulence all employed the standard statistical approach to turbulence \[12\], in which the properties of the flow under a suitable averaging procedure are studied. However, it is also possible to exploit the deterministic chaos exhibited by turbulent flows \[28, 139, 174, 193\] to investigate their behaviour. As could be predicted from the differences in dynamical behaviour across dimensions, the chaotic properties are also vastly different when comparing two and three dimensions. In particular, the scaling behaviour of the maximal Lyapunov exponent, Kolmogorov-Sinai entropy and attractor dimension (see \[174\] for definitions) in three dimensions was determined entirely by the Reynolds number of the flow \[19, 26, 155\], see also Chapter III of this Thesis. In two dimensions, we saw in Chapter III that for the entropy and attractor dimension, this scaling was found to be more complicated and non-universal, being influenced by the system size and forcing length scale. The strong contrast between these two cases then suggests that these chaotic properties may be utilised in the study of this transitional behaviour.

The use of the chaotic properties of a system in the study of phase transitions has seen a small amount of attention in the critical phenomena literature \[11, 40, 43\]. In such studies, it was found that the maximal Lyapunov exponent could be used as an indicator of a phase transition, showing differing behaviour either side of a critical point. This, combined with the aforementioned drastic differences in the scaling behaviour of chaotic properties of turbulent flows in two versus three dimensions, suggests the maximal exponent might provide a useful alternative viewpoint in the study of this transition in thin-layer flows. Furthermore, the Lyapunov exponent measured in numerical simulations of turbulent fluid flow is found to be a remarkably stable quantity, particularly against the effects of numerical resolution \[95\]. As such, it may be expected to be a robust measure of the transitional
behaviour seen in thin-layer turbulence. Finally, the Lyapunov exponent gives a measure of the predictability of a system. Given the observation of transitional behaviour seen in the Earth’s atmosphere understanding how the predictability of thin-layer turbulence changes across such transitions may then provide important information for the wider study of atmospheric predictability.

V.1.1 Turbulence in a Thin Layer

In this Chapter, we study the transition between two- and three-dimensional phenomenology in thin-layer turbulence via measurement of the maximal Lyapunov exponent in direct numerical simulations (DNS) of the incompressible Navier-Stokes equations

\[
\frac{\partial}{\partial t} u + u \cdot \nabla u = -\nabla P + \nu \nabla^2 u + \mu \nabla^{-2} u + f, \\
\nabla \cdot u = 0.
\]  

(V.1)

In the above \(u(x, t)\) is the velocity field, \(P(x, t)\) the pressure field, \(f(x, t)\) is an external force used to sustain the flow and \(\nu\) is the kinematic viscosity. To avoid the formation of a large scale condensate as a result of the inverse energy cascade, we also include a hypo-viscous
term with hypo-viscosity, $\mu$, which removes energy at the large scales. This is an important addition, as such a condensate is a form of self-organization, which can cause a reduction of chaos in the flow. In all cases $\mu$ is set such that a condensate is unable to form. We employ the standard pseudo-spectral method with full de-aliasing using the two-thirds rule. Our simulations are performed in a fully periodic box with side lengths $L \times L \times H$, with $L = 2\pi$ and $H < L$. Throughout we will consider the side of length $H$ to be in the $z$ direction. To facilitate comparison with previous studies, our external forcing function acts only on the $x$ and $y$ components of the velocity field and has the form $f = (-\partial_y \phi, \partial_x \phi, 0)$ such that it is solenoidal. The scalar field $\phi(x, t)$ is stochastic and delta correlated in time, which ensures that on average the energy injection rate is $\epsilon$, which can be set by the amplitude of the forcing. Additionally, the forcing is concentrated in Fourier space on modes with magnitude $k_f \approx 2\pi/l_f$. The initial conditions of the flow are such that the field is near zero, with the small amount of energy spread across a wide range of length scales. As such, the flow is essentially generated by the stochastic forcing. We maintain an even grid spacing in physical space, therefore, upon reducing $H$ we also reduce the total number of grid points needed in the vertical direction. In Fourier space this leads to a larger spacing between modes in the vertical direction than in the horizontal directions.

In Figure V.1 we show the energy spectrum from one of our thin-layer simulations. Here we can observe the mixed two- and three-dimensional phenomenology discussed in the introduction to this Chapter. The situation here is almost identical to what was seen in the famous Nastrom-Gage spectrum measured in the atmosphere. Although we do see a steeper than expected spectrum for the enstrophy cascade region, with an exponent of -4, this is not uncommon in numerical simulations at moderate resolution. Importantly, what this figure shows is that at least in terms of the energy spectrum this approximation to true atmospheric turbulence can capture the qualitative features of the real-world situation.

In Figure V.2 we show the phenomenological differences between two-dimensional, three-dimensional and thin-layer turbulence. In V.2(a) we see a fully two-dimensional vorticity field complete with a number of large coherent vortices. In V.2(c) we present the fully three dimensional case. Here we have a highly intermittent vorticity distribution, with areas of low vorticity and then long strands of high vorticity. Finally, in the centre image we see a thin-layer transitional case. Here we see a combination of both two- and three-dimensional phenomenology: long strands of vorticity amongst larger coherent vortices.
V.1.1.1 Dimensionless Parameters

As in \[217\], we find the system is described by a number of non-dimensional parameters. The first is the Reynolds number defined at the energy injection scale as

\[
\text{Re} = \frac{\varepsilon^{\frac{1}{3}} l_f^{\frac{4}{3}}}{v}. \tag{V.2}
\]

We also have the ratio of the forcing length scale and the side of length \( H \) defined as \( Q = l_f/H \) and the aspect ratio of the system given by \( A = H/L \). Of these two quantities, it has been observed that \( Q \) is the more important parameter for determining the transition points of the system \[15, 217\], thus, we formulate our results in terms of \( Q \). For \( Q \) much less than 1, at the length scale where energy is injected the system is fully three dimensional, and as such we expect three dimensional phenomenology to dominate. For \( Q \) much greater than 1 the system is expected to appear two-dimensional. In between these two extremes it is observed that both two- and three-dimensional behaviour coexist in the flow. Indeed, in \[15\] using a severe Galerkin truncation in the vertical direction, an \( \text{Re} \) independent critical value of \( Q \) was found, above which the flow transitions from three dimensional behaviour to mixed phenomenology. Additionally, a second critical value at which point the system moves to two-dimensional behaviour was found, although this point was found to have an \( \text{Re} \) dependence. Our simulations span the ranges \( \text{Re} \approx 90 – 1200 \) and \( Q \approx 0.1 – 16 \) with the forcing length scale either \( k_f = 4, 8 \). Finally, the number of grid-points used in the horizontal directions varied from 128 – 1024, such that the simulations remained well resolved.
V.1.2 Non-Integer Spatial Dimensions

The thin-layer turbulence situation we have just described has been referred to in the literature as turbulence between two and three dimensions or even 2.5-dimensional turbulence. Of course, these thin-layer systems are in fact three-dimensional but due to geometric confinement they effectively behave as two-dimensional above a certain length scale and three-dimensional below it. However, in Chapter II we noted that the spatial dimension in the EDQNM equation is simply another parameter which we are free to vary. This then allows us to consider an analytic continuation of turbulence in the EDQNM approximation to non-integer spatial dimensions.

This opens an interesting avenue for investigation, whereby we can compare the pseudo-non-integer spatial dimension of thin-layer turbulence with that of truly non-integer turbulence in the EDQNM approximation. As we discussed in Chapter II, non-integer dimensional turbulence between two and three dimensions was considered by Fournier [72], where a transition between forward and inverse energy cascades was found around \( d \approx 2.05 \). In this Chapter we will further study this transition using both statistical and chaotic properties. In doing so, we will be able to make a direct comparison with our thin-

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**Figure V.3** Ratio of average energy in the three dimensional modes to the average total energy across a range of \( Q \) and \( Re \) values. Conor gradient used to indicate \( Re \) value, becoming darker as \( Re \) increases.

---

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure_v3.png}
\caption{Ratio of average energy in the three dimensional modes to the average total energy across a range of \( Q \) and \( Re \) values. Conor gradient used to indicate \( Re \) value, becoming darker as \( Re \) increases.}
\end{figure}
layer turbulence results, thus allowing us to ascertain if the transitions between two- and three-dimensional in each case happen in a similar way.

To study this problem, we make use of the \( d \)-dimensional EDQNM equations derived in Chapter II for both the energy spectrum and correlated energy spectrum with the addition of a hypo-viscous term. We restrict the spatial dimension such that \( 2 \leq d \leq 3 \). The viscosity and hypo-viscosity are held constant with values of \( \nu = 1 \times 10^{-6} \) and \( \mu = 0.05 \). The forcing wave-number is also held constant and we have \( k_f = 8 \) in all cases.

V.2 Numerical Implementation of Thin Layer Turbulence

To study thin-layer turbulence using pseudo-spectral DNS requires some small but important modifications to the DNS code we have used throughout this Thesis. The bulk of the code is left unmodified, and therefore [236] provides the majority of the details. Here we will outline the changes that must be made to what is reported in [236] to allow the side lengths of the physical domain to be altered.

We consider our system to exist within a box with periodic boundary conditions. The box has dimensions \( L_x \times L_y \times L_z \), however for our purposes we will take \( L_x = L_y \) and \( L_z < L_x \). This differs from the standard case, where all three side lengths are equal to \( 2\pi \). As our system is periodic, we can expand the velocity field within the domain into a Fourier series. As we are in a finite domain we will have quantised momentum, that is

\[
k_{\alpha} = \frac{2\pi}{L} n_{\alpha}, \tag{V.3}
\]

in which the wave-number \( n_{\alpha} \) must be an integer. Then we see that the choice of \( L_x = L_y = L_z = 2\pi \) corresponds to the simplest case where the momentum vector \( \mathbf{k} \) and wave-vector \( \mathbf{n} \) are equal. The use of periodic boundary conditions here must be borne in mind when interpreting the results. We will touch upon this point again when discussing the findings of this Chapter.

To study a thin-layer system, clearly this relationship will no longer hold and we need to consider the horizontal and vertical directions separately. To retain some of the simplicity of the box with all side lengths equal to \( 2\pi \) we keep \( L_{x,y} = 2\pi \) and hence in the horizontal directions we have

\[
k_{x,y} = n_{x,y} \tag{V.4}
\]

In the vertical direction we find

\[
k_z = \frac{2\pi}{H} n_z, \tag{V.5}
\]
where to match the notation in the rest of this Chapter we rename $L_z$ to $H$, and then since $H < L_x = 2\pi$ there is no equality between the momentum and the wave-vector in this direction. An immediate consequence of this is that the lowest possible value for $k_z$ will be greater than for the horizontal directions restricting the size of eddies in this direction. Of course, given the geometric confinement of the system this makes sense as we cannot have structures larger than the domain.

We will now consider how this system is discretised to allow for numerical calculations. Since we are using the pseudo-spectral method, we define a lattice with $N_x \times N_y \times N_z$ collocation points. In the standard cubic domain we simply take $N_x = N_y = N_z = N$ where $N$ then sets the lattice spacing. Mirroring the discussion above concerning the domain size, we take our lattice to be formed of $N \times N \times M$ points with $M$ setting the number of points, and thus the lattice spacing, in the vertical direction. At this point we have two options: we could take $M = N$ which would result in a higher simulation resolution in the vertical direction, or we can take $M < N$ such that the lattice spacing in real space remains constant in all directions.

The second option is simplest and has the added advantage that we can use $M$ to set $L_z$, as such, it is our choice for this study. Furthermore, this method allows us to continue to make use of the standard approach to de-aliasing using the two-thirds rule that is already implemented in the code. The main code modifications are then to allow for a different number of lattice points in the vertical direction and to correctly compute the $k_z$ values needed for computing derivatives in the pseudospectral method. This results in the following expression being used in the code

$$k_z = \frac{N}{M} n_z$$

which gives correct spacing in Fourier space for a constant lattice spacing in all directions in real space.

### V.3 The Maximal Lyapunov Exponent Between Two and Three dimensions

The maximal Lyapunov exponent gives, to leading order, the rate of divergence of trajectories in the state space of the system. This state space in our simulations is of very high dimension, equal to the number of Fourier modes retained. Systems with a positive maximal Lyapunov exponent are said to exhibit deterministic chaos, a state
characterized by an extreme sensitivity to initial conditions. Turbulent fluid flow is known
to be deterministically chaotic and, given the ubiquity of turbulence in nature, this has
implications for the predictability of real world phenomena. This work is motivated in
part by atmospheric predictability, where the chaotic nature of turbulence, coupled with
finite experimental resolution, leads to small measurement errors growing exponentially
in time. As a result, if weather forecasts are to be accurate within a given error tolerance,
there is a finite predictability time before the error will grow to exceed any tolerance. This
predictability time is determined by the maximal Lyapunov exponent, at least for small scale
weather phenomena. For larger scale climate forecasting it is possible for the predictability
time to exceed that given by the Lyapunov exponent. We return to this point in Section V.5.

In our DNS, the maximal Lyapunov exponent is calculated using the procedure outlined in
Chapter III. Put briefly, we consider a reference field \(u_1\) and a perturbed field, \(u_2\), which at
the perturbation time \(t_0\) is defined as

\[ u_2(t_0) = u_1(t_0) + \delta_0, \quad (V.7) \]

in which \(\delta_0\) is a Gaussian random velocity field with zero mean. The variance of the
perturbation field is chosen such that the initial separation, \(\Delta\), between the two velocity
fields can be considered infinitesimal, i.e. \(|\delta_0| \approx \Delta \ll U\), where \(U\) is the RMS velocity.
Importantly, if, as in this work, a stochastic forcing is used, the force should only be
randomly generated once per iteration and the same force applied to both the reference
and perturbed fields. If a new random force is generated for each field, this acts like a new
perturbation at each iteration and destroys the exponential growth of separation between
the fields.

For the EDQNM calculations of turbulence in non-integer dimensions between two and
three we use the same process for computing the maximal exponent as we did in Chapter IV
for higher dimensional turbulence. As a reminder, this involves the numerical integration
of the equation for the correlation between two distinct velocity fields with are initially
completely uncorrelated below a certain length scale. This initial correlation spectrum
mimics the effect of finite experimental precision.

The expected scaling behavior of \(\lambda\) in both two- and three-dimensional turbulence can
be estimated on dimensional grounds by assuming it will be determined by the inverse
of the smallest timescale of the flow. In homogeneous and isotropic three-dimensional
turbulence, this is given by the Kolmogorov time scale, which then implies

\[ \lambda_{3D} \sim r^{-1} \sim \frac{1}{T} \sqrt{\text{Re}}, \quad (V.8) \]
Figure V.4  Maximal Lyapunov exponent, $\lambda$, scaled by the enstrophy dissipation rate timescale against $Re$. Conor gradient is set such that approaching the transition at $Q/\sqrt{Re}$ from either side results in a lighter color. Additionally points below the transition become more red whilst above they become more blue. Lower dashed line has a gradient of 0.29 whilst higher dashed line has a gradient of 0.14, and these give the corresponding $Re_E$ scaling exponents.
Figure V.5  Maximal Lyapunov exponent, $\lambda$, scaled by the Reynolds number and enstrophy dissipation rate against $Q/\sqrt{Re}$. Conor gradient used to indicate $Re_f$ value, becoming darker as $Re_f$ increases. An empirically determined exponent is used in scaling the maximal exponent. This is to highlight the discontinuous transition.
where $\tau$ is the Kolmogorov time and $T$ is the large eddy turnover time. Following similar arguments in two dimensions results in \[167\]

$$\lambda_{2D} \sim \sqrt{\eta}.$$ \hspace{1cm} (V.9)

in which $\eta$ is the enstrophy dissipation rate. In the two-dimensional case there is also the possibility of logarithmic dependence on $\text{Re}$ due to logarithmic corrections to the energy spectrum in the enstrophy scaling range \[124\]. It is not clear which of these dimensional estimates should be used in either thin-layer turbulence or non-integer turbulence between two and three dimensions. Furthermore, there is no theory in the literature to guide this choice.

\section*{V.4 The Transition between Two and Three Dimensional Turbulence}

\subsection*{V.4.1 In a Thin Layer}

We will focus on the second critical value mentioned in Section \[V.1.1.1\] which we denote as $Q_{2D}(\text{Re})$, this is the point at which the flow transitions from mixed two- and three-dimensional dynamics to pure two-dimensional behaviour. In \[15\] it was shown that by considering the interplay between the layer thickness $H$ and the shearing force driving 3D instabilities in the flow, this critical thickness should behave as

$$Q_{2D}(\text{Re}) \propto \sqrt{\text{Re}},$$ \hspace{1cm} (V.10)

see also \[80\] for further information. Before presenting results for the maximal Lyapunov exponent in thin-layer turbulence, we will first establish an approximate value in terms of $Q/\sqrt{\text{Re}}$ at which the transition to two-dimensional dynamics occurs using a standard indicator. We will consider the velocity field to be decomposed into two- and three-dimensional parts \textit{i.e.}

$$u(k, t) = u_{2D}(k, t) + u_{3D}(k, t)$$

$$= u(k : k_z = 0, t) + u(k : k_z \neq 0, t),$$ \hspace{1cm} (V.11)

such that the two-dimensional part is composed of all modes with vertical wave-number, $k_z = 0$. Using this decomposition, the total energy of the flow also becomes split into two-
and three-dimensional parts
\[ E(t) = E_{2D}(t) + E_{3D}(t). \] (V.12)

At the point \( Q_{2D}(Re) \), we expect the three dimensional energy to vanish. As such, we consider the ratio of the averaged three dimensional energy to the averaged total energy. In Figure [V.3] we plot this ratio for a range of \( Q \) and \( Re \) values. Here, we see a common curve across all \( Re \) values, with the possible exception of only the highest \( Re \) values. This is likely explained by the transition from three dimensional behaviour to mixed dynamics becoming \( Re \) independent at high enough \( Re \), as found in [217]. For all cases we find the transition point \( Q_{2D}(Re) \) to occur at \( Q/\sqrt{Re} \approx 0.25 \). This is consistent with what was found in the pre-condensate phase of the simulations in [217]. It should be noted that the value of \( Q_{2D}(Re) \) may be influenced by the form of forcing employed. In our case, the forcing is fully two-dimensional, however, it has been found that when using a three dimensional force the transition point is altered [177].

To investigate the transition from the viewpoint of predictability, we consider the \( Re \) dependence of \( \lambda \). Note that here, similarly to in [15], we use a second forcing scale Reynolds number defined as
\[ Re_f = \frac{l_f \sqrt{E}}{v}. \] (V.13)

The reason for this second definition is that the \( Re \) defined in Equation [V.2] does not contain any information about the dynamical properties of the underlying flow, making determination of scaling exponents difficult. Note, that we continue to use \( Re \) as defined in Equation [V.2] when discussing the location of \( Q_{2D}(Re) \) to facilitate comparison with the literature. In Figure [V.4] we clearly observe two distinct scaling laws, one for points below \( Q_{2D}(Re) \) and another for those above. It should be noted that, in contrast with these previous studies in three dimensions, we non-dimensionalize \( \lambda \) using the enstrophy dissipation rate timescale. This is justified by the relationship between enstrophy production and velocity derivative skewness which, if we assume K41 holds, gives this timescale the same \( Re \) scaling as \( \tau \). The scaling exponent below the critical point is in good agreement with the \( \lambda \tau \) scaling shown in [19] for purely three dimensional turbulence. In Chapter III the \( Re \) dependence of \( \lambda \) in two-dimensional turbulence was found to be \( \lambda \sim Re^{0.16} \) which is in line with what we find for beyond \( Q_{2D}(Re) \). Figure [V.4] highlights the possibility of an increase in \( Re \), seemingly paradoxically, causing an increase in predictability as the system moves from the inverse cascade branch to the bidirectional cascade branch. Given that the value of \( Q/\sqrt{Re} \) at which this change in scaling occurs is the same as for the energy indicator and that seen in the literature, it is clear that the Lyapunov exponent provides a robust measure of the transition.

In Figure [V.5] we show \( \lambda \) re-scaled by both \( Re \) and \( \eta \) for a range of \( Q \) and \( Re \) values. The
power of $\Re f$ chosen corresponds to the scaling exponent for points below the transition point in Figure V.4. This scaling gives an approximately constant value for points below $Q/\sqrt{\Re} \approx 0.25$. At and above this point, we observe what appears to be a discontinuous jump as the flow becomes two-dimensional and the scaling behaviour is changed.

Notably, in both Figure V.4 and V.5 there is no indication of a first transition from three dimensional to mixed dynamics. This suggests the leading chaotic properties of the flow remain effectively fully three dimensional until the point $Q_{2D}(\Re)$. This is in agreement with the idea that the maximal Lyapunov exponent should be related to the shortest timescale of the flow. In both the three dimensional and mixed states, a forward cascade of energy to the smallest scales is seen, only vanishing when we pass $Q_{2D}(\Re)$. Physically, we can understand this behaviour by considering the cascades and triadic interactions involved at each stage. As we increase $Q$, the triads corresponding to three dimensional dynamics are progressively removed. Upon reaching the transition point, a critical proportion of these triads have been lost, ending the forward energy cascade and rendering the flow two-dimensional. Unlike for $E_{3D}/E$, which decreases continuously as the forward cascade region is reduced, if $\varepsilon$ and $\nu$ are fixed, then as $Q$ is increased, and as long as a forward cascade exists, $\lambda$ will have the same value before changing discontinuously when no forward cascade remains. A similar discontinuous transition was observed in [195], where the energy cascade was reversed by altering the weighting of certain triadic interactions between helical modes. See also the appendix of [22] for further discussion of this triadic interpretation.

It is also possible to view the transition through a less abstract physical interpretation, albeit one that is intimately connected with the triadic interpretation. It is well known that in two dimensions the vorticity equation has no vortex stretching term [59]. As a result, since we consider the steady state case, the enstrophy dissipation timescale in our simulations in the $Q/\sqrt{\Re} > 0.25$ regime is set entirely by the rate of enstrophy injection by the forcing. On the other side of the transition vortex stretching is possible, and thus additional enstrophy is produced by non-linear interactions. This then leads to a higher rate of enstrophy dissipation at steady state for an equal rate of enstrophy injection. Hence, in the $Q/\sqrt{\Re} < 0.25$ regime the maximal Lyapunov exponent is determined by the smallest scales of the flow. The transition can then be understood as a consequence of the effect of geometric confinement on vortex stretching. The predictability of flows on either side of the transition can then be vastly different: on the three-dimensional side it is governed by action at the smallest scales of the flow and by the macroscopic energy injection scale on the two-dimensional side.

An important point to consider in the simulation results presented here is the influence of the form of geometric confinement used. It is clear that considering a periodic flow
in a box with a varying height is a very artificial kind of confinement. In atmospheric turbulence, flow confinement is typically seen as a result of stratification. As mentioned in the introduction, stratified flows are also known to exhibit transitions between two- and three-dimension turbulence, dependent on the degree of stratification. It would be interesting to study the predictability of the transition in these flows as a more representative approximation of atmospheric turbulence. It may be possible to make a connection between the Reynolds number dependent $Q$ criterion for the transition seen here and the Richardson stability criterion of stratified flows [187], particularly as both are related to vortex stretching.

An interesting area for investigation is then what happens to the behaviour of the chaotic properties of the flow which depend on all the active degrees of freedom in the flow, for example the Kolmogorov-Sinai entropy and attractor dimension [174]. These may show different transitional behaviour and reveal further information about the properties of such transitions. However, their calculation is beyond the scope of this work.

Since $\lambda$ represents the exponential rate at which two initially close fields diverge from each other, it provides a measure of the predictability time of the flow. Figure V.5 suggests that this predictability time will exhibit discontinuous jumps. This is of particular interest in real world thin-layer systems which are, in general, non-stationary. In such flows, as $Re$ varies and the flow transitions from one set of dynamics to another, predictability may be drastically altered.

Finally, we have also studied the temporal behaviour of $\lambda$. In [217], as the point $Q_{2D}(Re)$ was approached, intermittent bursts of three dimensional energy were observed and related to the idea of on-off intermittency in dynamical systems [79]. Such bursts should impact the behaviour of the finite time Lyapunov exponents and would be expected to cause large fluctuations. Indeed, in Figure V.6 the case with $Q/\sqrt{Re} \approx 0.25$ is seen to undergo large variations in time. As the cases shown are at comparable $Re$ values then we can be relatively confident these fluctuations are caused by proximity to the transition. Although we only show two cases, this behaviour is typical of points close to the transition point.

The appearance of large fluctuations as we approach the transition point is reminiscent of phase transitions in critical phenomena. It is then tempting to try to classify this transition from a bidirectional cascade to two-dimensional dynamics. Indeed, the abrupt change in behaviour of the Lyapunov exponent at $Q_{2D}(Re)$ suggests something similar to a first order phase transition may be occurring. Making a definitive statement on this issue will require further investigation and a wider range of the parameter space to be studied.
Figure V.6  Time series for the finite time Lyapunov exponent, $\gamma(t)$ re-scaled by the mean value. Time is measured from the point the exponent has stabilized. We show a case far from $Q/\sqrt{Re} \approx 0.25$ (blue) and one close to this point (black). These cases have similar Reynolds number values with $Re \approx 600$. 
V.4.2 In Non-Integer Spatial Dimensions

Before investigating the maximal Lyapunov exponent for non-integer dimensional turbulence between two and three dimensions we will first aim to distinguish the transition point via more conventional methods, as we did for our thin-layer DNS. There are a number of subtleties to performing these non-integer dimensional EDQNM calculations. Most notably is the choice of the free parameter. As in Chapter IV, for non-integer dimensions the parameter was set using a least-squares fit to the integer dimensional values obtained via DNS and analytical methods. However, since we are primarily concerned with the maximal Lyapunov exponent, which is defined in terms of the rate of growth of the total error, the choice of free parameter is almost arbitrary. Another important point is how the hypo-viscous term is handled. For instance, in [33] the eddy damping term is modified to account for the hypo-viscosity. However, here we simply introduce the hypo-viscous term as a means to achieve steady state. As such, it acts more like a scale dependent forcing term and we do not consider it in the eddy damping term, which remains as in Chapter III.

We exploit the changing cascade properties between two and three dimensions and the massive Reynolds numbers achievable using EDQNM. We know that for purely three dimensional turbulence at high Reynolds number the direct cascade of energy will ensure...
viscous dissipation removes most of the energy from the system. Whilst in purely two-dimensional turbulence with hypo-viscosity, the inverse energy cascade will result in hypo-viscous dissipation removing all injected energy.

In Figure V.7, we show the ratio of the hypo-viscous energy dissipation rate, $\varepsilon_\mu$, to the total energy dissipation rate, $\varepsilon_T = \varepsilon_\nu + \varepsilon_\mu$, where the subscripts refer to viscosity and hypo-viscosity respectively. We observe an abrupt change in this ratio as the dimension is lowered towards two dimensions. This figure suggests that the transition between three and two dimensions in this manner is a discontinuous one. This is then at odds with what was seen in Figure V.3 suggesting the transition in the non-integer dimensional system is likely of a different nature than in the thin-layer system. What we can say is that in the region around $d \approx 2.05 - 2.10$ the flow appears to become essentially two-dimensional. Note, that there is still some hypo-viscous dissipation at $d = 3$, this is due to the finite Reynolds number used in the calculations.

We can probe the nature of the transition between forward to inverse cascade by considering the energy spectrum. In Figure V.8 we present the energy spectra from a number of EDQNM calculations between two and three dimensions. To facilitate comparison across dimensions, we have normalised all spectra by their value at the forcing wave-number $k_f$.

Our results here are consistent with what was found in [72], as the spatial dimension is lowered past a critical dimension $d_{c_1}$ the forward cascade inertial range exponent begins to vary towards -3. This value is achieved at a second critical dimension $d_{c_2}$, for $d$ lower than this value we obtain a two-dimensional spectrum.

This situation is then different to what was seen in the thin-layer case. There, the energy spectrum can exhibit both two- and three-dimensional features simultaneously. This is due to the two-dimensionalization being a scale dependent process due to geometric confinement. Here, on the other hand, the two-dimensionalization happens at all scales. This suggests that thinking of thin-layer turbulence as being a state between two and three spatial dimensions should not be taken literally.

Turning now to the maximal Lyapunov exponent, in Figure V.9 we display the results obtained from our EDQNM calculations. Here we plot simply the maximal exponent versus the spatial dimension without attempting to determine the scaling behaviour like we did in the thin-layer case. Interestingly, we find a divergence in the Lyapunov exponent around $d = 2.08$, this is within the transitional region we determined from the statistical measures of the system presented above. As such, it is clear the Lyapunov exponent is capturing this transitional behaviour and gives a clear representation of the transition point.

In [72], it was suggested that the Kolmogorov constant would become infinite at the
Figure V.8  Energy spectra from EDQNM calculations in $d = 2.01, 2.05, 2.15, 2.2, 2.5, 3$ scaled by the value at the forcing wave-number, $k_f = 8$. Colouring goes from dark to light as the dimension decreases. We also show in red the transitional case of $d = 2.07$.

Figure V.9  Maximal Lyapunov exponent, $\lambda$, against spatial dimension, $d$, for a range of dimensions between $d = 2$ and $d = 3$. 
critical dimension at which the energy cascade reverses direction. Our findings here are in agreement with this prediction and it appears this divergence in the Kolmogorov constant is well captured through the Lyapunov exponent.

V.5 Conclusions and Final Remarks

Using Lyapunov exponents in systems of complexity to study phase transitions has received little attention in the literature. Therefore, their utilization in this work on fluid turbulence, and the clarity of the results achieved, suggests this method may be of particular use in extended non-equilibrium systems in general. A particular application of our results may well be found in the next generation of numerical weather prediction models. For systems with multiple timescales, the Lyapunov exponent is proportional to the smallest characteristic timescale, regardless of the size of the fluctuations in the different timescales. In the atmosphere, predictions can be made beyond the limit imposed by the Lyapunov timescale, which is associated with turbulence, as the predictability is imposed by the large scale dynamics [139, 140]. However, within the last decade, increases in computing power have allowed for large-eddy simulations to be nested within numerical weather prediction models, whereby three dimensional turbulence is resolved explicitly. These high-resolution simulations have important applications in many areas, such as particle transport dispersion modelling and wind turbine site profiling [154].

Furthermore, the advent of exascale computing will see operational global weather models run at far greater resolution (< 1km), which will allow regional models to operate at scales where turbulent phenomena are explicitly resolved [157]. Therefore, our finding of a discontinuous change in predictability at $Q_{2D}(Re)$ indicates that understanding the transition between two- and three-dimensional turbulent regimes in the atmosphere may be essential for determining predictability in different weather scenarios in these future high-resolution regional models. Forecast skill could be improved, particularly in severe convective thunderstorms, by more accurately resolving the atmosphere's transition from predominantly two to three-dimensional turbulent motion, which occurs in convection, as the error growth may change rapidly across this transition [98, 99, 108, 238]. Additionally, in-situ aircraft observations have shown that in the hurricane boundary layer, a height-dependent transition between two- and three-dimensional turbulence occurs, and that the large-scale hurricane vortex feeds directly from the small scales [42]. Given this, our result that the transition from two- to three-dimensional dynamics is accompanied by a discontinuous change in the Lyapunov exponent, means that correctly resolving this transition in hurricane models will be necessary for correctly predicting changes in
intensity.

Based on the results of this study into geometrically confined turbulence alone, it is not yet possible to make any definitive claims regarding real-world atmospheric turbulence. However, the potential applications discussed in the preceding paragraphs suggest that further study of transitions in turbulence through the lens of predictability should be carried out. It is known that transitions between two- and three-dimensional turbulence occur in both rotating and stratified flows. Considering the transition in these cases from the point of view of triadic interactions presents a different picture than in the problem studied here. In our simulations, the flow is geometrically confined in an artificial way by varying the height of the domain, resulting in certain triads being removed entirely from the flow. This is not the case in rotating and stratified flows, where the effects of rotation and stratification will progressively damp certain triads. It is then possible that the predictability of these flows around the transition will not show the same behaviour as seen in thin-layer turbulence. Given that these flows better approximate true atmospheric turbulence than the problem studied in this work, understanding their predictability will allow more concrete claims to be made regarding the potential applications discussed here.

Additionally, our use of periodic boundary condition should be considered, of course, our numerical method relies of the use of periodic boundary conditions but it is important to understand the limitations they impose. Real world flows in the atmosphere will have boundary layers between stratified layers and with the ground. These will likely change the dimensional features seen at given Reynolds numbers. However in the case of the hurricane boundary layer discussed above, the phenomenology of HIT is found hence our there are likely to be situations and length-scales where our choice of boundary condition are compatible with real world flows.

From our investigation into non-integer dimensional turbulence between two and three dimensions, we find a number of similarities and differences between this case and thin-layer turbulence. Notably, in both we find a discontinuous behaviour for the Lyapunov exponents as a critical point is passed. However, the transition in dynamics in these systems are profoundly different. The thin-layer case is caused by a scale dependent confinement altering the cascade behaviour. whilst in the non-integer dimensional case, the cascade behaviour is altered at the level of the weightings of the triadic interactions.

A final caveat to consider when interpreting the findings presented here relating to thin-layer turbulence is the range of applicability of the Lyapunov exponent in numerical weather forecasting. In the atmosphere it is the large-scale dynamics that determine long-term predictability, hence, predictability is found beyond the timescale defined by the maximal
However, for local, short-time scales, predictability is dominated by the non-linear dynamics of the system, and thus the Lyapunov exponent is a useful measure. Hence, for the next generation of high resolution numerical weather prediction models, particularly those used in now-casting, the transitions in predictability as measured via the maximal Lyapunov exponent may become important to accurately resolve. For a more in depth discussion of some of these points see [127] and [215].

To summarize, we have studied the behaviour of the maximal Lyapunov exponent in thin-layer turbulence through the use of direct numerical simulation and in non-integer dimensional turbulence using the EDQNM approximation. Using this exponent, we have measured the point at which the energy cascade within the flow transitions. In the thin-layer case, this point was found to occur at \( Q/\sqrt{Re} \approx 0.25 \) when measured from Lyapunov exponent data, which is in agreement with the value obtained via more standard methods [217]. The nature of the transition when viewed through the Lyapunov exponent is abrupt and discontinuous. As the maximal exponent is determined by the small scale features of the flow, it is not sensitive to the transition from a purely forward cascade to a bidirectional cascade. This suggests the short time predictability of such bidirectional cascade systems is as in three dimensions. However, near the transition to a purely inverse cascade, the potentially discontinuous nature of the transition leaves the possibility for dramatic changes of the predictability time in this region. For the non-integer dimensional case, a discontinuous transition was also observed, however here it was a full divergence of the exponent.

Our results demonstrate that, almost paradoxically, the predictability of a system can change discontinuously even when other quantities, such as the energy, vary smoothly. As such, studying this transition via the Lyapunov exponent provides a complementary approach, and highlights the importance of resolving these effects in future models of atmospheric predictability.
This Thesis has explored the effect of spatial dimension on the dynamics of homogeneous isotropic turbulence. It has long been known that two- and three-dimensional turbulence exhibit remarkably different behaviour. In the work presented within this Thesis we have considered not only two- and three-dimensional turbulence, but also higher dimensions as well as non-integer dimensions. These studies have been centred around the measurement of the chaotic properties of these flows in different dimensions.

The main motivating idea tying together all the work presented in this Thesis is the problem of anomalous scaling in turbulence. The marriage of a freely varying spatial dimension and the measurement of the chaotic properties of the flow provides a novel way of approaching this problem. By treating the spatial dimension as a free parameter, we can modify the energy transfer properties of the Navier-Stokes equations in a self-consistent manner. This allows for a direct comparison with the real world three-dimensional case.

Here, we give an overview of the main results of the work carried out for this Thesis. Following this, we close by considering future extensions to this work.
VI.1 The (Non)-Universality of Information Production

The work presented in Chapter III was centred around the computation of the Lyapunov spectrum in both two- and three-dimensional turbulence. These measurements can only feasibly be carried out numerically and at exceptional computational expense. By obtaining a sufficient portion of the Lyapunov spectrum, we were able to compute both the Kolmogorov-Sinai entropy, which measures the rate of information production, and attractor dimension.

In both dimensions, we were focused on the scaling behaviour of these quantities. For the three-dimensional case, they are found to be determined solely by the Reynolds number of the flow, in agreement with the universality suggested by the K41 theory. However, the scaling of these quantities with the Reynolds number was found to be different than predicted by K41. Notably, the KS-entropy was found to scale as

\[ h_{\text{KS}} \sim \text{Re}^\alpha, \]  

(VI.1)

with \( \alpha = 2.65 \), whereas \( \alpha_{\text{K41}} = 2.75 \).

There are numerous possible reasons for such discrepancies. In the literature, anomalous scaling exponents are typically said to be as a result of internal intermittency. However, it has also been argued that finite Reynolds number effects may be responsible. Indeed, in the results presented here, only modest Reynolds numbers were achieved due to the aforementioned computational expense. Hence, it is likely that the anomalous scaling found was predominantly a result of finite Reynolds number effects. Although, this is not to say that future high Reynolds number measurements will certainly agree with the K41 values.

The two-dimensional picture was found to be far more complicated. Most importantly, both the KS-entropy and the attractor dimension were found to exhibit non-universal behaviour. This was not expected from the analytical predictions which suggested simple Reynolds number scaling as in three dimensions. Instead, dependencies of both system size and energy injection scale were observed with possible implications for atmospheric predictability.
VI.2 Vanishing of Chaos

In Chapter IV we studied both the statistical and chaotic properties of turbulence in dimensions greater than three. Whilst some results are presented from four dimensional DNS, in general, it is not computationally feasible to study higher dimensional turbulence in this way. As such, the majority of the results in this Chapter were obtained from numerical calculations of the Eddy Damped Quasi Normal Markovian closure approximation.

In our study of the statistical properties of higher dimensional turbulence, we focused on integer dimensions between three and twenty. As we raise the spatial dimension, we observe an increased bottleneck effect in the energy spectrum. This is suggestive of a change in the balance between energy transfer and dissipation as the dimension is increased. This may be related to the idea that as the dimension is increased it becomes more likely for triad interactions to result in the forward transfer of energy.

Further exploring this idea, the enstrophy production, and thus the velocity derivative skewness, were also measured. Here, it was found that enstrophy production peaks somewhere near five or six dimensions. Additionally, the skewness was found to peak around eight dimensions, implying a finite value in the infinite dimensional limit.

Turning to the chaotic properties, we now allow the spatial dimension to take on non-integer values and we focus on dimensions between three and eight. We observe a remarkable critical transition at $d \approx 5.8$ to a regime in which turbulence is no longer chaotic, that is, the maximal Lyapunov exponent becomes negative at all Reynolds numbers.

Given the vastly different chaotic properties in two versus three dimensions, and the role played by enstrophy production in that case, it seems likely the transition to a non-chaotic regime and the peak in enstrophy production are related. If such a link exists, it may suggest a connection between the critical dimension for chaos seen in this Thesis and a critical dimension of anomalous scaling in turbulence.

VI.3 Critical Transitions of Predictability

The measurement of the chaotic properties of the flow also allows the predictability of turbulence to be studied. Naturally, this is connected to atmospheric predictability and weather forecasting. This is of particular interest for the work of this Thesis due to the observations of co-existing two- and three-dimensional turbulence in the atmosphere.
By performing DNS of a geometrically confined thin-layer turbulent system in Chapter V, we study the transition between two- and three-dimensional dynamics. Most notably, we find that as the system moves from a mixed dynamical state to purely two-dimensional dynamics, there is a discontinuous jump in predictability, as measured through the maximal Lyapunov exponent.

Now, with the caveat that true atmospheric turbulence is far more complex than the thin-layer system studied in this Thesis, it is possible that this discontinuous predictability may play a role in real world atmospheric predictability. Indeed, given that dimensional transitions have been observed in hurricanes, it is plausible that these are accompanied by transitions in predictability.

VI.4 Future Work

The results presented in this Thesis raise a number of questions that should be investigated further. Here we discuss the most interesting directions for future study.

VI.4.1 Isomorphic Systems

In Chapter III the ideas of Ornstein concerning connections between systems with the same entropy were briefly considered. In short, two systems with the same KS-entropy are indistinguishable at the level of information production. It may then be possible to find a simpler system that is isomorphic to HIT, then results obtained in this simpler system could be used to derive analogous results for HIT. This is very much in the same spirit as the use of the ADS/CFT fluid-gravity correspondence relating relativistic turbulence to higher dimensional black hole dynamics.

VI.4.2 Non-Chaotic Turbulence

The critical transition to a non-chaotic regime reported in Chapter IV is of exceptional interest given the possible connections to a true critical dimension. However, since the results we have presented in this Thesis are from the EDQNM approximation, it is possible they will not be replicated in true Navier-Stokes turbulence. It is not clear whether an analytic approach is possible, although it seems unlikely. However, what will eventually be possible is DNS of turbulence in five and six dimensions. Once the computing power is available, the measurements made in Chapter IV should be replicated in DNS to determine
whether a transition to non-chaos is a true phenomenon. From here, connections to anomalous scaling and critical dimensions can be explored.

VI.4.3 Transitions in True Atmospheric Turbulence

The simple thin-layer turbulence studied in Chapter V is merely an approximation of real world atmospheric turbulence. Indeed, effects such as rotation and stratification are also known to influence the effective dimensionality of the flow and both are important for atmospheric dynamics. It would be of interest to perform numerical studies of ever more realistic turbulent flows to determine whether discontinuous predictability transitions should be expected in real world turbulence.
A.1 Geometric Coefficients

In Chapter II during the derivation of the EDQNM equations, a number of geometric coefficients resulting from the contractions of various operators were defined. Here we will go through the (tedious) algebra involved in relating these coefficients to the geometry of the wavevector triad \( \{k,p,q\} \).

![Figure A.1](image.png)  
*Figure A.1*  Geometry of triad condition between wavevectors.
A.1.1 \( a_{kpq,d} \)

We define \( a_{kpq,d} \) as in Chapter [11]:

\[
a_{kpq,d} = \frac{P_{\alpha\beta\gamma}(k) P_{\alpha\sigma\delta}(k) P_{\rho\alpha}(p) P_{\gamma\delta}(q)}{4k^2}. \tag{A.1}
\]

To begin we will first consider the product \( P_{\alpha\beta\gamma}(k) P_{\alpha\sigma\delta}(k) \) which can be expanded to give

\[
P_{\alpha\beta\gamma}(k) P_{\alpha\sigma\delta}(k) = \left( k_{\beta} P_{\alpha\gamma}(k) + k_{\gamma} P_{\alpha\beta}(k) \right) \left( k_{\sigma} P_{\alpha\delta}(k) + k_{\delta} P_{\alpha\sigma}(k) \right)
= \left( k_{\beta} \delta_{\alpha\gamma} - \frac{k_{\alpha} k_{\beta} k_{\gamma}}{k^2} + k_{\gamma} \delta_{\alpha\beta} \right) \left( k_{\sigma} \delta_{\alpha\delta} - \frac{k_{\alpha} k_{\sigma} k_{\delta}}{k^2} + k_{\delta} \delta_{\alpha\sigma} \right)
= k_{\beta} k_{\sigma} \delta_{\gamma\delta} + k_{\beta} k_{\delta} \delta_{\gamma\sigma} + k_{\gamma} k_{\sigma} \delta_{\beta\delta} + k_{\gamma} k_{\delta} \delta_{\beta\sigma} - \frac{4 k_{\beta} k_{\gamma} k_{\delta}}{k^2}.
\tag{A.2}
\]

Our next step is to multiply this expression by \( P_{\rho\alpha}(p) \) which results in

\[
P_{\alpha\beta\gamma}(k) P_{\alpha\sigma\delta}(k) P_{\rho\alpha}(p) = \left( k_{\beta} k_{\sigma} \delta_{\gamma\rho} + k_{\beta} k_{\delta} \delta_{\gamma\rho} + k_{\gamma} k_{\sigma} \delta_{\beta\rho} + k_{\gamma} k_{\delta} \delta_{\beta\sigma} - \frac{4 k_{\beta} k_{\gamma} k_{\delta}}{k^2} \right) \times \left( \delta_{\rho\sigma} - \frac{p_{\rho} p_{\sigma}}{p^2} \right)
= k^2 \delta_{\gamma\sigma} + (d-3) k_{\gamma} k_{\sigma} - \frac{(k \cdot p)^2}{p^2} \delta_{\gamma\delta} - \frac{k \cdot p}{p^2} k_{\gamma} p_{\delta}
- \frac{k \cdot p}{p^2} k_{\gamma} p_{\delta} + 4 \frac{(k \cdot p)^2}{k^2 p^2} k_{\gamma} k_{\delta}.
\tag{A.3}
\]

In the above we have made use of the fact that \( \delta_{aa} = d \). Continuing, we take the product of this expression and \( P_{\gamma\delta}(q) \) to find

\[
P_{\alpha\beta\gamma}(k) P_{\alpha\sigma\delta}(k) P_{\rho\alpha}(p) P_{\gamma\delta}(q) = 2(d-2)k^2 - (d-3) \left( \frac{(k \cdot p)^2}{p^2} + \frac{(k \cdot q)^2}{q^2} \right)
+ 2 \frac{(k \cdot p)(k \cdot q)(p \cdot q)}{p^2 q^2} - 4 \frac{(k \cdot p)^2 (k \cdot q)^2}{k^2 p^2 q^2}.
\tag{A.4}
\]

Now, defining the angles \( \alpha, \beta \) and \( \gamma \) as in Figure A.1, we have the following results

\[
x = \cos \alpha = -\frac{p \cdot q}{pq}, \quad y = \cos \beta = \frac{k \cdot q}{k q} \quad \text{and} \quad z = \cos \gamma = \frac{k \cdot p}{k p}
\tag{A.5}
\]

which can be used to simplify Equation (A.4) to the form

\[
P_{\alpha\beta\gamma}(k) P_{\alpha\sigma\delta}(k) P_{\rho\alpha}(p) P_{\gamma\delta}(q) = 2(d-2)k^2 - k^2 (d-3) \left( y^2 + z^2 \right) - 2k^2 xyz - 4k^2 y^2 z^2.
\tag{A.6}
\]
Our final result is then
\[ a_{kpqd} = \frac{1}{2} \left( (d-2) - \frac{d-3}{2} \left( y^2 + z^2 \right) - xyz - 2y^2z^2 \right) \]  
(A.7)

### A.1.2 \( b_{kpqd} \) and \( b_{kqp,q} \)

We now turn to the two \( b \) coefficients. Since they only differ by an exchange of \( p \) and \( q \) we will only consider \( b_{kpqd} \) in detail. We first recall from Chapter II that
\[ b_{kpqd} = P_{\alpha\beta\gamma}(k)P_{\beta\sigma\delta}(p)P_{\alpha\sigma}(k)P_{\gamma\delta}(q). \]  
(A.8)

As with \( a_{kpqd} \) we will work term by term starting with \( P_{\alpha\beta\gamma}(k)P_{\alpha\sigma}(k) \) which results in
\[ P_{\alpha\beta\gamma}(k)P_{\alpha\sigma}(k) = \left( k_\beta P_{\alpha\gamma}(k) + k_\gamma P_{\alpha\beta}(k) \right) \left( \delta_{\alpha\alpha} - \frac{k_\alpha k_\alpha}{k^2} \right) \]
\[ = \left( k_\beta \delta_{\alpha\gamma} - 2 \frac{k_\alpha k_\beta k_\gamma}{k^2} + k_\gamma \delta_{\alpha\beta} \right) \left( \delta_{\alpha\alpha} - \frac{k_\alpha k_\alpha}{k^2} \right) \]  
(A.9)

We then take the product with \( P_{\beta\sigma\delta}(p) \) to obtain
\[ P_{\alpha\beta\gamma}(k)P_{\beta\sigma\delta}(p)P_{\alpha\sigma}(k)P_{\gamma\delta}(q) = kp \left( \left( d - 5 \right) z + 4z^2 + \left( d - 1 \right) xy \right) \]
\[ + 2yz^2 + 2x^2z^2 + 4xyz^2 \].  
(A.11)

We can simplify this expression by considering the relationship between the angles \( \alpha, \beta \) and \( \gamma \), namely the fact that
\[ \alpha + \beta + \gamma = \frac{\pi}{2}, \]  
(A.12)

which then means
\[ x^2 + y^2 + z^2 = 1 - 2xyz. \]  
(A.13)
Using this relationship, we find

\[ P_{\alpha\beta\gamma}(k)P_{\beta\alpha\delta}(p)P_{\alpha\delta}(q)P_{\gamma\sigma}(q) = kp\left((d-5)z + 4z^3 + (d-1)xy + 2z(y^2 + x^2 + 2xyz)\right) \]
\[ = kp\left((d-5)z + 4z^3 + (d-1)xy + 2z(1-z^2)\right) \]
\[ = kp\left((d-3)z + (d-1)xy + 2z^3\right). \quad \text{(A.14)} \]

We then have that

\[ b_{kpqd} = \frac{p}{2k} \left((d-3)z + (d-1)xy + 2z^3\right), \quad \text{(A.15)} \]
and therefore

\[ b_{kqpd} = \frac{q}{2k} \left((d-3)y + (d-1)xz + 2y^3\right). \quad \text{(A.16)} \]

### A.2 Relationship between Coefficients

In Chapter II we claimed that the \(a\) and \(b\) geometric coefficients are related such that

\[ a_{kpqd} = \frac{b_{kpqd} + b_{kqpd}}{2}, \quad \text{(A.17)} \]

that is, the \(a\) term is a symmetrization of \(b\) in \(p\) and \(q\). To show this we first need to consider the fractions \(p/k\) and \(q/k\) which appear in the expressions for the \(b\) coefficients. If we consider the sine rule we find that

\[ \frac{p}{k} = \frac{\sin \beta}{\sin \alpha} = \sqrt{\frac{1-y^2}{1-x^2}} = \frac{\sqrt{\left(1-y^2\right)\left(1-x^2\right)}}{1-x^2}. \quad \text{(A.18)} \]

By expressing this in the form of the last equality we can use Equation\texttt{A.13} to make progress

\[
(1-y^2)(1-x^2) = 1 - x^2 - y^2 + x^2 y^2
= z^2 + 1 - x^2 - y^2 - z^2 + x^2 y^2
= z^2 + 2xyz + x^2 y^2
= (z + xy)^2.
\]
\[ \text{(A.19)} \]

Hence, we have

\[ \frac{p}{k} = \frac{z + xy}{1-x^2}. \quad \text{(A.20)} \]
and similarly

\[ \frac{q}{k} = \frac{y + xz}{1 - x^2}. \]  
(A.21)

We can now exploit these expressions to prove the relationship between \(a\) and \(b\) coefficients

\[ \frac{b_{kpq,d} + b_{kqp,d}}{2} = \frac{1}{4} \left( \frac{z + xy}{1 - x^2} \right) \left( (d - 3)z + (d - 1)xy + 2z^3 \right) \]
\[ + \frac{1}{4} \left( \frac{y + xz}{1 - x^2} \right) \left( (d - 3)y + (d - 1)xz + 2y^3 \right). \]  
(A.22)

A useful method to tackle this algebra is to split the \(b\) terms into their three dimensional forms plus a remainder, that is

\[ b_{kpq,d} = \frac{d - 3}{2} \frac{p}{k} (z + xy) + \frac{p}{k} (xy + z^3). \]  
(A.23)

We then use our expression for \(p/k\) and find

\[ b_{kpq,d} = \frac{d - 3}{2} \left( 1 - y^2 \right) + \frac{z + xy}{1 - x^2} (xy + z^3) = \frac{d - 3}{2} \left( 1 - y^2 \right) + b_{kpq,3}. \]  
(A.24)

which of course means

\[ b_{kqp,d} = \frac{d - 3}{2} \left( 1 - z^2 \right) + \frac{y + xz}{1 - x^2} (xz + y^3) = \frac{d - 3}{2} \left( 1 - z^2 \right) + b_{kqp,3}. \]  
(A.25)

Let us now consider the three dimensional problem of relating \(a\) and \(b\) where we have

\[ \frac{b_{kpq,3} + b_{kqp,3}}{2} = \frac{1}{2} \left( \frac{z + xy}{1 - x^2} \right) \left( xy + z^3 \right) + \frac{1}{2} \left( \frac{y + xz}{1 - x^2} \right) \left( xz + y^3 \right) \]
\[ = \frac{1}{2} \left( x^2 + y^2 + z^2 + xyz \right) \left( y^2 + z^2 \right) + 2xyz - 2y^2 z^2 \]
\[ = \frac{1}{2} \left( x^2 + y^2 + z^2 + xyz \right) \left( y^2 + z^2 \right) + 2xyz - 2y^2 z^2 \]  
(A.26)

we now use Equation \(A.13\) on both terms in the first product to find

\[ \frac{b_{kpq,3} + b_{kqp,3}}{2} = \frac{1}{2} \left( 1 - xyz \right) \left( 1 - x^2 - 2xyz \right) + 2xyz - 2y^2 z^2 \]
\[ = \frac{1}{2} \left( 1 - xyz \right) \left( 1 - x^2 \right) \left( 1 - x^2 \right) + 2y^2 z^2 \]
\[ = \frac{1}{2} \left( 1 - xyz - 2y^2 z^2 \right) \]
\[ = a_{kpq,3}. \]  
(A.27)
From here, we now look at the difference between the $d$ dimensional $a$ coefficient and it’s three dimensional form

$$a_{kpqd} - a_{kpq;3} = \frac{1}{2} \left( (d-2) - \frac{d-3}{2} (y^2 + z^2) - xyz - 2y^2 z^2 - 1 - xyz - 2y^2 z^2 \right)$$

$$= \frac{1}{2} \left( \frac{d-3}{2} (2 - x^2 - y^2) \right)$$

$$= \frac{b_{kpqd} + b_{kqp;d}}{2} - \frac{b_{kpq;3} - b_{kqp;3}}{2}.$$  \hfill (A.28)

Therefore we have shown

$$a_{kpqd} = \frac{b_{kpqd} + b_{kqp;d}}{2}. \hfill (A.29)$$

### A.3 Quasi Normal Equation

Here we make use of the properties of the $a$ and $b$ coefficients to simplify Equation [II.24] which we reproduce here for convenience

$$\left( \frac{\partial}{\partial t} + \nu (k^2 + p^2 + q^2) \right) T(k) = \frac{A_d k^{d-1}}{2} \int dp \int dq \left[ P_{a\alpha \delta}(k) P_{\beta \delta}(p) P_{\gamma \delta}(q) C(p) C(q) 
- P_{\beta \alpha \delta}(p) P_{a \alpha \delta}(k) P_{\gamma \delta}(q) C(k) C(q) 
- P_{\gamma \alpha \delta}(q) P_{a \alpha \delta}(k) P_{\beta \delta}(p) C(k) C(p) \right] \delta(k - p - q).$$  \hfill (A.30)

We begin by replacing the various operator contractions by the corresponding geometric coefficient giving us

$$\left( \frac{\partial}{\partial t} + \nu (k^2 + p^2 + q^2) \right) T(k) = \frac{A_d k^{d+1}}{2} \int dp \int dq \left[ 4 a_{kpqd} C(p) C(q) 
- 2 b_{kpqd} C(k) C(q) - 2 b_{kqp;d} C(k) C(p) \right] \delta(k - p - q),$$  \hfill (A.31)

if we now replace $a$ we find

$$\left( \frac{\partial}{\partial t} + \nu (k^2 + p^2 + q^2) \right) T(k) = \frac{A_d k^{d+1}}{2} \int dp \int dq \left[ 2 b_{kpqd} (C(p) C(q) - C(k) C(q)) 
+ 2 b_{kqp;d} (C(p) C(q) - C(k) C(p)) \right] \delta(k - p - q).$$  \hfill (A.32)
Since both \( p \) and \( q \) are integration variables in the above we can perform a relabeling of \( p \rightarrow q \) in the second term of the integrand which gives us

\[
\left( \frac{\partial}{\partial t} + \nu (k^2 + p^2 + q^2) \right) T(k) = 2A_d k^{d+1} \int \, dp \int \, dq \left[ b_{kpq,d} \left( C(p)C(q) - C(k)C(q) \right) \right] \delta(k-p-q).
\]

(A.33)
In Chapter II in our derivation of the EDQNM equation we encounter an integral of the form

\[ I(k) = \int dp \int dq \, f(k, p, q) \delta(k - p - q), \]  

and claim it can be transformed into a two dimensional integral. In this Appendix, we show how this may be done. The delta function ensures that the integral is restricted to vectors such that \( k, p \) and \( q \) can form a triangle as in Figure A.1. We begin by performing the trivial \( q \) integral which results in

\[ I(k) = \int_{q = k - p} dp \, f(k, p, q). \]  

Since we are concerned with isotropic turbulence, the function \( f(k, p, q) \) is then a function of triangle shape but not of orientation due to isotropy. We can exploit this symmetry by making use of \( d \)-dimensional spherical polar coordinates then \( \int dp \) is made up of one radial integral and \( d - 1 \) angular integrals. As \( f(k, p, q) \) does not depend on triangle orientations we are free to rotate our coordinate system such that the azimuthal angle is the angle \( \gamma \) between \( k \) and \( p \) shown in Figure A.1. Then we have

\[ I(k) = A_{d-1} \int_0^\infty dp \, p^{d-1} \int_0^\pi d\gamma (\sin \gamma)^{d-2} f(k, p, q), \]

where the properties of \( f(k, p, q) \) mean it only depends on the magnitude of \( p \) and on the angle between \( k \) and \( p \) through \( q \). All other angular variables can be simply integrated to
give \( A_{d-1} \).

Our next step is to make a change of variables from \( \gamma \) to \( z = \cos \gamma \) with \( dz = -\sin \gamma \, dy \) giving us

\[
I(k) = A_{d-1} \int_0^\infty dp \, p^{d-1} \int_{-1}^1 dz \, (\sin \gamma)^{d-3} f(k, p, q). \quad (B.4)
\]

From here we invoke the law of cosines

\[
q^2 = k^2 + p^2 - 2kpz,
\]

(B.5)

to allow us to compute the Jacobian needed to transform from

\[
dpdz = \left| \frac{\partial(p, z)}{\partial(p, q)} \right| dpdq.
\]

(B.6)

It is then readily shown that the Jacobian is evaluated to give

\[
\left| \frac{\partial(p, z)}{\partial(p, q)} \right| = -\frac{q}{kp}.
\]

(B.7)

We make a final alteration using the law of sines to replace the \( \sin \gamma \) factor by \( \sin \alpha \) such that we find

\[
dp = -(\sin \alpha)^{d-3} \left( \frac{qp}{k} \right)^{d-2} dpdq.
\]

(B.8)

The limits on the \( q \) integration can be found from the cosine law and we obtain from \( z = 1 \) that \( q = |k - p| \) and from \( z = -1 \) we find \( q = k + p \) therefore we have

\[
I(k) = A_{d-1} \int_0^\infty dp \int_{|k-p|}^{k+p} dq \, (\sin \alpha)^{d-3} \left( \frac{qp}{k} \right)^{d-2} f(k, p, q),
\]

(B.9)

or as we write in the main text of Chapter II

\[
I(k) = A_{d-1} \int_{\Omega(k)} dpdq (\sin \alpha)^{d-3} \left( \frac{qp}{k} \right)^{d-2} f(k, p, q),
\]

(B.10)

where \( \Omega(k) \) is the integration region described in the previous integral. Using this relation we can go from equation [II.39] to [II.40]
APPENDIX

C

EDQNM VOLUME FRACTIONS

To numerically evaluate the integral over triads on the right hand side of the EDQNM equation requires a method of discretising the three dimensional integration region $\Omega$. Our discretisation process, as described in Chapter II, involves a truncation and partitioning of $(k,p,q)$-space such that $k_{\text{bot}} \leq k_i, p_m, q_n \leq k_{\text{top}}$. To each point $(k_i, p_m, q_n)$ there is an associated volume $V_{i mn} = \Delta k_i \Delta p_m \Delta q_n$. This is the volume of a cuboid whose sides are determined by a logarithmic partitioning of space, this is described in Chapter II.

It is possible for the volume $V_{imn}$ to be intersected by the boundary of $\Omega$. In such cases, we need to determine the fraction $v_{imn}$ of $V_{imn}$ that lies within $\Omega$ to correctly weight the points in the numerical integral. The determination of these volume fractions is a somewhat complex exercise in geometry. The most straightforward geometrical approach to this problem is to first consider the possible intersections a volume and $\Omega$ can have.

We recall that $\Omega$ is the interior of the planes $k - p - q = 0$, $p - q - k = 0$ and $q - k - p = 0$ with $k, p, q > 0$. This domain is completely symmetric in $k, p, q$ hence we only need to consider points with $k_i \geq p_m \geq q_n$. We first look at the case $k_i = p_m = q_n$, to determine if such volumes can have sub-volumes lying outwith $\Omega$ we consider the corners of the volume. These volumes have 8 corners given by the combinations of the points $(k_{i \pm 1/2}, p_{m \pm 1/2}, q_{n \pm 1/2})$, however we can use the symmetry of the problem to reduce the number of points we need to check. If we look at the points with $k = k_{i + 1/2}$ then we can check the plane $k - p - q = 0$ and by the symmetry of the domain and the corner points
same will apply to the $p - q - k = 0$ and $q - k - p = 0$ planes for the $p_{m+1/2}$ and $q_{n+1/2}$ points respectively.

To set the notation we will use throughout this Appendix, we define $X_1 = k_{l+1/2} - p_{m-1/2} - q_{n-1/2}$, $X_2 = k_{l+1/2} - p_{m-1/2} - q_{n+1/2}$, $X_3 = k_{l+1/2} - p_{m+1/2} - q_{n-1/2}$ and $X_4 = k_{l+1/2} - p_{m+1/2} - q_{n+1/2}$ such that for $k_i = p_m = q_n$ we have $X_1 > X_2 = X_3 > X_4$. These are simply the points representing the four corners inserted into the plane equation $k - p - q$. If $X_i > 0$ then that corner lies outwith the domain and thus there is an intersection, then by symmetry the volume intersects all three planes. We have

$$X_1 = 2^{1/2} - 2^{1/2} - 2^{1/2} = 2^{1/2} - 2 \times 2^{1/2} = 2^{1/2} - 2^{1/2} - 2^{1/2},$$

(C.1)

therefore if $F \geq 1$ then $X_1 \leq 0$ and the volume intersects all three planes. Given that $X_1$ is the largest term, and we are interested in $F > 2$, this means for $k_i = p_m = q_n$ we find $v_{imn} = 1$ and there are no intersections.

Let us now consider $k_i > p_m > q_n$, where we now have $X_1 > X_2 > X_3 > X_4$. Given what we have seen for the case of equality, for $F \geq 1$ it is only possible for these volumes to intersect the $k - p - q = 0$ plane as the distance to the other planes is greater in this case than in the equal case. If $0 > X_1 > X_2 > X_3 > X_4$ then $v_{imn} = 1$. Now, if only $X_i > 0$, we have a single corner outside of the domain. In Figure C.1 we present a graphical representation of this situation, here and in subsequent Figures the $x_i$ correspond to the points $(k, p, q)$ associated with each $X_i$. To compute the volume fraction we need to find the volume of the right angle triangular pyramid depicted with dashed line. This can be found as

$$V_{\text{pyramid}} = \frac{\Gamma \Lambda \Delta}{6},$$

(C.2)

To begin we find the distance from $x_1$ to the $k - p - q = 0$ plane in the $k$-direction, denoted as $\Gamma$. At $x_1$ we have $k = k_{l+1/2}$ and on the plane we have $k = p_{m-1/2} + q_{n-1/2}$. Hence, we find

$$\Gamma = |k_{l+1/2} - p_{m-1/2} - q_{n-1/2}| = |X_1|.$$

(C.3)

Following a similar procedure for $\Lambda$ in the $p$-direction and $\Delta$ in the $q$-direction we find $\Gamma = \Lambda = \Delta = |X_1|$ and thus for this case the volume fraction within the integration domain is

$$v_{imn} = \frac{V_{imn} - V_{\text{pyramid}}}{V_{imn}} = \frac{V_{imn} - X_1^3/6}{V_{imn}}$$

(C.4)

Now for the case $X_1 > X_2 > 0 > X_3 > X_4$ we have two corners outside the integration domain, as seen in Figure C.2. Once again we are interested in computing the volume of the region.
Figure C.1  Volume fraction for one corner outside of integration domain for $k_i > p_m > q_n$

Figure C.2  Volume fraction for two corners outside of integration domain for $k_i > p_m > q_n$
shown in dashed lines, however, in the case the volume is more complicated. We can consider it as the volume of the large right triangular pyramid, $V_{\text{pyramid}}^{(1)}$, represented by dashed and dotted lines minus the volume of the dotted line pyramid, $V_{\text{pyramid}}^{(2)}$. Following the steps for the single corner case we find

$$V_{\text{pyramid}}^{(1)} = \frac{X_3^3}{6},$$

and

$$V_{\text{pyramid}}^{(2)} = \frac{X_2^3}{6}.$$  

Thus, in this case we have

$$v_{i\text{mn}} = \frac{V_{i\text{mn}} - \left(X_1^3 - X_2^3\right)/6}{V_{i\text{mn}}}$$

Now we consider three corners outside the domain as depicted in Figure C.3. This corresponds to $X_1 > X_2 > X_3 > 0 > X_4$. Here we find the volume of the dashed domain as before, considering the volume of the large pyramid formed of both dashed and dotted lines, $V_{\text{pyramid}}^{(1)}$ minus the volumes of the smaller pyramids, $V_{\text{pyramid}}^{(2)}$ and $V_{\text{pyramid}}^{(3)}$. Doing so we find

$$V_{\text{pyramid}}^{(1)} = \frac{X_3^3}{6},$$

$$V_{\text{pyramid}}^{(2)} = \frac{X_2^3}{6}.$$
Figure C.4  Volume fraction for two corners outside of integration domain for $k_i = p_m > q_n$

and

$$V_{\text{pyramid}}^{(3)} = \frac{X_3^3}{6}. \quad (C.10)$$

Therefore, our volume fraction in this case is

$$v_{i,m,n} = \frac{V_{i,m,n} - (X_1^3 - X_2^3 - X_3^3)/6}{V_{i,m,n}}. \quad (C.11)$$

We now look at the case of $k_i > p_m > q_n$ and four corners outside the domain, that is $X_1 > X_2 > X_3 > X_4 > 0$. It would be straightforward to form the required volume in this case following the same procedure as the previous cases. However, what we find is that if $X_4 > 0$ then $k_i - p_i - q_i > 0$ and thus the central point lies outside the domain. This causes issues for our numerical scheme as quantities like $\sin \alpha$ in the EDQNM equation are not defined outside the domain. It is possible to introduce a symmetric averaging procedure to obtain values for the part of the volume that does lie in the domain [35,132]. However, for simplicity in our numerical scheme we simply set such volume fractions to be zero.

All that remains now is to consider the cases of $k_i > p_m = q_n$ and $k_i = p_m > q_n$. For the former case volumes can only be intersected by one domain plane boundary and thus has the same volume fractions as for $k_i > p_m > q_n$. However, volumes in the latter case can be intersected by two of the domain plane boundaries. These planes are $k - p - q = 0$ and $p - k - q = 0$.

There are only two cases to consider now. When there are two corners outside the domain, that is one outside the $k - p - q = 0$ plane and one outside the other, which occurs when only $X_1 > 0$. We can see this depicted in Figure C.4. This case is very similar to the first case.
Figure C.5  Volume fraction for four corners outside of integration domain for $k_i = p_m > q_n$

when $k_i > p_m > q_n$, but we have two equal volume pyramids to remove. Hence we have

$$v_{imn} = \frac{V_{imn} - X_1^3/3}{V_{imn}}. \quad (C.12)$$

Then in Figure C.5 we see the case $X_1, X_2 > 0$ when four corners are outside the domain. This case mirrors that of the two corner case when $k_i > p_m > q_n$ but with two of each small pyramid. Therefore,

$$v_{imn} = \frac{V_{imn} - (X_1^3 - X_2^3)/3}{V_{imn}}. \quad (C.13)$$
For completeness, here we present the values used for the Kolmogorov constant in each dimension alongside the corresponding value of the free parameter. For the Kolmogorov constant the values were obtained from DNS results for three and four dimensions \cite{20} whilst in higher dimensions results obtained in \cite{87} by using the Lagrangian renormalised approximation (LRA) \cite{109} are used. The appropriate value for the free parameter in each dimension is obtained using the method in Chapter \cite{1}.

\[
\begin{array}{ccc}
 d & C_d & A \\
 3 & 1.72 & 0.49 \\
 4 & 1.33 & 0.366 \\
 5 & 1.16 & 0.28 \\
 6 & 1.08 & 0.23 \\
 7 & 1.03 & 0.195 \\
 10 & 0.952 & 0.134 \\
 20 & 0.877 & 0.066 \\
\end{array}
\]

\textbf{Table D.1}  \textit{Free parameters for a number of different spatial dimensions and the Kolmogorov constant they correspond to.}


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