Stability, Hilbert Scheme and PT Moduli of Genus Four Curves and Failure of the MMP/Wall-Crossing Correspondence

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Abstract

Inspired by concepts in string theory, the notion of stability conditions on triangulated categories was introduced by Bridgeland in 2002. Its impact across mathematics includes the solutions of classical problems in algebraic geometry, which were hard to tackle directly. This concept leads to a wall-crossing machinery: there is a manifold of stability conditions, with a wall-and-chamber decomposition, such that the moduli space of stable objects only changes as we cross a wall. This has many geometrical applications.

In the first part, we show that wall-crossing transformations can be more involved than was previously known, by proving the existence of a wall-crossing with unexpected behaviour. In particular, it fails an expected correspondence between wall-crossing and birational transformations. This significantly complicates the overall picture in this fundamentally important correspondence to applications of stability conditions to algebraic geometry. In the second part, we apply the machinery to answer some basic questions about the classical Hilbert scheme of canonical genus four curves in $\mathbb{P}^3$ via an effective control over its wall-crossing. The strategy uses the space of PT-stable pairs as an intermediate step.

Lay Summary

Classical algebraic geometry studies the set of solutions of systems of polynomials. Classical birational geometry considers the classification problem in algebraic geometry. Since the revolution of algebraic geometry by Grothendieck, moduli spaces are one of its most fundamental tools: they parametrize geometric objects of some predetermined type. The most classical case are Hilbert schemes, parametrizing subspaces of a given space; these are very nasty spaces though: there is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme (Murphy's law, [13, Law 1.34]).

Stability conditions on Derived Categories developed by Bridgeland to give mathematical foundations to a concept of stability in modern physics and string theory. They allow us to modify moduli spaces via so-called wall-crossing transformation.

This thesis has found novel features and applications of wall-crossing: On the one hand, understanding the geometry of some non-previously known Hilbert scheme as well as some other relevant moduli spaces (important in enumerative geometry), describing the components and their intersections as much as possible, and on the other hand, finding a surprising fact in birational geometry which breaks down the expectation of existence of some well-known correspondence.

Contents

Chapter 1

Introduction

In geometry, one of the most interesting problems is related to the moduli spaces (or parameter spaces) which parametrize certain objects of our interest. For instance, one can consider the space of all curves of a given degree and genus in the 3-dimensional spaces.

In algebraic geometry, the geometric objects of our interest usually turn to some sheaf theoretical objects. To have a "nice" moduli space of algebro-geometric objects, we should restrict to the "stable" ones among them; i.e., for any kind of parametrization problem, we need a notion of "stability conditions".

Hilbert schemes are among the most interesting moduli space. As an example, the Hilbert scheme of curves parametrizes the ideal sheaves of our favorite curves in an ambient space; in this example, curves in an ambient space as geometric objects turn to the ideal sheaves of curves. Another important example of moduli spaces are moduli spaces of Pandharipande-Thomas stable pairs. Both can be thought as a sheaf theoretic interpretation of the problem of counting curves in the sense of Gromove-Witten invariants and stable maps.Working with sheaf theoretic objects means working with complexes of coherent sheaves: e.g. an ideal sheaf (in the case of the Hilbert scheme) can be thought as a one-term complex, or stable pairs can be thought as two-term complexes, and so on. Complexes will behave the best if we consider them as objects in the derived categories of coherent sheaves.

For derived categories, and more generally for triangulated categories, there is the notion of "Bridgeland stability conditions" ([8]). Therefore, if we can construct a Bridgeland stability condition on the derived category of coherent sheaves on the ambient space, $\mathrm{D}^d(X)$, then many sheaf theoretical moduli spaces can be detected as moduli spaces associated to chambers in the space of all Bridgeland stable objects, $\mathrm{Stab}(X)$, and one can reach from one to another just by crossing some walls.

The main goals of this project are:

(I) Introducing a (birationally) surprising wall-crossing on the way to the large volume limit in $\mathrm{Stab}(\mathbb{P}^3)$ for canonical genus four curves,

(II) Describing the corresponding space of PT stable pairs during the journey,

(III) Understanding the geometry of the Hilbert scheme $\mathcal{H}ilb^{6t-3}(\mathbb{P}^3)$ as much as possible.

Our method uses Bridgeland stability conditions on $\mathrm{D}^b(\mathbb{P}^3)$ constructed in [6] by Bayer, Macrì and Toda. They suggested a (conjectural) Bridgeland stability condition on threefolds, and for the case of $\mathbb{P}^3$, Macrì proved that the conjecture comes true ([19]). Existence of Bridgeland stability, and in particular, support property guarantees that there exists a wall-chamber decomposition of the stability manifold, and thus we can operate a wall-crossing on $\mathrm{Stab}(\mathbb{P}^3)$. By choosing a path in the stability space and crossing all the walls on our way, we will be able to start from an efficient compactification of our main component and end up with the moduli space of stable pairs as well as the Hilbert scheme. Furthermore, we will describe an unexpected birational phenomenon on the way which corresponds to a special wall-crossing. In chapter 2, we will introduce the tilt and Bridgeland stability conditions. Chapter 3 is a computational chapter listing the walls and computing the Ext-groups. In chapter 4, using a constructive way to understand the intersection of the components, we will prove the following Theorem which describes a new birational phenomenon and fails a well-known correspondence on the Minimal Model Program (MMP):

Theorem 1.0.1. *Fix $v = (1, 0, -6, 15)$. There is a wall-crossing with respect to Bridgeland stability conditions $\mathcal{N}_2 \to \mathcal{N}_3$ with the following properties:*

(i) $\mathcal{N}_2$ is a smooth and irreducible variety,

(ii) $\mathcal{N}_3 = \widetilde{\mathcal{N}_2} \cup \mathcal{N}_3'$, where $\widetilde{\mathcal{N}_2}$ is birational to $\mathcal{N}_2$ and $\mathcal{N}_3'$ is a new irreducible component,

(iii) There is a diagram

$$\begin{array}{ccc} \mathcal{N}_2 & & \widetilde{\mathcal{N}_2} \\ \text{small contraction } (\phi) \searrow & & \swarrow \text{ divisorial contraction } (\psi) \\ & \mathcal{W} & \end{array}$$

where both ϕ and ψ have relative Picard rank 1. In particular, $\widetilde{\mathcal{N}_2}$ is not $\mathbb{Q}$-factorial.

(Notice that we consider $\mathcal{N}_3$ as an algebraic space given by the reduced part of the moduli space defined by the union $\widetilde{\mathcal{N}_2} \cup \mathcal{N}_3'$.)

Furthermore, we give a precise description of the intersection of the components in Theorem 4.5.18. Finally, in chapter 5, after we the description of (the reduced part of) the moduli space of stable pairs in Theorem 5.6.1, we give a description of (the reduced part of) the Hilbert scheme:

Theorem 1.0.2. *The Hilbert scheme $\mathcal{H}ilb^{6t-3}(\mathbb{P}^3)$ has components birational to:*

1) The main component, $\mathscr{H}_{CM}$, which is a $\mathbb{P}^{15}$-bundle over $|\mathcal{O}(2)|$ (24-dimensional),

2) $\mathscr{H}'_{CM}$ which generically parametrizes the union of a plane quartic with a thickening of a line in the plane (28-dimensional),

3) $\mathscr{H}_1$ which generically parametrizes the disjoint union of a line in $\mathbb{P}^3$ and a plane quintic together with 1 floating point (30-dimensional),

4) $\mathscr{H}_2$ which generically parametrizes the union of a line in $\mathbb{P}^3$ and a plane quintic together with 2 floating points, and

5) $\mathscr{H}_6$ which generically parametrizes a plane sextic together with 6 floating points.

The first four components are irreducible.

Chapter 2

Bridgeland stability conditions on $\mathbb{P}^3$

2.1 Introduction and notation

Stability conditions on derived categories is a tool originally developed by Tom Bridgeland almost 20 years ago in order to give mathematical foundations to the concept of π-stability in string theory. In algebraic geometry, this machinery allows us to modify moduli spaces via so-called wall-crossing transformation. In this section, we will briefly introduce the notion of Bridgeland stability conditions and construct it precisely on $\mathbb{P}^3$, which is needed in this thesis.

Notation and convention. In this thesis, $\mathrm{D}^b(X)$ denotes the derived category of coherent sheaves on X. The notation $\otimes$ denotes derived tensor, unless otherwise is explicitly stated. Also, when there is no confusion, the subobject and the quotient of the defining short exact sequence of any wall will be denoted by A and B, respectively. In Chapter 4, we denote by σ_0, σ_- and σ_+, the stability conditions on the wall $\mathcal{W}$, in the chambers $\mathcal{N}_2$ and $\mathcal{N}_3$, respectively. Notice that as we will see $\mathcal{N}_1$ is a projective bundle, and hence $\mathcal{N}_2$ as its blow up at a locus, is a smooth reduced moduli space. We consider the spaces associated to the upper chambers, $\mathcal{N}_i$ as algebraic spaces given by the reduced part of the moduli spaces.

2.2 Stability on Abelian categories

Starting with the notion of slope-stability on curves, we would give some intuition on the abstract definition of the stability conditions. Let C be a smooth projective curve. For any vector bundle $\mathcal{E}$ on C, there are two associated numerical invariant: one is rank of $\mathcal{E}$, $\mathrm{rk}(\mathcal{E})$, and the other is the degree of $\mathcal{E}$, $\deg(\mathcal{E})$. Then we can define the ***slope*** of $\mathcal{E}$ as:

$$\mu(\mathcal{E})\colon = \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}.$$

We say that $\mathcal{E}$ is ***slope-(semi)stable*** if for each $\mathcal{F} \subset \mathcal{E}$, we have $\mu(\mathcal{F})(\leq) < \mu(\mathcal{E})$.

For the sake of generalization, we want to somehow abstract the notion of slope: having $\mathrm{rk}(\mathcal{E})$ and $\deg(\mathcal{E})$ in hand, instead of the slope, we could define the function $Z\colon Coh(C) \to \mathbb{C}$

as below:

$$Z(\mathcal{E})\colon = -\deg(\mathcal{E}) + i\mathrm{rk}(\mathcal{E}).$$

This time we can define the slope as

$$\mu(\mathcal{E})\colon = -\frac{\mathrm{Re}(Z)}{\mathrm{Im}(Z)}.$$

Inspired by this, we want to define stability conditions on an abelian category $\mathcal{A}$:

Definition 2.2.1. A pair $(\mathcal{A}, Z)$ is ***stability conditions*** if Z is a group homomorphism, called a central charge $Z\colon K_0(\mathcal{A}) \to \mathbb{C}$ where $K_0(\mathcal{A})$ is the Grothendieck group of $\mathcal{A}$, such that

- For each non-zero object E in $\mathcal{A}$, we have $Im(Z(E)) \geq 0$ and if $Im(Z(E)) = 0$, then $\mathrm{Re}(Z(E)) < 0$,
- For any non-zero object E in $\mathcal{A}$, there is a ***Harder-Narasimhan*** filtration

  $$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

  where E_i are objects in $\mathcal{A}$ and $A_i := E_i/E_{i-1}$ are semistable objects with $\mu(A_i) \geq \mu(A_{i-1})$ for each i.

Example 2.2.1. Let C be a projective curve, and define Z as above $Z(\mathcal{E}) := -\deg(\mathcal{E}) + i\mathrm{rk}(\mathcal{E})$. Then $(\mathcal{A} = Coh(C), Z)$ is a stability condition.

Remark 2.2.2. Let X be a projective variety of dimension $n \geq 2$. Note that for the abelian category $Coh(X)$, we cannot find any central charge with above properties such that factorize via Chern character. For example, if X is a surface, for any torsion sheaf, the generalised central charge (similar to the one in Example 2.2.1) $Z = -ch_1 + i.ch_0$ would vanish.

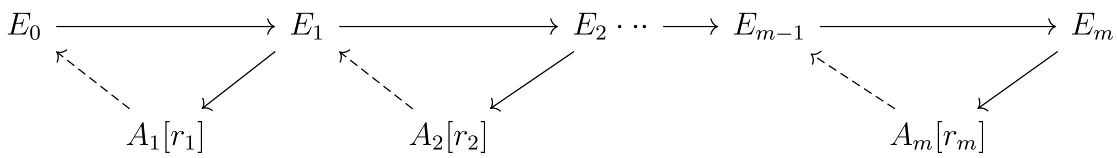
Having this Remark in mind, for higher dimensional varieties, instead of Coherent sheaves, we need to find another abelian category inside $\mathrm{D}^b(X)$ and introduce the Bridgeland stability conditions.

2.3 Stability on derived categories (Bridgeland stability)

Let X be an n-dimensional projective variety. In order to generalize the notion of stability conditions from the abelian category $Coh(X)$ in the derived category, we think of it as a heart of a bounded t-structure:

Definition 2.3.1. A ***heart of a bounded t-structure*** $\mathcal{A}$ on $\mathrm{D}^b(X)$ is a full additive subcategory of $\mathrm{D}^b(X)$ such that

- $Hom(A[i], B[j]) = 0$ for all $A, B \in \mathcal{A}$ and $i > j$.
- For any $E \in \mathcal{A}$, there are objects $E_i \in \mathrm{D}^b(X)$, $A_i \in \mathcal{A}$, and integers $r_1 > \cdots > r_m$ such that there are triangles



Remark 2.3.1. We can see that $\mathcal{A}$ is an abelian category. Also, it is easy to see that $K_0(\mathcal{A}) = K_0(X) = K_0(\mathrm{D}^b(X))$.

As we briefly mentioned in the previous section, in the case of curves we associate numerical invariant, or a number (or vector) in $\mathbb{C}$ to each vector bundle on a curve to define slope. The idea is to mimic this for a heart of a bounded t-structure. As in the case of abelian categories, we want to have a central charge $Z\colon K_0(\mathcal{A}) \to \mathbb{C}$ for any heart of a bounded t-structure $\mathcal{A}$. For higher dimension X, there are more than two numerical invariants and associating complex numbers is more tricky. The rough idea is that we want to associate a vector (depending on the numerical invariants), and then associate a complex number to that vector. Basically, we want to consider a vector in a finite rank lattice (usually the Chern character) as the associated vector; so we fix a finite rank lattice Λ and a group homomorphism $v\colon K_0(X) \twoheadrightarrow \Lambda$, such that the central charge factor via this morphism. We fix a norm on $\Lambda_{\mathbb{R}}$, and denote it by $\|\|$.

Definition 2.3.2. Let X be a variety of dimension n. A pair $\sigma = (\mathcal{A}, Z)$ is a Bridgeland stability conditions on $\mathrm{D}^b(X)$ if

- $\mathcal{A}$ is a heart of a bounded t-structure,
- The central charge $Z\colon \Lambda \to \mathbb{C}$, is an additive homomorphism,
- For any non-zero object E in the heart, we have $Z(v(E)) \in \mathbb{H} \cup \mathbb{R}_{<0}$, where $\mathbb{H}$ is the upper half plane in $\mathbb{C}$,
- Support property: we have

$$inf\left\{\frac{|Z(v(E))|}{\|v(E)\|} : E \textit{ non-zero semitable object in } \mathcal{A}\right\} > 0.$$

The main aspect of the support property is that it induces a wall-chamber structure on the stability manifold $\mathrm{Stab}(X)$: Let $\mathrm{Stab}(X)$ be the set of all (Bridgeland) stability conditions on $\mathrm{D}^b(X)$ with respect to the lattice Λ and v (we could use Z and Zov interchangeably, if there is no confusion as we need to do so to apply the following Theorem). Bridgeland gave this set the structure of a complex manifold as a consequence of the following Theorem:

Theorem 2.3.2 ([8, Theorem 1.2]). *Let* D *be a triangulated category. For each connected component* $\mathrm{Stab}(D)^\dagger \subset \mathrm{Stab}(D)$ *there are a linear subspace* $V(\mathrm{Stab}(D)^\dagger) \subset Hom_{\mathbb{Z}}(K_0(D), \mathbb{C})$*, with a well-defined linear topology, and a local homeomorphism* $\mathcal{Z} : \mathrm{Stab}(D)^\dagger \to V(\mathrm{Stab}(D)^\dagger)$ *which maps a stability condition* $(Z, \mathcal{A})$ *to its central charge* Z*.*

2.4 Tilt and Bridgeland stability on $\mathbb{P}^3$

In this section, we define stability conditions on the derived category of coherent sheaves of $\mathbb{P}^3$ (as the ambient space in this thesis), following the construction in [6]. We have already explained about the slope stability, and to construct Bridgeland stability on $\mathbb{P}^3$, we need to first construct a so-called tilt stability. Let H be an ample divisor on $\mathbb{P}^3$. Let $\mathrm{Coh}(\mathbb{P}^3)$ be the abelian category of coherent sheaves (as an initial heart of a bounded t-structure) on $\mathbb{P}^3$. Let $\alpha > 0, \beta < 0$ be two real numbers. Also define the ***twisted slope function*** μ_β by $\mu_\beta(E) \colon = \frac{c_1(E) - \beta c_0(E)}{c_0(E)}$ for $E \in \mathrm{D}^b(\mathbb{P}^3)$ if $c_0(E) \neq 0$, and $\mu_\beta(E) = +\infty$ otherwise. By ***tilting***, one can define a new heart of a bounded t-structure as follows: the torsion pair is defined by

$$\mathcal{T}_\beta = \{E \in \mathrm{Coh}(\mathbb{P}^3) \colon \mu_\beta(G) > 0 \quad for \quad all \quad E \twoheadrightarrow G\},$$
$$\mathcal{F}_\beta = \{E \in \mathrm{Coh}(\mathbb{P}^3) \colon \mu_\beta(F) \leq 0 \quad for \quad all \quad F \hookrightarrow E\}.$$

A new heart of a bounded t-structure can be defined as $\mathrm{Coh}^\beta(\mathbb{P}^3) =< \mathcal{F}_\beta[1], \mathcal{T}_\beta >$. We define the ***twisted Chern character*** $\mathrm{ch}^\beta(E) = e^{-\beta H}.\mathrm{ch}(E)$. For $v = (\mathrm{H}^3\mathrm{ch}_0, \mathrm{H}^2\mathrm{ch}_1, H\mathrm{ch}_2) = \mathrm{H}.\mathrm{ch}_{\leq 2}$ a fixed vector, the central charge and the corresponding slope function for the new heart can be defined as

$$Z^{tilt}_{\alpha,\beta} = -(\mathrm{ch}_2 - \beta\mathrm{ch}_1 + (\beta^2/2)\mathrm{ch}_0) + (\alpha^2/2)\mathrm{ch}_0 + i(\mathrm{ch}_1 - \beta\mathrm{ch}_0) = -(\mathrm{ch}^\beta_2) + (\alpha^2/2)\mathrm{ch}^\beta_0 + i(\mathrm{ch}^\beta_1),$$

and (using the twisted notation)

$$\nu_{\alpha,\beta} = -\frac{\mathrm{Re}(Z^{tilt}_{\alpha,\beta})}{\mathrm{Im}(Z^{tilt}_{\alpha,\beta})} = \frac{\mathrm{H}.\mathrm{ch}^\beta_2 - (\alpha^2/2)\mathrm{H}^3.\mathrm{ch}^\beta_0}{\mathrm{H}^2.\mathrm{ch}^\beta_1},$$

with $\nu_{\alpha,\beta}(E) = +\infty$ if $\mathrm{H}^2.\mathrm{ch}^\beta_1(E) = 0$. The pair $\sigma_{\alpha,\beta} := (\mathrm{Coh}^\beta(\mathbb{P}^3), Z^{tilt}_{\alpha,\beta})$ is called ***tilt-stability***. We denote by $\mathrm{Stab}^{tilt}(\mathbb{P}^3)$, the space of all tilt-stability conditions. It was conjectured in [6] for arbitrary three-folds, and proved in by Macrì in [19] for $\mathbb{P}^3$ that tilting again gives a Bridgeland stability condition, which we will now describe. Define

$$\mathcal{T}_{\alpha,\beta} = \{E \in \mathrm{Coh}^\beta(\mathbb{P}^3) \colon \nu_{\alpha,\beta}(G) > 0 \quad for \quad all \quad E \twoheadrightarrow G\},$$
$$\mathcal{F}_{\alpha,\beta} = \{E \in \mathrm{Coh}^\beta(\mathbb{P}^3) \colon \nu_{\alpha,\beta}(F) \leq 0 \quad for \quad all \quad F \hookrightarrow E\}.$$

Now, define a new heart, central charge, and slope respectively as follows:

$$\mathrm{Coh}^{\alpha,\beta}(\mathbb{P}^3) =< \mathcal{F}_{\alpha,\beta}[1], \mathcal{T}_{\alpha,\beta} >,$$

$$Z_{\alpha,\beta,s} = -\mathrm{ch}^\beta_3 + (s + (1/6))\alpha^2\mathrm{H}^2.\mathrm{ch}^\beta_1 + i(\mathrm{H}.\mathrm{ch}^\beta_2 - (\alpha^2/2)\mathrm{H}^3.\mathrm{ch}^\beta_0),$$

and

$$\lambda_{\alpha,\beta,s} = -\frac{Re(Z_{\alpha,\beta,s})}{Im(Z_{\alpha,\beta,s})}.$$

with $\lambda_{\alpha,\beta,s}(E) = +\infty$ if $\mathrm{Im}(Z_{\alpha,\beta,s})(E) = 0$. The pair $\sigma_{\alpha,\beta,s} = (\mathrm{Coh}^{\alpha,\beta}(\mathbb{P}^3), Z_{\alpha,\beta,s})$ (when exists) is called ***Bridgeland stability***. Before going further, we have a formal definition of a wall and chamber following [12]:

Definition 2.4.1. A *numerical wall* in Bridgeland stability with respect to a class $v \in \Lambda$ is a non trivial proper subset of the stability space which is defined as

$$\mathcal{W}_{v,v'} = \{(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R} : \lambda_{\alpha,\beta,s}(v) = \lambda_{\alpha,\beta,s}(v'), \text{ for any } v' \in \Lambda\}.$$

An *actual wall* is a subset $\mathcal{W}'$ of a numerical wall if the set of semistable objects with class v changes at $\mathcal{W}'$ (we can give a similar definition for tilt-stability). A *chamber* is defined as a connected component of the complement of the set of actual walls (Similarly, numerical an actual wall can be defined for tilt stability).

The main point to show that $(\mathrm{Coh}^{\alpha,\beta}(\mathbb{P}^3), Z_{\alpha,\beta,s})$ defines a Bridgeland stability condition (for all $s > 0$) is a Bogomolov-type inequality, which we will refer to it as ***BMT inequality***. Before stating that, we have the classical Bogomolov-Gieseker inequality:

Theorem 2.4.1 ([6, Corollary 7.3.2])**.** *Any $\nu_{\alpha,\beta}$-semistable object $E \in \mathrm{Coh}^{\beta}(\mathbb{P}^3)$ satisfies*

$$2(\mathrm{H}^3.\mathrm{ch}_0(E))(\mathrm{H}.\mathrm{ch}_2(E)) \leq (\mathrm{H}^2.\mathrm{ch}_1(E))^2.$$

Theorem 2.4.2 ([5, Lemma 8.8], [19, Theorem 1.1])**.** *Any $\nu_{\alpha,\beta}$-semistable object $E \in \mathrm{Coh}^{\beta}(\mathbb{P}^3)$ satisfies*

$$\alpha^2[(\mathrm{H}^2.\mathrm{ch}_1^{\beta}(E))^2 - 2(\mathrm{H}^3.\mathrm{ch}_0^{\beta}(E)(\mathrm{H}.\mathrm{ch}_2^{\beta}(E))] + 4(\mathrm{H}.\mathrm{ch}_2^{\beta}(E))^2 - 6(\mathrm{H}^2.\mathrm{ch}_1^{\beta}(E))\mathrm{ch}_3^{\beta}(E) \geq 0,$$

and Therefore $(\mathrm{Coh}^{\alpha,\beta}(X), Z_{\alpha,\beta,s})$ is a Bridgeland stability condition for all $s \geq 0$. The support property is also satisfied.

The support property implies the manifold $\mathrm{Stab}(\mathbb{P}^3)$ admits a chamber decomposition, depending on v, such that (i) for a chamber C, the moduli space $\mathcal{M}_{\sigma}(v) = \mathcal{M}_C(v)$ is independent of the choice of $\sigma \in C$, and (ii) walls consist of stability conditions with strictly semistable objects of class v ([3]). It turns out that there is a well-behaved wall-chamber structure in $\mathrm{Stab}^{tilt}(\mathbb{P}^3)$ (and $\mathrm{Stab}(\mathbb{P}^3)$). The last part of the following Theorem was proved for surfaces in [18]:

Theorem 2.4.3 ([5])**.** *The function $\mathbb{R}_{>0} \times \mathbb{R} \to \mathrm{Stab}^{tilt}(\mathbb{P}^3)$ which is defined by $(\alpha, \beta) \to (\mathrm{Coh}^{\beta}(X), Z_{\alpha,\beta})$ is continuous. Moreover, walls with respect to a class v in the image of this map are locally finite. In addition, the walls in the tilt-stability space are either nested semicircles or vertical lines.*

Remark 2.4.4. Note that the $Jordan - Holder$ factors of the objects on a wall are all stable along the wall.

For more details on Bridgeland stability conditions on $\mathbb{P}^3$, we refer to [26].

Chapter 3

Wall description and Ext-computation

In this Chapter, we list the walls (Theorem 3.1.13) and compute Ext-groups (Lemmas 3.2.2, 3.2.3, 3.2.5, 3.2.6, 3.2.7, and 3.2.8) which we need for the rest.

3.1 Walls

According to Theorem 2.4.3 there is a wall-chamber structure in the stability manifold. In this section, we numerically describe the walls in $\mathrm{Stab}^{tilt}(\mathbb{P}^3)$ with respect to $\mathrm{ch}(\mathcal{I}_C)$,where C is our canonical genus four curve, and give a geometric description of the walls.

First of all, we compute the Chern character of $\mathcal{I}_C$:

Proposition 3.1.1. For a canonical genus 4 curve C in $\mathbb{P}^3$, we have $\mathrm{ch}(\mathcal{I}_C) = (1, 0, -6, 15)$.

Proof. Since C is a (2,3)-complete intersection, we have the short exact sequence $\mathcal{O}(-5) \hookrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-3) \twoheadrightarrow \mathcal{I}_C$, from which the claim follows. □

Consider the hyperbola $\mathbb{H}$ in the (α, β)-plane defined by $\mathrm{Im}(Z_{\alpha,\beta,s}(v)) = 0$. For such α, β, s, semistable objects of Chern character v have phase 0. Moreover, these semistable objects have positive and negative phases with respect to the stability conditions on the left and right side of the hyperbola, respectively. Thus on the left we work with $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$ and $\mathrm{Coh}^{\alpha,\beta}(\mathbb{P}^3)$, for tilt and Bridgeland stability, respectively, and On the right side of the hyperbola we work in $\mathrm{Coh}^{\beta}(\mathbb{P}^3)[-1]$ and $\mathrm{Coh}^{\alpha,\beta}(\mathbb{P}^3)[-1]$. Theorem 2.4.3 gives an order for the walls or semicircles in $\mathrm{Stab}^{tilt}(\mathbb{P}^3)$. We refer to the semicircle with the smallest radius as the first wall, and so on. First, we have some Lemmas:

Lemma 3.1.2. Let β be an integer, and E a tilt semistable object in $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$.

1. If $\mathrm{ch}^{\beta}(E) = (1, 1, d, e)$, then $d - 1/2 \in \mathbb{Z}_{\leq 0}$. If $d = 1/2$, then $E \cong \mathcal{I}_Z(\beta + 1)$ for a zero dimensional subscheme Z in $\mathbb{P}^3$ of length $1/6 - e$. If $d = 1/2 - D$ where $D = 1, 2$, then we have $E \cong \mathcal{I}_{C_D}(\beta + 1)$ where C_D is a rational degree D curve, plus $D - e - 5/6$ (floating/embedded) points in $\mathbb{P}^3$.

2. If $\mathrm{ch}^{\beta}(E) = (0, 1, d, e)$, then $d + 1/2 \in \mathbb{Z}$ and $E \cong \mathcal{I}_{Z/P}(\beta + d + 1/2)$ in which Z is a zero dimensional subscheme supported in a plane in $\mathbb{P}^3$ and of length $1/24 + d^2/2 - e$.

Proof. The first case of 1 and 2 were proven in [26, Lemma 5.4]. For C_D, we notice that the ch_2 of an ideal sheaf of a curve is equal to $-$deg of the curve. Also, ch_3 can be computed using Riemann-Roch. The second case of 1 can be easily shown as in [26, Lemma 5.4] by suitably twisting E.

□

Proposition 3.1.3. Fix the class $v = (1, 0, -6, 15)$. The walls in $\mathrm{Stab}^{tilt}(1, 0, -6, 15)$ with respect to v and for $\beta < 0$ are given by the following equations of semicircles in the (β, α) plane, with $\mathrm{ch}^{-4}_{\leq 2}$ of either the sub-object or the quotient, F, given as follows:
1) $(\beta + 4)^2 + \alpha^2 = 4$, $\mathrm{ch}^{-4}_{\leq 2}(F) = (1, 2, 2)$,
2) $(\beta + 4.5)^2 + \alpha^2 = 8.25$, $\mathrm{ch}^{-4}_{\leq 2}(F) = (1, 3, 5/2)$,
3) $(\beta + 5.5)^2 + \alpha^2 = 18.25$, $\mathrm{ch}^{-4}_{\leq 2}(F) = (1, 3, 7/2)$,
4) $(\beta + 6.5)^2 + \alpha^2 = 30.25$, $\mathrm{ch}^{-4}_{\leq 2}(F) = (1, 3, 9/2)$.

Furthermore, the hyperbola which is defined by $\mathrm{Re}(Z_{\alpha,\beta}(\nu) = 0)$ where $v = \mathrm{ch}(\mathcal{I}_C)$ intersects all these semicircles at their top.

Proof. Given a short exact sequence in $\mathrm{Coh}^{-4}(\mathbb{P}^3)$ that defines a wall for $M^{tilt}_{\alpha,\beta}(v)$, either the subobject or the quotient of $E \in M^{tilt}_{\alpha,\beta}(v)$ will have positive rank. Let F be this object, G the other one, and write $\mathrm{ch}^{-4}_{\leq 2}(F) = (r, c, d)$ with $r \geq 1$. As $\mathrm{ch}^{-4}_{\leq 2}(E) = (1, 4, 2)$ and $F, G \in \mathrm{Coh}^{-4}(\mathbb{P}^3)$, we have $c \geq 0$ and $4 - c \geq 0$. Now, if either $c = 0$ or $c = 4$, then either F or G would have slope $+\infty$, a contradiction; Therefore $1 \leq c \leq 3$. We want to find all the possibilities for $\mathrm{ch}^{-4}_{\leq 2}(F)$ and $\mathrm{ch}^{-4}_{\leq 2}(G)$. The equation $\nu_{\alpha,-4}(E) = \nu_{\alpha,-4}(F)$ implies $\alpha^2 = (8d - 4c)/(4r - c)$ which has to be positive. As by our assumption $r \geq 1$, and also $c \leq 3$, this implies $d > c/2$, i.e. $d \geq c/2 + 1$. Combined with the Bogomolov-Gieseker inequality (Theorem 2.4.1), we get

$$\begin{cases} c^2 \geq 2rd \geq r(c+2), \\ r \geq 1. \end{cases}$$

This has no solution for $c = 1$. For $c = 2$, the only solution is $d = 2$, $r = 1$. For $c = 3$, we have $r = 1$ and $d \in \{5/2, 7/2, 9/2\}$. Plugging these into the equation $\nu_{\alpha,-4}(E) = \nu_{\alpha,-4}(F)$ gives the corresponding semicircles. The last part comes from Bertram's Nested Wall Theorem (Theorem 2.4.3) which is restated in [26, Theorem 3.3] as well. □

Lemma 3.1.4. Let $E \in Coh^{\beta}(\mathbb{P}^3)$ be a $\nu_{\alpha,\beta}$-semistable object with $\mathrm{ch}(E) = (0, 2, -8, e)$. Then $e \leq 49/3$. Moreover, if the equality holds, then $E = \mathcal{O}_Q(-3)$ for a (possibly singular) quadric surface Q in $\mathbb{P}^3$.

Proof. The first part is a special case of [25, Theorem 2.20]. For the second part, the proof of [25, Theorem 2.20] shows that in the case of equality, E becomes unstable at the wall with radius one with same center as our first wall, i.e. the wall $(\beta + 4)^2 + \alpha^2 = 1$, and also the destabilizing subobject of E must have rank one. Thus, the destabilizing short exact sequence is of the form $\mathcal{O}(-3) \hookrightarrow E \twoheadrightarrow \mathcal{O}(-5)[1]$, and so we have $E = \mathcal{O}_Q(-3)$ for a quadric surface Q in $\mathbb{P}^3$. □

To describe the walls in Bridgeland stability space, we need the following result and Remarks:

Lemma 3.1.5 ([5, Lemma 8.9]). Let $E \in \mathrm{Coh}^{\alpha,\beta}(\mathbb{P}^3)$ be a $\lambda_{\alpha,\beta,s}$-semistable object, for all $s \gg 1$ sufficiently big. Then it satisfies one of the following conditions:
(a) $H^{-1}_\beta(E) = 0$ and $H^0_\beta(E)$ is $\nu_{\alpha,\beta}$-semistable.
(b) $H^{-1}_\beta(E)$ is $\nu_{\alpha,\beta}$-semistable and $H^0_\beta(E)$ is either 0 or supported in dimension 0. Moreover, if $H^{-1}_\beta(E)$ is $\nu_{\alpha,\beta}$-stable, $H^0_\beta(E)$ is either 0 or zero dimensional torsion sheaf, and $\mathrm{Hom}(\mathcal{O}_p, E) = 0$ for all points $p \in \mathbb{P}^3$, then E is $\lambda_{\alpha,\beta,s}$-stable, for all $s \gg 1$ sufficiently big.

Remark 3.1.6. Note that (a) and (b) correspond to semistable objects with respect to the stability conditions on the left and right of the hyperbola $\mathbb{H}$, respectively.

Lemma 3.1.5 and Remark 3.1.6 allow us to transfer everything from $\mathrm{Stab}^{tilt}(\mathbb{P}^3)$ to $\mathrm{Stab}(\mathbb{P}^3)$.

Lemma 3.1.7 ([15, Corollary 11.4]). Let $j\colon Y \hookrightarrow X$ be a smooth hypersurface. Then for any $\mathcal{F} \in \mathrm{D}^b(Y)$ there exists a distinguished triangle

$$\mathcal{F} \otimes \mathcal{O}_Y(-Y)[1] \to j^*j_*\mathcal{F} \to \mathcal{F}.$$

Lemma 3.1.8 ([15, Corollary 3.34]). Let $f\colon X \to Y$ be a morphism of smooth schemes over a field k. For any $\mathcal{F} \in \mathrm{D}^b(X)$ and $\mathcal{G} \in \mathrm{D}^b(Y)$ there exists a functorial isomorphism (which is called Grothendieck-Verdier duality)

$$f_*Hom(\mathcal{F}, f^!\mathcal{G}) = Hom(f_*\mathcal{F}, \mathcal{G}).$$

Corollary 3.1.9. Let $\iota_P\colon P \hookrightarrow \mathbb{P}^3$ be the inclusion map. For any $\mathcal{F} \in \mathrm{D}^b(P)$ and $\mathcal{G} \in \mathrm{D}^b(\mathbb{P}^3)$, we have $({\iota_P}_*\mathcal{F})^\vee = {\iota_P}_*\mathcal{F}^\vee(1)[-1]$ and $(\iota_P^*\mathcal{G})^\vee = \mathcal{G}^\vee$.

Proof. Apply Lemma 3.1.8 to the inclusion map. □

Lemma 3.1.10. Fix a vector $w = (0, 1, -i - 1/2, e)$. The stable objects of class w for stability conditions with $\mathrm{Im}(Z_{\alpha,\beta,s}(w)) < 0$ near the hyperbola $\mathrm{Im}(Z_{\alpha,\beta,s}(w)) = 0$ are of the form ${\iota_P}_*(\mathcal{I}_Z^\vee(-i))$, where Z is a zero dimensional subscheme of length $l = 1/6+(i/2)(i+1)-e$, and P is a plane.

To list the walls, we need the following three Lemmas to describe the objects with respect to their Chern characters.

Proof. Let E be such an object of Chern character w, i.e. E is (semi)stable near the hyperbola, which means it is still (semi)stable on the hyperbola when we reach the hyperbola from the right or left. For objects with $\mathrm{Im}\, Z_{\alpha,\beta,s} = 0$, semistability doesn't change as s varies; in particular, we can let $s \to +\infty$ and apply Lemma 3.1.5 to $E[1]$. Therefore, $\mathcal{H}^0_\beta(E)$ is $\nu_{\alpha,\beta}$-semistable, and $\mathcal{H}^1_\beta(E)$ is a torsion sheaf T with zero-dimensional support of length m. Notice that we have $\mathrm{ch}(\mathcal{H}^0_\beta(E)) = (0, 1, -i - 1/2, e + m)$. Thus, from Lemma 3.1.2, and varying

α, β along the wall we have $\mathcal{H}^0_\beta(E) = \mathcal{I}_{Z/P}(-i)$, where Z is zero dimensional subscheme of P. Notice that E is semistable for $\operatorname{Im} Z_{\alpha,\beta,s}(w) < 0$; this implies $\operatorname{Hom}(\mathcal{O}_Z[-1], E) = 0$, as $\mathcal{O}_Z[-1]$ is semistable of phase 0. Therefore $\operatorname{Hom}(\mathcal{O}_Z[-1], \mathcal{I}_{Z/P}(-i)) = 0$, which is possible only if $Z = \emptyset$, and thus $\mathcal{H}^0_\beta(E) = \mathcal{O}_P(-i)$. Having $\mathcal{H}^0_\beta$ and $\mathcal{H}^1_\beta$, E fits into a short exact sequence $\mathcal{O}_P(-i) \to E \to T[-1]$. Dualizing this sequence, noting that $(\mathcal{O}_P(-i))^\vee = \mathcal{O}_P(i+1)[-1]$ (by Corollary 3.1.9), and letting $T' := T^\vee[3]$, we get an exact triangle $T'[-2] \to E^\vee \to \mathcal{O}_P(i+1)[-1]$. Hence we have $E^\vee[1] = (\mathcal{O}_P(i+1) \to T')$. Now, $\operatorname{Hom}(\mathcal{O}_p[-1], E) = 0$, for all $p \in P$ implies $\operatorname{Hom}(E^\vee[1], \mathcal{O}_p[-1]) = 0$, which is equivalent to the map $\mathcal{O}_P(i+1) \to T'$ being surjective. This implies $T' = \mathcal{O}_Z$ for a 0-dimensional subscheme Z of P, and hence $E^\vee[1] = \iota_{P*}\mathcal{I}_{Z/P}(i+1)$, where $\iota_P \colon P \hookrightarrow \mathbb{P}^3$. Now, Corollary 3.1.9 implies $E = \iota_{P_*}(\mathcal{I}^\vee_Z(-i))$. As for the length of Z, using Lemma 3.1.2 implies $l = 1/6 + (i/2)(i+1) - e$.

□

Lemma 3.1.11. Fix a vector $w = (1, -1, -D + 1/2, e)$ for $D = 1, 2$. The stable objects of Chern character w for stability conditions with $\operatorname{Im}(Z_{\alpha,\beta,s}(w)) < 0$ near the hyperbola $\operatorname{Im}(Z_{\alpha,\beta,s}(w)) = 0$ are complexes $\mathcal{O}_{\mathbb{P}^3} \to \mathcal{F}$ given by the section of pure one-dimensional sheaf $\mathcal{F}$ supported on a curve of degree D with cokernel of length $l = 3D - e - 7/6 \in \mathbb{Z}_{\geq 0}$.

Proof. Let E be such an object of Chern character w. A similar argument as in the proof of Lemma 3.1.10 implies $\mathcal{H}^1_\beta(E)$ is a torsion sheaf T with zero-dimensional support of length l, $\mathcal{H}^0_\beta(E)$ is $\nu_{\alpha,\beta}$-semistable, and $\operatorname{ch}(\mathcal{H}^0_\beta(E)) = (1, -1, -D + 1/2, e + l)$. Therefore, from Lemma 3.1.2, and varying α, β along the wall we have $\mathcal{H}^0_\beta(E) = \mathcal{I}_{C_D}(-1)$ where C_D is a rational degree D curve possibly with $3D - e - l - 7/6$ embedded or floating points. With the same reasoning as in the proof of Lemma 3.1.10, we have $\operatorname{Hom}(\mathcal{O}_p[-1], E) = 0$, for all $p \in \mathbb{P}^3$, and so $\operatorname{Hom}(\mathcal{O}_p[-1], \mathcal{I}_{C_D}(-1)) = 0$; if $3D - e - l - 7/6 \neq 0$, then we have $\operatorname{Hom}(\mathcal{O}_{Z_{3D-e-l-7/6}}[-1], \mathcal{I}_{C_D}(-1)) \neq 0$, which is a contradiction, and hence we must have $l = 3D - e - 7/6$. Therefore, E fits into $\mathcal{I}_{C_D}(-1) \to E \to T[-1]$, with $\dim(T) = l$, and hence by definition of stable pairs, we get the result.

□

Lemma 3.1.12. Fix a vector $w = (1, -1, 1/2, e)$. Then for the stable objects E of Chern character w for stability conditions with $\operatorname{Im}(Z_{\alpha,\beta,s}(w)) < 0$ near the hyperbola $\operatorname{Im}(Z_{\alpha,\beta,s}(w)) = 0$, we have $e = -1/6$, and $E \cong \mathcal{O}(-1)$.

Proof. Let E be such an object with Chern character w, i.e. E is (semi)stable near the hyperbola, which means it is still (semi)stable on the hyperbola when we reach the hyperbola from the right or left. For objects with $\operatorname{Im} Z_{\alpha,\beta,s} = 0$, semistability doesn't change as s varies; in particular, we can let $s \to +\infty$ and apply Lemma 3.1.5 to $E[1]$. Therefore, $\mathcal{H}^1_\beta(E)$ is a torsion sheaf T with zero-dimensional support, and $\mathcal{H}^0_\beta(E)$ is $\nu_{\alpha,\beta}$-semistable. Notice that E is semistable for $\operatorname{Im} Z_{\alpha,\beta,s}(w) < 0$; this implies $\operatorname{Hom}(\mathcal{O}_z[-1], E) = 0$ and so $\operatorname{Hom}(\mathcal{O}_z[-1], \mathcal{H}^0_\beta(E)) = 0$ for all $z \in \mathbb{P}^3$, as $\mathcal{O}_z[-1]$ is semistable of phase 0. But $(1, -1, 1/2, e)$ is the Chern character of a tilt-semistable sheaf of the form $\mathcal{I}_Z(-1)$ where Z is a zero dimensional subscheme of length $-1/6 - e$ by Lemma 3.1.2. But if

$-1/6-e \neq 0$, then we would have $\mathrm{Hom}(\mathcal{O}_z[-1], \mathcal{H}^0_\beta(E)) \neq 0$ for $z \in Z$, which is a contradiction. Therefore $e = -1/6$ and thus $\mathcal{H}^0_\beta(E) \cong \mathcal{O}(-1)$. Now by Serre duality, we have $\mathrm{Ext}^1(\mathcal{H}^1_\beta(E)[-1], \mathcal{H}^0_\beta(E)) = \mathrm{Ext}^2(T, \mathcal{O}(-1)) = \mathrm{H}^1(T)^\vee = 0$. This means that we must have $\mathcal{H}^1_\beta(E) = 0$ (otherwise, E would be a direct sum and hence unstable), and therefore $E \cong \mathcal{H}^0_\beta(E) \cong \mathcal{O}(-1)$.

□

Theorem 3.1.13. *For the walls in $M^{tilt}_{\alpha,\beta}(1, 0, -6, 15)$, the walls on the both sides of $\mathbb{H}$ are given by the following pairs:*

walls in tilt stability	walls on the left side of $\mathbb{H}$	walls on the right side of $\mathbb{H}$
$(\beta+6.5)^2+\alpha^2 = 30.25$	$\langle \mathcal{O}(-1), \mathcal{I}_{Z'_6/P}(-6)\rangle$ $\langle \mathcal{I}_{Z_1}(-1), \mathcal{I}_{Z'_5/P}(-6)\rangle$ $\langle \mathcal{I}_{Z_2}(-1), \mathcal{I}_{Z'_4/P}(-6)\rangle$ $\langle \mathcal{I}_{Z_3}(-1), \mathcal{I}_{Z'_3/P}(-6)\rangle$ $\langle \mathcal{I}_{Z_4}(-1), \mathcal{I}_{Z'_2/P}(-6)\rangle$ $\langle \mathcal{I}_{Z_5}(-1), \mathcal{I}_{Z'_1/P}(-6)\rangle$ $\langle \mathcal{I}_{Z_6}(-1), \mathcal{O}_P(-6)\rangle$	$\langle \mathcal{O}(-1), \iota_{P_*}\mathcal{I}^\vee_{Z_6}(-6)\rangle$
$(\beta+5.5)^2+\alpha^2 = 18.25$	$\langle \mathcal{I}_{L_0}(-1), \mathcal{I}_{Z_2/P}(-5)\rangle$ $\langle \mathcal{I}_{L_1}(-1), \mathcal{I}_{Z_1/P}(-5)\rangle$ $\langle \mathcal{I}_{L_2}(-1), \mathcal{O}_P(-5)\rangle$	$\langle (\mathcal{O}(-1) \to \mathcal{O}_L(1)), \mathcal{O}_P(-5)\rangle$ $\langle (\mathcal{O}(-1) \to \mathcal{O}_L), \iota_{P_*}\mathcal{I}^\vee{}_{Z_1}(-5)\rangle$ $\langle \mathcal{I}_{L_0}(-1), \iota_{P_*}\mathcal{I}^\vee{}_{Z_2}(-5)\rangle$
$(\beta+4.5)^2+\alpha^2 = 8.25$	$\langle \mathcal{I}_{C_2}(-1), \mathcal{O}_P(-4)\rangle$	$\langle \mathcal{I}_{C_2}(-1), \mathcal{O}_P(-4)\rangle$
$(\beta+4)^2+\alpha^2 = 4$	$\langle \mathcal{O}(-2), \mathcal{O}_Q(-3)\rangle$	$\langle \mathcal{O}(-2), \mathcal{O}_Q(-3)\rangle$

where Z_i's and Z'_i's are zero dimensional subschemes of length i, L_i's are lines plus i extra embedded/floating points, C_2 is a conic, Q is a quadric in $\mathbb{P}^3$, C a curve supported on Q, and $\iota_P\colon P \hookrightarrow \mathbb{P}^3$ is the inclusion map. All the walls in $\mathrm{Stab}(\mathbb{P}^3)$ induced by walls in $\mathrm{Stab}^{tilt}(\mathbb{P}^3)$ intersect the hyperbola.

Proof. First, we describe the corresponding walls on the left side of the left branch of the hyperbola. For the wall $(\beta+4)^2+\alpha^2 = 4$, we have $\mathrm{ch}^{-4}(F) = (1, 2, 2, e)$, and thus $\mathrm{ch}^{-3}(F) = (1, 1, 1/2, e-7/6)$. Using Lemma 3.1.2 implies $F \cong \mathcal{I}_Z(-2)$, in which Z is a zero dimensional sub-scheme of length $1/6-e+7/6 = 4/3-e$. As the length is a non-negative integer, we have $e \leq 4/3$. On the other hand, $\mathrm{ch}(G) = (0, 2, -8, 53/3-e)$. Applying Lemma 3.1.4 implies $53/3-e \leq 49/3$ or $e \geq 4/3$. Therefore $e = 4/3$ and Z is a zero dimensional sub-scheme of length 0, i.e. $F \cong \mathcal{O}(-2)$. We have $\mathrm{ch}(G) = (0, 2, -8, 49/3)$, and hence Lemma 3.1.4 implies the result.

For the wall $(\beta+4.5)^2+\alpha^2 = 8.25$, Proposition 3.1.3 implies $\mathrm{ch}^{-4}(F) = (1, 3, 5/2, e)$ and $\mathrm{ch}^{-4}(G) = (0, 1, -1/2, 5/3-e)$. The former implies $\mathrm{ch}^{-2}(F) = (1, 1, -3/2, e-1/3)$. As the wall intersects the line $\beta = -2$, Remark 2.4.4 and Lemma 3.1.2 implies $F \cong \mathcal{I}_{C_2}(-1)$, where C_2 is a degree two curve plus $7/6-e+1/3 = 3/2-e$ points. Lemma 3.1.2 shows $G \cong \mathcal{I}_{Z/P}(-4+(-1/2)+1/2) = \mathcal{I}_{Z/P}(-4)$, in which Z is a zero dimensional sub-scheme of length $e-3/2$, and P is a plane in $\mathbb{P}^3$, containing Z. Non-negativity of lengths implies $e=3/2$, and so $F=\mathcal{I}_{C_2}(-1)$ and $G=\mathcal{O}_P(-4)$, in which C_2 is a connected conic Cohen-Macaulay curve.

For the wall $(\beta+5.5)^2+\alpha^2=18.25$, Proposition 3.1.3 implies the semistable objects on this semicircle are given by a pair (F,G) such that $\mathrm{ch}^{-4}(F)=(1,3,7/2,e)$ and $\mathrm{ch}^{-4}(G)=(0,1,-3/2,5/3-e)$. Therefore we have $\mathrm{ch}^{-4}(F\otimes\mathcal{O}(-3))=(1,0,-1,e-3/2)$, which means $F\otimes\mathcal{O}(-3)\cong\mathcal{I}_L(-4)$, where L is a line with $l=5/2-e$ points (Lemma 3.1.2), and so $F\cong\mathcal{I}_L(-1)$. A similar argument, using Lemma 3.1.2 shows $G\cong\mathcal{I}_{Z'/P}(-5)$, where Z' is a zero dimensional subscheme of length $l'=-1/2+e$, supported on a plane $P\subset\mathbb{P}^3$. As the lengths l,l' are non-negative, we get $1/2\leq e\leq 5/2$. As l,l' are integers, the following possibilities for e,l and l' remain: (1) $e=5/2$, and so $l=0,l'=2$. (2) $e=3/2$, and so $l=1,l'=1$. (3) $e=1/2$, and so $l=2,l'=0$. Thus all the possibilities for the pair (F,G) are

$$(F,G)\in\{(\mathcal{I}_{L_0}(-1),\mathcal{I}_{Z_2'/P}(-5)),(\mathcal{I}_{L_1}(-1),\mathcal{I}_{Z_1'/P}(-5)),(\mathcal{I}_{L_2}(-1),\mathcal{I}_{Z_0'/P}(-5))\},$$

where Z_i''s are zero dimensional subschemes of length i, supported on plane P in $\mathbb{P}^3$, L_i's are lines plus i extra points in $\mathbb{P}^3$. Notice that we have $\mathcal{I}_{Z_0'/P}(-5)=\mathcal{O}_P(-5)$.

For the wall $(\beta+6.5)^2+\alpha^2=30.25$, Proposition 3.1.3 implies that the semistable objects on this semicircle are given by a pair (F,G) such that $\mathrm{ch}^{-4}(F)=(1,3,9/2,e)$ and $\mathrm{ch}^{-4}(G)=(0,1,-5/2,5/3-e)$. Therefore, we have $\mathrm{ch}^{-4}(F\otimes\mathcal{O}(-3))=(1,0,0,e-9/2)$, which means $F\otimes\mathcal{O}(-3)\cong\mathcal{I}_Z(-4)$, where Z is a zero dimensional sunscheme of length $l=9/2-e$ (Lemma 3.1.2), and so $F\cong\mathcal{I}_Z(-1)$. A similar argument, using Lemma 3.1.2 shows $G\cong\mathcal{I}_{Z'/P}(-6)$, where Z' is a zero dimensional subscheme of length $l'=e+3/2$, supported on a plane $P\subset\mathbb{P}^3$. As the lengths l,l' are non-negative integers, we must have $-3/2\leq e\leq 9/2$ with $e+1/2\in\mathbb{Z}$. This leaves the following possibilities for the pair (F,G):

$$(\mathcal{I}_{Z_0}(-1),\mathcal{I}_{Z_6'/P}(-6)),(\mathcal{I}_{Z_1}(-1),\mathcal{I}_{Z_5'/P}(-6)),(\mathcal{I}_{Z_2}(-1),\mathcal{I}_{Z_4'/P}(-6)),(\mathcal{I}_{Z_3}(-1),\mathcal{I}_{Z_3'/P}(-6)),\\(\mathcal{I}_{Z_4}(-1),\mathcal{I}_{Z_2'/P}(-6)),(\mathcal{I}_{Z_5}(-1),\mathcal{I}_{Z_1'/P}(-6)),(\mathcal{I}_{Z_6}(-1),\mathcal{I}_{Z_0'/P}(-6))$$

where Z_i's and Z_i''s are zero dimensional subschemes of length i, and Z_i''s are supported on plane P in $\mathbb{P}^3$. But we have $\mathcal{I}_{Z_0}(-1)=\mathcal{O}(-1)$, and $\mathcal{I}_{Z_0'/P}(-6)=\mathcal{O}_P(-6)$. Hence the proof for the left side of the hyperbola is completed.

As for the right side of the hyperbola, for the walls $(\beta+4)^2+\alpha^2=4$ and $(\beta+4.5)^2+\alpha^2=8.25$, we get the same results as on the left side (as there is no point involved). For the wall $(\beta+6.5)^2+\alpha^2=18.25$, from Proposition 3.1.3, we have $\mathrm{ch}^{-4}_{\leq 2}(F)=(1,3,7/2)$ for a subobject or quotient of semistable objects with respect stability conditions on the semicircle. Therefore we have $\mathrm{ch}(F)=(1,-1,-1/2,e)$. Lemma 3.1.11 implies $F\cong(\mathcal{O}(-1)\to\mathcal{O}_L(l-1))$, where L line, with $l=11/6-e\in\mathbb{Z}^{\geq 0}$ points on it. On the other hand, we can see that the other corresponding JH factor is given by $\mathrm{ch}(G)=(0,1,-11/2,15-e)$. Now using Lemma 3.1.10, we have $G\cong\iota_{P_*}(\mathcal{I}^\vee{}_{Z_{l'}})(-5)$, where $Z_{l'}$ is a zero dimensional subscheme of length $l'=1/6+e\in\mathbb{Z}^{\geq 0}$. The conditions on the lengths l,l' gives three possibilities for e, i.e. $e\in\{-1/6,5/6,11/6\}$. This gives the three pairs stated above.

As for the wall $(\beta+6.5)^2+\alpha^2=30.25$, from Proposition 3.1.3, we have $\mathrm{ch}^{-4}_{\leq 2}(F)=(1,3,9/2)$ for a subobject or quotient of semistable objects with respect stability conditions on the semicircle. Therefore we have $\mathrm{ch}(F)=(1,-1,1/2,e)$. Lemma 3.1.12 implies $e=-1/6$, and $F\cong\mathcal{O}(-1)$. On the other hand, we can see that the Chern character of the other corresponding JH factor is given by $\mathrm{ch}(G)=(0,1,-6-1/2,15-e)$. Now using Lemma 3.1.10, we have $G\cong \iota_{P_*}\mathcal{I}^{\vee}{}_{Z_{l'}}(-6)$, where $Z_{l'}$ is a zero dimensional subscheme of length $l'=1/6+6+e=6$.

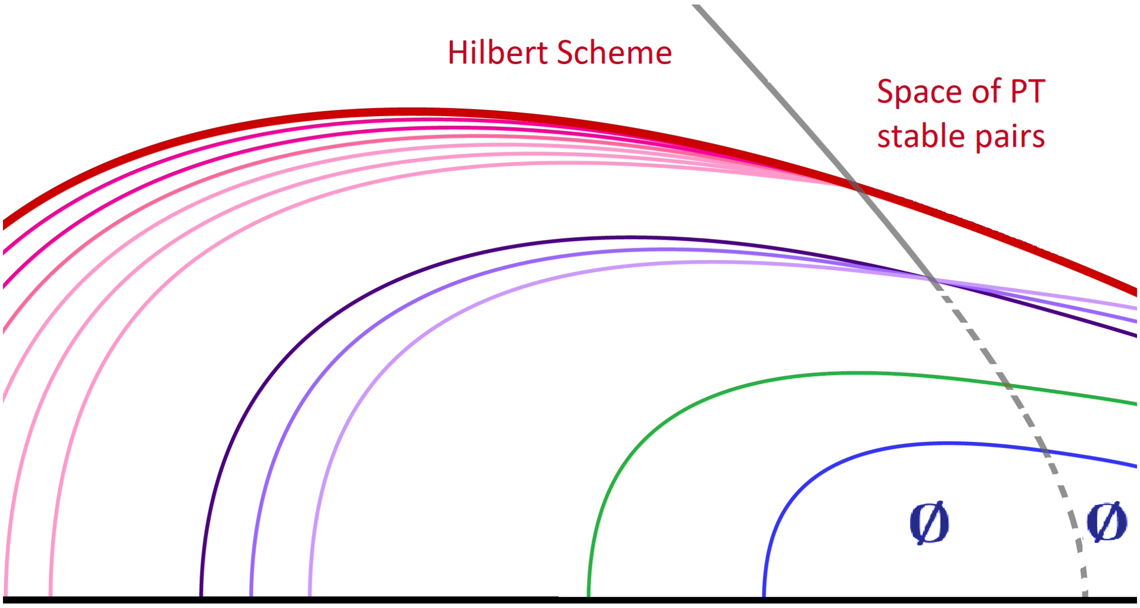


Figure 3.1: walls

□

3.2 Ext computation of the walls on the stable pairs side

So far, we have described all the possible walls in $\mathrm{Stab}(\mathbb{P}^3)$ close to the left branch of the hyperbola $\beta^2-\alpha^2=12$. To study the wall-crossing and describe the chambers (on the right side of the hyperbola) in section 5.4, we need to compute all the necessary Ext^1-groups related to the walls. When there is no confusion, we use the notation A and B for the subobject and the quotient of the objects on the walls.

First, we have the following Lemma to compute the pull-backs:

Lemma 3.2.1. For a plane P, a line L, $p=L\cap P$, a line $L'\subset P$, zero-dimensional subschemes $q,q'\subset\mathbb{P}^3$, $L_1=L\cup q$, $L_2=L\cup q\cup q'$, a conic $C_2=L\cup L'$, a zero-dimensional subscheme Z, and $\iota_P\colon P\hookrightarrow\mathbb{P}^3$ we have

$$\iota_P^*(\mathcal{I}_L)=\begin{cases}\mathcal{I}_{p/P}, & L\not\subset P\\ \mathcal{O}_P(-1)\oplus\mathcal{O}_L(-1), & L\subset P\end{cases}$$

$$\iota_P^*(\mathcal{I}_{L_1}) = \iota_P^*(\mathcal{I}_{L\cup q}) = \begin{cases} \mathcal{I}_{p/P}, & L \not\subset P, q \not\subset P \\ \mathcal{I}_{p/P}, & L \not\subset P, p = q \\ \mathcal{I}_{p\cup q} \oplus \mathcal{O}_q, & L \not\subset P, p \neq q \subset P \\ \mathcal{O}_P(-1) \oplus \mathcal{O}_L(-1), & L \subset P, q \not\subset P \\ \mathcal{O}_P(-1) \oplus \mathcal{O}_L(-2) \oplus \mathcal{O}_q, & L \subset P, q \subset L \\ \mathcal{I}_q(-1) \oplus \mathcal{O}_q, & L \subset P, q \subset P, q \not\subset L \end{cases}$$

$$\iota_P^*(\mathcal{I}_{L_2}) = \iota_P^*(\mathcal{I}_{L\cup q\cup q'}) = \begin{cases} \mathcal{I}_{p/P}, & L \not\subset P, q, q' \not\subset P \\ \mathcal{I}_{p\cup q} \oplus \mathcal{O}_q, & L \not\subset P, q \subset P, q' \not\subset P \\ \mathcal{I}_{p\cup q\cup q'} \oplus \mathcal{O}_q \oplus \mathcal{O}_{q'}, & L \not\subset P, q, q' \subset P; q, q' \neq p \\ \mathcal{I}_{p/P} \oplus \mathcal{I}_{p\cup q'} \oplus \mathcal{O}_{q'}, & L \not\subset P; q, q' \subset P; \\ & q' \neq p = q \\ \mathcal{I}_{p/P}, & L \not\subset P; q, q' \subset P; \\ & q' = p = q \\ \mathcal{O}_P(-1) \oplus \mathcal{O}_L(-1), & L \subset P, q, q' \not\subset P \\ \mathcal{O}_P(-1) \oplus \mathcal{O}_L(-2) \oplus \mathcal{O}_q, & L \subset P, q \subset L, q' \not\subset P \\ \mathcal{O}_P(-1) \oplus \mathcal{O}_L(-2) \oplus \mathcal{O}_q \oplus \mathcal{O}_{q'}, & L \subset P, q, q' \subset L \\ \mathcal{I}_q(-1) \oplus \mathcal{O}_q \oplus \mathcal{I}_{q'}(-1) \oplus \mathcal{O}_{q'}, & L \subset P, q, q' \subset P, q, q' \not\subset L \\ \mathcal{O}_P(-1) \oplus \mathcal{O}_L(-2) \oplus \mathcal{O}_q \oplus \mathcal{I}_{q'} \oplus \mathcal{O}_{q'}, & L \subset P, q, q' \subset P, q \subset L, \\ & q' \not\subset L \end{cases}$$

$$\iota_P^*(\mathcal{O} \xrightarrow{s} \mathcal{O}_L(1)) = \begin{cases} \mathcal{I}_{L\cap P}, & L \not\subset P, \\ & \text{zero locus of } s \text{ is not } L \cap P \\ \mathcal{O}_P \oplus \mathcal{O}_{L\cap P}[-1], & L \not\subset P, \\ & \text{zero locus of } s \text{ is } L \cap P \\ (\mathcal{O}_P \to \mathcal{O}_L(1)) \oplus \mathcal{O}_L, & L \subset P \end{cases}$$

$$\iota_P^*(\mathcal{O} \xrightarrow{s} \mathcal{O}_L(2)) = \begin{cases} \mathcal{I}_{L\cap P}, & L \not\subset P, \\ & \text{zero locus of } s \text{ does not contain } L \cap P \\ \mathcal{O}_P \oplus \mathcal{O}_{L\cap P}[-1], & L \not\subset P, \\ & \text{zero locus of } s \text{ contains } L \cap P \\ (\mathcal{O}_P \to \mathcal{O}_L(2)) \oplus \mathcal{O}_L(1), & L \subset P \end{cases}$$

Proof. For $\iota_P^*(\mathcal{I}_L)$, first assume $L \not\subset P$; the exact sequence $\mathcal{I}_L \to \mathcal{O} \to \mathcal{O}_L$ implies $\iota_P^*\mathcal{I}_L = \mathcal{I}_{p/P}$. Now assume $L \subset P$. Assume that $P \cap P' = L$ where P' is a plane containing L. From the resolution $\mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{I}_L$ induced by P and P', we have $\iota_P^*\mathcal{I}_L = \iota_P^*(\mathcal{O}(-2) \to \mathcal{O}(-1)) \oplus \iota_P^*(\mathcal{O}(-1)) = (\mathcal{O}_P(-2) \to \mathcal{O}_P(-1)) \oplus \mathcal{O}_P(-1) = \mathcal{O}_L(-1) \oplus \mathcal{O}_P(-1)$. Arguments for For $\iota_P^*(\mathcal{I}_{L_1})$ and For $\iota_P^*(\mathcal{I}_{L_2})$ are similar.

Now, let us compute $\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1))$. First, assume that $L \not\subset P$, and the zero locus of s is not $L \cap P$. Then we have $\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) = (\mathcal{O}_P \xrightarrow{s|_{L\cap P}} \mathcal{O}_{L\cap P})$, from which the claim follows in both cases. Now, let us assume $L \subset P$. We have $\mathcal{O}_L(1) = (\mathcal{I}_L(1) \to \mathcal{O}(1))$, and also $\mathcal{O} \to \mathcal{O}_L(1)$ factors via $\mathcal{O}(1)$; therefore we have $(\mathcal{O} \to \mathcal{O}_L(1)) = (\mathcal{O} \oplus \mathcal{I}_L(1)) \to \mathcal{O}(1)$. Thus using the first part of this Lemma, we have

$$\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) = (\mathcal{O}_P \oplus \mathcal{O}_P \oplus \mathcal{O}_L) \to \mathcal{O}_P(1) = \mathcal{O}_L \oplus (\mathcal{O}_P \oplus \mathcal{O}_P \to \mathcal{O}_P(1)).$$

Notice that the later map is the extension of s in the first factor, and the equation of L in the second factor. Therefore we get $\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) = \mathcal{O}_L \oplus (\mathcal{O}_P \xrightarrow{s|_P} \mathcal{O}_L(1))$, as desired.

A similar argument implies the result for $\iota_P^*(\mathcal{O} \to \mathcal{O}_L(2))$.

□

We have the following Lemmas on the Ext^1's:

Lemma 3.2.2. For the wall $\langle \mathcal{O}(-2), \mathcal{O}_Q(-3) \rangle$, we have:

$$\mathrm{Ext}^1(\mathcal{O}(-2), \mathcal{O}(-2))) = 0, \qquad \mathrm{Ext}^1(\mathcal{O}_Q(-3), \mathcal{O}_Q(-3)) = \mathbb{C}^9,$$
$$\mathrm{Ext}^1(\mathcal{O}_Q(-3), \mathcal{O}(-2)) = \mathbb{C}^{16}, \qquad \mathrm{Ext}^1(\mathcal{O}(-2), \mathcal{O}_Q(-3)) = 0.$$

Proof. This is just a straightforward computation. □

Lemma 3.2.3. For the wall $\langle \mathcal{I}_{C_2}(-1), \mathcal{O}_P(-4) \rangle$, we have:

$$\mathrm{Ext}^1(\mathcal{I}_{C_2}(-1), \mathcal{I}_{C_2}(-1)) = \mathbb{C}^8, \qquad \mathrm{Ext}^1(\mathcal{O}_P(-4), \mathcal{O}_P(-4)) = \mathbb{C}^3,$$
$$\mathrm{Ext}^1(\mathcal{O}_P(-4), \mathcal{I}_{C_2}(-1)) = \mathbb{C}^{13}, \qquad \mathrm{Ext}^1(\mathcal{I}_{C_2}(-1), \mathcal{O}_P(-4)) = \mathbb{C}.$$

Proof. Similarly, this is just a straightforward computation. □

Before the Ext computation for the other walls, we need the following Lemma:

Lemma 3.2.4. For two different points $p \neq q$ in a plane P and a point of length 2, $Z = Y_1 \cup Y_2 \subset P$, and $L \subset P$ a line, such that $p \subset l$, $p \not\subset l'$, for lines l, l' in P, we have (all tensor products are in $\mathrm{D}^b(P)$)

$$\mathcal{I}_p \otimes \mathcal{I}_p = \mathcal{O}_p \oplus \mathcal{I}_p^2, \quad \mathcal{I}_p \otimes \mathcal{O}_p = \mathcal{O}_p[1] \oplus \mathcal{O}_p^{\oplus 2}, \quad \mathcal{I}_p \otimes \mathcal{O}_q = \mathcal{O}_q,$$
$$\mathcal{I}_p \otimes \mathcal{I}_q = \mathcal{I}_{p\cup q}, \quad \mathcal{I}_p \otimes \mathcal{I}_{p\cup q} = \mathcal{O}_p \oplus \mathcal{I}_{p^2\cup q}, \quad \mathcal{I}_p \otimes \mathcal{O}_l = \mathcal{O}_l(-1) \oplus \mathcal{O}_p, \quad \mathcal{I}_p \otimes \mathcal{O}_{l'} = \mathcal{O}_{l'},$$

$$\mathcal{I}_Z \otimes \mathcal{O}_p = \begin{cases} \mathcal{O}_p[1] \oplus \mathcal{O}_p^{\oplus 2}, & p \subset Z \\ \mathcal{O}_p, & p \not\subset Z \end{cases}$$

$$\mathcal{I}_Z \otimes \mathcal{O}_L = \begin{cases} \mathcal{O}_Z \oplus \mathcal{O}_L(-2), & Z \subset L \\ \mathcal{O}_{Y_1} \oplus \mathcal{O}_L(-1), & Y_1 \subset L, \text{and } Y_2 \not\subset L \\ \mathcal{O}_L, & Y_1, Y_2 \not\subset L \end{cases}$$

Proof. Tensoring the short exact sequence $\mathcal{I}_p \hookrightarrow \mathcal{O}_P \twoheadrightarrow \mathcal{O}_p$ by $\mathcal{I}_p$, we have

$$0 \to \mathcal{T}or^1(\mathcal{I}_p, \mathcal{O}_p) \to \mathcal{I}_p \otimes \mathcal{I}_p \to \mathcal{I}_p \to \mathcal{O}_p \otimes^u \mathcal{I}_p \to 0,$$

where $\otimes^u$, is the underived tensor. We know that $\mathcal{I}_p \cong \mathcal{O}_P(-2) \hookrightarrow \mathcal{O}_P(-1)^{\oplus 2}$, tensoring this by $\mathcal{O}_p$, gives $\mathcal{I}_p \otimes \mathcal{O}_p \cong \mathcal{O}_p \xrightarrow{0} \mathcal{O}_p^{\oplus 2}$, and so we have $\mathcal{I}_p \otimes \mathcal{O}_p = \mathcal{O}_p[1] \oplus \mathcal{O}_p^{\oplus 2}$.

Now, for $\mathcal{I}_p \otimes \mathcal{I}_p$, the above sequence will be

$$\mathcal{O}_p \to \mathcal{I}_p \otimes \mathcal{I}_p \xrightarrow{f} \mathcal{I}_p \xrightarrow{g} \mathcal{O}_p^{\oplus 2} = \mathcal{I}_p/\mathcal{I}_p^2 \to 0.$$

But $im(f) = ker(g) = \mathcal{I}_p^2$, so we have

$$\begin{array}{ccccccc} \mathcal{O}_p & \longrightarrow & \mathcal{I}_p \otimes \mathcal{I}_p & \longrightarrow & \mathcal{I}_p & \longrightarrow & \mathcal{I}_p/\mathcal{I}_p^2 \\ & & \downarrow & \nearrow & & & \\ & & \mathcal{I}_p^2 & & & & \end{array}$$

Notice that the sequence $\mathcal{O}_p \hookrightarrow \mathcal{I}_p \otimes \mathcal{I}_p \xrightarrow{a} \mathcal{I}_p^2$ splits as there is a natural map $b \colon \mathcal{I}_p^2 \to \mathcal{I}_p \otimes \mathcal{I}_p$, such that $a \circ b = id$. Thus we have $\mathcal{I}_p \otimes \mathcal{I}_p \cong \mathcal{O}_p \oplus \mathcal{I}_p^2$.

Now, for $\mathcal{I}_p \otimes \mathcal{O}_l$, again considering $\mathcal{I}_p \cong \mathcal{O}_P(-2) \hookrightarrow \mathcal{O}_P(-1)^{\oplus 2}$ and tensoring it by $\mathcal{O}_l$ gives $\mathcal{O}_l(-2) \to \mathcal{O}_l(-1) \oplus \mathcal{O}_l(-1)$, which is $\mathcal{O}_p \oplus \mathcal{O}_l(-1)$.

For $\mathcal{I}_Z \otimes \mathcal{O}_p$, if $p \subset Z$, then tensoring $\mathcal{I}_Z \cong \mathcal{O}_P(-3) \hookrightarrow \mathcal{O}_P(-2) \oplus \mathcal{O}_P(-1)$ by $\mathcal{O}_p$, gives $\mathcal{I}_Z \otimes \mathcal{O}_p \cong \mathcal{O}_p \xrightarrow{0} \mathcal{O}_p^{\oplus 2}$, and so we have $\mathcal{I}_Z \otimes \mathcal{O}_p = \mathcal{O}_p[1] \oplus \mathcal{O}_p^{\oplus 2}$. If $p \not\subset Z$, then tensoring $\mathcal{I}_p \hookrightarrow \mathcal{O}_P \twoheadrightarrow \mathcal{O}_p$ by $\mathcal{I}_Z$, gives

$$\mathcal{I}_Z \otimes \mathcal{I}_p \hookrightarrow \mathcal{I}_Z \twoheadrightarrow \mathcal{I}_Z \otimes \mathcal{O}_p. \tag{3.1}$$

To compute $\mathcal{I}_Z \otimes \mathcal{I}_p$, we notice that $\mathcal{I}_Z \otimes \mathcal{I}_p = (\mathcal{O}_P \to \mathcal{O}_Z) \otimes (\mathcal{O}_P \to \mathcal{O}_Z) = \mathcal{O}_P \to \mathcal{O}_Z \oplus \mathcal{O}_p$, which is quasi-isomorphic to $\mathcal{I}_{Z \cup p}$. Therefore using 3.1 gives the result.

Now, for $\mathcal{I}_Z \otimes \mathcal{O}_L$, if $Z \subset L$, again considering $\mathcal{I}_Z \cong \mathcal{O}_P(-3) \hookrightarrow \mathcal{O}_P(-2) \oplus \mathcal{O}_P(-1)$ and tensoring it by $\mathcal{O}_L$ gives $\mathcal{O}_L(-2) \to \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1)$, which is $\mathcal{O}_Z \oplus \mathcal{O}_L(-2)$. If $Y_1 \subset L$, but $Y_2 \not\subset L$, then tensoring the same sequence by $\mathcal{O}_{Y_1}$ gives the result. If $Y_1, Y_2 \not\subset L$, then tensoring $\mathcal{I}_L \hookrightarrow \mathcal{O}_P \twoheadrightarrow \mathcal{O}_L$ by $\mathcal{I}_Z$, and noticing that $\mathcal{I}_Z \otimes \mathcal{I}_L = \mathcal{I}_{Z \cup L}$ in this case, imply the claim.

Similarly, the other equations will be obtained.

□

Lemma 3.2.5. For the wall $\langle \mathcal{I}_L(-1), \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5) \rangle$, we have:

$$\operatorname{Ext}^1(\mathcal{I}_L(-1), \mathcal{I}_L(-1)) = \mathbb{C}^4, \quad \operatorname{Ext}^1(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5), \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)) = \mathbb{C}^7,$$

$$\operatorname{Ext}^1(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5), \mathcal{I}_L(-1)) = \mathbb{C}^{18},$$

$$\mathrm{Ext}^1(\mathcal{I}_L(-1), \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)) = \begin{cases} 0, & \langle Z_2\rangle \cap L = \emptyset \\ \mathbb{C}, & \langle Z_2\rangle \cap L \neq \emptyset \text{ but } \langle Z_2\rangle \neq L \\ \mathbb{C}^2, & \langle Z_2\rangle = L \end{cases}$$

where $\langle Z_2\rangle$ is the line spanned by Z_2. Furthermore, we have

$$\mathrm{Ext}^2(\mathcal{I}_L(-1), \mathcal{I}_L(-1)) = 0, \quad \mathrm{Ext}^2(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5), \mathcal{I}_L(-1)) = 0,$$
$$\mathrm{Ext}^2(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5), \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)) = \mathbb{C}^4.$$

Proof. As $\mathrm{Ext}^1(\mathcal{I}_L(-1), \mathcal{I}_L(-1))$ is the tangent space of $\mathbb{G}r(2,4)$, we have $\mathrm{Ext}^1(\mathcal{I}_L(-1), \mathcal{I}_L(-1)) = \mathbb{C}^4$. We know that $\mathrm{Ext}^1(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5), \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5))$ is the tangent space to the space parametrizing two points in a plane in $\mathbb{P}^3$. This is a bundle over $(\mathbb{P}^3)^*$ with fibers isomorphic to $(\mathbb{P}^2)^{[2]}$, thus it is smooth of dimension 7, and hence $\mathrm{Ext}^1(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5), \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)) = \mathbb{C}^7$.

For $\mathrm{Ext}^1(A, B)$, using Serre duality we have:

$$\mathrm{Ext}^1(\mathcal{I}_L(-1), \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)) = \mathrm{Ext}^1(\iota_P^*(\mathcal{I}_L(-1)), (\mathcal{I}_{Z_2})^\vee(-5)) = \mathrm{H}^1(\iota_P^*(\mathcal{I}_L) \otimes \mathcal{I}_{Z_2}(1))^\vee.$$

We use Lemma 3.2.1 for $\iota_P^*(\mathcal{I}_L(-1))$; so there are two cases:

1) $L \subset P$**.** Let $Z_2 = Y_1 \cup Y_2$, for Y_1 and Y_2 single points. By Lemma 3.2.4, we have

$$\mathrm{Ext}^1(A, B) = \mathrm{H}^1(\mathcal{I}_{Z_2})^\vee \oplus \mathrm{H}^1(\mathcal{O}_L \otimes \mathcal{I}_{Z_2})^\vee$$
$$= \begin{cases} \mathrm{H}^1(\mathcal{I}_{Z_2})^\vee \oplus \mathrm{H}^1(\mathcal{O}_L(-2) \oplus \mathcal{O}_{Z_2})^\vee = \mathbb{C} \oplus (\mathbb{C} \oplus 0) = \mathbb{C}^2, & L \subset P \text{ and } Z_2 \subset L \\ \mathrm{H}^1(\mathcal{I}_{Z_2})^\vee \oplus \mathrm{H}^1((\mathcal{O}_L(-1) \oplus \mathcal{O}_{Y_1}) \otimes \mathcal{I}_{Y_2})^\vee & L \subset P,\ Y_1 \subset L, \\ = \mathbb{C} \oplus \mathrm{H}^1(\mathcal{O}_L(-1) \oplus \mathcal{O}_{Y_1})^\vee = \mathbb{C} \oplus (0+0) & \text{and } Y_2 \not\subset L \\ \mathrm{H}^1(\mathcal{I}_{Z_2})^\vee \oplus \mathrm{H}^1(\mathcal{O}_L)^\vee = \mathbb{C} \oplus 0, & L \subset P, \text{ and } Y_1, Y_2 \not\subset L \end{cases}$$

2) L **is not contained in** P**.** Recall that $p = P \cap L$. In this case, again using Lemma 3.2.1 ($\iota_P^*(\mathcal{I}_L) = \mathcal{I}_p$) and Serre duality we have $\mathrm{Ext}^1(A, B) = \mathrm{H}^1(\mathcal{I}_p \otimes \mathcal{I}_{Z_2}(1))^\vee$. Now, we consider the exact triangle

$$\mathcal{I}_p \otimes \mathcal{I}_{Z_2}(1) \to \mathcal{I}_{Z_2}(1) \to \mathcal{O}_p \otimes \mathcal{I}_{Z_2}(1).$$

- If $p \not\subset Z_2$, then using Lemma 3.2.4, we have $\mathcal{O}_p \otimes \mathcal{I}_{Z_2}(1) = \mathcal{O}_p$ and so taking the long exact cohomology sequence of the above triangle, and noticing that the unique global section of $\mathcal{I}_{Z_2}(1)$ vanishes exactly along $\langle Z_2\rangle$, implies that the map $\mathrm{H}^0(\mathcal{I}_{Z_2}(1)) = \mathbb{C} \to \mathrm{H}^0(\mathcal{O}_p) = \mathbb{C}$ is non-zero if and only if p is not colinear with Z_2. Therefore in this case, $\mathrm{Ext}^1(A, B) = \mathrm{H}^1(\iota_P^*(\mathcal{I}_L) \otimes \mathcal{I}_{Z_2}(1))^\vee = \mathbb{C}$ if p is colinear with Z_2, and $\mathrm{Ext}^1(A, B) = \mathrm{H}^1(\iota_P^*(\mathcal{I}_L) \otimes \mathcal{I}_{Z_2}(1))^\vee = 0$ otherwise.
- If $p \subset Z_2$, then using Lemma 3.2.4, we have $\mathcal{O}_p \otimes \mathcal{I}_{Z_2}(1)$ is a two term complex with $\mathcal{H}^0(\mathcal{O}_p \otimes \mathcal{I}_{Z_2}(1)) = \mathcal{I}_{Z_2}/m_p.\mathcal{I}_{Z_2} = (\mathcal{O}_p)^{\oplus 2}$ and $\mathcal{H}^{-1}(\mathcal{O}_p \otimes \mathcal{I}_{Z_2}(1)) = \mathcal{T}or^1(\mathcal{O}_p \otimes \mathcal{I}_{Z_2}(1)) = \mathcal{O}_p$. The map $\mathrm{H}^0(\mathcal{I}_{Z_2}(1)) = \mathbb{C} \to \mathrm{H}^0(\mathcal{O}_p \otimes \mathcal{I}_{Z_2}(1)) = \mathrm{H}^0((\mathcal{O}_p)^{\oplus 2}) = \mathbb{C}^2$ is always non-zero, and as $\mathrm{H}^1(\iota_P^*(\mathcal{I}_{Z_2}(1)) = 0$, we will have $\mathrm{Ext}^1(A, B) = \mathrm{H}^1(\iota_P^*(\mathcal{I}_L) \otimes \mathcal{I}_{Z_2}(1))^\vee = \mathbb{C}$.

Now we compute $\mathrm{Ext}^1(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5), \mathcal{I}_L(-1))$. In case $L \subset P$ we use Lemma 3.2.1 to obtain

$$\mathrm{Ext}^1(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5), \mathcal{I}_L(-1)) = \mathrm{Ext}^1((\mathcal{I}_{Z_2})^\vee(-5), \iota_P^!(\mathcal{I}_L(-1))) = \mathrm{H}^0(\iota_P^*(\mathcal{I}_L) \otimes \mathcal{I}_{Z_2} \otimes \mathcal{O}(5))$$

$$= \mathrm{H}^0\big((\mathcal{O}(-1) \oplus \mathcal{O}_L(-1)) \otimes \mathcal{I}_{Z_2}(5)\big) = \mathrm{H}^0(\mathcal{I}_{Z_2}(4)) \oplus \mathrm{H}^0(\mathcal{I}_{Z_2} \otimes \mathcal{O}_L(4)) = \mathbb{C}^{13} \oplus \mathrm{H}^0(\mathcal{I}_{Z_2} \otimes \mathcal{O}_L(4)).$$

Tensoring $\mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_L$ by $\mathcal{I}_{Z_2}(4)$ implies $\mathrm{H}^0(\mathcal{I}_{Z_2} \otimes \mathcal{O}_L(4)) = \mathbb{C}^5$, and therefore $\mathrm{Ext}^1(B, A) = \mathbb{C}^{18}$. Now let $L \not\subset P$. As in this case we have $\iota_P^*(\mathcal{I}_L) = \mathcal{I}_p$, a similar computation as before shows that $\mathrm{Ext}^1(B, A) = \mathrm{H}^0(\mathcal{I}_p \otimes \mathcal{I}_{Z_2}(5)) = \mathbb{C}^{18}$ for all the possible configurations, by Lemma 3.2.4.

We can easily see that $\mathrm{Ext}^2(\mathcal{I}_L(-1), \mathcal{I}_L(-1)) = 0$. As for $\mathrm{Ext}^2(B, A)$, using Lemma 3.2.4,in a similar way as for $\mathrm{Ext}^1(B, A)$, we can see that when $L \subset P$, we have

$$\mathrm{Ext}^2(B, A) = \mathrm{H}^1((\mathcal{O}(-1) \oplus \mathcal{O}_L(-1)) \otimes \mathcal{I}_{Z_2} \otimes \mathcal{O}(5)) = \mathrm{H}^1(\mathcal{I}_{Z_2}(4)) \oplus \mathrm{H}^1(\mathcal{I}_{Z_2} \otimes \mathcal{O}_L(4)) = 0,$$

and when $L \not\subset P$, we have $\mathrm{Ext}^2(B, A) = \mathrm{H}^1(\mathcal{I}_p \otimes \mathcal{I}_{Z_2}(5)) = 0$.

Finally, for $\mathrm{Ext}^2(B, B)$, applying Lemma 3.1.7 to $j = \iota_P \colon P \to \mathbb{P}^3$ and $\mathcal{F} := \mathcal{I}_{Z_2}$, we get $\mathcal{I}_{Z_2}[1] \to \iota_P^*(\iota_{P_*}\mathcal{I}_{Z_2}(1)) \to \mathcal{I}_{Z_2}(1)$, and so applying $\mathrm{Hom}(\mathcal{I}_{Z_2}, -)$ to this exact triangle gives

$$\mathrm{Ext}^2(B, B) = \mathrm{Ext}^2(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5), \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)) = \mathrm{Ext}^2(\iota_{P_*}\mathcal{I}_{Z_2}, \iota_{P_*}\mathcal{I}_{Z_2})$$

$$= \mathrm{Ext}^2(\mathcal{I}_{Z_2}, \iota_P^!(\iota_{P_*}\mathcal{I}_{Z_2})) = \mathrm{Ext}^1(\mathcal{I}_{Z_2}, \iota_P^*(\iota_{P_*}\mathcal{I}_{Z_2}(1))) = \mathrm{Ext}^1(\mathcal{I}_{Z_2}, \mathcal{I}_{Z_2}(1)) = \mathbb{C}^4.$$

□

Lemma 3.2.6. For the wall $\langle \mathcal{O}(-1) \to \mathcal{O}_L, \iota_{P_*}\mathcal{I}_{Z_1}^\vee(-5)\rangle$, we have

$$\mathrm{Ext}^1((\mathcal{O}(-1) \to \mathcal{O}_L), (\mathcal{O}(-1) \to \mathcal{O}_L)) = \mathbb{C}^5, \quad \mathrm{Ext}^1(\iota_{P_*}\mathcal{I}_{Z_1}^\vee(-5), \iota_{P_*}\mathcal{I}_{Z_1}^\vee(-5)) = \mathbb{C}^5,$$

$$\mathrm{Ext}^1(\iota_{P_*}\mathcal{I}_{Z_1}^\vee(-5), \mathcal{O}(-1) \xrightarrow{s} \mathcal{O}_L) = \begin{cases} \mathbb{C}^{21}, & \text{the zero locus of } s \text{ is at } Z_1, \\ \mathbb{C}^{20}, & \text{the zero locus of } s \text{ is in } P \text{ but} \neq Z_1, \\ \mathbb{C}^{19}, & \text{the zero locus of } s \text{ is not in } P \end{cases}$$

$$\mathrm{Ext}^1(\mathcal{O}(-1) \xrightarrow{s} \mathcal{O}_L, \iota_{P_*}\mathcal{I}_{Z_1}^\vee(-5)) = \begin{cases} \mathbb{C}^2, & \text{the zero locus of } s \text{ is at } Z_1, \\ \mathbb{C}, & \text{the zero locus of } s \text{ is in } P \text{ but} \neq Z_1, \\ 0, & \text{the zero locus of } s \text{ is not in } P \end{cases}$$

Proof. We notice that $\mathrm{Ext}^1(\mathcal{O}(-1) \to \mathcal{O}_L, \mathcal{O}(-1) \to \mathcal{O}_L)$ is the parameter space of lines in $\mathbb{P}^3$ (which is $\mathbb{G}r(2,4)$) plus one point on the line, so we have $\mathrm{Ext}^1(\mathcal{O}(-1) \to \mathcal{O}_L, \mathcal{O}(-1) \to \mathcal{O}_L) = \mathbb{C}^5$. Also, $\mathrm{Ext}^1(\iota_{P_*}\mathcal{I}_{Z_1}^\vee(-5), \iota_{P_*}\mathcal{I}_{Z_1}^\vee(-5))$ is the parameter space of one point in a plane, which gives the flag variety $\mathfrak{Fl}_1$. Therefore we have $\mathrm{Ext}^1(B, B) = \mathbb{C}^5$.

Let q be the zero locus of s. Now we compute $\mathrm{Ext}^1(B, A)$ as follows:

$$\mathrm{Ext}^1(\iota_{P_*}\mathcal{I}_{Z_1}^\vee(-5), \mathcal{O}(-1) \to \mathcal{O}_L) = \mathrm{H}^0\big(\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(5)\big).$$

First, assume that $L \not\subset P$. There are two cases:

- (I) Zero locus of s is not at $L \cap P$: Using Lemmas 3.2.1 and 3.2.4 we have:
  (i) if $L \cap P = Z_1$, then similarly we have $\mathrm{H}^0(\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(5)) = \mathrm{H}^0(\mathcal{I}_{L\cap P} \otimes \mathcal{I}_{Z_1}(5)) = \mathrm{H}^0(\mathcal{I}_{Z_1} \otimes \mathcal{I}_{Z_1}(5)) = \mathrm{H}^0(\mathcal{O}_{Z_1} \oplus \mathcal{I}^2_{Z_1}(5)) = \mathbb{C}^{19}$.
  (ii) if $L \cap P \neq Z_1$, then we have $\mathrm{H}^0(\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(5))) = \mathrm{H}^0(\mathcal{I}_{L\cap P} \otimes \mathcal{I}_{Z_1}(5)) = \mathrm{H}^0(\mathcal{I}_{(L\cap P)\cup Z_1}(5)) = \mathbb{C}^{19}$.
- (II) Zero locus of s is $L \cap P$: By Lemma 3.2.1, we have $\mathrm{H}^0(\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(5)) = \mathrm{H}^0((\mathcal{O}_P \oplus \mathcal{O}_{L\cap P}[-1]) \otimes \mathcal{I}_{Z_1}(5)) = \mathrm{H}^0(\mathcal{O}_P \otimes \mathcal{I}_{Z_1}(5)) \oplus \mathrm{H}^{-1}(\mathcal{O}_{L\cap P} \otimes \mathcal{I}_{Z_1}(5)) = \mathrm{H}^0(\mathcal{I}_{Z_1}(5)) \oplus \mathrm{H}^{-1}(\mathcal{O}_{L\cap P} \otimes \mathcal{I}_{Z_1}(5)) = \mathbb{C}^{20} \oplus \mathrm{H}^{-1}(\mathcal{O}_{L\cap P} \otimes \mathcal{I}_{Z_1}(5))$. Now there are two subcases:
  (i) if $Z_1 \neq P \cap L$, by Lemma 3.2.4 we have $Ext^1(B, A) = \mathbb{C}^{20} \oplus \mathrm{H}^{-1}(\mathcal{O}_{L\cap P} \otimes \mathcal{I}_{Z_1}(5)) = \mathbb{C}^{20} \oplus \mathrm{H}^{-1}(\mathcal{O}_{L\cap P}) = \mathbb{C}^{20} \oplus 0 = \mathbb{C}^{20}$.
  (ii) if $Z_1 = P \cap L$, by Lemma 3.2.4 we have $\mathcal{O}_{L\cap P} \otimes \mathcal{I}_{Z_1}(1) = \mathcal{O}_{Z_1}[1] \oplus \mathcal{O}_{Z_1}^{\oplus 2}$, and therefore $Ext^1(B, A) = \mathrm{H}^0(\mathcal{I}_{Z_1}(5)) \oplus \mathrm{H}^0(\mathcal{O}_{Z_1}) = \mathbb{C}^{20} \oplus \mathbb{C}^1 = \mathbb{C}^{21}$.

Second, assume that $L \subset P$. In this case we have $\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) = (\mathcal{O}_P \to \mathcal{O}_L(1)) \oplus \mathcal{O}_L$. Therefore we have

$$\mathrm{Ext}^1(B, A) = \mathrm{H}^0(\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(5)) = \mathrm{H}^0((\mathcal{O}_P(5) \to \mathcal{O}_L(6)) \otimes \mathcal{I}_{Z_1}) \oplus \mathrm{H}^0(\mathcal{O}_L \otimes \mathcal{I}_{Z_1}(5))$$
$$= \mathrm{H}^0(\mathcal{I}_{Z_1}(5) \to \mathcal{O}_L \otimes \mathcal{I}_{Z_1}(6)) \oplus \mathrm{H}^0(\mathcal{O}_L \otimes \mathcal{I}_{Z_1}(5)).$$

But using 3.2.4 we have

$$\mathrm{H}^0(\mathcal{O}_L \otimes \mathcal{I}_{Z_1}(5)) = \begin{cases} \mathrm{H}^0(\mathcal{O}_L(5)) \cong \mathbb{C}^6, & Z_1 \not\subset L, \\ \mathrm{H}^0(\mathcal{O}_L(4) \oplus \mathcal{O}_{Z_1}) = \mathbb{C}^5 \oplus \mathbb{C} \cong \mathbb{C}^6, & Z_1 \subset L. \end{cases}$$

On the other hand, to compute $\mathrm{H}^0(\mathcal{I}_{Z_1}(5) \xrightarrow{s} \mathcal{O}_L \otimes \mathcal{I}_{Z_1}(6)) = \mathrm{H}^0(\ker(t))$, we notice that s factors as follows: $\mathcal{I}_{Z_1}(5) \overset{s_1}{\twoheadrightarrow} \mathcal{O}_L \otimes \mathcal{I}_{Z_1}(5) \xrightarrow{s_2} \mathcal{O}_L \otimes \mathcal{I}_{Z_1}(6)$. Note that $\ker(s_1) = \mathcal{I}_{Z_1}(4)$. Now there are three cases:

- (1) If $Z_1 \not\subset L$, then by Lemma 3.2.4 we have $s_2 \colon \mathcal{O}_L(5) \to \mathcal{O}_L(6)$, which is injective.
- (2) If $Z_1 \subset L$, but $Z_1 \neq$ zero locus of s, then by Lemma 3.2.4 we have $s_2 \colon \mathcal{O}_L(4) \oplus \mathcal{O}_{Z_1} \to \mathcal{O}_L(5) \oplus \mathcal{O}_{Z_1}$, which is injective.
- (3) If $Z_1 =$ zero locus of $s \subset L$, then $s_2|_{\mathcal{O}_{Z_1}} = 0$.

For cases (1) and (2), where s_2 is injective, we have

$$\mathrm{H}^0(\ker(s)) = \mathrm{H}^0(\ker(s_1)) = \mathrm{H}^0(\mathcal{I}_{Z_1}(4)) = \mathbb{C}^{14}.$$

For case (3), we have

$$\mathrm{H}^0(\ker(s)) = \mathrm{H}^0(\ker(\mathcal{I}_{Z_1}(5) \to \mathcal{O}_L(4))) = \mathrm{H}^0(\mathcal{O}_P(4)) = \mathbb{C}^{15}.$$

Thus $\mathrm{Ext}^1(B, A)$ is

$$\mathrm{H}^0(\mathcal{I}_{Z_1}(5) \to \mathcal{O}_L \otimes \mathcal{I}_{Z_1}(6)) \oplus \mathrm{H}^0(\mathcal{O}_L \otimes \mathcal{I}_{Z_1}(5)) = \begin{cases} \mathbb{C}^{14} \oplus \mathbb{C}^6 = \mathbb{C}^{20}, & Z_1 \neq q, \\ \mathbb{C}^{15} \oplus \mathbb{C}^6 = \mathbb{C}^{21}, & Z_1 = q \subset L. \end{cases}$$

For the last part, using Serre duality we have

$$\mathrm{Ext}^1(A, B) = \mathrm{Ext}^1(\mathcal{O}(-1) \to \mathcal{O}_L, \iota_{P_*}\mathcal{I}_{Z_1}^\vee(-5)) = \mathrm{Ext}^1((\iota_P^*(\mathcal{O}(-1) \to \mathcal{O}_L), \mathcal{I}_{Z_1'}^\vee(-5))$$

$$= \mathrm{Ext}^1(\mathcal{I}_{Z_1'}^\vee(-5), (\iota_P^*(\mathcal{O}(-1) \to \mathcal{O}_L) \otimes \mathcal{O}(-3))^\vee = \mathrm{H}^1((\iota_P^*(\mathcal{O}(-1) \to \mathcal{O}_L) \otimes \mathcal{I}_{Z_1}(2))^\vee.$$

There are three cases:

(1) $L \not\subset P$ **and** $L \cap P$ **is the zero locus of** s : From Lemmas 3.2.1 and 3.2.4, we have

$$\mathrm{Ext}^1(A, B) = \mathrm{H}^1((\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(1))^\vee = \mathrm{H}^1(\mathcal{I}_{Z_1}(1))^\vee \oplus \mathrm{H}^0(\mathcal{O}_{L\cap P} \otimes \mathcal{I}_{Z_1}(1))^\vee$$

$$= 0 \oplus \mathrm{H}^0(\mathcal{O}_{L\cap P} \otimes \mathcal{I}_{Z_1}(1))^\vee = \begin{cases} \mathrm{H}^0(\mathcal{O}_{P\cap L})^\vee = \mathbb{C}, & L \cap P \neq Z_1 \\ \mathrm{H}^0(\mathcal{O}_{L\cap P}[1] \oplus \mathcal{O}_{L\cap P}^{\oplus 2})^\vee = 0 \oplus \mathbb{C}^2 = \mathbb{C}^2, & L \cap P = Z_1 \end{cases}$$

(2) $L \not\subset P$**, and** $L \cap P$ **is not the zero section of** s : From Lemmas 3.2.1 and 3.2.4, we have

$$\mathrm{Ext}^1(A, B) = \mathrm{H}^1(\mathcal{I}_{L\cap P} \otimes \mathcal{I}_{Z_1}(1))^\vee$$

$$= \begin{cases} \mathrm{H}^1(\mathcal{I}_{(P\cap L)\cup Z_1}(1))^\vee = 0, & L \cap P \neq Z_1 \\ \mathrm{H}^1(\mathcal{O}_{L\cap P})^\vee \oplus \mathrm{H}^1(\mathcal{I}_{L\cap P}^2(1))^\vee = 0 \oplus 0 = 0, & L \cap P = Z_1 \end{cases}$$

(3) $L \subset P$: Again, using Lemmas 3.2.1, we have

$$\mathrm{Ext}^1(A, B) = \mathrm{H}^1((\mathcal{O}_P \xrightarrow{s} \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(1))^\vee \oplus \mathrm{H}^1(\mathcal{O}_L \otimes \mathcal{I}_{Z_1}(1))^\vee$$

$$= \mathrm{H}^1((\mathcal{O}_P \xrightarrow{s} \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(1))^\vee \oplus 0.$$

Exactly the same argument as above (just twisting everything by -4), implies

$$\mathrm{H}^0(\mathcal{I}_{Z_1}(1) \to \mathcal{O}_L \otimes \mathcal{I}_{Z_1}(2)) = \begin{cases} 0, & Z_1 \neq \text{zero locus of } s, \\ \mathbb{C}, & Z_1 = \text{zero locus of } s \subset L. \end{cases}$$

Now, taking the cohomology long exact sequence of $\mathcal{O}_L \otimes \mathcal{I}_{Z_1}(2)[-1] \to (\mathcal{O}_P \to \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(1) \to \mathcal{O}_P \otimes \mathcal{I}_{Z_1}(1)$ and noticing that $\mathrm{H}^0(\mathcal{O}_P \otimes \mathcal{I}_{Z_1}(1)) = \mathrm{H}^0(\mathcal{I}_{Z_1}(1)) = \mathbb{C}^2$, and (by Lemma 3.2.4)

$$\mathrm{H}^0(\mathcal{O}_L \otimes \mathcal{I}_{Z_1}(2)) = \begin{cases} \mathrm{H}^0(\mathcal{O}_L(2)) \cong \mathbb{C}^3, & Z_1 \not\subset L, \\ \mathrm{H}^0(\mathcal{O}_L(1) \oplus \mathcal{O}_{Z_1}) = \mathbb{C}^2 \oplus \mathbb{C} \cong \mathbb{C}^3, & Z_1 \subset L, \end{cases}$$

we will have

$$0 \longrightarrow \mathrm{H}^0((\mathcal{O}_P \to \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(1)) \longrightarrow \mathbb{C}^2 \longrightarrow \mathbb{C}^3 \longrightarrow \mathrm{H}^1((\mathcal{O}_P \to \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1}(1)) \longrightarrow 0.$$

Therefore, in this case we have

$$\mathrm{Ext}^1(A,B) = \mathrm{H}^1(\mathcal{I}_{Z_1}(1) \to \mathcal{O}_L \otimes \mathcal{I}_{Z_1}(2))^\vee = \begin{cases} \mathbb{C}, & Z_1 \neq \text{zero locus of } s, \\ \mathbb{C}^2, & Z_1 = \text{zero locus of } s \subset L. \end{cases}$$

□

Lemma 3.2.7. For the wall $\langle \mathcal{O}(-1) \to \mathcal{O}_L(1), \mathcal{O}_P(-5) \rangle$, we have:

$$\mathrm{Ext}^1(\mathcal{O}(-1) \to \mathcal{O}_L(1), \mathcal{O}(-1) \to \mathcal{O}_L(1)) = \mathbb{C}^6, \quad \mathrm{Ext}^1(\mathcal{O}_P(-5), \mathcal{O}_P(-5)) = \mathbb{C}^3,$$

$$\mathrm{Ext}^1(\mathcal{O}_P(-5), \mathcal{O}(-1) \xrightarrow{s} \mathcal{O}_L(1)) = \begin{cases} \mathbb{C}^{22}, & L \subset P \\ \mathbb{C}^{21}, & L \not\subset P, \text{ and the zero locus of } s \supset L \cap P, \\ \mathbb{C}^{20}, & L \not\subset P, \text{ and the zero locus of } s \not\supset L \cap P \end{cases}$$

$$\mathrm{Ext}^1(\mathcal{O}(-1) \xrightarrow{s} \mathcal{O}_L(1), \mathcal{O}_P(-5)) = \begin{cases} \mathbb{C}^2, & L \subset P, \\ \mathbb{C}, & L \not\subset P \text{ and the zero locus of } s \supset L \cap P \\ 0, & L \not\subset P, \text{ and the zero locus of } s \not\supset L \cap P \end{cases}$$

Proof. We notice that $\mathrm{Ext}^1((\mathcal{O} \to \mathcal{O}_L(2)) \otimes \mathcal{O}(-1), (\mathcal{O} \to \mathcal{O}_L(2)) \otimes \mathcal{O}(-1))$ is the parameter space of lines in $\mathbb{P}^3$ (which is $\mathbb{G}r(2,4)$) plus two points on the line, so we have $\mathrm{Ext}^1(\mathcal{O}(-1) \to \mathcal{O}_L(1), \mathcal{O}(-1) \to \mathcal{O}_L(1)) = \mathbb{C}^6$. Also, $\mathrm{Ext}^1(\iota_{P_*}(\mathcal{O}_P)(-5), \iota_{P_*}(\mathcal{O}_P)(-5))$ is the parameter space of a plane in $\mathbb{P}^3$ which is given by $(\mathbb{P}^3)^*$, and thus we have $\mathrm{Ext}^1(B,B) = \mathbb{C}^3$.

Let $q \cup q'$ be the zero locus of s. Now we compute $\mathrm{Ext}^1(B,A)$ as follows:

$$\mathrm{Ext}^1(B,A) = \mathrm{Ext}^1(\iota_{P_*}(\mathcal{O}_P)(-5), \mathcal{O}(-1) \to \mathcal{O}_L(1)) = \mathrm{Ext}^1(\mathcal{O}_P(-5), \iota_P^!(\mathcal{O}(-1) \to \mathcal{O}_L(1)))$$

$$= \mathrm{Hom}(\mathcal{O}_P(-5), \iota_P^*(\mathcal{O} \to \mathcal{O}_L(2))) = \mathrm{H}^0(\iota_P^*(\mathcal{O} \to \mathcal{O}_L(2)) \otimes \mathcal{O}(5)).$$

First, assume that $L \not\subset P$. By Lemma 3.2.1 we have:

$$\mathrm{Ext}^1(B,A) = \begin{cases} \mathrm{H}^0((\mathcal{O}_P(5) \oplus \mathcal{O}_{L\cap P}[-1])) = \mathbb{C}^{21} \oplus 0, & q \cup q' \supset L \cap P, \\ \mathrm{H}^0(\mathcal{I}_{L \cap P}(5)) = \mathbb{C}^{20}, & q \cup q' \not\supset L \cap P. \end{cases}$$

Second, assume that $L \subset P$**.** Using Lemma 3.2.1, we have $\mathrm{H}^0(\iota_P^*(\mathcal{O} \to \mathcal{O}_L(2)) \otimes \mathcal{O}(5)) = \mathrm{H}^0((\mathcal{O}_P \to \mathcal{O}_L(2)) \otimes \mathcal{O}(5)) \oplus \mathrm{H}^0(\mathcal{O}_L(1) \otimes \mathcal{O}(5)) = \mathrm{H}^0((\mathcal{O}_P(5) \to \mathcal{O}_L(7))) \oplus \mathrm{H}^0(\mathcal{O}_L(6))$.

We have $\mathrm{H}^0((\mathcal{O}_P(5) \xrightarrow{s} \mathcal{O}_L(7))) = \mathrm{H}^0(\ker(s)) = \mathrm{H}^0(\mathcal{I}_L(5)) = \mathbb{C}^{15}$. Also we can easily see $\mathrm{H}^0(\mathcal{O}_L(6)) = \mathbb{C}^7$. Therefore, $\mathrm{H}^0(\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) \otimes \mathcal{I}_{Z_1'}(5)) = \mathbb{C}^{15} \oplus \mathbb{C}^7 = \mathbb{C}^{22}$.

Finally, for $\mathrm{Ext}^1(A, B)$ using Serre duality, we have

$$\mathrm{Ext}^1((\mathcal{O}(-1) \to \mathcal{O}_L(1)), \mathcal{O}_P(-5)) = \mathrm{H}^1(\iota_P^*(\mathcal{O} \to \mathcal{O}_L(2)) \otimes \mathcal{O}(1))^\vee.$$

1) If $L \subset P$:

$$\mathrm{Ext}^1(A, B) = \mathrm{H}^1(\mathcal{O}_P(1) \xrightarrow{s} \mathcal{O}_L(3))^\vee \oplus \mathrm{H}^1(\mathcal{O}_L(2))^\vee = \mathrm{H}^1(\mathcal{O}_P(1) \xrightarrow{s} \mathcal{O}_L(3))^\vee \oplus 0.$$

We have $\mathrm{H}^0(\mathcal{O}_P(1) \xrightarrow{s} \mathcal{O}_L(3)) = \mathrm{H}^0(\ker(s)) = \mathrm{H}^0(\mathcal{I}_L(1)) = \mathbb{C}$. Now, taking the long exact sequence of $\mathcal{O}_L(3)[-1] \to (\mathcal{O}_P(1) \xrightarrow{s} \mathcal{O}_L(3)) \to \mathcal{O}_P(1)$, gives

$$\mathrm{H}^{-1}(\mathcal{O}_L(3)) = 0 \longrightarrow \mathrm{H}^0(\mathcal{O}_P(1) \xrightarrow{s} \mathcal{O}_L(3)) \longrightarrow \mathrm{H}^0(\mathcal{O}_P(1)) = \mathbb{C}^3 \longrightarrow$$
$$\mathrm{H}^0(\mathcal{O}_L(3)) = \mathbb{C}^4 \longrightarrow \mathrm{H}^1(\mathcal{O}_P(1) \xrightarrow{s} \mathcal{O}_L(3)) \longrightarrow \mathrm{H}^1(\mathcal{O}_P(1)) = 0,$$

which implies $\mathrm{H}^1(\mathcal{O}_P(1) \to \mathcal{O}_L(3)) = \mathbb{C}^2$.

2) If $L \not\subset P$, using Lemma 3.2.1, we have

$$\mathrm{Ext}^1(A, B) = \begin{cases} \mathrm{H}^1(\mathcal{O}_P(1))^\vee \oplus \mathrm{H}^0(\mathcal{O}_{L\cap P})^\vee = 0 \oplus \mathbb{C}, & q \cup q' \supset L \cap P, \\ \mathrm{H}^1(\mathcal{I}_{L\cap P}(1))^\vee = 0, & q \cup q' \not\supset L \cap P. \end{cases}$$

□

Lemma 3.2.8. For the wall $\langle \mathcal{O}(-1), \iota_{P_*}\mathcal{I}_{Z_6}^\vee(-6)\rangle$, we have:

$$\mathrm{Ext}^1(\mathcal{O}(-1), \mathcal{O}(-1)) = 0, \quad \mathrm{Ext}^1(\iota_{P_*}\mathcal{I}_{Z_6}^\vee(-6), \iota_{P_*}\mathcal{I}_{Z_6}^\vee(-6)) = \mathbb{C}^{15},$$

$$\mathrm{Ext}^1(\iota_{P_*}\mathcal{I}_{Z_6}^\vee(-6), \mathcal{O}(-1)) = \mathbb{C}^{22},$$

$$\mathrm{Ext}^1(\mathcal{O}(-1), \iota_{P_*}\mathcal{I}_{Z_6}^\vee(-6)) = \begin{cases} \mathbb{C}^3, & \text{6 points on a line} \\ \mathbb{C}^2, & \text{5 points on a line} \\ \mathbb{C}, & \text{6 points on a conic} \\ 0, & \text{generic points} \end{cases}$$

Proof. The fist statement is obvious. We notice that $\mathrm{Ext}^1(\iota_{P_*}\mathcal{I}^\vee_{Z_6}(-6), \iota_{P_*}\mathcal{I}^\vee_{Z_6}(-6))$ is a parameter space of six points in a plane which is of dimension 15 as claimed.

Now we compute $\mathrm{Ext}^1(B, A)$:

$$\mathrm{Ext}^1(\iota_{P_*}\mathcal{I}^\vee_{Z_6}(-6), \mathcal{O}(-1)) = \mathrm{Hom}(\mathcal{I}^\vee_{Z'_6}(-6), \iota_P^*(\mathcal{O})) = \mathrm{H}^0(\mathcal{I}_{Z'_6}(6)) = \mathbb{C}^{22}.$$

For the last part, using Serre duality we have

$$\mathrm{Ext}^1(\mathcal{O}(-1), \iota_{P_*}\mathcal{I}^\vee_{Z_6}(-6)) = \mathrm{Ext}^1(\iota_P^*\mathcal{O}(-1), \mathcal{I}^\vee_{Z'_6}(-6)) = \mathrm{H}^1(\mathcal{O}_P \otimes \mathcal{I}_{Z'_6}(2))^\vee = \mathrm{H}^1(\mathcal{I}_{Z'_6}(2))^\vee$$

$$= \begin{cases} \mathbb{C}^3, & \text{6 points on a line} \\ \mathbb{C}^2, & \text{5 points on a line} \\ \mathbb{C}, & \text{6 points on a conic} \\ 0, & \text{generic points} \end{cases}$$

as $\dim \mathrm{H}^1(\mathcal{I}_{Z'_6}(2))^\vee = \dim \mathrm{H}^0(\mathcal{I}_{Z'_6}(2))^\vee$ by taking a long exact sequence of $\mathcal{I}_{Z'_6}(2) \to \mathcal{O}_P(2) \to \mathcal{O}_{Z'_6}$.

□

3.3 Ext computation of the walls on the Hilbert scheme side

In this section, we compute Ext groups for walls $\langle \mathcal{I}_{L_2}(-1), \mathcal{O}_P(-5)\rangle$, $\langle \mathcal{I}_{L_1}(-1), \mathcal{I}_{Z_1/P}(-5)\rangle$ and $\langle \mathcal{I}_L(-1), \mathcal{I}_{Z_2/P}(-5)\rangle$.

Lemma 3.3.1. Let $L_2 = L \cup q \cup q'$, and $L \cap P = p$. For the wall $\langle \mathcal{I}_{L_2}(-1), \mathcal{O}_P(-5)\rangle$, we have:

$$\mathrm{Ext}^1(\mathcal{O}_P(-5), \mathcal{I}_{L_2}(-1)) = \begin{cases} \mathbb{C}^{22}, & L \subset P, q, q' \subset P \\ \mathbb{C}^{21}, & L \subset P, q \subset P, q' \not\subset P \\ \mathbb{C}^{20}, & (L \not\subset P) \text{ or } (L \subset P, q, q' \not\subset P) \end{cases}$$

Proof. We only consider the most generic cases. Other cases are similar by using Lemma 3.2.1. Let $L \not\subset P$ and $q, q' \not\subset P$. Using Lemma 3.2.1, we have:

$$\mathrm{Ext}^1(\mathcal{O}_P(-5), \mathcal{I}_{L_2}(-1)) = \mathrm{Ext}^1(\mathcal{O}_P(-5), \iota^!\mathcal{I}_{L_2}(-1)) = \mathrm{Ext}^1(\mathcal{O}_P(-5), \mathcal{I}_p[-1])$$
$$= \mathrm{Hom}(\mathcal{O}_P(-5), \mathcal{I}_p) = \mathbb{C}^{20}.$$

Similarly, for the cases $q \subset P, q' \not\subset P$ and $q, q' \subset P$, by using Lemma 3.2.1, we get $\mathbb{C}^{20}$.

Now let $L \subset P$, and $q, q' \not\subset P$. Using Lemma 3.2.1, we have:

$$\mathrm{Ext}^1(\mathcal{O}_P(-5), \mathcal{I}_{L_2}(-1)) = \mathrm{Ext}^1(\mathcal{O}_P(-5), \iota^!\mathcal{I}_{L_2}(-1)) = \mathrm{Ext}^1(\mathcal{O}_P(-5), \mathcal{O}_P(-1) \oplus \mathcal{O}_L(-1)[-1])$$
$$= \mathrm{Hom}(\mathcal{O}_P(-4), \mathcal{O}_P) \oplus \mathrm{Hom}(\mathcal{O}_P(-4), \mathcal{O}_L) = \mathbb{C}^{15} \oplus \mathbb{C}^5 = \mathbb{C}^{20}.$$

Similarly, for $q \subset P, q' \not\subset P$ and $q, q' \subset P$, we get $\mathbb{C}^{21}$ and $\mathbb{C}^{22}$, respectively. □

Lemma 3.3.2. For the wall $\langle \mathcal{I}_{L_1}(-1), \mathcal{I}_{Z_1/P}(-5)\rangle$, with $L_1 = L \cup q$ and $p = L \cap P$, we have:

$$\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1)) = \begin{cases} \mathbb{C}^{22}, & L \not\subset P, p \neq q = Z_1 \subset P \\ \mathbb{C}^{21}, & (L \not\subset P, Z_1 = p) \text{ or } (L \subset P) \\ \mathbb{C}^{20}, & L \not\subset P, Z_1 \neq p \end{cases}$$

Proof. We only consider the most generic cases. Other cases are similar by using Lemma 3.2.1. First let $L \not\subset P, q \neq Z_1 \neq p$. We have

$$\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_1}(-5), \iota^! \mathcal{I}_{L_1}(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_1}(-5), \mathcal{I}_p[-1])$$
$$= \mathrm{Hom}(\mathcal{O}_P(-5), \mathcal{I}_p) = \mathbb{C}^{20}.$$

Now, let $L \not\subset P, Z_1 = p$ (the case $L \subset P$ is a degenerate case). We have

$$\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_1}(-5), \iota^! \mathcal{I}_{L_1}(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_1}(-5), \mathcal{I}_p[-1])$$
$$= \mathrm{Ext}^1(\mathcal{I}_p(-5), \mathcal{I}_p[-1]) = \mathrm{Hom}(\mathcal{O}(-5), \mathcal{O}) = \mathbb{C}^{21}.$$

Finally, let $L \not\subset P, p \neq q = Z_1 \subset P$. We have

$$\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_1}(-5), \iota^! \mathcal{I}_{L_1}(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_1}(-5), (\mathcal{I}_{p\cup q} \oplus \mathcal{O}_p)[-1])$$
$$= \mathrm{Hom}(\mathcal{I}_{Z_1}(-5), \mathcal{I}_{p\cup q}) \oplus \mathrm{Hom}(\mathcal{I}_{Z_1}(-5), \mathcal{O}_p) = \mathrm{Hom}(\mathcal{O}(-5), \mathcal{I}_p) \oplus \mathrm{Hom}(\mathcal{I}_{Z_1}(-5), \mathcal{O}_p)$$
$$= \mathbb{C}^{20} \oplus \mathbb{C}^2 = \mathbb{C}^{22}.$$

□

Lemma 3.3.3. Let $Z_2 = Z \cup Z'$ and $p = L \cap P$. For the wall $\langle \mathcal{I}_L(-1), \mathcal{I}_{Z_2/P}(-5)\rangle$, we have:

$$\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1)) = \begin{cases} \mathbb{C}^{22}, & Z_2 \subset L \subset P \\ \mathbb{C}^{21}, & (L \not\subset P) \text{ and (either } Z \text{ or } Z' = p), \\ \mathbb{C}^{20}, & L \cap Z_2 = \emptyset \end{cases}$$

Proof. We only consider the most generic cases. Other cases are similar by using Lemma 3.2.1. First, let $L \cap Z_2 = \emptyset$. We only need to look at the generic case when $L \not\subset P$:

$$\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_2}(-5), \iota^! \mathcal{I}_L(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_2}(-5), \mathcal{I}_p[-1])$$
$$= \mathrm{Hom}(\mathcal{I}_{Z_2}(-5), \mathcal{I}_p) = \mathbb{C}^{20}.$$

Now, let $L \not\subset P$ and $Z = p$ (the case $Z_2 = p$ is similar):

$$\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_2}(-5), \iota^! \mathcal{I}_L(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_2}(-5), \mathcal{I}_p[-1])$$

$$= \mathrm{Hom}(\mathcal{I}_{Z\cup Z'}(-5), \mathcal{I}_Z) = \mathrm{Hom}(\mathcal{I}_{Z'}(-5), \mathcal{O}) = \mathbb{C}^{21}.$$

Finally, let $Z_2 = Z \cup Z' \subset L \subset P$ and Z, Z' are disjoint ($Z = Z'$ case is similar):

$$\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_2}(-5), \iota^! \mathcal{I}_L(-1)) = \mathrm{Ext}^1(\mathcal{I}_{Z_2}(-5), (\mathcal{O}_P(-1) \oplus \mathcal{O}_L(-1))[-1])$$

$$= \mathrm{Hom}(\mathcal{I}_{Z\cup Z'}(-4), \mathcal{O}_P) \oplus \mathrm{Hom}(\mathcal{I}_{Z\cup Z'}(-4), \mathcal{O}_L) = \mathbb{C}^{15} \oplus \mathbb{C}^7 = \mathbb{C}^{22}.$$

□

Chapter 4

Novel Wall-Crossing Phenomenon: Wall-Crossing/MMP correspondence Fails

4.1 Introduction

In this Chapter, we prove one of the main Theorems of this thesis (Theorem 1.0.1).

We show that the wall-crossing with respect to Bridgeland stability conditions fails to completely explain and be recovered from the birational geometry of stable sheaves. We give an example of a wall-crossing in $\mathrm{D}^b(\mathbb{P}^3)$ that behaves in a manner that is surprising from the birational geometry: the wall induces a small contraction of the moduli space of stable objects associated to one of the adjacent chambers, but for the other adjacent chamber has a divisorial contraction. This profoundly complicates the overall picture, as this correspondence is substantial to applications of stability conditions in birational/algebraic geometry.

There are many examples of moduli spaces of sheaves on surfaces whose entire MMP can explain and be explained by wall-crossing: each wall-crossing induces a birational map (as a MMP step in the movable cone), and every birational model associated to a movable divisor D on the moduli space appears as a moduli space $\mathcal{M}_\sigma(v)$ of σ-stable objects for some fixed vector v. If we replace surfaces by $\mathbb{P}^3$, it turns out that this picture breaks down; indeed, our main result gives an example of a wall-crossing from a smooth moduli space to a space whose main component is not even $\mathbb{Q}$-factorial.

Strategy of the proof. To prove Theorem 1.0.1 and make a birational description of our wall, we need to describe the exceptional locus of the birational map associated with the wall (Theorem 4.5.18). We observe that the morphism $\widetilde{\psi}\colon \mathcal{M}_{\sigma_+}(v) \to \mathcal{M}_{\sigma_0}(v)$ identifying S-equivalent objects with respect to stability conditions on the wall, contracts the new component $\mathcal{M}'$ as a $\mathbb{P}^{17}$-bundle. Therefore, in order to understand $\psi = \widetilde{\psi}|_{\widetilde{\mathcal{M}_{\sigma_-}(v)}}$, we first need to understand the intersection of each $\mathbb{P}^{17}$ with $\widetilde{\mathcal{M}_{\sigma_-}(v)}$. There are a couple of steps to reach this goal. First, after some heavy Ext computations, we determine the singular locus of $\mathcal{M}_{\sigma_+}(v)$ along $\mathcal{M}'$ (which will eventually be the intersection of the two components). Secondly, we will explicitly construct enough degenerations of objects in $\mathcal{M}' \cap \widetilde{\mathcal{M}_{\sigma_-}(v)}$ to objects in $\mathcal{M}'$ to recover the 14-dimensional cone. The moduli space $\mathcal{M}'$ contains stable pairs whose underlying curve is the union of a plane quintic with a line intersecting this quintic, along with two marked points on the quintic. We show that such stable pairs arise as the degeneration of the ideal sheaf of (2,3)-complete intersection curves if and only if the quintic has two nodes that are colinear with the intersection point with the line, and if the two marked points are the nodes. In this case, the plane quintic arises as the projection of a (2,3)-complete intersection curve in $\mathbb{P}^3$ from the intersection point with the line. The next step is to degenerate the union to a plane quartic union a thick line passing through those nodes, via Lemma 4.5.15.

After all, we are able to analyse our wall-crossing in Section 4.6 and prove our main Theorem (see Theorem 4.6.3). We explain the birational situation and the relation between wall-crossing in the stability manifold and in the movable cone of some intermediate spaces birational to our component.

Notations. Recall that in this Chapter, σ_0, σ_- and σ_+ denote the stability conditions on the wall $\mathcal{W}$, in the chambers $\mathcal{N}_2$ and $\mathcal{N}_3$, respectively.

4.2 Bayer-Macrì linearization map

Let S be a K3 surface, and v a primitive algebraic class in the Mukai lattice with self-intersection with respect to the Mukai pairing. Let $\mathrm{Stab}(S)$ be the space of stability conditions on S. In [9], Bridgeland described a connected component $\mathrm{Stab}^\dagger(S)$ of $\mathrm{Stab}(S)$ which admits a chamber decomposition. Also as before, when there is no confusion, A and B will be used for the subobject and the quotient of the defining short exact sequence of any wall.

Theorem 4.2.1 ([3, Theorem 1.1]). *Let σ, η be generic stability conditions with respect to v. Then the two moduli spaces $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_\eta(v)$ of Bridgeland-stable objects are birational to each other.*

As a consequence, we can canonically identify the Néron-Severi groups of $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_\eta(v)$. Now consider the chamber decomposition of $\mathrm{Stab}(S)^\dagger$ with respect to v as above, and let C be a chamber. The main result of [4] gives a natural map

$$l_C \colon C \to \mathrm{NS}(\mathcal{M}_C(v)),$$

to the Néron-Severi group of the moduli space, whose image is contained in the ample cone of $\mathcal{M}_C(v)$. Their main result describing the global behavior of this map is the following:

Theorem 4.2.2 ([3, Theorem 1.2]). *Fix a base point $\sigma \in \mathrm{Stab}(S)^\dagger$.*

(a) Under the identification of the Néron-Severi groups, the maps l_C glue to a piece-wise analytic continuous map

$$l\colon \mathrm{Stab}(S)^\dagger \to \mathrm{NS}(\mathcal{M}_\sigma(v)).$$

(b) The map l is compatible, in the sense that for any generic $\sigma' \in \mathrm{Stab}(S)^\dagger$, the moduli space $\mathcal{M}_{\sigma'}(v)$ is the birational model corresponding to $l(\sigma')$. In particular, every smooth K-trivial birational model of $\mathcal{M}_\sigma(v)$ appears as a moduli space $\mathcal{M}_C(v)$ of Bridgeland stable objects for some chamber $C \subset \mathrm{Stab}(S)^\dagger$.

Claim (b) is the precise version of their claim that MMP can be run via wall-crossing: any minimal model can be reached after wall-crossing as a moduli space of stable objects. Extremal contractions arising as canonical models are given as coarse moduli spaces for stability conditions on a wall.

4.3 Previous work

4.3.1 Bridgeland Wall-crossing versus Mori wall-crossing

Consider the following principles:

(1) Every wall-crossing is birational (in the sense that it is a step in the Minimal Model Program), or inducing a Mori fibration.

(2) These birational transformations/Mori fibrations are induced by a continuous map from $\mathrm{Stab}(S)$ to the movable cone of $\mathcal{N}_2$. Its image includes every chamber of the movable cone, and thus every birational model isomorphic in codim 2 appears as a moduli space.

These principles hold for a number of cases: Moduli of sheaves on K3 surfaces ([3]), which is briefly explained in Section 4.2. For the Hilbert scheme of points on $\mathbb{P}^2$, some results can be found in [2, 17]. A similar argument for Enriques surface can be found in [21] and [7], and for abelian surfaces in [31]. The relation between Bridgeland stability and MMP for general smooth projective surfaces is discussed in [28] and [27]. A relation between the geometry of a variety and Bridgeland stability conditions on derived category of coherent sheaves of the variety for surfaces with rational curves of negative self-intersection is investigated in [29].

4.4 The first wall-crossing

In this section, we describe the moduli spaces for the first two chambers, and analyse the corresponding wall-crossing. Before this we need the following Lemma which gives a condition under which we can realize what components can survive all the way to the large volume limit:

Lemma 4.4.1. Suppose that $\mathcal{H}^0$ of a general object in an irreducible component created by a wall is an ideal sheaf. Then this irreducible component survives all the way up to the moduli space of stable pairs.

Proof. As stability is an open condition, we only need to show this for one object which is created after each wall. If an object E destabilized by a subobject E', there is an injection $E' \hookrightarrow E$ which induces an injection $\mathcal{H}^0(E') \hookrightarrow \mathcal{H}^0(E)$. By assumption, $\mathcal{H}^0$ of a general object created by the wall is an ideal sheaf $\mathcal{I}$. On the other hand, $\mathcal{H}^0$ of the destabilizing subobjects are all of the form $\mathcal{O}_P(-i)$, by Theorem 3.1.13. But we have $\mathrm{Hom}(\mathcal{O}_P(-i), \mathcal{I}) = 0$, therefore the induced map on $\mathcal{H}^0$ is not injective, which is a contradiction. This implies the result. □

We also need the following Lemma:

Lemma 4.4.2. Suppose that E is a sheaf with $c_0(E) = 1$ and $c_1(E) = 0$, such that it fits into $\mathcal{I}_D(-1) \hookrightarrow E \twoheadrightarrow \mathcal{O}_P(-k)$, for $\mathcal{I}_D$ an ideal sheaf of a subscheme D transverse to the plane P, and k a positive integer. Then E is an ideal sheaf of a curve.

Proof. First, we observe that taking ι_P^* of $\mathcal{I}_D \hookrightarrow \mathcal{O} \twoheadrightarrow \mathcal{O}_D$ implies $\iota_P^*(\mathcal{I}_D) = \mathcal{I}_{D\cap P/P}$. In order to show E is an ideal sheaf of a curve, we need to show that it is torsion-free. We know that any subsheaf of $\mathcal{O}_P(-k)$ contains a subsheaf of the form $\mathcal{O}_P(-i)$ for some $k < i$. It is enough to show that such $\mathcal{O}_P(-i)$ does not lift to a subsheaf of E. However, using the identification $\mathrm{Ext}^1(\mathcal{O}_P(-i), \mathcal{I}_D(-1)) = \mathrm{Ext}^1(\mathcal{O}_P(-i), \iota_P^*\mathcal{I}_D(-1)(1)[-1]) = \mathrm{H}^0(\mathcal{I}_{D\cap P/P}(i))$, one can see that $\mathrm{Ext}^1(\mathcal{O}_P(-k), \mathcal{I}_D(-1)) \hookrightarrow \mathrm{Ext}^1(\mathcal{O}_P(-i), \mathcal{I}_D(-1))$ is injective. Hence $\mathcal{O}_P(-i)$ does not lift to a subsheaf of E.

□

Proposition 4.4.3. The moduli space for the first chamber below the first wall, which is given by all the extensions of two objects $\mathcal{O}(-2), \mathcal{O}_Q(-3)$, is empty.

Proof. These two objects define the semicircle $(\beta + 4)^2 + \alpha^2 = 4$ (Proposition 3.1.3) which is above the semicircle $(\beta + 3.75)^2 + \alpha^2 = 2.06$ given by the BMT inequality (Theorem 2.4.2). We know that the moduli space is empty below the BMT semicircle. On the other hand, the moduli space can become non-empty only as we cross a wall as below the BMT semicircle there is no stable object. Combining these two, the claim is concluded. □

Let $\mathcal{N}_1$ be the first non-empty moduli space for the next chamber, which appears after crossing the smallest wall $\langle \mathcal{O}(-2), \mathcal{O}_Q(-3)\rangle$. In the following Proposition we will see that this moduli space gives a compactification of objects corresponding to $(2,3)$-complete intersection curves in $\mathbb{P}^3$:

Proposition 4.4.4. The moduli space $\mathcal{N}_1$ is a $\mathbb{P}^{15}$-bundle over $\mathbb{P}^9$. More precisely, the complement of (2,3)-complete intersections in $\mathcal{N}_1$ are parametrized by the pairs (Q, C) where $Q = P \cup P'$ is a union of two planes and C is a conic in one of the two planes. The associated objects are non-torsion free sheaves E given by $\mathcal{O}_P(-4) \hookrightarrow E \twoheadrightarrow \mathcal{I}_{C_2}(-1)$, for a conic C_2.

Proof. We use Lemma 3.2.2. Any object E in $\mathcal{N}_1$ is generated by a short exact sequences $\mathcal{O}(-2) \hookrightarrow E \twoheadrightarrow \mathcal{O}_Q(-3)$. The parameter space of $\mathcal{O}(-2)$ is just given by a point, whereas objects of the form $\mathcal{O}_Q(-3)$ are parametrized by $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(2))) = \mathbb{P}(\mathbb{C}^{10}) = \mathbb{P}^9$. Given Q, the extensions parametrized by $\mathrm{H}^0(Q, \mathcal{O}(3)) = \mathbb{C}^{16}$ up to rescaling, i.e. $\mathbb{P}^{15}$. Therefore $\mathcal{N}_1$ will be a $\mathbb{P}^{15}$-bundle over $\{pt\} \times \mathbb{P}^9 \cong \mathbb{P}^9$. Description of generic element as an ideal sheaf is easily obtained by noticing that $\mathcal{H}^1(\mathcal{O}(-2)) = 0$, and then taking long exact sequence of the defining sequence and applying Lemma 4.4.2 to $D = P'$ for another plane P'. Notice that a cubic surface, S, and a quadric, Q, define a complete intersection curve if and only if S does not contain any irreducible component of Q. Thus being (2,3)-complete intersection is an open condition. Let us denote by CI the set of (2,3)-complete intersections in $\mathcal{N}_1$. We notice that $\mathcal{N}_1$ compactifies CI with the pairs of a quadric Q, plus a cubic equation that does not vanish entirely on Q; i.e., it can be zero on one of the two components of Q, when Q is reducible. Therefore in the complement $\mathcal{N}_1 \setminus CI$ we have (S, Q) such that $Q = P \cup P'$ is a union of two planes. This gives us the sequence $\mathcal{O}_P(-1) \hookrightarrow \mathcal{O}_Q \twoheadrightarrow \mathcal{O}_{P'}$. Now if C_2 is a degree 2 curve in P', we have $\mathcal{O}(-1) = \mathcal{I}_{P'} \hookrightarrow \mathcal{I}_{C_2} \twoheadrightarrow \mathcal{O}_{P'}(-2)$. Combining these two sequences with the defining sequence of E, we get the lift of $\mathcal{O}_P(-4) \hookrightarrow \mathcal{O}_Q(-3)$ to the subobject of the claimed extension as follows:

$$\begin{array}{ccccc}
0 & \hookrightarrow & \mathcal{O}_P(-4) & \xrightarrow{\cong} & \mathcal{O}_P(-4) \\
\downarrow & & \vdots & & \downarrow \\
\mathcal{O}(-2) & \longrightarrow & E & \longrightarrow & \mathcal{O}_Q(-3) \\
\| \cong & & \vdots & & \downarrow \\
\mathcal{O}(-2) & \hookrightarrow & \mathcal{I}_{C_2}(-1) & \twoheadrightarrow & \mathcal{O}_{P'}(-3).
\end{array}$$

As $\mathrm{Ext}^1(\mathcal{I}_{C_2}(-1), \mathcal{O}_P(-4)) = \mathbb{C}$, we will have the result.

□

Before describing the next chambers, we need the following Lemmas:

Lemma 4.4.5 ([12, Lemma 4.4])**.** Let $F \hookrightarrow E \twoheadrightarrow G$ be an exact sequence at a wall in Bridgeland stability with E semistable to one side of the wall and F, G distinct stable objects of the same (Bridgeland) slop. Then we have:

$$\mathrm{ext}^1(E, E) \leq \mathrm{ext}^1(F, F) + \mathrm{ext}^1(G, G) + \mathrm{ext}^1(F, G) + \mathrm{ext}^1(G, F) - 1.$$

Lemma 4.4.6 ([20], [12, Theorem 4.7])**.** Any birational morphism $f\colon X \to Y$ between smooth proper algebraic spaces of finite type over complex numbers s.t. the contracted locus E is irreducible, and $f(E)$ is smooth, is the blow up of Y in $f(E)$.

Now, we want to describe the moduli space $\mathcal{N}_2$ for the next chamber, which comes after the wall $\langle \mathcal{I}_{C_2}(-1), \mathcal{O}_P(-4) \rangle$. Since $\mathrm{Ext}^1(\mathcal{I}_{C_2}(-1), \mathcal{O}_P(-4)) = \mathbb{C}$ for all $\mathcal{I}_{C_2} \in \mathcal{H}ilb^{2t+1}(\mathbb{P}^3)$ and all hyperplanes $P \in (\mathbb{P}^3)^\vee$, there is a unique object in $\mathcal{N}_1$ destabilized at the wall $\langle \mathcal{I}_{C_2}(-1), \mathcal{O}_P(-4) \rangle$, identifying the destabilized locus with $\mathcal{H}ilb^{2t+1}(\mathbb{P}^3) \times (\mathbb{P}^3)^\vee$.

Proposition 4.4.7. The moduli space $\mathcal{N}_2$ for the next chamber is a blow up of $\mathcal{N}_1$ in the locus $\mathcal{H}ilb^{2t+1}(\mathbb{P}^3) \times (\mathbb{P}^3)^\vee$. The generic element in the exceptional locus is given by an ideal sheaf $\mathcal{I}_{C_2 \cup C_4}$, for C_4 a plane quartic, and C_2 a conic not in the plane.

Proof. Lemma 3.2.3 implies $\mathrm{Ext}^1(\mathcal{I}_{C_2}(-1), \mathcal{O}_P(-4)) = \mathbb{C}$, and $\mathrm{Ext}^1(\mathcal{O}_P(-4), \mathcal{I}_{C_2}(-1)) = \mathbb{C}^{13}$. We know that the parameter space for $\mathcal{I}_{C_2}(-1)$ is $\mathcal{H}ilb^{2t+1}(\mathbb{P}^3)$, and the parameter space for $\mathcal{O}_P(-4)$, is $Gr(3,4) \cong (\mathbb{P}^3)^\vee$, and we have $dim(\mathcal{H}ilb^{2t+1}(\mathbb{P}^3)) = 8$, and $dim((\mathbb{P}^3)^\vee) = 3$. Therefore the locus of extensions in $\mathrm{Ext}^1(\mathcal{I}_{C_2}(-1), \mathcal{O}_P(-4))$ is isomorphic to $\mathcal{H}ilb^{2t+1}(\mathbb{P}^3) \times (\mathbb{P}^3)^\vee$. Lemma 4.4.5 implies $\mathrm{ext}^1(E,E) \leq 8+3+1+13-1 = 24$ for each E given by a class in $\mathrm{Ext}^1(\mathcal{O}_P(-4), \mathcal{I}_{C_2}(-1))$. This implies that $\mathcal{N}_2$ is smooth. The dimension of the locus of extensions in $\mathrm{Ext}^1(\mathcal{O}_P(-4), \mathcal{I}_{C_2}(-1))$ (or the exceptional locus) is $12 + (8+3) = 23$, so it is a divisor in $\mathcal{N}_2$. Using Lemma 4.4.6 induces the result. Description of generic element as an ideal sheaf is easily obtained by noticing that $\mathcal{H}^1(\mathcal{I}_{C_2}(-1)) = 0$, and then taking long exact sequence of the defining sequence and applying Lemma 4.4.2 to $D = C_2$ in the general case. □

The description of the next wall-crossing is the main content of this Chapter.

4.5 Exceptional locus in $\mathcal{N}_3$

In this section, we will describe the moduli space $\mathcal{N}_3$ for the next chamber as $\mathcal{N}_3 = \widetilde{\mathcal{N}_2} \cup \mathcal{N}'_3$, where $\widetilde{\mathcal{N}_2}$ is birational to $\mathcal{N}_2$, and $\mathcal{N}'_3$ is a new irreducible component. As the morphism $\mathcal{N}_3 \dashrightarrow \mathcal{W}$ contracts $\mathcal{N}'_3$ (projecting the $\mathbb{P}^{17}$-bundle to its base), to understand its restriction $\widetilde{\mathcal{N}_2} \to \mathcal{N}_2$, we first need to understand the intersection $\mathcal{N}_2 \cap \mathcal{N}'_3$; this is the main goal of this section. This intersection is closed in $\mathcal{N}'_3$. Also, we can stratify the base of $\mathcal{N}'_3$ via $\mathrm{Ext}^1(A,B)$ (Lemma 3.2.5). We know that the intersection lies over the strata where $\mathrm{ext}^1(A,B) \geq 1$.

We need to prove the surjectivity of the map $\delta \colon \mathrm{Ext}^1(B,A) \to \mathrm{Hom}(\mathrm{Ext}^1(A,B), \mathrm{Ext}^2(B,B))$ (Lemmas 4.5.4 and 4.5.5). We show that $\ker(\delta)$ gives the precise description of the singularity locus of $\mathcal{N}_3$. Yo recover the 14-dimensional cone, we construct enough degenerations of objects in the intersection to objects in $\mathcal{N}'_3$, by projecting the curves in the intersection to the plane quintics with two nodes union a line, and then degenerate the later to plane quartics union lines meeting those nodes(Lemma 4.5.15). This will imply that the singularity locus is the same as the intersection of the two components (which is the same as the exceptional locus of ψ). Having all these, Theorem 4.5.18 will be proved. First, we describe the moduli space.

Let $\mathfrak{Fl}_2$ is the space parametrizing flags $Z_2 \subset P \subset \mathbb{P}^3$ where P is a plane and Z_2 a zero-dimensional subscheme of length 2. The next wall crossing was considered in [24]:

Proposition 4.5.1. The moduli space $\mathcal{N}_3$ for the next chamber consists of two irreducible components: one is $\widetilde{\mathcal{N}_2}$ which is birational to $\mathcal{N}_2$; the other is a new component, $\mathcal{N}'_3$ which is a $\mathbb{P}^{17}$-bundle over $\mathbb{G}r(2,4) \times \mathfrak{Fl}_2$. The later generically parametrizes union of a line and a plane quintic together with choice of two points on the quintic. For any general element E in the new component, $\mathcal{H}^0(E)$ is of the form of an ideal sheaf, and $\mathcal{H}^1(E) \neq 0$.

Proof. Any object in $\mathcal{N}'_3$ fits into a short exact sequence $\mathcal{I}_L(-1) \to E \to \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)$. Now we just notice that from Lemma 3.2.5, we have $\mathrm{Ext}^1(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5), \mathcal{I}_L(-1)) = \mathbb{C}^{18}$.

This gives the description of $\mathcal{N}_3'$ as $\mathbb{P}^{17}$-bundle. As the set of objects in $\mathcal{N}_2$ that are also σ_+-stable is open, its closure $\widetilde{\mathcal{N}_2}$ is birational to $\mathcal{N}_2$ (we notice that by Proposition 4.4.7 and Lemma 4.4.1 or Lemma 4.5.2 below, the whole $\mathcal{N}_2$ is not destabilized by crossing the wall). As $\dim \mathcal{N}_3' = 28 > 24 = \dim \widetilde{\mathcal{N}_2}$, the locus $\mathcal{N}_3'$ is its own irreducible component. Description of generic element as an ideal sheaf is easily obtained by noticing that $\mathcal{H}^1(\mathcal{I}_L(-1)) = 0$, and then taking long exact sequence of the defining sequence of any class E in $\mathrm{Ext}^1(B, A)$, and applying Lemma 4.4.2 to $D = L$ in the general case. The claim $\mathcal{H}^1$ being non-zero is obtained in a similar way and noticing that $\mathcal{H}^1(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)) \neq 0$. □

4.5.1 Intersection in $\mathcal{N}_3$

To describe the intersection of $\widetilde{\mathcal{N}_2}$ with the new component $\mathcal{N}_3'$ in $\mathcal{N}_3$, let $\mathcal{U}_{-,+}$ be the destabilizing locus in $\mathcal{N}_2$. More precisely, we have

$$\mathcal{U}_{-,+} = \{E \colon \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5) \to E \to \mathcal{I}_L(-1)\}, \quad \mathcal{N}_3' = \{E \colon \mathcal{I}_L(-1) \to E \to \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)\}.$$

Lemma 4.5.2. The locus $\mathcal{U}_{-,+}$ is 10-dimensional; it contains the exceptional locus of $\phi \colon \mathcal{N}_2 \to \mathcal{W}$ of dimension 8 which is a $\mathbb{P}^1$-bundle over its 7-dimensional image under ϕ.

Proof. From Lemma 3.2.5, we have

$$\mathrm{Ext}^1(\mathcal{I}_L(-1), \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)) = \begin{cases} 0, & \langle Z_2 \rangle \cap L = \emptyset \\ \mathbb{C}, & \langle Z_2 \rangle \cap L \neq \emptyset \text{ but } \langle Z_2 \rangle \neq L \\ \mathbb{C}^2, & \langle Z_2 \rangle = L \end{cases}$$

Considering the case where $\mathrm{Ext}^1(A, B) \neq 0$, the generic case $L \not\subset P$ and $p \cup Z_2$ colinear is given by a 10-dimensional stratum ($\mathbb{P}^0$-bundle over the parameter space of the configuration).

When $\mathrm{Ext}^1(A, B) = \mathbb{C}^2$, the parameter space of all the points A, B is given by the configurations $Z_2 \subset L \subset P$ which gives a 8-dimensional stratum ($\mathbb{P}^1$-bundle over the parameter space of the configuration). By the positivity Lemma, ϕ contracts exactly the $\mathbb{P}^1$ coming from $\mathrm{Ext}^1(A, B) = \mathbb{C}^2$. □

Remark 4.5.3. Recall that $A = \mathcal{I}_L(-1)$, $B = \iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)$. Let $E \in \mathcal{N}_3'$ fitting into a short exact sequence $A \hookrightarrow E \twoheadrightarrow B$. Deformations of E within the exceptional locus corresponds to maps $E \to E[1]$ fitting into a map of exact triangles

$$\begin{array}{ccccc} A & \xrightarrow{a} & E & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ A[1] & \longrightarrow & E[1] & \xrightarrow{b} & B[1] \end{array}$$

This is equivalent to $b \circ f \circ a = 0$. In particular, if there are deformations of E that do not remain in the exceptional locus, then $\mathrm{Ext}^1(A, B) \neq 0$.

Surjectivity. The goal of this subsection is to show that

$$\delta\colon \mathrm{Ext}^1(B,A) \to \mathrm{Hom}(\mathrm{Ext}^1(A,B), \mathrm{Ext}^2(B,B))$$

(which is induced by the natural map $\delta'\colon \mathrm{Ext}^1(B,A) \otimes \mathrm{Ext}^1(A,B) \to \mathrm{Ext}^2(B,B)$) is surjective.

Before proving the surjectivity, we begin with the following Lemma:

Lemma 4.5.4. If δ is surjective when $\mathrm{Ext}^1(A,B) = \mathbb{C}^2$, then it would be surjective when $\mathrm{Ext}^1(A,B) = \mathbb{C}$ as well.

Proof. As the domain $\mathrm{Ext}^1(B,A) = \mathbb{C}^{18}$ is constant for all the configurations, surjectivity of δ is an open condition. Since all $PGL(3)$-orbits of configurations of $Z_2 \subset P$ and L contain the case $Z_2 \subset L \subset P$ in its closure, for which $\mathrm{Ext}^1(A,B) = \mathbb{C}^2$, this proves the claim. □

Hence we only need to prove the surjectivity when $\mathrm{Ext}^1(A,B) = \mathbb{C}^2$:

Lemma 4.5.5. Assume that $\mathrm{Ext}^1(A,B) = \mathbb{C}^2$. Then δ is surjective.

Proof. First, we simplify the map

$$\delta'\colon \mathrm{Ext}^1(\iota_{P_*}\mathcal{I}^\vee_{Z_2}(-5), \mathcal{I}_L(-1)) \otimes \mathrm{Ext}^1(\mathcal{I}_L(-1), \iota_{P_*}\mathcal{I}^\vee_{Z_2}(-5)) \to \mathrm{Ext}^2(\iota_{P_*}\mathcal{I}^\vee_{Z_2}(-5), \iota_{P_*}\mathcal{I}^\vee_{Z_2}(-5)).$$

***Claim* 1** : δ' is isomorphic to

$$\mathrm{Hom}(\mathcal{O}_P(2) \oplus \mathcal{O}_L(3)[-1], \mathcal{I}_{Z_2}(6)) \otimes \mathrm{Ext}^1(\mathcal{I}_{Z_2}(5), \mathcal{O}_P(2) \oplus \mathcal{O}_L(3)[-1]) \to$$
$$\to \mathrm{Ext}^1(\mathcal{I}_{Z_2}(5), \mathcal{I}_{Z_2}(6)).$$

Proof of Claim 1: Applying ι_P^* to δ' gives the following map:

$$\iota_P^*\delta'\colon \mathrm{Ext}^1(\iota_P^*\iota_{P_*}\mathcal{I}^\vee_{Z_2}(-5), \iota_P^*\mathcal{I}_L(-1)) \otimes \mathrm{Ext}^1(\iota_P^*\mathcal{I}_L(-1), \iota_P^*\iota_{P_*}\mathcal{I}^\vee_{Z_2}(-5)) \to$$
$$\to \mathrm{Ext}^2(\iota_P^*\iota_{P_*}\mathcal{I}^\vee_{Z_2}(-5), \iota_P^*\iota_{P_*}\mathcal{I}^\vee_{Z_2}(-5)).$$

We note that $\iota_P^*\mathcal{I}_L(-1) = \mathcal{I}_{L/P}(-1) \oplus \mathcal{O}_L(-2)$. Using this and Lemma 3.1.7 implies that the map $\iota_P^*\delta'$ projects to:

$$\mathrm{Hom}((\mathcal{I}^\vee_{Z_2}(-6), \mathcal{O}_P(-2) \oplus \mathcal{O}_L(-2)) \otimes \mathrm{Ext}^1(\mathcal{O}_P(-2) \oplus \mathcal{O}_L(-2), \mathcal{I}^\vee_{Z_2}(-5)) \to$$
$$\to \mathrm{Ext}^1(\mathcal{I}^\vee_{Z_2}(-6), \mathcal{I}^\vee_{Z_2}(-5)),$$

which can be written as (noting that $\mathcal{O}_L^\vee = (\mathcal{O}_P(-1) \to \mathcal{O}_P)^\vee$ is isomorphic to $(\mathcal{O}_P \to \mathcal{O}_P(1))^\vee = \mathcal{O}_L(1)[-1]$, using the degree change):

$$\mathrm{Hom}(\mathcal{O}_P(2) \oplus \mathcal{O}_L(3)[-1], \mathcal{I}_{Z_2}(6)) \otimes \mathrm{Ext}^1(\mathcal{I}_{Z_2}(5), \mathcal{O}_P(2) \oplus \mathcal{O}_L(3)[-1]) \to$$
$$\to \mathrm{Ext}^1(\mathcal{I}_{Z_2}(5), \mathcal{I}_{Z_2}(6)),$$

and so the Claim 1 is proved.

Now, using Serre duality for $\mathrm{Ext}^1(\mathcal{I}_{Z_2}(5), \mathcal{O}_P(2) \oplus \mathcal{O}_L(3)[-1])$ we have

$$\mathrm{Ext}^1(\mathcal{I}_{Z_2}(5), \mathcal{O}_P(2)) = \mathrm{Ext}^1(\mathcal{O}_P(2), \mathcal{I}_{Z_2}(2))^\vee = \mathbb{C},$$

and

$$\mathrm{Ext}^1(\mathcal{I}_{Z_2}(5), \mathcal{O}_L(3)[-1]) = \mathrm{Hom}(\mathcal{I}_{Z_2}(5), \mathcal{O}_L(3)) = \mathbb{C}.$$

Also for $\mathrm{Hom}(\mathcal{O}_P(2) \oplus \mathcal{O}_L(3)[-1], \mathcal{I}_{Z_2}(6))$, component-wise we have (after applying $\mathrm{RHom}(-, \mathcal{I}_{Z_2}(3))$ to $\mathcal{O}_P(-1) \to \mathcal{O}_P \to \mathcal{O}_L$):

$$\mathrm{Hom}(\mathcal{O}_P(2), \mathcal{I}_{Z_2}(6)) = \mathrm{H}^0(\mathcal{I}_{Z_2}(4)) = \mathbb{C}^{13},$$

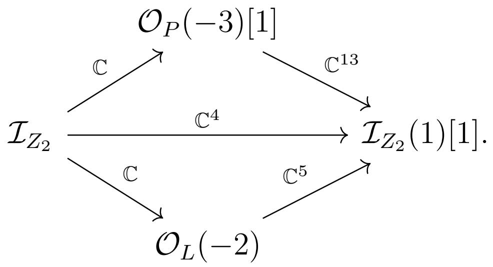
and

$$\mathrm{Hom}(\mathcal{O}_L(3)[-1], \mathcal{I}_{Z_2}(6)) = \mathrm{Ext}^1(\mathcal{O}_L, \mathcal{I}_{Z_2}(3)) = \mathbb{C}^5,$$

and as $\mathrm{Ext}^1(\mathcal{I}_{Z_2}(5), \mathcal{I}_{Z_2}(6)) = \mathbb{C}^4$, the above map is the same as the following two maps:

$$\mathbb{C}^{13} \otimes \mathbb{C} = \mathbb{C}^{13} \to \mathbb{C}^4, \quad \text{and} \quad \mathbb{C}^5 \otimes \mathbb{C} = \mathbb{C}^5 \to \mathbb{C}^4,$$

i.e. we have the following diagram



Claim 2 : $\mathbb{C} \otimes \mathbb{C}^{13} \to \mathbb{C}^4$ is surjective.

Proof of Claim 2: For $\mathbb{C} \otimes \mathbb{C}^{13} \to \mathbb{C}^4$ or $\mathrm{Ext}^1(\mathcal{I}_{Z_2}, \mathcal{O}_P(-3)) \otimes \mathrm{Hom}(\mathcal{O}_P(-3), \mathcal{I}_{Z_2}(1)) \to \mathrm{Ext}^1(\mathcal{I}_{Z_2}, \mathcal{I}_{Z_2}(1))$, we consider $E \in \mathrm{Ext}^1(\mathcal{I}_{Z_2}, \mathcal{O}_P(-3))$ and apply $\mathrm{RHom}(-, \mathcal{I}_{Z_2}(1))$ to $\mathcal{O}_P(-3) \to E \to \mathcal{I}_{Z_2}$, and so we have

$$0 \to \mathrm{Hom}(\mathcal{I}_{Z_2}, \mathcal{I}_{Z_2}(1)) \to \mathrm{Hom}(E, \mathcal{I}_{Z_2}(1)) \to \mathrm{Hom}(\mathcal{O}_P(-3), \mathcal{I}_{Z_2}(1)) \to$$
$$\to \mathrm{Ext}^1(\mathcal{I}_{Z_2}, \mathcal{I}_{Z_2}(1)) \to \mathrm{Ext}^1(E, \mathcal{I}_{Z_2}(1)) \to 0$$

For computing $\mathrm{Ext}^1(E, \mathcal{I}_{Z_2}(1))$, as $\mathrm{Ext}^1(\mathcal{O}_P(-1), \mathcal{O}_P(-3)) = 0$ from the top and the middle row of diagram below, we get $\mathcal{O}_P(-1) \hookrightarrow E$, and so we can complete the diagram:

$$\begin{array}{ccccccc}
0 & \hookrightarrow & \mathcal{O}_P(-1) & \overset{\cong}{=\!=} & \mathcal{O}_P(-1) & \overset{0}{\dashrightarrow} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}_P(-3) & \longrightarrow & E & \longrightarrow & \mathcal{I}_{Z_2} & \longrightarrow & \mathcal{O}_P(-3)[1] \\
\| \cong & & \downarrow & & \downarrow & & \\
\mathcal{O}_P(-3) & \hookrightarrow & Q & \twoheadrightarrow & \mathcal{O}_L(-2), & &
\end{array}$$

where Q is the quotient of $\mathcal{O}_P(-1) \to E$. Therefore, the bottom row implies $Q = \mathcal{O}_P(-2)$. But as $\mathrm{Ext}^1(\mathcal{O}_P(-2), \mathcal{O}_P(-1)) = 0$, we are left with the trivial extension $E = \mathcal{O}_P(-1) \oplus \mathcal{O}_P(-2)$. Therefore we have

$$\mathrm{Ext}^1(E, \mathcal{I}_{Z_2}(1)) = \mathrm{Ext}^1(\mathcal{O}_P(-1) \oplus \mathcal{O}_P(-2), \mathcal{I}_{Z_2}(1)) = \mathrm{H}^1(\mathcal{I}_{Z_2}(2)) \oplus \mathrm{H}^1(\mathcal{I}_{Z_2}(3)) = 0,$$

i.e. from the above long exact sequence we have the surjection

$$\mathrm{Hom}(\mathcal{O}_P(-3), \mathcal{I}_{Z_2}(1)) = \mathbb{C}^{13} \twoheadrightarrow \mathrm{Ext}^1(\mathcal{I}_{Z_2}, \mathcal{I}_{Z_2}(1) = \mathbb{C}^4,$$

***Claim* 3** : $\mathbb{C} \otimes \mathbb{C}^5 \to \mathbb{C}^4$ is surjective.

Proof of Claim 3: Applying $R\mathrm{Hom}(-, \mathcal{I}_{Z_2}(1))$ on $\mathcal{O}(-1) = \mathcal{I}_L \hookrightarrow \mathcal{I}_{Z_2} \twoheadrightarrow \mathcal{I}_{Z_2/L} = \mathcal{O}_L(-2)$ gives the surjection

$$\mathrm{Ext}^1(\mathcal{O}_L(-2), \mathcal{I}_{Z_2}(1)) = \mathbb{C}^5 \twoheadrightarrow \mathrm{Ext}^1(\mathcal{I}_{Z_2}, \mathcal{I}_{Z_2}(1)) = \mathbb{C}^4,$$

Therefore, δ is surjective when $\mathrm{Ext}^1(A, B) = \mathbb{C}^2$.

□

Intersection of the components. In this subsection, we will describe the intersection $\widetilde{\mathcal{N}_2} \cap \mathcal{N}_3'$. So far, we have proved that the map $\delta\colon \mathrm{Ext}^1(B, A) \to \mathrm{Hom}(\mathrm{Ext}^1(A, B), \mathrm{Ext}^2(B, B))$ is surjective. Consider the map $\zeta\colon \mathrm{Ext}^1(A, B) \to \mathrm{Ext}^2(B, B)$, which is defined by composition with a class in $\mathrm{Ext}^1(B, A)$.

Lemma 4.5.6. For any $E \in \mathcal{N}_3'$, we have

$$\mathrm{ext}^1(E, E) = 28 + dim(\ker(\zeta)).$$

Proof. We consider the long exact Hom-sequence associated to the short exact sequence $A \to E \to B$. Using Lemma 3.2.5, we get

$$
\begin{array}{ccccccccc}
 & & \operatorname{Ext}^2(A,A)=0 & \longleftarrow & 0 & \longleftarrow & \operatorname{Ext}^2(B,A)=0 & & \\
 & & & & \uparrow & & & & \\
\mathbb{C}^4 \supset \operatorname{Im}(\zeta) & \longleftarrow & \operatorname{Ext}^1(A,B) & \longleftarrow & \operatorname{Ext}^1(E,B) & \longleftarrow & \mathbb{C}^7 & \longleftarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & \operatorname{Ext}^1(E,E) & & & & \\
 & & & & \uparrow & & & & \\
0 & \longleftarrow & \mathbb{C}^4 & \longleftarrow & \operatorname{Ext}^1(E,A) & \longleftarrow & \mathbb{C}^{18} & \longleftarrow & \mathbb{C} \leftarrow 0 \\
 & & & & \uparrow & & & & \\
 & & 0 & \longleftarrow & \operatorname{Hom}(E,B)=\mathbb{C} & \longleftarrow & \mathbb{C} & \longleftarrow & 0 \\
 & & & & \uparrow & & & & \\
0 & \longleftarrow & \operatorname{Hom}(A,E)=\mathbb{C} & \leftarrow & \operatorname{Hom}(E,E)=\mathbb{C} & \longleftarrow & 0 & & \\
 & & & & \uparrow & & & & \\
 & & & & \operatorname{Hom}(E,A)=0. & & & &
\end{array}
$$

The second row gives $\operatorname{ext}^1(E,B) = 7 + \dim(\ker(\zeta))$. On the other hand, the fourth row implies $\operatorname{ext}^1(E,A) = 21$. Therefore, the column of $\operatorname{Ext}^1(E,E)$ gives $[7 + \dim(\ker(\zeta))] - \operatorname{ext}^1(E,E) + 21 = 0$, which implies the claim. □

As $\dim(\mathcal{N}_3') = 28$, the locus where $\operatorname{ext}^1(E,E) > 28$ is the singular locus in $\mathcal{N}_3'$, which we will now describe in more detail in Proposition 4.5.7.

Proposition 4.5.7. Let $\mathcal{R}$ be the locus in $\mathcal{N}_3'$ where $\mathcal{N}_3$ is singular. Let $\widetilde{\psi}\colon \mathcal{N}_3 \to \mathcal{W}$. Then the restriction $\widetilde{\psi}|_{\mathcal{R}}$ is generically a $\mathbb{P}^{13}$-bundle over a 10-dimensional base, degenerating to a bundle over a 7-dimensional base whose fibers are a 14-dimensional cone with $\mathbb{P}^9$ as vertex.

Proof. Given A and B, the singular locus is a subset of $\mathbb{P}\operatorname{Ext}^1(B,A)$-bundle, such that $\ker(\zeta)$ is non-zero (by Lemma 4.5.6). In other words, we want to find all extension classes $\vartheta \in \operatorname{Ext}^1(B,A)$ such that the induced map $\zeta\colon \operatorname{Ext}^1(A,B) \to \operatorname{Ext}^2(B,B)$ has non-trivial kernel. Having a non-trivial kernel means that ζ cannot have full rank and hence the locus is given by the projectivization of those ϑ for which $\zeta = \delta(\vartheta)$ drops rank.

Now there are two non-zero cases of $\operatorname{Ext}^1(A,B)$:

Case(I) $\operatorname{Ext}^1(A,B) = \mathbb{C}$. In this case, dropping rank means $\zeta = \delta(\vartheta) = 0$. Also as in this case we have $\delta\colon \mathbb{C}^{18} \to \mathbb{C}^4$, and by Lemmas 4.5.4 and 4.5.5, δ is surjective, we will have $\ker(\delta) = \mathbb{C}^{14}$. Therefore in this fiber we get $\mathbb{P}\ker(\delta) = \mathbb{P}^{13}$.

The base depends on the configurations of Z_2, L and P: The generic case is when $L \not\subset P$ and $p \cup Z_2$ colinear, which gives a 10-dimensional base (3+4+3 in which 3 is for the plane, 4 for two points in the plane, and 3 for the spatial line with one condition).

The base for this $\mathbb{P}^{13}$-bundle for the generic case is 10-dimensional as above, and so the objects E with ζ non-injective correspond to a 23-dimensional space.

Case(II) $\mathrm{Ext}^1(A,B)=\mathbb{C}^2$**.** In this case, dropping rank means we have to consider

$$\mathcal{V}=\mathbb{P}(\delta^{-1}(\text{rank}\leq 1 \text{ matrices in } \mathrm{Hom}(\mathrm{Ext}^1(A,B),\mathrm{Ext}^2(B,B))\cong \mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^4))).$$

Lemma 3.2.5 says that we are in the case $Z_2\subset L\subset P$, so the base would be a 7-dimensional locus (3 for the plane, and 4 for two points in the plane). Now we want to find the fiber locus. There are two non-injective possibilities for $\ker(\zeta)$:

(1) $\ker(\zeta)=\mathbb{C}$. In this case, as a cone over $\mathbb{P}(Im\delta\cap$ (rank 1 matrices)), the corresponding part of $\widetilde{\mathcal{N}_2}\cap\mathbb{P}(\mathrm{Ext}^1(B,A))$ as a fiber over the base, would be $\mathcal{V}$ which is a subset of $\mathbb{P}(\mathrm{Ext}^1(B,A))=\mathbb{P}^{17}$. Consider the map $\mathbb{C}^2\to\mathbb{C}^4$ and note that $\ker(\zeta)=\mathbb{C}$ is equivalent to choose 2 linearly dependent vectors in $\mathbb{C}^4$, which is a condition of codimension 3. Therefore our desired preimage would be of codimension 3 as well. Therefore, the intersection $\widetilde{\mathcal{N}_2}\cap\mathbb{P}(\mathrm{Ext}^1(B,A))$ would be a (projective) 14-dimensional subset of $\mathbb{P}^{17}$. We can see that as the dimension of the fiber increases in this case, the corresponding fiber would be the degeneration of the $\mathbb{P}^{13}$.

(2) $\ker(\zeta)=\mathbb{C}^2$. In this case, the codomain of δ is the space of 4 by 2 matrices, so we have $\dim(\ker(\delta))=10$ as δ is surjective by Lemma 4.5.5, which means that $\mathbb{P}(\ker\delta)$ is 9-dimensional as a subset of $\widetilde{\mathcal{N}_2}\cap\mathbb{P}(\mathrm{Ext}^1(B,A))$. As ζ is trivial in this case, this locus in the preimage of δ is the singularity locus (or the vertex) of the 14-dimensional cone.

Therefore as the degeneration of the $\mathbb{P}^{13}$-bundle, we have a 14-dimensional fiber cone over a 7 dimensional base (with vertex a $\mathbb{P}^9$-bundle over the 7-dimensional base).

□

Because $\mathcal{N}_3'$ is smooth of dimension 28, the intersection $\widetilde{\mathcal{N}_2}\cap\mathcal{N}_3'$ is contained in the singular locus $\mathcal{R}$. We now want to prove the converse $\mathcal{R}\subset\widetilde{\mathcal{N}_2}\cap\mathcal{N}_3'$. We first consider the following subsets of $\widetilde{\psi}(\mathcal{R})$:

- Let $U_1\subset\mathbb{G}r(2,4)\times\mathfrak{Fl}_2$ be the locus where $L\not\subset P$ and $p=L\cap P$ is colinear with Z_2, but p is disjoint from Z_2, and Z_2 consists of two distinct points.
- Let $U_2\subset\mathbb{G}r(2,4)\times\mathfrak{Fl}_2$ be the locus where Z_2 consists of two distinct points, and L is the line spanned by Z_2.

We observe that U_1 and U_2 are open and dense in the loci of $\mathbb{G}r(2,4)\times\mathfrak{Fl}_2$ where $\mathrm{Ext}^1(A,B)=\mathbb{C}$ and $\mathrm{Ext}^1(A,B)=\mathbb{C}^2$, respectively. Hence $\mathcal{R}_1:=(\widetilde{\psi}|_{\mathcal{R}})^{-1}(U_1)$ and $\mathcal{R}_2:=(\widetilde{\psi}|_{\mathcal{R}})^{-1}(U_2)$ are dense in the loci where $\widetilde{\psi}|_{\mathcal{R}}$ is a $\mathbb{P}^{13}$-bundle or a 14-dimensional cones bundle, respectively; as the intersection $\widetilde{\mathcal{N}_2}\cap\mathcal{N}_3'$ is closed, it is therefore enough to show that $\mathcal{R}_1$ and $\mathcal{R}_2$ are contained in $\widetilde{\mathcal{N}_2}\cap\mathcal{N}_3'$.

Lemma 4.5.8. Let $E \in \mathcal{R}_1$ (i.e., E is the general element of the $\mathbb{P}^{13}$-bundle). We have

- $\mathcal{H}^0(E) = \mathcal{I}_{C_5 \cup L}$, where $C_5 \subset P$ is a plane quintic containing $L \cap P$ and having a node at each point of Z_2, and
- $\mathcal{H}^1(E) = \mathcal{O}_{Z_2}^{\vee}$.

Proof. We want to describe the $\mathbb{P}^{13}$-bundle in Proposition 4.5.7 more precisely. Imitating the beginning of the proof of Lemma 4.5.5 for the generic case $\langle Z_2 \rangle \cap L \neq \emptyset$ and $\langle Z_2 \rangle \neq L$, and applying ι_P^* to δ' gives the following map:

$$\iota_P^*\delta' \colon \mathrm{Ext}^1(\iota_P^*\iota_{P_*}(\mathcal{I}_{Z_2})^{\vee}(-5), \iota_P^*\mathcal{I}_L(-1)) \otimes \mathrm{Ext}^1(\iota_P^*\mathcal{I}_L(-1), \iota_P^*\iota_{P_*}(\mathcal{I}_{Z_2})^{\vee}(-5)) \to$$
$$\to \mathrm{Ext}^2(\iota_P^*\iota_{P_*}(\mathcal{I}_{Z_2})^{\vee}(-5), \iota_P^*\iota_{P_*}(\mathcal{I}_{Z_2})^{\vee}(-5)).$$

We notice that $\iota_P^*\mathcal{I}_L(-1) = \mathcal{I}_p(-1)$ in the generic case, and so using Lemma 3.1.7 implies that the map $\iota_P^*\delta'$ projects to

$$\mathrm{Hom}((\mathcal{I}_{Z_2})^{\vee}(-6), \mathcal{I}_p(-1)) \otimes \mathrm{Ext}^1(\mathcal{I}_p(-1), (\mathcal{I}_{Z_2})^{\vee}(-5)) \to \mathrm{Ext}^1((\mathcal{I}_{Z_2})^{\vee}(-6), (\mathcal{I}_{Z_2})^{\vee}(-5)),$$

which can be written as :

$$\mathrm{Hom}((\mathcal{I}_p)^{\vee}(1), \mathcal{I}_{Z_2}(6)) \otimes \mathrm{Ext}^1(\mathcal{I}_{Z_2}(5), (\mathcal{I}_p)^{\vee}(1)) \to \mathrm{Ext}^1(\mathcal{I}_{Z_2}(5), \mathcal{I}_{Z_2}(6)),$$

applying Serre Duality we will have

$$\mathrm{Hom}((\mathcal{I}_p)^{\vee}(1), \mathcal{I}_{Z_2}(6)) \otimes \mathrm{Ext}^1((\mathcal{I}_p)^{\vee}(1), \mathcal{I}_{Z_2}(2))^{\vee} \to \mathrm{Ext}^1(\mathcal{I}_{Z_2}(6), \mathcal{I}_{Z_2}(2))^{\vee},$$

rearranging this will give

$$\mathrm{Hom}((\mathcal{I}_p)^{\vee}(1), \mathcal{I}_{Z_2}(6)) \otimes \mathrm{Ext}^1(\mathcal{I}_{Z_2}(6), \mathcal{I}_{Z_2}(2)) \to \mathrm{Ext}^1((\mathcal{I}_p)^{\vee}(1), \mathcal{I}_{Z_2}(2)).$$

Applying $\mathrm{Hom}(\mathcal{I}_{Z_2}(6), -)$ and $\mathrm{Hom}((\mathcal{I}_p)^{\vee}(1), -)$ to $\mathcal{I}_{Z_2}(2) \hookrightarrow \mathcal{O}(2) \twoheadrightarrow \mathcal{O}_{Z_2}$ implies the map can be written as

$$\mathrm{Hom}((\mathcal{I}_p)^{\vee}(1), \mathcal{I}_{Z_2}(6)) \otimes \mathrm{Hom}(\mathcal{I}_{Z_2}(6), \mathcal{O}_{Z_2}) \to \mathrm{Hom}((\mathcal{I}_p)^{\vee}(1), \mathcal{O}_{Z_2})/\mathrm{Im}(\mathrm{Hom}(\mathcal{O}(2), \mathcal{O}_{Z_2})),$$

but as $\mathrm{Hom}(\mathcal{I}_{Z_2}(6), \mathcal{O}_{Z_2}) = \mathrm{H}^0(\mathcal{I}_{Z_2}/\mathcal{I}_{Z_2}^2(6))^{\vee}$, the map $\eta \colon \mathrm{Ext}^1(B, A) \to \mathrm{Ext}^2(B, B)$ (which is induced by the above map) can be written as

$$\mathrm{H}^0(\mathcal{I}_{Z_2} \otimes \mathcal{I}_p(5)) \to \mathrm{H}^0(\mathcal{I}_{Z_2}/\mathcal{I}_{Z_2}^2(6)),$$

therefore $\ker(\eta) = \mathrm{H}^0(\mathcal{I}_{Z_2}^2 \otimes \mathcal{I}_p(5)) = \mathbb{C}^{14}$ (notice that we have $\mathrm{H}^0(\mathcal{O}(5)) = \mathbb{C}^{21}$, $\mathrm{H}^0(\mathcal{O}_{Z_2}^2) = \mathbb{C}^6$, and $\mathrm{H}^0(\mathcal{O}_p) = \mathbb{C}$). This exactly gives the $\mathbb{P}^{13}$-bundle above.

The cohomology long exact sequence of the defining short exact sequence of E is

$$0 \longrightarrow \mathcal{H}^0(\mathcal{I}_L(-1)) = \mathcal{I}_L(-1) \longrightarrow \mathcal{H}^0(E) \longrightarrow \mathcal{H}^0(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)) = \mathcal{O}_P(-5)$$
$$\longrightarrow \mathcal{H}^1(\mathcal{I}_L(-1)) = 0 \longrightarrow \mathcal{H}^1(E) \longrightarrow \mathcal{H}^1(\iota_{P_*}(\mathcal{I}_{Z_2})^\vee(-5)) = \mathcal{O}_{Z_2}^\vee$$
$$\longrightarrow \mathcal{H}^2(\mathcal{I}_L(-1)) = 0.$$

The second row implies $\mathcal{H}^1(E) = \mathcal{O}_{Z_2}^\vee$. The first row $\mathcal{I}_L(-1) \to \mathcal{H}^0(E) \to \mathcal{O}_P(-5)$ implies $\mathrm{ch}_{\leq 2}(\mathcal{H}^0(E)) = (1, 0, -6)$. Therefore applying Lemma 4.4.2 to $D = L$ implies $\mathcal{H}^0(E)$ is an ideal sheaf of curves. Hence the argument above implies that this ideal sheaf is of the form $\mathcal{I}_{C_5 \cup L}$.

□

By Lemma 4.5.8, we assume that C_5 is smooth outside Z_2. Let C' be the normalization of C_5 which is a smooth curve of genus 4. Let $\mathcal{L} := \mathcal{O}_{C'}(1)$, a line bundle of degree 5 with at least 3 global sections. By Riemann-Roch we have $\mathrm{h}^0(\mathcal{L}) - \mathrm{h}^1(\mathcal{L}) = 1 + \deg(\mathcal{L}) - \mathrm{genus}(\mathcal{L}) = 2$, and therefore $\mathrm{h}^0(\omega_{C'} \otimes \mathcal{L}^\vee) = \mathrm{h}^1(\mathcal{L}) \geq 1$ for canonical bundle $\omega_{C'}$, and so $\omega_{C'} \otimes \mathcal{L}^\vee$ has section. Thus, as $\omega_{C'} \otimes \mathcal{L}^\vee$ has degree one, it is of the form $\mathcal{O}_{C'}(x)$, for some point $x \in C'$. Therefore, $\mathcal{L} = \omega_{C'}(-x)$ and it has exactly 3 global sections.

Lemma 4.5.9. *C' is not hyper-elliptic.*

Proof. By construction, since $\omega_{C'}(-x) = \mathcal{O}_{C'}(1)$ is globally generated (as the pull-back of $\mathcal{O}_{\mathbb{P}^3}(1)$), we have $\mathrm{H}^0(\omega_{C'}(-x)) = \mathbb{C}^3$. Therefore, $\mathrm{H}^0(\omega_{C'}(-x-y)) = \mathbb{C}^2$, for any $y \in C'$. This means that C' cannot be hyper-elliptic. □

Lemma 4.5.10. *x is the same as $\langle Z_2 \rangle \cap L$.*

Proof. By Lemma 4.5.9, C' is non-hyperelliptic, so embedding via $\mathrm{H}^0(\omega_{C'}(-x))$ is the same as the canonical embedding via $\mathrm{H}^0(\omega_{C'})$ followed by projecting from the point x. The nodal points Z_2 correspond to two trisecant lines l, l' containing x. Let P_0 be the plane spanned by l, l'. As C' is a canonical genus four curve, thus a $(2,3)$-complete intersection curve, the intersection $C' \cap P_0$ is also a $(2,3)$ complete intersection. But that is only possible if it contains x as a double point (the quadratic equation has to be exactly the equation defining $l \cup l'$, and the cubic equation has to vanish at x), and thus its projection must be colinear to Z_2, i.e. $x = p$. □

For deforming $C' \cup L$, let us look at the family $\mathcal{C} := Bl_0(C' \times \mathbb{A}^1)$. We have the projection $\mathsf{q} \colon \mathcal{C} \to \mathbb{A}^1$; its fibers $C_t = \mathsf{q}^{-1}(t)$ are $C_t = C'$ for $t \neq 0$, and $C_0 = C' \cup L$.

Lemma 4.5.11. *Let ω_{q} be the relative canonical bundle, and $\mathcal{L}' := \omega_{\mathsf{q}}(-2L)$. For all $t \in \mathbb{A}^1$, we have $\mathrm{H}^0(\mathcal{L}'|_{C_t}) = \mathbb{C}^4$, and $\mathcal{L}'|_{C_t}$ is globally generated.*

Proof. For $t \neq 0$, $\mathcal{L}'|_{C_t} = \omega_{C_t}$ implies the claim as $C_t = C'$ is canonical. For C_0, we observe

$$\mathcal{L}'|_{C'} = \omega_{\mathsf{q}}(-2L)|_{C'} = \omega_{C'}(-x) = \mathcal{L}, \quad \text{and} \quad \mathcal{L}'|_L = \omega_{\mathsf{q}}(-2L)|_L = \omega_{\mathsf{q}}|_L(2) = \mathcal{O}_L(1).$$

From the exact sequence $\mathcal{O}_L \to \mathcal{L}'|_{C_0} \to \mathcal{L}'|_{C'}$ and noticing that $\mathrm{H}^0(\mathcal{O}_L) = \mathbb{C}^1$ and $\mathrm{H}^0(\mathcal{L}'|_{C'}) = \mathrm{H}^0(\mathcal{L}) = \mathbb{C}^3$, we have the result.

□

We will show that any $E \in \mathcal{R}_1$ is a limit of an ideal sheaf of canonical genus four curves in $\mathcal{N}_2 \cap \mathcal{N}_3$:

Corollary 4.5.12. All E in $\mathbb{P}^{13} = \mathbb{P}(\mathrm{H}^0(\mathcal{I}_{Z_2}^2 \otimes \mathcal{I}_p(5)))$ are limits of objects in $\mathcal{N}_2 \cap \mathcal{N}_3$.

Proof. Recall that as in Lemma 4.5.8, we can assume that E is generic. By Lemma 4.5.11, $\mathbb{P}(\mathfrak{q}_*\mathcal{L}')$ is a $\mathbb{P}^3$-bundle over $\mathbb{A}^1$, and $\omega_{\mathfrak{q}}|_{C_t}$ gives the morphism $\mathfrak{g}\colon \mathcal{C} \to \mathbb{P}(\mathfrak{q}_*\mathcal{L}')$. Then $\mathfrak{g}_t := \mathfrak{g}|_{C_t}$ is the canonical embedding for $t \neq 0$, and the partial normalization $\mathfrak{g}_0\colon C' \cup L \to C_5 \cup L$ for $t = 0$. Let $E^\bullet$ be the two term complex $\mathcal{O}_{\mathbb{P}(\mathfrak{q}_*\mathcal{L}')} \to \mathfrak{g}_*\mathcal{O}_{\mathcal{C}}$, and E_t, the restriction of $E^\bullet$ to the fiber over $t \in \mathbb{A}^1$. For $t \neq 0$, the map defining E_t is surjective, and so $E \cong \mathcal{I}_{\mathfrak{g}(C_t)}$, an object of $\mathcal{N}_2 \cap \mathcal{N}_3$. For $t = 0$, the map fails to be surjective at the nodes, and so $\mathcal{H}^0(E_0) = \mathcal{I}_{C_5 \cup L}$ and $\mathcal{H}^1(E_0) = \mathcal{O}_{Z_2}^\vee$, and therefore E_0 is quasi-isomorphic to $E \in \mathcal{N}_3'$ itself.

□

Lemma 4.5.13. We have $\mathcal{R}_1 \subset \widetilde{\mathcal{N}_2} \cap \mathcal{N}_3'$.

Proof. We just constructed the set of ideal sheaves of canonical genus four curves in the intersection which is dense in R_1 (Corollary 4.5.12). As the base of $\mathcal{R}_1$ is U_1, and the intersection is proper (and so every fiber will be contained in the intersection), then $\mathcal{R}_1$ will be contained in the intersection. □

Remark 4.5.14. We want to show that the intersection $\widetilde{\mathcal{N}_2} \cap \mathcal{N}_3'$ is exactly $\overline{\mathcal{R}_1} = \mathcal{R}$. In order to do so, we have already shown that any object in $\mathcal{R}_1$ (corresponding to U_1 or $\mathrm{Ext}^1(A, B) = \mathbb{C}$) is the intersection (as the specialization of an ideal sheaf of canonical genus 4 curves); i.e. $\mathcal{R}_1 \subset \widetilde{\mathcal{N}_2} \cap \mathcal{N}_3' \subset \mathcal{R}$ (Lemma 4.5.13). As $U_2 \subset \overline{U_1}$, the question is whether the objects in $\mathcal{R} \setminus \mathcal{R}_1$ (corresponding to U_2 or $\mathrm{Ext}^1(A, B) = \mathbb{C}^2$) are also in the intersection. We will show this by explicitly degenerating objects in $\mathcal{R}_1$ to get a 14-dimensional family of objects in $\overline{\mathcal{R}_1}$ lying over a given point of U_2. As the fiber in $\mathcal{R}$ over the given point, is 14-dimensional and irreducible, and as the intersection is closed, this will show that the whole fiber is contained in the closure. As U_2 is dense in the locus over which $\mathcal{R}$ is a cone bundle, this will prove that the entire cone bundle is contained in the intersection.

We will simultaneously deform L to $\langle Z_2 \rangle$ and C_5 to $C_4 \cup \langle Z_2 \rangle$ for a plane quartic C_4 containing Z_2; the limit of the corresponding objects in $\mathcal{R}_1$ will be $\mathcal{H}^0(E_0) = \mathcal{I}_{C_4 \cup D}$ for a thickening D of $\langle Z_2 \rangle$.

Lemma 4.5.15. We have $\overline{\mathcal{R}_1} = \mathcal{R}$, and thus $\mathcal{R} \subset \widetilde{\mathcal{N}_2} \cap \mathcal{N}_3'$.

Proof. To construct objects in the closure, we consider a family of objects in $\mathcal{R}_1$ parametrized by $\mathbb{A}^1 \setminus \{0\}$, and then fill in the central fiber to an additional object in $\widetilde{M_{\sigma_-}(v)}$. Let $C = Spec\mathbb{C}[t]$, and $E \in \mathcal{R}_1$. As such objects are uniquely determined by their $\mathcal{H}^0$, they are ideals $\mathcal{I}$ in $\mathbb{C}[x,y,z,w][t,t^{-1}]$ and we want to find their flat limit as t goes to zero (notice that to distinguish different limits, it is enough to distinguish them after restriction to $\mathbb{A}^3$, so in what follows we consider the ideals in $\mathbb{A}^3$ instead of $\mathbb{P}^3$).

By Remark 4.5.14, we want to understand the infinitesimal direction of the thickening. There are two possibilities to deform $C_5 \cup L$ to $C_4 \cup D$:

(1) Simultaneously, (in the plane P' of L) deform L directly to $\langle Z_2 \rangle$ with parameter t (L and all its deformations pass through $p = L \cap P$), and deform C_5 to $C_4 \cup \langle Z_2 \rangle$ with parameter bt for some $b \in \mathbb{C}$.

(2) Deform the plane P' (which always will contain L and its deformations L_t's) with parameter c.

For each class E fitting in $\mathcal{I}_{C_5 \cup L} \hookrightarrow E \twoheadrightarrow \mathcal{O}_{Z_2}[-1]$, we want to figure out how the deformations above (or deforming b, c) would affect it. To be more precise, let P is given by $z = 0$, and Z_2 given by $(0,0,0)$ and $(0,-1,0)$, and so $\langle Z_2 \rangle$ is given by $x = z = 0$. Also, let the deformations of L, i.e. L_t's are given by $x = z - t(y-1) = 0$, and we want to have L_0 identical to $\langle Z_2 \rangle$ in the end. Let us denote a branch of C_4 passing through p_1 by L'' and assume that C_4 is given by f_4. Furthermore, we set $z = bty^2(y+1)^2(y-1) = 0$ to denote $(C_5)_{bt}$, i.e. deformations of C_5 to $C_4 \cup \langle Z_2 \rangle$. Therefore, (locally) we have

$$\mathcal{I} := \mathcal{I}_{L_t \cup (C_5)_{bt}} = (z, f_4.x + bty^2(y+1)^2(y-1)) \cap (x, z - t(y-1))$$

$$= (zx, z^2 - tz(y-1), f_4.x^2 + btxy^2(y+1)^2(y-1), -tf_4x + btzy^2(y+1)^2 - bt^2y^2(y+1)^2(y-1)).$$

Dividing the last term by t and then letting t go to zero, we have

$$(zx, z^2, f_4.x^2, -f_4x + bzy^2(y+1)^2) \subset lim_{t\to 0}\mathcal{I}. \tag{4.1}$$

The closure of this ideal in $\mathbb{P}^3$ represents a degree four curve, C_4, plus a degree two infinitesimal thickening around L'. We also notice that the direction of the infinitesimal thickening is determined by the ratio of f_4x and $bzy^2(y+1)^2$.

To determine the thickening direction at each point $(x = 0, y, z = 0)$ on L', we can write the normal vectors to L' as

$$w_{\mathfrak{A},\mathfrak{B}} = \mathfrak{A}(y)\frac{\partial}{\partial x} + \mathfrak{B}(y)\frac{\partial}{\partial z}.$$

The question is that for which $\mathfrak{A}, \mathfrak{B}$ we have

$$w_{\mathfrak{A},\mathfrak{B}}(\mathfrak{f}(0,y,0)) = 0 \quad \text{for all } \mathfrak{f} \in lim_{t\to 0}\mathcal{I}.$$

As we mentioned above, we deform the plane P' with the parameter c, by replacing P' by another plane containing L', i.e. replacing x by $x - cz$ for some $c \in \mathbb{C}$.

Now we could ask how $\mathfrak{A}(y)$ and $\mathfrak{B}(y)$ depend on the choice of the parameters b and c. We apply $w_{\mathfrak{A},\mathfrak{B}}$ to the last term of the ideal in 4.1, replace x by $x - cz$, and then plug in $(0, y, 0)$ to solve the following equation for $\mathfrak{A}(y)$ and $\mathfrak{B}(y)$

$$0 = w_{\mathfrak{A},\mathfrak{B}}(-f_4(0,y,0)(x - cz) + bzy^2(y+1)^2)$$

$$= -\mathfrak{A}(y)f_4(0,y,0) + c\mathfrak{B}(y)f_4(0,y,0) + b\mathfrak{B}(y)y^2(y+1)^2.$$

Therefore, we will get

$$\frac{\mathfrak{A}(y)}{\mathfrak{B}(y)} = b\frac{y^2(y+1)^2}{f_4(0,y,0)} + c.$$

We notice that $lim_{y\to 0}(\frac{\mathfrak{A}(y)}{\mathfrak{B}(y)}) = c$, and then b also can be recovered from $\frac{\mathfrak{A}(y)}{\mathfrak{B}(y)}$ at any point $y \neq 0$.

Notice that we have 12 dimension for the choice of C_4 in the plane P, and two more dimensions for the parameters b and c, corresponding to the proportion of the deformations of L and C_5, and the deformation of the the plane P' (containing L), respectively. Thus we have a 14-dimensional locus, which is open in the irreducible 14-dimensional cone and so it has to be dense in each fiber; therefore, its closure has to be the entire cone. This means that the closure of $\mathcal{R}_1$ is the whole $\mathcal{R}$. By Lemma 4.5.13, we have $\mathcal{R}_1 \subset \widetilde{\mathcal{N}_2} \cap \mathcal{N}_3'$, and as the intersection is closed, we will have $\overline{\mathcal{R}_1} \subset \widetilde{\mathcal{N}_2} \cap \mathcal{N}_3'$; but we just proved $\overline{\mathcal{R}_1} = \mathcal{R}$, so we have $\mathcal{R} \subset \widetilde{\mathcal{N}_2} \cap \mathcal{N}_3'$.

□

Remark 4.5.16. We notice that, once we are given the infinitesimal thickening direction at each point on L', we can have the speed of the deformation of the plane P' as well as the proportion of the speeds of the deformations of C_5 and L. The normal direction points in the direction of P' near Z_2 (i.e. the (deformations of) plane P' could be recovered from the normal direction near Z_2), in the direction of P near the other intersection points $C_4 \cap L'$, and varies in between depending on b.

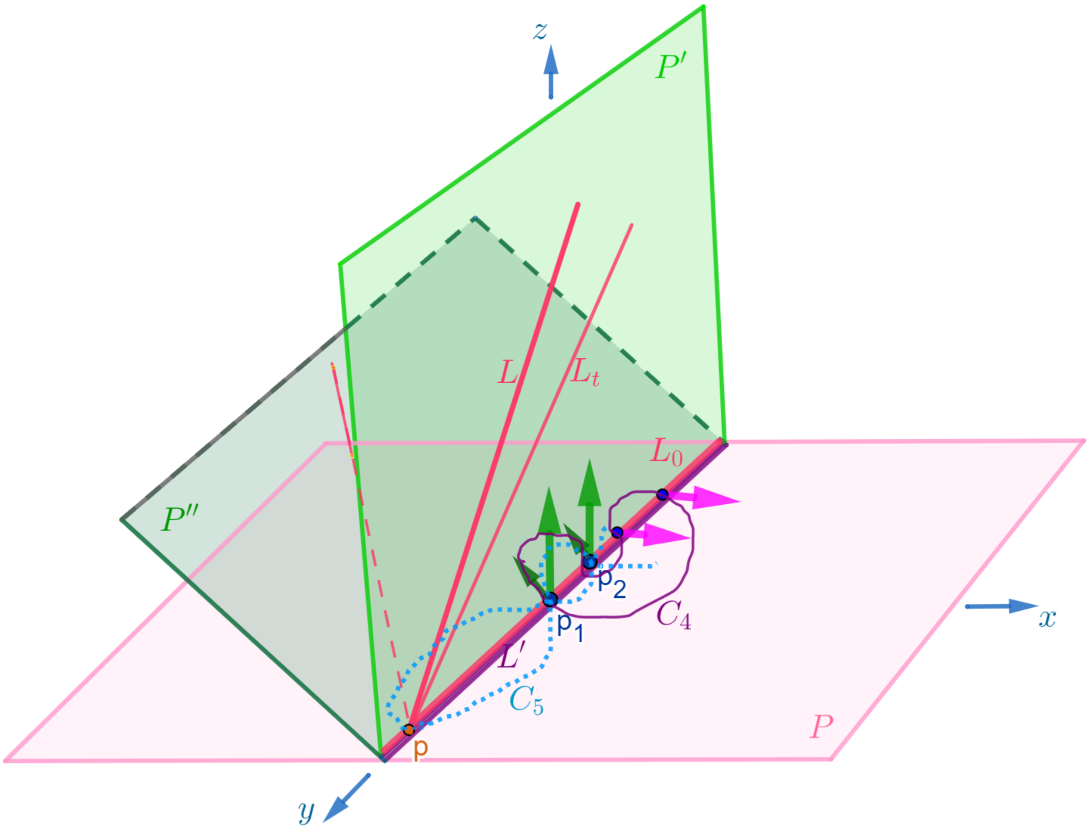


Figure 4.1: Tangent direction of the infinitesimal thickening (P'' is a deformation of P')

Now we can show that $\mathcal{R}$ is exactly the intersection:

Corollary 4.5.17. Let $E \in \mathcal{N}_3'$. E is in the intersection $\widetilde{\mathcal{N}_2} \cap \mathcal{N}_3'$ if and only if $ext^1(E,E) > 28$. In other words, the singularity locus of $\mathcal{N}_3'$ is exactly the intersection.

Proof. Deduced from Proposition 4.5.7 and Lemma 4.5.15. □

At this point, we can complete the proof of Theorem 1.0.1:

Theorem 4.5.18. *We have*

$$\widetilde{\mathcal{N}_2} \cap \mathcal{N}_3' = \{E \colon \zeta \text{ associated to } E \text{ is non-injective}\}.$$

More precisely, we have $\widetilde{\mathcal{N}_2} \cap \mathcal{N}_3' = \mathcal{R}$ which is equal to the exceptional divisor of the contraction map ψ; i.e. the intersection contains an open subset $\mathcal{R}_1$ such that $\psi|_{\mathcal{R}_1}$ is a $\mathbb{P}^{13}$-bundle over a 10-dimensional base. Over a 7-dimensional subset of the base, the fibers degenerate to a 14-dimensional cone with $\mathbb{P}^9$ as vertex over the 5-dimensional variety of rank one 2×4 matrices.

Proof. The first part is obtained from Lemma 4.5.6 and Corollary 4.5.17, and noticing that $\widetilde{\mathcal{N}_2} \to \mathcal{W}$ is the restriction of $\mathcal{N}_3 \dashrightarrow \mathcal{W}$, and the later contracts $\mathcal{N}_3'$. The second part is obtained from Lemma 4.5.15.

□

4.6 Birational morphism corresponding to the wall-crossing

In this section, we will finally prove our main Theorem, i.e. Theorem 1.0.1, which explains the birational relation between $\mathcal{N}_2$ and $\widetilde{\mathcal{N}_2}$. From the construction we know that $\mathcal{N}_2 \setminus \mathcal{U}_{-,+}$ is isomorphic to $\mathcal{N}_3 \setminus \mathcal{N}_3'$, i.e. they both parametrize strictly stable objects with respect to stability conditions on the wall $\mathcal{W}$, and stable with respect to stability conditions on the both sides of $\mathcal{W}$. Now the question is how we can describe the birational relationship between their closures. First, we have the following Lemma and Corollary:

Lemma 4.6.1. For a stability condition σ_0 on the wall $\mathcal{W}$, the moduli space $\mathcal{W}$ exists. Furthermore, there are morphisms $\mathcal{N}_2 \to \mathcal{W}$ and $\widetilde{\mathcal{N}_2} \to \mathcal{W}$ which contract the loci of S-equivalent objects, which are a $\mathbb{P}^1$-bundle and a 23-dimensional locus, respectively.

Proof. It follows from [1], Lemma 4.5.2, and Theorem 4.5.18. □

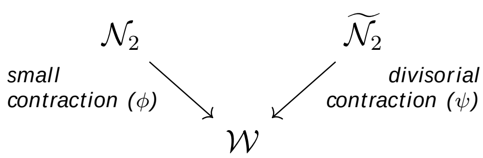
Corollary 4.6.2. The map ϕ is a small contraction and ψ is a divisorial contraction.

Proof. Since the 8 dimensional component of $\mathcal{U}_{-,+}$, as the exceptional locus of ϕ is a $\mathbb{P}^1$-bundle (Lemma 4.5.2), it turns out that there is a small contraction from $\mathcal{N}_2$ to the good moduli space for the wall, $\mathcal{W}$ (Lemma 4.6.1). On the other hand, as we have seen in Theorem 4.5.18, the exceptional locus of ψ in $\widetilde{\mathcal{N}_2}$ is of dimension 23 (and so of codimension one), so we have a divisorial contraction from $\widetilde{\mathcal{N}_2}$ to $\mathcal{W}$. □

In the following theorem, notice that $\mathcal{N}_2$ is a blow up of a projective bundle and so it will be a smooth reduced moduli space. We consider $\mathcal{N}_3$ as an algebraic space given by the reduced part of the moduli space defined by the union $\widetilde{\mathcal{N}_2} \cup \mathcal{N}_3'$.

Theorem 4.6.3. *Fix $v = (1, 0, -6, 15)$. There is a wall-crossing with respect to Bridgeland stability conditions $\mathcal{N}_2 \to \mathcal{N}_3$ between two moduli spaces separated by the wall $\mathcal{W}$ with the following properties:*

- *$\mathcal{N}_2$ is a smooth and irreducible variety,*
- *$\mathcal{N}_3 = \widetilde{\mathcal{N}_2} \cup \mathcal{N}_3'$, where $\widetilde{\mathcal{N}_2}$ is birational to $\mathcal{N}_2$ and $\mathcal{N}_3'$ is a new irreducible component,*
- *There is a diagram*



where both ϕ and ψ have relative Picard rank 1. Furthermore, $\widetilde{\mathcal{N}_2}$ is not $\mathbb{Q}-$factorial.

Proof. In order to show that the relative Picard rank of ψ is 1, it is enough to show that the fibers have $1-$dimensional N_1. To show this for the 14-dimensional cone, we extend the method in [11, Example 2.8.] from a cone with point vertex to our cone with the $\mathbb{P}^9$ vertex: Let $X \subset \mathbb{P}(\mathbb{C}^{10} \oplus \mathbb{C}^8)$ be the projective cone with vertex $\mathbb{P}^9 = \mathbb{P}(\mathbb{C}^{10})$ over Y, the 4-dimensional variety corresponding to 2×4-matrices of rank ≤ 1 in $\mathbb{P}^7$. We want to describe the blow-up of X at the vertex $\mathbb{P}^9$: Let Z be the $\mathbb{P}^{10}$-bundle over Y with the fiber

$$\mathbb{P}(Z_y) = \mathbb{P}\Big(\mathbb{C}^{10} \oplus \text{one-dimensional subspace in } \mathbb{C}^8 \text{ corresponding to y}\Big)$$

over $y \in Y$. Let us call this blow-up $\pi\colon Z \to X$. There is a natural map from Z to $\mathbb{P}^{17}$ (with image contained in X) by forgetting the point y and sending a point in the fiber $\mathbb{P}^{10} = \mathbb{P}(Z_y)$, corresponding to a one-dimensional subspace of Z_y, to the corresponding one-dimensional subspace of $\mathbb{C}^{18}$. On the other hand, there is a fibration $f : Z \to Y$. Thus we have

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{\pi} \\ Y & & X. \end{array}$$

Now, let $E := Y \times \mathbb{P}^9$ be the exceptional locus of π, i.e. we have $\pi(E) = \mathbb{P}^9$. Following [11, Example 2.8.], as the relative dimension of f is 10, we have $N_1(Z) = E^9.f^*N_0(Y) \oplus E^{10}.f^*N_1(Y)$. Let i be the embedding of Y into $Y \times \mathbb{P}^9$ followed by embedding in Z. Then as $\pi \circ i$ is constant, we will have $\pi_*(E^{10}.f^*N_1(Y)) = 0$. Now, taking π_* of $N_1(Z) = E^9.f^*N_0(Y) \oplus E^{10}.f^*N_1(Y)$, we have $\pi_*N_1(Z) = \pi_*E^9.f^*N_0(Y)$, and we know that $N_1(X)$ is generated by $\pi_*N_1(Z)$. But the equality indicates that $\pi_*N_1(Z)$ has rank at most one, and on the other hand, cannot have rank zero. Therefore $\mathrm{rk}(N_1(X)) = 1$. Finally, we need to show that $\widetilde{\mathcal{M}_{\sigma_-}}(v)$ is not $\mathbb{Q}-$factorial. If it was $\mathbb{Q}-$factorial, then $\mathcal{M}_{\sigma_0}(v)$ would also be $\mathbb{Q}-$factorial, since it is the image of $\widetilde{\mathcal{M}_{\sigma_-}}(v)$ under a divisorial contraction of relative Picard rank 1 (see the proof of [16, Corollary 3.18]). On the other hand, $\mathcal{M}_{\sigma_0}(v)$ is the image of $\mathcal{M}_{\sigma_-}(v)$ (which is smooth and inparticular, $\mathbb{Q}-$factorial) under a small contraction, and hence cannot be $\mathbb{Q}-$factorial; this is a contradiction.

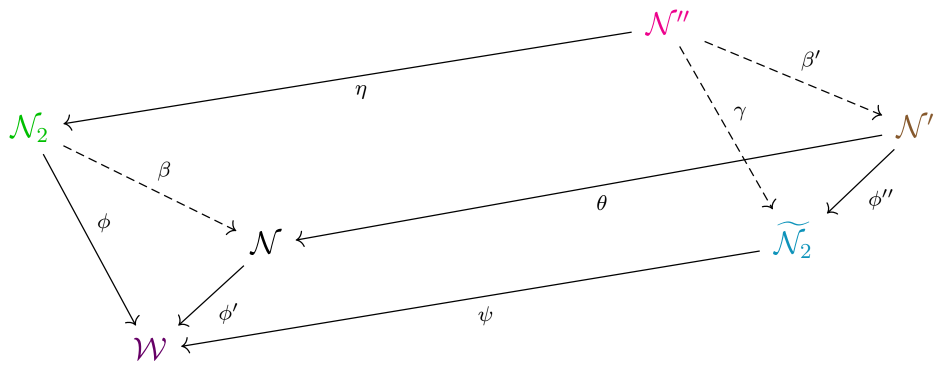
□

We can see that it wouldn't be possible to explain the diagram in the statement of the above Theorem in $Mov(\mathcal{N}_2)$ or in $Mov(\widetilde{\mathcal{N}_2})$. To explain the birational behaviour of our wall-crossing more, we start with the following wall-crossing between the two chambers in $\mathrm{Stab}(\mathbb{P}^3)$, i.e. the wall-crossing $\mathcal{N}_2 \to \mathcal{N}_3$.

In what follows, by abuse of terminology, we use the terminology "flip" for the small birational map with respect to the relevant small contractions. Let $\mathcal{N}$ be the flip of $\mathcal{N}_2$ with respect to the small contraction ϕ. Let $\mathbb{E}$ be the exceptional divisor in $\widetilde{\mathcal{N}_2}$, and consider $\mathbb{E}' = \psi(\mathbb{E})$; then ψ is the blow-up of $\mathcal{W}$ along $\mathbb{E}'$. Now define $\phi'\colon \mathcal{N} \to \mathcal{W}$; then we can blow up $\mathcal{N}_2$ and $\mathcal{N}$ along $\phi^{-1}(\mathbb{E}')$ and $\phi'^{-1}(\mathbb{E}')$ and call the result $\mathcal{N}''$ and $\mathcal{N}'$, respectively. Notice that $\mathcal{N}'$ is the flip of $\mathcal{N}''$ with respect to the small contraction γ.

Now, under the map from $\mathrm{Stab}(\mathbb{P}^3)$ to the movable cone $Mov(\mathcal{N}') = Mov(\mathcal{N}'')$, the wall $\mathcal{W}$ is not sent to a single wall in the movable cone, but it is sent to walls corresponding to the intersection of two, the divisorial and the flipping contractions. Furthermore, this map sends the two adjacent chambers not to the chambers but to a subset on the divisorial and flipping walls; in other words, crossing the wall $\mathcal{W}$ in the stability world is equivalent to remaining and walking on the broken line passing the point which is the image of the wall $\mathcal{W}$ in the birational world (see Figure 4.2). Notice that $\mathcal{N}$ does not seem to appear as a moduli space of Bridgeland-stable objects. Similarly, $\mathcal{N}', \mathcal{N}''$ do not seem to show up in the stability space.

The wall-crossing in the stability space is equivalent to the combination consists of the arrow starting from $\mathcal{N}_2$ and heading to $\mathcal{W}$, followed by the arrow starting from $\mathcal{W}$ and heading to $\widetilde{\mathcal{N}_2}$ in the movable cone (see Figure 4.2). To summarize, we have the following diagram which explains the relation between $\mathcal{N}_2$ and $\widetilde{\mathcal{N}_2}$:



Existence of the small maps γ and ϕ'' is implied by the universal property of blow-ups as the maps from $\mathcal{N}''$ and $\mathcal{N}'$ to $\mathcal{W}$ must be factored via ψ (notice that as E' is closed in $\mathcal{W}$, the blow-up of its inverse image in $\mathcal{N}''$ and $\mathcal{N}'$ must be Cartier). Pictorially, the movable cone of $\mathcal{N}'$ (or $\mathcal{N}''$) looks like the following:

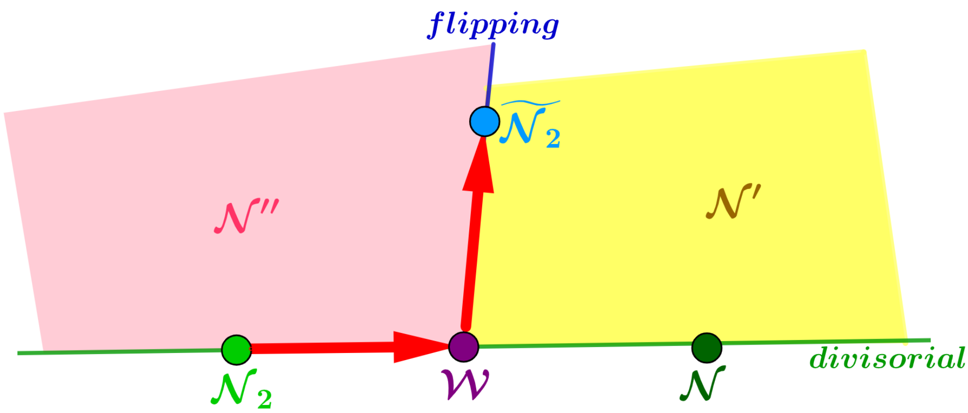


Figure 4.2: Birational models in the movable cone

Chapter 5

Geometry of stable pairs and the Hilbert Scheme

5.1 Introduction

In this chapter, we describe the Hilbert scheme by proving Theorem 1.0.2 and also in the way, we describe the moduli of stable pairs in Theorem 5.6.1.

Despite their very natural definition as the parameter spaces of embedded sub-varieties in the projective spaces, Hilbert schemes are one of the worst-behaved moduli spaces in algebraic geometry, given that the Murphy's Law holds for them ([13, Law 1.34]). For $\mathbb{P}^2$ as an ambient space, many facts are known about the geometry of the Hilbert schemes of points, but already in the case of $\mathbb{P}^3$, few general results are known, and not many examples are fully understood. The first examples of wall-crossing for the Hilbert scheme of curves in $\mathbb{P}^3$ can be found in [26, 30] for twisted cubics, and in [12] for elliptic quartics. Our goal is to understand the geometry of the Hilbert scheme of canonical genus four curves as much as possible as well as the associated moduli space of PT stable pairs of those curves. The Hilbert scheme of canonical genus 4 curves in $\mathbb{P}^3$ parametrizes such curves is denoted by $\mathcal{H}ilb^{6t-3}(\mathbb{P}^3)$, as the Hilbert polynomial is given by $6t-3$. This space parametrizes more objects than only the canonical genus four pure curves; considering the formula $p(t) = dt + 1 - g$ for the Hilbert polynomial of genus g and degree d in $\mathbb{P}^3$, we can imagine that some curves of higher genera together with some extra points could have the same Hilbert polynomial, and therefore they should be counted as the objects parametrized by $\mathcal{H}ilb^{6t-3}(\mathbb{P}^3)$ as well.

A smooth non-hyperelliptic genus 4 curve C embeds into $\mathbb{P}^3$ as a (2,3)-complete intersection curve. The question is how to compactify this 24-dimensional space of all such curves. As we have embedding in the projective space, a classical answer would be considering the Hilbert scheme of such curves. But this has Many irreducible components, e.g. plane sextics with 6 floating points added, yield a component of dimension 48. It would be hard to even list all the irreducible components. Therefore, instead of studying the Hilbert scheme directly, we consider Bridgeland stability conditions on $\mathrm{D}^b(\mathbb{P}^3)$ which gives better compactifications:

depending on a choice of a stability condition $\sigma \in \mathrm{Stab}(\mathbb{P}^3)$, it gives $\mathcal{M}_\sigma(1, 0, -6, 15)$, the moduli space of σ-stable complexes E with $\mathrm{ch}(E) = \mathrm{ch}(\mathcal{I}_C)$. Therefore, following a path along the space of stability conditions, we want to to understand how $\mathcal{M}_\sigma(1, 0, -6, 15)$ changes: At the beginning of the path, we will get a very efficient compactification, given by a $\mathbb{P}^{15}$-bundle (choice of cubic) over $\mathbb{P}^9$ (choice of quadric), parametrizing some non-torsion free sheaves in addition to ideal sheaves. On the other hand, at the large-volume limit, we (partially) recover the Hilbert scheme; rough control over wall-crossing gives the least number of components (5). Right before this chamber we have the corresponding PT-moduli space of stable pairs (with 8 components).

Strategy of proof. For each wall $\langle A, B\rangle$ on the right side as in Theorem 3.1.13, the newly stable objects are given by an exact triangle $A \to E \to B$. For each A, B this gives a locus $\mathbb{P}(\mathrm{Ext}^1(B, A))$ inside the moduli space after crossing the wall. We stratify the space of pairs (A, B) by $\dim(\mathbb{P}(\mathrm{Ext}^1(B, A)))$; for each stratum, we describe a general element and decide whether it is in the closure of other strata.

5.2 Previous works

In general, we do not know much about the geometry of Hilbert schemes of curves in $\mathbb{P}^3$, $\mathcal{H}ilb^{dt+1-g}(\mathbb{P}^3)$ (d=degree of the curve, g= genus of the curve). One of the few positive well-know results about the global geometry of Hilbert scheme, is the following:

Theorem 5.2.1 ([14], [13]). *For any non-negative integer r, and any polynomial $p(t)$, the Hilbert scheme $\mathcal{H}ilb^{p(t)}(\mathbb{P}^r)$ is connected.*

Regarding the geometry of the Hilbert scheme of curves in $\mathbb{P}^3$ (via wall crossing), the following two cases are known:

(i) twisted cubics: for genus 0 degree 3 curves in $\mathbb{P}^3$, the Hilbert scheme $\mathcal{H}ilb^{3t+1}(\mathbb{P}^3)$ is considered by Schmidt in [26], and Xia in [30]. They prove the following theorem:

Theorem 5.2.2 ([30], [26]). *There is a path γ in $\mathrm{Stab}(\mathbb{P}^3)$ that crosses three walls and four chambers for a fixed Chern character $\mathrm{ch}(\mathcal{I}_C)$, where C is a twisted cubic. We have the following list of the moduli space of semi-stable objects in each chamber with respect to the path γ:*
(1) The empty space.
(2) A 12-dimensional smooth projective integral variety $\mathcal{M}_1$.
(3) A projective variety $\mathcal{M}_2$ with two irreducible components : $\tilde{\mathcal{M}}_1$ and M_2' .$\tilde{\mathcal{M}}_1$ is a blow-up of $\mathcal{M}_1$ along a smooth center of dimension 5, and M_2' is a $\mathbb{P}^9$-bundle over $\mathbb{P}^3 \times (\mathbb{P}^3)^*$.
(4) The Hilbert scheme of twisted cubics, which is a blow-up of $\mathcal{M}_2$ along a 5-dimensional smooth center contained in $M_2' - \tilde{\mathcal{M}}_1$.

(ii) elliptic quartics: for genus 1 degree 4 curves in $\mathbb{P}^3$, the Hilbert scheme $\mathcal{H}ilb^{4t}(\mathbb{P}^3)$ is considered by P. Gallardo, C. Lozano Huerta, and B. Schmidt in [12].They prove the following theorem:

Theorem 5.2.3 ([12]). *There is a path γ in $\mathrm{Stab}(\mathbb{P}^3)$ that crosses five walls and six chambers for a fixed Chern character $\mathrm{ch}(\mathcal{I}_C)$, where C is a elliptic quartics. We have the following list of the moduli space of semistable objects in each chamber with respect to the path γ:*
(1) The empty space.
(2) The Grassmannian $\mathcal{M}_1 = \mathbb{G}r(1, 9)$.
(3) $\mathcal{M}_2$ is the blow up of $\mathcal{M}_1$ along a smooth locus isomorphic to $\mathbb{G}r(1, 3) \times (\mathbb{P}^3)^*$
(4) $\mathcal{M}_3$ with two irreducible components : $\tilde{\mathcal{M}}_2$ and M_3' . $\tilde{\mathcal{M}}_2$ is a blow-up of $\mathcal{M}_2$ long the smooth incidence variety parametrizing length two sub- schemes in a plane in $\mathbb{P}^3$, and M_3' is a $\mathbb{P}^{14}$-bundle over $(\mathbb{P}^3)^{[2]} \times (\mathbb{P}^3)^*$.
(5) $\mathcal{M}_4$ with two irreducible components : $\tilde{M}_3'$ and M_4' . $\tilde{M}_3'$ is birational to M_3', and M_4' is equal to $\tilde{\mathcal{M}}_2$.
(6) The Hilbert scheme $\mathcal{H}ilb^{4t}(\mathbb{P}^3)$, with two components: the principal component contains an open subset of elliptic quartics and is equal to $\tilde{\mathcal{M}}_2$, and the second component is of dimension 23 and is birational to M_3'. Moreover, the two components intersect transversally along a locus of dimension 15.

5.3 Pandharipande-Thomas stable pairs

In this section, we introduce the notion of stable pairs in the sense of Pandharipande and Thomas. In [22], Pandharipande and Thomas introduced the notion of stable pairs for a nonsingular 3-fold X: it is defined as a pair (C, D) where C is a curve in X, and D is a divisor of C. In fact, they needed this pair to define a new invariant on 3-folds modifying DT invariant. The disadvantage of the moduli space of the ideal sheaves of curves in 3-folds (or the Hilbert scheme of curves in the 3-fold) is that the subschemes it parametrizes might be non-reduced, or have some free points. Moving from DT invariants to PT invariants, the free points are just removed and only the points on curves (as divisors) are kept. It means that for counting, the ideal sheaves of curves in X which might have some extra free (or floating/embedded) points are replaced by sheaves scheme theoretically supported on the corresponding Cohen-Macaulay curves with an extra condition:

Definition 5.3.1 ([22]). A *stable pair* $(\mathcal{F}, s)$ consists of a 1-dimensional pure sheaf $\mathcal{F}$ on X with zero-dimensional cokernel of the sections $s\colon \mathcal{O}_X \to \mathcal{F}$.

Equivalently, we have the following characterization of stable pairs:

Lemma 5.3.2 ([22]). An object $E \in \mathrm{D}^b(X)$ is a stable pair if: i) $\mathcal{H}^0(E)$ is an ideal sheaf, ii) $\mathcal{H}^1(E)$ is a sheaf with 0-dimensional support, and iii) $\mathrm{Hom}(\mathcal{O}_p[-1], E) = 0$, for all $p \in X$.

5.4 Chambers on the stable pairs side

In this section, we describe the rest of the corresponding moduli spaces to the chambers close to the hyperbola from the right, starting from the moduli space $\mathcal{N}_4$.

Let $\mathcal{N}_4$, be the moduli space for the next chamber. Let $\mathcal{U}$ be the universal line over $\mathbb{G}r(2,4)$, and $\mathfrak{Fl}_l$ is the space parametrizing flags $Z_l \subset P \subset \mathbb{P}^3$ where P is a plane and Z_l a zero-dimensional subscheme of length l.

Proposition 5.4.1. The moduli space $\mathcal{N}_4$ has four irreducible components: $\widetilde{\widetilde{\mathcal{N}_2}}$, $\widetilde{\mathcal{N}_3'}$, $\mathcal{N}_4'$ and $\mathcal{N}_4''$. The first two are birational to their counterparts in $\mathcal{N}_3$. The component $\mathcal{N}_4'$ is a $\mathbb{P}^{18}$-bundle over $\mathcal{U} \times \mathfrak{Fl}_1$, and it generically parametrizes the union of a line in $\mathbb{P}^3$ together with a choice of a point on it, and a plane quintic intersecting the line, together with choice of a point on it. The component $\mathcal{N}_4''$ is a $\mathbb{P}^{19}$-bundle over $\mathbb{G}r(2,4) \times \mathfrak{Fl}_1$, and it generically parametrizes disjoint unions of a line in $\mathbb{P}^3$ and a plane quintic together with a choice of a point on it. For both new components, $\mathcal{H}^0$ of the generic element is given by an ideal sheaf, and $\mathcal{H}^1 \neq 0$.

Proof. The first part comes from Lemma 4.4.1 and Propositions 4.4.7 and 4.5.1. The fourth wall on the right side of the hyperbola is $\langle(\mathcal{O}(-1) \xrightarrow{s} \mathcal{O}_L), {\iota_P}_* \mathcal{I}_{Z_1}^\vee(-5)\rangle$ (Theorem 3.1.13). Using Lemma 3.2.6 we will get two new components, $\mathcal{N}_4'$ and $\mathcal{N}_4''$ which are a $\mathbb{P}^{18}$-bundle over $\mathcal{U} \times \mathfrak{Fl}_1$ and a $\mathbb{P}^{19}$-bundle over $\mathbb{G}r(2,4) \times \mathfrak{Fl}_1$, respectively. We will show that the $\mathbb{P}^{20}$-bundle is contained in the closure of the $\mathbb{P}^{18}$-bundle.

To understand the objects in the new components more precisely, for any $E \in \operatorname{Ext}^1(B,A)$, we first want to understand its image in $\operatorname{Hom}(\mathcal{H}^0(B), \mathcal{H}^1(A))$ to see for which elements we get a non-zero map. Let q be the zero locus of s. We notice that we have the short exact sequence $\mathcal{H}^0(A) = \mathcal{I}_L(-1) \to A \to \mathcal{H}^1(A)[-1] = \mathcal{O}_q[-1]$. Taking $\operatorname{RHom}(\mathcal{H}^0(B) = \mathcal{O}_P(-5), -)$ of this, we have:

$$0 \to \operatorname{Ext}^1(\mathcal{O}_P(-5), \mathcal{I}_L(-1)) \to \operatorname{Ext}^1(\mathcal{O}_P(-5), A) \to \operatorname{Hom}(\mathcal{O}_P(-5), \mathcal{H}^1(A) = \mathcal{O}_q) \to 0.$$

We consider the stratification of $\mathcal{U} \times \mathfrak{Fl}_1$, given by $\dim(\operatorname{Ext}^1(B,A))$, and consider a general object in each stratum:

(1) $L \not\subset P$**, the zero locus of** s **is not** $L \cap P$ **and** $Z_1 \neq L \cap P$**.** In this case using Lemma 3.2.1, we have

$$\operatorname{Ext}^1(\mathcal{H}^0(B), A) = \operatorname{Ext}^1({\iota_P}_*\mathcal{O}(-5), \mathcal{O}(-1) \to \mathcal{O}_L)$$

$$= \operatorname{Hom}(\mathcal{O}(-5), \iota_P^*(\mathcal{O} \to \mathcal{O}_L(1))) = \mathrm{H}^0(\iota_P^*(\mathcal{O} \to \mathcal{O}_L(1)) \otimes \mathcal{O}(5)) = \mathrm{H}^0(\mathcal{I}_{L\cap P}(5)) = \mathbb{C}^{20},$$

and $\operatorname{Hom}(\mathcal{H}^0(B), \mathcal{O}_q) = 0$, and therefore any class in $\operatorname{Ext}^1(\mathcal{H}^0(B), A)$ induces zero map in $\operatorname{Hom}(\mathcal{H}^0(B), \mathcal{H}^1(A))$. Moreover, we have

$$\operatorname{Ext}^1(B,A) = \mathrm{H}^0(\mathcal{I}_{(L\cap P)\cup Z_1}(5)) \hookrightarrow \operatorname{Ext}^1(\mathcal{H}^0(B), A) = \mathrm{H}^0(\mathcal{I}_{L\cap P}(5)),$$

and therefore the image is exactly given by quintics containing $L \cap P$ and Z_1. Now, from

$$\begin{array}{ccccc} \mathcal{H}^0(A) = \mathcal{I}_L(-1) & \longrightarrow & \mathcal{H}^0(E) & \longrightarrow & \mathcal{H}^0(B) = \mathcal{O}_P(-5) \\ \mathcal{H}^1(A) = \mathcal{O}_q & \longrightarrow & \mathcal{H}^1(E) & \longrightarrow & \mathcal{H}^1(B) = \mathcal{O}_{Z_1}, \end{array}$$

as the connecting map is zero, $\mathcal{H}^1(E)$ will have length 2. Moreover, the top row is a short exact sequence $0 \to \mathcal{I}_L(-1) \hookrightarrow \mathcal{H}^0(E) \twoheadrightarrow \mathcal{O}_P(-5) \to 0$. The induced extension class

$$\mathrm{Ext}^1(\mathcal{O}_P(-5), \mathcal{I}_L(-1)) = \mathrm{Ext}^1(\mathcal{O}_P(-5), \iota_P^*\mathcal{I}_L(-1)(1)[-1]) = \mathrm{H}^0(\mathcal{I}_{L\cap P}(5))$$

corresponds to quintics containing the intersection point.

Using Lemma 4.4.2, induces an ideal sheaf of the union of a line in $\mathbb{P}^3$ and a plane quintic intersecting the line for $\mathcal{H}^0(E)$, i.e. for a generic object in $\mathrm{Ext}^1(B, A)$, the sheaf $\mathcal{H}^0(E)$ is ideal sheaf of the union of a line in $\mathbb{P}^3$ and a plane quintic intersecting in $L \cap P$. This together with the description of $\mathcal{H}^i(E)$ gives the result about the description of the component $\mathcal{N}_4'$.

(2) $L \not\subset P$ **and zero locus of** s **is** $L \cap P$**, but** $\neq Z_1$**.** Using Lemma 3.2.1, we have

$$\begin{array}{ccc} & & \mathrm{Ext}^1(\mathcal{H}^0(B), \mathcal{H}^0(A)) = \mathrm{H}^0(\mathcal{I}_q(5)) \\ & & \downarrow b \\ \mathrm{Ext}^1(B, A) = \mathrm{H}^0(\mathcal{I}_{Z_1}(5)) & \xrightarrow{a} & \mathrm{Ext}^1(\mathcal{H}^0(B), A) = \mathrm{H}^0(\mathcal{O}_P(5)) \\ & & \downarrow \\ & & \mathrm{Hom}(\mathcal{H}^0(B), \mathcal{H}^1(A)) = \mathrm{H}^0(\mathcal{O}_q) = \mathbb{C}. \end{array}$$

As $\mathrm{im}(a) \not\subset \mathrm{im}(b)$, the general element in $\mathrm{Ext}^1(B, A)$ induces a non-zero morphism $\mathcal{H}^0(B) \to \mathcal{H}^1(A)$, and so the connecting map in the above diagram is non-zero, which means $\mathcal{H}^1(E)$ has length 1. From the above diagram we have $\mathrm{im}(\mathcal{H}^0(E) \to \mathcal{H}^0(B) = \mathcal{O}_P(-5)) = \ker(\mathcal{H}^0(B) = \mathcal{O}_P(-5) \to \mathcal{H}^1(A) = \mathcal{O}_q) = \mathcal{I}_q(-5)$, which induces $0 \to \mathcal{I}_L(-1) \hookrightarrow \mathcal{H}^0(E) \twoheadrightarrow \mathcal{I}_q(-5) \to 0$. This means that the extension class

$$\mathrm{Ext}^1(\mathcal{I}_q(-5), \mathcal{I}_L(-1)) = \mathrm{Ext}^1(\iota_{P_*}\mathcal{I}_{L\cap P}(-5), \mathcal{I}_L(-1))$$

$$= \mathrm{Ext}^1(\mathcal{I}_{L\cap P}(-5), \iota_P^*\mathcal{I}_L(-1)(1)[-1]) = \mathrm{Hom}(\mathcal{I}_{L\cap P}(-5), \mathcal{I}_{L\cap P}) = \mathrm{H}^0(\mathcal{O}_P(5))$$

corresponds to quintics not necessarily containing the intersection point.

This induces an ideal sheaf of the union of a line in $\mathbb{P}^3$ and a plane quintic not intersecting the line for $\mathcal{H}^0(E)$, i.e. for a generic object in $\mathrm{Ext}^1(B, A)$, the sheaf $\mathcal{H}^0(E)$ is the ideal sheaf of the disjoint union of a line in $\mathbb{P}^3$ and a plane quintic. This together with the description of $\mathcal{H}^i(E)$ gives the result about the description of the component $\mathcal{N}_4''$.

(3) $L \not\subset P$ **and zero locus of** s **is** $L \cap P = Z_1$**.** Consider the diagram (by Lemma 3.2.6)

$$\begin{array}{ccc} & & \mathrm{Ext}^1(\mathcal{H}^0(B),\mathcal{H}^0(A)) = \mathrm{H}^0(\mathcal{I}_q(5)) \\ & & \downarrow b \\ \mathrm{Ext}^1(B,A) = \mathrm{H}^0(\mathcal{I}_q(5)\oplus(\mathcal{I}_q\otimes\mathcal{O}_q[-1])) & \xrightarrow{a} & \mathrm{Ext}^1(\mathcal{H}^0(B),A) = \mathrm{H}^0(\mathcal{O}(5)\oplus\mathcal{O}_q[-1]) \\ & & \downarrow c \\ & & \mathrm{Hom}(\mathcal{H}^0(B),\mathcal{H}^1(A)) = \mathrm{H}^0(\mathcal{O}_q) = \mathbb{C}. \end{array}$$

As $\mathrm{im}(a) = \mathrm{im}(b)$, the composition $c \circ a$ is zero. Therefore, general $E \in \mathrm{Ext}^1(B,A)$ induces the zero connecting map in the above diagram, i.e. it gives the following short exact sequences

$$\mathcal{H}^0(A) = \mathcal{I}_L(-1) \hookrightarrow \mathcal{H}^0(E) \twoheadrightarrow \mathcal{H}^0(B) = \mathcal{O}_P(-5)$$

and

$$\mathcal{H}^1(A) = \mathcal{O}_q \hookrightarrow \mathcal{H}^1(E) \twoheadrightarrow \mathcal{H}^1(B) = \mathcal{O}_q.$$

Therefore, the plane quintic is smooth, and using Lemma 4.4.2, for $E \in \mathrm{Ext}^1(B,A)$ general, we have $\mathcal{H}^0(E) = \mathcal{I}_{L\cup C_5}$ corresponds the connected union of the quintic with L intersecting in a node, and so the curve $L \cup C_5$ is Gorenstein. Thus, we can first apply Lemma 5.3.2 to realize E as a stable pair, and then apply [23, Proposition B.5] to identify the stable pair E with $\mathcal{H}^0(E) = \mathcal{I}_{L\cup C_5}$ with length 2 subschemes of $L \cup C_5$. But from the second short exact sequence above, $\mathcal{H}^1(E)$ is supported at q, and therefore this length 2 subscheme is supported at q. On the other hand, any such subscheme is in the closure of the locus of $\mathcal{H}ilb^{[2]}(L \cup C_5)$ corresponding to one point on L and one point on C_5. This implies that every object in the $\mathbb{P}^{20}$-bundle is in the closure of the $\mathbb{P}^{18}$-bundle over $\mathcal{U} \times \mathfrak{Fl}_1$.

□

Let $\mathcal{N}_5$, be the moduli space for the next chamber. Let $C_4 \subset P$ be a plane quartic and $\mathbb{L}$ a thickening of a line $L \subset P$. The other notations as defined right before Proposition 5.4.1:

Proposition 5.4.2. The moduli space $\mathcal{N}_5$ has seven irreducible components: $\widetilde{\widetilde{\widetilde{\mathcal{N}_2}}}$, $\widetilde{\widetilde{\mathcal{N}'_3}}$, $\widetilde{\mathcal{N}'_4}$, $\widetilde{\mathcal{N}''_4}$, $\mathcal{N}'_5$, $\mathcal{N}''_5$ and $\mathcal{N}'''_5$. The first four are birational to their counterparts in $\mathcal{N}_4$. The component $\mathcal{N}'_5$ is a $\mathbb{P}^{19}$-bundle over $(\mathcal{U} \times_{\mathbb{G}r(2,4)} \mathcal{U}) \times (\mathbb{P}^3)^*$, and it generically parametrizes the union of a line in $\mathbb{P}^3$ together with a choice of two points on it and a plane quintic intersecting the line. The component $\mathcal{N}''_5$ is a $\mathbb{P}^{20}$-bundle over $\mathcal{U} \times (\mathbb{P}^3)^*$, and it generically parametrizes the disjoint unions of a plane quintic and a line in $\mathbb{P}^3$ together with a choice of a point on it. The component $\mathcal{N}'''_5$ is a $\mathbb{P}^{21}$-bundle over $\mathfrak{Fl}_2$, and it generically parametrizes the union of a plane quartic with a thickening of a line in the plane. For $\mathcal{N}'_5$ and $\mathcal{N}''_5$, any generic element has $\mathcal{H}^0$ an ideal sheaf, and $\mathcal{H}^1$ non-zero. In $\mathcal{N}'''_5$, any generic element is of the form $\mathcal{I}_{\mathbb{L}\cup C_4}$.

Proof. The first part comes from Lemmas 4.4.1 and Propositions 4.4.7, 4.5.1 and 5.4.1. The fifth wall on the right side of the hyperbola is $\langle(\mathcal{O}(-1) \xrightarrow{s} \mathcal{O}_L(1)), \mathcal{O}_P(-5)\rangle$ (Theorem 3.1.13). Using Lemma 3.2.7, we get two new components, $\mathcal{N}_5'$ and $\mathcal{N}_5''$, which are a $\mathbb{P}^{19}$-bundle over $(\mathcal{U} \times_{\mathbb{G}r(2,4)} \mathcal{U}) \times (\mathbb{P}^3)^*$ and a $\mathbb{P}^{20}$-bundle over $\mathcal{U} \times (\mathbb{P}^3)^*$.

To understand the objects in the new components more precisely, for any $E \in \operatorname{Ext}^1(B, A)$, we want to understand its image in $\operatorname{Hom}(B, \mathcal{H}^1(A))$ to see for which elements we get a non-zero map. Let $q \cup q'$ be the zero locus of s. We notice we have the short exact sequence $\mathcal{H}^0(A) = \mathcal{I}_L(-1) \to A \to \mathcal{H}^1(A)[-1] = \mathcal{O}_{q\cup q'}[-1]$. Taking $\operatorname{RHom}(\mathcal{H}^0(B) = B, -)$ of this, we have

$$0 \to \operatorname{Ext}^1(B, \mathcal{I}_L(-1)) = \mathbb{C}^{20} \to \operatorname{Ext}^1(B, A) \to \operatorname{Hom}(B, \mathcal{H}^1(A) = \mathcal{O}_{q\cup q'}) \to 0.$$

Now, there are three cases:

(1) $L \not\subset P$ **and zero locus of** s **does not contain** $L \cap P$**.** In this case we have $\operatorname{Ext}^1(B, A) = \mathrm{H}^0(\mathcal{I}_{L\cap P}(5)) = \mathbb{C}^{20}$ (Lemma 3.2.7) and $\operatorname{Hom}(B, \mathcal{H}^1(A)) = 0$, and therefore any class in $\operatorname{Ext}^1(B, A)$ induces zero map in $\operatorname{Hom}(B, \mathcal{H}^1(A))$. Now from

$$\begin{array}{ccccc} \mathcal{H}^0(A) = \mathcal{I}_L(-1) & \longrightarrow & \mathcal{H}^0(E) & \longrightarrow & \mathcal{H}^0(B) = \mathcal{O}_P(-5) \\ \hookrightarrow \mathcal{H}^1(A) = \mathcal{O}_{q\cup q'} & \longrightarrow & \mathcal{H}^1(E) & \longrightarrow & \mathcal{H}^1(B) = 0, \end{array}$$

as the connecting map is zero, $\mathcal{H}^1(E)$ will have length 2, and the top row is a short exact sequence $0 \to \mathcal{I}_L(-1) \hookrightarrow \mathcal{H}^0(E) \twoheadrightarrow \mathcal{O}_P(-5) \to 0$. The extension class

$$\operatorname{Ext}^1(\mathcal{O}_P(-5), \mathcal{I}_L(-1)) = \mathrm{H}^0(\mathcal{I}_{L\cap P}(5))$$

corresponds to quintics containing the intersection point. Using Lemma 4.4.2, this induces an ideal sheaf of the union of a line in $\mathbb{P}^3$ and a plane quintic intersecting the line for $\mathcal{H}^0(E)$, i.e., for a generic choice in $\operatorname{Ext}^1(B, A)$, the sheaf $\mathcal{H}^0(E)$ is the ideal sheaf of the union of a line in $\mathbb{P}^3$ and a plane quintic intersecting the line. This together with the description of $\mathcal{H}^1(E)$ gives the result about the description of the component $\mathcal{N}_5'$.

(2) $L \not\subset P$ **and zero locus of** s **contains** $L \cap P$**.** In this case, we have

$$\operatorname{Ext}^1(B, \mathcal{H}^0(A)) = \mathrm{H}^0(\mathcal{I}_{L\cap P}(5)) \hookrightarrow \operatorname{Ext}^1(B, A) = \mathrm{H}^0(\mathcal{O}_P(5)) \to \operatorname{Hom}(B, \mathcal{H}^1(A)) = \mathbb{C},$$

and therefore any general class in $\operatorname{Ext}^1(B, A)$ induces a non-zero map in $\operatorname{Hom}(B, \mathcal{H}^1(A))$. This time as the connecting map in the above diagram is non-zero, $\mathcal{H}^1(E)$ will have length 1. Similarly as in the previous Lemma, we have $\operatorname{im}(\mathcal{H}^0(E) \to B = \mathcal{O}_P(-5)) = \ker(B = \mathcal{O}_P(-5) \to \mathcal{H}^1(A) = \mathcal{O}_{q\cup q'}) = \iota_{P_*}\mathcal{I}_{L\cap P}(-5)$. Therefore the diagram induces a short exact sequence $0 \to \mathcal{I}_L(-1) \hookrightarrow \mathcal{H}^0(E) \twoheadrightarrow \iota_{P_*}\mathcal{I}_{L\cap P}(-5) \to 0$. This means that the extension class

$$\begin{aligned}\mathrm{Ext}^1(\iota_{P_*}\mathcal{I}_{L\cap P}(-5), \mathcal{I}_L(-1)) &= \mathrm{Ext}^1(\mathcal{I}_{L\cap P}(-5), i_P^*\mathcal{I}_L(-1)(1)[-1]) \\ &= \mathrm{Hom}(\mathcal{I}_{L\cap P}(-5), \mathcal{I}_{L\cap P}) = \mathrm{H}^0(\mathcal{O}_P(5))\end{aligned}$$

corresponds to quintics not necessarily containing the intersection point. This induces an ideal sheaf of the disjoint union of a line in $\mathbb{P}^3$ and a plane quintic for $\mathcal{H}^0(E)$, i.e., for a generic class in $\mathrm{Ext}^1(B, A)$, the sheaf $\mathcal{H}^0(E)$ is the ideal sheaf of the disjoint union of a line in $\mathbb{P}^3$ and a plane quintic. This together with the description of $\mathcal{H}^1(E)$ gives the result about the description of the component $\mathcal{N}_5''$.

(3) $L \subset P$**.** Recall that from 3.2.7, we have $\mathrm{Ext}^1(B, A) = \mathrm{H}^0(\mathcal{O}_L(6)) \oplus \mathrm{H}^0(\mathcal{I}_L(5)) = \mathbb{C}^6 \oplus \mathbb{C}^{15} = \mathbb{C}^{22}$. Therefore in this case, we have a $\mathbb{P}^{21}$-bundle over a $3 + 4 = 7$-dimensional (3 for the choice of a plane and 3 for two points in the plane) locus, which gives a stratum of dimension 28, and therefore it corresponds to a new component, $\mathcal{N}_5'''$. Notice that we have $\mathrm{Ext}^2(B, \mathcal{H}^0(A)) = 0$; thus we have

$$\mathrm{Ext}^1(B, A) \overset{\gamma}{\twoheadrightarrow} \mathrm{Hom}(B, \mathcal{H}^1(A)).$$

Surjectivity of γ implies that a general class in $\mathrm{Ext}^1(B, A)$ induces a surjective connecting map $B \to \mathcal{H}^1(A)$. This means that $\mathcal{H}^1(E) = 0$, for any general element E in the component. Therefore we have $\ker(\text{connecting map}) = \ker(\mathcal{O}_P(-5) \twoheadrightarrow \mathcal{O}_{q\cup q'}) = \mathcal{I}_{q\cup q'/P}(-5)$, and thus any general element E in the component fits into

$$\mathcal{I}_L(-1) \hookrightarrow E \twoheadrightarrow \mathcal{I}_{q\cup q'/P}(-5).$$

Now, let $\mathbb{L}$ be the double line obtained by thickening L, with tangent direction of infinitesimal thickening contained in the plane at $(L\cap C_4)\cup q\cup q'$. From $\mathbb{L}\cup C_4 \subset \mathbb{L}\cup P$, we get

$$\mathcal{I}_L(-1) = \mathcal{I}_{\mathbb{L}\cup P} \hookrightarrow \mathcal{I}_{\mathbb{L}\cup C_4} \twoheadrightarrow \mathcal{I}_{(\mathbb{L}\cup C_4)\cap P/P} = \mathcal{I}_{L\cup C_4\cup q\cup q'/P} = \mathcal{I}_{q\cup q'/P}(-5).$$

On the other hand, considering the composition $E = \mathcal{I}_{\mathbb{L}\cup C_4} \twoheadrightarrow \mathcal{I}_{q\cup q'/P}(-5) \hookrightarrow \mathcal{O}_P(-5)$ gives the exact triangle

$$\big(E \to \mathcal{O}_P(-5)\big) \to E \to \mathcal{O}_P(-5).$$

But $\big(E \to \mathcal{O}_P(-5)\big) \cong \big(\mathcal{O}(-1) \to \mathcal{O}_L(1)\big) = A$: Let $K[1]$ be the cone of the composition $E \to \mathcal{O}_P(-5)$. From the octohedral axiom, we get an exact triangle $\mathcal{I}_L(-1) \to K \to \mathcal{O}_{q\cup q'}[-1]$. But we have $Ext^1(\mathcal{O}_{q\cup q'}[-1], \mathcal{I}_L(-1)) = 0$, and therefore we get $K = \mathcal{O}_{q\cup q'}[-1] \oplus \mathcal{I}_L(-1)$. Then the original exact triangle we started with implies $[E \to \mathcal{O}_P(-5)] \cong \mathcal{O}_{q\cup q'}[-1] \oplus \mathcal{I}_L(-1)$. Therefore $\mathcal{H}^0([E \to \mathcal{O}_P(-5)]) = \mathcal{I}_L(-1) = \mathcal{H}^0(A)$, and $\mathcal{H}^1([E \to \mathcal{O}_P(-5)]) = \mathcal{O}_{q\cup q'} = \mathcal{H}^1(A)$.

Therefore our E also arises via a class in $\mathrm{Ext}^1(B, A)$. This means for a generic class in $\mathrm{Ext}^1(B, A)$, the sheaf E is the ideal sheaf of a plane quartic with a thickening of the line L. This completes the result about $\mathcal{N}_5'''$.

□

Let $\mathcal{N}_6$, be the moduli space for the next chamber. The other notations as defined right before Proposition 5.4.1:

Proposition 5.4.3. The moduli space $\mathcal{N}_6$ has eight irreducible components: $\widetilde{\widetilde{\widetilde{\widetilde{\mathcal{N}_2}}}}$, $\widetilde{\widetilde{\widetilde{\mathcal{N}'_3}}}$, $\widetilde{\widetilde{\mathcal{N}'_4}}$, $\widetilde{\widetilde{\mathcal{N}''_4}}$, $\widetilde{\mathcal{N}'_5}$, $\widetilde{\mathcal{N}''_5}$, $\widetilde{\mathcal{N}'''_5}$, and $\mathcal{N}'_6$. The first seven are birational to their counterparts in $\mathcal{N}_5$. The component $\mathcal{N}'_6$ is a $\mathbb{P}^{21}$-bundle over $\mathfrak{Fl}_6$, and it generically parametrizes plane degree 6 curves together with a choice of 6 points on it. A general element of the new component has $\mathcal{H}^0$ of the form of an ideal sheaf $\mathcal{I}_{C_6}$, where C_6 is a plane sextic curve, and $\mathcal{H}^1 \neq 0$.

Proof. The first part comes from Lemma 4.4.1, and Propositions 4.4.7, 4.5.1, 5.4.1 and 5.4.2. For the new component, we notice that the sixth wall on the right side of the hyperbola is given by $\langle \mathcal{O}(-1), \iota_{P_*}\mathcal{I}^\vee_{Z_6}(-6)\rangle$ (Theorem 3.1.13). Using Lemma 3.2.8, the new component, $\mathcal{N}'_6$, is a $\mathbb{P}^{21}$-bundle over $\mathfrak{Fl}_6$ generically parametrizes plane degree 6 curves together with a choice of 6 points on it. Description of generic element as an ideal sheaf is easily obtained by noticing that $\mathcal{H}^1(\mathcal{O}(-1)) = 0$, and then taking long exact sequence of the defining sequence of any class E in $\mathrm{Ext}^1(B, A)$, and applying Lemma 4.4.2 to $D = \emptyset$. The claim $\mathcal{H}^1$ being non-zero is obtained in a similar way and noticing that $\mathcal{H}^1(\iota_{P_*}\mathcal{I}^\vee_{Z_6}(-6)) \neq 0$.

□

5.5 Chambers on the Hilbert scheme side

In this section, we look at the loci created by walls $\langle \mathcal{I}_{L_2}(-1), \mathcal{O}_P(-5)\rangle$, $\langle \mathcal{I}_{L_1}(-1), \mathcal{I}_{Z_1/P}(-5)\rangle$ and $\langle \mathcal{I}_L(-1), \mathcal{I}_{Z_2/P}(-5)\rangle$. We denote by $\mathfrak{PL}_i$, the parametrizatin space of a plane, a line, and i floating points.

Proposition 5.5.1. The locus $\mathscr{H}_2$ created by the wall $\langle \mathcal{I}_{L_2}(-1), \mathcal{O}_P(-5)\rangle$ is a $\mathbb{P}^{19}$-bundle over $\mathfrak{PL}_2$, of dimension 32 and generically parametrizes the union of a line in $\mathbb{P}^3$ and a plane quintic together with 2 floating points.

Proof. By the proof of Lemma 3.3.1, when $L \not\subset P$, then for the generic case we have

$$\mathrm{Ext}^1(\mathcal{O}_P(-5), \mathcal{I}_{L_2}(-1)) = \mathrm{Hom}(\mathcal{O}_P(-5), \mathcal{I}_p) = \mathbb{C}^{20}.$$

This corresponds to a plane quintic in P containing $p = L \cap P$; this induces a $\mathbb{P}^{19}$-bundle over a locus which paramerrizes a plane, a line and two floating points. Hence the component has dimension $19 + (3 + 4 + 2 \times 3) = 32$. Similarly, the non-generic cases give loci of dimensions $28, 29$ in the closure of the generic case.

On the other hand, when $L \subset P$ and $q, q' \not\subset P$, by the proof of Lemma 3.3.1 we have $\mathrm{Ext}^1(\mathcal{O}_P(-5), \mathcal{I}_{L_2}(-1)) = \mathrm{Hom}(\mathcal{O}_P(-4), \mathcal{O}_P) \oplus \mathrm{Hom}(\mathcal{O}_P(-4), \mathcal{O}_L) = \mathbb{C}^{15} \oplus \mathbb{C}^5 = \mathbb{C}^{20}$. This corresponds to a plane quartic and a line in the plane together with two floating points. This gives a locus of dimension $19 + (3 + 2 + 6) = 30$ in the closure of the generic case.

Similarly, for $q \subset P, q' \not\subset P$ and $q, q' \subset P$, as in Lemma 3.3.1, we get $\mathbb{C}^{21}$ and $\mathbb{C}^{22}$, respectively. The first one correspond to a plane quartic and a line in the plane together with one point on the line and one floating point, which gives a locus of dimension $20+(3+2+2+3) = 30$. The second one correspond to a plane quartic and a line in the plane together with two points on the line, which gives a locus of dimension $21+(3+2+2+2) = 30$. In both cases as we have plane curves, we have local complete intersections, and hence by using [10, Theorem 1.3], we can detach the point from the curve and deform it to two floating points; hence both loci are in the closure of the generic case, i.e., in $\mathscr{H}_2$. □

Proposition 5.5.2. The locus $\mathscr{H}_1$ created by the wall $\langle \mathcal{I}_{L_1}(-1), \mathcal{I}_{Z_1/P}(-5)\rangle$ is a $\mathbb{P}^{20}$-bundle over $\mathfrak{PL}_1$, of dimension 30 and generically parametrizes the disjoint union of a line in $\mathbb{P}^3$ and a plane quintic together with 1 floating point.

Proof. By Lemma 3.3.2, there are three possibilities for $\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1))$, which we only consider generic case of each:

1) $\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1)) = \mathbb{C}^{22}$: in this case, as in the proof of Lemma 3.3.2,

$$\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1)) = \mathrm{Hom}(\mathcal{O}(-5), \mathcal{I}_p) \oplus \mathrm{Hom}(\mathcal{I}_{Z_1}(-5), \mathcal{O}_p \oplus \mathcal{O}_p) = \mathbb{C}^{20} \oplus \mathbb{C} \oplus \mathbb{C} = \mathbb{C}^{22}.$$

This corresponds to a plane quintic in P containing $p = L \cap P$; this induces a $\mathbb{P}^{21}$-bundle over a locus which paramerrizes a plane, a line and a double point in the plane. Hence the component is contained in the closure of $\mathscr{H}_2$, and has dimension $21 + (3 + 4 + 1 \times 2) = 30$.

2) $\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1)) = \mathbb{C}^{21}$: in this case, as in the proof of Lemma 3.3.2,

$$\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1)) = \mathrm{H}^0(\mathcal{O}_P(-5)) = \mathbb{C}^{21}.$$

This corresponds to a plane quintic in P not containing $p = L \cap P$; this induces a $\mathbb{P}^{20}$-bundle over a locus which paramerrizes a plane, a line and a floating point. Hence this is a new component $\mathscr{H}_1$ which is not contained in the closure of $\mathscr{H}_2$, and has dimension $20 + (3 + 4 + 3) = 30$.

3) $\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1)) = \mathbb{C}^{20}$: in this case, as in the proof of Lemma 3.3.2,

$$\mathrm{Ext}^1(\mathcal{I}_{Z_1/P}(-5), \mathcal{I}_{L_1}(-1)) = \mathrm{Hom}(\mathcal{I}_{Z_1}(-5), \mathcal{I}_p) = \mathbb{C}^{20}.$$

This corresponds to a plane quintic in P containing $p = L \cap P$ and not containing Z_1; this induces a $\mathbb{P}^{19}$-bundle over a locus which paramerrizes a plane, a line a double point in the plane and another floating point. Hence the component is contained in the closure of $\mathscr{H}_2$, and has dimension $19 + (3 + 4 + 2 + 3) = 31$.

□

Proposition 5.5.3. The locus $\mathscr{H}'_{CM}$ created by the wall $\langle \mathcal{I}_L(-1), \mathcal{I}_{Z_2/P}(-5)\rangle$ is of dimension 28 and generically parametrizes the union of a plane quartic with a thickening of a line in the plane, which is the same as $\mathcal{N}_5'''$ (Proposition 5.4.2).

Proof. By Lemma 3.3.3, there are three possibilities for $\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1))$, which we only consider generic case of each:

1) $\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1)) = \mathbb{C}^{22}$: in this case, as in the proof of Lemma 3.3.3,

$$\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1)) = \mathrm{Hom}(\mathcal{I}_{Z\cup Z'}(-4), \mathcal{O}_P) \oplus \mathrm{Hom}(\mathcal{I}_{Z\cup Z'}(-4), \mathcal{O}_L).$$

This generically parametrizes the union of a plane quartic with a thickening of a line in the plane, which is the same as $\mathcal{N}_5'''$ in Proposition 5.4.2, which was proved to be irreducible and has dimension $21 + (3 + 2 + 2) = 28$.

2) $\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1))$: in this case, as in the proof of Lemma 3.3.3,

$$\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1)) = \mathrm{Hom}(\mathcal{I}_{Z'}(-5), \mathcal{O}) = \mathbb{C}^{21}.$$

This corresponds to a plane quintic in P not containing $p = L \cap P$; this induces a $\mathbb{P}^{20}$-bundle over a locus which paramerrizes a plane, a line and a point in the intersection and another point in the plane. Hence this is contained in the closure of $\mathscr{H}_1$, and has dimension $20 + (3 + 4 + 2) = 29$.

3) $\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1)) = \mathbb{C}^{20}$: in this case, as in the proof of Lemma 3.3.3,

$$\mathrm{Ext}^1(\mathcal{I}_{Z_2/P}(-5), \mathcal{I}_L(-1)) = \mathrm{Hom}(\mathcal{I}_{Z_2}(-5), \mathcal{I}_p) = \mathbb{C}^{20}.$$

This corresponds to a plane quintic in P containing $p = L \cap P$ and not containing Z_2; this induces a $\mathbb{P}^{19}$-bundle over a locus which paramerrizes a plane, a line, and two points in the plane. Hence the component is contained in the closure of $\mathscr{H}_2$, and has dimension $19 + (3 + 4 + 2 \times 2) = 30$. □

Corollary 5.5.4. $\mathscr{H}_2$ is irreducible.

Proof. We note that Propositions 5.5.2, 5.5.1, and 5.5.3 imply that all the potential loci are contained in the closure of $\mathscr{H}_2$. □

5.6 Space of stable pairs

In this section, we give a full description of (the reduced part of) the moduli space of PT stable pairs by summarizing the results in Previous Chapters. As we have seen in the previous sections this space has 8 irreducible components. Considering the description of chambers in Section 5, we summarize the result in the following Theorem:

Theorem 5.6.1. *The moduli space of PT stable pairs $\mathcal{O}_{\mathbb{P}^3} \to \mathcal{F}$, where the support of $\mathcal{F}$ is on non-hyperelliptic genus 4 curves in $\mathbb{P}^3$ has components birational to the following eight irreducible components (one 24-dimensional, six 28-dimensional, and one 36-dimensional)::*

(1) A $\mathbb{P}^{15}$-bundle over $|\mathcal{O}(2)|$, which generically parametrizes (2,3)-complete intersections,

(2) a $\mathbb{P}^{17}$-bundle over $\mathbb{G}r(2,4) \times \mathfrak{Fl}_2$, which generically parametrizes the union of a line and a plane quintic intersecting the line, together with a choice of two points on the quintic,

(3) a $\mathbb{P}^{18}$-bundle over $\mathcal{U} \times \mathfrak{Fl}_1$, which generically parametrizes the union of a line in $\mathbb{P}^3$ together with a choice of a point on it, and a plane quintic intersecting the line, together with choice of a point on it,

(4) a $\mathbb{P}^{19}$-bundle over $\mathbb{G}r(2,4) \times \mathfrak{Fl}_1$, which generically parametrizes the disjoint union of a line in $\mathbb{P}^3$ and a plane quintic together with a choice of a point on it,

(5) a $\mathbb{P}^{19}$-bundle over $(\mathcal{U} \times_{\mathbb{G}r(2,4)} \mathcal{U}) \times (\mathbb{P}^3)^\vee$, which generically parametrizes the union of a line in $\mathbb{P}^3$ together with a choice of two points on it, and a plane quintic intersecting the line,

(6) a $\mathbb{P}^{20}$-bundle over $\mathcal{U} \times (\mathbb{P}^3)^\vee$, which generically parametrizes the disjoint union of a line in $\mathbb{P}^3$ together with a choice of a point on it, and a plane quintic,

(7) a $\mathbb{P}^{21}$-bundle over $\mathfrak{Fl}_2$, which generically parametrizes the union of a plane quartic with a thickening of a line in the plane.

(8) a $\mathbb{P}^{21}$-bundle over $\mathfrak{Fl}_6$, which generically parametrizes a plane degree 6 curve together with a choice of 6 points on it.

Here, $|\mathcal{O}(2)|$ parametrizes quadric surfaces in $\mathbb{P}^3$, $\mathcal{U}$ is the universal line over $\mathbb{G}r(2,4)$, and $\mathfrak{Fl}_j$ is the space parametrizing flags $Z_j \subset P \subset \mathbb{P}^3$ where P is a plane and Z_j a zero-dimensional subscheme of length j ($j = 1, 2, 6$).

Proof. This is obtained from Propositions 4.4.4, 4.4.7, 5.4.1, 5.4.2, and 5.4.3 and Lemma 4.4.1. We notice that all the loci appearing after each wall are new irreducible components as they have the maximal dimensions and cannot be considered as a subset of the previous components. □

Now, we have the birational description of the components of the intermediate moduli spaces:

Theorem 5.6.2. *Consider a path $\gamma\colon [0,1] \to \mathbb{R}^+ \times \mathbb{R} \subset \mathrm{Stab}(\mathbb{P}^3)$. The image of this path (outside of the walls) describes the moduli spaces of semistable objects with the fixed Chern character $(1, 0, -6, 15)$ for the corresponding chambers which are as follows:*

($\mathcal{N}_0$): the empty space,

($\mathcal{N}_1$): only contains the main component.

($\mathcal{N}_2$): the blow up of $\mathcal{N}_1$ along the smooth locus $(\mathbb{P}^3)^* \times (\mathbb{P}^3)^* \times \mathcal{H}ilb(C_2)$.

($\mathcal{N}_3$): has two irreducible components: $\widetilde{\mathcal{N}_2}$ and $\mathcal{N}_3'$. The component $\widetilde{\mathcal{N}_2}$ is obtained from $\mathcal{N}_2$ by a small contraction followed by a divisorial extraction. The component $\mathcal{N}_3'$ is a new component described in (2) in Theorem 5.6.1.

($\mathcal{N}_4$): has four irreducible components: $\widetilde{\widetilde{\mathcal{N}_2}}$, $\widetilde{\mathcal{N}_3'}$, $\mathcal{N}_4'$ and $\mathcal{N}_4''$. The component $\widetilde{\widetilde{\mathcal{N}_2}}$ is birational to $\mathcal{N}_2$, $\widetilde{\mathcal{N}_3'}$ is birational to $\mathcal{N}_3'$. The components $\mathcal{N}_4'$ and $\mathcal{N}_4''$ are two new components described in (3) and (4) in Theorem 5.6.1, respectively.

($\mathcal{N}_5$): has seven irreducible components: $\widetilde{\widetilde{\widetilde{\mathcal{N}_2}}}$, $\widetilde{\widetilde{\mathcal{N}_3'}}$, $\widetilde{\mathcal{N}_4'}$, $\widetilde{\mathcal{N}_4''}$, $\mathcal{N}_5'$, $\mathcal{N}_5''$, and $\mathcal{N}_5'''$. The component $\widetilde{\widetilde{\widetilde{\mathcal{N}_2}}}$ is birational to $\mathcal{N}_2$, $\widetilde{\widetilde{\mathcal{N}_3'}}$ is birational to $\mathcal{N}_3'$, $\widetilde{\mathcal{N}_4'}$ is birational to $\mathcal{N}_4'$. The

components $\mathcal{N}_5'$, $\mathcal{N}_5''$, and $\mathcal{N}_5'''$ are three new components described in (5), (6) and (7) in Theorem 5.6.1, respectively.

($\mathcal{N}_6$): has eight irreducible components: $\widetilde{\widetilde{\widetilde{\widetilde{\mathcal{N}_2}}}}$, $\widetilde{\widetilde{\widetilde{\mathcal{N}_3'}}}$, $\widetilde{\widetilde{\mathcal{N}_4'}}$, $\widetilde{\widetilde{\mathcal{N}_4''}}$, $\widetilde{\mathcal{N}_5'}$, $\widetilde{\mathcal{N}_5''}$, $\widetilde{\mathcal{N}_5'''}$, and $\mathcal{N}_6'$. The component $\widetilde{\widetilde{\widetilde{\widetilde{\mathcal{N}_2}}}}$ is birational to $\mathcal{N}_2$, $\widetilde{\widetilde{\widetilde{\mathcal{N}_3'}}}$ is birational to $\mathcal{N}_3'$, $\widetilde{\widetilde{\mathcal{N}_4'}}$ is birational to $\mathcal{N}_4'$, $\widetilde{\mathcal{N}_5'}$ is birational to $\mathcal{N}_5'$, $\widetilde{\mathcal{N}_5''}$ is birational to $\mathcal{N}_5''$, and $\widetilde{\mathcal{N}_5'''}$ is birational to $\mathcal{N}_5'''$. The component $\mathcal{N}_6'$ is a new component described in (8) in Theorem 5.6.1.

Proof. This is obtained from Propositions 4.4.4, 4.4.7, 5.4.1, 5.4.2, and 5.4.3, and Lemma 4.4.1. The birational description of the wall between $\mathcal{N}_2$ and $\mathcal{N}_3$ comes from Chapter 4. □

Remark 5.6.3. From Lemmas 3.2.3, 3.2.5, 3.2.6, 3.2.7, 3.2.8, the destabilizing loci for each of the moduli spaces corresponding to a chamber along the path on the right hand side in Figure 3.1 are described as follows:

($\mathcal{N}_1$): There is an 11 dimensional destabilizing locus $(\mathbb{P}^3)^* \times (\mathbb{P}^3)^* \times \mathcal{H}ilb(C_2)$ in $\mathcal{N}_1$.

($\mathcal{N}_2$): There is a destabilizing locus of two strata in $\mathcal{N}_2$: one is $\mathbb{G}r(2,4) \times \mathfrak{Fl}_2$ of dimension 11, and the other is a $\mathbb{P}^1$-bundle over $\mathfrak{Fl}_2$ of dimension 8.

($\mathcal{N}_3$): There is a destabilizing locus of two strata in $\mathcal{N}_3$: one is $\mathbb{G}r(2,4) \times \mathfrak{Fl}_1$ of dimension 9, and the other is a $\mathbb{P}^1$-bundle over $(\mathbb{P}^2)^* \times \mathfrak{Fl}_1$ of dimension 8.

($\mathcal{N}_4$): There is a destabilizing locus of two strata in $\mathcal{N}_4$: one is $\mathcal{U} \times (\mathbb{P}^3)^*$ of dimension 8, and the other is a $\mathbb{P}^1$-bundle over $\mathfrak{Fl}_2$ of dimension 8.

($\mathcal{N}_5$): There is a destabilizing locus of three strata in $\mathcal{N}_5$: one is a 14-dimensional locus parametrizing 6 points on a conic, the second locus is a $\mathbb{P}^1$-bundle over a 12-dimensional locus parametrizing 5 points on a line, and the third locus is a $\mathbb{P}^2$-bundle over a 11-dimensional locus parametrizing 6 points on a line.

Remark 5.6.4. The exceptional locus in $\mathcal{N}_2$ is a $\mathbb{P}^{12}$-bundle over $\mathcal{H}ilb^{2t+1}(\mathbb{P}^3) \times (\mathbb{P}^3)^\vee$ which generically parametrizes union of a plane quartic and a conic intersecting in two points (by Lemma 3.2.3 and Proposition 4.4.7).

Example 5.6.5. We give examples of two objects in the space of PT stable pairs, such that when we move from the large volume limit to the empty set on the right side of the hyperbola, those objects get destabilized at the walls $\langle (\mathcal{O}(-1) \to \mathcal{O}_L(1)), \mathcal{O}_P(-5) \rangle$ and $\langle (\mathcal{O}(-1) \to \mathcal{O}_L), \iota_{P_*}\mathcal{I}^\vee{}_{Z_1}(-5) \rangle$. First we consider $E := \mathcal{O} \to \mathcal{O}_L(1) \oplus \mathcal{O}_{C_5}$ which is defined by a sections of the object $\mathcal{O}_L(1) \oplus \mathcal{O}_{C_5}$, for C_5 a plane quintic. This means that $\mathcal{O}(-1) \to \mathcal{O}_L(1)$ destabilizes E. More precisely, we have (the right column comes from the section $\mathcal{O} \to \mathcal{O}_L(2)$)

$$\begin{array}{ccccc} \mathcal{O}(-1) & \hookrightarrow & \mathcal{O} & \twoheadrightarrow & \mathcal{O}_P \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_L(1) & \hookrightarrow & \mathcal{O}_L(1) \oplus \mathcal{O}_{C_5} & \twoheadrightarrow & \mathcal{O}_{C_5} \end{array}$$

or

$$(\mathcal{O}(-1) \to \mathcal{O}_L(1)) \hookrightarrow E \twoheadrightarrow \mathcal{O}_P(-5),$$

which means that E get destabilized at the walls $\langle (\mathcal{O}(-1) \to \mathcal{O}_L(1)), \mathcal{O}_P(-5) \rangle$. Now we consider $E' := \mathcal{O} \to \mathcal{O}_L \oplus \mathcal{O}_{C_5}(a)$ which is defined by a sections of the object $\mathcal{O}_L \oplus \mathcal{O}_{C_5}(a)$, for C_5 a plane quintic and $a \in C_5$. This means that $\mathcal{O}(-1) \to \mathcal{O}_L$ destabilizes E'. More precisely, we have (the right column comes from the section $\mathcal{O} \to \mathcal{O}_L(1)$)

$$\begin{array}{ccccc} \mathcal{O}(-1) & \hookrightarrow & \mathcal{O} & \twoheadrightarrow & \mathcal{O}_P \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_L & \hookrightarrow & \mathcal{O}_L \oplus \mathcal{O}_{C_5}(1) & \twoheadrightarrow & \mathcal{O}_{C_5}(1) \end{array}$$

or

$$(\mathcal{O}(-1) \to \mathcal{O}_L) \hookrightarrow E' \twoheadrightarrow \iota_{P_*} \mathcal{I}^\vee{}_{Z_1}(-5).$$

(as $\mathcal{H}^0(\mathcal{O}_P \to \mathcal{O}_{C_5}(1)) = \mathcal{H}^0(\iota_{P_*}\mathcal{I}^\vee{}_{Z_1}(-5)) = \mathcal{O}_P(-5)$ and $\mathcal{H}^1(\mathcal{O}_P \to \mathcal{O}_{C_5}(1)) = \mathcal{O}_{Z_1})$. This means that E' get destabilized at the walls $\langle (\mathcal{O}(-1) \to \mathcal{O}_L), \iota_{P_*}\mathcal{I}^\vee{}_{Z_1}(-5) \rangle$.

5.7 The Hilbert scheme

In this section, we describe (the reduced part of) the Hilbert scheme $\mathcal{H}ilb^{6t-3}(\mathbb{P}^3)$ as much as possible.

Proposition 5.7.1. Let $C \subset \mathbb{P}^3$ be a curve of degree 6. Then we have $g_{arith}(C) \leq 10$.

Proof. This is obtained from Castelnuovo inequality. □

This means that a curve of Hilbert polynomial $6t-3$ has an associated Cohen-Macaulay curve of degree 6 and arithmetic genus g_{arith} satisfying $4 \leq g_{arith} \leq 10$, and has $g_{arith} - 4$ floating or embedded points.

Description of the hyperbola as an actual wall. Recall that the hyperbola is given by $\operatorname{Im}(Z_{\alpha,\beta,s}) = 0$, and so to describe the hyperbola as an actual wall, we need to find some objects E such that $\operatorname{Im}(Z_{\alpha,\beta,s}\operatorname{ch}(E)) = 0$.

Lemma 5.7.2. As an actual wall, the hyperbola $\operatorname{Im}(Z_{\alpha,\beta,s}) = 0$ is given by $\langle \mathcal{I}_{C_i}, \mathcal{T}_i[-1] \rangle$ for $1 \leq i \leq 6$, where C_i is a Cohen-Macaulay curve of degree 6 and genus $4+i$, and $\mathcal{T}_i$ is a torsion sheaf of length i.

Proof. For objects with $\operatorname{Im}(Z_{\alpha,\beta,s}\operatorname{ch}(E)) = 0$, semistability doesn't change as s varies; in particular, we can let $s \to +\infty$ and apply Lemma 3.1.5 to $E[1]$. Therefore, $\mathcal{H}^0_\beta(E)$ is $\nu_{\alpha,\beta}$-semistable, and $\mathcal{H}^1_\beta(E)$ is a torsion sheaf $\mathcal{T}_i$ of length i. Notice that we have $\operatorname{ch}(\mathcal{H}^0_\beta(E)) = (1, 0, -6, 15+i)$, which is the Chern class of the ideal sheaf of a genus 4+i sextic curve C_i. By Proposition 5.7.1, the length cannot be more than 6.

□

Finally, we give a description of (the reduced part of) the Hilbert scheme:

Theorem 5.7.3. *The Hilbert scheme $\mathcal{H}ilb^{6t-3}(\mathbb{P}^3)$ has components birational to:*

1) The main component, $\mathscr{H}_{CM}$, which is a $\mathbb{P}^{15}$-bundle over $|\mathcal{O}(2)|$ (24-dimensional),

2) $\mathscr{H}'_{CM}$ which generically parametrizes the union of a plane quartic with a thickening of a line in the plane (28-dimensional),

3) $\mathscr{H}_1$ which generically parametrizes the disjoint union of a line in $\mathbb{P}^3$ and a plane quintic together with 1 floating point (30-dimensional),

4) $\mathscr{H}_2$ which generically parametrizes the union of a line in $\mathbb{P}^3$ and a plane quintic together with 2 floating points, and

5) $\mathscr{H}_6$ which generically parametrizes a plane sextic together with 6 floating points.

The first four components are irreducible.

Proof. Theorem 5.6.1 describes the eight components of the space of stable pairs. We notice that crossing the DT/PT wall (described in Lemma 5.7.2) from the right side to the left side changes the heart from $\mathrm{Coh}^{\alpha,\beta}(\mathbb{P}^3)$ to $\mathrm{Coh}^{\alpha,\beta}(\mathbb{P}^3)[-1]$. The component $\mathscr{H}_{CM}$ (which is birational to the main component) and the component $\mathscr{H}'_{CM}$ (which is birational to $\mathcal{N}_5''')$ survive after crossing this wall: we just need to check this for one object in each; let $\mathcal{I}$ be either an ideal sheaf of a (2,3)-complete intersections or an ideal sheaf of the union of a plane quartic with a thickening of a line in the plane (see Propositions 4.4.4 and 5.4.2). We have $\mathrm{Hom}(\mathcal{I}_{C_i}, \mathcal{I}) = 0$, for C_i a Cohen-Macaulay curve of degree 6 and genus $4+i$, where $1 \leq i \leq 6$. Therefore $\mathcal{I}$ cannot get destabilized at this wall.

We notice that apart from these two components of genus 4 Cohen-Macaulay curves, other six components in the space of stable pairs are destabilized once we cross the DT/PT wall: each object E in those components has $\mathcal{H}^1(E) \neq 0$ (see Propositions 4.5.1, 5.4.1, 5.4.2 and 5.4.3), and the short exact sequence $\mathcal{H}^0(E) \to E \to \mathcal{H}^1(E)[-1]$ destabilizes any object in those components.

Note that the underlying Cohen-Macaulay curves which are parametrized by the generic elements of the components remain the same in both the Hilbert scheme and the space of stable pairs (as the support of $\mathcal{F}$, for a given stable pair $(\mathcal{F}, s)$). Therefore, the new components in the Hilbert scheme generically parametrize ideal sheaves of curves with underlying Cohen-Macaulay curves either the disjoint union of a line in $\mathbb{P}^3$ and a plane quintic, or the union of a line in $\mathbb{P}^3$ and a plane quintic intersecting the line, or a plane sextic. They have 1,2, or 6 floating/embedded points, respectively (see and Propositions 5.5.2, 5.5.1, 5.5.3, 5.4.3). The dimension of $\mathscr{H}_1$ is obtained from Proposition 5.5.2.

The dimensions and irreducibility of $\mathscr{H}_{CM}$ and $\mathscr{H}'_{CM}$ are clear from the stable pairs side. For $\mathscr{H}_1$, there is only one point involved and the underlying curve is a disjoint union of a line and a plane quintic which is a locally complete intersection curve of codimension 2; hence [10, Theorem 1.3] implies that we can always detach the point from the underlying curve in this case. Therefore $\mathscr{H}_1$ is irreducible.

As for $\mathscr{H}_2$, Corollary 5.5.4 implies the irreducibility. □

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