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Two Models of Institutional Design in Crime and Education

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September 26, 2022
For my nana AVM Surinder Singh, who taught me to wrestle and to be persistent.
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Edinburgh
06 December 2021
Chapter 1

Introduction

This thesis consists of two chapters looking at elements of optimal institutional design. In Chapter two I present a model of Crime Deterrence.

One of the most ubiquitous aspects of modern society is crime. Adherence to the law is an important component of the social contract. Not impinging on the rights of others is the price all citizens pay for the protection of their own rights.

I consider the problem of choosing rewards and punishments to incentivize adherence to the law and deter crime. From the point of view of the policy maker this is a moral hazard problem. Whether or not someone has committed a crime can never be perfectly observed. This necessarily implies that some criminals may be falsely acquitted and some innocent citizens falsely convicted. This fact must necessarily temper the magnitudes of both rewards and punishments as well as the level of suspicion regarding unobserved citizens. This is similar to such classic moral hazard problems as of providing insurance when the level of caution is unobserved or of choosing wages for the manager of a firm when their effort is unobservable. However, unlike most classic moral hazard problems, crime deterrence
pertains to multiple agents whose actions are unobservable.

The first thing I do in this paper is extend the workhorse moral hazard model of Grossman and Hart [1983] to consider the scenario with many agents. I find that the optimal contracts chosen in this scenario are identical to those in the single agent case in every important respect. My framework considers a very simple form of crime. I use the example of fare evasion on a train. The train firm would like everyone to buy a ticket however they can’t monitor everyone’s action. They choose rewards for agents who they find bought tickets, punishments for those caught evading and rewards/punishments for those whom they are unable to monitor. These must be such that every traveller would like to buy a ticket rather than risk getting caught.

Another key difference between crime and many of the other issues studied in the moral hazard literature is that of externalities. The actions of any criminal agent induces externalities on other agents. In a society with a large amount of crime, individual criminals may be hard to find, and vice versa. This may induce various agents to collude with each other to commit crimes together. Furthermore, there may be complementaries between different agents, i.e. an agents payoffs from crime may be increasing in the intensity of crime in society. This too would induce collusion. If the policy maker fails to take this into account when choosing rewards and punishments, they may fail to deter crime efficiently.

I explore the implications of this by allowing for collusion in my model. Due to the monitoring structure I choose, there are very strong externalities that arise from an agents action. If an agent chooses not to buy a ticket, they make it less likely that any other agent gets fined and conversely, buying a ticket makes it more likely
for other agents to be rewarded. Given this, it is possible for agents to collude with each other. I demonstrate that the scope of collusion changes the magnitudes of rewards and punishments, but that the nature of the optimal contracts remains unchanged. Implying that, fixing the number of agents, if the scope for collusion exists, it is costlier to deter wrongdoing.

The above conclusion leads naturally to the question of how the number of agents affects the nature and magnitude of the optimal incentives. This is the final thing I examine in this paper. I find that the magnitude of the incentives must increase as the number of agents increases.

The third chapter deals with the question of how agents make decisions in a dynamic framework when their preferences are reference dependent.

We consider the case of a student who must choose how much effort to put into studying for a course that is evaluated on the basis of two exams. The student chooses how much effort to put in to studying for the mid term, observes the midterm score and then chooses how much effort to put in to studying for the final.

The result depends on the ability of the student, the effort they put into studying as well as luck. The student initially knows only the parameters of the distribution from which their ability is drawn, not the ability itself. They also know the distribution of the random component.

We have used a considerably simplified version of aspirations in our model. We assume that exerting greater effort leads to higher aspirations, i.e. that aspirations
are a function of effort. This seems reasonable on the grounds that someone who has worked very hard would feel disappointed with a low result.

The utility framework we consider has two components. The first component which we term the ‘achievement effect’ depends on the score on the exam only. Higher scores give higher utility. The second component, the ‘aspiration effect’ is reference dependent, and depends on the ‘aspiration gap’ i.e. the difference between the aspiration and the actual result. This framework admits two possibilities, that the student is disappointed (a negative aspiration gap) or that they are elated (a positive aspiration gap).

This paper looks at what happens in the second period, once the result of the midterm is observed. We begin by deriving the students newly induced distribution over their ability. We assume that everything is normally distributed, for computational ease as well as because this seems like a reasonable assumption. We use a reference dependent utility function. We then set up and solve an expected utility maximization problem on the basis of these beliefs.

We find that the optimal effort is unique, given parameter values. We compare the choice of optimal effort in our reference dependent framework with one that would arise from a ‘fully rational’ set up with no reference dependence. We find that the effect of aspirations on the optimal effort depends entirely on whether exerting more effort grows aspirations faster or (expected) results.

We then investigate the effect of a better midterm result on the optimal effort in the second exam. We find that it is almost always the case that a better first result leads to decreased effort. So, in the majority of circumstances, a fear of
failure trumps ambition.

The cases when the opposite is true are quite interesting as well. A better first result only induces greater effort if two conditions are true:
Firstly, the marginal expected aspirations gap must be positive, i.e. it must be the case that exerting effort increases the expected result more than it increases the aspiration. Secondly, the aspiration gap must be \textbf{small and negative}. So, if the aspiration induced by the optimal effort is slightly higher than the expected result then a better first result would induce more effort. Another interesting finding is that there is a magnitude effect. The higher the aspiration, the larger the set of aspiration gaps that correspond to increased optimal effort.
Chapter 2

Deterring Many Colluding Criminals

Abstract

I present a model of Crime Deterrence. I consider the problem of choosing rewards and punishments to incentivize adherence to the law and deter crime. From the point of view of the policy maker this is a principal-(many) agent problem with moral hazard and collusion. I first set up and solve the no-collusion principal-(many) agent problem which I use as a baseline for the model with collusion. I find that holding fixed the number of agents, the possibility of collusion makes the optimal punishments(or rewards) more severe. I then consider what happens to the optimal punishments(or rewards) when we change the number of agents. A larger group of agents implies more severe punishments(or rewards) suggesting that crime policy based on the principle of marginal deterrence becomes increasingly costly(and hence undesirable) as the pool of potential offenders grows.
2.1 Introduction

2.1.1 Introduction

One of the most ubiquitous aspects of modern society is crime. Adherence to the law is an important component of the social contract. Not impinging on the rights of others is the price all citizens pay for the protection of their own rights. The question of why a crime is committed is of course extremely complicated and multifaceted and a complete examination would look into various aspects relating to agency, equality, access, psychology etc. The most important consequence of the nuanced nature of crime is that the motivation behind a crime must necessarily be unique to the criminal, implying that the impact of any one size fits all policy is unpredictable. Owing to the large size of a typical society however, there may nonetheless be some merit in considering fairly general crime deterrence policies and relying on the law of large numbers to ensure that they are effective on average. This paper considers crime deterrence in precisely that spirit.

I consider the problem of choosing rewards and punishments to incentivize adherence to the law and deter crime. From the point of view of the policy maker this is a moral hazard problem. Whether or not someone has committed a crime can never be perfectly observed. This necessarily implies that some criminals may be falsely acquitted and some innocent citizens falsely convicted. This fact must necessarily temper the magnitudes of both rewards and punishments as well as the level of suspicion regarding unobserved citizens. This is similar to such classic moral hazard problems as of providing insurance when the level of caution is unobserved or of choosing wages for the manager of a firm when their effort is unobservable. However, unlike most classic moral hazard problems, crime deterrence pertains to multiple agents whose actions are unobservable.
The first thing I do in this paper is extend the workhorse moral hazard model of Grossman and Hart [1983] to consider the scenario with many agents. I find that the optimal contracts chosen in this scenario are identical to those in the single agent case in every important respect. My framework considers a very simple form of crime. I use the example of fare evasion on a train. The train firm would like everyone to buy a ticket however they can’t monitor everyone’s action. They choose rewards for agents who they find bought tickets, punishments for those caught evading and rewards/punishments for those whom they are unable to monitor. These must be be such that every traveller would like to buy a ticket rather than risk getting caught.

Another key difference between crime and many of the other issues studied in the moral hazard literature is that of externalities. The actions of any criminal agent induces externalities on other agents. In a society with a large amount of crime, individual criminals may be hard to find, and vice versa. This may induce various agents to collude with each other to commit crimes together. Furthermore, there may be complementaries between different agents, i.e. an agent’s payoffs from crime may be increasing in the intensity of crime in society. This too would induce collusion. If the policy maker fails to take this into account when choosing rewards and punishments, they may fail to deter crime efficiently.

I explore the implications of this by allowing for collusion in my model. Due to the monitoring structure I choose, there are very strong externalities that arise from an agent’s action. If an agent chooses not to buy a ticket, they make it less likely that any other agent gets fined and conversely, buying a ticket makes it more likely for other agents to be rewarded. Given this, it is possible for agents to collude
with each other. I demonstrate that the scope of collusion changes the magnitudes of rewards and punishments, but that the nature of the optimal contracts remains unchanged. Implying that, fixing the number of agents, if the scope for collusion exists, it is costlier to deter wrongdoing.

The above conclusion leads naturally to the question of how the number of agents affects the nature and magnitude of the optimal incentives. This is the final thing I examine in this paper. I find that the magnitude of the incentives must increase as the number of agents increases.

The rest of the paper is structured as follows. I review the relevant literature in section 2.1.2 and relate my framework and results to it. In section 2.2 I introduce the no collusion model and solve it. In section 2.3 I discuss the idea of collusion generally, before introducing it into the model and solving it. I then consider the effect that adding in more agents has on the incentives in section 2.4.

## 2.1.2 Literature Review

This paper contributes to a few different literatures. It contributes to the literature on moral hazard by extending the workhorse principal agent model from Grossman and Hart [1983] to consider the case with multiple homogeneous agents. This paper also relates to many papers on law enforcement and crime deterrence by discussing how to choose punishments when there are externalities in criminal activity. It also discusses the impact of collusion between agents on the nature of the optimal contract as well as the implications on efficiency.

The modelling framework I have adopted is based on that of Grossman and Hart [1983]. They showed how to solve the contracting problem between a (potentially
risk averse) principal and a single risk averse agent who takes an unobservable action. The key change that I have made is that I consider the case of many agents. They split the moral hazard problem of choosing an optimal action (profile) and transfers to implement it into two separate problems and solve it using concave programming. I have abstracted away the optimal action problem by choosing one particular action profile to incentivize. The action profile I focus on is the one where everyone takes the principal’s preferred action. I focus exclusively on the problem of choosing optimal transfers to implement that action profile. Other differences between my problem and that of Grossman and Hart [1983] are that I consider a binary action and my agents have additively separable utility. My model also has a fairly specialized monitoring structure.

The principal-many agent model has also been studied by Mookherjee [1984], Ma et al. [1988], Ma [1988] and others. My benchmark no collusion model is quite similar to all of the above. It is possible to derive my results as a special case of Mookherjee [1984]. The key differences between my main model and the one in Mookherjee [1984] is that I have assumed that the action profile is common knowledge among the agents and that the principal chooses each agents transfers on the basis of their own signal, as opposed to the entire signal profile. Ma et al. [1988] and Ma [1988] both consider the problem of choosing incentives to induce a particular action profile using different equilibrium concepts such as Nash Equilibrium or subgame perfect equilibrium. The mechanisms presented in the cited papers are prone to manipulation by groups of agents, which is what motivates this chapter.

This paper also contributes to the literature on coalitions and equilibrium refinements to make incentives robust to cooperation and deviations from desired
outcomes by groups of agents. Papers such as Aumann [1960] and Bernheim et al. [1987] are concerned with proposing equilibrium concepts that apply to a wide set of games and make them robust against deviations by coalitions. The goal of this paper is different. We propose a particular game and show that the Nash equilibrium is vulnerable to coalitional deviations.

This paper also draws on the literature on collusion. The form of collusion I have used is quite similar to that of Tirole [1986] in that I assume that all side contracts must be Pareto optimal and that I use a similar methodology to set up the collusion problem and solve it. My paper differs from his in that in my paper the (side) contracting parties all have the same information and so I focus on the hidden action problem only. Furthermore my model only contains a principal and agents, not a supervisor. My model does have a monitor, however this is merely a personification of the principal’s information generating technology and the monitors incentives are aligned with the principal’s. In this respect my paper also differs from a number of other papers about collusion such as Laffont and Martimort [1997], Baliga and Sjöström [1998] and Ortner and Chassang [2018].

This paper also contributes to the literature on the economics of crime. The prescriptions I make are along the lines of the theory of deterring the marginal criminal along the lines of Becker [1968]. Following Becker [1968], Ehrlich [1973] wrote a model that provided insights into criminal behaviour on the bases of the state preference approach to behaviour under uncertainty. They tested their theory against data on variations in index crimes across states in the US. Their findings indicate the existence of a deterrent effect of law enforcement activities on all crimes. This acts as the justification for the efficacy of the prescriptions of my model: i.e. that they are likely to work. Kemp and Ng [1979] derive optimality conditions for the
amount of resources to devote to law enforcement. Their optimal expenditure solution is of the same type as ours in that it is proportional to the ratio of the cost of offending and the probability of capture. Similar problems have been considered innumerable times in the literature and the solutions tend to be conformable with my own. Recent examples include [Ortner and Chassang 2018, Varas et al. 2020] and [Banerjee et al. 2019].

2.2 Main

I begin by introducing some notation and discussing the model environment in some detail.

2.2.1 Model Preliminaries

There are $n$ agents (train travellers). Agent $i$ chooses an action $a_i \in \{0, 1\}$ where $a_i = 1$ implies they didn’t commit a crime i.e. they bought a ticket. $a$ (no subscript) is an action profile and $a_{-i}$ is the profile of all agents except agent $i$. The principal issues signal contingent transfers $\{t_i(\theta)\}_{\theta \in \Theta}$. The signals are discussed a little later. Agents $i$’s utility is

$$U(a_i, t(\theta_i)) = -a_i c + W(t(\theta_i)). \quad (2.1)$$

I have assumed that the agents are risk averse implying that $W(\cdot)$ is increasing and concave. $c$ is the cost of the ticket. I have ignored the possibility of treating this as a policy instrument for now.

The signal about each agent is relayed to the principal by a monitor (the ticket collector). The monitor checks agents one by one at random and stops checking
once she finds one agent guilty of fare evasion. This assumption is justifiable, as the agents can disembark from the train to avoid being checked in the time the monitor takes to issue a fine. This is especially true in the context of commuter trains as the comparatively short gap between stops makes escaping easy.

As a consequence of this restriction three possible signals may arise about a particular agent. They may be checked and found guilty, checked and exonerated or not checked at all. These can be thought of as the ‘bad’, ‘good’ and ‘null’ signals respectively.

The signal profile observed by the principal is $\theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ where $\theta_i$ is the signal that the monitor sent about agent $i$. $\theta_i \in \{0, 1, \emptyset\}$, where like the action, $\theta_i = 1$ is the good signal.

Let us discuss each signal in turn. Agent $i$ is given the signal $\emptyset$ and the transfer $t(\emptyset)$ if they are not monitored, i.e. if the ticket collector has already penalized an agent and stopped checking before they reached agent $i$. Since each agent’s monitoring probability depends only on the report sent by the ticket collector and actions of the other agents and not on their own, the probability of receiving the signal $\emptyset$ is the same whether or not agent $i$ buys a ticket.

If the agent is monitored, which signal they get depends on two things: whether or not they bought a ticket and whether or not the ticket collector made a mistake\(^1\). With some probability, $\varepsilon < \frac{1}{2}$ the ticket collector can make a mistake and

\(^1\)The reason for including these error probabilities is to rule out zero probability events as their existence allows the principal to implement the forcing contract. See Bolton and Dewatripont 2005
penalize someone who bought a ticket. Similarly, with probability $(1 - \delta) < \frac{1}{2}$, someone who didn’t buy a ticket may be accidentally let off by the ticket conductor.

Hence the probability of observing each possible signal about some agent can be split into two components. The first component is the probability of being monitored by the ticket collector. The second component corresponds to whether or not the ticket collector makes a mistake. The signal probabilities are given below. Let $P(m(i)|a_{-i})$ be the probability of agent $i$ being monitored by the ticket collector.

\[
P(\theta_i = 0|a_i = 0, a_{-i}) = P(m(i)|a_{-i})\delta \quad (2.2)
\]
\[
P(\theta_i = 1|a_i = 0, a_{-i}) = P(m(i)|a_{-i})(1 - \delta) \quad (2.3)
\]
\[
P(\theta_i = 0|a_i = 1, a_{-i}) = P(m(i)|a_{-i})\varepsilon \quad (2.4)
\]
\[
P(\theta_i = 1|a_i = 1, a_{-i}) = P(m(i)|a_{-i})(1 - \varepsilon) \quad (2.5)
\]
\[
P(\theta_i = \emptyset|a_i = 0, a_{-i}) = P(\theta_i = \emptyset|a_i = 1, a_{-i}) = 1 - P(m(i)|a_{-i}) \quad (2.6)
\]

### 2.2.2 The No Collusion Model

In this section I set up the principal’s problem in the case when agents can’t collude. The conditions established will be a simple binary action version of the classic Grossman and Hart [1983] framework extended to multiple identical agents. This will serve as a benchmark for the case when collusion is possible.

\footnote{The sample space and fundamental events for this are discussed in greater detail in Appendix A.1}
The principal’s objective is to choose an optimal action profile along with transfers that implement it to maximize the total welfare:

\[ V(a) + \sum_{\theta} \sum_{i=1}^{n} \lambda U(a_i, t(\theta_i))P(\theta|a_i, a_{-i}) - t(\theta_i)P(\theta|a_i, a_{-i}). \]

That is, the sum of the agent’s expected utilities, the benefit to the principal and the expected cost of inducing the action profile chosen. \( V(a) \) is the net benefit that the principal gets from agents purchasing tickets and \( \lambda \) is the weight that the principal places on the sum of agent’s utilities.

In order to implement action \( a \), the principal’s chosen transfers must satisfy \( n \) incentive compatibility constraints. For each agent, the expected utility from acquiescing and playing the principal’s preferred action, \( a_i \), must exceed the expected utility from ‘defecting’ and playing \( \neg a_i \), the opposite action. Therefore, the constraint for agent \( i \) is

\[ -a_i c + \sum_{\theta} W(t_i(\theta))P(\theta|a_{-i}, a_i) \geq -\neg a_i c + \sum_{\theta} W(t_i(\theta))P(\theta|a_{-i}, \neg a_i). \]

The constraint set would ordinarily include one such constraint for every agent and the principal would have to choose a bespoke transfer for each agent. However, because all the agents are identical, a single constraint will suffice as transfers that are incentive compatible for one agent are incentive compatible for all. We now drop the \( i \) subscript on everything except the signal as this is now the only factor that differs from agent to agent.

As discussed above, the principal may like to choose both the optimal action and transfers that implement it. However, in this paper I will ignore the optimal action problem and focus only on the problem of optimally selecting transfers to implement some fixed action profile; in this case the action profile \( 1_n \), that is, \( a_i = 1 \ \forall i \). This is, to an extent justified as it’s safe to assume that the train
company would like every passenger to buy a ticket. The problem to be solved then is,

$$\max_{t(0), t(1), t(\emptyset)} V(a) + \sum_{i=1}^{n} -\lambda c + \sum_{\theta} \left[ (\lambda W(t(\theta_i)) - t(\theta_i)) P(\theta_i | 1_n) \right]$$  \hspace{1cm} (O)$$

subject to,

$$\sum_{\theta} W(t(\theta_i)) \left[ P(\theta_i | a_i = 1, 1_{n-1}) - P(\theta_i | a_i = 0, 1_{n-1}) \right] - c. \geq 0$$ \hspace{1cm} (IC)$$

This formulation is isomorphic to another formulation used by a number of other papers including Grossman and Hart [1983]: The one where the objective doesn’t include the agent’s utility, instead including a participation constraint for every agent. My formulation makes more sense in the context of crime as it isn’t very intuitive to consider the possibility of agents opting out of participating in society. On the other hand it stands to reason that the principal in such a problem would be concerned about the agent’s utility and would not like to punish them too harshly.

I now examine the first order conditions of the problem. Let us first consider the FOC with respect to to $t(\theta)$ generically. Let the optimal contract in the no collusion case be $\{t^{nc}(\theta)\}_{\theta \in \Theta}$. Let $\mu$ be the Lagrange multiplier on the IC constraint for agent $i$.

$$F.O.C \ t(\theta) : \frac{1}{W'(t^{nc}(\theta))} = \lambda + \mu \left[ 1 - \frac{P(\theta | a = (0, 1_{n-1}))}{P(\theta | a = 1_{n})} \right]$$  \hspace{1cm} (2.7)$$

Equation (2.7) shows that the transfers that the principal offers depend on the likelihood of the observed signal arising from the agent doing their preferred action ($a_i = 1$). For the transfers to make intuitive sense, it should be the case that they increase in this likelihood, as this would mean that the incentives line up with the principal’s objective.
The FOC’s with respect to the actual transfers shed further light on this.

\[
\text{F.O.C } t(0) \quad \frac{1}{W'(t_{nc}(0))} = \lambda + \mu \left[ 1 - \frac{\delta}{\varepsilon} \right] \quad (2.8)
\]

\[
\text{F.O.C } t(1) \quad \frac{1}{W'(t_{nc}(1))} = \lambda + \mu \left[ 1 - \frac{1 - \delta}{1 - \varepsilon} \right] \quad (2.9)
\]

\[
\text{F.O.C } t(\emptyset) \quad \frac{1}{W'(t_{nc}(\emptyset))} = \lambda \quad (2.10)
\]

Since \( W(\cdot) \) is increasing and concave, \( W'(t(\cdot)) \) is a decreasing function and \( \frac{1}{W'(t(\cdot))} \) is an increasing function. This implies that \( t_{nc}(1) > t_{nc}(0) \) whenever

\[
1 - \frac{1 - \delta}{1 - \varepsilon} > 1 - \frac{\delta}{\varepsilon}.
\]

The above condition is met so long as \( \varepsilon < \delta^3 \), i.e. as long as the ticket conductor is not absolutely awful at her job. Thus the principal’s incentive policy is quite sensible.

Since the problem is only subject to one constraint, we know from the Kuhn-Tucker theorem that the constraint binds at the optimum, hence we can rearrange the constraint to get

\[
W(t_{nc}(1)) - W(t_{nc}(0)) = \frac{c}{(\delta - \varepsilon)P(m|(0_0, 1_{n-1})}) \quad (2.11)
\]

Since everything on the right hand side of \( (2.11) \) is positive, the ‘good’ transfer is greater than the bad transfer. The difference between the transfers can be thought of as the severity of the punishment(or the richness of the reward). This is increasing in the cost of the ticket. This means that the higher the cost of the ticket, the stronger the incentives need to be. The right hand side is in decreasing in the probability of being monitored. So, the harder it is to find offenders, the more severe the punishments are.

\[\text{this is true by assumption}\]
2.3 Collusion

2.3.1 Introduction

Having established the benchmark case with no collusion, I can now turn to the collusion problem. Scope for collusion exists if it is possible for agents to gain utility by mutually agreeing to take actions that aren’t in the optimal profile that the principal wants. If this scope exists then the transfers chosen by the principal in the way above, under the assumption that agents act independently, will not induce the optimal action profile as every agent can gain utility by colluding and not buying a ticket. I will begin by discussing the nature of collusion in my model before going on to find the optimal collusion proof contract.

2.3.2 Side Contracts

Collusion among agents can generally be represented by a side contract which consists of three components. One component is the side contracted action. We know that since the agents are all identical that no coalition would form where both actions exist in the contracted action profile. I will show below, on the basis of Theorem 1, that the side contracted action will be to collectively defect and choose not to buy a ticket.

The second important thing to consider is side payments. These are transfers between colluding agents, to incentivize them to take the side contracted action. The first case I will consider is the case without side payments, Where agents mutually agree not to buy a ticket. I will show that this is optimal and that no agent has any incentive to defect from this contract. Another type of side contract that it is natural to examine is the insurance contract- where every agent pays money into
a common pool and any agent that gets the bad signal gets reimbursed. This will be examined in future work.

The final thing to consider is the coalition. An agent may wish to collude with only one other agent, multiple other agents or even every other agent on the train. Since every agent is the same; they pay the same ticket price, face the same probability of being monitored and receive the same signal dependent transfers; no agent has any specific preferences over which agent to collude with. However, since the probability of being monitored by the ticket collector depends on the number of other agents who buy a ticket (see Lemma 1), the question of how many agents to include in the side contract is non-trivial.

2.3.3 Collusion Without Side Payments

In this section I discuss the optimal contracting problem in the face of collusion. The train company still wants to induce every single agent to buy a ticket, however, in this case the incentives must be strong enough to discourage not only individual deviations, but also group deviations from the contracted action. As discussed in the last section, it is not immediately clear what kinds of coalitions agents would like to enter in. This is the first question I tackle. I demonstrate that the profitability of deviating and not buying a ticket increases when a larger group of passengers also deviate. This implies that the efficient coalition is the Grand Coalition- containing every single agent. I subsequently discuss the problem that the principal must solve and its solution.

Lemma 1. An agent’s probability of being monitored is increasing in the number of other agents that have a ticket

The ticket collector, who is generally good at her job (she makes mistakes with low
probabilities $\varepsilon$ and $1 - \delta$), checks agents until she finds someone ticketless. An agent is only checked if all the agents checked before him receive the good signal. As a result, more agents having tickets means that there is a greater likelihood that the agents checked before all had tickets and were cleared and hence a greater likelihood of getting checked. A formal proof of this lemma is available in the appendix.

This lemma has one very important implication, i.e. that the efficient coalition is the grand coalition - consisting of all the agents. This implication is proved as the first step to proving theorem [1].

2.3.4 Optimal Collusion Proof Contract

The principal must now choose transfers that induce each agent to buy a ticket instead of colluding with the other agents and defecting. The principal’s objective remains unchanged, they still want to choose transfers to induce the action profile $a = 1_n$. The constraint set however is a very different.

It is reasonable to expect that it would be very different to the no collusion case. The transfers chosen must satisfy a large set of constraints: one for each possible agent, both in and out of every possible coalition. However, using lemma [1] it is possible to considerably reduce the scope of the problem.

I will place some restrictions on the coalition formation process. I assume that only one coalition can form at a time. For e.g. if $x$ agents decide to form a coalition then the other $n - x - 1$ agents by assumption are forced to act individually. Let $\Gamma_x$ be a coalition of $x$ agents, where $0 \leq x \leq n$. There are two types of constraints that apply for every value of $x$: one for the members of the coalition, and one for every other agent. The reason for this is that the composition of the action profile
of every other agent, $a_{-i}$, varies depending on whether or not agent $i$ is a member of the coalition. The profile $a_{-i}$ can be represented in the following way.

- $a_{-i} = (a_{\Gamma-x}, a_{-\Gamma_x})$ if agent $i$ is in the coalition.
- $a_{-i} = (a_{\Gamma_x}, a_{-\Gamma-x})$ if agent $i$ is not in the coalition.

When agent $i$ is in the coalition, $a_{\Gamma-x}$ is the action profile of the $x - 1$ other collusive agents and $a_{-\Gamma_x}$ is the profile of the $n - x$ non collusive agents. Similarly, when agent $i$ is not in the coalition, $a_{\Gamma_x}$ is the profile of the $x$ collusive agents and $a_{-\Gamma-x}$ is the profile of the $n - x - 1$ other non collusive agents.

On this basis, we can formulate the incentive compatibility constraints for the coalition $\Gamma_x$ as follows:

$$E_{\theta} U(a_i = 0, t(\theta_i), (a_{\Gamma-x}, a_{-\Gamma_x})) \leq E_{\theta} U(a_i = 1, t(\theta_i), (a_{\Gamma-x}, a_{-\Gamma_x}))$$  \hspace{1cm} (2.12)

$$E_{\theta} U(a_i = 0, t(\theta_i), (a_{\Gamma_x}, a_{-\Gamma-x})) \leq E_{\theta} U(a_i = 1, t(\theta_i), (a_{\Gamma_x}, a_{-\Gamma-x}))$$  \hspace{1cm} (2.13)

The above constraints induce agents who are and aren’t in the coalition, respectively, to choose the good action, taking as given what the other agents are doing. The constraint set consists of these two constraints\(^4\) for all $n$ coalitions. The reason is that the transfers chosen by the principal must deter all possible coalitions.

These constraints aren’t currently in a form that lends itself to analysis well. The transfers chosen depend on the precise composition of $a_{-i}$ which depend on what agent $i$ believes the other agents will do. In order to proceed with the analysis it would help to be explicit about the profile $a_{-i}$. Since the agents are

\(^4\)Except in the cases where $x = 0$ and $x = n$ as the counter factual constraint in either case pertains to zero members
all homogeneous, the identity of those taking the good or the bad action doesn’t matter. What is relevant is the number of ones and zeros in the profile. It is thus desirable to represent a profile in the following way:

\[ a_{-i} = (0_p, 1_{n-p-1}), \]

implying that the profile contains \( p \) zeros and \( n - p - 1 \) ones.

I first consider the beliefs of an agent who is in the coalition. For the chosen transfers to be truly collusion proof, the incentives chosen must be strong enough to induce the agent to leave the coalition by choosing the good action even if they are the only member doing so. Their beliefs about the non-collusive agents is that they will take the incentive compatible action. Similarly, an agent who isn’t in the coalition assumes that the other non-collusive agents take their incentive compatible action and the collusive agents take the side contracted action. We can then represent the action profiles as

- \( a_{-i} = (a_{\Gamma^-x}, a_{-\Gamma^x}) = (0_{x-1}, 1_{n-x}) \) if agent \( i \) is in the coalition.
- \( a_{-i} = (a_{\Gamma^x}, a_{-\Gamma^-x}) = (0_x, 1_{n-x-1}) \) if agent \( i \) is not in the coalition.

and the constraints as

\[
\mathbb{E}_\theta \left( U(a_i = 0, t(\theta_i), (0_{x-1}, 1_{n-x})) \right) \leq \mathbb{E}_\theta \left( U(a_i = 1, t(\theta_i), (0_{x-1}, 1_{n-x})) \right) \quad (\Gamma_x : C)
\]

\[
\mathbb{E}_\theta \left( U(a_i = 0, t(\theta_i), (0_x, 1_{n-x-1})) \right) \leq \mathbb{E}_\theta \left( U(a_i = 1, t(\theta_i), (0_x, 1_{n-x-1})) \right) \quad (\Gamma_x : NC)
\]

Where \( C \) and \( NC \) reflect the fact that the constraint applies to collusive and non-collusive agents respectively.
We now move on to a discussion of the entire constraint set. This discussion will consist of the following theorem and its proof.

**Theorem 1.** The constraint set for the problem of optimal collusion proof contracting consists of a single constraint, the one pertaining to the grand coalition:

\[ \mathbb{E}_\theta \left( U \left( a_i = 0, t(\theta_i), (0_{n-1}, 1_0) \right) \right) \leq \mathbb{E}_\theta \left( U \left( a_i = 1, t(\theta_i), (0_{n-1}, 1_0) \right) \right) \quad (\Gamma_n : C) \]

*Proof.* The proof proceeds by stating the constraints for all the possible coalitions and showing that all but one of them are redundant.

I denote by \( \Gamma_n \), the grand coalition consisting of all the agents and by \( \Gamma_0 \) the null coalition, i.e the case when no coalitions form. In both of these cases only one constraint applies, as the counter-factual set of agents is empty.

The constraint set can be represented by the following series of constraints.

\[ \mathbb{E}_\theta \left( U \left( a_i = 0, t(\theta_i), (0_{n-1}, 1_0) \right) \right) \leq \mathbb{E}_\theta \left( U \left( a_i = 1, t(\theta_i), (0_{n-1}, 1_0) \right) \right) \quad (\Gamma_n : C) \]
\[ \mathbb{E}_\theta \left( U \left( a_i = 0, t(\theta_i), (0_{n-1}, 1_0) \right) \right) \leq \mathbb{E}_\theta \left( U \left( a_i = 1, t(\theta_i), (0_{n-1}, 1_0) \right) \right) \quad (\Gamma_{n-1} : NC) \]
\[ \mathbb{E}_\theta \left( U \left( a_i = 0, t(\theta_i), (0_{n-2}, 1_1) \right) \right) \leq \mathbb{E}_\theta \left( U \left( a_i = 1, t(\theta_i), (0_{n-2}, 1_1) \right) \right) \quad (\Gamma_{n-1} : C) \]

\[
\vdots
\]

\[ \mathbb{E}_\theta \left( U \left( a_i = 0, t(\theta_i), (0_0, 1_{n-1}) \right) \right) \leq \mathbb{E}_\theta \left( U \left( a_i = 1, t(\theta_i), (0_0, 1_{n-1}) \right) \right) \quad (\Gamma_1 : C) \]
\[ \mathbb{E}_\theta \left( U \left( a_i = 0, t(\theta_i), (0_0, 1_{n-1}) \right) \right) \leq \mathbb{E}_\theta \left( U \left( a_i = 1, t(\theta_i), (0_0, 1_{n-1}) \right) \right) \quad (\Gamma_0 : NC) \]

It is immediately evident from the above that we can eliminate all the non collusive agent constraints as the constraint \( \Gamma_x : NC \) is identical to the constraint \( \Gamma_{x+1} : C \)
for every $x$.

We can expand the collusive constraints above by evaluating the expected utility as follows,

$$\sum_{\theta} W(t_i(\theta)) P(\theta|(0_{n-1}, 1_0)) \leq -c + \sum_{\theta} W(t(\theta)) P(\theta|(0_{n-1}, 1_0)) \quad (\Gamma_n : C)$$

$$\vdots$$

$$\sum_{\theta} W(t_i(\theta)) P(\theta|(0_{x-1}, 1_{n-x})) \leq -c + \sum_{\theta} W(t(\theta)) P(\theta|(0_{x-1}, 1_{n-x})) \quad (\Gamma_x : C)$$

$$\vdots$$

$$\sum_{\theta} W(t_i(\theta)) P(\theta|(0_0, 1_{n-1})) \leq -c + \sum_{\theta} W(t(\theta)) P(\theta|(0_0, 1_{n-1})) \quad (\Gamma_1 : C)$$

We can now expand the summation and rearrange by grouping terms to get

$$(\delta - \varepsilon) (W(t(1)) - W(t(0))) \geq \frac{c}{P(m|(0_{n-1}, 1_0))} \quad (\Gamma_n : C)$$

$$\vdots$$

$$(\delta - \varepsilon) (W(t(1)) - W(t(0))) \geq \frac{c}{P(m|(0_{x-1}, 1_{n-x}))} \quad (\Gamma_x : C)$$

$$\vdots$$

$$(\delta - \varepsilon) (W(t(1)) - W(t(0))) \geq \frac{c}{P(m|(0_0, 1_{n-1}))} \quad (\Gamma_1 : C)$$

We can see that the LHS of all the constraints is now identical. Furthermore, from lemma 1 we know that $P(m|(0_0, 1_{n-1}) > P(m|(0_{x-1}, 1_{n-x}) > P(m|(0_0, 1_{n-1}))$ for all $n > x > 1$. Hence, we can order the RHSs as follows.

$$\frac{c}{P(m|(0_{n-1}, 1_0))} > \frac{c}{P(m|(0_{x-1}, 1_{n-x}))} > \frac{c}{P(m|(0_0, 1_{n-1}))}$$

Therefore whenever the constraint $\Gamma_n : C$ holds, the other constraints do as well.
The final collusion proof contract is found as a solution to the following problem.

\[
\max_{t(0), t(1), t(\emptyset)} V(a) + \sum_{i=1}^{n} -\lambda c + \sum_{\theta} \left( \lambda W(t(\theta_i)) - t(\theta_i) \right) P(\theta_i|a = (0_0, 1_n))
\]  

subject to

\[
\sum_{\theta} W(t(\theta_i)) \left[ P(\theta_i|a_i = 1, a_{-i} = (0_{n-1}, 1_0)) - P(\theta_i|a_i = 0, a_{-i} = (0_{n-1}, 1_0)) \right] - c \geq 0 \forall i
\]

It’s FOCs are

\[
\text{F.O.C } t(\theta) \quad \frac{1}{W'(t^c(\theta))} = \lambda + \mu \left[ \frac{P(\theta|a_i = 1, (0_{n-1}, 1_0))}{P(\theta|a_i = 1, (0_0, 1_n))} - \frac{P(\theta|a_i = 0, (0_{n-1}, 1_0))}{P(\theta|a_i = 1, (0_0, 1_n))} \right]
\]  

(2.14)

\[
\text{F.O.C } t(0) \quad \frac{1}{W'(t^c(0))} = \lambda + \mu \left[ \frac{P(m|(0_{n-1}, 1_0))}{P(m|(0_0, 1_n))} \right] \left( 1 - \frac{\delta}{\varepsilon} \right)
\]  

(2.15)

\[
\text{F.O.C } t(1) \quad \frac{1}{W'(t^c(1))} = \lambda + \mu \left[ \frac{P(m|(0_{n-1}, 1_0))}{P(m|(0_0, 1_n))} \right] \left( 1 - \frac{1 - \delta}{1 - \varepsilon} \right)
\]  

(2.16)

\[
\text{F.O.C } t(\emptyset) \quad \frac{1}{W'(t^c(\emptyset))} = \lambda
\]  

(2.17)

Here \( \{t^c(\theta)\}_{\theta \in \Theta} \) is the optimal collusion proof contract. We see that the first order conditions are nearly identical to the no collusion case. The likelihood ratios in (2.15)-(2.17) are just the same as those in the no collusion problem multiplied by a constant, implying that the collusion proof transfers too are monotone in the desirability of the action.

We know that the constraint binds at the optimal contract, so we can rearrange the constraint to get

\[
W(t^c(1)) - W(t^c(0)) = \frac{c}{(\delta - \varepsilon) P(m|(0_{n-1}, 1_0))}
\]  

(2.18)

Comparing this with (2.11) it is clear to see that the difference between the good
and bad payoff in the collusion case would be bigger than in the no collusion case as, the probability of being monitored is at its highest point in the no collusion case and at its lowest in \(2.18\), above.

### 2.4 The Effect of Multiple Agents

We have seen that adding multiple agents and allowing for collusion doesn’t seem to make much of a difference to the types of transfers as the First Order Conditions are of the same sort as in the canonical Moral Hazard model. However, increasing the number of agents does have an effect on the magnitude of the transfers. As I will show below.

We can see from equation \(2.18\), that since \(c\), \(\delta\) and \(\epsilon\) are fixed, what determines the magnitude of transfers is the agents probability of being monitored. So, we need to consider what would happen to an agent’s probability of being monitored if we increase the number of agents. The following lemma(theorem?) shows that it would decrease.

**Lemma 2.** The probability of each agent being monitored falls as the number of agent’s increases.

The proof of this lemma is given in appendix A.2.

The intuition of the proof is as follows. I introduce an extra agent and consider what happens to agent \(i\)’s monitoring probability. I first consider what happens if we add in one more agent that has a ‘Golden Ticket’, i.e. has irrefutable evidence that they did the correct action. I show that this does not change the probability of agent \(i\) being monitored at all as the monitor does not need to spend any time
at all on the additional agent. Hence it is straightforward to see that if the extra agent has a regular ticket or no ticket then the evidence available to the monitor is not as strong and hence, they must spend more time on the extra agent implying that agent \(i\)’s monitoring probability falls.

We can see that since the magnitude of transfers is inversely related to the probability of monitoring, which falls, that the optimal transfers must become more severe. To be more precise, the distance between the good and bad transfers must increase.

### 2.5 Conclusion

This paper has the following conclusions. Firstly I have shown that adding in additional identical agents into the canonical moral hazard model doesn’t change the shape of the optimal transfer function even when those agents can collude with each other. Additionally I have shown that the cost to the principal of inducing compliance or deterring collusion increases as the number of agents increases.

I have used a sequential monitoring structure, which despite it being in line with how monitoring typically works is still a stylized assumption. Further work on this topic could try and generalize this. I have also assumed that the monitor stops checking once they find a single non complier. Further work could generalize this as well.

Another important thing to consider is what happens if there is heterogeneity among the agents. This would probably make the problem more complicated as collusion is likely to be more appealing for certain agents. Furthermore if it is
not possible for the incentives to depend on the agents types, then the problem is likely to be even harder.
Chapter 3

A Dynamic Model of Educational Achievement and Aspirations

Abstract

The second chapter deals with the question of how agents make decisions in a dynamic framework when their preferences are reference dependent. I specifically consider the case of a student that must decide how much effort to put in to a course that is evaluated by the means of two exams. The student studies for the first exam, observes their result and then chooses how hard to study for the final. Exerting additional effort raises the expected final result, but also raises the aspiration, increasing the chance of accruing disutility from underperforming. The core question of this chapter pertains to the effect of raising the first result on the effort exerted. I find that in most cases, the student responds to an increase in the first result by reducing the amount of effort they put in. This is due to the fact that raising the first result has the effect of subsidizing the final result. This coupled with the fact that students are both risk and regret averse means they prefer to exert lower effort. This result highlights the importance of communication.
to course design. Designers need to be careful not to demotivate students and to incentivize effort.
3.1 Introduction

We introduce a model to describe how an individual’s educational aspirations form and update over time. There are already a number of models that discuss the formation of aspirations and the welfare effects of the same. However, to the best of our knowledge, this is one of the few projects to endogenise aspirations and consider the dynamic nature of this process.

In our view, educational achievement is not a one-shot process, instead, a person’s final level of education results from a sequence of tasks which must be performed. In order to earn a master’s degree, a candidate must have earned a bachelor’s degree in a relevant field, which requires a candidate to have studied subject specific courses in high school and so on.

Consequently, a Master’s degree is not something that an individual must and indeed typically will aspire to at the beginning of their educational careers. The aspirations of a student at any point in time depends on the information they have about themselves, their interests, abilities, etc. Furthermore, these aspirations will change as new information arrives.

Educational aspirations are initially formed as a result of an individuals initial exposure to their environment and influenced by factors like family income, parental education, distance from educational institutions, time invested in the child’s cognitive development etc. These, along with their intrinsic ability, which is typically unobserved, is what determines their educational achievement early in life. However, as their careers progress, they form new aspirations, on the basis of what they have inferred about their ability from their past achievements. Their ability too, cannot remain intransigent. It must evolve over time as their aspirations adjust.
If a person finds that they are good at mathematics, they will choose to study more mathematics, which in turn will make them better at mathematics.

In the literature we find models that treat aspirations as exogenous (Appadurai, 2004; 129 Ray, 2006; Lybbert and Wydick, 2018), but in our view, one of the most interesting mechanisms is the aspirations formation and adjustment process itself. The primary aim of this project is to formulate the simplest possible model that features an educational choice on the basis of ‘responsive’ aspirations.

We consider the case of a student who must choose how much effort to put into studying for a course that is evaluated on the basis of two exams. The student chooses how much effort to put in to studying for the mid term, observes the midterm score and then chooses how much effort to put in to studying for the final.

The result depends on the ability of the student, the effort they put into studying as well as luck. The student initially knows only the parameters of the distribution from which their ability is drawn, not the ability itself. They also know the distribution of the random component.

We have used a considerably simplified version of aspirations in our model. We assume that exerting greater effort leads to higher aspirations, i.e. that aspirations are a function of effort. This seems reasonable on the grounds that someone who has worked very hard would feel disappointed with a low result.

The utility framework we consider has two components. The first component which

\[1\] Consisting of two papers
we term the ‘achievement effect’ depends on the score on the exam only. Higher scores give higher utility. The second component, the ‘aspiration effect’ is reference dependent, and depends on the ‘aspiration gap’ i.e. the difference between the aspiration and the actual result. This framework admits two possibilities, that the student is disappointed (a negative aspiration gap) or that they are elated (a positive aspiration gap).

This paper looks at what happens in the second period, once the result of the midterm is observed. We begin by deriving the students newly induced distribution over their ability. We assume that everything is normally distributed, for computational ease as well as because this seems like a reasonable assumption. We use a reference dependent utility function. We then set up and solve an expected utility maximization problem on the basis of these beliefs.

We find that the optimal effort is unique, given parameter values. We compare the choice of optimal effort in our reference dependent framework with one that would arise from a ‘fully rational’ set up with no reference dependence. We find that the effect of aspirations on the optimal effort depends entirely on whether exerting more effort grows aspirations faster or (expected) results.

We then investigate the effect of a better midterm result on the optimal effort in the second exam. We find that it is almost always the case that a better first result leads to decreased effort. So, in the majority of circumstances, a fear of failure trumps ambition.

The cases when the opposite is true are quite interesting as well. A better first result only induces greater effort if two conditions are true:
Firstly, the marginal expected aspirations gap must be positive, i.e. it must be the case that exerting effort increases the expected result more than it increases the aspiration. Secondly, the aspiration gap must be small and negative. So, if the aspiration induced by the optimal effort is slightly higher than the expected result then a better first result would induce more effort. Another interesting finding is that there is a magnitude effect. The higher the aspiration, the larger the set of aspiration gaps that correspond to increased optimal effort.

The final thing we investigate in this paper is the effect on the maximized utility of a better first result. As expected, a better first result always leads to higher maximized utility.

3.2 Literature Review- Joint with Patricio Valdivieso (Heriot-Watt University)

3.2.1 Aspirations in Philosophy

The study of aspirations belongs to the philosophical discussion about hope. Other sentiments are also strongly attached to hope, such as desire, optimism and happiness. Even consciousness itself has been linked to hope, this includes the problem of time and decision making, the construction of ideas about oneself and what becomes real to a person.

A Short History of Hope

The origins of the study of hope can be traced back to ancient Greek philosophers who assigned a primarily negative connotation to hope, due to its potential
to mislead the actions of agents. Both, Thucydides and Plato, reflected upon the gullibility of hopeful agents (Gregory and Waterfield 2008; Hammond et al. 2009; Bloeser and Stahl 2017). Plato did allow for a positive consideration of hope by recalling the "pleasures of anticipation, expectations of future pleasures that are called hopes" (Fowler and Lamb 1925). Similarly, Aristotle recognized some positive attributes of hope and reflected upon the relationship between hope and courage. Aristotle elaborated upon the difference between hope that requires courage[2] and non-courageous hope[3] (Ross 2009).

Overtime, the relationship between hope and faith became central to the philosophical discussion about hope. Martin Luther in his famous discourse considers that:

"the highest wisdom of the world is faith and faith is just a future state or expectation of what a man can hope for" (Luther [1566/1840], P.79).

According to his discussion, the main difference between both concepts is that faith has to do with a person’s understanding, while hope has to do with a person’s will. Luther also recognized the positive nature of hope as a "word of promise for that which is good" (Luther [1566/1840], P.219) and the inseparable relationship between hope and trouble, "Faith without hope is nothing worth; for hope endureth and overcometh misfortune and evil" (Luther [1566/1840], P.220) where hope is a natural reaction to trouble. He finalizes his elaboration on hope, with a strong verdict "Everything that is done in the world is done in hope" (Luther [1566/1840], P.220) which to us is a first important recognition of the central role hope plays

---

2 Two examples of non-courageous hope include: "Hopeful at sea and in disease" (Ross 2009, 3.6, 1115a); "Hopeful based on one’s experience of good fortune" (Ross 2009, 3.8, 1117a).

3 "The brave man, on the other hand, has the opposite disposition; for confidence is the mark of a hopeful disposition" (Ross 2009, 3.7, 116a2).
Philosophers of the 17\textsuperscript{th} and 18\textsuperscript{th} century were optimistic about the nature of hope. Hope was believed to be closely related to confidence or courage\footnote{“Hope is a weaker form of confidence” (Descartes 1649/1989)}. Both, Descartes and Hobbes referred to the Aristotelian relationship between hope and courage, where hope acted as a driver for courage (Descartes 1649/1989; Hobbes 1651/1998). During this period a different strand of thought about hope emerged in the works of Spinoza and Hume. They discussed the passion inherent in hope with its uncertain and uncontrollable nature. They recognized the irrational and reactionary nature of hope. In fact, Spinoza recognizes hope as the cause of superstition, especially when it is accompanied by fear (Spinoza 1670/2002). This Lutheran view gave a political significance to hope, granting both, hope and fear as the basic elements of political power. The idea that men are governed by these two forces highlighted the fragility of men as victims of false belief. If this is true then good laws can also be used to ”motivate people by arranging outcomes such that they can be motivated by hope” (Spinoza 1670/2002). This recognition of the strength of hope as a source of political power was accompanied by the psychological approach of Hume, who claimed that hope was a ”direct passion produced when the mind considers events between absolute certainty and absolute impossibility” (Hume 1739/2003).

Following these disparate attempts to address the problem of hope, Emmanuel Kant embarked on an ambitious study on the relationship between hope and reason. He viewed hope as one of the three fundamental questions of philosophy (Kant 1781/1999) and adopted the Lutheran view of a connection between rational hope and religious faith. He claimed that moral progress was an object of hope. He claimed that due to the ”evil nature of humans” we must aim to bat-
tle against hope through the exercise of will (Kant et al., 1793/1960). This is a recognition of the importance of hope as an impulse to agency. He addressed the uncertain nature of hope together with the requirement that a person "must wish for it" (Bloeser and Stahl, 2017). He reflected upon the irrationality of hope and established that as long as the object of hope cannot be proven to be impossible, one can not assign irrationality to a certain hope. His work was one of the most crucial steps towards a consensus on the definition and theory of hope.

Post-Kantian studies tend to disagree on the role of hope. Schopenhauer, Nietzsche and Camus reject hope for its misleading nature. Schopenhauer notes that hope can lead to disappointment in two ways, when it is not realized or when the outcome does not provide as much satisfaction as expected (Schopenhauer, 1818/2016). This invokes the idea of the "pleasures of anticipation" developed by Plato (Fowler and Lamb, 1925). Nietzsche puts forth one of the most famous critiques of hope, calling it "the worst of all evils because it prolongs the torment of man" (Nietzsche, 1908/1994). Despite his critical views, he recognizes the ambivalent nature of hope, using the metaphor of the rainbow that bridges to the "overman", while at the same time being an illusory bridge (Nietzsche, 1885/2003).

Despite the negative atmosphere of the times, Kierkegaard and Marcel recognized the transformative power of hope to overcome everyday experience (Kierkegaard, 1847/1946).

Ernst Bloch's modern treatise on hope, explores the dynamic relationship between past realizations which are accepted with certainty and the transition to future states which must be continuous. The author describes this process as:

"dynamic relationships in which 'become' has not completely triumphed.

The real is process; the latter is widely ramified mediation between
present, unfinished past, and above all: possible future.” [Bloch and Plaice, 1986].

Bloch and Plaice’s description accurately captures the vision for our model of aspirations formation in which the aspired future state has a deterministic component from the past, but also has some partially-random component that feeds from current states of reality, where reality is a process that incorporates possible futures. It is essential to understand the core of such process and it is our intention to partially model that mechanism.

3.2.2 Aspirations in Psychology

The study of aspirations is not exclusive to any one science. Psychologists have made some of the most interesting contributions to explain the intricacies of the aspirations formation process. Unlike philosophers, psychologists value empirical evidence over logical reasoning.

There is extensive literature related to goal setting in psychology. The availability of several experiments allows us to recognize a set of trends already pointed out by the philosophy literature. In particular, Locke and Latham [1990], using the results of hundreds of studies developed the theory of goal setting and task performance which deals with goal directed actions and their formation process.

In an attempt to define goal-oriented actions, Binswanger [1986], argued that all goal-oriented actions shared a set of common characteristics such as self-generation, value-significance and goal-causation. These three characteristics point to the multidimensional nature of goal-oriented action. Also, the recognition of the link between past and future is captured by Bandura’s ”reinforcement effect”, which states that every effect generated by an action becomes an antecedent of the future in that it is related to new expectations regarding the future possibility of such an
The other relevant discussion has to do with the term "level of aspirations" itself and its location within the universe of goal-related concepts. A natural question should be why do we focus on aspirations and not on hope or on goals? or more generally, why do the philosophers study hope, psychologists goals and economists aspirations? The answer is that each of these terms imply a different dimension of the universe of goal related concepts.

In particular, the concept "level of aspirations" pertains to the type of goal related concept that stresses the conscious aspect and leaves the external aspect as an implication. Similarly, the "level of aspirations" is a concept that places emphasis on the aim of the action. A comparable term such as "goal" which also emphasizes the aim of the action, can be distinguished by its lack of clarity regarding its emphasis between conscious and external aspects. Ultimately, both terms and other goal related concepts will reflect some level of interaction of the underlying goal setting process.

Main Findings

A set of findings about the relationship between goals and performance are presented in the psychology literature. Possibly the most well known property is that of the positive linear association between performance and goal difficulty. This association is found in more than 12 different studies. Linear association was found in most cases, except when a person reached the upper limit of their ability.
tween goal specificity and performance. When a person is left alone to choose their performance goal, their performance level is low compared to performance after specific and externally assigned high goals [Mosholder, 1980]. Apparently, the mechanism in place is that of overvaluation of performance by individuals who are set free to choose their goal. Here, the anticipation of satisfaction, referred to as "valence" in the psychology literature⁶, plays a cumbersome role in the explanation of this particular result. Empirical results show that those individuals with "do best" goals experience a higher level of satisfaction (valence) for every level of performance than those with high specific goals [Mento et al., 1992]. Also, those with specific goals are significantly less likely to fall behind their previous performance on a learning task, than "do best" subjects [Locke and Bryan, 1966]. Another relationship of interest is that of goals and performance variance. In particular higher specificity in goals has been associated with lower variance in performance [Locke et al., 1989]. Also, "ceiling and floor" effects have been reported to affect variance in performance, where subjects with specific difficult goals showed less variance than those with moderate or easy goals. Arguably, this can be accounted for by the fact that those aiming for the higher threshold can only fail to achieve the goal, while those in the moderate or easy goal groups can miss the target to either side of the goal [Locke and Latham, 1990].

Finally, there is inconclusive experimental evidence about a set of questions including: goals and intrinsic motivation, personal, distal and proximal goals on performance and the role of sub-goals on performance. We could argue that there is evidence that proximal goals have a higher impact on performance than distal goals, within certain thresholds (e.g. for weight loss interventions, daily goals are

⁶This is arguably a similar idea as the aforementioned "pleasures of anticipation" [Schopenhauer, 1818/1818/2016] in the philosophy literature.
approximately equal to weekly goals and both show strictly better results than monthly goals [Balcazar et al., 1985]). Under the previous assumptions, sub-goals would work as short-term mechanisms that support long-term strategies by adding valence to short-term achievement. Overall, the s-shape of the utility function suggested by prospect theory (to be discussed in section 3.3) is to the very least supported by experimental results in the psychology literature.

**Models in Psychology**

A set of interactions between goal difficulty, expectancy and performance have been clearly identified in the psychology literature and basic models about such interactions have been put in place. Three somewhat different models attempt to capture this relationship, expectancy theory [Vroom, 1964], goal theory [Locke and Latham, 1990] and Atkinson theory [Atkinson, 1958].

Expectancy theory argues for a positive linear relationship between expected success and performance. Conversely, Goal theory offers an explanation based on a positive linear relationship between goal difficulty and performance. Given the negative correlation between probability of success and goal difficulty, these two theories conflict in their predictions.

The third alternative is Atkinson’s theory, which states that the relationship between probability of success and performance is a non-linear concave curve with its maximum at a probability of success of $P_s = 0.50^7$.

One way around the conflict in the theories was to recognize two type of interac-

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7The model stated $I = 1 - P_s$, where $I$ = incentive value of success and $P_s =$ probability of success
tions, "between group" and "within group" interactions, while measuring expectations with respect to a specific assigned or chosen goal level [Garland, 1984]. This solution suggested some trends such that higher goals were associated with lower success expectancy ratings and the opposite was true for those with lower goals. This can be formalized as the recognition of a negative correlation between performance and goal difficulty, exclusively from a between group perspective, while the within group analysis remains consistent with goal theory prediction of a positive association between goal difficulty and performance [Locke et al., 1986].

A second solution has to do with the measurement of expectancies. It is essential for correct identification, to acknowledge a common set of possible goals and within that range, each person can allocate probabilities to each outcome, which should capture actual comparable expectancies regardless of personal ability (which is assumed to be strongly correlated with goal selection).

The way to implement this is through the concept of "Self-efficacy" a concept introduced in Bandura’s social learning theory [Bandura and Simon, 1977], which is defined as "how well one can execute courses of action required to deal with prospective situations" [Bandura, 1982, p.122]. This concept has been shown to be positively and strongly related to future performance (Bandura, 1982; 1986). In an effort to simplify the definition of self-efficacy we express it as: A personal holistic assessment of one’s performance. This self-evaluation should include all factors one considers relevant to future performance.

Using this conceptual framework where self-efficacy is included as a more comprehensive alternative for expectancy, and incorporating a common range of performance options as the baseline for this measurement, a set of trends were estab-
lished, and both, between and within group correlations turned positive for both measurements, goal difficulty and expectancy against performance, also, a positive correlation between self-efficacy (expectancy) and goal difficulty was observed.

There is a collection of thirteen studies about self-efficacy and performance which report an overall positive correlation of 0.39 between self-efficacy and performance. The previous positive relationships work as a direct effect, but also there is an indirect effect through the channel of (self-set) personal goals, which according to the literature is slightly stronger 0.42 [Feltz, 2007].

A third effect of interest is that of assigned goals, which interestingly have been proven to affect personal goals (a well established fact) but also self-efficacy, just by the mere act of setting such external goals. This is found to exist even before any measurement of performance [Meyer and Gellatly, 1988; Earley and Lituchy, 1991]. This is aligned with concepts such as those of the "intrinsic value of aspirations" previously introduced in the philosophy literature.

Finally, within the relationships modelled in the psychology literature one is of particular interest to us, that is, the relationship between valence, goals and performance which relates to our early discussion pertaining to goals and intrinsic motivation in section 3.2.2. In particular, there is evidence of a positive relationship between valence and performance [Locke and Shaw, 1984; Matsui et al., 1981].Similarly, there is evidence that subjects with higher goals have lower valence for every performance level. This implies that those with lower goals tend to attain their mean valence at lower performance levels than subjects with medium or high goals. This ties neatly with the initial claim (in section 3.2.2) that individuals who

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*See table 3 – 2 of Locke and Latham 1990, p.71, for the detailed results of the 13 studies.*
are free to choose their goal, tend to the lower limit and that this is a consequence of a valuation problem. Hopefully now, the idea of the myopic valuation due to the disparity in goals and their associated valence functions fits clearly with the theoretical framework presented so far.

Determinants of Goal Selection

A final aspect of interest to us is that of what drives goal selection? what are the determinants of goal choice? This is essential to us, since it will work as a theoretical reference, from the psychology literature, for our modelling exercise.

The first approach to this topic was the study of levels of aspirations in the 30’s and 40’s [Campbell, 1982]. The level of aspiration was defined by Jerome Frank:

"The level of aspiration ... is defined as the level of future performance in a familiar task which an individual, knowing his level of past performance in the task, explicitly undertakes to reach" (Frank, 1935, p.119).

This definition acknowledges internal calculations and mentions the role of past information about the particular task, implying a probabilistic estimation of what is and is not feasible.

Additionally, the possibility of a goal structure was suggested [Gould, 1939; Lewin et al., 1944], where three type of goals were included, ideal goals, action goals (a concept similar to that of expected goals) and minimum goal (minimum threshold). These last two types, showed the highest predictive power for performance [Locke and Bryan, 1966].
The two basic determinants of the level of aspirations, according to the psychology literature are the expectancy of success and the valence of success (Frank, 1935; Gould, 1939; Festinger, 1942). These two categories are relatively general and include a set of factors, but these groupings are used, from a conceptual point of view, as the two main general criteria for the selection of a certain level of aspirations.

The first set of determinants, those that affect the expectancy of success, include previous performance, ability (measured by past performance), success and failure and expectancy and self-efficacy.

Previous performance and ability are self-explanatory. Success and failure experiences showed a set of interesting interactions such as, raised level of aspirations after failure (Lewin et al., 1944), a string of failure events proved to have a greater impact on lowering aspirations than ”one shot” failure events (Gould, 1939). Also, immediately previous trials had the greatest effect on choice (Lopes, 1976) and the effect of failure events on the level of aspirations was less uniform than the effect of successful events (Frank, 1941).

These findings, raised questions and theories about the mechanisms at work. Why would one raise their goal after failing to attain it? Maybe it is a defensive strategy to protect self-esteem (Frank, 1941)? A set of three possible interacting motives behind the choice of aspirations was proposed (Bayton, 1943). First, being ambitious enough, second, trying to maintain predictive accuracy (with respect to self-performance) and finally the natural desire to avoid failure. These group of conflicting motives, related to the success and failure condition, are likely to affect the aspiration selection procedure.
Finally, expectancy and self-efficacy, which we have previously introduced, will affect the beliefs about personal ability and consequently the amount of effort required for a particular level of aspirations.

The second set of determinants have to do with factors affecting valence or perceived desirability of any given goal or level of aspiration. Given the nature of the problem of perceived desirability, most factors affecting this will be of an external or environmental nature.

Things like group norms and normative information, "the norms of one’s own group, the norms of other groups and one’s own standing in the group" [Lewin et al., 1944]. Role modelling, which was shown to have a positive effect on subjects higher personal goals [Rakestraw Jr and Weiss, 1981]. Higher competition was also positively associated with goals [Mueller, 1983] and factors such as group goals and group pressure worked as goal commitment devices and showed positive correlation with goal levels [Forward and Zander, 1971; Matsui et al., 1987]. Interestingly, two relevant social technologies, goal assignment and feedback showed a significant positive association with personal goals.

Finally, factors pertaining to some state of mind or feeling such as mood, dissatisfaction and valence are also relevant to perceived desirability. Positive mood was shown to be positively associated with goals, while the opposite was true for negative mood [Hom and Arbuckle, 1988]. Valence, the emotional value attached to a certain goal or aspiration, also proved to be significantly associated with personal goals [Locke and Shaw, 1984] in a way such that high goal subjects presented consistently lower levels of valence, than low goal subjects, for every level of per-
formance. Then, the natural question would be, why would any one choose high goals? The answer proposed by Locke is that indeed they would not, most would choose moderate goals in the absence of any external factor (Locke, 1965; 1966).

A phrase that captures neatly the overall aspirations setting process presented so far, is that of Jerome Frank:

"The goal a person chooses is a compromise between what the person desires, all things considered and what he or she judges to be possible” [Frank, 1941].

As we have briefly presented here, the psychology literature has provided a set of meaningful experimental results to backup a group of models and interactions that will enable us to build a comprehensive economic behavioural model.

### 3.2.3 Aspirations in Economics

We could argue that in economics, the interest for aspirations and goal setting theory in general, is comparatively recent, as opposed to the philosophy and psychology literature. In particular, beginning in the 90’s and early 2000’s a set of models introduced the concept of reference points and detailed a group of interactions between reference points, goals or aspirations and different types of outcomes (e.g. educational, consumption, labour, etc.) (Lant, 1992; Heath et al., 1999; Heckman et al., 2006; Kőszegi and Rabin, 2006; Dalton et al., 2016; Lybbert and Wydick, 2018).

In particular, Heath et al. (1999), presented a model that employed goals as reference points and argued that goals, working as reference points, affected outcomes...
(e.g. educational) in a similar way as the value function of prospect theory [Kahneman and Tversky, 1979], implying loss aversion and diminishing sensitivity with respect to the reference point or goal of the value function.

This research identifies phenomena such as that of “the starting problem” which refers to the idea that initial endowments such as inherited socioeconomic status could affect, positively or negatively the willingness to exert more or less effort towards a fixed goal. Consequently, the initial standing of an individual relative to a reference point or goal would matter. Their research also reconciles other previously documented mechanisms by the psychology literature such as ”subgoaling”. Perhaps the most valuable contribution from this paper to our model is the link between the goal setting literature and the value function. By doing this, they provide a frame of reference in terms of the behaviour of the value function in the vicinity of a reference point, which we partially adopt in our model. It must be noted that this model [Heath et al., 1999] is mainly a static one and does not deal with the dynamic aspect of aspirations. Conversely, we do try to incorporate the dynamic aspect of the process in our modelling exercise.

Heckman worked on a series of econometric specifications that included cognitive and non-cognitive ability measures as explanatory variables on schooling decisions and wages. They link this same vector of ability with a set of risky behaviours, highlighting the relevance of non-cognitive skills on educational attainment and labour market outcomes [Heckman et al., 2006].

Also in 2006, Köszegi and Rabin introduced a model of reference-dependent preferences and loss aversion [Köszegi and Rabin, 2006]. This model is one of the closest attempts to a general theory of reference dependent utility in economics,
nevertheless, it is framed as a consumption and labour model which differs substantially from our educational attainment framework. Despite the differences we find considerable value in Kőszegi and Rabin [2006]. First, they made explicit the distinction between "consumption utility” which in our modelling framework is called the ‘achievement’ and pertains to direct utility from a certain level of educational attainment and "gain-loss utility” which is the utility from either attaining or not a certain reference point. This second element of utility, the one emerging from the result with respect to a reference point, is captured in our model by what we have termed as the "aspiration premium”. This effect captures the distance between the actual result and the chosen aspiration in a given period.

Their second contribution is the introduction of a framework incorporating a random component, which to our understanding is an important step towards a more realistic model, but their model is tailored for consumption decisions and fails to explain the transition over time from one reference point to the next. Lastly, they also discuss the nature of the reference point itself. The authors mention in their paper the understudied nature of this matter, which provoked our interest in this problem. In particular, they assume that the reference point is fully determined by recent past expectations, including probabilistic beliefs about both, consumption bundles and reference points.

Interestingly, this precise problem, the nature and evolution of reference points, has also been studied by organisational behaviour scholars, in particular Lant [1992], presents the results from an empirical study about how groups (organisations) form aspirations about profits and how this process evolves over time. Lant [1992] find evidence for adaptative history dependent models as good descriptors of aspirations formations. Lant [1992] also claims to find a systematic optimistic bias
in the aspirations formation process, which resonates with findings from the philosophy and psychology literature. They also provide evidence to support the idea that adaptative learning might lead to behavioural outcomes consistent with rationality, which would imply eventual convergence towards some expected outcome. Unfortunately all of these results correspond to a study over group aspirations and can not be assumed for the individual aspirations formation process which is at the core of our model.

A more recent modelling attempt by Dalton et al. [2016], presents a model of internal constraints, where those from a disadvantaged background, which we could consider as individuals with low initial endowments, have a higher probability of low aspirations and low achievement. This higher likelihood of comparatively worse outcomes is what they claim as the consequence of "internal constraints". Their model aims to demonstrate that internal constraints such as low aspirations or beliefs of despair will act as perpetrators of behavioural poverty traps. They demonstrate the effect of aspirations by comparing the outcome of their 'behavioural' model with that from a 'fully rational' one. We do this as well.

Their model uses three central premises, first, aspirations act as reference points, second, aspirations are affected by the individual’s actions and finally, individuals do not fully internalise the interaction between actions and aspirations. We use similar premises for our modelling exercise, in particular we also use aspirations as reference points and we also assume some level of noise in the individuals' understanding of the process. However, unlike their agent, ours is always fully maximising and their deviations from the outcomes of a 'fully rational' model are a result of their aspiration being a component of their utility, and not their inability to incorporate (anticipate) the impact of their current actions on their future
aspirations. This model [Dalton et al., 2016], just like that of [Heath et al., 1999] is also static.

Another recent modelling effort was presented by [Genicot and Ray, 2017]. Their model is based on socially determined aspirations and its bidirectional interaction with growth and inequality. Their model addresses 3 main aspects, aspirations formation, individual reactions to aspirations and aggregate societal outcomes. In particular, their approach to aspiration formation includes the use of ”threshold-based utility”, which is comparable with our reference-dependent utility framework. In their setup, aspirations are endogenous and affected by the individual’s prospects and by the environmental outlook, in particular by the overall income distribution. Their results are mainly driven by the distance between aspirations and the current standards of living, which as they mention, can ”either encourage or frustrate investment” [Genicot and Ray, 2017].

Despite the meaningful contribution of this model, their approach is quite different from ours in some respects. First, their main interest is about the interaction between the overall income distribution and aspirations, while we are more concerned with individual decision making in a simple educational setup, where the position of the individual in the distribution plays no explicit role. Second, we are mainly concerned about the information problem regarding the updated beliefs about individual ability while their model is mainly focused on the impact of the overall societal setup in individual outcomes, quite a different aim. Despite the great value that this model adds we find that ours belongs to a distinct area of microeconomics as we are more concerned about the individual decision making in isolation. As the authors mention in their paper, they ”do not provide philosophical or evolutionary foundations for aspirations formation” [Genicot and Ray].
2017] while we do attempt to address a question that is mainly contained within this domain.

Finally, the most recent work by [Lybbert and Wydick 2018] provides an up to date, comprehensive review of the literature in economics, philosophy and psychology related to aspirations and hope. They introduce a model of hope that incorporates elements such as aspirations, agency and pathways. Their model, in a fashion similar to that of [Dalton et al. 2016], uses an aspirations based utility framework and links poverty outcomes to failed aspirations due to a set of initial conditions.

The authors use exogenous aspirations\footnote{Aspirations are not a choice variable in their model.} to simplify their analysis and base their model on an aspirations-dependent utility function. Despite this particular modelling decision, they do discuss ”optimal aspirations” and departures from optimality.

Other modelling artefacts are cleverly used in this paper and we will partially adopt\footnote{Only at a conceptual level, instrumentally we use our very own framework.} some of them from a conceptual point of view. The authors include a perception parameter, $\rho$ which affects perceived agency $\tilde{\pi}$ in their model, which in our model translates, somewhat, to our unknown true ability measure $\omega$. Importantly, the agents knowledge about their unknown true ability is affected by information received after period 1 and captured by the distribution $\Omega_2$ in period two. In other words, in our model we introduce the concept of unknown ability, using beliefs about ability, which is conditioned on true ability and the first period information.
They also include the concept of the locus of control [Rotter, 1954] which captures the extent to which an individual has an external (internal) locus of control. This relates to the problem of control (internal) or lack of control (external) over future outcomes. This may affect their perceived agency and thus produce low aspirations and despair [Lybbert and Wydick, 2018]. Both of these concepts, the locus of control and self-esteem are the two grand components of what Heckman terms as "cognitive ability".

In a series of related papers (Thakral [2022], Thakral and Tô [2021a], Thakral and Tô [2020] and Thakral and Tô [2021b]) Thakral and Tô develop a model in which the decision maker chooses their reference point optimally. In Thakral and Tô [2021b] the study the mechanics of how the reference point is determined. They discuss how reference points adjust by examining the context of cabdrivers’ daily labor-supply behavior. They find that drivers work less in response to higher accumulated income, with a strong effect for recent earnings that gradually diminishes for earlier earnings. This set up is similar to the one we would use to analyse the first period in our model. The use this as the basis of a model of Anticipatory Utility in Thakral [2022]. They construct a model in which the agent accrues utility from anticipation of future consumption. Their model posits that decision makers (i) initially focus on the most tempting alternative in their choice set, and (ii) experience gain-loss utility from looking forward to future consumption. In the model, when evaluating a consumption stream, the decision maker chooses a level of anticipation each period, and anticipatory utility exhibits reference dependence with respect to the decision maker’s previous level of anticipation. They employ this framework in the papers Thakral and Tô [2021a] and Thakral and Tô [2020] to consider a variety of interesting questions. In Thakral and Tô [2021a] they look at how the timing of information affects consumption decisions. This is related
to questions that we attempt to answer in our model as well. In Thakral and Tö 2020 they look at how the predictions of their model change when they introduce ‘temptation goods’ which decision makers do not fully value prospectively.

Our model is also quite similar to the effort game modelled in Tincani 2018. The model specifics are presented in an online appendix to that paper. Their model consists of students choosing how much costly effort to exert. Effort increases achievement. The students are heterogeneous in the cost to them of exerting effort and belong to a reference group. In their model each student knows their own type and compare themselves to other students. In our model the students type (their ability) is unknown and they are concerned only about their own outcome.

In Abeler et al. 2011, the authors design and administer an experiment in order to determine if and to what extent expectations determine an agents reference point. They find that when subjects expectations are manipulated, their effort provision varies in line with the predictions of models of expectations based reference dependent preferences. They find that if expectations are high subjects work longer and earn more money. My model takes this relationship between effort and aspirations as an assumption.

Ayllón and Fusco 2017 is extremely relevant for my paper. The aim of their paper is to determine the extent of the dynamic cross-effects between an individuals subjective and objective perception of their financial situation. An individual’s economic fare can be assessed both objectively, looking at one’s income with reference to a poverty line, or subjectively, on the basis of the individual’s perceived

\begin{footnote}
  \url{http://www.homepages.ucl.ac.uk/~uctpmt1/Tincani_heterogeneous_online_supplementary_material.pdf}
\end{footnote}
experience of financial difficulties. This is comparable to my setting where the midterm result is objective and the aspiration it induces is subjective. Their main result highlights the existence of a feedback effect from past perceived financial difficulties on income poverty. Their paper explores the effect of the subjective on the objective, whereas ours goes in the opposite direction. In our paper the objective (midterm result) influences the subjective (aspiration).

Mazur [2021] looks at the economic consequences of pessimism in economics. They find that pessimistic beliefs are self-fulfilling. Their paper considers this question in the light of educational investment based on beliefs about the returns. The setting therefore is fairly far removed from ours, but the conclusions are quite relevant. They find that pessimism leads to underinvestment and hence is self-fulfilling. This is true in my model.

There is a wide literature on the effect of feedback on education (see Villeval [2020] for a survey). Azmat et al. [2019] studies the effect of providing feedback to college students on their position in the grade distribution by using a natural field experiment. They find that greater grades transparency decreases educational performance. Specifically, they posit the following hypothesis: ..individually who receive positive news (who were underestimating their relative position in the absence of information) reduce their output, while individually who receive negative news (who were overestimating their relative position in the absence of information) increase their output.. This is in line with my findings as well.

Blanes i Vidal and Nossol [2011] use a quasi-experimental research design to study the effect of giving workers feedback on their relative performance. They find that merely providing this information leads to a large and long-lasting increase in pro-
ductivity. This is similar to my finding that a better first result weakly increases the students utility. Similarly Tran and Zeckhauser [2012] find that Vietnamese students enrolled in an English course performed significantly better on the official standardized international final test when they were told their rankings on practice tests than when they were not.

In contrast, Barankay [2011] find that when compared to a control group with no rank feedback, employees who received feedback about their rank; relative to others performing the same task; were less likely to return to work and also less productive on the job. This is in keeping with my findings that in most cases the student responds to an increase in the first result by reducing the amount of effort they put in.

3.3 The Mechanics of Aspirations as Reference Points

3.3.1 Aspirations as Reference Points

As mentioned above, Heath et al. [1999] put forth the idea that goals work as reference points and that such goals affect outcomes according to the predictions of prospect theory [Kahneman and Tversky, 1979]. A set of stylized facts that are relevant for our model are depicted by the authors.

The first important claim is that goals act as reference points. In our model this translates to aspirations which act as reference points for educational achievement. Consistent with the prospect theory value function approach, we adopt a utility function that allows for the existence of reference points embodied by aspirations
and which exhibits loss aversion\footnote{although in our model this could more accurately be termed regret aversion} and diminishing sensitivity. These three characteristics imply a set of properties.

The existence of a reference point determined by aspirations, and the curvature of the value function before and after this point (S-shape) means that the reference point marks the difference between success and failure, which we consider as a binary approach to outcomes. Those who struggle to achieve a certain goal will fall under the reference point and will experience increasing gains the closer they get to their objective. Similarly, those who achieve such a goal will start to experience decreasing gains for every additional unit of achievement with respect to their reference point.

3.3.2 Loss Aversion

Loss aversion implies a set of predictions. First, it means that those who are closer to their reference points will exert additional effort. Also, it implies that a loss of \( X \) provides a higher loss sentiment than the gain sentiment provided by a proportional gain. Mathematically, \(|V'(−X)| > V'(X)\), where \( V'(X) \) is the value of the slope of the value function evaluated at \( X \). Also, importantly, the reference point will be relevant in determining the value of the gain or loss. Consider an individual with a goal of \( X \) who overachieves by 5 units: \( b'_X(X + 5) = V'(5) \) and compare it with an individual who attains the same result, but who had a more ambitious goal \( b'_{X+10}(X + 5) = V'(−5) \). Then by loss aversion, \( V'(5) < |V'(−5)| \).

From the previous property we can deduce that the effort exerted given a negative distance (loss) to the reference point, will be higher than the comparable effort from a proportional positive distance (gain). Another important implication
is that the distance from the reference point matters quite a lot. If we were to artificially raise the aspiration of an individual who is currently failing to attain their aspiration, the amount of effort the individual is willing to exert would fall. However, if the individual who’s aspiration we raised was instead exceeding their current aspiration then the effort they are willing to put in would go up.

3.3.3 Diminishing Sensitivity

Diminishing sensitivity implies that the effort individuals are willing to make increases when they come closer to their objective, while it could pose a problem when a person is too far away from their aspirations or when someone aspires to something too far from their current state, since it will imply low valence and effort. One way the literature approaches this problem is by “subgoaling”, so that it is possible to overcome the ”initial problem” with more immediate rewards from ”sub-goals”.

3.3.4 The Initial Problem

”The initial problem” is the term the authors [Heath et al., 1999] use to characterise the implications of the effort exerted when a person is too far or too close from a goal. If the starting point is too far away from the objective, then the endogenous effect of the reference point on self-efficacy could be very low, while the opposite would be true for short distances from the reference point.

3.3.5 Aspirations, Agency and Optimal Aspirations

The previous paragraph suggests that there is an interaction between reference points and agency. Most importantly it also suggests that in most situations,
there is scope for a change in the reference point that would generate (endoge- 
nously) a better outcome. In particular we can introduce the idea of an optimal 
aspiration or reference point, which is the reference point (e.g. aspired educational 
level) that maximizes the utility of the individual.

Note that the idea that higher goals could increase performance comes from the 
assumption that the marginal cost of a certain level of achievement is independent 
of the reference point\(^{13}\) but we relax this and assume instead that educational 
achievement is a function of both, aspirations and ability, but also, that the beliefs 
about true ability are not fixed and could evolve overtime.

At this stage, we must recognize that the decision of achieving an extra unit of 
educational attainment, is not as simple as adding units until the marginal benefit 
equals the marginal cost of effort, but will depend upon the level of aspirations 
itself. It is our understanding that the cost of this achievement is not independent 
from the reference point itself and as argued by the philosophical literature, the 
sole exercise of aspiring for an outcome will produce some level of agency that 
could reduce, even in an infinitesimal amount, the cost of achievement.

Furthermore, exerting high effort affects the marginal benefit as it produces bet-
ter results, however it also entails a cost in its effect on the aspiration. This is 
probably one of the most important implications of our model, since higher effort 
implies higher expected results and a higher probability of failing to achieve such 
results, hence it is directly associated with a higher cost in terms of the aspiration.

\(^{13}\)Which is backed up by plenty of experimental evidence in social psychology (Locke and 
Bryan 1968; Tubbs 1986; Locke and Latham 1990)
So, to summarise, we assume that exerting effort entails both a direct cost and an indirect one via an increase in aspirations. This implies that the probability of achieving a particular educational result increases as one increases their effort, but also, the probability of failing to achieve the implied aspiration also increases.

3.4 Model

3.4.1 Definitions and Preliminaries

We consider the scenario of a student who has enrolled in a course that is evaluated by the means of two examinations. The student chooses how much effort to put into studying for each exam in order to maximize their expected utility.

Aspiration

$e_1$ and $e_2$ denote the effort put into the midterm and final exam respectively. The student accrues utility based on their final result and how it compares to the aspiration they formed over the course of the two exams. From the literature it is clear that aspirations are fairly complex objects. For our purposes we have modelled the aspiration, $a$ as an increasing function of the effort, $e_1 + e_2$ that the student puts in to studying for the two exams. We feel this is a good modelling choice for two reasons.

Firstly, this assumption along with a few others that we make regarding the utility function later allow us to match the loss aversion and diminishing sensitivity of prospect theory and to do so parsimoniously.
Secondly, this assumption is in line with the psychological discourse on valence. Greater effort leads to greater anticipation of success and hence greater loss of utility if the aspiration is not met.

Results

There are three ‘results’ that matter. \( R \) is the final grade on the course and \( r_1, r_2 \) are the results on the first and final examinations respectively. We denote by ‘\( p \)’ and ‘\((1 - p)\)’ the contribution of the first and final examinations respectively on the final grade for the course.

\[
R = pr_1 + (1 - p)r_2 \tag{3.1}
\]
\[
r_i = \omega + G(e_i) + \varepsilon_i \tag{3.2}
\]

\( \omega \) is the persons intrinsic ability. We interpret this ability as being the score that a student would get on a test if they were to put in no effort at all. \( e_i \) is the effort that the student puts in period \( i \). Effort translates directly into results via the function \( G(\cdot) \). i.e. effort \( e_i \) raises the result, \( r_i \) by the amount \( G(e_i) \). \( G(\cdot) \) is increasing and concave. The students result is also subject to a random shock, \( \varepsilon_i \) which we assume follows the standard normal distribution (i.e. \( \varepsilon_i \sim \mathcal{N}(0,1) \)), by assumption.

The students intrinsic ability is unknown to them. In the first period the student knows that their ability is a random draw from the distribution \( \mathcal{N}(\mu, \sigma^2) \), which we will refer to as \( \Omega_1(\omega) \). In the second period the student can use their result \( r_1 \)

\[\text{(This assumptions is based on the usual diminishing returns in economics.)}\]
to form more accurate beliefs about their ability. We call this updated distribution $\Omega_2(\omega|r_1)$.

**Utility Function**

We now define the utility function we will use.

We first define an indicator function $D(x)$ as follows

$$D(x) = \begin{cases} 
1 & \text{if } x \text{ is true} \\
-1 & \text{otherwise}
\end{cases} \quad (3.3)$$

We also define a CARA utility function $U(x; b)$ as follows

$$U(x; b) = 1 - e^{-bx} \quad (3.4)$$

We denote by $U_1(x; b)$ the first partial derivative with respect to the first argument.

$$U_1(x; b) = be^{-bx} \quad (3.5)$$

On this basis, we construct the utility function $V(R, a, \beta)$ that is appropriate for our model as follows.

$$V(R, a, \beta) = U(R; \beta) + kD(R \geq a)U(R - a; D(R \geq a)\beta) \quad (3.6)$$

$D(R \geq a)$ is the disappointment function. It takes the value 1 when the student has exceeded their aspiration i.e. $R \geq a$ is true and hence the student is 'elated'
and $-1$ when the statement is false i.e. when the student is ‘disappointed’.

The utility function we have constructed has two parts. The first portion $U(R; \beta)$ is called the ‘achievement’. This is increasing and concave reflecting the fact that the student accrues higher utility from higher results with a diminishing marginal effect. The second part is called the ‘aspiration premium’. This portion is reference dependent to reflect the fact that the utility derived from a particular result depends on the result that was aspired to before the exam took place. $k$ is a constant reflecting the importance of the aspiration premium relative to the achievement to the student.

We can see that if the result exceeds the aspiration then the student receives a ‘boosted’ utility of

$$U(R; \beta) + kU(R - a; \beta).$$

The size of the boost depends on $R - a$ which we refer to as the aspirations gap and denote $\Delta$. We can see that the boost increases at a diminishing rate implying that the student is risk averse in the elation zone. What this means is that once the students goal is met they aren’t affected very much by the extent to which the goal is exceeded.

Similarly, if the result is below the aspiration then the utility is dampened:

$$U(R; \beta) - kU(R - a; -\beta).$$

The aspiration effect in this case is increasing and convex. The student is risk loving in this region.
Expected Utility Maximisation Problem

We can now state the problem to be solved. We denote by $C(e_i)$ the cost of exerting effort $e$ in period $i$. As discussed earlier, this is not the only ‘cost’ to exerting effort, there is also a utility cost owing to the fact that effort raises aspirations. $C(e_i)$ maybe interpreted as the physical cost e.g. fatigue or the opportunity cost e.g. foregone leisure.

In the second period the student knows the realization of $r_1$ and chooses effort $e_2$ to solve the following problem.

$$
\max_{e_2} \int V(R, a(e_2), \beta) f(R|r_1) dR - C(e_2) \tag{3.7}
$$

$$
= \max_{e_2} \int U(R; \beta) f(R|r_1) dR + k \left( \int D(R \geq a) U(R \geq a; D(R \geq a)\beta) f(R|r_1) dR \right) - C(e_2) \tag{3.8}
$$

$f(R|r_1)$ is the probability distribution of the final result $R$, induced by the realization $r_1$. The student knows that $r_1 = \omega + G(e_1) + \varepsilon$, and so, upon observing $r_1$, they can reverse engineer this to update their beliefs regarding their own ability $\omega$. The first thing we must hence do to solve (3.8) is to derive the distribution $f(R|r_1)$. We do so in section (3.4.2).

We can then take the first order condition for the second period problem and simplify it. This is done in sections (3.4.3) and (3.4.4). In section (3.5.1) we compare the first order condition for our behavioural problem with its analogue from a fully rational framework to see what effect the existence of aspirations has on optimal behaviour.

In the next section, (3.5.2), we investigate the key question of this chapter. What is the effect of a better first result on effort. We find that it is almost always the
case that a better first result leads to the student exerting less effort.

We follow this up with section (3.6.1) where we investigate the effect of changing $r_1$ on the maximized utility of the student.

The key questions of this chapter are addressed through analysis of the second period problem, however, for completeness sake we do address the first period problem in section (3.7).

We can define a value function based on optimally chosen effort in the second period and then address the first period problem using backward induction. We call the second period value function $W(e_1, r_1)$ and define it as follows.

$$W(e_1, r_1) = \max_{e_2} \int U(R; \beta) f (R|r_1) \, dR$$

$$+ k \left( \int D(R \geq a)U(R - a; D(R \geq a) \beta) f (R|r_1) \, dR \right) - C(e_2)$$

$$W(e_1, r_1) = \max_{e_2} \int U(R; \beta) f (R|r_1) \, dR$$

$$+ k \left( \int D(R \geq a)U(R - a; D(R \geq a) \beta) f (R|r_1) \, dR \right) - C(e_2)$$

Effort in the first period shifts the mean of the distribution of the first result $r_1$, this is taken as the continuation value in the second period value function. The first period problem is as follows.

$$\max_{e_1} -C(e_1) + \int W(e_1, r_1) f(r_1) \, dr_1$$

The student chooses $e_1$ to maximize the expected second period value. The first order condition for this problem is presented in section (3.7).
3.4.2 Second Period Beliefs and the Distribution of the Results

We now consider how the student updates their beliefs about their own ability and then forms beliefs about the possible final result that they may get. From equation (3.2) we can see that the first result, $r_1$, is determined by both known and unknown quantities. This implies that once the student observes $r_1$, they can refine their beliefs about their own ability. They do this via Bayesian updating.

The distribution $\Omega_2(\omega|r_1)$ is the updated belief in the second period. From Bayes Theorem we get

$$\Omega_2(\omega|r_1) = \frac{f(r_1|\omega)\Omega_1(\omega)}{f(r_1)}.$$  \hfill (3.11)

Now, we consider each component of (3.11) in turn.

$$r_1 = \omega + G(e_1) + \varepsilon$$

We begin with $f(r_1|\omega)$. $G(e_1)$ is not a random variable, $\varepsilon$ follows the standard normal distribution and as we are conditioning on it, $\omega$ is treated as non stochastic. Hence, the distribution of $r_1$ conditioned on $\omega$ is normal with mean $(0 + \omega + G(e_1))$ and variance 1.

We now turn to the unconditional distribution of $r_1$. $\omega$ must now be treated as a random variable, however, as it is normally distributed, so is the sum $G(e_1) + \omega + \varepsilon$. It has mean $G(e_1) + \mu + 0$ and variance $1 + \sigma^2$.

Since both the prior over ability $\Omega(\omega)$ and the distribution of the signal $f(r_1)$ are
normal, we know that the posterior distribution $\Omega_2(\omega|r_1)$ will also be normal\textsuperscript{15}. Some algebra, or the straightforward application of the formula given in Chamley \textsuperscript{2004} and Grimmett et al. \textsuperscript{2001} gives us the following for the posterior distribution of beliefs about ability.

$$\omega|r_1 \sim N\left(\frac{\sigma^2(r_1 - G(e_1)) + \mu}{1 + \sigma^2}, \frac{\sigma^2}{1 + \sigma^2}\right)$$

We now turn to the distribution of $R$ which is the combined result of both examinations. Combining (3.1) and (3.2) we get

$$R = pr_1 + (1 - p) (\omega + G(e_2) + \varepsilon)$$

$r_1$ and $G(e_2)$ are non-stochastic and hence they only shift the mean of $f(R|r_1)$. $r_1$ has already happened and for every possible value of $e_2$, $G(e_2)$ is known and consequently non-stochastic.

Since both the beliefs about ability and the random shock are normally distributed, $R$ is normally distributed as well. The mean of this distribution, which we call $\bar{R}$ is

$$\bar{R} = pr_1 + (1 - p) (E(\omega|r_1) + G(e_2) + E(\varepsilon))$$

$$= \frac{(p + \sigma^2) r_1 + (1 - p) \{\mu + G(e_2) + \sigma^2 (G(e_2) - G(e_1))\}}{1 + \sigma^2} \quad (3.12)$$

Similarly, the variance of $R$, $\sigma^2_R$ is as follows

\textsuperscript{15}As the normal distribution is conjugate prior with itself.
\[ \sigma_R^2 = (1 - p)^2 \left( \frac{\sigma^2}{1 + \sigma^2} + 1 \right) = \frac{(1 - p)^2(1 + 2\sigma^2)}{1 + \sigma^2}. \] (3.13)

Note that the choice variables \( e_1 \) and \( e_2 \) shift the mean of the distribution. To highlight this fact we will henceforth refer to the conditional distribution of \( R \) as \( f(R - \bar{R} | r_1) \). This is not a meaningful change, merely a relabelling. Note also that that the variance does not depend on any endogenous variables.

### 3.4.3 The Second Period First Order Condition

Having derived the probability distribution of the final result \( R \) we can evaluate the first order condition of the second period problem, which is restated below.

\[
\int U(R; \beta) f(R - \bar{R} | r_1) \, dR + k \int D(R \geq a) U(R - a; D(R \geq a)\beta) f(R - \bar{R} | r_1) \, dR - C(e_2)
\] (3.14)

Taking the first order condition of the above is complicated by the fact that \( e_2 \) appears in both the utility and distribution functions. We can simplify matters by restating the problem in terms of the difference between the result and aspiration, \( \Delta = R - a(e_2) \), which we call the aspiration gap. The mean of this term, \( \mathbb{E}(\Delta) = \bar{R} - a(e_2) \), we call the expected aspiration gap.

The expected aspiration gap and its derivative, the marginal expected aspiration gap have an important roll to play in our model. The marginal expected aspiration gap, \( \frac{\partial \Delta}{\partial e_2} = \frac{\partial \bar{R}}{\partial e_2} - a'(e_2) \) represents the key trade-off of our model. Exerting effort increases the expected result, but also raises the aspiration. This and related terms will turn out to be the driver of most of our results.
The restated problem is given below.

\[
\max_{e_2} \int U(R; \beta) f(R - \bar{R}|r_1) \, dR + k \left( \int D(\Delta \geq 0) U(\Delta; D(\Delta \geq 0) \beta) f(\Delta - \bar{\Delta}|r_1) \, d\Delta \right) - C(e_2)
\]

(3.15)

This formulation is simpler as it has a much more tractable first order condition. This stems from the fact that as we now integrate over \( \Delta \) we can ignore its dependence on \( e_2 \) meaning that we need only differentiate the distribution and not the utility function. This eliminates the need to do the product rule.

The first order condition with respect to \( e_2 \) is as follows

\[
- \frac{\partial \bar{R}}{\partial e_2} \int U(R; \beta) f'(R - \bar{R}|r_1) \, dR \\
+ k \left\{ - \frac{\partial \bar{\Delta}}{\partial e_2} \right\} \int D(\Delta \geq 0) U(\Delta; D(\Delta \geq 0) \beta) f'(\Delta - \bar{\Delta}|r_1) \, d\Delta \\
= C'(e_2).
\]

(3.16)

The integrals above cannot be evaluated as is because their behaviour at the limits is undefined. However, in the next section we present a lemma that we can use to overcome this problem. We then use integration by parts to simplify the first order condition.

### 3.4.4 Simplifying the First order Condition

In order to simplify the first order condition we make use of the following lemma.

**Lemma 3.** If \( x \sim \mathcal{N}(\bar{x}, \sigma^2) \) with p.d.f \( f(x) \) and \( U(x; a) = 1 - e^{-ax} \) then,
1. \( U(x - b; a) f(x) = f(x) - \left( \frac{U_i(x - b - \frac{\sigma^2}{2}; a)}{a} \right) f(x + a\sigma^2) \)

2. \( U_1(x - b; a) f(x) = U_1(\bar{x} - b - \frac{\sigma^2}{2}; a) f(x + a\sigma^2) \)

The proof is in the Appendix (B.1).

The first integral in (3.16) can be evaluated using integration by parts and the lemma as follows.

\[
\int U(R; \beta) f'(R - \bar{R}|r_1) dR = [U(R; \beta) f (R - \bar{R}|r_1)]_{-\infty}^{\infty} - \int U_1(R; \beta) f (R - \bar{R}|r_1) dR = \left[ f (R - \bar{R}|r_1) - \frac{U_1 \left( R(e_2) - \frac{\beta\sigma^2}{R^2}; \beta \right)}{\beta} f (R - \bar{R}|r_1) \right]_{-\infty}^{\infty} \tag{3.17}
\]

\[
= -U_1 \left( R(e_2) - \frac{\beta\sigma^2}{R^2}; \beta \right) \int f (R - \bar{R}|r_1) dR
\]

\[
= 0 - U_1 \left( R(e_2) - \frac{\beta\sigma^2}{R^2}; \beta \right) \tag{3.18}
\]

What we have done above is transform the expected marginal utility from a lottery into its certainty equivalent. It is well known that with CARA utility, the certainty equivalent of the uncertain outcome \( x \) is \( \bar{x} - \frac{\beta\sigma^2}{2} \), where \( \bar{x} \) is the expected value of \( x \) and \( \frac{\beta\sigma^2}{2} \) is the risk premium. We do the same for the second integral.

\[
\int D(\Delta \geq 0) U(\Delta; D(\Delta \geq 0)\beta) f'(\Delta - \Delta|r_1) d\Delta
\]

\[
= [D(\Delta \geq 0) U(\Delta; D(\Delta \geq 0)\beta) f (\Delta - \Delta|r_1)]_{-\infty}^{\infty}
\]

\[
- \int D(\Delta \geq 0) U_1 (\Delta; D(\Delta \geq 0)\beta) f (\Delta - \Delta|r_1) d\Delta
\]

\[
= - \int D(\Delta \geq 0) U_1 \left( \bar{\Delta} - D(\Delta \geq 0)\frac{\beta\sigma^2}{2}; D(\Delta \geq 0)\beta \right) f (\Delta - \bar{\Delta}|r_1) d\Delta
\]

\[
= F (-\beta\sigma^2) U_1 \left( \frac{\beta\sigma^2}{2}; -\beta \right) - (1 - F (\beta\sigma^2)) U_1 \left( \Delta - \frac{\beta\sigma^2}{2}; \beta \right) \tag{3.19}
\]
\( F(\cdot) \) is the cumulative distribution function. Putting together the expressions in (3.18) and (3.19) yields the following for the first order condition.

\[
\frac{\partial \bar{R}}{\partial e_2} U_1 \left( \bar{R}(e_2) - \frac{\beta \sigma_R^2}{2}; \beta \right) + k \{ \frac{\partial \bar{\Delta}}{\partial e_2} \} \\
\left( (1 - F (\beta \sigma_R^2)) U_1 \left( \bar{\Delta} - \frac{\beta \sigma_R^2}{2}; \beta \right) - F (-\beta \sigma_R^2) U_1 \left( \bar{\Delta} + \frac{\beta \sigma_R^2}{2}; -\beta \right) \right)
\]

\[
= C'(e_2)
\]

(3.20)

In the next section we interpret this first order condition and then proceed to some comparative statics.

### 3.5 Interpretation of The Results

#### 3.5.1 Interpreting the First Order condition

The first question that we can answer using this framework is what effect do aspirations have on the effort exerted by the student. We do this by comparing the optimal effort from a fully rational version of the model to the one that arises in our setting.

We know that \( \beta \), the risk aversion parameter, is positive, by assumption. Differ-
entiating $\tilde{R}$ from (3.12) we can see that
\[
\frac{\partial \tilde{R}}{\partial e_2} = (1 - p)G'(e_2)
\]
which is also positive by assumption. So, (I) above is positive. This means that exerting more effort yields higher utility via the 'achievement effect'. Furthermore, since $U(x; a)$ is concave the marginal utility is diminishing. One important implication of this is that in a fully 'rational' world; i.e. if the aspiration effect didn’t exist; we would have a unique solution since we have assumed that the cost function $C(e_2)$ is convex.

If the remaining portion of the first order condition, $(II) \times (III) + (IV)$, is positive(negative) then the aspiration effect increases(decreases) the optimal effort relative to the rational one.

The following lemma helps us analyse (III) and (IV).

**Lemma 4.** $\forall \beta \geq 0 \ U_1(x - d, \beta) > 0$ and $U_1(x + d, -\beta) < 0$. This is true for all values of $x$; positive or negative.

**Proof.** This lemma concerns the sign of the marginal utility. From (3.5), $U_1(x; b) = be^{-bx}$. As $e$ raised to any power is positive, the sign of the marginal utility is the same as the sign of $b$. $\square$

From Lemma (4), since $U_1 \left( \Delta + \frac{\beta \sigma^2_R}{2}; -\beta \right)$ is negative, we can see that
\[
U_1 \left( \Delta - \frac{\beta \sigma^2_R}{2}; \beta \right) - U_1 \left( \Delta + \frac{\beta \sigma^2_R}{2}; -\beta \right)
\]
is the sum of two positive expressions and hence is also positive.
(1 − \(F(\beta \sigma_{R}^{2})\)) and \(F(-\beta \sigma_{R}^{2})\) are both probabilities and hence are positive. So, (III) + (IV) is the weighted sum of two positive expressions and hence is positive.

The effect of aspirations on the optimal \(e_{2}\) thus depends entirely on the sign of expression (II), namely, the marginal expected aspiration gap \(\frac{\partial \bar{\Delta}}{\partial e_{2}}\).

\[
\frac{\partial \bar{\Delta}}{\partial e_{2}} = \frac{\partial R}{\partial e_{2}} - a'(e_{2})
\]

has two components that reflect the marginal contribution of effort to the result and the marginal change in aspirations induced by effort, respectively.

Given that we have assumed that the effort function \(G(e_{2})\) is positive and concave, it’s derivative is positive as well. Similarly, we have posited that increasing effort would be accompanied by increased aspirations and so \(a'(e_{2})\) is also positive.

Which of these two effects is greater however depends on the characteristics of the individual. If \(\frac{G'(e_{2})}{2} \geq a'(e_{2})\) this means that exerting effort has a greater effect on the result than it does on the student’s aspiration. This has the effect of causing the student to put in greater effort than they would in the ‘rational’ case. On the other hand, if the marginal effect on aspirations exceeds the marginal increase in the result, the optimal effort to exert is lower than it would have been had the student been concerned exclusively with the physical cost of effort. One explanation for this pattern of behaviour emerges if we view the marginal expected aspiration gap as resulting from the student’s level of optimism. If \(\Delta' < 0\) then the student is over optimistic about how effective their effort will be, leading them to put in less effort, and vice versa.
3.5.2 Comparative Statics

The Effect of Changing $r_1$

We now turn to one of the main questions of this paper. What effect does the first result have on the optimal effort in the second period. A few possible answers exist for this question. If the result is low, the student might like to cut their losses by exerting little extra effort and keeping their aspirations low. On the other hand, it is equally plausible that the student would redouble their efforts in order to raise their grade on the course by performing well on the final exam. It may also be the case that the student responds to a higher result by resting on their laurels and exerting less effort on the final.

We attempt to answer this question by using the implicit function theorem to investigate the sign of $\frac{de_2^*}{dr_1}$.

Implicit Function Theorem

Denote by $\varphi(r_1, e_2)$ the derivative of the objective function with respect to $e_2$. When the optimal effort is exerted then the first order condition is satisfied i.e. $\varphi(r_1, e_2^*(r_1)) = 0$. Furthermore, it is plain to see that $\varphi(\cdot, \cdot)$ is continuously differentiable everywhere on $\mathbb{R}^2$. If the partial derivative, $\varphi_{e_2}(r_1, e_2^*(r_1)) \neq 0$ then all the conditions for the IFT will be satisfied and hence

$$\frac{de_2^*(r_1)}{dr_1} = -\frac{\varphi_{r_1}(r_1, e_2^*)}{\varphi_{e_2}(r_1, e_2^*)}.$$

The effect of the result on effort is summarised in the following theorem.

**Theorem 2.** $\frac{de_2^*(r_1)}{dr_1} > 0$ if and only if,

1. $\frac{\partial \Delta}{\partial e_2} > 0$
2. $\Delta < 0$

3. $\frac{1}{k} \leq -\frac{\partial \Delta}{\partial e_2} \frac{(1-F(\beta \sigma_k^2)) e^{2\beta a} - F(-\beta \sigma_k^2) e^{2\beta R}}{e^{\beta a}}$

A more useful corollary of this theorem is

**Corollary 1.** For almost all possible parameter values, the optimal effort exerted in the second period is inversely related to the first period result. i.e.

$$\frac{de_2^*(r_1)}{dr_1} < 0.$$ 

The proof of the theorem is in appendix (B.2). The theorem tells us that it is almost always the case that the student responds to an increase(decrease) in the first result, by decreasing(increasing) the amount of effort they would like to exert. These results, for the most part are in keeping with and can be explained by prospect theory.

We first consider the scenarios where $\frac{de_2^*(r_1)}{dr_1} < 0$. The table below gives the configurations of the expected and marginal expected aspiration gap in which the optimal response to a better result is to reduce effort in the second period.

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$\frac{de_2^*(r_1)}{dr_1}$</th>
<th>$\frac{de_1^*(r_1)}{dr_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>-</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
</tr>
</tbody>
</table>

From the second row of the table we can see that whenever the marginal expected aspiration gap is negative the effort exerted will fall. When the marginal expected aspiration gap is negative, a change in effort has a greater effect on the aspirations than on the result. In this scenario, the benefits from exerting greater effort are
out weighed by the increased probability of being disappointed. Furthermore, in this case the increased first result acts as a subsidy by making it cheaper in terms of both aspiration and physical cost to acquire a given final result. As a result, the optimal effort exerted falls.

When the marginal expected aspiration gap is positive and the returns to effort from a better expected result exceed the increased risk of disappointment, so long as the aspiration is below the expected result, i.e. $\Delta > 0$, the optimal effort falls when the first result is better. This is due to the fact that in the elation zone the marginal utilities from both the achievement and aspiration effects are diminishing. As a result, the student is exceptionally risk averse.

In the case where the aspiration exceeds the expected result, condition 3 of Theorem (2) determines when the effort increases or decreases. The right hand side of this condition behaves in an extremely interesting manner. It approaches zero as $\Delta$ approaches $-\infty$ as well as when it approaches 0 and attains a unique maximum at a small negative value. The value of the RHS at its peak depends on the magnitude of the aspiration. Since the left hand side is a constant, if the aspiration is small then the condition is never satisfied.

When it is, the condition is satisfied by a range of negative expected aspiration gap values close to zero. This is in accordance with the diminishing sensitivity property of prospect theory, which predicts that the student is willing to exert additional effort when they are (at risk of) missing out on their goal by a small amount.

It can be shown that as $\Delta$ approaches zero, so does, \( (1 - F (\beta \sigma_R^2)) e^{2\beta a} - F (-\beta \sigma_R^2) e^{2\beta R}, \)}
and hence, so does the RHS of the condition 3 in Theorem (2). So, the incentive to increase effort falls off before the reference point, contrary to the predictions of prospect theory. This is due to the fact that the utility accruing from the aspiration effect is falling while the utility from the achievement effect is going up.

These results show that an assessment system of the type we have modelled is ineffective at eliciting effort and engagement from the students. The stems from the risk and regret aversion of the student and from the fact that at the time the effort decision is made the student already knows with certainty a percentage of their final grade.

3.6 Effect on Maximised Utility

3.6.1 Effect of $r_1$ on Maximised Utility.

We now consider what effect the disclosure of the first result has on the students maximized utility. The maximized utility can be expressed as a value function and its derivative evaluated via the envelope theorem.

We define the value function as follows.

$$W(r_1, \beta) = \max_{e_2} J(r_1, \beta, e_2)$$

Where,

$$J(r_1, \beta, \beta, e_2) = \int U(R; \beta) f(R - R|r_1) \, dR$$

$$+ k \left( \int D(\Delta \geq 0) U(\Delta; D(\Delta \geq 0)) f(\Delta - \Delta|r_1) \, d\Delta \right) - C(e_2)$$

If $W(\cdot)$ is differentiable then we can write its derivative in the following way, using the envelope theorem.

$$W'(r_1, \beta, \beta) = \left. \frac{\partial J(r_1, \beta, \beta, e_2)}{\partial r_1} \right|_{e_2 = e_2^*(r_1)}$$
It is already evident that the derivative of the objective function \( J(\cdot) \) will be quite similar to the expression obtained in the first order condition \( (3.16) \). We evaluate this derivative and present it below.

\[
\begin{align*}
- \frac{\partial \bar{R}}{\partial r_1} & \int U(R; \beta) f'(R - \bar{R}|r_1) \, dR \\
+ k \left\{ - \frac{\partial \bar{\Delta}}{\partial r_1} \right\} & \int D(\Delta \geq 0) U(\Delta; D(\Delta \geq 0) \beta) f'(\Delta - \bar{\Delta}|r_1) \, d\Delta
\end{align*}
\] (3.25)

The integrals above are identical to those in \( (3.16) \) and can be simplified in the same way. The simplified expression is

\[
\begin{align*}
\frac{\partial \bar{R}}{\partial r_1} & U_1 \left( \bar{R}(e_2^*, r_1) - \frac{\beta \sigma_R^2}{2}; \beta \right) \\
+ k \frac{\partial \bar{\Delta}}{\partial r_1} & \left( 1 - F(\beta \sigma_R^2) \right) U_1 \left( \bar{\Delta} - \frac{\beta \sigma_R^2}{2}; \beta \right) - F(-\beta \sigma_R^2) U_1 \left( \bar{\Delta} + \frac{\beta \sigma_R^2}{2}; -\beta \right)
\end{align*}
\] (3.26)

This expression is positive. To see that this is the case notice that \( \frac{\partial \bar{\Delta}}{\partial r_1} = \frac{\partial \bar{R}}{\partial r_1} \) and from \( (3.12) \) \( \frac{\partial \bar{R}}{\partial r_1} = \frac{(p + \sigma^2)}{(1 + \sigma^2)} \), which is positive. The expression in the brackets on the second line in \( (3.26) \) is identical to the expression in lines (III) and (IV) of the first order condition, which, we have already established is positive.

This means that the maximized value of utility is positively related to the value of the first result. This is due to the the effect that the first result has of subsidizing better final results. It does so by making it cheaper in terms of both aspirations and physical cost to acquire a better result.

### 3.7 First Stage Problem

In section \( (3.4.1) \) we set up the following problem.

\[
\max_{e_1} -C(e_1) + \int W(e_1, r_1) f(r_1) \, dr_1
\] (3.27)
Where $W(e_1, r_1)$ is the second period value function.

$$W(e_1, r_1) = \max_{e_2} \int U(R; \beta) f \left( R - \bar{R} | r_1 \right) dr + k \left( \int D(\Delta \geq 0) U(\Delta; D(\Delta \geq 0) \beta) f(\Delta - \bar{\Delta} | r_1) d\Delta \right) - C(e_2)$$  \hspace{1cm} (3.28)

The problem in its current form is a bit tricky to solve, but we can implement a similar reformulation to the one we used in the second period problem. We define $\delta(e_1) = r_1 - \bar{r}_1$. $\bar{r}_1 = \mu + G(e_1)$ is the mean of the first result. The reformulated maximization problem and value function are as follows.

$$\max_{e_1} -C(e_1) + \int W(\delta, e_1) f(\delta) d\delta$$  \hspace{1cm} (3.29)

$$W(\delta, e_1) = \max_{e_2} J(\delta, e_1)$$

$$= \max_{e_2} \int U(R; \beta) f \left( R - \bar{R}(\delta, e_1) \right) dr + k \left( \int D(\Delta \geq 0) U(\Delta; D(\Delta \geq 0) \beta) f(\Delta - \bar{\Delta}(\delta, e_1)) d\Delta \right) - C(e_2)$$  \hspace{1cm} (3.30)

The first order condition for (3.29) is

$$-C'(e_1) + \int W_{e_1}(\delta, e_1) f(\delta) d\delta + \int W(r_1, e_1) f'(r_1) \frac{\partial r_1}{\partial e_1} dr_1 = 0.$$  \hspace{1cm} (3.31)

From the envelope theorem,

$$W_{e_1}(\delta, e_1) = \left. \frac{\partial J(\delta, e_1, e_2)}{\partial e_1} \right|_{e_2 = e_2^*}.$$  \hspace{1cm} (3.32)

Using this in the first order condition (3.31) yields.

$$\int \frac{\partial \bar{R}}{\partial e_1} U_1 \left( \bar{R}(e_2) - \frac{\beta \sigma_R^2}{2}; \beta \right) f(\delta) d\delta$$

$$+ k \left[ \int \frac{\partial \Delta}{\partial e_1} \left( 1 - F(\beta \sigma_R^2) \right) U_1(\Delta - \frac{\beta \sigma_R^2}{2}; \beta) f(\delta) d\delta - \int \frac{\partial \Delta}{\partial e_1} F(-\beta \sigma_R^2) U_1(\Delta + \frac{\beta \sigma_R^2}{2}; -\beta) f(\delta) d\delta \right]$$

$$= C'(e_1)$$  \hspace{1cm} (3.33)

In section (3.4.2), in equation (3.12) we found an expression for the mean of the result $\bar{R}$. We now rephrase it in terms of $\delta$ and present it below.
\[ \bar{R} = \frac{p + \sigma^2}{1 + \sigma^2} \delta + \mu + pG(e_1) + (1 - p)G(e_2) \]

On this basis we find the following

\[ \frac{\partial \Delta}{\partial e_1} = \frac{\partial \bar{R}}{\partial e_1} - 0 = (1 - p) \left\{ G'(e_2) \frac{\partial e_2^*}{\partial e_1} - \frac{\sigma^2}{1 + \sigma^2} \right\}. \] (3.34)

We can use (3.34) as well as Lemma 1 to simplify the first order condition (3.33). We shall not be doing so in this paper however and will defer it to a future endeavour.

### 3.8 Summary and Concluding Remarks

In this chapter we have aimed to provide the theoretical basis for a model that describes how an agent with reference dependent preferences chooses how much effort to exert upon receiving a signal about their ability.

After providing a comprehensive review of the aspirations literature in 3 of its main areas of study, we have made an attempt to build a model that incorporates educational aspirations, individual effort, beliefs about ability and information updating.

We believe that the setup of the model itself is one of our main contributions. Similarly, we have managed to provide the closed form solution to the problem of choosing optimal effort based on the information obtained in period 1. This is our second and probably our most valuable contribution. This solution will open routes for many additional queries in future research.
Our third contribution has to do with contrasting the choices made in a 'fully rational' setting with those that emerge from our behavioural model. Whether the optimal effort in the presence of aspirations is higher or lower than in the rational setting is determined by the marginal expected aspiration gap. If the marginal expected aspiration gap is positive then additional effort increases the expected result more than it does the aspiration. This causes the student to exert greater effort than they would in a rational setting. This is due to the fact that in our model the student enjoys a premium from exceeding their aspiration. When the marginal expected aspiration gap is positive exerting effort makes it more likely that the student ends up exceeding their aspiration.

Our fourth contribution is to determine what happens to optimal effort when the first result increases. We find that the optimal response to a better first result is almost always to exert less effort in the second period. These results show that an assessment system of the type we have modelled is ineffective at eliciting effort and engagement from the students. The stems from the risk and regret aversion of the student and from the fact that at the time the effort decision is made the student already knows with certainty a percentage of their final grade. The first result hence has the effect of subsidizing the final result by making a better result cheaper in terms of both aspirations and physical costs.

Our fifth contribution is to determine the impact of the first result on the maximized value of utility. We find that it is positively related to the value of the first result. This is due to the effect that the first result has of subsidizing better final results. It does so by making it cheaper in terms of both aspirations and physical cost to acquire a better result.
Our final contribution is to set up the decision problem faced by the student in the first period. The student chooses $e_1$ to influence $r_1$, the continuation value in the second period. We state the first order condition of this problem, but defer analysing it to a later paper.
Chapter 4

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Appendices
Appendix A

Chapter 2

Before beginning the proofs I briefly discuss the probability space I will be working with. The 'experiment' that these lemmas deal with pertains to the monitor who goes around checking tickets. The sample space for this experiment has two ingredients; A permutation, which is a function that describes which order the passengers are checked in and a monitoring outcome vector which describes whether an agent received a good or bad signal. In addition, I need to know about the purchase vector which is a vector describing which agents bought a ticket. The proof of the lemma proceeds by showing that the event \( m(i) \), that agent \( i \) is monitored is more likely when more people have tickets.

Definition 1. The sample space is \( \Omega = \{(\pi, S) \in \mathcal{P}(n) \times \{0,1\}^n \} \), where

- \( \mathcal{P}(n) \) is the set of all permutations of the first \( n \) integers.
- \( S \in \{0,1\}^n \) is a monitoring outcome vector.

I assume that every agent on the train is indexed by an integer between 1 and \( n \). A permutation determines the order in which the agents will be checked. So, for some permutation \( \pi \in \mathcal{P}(n) \), \( \pi_j \) tells us who the \( j^{th} \) person in the checking order
is. Conversely, $\pi^{-1}(j)$ tells us where in the order agent $j$ is.

$S = \{S_1, S_2, S_3, \ldots, S_n\}$ is some monitoring outcome vector. The value of $S_j$ tells us what would happen if person $j$ were to be checked. They could get the good signal ($S_j = 1$) or the bad signal ($S_j = 0$). The difference between $S$ and $\theta$ is that $S$ consists only of 1’s and 0’s whereas $\theta$ contains the null signal as well. The permutation $\pi$ can be applied to the outcome vector to give the outcomes for each agent in the order of checking, i.e, $\pi_j(S)$ tells us what the outcome was for the $j^{th}$ agent in the order.

Using the above definition of a sample space I can define the event(s) $m(i)$ as follows

**Definition 2.** The event that person $i$ is checked is

$$m(i) = \{(\pi, S) \in \Omega ; \quad \pi_t(S) = 1 \quad \text{for all} \quad 1 \leq t \leq \pi^{-1}(i) - 1\}$$

In other words, the event $m(i)$ consists of those $(\pi, S)$ tuples in which the monitoring outcome for all agents before $i$ in the checking order is favourable.

I also define a probability measure $P_u(\cdot)$ as follows

**Definition 3.** $P_u(A)$ gives the probability of some event $A$ occurring if the ticket purchase vector is $u = \{u_1, u_2, u_3, \ldots, u_n\}$. $P_u(\cdot)$ has the following properties.

1. $S_j$ is i.i.d with $P_u(S_j = u_j) > \frac{1}{2}$
2. $\pi$ and $S$ are independent.
3. $P_u((\pi, \cdot)) = \frac{1}{n!}$ for all $\pi, u$

The first property follows from the assumption that the monitor is pretty good at her job, i.e. that the error rates $\varepsilon, (1 - \delta) > \frac{1}{2}$. The second assumption is pretty
reasonable as there is no reason to believe that the order in which agents are checked has any influence on the likelihood of an error. Similarly, it’s reasonable to believe that every permutation is equally likely.

### A.1 Lemma One

I define the mapping $\Phi_j(\cdot)$ in the following way.

**Definition 4.** For some purchase vector $v$, $\Phi_j(v) = [v_1, v_2, v_3, \ldots, \max(1, v_j), \ldots, v_n]$

If the purchase vector contains a zero in the $j^{th}$ position then the above mapping transforms it into a one, leaving it unchanged otherwise. So, the new purchase vector will contain at most one additional ticket holder.

Lemma 1 will hold whenever the following proposition is true.

**Proposition 1.** For any purchase vector $v$ and any $j$, $\mathbb{P}_v(m(i)) \leq \mathbb{P}_{\Phi_j(v)}(m(i))$

Or, in other words, taking any purchase vector and swapping out one non-ticket holder with a ticket holder makes everyone more likely to get checked. This is a more restrictive statement than Lemma 1 as it deals with a change in only one agent’s action whereas Lemma 1 pertains to general changes. However, we can compare any two purchase vectors through repeated applications of the mapping $\Phi(\cdot)$ and hence the following may be regarded as sufficient proof for Lemma 1.

**Proof.** To show that the probability of the event that agent $i$ is monitored, $m(i)$, increases upon the application of the mapping $\Phi_j(\cdot)$, it is sufficient to show this is true for some generic elementary event in that event. An elementary event $(\pi, S)$ is included in the event $m(i)$ if and only if the monitoring outcome is favourable
for every agent that appears before agent $i$ in the permutation $\pi$ (Regardless of their ticket purchase status). I.e. for any $i$, $(\pi, S) \in m(i)$ iff $\pi_t(S) = 1$ for all $1 \leq t \leq \pi^{-1}(i) - 1$. So, pick any $(\pi, S) \in m(i)$. I assume that $\pi^{-1}(j) \leq \pi^{-1}(i) - 1$, as the proof is trivial otherwise. I also assume that $v_j = 0$. Then,

\[
P_v((\pi, S)) = P_v(S)P_v(\pi)
\]

\[
= \left[ \prod_{t=1}^{\pi^{-1}(i)-1} P_v(S_{\pi_t} = 1) \right] P_v(\pi)
\]

\[
= \left[ \prod_{t=1}^{\pi^{-1}(i)-1} P_v(S_{\pi_t} = 1) \right] P_{\phi_j(v)}(\pi)
\]

\[
= \left[ P_v(s_j \neq v_j) \prod_{t=1, t \neq \pi^{-1}(j)}^{\pi^{-1}(i)-1} P_{\phi_j(v)}(S_{\pi_t} = 1) \right] \frac{1}{n!}
\]

\[
\leq \left[ P_{\phi_j(v)}(S_j = v_j) \prod_{t=1, t \neq \pi^{-1}(j)}^{\pi^{-1}(i)-1} P_{\phi_j(v)}(S_{\pi_t} = 1) \right] \frac{1}{n!}
\]

\[
= P_{\phi_j(v)}((\pi, S))
\]

A.2 Lemma Three

I amend the notation from the previous section slightly by adding a superscript denoting the number of agents, i.e $m^k(i)$ is the event that agent $i$ is monitored when there are a total of $k$ agents. Note that the sample space corresponding to each distinct number of agents is different and as such we cannot compare events between them. However, as we will show we can in fact compare the probabilities. Call the sample space with $n$ agents $\Omega_n$. 

\[ \square \]
Let \( v' \) be the ticketing status of the \((n+1)\)th agent. The agent may have no ticket (0), an ordinary ticket (1) or a golden ticket (G), so \( v' \in \{0, 1, G\} \). Let \( v, v' \) be the \((n+1)\) dimensional ticket purchase vector obtained from the \( n \) dimensional \( v \) by appending \( v' \) at the end.

For every \((\pi, S) \in \Omega_n\) there is a corresponding set of events in \( \Omega_{n+1} \). We call this set \( \Omega'(\pi, S) = \varphi(\pi) \times (\{S\} \times \{0, 1\}) \). I.e. Given any fundamental event in \( \Omega_n \), we can define a set of fundamental events \( \Omega'(\pi, S) \subset \Omega_{n+1} \).

We begin by defining the set \( \varphi(\pi) \), where
\[
\varphi(\pi) = \{ \varphi_1(\pi), \varphi_2(\pi), \ldots, \varphi_{n+1}(\pi) \}.
\]
Each element of \( \varphi(\pi) \) is a permutation in \( \mathcal{P}(n+1) \). \( \varphi_j \) is the permutation where the new agent is the \( j \)th person checked and every one else is checked in the same order as in \( \pi \). \( \varphi_{j,k} \) is the \( k \)th element of \( \varphi_j \) (i.e. this is the identity of the \( k \)th person checked as per the permutation \( \varphi_j \)). We define \( \varphi_j \) as follows
\[
\varphi_{j,k} = \begin{cases} 
\pi_k & \text{for } k < j \\
(n + 1) & \text{for } k = j \\
\pi_{k-1} & \text{for } k > j.
\end{cases}
\]

The extension of the monitoring outcome vector to the case with \( n + 1 \) agents is straightforward. The new outcome vector is \( S^{n+1} \in (\{S\} \times \{0, 1\}) \). I.e., We keep the outcome for the first \( n \) agents the same as in \( S \) and then consider a zero or a one as the outcome for the final agent. Note the following

1. We are only concerned about \( S^{n+1} = (\{S\}, 1) \) as fundamental events where \( S^{n+1}_{n+1} = 0 \) are never a part of the event \( m^{n+1}(i) \).
2. $\Pr_{v,G}(S^{n+1}_{n+1} = 1) = 1$, since the golden ticket is beyond reproach

3. $1 > \Pr_{v,1}(S^{n+1}_{n+1} = 1) > \Pr_{v,0}(S^{n+1}_{n+1} = 1) > 0$

We restate lemma 2 in the following way

**Proposition 2.** For each $(\pi, S) \in \Omega_n$ there exists a corresponding $\Omega'(\pi, S) \subset \Omega_{n+1}$ such that:

1. $\Omega'(\pi, S)$ is unique
2. $\Pr_v((\pi, S) \in m^n(i)) \geq \Pr_{v,v'}(\Omega'(\pi, S) \subset m^{n+1}(i)) \quad \forall v' \in \{0, 1, G\}$

**Proof.**

1. This is true because we have constructed it in such a way.

2. We know that the probability that any pair $(\pi, S)$ is in $m^n(i)$ is given by

   $$\Pr_v((\pi, S) \in m^n(i)) = \prod_{t=1}^{n+1} \Pr_v(S_{\pi_t} = 1) \Pr_v(\pi).$$

   We first consider the case when $v' = G$

   (a)

   $$\Pr_{v,G}(\Omega'(\pi, S) \subset m^{n+1}(i)) = \sum_{j=1}^{n+1} \prod_{t=1}^{\varphi_j^{-1}(i)-1} \Pr_{v,G}(S^{n+1}_{\varphi_{j,t}} = 1) \frac{1}{(n + 1)!}\quad(A.7)$$

   We know that there are $(n + 1)$ permutations in the set $\varphi(\pi)$, as there are $(n + 1)$ different positions that the extra agent can occupy. Each of these permutations is equally likely.

   If the new agent does not appear before agent $i$ in the permutation $\varphi_j$, then by definition.
\[
\prod_{t=1}^{\pi^{-1}(i)-1} \mathbb{P}_{v,G}(S_{\phi_{j,t}}^{n+1} = 1) = \prod_{t=1}^{\pi^{-1}(i)-1} \mathbb{P}_{v}(S_{\pi_t} = 1) . \tag{A.8}
\]

However, since \(\mathbb{P}_{v,G}(S_{n+1}^{n+1} = 1) = 1\), even if the extra agent does appear before \(i\) in \(\phi_j\),

\[
\prod_{t=1}^{\phi_{j}^{-1}(i)-1} \mathbb{P}_{v,G}(S_{\phi_{j,t}}^{n+1} = 1) = \left[ \prod_{t=1, t \neq n+1}^{\phi_{j}^{-1}(i)-1} \mathbb{P}_{v,G}(S_{\phi_{j,t}}^{n+1} = 1) \right] \mathbb{P}_{v,G}(S_{n+1}^{n+1} = 1) = \left[ \prod_{t=1, t \neq n+1}^{\pi^{-1}(i)-1} \mathbb{P}_{v}(S_{\pi_t} = 1) \right] \mathbb{P}_{v,G}(S_{n+1}^{n+1} = 1)
\]

So, equation (A.8) is true for all \(\phi_j \in \phi(\pi)\). We can thus rewrite equation (A.7) as follows

\[
\mathbb{P}_{v,G}(\Omega'(\pi, S) \subset m^{n+1}(i)) = \sum_{j=1}^{n+1} \left[ \prod_{t=1}^{\phi_{j}^{-1}(i)-1} \mathbb{P}_{v,G}(S_{\phi_{j,t}}^{n+1} = 1) \right] \frac{1}{(n+1)!} \\
= \left[ \prod_{t=1}^{\pi^{-1}(i)-1} \mathbb{P}_{v}(S_{\pi_t} = 1) \right] \frac{(n+1)}{(n+1)!} \\
= \left[ \prod_{t=1}^{\pi^{-1}(i)-1} \mathbb{P}_{v}(S_{\pi_t} = 1) \right] \frac{1}{n!} \\
= \mathbb{P}_v((\pi, S) \in m^n(i)) . \tag{A.9}
\]

So, Proposition 2 has been proved for the \(v' = G\) case

We now turn to the case when the extra agent has a regular ticket.
(b) We compare this case with the case of the golden ticket. Like in equation \( (A.8) \), when agent \( n + 1 \) appears after agent \( i \) in the permutation the following equation holds.

\[
\prod_{t=1}^{\varphi_j^{-1}(i)-1} \mathbb{P}_{v,1}(S_{\varphi_j,t}^{n+1} = 1) = \prod_{t=1}^{\varphi_j^{-1}(i)-1} \mathbb{P}_{v,G}(S_{\varphi_j,t}^{n+1} = 1) \\
= \prod_{t=1}^{\pi^{-1}(i)-1} \mathbb{P}_v(S_{\pi_t} = 1).
\]

We know that \( \mathbb{P}_{v,1}(S_{n+1}^{n+1} = 1) < 1 \), Hence, when the extra agent appears before agent \( i \),

\[
\prod_{t=1}^{\varphi_j^{-1}(i)-1} \mathbb{P}_{v,1}(S_{\varphi_j,t}^{n+1} = 1) = \left[ \prod_{t \neq n+1}^{\varphi_j^{-1}(i)-1} \mathbb{P}_{v,1}^{n+1}(S_{\varphi_j,t}^{n+1} = 1) \right] \mathbb{P}_{v,1}^{n+1}(S_{n+1}^{n+1} = 1) \\
< \left[ \prod_{t \neq n+1}^{\varphi_j^{-1}(i)-1} \mathbb{P}_{v,1}^{n+1}(S_{\varphi_j,t}^{n+1} = 1) \right] \mathbb{P}_{v,G}^{n+1}(S_{n+1}^{n+1} = 1)
\]

Therefore,

\[
\mathbb{P}_{v,1}\left(\Omega'(\pi, S) \subset m^{n+1}(i)\right) = \sum_{j=1}^{n+1} \left[ \prod_{t=1}^{\varphi_j^{-1}(i)-1} \mathbb{P}_{v,1}(S_{\varphi_j,t}^{n+1} = 1) \right] \frac{1}{(n + 1)!} \\
\leq \sum_{j=1}^{n+1} \left[ \prod_{t=1}^{\varphi_j^{-1}(i)-1} \mathbb{P}_{v,G}(S_{\varphi_j,t}^{n+1} = 1) \right] \frac{1}{(n + 1)!} \\
= \mathbb{P}_{v,G}\left(\Omega'(\pi, S) \subset m^{n+1}(i)\right) \quad (A.10)
\]

Combining equations \( (A.9) \) and \( (A.10) \) we see that the probability of being monitored falls when you add an extra ticketed agent. We can repeat the same logic to establish the case when the extra agent does not have a ticket and hence complete the proof.
Appendix B

Chapter 3

B.1 Lemma 1

We first prove the following Lemma which we will use in the proof of Lemma 1.

Lemma 5. If $x \sim \mathcal{N}(\mu, \sigma_R^2)$ with p.d.f $f(x)$ then,

$$e^{ax} f(x) = e^{a(\mu + \frac{\alpha^2}{2})} f(x - a\sigma_R^2)$$  \hspace{1cm} (B.1)

Proof. Using the functional form of the normal density function for $X$ and combining it with $e^{ax}$. We get a new distribution over $x$ ($f_2(x)$), with mean $\mu + a\sigma^2$, multiplied by a constant. i.e.
\[ e^{ax}f(x) = \frac{1}{\sqrt{2\pi \sigma_R^2}} e^{-\frac{(x-x)^2}{2\sigma^2}} e^{ax} \]
\[ = \frac{1}{\sqrt{2\pi \sigma_R^2}} e^{-\frac{(x-x+\sigma^2 a^2 R^2)}{2\sigma^2}} e^{a(x+x)} \]
\[ = \frac{1}{\sqrt{2\pi \sigma_R^2}} e^{-\frac{(x-(x-a)^2)}{2\sigma^2}} e^{a+x+\frac{a^2 \sigma^2 R^2}{2}} \]
\[ = f_2(x)e^{a+x+\frac{a^2 \sigma^2}{2}} \quad (B.2) \]

However, we can express \( f_2(\cdot) \) using the distribution \( f(\cdot) \) as follows,
\[ f_2(x) = f(x - a \sigma^2 R). \]
Doing so gives the desired result \( (B.1) \). \qed

We prove Lemma 1 in parts as follows.

**Proof.** Consider the following:

1. Using (3.3)
\[ U(x - b; a)f(x) = f(x) - (e^{-a(x-b)} f(x)) \quad (B.3) \]

2. Using Lemma (B.1) to evaluate \( e^{-a(x-b)} f(x) \)
\[ e^{-a(x-b)} f(x) = e^{ab} (e^{-ax} f(x)) \]
\[ = e^{ab} e^{-a(\pi - a^2 b \sigma^2)} f(x + a \sigma^2) = e^{-a(\pi - b - a^2 \sigma^2 b \sigma^2)} f(x + a \sigma^2) \quad (B.4) \]

3. From (3.5),
\[ U_1(x : a) = ae^{-ax}, \]
which implies that,
\[ e^{-a(\pi - b - a^2 \sigma^2 b \sigma^2)} = \frac{U_1(\pi - b - a \sigma^2; a)}{a}. \]
4. So,

\[ U(x-b; a)f(x) = f(x) - (e^{-a(x-b)}f(x)) = f(x) - \left( \frac{U_1(\pi - b - a\sigma^2; a)}{a} f(x + a\sigma^2_R) \right) \]

as required.

5. Similarly, using Lemma 2, \( U_1(x-b; a)f(x) = U_1(x-b - \frac{a\sigma^2_2}{2}; a)f(x + a\sigma^2_R) \).

\[ \square \]

**B.2 Theorem 1**

For almost all possible parameter values, the optimal effort exerted in the second period is inversely related to the first period result. i.e.

\[ \frac{d\epsilon_2^*(r_1)}{dr_1} < 0 \]

**Proof.** We begin by noting that \( \varphi_{\epsilon_2}(r_1, \epsilon_2^2) \) is the second derivative of the objective function with respect to \( \epsilon_2^2 \) evaluated at the optimal value \( \epsilon_2^2 \) and hence, owing to the second order condition for a maximum, \( \varphi_{\epsilon_2}(r_1, \epsilon_2^2) < 0 \). The sign of \( \frac{d\epsilon_2^*(r_1)}{dr_1} \) then depends on the numerator of 3.21 which we evaluate below:

\[ \varphi_{\epsilon_2}(r_1, \epsilon_2^2) = \frac{\partial^2 \bar{R}(\epsilon_2^2)}{\partial \epsilon_2^2} U_1 \left( \bar{R}(\epsilon_2^2) - \frac{\beta\sigma^2_R}{2}; \beta \right) + \left( \frac{\partial \bar{R}}{\partial \epsilon_2^2} \right)^2 U_2 \left( \bar{R}(\epsilon_2^2) - \frac{\beta\sigma^2_R}{2}; \beta \right) 
+ \frac{k}{2} \left( \frac{\partial^2 \Delta(\epsilon_2^2)}{\partial \epsilon_2^2} \right) \left( \left( 1 - F \left( \frac{\beta\sigma^2_R}{2} \right) \right) U_1 \left( \bar{A} - \frac{\beta\sigma^2_R}{2}; \beta \right) - F \left( -\beta\sigma^2_R \right) U_2 \left( \bar{A} + \frac{\beta\sigma^2_R}{2}; \beta \right) \right) 
+ \left( f \left( \frac{\beta\sigma^2_R}{r_1} \right) U_1 \left( \bar{A} - \frac{\beta\sigma^2_R}{2}; \beta \right) + f \left( -\beta\sigma^2_R | r_1 \right) U_1 \left( \bar{A} + \frac{\beta\sigma^2_R}{2}; \beta \right) \right) \right) 
- C''(\epsilon_2^2) \]
\[-\varphi_{r}(r_1, e_2^*)\]
\[
= - \left[ \frac{\partial R}{\partial e_2} U_2 \left( R(e_2^*) - \frac{\beta \sigma_R^2}{2}; \beta \right) \frac{\partial R}{\partial r_1} \right. \\
+ k \frac{\partial \Delta}{\partial e_2} \frac{\partial R}{\partial r_1} \left\{ \left( 1 - F(\beta\sigma_R^2) \right) U_2 \left( \Delta - \frac{\beta \sigma_R^2}{2}; \beta \right) - F(-\beta\sigma_R^2) U_2 \left( \Delta + \frac{\beta \sigma_R^2}{2}; -\beta \right) \right\} \\
\left. + \left( f(\beta\sigma_R^2|r_1) U_1 \left( \Delta - \frac{\beta \sigma_R^2}{2}; \beta \right) + f(-\beta\sigma_R^2|r_1) U_1 \left( \Delta + \frac{\beta \sigma_R^2}{2}; -\beta \right) \right) \right\} \right]
\end{align*}

The following result allows us to eliminate the term in (B.7).

**Result 1.** The term in (B.7)
\[
\left( f(\beta\sigma_R^2|r_1) U_1 \left( \Delta - \frac{\beta \sigma_R^2}{2}; \beta \right) + f(-\beta\sigma_R^2|r_1) U_1 \left( \Delta + \frac{\beta \sigma_R^2}{2}; -\beta \right) \right) \right\} = 0
\end{align*}

This can be proved by substituting the functional forms for \( f(\cdot) \) and \( U_1(\cdot) \).

We are left with the following:
\[
\begin{align*}
- \frac{\partial R}{\partial r_1} \left[ \frac{\partial R}{\partial e_2} U_2 \left( R(e_2^*) - \frac{\beta \sigma_R^2}{2}; \beta \right) \right. \\
\left. + k \frac{\partial \Delta}{\partial e_2} \left\{ \left( 1 - F(\beta\sigma_R^2) \right) U_2 \left( \Delta - \frac{\beta \sigma_R^2}{2}; \beta \right) - F(-\beta\sigma_R^2) U_2 \left( \Delta + \frac{\beta \sigma_R^2}{2}; -\beta \right) \right\} \right]
\end{align*}

The following results allow us to simplify the above further.

**Result 2.** \( U_2(x; b) = -bU_1(x; b) \)

**Result 3.** From the First Order Condition (I)-(IV),
\[
\begin{align*}
\frac{\partial R}{\partial e_2} U_1 \left( R(e_2) - \frac{\beta \sigma_R^2}{2}; \beta \right) &= \\
C'(e_2) - k \frac{\partial \Delta}{\partial e_2} \left\{ \left( 1 - F(\beta\sigma_R^2) \right) U_1 \left( \Delta - \frac{\beta \sigma_R^2}{2}; \beta \right) - F(-\beta\sigma_R^2) U_1 \left( \Delta + \frac{\beta \sigma_R^2}{2}; -\beta \right) \right\}
\end{align*}
\]
Applying Result 2 to (B.9) yields

\[ \beta \frac{\partial \bar{R}}{\partial r_1} \left[ \frac{\partial \bar{R}}{\partial e_2} U_1 \left( \bar{R} - \frac{\beta \sigma_R^2}{2} ; \beta \right) \right] + k \frac{\partial \bar{\Delta}}{\partial e_2} \left\{ (1 - F (\beta \sigma_R^2)) U_1 \left( \Delta - \frac{\beta \sigma_R^2}{2} ; \beta \right) + F (-\beta \sigma_R^2) U_1 \left( \Delta + \frac{\beta \sigma_R^2}{2} ; -\beta \right) \right\} \]

(B.10)

Result 3 allows us to substitute for \( \frac{\partial \bar{R}}{\partial e_2} U_1 \left( \bar{R} - \frac{\beta \sigma_R^2}{2} ; \beta \right) \) in (B.10), leaving us with

\[ -\varphi_{r_1}(r_1, e^*_2) = \beta \frac{\partial \bar{R}}{\partial r_1} \left[ C'(e^*_2) + k \frac{\partial \bar{\Delta}}{\partial e_2} \left\{ 2F (-\beta \sigma_R^2) U_1 \left( \bar{\Delta} + \frac{\beta \sigma_R^2}{2} ; -\beta \right) \right\} \right] \]

(B.11)

We know that \( \beta, \frac{\partial \bar{R}}{\partial r_1}, C'(e^*_2), k, \sigma_R^2 \) and \( F (-\beta \sigma_R^2) \) are all positive and that \( U_1 \left( \bar{\Delta} + \frac{\beta \sigma_R^2}{2} ; -\beta \right) \) is negative. \( \bar{\Delta}'(e^*_2) \) may be either positive or negative.

If \( \bar{\Delta}'(e^*_2) \) is negative then \( C'(e^*_2) + k \frac{\partial \bar{\Delta}}{\partial e_2} \left\{ 2F (-\beta \sigma_R^2) U_1 \left( \bar{\Delta} + \frac{\beta \sigma_R^2}{2} ; -\beta \right) \right\} \) is the sum of two positive expressions, meaning that \( -\varphi_{r_1}(r_1, e^*_2) \) is positive and hence \( \frac{de^*_2(r_1)}{dr_1} < 0 \).

Things get a little more complicated when \( \bar{\Delta}'(e^*_2) \) is positive. To analyse this case, we return to (B.10). Like in the earlier case we can see that if the expression in the curly brackets,

\[ \left\{ (1 - F (\beta \sigma_R^2)) U_1 \left( \Delta - \frac{\beta \sigma_R^2}{2} ; \beta \right) + F (-\beta \sigma_R^2) U_1 \left( \Delta + \frac{\beta \sigma_R^2}{2} ; -\beta \right) \right\}, \]

is positive then the entire expression is positive implying that \( \frac{de^*_2(r_1)}{dr_1} < 0 \).

It turns out that the sign of the curly brackets depends entirely on the value of \( \bar{\Delta} \).

When \( \bar{\Delta} \geq 0 \) i.e. when the aspiration, \( a(e_2) \leq \bar{R} \) then \( \frac{de^*_2(r_1)}{dr_1} < 0 \).
Even when $\bar{\Delta} \leq 0$, i.e. when the aspiration $a(e_2) \geq \bar{R}$, for sufficiently large aspiration gaps, $\frac{de^*_2(r_1)}{dr_1} < 0$ still holds.

However, when the gap is very small, the opposite is true, i.e. that $\frac{de^*_2(r_1)}{dr_1} > 0$.

We also find that as the magnitude of the aspiration increases, the range of aspiration gaps for which $\frac{de^*_2(r_1)}{dr_1} > 0$ grows.