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Closing the Loop:
The integration of long-term ambient vibration monitoring in Structural Engineering design

Zachariah Wynne

Doctor of Philosophy
University of Edinburgh
2022
Declaration

I declare that this thesis is an original report of my research, has been written by me and has not been submitted for any previous degree. The experimental work is almost entirely my own work; the collaborative contributions have been indicated clearly and acknowledged in full below. Due references have been provided on all supporting literature and resources. I confirm that the work submitted is my own, except where work which has formed part of jointly-authored publications has been included. My contribution and those of the other authors to this work have been explicitly indicated below. I confirm that appropriate credit has been given within this thesis where reference has been made to the work of others.

The work presented in Chapter 3 was previously published in Structures as Perceptions of long-term monitoring for civil and structural engineering by Zachariah Wynne (student), Tim Stratford (co-supervisor) and Thomas P.S. Reynolds (primary supervisor). This study was conceived by all of the authors. I carried out the development of the survey used in the research, the dissemination of the survey and analysis of the survey responses, and was the primary and corresponding author of the paper.

The work presented in Chapter 6 was previously published in Structural Control and Health Monitoring as Reducing coherent filtering artefacts in time-domain operational modal analysis by Zachariah Wynne (student), James R. Hopgood, Tim Stratford (co-supervisor) and Thomas P.S. Reynolds (primary supervisor). This study was conceived by all of the authors. I made the initial observations which prompted the research, carried out the development of the methods used in the research, the analysis of the data and was the primary and corresponding author of the paper.

Data collection for the work on the Aberfeldy footbridge, presented in Chapter 7, was assisted by Rich May and facilitated by the Aberfeldy Golf Club. Analysis of this data was published previously as a chapter in Civil Structural Health Monitoring as Operational modal analysis of a historic GRP structure by Zachariah Wynne (student), Tim Stratford (co-supervisor) and Thomas P.S. Reynolds (primary supervisor). I carried out the collection of the data, analysis of the data and was the primary and corresponding author of the paper. Additional data used in this study was provided by Tim Stratford (co-supervisor).

Data collection for the work on the Whitmore Building, presented in Chapter 7, was carried out by Thomas P.S. Reynolds (primary supervisor).

The work on lake seiches presented in Chapter 7 builds on work that was previously published in Environmental Fluid Mechanics as A novel technique
for experimental modal analysis of barotropic seiches for assessing lake energetics by Zachariah Wynne (student), Thomas P.S. Reynolds (primary supervisor), Damien Bouffard, Geoffrey Schladow and Danielle Wain. This work was also published by the same authors in proceedings from the 8th International Operational Modal Analysis Conference as Operational modal analysis of low frequency surface waves in lakes and reservoirs. This study was conceived by all of the authors. I carried out the analysis of the data and was the primary and corresponding author of both papers. Data from Lake Geneva is provided by the Swiss Federal Office for the Environment. Data for Lake Tahoe is provided by Heather Sprague of the University of California at Davis.

The work on the vibration response of the MX3D Bridge presented in Chapter 7 is currently under review for publication in Case Studies in Construction Materials as Dynamic testing and analysis of the world’s first metal 3D printed bridge by Zachariah Wynne (student), Craig Buchanan, Pinelopi Kyvelou, Leroy Gardner, Rolands Kromanis, Tim Stratford (co-supervisor) and Thomas P.S. Reynolds (primary supervisor). This study was conceived by all of the authors. I carried out the collection of data for the analysis of the impact hammer response, designed the methodology for collection of the pedestrian crossing data, conducted the data analysis and was lead and corresponding author of the paper. The description of the finite element model in Chapter 7 was written in collaboration with Pinelopi Kyvelou.

The data from the MX3D Bridge, used in Chapter 7 and Chapter 9, is provided by the Smarter Bridge consortium which I have been an active member of since 2019, advising on the installation, testing and verification of the sensor network. A description of the sensor network used on the MX3D Bridge is currently being prepared for submission as Design and installation of the sensor network for the world’s first metal 3D printed bridge by Craig Buchanan, Theo Glashier, Thomas P.S. Reynolds (primary supervisor), Zachariah Wynne (student), Sage Cammers-Goodwin, Rolands Kromanis, Farid Vahdatikhaki, Alex Tessier, Alec Shuldiner, Kean Walmsley, Josh Cameron, Farhad Javid, Jacky Bibliowicz, Pan Zhang, Liviu Calin, Nigel Morris, Mike Lee, Adrian Butscher, Arne Vogt, Eric Putnam, and Eric Daub.

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The work on quantifying the change in natural frequencies of a beam structure with temperature and mass presented in Chapter 9 was previously published as a chapter in Civil Structural Health Monitoring as Mass and temperature changes in operational modal analysis by Zachariah Wynne (student), George Kanellopoulos, Vasileios Koutsomarkos, Angus Law, Tim Stratford (co-supervisor) and Thomas P.S. Reynolds (primary supervisor). This study was conceived by all of the authors. I carried out the collection of all acceleration data used in these studies, designed the methodology for collection of the data, conducted the data analysis and was lead and corresponding author of the paper.

The work on random sampling mass estimation presented in Chapter 9 was previously published in Engineering Structures as Quantifying mass changes with ambient vibration measurements by Zachariah Wynne (student), Tim Stratford (co-supervisor) and Thomas P.S. Reynolds (primary supervisor). I carried out the
collection of all data used in this study, designed the methodology for collection
of the data, developed the methods presented in the paper, conducted all data
analysis and was lead and corresponding author of the paper.

(Zachariah Wynne)
Abstract

This study investigated the integration of long-term monitoring into the structural engineering design process to improve the design and operation of civil structures. A survey of civil and structural engineering professionals, conducted as part of this research, identified the cost and complexity of in-situ monitoring as key barriers to their implementation in practice. Therefore, the research focused on the use of ambient vibration monitoring as it offers a low cost and unobtrusive method for instrumenting new and existing structures.

The research was structured around the stages of analysing ambient vibration data using operational modal analysis (OMA), defined in this study as: i) pre-selection of analysis parameters, ii) pre-processing of the data, iii) estimation of the modal parameters, iv) identification of modes of vibration within the modal estimates, and v) using modal parameter estimates as a basis for understanding and quantifying in-service structural behaviour.

A method was developed for automating the selecting of the model order, the number of modes of vibrations assumed to be identifiable within the measured dynamic response. This method allowed the modal estimates from different structures, monitoring periods or analysis parameters to be compared, and removed part of the subjectivity identified within current OMA methods.

Pre-processing of ambient acceleration responses through filtering was identified as a source of bias within OMA modal estimates. It was shown that this biasing was a result of filtering artefacts within the processed data. Two methods were proposed for removing or reducing the bias of modal estimates induced by filtering artefacts, based on exclusion of sections of the response corrupted by the artefacts or fitting of the artefacts as part of the modal analysis.

A new OMA technique, the short-time random decrement technique (ST-RDT) was developed on the basis of the survey of industry perceptions of long-term monitoring and limitations of existing structural monitoring techniques identified within the literature. Key advantages of the ST-RDT are that it allows the uncertainty of modal estimates and any changes in modal behaviour to be quan-
tified through subsampling theory. The ST-RDT has been extensively validated with numerical, experimental and real-world case studies including multi-storey timber buildings and the world’s first 3D printed steel bridge.

Modal estimates produced using the ST-RDT were used as a basis for developing an automated method of identifying modes of vibration using a probabilistic mixture model. Identification of modes of vibration within OMA estimates was previously a specialized skill. The procedure accounts for the inherent noise associated with ambient vibration monitoring and allows the uncertainty within the modal estimates associated with each mode of vibration to be quantified.

Methods of identifying, isolating and quantifying weak non-linear modal behaviour, changes in dynamic behaviour associated with changes in the distributions of mass or stiffness within a structure have been developed based on the fundamental equations of structural dynamics. These methods allow changes in dynamic behaviour associated with thermally-induced changes in stiffness or changes in static loading to be incorporated within the automated identification of modes of vibration. These methods also allow ambient vibration monitoring to be used for estimating structural parameters usually measured by more complex, expensive or delicate sensors. Examples of this include estimating the change in elastic modulus of simple structures with temperature or estimating the location and magnitude of static loads applied to a structure in-service.

The methods developed in this study are applicable to a wide range of structural monitoring technologies, are accessible to non-specialist audiences and may be adapted for the monitoring of any civil structure.
Lay summary

To address the climate emergency, the way in which bridges, buildings and other civil infrastructure are designed, built and maintained must evolve.

The current design process is linear: a structure is designed using the available information, it is built, and that is typically the end of the designers’ involvement. Measuring the way structures respond to loading in use may allow more efficient structural designs to be created, damage to be detected and repaired earlier, and might help the performance of new materials to be verified once they are installed.

One method of measuring the behaviour of structures is through ambient vibration monitoring using accelerometers. Accelerometers measure how a structure moves when a force is applied to it and are popular for measuring structural behaviour as they are relatively low cost, durable, and can be easily installed on new and existing structures. However, for engineers to understand accelerometer data it must be converted into the modal parameters: the natural frequencies (the speeds at which a structure vibrates), damping ratios (how quickly vibrations die away) and mode shapes (the relative size of vibrations at different parts of the structure). This can be achieved using operational modal analysis (OMA) which makes use of the underlying physics of vibrating systems and the statistical properties of the measured acceleration data.

Existing methods of OMA are complex and time consuming, requiring expert knowledge and subjective judgements. They also do not give clear indications of how the modal parameters might change over time due to changes in temperature, loading or damage to the structure, or how confident we can be in our estimates of them.

This research developed new methods which automate the OMA procedure, provide indications of how modal parameters change over time, and communicate to engineers the confidence in modal parameter estimates. The methods are demonstrated using data from a wide range of real structures including a 3D printed steel bridge and multi-storey timber buildings. The methods developed in this research can be applied to a wide range of other measurements and are designed to be easily understood and applied by engineers.
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Nomenclature

\( \Delta L \)  Sum change in static loading
\( \Delta T \)  Average change in temperature
\( \Delta k \)  Real valued average change in effective modal stiffness
\( \Delta m \)  Change in mass
\( \Delta t \)  Time step
\( \Delta \)  Non-Gaussian mixture model parameters
\( \Delta_{GMM} \)  Gaussian mixture model parameters
\( \Sigma_c \)  Diagonal covariance matrix of mixture model distribution
\( \Theta \)  Vector of combined static load and scaled static load location factors
\( \alpha \)  General mode shape scaling factor
\( \alpha_0 \)  Decay rate
\( \alpha_K \)  Temperature change factor
\( \alpha_L \)  Static load change factor
\( \alpha_M \)  Mass-normalized mode shape scaling factor
\( \alpha_c \)  Confidence intervals of subsampled data
\( \alpha_{max} \)  Amplitude of \( \omega_{peak} \)
\( \beta_L \)  Load location weighting factor
\( \beta_m \)  Scaled load location factor
\( \delta_t \)  Time lag relative to start of data segment
\( \eta(\tau) \) Ratio of damping envelope and expected variance of noise within random decrement signature at time lag \( \tau \)

\( \gamma \) Modal parameters of a system at a specific point in time

\( \hat{\theta} \) Parameters of a subsample

\( \perp \perp \) Variables are independent

\( \lambda \) Complex eigenvalues

\( \lambda_c \) Confidence intervals of parameterized model

\( \lambda_\zeta \) Rayleigh damping equation factor

\( \mu \) Means of mixture model distribution

\( \mu_c \) Mean average value of subsampled data

\( \mu_k \) Mean of data objects assigned to cluster

\( \mu_{\zeta,k} \) Mean of damping estimates assigned to cluster

\( \mu_\zeta \) Rayleigh damping equation factor

\( \mu_{e,m} \) Amplitude of the unity-scaled mode shape at the excitation location

\( \mu_{f,k} \) Mean of frequency estimates assigned to cluster

\( \mu_{r,m} \) Amplitude of the unity-scaled mode shape at the response measurement location

\( \not\perp \not\perp \) Variables are not independent

\( \omega \) Natural frequency, radians per second

\( \omega_0 \) Baseline natural frequency of a mode, radians per second

\( \omega_a \) Frequency used in half-power bandwidth method, radians per second

\( \omega_b \) Frequency used in half-power bandwidth method, radians per second

\( \omega_c \) Filter cutoff frequency, radians per second

\( \omega_d \) Damped natural frequency, radians per second

\( \omega_{\text{peak}} \) Peak of frequency spectrum, radians per second
\( \phi \) Unity-scaled mode shape

\( \phi_M \) Mass-normalized mode shape

\( \phi_{j,\text{lag}} \) Phase lag of modal component

\( \psi \) Arbitrarily scaled mode shape

\( \rho_{h,m} \) Factor relating stride length to length of floor span

\( \rho_{x,y} \) Time-lagged correlation function of data sets \( x \) and \( y \)

\( \sigma_k^2 \) Variance of Gaussian distribution

\( \sigma_n \) Standard deviation of the frequency estimates for mode \( n \)

\( \sigma_n^2 \) Variance of a white noise stochastic process

\( \sigma_x^2 \) Variance of data set \( x \)

\( \sigma_{A_0}^2 \) Variance of the initial amplitude component of a random decrement signature

\( \sigma_{A_F}^2 \) Variance of the forced response of a random decrement signature

\( \sigma_{A_R}^2 \) Variance of free-response components of a random decrement signature

\( \sigma_{A_d}^2 \) Variance of the first derivative response of a random decrement signature

\( \sigma_\varepsilon^2 \) Variance of noise in a random decrement signature

\( \sigma_{\varepsilon,n}^2 \) Variance of damping ratio estimates for mixture model component \( n \)

\( \sigma_{f,n}^2 \) Variance of natural frequency estimates for mixture model component \( n \)

\( \tau \) Time lag

\( \tau_Q \) Quantile level for quantile regression

\( \theta \) Estimated modal parameters

\( \theta_0 \) True distribution of parameters

\( \theta_n \) Noise mode modal parameters

\( \theta_\varepsilon \) Corruption of modal parameters

\( \varepsilon \) Function defining measurement noise
\( \varepsilon_D \) Distance from cluster boundary in DBSCAN
\( \zeta \) Damping ratio
\( \zeta_0 \) Mean damping ratio
\( \zeta_m \) Damping ratio of mode \( m \)
\( \zeta_{\text{max}} \) Maximum observed damping ratio
\( \zeta_{\text{min}} \) Minimum observed damping ratio
\( \{L\} \) Vector of static load measurements
\( \{T\} \) Vector of temperature measurements
\( \{\omega\} \) Vector of frequency estimates
\( \{\psi\}^* \) Complex conjugate of mode shape vector
\( \{\psi\}^T \) Transpose of mode shape vector
\( A \) Amplitude of mode shape
\( A_T \) Random decrement trigger amplitude
\( A_j \) Amplitude of mode shape at location \( j \)
\( A_s \) State matrix
\( A_x \) Analysis channel of data
\( A_0 \) Amplitude of initial amplitude response components of random decrement signature
\( A_F \) Amplitude of forced response components of random decrement signature
\( A_R \) Free response components combined amplitude
\( A_d \) Amplitude of the initial first derivative response components of random decrement signature
\( A_{m,hd} \) Amplitude of mobility mode shape at accelerometer location
\( A_{m,\text{max}} \) Maximum absolute amplitude of mobility mode shape
\( B \) Regression coefficient
\( E(X) \) Expected value of data set \( X \)

\( F \) Force

\( F_h \) Harmonic force applied by pedestrian footfall

\( G_{xx\omega} \) Power spectral density of measured response \( x \) at frequency \( \omega \)

\( H[f] \) Frequency roll-off of moving average filter

\( H_a, H_b, H_c \) Scale factors for piecemeal approximation of probability density function

\( H_{\text{ideal}} \) System frequency response for an ideal filter

\( L \) Static load

\( L_b \) Overlap between ST-RDT windows

\( L_w \) ST-RDT window length

\( L_{\text{higher}} \) Higher limit of uniform distribution

\( L_{\text{lower}} \) Lower limit of uniform distribution

\( L_{rd} \) Length of random decrement signature

\( MAC \) Modal assurance criterion

\( M_e \) Eigenvalues

\( M_k \) Rank of singular value decomposition

\( N \) Number of modes of vibration

\( N_0 \) Size of subsample

\( N_s \) Number of samples

\( N_{rd} \) Number of segments in RDS

\( N_{\text{total}} \) Total length of data set

\( N_{\text{windows}} \) Number of windows in subsampling analysis

\( R \) Vibration response factor

\( R^2 \) Coefficient of determination
$R_{filt}$ Scale factor for filtering artefact removal

$R^2_{\text{arg}}$ Target coefficient of determination

$S_j$ Singular value $j$

$S_{\text{res}}$ Sum of squared residuals

$S_{\text{tot}}$ Total Sum of squares

$T$ Temperature

$T_j$ Block Toeplitz matrix

$T_j$ Period of oscillation of mode $j$

$T_r$ Minimum data set length for estimating modal parameters

$T_s$ Sampling period of discrete-time process

$T_x$ Trigger channel of data

$T_{x(t)}$ Binary random decrement trigger condition for data set $x$ at time $t$

$U_j$ Singular vector $j$

$VI$ Mean absolute volatility of data

$\text{Var}[^{\hat{D}}_{xT}]$ Variance of random decrement signature $^{\hat{D}}_{xT}$

$W_t$ Weighting factor used in ST-RDT

$W_x$ Weighting channel of data

$X$ Response of a system

$X_0$ Response of a system due to the initial amplitude of the response at $\tau = 0$

$X_F$ Response of a system due to forcing applied at $\tau > 0$

$X_d$ Response of a system to the initial first derivative of the response at $\tau = 0$

$X_k$ Discrete Fourier coefficients of data $x$

$X_n$ Noise in the sensor used to measure the response of a system

$X_{\text{data}}$ Measured modal estimates

$\dot{X}$ First derivative response of a system
\( \hat{D}_{M_k} \) Rank \( M_k \) approximation of random decrement signature

\( \hat{D} \) Random decrement signature

\( \hat{D}^{\text{INP\ filtered}} \) Random decrement signature for filtered indicative noise profile

\( \hat{D}^{\text{INP\ unfiltered}} \) Random decrement signature for unfiltered indicative noise profile

\( \hat{D}^{\text{filtered}} \) Random decrement signature for filtered analysis data

\( \hat{D}^{\text{unfiltered}} \) Random decrement signature for unfiltered analysis data

\( \hat{D}_{A_x T_x W_x} \) Random decrement signature for analysis channel \( A_x \), trigger channel \( T_x \) and weighting channel \( W_x \)

\( \hat{D}_{X T, 0} \) Free response component of random decrement signature

\( \hat{D}_{X T, n} \) Non-free response component of random decrement signature

\( \hat{D}_{x T} \) Random decrement signature for data set \( x \) with triggering condition \( T \)

\( \hat{R}_j \) Auto- and cross-correlation functions of a multi-channel data set at sample \( j \)

\( \text{MinPts} \) Minimum number of points to define a valid DBSCAN cluster

\( [H_1], [H_2] \) Hankel matrices

\( [K] \) Stiffness matrix

\( [M] \) Mass matrix

\( [R] \) Complex magnitudes of residues of an oscillation

\( [\Delta K] \) Change in stiffness matrix

\( [\Delta L] \) Change in static loading matrix

\( [\Delta M] \) Change in mass matrix

\( \rho'_{x T} \) First order time derivative of correlation function of \( x \) conditional on triggering condition \( T \)

\( \rho_{x T} \) Correlation function of \( x \) conditional on triggering condition \( T \)

\{0\} Vector of zeros

\{X\} Vector of displacement responses
\{\ddot{X}\} \quad \text{Vector of acceleration responses}

\ddot{y} \quad \text{Mean average of measured response}

\dot{x} \quad \text{First differential of discrete time domain data}

\dot{m}_m \quad \text{Modal mass of mode } m

\text{erf} \quad \text{Error function}

\ddot{a} \quad \text{Probability density function between } a_1 \text{ and } a_2

\ddot{b} \quad \text{Probability density function between } b_1 \text{ and } b_2

\hat{y}_N \quad \text{Order } N \text{ approximated structural response}

a \quad \text{Regression intercept}

a_1, a_2 \quad \text{Random decrement technique amplitude limits}

a_{\text{imag}, h, m} \quad \text{Imaginary component of acceleration response due to footfall excitation of mode } m

a_{\text{real}, h, m} \quad \text{Real component of acceleration response due to footfall excitation of mode } m

b_1, b_2 \quad \text{Random decrement technique first differential limits}

c \quad \text{Damping coefficient}

c_{\text{critical}} \quad \text{Critical damping}

d_f \quad \text{Frequency resolution of FFT coefficients}

f \quad \text{Natural frequency, hertz}

f_0 \quad \text{Mean natural frequency, hertz}

f_c \quad \text{Filter cutoff frequency, hertz}

f_s \quad \text{Sample frequency, hertz}

f_w \quad \text{Harmonic of pedestrian walking frequency, hertz}

f_w \quad \text{Pedestrian walking frequency, hertz}

f_{\text{max}} \quad \text{Maximum observed frequency, hertz}
\( f_{\min} \) Minimum observed frequency, hertz

\( h(t) \) Impulse response at time \( t \)

\( h \) Pedestrian walking harmonic

\( k \) Number of singular values used in approximation

\( k_{\text{clusters}} \) Number of clusters

\( m \) Mass of a single degree of freedom system

\( m_f \) Mixing fraction of mixture model components

\( m_k \) Modal stiffness

\( m_m \) Modal mass

\( p(t) \) Continuous force across time span \( t \)

\( t \) Time

\( t_{pk} \) Time at which data reaches a peak value

\( x(t) \) Continuous response of a system across time span \( t \)

\( x \) Analysis data

\( x \) Data series

\( x_0 \) Response of a system due to the initial amplitude of the response at \( \tau = 0 \)

\( x_F \) Response of a system due to forcing applied at \( \tau > 0 \)

\( x_d \) Response of a system to the initial first derivative of the response at \( \tau = 0 \)

\( x_n \) Noise in the sensor used to measure the response of a system

\( x_{\text{data}} \) Data object

\( y \) Data series
Chapter 1

Introduction

Civil engineering fundamentally shapes people’s everyday lived experience, from where they live and work, how they move through the world, and how their basic needs are met. While civil engineering as a distinct profession has only existed since the mid-eighteenth century [10], the process of designing and constructing bridges, buildings, and roads is old as civilization itself [11]. However, the last 100 years has seen the emergence of a disconnect between civil engineers and the structures they design. While other industries have seen the rapid development of technologies to learn from the performance of products in-service [12], the civil engineering design process remains linear. A structure is designed using the best available information, it is built, and that is typically the end of the designers’ involvement.

This project was motivated by a desire to allow engineers to learn from the performance of civil structures in-service. To combat the threats posed by climate change, the environmental impact of new civil structures must be reduced through the adoption of more efficient construction materials and structural forms. However, the lack of historical precedent for such new materials and construction techniques creates risks, as their long-term behaviour in-service is often untested. In parallel to the need to reduce the climate impact of the construction and operation of civil structures, large parts of the built environment are reaching the end of their design life, requiring in-service assessment of their structural integrity, and the rehabilitation, re-use or removal of the asset as appropriate.

It is hoped that establishing a feedback loop within the design process can aid the adoption of new materials and structural forms, and allow engineers to create more robust and efficient designs. This might be achieved through “smart structures”, structures instrumented with sensors in-service [12]. However, in order to integrate in-service measurements of structural behaviour into the design process
it is necessary to understand how civil structures are designed and the needs of practicing engineers. It is essential that any tools developed for integrating in-service measurements into civil engineering design present insights from the data in a way which can be easily interpreted by engineers and that any uncertainty in the results is accurately and clearly conveyed. Alongside this, if such tools are to be widely adopted, it is essential that the technology used is relatively low-cost and that the analysis methods require minimal user input, limited time demands from the engineers, and can make efficient use of the continuous stream of data produced by in-service monitoring.

1.1 In-service monitoring

In-service monitoring, the process of collecting measurements of a structure’s behaviour in-situ under ambient or real-world loadings, has seen rapid growth in the last fifty years. A wide range of technologies for monitoring structural behaviour in-service have been developed to measure all types of structural behaviour, from the settlement of bridge foundations to the wind-induced movement of skyscrapers [13]. Despite the installation of sensors on a wide variety of civil structures, the analysis of in-service structural measurements has remained a highly specialized and subjective process [14]. To aid the adoption of long-term structural monitoring within the civil and structural engineering design process it is necessary to develop efficient, accurate, robust and intuitive methods for analysing in-situ monitoring data.

Understanding the behaviour of civil structures in-situ is a multifaceted problem, as changes in behaviour can be driven by a wide range of different factors, such as changes in the loading applied to the structure, changes in temperature, degradation of structural materials, or damage to the structure [15] [13]. Therefore, robust methods to separate different structural behaviours must be developed. If long-term monitoring is to be implemented within the existing civil and structural engineering design processes, these methods must be generalizable to new structures without the need for extensive tuning of analysis parameters.

Due to the variability of structural materials, the limitations of in-service measurement technology, and the limitations of our current understanding of structural behaviour, there will always be uncertainty in both the measurements of structural behaviour in-service and predictions of structural behaviour based on those measurements. If engineering decisions are to be based on long-term monitoring, it is crucial that the uncertainty inherent in the analysis of monitoring
data collected in-service be clearly communicated to designers and operators of
civil structures [16]. This requires both the development of intuitive methods for
conveying the uncertainty in the approximations of structural behaviour, and the
development of methods which allow the uncertainty in the measurements to be
projected to any predictions made about the state of the structural system.

A further challenge in the integration of long-term monitoring into the engi-
neering design process is that the parameters measured in-service are typically
different from those used when designing a structure. For example, while a struc-
tural element may be designed for an allowable load or stress, in-service mea-
surements collected from that element may have units of strain, displacement or
acceleration [13]. This may be a barrier to engineers’ understanding the perform-
ance of structures in-service. To streamline the integration of in-situ monitoring
into the structural design process, methods are required which produce insights
into the structural behaviour which may be easily accessed and acted on by civil
and structural engineers.

1.2 Operational modal analysis

While there are a broad range of technologies for monitoring the behaviour of
structures in-situ, this study primarily focuses on accelerometers as they are rela-
tively low cost, robust and may be installed on both new and existing structures.
Therefore, they offer particular promise for long-term structural monitoring. Un-
like technologies such as a strain gauges, which provide measurements which
may be dominated by the local behaviour of the structure, accelerometers allow
the global behaviour of the structure to be assessed through quantification of
dynamic structural behaviour, the oscillation of structures due to time-varying
loading [15] [17].

Dynamic structural behaviour is quantified through the modal parameters of
a structure including:

1. **Natural frequencies** - The frequency at which a structure naturally oscillates
   in free vibration,

2. **Damping ratios** - How quickly the energy of the oscillations at each natural
   frequency are dissipated,

3. **Mode shapes** - The relative movement of different parts of the structure at
   each natural frequency [15] [17].
For small structures it is possible to apply a known force to the structure, measure
the dynamic response, and estimate the modal parameters through experimental
modal analysis.

Due to the size and complexity of civil structures and the disruption or risks
associated with applying known forces to critical infrastructure assets, experi-
mental modal analysis is not suitable for many civil structures. Operational modal
analysis (OMA), where the dynamic response of a structure is measured under
ambient or in-service loadings, provides a robust alternative for the monitoring
of civil structures in-situ [17].

A challenge in OMA is that as the force is unmeasured, assumptions must
be made about the statistical properties of the applied forcing and the dynamic
response of the structure [17]. On the basis of these assumptions, the modal
parameters of the system can be estimated. While a wide range of OMA methods
have been developed, there remains significant barriers to the use of OMA for
long-term in-situ structural monitoring including quantifying the uncertainty in
the modal estimates, isolating changes in the modal parameters associated with
changes in the mass or stiffness of the structure, and automation of the OMA
process.

In this thesis, advances in statistical pattern recognition and machine learning
are applied to overcome the limitations of existing OMA methods and provide
robust methodologies for long-term in-service structural monitoring using ambient
vibration measurements. While these applications are primarily demonstrated
using modal parameters estimated through OMA, they are also widely applicable
to other structural monitoring technologies.

1.3 Summary

Long-term in-service monitoring has the potential to change the way civil struc-
tures are designed and operated through allowing the in-situ performance of
structures under in-service loadings to be quantified. This may allow for the
development of more robust and efficient structural designs through the compari-
son of as-built behaviour with design stage predictions, aid the adoption of novel
materials and construction techniques for which there is a limited history of use
in-service, and prolong the design life of existing infrastructure assets through
early damage detection and proactive maintenance.

This study seeks to understand the potential of long-term structural mon-
itoring to inform the design of future civil structures. To integrate in-service
monitoring within the existing design process, methods must be developed which quantify the uncertainty within estimates of structural behaviour derived from in-situ measurements, alongside techniques which allow for the quantification and separation of different drivers of changes in structural behaviour. If long-term monitoring is to form part of the engineering design process, it is crucial that the methods developed are easily accessible by non-specialists and that they meet the needs of practicing engineers. To achieve this, the historical development of the civil engineering design process and the limitations of existing methods for analysing in-situ structural monitoring data must be understood.
Chapter 2

Background and Literature Review

At midnight on the 3rd of December 2015, the Forth Road Bridge, the primary road link crossing the Firth of Forth, Scotland, was closed following the discovery of a 2 cm fatigue crack in a primary truss [18]. The bridge would never fully reopen. Nine days later at COP 21, the United Nations Climate Change Conference, the Paris Agreement, a wide-ranging international commitment to combating climate change, was signed [19]. In agreeing to this text, 196 countries committed to reducing their greenhouse gas emissions and improving their resilience to the impacts of climate change. The construction industry will play a central role if countries are to meet these commitments, by reducing embodied and operational carbon within new-build structures through the use of novel materials and structural forms, and improving the maintenance, reuse and strengthening of existing infrastructure assets. A radical example of what the future of the built environment might look like, the Treet, a 14-storey engineered timber building, had opened in Bergen, Norway, earlier in the Autumn of 2015. With a height of 48 m, the timber apartment building was the tallest timber structure in the world at the time of completion [20, 21]. Due to the higher stiffness-to-mass ratio of the engineered timber compared to traditional construction materials, and the structural and environmental efficiency of the design, the Treet is expected to have a significantly lower whole-life embodied carbon compared to an equivalent concrete, steel or masonry structure [20].

The closure of the Forth Road Bridge, the signing of the Paris Agreement, and the opening of the Treet timber building, pictured in Figure 2.1, illustrate the three key challenges faced by the structural engineering industry: i) the need to maintain, strengthen and reuse existing infrastructure, ii) the need to reduce
carbon emissions associated with the built environment through more efficient and resilient structural designs, and iii) the need to embrace new materials and structural forms as a method of reducing embodied carbon in the built environment.

Figure 2.1: Left) The Forth Road Bridge. Photo credit: Stuart Halliday [22]. Right) The Treet, a 14-storey timber building. Photo credit: Sparrow [23].

In-situ structural monitoring may allow engineers to adapt to these challenges through measuring and understanding the behaviour of structures in-service. However, robust and efficient methods must be developed to infer engineering design parameters from the in-service behaviour of structures so that engineers may learn from past designs and make evidence-based judgements on the condition and performance of both new and existing infrastructure assets.

2.1 Ageing infrastructure and the climate emergency

The 2020 “Report Card for America’s Infrastructure” published by the American Society of Civil Engineers [24] states the problem posed by ageing infrastructure. Over 259,000 of United States (U.S.) highway bridges (42%) are over 50 years old, with the expected repair cost to address existing structural issues estimated at $125 billion [24]. 40% of U.S. roads are deemed to be in poor or mediocre condition [24]. In 2021 $5 million in funds were approved for the maintenance of federally owned levees within the U.S., compared to the $21 billion of investment required for maintenance of these levees as estimated by the U.S. Army Core
of Engineers [24]. This issue is not limited to the United States. Countries across the globe are facing the challenges brought on by the ageing of the vast amounts of infrastructure built in the twentieth-century [25]. In the UK, the RAC Foundation found that local councils self-reported that there were 3,105 bridges in substandard condition, with a backlog of bridge maintenance work in excess of 12 years [26]. Meanwhile, the BRE Trust estimates that 82.8% of the UK housing stock was constructed more than 30 years ago, with the majority (54.8%) constructed more than 50 years ago [27]. Many post-war structures built in Western Europe and North America in the 1950s and 1960s are approaching the end of their design life, presenting an ever-growing pool of infrastructure assets requiring urgent maintenance and replacement. This issue is compounded by the ageing of older civil infrastructure, such as the Victorian railway and sewage systems in the UK, and the late 19th and early 20th century steel bridges in the U.S..

This issue is not limited to Western Europe and North America. In China and India, which both experienced rapid urbanisation and infrastructure construction starting in the mid-1980s, a lack of legislative oversight combined with poor building practice has resulted in reduced design lives and an increased likelihood of serviceability or ultimate limit state failures [28, 29, 30, 31]. This pattern of rapid urbanisation and expansion resulting in poor quality infrastructure is now being mirrored across much of the developing world, further spreading the risks posed by the accelerated failure of infrastructure assets [32, 33, 34, 35, 36].

The challenges posed by ageing infrastructure are compounded by the need to reduce global greenhouse gas emissions to tackle climate change. The UN Environment Programme [37] have reported that building and infrastructure operation are responsible for 28% of global energy-related carbon dioxide (CO₂) emissions, with the construction industry responsible for a further 10% of global energy-related CO₂ emissions. As highlighted by Abergel et al. [38], to reach net-zero carbon buildings and infrastructure by 2050, indirect building sector emissions associated with material use and construction must fall by 60%, while direct CO₂ emissions associated with infrastructure operation must fall by 50%. If these reductions are to be met it is not viable to replace existing infrastructure with like-for-like structures. The viability of such an approach without considering the required reduction in CO₂ emissions is further called into question by the changing climate itself. The last twenty years have seen early indications that the severity and frequency of extreme weather events such as hurricanes, flooding, and wildfires are increasing, likely as a result of climate change, as summarized by Stott [39] and the National Academies of Sciences, Engineering, and Medicine.
Even in the absence of an increase in the number and scale of extreme weather events, the damage induced by such events would still increase, due to a growing global population, increased urbanization, and global economic and structural development. Numerous researchers have highlighted the need to ensure existing infrastructure assets are resilient to extreme weather events and a wider range of climatic conditions (e.g. [42, 43, 44, 45]), and recommended that the increased likelihood of such events be reflected in the design and construction of new structures (e.g. [46, 47, 48, 49]).

A wide range of partial solutions have been proposed for reducing the CO₂ emissions associated with material use and the construction of civil infrastructure assets, here referred to as construction phase emissions. One method for reducing such emissions is through the use of novel materials. Sathre and Gustavsson [50] compared the carbon emissions associated with material production for a 4-storey engineered timber apartment building in Sweden to an equivalent concrete frame building, finding that the wood-framed structure had resulted in a 55% reduction in the construction phase emissions and with the energy use associated with the manufacture of the materials reduced by 28%. These findings are supported by the work of Skullestad et al. [51] who demonstrated that, compared to equivalent steel and concrete structures, high-rise timber buildings of a range of dimensions can result in a 34% to 84% reduction in their life-cycle climate change impact, as quantified through material embodied carbon, material need and whole-life energy costs.

In the context of traditional building materials, large amounts of research have focused on how the construction phase emissions of concrete might be reduced. While concrete itself has a relatively low embodied carbon and production energy requirements as a material [52], the quantity of concrete used across the globe (approximately 4.4 billion tonnes per year [53], equivalent to slightly over 500 kg per person on earth) makes it one of the largest single sources of carbon emissions [54, 52, 55], with the emissions from the production of concrete alone estimated to be 8% of the total global CO₂ emissions [53]. Methods for reducing the construction phase emissions of concrete include the use of recycled aggregate, which may offer a way to reduce the energy required in the excavation and transport of virgin aggregate. Xiao [56] estimated that the use of recycled aggregate resulted in a 12.79% reduction in the construction phase emissions for a 12-storey building when compared to an identical building using non-recycled aggregate. Apart from the direct emissions savings associated with the use of recycled material, reduced transportation of the materials contributed significantly to this saving. However, the use of recycled aggregate did result in a meaningfully lower elastic modulus of
the concrete in-service, and worse shrinkage and creep performance, which may negatively impact the long-term behaviour of the building [56]. These findings on the lower elastic modulus of recycled aggregate concrete agree with previous findings by Ozbakkaloglu et al. [57] and Casuccio et al. [58] as well as previous observations by Xiao [56, 59, 60], highlighting some of the risks associated with the use of novel construction materials.

Alongside reducing the construction phase emissions of the material, more efficient use can be made of the material. An example of recent advances in this area include the use of additively manufactured or 3D printed structural elements and connections which allow for more efficient transfer of forces by concentrating material only where needed [61, 62, 63, 64]. A radical example of this approach is the MX3D Bridge, the world’s first 3D printed steel bridge [9], pictured in Figure 2.2. Similarly, fabric formwork concrete can allow for material to be located in areas of maximum shear and moment through non-linear flexible falsework, reducing material wastage [65, 66, 67, 68]. Further examples of the efficiency driven approach include organic structural forms using engineered timber products [69, 70, 71], such as those shown in Figure 2.3, innovative concrete construction utilizing novel admixtures [72, 73, 74], and new steel construction methods [75, 76, 77, 78], often designed through parametric modelling [79, 80, 81, 82]. For existing structures, where extending the operational life of an asset can delay the need for replacement and reduce the life-cycle embodied carbon of the existing asset, carbon or glass fibre reinforced polymer strengthening has seen widespread adoption for the repair of cracked and damaged structures [83, 84, 85, 86], while post-tensioning of concrete and masonry structures has been shown to be an effective method for repairing damaged bridges [87, 88, 89].
The rapid pace of development within structural engineering, driven by the need to reduce carbon emissions and maintain or replace existing infrastructure is pushing the limits of existing design guidance and best-practice. To allow greater adoption of advances in structural form and the use of novel materials requires an evolution of the structural engineering design process itself, the precedent for which can be seen the historical predecessors to civil engineering design.

2.2 The structural engineering design process

To understand the need for change within the structural engineering design process, the evolution of the civil engineer’s role in the design process and the structural engineering design process itself must be understood.

While the term “civil engineer” dates from the mid-eighteenth century [10], the practice of a designated individual overseeing the design and construction of large projects is likely to be as old as the division of labour [11]. In the eyes of the early civil engineers, what set them apart from earlier engineers, architects, and master builders, was the “application of physical and scientific principles” in the construction of roads, bridges, ports and buildings [91]. However, the practice of observing the behaviour of structures in-service and modifying future designs existed long before civil engineering was recognised as a distinct profession.

Developments in civil engineering have consistently been driven by failure or fear of it. This is illustrated by one of the earliest surviving legal texts, the Code of Hammurabi, a Babylonian legal text dated to 2250 BC [92] pictured in Figure 2.4 which included the following laws:
If a builder build a house for someone, and does not construct it properly, and the house which he built fall in and kill its owner, then that builder shall be put to death.

If it ruin goods, he shall make compensation for all that has been ruined, and inasmuch as he did not construct properly this house which he built and it fell, he shall re-erect the house from his own means.

If a builder build a house for someone, even though he has not yet completed it; if then the walls seem toppling, the builder must make the walls solid from his own means. [92]

While the severity of the punishments may have changed, the underlying principles of ensuring that structures are safe for occupants, fit for purpose, and structurally sound at all stages of construction and use, have remained the core principles of all modern building codes and design guidance. The reactive nature of design codes is evident in other surviving examples of early building regulations. Chapter 22, Verse 8 of the book of Deuteronomy, the fifth book of the Jewish Torah, Christian Old Testament or Islamic Quran [95], believed to have been written in the 7th century BC [96], includes the following stipulation:

When you build a new house, then you shall make a parapet for your roof, that you may not bring guilt of bloodshed on your household if anyone falls from it. [95]

While both texts concern the design of civil structures, a key difference between Deuteronomy 22:8 when compared to the Code of King Hammurabi is that while the Code of King Hammurabi is interpretative, it is left up to the builder to

Figure 2.4: The Code of King Hammurabi. Photo credits: left) Mbzt [93], right) Steven Zucker [94].
decide how to meet the legal requirements, the law laid out in Deuteronomy is *prescriptive*: do X to avoid Y. This prescriptivism is seen in other early building codes, such as the Sanskrit writings in the Vastu shastra, which provide detailed guidance on the planning, design and construction of Indian towns and temple complexes developed across generations [97, 98], and the Rebuilding of London Act of 1666 which laid out extensive requirements for the building materials to be used, the required thickness of walls and widths of streets, and the maximum height of private residences [99]. This prescriptive design guidance would come to dominate the codified design standards of the 20th century, as discussed later, but these early examples are relatively scarce, mostly concerning the rare higher density settlements or larger religious structures [100].

Alongside the prescriptive nature of the Rebuilding of London Act of 1666, it is also an example of the reactive nature of design standards, being introduced in direct response to devastation caused by the Great Fire of London of the same year. This reactivity is a key driver of modern building codes, as can be seen in response to the Ronan Point collapse (1968, Figure 2.5) which led to the addition of legislation on preventing disproportionate collapse to UK building standards [101], the Cleddau Bridge disaster (1970), which led to the creation of the engineering standard BS 5400 [102, 103], the Potters Bar rail accident (1988), which led to changes to the design of railway points and the end of subcontracted routine rail maintenance in the UK, or the Grenfell fire (2018), which resulted in the ongoing revision of the UK building regulations and regulatory system [104].

The reactivity of building standards is part of a long tradition of learning from mistakes that predate modern engineering design guidance. Early civil engineering design practice was characterized by the direct master-to-student oral transfer of knowledge through apprenticeships [106]. This knowledge was largely informal, based on the master’s direct experience in the field, or that which had been passed to them through communication with their own master or other practitioners, likely local to the area [107]. While individual practitioners could revise designs to avoid past mistakes, a limitation of the informal approach was that it made the spread of knowledge slow, with the lack of a theoretical or scientific understanding of structural behaviour hindering the transfer of techniques to new materials and structural forms.

The informal transfer of knowledge also contributed to a regional fragmentation of knowledge, worsened in Europe by the cloistering of material specific knowledge, such as masonry or carpentry, within the powerful guild systems to which many of the tradesmen belonged [108]. While these guilds might have acted to disseminate knowledge among the members in the same way as modern
professional institutions, work by Kieser [108] suggests that the guilds’ economic philosophy stifled innovation in medieval Europe. Medieval guilds exercised tight control over the purchase of raw goods, the standardization of construction methods members were allowed to use, the salaries paid to tradesmen, and the price and conditions of the sale of all goods and services. While this prescriptivism is similar to that which would develop into modern construction standards in the 20th century, the primary objective of these restrictions was to ensure equal opportunities for all guild members and market stability for the consumer. The guilds held a legal monopoly over their specialized marketplace such that it was not possible for a non-guild member to compete through utilizing more efficient or innovative construction practices. Furthermore, the European guilds’ practices were deeply rooted in Christian tradition, with the pursuit of profit deemed as sinful. For this reason guilds took active steps to discourage innovation, with harsh punishments imposed for the smallest of deviations from tradition, as explored extensively by Kieser [108]. Where innovation did occur during this period, it primarily occurred in domains beyond the guilds sphere of influence, such as monasteries [109, 110, 111], or warfare [112, 113]. The guilds’ political and eco-

Figure 2.5: Collapse of Ronan Point towerblock, 1968. Photo credit: Derek Voller [105].
nomic power led to a stagnation of engineering design which would extend until the end of the 15th century, where guilds would come to be replaced in Europe by a profit-driven mercantile class who embraced the opportunities brought by international trade, new technologies, and mass communication [114].

Up until the 15th century, aside from piecemeal local planning legislation, the informal transfer of knowledge was the primary driver of what would now be considered as structural engineering design. However, the introduction of the printing press in the early 15th century created a new market for engineering design guidance. As discussed by Yeomans [115], printed media represented a dramatic shift in how knowledge was shared, with the formalization of the knowledge transfer allowing for the adoption of new construction techniques and materials. This can be seen in the evolution of carpentry manuals, as described by Yeomans [115], from simple pocketbooks of geometric formulas to methods of disseminating more efficient or novel designs. The carpenter was the cornerstone of Western European construction, providing not only the timber elements of the finished structure, such as floors and roofing, but also the temporary works for foundations, masonry or brick construction, and plastering [115]. An example of the importance of these carpentry manuals in the dissemination of knowledge can be seen in the spread of king and queen post truss systems, one of the most efficient forms of triangular truss, across the UK in the 17th century [116]. While this form of construction was rarely seen in the UK prior to the 1600s, it spread rapidly after its inclusion in a number of popular carpentry manuals of the time [115], with the more efficient structural form allowing for larger spans using less material. The demand for such design guidance only increased following the inclusion of designs for fashionable structural features, largely inspired by Gothic and Palladian architecture, such as elliptical arches, domed vaults, and curving staircases, which were attractive to a newly wealthy urban middle-class clientele [115]. Aspects of the early design guidance found in carpentry manuals continue to influence modern design standards, such as the UK National Annex to BS EN 1993-1-1 recommendation of an allowable deflection of the length of span/360 for “Beams carrying plaster or other brittle finish” [117], a direct descendant of the recommendation put forward in 1871 by Tredgold [118] of a maximum floor deflection criteria of span/480 to prevent the cracking of plaster partitions.

Alongside allowing the dissemination of ideas, the carpentry manuals also changed the dynamics of engineering design by allowing the designs themselves to be bought and sold. This introduced an incentive for researching and validating new methods and construction techniques [116], an incentive which was further strengthened in Europe by improved patent law [119]. This incentive was
capitalized on during the development of structural steel and early reinforced concrete, with early innovators patenting their designs and selling them as design “systems” [119, 120, 121], with the proceeds of such sales and a competitive market economy driving further research and innovation. This marked a key turning point in the evolution of structural engineering design: the move from the sharing of knowledge which was largely tacit, such as that in the carpentry manuals where the physical reasoning for the design efficiency was primarily based on experience and intuition, to a research-based knowledge economy, in which the most reliable and efficient designs rose to prominence, buoyed by market forces.

The proliferation of new materials, structural systems and construction techniques were not without its drawbacks. Compatibility of structural components was not guaranteed, nor were the skills needed to design and construct structures from those components. Not only did the lack of compatibility complicate the structural design process and procurement of materials, but it also required the manufacturing industry to produce wide ranges of often relatively niche components and sizes, thus missing the economies of scale that industrialization had once promised [100]. These issues were most apparent in large scale public works undertaken by the British Empire. Construction materials that were manufactured in the UK were shipped to all corners of the globe, where it was often discovered that many of the parts were incompatible. Therefore, as documented in extensive detail by McWilliam [100], the British Government took a keen interest when an ad-hoc subcommittee of the Institute of Civil Engineers (ICE) met in 1901 to consider whether rolled steel sections should be standardized [100]. As highlighted by McWilliam [100], early construction standards were neither prescriptive nor enforced by the British government. Instead, it was the reliance of engineers on such standards, purchased for a small fee from the newly formed Engineering Standards Committee (later renamed to the more familiar British Standard Institution (BSI) in 1931), which drove market forces and the de-facto standardization in many areas of construction. The success of this approach is a testament to the influence which the ICE held at the beginning of the 20th century, when it was the largest professional institution in the world, boasting more than 50% more members than any other professional institution [100].

Early construction standards were limited in scope, primarily being focused on ensuring standardization of construction components, and not on the engineering designs themselves. The transition to more prescriptive codes of practice was gradual, fragmented, and marked by infighting between the ICE and BSI. While codes of practice had been considered in early meetings of the ICE, the general sentiment was that this would remove the flexibility and adaptability required
of many civil engineering designs [100]. The earliest BSI codes of practice concerned the uses of reinforced concrete and structural steel in buildings, followed by more prescriptive rules for loading of rail and road bridges, prepared at the request of the British government in the 1930s and 1940s. The development of such codes was piecemeal, based on a mixture of research and learnt-experience [100]. The latter half of the twentieth century saw a proliferation of codes of practice under the British Standards in the UK, as well as the emergence of the International Organisation for Standardization (ISO) codes across Europe. In the United States, the American Society of Civil Engineers (ASCE), the American Institute of Steel Construction (AISC), and the American Concrete Institute (AIC) retained a tighter control over the writing of codes of practice, avoiding the split which had befallen the BSI and ICE. However, the progression of standardization and the move from interpretive design guidance to prescriptive codes of practice followed broadly similar lines, and was largely driven by the same reactivity to engineering failures discussed earlier.

The key criticism of codes of practice, as well as design standards more broadly, is that their development is driven in large part by the reaction to catastrophic events and that they stifle innovation. The reactive nature of codes of practice, as has already been touched upon, can be seen in the publication dates of the standards themselves. As highlighted by McWilliam [100], the British Standard for temporary works (BS 5975 [122]) was introduced in response to a series of high profile formwork collapses in the 1970s. The code of practice covering ship-to-shore maritime structures (BS 6349 [123]) was created following a gangway collapse which caused the death of six people [100]. As demonstrated by Soane [124], this pattern of high-profile collapses driving the development and revision of design standards is not limited to the UK but is observed across the world. The theme of learning through failure as a driving force in structural engineering design has been extensively studied in the work of Henry Petroski. As stated by Petroski [125] “It has long been practically a truism among practising engineers and designers that we learn much more from failures than from successes” (Pg 1.). The argument put forward by Petroski [126, 125] is that failure is a primary driver of improvement as it forces engineers to reflect on past designs in order to understand their failure. Within this framework, the prescriptive codified approach of building standards is therefore the public output of this reflection, created to allow engineers to avoid the mistakes of others. However, this approach still has its limitations, especially within the confines of prescriptive codes of practice. It is reliant on the identification that failure has occurred, and that failure being public in a way that drives institutional change [100]. More
broadly it can impede the adoption of new materials and technology, as the lack of legislative guidance creates unquantifiable risks for the innovators, or bars their implementation entirely. One example of this is highlighted in a case study by Suprun and Stewart [127] examining innovation in the Russian construction industry, where they identify the requirements for centralized government approval of innovative construction strategies as a key barrier to their adoption. A further example of how design standards can impede the adoption of new technologies are engineered timber products such as cross-laminated timber. A study of structural engineering timber experts across Europe conducted by Espinoza et al. [128] found that the majority (51%) viewed the incompatibility of engineered timber with existing design standards as a large barrier to its adoption, far greater than for any other potential barrier explored.

Considering the challenges of ageing infrastructure and the climate emergency, there has been a growing recognition within the structural engineering industry that how structures are designed and maintained has to change. The prescriptive nature of codes of practice are impeding the adoption of the solutions necessary to combat these dual crises, while the reactive nature of their development is poorly suited to the challenges of the current moment. This sentiment is driving the development of the structural engineering design process itself, characterized by the emergence of proactive and interpretive design procedures which maintain the advantages of standardization and simplicity offered by codes of practice, but allow for the flexibility and free-market forces which drove pre-20th century innovation.

2.3 New frontiers in structural engineering design

The last 30 years have seen the digitization of the structural engineering design process, through computer-aided design (CAD), computer modelling of structures, and building information modelling (BIM). This digital revolution is allowing for more complex structural analyses and the integration of in-service measurements of structural performance to influence the design, maintenance and operation of civil infrastructure.
2.3.1 Performance based design

Codified design standards provide prescriptive guidance on the strength and stiffness of structural elements and connections to meet an, often unspecified, implicit design performance criterion. There are numerous drawbacks to this approach, as discussed in the previous section, including the lack of design standards for novel materials and structural forms, a lack of flexibility within the design standards, the reliance on an implicit relationship between the structural behaviour as described by design standards and the performance criteria to be met, and the complex interplay of different structural behaviours which may impact in-service structural performance.

An alternative approach that has gained traction in the last thirty years is limit state or performance based design (PBD), in which structures are designed to meet a specific set of performance criteria with minimal prescriptive requirements for the properties of individual materials, structural elements and connections [129]. This approach marks a move towards classical methods of engineering design in which the form of the structure was dictated purely by the requirements of its functions under the limitations of the context. This was succinctly contextualized in the context of architectural design by Kalay [130] who described performance as “the confluence of the form, function and context”. While there are numerous early examples in the literature of PBD or limit state design criteria for areas such as the design of retaining walls [131, 132] and fire safety design [133], the earliest widespread adoption of PBD in structural engineering was in the field of seismic design [134]. The core publications which incorporated performance-based design into the seismic design of civil structures, as reported by Ahmed [134], were:

- the Structural Engineers Association of California (SEAOC) Vision 2000 report published in 2000 [135],
- the Applied Technology Council (ATC) methodology for the seismic evaluation and retrofit of concrete buildings (ATC-40), published in 1996 [136],

The development of these documents was spurred by the October 17th 1989 Loma Prieta earthquake in California which caused $6 billion in damage in the California Bay Area [139] and led to 63 deaths, 42 of which occurred during the
collapse of the Cypress Street viaduct, pictured in Figure 2.6. While the Cypress Street viaduct had been retrofitted with structural elements to resist longitudinal seismic movement in 1977, the emphasis at the time was on following prescriptive codes of practice. As a result of this, no seismic analysis of the structure was conducted, an issue identified by Housner and Thiel Jr [139] in the report into the engineering implications of the earthquake as having contributed to the 1989 collapse. Housner and Thiel Jr [139] identified this reliance on prescriptive solutions and a lack of understanding of the seismic performance of civil structures in-service as a risk to infrastructure assets and strongly recommended the development of design practices based on structural modelling and in-situ measurements of seismic performance to guide future seismic retrofitting (Page 50, [139]).

![Figure 2.6: Remains of the Cypress Street Viaduct following its collapse in the Loma Prieta earthquake. Photo credit: USGS [140].](image)

PBD is primarily based on the concept of performance targets [134], often specified displacement, acceleration, stress or strain limits. These may be expanded to consider different use cases or scenarios such that the performance targets for day-to-day use of a structure are different from those used for extreme events such as earthquakes. The key challenges faced in PBD are selecting appropriate performance targets, identifying appropriate design scenarios, and balancing the multi-faceted behaviour of structures in-service [134].

PBD of structures for seismic loading has continued to develop and PBD has expanded into a range of structural engineering sub-disciplines. For example, Mokarram and Banan [141] proposed a multi-objective approach that balanced the initial cost of construction with the life-cycle costs of damage to the structure due to seismic events. The areas of structural design which have seen the most rapid development and effective deployment of PBD methodologies are in the design of civil structures for extreme loading events. Outside of seismic events, PBD was expanded to the design of bridge piers during vehicle impact by Auyeung
et al. [142], who proposed the use of a damage ratio index which defined the allowable damage to the bridge pier under different collisions, and in the work of Ma et al. [143], who developed a framework for the design of bridge structures under vehicle collision-induced fire events. Similarly, Ashkezari [144] developed PBD criteria for the design of steel moment frames under blast loading. A key theme in current work in PBD is the development of simplified design procedures which can negate the need for complex finite element modelling. Analysis of the development of PBD standards in seismic engineering has demonstrated that this is a key criterion if PBD approaches are to be adopted in practice [145].

More recent work has looked at the development of PBD methodologies for serviceability events such as the design of tall buildings under wind loading, as discussed by Van de Lindt and Dao [146] and Alinejad et al. [147], among others, and the design of bridges under wind loading, discussed by Ciampoli et al. [148]. The key difference between wind and seismic related PBD is that while seismic PBD focuses primarily on limiting damage and collapse, wind PBD places a far greater emphasis on the serviceability limit state: ensuring that a structure remains operational and fit for use during high winds. In the field of offshore energy infrastructure, Ciampoli and Petrini [149] expanded this approach to consider the serviceability and maintenance requirements of offshore wind turbines, as well as the additional complexity introduced by the combined wave and wind loading on the structures. The serviceability driven approach for PBD, first seen in the field of wind engineering, has also expanded to areas such as user-comfort of footbridges under footfall vibration [150, 151] and the development of more energy-efficient mixed-use building designs [152].

The key challenge facing the continued use and development of PBD, as highlighted by Poland and Horn [145], is the proactive real-world verification, calibration and validation of the design techniques developed, something which is dependant on the analysis and interpretation of data from structures collected in-service.

2.3.2 Digital Shadows and Digital Twins

Advances in computing power, statistical algorithms, and the blending of virtual and real-world assets, referred to in the literature as Construction 4.0 or Industry 4.0, are allowing for this integration of real-world data into engineering design, and is expected to have wide-ranging impacts on the design, operation and maintenance of civil structures [153, 154].

One technology that is seeing growing interest are digital shadows, virtual
models of physical assets updated with real-world measurements [12], and digital twins, where the operation and maintenance of the physical assets are automatically scheduled and controlled by the digital shadow [155, 12]. The first description of what would come to be known as digital twins is often attributed to Gelernter [156] in which they describe “software models of some chunk of reality, some piece of the real world”. The first implementation of a digital twin was by Michael Grieves in 2002 for product life-cycle management, as reported by Kritzinger et al. [157] and Grieves [12]. A digital twin was described by Grieves [12] as fulfilling two key criteria, i) it contains information about a physical object or system, and ii) the information was linked with the object or system in question. This definition was expanded on by Fei et al. [158] who defined digital twins as:

an integrated multi-physics, multi-scale, probabilistic simulation of a complex product and uses the best available physical models, sensor updates, etc., to mirror the life of its corresponding twin

As highlighted by Grieves [12], the flow of information in a digital twin must be bi-directional. That is, the virtual model is updated using data collected from the physical object, and the control/operation of the physical object is automatically controlled dependant on the simulations performed by the virtual model. Where the flow of data is uni-directional, where the data from the physical object is used to update the virtual object, but the virtual object exhibits no automatic control over the physical object or vice versa, the virtual model is referred to by Grieves [12] as a “digital shadow”. Similarly, if there is no automated flow of information between the virtual and physical objects, the virtual representation of the physical object is not a “digital twin” but a “digital model”.

Digital twins have been applied across a broad range of fields for the management, maintenance and improvement of a variety of systems. Early applications focused on the manufacturing industry, with digital shadows and digital twins having been employed to optimize the maintenance and operation of assembly lines [157, 159].

As summarized by Silva et al. [160] there has been growing interest from municipalities in “Smart Cities” or city-level digital twins. An example of this approach in the civil infrastructure sector is presented by Conejos et al. [161] who discussed the design, development and deployment of one of the most ambitious digital twins to date, a digital twin of the water distribution network in Valencia, Spain. This digital twin incorporated the ability to make use of current water usage patterns and demand forecasting to allow optimization of water network
elements, target proactive maintenance to reduce water supply issues, model the likely areas of leaks within the water network, and predict the water quality based on historic data.

The development of digital twins for civil infrastructure at the building and city level are discussed by Lu et al. [162, 163]. Within this work, a digital twin of the West Cambridge Site, University of Cambridge, was used as a case study for the implementation of multi-use digital twins. Uses for these digital twins included anomaly detection in HVAC systems based on vibration monitoring, monitoring of temperature and humidity for occupant user comfort, detection of temperature drops associated with boiler malfunctions, and automation of maintenance and repair prioritization. It should be noted that the flow of data within this digital twin was primarily from the physical asset to the virtual model, with the virtual model having little or no direct control on the operation and maintenance of the structure, and instead mainly providing detailed analytics for the estate’s management team. However, the project does demonstrate some of the potential benefits of digital twins, such as the centralization of information on the current state of the system, and how real-world data might be used to inform engineering decision making in areas such as the control of internal temperatures and ventilation. This work highlights that many of the practical challenges associated with the real-world implementation of digital twins for civil infrastructure, such as the collection, transfer and storage of large amounts of data, and the hierarchical synchronization and unification of data from heterogeneous data sources, can be overcome. However, this work further reiterates the need for near real-time big data analysis techniques, as previously highlighted by Al-Fuqaha et al. [164] and Krylovskiy et al. [165], so that digital twins might make the best use of the data available to them, and the need for robust machine-learning-based decision methods to assist the operation and maintenance of the site.

A key challenge to the successful implementation of digital shadows and digital twins highlighted by Brovkova et al. [166] is interpreting the data from a wide array of different sensors so that data from the physical assets may be contextualized correctly within the virtually modelled behaviour. The scope of these challenges varies depending on the degrees of interpretation required to relate the input data and the desired model updating/physical system control. In some instances, such as controlling the internal environment within a single building, where the input and output data are strongly related, measurements have low levels of uncertainty and there are low risks of adverse outcomes [167], this is simpler to achieve than for more complex systems, where the measured data must first be interpreted before being integrated with the digital shadow or
Wagg et al. [168] highlighted the need to quantify the uncertainty in the data, as well as the need for methods of ascribing trust to the outputs of a digital twin to support engineering decision-makers. A further challenge is the long-term stability of digital twins or their robustness to unseen conditions and events [169]. Alongside these technical challenges, there is a range of commercial and contractual challenges, such as the identification of data ownership and the liabilities associated with digital twin based design, maintenance and operation of civil structures [170][171].

2.4 In-service monitoring for structural engineering design

In Section 2.1, the need for changes in the way civil structures are designed, operated and maintained to combat the twin perils of ageing infrastructure and the climate emergency was introduced. It was shown in Section 2.2 that the design process has historically been reactive, driven by failure and collapse. Performance-based design, discussed in Section 2.3.1 where structures are designed to meet specific criteria, offers a more proactive approach to design but is dependant on real-world validation, while digital shadows and digital twins, defined in Section 2.3.2 offer a technological basis through which real-world measurements might be used to inform the structural design process. However, the success of performance-based design and digital shadows/twins is dependant on the accuracy, efficacy and interpretability of data collected from civil structures in-service.

There is a wide range of technologies for collecting structural data in-service. Load cells or weigh-in-motion sensors, which measure the force transmitted through them utilizing the change in strain measured using a Wheatstone bridge [172], offer a means of quantifying the forces applied to structures in-service [173]. However, they are expensive to install and maintain, and are not suited to all structures as accurate in-service measurements of loading can only be achieved where the only load paths pass through the load cells themselves. Therefore they are poorly suited for measuring the loading on structures such as floor slabs, where there are often continuous or multiple points of contact between the transverse and vertical structural elements.

A widely used monitoring technology are electronic strain gauges, which measure the differential movement of structural elements. Vibrating wire strain gauges achieve this through measuring the changes in transmissibility of elec-
trical current which occurs when a length of wire changes length. A key problem with vibrating wire strain gauges is that the impact of temperature, which leads to thermal expansion/contraction of the wire element itself, must be accounted for in the post-processing of data. Vibrating wire strain gauges are also sensitive to electromagnetic interference, such as may be induced by power cables, and measure only local effects, making interpolation of the data across structures reliant on high numbers of sensors [174]. An alternative type of strain gauges, Fibre Bragg Grating (FBG) sensors, replace the vibrating wire element with an optical fibre embedded in or attached to the structural elements of interest. As the length of the structural element changes, causing a concurrent change in length of the optical fibre, the wavelength of light passed through the fibre changes [175]. FBGs have the advantage that they can also be used as a temperature sensor, as the refractive index of the optical fibre changes in response to changes in thermal conditions. One of the key benefits of FBGs is multiplexing, the bundling of hundreds of FBGs within a single optical fibre, each developed for the sensing of specific parameters such as strain, temperature, pressure, acceleration or displacement [175, 176]. FBGs are insensitive to electromagnetic interference, a common issue encountered in other sensing technologies, and are lightweight, durable, and highly sensitive [175]. They also allow for continuous monitoring of large structures as shown by Cheng et al. [177] who demonstrated the use of FBGs for collecting strain measurements from the 2,160m span of the Tsing Ma Bridge, China. Experimental measurements have demonstrated that measurements can be collected from FBGs of hundreds of kilometres in length [178, 179], potentially allowing for the instrumentation of continuous infrastructures such as roads and railway lines. The traditional barrier to widespread use of FBGs in-service has been their cost, although as noted by Rodrigues et al. [180] their cost has become broadly similar to conventional strain gauges when quantified based on the number of measurement points on a structure.

A similar technology to strain gauges is displacement gauges which are designed to measure the differential movement between two fixed points. These are typically installed in bridge decks to measure the movement between elements such as the deck and piers at the bridge bearings. Their design and operation are broadly similar to strain gauges, with the change in resistance of a length of wire used as a basis for calculating an equivalent displacement value. Like strain gauges, displacement gauges are sensitive to both electromagnetic interference and thermal effects [181].

Two other broadly used sensing technologies for civil infrastructure are inclinometers, which measure the absolute rotation of a structure or structural element
using a piezoelectric sensor, and accelerometers, which measure the acceleration of a point on a structure through a piezoelectric or micro-electromechanical system (MEMS) sensor. Inclinometers are primarily used for quantification of thermally induced movements, which will often induce a rotation in structural elements due to differential heating, and as an indicator of damage to the structure, such as where subsidence of a bridge pier has occurred [182].

Accelerometers offer a low-cost, durable method of instrumenting civil structures in-service, and may be applied to both new and existing structures. Unlike conventional strain gauge technologies, which measure the behaviour of a structure at specific locations, accelerometers can allow for a global understanding of the behaviour of a structure, as the acceleration or vibration response of a structure is dependant on its dynamic structural behaviour [15] which may be quantified through modal analysis, discussed in Section 2.4.2. Excessive oscillation of structures may pose a risk to their structural integrity as famously demonstrated by the collapse of the Tacoma Narrows Bridge in November 1940, pictured in Figure 2.7. The exact cause of the collapse is unknown, but a widely cited theory put forward by Amman et al. [183], the committee who first investigated the cause of the collapse, is that wind stress across the bridge deck caused an aeroelastic flutter effect, inducing a torsional mode of vibration. It is theorized that the specific wind speed and direction at the time of the collapse induced resonance of this torsional mode, such that the frequency of the applied wind forcing was the same as the frequency of the movement of the bridge deck causing increasing amplitudes of oscillation until the deck itself collapsed.

Figure 2.7: Remains of the Tacoma Narrows Bridge following its collapse due to resonant oscillation. Photo credit: Stillman Fires Collection [184].

Figure 2.8: The Millennium Bridge, London, which experienced noticeable amplitudes of transverse oscillations under footfall loading by crowds of pedestrians. Photo credit: Paul Lomax [185].

Alongside the structural issues which may result from the oscillation of civil
structures, these oscillations may be uncomfortable for the users of a structure. Examples of this effect include when a tall building sways in the wind or when a footbridge oscillates under pedestrian footfall, as occurred after the opening of the Millennium Bridge, London, pictured in Figure 2.8 which experienced noticeable transverse oscillations under footfall loading. While the oscillation of the Millennium Bridge was not believed to pose a risk to its structural integrity, the unease it induced in pedestrians led to its closure two days after opening, and 18 months of works to install 89 dampers to the structure to dissipate the unwanted vibrations at a cost of £5 million [186, 187].

The perception of vibrations has been widely studied and is typically compared to acceptability criteria, with oscillations with frequencies below 15Hz being more easily noticed by humans than higher frequency vibrations. An example of the vibration acceptability criteria base curve given in ISO 10137 - Serviceability of buildings against vibration [188] is presented in Figure 2.9. Vibration base curves are typically combined with multiplication factors representing the sensitivity of users and occupants to vibrations in different environments, as well as user’s higher sensitivity to vertical vibration compared to transverse vibrations [189].

![Figure 2.9: Acceptable vibration values given in ISO 10137](image)

To allow the quantification of environmentally induced changes in structural behaviour, structural sensors are often installed alongside a range of environmental sensors, such as thermistors for measuring temperature, humidity sensors, alkalinity sensors, photo-resistors or light sensors, or more generalized weather stations. Combined analysis of these data sets can allow the separation of envi-
vironmental effects from changes in structural behaviour as might be induced by
damage to the structure. For many structural sensors, such as strain gauges,
temperature measurements are an essential requirement in order to correct for
thermally induced changes in the output of the sensor itself [174].

A key part of in-service structural monitoring is the selection, development
and deployment of the wider network of supporting technologies. Aspects of this
include the selection of an appropriate sample rate for the data. Some parameters
such as temperature are typically slowly varying and a low sample rate can help
minimize the data transfer rate and data storage required. However, parameters
such as acceleration require much higher sample rates to capture the behaviour
of interest and careful selection of the parameters used in the analogue-to-digital
conversion.

There are also a wide range of developing novel technologies for collecting
information about the performance of structures in-situ, ranging from the use
of distributed networks of mobile phones or unmanned aerial drones to measure
bridge excitation [190] [191] to the use of digital image correlation or satellite
images to identify the settlement of buildings, bridges and railways [192] [193].
Alongside this, there have been advances in the use of wireless sensor networks,
as discussed by Noel et al. [194], the use of low-cost sensors, as discussed by
Swartz et al. [195], and analysis of structural behaviour which occurs below the
Nyquist frequency (half of the sample rate), as discussed by Mishali and Eldar
[196].

2.4.1 Structural health monitoring

The traditional use of in-service monitoring has been as a basis for structural
health monitoring (SHM) or structural condition monitoring (SCM), the aims
for which are defined by Balageas et al. [197] as to give:

at every moment during the life of a structure, a diagnosis of the
“state” of the constituent materials, of the different parts, and the
full assembly of these parts constituting the structure as a whole

SHM could be considered as a specific type of digital shadow, introduced previ-
ously in Section 2.3.2 in which the objective is to use the data from the physical
structure to update a virtual model of the structural behaviour which can then be
used to inform the maintenance, operation and inspection of the structure. Ryt-
ter [198] proposed a taxonomy of four levels of SHM, each of which corresponds
to the level of information which has been extracted from the in-situ monitoring
data. These levels are:
1. Determining whether a structure is damaged.

2. Locating the damage on a structure.

3. Quantifying the severity of the damage.


To reach these levels, several key challenges must be overcome. The first is ensuring that there is a baseline understanding of the condition of the structure in the undamaged condition, often referred to as the reference or baseline data set [199]. This data set should include a wide range of different operational and environmental conditions, to allow robust and reliable identification of changes in behaviour associated with damage. The detection of such changes in behaviour requires a robust understanding of the impact of the environmental and operational conditions on the behaviour of a structure, as discussed in greater detail in the context of vibration monitoring in Section 2.4.2.6.

Current research in SHM techniques can be split into three broad, in some cases overlapping, categories. The first of these are black box or data-driven SHM methods, examples of which include neural networks (e.g. [200, 201, 202, 203]) and generative adversarial models (e.g. [204, 205]). These identify features of the data that are reliable damage indicators through the use of training data, with robustness and accuracy of such indicators then assessed on previously unseen validation or hold-back data. The key drawbacks to this approach are that as the indicators are derived purely from the data they may not be interpretable by engineers, the indicators identified for one structure or data type cannot be directly applied to other similar structures, and there is a general scarcity of data from civil structures in both the damaged and undamaged condition [206, 207]. The second category of SHM techniques are based on physical relationships between the measured data and the behaviour of the structure, sometimes referred to as model-based approaches [208]. These methods seek to relate changes in parameters such as strain, load distribution, or dynamic behaviour, to damage to the structure [208]. As these relationships are based on the fundamental behaviour of the structure they have the advantage that methods or models developed for one structure may be adapted and applied to other similar structures. The physics-based approach has been successfully demonstrated in a number of case studies such as the work by Gatti [209], Liu et al. [210] and Chen et al. [211]. The third SHM approach is referred to in the literature as physics-based artificial intelligence or interpretable machine learning and seeks to combine the advances in statistical analysis of large data sets used in the data-driven SHM approaches,
with the physical basis and interpretability of the model-based SHM approaches [212]. This is a key area of development if the advances in SHM are to form a basis for the development of digital shadows and digital twins which often implicitly require a physical basis to the model updating using real-world data [213].

One closely related discipline to SHM which had generated considerable interest within the SHM literature is operational modal analysis, which seeks to understand the performance and condition of a structure based on its dynamic behaviour. Operational modal analysis is primarily implemented through in-situ monitoring using accelerometers, as they are low-cost, durable and simple to install on both new and existing structures, but the developments in the field have been applied to a broad range of in-situ monitoring technologies including electronic strain gauges [214], FBG strain gauges [215], video image tracking [216] and acoustic sensors [217].

### 2.4.2 Operational modal analysis

When any force is applied to a linear dynamic structure it causes the structure to accelerate. This acceleration causes a displacement of the structure, after which, if the force is removed, the structure will oscillate. Every structure oscillates at distinct frequencies known as natural frequencies, which are dependant on the distribution of mass in a structure, given by a structure’s mass matrix $[M]$, the distribution of stiffness within the structure, known as the structures stiffness matrix $[K]$, and the vector mode shape of the structure $\{\psi\}$. The $j$ natural frequencies at which an undamped linear dynamic system oscillates, $\omega$, measured in radians per second, can be found from the solutions to the generalized relationship $\omega_j^2[M]\{\psi_j\} - [K]\{\psi_j\} = 0$. The natural frequency $\omega$ is converted to $f$, measured in hertz, through $f = \omega/(2\pi)$.

The rate at which the energy from the oscillation of a mode is dissipated is governed by the damping ratio of the mode $\zeta$ and is typically given as a fraction of the critical damping $c_{\text{critical}}$, the damping which would cause the structure to freely return to its initial position without overshoot. For civil structures with low damping ratios the damping is typically assumed to be linear viscous damping, where the damping force is proportional to the velocity of the system, with a constant damping coefficient $c$. The critical damping of a simple harmonic oscillator with linear viscous damping is given by $c_{\text{critical}} = 2\sqrt{km}$ [15].

The relative movement of different parts of a structure during oscillation at the natural frequency is referred to as the normalized mode shape, $\{\phi\}$, with the amplitude $A$ of this oscillation dependant on both the applied force, modal
damping and the modal mass, the product of the mass matrix of the system with the mode shape. A distinct set of natural frequency, damping ratio, mode shape, and modal mass, referred to collectively as a set of modal parameters, defines a mode of vibration of the structure [15]. The acceleration dynamic response $y$ of a linear dynamic system measured at time $t$ after an impulse force $F$, measured at any arbitrary point on the system is given by [Equation 2.1]

$$y = F \sum_{j=1}^{\infty} A_j e^{-\zeta_j \omega_j t} \cos(\omega_j t)$$ (2.1)

In [Equation 2.1] $j$ is one of the infinite modes of vibration of the system, and $A_j$ is the amplitude of the mode $\{\phi_j\}$ at the measurement location. For most real structures, the dynamic response is dominated by a small number of modes of vibration, such that the free-oscillation response can be approximated as a sum of the responses in $N$ modes of vibration through [Equation 2.2]

$$y \approx F \sum_{j=1}^{N} A_j e^{-\zeta_j \omega_j t} \cos(\omega_j t)$$ (2.2)

**Modal analysis** is the process of estimating the modal parameters of a structure from measurements of its displacement, velocity, or acceleration response under forced excitation. If the force applied to the structure, $F$, is known, *experimental modal analysis* (EMA) can be used to estimate the structure’s modal parameters [15, 17]. However, for the analysis of civil structures it is often not viable to apply a known force to a structure, due to the associated costs, disruption, and risk which may be involved. Instead, *operational modal analysis* (OMA) whereby the dynamic behaviour of civil structures under ambient or in-service loadings, such as wind, footfall, or vehicle traffic, may be used. Unlike experimental modal analysis, where a known force is applied to a structure, the loading applied to structures in-service is unmeasured. Therefore assumptions must be made about the forcing during OMA, namely that the forcing has no dominant frequency component within the frequency range of interest, and that the frequency spectrum of the loading is sufficiently broad as to excite all modes of interest, sometimes referred to as the white-noise or broadband forcing assumptions [17]. Alongside the assumptions about the forcing, Rainieri and Fabbrocino [17] identify three key assumptions in OMA about the structural behaviour:

1. **Stationarity** - The modal parameters of the system are constant over time.
2. **Linearity** - Any combination of input forces results in a fixed combination
of output responses.

3. **Observability** - The data from the structure has sufficient fidelity, accuracy and temporal resolution to observe the structural modes.

In addition to this, as identified by Brincker and Ventura [218], there is the assumption of *ergodicity*: if given an infinite data set, all possible responses would occur, and the statistical characteristics of the response can be well approximated by a large selection of points.

There are two broad categories of OMA, frequency-domain OMA, where the frequency spectra of the measured structural response are used to estimate the modal parameters, and time-domain OMA, where a time-lagged autocorrelation function of the data is used to estimate the modal parameters.

### 2.4.2.1 Frequency-domain OMA

Frequency-domain OMA estimates the modal parameters of the structure through analysis of the frequency spectra or Fourier transform of the data. As data from real structures is finite and sampled at discrete intervals, the discrete Fourier Transform $X_k$, given by Equation 2.3 or 2.4 is used.

$$X_k = \sum_{n=0}^{N_s-1} x_n e^{-\frac{2\pi i}{N_s} kn} \quad (2.3)$$

$$X_k = \sum_{n=0}^{N_s-1} x_n \left( \cos \left( \frac{2\pi}{N_s} kn \right) - i \sin \left( \frac{2\pi}{N_s} kn \right) \right) \quad (2.4)$$

Within Equations 2.3 and 2.4, $x$ is the discrete time-domain data, $N_s$ is the number of samples, and $k/N_s$ is the frequency. $X_k$ are the discrete Fourier coefficients, indicating the amplitude and phase of each frequency component of the data. As discussed by He and Fu [219], the development of the Fast Fourier Transform (FFT) by Cooley and Tukey [220] in 1965, in which each DFT calculation is divided into two or more separate calculations at each step, laid the foundation for modern modal analysis by allowing for fast, efficient and accurate calculation of Fourier coefficients.

The earliest frequency-domain OMA method is the peak picking method, also referred to as the basic frequency-domain method [17], whereby natural frequencies of the system are found by identifying peaks, $\omega_{peak}$, within the frequency spectra. These peaks indicate the higher amplitude of oscillation of the system at those frequencies, an indicator of a mode of vibration. Once the peaks
are identified, the damping ratio of the mode ($\zeta$) can be found using the half-power bandwidth method. The frequencies $\omega_a$ and $\omega_b$ which correspond to where the amplitude $\alpha_{\text{max}}$ of mode $\omega_{\text{peak}}$ has reduced to $\alpha_{\text{max}}/\sqrt{2}$ either side of $\omega_{\text{peak}}$ are first identified. The damping ratio of the mode is then estimated using $\zeta = (\omega_b^2 - \omega_a^2) / (4\omega_{\text{peak}}^2)$.

The amplitude and phase of the mode can be estimated using the real and imaginary parts of the Fourier coefficients at $\omega_{\text{peak}}$. The key limitation of peak picking and the half-power bandwidth method is that it fails if there is more than one peak in close proximity, as the values of $\omega_a$ and $\omega_b$ cannot be identified. It is also sensitive to the frequency resolution used when calculating the FFT and is highly sensitive to noise in the data as it relies on a small number of Fourier coefficients for estimating the damping [17].

Estimating modal parameters using the frequency spectrum calculated through the FFT can be challenging due to noise within the data. To overcome this, Welch’s method [221], where the frequency spectrum generated from overlapping windows of data are averaged together, may be used. If the modal parameters are known to vary over time, the short-time Fourier Transform method [221], where independent frequency spectra are generated for each overlapping window of data, can give an indication of variations in the modal parameters. Despite the wide spread use of the short-time Fourier transform in the broader vibration literature (e.g. [224, 223, 222, 225, 226]), very little research has applied it to the analysis of civil structures. However, recent work by Ahmadi et al. [227] has highlighted that the method may provide a robust and efficient method for identifying damage to bridges and buildings [228].

A more robust frequency-domain OMA method than peak picking, developed by Brincker et al. [229], is Frequency Domain Decomposition (FDD). Frequency-domain decomposition uses the power spectral density (PSD) of the measured structural response, $G_{xx}\omega$, an example of which is shown in Figure 2.10. The PSD is related to the Fourier coefficients of the FFT $X_k$ by $G_{xx}\omega = (X_k) / (2df)$, where $df$ is the frequency resolution of the FFT coefficients, typically the sample frequency $f_s$ unless averaging of the frequency spectra through methods such as Welch’s method [221] has been applied. Singular value decomposition (SVD) [230] is then applied to the data to identify the dominant peaks in the frequency spectra [17] as $G_{xx}\omega = U_j S_j U_j^H$. In this relationship, $S_j$ are the singular values of the data, the complex frequencies of the system from which the natural frequencies and damping of modes of vibration can be calculated using peak-picking and the half-power method, and $U_j$ are the singular vectors of the system, containing the complex amplitude and phase information of the mode shape $\psi_j = U_j$ [229].
As highlighted by Rainieri and Fabbrocino [17], a key limitation of FDD, and enhanced-FDD, in which the mode shape estimate is used to refine the identification of the natural frequency [218], is that the modes shape estimates may be biased for close modes (modes with similar natural frequencies) due to the implicit assumption of independence in the SVD.

![Figure 2.10: Examples of power spectral density (PSD) plots of acceleration data from a highway bridge, reproduced from Elhattab et al. [231]. Modes of vibration manifest as peaks of the spectrum, with the magnitude of each peak corresponding to the amplitude of the oscillation, and the breadth of the peak being related to the damping of the mode.](image)

The final set of frequency-domain OMA methods are based on curve-fitting of the PSD of the data. The first of these approaches was the Least-Squares Frequency Domain (LSFD) method which minimizes the squared difference between a fitted PSD and the measured PSD [232]. The key limitation in this approach was the non-linearity of the minimization and the selection of an appropriate number of poles or modes to include within the fitted PSD. These issues were partially resolved by the Least-Squares Complex Frequency (LSCF) method, in which the error is weighted according to the magnitude of the PSD [233], and by Poly-Reference LSCF or the PolyMAX method in which the output from multiple sensors is used as a basis for the analysis allowing the modal participation factor (the contribution of each mode to the overall structural response) to be used as a basis for the selection of the number of real modes in the data [234].

### 2.4.2.2 Time-domain OMA

While frequency-domain OMA methods primarily use the FFT or PSD of the data, many time-domain OMA methods use the time-lagged correlation function of the data, or variations on it, as a basis for modal analysis. The time-lagged
correlation function \( \rho_{x,y} \) for two data sets of length \( N_s \), \( x \) and \( y \), at a time-lag \( \tau \) is defined through Equation 2.5

\[
\rho_{x,y}(\tau) = \frac{1}{N_s} \sum_{j=1}^{j=N_s} x(j)y(j+\tau)
\]  

(2.5)

If data sets \( x \) and \( y \) come from a linear dynamic system under broadband random excitation, the correlation function is an estimate of the free response of the system in data set \( y \) weighted by data set \( x \), as it is assumed no correlation exists between non-deterministic components of \( x \) and \( y \), while a correlation does exist between the deterministic components of \( x \) and \( y \) through the system’s free-response. The correlation function where \( x = y \) is referred to as the auto-correlation. Where \( x \neq y \), the correlation function is referred to as a cross-correlation.

An early use of correlation functions for estimating dynamic properties of structural systems is presented by Cole Jr. [235, 236], which introduces the random decrement technique (RDT), the development and variations of which are explored in detail in Section 2.4.2.3. In the RDT, short segments of data are collected where a triggering condition, such as the amplitude of the data being in a particular range, is met and averaged together to form a random decrement signature (RDS). This RDS is an estimate of the free-response of the system to some initial amplitude and velocity of response. The RDT is not a “full” OMA method, as it does not provide the modal parameters of the system. However, the modal parameters may be estimated from the RDS through least-squares curve fitting or through SVD based methods such as the Ibrahim time-domain method [237] and matrix pencil method [238], both discussed in detail in Section 2.4.2.3.

A widely used time-domain OMA method, developed by Peeters et al. [239], is covariance driven stochastic-subspace identification (SSI-Cov). In SSI-Cov, the auto- and cross-correlation functions of a multi-channel data set \( X \) denoted by Rainieri and Fabbrocino [17] as \( [\hat{R}_j] \), are used to form a block Toeplitz matrix \( [T_j] \) using Equation 2.6 [17].

\[
[T_j] = \begin{bmatrix}
[\hat{R}_j] & [\hat{R}_{j-1}] & \ldots & [\hat{R}_1] \\
[\hat{R}_{j+1}] & [\hat{R}_j] & \ldots & [\hat{R}_2] \\
\ldots & \ldots & \ldots & \ldots \\
[\hat{R}_{2j-1}] & [\hat{R}_{2j-2}] & \ldots & [\hat{R}_j]
\end{bmatrix}
\]  

(2.6)

SVD can then be performed on \( T_j \) to find the \( k \) singular values and singular vectors \( [T_j] = U_j S_j V_j^T \).
Using the singular values and vectors a $k$ order state matrix, $[A_s]$, can be formed through Equation 2.7:

$$[A_s] = \sqrt{S[1:k]}^{-1}U[:,1:k]^HT_jV[:,1:k]\sqrt{S[1:k]}^{-1}$$  (2.7)

Calculating the eigenvalue solution of $A_s$ gives the eigenvector, which is equal to the $k$ mode shapes $\psi_k$, and the eigenvalues $M_e$ which are the complex frequencies of the system. The $k$ natural frequencies, $f_k$, and damping ratios, $\zeta_k$, can be found from the eigenvalues $M_e$ as $f_k = |f_s \log(\text{diag}(M_e))|$ and $\zeta_k = (\text{real}(f_s \log(\text{diag}(M_e)))) / (f_k)$, where $f_s$ is the sample rate of the data. The order $k$ of the SVD should be equal to the number of modes in the data. However, for civil structures it is rare to know the number of modes of vibration which are identifiable from the data. Therefore the frequency, damping and mode shapes are often calculated over a range of model orders, with the results presented as stability diagrams, an example of which is shown in Figure 2.11.

An extension to SSI-Cov, data-driven stochastic subspace identification (DD-SSI), was developed by Brincker et al. [242]. Unlike SSI-Cov, DD-SSI is performed directly on the raw data, not on correlation functions, and utilizes elements of Kalman filter theory to estimate the state matrix of the system [242].

Other time-domain OMA methods include second-order blind identification (SOBI) methods, autoregressive (AR) and autoregressive-moving average (ARMA) methods, and the Natural Excitation Technique (NExT) methods. However the successful implementation of these methods in practice has been limited, and so they will not be discussed in detail. The interested reader is referred to Rainieri and Fabbrocino [17] and Brincker and Ventura [218] for further details.

Wavelet analysis, where a wave function with known characteristics is compared with the measured data, has also been applied within time-domain OMA. However, as wavelet analysis does not directly result in modal parameter estimates for the system, and has specific considerations in its application and use which are beyond the scope of this research, it will not be presented in detail. The interested reader is directed to Masjedian and Keshmiri [243] and Ulriksen et al. [244] for greater detail on the development of wavelet analysis for SHM and OMA.

2.4.2.3 Random Decrement Technique

The random decrement technique (RDT) forms the basis for much of the work presented in this thesis as it is computationally efficient and allows greater flexibility than other time-domain OMA methods for the analysis of data which does
Figure 2.11: Example of a stability diagram for SSI-Cov analysis, reproduced from Peeters and De Roeck [241], showing frequency estimates with increasing model order (number of poles). The “[oplus]” symbol is used for modal estimates with stable frequency, damping and mode shape; “.v” for a modal estimate with stable frequency and mode shape; “.d” for a modal estimate with stable frequency and damping; “.f” for a modal estimate with stable frequency and “.” for other modal estimates. The two embedded subplots are used to show the close modes around 2.4 Hz and 7 Hz. Frequencies are considered stable if the difference between the frequency estimate at order $k$ exhibits less than 0.5% variation from that observed at order $k - 1$. Damping estimates are considered stable if the damping at order $k$ exhibits less than 5% variation from that observed for the same frequency at order $k - 1$. Mode shapes are considered stable if the modal assurance criteria, discussed in Section 2.4.2.6, is greater than 0.98 between the mode shapes at order $k$ and order $k - 1$.

not completely fulfill the assumptions of OMA introduced at the start of this section, notably the requirements of stationarity (constant modal parameters) and the broadband forcing assumption. As many other OMA methods are closely related to the RDT, being either based on correlation functions or Welch averaged frequency spectra, facilitating improvements to the RDT and a greater understanding of its theoretical basis can benefit OMA methods more generally.

The RDT was originally developed by Henry J. Cole Jr for identifying damage induced flutter of space shuttle wings during flight [235, 236, 245]. The application of the RDT was defined by Cole Jr [245] as follows:

i. Collect acceleration data from the structure.

ii. Where the acceleration data crosses a specific amplitude-threshold collect
iii. Average the segments of data together to form a random decrement signature (RDS).

This approach is referred to in more recent literature as a level-crossing triggering condition \[246\] and may be expressed through two equations. Using notation adapted from Asmussen \[246\], let \( x \) be the acceleration data recorded at time steps \( j \) and \( T \), a binary true-false operator, be the level-crossing triggering condition for a trigger amplitude \( A_T \) defined at each time step \( j \) using Equation 2.8.

\[
T_j = \begin{cases} 
\text{True} & \text{if } x_{j-1} < A_T \text{ and } x_j > A_T \\
\text{True} & \text{if } x_{j-1} > A_T \text{ and } x_j < A_T \\
\text{False} & \text{otherwise}
\end{cases}
\]  

(2.8)

The RDS for \( N_{rd} \) instances of \( T_j = \text{True} \) at a time-lag \( \tau \) before or after the trigger condition is met, \( \hat{D}_{xT}(\tau) \), is then given by Equation 2.9:

\[
\hat{D}_{xT}(\tau) = \frac{1}{N_{rd}} \sum_{n=0}^{N_{rd}} (x(t_j + \tau)|T_j = \text{True})
\]

(2.9)

Cole Jr \[236\] developed the RDT to aid in the detection of damage in the presence of non-linear dynamic behaviour, specifically the amplitude-dependant oscillation of space shuttle wings. By collecting segments of data which began where the amplitude of the oscillations crossed a threshold, estimates of the system’s dynamic response under a specific initial amplitude of response could be isolated. Unlike the auto- or cross-correlation of the data, the RDS was thought independent of the input excitation. However, as noted by Cole Jr \[245\], this is not correct, as the RDS is only independent of the input excitation if the spatial distribution of the input forces is constant or if the system is a single degree-of-freedom (SDOF) system. If the spatial distribution of the forcing is not constant, the relative magnitude of different modes of vibration within the RDS will vary, due to the differing magnitudes of the modal responses.

The RDT as applied by Cole Jr \[236\] was not used for estimating the modal parameters of the structure, as the dynamic response of the shuttle wings was known to exhibit amplitude-dependant non-linear dynamic characteristics. Instead the RDSs for specific level-crossing triggering conditions collected from different flight tests were directly compared on the basis that any damage to the shuttle wings was likely to result in change in the dynamic behaviour. In Cole Jr’s
research [245], a method for estimating structural damping is presented based on fitting a damping envelope to the RDS. Cole Jr [245] noted that where the RDS contained multiple modes of vibration, the signal must be filtered such that the data contains oscillations from a single mode of vibration to allow the damping envelope for that mode to be estimated. Cole Jr and Reed Jr [247] expanded the RDT analysis to in-situ damage detection of civil structures in-service through application of the RDT to a steel-girder bridge and concrete beams subjected to fatigue cracking. In both sets of experiments, direct comparison of the RDSs was used a metric for damage detection. Direct comparison of RDSs was further applied by Brignac et al. [248] for detection of damage in-flight for the YF-16 fighter aircraft, by Yang and Caldwell [249] for detection of damage to pipes in nuclear power stations, and by Yang et al. [250] for damage detection in a model of an offshore platform. However, as discussed above, direct comparison of RDSs for detecting changes in modal parameters is only possible where either the RDS only contains oscillations associated with a single mode of vibration, or where the relative amplitudes of different modes of vibration within the RDS is constant.

Chang [251] provided a mathematical basis for the RDT as an approximation of the free-response of a structural system, and expanded the RDT to include direct curve-fitting of the RDS for estimating the natural frequencies and damping of modes of vibration. This curve fitting was achieved through minimization of the least-squared error or euclidean distance between the RDS and a fitted sum of free-response signals, and was limited to the analysis of one- and two-degree-of-freedom systems with an initial amplitude of zero. An example of this early curve-fitting work is presented in Figure 2.12. This work provided a basis for the development of the Ibrahim Time Domain (ITD) method by Ibrahim [252]. In the ITD method, a series of three time-lagged RDSs are used to form a block Hankel matrix, which provides the basis for a general Eigenvalue problem to solve for the modal parameters. While Ibrahim and Goglia [253] used regression to solve the generalized eigenvalue problem, the solution may also be found through singular-value decomposition, through a similar solution as utilized in the matrix pencil method [254] discussed in Section 2.4.2.4. While the initial ITD tests were limited to small numbers of analysis channels, the technique was greatly expanded for consideration of over-determined models (where the number of analysis channels is greater than the number of components to be estimated from the data) through the use of large numbers of analysis channels by Ibrahim and Pappa [255, 256].
The next major theoretical developments of the RDT can be found in the works of Asmussen et al. [257], Asmussen [246], Ibrahim et al. [258] and Asmussen et al. [259].

A statistical definition was introduced by Asmussen et al. [257] which links the correlation functions of stationary zero-mean distributed Gaussian processes and the RDS. The full derivations of the RDT equations, summarized below, is presented in Appendix A of Asmussen et al. [257]. For consistency with the rest of the thesis, the notation used in this section has been adapted from that originally used by Asmussen et al. [257]. Asmussen et al. [257] introduced a generalized triggering condition $T$ given by Equation 2.10.

$$T_j = \begin{cases} 
\text{True} & \text{if } a_1 < x(t_j) < a_2 \text{ and } b_1 < \dot{x}(t_j) < b_2 \\
\text{False} & \text{otherwise} 
\end{cases} \quad (2.10)$$

In Equation 2.10, $x$ is the analysis data measured at discrete time intervals $j$, $\dot{x}$ is the first differential of the analysis data, $a_1$ and $a_2$ are amplitude limits, and $b_1$ and $b_2$ are first differential limits. It is shown by Asmussen et al. [257] that the RDS ($\hat{D}_{xT}$) at a time-lag $\tau$ for this triggering condition is the expected value of the data set $x$ conditional on the triggering condition $T$: $\hat{D}_{xT}(\tau) = E(x(t_j + \tau)|T_j = \text{True})$. 

Figure 2.12: Early modal analysis of an RDS generated using numerical simulations of a single degree-of-freedom system reproduced from Chang [251]. Top plot) Measured RDS, lower plot) Least-squares approximation of RDS.

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For a linear system excited by a white-noise stationary process Asmussen et al. \[257\] demonstrated that the RDS is an approximation of a weighted sum of the correlation function $\rho_{xT}(\tau)$ of $x$ and $T$ and the first order time-derivative of the correlation function $\rho'_x(\tau)$, as given by Equation 2.11 where $\rho'_x(\tau)$ is also given by the correlation function between $\dot{x}$ and $T$.

$$\hat{D}_{xT} \approx \frac{\rho_{xT}(\tau)}{\sigma_x^2} \hat{a} - \frac{\rho'_x(\tau)}{\sigma_{\dot{x}}^2} \hat{b} \quad (2.11)$$

In Equation 2.11, $\hat{a}$ and $\hat{b}$ are related to the probability density functions between the trigger bounds $a_1$ and $a_2$, and $b_1$ and $b_2$ through Equations 2.12 and 2.13, where $x_0$ are the possible values of $x$.

$$\hat{a} = \frac{\int_{a_1}^{a_2} x P(x(x_0))dx_0}{\int_{a_1}^{a_2} P(x(x_0))dx_0} \quad (2.12)$$

$$\hat{b} = \frac{\int_{b_1}^{b_2} \dot{x} P(\dot{x}(\dot{x}_0))d\dot{x}_0}{\int_{b_1}^{b_2} P(\dot{x}(\dot{x}_0))d\dot{x}_0} \quad (2.13)$$

The expected variance of the RDS, $Var[\hat{D}_{xT}(\tau)]$ is also derived for the generalized triggering condition and is given by Equation 2.14, with $\sigma_x^2$ being the variance of data set $x$.

$$Var[\hat{D}_{xT}(\tau)] \approx \frac{\sigma_x^2}{N} \left( 1 - \left( \frac{\rho_{xT}(\tau)}{\sigma_x^2} \right)^2 \right) \quad (2.14)$$

Equations 2.11 and 2.14 are conditional on the assumption that both the forcing applied to the system and any noise present within the measurements are white-noise processes which are uncorrelated with the triggering condition.

Asmussen et al. \[257\] \[259\] introduced the vector random decrement technique as an effective means by which the modes of vibration can be estimated from ambient vibration data. The vector RDT utilizes a cross-correlation or vector implementation of the RDT in which the phase relationship between two channels of analysis data $x$ and $y$ is maintained through vector triggering, that is collection of a segment of data from the analysis channel $x$ where the triggering condition $T$ is met in channel $y$. This approach is again conditional on the triggering condition being independent of the forcing applied to the system and any noise within the data sets $x$ and $y$.

More recent work has focused on the use of the RDT for analysis of a wide range of weak non-linear dynamic behaviour, where the modal parameters of a system vary with external factors. The majority of this work utilizes different...
triggering conditions for quantifying amplitude-dependant damping. Reynolds et al. [260, 261, 262] have extensively explored the use of the RDT for quantifying amplitude-dependant behaviour of timber structures. Marukawa et al. [263], Tamura and Suganuma [264], Quan et al. [265] and Kim et al. [266], among others, have utilized the RDT for estimating the dynamics of tall structures under wind excitation, with a particular focus on quantifying the effectiveness of tuned mass damper systems and estimation of amplitude dependant damping, an example of which is reproduced from Tamura and Suganuma [264] in Figure 2.13. A variety of techniques are used within these works for separating non-linear dynamic behaviour including close filtering of the data to isolate the responses of individual modes of vibrations prior to the RDT analysis, and estimating time-lagged damping ratios for individual sections of the RDS. A variation of triggering of the RDT based on the Hilbert amplitude envelope of the data, as demonstrated by Huang and Gu [267], has been shown as potentially being a more effective alternative to range-crossing triggering for quantifying amplitude dependant behaviour, with the results estimated for numerical systems exhibiting a lower variance.

A similar use of the RDT for quantifying amplitude dependant behaviour with a range-crossing triggering level was adapted by Vesterholm et al. [268] for estimating the parameters of a duffing oscillator simulated using synthetic data, in which the forcing of the structure is described by a second-order non-linear differential equation. Similar problems were previously tackled in Mahfouz [269] and more recently by Hou et al. [270] in which the RDT was used to estimate the non-linear damping and restoring force acting on a ship rolling in high seas.

While the majority of recent work has used the RDT to estimate the modal parameters of structures, there remains interest in the use and comparison of the RDSs themselves without the estimation of modal parameters. Shiryayev and Slater [271] demonstrated the use of RDSs for detection of fatigue cracking in a similar manner to that originally proposed by Cole Jr [235], however they also highlight that using the RDS as a damage indicator remained sensitive to the excitation of the structure. Vesterholm et al. [272] presented a comparison of RDSs as a basis for detecting non-linear behaviour. In this work the principle components of fitted RDSs for different triggering levels is used as an indicator of non-linear dynamic behaviour. However, this method was limited to the analysis of single-degree of freedom systems under white noise forcing and with additive white noise corruption of the data, and its general applicability to multi-degree of freedom systems and more complex forcing is likely to be limited by the same issues faced in direct comparison of RDSs, as described by Cole Jr [235].
Figure 2.13: Estimates of amplitude dependant natural frequency (top subplot) and damping ratio (lower subplot) of a tall tower under wind excitation estimated using the RDT, reproduced from Tamura and Suganuma [264].

The application of the RDT for modal analysis, as well as OMA methods more generally, requires the selection of a number of parameters including the length of the data set required for accurate modal parameter estimates, the number of segments of data included within the RDS, and the number of time-lags or RDS length.

There has been limited work on the data set length required for accurate modal estimates using OMA. The BSI [273] and the ISO Technical Committee [274] reference the equation given by Oppenheim [275] and reproduced in Equation 2.15 for a minimum data set length $T_r$ for estimating modal parameters with...
a 4% bias and 10% variance.

\[ T_r = \frac{200}{\zeta f_j} \] (2.15)

In Equation 2.15, \( f_j \) is the natural frequency of mode \( j \) and \( \zeta_j \) is the damping ratio of mode \( j \). However, this equation is valid only for linear systems excited with independent white-noise forcing and corrupted with additive white-noise noise [275]. A minimum data set length of twenty half oscillations, or ten times the lowest frequency of the system, as given by Equation 2.16 is proposed by Brincker and Ventura [218].

\[ T_r = \frac{10}{\zeta f_j} \] (2.16)

This duration is twenty times lower than that recommended by Oppenheim [275] and is provided without explanation. A comparison of the recommended window lengths based on Equation 2.15 and 2.16 for a range of natural frequencies and damping ratios is presented in Figure 2.14.

![Figure 2.14: Duration of data set length to achieve bias of 4% and variance of 10% for a variety of natural frequencies and damping ratios as given by Equations 2.15 and 2.16.](image)

The applicability of these minimum data set lengths to real-world systems, which may exhibit correlated errors and non-stationary modal behaviour is largely untested. For the RDT, these data set lengths would also have to be adapted if a triggering condition which does not use every data point is to be used.

Further indications of the data set length required for accurate modal estimates can be found by considering research on the number of segments of data to be included within the RDS for accurate modal estimates. Within the literature a range of numbers of segments have been considered, with Chang [251] recommending 2,000 independent segments for accurate damping estimation, based on the results presented in Figure 2.15, and a minimum of 500 segments for accurate frequency estimation, a finding supported by Asmussen et al. [259]. However
both of these numbers are based on analysis of systems with uncorrelated errors, white noise forcing and additive white noise corruption of the data, and analysis of RDSs containing small numbers of modes.

![Graph showing comparison of number of segments and accuracy of damping estimation](image)

Figure 2.15: Comparison of number of segments and accuracy of damping estimation, reproduced from Chang [251].

Research on the number of time-lags or the RDS length to be used is also limited. Chang [251] presented an early analysis of the impact of RDS length on estimated modal parameters, as previously plotted in Figure 2.15. This analysis was limited to one- and two-DOF systems under white-noise forcing and excitation, with least-squares curve fitting used for the estimation of modal parameters.
The results for this analysis suggested an optimal RDS length of approximately 125% the beat period of the two frequencies present within the RDS for a two DOF system. The applicability of this result to higher-order systems, or for systems analysed through SVD based modal analysis rather than direct curve fitting, is unknown. A wide range of RDS lengths have been used within the literature from fractions of a single oscillation to ten or more complete oscillations of a mode of interest \[276, 277\], which may be evidence of the system specific nature of RDS length.

2.4.2.4 Estimating modal parameters from the RDS

As previously discussed, the RDT is not a full modal analysis procedure. In order to estimate the modal parameters of the system from the RDS, it is necessary to analyse the RDS with a curve-fitting or low rank-approximation technique. The matrix pencil method (MPM) \[254\] is one method to decompose a signal into a lower rank approximated sum of damped sinusoids. It has been shown to be more robust to noise than other singular value decomposition (SVD) based modal analysis methods \[278\] and more computationally efficient than direct least-squares fitting of damped sinusoids to a signal through Prony analysis \[279, 278\]. It is closely related to the Ibrahim Time Domain (ITD) method \[255, 256\].

To apply the MPM, each RDS (\(\hat{D}\)) of length \(L_{rd}\) is used to form two Hankel matrices, \([H_1]\) and \([H_2]\), through splitting and wrapping the RDS. In these Hankel matrices, defined through Equations 2.17 and 2.18, the subscript \(0\) refers to the first time step of the RDS.

\[
[H_1] = \begin{bmatrix}
\hat{D}_0 & \hat{D}_1 & \hat{D}_2 & \ldots & \hat{D}_{L_{rd}/3-1} \\
\hat{D}_1 & \hat{D}_2 & \hat{D}_3 & \ldots & \hat{D}_{1+L_{rd}/3-1} \\
\hat{D}_2 & \hat{D}_3 & \hat{D}_4 & \ldots & \hat{D}_{2+L_{rd}/3-1} \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
\hat{D}_{2L_{rd}/3} & \hat{D}_{1+2L_{rd}/3} & \hat{D}_{2+2L_{rd}/3} & \ldots & \hat{D}_{L_{rd}-1}
\end{bmatrix}
\] (2.17)

\[
[H_2] = \begin{bmatrix}
\hat{D}_1 & \hat{D}_2 & \hat{D}_3 & \ldots & \hat{D}_{L_{rd}/3} \\
\hat{D}_2 & \hat{D}_3 & \hat{D}_4 & \ldots & \hat{D}_{1+L_{rd}/3} \\
\hat{D}_3 & \hat{D}_4 & \hat{D}_5 & \ldots & \hat{D}_{2+L_{rd}/3} \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
\hat{D}_{1+2L_{rd}/3} & \hat{D}_{2+2L_{rd}/3} & \hat{D}_{3+2L_{rd}/3} & \ldots & \hat{D}_{L_{rd}}
\end{bmatrix}
\] (2.18)

The Hankel matrices are used to form a generalized eigenvalue problem \([H_1] - \lambda[H_2] = 0\), where \(\lambda\) are the unknown eigenvalues of the system in the complex
form $\lambda_j = e^{(-\alpha_{0,j} + i\omega_j)\Delta t}$. $\Delta t$ is the time step separating each corresponding value of $[H_1]$ and $[H_2]$, and is equal to one over the sample rate ($f_s$). The physical interpretation of the $\lambda$ values are that they are equal to the change in each of the damped sinusoidal components of the system across the time step $\Delta t$.

In order to identify the $M_k$ largest damped sinusoidal components a rank $M_k$ singular value decomposition (SVD) of matrix $[H_2]$ (denoted with subscript $M$) is used where it is assumed $[H_1] - \lambda [H_2] \cong [H_1] - \lambda [H_{2,M}] = 0$. The SVD of this equation is given through Equation 2.19.

$$[H_1] - \lambda [U_2][S_2][V_2^H] \cong 0 \quad (2.19)$$

Left multiplying Equation 2.19 by $U_2^H$, and right multiplying by $[V_2]$ gives Equation 2.20.

$$[U_2^H][H_1][V_2] - \lambda [U_2^H][U_2][S_2][V_2^H][V_2] \cong 0 \quad (2.20)$$

Equation 2.20 may be simplified as Equation 2.21.

$$[U_2^H][H_1][V_2] - \lambda [S_2] \cong 0 \quad (2.21)$$

$[U_2^H][H_1][V_2] - \lambda [S_2] \cong 0$ now forms a rank $M_k$ generalized eigenvalue problem of the form $Ax - \lambda Bx = 0$ which can be solved to find the eigenvalues of the system $\lambda$.

As stated above, the values of $\lambda$ are the eigenvalues of the dynamic system for each of the $M_k$ components in the form $\lambda_j = e^{(-\alpha_{0,j} + i\omega_j)\Delta t}$. Taking the log of both sides of $\lambda_j = e^{(-\alpha_{0,j} + i\omega_j)\Delta t}$, and scaling by the time step ($\Delta t$) between $[H_1]$ and $[H_2]$ gives Equation 2.22.

$$\log(\lambda_j) / \Delta t = -\alpha_{0,j} + i\omega_j \quad (2.22)$$

Hence the decay rate ($\alpha_0$) is the real part of Equation 2.22 and the angular frequency ($\omega$) is the imaginary part of Equation 2.22.

The decay rate (rate of exponential decay of the oscillations) is converted to the damping ratio (energy loss per oscillation) by dividing through by the angular frequency $\zeta = \alpha_0 / \omega$.

With the $M_k$ values of $\alpha_0$ and $\omega$ known, the residues, $[R]$, of the system can be calculated through finding the pseudo-inverse solution to $\hat{D}(t) = [Z(t)][R]$, 48
where $[Z(t)]$ is given by Equation 2.23 and $[R]$ is given by Equation 2.24.

$$
[Z(t)] =
\begin{bmatrix}
e^{(-\alpha_0,1+j\omega_1)t(0)} & e^{(-\alpha_0,2+j\omega_2)t(0)} & \cdots & e^{(-\alpha_0,M_k+j\omega_{M_k})t(0)} \\
e^{(-\alpha_0,1+j\omega_1)t(1)} & e^{(-\alpha_0,2+j\omega_2)t(1)} & \cdots & e^{(-\alpha_0,M_k+j\omega_{M_k})t(1)} \\
\vdots & \vdots & \ddots & \vdots \\
e^{(-\alpha_0,1+j\omega_1)t(L_{rd})} & e^{(-\alpha_0,2+j\omega_2)t(L_{rd})} & \cdots & e^{(-\alpha_0,M_k+j\omega_{M_k})t(L_{rd})}
\end{bmatrix}
(2.23)
$$

$$
[R] =
\begin{bmatrix}
R_1 \\
R_2 \\
\vdots \\
R_{M_k}
\end{bmatrix}^T
(2.24)
$$

The magnitudes of the residues $R$ (complex amplitudes) are the amplitudes of each of the $M_k$ damped sinusoidal components of the rank reduced approximation of $\hat{D}$, whilst the angle of the residues are the phase of each damped sinusoidal component. The RDS approximated with the rank $M_k$ decomposition, $\hat{D}_{M_k}$ is then reconstructed as the solution to $\hat{D}_{M_k} = [Z(t)][R]$.

It should be noted, that the MPM gives no indications of the suitability of the rank-$M_k$ or model order used. Therefore, as described previously for the stochastic subspace methods, the MPM is typically applied with a range of model orders and the results presented as stability diagrams.

### 2.4.2.5 Current best practice for OMA

When estimating modal parameters through OMA there are a range of practical factors which should be considered to ensure the estimated modal parameters offer an accurate approximation of the dynamic structural behaviour. Alongside this, the OMA technique selected for analysis of the data should be carefully selected based on the characteristics of the structure and the objective of the OMA.

**Sensor locations**

The sensors used for the data collection, typically accelerometers, should be located at points on the structure where the maximum amplitude of each mode shape will occur to increase the ratio of measured dynamic response to background noise within the measured signal. However, where the number of sensors used to measure the structural response is limited, this objective must be balanced with the requirement to accurately identify all modes of vibration within the frequency range of interest. Simple numerical models of the structure or more complex finite element models can be used to guide the location of structural sensors [15].
Sampling frequency and signal post-processing

The theoretical maximum frequency from which a signal may be reconstructed is given by the Nyquist frequency, half of the sampling frequency. In practice frequencies close to the Nyquist frequency may be attenuated or corrupted by the analogue to digital conversion [280]. Therefore, it is recommended that the sampling frequency selected is greater than twice the maximum frequency of interest. Typically a minimum sampling frequency of two and a half times the maximum frequency of interest is used [15]. It is common to use a sampling frequency much greater than the maximum frequency of interest with the signal post-processed after collection to remove frequencies outside of the range of interest and the signal downsampled to reduce the computational burden of OMA as appropriate.

Prior to time-domain OMA, the data is often subjected to low-, high- or band-pass filtering to remove responses outside of the frequencies of interest. Popular filters include the Butterworth filter, which has a flat response in the filter pass-band, and the Bessel filter, which minimizes filtering artefacts associated with truncation of the frequency spectrum of the data [281].

Frequency-domain OMA methods

As discussed in Section 2.4.2.1 and Section 2.4.2.2 there is a range of OMA methods which may be used for estimating modal parameters from the data. For greater detail on the development, theoretical background and application of these methods see He and Fu [219], Rainieri and Fabbrocino [17] and Brincker and Ventura [218].

The peak-picking method combined with the half-power bandwidth method for estimating damping is the most computationally simple analysis method, requiring only calculation of the Fourier transform or power spectral density of the data and identification of peaks within the spectra. However, this method is highly sensitive to noise in the analysis data and cannot be used for the analysis of close modes of vibration [219]. Other frequency-domain methods, such as circle-fit methods, FDD, enhanced-FDD, the LSFD method and variants of the LSCF method resolve the issues posed by noise within the data through fitting the Fourier/PSD coefficients using a broader range of the frequency/PSD spectrum [219]. Key issues with these methods include the challenges of separating the spectra of close modes of vibrations, the computational challenges posed by fitting non-linear frequency spectra and the need to select an appropriate number of poles or modes to include within the fitted PSD [17]. More broadly, frequency-domain OMA methods may obscure deviations from the OMA assumptions about the dynamic behaviour of the structure, such as the stationarity and linearity of
the structural response, and may not be intuitive to non-specialist users as they requires the user to understand the frequency-domain representation of time domain signals.

**Stochastic subspace OMA methods**

Stochastic subspace identification (SSI) techniques such as SSI-Cov and DD-SSI are popular time-domain OMA methods due in part to their inclusion within a number of commercially available analysis programs such as ARTeMIS \[252\] and PULSE \[283\]. Key limitations in the use of SSI methods include the need to manually interpret the modal analysis results using stability diagrams, the challenges of selecting an appropriate model order when approximating the system response and a lack of transparency in the analysis which may reduce user confidence or which may obscure when the OMA assumptions of the dynamic behaviour of the structure have been violated.

**Auto-regressive OMA methods**

The auto-regressive (AR) method of modal identification, and extensions to it such as auto-regressive with moving averages (ARMA) and auto-regressive with exogenous input (ARX) methods, are robust methods for damage detection and dynamic analysis of structures which are well approximated using a small number of modes of vibration \[17\]. These methods can give indications of when the OMA assumptions of stationarity and linearity may have been violated \[284\]. However they perform poorly when there are large numbers of modes of vibration and require careful initialization of the analysis parameters.

**Random decrement OMA methods**

In comparison to SSI or AR methods, the RDT (in conjunction with a modal identification technique such as the ITD method or MPM) may appear to non-specialist users as be a more intuitive and accessible OMA method. The fundamental assumption of the RDT, that the response of a linear dynamic system to an unknown stochastic input force can be approximated through ensemble averaging of segments of the analysis data which meet a defined triggering condition, can be simply demonstrated through visually accessible examples. Alongside this, it is simple to plot and compare the measured RDS and ITD/MPM approximated system response and the use of different triggering conditions within the RDT allows for the analysis weak non-linear dynamic behaviour, as discussed in Section 2.4.2.3. However, for systems containing multiple modes of vibration, or where the OMA assumptions are violated, the intuitive appeal of the RDT may be misleading, with the RDS dependant on complex relationships between the applied forcing, measurement noise and dynamic system response. Current
limitations to the use of the RDT include a limited understanding of uncertainty within the RDT modal parameter estimates, the complex relationship between the RDS and components of dynamic response, and a lack of efficient methods for post-processing of the RDT modal parameters estimates.

2.4.2.6 Unresolved challenges in OMA

The OMA literature highlights five key unresolved challenges for the accurate and efficient application of OMA techniques:

i. Reducing and eliminating sources of biasing in OMA

ii. Quantifying the uncertainty in modal estimates,

iii. Automating the identification of modes of vibration in OMA modal estimates,

iv. Identifying and quantifying non-linear dynamic behaviour,

v. Inferring structural parameters from OMA modal estimates.

Biasing in OMA

Many researchers have noted that OMA is susceptible to biasing of modal estimates due to a variety of different factors.

Aquino and Tamura [285] highlighted that the filtering of ambient vibration data prior to analysis with the RDT led to changes in the damping ratios; a key parameter in the design of civil structures which affects how long unwanted oscillations may persist, the amplitude which oscillations may reach, and the likelihood of resonance being induced in the system. Other observations on the impact of filtering on the RDT include earlier work by Tamura [286] and the original observations made by Cole Jr [287] during the development of the RDT. For both covariance-driven and reference based stochastic subspace identification methods, filtering of the data leads to the introduction of spurious modes as noted by Reynders et al. [288] and Peeters and De Roeck [289] [241]. This complicates the identification of structural modes within the stability diagram [289, 241]. While the impact of filtering artefacts on experimental modal analysis [290, 291, 292] and on signal processing for system identification [293, 281] is well understood, the impact of filtering on ambient vibration data analysis, where it
is the coherence of filtering artefacts within the data which distorts modal parameter estimates, has not previously been addressed.

**Uncertainty in OMA**

In order for in-service monitoring to inform the design and maintenance of civil structures, it is crucial to understand and convey the uncertainty within the results.

An emerging field in the OMA literature, introduced by Au et al. [294], is Bayesian OMA (BOMA) which seeks to account for the underlying uncertainty inherent when working with ambient vibration data. Under BOMA, modal identification is framed as the probability of observing the measured output given a model of the system through Bayes theorem, the formula for which is given in Equation 2.25.

\[
P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}
\]  

(2.25)

In Equation 2.25, \(P(A|B)\) is the conditional probability of event \(A\) given that event \(B\) has occurred, usually referred to as the posterior probability. Similarly, \(P(B|A)\) is the probability of event \(B\) given that event \(A\) has occurred. The probability of the independent events, \(P(A)\) and \(P(B)\) are the marginal or prior probabilities, the probabilities of event \(A\) or event \(B\) occurring. Brownjohn et al. [295] demonstrated a frequency-domain BOMA method based on the use of raw (unaveraged) frequency spectra. A Bayesian approach is used to find the most probable value (MPV) for a model of the structure’s response, and by extension its frequency spectra, within pre-selected frequency bands of the data. Several computational issues with the approach are identified, such as the growth in the number of dimensions for optimization within the MPV calculations, the challenges associated with interpreting the results, and the trade-off between identification precision and modelling error. Further issues, as identified by Au et al. [294] include the identification of close-modes and identification of suitable frequency bands for the analysis. A detailed explanation of the theory and implementation of BOMA, alongside selected case studies, is provided by Au [16].

Quantifying the uncertainty within OMA modal estimates can also be framed as a sampling problem: estimating the statistics of a population, in this case the modal parameters of a system under all viable real-world conditions, given a sample of the population, here the measured data from the system. For independent and identically distributed (i.i.d.) variables, the jackknife [296, 297]
and bootstrap resampling methods has been shown to be an effective method for estimating statistics of the population given a finite sample of the data.

In jackknife resampling, $n$ calculations of the statistics of interest are calculated from a sample of length $n$ with the $n^{th}$ observation excluded in each calculation. This is commonly referred to as sampling without replacement. Given the $n$ observations of the statistic of interest, the variance of this statistic can then be calculated from the resampled population. In bootstrap resampling, a schematic of which is presented in Figure 2.16, the observations are resampled with replacement such that each observation may appear more than once in the bootstrapped sample. Book length discussions of the theory of bootstrap and jackknife resampling are given in Chernick, Davison, and Efron. Bootstrapping has been explored as a method for estimating the variation in the correlation functions which form the basis for many time-domain OMA methods. Kijewski and Kareem, Kijewski and Kareem, and Zhou and Li have all demonstrated bootstrapping approaches to the RDT through the creation of bootstrapped RDSs by resampling of the segments used to form the RDSs. This technique was expanded on by Silva et al. who used bootstrapped RDSs as the basis for DD-SSI analysis.

![Figure 2.16: Schematic for application of bootstrap resampling, reproduced from Husnain et al.](image)

The key limitation of bootstrap and jackknife resampling is that they require the observations to be independent. However, any correlation of the forced and
noise components of the structural response with the triggering condition make
the segments of data used to form the RDS non-independent. By extension,
the modal parameters estimated from the bootstrapped segments are also non-
independent, making classical bootstrap and jackknife resampling poorly suited
for estimating the variance of modal parameters.

Within the wider statistics literature, several methods to overcome the limitation
of independent samples have been suggested. One approach is to transform
the data to impose independence on the observations [300], however this is reliant
on understanding the correlation present within the samples, for which no
methods currently exist within the OMA literature. An alternative approach is
resampling blocks of the data within which any correlation between observations
is preserved. Where the blocks are non-overlapping this is referred to as a simple
block bootstrap [308, 300], however the key limitation of this approach is that
the choice of large block sizes limits the number of blocks available for resampling,
while the choice of smaller blocks reduces the length of correlation between
samples which may be preserved within the blocks. A method to overcome this
is through the use of overlapping blocks, referred to as a moving block bootstrap
[300]. However, where a fixed window length is used the stationarity of the boot-
strapped observations is not maintained, as the length of observed correlation
is dependant on the window length selected [309]. This issue can be partially
resolved through the stationary bootstrap, in which the block length is randomly
varied [310].

There are several situations in which the jackknife and bootstrap sampling
methods fail, discussed at length in Chernick [300]. In the context of OMA,
the most pressing of these are the presence of correlated or non-independent
behaviour. This correlation may exist within the data due to a lack of stationarity
or correlated errors. The temporal order of the vibration data is crucial in OMA,
as the current state of the system is dependant on the previously applied forcing.
Similarly, it is likely that the current modal parameters of the system are more
strongly correlated with near past and near future observations of the modal
parameters, such as in the case of temperature induced changes in natural fre-
quencies. While the stationary bootstrap block resampling method can be used
to maintain some of these dependencies, it has limited applicability where the
data exhibits significant heteroskedastic behaviour [311], the condition where the
variance of the error in the statistic of interest varies widely.

This can be illustrated in the context of the RDT through a simple example.
Consider data collected from a footbridge excited by footfall excitation over a
24-hour period. It is likely that the natural frequencies of the bridge will vary
with the temperature due to changes in stiffness, and any confidence intervals generated for the modes of vibration should reflect this expected variation. However, the excitation applied to the bridge is likely to be higher during daylight hours, when the temperature is higher, than at night. Therefore if an RDS is formed using 24 hours of data and block bootstrapping is applied to resample the segments included within the RDS, segments corresponding to higher daytime temperatures are likely to have higher amplitudes and will dominate the resampled RDSs. This will lead to a strong bias in the expected distribution of natural frequencies for each mode of vibration towards higher temperatures.

A solution to the issues posed by correlated errors, non-linear correlated behaviour, and heteroskedasticity is subsampling [309]. Subsampling can be summarized as estimation of the statistics of interest, in this case the modal parameters of the structure, on small overlapping subseries or subsamples of the data. Key to its success is that, unlike bootstrapping or jackknife methods, the original structure of the data is preserved, therefore preserving any correlation or heteroskedasticity in the original data set associated with the order in which samples are observed within the subsamples.

The application of subsampling can be summarized as follows:

i. Divide the data into overlapping windows.

ii. Calculate the statistic of interest for each window of data.

iii. Fit a model to the subsample statistics to describe the behaviour of the system conditional on the window length selected.

Wolf [312] summarized the theoretical basis of subsampling as follows: as each subseries of the data was generated by the true probability mechanism of the system, information about the distribution of any given statistic across the full population can be gained by evaluating it on all subsamples of the sample data. The only assumption that is required for the application of subsampling is that the distribution of parameters for any given subsample, denoted here as \( \hat{\theta}_{N_0} \), where \( N_0 \) is the size of the subsample, will approach the true or limiting distribution of the parameters as \( N_0 \to \infty \). It therefore follows that there is some unknown relationship between the distribution of \( \hat{\theta}_{N_0} \), the size of the subsample \( N_0 \), and the true distribution of the parameter \( \theta_0 \). Note that, as no assumptions are made about the true distribution of \( \theta_0 \), this inference holds for correlated (non-independent) and heteroskedastic (non-identically distributed) data [309] [312]. Despite the robustness of subsampling to the issues of correlated and heteroskedastic behaviour.
within in-service measurements of structural behaviour, no evidence of its application to OMA or SHM was found in the review of the literature.

**Automated mode detection in OMA**

Identifying modes of vibration within OMA modal estimates is primarily carried out manually. In order to promote widespread uptake of OMA, methods for automating this process are required. Previous work on automating the identification of modal estimates has primarily focused on identifying stable modes within stability diagrams. Methods which have been explored in detail for automated mode identification in stability diagrams are k-means clustering [313, 314, 315, 316, 317], Density based spatial clustering of applications with noise (DBSCAN) [318, 319] and Gaussian mixture models [320, 321, 322, 323, 324].

K-means clustering is an unsupervised machine learning technique, where the data is divided into $k$ clusters of data by minimizing the squared or euclidean difference between the value of a data point and the mean value of all points assigned to the nearest cluster [325].

There has been some past success using k-means clustering for automated identification of modes of vibration in stability diagrams. Tronci et al. [317] demonstrated the use of k-means clustering in conjunction with a Mahalanobis distance (scaling of the euclidean distance between points by the covariance of the data set) error term alongside a minimum assignment distance for removal of noise and outliers for model order selection and clustering of modal estimates obtained using data-driven stochastic subspace identification (DD-SSI). Sun et al. [315] and Reynders et al. [313] explored the use of k-means clustering for identification of stable modes in stability diagrams of modal estimates from covariance-driven stochastic subspace identification (SSI-Cov), incorporating a variety of thresholds for identifying stable poles and the assignment of poles to a cluster of similar modal estimates. An iterative multi-stage clustering approach was developed by Neu et al. [314] which incorporated k-means clustering for the separation of noise modes and physical modes within modal estimates from DD-SSI, with smaller clusters deemed as likely noise modes and removed from the results. K-means clustering was compared to a variety of other clustering techniques for the identification of modes within stability diagrams of SSI-Cov modal estimates by [316].

There are some limitations of k-means clustering in the context of OMA including the selection of an appropriate value of $k$, the problems posed by close modes which may result in overlapping clusters of modal estimates, and the in-
ability of k-means clustering to account for the broadband noise in modal estimates which may result from spurious or non-modal components of the frequency spectra/correlation functions.

Density based spatial clustering of applications with noise (DBSCAN), where data points are assigned to a cluster based on where a minimum distance criterion is met, remove the need to pre-specify the number of clusters to be detected within the data. DBSCAN has previously been explored as a method of identifying stable modes by Li et al. [318] using numerically generated data, an example of which is reproduced in Figure 2.17, and by Ye and Zhao [319] using data collected from the Z24 road bridge, an example of which is reproduced in Figure 2.18. As they highlight, a key benefit of DBSCAN is that the stability criteria for identifying modes of vibration may be used as a basis for the minimum distance criteria. However, key limitations of DBSCAN include the need for distinct, well-defined continuous clusters which are clearly distinct from noise within the data [326], such as the data presented in Figure 2.17. This can limit the use of DBSCAN for close modes of vibration, or data sets containing high levels of spurious or noise modal estimates. DBSCAN can also require extensive tuning of parameters, as the assignment of points to a cluster is highly sensitive on the DBSCAN parameters selected. Within OMA estimates, the optimal value of the DBSCAN parameters is often unclear, as indicated by the complex results presented in Figure 2.18.

Figure 2.17: Application of DBSCAN for identification of modes within a stability diagram, reproduced from Li et al. [318].
Gaussian mixture models (GMMs) allow for the analysis of close modes of vibration through modelling the set of modal estimates as a combination of \( n \) Gaussian distributions \cite{325}. The probability of each frequency estimate belonging to a given distribution is given by the probability density function (PDF) defined in Equation 2.26, here defined in the context of a single OMA frequency estimate \( f_j \) for a Gaussian distribution \( n \) as:

\[
PDF(f_j) = \frac{1}{\sigma_n \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{f_j - f_{0,n}}{\sigma_n} \right)^2}
\]  

(2.26)

In Equation 2.26, \( f_{0,n} \) is the natural frequency of mode \( n \), and \( \sigma_n \) is the standard deviation of the frequency estimates drawn from that mode. Limitations to this approach, include selection of an appropriate value of \( n \), initialization of the GMM with appropriate values, and separation of spurious or noise modes from real-modes of vibration.

As with k-means clustering and DBSCAN, the primary use of GMMs in the OMA literature has been for the analysis of stability diagrams. For example, Cheema et al. \cite{323} demonstrated the use of Dirichlet-Process GMMs, an extension to GMMs in which the number of Gaussian distributions may grow or shrink, for identifying stable modes within stability diagrams of modal estimates from SSI-Cov. GMMs have also been explored by Chiplunkar and Morlier \cite{327} as a standalone method of frequency-domain OMA, by fitting the frequency power-spectrum. This method is presented as an automated variant on the half-power method \cite{328}, with the natural frequencies and damping ratios derived based on the means and variances of the fitted Gaussian’s. Santos et al. \cite{322}, Zhang et al.
and Neu et al. [321] used GMMs as a method of identifying when damage has occurred to a structure, indicated by a change in the distribution of natural frequencies. However in this work the GMM was not used to separate the modes of vibration. Favarelli and Giorgetti [324] built on this work through comparing GMMs to a range of other methods for damage detection using data from for the well-separated modes of the Z24 road bridge. However again the primary objective here was identification of damage, not separation of modes of vibration.

Non-linear structural behaviour in OMA
A key assumption within OMA is that the behaviour of the structure is linear, for a given input force the response of the system is constant. However, this is rarely true for real-world structures, due to environmentally induced changes in structural behaviour, or changes in structural behaviour associated with factors such as the amplitude of movement. This is referred to by Ewins [15] as weak non-linear behaviour: while the structure still has defined modes of vibration, the exact parameters of these modes of vibration are liable to change over time. This non-linear behaviour may be divided into two categories:

1. Weak non-linear modal behaviour - The change in structural behaviour induced by changes in the internal or external environment of the structure which occurs over longer time frames.

2. Weak non-linear dynamic behaviour - The change in structural behaviour which results over very short-time frames typically induced by the oscillation of the structure itself [15].

Weak non-linear modal behaviour is induced by a change in the mass, stiffness or damping distribution of the structure. A change in any of these distributions will result in a change in the natural frequencies, damping ratios, mode shapes and modal masses of the structure, as evidenced by the definitions of these properties given at the start of this section. A wide range of factors can induce non-linear modal behaviour, but the most commonly encountered forms of weak non-linear modal behaviour are temperature-induced, mass-induced, and damage-induced changes. As the temperature of a structure changes, there is a corresponding change in the stiffness of the structural elements, resulting in a change in the modal parameters. This effect may also cause localized effects, such as the change in frequency and damping which may result from thermal expansion causing the closure of movement joints, or changes in the contact area between structural elements. Changes in the mass of a structure can also cause
meaningful changes in the modal parameters, particularly for lightweight structures where the mass of occupants, vehicles or temporary fixtures and furnishings, may lead to a meaningful change in the mass distribution of the structure.

For mass-induced and temperature-induced weak non-linear modal behaviour the change in parameters is temporary, the mass-, stiffness-, and damping-distributions will revert to a baseline value after the removal of the environmental effects. Damage-induced weak non-linear modal behaviour is different as it represents a permanent step change in the mass-, stiffness-, and/or damping-distributions. This will manifest in the modal parameters as a step-change event, a feature used for damage identification and localization in many SHM techniques through anomaly detection [329].

Regression methods have been presented as a method for quantifying environmentally induced variation in natural frequencies such that they may be removed from the modal estimates to allow damage to be detected. An early example of the use of regression methods in OMA can be found in the analysis of natural frequencies of the Z24 road bridge presented in Peeters and De Roeck [241], who compared multivariable linear regression and autoregressive with extra input (ARX) models for quantifying thermally-induced non-linear modal behaviour. The same techniques were applied more recently by Ramos et al. [330] who attempted to quantify the variation in natural frequencies of two historic masonry structures due to changes in air temperature, humidity and root-mean-square excitation level of the structures. An important finding from Ramos et al. [330] is the apparent dependence of the natural frequencies on the relative humidity of the structure, indicating that other environmental variables apart from temperature variation may induce non-linear modal behaviour. However, a key limitation of the findings presented by Ramos et al. [330], reproduced in Figure 2.19, is that the fitted regression parameters are not assessed on unseen data and there is no discussion of any indications of overfitting within the results.

A similar approach of assuming that the natural frequency of a structure is linearly related to the temperature of the structure is presented in Anastasopoulos et al. [215] for the analysis of one year of data from a steel girder arch bridge. Again, the applicability of the fitted regression coefficients to unseen data is not addressed within the work. The results presented in this work would suggest that the temperature may be more strongly correlated with the squared frequency of the mode, as evidenced by the relationship in the plotted comparisons of temperature and frequency in this work, reproduced in Figure 2.20.

Ceravolo et al. [331] also presents linear regression analysis results for long-term monitoring of a masonry structure which assume a linear relationship be-
between natural frequencies and temperature, despite the plotted relationships between the natural frequencies and temperature, reproduced in Figure 2.21, suggesting a squared relationship. The regression models developed within this work are used as a preliminary basis for damage detection using non-linear modal behaviour, with a deviation from the predicted variation in natural frequencies assumed to be an indicator of damage. This approach is also directly adopted by Yang et al. [332] for analysis of data collected from a high-speed railway bridge. However, the results from this analysis suggest that a linear approximation of the relationship between natural frequencies and temperature may be appropriate for this structure.

Cury et al. [333] presents the use of neural networks as a means of fitting what
they refer to as a non-linear regression model of temperature. This model is not a regression model in the traditional sense, instead, it is more closely related to logistic regression methods as the thermal behaviour is modelled through a series of interconnected classifications of non-linear modal behaviour based on measured temperature. Classification algorithms such as logistic regression and artificial neural networks have seen widespread interest within the SHM literature as methods for damage detection and classification, as discussed by Abdeljaber et al. [334] and Azimi et al. [335].

Ridge regression, a weighted variation on linear regression [336], is presented as a method for removal of environmental variability in natural frequency measurements by Zhang et al. [337]. The use of linear regression within a two-step automated OMA method is discussed in Cabboi [329]. This method also assumes a linear relationship between temperature and natural frequencies. No examples of work that utilized regression methods for assessing damage based on changes in natural frequencies were found in the literature review, with damage detection primarily being addressed through classification methods such as logistic regression and artificial neural networks. No examples have been found for the use of regression methods for quantifying non-linear modal behaviour associated with changes in static loading.

In the wider context of in-service structural monitoring, there has been a great deal of work on the use of linear regression methods for the separation of thermally induced variations in strain from the strain measured in service, an overview of which is presented in Kromanis and Kripakaran [338]. Alongside the linear regression methods discussed above, the use of time-lagged temperature measurements as a method for modelling the time-lagged thermal behaviour of structures is presented, work which is likely to have wider applications within the
analysis of non-linear modal behaviour.

Quantile regression methods are largely overlooked within both the OMA and SHM literature. This has been commented on by Tee et al. [339], who developed a method that used distributed quantiles of auto-regressive time-series models of vibration amplitude as a metric for damage detection and localization. Further details of the same work can be found in Tee and Cai [340].

Two key issues which must be addressed when applying regression methods to the analysis of OMA and SHM data are multicollinearity and overfitting. Multicollinearity occurs when the inputs to the regression analysis are strongly correlated, such as temperature measurements from different parts of a structure. This correlation between input data can negatively impact the interpretability of the regression parameters, as a small decrease in one regression coefficient may be accounted for by an increase in a regression coefficient for a different input data set. Multicollinearity may also increase the risks of overfitting of the data, where spurious components such as noise within the input data are explained by the fitted regression coefficients. Overfitting negatively impacts the generalizability of the fitted model to new and unseen data, where the spurious components are not present, and affects both multicollinear and independent data sets [341].

Non-linear dynamic behaviour poses a challenge for OMA as the change in modal parameters occurs over short-time periods. An example of this is Coulomb damping, in which the damping is induced by the friction between elements [15]. A closely related phenomenon is amplitude dependant damping, in which the damping is proportional to the amplitude of the motion, as described by Tamura and Suganuma [264]. Non-linear dynamic behaviour pose design problems for tall buildings, which may employ tuned-mass or tuned-stiffness dampers to control the amplitude of oscillations, as the magnitude of movement to be expected is dependant on the rate at which energy is dissipated from the structure [342]. This amplitude dependence has also been observed to extend to the natural frequencies of structures, as demonstrated by Tamura and Suganuma [264] and discussed in Section 2.4.2.3. In the context of design, non-linear dynamic behaviour can complicate the selection of suitable design parameters, as parameters such as the damping may experience large variations during different periods of use. Apart from the use of the RDT for identifying and quantifying amplitude dependant behaviour, other techniques which have previously been utilized for quantification of non-linear dynamic behaviour include wavelet analysis [333], non-linear curve fitting and non-linear modal expansions [341].
Inferring structural parameters from OMA modal estimates

If OMA is to be used as a basis for SHM or PBD, the relationship between the measured dynamic behaviour of a structure and the physical structure must be understood. One approach to deriving this relationship is through model updating, in which parameters of a digital model of the structure such as a finite element model is iteratively updated such that the modal parameters of the modelled structure match those measured in-service [345]. A key metric used in this updating is the modal assurance criterion (MAC) [346], defined through Equation 2.27, which measures the degree of consistency between two mode shape vectors, \( \{ \psi_0 \} \) and \( \{ \psi_1 \} \).

\[
MAC = \frac{\vert \{ \psi_0 \}^T \{ \psi_1^* \} \vert^2}{\{ \psi_0 \}^T \{ \psi_0 \}^* \{ \psi_1 \}^T \{ \psi_1^* \}} \tag{2.27}
\]

In Equation 2.27, \( \{ \psi \}^T \) is the transpose of the mode shape vector, and \( \{ \psi \}^* \) is the complex conjugate of the mode shape vector. Factors such as the non-linear behaviour of civil structures, the biasing of modal estimates, and the uncertainty in the modal estimates, can complicate model updating. The process of model updating can be highly time intensive, although recent work has explored the automation of this process such as Sanayei et al. [347], Altunişik et al. [348], and Ferrari et al. [349].

An alternative to model updating for relating structural and dynamic behaviour is to make use of the weak non-linear behaviour exhibited by the structure itself as a basis for the inference. A method that utilizes this approach was put forward by Parloo et al. [350] which uses the change in frequency which occurs when a known change is made to the mass matrix of a structure, such as the addition of a known mass, as a means to estimate the modal mass of a structure. Let \( \omega_0 \) be the original natural frequency of a mode with mass matrix \([M]\), stiffness matrix \([K]\) and mode shape \(\{\psi_0\}\), and let \( \omega_1 \) and \( \{\psi_1\} \) be the natural frequency and mode shape of the same mode after the addition of a known mass \([\Delta M]\). Therefore the natural frequencies of the system before and after the addition of the known mass is given by the solutions to Equations 2.28 and 2.29 respectively.

\[
[K]\{\psi_0\} = \omega_0^2[M]\{\psi_0\} \tag{2.28}
\]

\[
[K]\{\psi_1\} = \omega_1^2 ([M] + [\Delta M]) \{\psi_1\} \tag{2.29}
\]

As derived by Khatibi et al. [351], if the change in the mode shape is considered negligible such that \( \{ \psi_0 \} \approx \{ \psi_1 \} \), a mode shape scaling factor \( \alpha \) can be derived.
which relates the modal mass normalized and amplitude normalized mode shapes through [Equation 2.30]

\[
\alpha = \sqrt{\frac{\omega_0^2 - \omega_1^2}{\omega_1^2 \cdot \{\psi_0\}^T \cdot [\Delta M] \cdot \{\psi_n\}}}
\] (2.30)

The primary use of the mode shape scaling factor in OMA, as discussed in depth by Parloo et al. [350], Parloo et al. [352], Brownjohn and Pavic [353], and Khatibi et al. [351], is for scaling of mode shapes such that the relative amplitude of the modes for a given applied force may be calculated. This is an important step in the updating of finite element models of the structure, computational models designed to simulate the dynamic behaviour of real-world structures [354].

While there has been some limited progress in estimating in-situ dynamic loading (e.g. [355, 356, 357]) and estimating tensile cable loading [358], less work has explored the potential of using OMA for estimating static loading, the loading applied to structures by the action of added mass under gravity. Previous work which has explored predicting static loading includes Hansen et al. [359], who used finite-element model simulations of various loading scenarios to predict likely mass locations and quantities. However, as noted in Vigsø et al. [360], key limitations of this approach include the need for an accurate finite-element model of the structure, and the need for engineering judgement when selecting likely loading scenarios.

However, a greater understanding of the relationship between weak non-linear modal behaviour and changes in stiffness or mass, alongside robust methods for quantifying and tracking modal parameters for individual modes of vibration, has the potential to greatly expanded the information available to structural engineers about the in-service performance of structures. While tracking structural behaviour has formed the basis for many SHM techniques, little research has explored the use of OMA as a means of informing structural engineering design through comparing in-service behaviour with design stage predictions.

2.5 Summary

In this chapter the key challenges in structural engineering posed by ageing infrastructure assets and the climate emergency have been identified. The existing design process, which is dominated by largely reactive and prescriptive codes of practice, is poorly placed to allow the adaptation of civil structures necessary to face these challenges.

66
Performance based design offers an interpretative and proactive approach to structural engineering design which may allow the adoption of novel materials, structural forms, and construction techniques to create more efficient and resilient civil structures. However, the long-term success and sustainability of this approach is dependant on validation of the performance based design methodologies in-service through comparison of in-situ behaviour with design stage predictions. Digital shadows and digital twins offer a technological basis for the integration of long-term structural monitoring into the structural engineering design process, but are reliant on efficient algorithms for updating of virtual models with data collect from in-service monitoring.

Operational modal analysis, where the dynamic behaviour of structures under ambient or in-service loadings is used to estimate the modal parameters of a structure, offers a particularly promising method for in-situ monitoring as it can relay information about the global behaviour of the structure and utilizes low-cost and durable structural sensors such as accelerometers.

However, there are a number of challenges to be overcome if operational modal analysis is to form the basis for improving the structural engineering design process, including understanding the causes of biasing in modal estimates, quantifying non-linearity and uncertainty in estimates of dynamic behaviour, and automating the detection of modes of vibration within sets of modal estimates. Despite these challenges, operational modal analysis has the potential to allow the estimation of a range of structural parameters in-situ, such as the in-service mass, stiffness and loading of civil structures, alongside the insights which can be gained from the analysis and comparison of in-service estimates of dynamic behaviour with design stage predictions.
Chapter 3

Perceptions of the engineering design process

To supplement the literature review and guide the research, a multi-national survey of civil engineering practitioners was carried out to gather information on the existing usage of long-term monitoring to inform civil and structural design, its potential to provide feedback to engineers and barriers to implementation within the existing design process. This chapter is adapted from work by Wynne et al. originally published as Perceptions of long-term monitoring for civil and structural engineering in Structures [1].

As discussed in the literature review, in-service monitoring, measuring the behaviour of structures under in-service loads and environmental conditions, has undergone rapid development over the last 20 years due to decreasing hardware costs, increases in the durability and sensitivity of the monitoring technology, and increased computation and data storage capabilities. These developments have largely been focused on the potential benefits in-situ monitoring may offer for condition assessment, damage-detection and structural health monitoring [361, 362, 363]. Recent research has started to explore the potential for long-term monitoring to inform the design of civil structures. Model updating, updating finite-element models based on in-service measurements, offers a direct way in which the predicted behaviour of a structure may be compared to its in-service behaviour [364]. Building-information modelling (BIM) allows engineers to integrate design calculations, assumptions, and material and construction information within a virtual model of an asset [365]. More recently, digital twins, virtual models of assets updated and integrated with real-world measurements and data [366], have allowed long-term monitoring to inform the operation and maintenance of large infrastructure assets [167] and individual structures [366].
Digital twins form a crucial part of Construction or Industry 4.0, the automation and digitization of the construction industry [153, 154]. However, little work has explored industry perceptions of how long-term in-service monitoring could influence and improve the civil and structural engineering design process and the practical, cultural and social barriers which must be overcome if engineers are to extract the greatest value from in-situ structural monitoring.

Alongside these barriers, perceived uncertainties in the design process must be characterized to identify where long-term monitoring may offer the greatest benefits when informing future designs. Alongside this, the perceptions and current use of long-term monitoring within industry must be understood to identify areas of successful implementation and barriers to future use. Extensive research into perceptions of BIM (e.g. [74, 367, 368]) have shown how BIM’s successful implementation and dissemination within civil and structural engineering are due in large part to understanding the demands and functionalities required by industry. Building on this work, the study presented in this chapter explores current perceptions of unknowns and uncertainties in the engineering design process to identify areas where long-term monitoring may offer the greatest benefit to engineers. The study also seeks to understand some of the issues which may help or hinder the integration of long-term monitoring within civil and structural design.

3.1 Methods

Quantitative and qualitative information about industry perceptions of long-term monitoring was collected through a questionnaire covering three topics:

- Perceptions of unknowns and uncertainties in the existing civil/structural engineering design process.
- Perceptions and current uses of long-term monitoring within industry.
- Future potential and barriers to adoption of long-term monitoring within civil and structural engineering.

Full details of the design, development and dissemination of the questionnaire, alongside the ethical and data privacy considerations, are presented in Appendix A. A copy of the questionnaire used in the research is provided in Appendix B.

A total of 146 responses to the questionnaire were collected from participants spread across the globe. The respondents included a cross-section of levels of engineering experience and a broad range of self-reported areas of civil and...
3.2 Results

3.2.1 Perceptions of uncertainty within design

The compiled results from a series of categorical questions exploring respondents’ confidence in characteristic design loads, dynamic loads, material parameters and performance limits are presented in Figures [3.1 to 3.4]. In these questions respondents were given the option to select “not applicable” if they felt they were not sufficiently familiar with the relevant design codes, standards or guidelines as to comment. The results show that participants view the majority of codified guidance on the engineering parameters as either being close to the true value (41.0% of total applicable responses) or an appropriately conservative value (36.8% of total applicable responses).

![Figure 3.1: Respondents’ reported confidence that the characteristic design loads (given or calculated) available in codes and standards reflect reality.](image)

Figure 3.1: Respondents’ reported confidence that the characteristic design loads (given or calculated) available in codes and standards reflect reality.

The most concerning responses relating to the characteristic design loads (Figure 3.1), are those which suggest that respondents perceive current guidance as underestimating or substantially underestimating design loads. More than 10% of the respondents view current guidance as underestimating design loads for all categories except the mass of structural elements. A limitation of the data presented here is that it is a broad generalization of multi-faceted and complex categories. However, future work in long-term monitoring should prioritize addressing these
concerns due to the devastating consequences which may accompany the under-
estimation of structural loads. Of the overestimation of design loads, the most
notable result is that 20.5% of respondents viewed current guidance on floor oc-
cupancy as an excessive overestimate of the true value after removal of the “not
applicable responses”. Floor occupancy is a topic of long-standing debate within
the engineering literature (for example [369, 370, 371]). Significant values for ex-
cessive overestimates are also reported for non-structural elements (8.6%), vehicle
loading (5.7%), wind loading (8.1%) and snow loading (8.2%), after the removal
of non-applicable responses. Characteristic values for these loads may be greatly
supplemented by long-term in-situ monitoring to inform design codes and guid-
ance as these are parameters which are difficult to estimate through conventional
surveys and short-term measurement methods. However, it is crucial the engi-
neering community understands the basis for the design values to address any
misconceptions regarding the origin of the values specified in design codes and
standards.

For the perceptions of dynamic loading [Figure 3.2] there is a decrease in
the percentage of respondents who feel they have the experience necessary to
respond to the question. This decrease in participation may partially explain the
increased percentage of responses who perceive substantial underestimates in the
existing design guidance. Those who are more technically familiar with specific
areas of structural or civil engineering may be concerned with the higher con-
sequences which would accompany underestimation of design loads, an example
of rational cognitive biases due to over representation of extreme events [372].
Conversely it could be that familiarity with the design standards and guidance

![Figure 3.2: Respondents' reported confidence that the characteristic dynamic
design loads (given or calculated) available in codes and standards reflect reality.](image)
causes an implicit bias towards over-estimating its flaws, a possible example of expert overconfidence such as described in detail within the work of Lin and Bier [373].

Figure 3.3: Respondents’ reported confidence that the design parameters (given or calculated) available in codes and standards reflect reality.

Within the responses for the material design parameters (Figure 3.3) many participants who had not previously stated any specific geotechnical expertise felt confident that existing guidelines on geotechnical/foundation strength/stiffness were either an excessive underestimate or a substantial overestimate of the true value, with both geotechnical categories representing the most polarizing topic explored within the questionnaire. This may be explained in several ways. At the most basic level, all civil structures must interact with geotechnical engineering and foundations in some manner, a nexus point of design not found in any other major subdivision of civil and structural engineering, and most civil engineers will have experienced project delays and contractual conflicts due to unforeseen ground conditions [374]. Geotechnical engineering may also offer the greatest range of uncertainties within the engineering design, due to both the lack of information available at the design stage and the complexities of soil-structure interactions [375]. Greater uncertainties translates to greater numbers of assumptions and assumptions which are broader in scope. Therefore, given the polarization of the area, geotechnical and foundation behaviour may be an area in which long-term monitoring may help narrow the scope of design assumptions through access to data about the in-situ behaviour of existing structures.

While smaller in scale, the polarization in perception of current guidance on static deflection and acceptable acceleration limits (Figure 3.4) may stem from
similar reasons. However, unlike geotechnical/foundation stiffness and strength, these values are largely subjective and context specific as reflected in standards and design guidance around the world. The benefit which long-term monitoring may offer with regard to this polarization of opinion is a move away from codefied limits and towards performance-based design [134, 148] in which specific design objectives and performance criteria, such as limiting occupant complaints or acceleration- and deflection- induced serviceability issues such as cracking, are met.

When asked “If you had access to a single set of perfect information about a structure you designed in the past to inform your future designs, what would be of most use to you?” a common theme in the responses was the need for greater information on the load history applied to structures, particularly with regard to the interactions and co-occurrence of loading types. Also highlighted within these responses were the impact of climate change on both the severity and frequency of extreme loading events, fitting with previous observations on the cognitive biases towards extreme events [372] within the responses. Also noted were the importance of material degradation and fatigue, both with respect to how this will impact the long-term behaviour of structures in-service, and how they might be better mitigated against. However the responses to the subsequent question, “In which area of civil/structural design do you think there is the greatest uncertainty about design assumptions?”, highlighted a very different selection of factors, with greater emphasis placed on the impact of factors which are difficult to recreate in controlled experiments, such as environmental loading, changes in environmental conditions or the human-structure and human-infrastructure interactions. This difference in the responses to the two questions can be interpreted in several ways. It could be due to the reluctance of participants to repeat previously given answers. Alternatively there may be a disconnect between the engineering

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Figure 3.4: Respondents’ reported confidence that the performance limits available in codes, standards or other guidance are appropriate.
parameters with the greatest uncertainty and those which would have the greatest impact on the design. Conversely, it may be due to limitations in the existing design process and the challenges of incorporating factors such as the degradation of structural materials, climate change or changing patterns of usage. However, careful consideration should be given as to whether long-term monitoring offers the greatest benefit solely by reducing the largest causes of uncertainty, or whether improved accuracy of better understood behaviour may result in greater gains in design efficiency and resilience.

### 3.2.2 Current use of long-term monitoring

Of the 146 responses, only 39.7% of respondents reported that they had worked on projects which had been evaluated by the designers following construction, 45.9% of respondents had not, and 14.4% responded that the question was either not applicable or did not fill in a response. This highlights one of the largest hurdles to be overcome if long-term monitoring is to be used to guide the design of future structures: whether the format of the existing design process allow for reflection on the successes or failures of past designs. The reported extent of assessment post-construction is largely informal, such as lessons-learnt and site walkthroughs. Where more extensive evaluations do take place these were identified by respondents as being primarily driven by two factors, paraphrased as: i) where there is a problem with the finished structure, ii) where it is mandated by the client as part of the contractual obligations. However the extent of evaluations is one of the few areas of the survey where there is meaningful difference in responses depending on fields of expertise, with respondents working on sensitive and critical infrastructure, such as bridges and power generation, reporting that evaluations are more detailed and longer-term than is reported for respondents working primarily on other types of civil structures. As with the implementation of BIM, this may suggest that intervention and leadership from governments and large clients may significantly increase the use and effectiveness of post-construction evaluation [376].

When asked how long-term monitoring data from previous projects could be used within the design process the predominant interest was in creating more efficient designs, for example “[long-term monitoring data] could be used for future projects with similar occupation and similar structural schemes to avoid common failures and utilise more of the capacity of the materials/structure”, and reducing long-term risk to the structures, for example “[confirming] when major upgrades are required and the reasons behind them” and “[i]n the prediction of fatigue
development; particularly in areas which cannot be easily inspected”. These aspects are in line with the previous observations made on current perceptions of uncertainty within the design process. Numerous respondents also highlighted that for rapidly developing technologies such as ultra-slender buildings, timber engineering, and reclamation of industrial brownfield land, the long-term efficacy of the design guidance is unknown, and in-situ monitoring may bring immediate and specific benefits to their continued development, summarized by one respondent as: “[long-term monitoring] data could help code and standards writers to modify and improve design requirements”.

The final questions on the current integration of long-term monitoring into the engineering design process examined the knowledge and literacy of the respondents with respect to digital twins, virtual models of assets updated with real-world data. Digital twins by definition will require the use of long-term monitoring data and are considered to form a key next-step in “Construction 4.0” or the digitization of the construction industry [377, 378]. Of the 146 respondents, 38.4% reported that they were familiar with the concept of a digital twin. This result in particular may be affected by some sampling bias, as it is likely that those who choose to take part in the survey have a prior interest in the application of new technologies within civil and structural engineering. Of those who reported familiarity with digital twins, 37.5% reported that their current or most recent employer was actively using digital twins. However, when the respondents who were familiar with digital twins were asked to define it in their own words, many respondents failed to make the distinction between BIM, “a digital representation of physical and functional characteristics of a facility” as defined by National institute of building sciences [379], and digital twins, “which is characterized by the cyber–physical integration” [155] or the updating of virtual models using real-world monitoring data [167]. Examples of responses where participants conflated BIM and digital twins, either explicitly or implicitly, included:

- “A digital representation of a structure storing data about each component”
- “Computer (BIM) model of actual structure”
- “It is a methodology for design; material specs; construction procedures; time delivery; cost and collect information of the projects stages along the life cycle of the projects and collaborate the stakeholders for share the same information in real time”
- “A geometrically accurate 3D computer model linked to a database of information containing details of the structures components”
Selected examples of responses which fully or partially captured key aspects of digital twins included:

- “A digital analytical model that can simulate/reproduce all the responses of interest given the parameters representing the real environment”
- “Using a digital (simulated) version of a system or piece of infrastructure to evaluate changes in performance with changing inputs.”
- “A digital (BIM or otherwise) model of the as-built structure to which, in theory, perturbations associated with the real structure can be applied in an attempt to either estimate and effect from a cause or ‘reverse engineer’ a cause from an observed effect”
- “A multiphysics models allowing simulation of an as-built structure; component; infrastructure; etc. fed with data measured in a real component; infrastructure; etc.; to mirror the life of the real structure”

### 3.2.3 Future potential of long term monitoring

The final section of the questionnaire explored the possible benefits and challenges of integrating long-term monitoring into the future civil/structural design process. The respondents were predominantly in favor of this integration, with 78.3% reporting that they thought that long-term monitoring data could be of use to them in future projects, with large proportions of respondents identifying that better understanding in-situ behaviour could reduce the risk in future projects. 10.9% of respondents left the question blank or had a non-committal response. Of the 9.5% who responded that long-term monitoring would not be of use in future designs a wider range of reasons were given, these are summarized as:

- the lack of direct financial benefit to the engineer,
- the difficulty and risks involved with interpreting in-service monitoring data,
- the difficulty of extrapolating from past projects to future projects,
- the applicability of current behaviour under a changing climate,
- the reliance on highly standardized designs and codefied standards not allowing for design flexibility,
- the lack of financial incentive to create more efficient designs when working on small domestic projects.
These responses were more generally reflected when respondents were asked about the barriers for the use of long-term monitoring data within the design process. The largest barriers to adoption identified within the results were the economic costs associated with long-term monitoring (46.9% of responses), the challenges presented by curating and maintaining long-term monitoring data or data privacy issues (23.8% of responses) and the challenges of using monitoring data to inform future designs (23.1% of responses). When asked whether the respondents current/most recent employer should engage in long-term monitoring of a structure if an opportunity arose, 65.0% of respondents responded positively, whilst 14.6% of respondents felt that they should not, and 20.3% respondents believed that it should only be undertaken if there was a financial incentive to do so. Those who felt that their employer should not become involved in long-term monitoring highlighted the “potential [legal] liabilities that is could expose”, the time commitment necessary, and the “lack of knowledge regarding what data to collect and why” and “what is [a] normal response and what is a response that may indicate problems”.

There was a slight difference in the responses when respondents were asked would their employer engage in such an opportunity, with the number of positive responses dropping to 45.5%, the number of negative responses increasing to 22.8% and the number of respondents highlighting the need for a financial incentive increasing to 31.7%. This highlights that the lack of current financial incentive, the perceived risks, lack of expertise, and time costs of long-term monitoring are likely to be major barriers to its widespread adoption within industry. However, it does indicate that if there is a client demand for such monitoring the industry is likely to adapt to meet it. This is reflected when respondents were asked whether their company should use long-term monitoring data from a project designed by another company, with 78.2% of respondents believing that this data should be used, 13.4% reporting that its use was dependant on the completeness of the data and the availability of other information about the project, and only 8.4% of respondents feeling that this data should not form a part of the design of other structures. This strongly suggests that when the costs and complexities associated with implementing long-term monitoring are removed as factors, there is majority support for using in-service monitoring to inform future designs. However, as highlighted in the responses, any legal liabilities associated with the use of such data much be understood and addressed if it is to anyway inform the design of future structures.

The challenges of implementing long-term monitoring as part of the civil/structural design process identified through the survey are broadly similar to
those of BIM [380, 381, 382]. However, integrating long-term monitoring, as well as the move to Construction 4.0 more broadly, has its own set of unique challenges as highlighted within the questionnaire responses. These include the difficulty of interpreting in-situ monitoring data, the risks associated with its use within design, the lack of clear legislative guidance, and the lack of appropriate experience and data analysis skills within civil and structural engineering. These additional challenges may limit the uptake of long-term monitoring. However, a growing interest from clients for in-situ monitoring to inform the maintenance and operation of civil structures may create the financial incentives needed to prompt rapid uptake of long-term monitoring within the industry, without the need for the government intervention required to encourage the uptake of BIM [383].

3.3 Summary

The key findings from this study are:

- 78.3% of respondents highlighted demand within civil and structural engineering for long-term monitoring data to validate design assumptions and create more efficient designs.

- There is disagreement on where this data may be of greatest use within the design process, with 51.6% of respondents identifying an interest in the use of long-term monitoring for reducing the risk of adverse outcomes, while 48.4% of respondents identified the key use as being to support the uptake of more efficient materials and designs.

- Barriers to the adoption of long-term monitoring identified within the survey include the cost of implementation, identified by 46.9% of respondents, and a lack of client demand, identified by 31.7% of respondents.

The broad-range of respondents’ areas of expertise, experience, geographical location, and employment lends weight to the conclusions drawn in this study.

The survey results highlight the financial barriers to implementing long-term monitoring of civil structures. To address this challenge, the research in this thesis will focus on ambient vibration monitoring with accelerometers, which is both low-cost and may be implemented on a temporary or permanent basis. The findings from the survey identify several challenges in implementing long-term monitoring to guide the design of future structures. A consistent theme in the responses is the lack of legislative guidance and the possibility of implicit legal
liability which might result from using in-situ monitoring data. To overcome these challenges requires a greater understanding of the uncertainty, biasing and correlated errors which might arise when analysing ambient vibration data, and development of methods to quantify and communicate the uncertainty in estimates of structural parameters to engineers.

A key finding from the survey is that the complexities associated with analysing long-term monitoring data present a barrier to entry for many structural engineering firms. This suggests that the adoption of long-term monitoring to inform structural design may be aided by the development of analysis methods which require minimal user input and may be used by non-specialist operators. To facilitate the integration of long-term monitoring within the existing design process, these methods should link the outputs of in-service monitoring with the structural parameters used within engineering design, and be able to identify and separate the various effects which result in variations of in-situ structural behaviour.
Chapter 4

Methodology

The aim of this research is to:

“Facilitate the integration of long-term monitoring within the engineering design process to improve the design and operation of civil structures”.

This chapter provides an overview of the motivation for the study and the overarching aim, the objectives designed to achieve that aim and the methodology associated with those objectives.

4.1 Motivation

As summarized above, the aim of the research is to use in-service monitoring to establish a feedback loop within the design process, through which engineers can learn from the performance of civil structures in-situ under in-service loadings. Unlike other engineering projects, civil structures are rarely replicated due to different client demands alongside site conditions and construction limitations, making a linear transfer of knowledge between projects challenging. Despite these limitations, the industry is highly innovative, with new structural forms, materials, construction techniques and site-specific modifications implemented across the world on a regular basis.

Existing research into the structural engineering design process largely focused on wider questions; encouraging innovation, limiting risk within the design process or improving the efficiency of the design process, yet relatively little research had looked at how engineers might objectively assess past successes and challenges, and how these assessments might be used to improve future designs. Long-term monitoring of structures may provide a useful tool to enable these
assessments and improvements by establishing more effective feedback within the
design process. This feedback would allow engineers to learn from past projects
through quantifying the in-situ behaviour of structures over their full life-cycle,
as well as formalizing the transfer of knowledge to future generations of engineers.
By understanding the behaviour of structures in-situ, the design stage assumptions
can be compared to in-service behaviour and iterated on to create more
efficient, resilient and economic designs.

4.2 Research aim and objectives

Long-term monitoring may allow the in-service performance of civil structures
to inform future structural designs, enabling the development of more efficient,
durable and elegant design solutions, as well as facilitating a proactive approach
to management and operation of civil structures.

The survey of industry perceptions and the literature review identified the
complexity and cost of in-situ monitoring as barriers to the adoption of long-term
monitoring. Therefore, this research will focus on ambient vibration monitoring
and operational modal analysis (OMA).

Ambient vibration monitoring offers a low-cost and unobtrusive method through
which new and existing structures may be instrumented. By analysing the vi-
bration responses of structures, insight is gained into a wide range of structural
behaviour, including the loading and mass applied to a structure, the response
of the structure to damage or changing external conditions, and the fundamental
structural behaviour of large dynamical systems. In-situ monitoring of ambient
vibration responses removes the requirement for expensive, intrusive and often
impractical forced excitation, and may offer greater insight into structural be-
haviour by quantifying behaviour under in-service loads. However, there are
significant barriers to the widespread adoption of ambient vibration monitoring
which must be addressed if long-term monitoring is to inform the design and
operation of civil structures.

These barriers to adoption are addressed through research objectives RO 1 to
RO 5:

RO 1. Establish a standardized method for selection of model order in OMA.

RO 2. Identify and remove biasing of OMA modal parameters estimates in-
    troduced by pre-processing of ambient vibration data.

RO 3. Quantify uncertainty in OMA modal parameter estimates.
RO 4. Automate the detection and separation of modes of vibration within OMA modal estimates.

RO 5. Relate changes in dynamic behaviour measured using ambient vibration monitoring to the physical changes of a structure and its environment.

4.3 Addressing research objectives

The research and objectives are structured around the stages of analysing ambient vibration data, defined here as: i) pre-selection of analysis parameters, ii) preprocessing of the data, iii) estimation of the modal parameters, iv) identification of modes of vibration within the modal estimates, and v) using modal parameter estimates as a basis for understanding and quantifying in-service structural behaviour.

A standardized method for selecting the model order used when estimating modal parameters through OMA is presented in Chapter 5. Standardization of the model order selection allows modal parameter estimates from different structures, different data sets, or different analysis parameters to be compared. The biasing of modal parameters introduced by filtering of the ambient vibration data, identified in the early stages of the research, is addressed in Chapter 6. The short-time random decrement technique (ST-RDT), a novel OMA method which overcomes some of the limitations of existing OMA methods, is introduced in Chapter 7. The distributions of modal parameter estimates produced using the ST-RDT are used as the basis for automated detection of modes of vibration, described in Chapter 8. Methods for quantifying and tracking changes in the modal parameters of a structure, referred to as weak non-linear modal behaviour, are presented in Chapter 9. It is shown that weak non-linear modal behaviour allows for a more detailed understanding of the global dynamic behaviour of a structure and the estimation of structural parameters usually measured by more costly or delicate sensors.

The methodology for addressing each of the research objectives introduced in Section 4.2 is presented in Sections 4.3.1 to 4.3.5. An overview of the numerical modelling, experimental and real-world case studies used to assess and validate the potential for long-term ambient vibration monitoring to inform structural engineering design throughout this research is presented in Section 4.3.7.
4.3.1 Research objective 1: Standardized model order selection

A key parameter to be selected when estimating modal parameters through OMA is the model order, the number of modes of vibration assumed to be identifiable within the data. This parameter dictates both the computational time and the computer storage required for calculating and storing modal estimates, with higher model orders requiring more intensive calculations and a greater number of modal estimates to be stored and processed.

The existing approach in OMA is to fit the data with a wide range of model orders and identify modes of vibration by identifying modal parameters which exhibit only small variations as the model order is increased, something which is typically achieved through inspection of stability diagrams. Interpreting these stability diagrams is labour intensive and is highly subjective, especially when distinguishing modal estimates associated with the fitting of noise within the data from structural modes of vibration. Another drawback of stability diagrams is that there is no way to compare representative results from separate data sets, as the modal estimates themselves give no direct indication of the uncertainty within the modal estimates. An appropriate model order for one data set may not be appropriate elsewhere as the number of modes identifiable within the data may change.

An alternative approach for model order selection was developed based on the wider data science and machine learning literature to address research objective 1: RO 1. Establish a standardized method for selection of model order in OMA.

In this approach, detailed in Chapter 5, the model order is systematically increased until a target explained variance, the variance within the observed data which can be explained by the fitted model, is achieved. Only the modal estimates corresponding to the lowest model order which achieves the target explained variance are stored for further analysis.

Using numerically generated vibration data it was shown that as the signal-to-noise ratio of the data decreases, as might be observed in periods of lower excitation, the model order required to achieve the target explained variance increases, due to a higher proportion of noise components within the data. This provides a visual indicator of the uncertainty in the modal estimates which lowers the barrier for entry when interpreting OMA modal parameter estimates. Furthermore, it aids in automating the identification of modes of vibration within the modal parameter estimates, as discussed in Chapter 8.
4.3.2 Research objective 2: Removing bias due to pre-processing of vibration data

During the early stages of the research, it was noted that the filtering of ambient vibration data prior to analysis introduced non-sinusoidal oscillations into the correlation functions and estimates of free-oscillation response used in time-domain OMA. Filtering of ambient vibration data before OMA is common within the literature, as the data often has a low signal-to-noise ratio, as well as being commonly used to remove components of the response outside of the frequency range of interest, and for the analysis of modes with similar natural frequencies, referred to as close modes. The non-sinusoidal oscillations were identified as filtering artefacts associated with Gibbs phenomena and resulted in corruption and biasing of the modal estimates. Counter-intuitively, the corruption introduced by the filtering artefacts was as severe for modes of vibration with frequencies far from the filter cut-off frequency as it was for those that were closer to the filter cut-off frequency. While the impact of filtering was well understood for forced vibrations, little work had looked at how filtering impacts time-domain OMA.

Two solutions were developed to address research objective 2:

RO 2. Identify and remove biasing of OMA modal parameters estimates introduced by pre-processing of ambient vibration data.

The first of these solutions relied on the trimming of filtering artefacts from the start of correlation functions and estimates of the systems free-response, as discussed and demonstrated in Chapter 6. The second solution uses an indicative noise profile, a section of the data containing only noise, analysis of which is integrated into the modal analysis. It was shown that this integration is an efficient means of reducing the corruption and biasing introduced by filtering artefacts without the need to trim the correlation functions or estimated free-response, which will inevitably result in the loss of some information about the behaviour of the dynamic system. The use of an indicative noise profile to remove filtering artefacts was demonstrated using real-world data from a concrete slab subject to footfall excitation, discussed further in Section 4.3.7.3.

4.3.3 Research objective 3: Quantifying uncertainty in modal parameter estimates

A key limitation identified in existing OMA methods which impacted their ability to form a meaningful part of the structural engineering design process was a
failure to quantify uncertainty in the modal estimates. As discussed in the literature review, one source of uncertainty arises due to the approximation of the dynamic response as a sum of damped sinusoids. Further sources of uncertainty are introduced due deviations of the structural response and forcing from the OMA assumptions of stationarity, linearity and observability.

To overcome these limitations the short-time random decrement technique (ST-RDT) was developed to address research objective 3:

RO 3. Quantify uncertainty in OMA modal parameter estimates.

The ST-RDT is a time-domain equivalent to the short-time Fourier transform, introduced in the literature review. In ST-RDT analysis the data is divided into short, overlapping windows from which a random decrement signature is produced using a modified version of the conventional random decrement technique (RDT). This provides a time-history of modal estimates from which weak nonlinear modal behaviour can be identified and the uncertainty and volatility of the modal estimates quantified. A detailed description of the ST-RDT is provided in Chapter 7 alongside how it is applied and a mathematical definition for the approach based on subsampling theory.

The ST-RDT was tested and verified through experimental and real-world case studies, discussed in greater detail in Section 4.3.7. Direct comparisons of the ST-RDT with experimental modal analysis and existing OMA methods are presented in Chapter 7. These comparisons highlight benefits of the ST-RDT such as the incorporation of real-world loading when evaluating structural behaviour, and the ability to identify and quantify different sources of uncertainty.

4.3.4 Research objective 4: Automated mode identification

The survey of industry perceptions highlighted that if long-term monitoring is to form part of the engineering design process, methods must be developed to automate the analysis and interpretation of in-situ monitoring data. Within OMA, a key barrier to entry is identifying modes of vibration within OMA modal estimates, as summarized in research objective 4:

RO 4. Automate the detection and separation of modes of vibration within OMA modal estimates.

Previous research on automated identification of modes of vibration has primarily focused on how modes might be identified within a stability diagram.
However, the ST-RDT produces a continuous time-history of modal estimates. This allowed clustering analysis of those modal estimates, described in [Chapter 8].

A range of hard and soft-classification algorithms, adapted for automated identification of modes using ST-RDT modal estimates, were implemented and are summarized in [Chapter 8]. Synthetic data designed to mimic the distributions of modal parameter estimates from ST-RDT, discussed further in [Section 4.3.7.1], were used to identify limitations of these algorithms.

A new method for automated identification of modes of vibration, using a Bayesian approach utilising probabilistic mixture models, was developed. This approach overcomes the limitations of existing clustering methods for automated identification of modes, and is demonstrated using numerically generated data and ST-RDT modal estimates derived using acceleration data collected from timber buildings, described in [Section 4.3.7.3].

4.3.5 Research objective 5: Modal identification in the presence of weak non-linear modal behaviour

As discussed in the literature review, a challenge in long-term ambient vibration monitoring is that civil structures may exhibit weak non-linear modal behaviour, changes in the modal parameters due to changes in loading applied to a structure, damage, or other changes in a structure’s environment.

In [Chapter 9], methods are developed to address research objective 5:

RO 5. Relate changes in dynamic behaviour measured using ambient vibration monitoring to the physical changes of a structure and its environment.

Two experimental case studies, detailed in [Section 4.3.7.2], were conducted to explore commonly encountered drivers of weak non-linear modal behaviour, temperature-induced changes in stiffness and changes in the static loading applied to a structure. These experiments allowed for the relationship between natural frequencies and temperature and/or applied static loading to be derived and verified. The relationship between damage induced changes in stiffness and load induced changes in natural frequency is described.

These derivations were incorporated into the probabilistic mixture models introduced in the previous section to allow changes in temperature and/or mass to be accounted for in the automated identification of modes of vibration. This greatly increased the robustness and accuracy of the modal identification, as
the physical basis for the models allow structural behaviour to be predicted for previously unseen temperatures and changes in applied mass.

A key driver for tracking weak non-linear modal behaviour beyond increasing the usability of OMA as a method for long-term structural monitoring is the possibility of using changes in dynamic behaviour to predict other structural parameters. This was demonstrated in the static loading experiment introduced above, in which the changes in natural frequencies alongside in-situ estimates of a mass-change parameter and mode shapes, estimated using a calibration data set, were used to estimate the magnitude and location of mass changes on a beam-structure, discussed further in Section 4.3.7.2.

The method for estimating changes in mass is expanded to allow predictions of changes in static loading or damage applied to a structure to be estimated using the probabilistic mixture model used for automated identification of modes, described above. Unlike previous research in this area, this method does not require a numerical model of the structure or knowledge of the mode shapes. The method is demonstrated using data collected from the MX3D Bridge, introduced in Section 4.3.7.3.

The potential benefits of estimating structural parameters using ambient vibration monitoring are that there are many structural parameters, such as in-service loadings and structural stiffness, which are challenging to measure in-situ. In this research short-time OMA methods such as the ST-RDT are demonstrated to be low-cost and effective methods for estimating these structural parameters. A mixture of numerically generated data, experimental and real-world case studies were used to validate the accuracy, robustness and efficiency of these methods, discussed further in the following section.

4.3.6 Contribution to knowledge

In relation to research objective 1, “Establish a standardized method for selection of model order in OMA”, the contributions to knowledge arising from this work are:

- Demonstrating the challenges of identifying spurious modal estimates associated with fitting of noise or non-dynamic responses within the measured structural response.

- Introducing the coefficient of determination, $R^2$, as a robust and established metric from the data-science literature which can be used for comparing approximated and measured structural responses.
• Introducing target explained variance as a method for comparing approximated structural responses from different structures, experimental setups and analysis parameters selected.

In relation to research objective 2, “Identify and remove biasing of OMA modal parameters estimates introduced by pre-processing of ambient vibration data”, the contributions to knowledge arising from this work are:

• Demonstrating the link between the biasing of modal estimates introduced by filtering and the characteristics of impulse response functions, and quantifying their impact on modal estimates using simulated and real-world data.

• Providing practical guidance on how to limit the bias introduced by filtering in OMA.

• Introducing a practical alternative to filtering for improving time-domain modal estimates through trimming of the autocorrelation of the noise from unfiltered time-domain correlation/free-response signals.

• Providing a method through which filtering artefacts may be removed or minimised using an indicative noise profile and fitting of the filtering artefacts as part of the modal analysis for use where filtering of the data is unavoidable.

In relation to research objective 3, “Quantify uncertainty in OMA modal parameter estimates”, the contributions to knowledge arising from this work are:

• Deriving a new definition of the random decrement signature (RDS) as a conditional correlation formed from the expected value of the analysis data conditional on the triggering data.

• Describing how noise, non-free-oscillation components of the RDS and non-linear dynamic responses impact the modal analysis procedure and selection of analysis parameters.

• Introducing the short-time random decrement technique (ST-RDT) as a practical OMA method which allows the uncertainty, volatility and weak non-linear variation of modal estimates to be quantified.

• Providing practical guidance on the selection of analysis parameters for use in the ST-RDT.
• Demonstrating the relationship between quantification of uncertainty and weak non-linear dynamic behaviour using the ST-RDT, subsampling theory and the theory of moving average filters.

In relation to research objective 4, “Automate the detection and separation of modes of vibration within OMA modal estimates”, the contributions to knowledge arising from this work are:

• Describing the limitations of established clustering techniques for automatic detection of modes of vibration within OMA modal estimates.

• Deriving and demonstrating the use of non-Gaussian mixture models for automated detection of modes of vibration within OMA modal estimates including the separation of spurious modal estimates and close modes of vibration.

In relation to research objective 5, “Relate changes in dynamic behaviour measured using ambient vibration monitoring to the physical changes of a structure and its environment”, the contributions to knowledge arising from this work are:

• Deriving the relationship between changes in natural frequency, changes in temperature and changes in static loading.

• Demonstrating the use of regression methods for estimating the parameters of these relationships.

• Relating the changes in natural frequency due to changes in static loading from those due to damage to a structure and providing practical guidance on distinguishing the two causes.

• Deriving and demonstrating the use of probabilistic mixture models for automated detection of modes of vibration in the presence of changes in temperature and changes in static loading.

• Introducing an empirical method and a probabilistic method for estimating changes in static loading based on changes in natural frequency.

• Demonstrating how changes in dynamic behaviour may be used to estimate structural parameters usually measured by more expensive, complex or delicate sensors.
4.3.7 Case studies

To address the research objectives, a mixture of numerically generated, experimental, and real-world case studies were used. Each case study was designed to test, validate or demonstrate a specific technique, or to quantify the potential benefits and challenges of ambient vibration data to inform the structural design process. A mixture of data types were used to ensure all methods are:

- accurate - the estimates the method produce are reasonable approximations of the true parameters of a system,
- robust - the estimates produced are accurate under a broad range of conditions with varying levels of noise or unknowns,
- efficient - the estimates balance the computational cost associated with producing them with the required resolution of the results to accurately interpret or describe the system’s behaviour.

In the interest of brevity, only key findings from the case studies are presented within the main body of the text. Further information on all case studies, including analysis parameters and additional results, are provided in the appendices.

4.3.7.1 Numerical case-studies

The advantage of using numerically generated data is that the true parameters of the underlying model of the system are known. This allowed for the accuracy of the methods to be quantified across a range of different model parameters. Within the research, a variety of numerical case studies are used.

Python scripts were created which utilised the Newmark-Beta numerical integration method \[384\] for solving the equations of motion to calculate displacement, velocity, and acceleration time-histories of single- and multi-degree of freedom (DOF) dynamic systems. These scripts allowed flexibility when selecting the number of DOFs, the modal parameters of the system, and the time-history of the applied forcing. This allowed the accuracy and robustness of the analysis methods to be tested across a wide range of system complexities and parameters, whilst maintaining the benefits of known system properties.

Synthetic data generated from a range of probability distributions were used when testing the automated mode identification methods and statistical tools in Chapters 8 and 9. These synthetic data sets mimicked the distributions of modal parameters which were observed in the experimental and real-world case studies. The advantage of the synthetic data was that the true classification of each set of
pseudo-modal estimates as either representing modal estimates for a given mode of vibration or as background noise was known. This allowed accurate, robust and efficient methods to be generated for automated mode identification and estimating structural behaviour from weak non-linear modal behaviour.

4.3.7.2 Experimental case-studies

Two experimental case studies were conducted as part of the research to explore key causes of weak non-linear modal behaviour: thermally-induced changes in dynamic behaviour and mass-induced changes in dynamic behaviour. The experiments used simple real-world structures, allowing validation that the expected dynamic behaviour occurs while introducing a limited range of unknowns, such as the forcing applied to the structure or small deviations from the expected behaviour due to discrepancies in the boundary conditions and material behaviour.

The first experimental case study examined the use of the ST-RDT, introduced in Chapter 7, for quantifying thermally induced changes in the dynamic behaviour of a beam structure. Thermal variations in dynamic behaviour are commonly encountered in long-term monitoring of civil structures and are a key factor that must be accounted for when estimating other structural parameters, identifying damage to structures, or automating the identification of modes of vibration. An equation was derived which allowed the change in temperature of the structure to be related to the change in natural frequencies without prior knowledge of the materials coefficient of thermal expansion or the modal stiffness of the structure. The results for this experiment provided a basis for the removal of thermally induced weak non-linear behaviour from the modal estimates of real-world structures, discussed in Chapter 9.

The second experimental case study tested the ST-RDT as a method for estimating the magnitude and location of changes in the static load applied to a beam-structure. The static load or mass applied to structures in-service is a key design consideration. However in-service measurements of load are challenging, with many existing design guidelines instead being based on visual surveys of applied loading. The experiment was used to develop and validate random sampling mass estimation (RSME), which allows the static load applied to a structure to be estimated based on changes in the natural frequencies of the structure by utilising a calibration data set. Unlike existing methods for mass estimation, the uncertainty in the modal estimates is projected to the mass estimates, allowing for more accurate identification of the most likely mass magnitudes and locations. The experimental case study examined the challenges of identifying
the total applied static loading in the presence of multiple applied loads, work which was expanded in the real-world case studies to develop statistical methods for estimating the magnitude of in-situ static loading based on weak non-linear modal behaviour without prior knowledge of the static load locations.

4.3.7.3 Real-world case studies

Real-world case studies represent the greatest challenge for OMA due to the range of unknowns, from the structural parameters of the system under observation to the true variation in those parameters due to fluctuations in the systems environment. As the focus of the research is on how long-term monitoring of real-world structures might be incorporated into the structural design process, all findings and methods developed must be robust, accurate and efficient when analysing data from real-world structures. The real-world case studies were used to highlight a range of areas where long-term monitoring might be used to inform the structural design process, as well as to emphasise the additional information which short-time OMA methods such as the ST-RDT can offer to designers when compared to conventional OMA techniques. This additional information includes indicators of the uncertainty in the modal estimates, discussed in Chapter 7, as well as information that allows structural parameters usually measured by more expensive, complex and delicate sensors to be estimated through the quantification of weak non-linear modal behaviour. An overview of the real-world case studies which form part of the research and why they were selected is presented below.

The Aberfeldy Footbridge - The Aberfeldy Footbridge, discussed in Chapter 7, is a historic glass-fibre reinforced polymer footbridge that has suffered extensive damage and is extremely dynamically active. Measurements taken as part of this research were used to compare the current dynamic behaviour of the bridge to historical forced and ambient vibration testing. It was shown that the reduction in natural frequencies which had previously been ascribed to damage to the structure was likely due to the reduction in frequency associated with the weight of pedestrians on the bridge.

Engineered timber structures - Multi-storey engineered timber buildings often experience higher peak accelerations in wind loading due to their relatively low ratio of mass to stiffness. These structures often exhibit highly non-linear dynamic behaviour such as a variation in damping with the amplitude of the response. Short periods (30 minutes to two hours) of ambient vibration data from ten engineered timber buildings, details of which are provided in Appendix I, was
used to quantify the magnitude of non-linear behaviour exhibited, and to relate
the natural frequency and damping ratio of the fundamental mode of vibration
to the height and slenderness of the structure. To guide future designs of timber
buildings the relationship between the fundamental natural frequency and damp-
ing ratio and the slenderness of the buildings was derived which incorporates the
uncertainty in the modal estimates.

The Whitmore Building - Closely related to the analysis of the engineered
timber structures was the analysis of data collected from the Whitmore Road
building, London. Whitmore Road is a seven-storey building comprising six tim-
ber storeys built above a reinforced concrete first storey. One week of ambient
vibration data was collected from the top floor of the building. The analysis of
this data is presented separately to the engineering timber structures data as the
greater data set length allowed for a more nuanced interpretation of the dynamic
behaviour exhibited.

The MX3D Bridge - The MX3D Bridge is the world’s first additively man-
ufactured or 3D printed metal bridge and is instrumented with a wide range of
structural and environmental sensors in-service [9]. The bridge was subjected to
a range of forced and ambient vibration testing regimes as part of the validation
of the structure while it was located at the University of Twente, before instal-
lation at its permanent location in Amsterdam, the Netherlands. Part of this
validation was a comparison between the measured vibration response and that
predicted using published design guidance in conjunction with modal parameters
from a finite element model of the bridge developed by Pinelopi Kyvelou at Im-
perial College London, and modal parameters estimated through experimental
and operational modal analysis. The wide range of structural sensors fitted to
the bridge allowed the methods developed for estimating structural parameters
through changes in dynamic behaviour to be tested, as discussed in Chapter 9.
Data from the bridge was also used to verify the methods developed for removing
temperature-induced fluctuations in natural frequencies, as previously introduced
in the experimental case studies section above.

Lake seiches - Seiches or standing waves are a key part of the behaviour
of lakes influencing the lake ecosystem, shoreline erosion, and how pollutants
are dispersed around the body of water. They represent a unique challenge for
OMA due to the low sample rates used and the very low frequencies of the se-
iche oscillations. As such they present an excellent case study for testing the
robustness of OMA methods. Water surface-level data from two lakes were used
to identify the fundamental seiche modes using the ST-RDT. This work builds
on previous analyses of these lake seiches using the conventional RDT. The ST-
RDT analyses identify the correlation between the wind speed/direction, and the frequency/damping of the seiche modes, and further demonstrated the additional information which short-time OMA methods can provide when compared to conventional OMA techniques.

**One-way spanning concrete floor slab** - Data from a one-way spanning concrete slab located in the stairwell of the Alexander Graham Bell building at the University of Edinburgh King’s Buildings campus was used to demonstrate the removal of filtering artefacts discussed in Chapter 6. This data was used as the signal-to-noise ratio of the data was relatively low and the accelerometer used exhibited a non-white noise profile, making removal of noise-induced components of correlation functions of the data challenging.

### 4.3.8 Implementation of data analysis

All data analysis was carried out using Python 3.6 [385] as it is a widely-used programming language, has no software costs associated with its use and has a wealth of existing libraries and modules. These factors are likely to aid any industry uptake of the methods developed and make them readily accessible to a wide audience.
Chapter 5

Model order selection in time-domain operational modal analysis

A challenge in existing operational modal analysis (OMA) techniques is selecting the model order used to describe the dynamic system as captured in the long-term monitoring data. In this chapter, an alternative to the current method of reporting the modal parameter estimates from a range of model orders is presented to address research objective 1:

RO 1. Establish a standardized method for selection of model order in OMA.

This method of selecting the model order is based on established practice within the machine learning and wider data science literature. The work presented in this chapter represents a key step towards efficient extraction of modal parameters, standardization of the reporting of modal parameters estimated from vibration monitoring data and automation of the OMA process. The metric for assessing modal estimates introduced in this chapter provides a benchmark that may allow long-term ambient vibration monitoring to guide the design of future structures.

5.1 Model order evaluation in operational modal analysis

The objective of OMA is to estimate the modal parameters of a dynamic system under ambient excitation. As was introduced in Chapter 2, estimating the modal parameters through time-domain OMA is based on the approximation of correlation functions or estimates of the free-responses of a dynamic system, denoted
\( y \), as the sum of \( N \) damped sinusoids given by \[ y \approx \sum_{j=1}^{j=N} A_j e^{\zeta_j \omega_j t} \cos(\omega_j t + \phi_{j,\text{lag}}) \] (5.1)

In Equation 5.1, \( t \) is a vector of time-lags, \( A_j \) is the amplitude of the sinusoidal component \( j \), \( \zeta_j \) is the damping ratio of component \( j \), \( \omega_j \) is the natural frequency of component \( j \) in radians per second and \( \phi_{j,\text{lag}} \) is the phase lag of component \( j \). Similarly, estimating the modal parameters of the system through frequency-domain OMA is based on approximation of the frequency or power spectra of the structural response, denoted \( X_k \), through Equation 5.2

\[ X_k \approx \sum_{j=1}^{j=N} X_j (A_j, \omega_j, \zeta_j, \phi_{j,\text{lag}}) \] (5.2)

A key unresolved issue within OMA is the selection of the model order \( N \), the number of modal components to be fitted to the data.

The conventional approach in the OMA literature is to estimate the modal parameters over a range of model orders, with the results presented as stability diagrams, an example of which is presented in Figure 5.1. However, there are key limitations and drawbacks to the use of stability diagrams within long-term structural monitoring.

Figure 5.1: Example of a stability diagram for SSI-Cov analysis, reproduced from Peeters and De Roeck [211]. For further details refer to caption of Figure 2.11.
The first set of issues with stability diagrams are associated with the computational intensity of approximating the system response at high model orders. For each unit increase in the model order, the number of parameters to be fitted to the data increases by a factor of four, associated with the four modal parameters ($\zeta_j, \omega_j, A_j, \phi_{j,lag}$). While some modal analysis methods, such as the matrix pencil method discussed in Chapter 7, greatly reduce the computational intensity of the modal analysis for higher model orders through the use of singular value decomposition, high model orders remain a barrier to the efficient analysis of ambient vibration data collected in-service. A related issue is the computational storage needed to store the modal parameter estimates from a range of model orders. For a single channel of data evaluated at model orders $N = 1$ to $N = M_k$, the total number of modal estimates $n$ is given by Equation 5.3.

$$n = 4 \left( \frac{M_k(M_k + 1)}{2} \right)$$

(5.3)

If the analysis channels of data are analysed independently, the value of $n$ scales linearly with the number of channels to data to be evaluated. However, if channels are analysed through cross-correlations, where each channel is analysed based on the response from all other channels, there is a squared relationship between the value of $n$ and the number of channels to be analysed. For example, if cross-correlation functions were formed using the output from the 22 channels of accelerometer data from the MX3D Bridge, introduced in Chapter 4, there are $22^2 = 484$ correlation functions for analysis. If each of these correlation functions is analysed using a model order of 1 to 50 there are 4,080 modal estimates per correlation function. Therefore, there are a total of 1,974,720 modal estimates for analysis, interpretation and storage. While this number of modal estimates is not an issue for analysing individual blocks of data, when smaller blocks of data are to be analysed through short-time OMA methods, such as the short-time random decrement technique introduced in Chapter 7, or data is to be analysed at regular intervals over the lifetime of a structure to detect signs of damage, the quantity of data to be analysed and stored may become infeasible.

The interpretation of stability diagrams is challenging as the modal estimates themselves, and how the modal estimates vary with increasing model order, provide relatively little information on the suitability of the model order selected. This complicates the comparison of modal estimates from different structures or the same structure under different levels of excitation or sensor layouts. The challenge of interpreting stability diagrams is demonstrated in Figure 5.2 which presents a comparison of the stability diagrams for the accompanying simulated
structural free-responses. For this demonstration, the modal parameter estimates are calculated using the matrix pencil method \[254\]. Modal estimates are classified based on the stability criteria described by Brincker and Ventura \[218\]. A frequency estimate is defined as stable if the variation in frequency between model order \(n\) and \(n+1\) is less than 0.5\% of the frequency at model order \(n\). Modal estimates are classified as having a stable damping ratio if the variation in damping estimates between model order \(n\) and \(n+1\) is less than 5\% of the damping at model order \(n\). Despite the second simulated free-response having been corrupted with additive white noise with a standard deviation five times greater than that of the first simulated response, the stability diagrams are visually highly similar. While there are four stable columns of modal estimates within the results, the exact model order to be selected to ensure accurate modal estimates which are unbiased by noise within the structural responses is unclear. For higher model orders, the variation in the modal parameters of the stable columns with changes in model order will be small, however the values and number of non-stable modal estimates will exhibit wide variations with changes in model order. Alongside this issue, there is no clear metric for comparing the modal estimates from the two structural responses.

In addition to the challenges associated with interpreting the stability diagram, it should be highlighted that once the model order exceeds the number of modes of vibration which are detectable within the structural response, each subsequent model order used in the analysis contains a higher proportion of spurious or noise estimates. For example, at model order 50 of the results presented in Figure 5.2, there are four real modes of vibration and 46 spurious modal estimates associated with the fitting of noise within the structural response. Therefore, 92\% of the modal estimates for model order 50 have no meaningful value for interpreting the dynamic response of the system, as illustrated in Figure 5.3. This high proportion of spurious modal estimates complicates automated modal identification, as discussed in greater detail in Chapter 8, and may mask changes in the modal parameters associated with damage to the structure.

### 5.2 Coefficient of determination

A robust method for comparing the modal estimates from different systems or at different model orders is to calculate the coefficient of determination, also known as the \(R^2\) score \[380\], between the measured time-domain structural response \(y\) and the model order \(N\) approximation of the time-response, denoted \(\hat{y}_N\), as given
Figure 5.2: Comparison of simulated free-responses and accompanying stability diagrams from a four-degree of freedom system, sampled at 200 Hz, corrupted independently with normally distributed white noise with varying standard deviation, $\sigma_n$. Frequency estimates are considered stable if there is less than 0.5% variation in the frequency estimates between model order $n$ and $n + 1$. Damping estimates are considered stable if there is less than 5% variation in damping estimates between model order $n$ and $n + 1$.

Figure 5.3: Minimum percentage of modal estimates corresponding to a true mode of vibration based on four identifiable modes within the structural response. Results presented as percentage of the modal estimates for individual model orders (dots) and as a percentage of the total number of modal estimates (crosses).
by Equation 5.4
\[
\hat{y}_N \approx \sum_{j=1}^{N} A_j e^{i\omega_j t} \cos (\omega_j t + \phi_{j,\text{lag}})
\] (5.4)

The $R^2$ score is the complement of the fraction given by the ratio of sum of squared residuals, $S_{\text{res}} = \sum (y - \hat{y}_N)^2$, and the total sum of squares, $S_{\text{tot}} = \sum (y - \bar{y})^2$, $R^2 = 1 - (S_{\text{res}} / S_{\text{tot}})$, where $\bar{y}$ is the mean average of the measured response $y$. While the $R^2$ score may be calculated for comparing the measured and approximated frequency-domain structural response, in practice, it is simpler to convert frequency-domain responses to the time-domain to account for the approximated amplitude and phase relationships within the $R^2$ score. For multi-channel OMA methods such as SSI-Cov, the $R^2$ score may either be calculated through concatenation of the correlation functions and the approximated correlation functions or calculated individually for each correlation function.

The strength of the coefficient of determination as a metric for comparing the measured and approximated dynamic responses is that, as it is scaled by the total sum of squares of the signal, $R^2$ is insensitive to the raw amplitude of the signals. As such, the maximum value of $R^2$ is $R^2 = 1$ when the approximated signal is exactly equal to the measured signal ($S_{\text{res}} = 0$). A value of $R^2 = 0$ indicates that $S_{\text{res}} = S_{\text{tot}}$, and that an equivalent approximation of the structural response could be achieved with a line with zero slope equivalent to a single sinusoidal component with an amplitude of zero. By extension, a negative $R^2$ score indicates that $S_{\text{res}} < S_{\text{tot}}$, and that the fitted response is a worse approximation than would be given by a line with zero slope.

$R^2$ scores provide an established method for describing the “goodness-of-fit” of the approximated response or the fraction of the variance of the measured system response $y$ which has been explained by the order $N$ approximated response $\hat{y}_N$ \[387\]. $R^2$ scores are an established metric for evaluating regression analysis, as discussed in Chatterjee \[388\] and in Chapter 9 and for informing the number of components to be used within principal component analysis and independent component analysis, as discussed in Maćkiewicz and Ratajczak \[389\] and Lee et al. \[390\].

The indicative power of the coefficient of determination for comparing different data sets is demonstrated in Figure 5.4, where the $R^2$ scores of the order $N$ approximated responses of the data introduced in Figure 5.2 are compared. The gap in the $R^2$ between the low noise data ($\sigma_n = 0.02m/s^2$) and the high noise data ($\sigma_n = 0.1m/s^2$) gives an immediate indication of which of the results come from a system with far higher spurious or non-modal components. The elbow
in the $R^2$ score indicates the point at which the addition of further components provides no meaningful difference in the approximated system response, the same point at which the approximated model order is close to that of the true model order of the system.

Figure 5.4: Comparison of coefficients of determination ($R^2$ scores) at varying model orders for approximations of the two system responses presented in Figure 5.2.

### 5.3 Target explained variance

To allow for the comparison of equivalent approximations of the structural response as quantified through the coefficient of determination, the model order can be systematically increased from unity until a specific $R^2$ score, referred to in this work as a target explained variance (TEV) or $R^2_{targ}$, is met. Only the modal estimates from the lowest model order at which the TEV is met are retained for future analysis and interpretation. The advantage of this approach is that it minimizes the number of modal estimates to be computed and stored, and it allows modal estimates from different structures or different periods of time to be directly compared.

The modal estimates for a specified TEV provide an intuitive basis for quantifying uncertainty within the modal estimates from a variety of sources. As the number of modes of vibration which are detectable within the structural response increases, it is reasonable to assume that the uncertainty in the modal estimates for a specific mode will be higher than modal estimates from a structural response which is dominated by a small number of modes, as discussed further in
This uncertainty is explicitly accounted for in the TEV, as the model order is systematically increased to that required to achieve the TEV, regardless of the number of modes present within the structural response. If modes are only weakly excited and the structural response is dominated by noise, the model order required to reach a specified TEV is likely to increase. This will result in a larger number of modal estimates, providing an indication of the uncertainty in the modal estimates for that data set when compared to modal estimates from a structural response that is not dominated by noise.

The selection of the TEV must be driven by the intended use of the modal estimates. If the primary need is for highly accurate damping and mode shape estimates or the detection of weakly-excited modes of vibration, a higher TEV is required. However, a higher TEV will result in larger numbers of spurious modal estimates associated with the fitting of noise within the structural response. If the primary interest is in minimizing the number of spurious modal estimates, such as for automated mode detection and tracking of frequency estimates of the dominant modes of vibration, a lower TEV will allow for reasonable modal estimates and fewer spurious components in the modal estimates. Alternatively, the TEV may be varied according to a pre-specified criterion, such as through identification of the elbow within the $R^2$ scores, the point at which each subsequent increase in model order results in only marginal increases in the explained variance.

5.4 Summary

In this chapter, the issues associated with the conventional approach of analysing dynamic responses over a range of model orders have been identified. It has been shown that the lack of a comparative metric for assessing structural responses leads to a proliferation of modal estimates associated with the fitting of noise and non-dynamic responses within the measured structural response. These spurious modal estimates provide no information about the dynamic behaviour of the structural system but greatly increase the computational cost of analysing and storing modal estimates, creating barriers for automated modal identification and interpretation of modal responses.

The coefficient of determination, $R^2$, provides an established and robust metric for comparing the approximated and measured structural responses through quantifying the variance of the measured response which is explained by the order $N$ approximation of the system. This metric allows the identification of a
suitable model order for approximating the system response, above which each additional sinusoidal component results in only marginal improvements in the explained variance and negligible variations in the fitted modal parameters of non-spurious dynamic responses. Increasing the model order until a target explained variance is met has been introduced as a method that allows comparison of modal estimates collected from different structures or different periods of time whilst accounting for the different sources of uncertainty within the measured structural response, such as varying numbers of modes or excitation levels.
Chapter 6

Pre-processing of data for time-domain operational modal analysis

This chapter is adapted from work originally published as Reducing coherent filtering artefacts in time-domain operational modal analysis in Structural Control & Health Monitoring [2] and addresses research objective 2:

RO 2. Identify and remove biasing of OMA modal parameters estimates introduced by pre-processing of ambient vibration data.

Signals collected for modal analysis are often filtered in order to reduce sensor noise, out-of-band oscillations, or for dealing with closely-spaced modes. This filtering introduces filtering artefacts into the data due to the non-ideal filter response and is well understood for impulse excitation. However, for ambient vibration data filtering artefacts become superimposed, preventing their visual identification, and will corrupt the correlation function of the data used in time-domain operational modal analysis (OMA) techniques such as covariance-driven stochastic subspace identification and the random decrement technique. This corruption leads to inaccurate, misleading, biased or spurious frequency and damping estimates, with the inaccuracy increasing for systems with higher damping, or lower signal-to-noise ratios. Counter-intuitively, the error in damping estimates is as large for modes with natural frequencies far from the filter cutoff frequency as for modes which are close to the cutoff frequency.

In this chapter, an alternative to filtering for time-domain OMA, trimming of the correlation of noise from unfiltered correlation functions, is introduced and tested using 10,000 numerically generated ambient vibration data sets. It has been shown that this technique reduces the mean absolute error in the frequency
estimates by over 200% and the mean absolute error in the damping estimates by over 400%. Additionally, a new technique which incorporates fitting of filtering artefacts as part of the modal analysis is introduced for where filtering of ambient vibration data is unavoidable, and is demonstrated using real-world acceleration data collected from a two-way spanning concrete slab subject to footfall excitation.

6.1 Introduction

Due to the size and complexity of many civil engineering structures it is not possible to measure their dynamic properties using conventional forced vibration analysis. Operational modal analysis (OMA), whereby the ambient vibrations induced in a structure during everyday use form the excitation force, has therefore become an essential tool for the modern structural engineer. It is common practice for ambient vibration data collected for use in OMA to be filtered prior to analysis in order to remove sensor noise, interfering signals such as the mains electricity frequency, vibrations caused by heating and ventilation systems, modal oscillations outside of the frequency range of interest, or to allow analysis of closely-spaced modes. However, as discussed in Chapter 2, a number of researchers have noted that the results obtained through time-domain OMA techniques are sensitive to the filtering of the data.

In this chapter, it is demonstrated that filtering artefacts associated with Gibbs phenomena are the cause of these variations in estimates of modal properties and that the impact of filtering on damping estimates may be in the order of hundreds of percent of the true damping value.

Counter-intuitively, the corruption of damping estimates is largely independent of how far the filter cutoff is from the frequency of the mode. Instead, the magnitude of the corruption of damping estimates is driven by the ratio between the amplitude of the filtering artefacts and the amplitude of the modal component of the time-domain signal to be fitted, a ratio which is dependant on both the signal-to-noise ratio of the data, and the relative amplitudes of different modes of vibration.

The chapter is organised as follows. The cause of filtering artefacts is presented and their impact on two common time-domain OMA methods, the random decrement technique and covariance-driven stochastic subspace identification, are demonstrated in Section 6.2 using numerically generated data. A new
method is demonstrated for the identification of filtering artefacts using the random decrement technique. Using this technique it is shown that the use of a filtered data set within the vector random decrement technique [246] may lead to filtering artefacts appearing in unfiltered data sets due to the coherence of segments across data channels. A new technique, based on "trimming" the noise component/filtering artefacts from the unfiltered time-domain correlation/free-response signals is introduced in Section 6.3. Using numerically generated data from a 2 degree-of-freedom (DOF) system it is shown in Section 6.4 that this technique drastically reduces the errors in both frequency and damping estimates when compared to the untrimmed signals or the filtered signals. Using the numerically generated data it is shown that the impacts of filtering artefacts are more acute for systems with higher damping and for systems with lower signal-to-noise ratios. Furthermore, it is shown that filtering artefacts introduce significant errors in the damping ratio even when the filter cutoff frequency is far from the natural frequency of the mode of vibration. Finally, a new method for removing the effects of filtering artefacts through the use of an indicative noise profile in the time-domain OMA is presented in Section 6.3.3 for use when filtering of the ambient vibration data is unavoidable. This procedure is demonstrated using real-world data collected from a two-way spanning concrete slab in Section 6.5.

6.2 Background and theoretical basis

6.2.1 Filtering Artefacts

In OMA it is assumed that, within the frequency range of interest, the noise and the forcing may be modelled as a broadband stationary process also known as a second order white-process [399]. This process can be approximated as a collection of uncorrelated impulses. In the frequency domain this translates to a power spectral density (PSD) of constant amplitude. If an individual impulse is filtered using a low-pass filter, the power spectrum associated with frequencies higher than the the low-pass cutoff frequency are removed or attenuated. In the time domain, this results in a sinc function or ringing artefacts, which relates to Gibbs phenomena also referred to as Gibbs oscillations [400]. This artefact and 'ringing' relates to the overshoot which occurs when a step-function is approximated by a truncated Fourier series [401]. For example, for an ideal low-pass filter with cutoff frequency $\omega_c$, the system frequency response $H_{\text{ideal}}(\omega)$ is given
by Equation 6.1

\[ H_{\text{ideal}}(\omega) = \begin{cases} 
1, & |\omega| \leq \omega_c \leq \frac{\pi}{T_s} \\
0, & \text{otherwise} 
\end{cases} \]  

(6.1)

In Equation 6.1, \( T_s \) is the sampling period of the discrete-time process, such that \( \frac{\pi}{T_s} \) is half the sampling frequency in radians per second. The corresponding impulse response, \( h(t) \), at time \( t = nT_s \), where \( n \) is the sample index, is given by Equation 6.2

\[ h_{\text{ideal}}(t) \big|_{t=nT_s} = h_{\text{ideal}}(nT_s) = \omega_c \text{sinc} (\omega_c nT_s) \]  

(6.2)

The effect of low-pass filtering of an impulse is illustrated in the frequency domain in Figure 6.1a. The impulse has a uniform frequency spectrum which is attenuated above the cutoff \( f_c = 25 \text{ Hz} \) by the low-pass filter. In the time-domain, shown in Figure 6.1b, the off-center impulse has an amplitude of 1 at \( t = 5 \text{ s} \) and 0 elsewhere. Due to the attenuation of frequencies above the low-pass cutoff frequency, filtering artefacts, the oscillations visible in Figure 6.1b are present in the filtered data.

![Figure 6.1: Example of filtering artefacts induced by Gibbs phenomena for filtering of an impulse response, \( f_s =100 \text{ Hz} \), with a Butterworth filter of order 7 and a low-pass cutoff \( f_c = 25 \text{ Hz} \). Unfiltered refers to the raw impulse data, and filtered refers to the same data after filtering with low-pass filter.](image)

(a) Frequency spectrum for filtering of an impulse. 
(b) Time-domain response of filtering an impulse.

As shown by Equation 6.2, the duration of the ‘ringing’, here called the length of a single filter artefact oscillation, is dependant on the frequency cutoff of the filter, \( \omega_c = 2\pi f_c \), with the amplitude of the filter artefact oscillations related to the filter order (the sharpness of the frequency roll-off), the type of filter used and amplitude of the initial discontinuity. 

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A factor which complicates the initial identification of filtering artefacts in ambient vibration data is that filtering of noise leads to a smoothing effect of the underlying signals, making visual distinction between filtering artefacts and real dynamic oscillations in the signals more difficult.

6.2.2 Artificial data generation

Simulated displacement data generated using the Newmark-beta numerical integration method \[384\] for a 2DOF spring-mass-damper is used to demonstrate and quantify the impact of filtering artefacts on time-domain OMA throughout this chapter. The spring-mass-damper system has a mass matrix \([M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\) and stiffness matrix \([K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\). The damping matrix was generated using the classical Rayleigh damping equations \(\zeta = \frac{1}{2} \left( \frac{\mu}{\omega} + \lambda \omega \right)\).

For the purpose of demonstrating the origin and impact of filtering artefacts in Sections 6.2.3 and 6.2.4, \(m_1 = m_2 = 1\) kg, \(k_1 = k_2 = 500\) N/m, \(\lambda \zeta = 5 \times 10^{-5}\) and \(\mu \zeta = 1\). This gives natural frequencies and damping for the 2DOF system of 5.69 Hz with 1.4% of critical damping, and 2.19 Hz with 3.64% of critical damping.

The data used in Sections 6.2.3 and 6.2.4, referred to as the 2DOF example data, consisted of 60 minutes of displacement data for each DOF with mass values of \(m_1 = m_2 = 1\) kg, generated at 100Hz using the Newmark-beta method, resulting in 360,000 samples of data for each DOF. The 2DOF system was excited with a Gaussian white-noise process independently at both DOFs with a mean value of 0N and a standard deviation of 20N.

The Newmark-Beta method does not attenuate the instantaneous force response of a system as a real structure does, resulting in instantaneous changes in the acceleration and velocity in the generated data which are not present in data collected from real-world systems. Therefore, displacement data was used to avoid the generation of the component of the autocorrelation function associated with the input force \[403\].

Additive white noise was independently added to the displacement data from both DOFs to achieve a modal signal-to-noise ratio (SNR \[16\]) of 3. The modal SNR of the data is here defined as the ratio between the amplitude of the frequency spectrum of the uncorrupted data and the noise data, averaged across both natural frequencies of the system and across both of the DOFs. Additive white noise is used to demonstrate the impact of filtering artefacts on time-domain OMA, with non-white noise addressed in Section 6.5.
6.2.2.1 Filter parameters

Within this chapter, the Butterworth filter with an order of 7 is used to demonstrate the biasing introduced in modal estimates due to filtering artefacts. Butterworth filters are widely used in practice as they maximise the flatness of the filter magnitude in the frequency passband, a key requirement when analysing dynamic systems [15].

6.2.3 Random decrement technique

In this section the impact of filtering artefacts on the random decrement technique (RDT), introduced in Chapter 2, is demonstrated for three common RDT variations: level-crossing RDT, peak-picking RDT, and the vector RDT. The RDT produces an estimate of the free-response of a linear system, as shown by Vandiver et al. [404] and Brincker et al. [405] and expanded upon by Asmussen et al. [259], through an ensemble average of segments of data drawn wherever a user defined triggering condition, $T_x(t)$, is met. The ensemble average is referred to as the random decrement signature (RDS), $\hat{D}$, from which allows the modal properties of the system may be estimated.

A number of different triggering conditions has been proposed for the RDT with two common examples being level-crossing and peak-picking/local-extremum.

6.2.3.1 Level-crossing random decrement technique

The level-crossing RDT triggering condition is defined as where the data is equal to $a$. As the data is unlikely to exactly match the level $a$, a narrow range is introduced between $a_1$ and $a_2$, with the triggering condition defined by Asmussen and Brincker [246] as $T_x(t) = \{a_1 \leq x(t) < a_2\}$, $a_2 - a_1 \to 0$. This is also referred to as a range-crossing triggering condition. In order to increase the number of trigger points, segments are also collected when the negative triggering condition, $T_{-x(t)}$, is met. These segments are subsequently multiplied by $-1$ prior to the ensemble averaging. The negative trigger level is defined as $T_{x(t)} = \{-a \leq x(t) < \Delta a\}$, $\Delta a \to 0$.

When the level-crossing is applied to noisy displacement data, the RDS is formed of two components: the free-response of the system associated with the displacement trigger level; and a noise impulse, the component of the RDS associated with triggering of the level-crossing RDT by noise. This noise impulse, the sharp peak at $t = 0$ shown in Figure 6.2a, represents the autocorrelation of the additive noise, as demonstrated by Orlowitz and Brandt [406]. This autocorrelation
function takes the form of a positive impulse for white noise.

The conceptual explanation for the positive noise impulse in the RDS is that, if the acceleration response has a mean of zero, then for all positive triggering conditions there is a higher likelihood that the amplitude of the dynamic response of the system is below the trigger range rather than above the trigger range. Therefore, it is more likely that triggering of the RDT occurs due to a dynamic response with a lower amplitude than the trigger range plus a positive noise impulse. It is less likely that there is a dynamic response with an amplitude higher than the trigger range plus a negative noise impulse. This causes coherence of the noise where triggering of the level-crossing RDT occurs (the start of each data segment in the RDS) and a net positive noise impulse. If the noise is stochastic this coherence does not occur anywhere else in the segments.

When the data is filtered prior to application of the RDT, the component of the RDS associated with the free-response of the system is amplified due to the higher signal to noise ratio. However, the autocorrelation of the filtered noise impulse remains, as shown by the filtering artefacts: the additional oscillations present in Figures 6.2 to 6.4. If the noise is non-white, as is common for accelerometer data [407], then, as shown by Orlovitz and Brandt [406], similar artefacts are present within the data and the autocorrelation functions or system free-response will display the autocorrelations of these artefacts as discussed in detail in Section 6.5. For time-domain OMA, the filtering artefact autocorrelations result in the estimation of an incorrect damping ratio as they are not a constantly damped sinusoid and cannot be accounted for in common modal analysis techniques.

The cause of this noise impulse and its relationship to the autocorrelation of a filtered impulse can be demonstrated by analysing a data set of white noise only with the level-crossing RDT, as shown by the top subplot of Figure 6.3a. The RDS of the unfiltered white noise data is the filter impulse response, with a normalized amplitude of 1 at $t = 0$ and an amplitude close to zero at $t \neq 0$. The RDS of the filtered data is an estimate of the autocorrelation of the filtered impulse, which is equivalent to filtering the RDS of the unfiltered data with the same low-pass filter used to filter the white noise data set prior to application of RDT, as shown by the lower subplot of Figure 6.3a.

### 6.2.3.2 Peak-picking random decrement technique

For the peak-picking RDT, the positive triggering condition, $T_x(t)$, is $T_x(t) = \{x(t - T_s) < x(t) \text{ and } x(t + T_s) < x(t)\}$, where $T_s$ is the sampling interval.
Figure 6.2: RDSs obtained from 2DOF example data detailed in Section 6.2.2. The level-crossing RDT trigger value used was 1.5 multiples of the standard deviation of the unfiltered data set, with a width of the range set at ±0.005m, prior to the addition of noise.

(a) Level-crossing triggering condition.  
(b) Peak-picking triggering condition.

Figure 6.3: Analysis of white noise data using random decrement technique with level-crossing and peak-picking triggering condition. Data in lower plots filtered using a 10 Hz Butterworth low-pass filter (filter order = 7).
When the peak-picking RDT is applied to a filtered white noise data set the noise impulse remains at time \( t = 0 \), as shown in the top subplot of Figure 6.3b. However, a further impulse is introduced at the second time step due to the formulation of the triggering condition. For a peak at time \( t_{pk} \), the value of the amplitude at the next time step, \( t_{pk+1} \), must be less than the amplitude at \( t_{pk} \). The inverse is true for troughs (the negative triggering condition, \( T_{-x(t)} \)). The effect of applying these conditions is that the data segments used to form the RDS are correlated at \( t_{pk+1} \). The value of the amplitude at \( t_{pk+1} \) may be calculated by considering the probability of a given amplitude \( (P(x \leq X \leq x + (a_2 - a_1))) \) multiplied by the cumulative probability of a lower amplitude \( (\sum P(X < x)) \) for all unique amplitudes.

The peak-picking RDS is a mixture of the autocorrelation function starting at \( t = 0 \), associated with the first peak of the filtering artefact, and the autocorrelation function at a phase lag equal to half the period of the filtering artefact oscillation, associated with all peaks of the filtering artefact not at \( t = 0 \). Therefore, the result produced by applying peak-picking RDT to filtered data is not equal to the RDT of the unfiltered data passed through an equivalent bandpass filter, as shown by the lower subplot of Figure 6.3b.

6.2.3.3 Vector random decrement technique

A common expansion of the random decrement technique as described by Asmussen and Brincker [246] relies on vector-triggering; collecting samples from multiple data sets at times where the triggering condition is met within a single data set. This concept has been expanded upon here to investigate the impact of filtering on the vector RDT.

Two copies of the 2DOF example data are created, one which has been corrupted with the additive white noise, and one which has been corrupted with the same additive white noise and then filtered using a Butterworth low-pass filter with a filter order of 7 and a cutoff of 10 Hz. By using combinations of these data sets as the data for analysis and the data which defines the triggering of the RDT, distinctions can be made between false-triggering of the RDT (triggering of RDT by noise spikes) and correlations between filtering artefacts and the data itself.

Figure 6.4 presents the RDSs obtained for different triggering data sets (rows) and analysis data sets (columns). For comparison, all RDSs have been scaled to have an amplitude of 1 at \( t = 0.9 \) seconds, with the vector RDT applied with a range-crossing trigger level of 1.5 times the standard deviation of the original
Figure 6.4: Comparison of scaled cross-correlation RDSs collected using the vector level-crossing RDT approach for the uncorrupted data ("original"), the data after corruption with additive white noise ("corrupted"), and when the data is corrupted and then filtered with a Butterworth low-pass filter ("corrupted & filtered"). RDS of original data shown in all subplots in grey.
data ±0.005. These results demonstrate that when the triggering is based on uncorrupted and unfiltered data (original data), no filtering artefacts are induced in either corrupted or corrupted and filtered data.

However, when the trigger data is corrupted the noise impulse is present in the RDS of corrupted data. Filtering artefacts are present in the corrupted and filtered data, due to additional triggering of the level-crossing RDT by filtering artefacts.

Crucially, when the trigger data has been corrupted and filtered, filtering artefacts are induced in the corrupted (non-filtered) data. These filtering artefacts are induced due to the additional triggering of the RDT by the oscillations in the filtering artefacts in the trigger data and the increased likelihood of triggering by noise when the amplitude of the system response, which is correlated across the data sets, is close to the trigger level. The implications of this in practice are that where a vector RDT approach is to be utilized, or any OMA method reliant on cross-correlations, it is essential that the trigger channel is subjected to minimal filtering to prevent the dispersion of filtering artefacts to unfiltered data sets.

### 6.2.4 Covariance-driven stochastic subspace analysis

Other time-domain OMA methods are also sensitive to the filtering of the data. A widely used OMA method is covariance-driven stochastic subspace identification (SSI-Cov) [408]. This takes the autocorrelation and cross-correlation matrices for several different data sets at varying levels of starting time lag and forms a Toeplitz Matrix before modal analysis through Singular Value Decomposition [241]. The results are typically presented in stability diagrams, with increasing model order representing the number of singular values fitted. Figure 6.5, which compares the SSI-Cov results for the 2DOF example data before the addition of white noise (uncorrupted data), after the addition of white noise (unfiltered data), and after the addition of white noise and filtering with a variety of filter cutoffs, show that the results for filtered data are typically much worse than unfiltered data. The damping estimates for mode 1 exhibit variation in excess of 30% of the true damping value, with smaller fluctuations observed in the natural frequency estimates of both modes as the model order increases. Alongside these fluctuations and outliers, there is significant biasing of the damping estimates of the second mode of vibration for both the unfiltered and filtered data of approximately 5% across all model orders.

The cause of the biasing and increased variation in SSI-Cov modal estimates in the unfiltered and filtered columns of Figure 6.5 is corruption of the autocor-
Figure 6.5: SSI-Cov estimates of frequency and damping (shown in black dots) for 2DOF example data, detailed in Section 6.2.2, for the uncorrupted data, after addition of additive white noise but without filtering (unfiltered data), and when the data has been corrupted and then filtered with a Butterworth low-pass filter of frequency cutoffs of 40Hz, 20Hz or 6Hz (filter order = 7). Maximum time-lag of 2 seconds used in SSI-Cov alongside model orders of 1 to 100. True frequency and damping values for each mode shown by the red-dashed line.
relation function by the noise impulse or filtering artefacts, shown by the colour variation on the left hand side of Figure 6.6a which varies with the low-pass filter cutoff. These artefacts obscure the modal behaviour of the system, the low frequency colour variation in Figure 6.6a.

The cross-correlations, shown in Figure 6.6b, are unaffected by filtering of the data. As SSI-Cov makes simultaneous use of both the auto- and cross-correlations for modal analysis its results are less impacted by filtering artefacts than RDT, which typically fits each RDS independently.

Figure 6.6: Presence of filtering artefacts in autocorrelation functions and its absence in cross-correlation functions for 2DOF example data. The x-axis shows the time-lag of the correlation function, while each row of the y-axis corresponds to the corresponding correlation function for either the data before corruption with white noise (noise free), the data after addition of white noise but without filtering (unfiltered), or when the data has been corrupted with additive white noise and filtered with a Butterworth low-pass filter with frequency cutoffs between 40Hz and 4Hz (filter order = 7).

6.2.5 Other causes of filtering artefacts

The dominant cause of filtering artefacts in time-domain OMA is almost always sensor noise, discussed at length in the previous sections, due to the low SNR of ambient vibration data. However there are several other causes of filtering artefacts which may be encountered in OMA.

Ku et al. [403] demonstrated that for simulated acceleration data for a system excited with white noise and analysed with RDT, the RDS is a mixture of a systems free-response and a ‘singular point’ which is dependant on the autocorrelation matrix of the input force and the diagonal mass matrix. This ‘singular
point’ is not present for displacement data \[403\] and for real structures is usually insignificant due to the dissipation of the force through the structure \[409\]. However any force impulses occurring close to an accelerometer, such as is induced by heel loading on floors, will cause a measurable instantaneous acceleration response. This response will induce filtering artefacts when the data is low-pass filtered.

While the preceding sections have focused on the filtering of high frequencies using a low-pass filter, filtering artefacts are also induced by high-pass filtering of the data. When data is filtered with a high-pass filter, such as to remove long term trends within the data; or diurnal, seasonal and annual results variation, artefacts will typically occur at the start of the entire data set. This is less complex to remove as the section of the data where filtering artefacts are visible may be excluded from further analysis.

Anti-aliasing filters may introduce filtering artefacts, however for OMA applications these artefacts are often of insignificant amplitude due to the lower power concentrated in the high frequency range.

### 6.3 Methods for reducing and removing filtering artefacts

In the previous section the causes of filtering artefacts in ambient vibration data for two common time-domain OMA methods were demonstrated. In this section methods of reducing filtering artefacts through the selection of appropriate filters are presented and an alternative to filtering, the removal of the noise impulse from correlation functions, is introduced. This technique is expanded to allow removal of filtering artefacts from correlation functions. Additionally, a method of incorporating the fitting of filtering artefacts as part of the modal analysis, for use when filtering of the data is unavoidable, is presented.

#### 6.3.1 Minimising the effects of filtering artefacts

All filtering of data which contains discontinuities, such as impulse responses, stochastic forcing, stochastic noise or step changes, will result in filtering artefacts. Therefore the only way to avoid these effects is to analyse unfiltered data. However, the amplitude of the overshoot induced by filtering is related to the type of filter used. A Bessel filter results in less ringing at the expense of a less sharp filter cutoff. Similarly a lower order Butterworth filter will result in a shorter
duration of ringing but with a greater amplitude of ringing [410]. More complex methodologies have been developed for minimising the effects of filtering artefacts such as the work of Gottlieb and Shu [411].

Bayesian time-domain OMA methods, as described by Yuen and Katafygiotis [412]; Yuen, Beck and Katafygiotis [413], and Yang, Lam and Beck [414]; directly avoid the introduction of filtering artefacts into the time-domain OMA as raw data is analysed without any form of pre-processing or filtering. Alongside this, Bayesian OMA methods are robust to non-stationarity in the structural response and applied forcing and may be a suitable alternative which offers greater insight into uncertainty within the modal estimates than conventional time-domain OMA methods, as discussed in detail by Yang, Lam and Beck [414].

6.3.2 Trimming Correlation functions

An effective alternative to filtering of the ambient vibration data is to remove or trim the component of a correlation function or an estimate of the system’s free response, such as the RDS, associated with the noise impulse. The modal parameters of the system are then estimated using only the remaining component of the correlation function or estimate of the free-response after removal of the noise impulse. If the data is corrupted with white noise, the noise impulse is limited to the first data point of the correlation function or range-crossing RDS, as shown previously in Figure 6.2. For the peak-picking RDT triggering condition, the noise impulse is spread across the first two data points of the RDS, as discussed in Section 6.2.3.2.

Where the data is corrupted with non-white noise, or if the data has been filtered with a low-pass filter, the noise has a non-flat frequency spectrum. This manifests in the time-domain as either a broad impulse (a peak extending across multiple time steps [281]) or as filtering artefacts extending across multiple time steps. These may also be trimmed by first identifying the section of the correlation function or RDS corrupted by the noise peak or filter artefacts, either visually or through knowledge of the likely duration of the artefact ringing, and then limiting the modal analysis to where the amplitude of the noise peak or filter artefacts have decreased to a negligible amplitude, as shown in Figure 6.7. The modal parameters of the system may then be estimated through analysis of the remaining section of the free-response estimate or correlation function, the non-shaded section within Figure 6.7, without biasing of the modal parameters induced by the noise impulse.

An amplitude threshold of 5% of the initial amplitude of the filtering arte-
fact is recommended to reduce the biasing which the filtering artefacts introduce. However, the selection of an appropriate length of the correlation function or RDS to be trimmed is dependant on a wide range of factors. If a mode contains large numbers of modes with low amplitudes, a lower amplitude threshold of 1% is advised to ensure that the ratio between the amplitude of each modal component and the amplitude of filtering artefacts is reduced as far as is reasonably practicable. This approach is not suitable if a system contains heavily-damped modes, as the longer trimming length required to meet the amplitude threshold for reduction of the filtering artefact may negatively impact the modal identification. In such situations it is advisable to alter the filter parameters so as to minimize the ringing which occurs as a result of the filtering, or use the indicative noise profile approach presented in the next section.

6.3.3 Removal of filtering artefacts through use of an indicative noise profile

Where filtering of the data for OMA is required, filtering artefacts are unavoidable. Alongside the selection of a filter to minimise ringing, a novel approach is proposed in this section whereby the autocorrelation of a filtered impulse is fitted as part of the modal analysis. If the amplitude and shape of the noise impulse is known, a filtered noise impulse can be numerically generated based on the expected filter response and subtracted from the autocorrelation functions, or RDS free-responses, prior to the modal analysis. However, a key limitation of this approach is that the exact parameters of the filtered noise impulse may not be known prior to analysis.

A more robust approach in practice is to generate an equivalent noise impulse autocorrelation (or noise RDS) using an indicative noise profile (INP), a section of sensor data containing only sensor noise. Through analysis of an INP containing
only sensor noise, the autocorrelation or RDS of the noise after filtering may be approximated. This approximated noise RDS or correlation function will account for the specific noise characteristics of the sensor to be analysed, different sources of background noise which may be present within the data resulting in a non-white noise profile, and the filter parameters selected for pre-processing of the data. Two correlation functions or RDSs are generated from the data in the INP, a correlation function or RDS for the unfiltered INP data ($\hat{D}_{\text{INP unfiltered}}$) and a correlation function or RDS for the INP filtered data ($\hat{D}_{\text{INP filtered}}$). Two correlation functions or RDSs are also generated for the analysis data set: the data set from which modal parameters are to be estimated. These are the correlation function or RDS of the unfiltered analysis data ($\hat{D}_{\text{unfiltered}}$) and the correlation function or RDS of the filtered analysis data ($\hat{D}_{\text{filtered}}$). As has been described in previous sections, the difference between the analysis correlation functions or RDSs in the unfiltered and filtered analysis data sets is linearly proportional to the scaled difference between the noise impulse and the filtered noise impulse through a constant scalar value $R_{\text{filt}}$, as given by Equation 6.3.

$$\hat{D}_{\text{unfiltered}} - \hat{D}_{\text{filtered}} = R_{\text{filt}} \left( \hat{D}_{\text{INP unfiltered}} - \hat{D}_{\text{INP filtered}} \right)$$ (6.3)

The maximum noise impulse response and correlated dynamic response of the system both occur at time-lag $\tau = 0$. Therefore an accurate estimate of $R_{\text{filt}}$ can be found through considering only the values of the INP and analysis correlation functions or RDSs at the zeroth time lag, as described by Equation 6.4:

$$R_{\text{filt}} = \frac{\hat{D}_{\tau=0}^{\text{unfiltered}} - \hat{D}_{\tau=0}^{\text{filtered}}}{\hat{D}_{\tau=0}^{\text{INP unfiltered}} - \hat{D}_{\tau=0}^{\text{INP filtered}}}$$ (6.4)

Once the scale factor $R_{\text{filt}}$ is known, the filtering artefacts may be removed from the filtered correlation function or RDS $\hat{D}_{\tau=0}^{\text{filtered}}$ through subtracting $R_{\text{filt}} \left( \hat{D}_{\tau=0}^{\text{INP unfiltered}} - \hat{D}_{\tau=0}^{\text{INP filtered}} \right)$. The intuitive explanation for this approach is that the attenuation of the noise autocorrelation component due to filtering, the denominator in Equation 6.4, is directly proportional to the attenuation due to filtering of the noise impulse within the correlation function or RDS through the scale factor $R_{\text{filt}}$. As the filtering artefacts may be approximated through analysis of the INP data, knowledge of the filtering artefacts and the filtering attenuation factor $R_{\text{filt}}$ can be used to remove filtering artefacts from the filtered RDS. This approach is simple to implement in practice, requiring no prior knowledge of the dynamic response of the system or specific filtering artefacts, and accounts for any frequency roll-off due to anti-aliasing filtering or for a non-white
noise profile. The INP is likely to be broadly similar in-situ and under controlled conditions, allowing the filtering artefacts to be predicted prior to in-situ testing.

A similar approach can be employed to remove the effects of bandpass limited stochastic forcing by estimating the frequency spectrum of the forcing, using the inverse Fourier transform to generate an equivalent time-domain forcing function and fitting this to the autocorrelations or RDSs. The scaling of INP can be improved upon during the modal analysis by scaling based on multiple points, through repeated iteration of the process of estimating $R_{\text{filt}}$ and modal analysis of the correlation function, or through directly estimating the scale factor $R_{\text{filt}}$ as part of the modal analysis. However for systems dominated by a single mode of vibration, the simple one-step scaling presented here produces satisfactory results.

### 6.4 Simulated data testing

#### 6.4.1 Data generation

To quantify and separate the uncertainty inherent in analysing systems under ambient excitation due to the unknown forcing of the system from systematic biasing of the modal estimates introduced due to filtering of the data, 10,000 numerical data sets were independently generated using the Newmark Beta method for each DOF of the 2DOF system the mass and stiffness matrices for which were presented in Section 6.2.2. For each 60 minute data set, sampled at 100Hz ($3.6 \times 10^5$ samples per DOF), the values of $k_1$ and $k_2$ were randomly selected from a uniform distribution between 0.1N/m and 20,000N/m with the constraint that both natural frequencies of the system must fall between 0.5 Hz and 25 Hz. The values of $\mu_\zeta$ were randomly generated from a uniform distribution between 0 and 10. The values of $\lambda_\zeta$ were randomly generated from a uniform distribution between 0 and 0.1. An additional condition was imposed which ensured the values of $\mu_\zeta$ and $\lambda_\zeta$ selected resulted in damping values for both modes less than 10% of the critical damping. Through the methodology described above, 10,000 independent data sets were generated each with a unique set of frequency and damping estimates, histograms of which are shown in Figure 6.8.

Additive white noise was independently added to the displacement data from both DOFs to achieve a modal signal-to-noise ratio (SNR) randomly selected with equal probability between 0.5 and 100. A wide SNR range was selected so as to assess the impact of filtering on modal parameter estimates when the level of noise within the data is low and it might be expected that the any biasing due to the filtering artefacts would be minimal.
Figure 6.8: Histograms of natural frequencies and damping ratios for the 10,000 independently generated numerical data sets of a 2DOF system. Histograms of frequency estimates based on a bin width of 0.2 Hz. Histograms of damping estimates based on a bin width of 0.2%.

A 2DOF system was selected for the simulated data testing to allow the variation in modal parameters due to filtering artefacts to be isolated from other sources of uncertainty within time-domain operational modal analysis. These sources of uncertainty for higher order systems include the selection of an appropriate model order during the modal analysis, and the selection of a random decrement signature length which balances the competing interests of modal identifiability with the higher ratio of noise to system response which is typical for longer random decrement signatures or correlation functions [246]. While filtering artefacts can be isolated for higher order systems, parsing the impact which these filtering artefacts have on the uncertainty and biasing for the modal estimates of individual modes from the other sources of uncertainty and biasing within the analysis is complex and beyond the scope of this current study.

### 6.4.2 Data analysis

Copies of each data set were filtered using a butterworth low-pass filter (order = 7) with a range of cutoff frequencies. Each filtered data set, alongside the unfiltered data sets, were analysed with the vector RDT with a range-crossing triggering condition [246] of one to two standard deviations of the trigger data. Each RDS was analysed with the Matrix Pencil method [254], with the model order increased until the coefficient of determination ($R^2$ [15]) between the RDS and the fitted signal was greater than 0.99. Separately, the unfiltered and trimmed RDS (the unfiltered RDS with the data point at time $t = 0$ excluded) was also analysed.
6.4.3 Simulated data - Mode detection efficacy

Filtering artefacts may significantly impact the ability of time-domain operational modal analysis to detect all modes of vibration within correlation functions and random decrement signatures, as shown in Figure 6.9. Presented in this figure are the number of modal estimates where: i) the true natural frequency of the mode is below the low-pass filter cutoff, and ii) the error in the frequency estimate is below 7.5Hz, and iii) the error in the damping estimate is less than 2% of the critical damping. The values for the allowable frequency and damping error are based on the 95% confidence interval for the distributions of modal estimates presented in Figure 6.8. As expected, the number of first and second order modes detected decreases as the low-pass filter frequency is decreased due to modal components being filtered out. However, there is a significant drop in the number of second order modes detected when the low-pass filter frequency is between 10 Hz and 20 Hz. This is driven by the filtering artefacts dominating the RDSs generated with data from the second DOF. Therefore the second largest component fitted to the RDS corresponds to components fitted to the filtering artefact, not the second mode of vibration.

Figure 6.9: Percentage of modes detected using the RDT across 10,000 numerically generated data sets for varying low-pass filter cutoff frequencies. Results obtained from RDSs of unfiltered data shown as a grey dashed line. Results for the unfiltered and trimmed RDSs shown as a solid black line.

6.4.4 Simulated data - Frequency estimates

The error in the frequency estimates of the simulated data for the first mode (the fitted modal component with the largest amplitude) and the second mode (the fitted modal component with the second largest amplitude) are plotted in Figures 6.10 and 6.11. In each figure the results are presented for a range of low-pass filter cutoffs, the unfiltered condition, and the unfiltered condition where
the first point of the RDS is trimmed prior to the matrix pencil fitting. For each fitted mode, the error is based on the closest natural frequency of the system. Results where the true natural frequency of the mode is above the low-pass filter cutoff are excluded from these plots. The color of the data point corresponds to the SNR of the data prior to filtering, with darker colors corresponding to a lower SNR. A moving average of the frequency estimates across all SNRs, based on a window width of +/-1.5Hz, is shown with red markers, with two standard deviations of the windowed results shown as orange markers.

![Error in the frequency (Hz) estimate for the largest component of the fitted RDS. The color of each data point corresponds to the SNR of the data prior to filtering, with darker colors corresponding to a lower SNR. A moving average of the frequency estimates across all SNRs, based on a window width of +/-1.5Hz, is shown with red markers, with two standard deviations of the windowed results shown as orange markers.](image)

**Figure 6.10**: Error in the frequency (Hz) estimate for the largest component of the fitted RDS. The color of each data point corresponds to the SNR of the data prior to filtering, with darker colors corresponding to a lower SNR. A moving average of the frequency estimates across all SNRs, based on a window width of +/-1.5Hz, is shown with red markers, with two standard deviations of the windowed results shown as orange markers.

It is clear that the unfiltered and trimmed RDS is the most consistently accurate approach, with the mean absolute frequency error of the unfiltered data in the plotted range (0.084 Hz) being 235% higher than for the unfiltered and trimmed data (0.025 Hz) for the largest fitted component, and 198% higher for the second largest fitted component (0.36 Hz versus 0.12 Hz). The error is consistently higher for the second largest fitted component, which usually corresponds to the second mode of vibration. This is likely due to the lower amplitude of
Figure 6.11: Error in the frequency (Hz) estimate for the second largest component of the fitted RDS. The color of each data point corresponds to the SNR of the data prior to filtering, with darker colors corresponding to a lower SNR. A moving average of the frequency estimates across all SNRs, based on a window width of +/-1.5Hz, is shown with red markers, with two standard deviations of the windowed results shown as orange markers.
movement for the second mode of vibration. This leads to a higher ratio of the filtering artefact amplitude (which is primarily dependant on the noise level of the signal) and the modal amplitude, which causes larger errors in the frequency of the fitted signal component. For the same reason the data sets with lower SNR have higher errors across all filtered and unfiltered (non-trimmed) results; a lower SNR leads to a higher ratio of the filtering artefact amplitude to the amplitude of the modal component within the RDS. However, as the noise impulse from a white noise process is only present in the first data point of the RDS and is therefore excluded from the unfiltered and trimmed RDS analysis, there is a much weaker correlation between the SNR and the error in the frequency estimates.

As might be expected, the largest errors in the frequency estimates occur at the higher frequencies and when the natural frequency of the system is close to the low-pass frequency cutoff. The filtering artefacts cause the distinctive linear variation in error as the natural frequency approaches the cutoff frequency. However there is no clear biasing in the frequency estimates if the cutoff frequency is more than a few Hz from the natural frequency of interest.

### 6.4.5 Simulated data - Damping estimates

The error in the damping estimates, provided as the percentage of critical damping $\zeta$, is presented in Figures 6.12 and 6.13. For each fitted mode, the error is based on the closest natural frequency of the system, with results where the true natural frequency of the mode is above the low-pass filter cutoff excluded from the analyses.

As with the frequency estimates, the unfiltered and trimmed results leads to damping estimates which have a mean absolute error in the plotted ranges which is 56% lower than the unfiltered results (0.11% error versus 0.18% error) for the largest component of the fitted RDS and 774% lower for the second largest fitted component (0.14% error versus 1.22% error). The error in the unfiltered and trimmed damping estimates are not correlated with the SNR, while the unfiltered and low-pass filtered results show increased variance in the damping error with decreasing SNR. In practice this increased variance with decreased SNR may easily be mistaken for amplitude-dependant damping behaviour, as high amplitude signals will have a higher SNR and may be expected to have a lower damping error.

Significant biasing is present in the low-pass filtered and unfiltered results for the second largest component of the fitted RDS, as shown by the non-zero mean error in the damping estimate plotted in Figure 6.13. These results show a
Figure 6.12: Error in the damping estimate (% of damping ratio $\zeta$) for the largest component of the fitted RDS. The color of each data point corresponds to the SNR of the data prior to filtering, with darker colors corresponding to a lower SNR. A moving average of the damping estimates across all SNRs, based on a window width of +/-0.25%, is shown with red markers, with two standard deviations of the windowed results shown as orange markers.
Figure 6.13: Error in the damping estimate (% of damping ratio $\zeta$) for the second largest component of the fitted RDS. The color of each data point corresponds to the SNR of the data prior to filtering, with darker colors corresponding to a lower SNR. A moving average of the damping estimates across all SNRs, based on a window width of +/-0.25%, is shown with red markers, with two standard deviations of the windowed results shown as orange markers.
consistent overestimation in the damping of the system which increases as the true system damping increases. Significantly, as shown in Figures 6.14 and 6.15, the error in the damping estimate induced by the filtering artefact is independent of how far the natural frequency of the mode is from the low-pass filter cutoff for both modes of vibration. The implications of this in practice is that damping estimates may be negatively impacted by filtering even when the mode of vibration is far from the filter cutoff frequency. For example it can be seen that there is significant biasing and errors in the damping estimates for modes of vibration at 10Hz, even when the low-pass filter cutoff is at 25Hz, with these errors being of a similar magnitude to that of the unfiltered data and having a variance which is twice as large as the errors of the damping estimates from the unfiltered and trimmed RDS.

Figure 6.14: Error in the damping estimate (% of damping ratio $\zeta$) for the largest component of the fitted RDS in comparison to the true natural frequency of the component. The color of each data point corresponds to the SNR of the data prior to filtering, with darker colors corresponding to a lower SNR. A moving average of the damping estimates across all SNRs, based on a window width of +/-0.25%, is shown with red markers, with two standard deviations of the windowed results shown as orange markers.
Figure 6.15: Error in the damping estimate (% of damping ratio \( \zeta \)) for the second component of the fitted RDS in comparison to the true natural frequency (Hz) of the component. The color of each data point corresponds to the SNR of the data prior to filtering, with darker colors corresponding to a lower SNR. A moving average of the damping estimates across all SNRs, based on a window width of \( \pm 0.25\% \), is shown with red markers, with two standard deviations of the windowed results shown as orange markers.
6.5 Case study - Removal of filtering artefacts from floor slab accelerations

To verify that filtering artefacts impact all data, and not just simulated data, 100 minutes of vertical acceleration data, sampled at 256 Hz, was collected from the underside of a two-way spanning concrete slab subject to footfall excitation. While this structure is dynamically simple, it is indicative of a typical application of OMA for the analysis of floor vibrations at the serviceability limit state \[415\] and allows the impact of filtering artefacts on the modal estimates to be robustly and accurately isolated. The range-crossing RDT, with a trigger range of 0 to \(+\infty\), was used for the analysis after the application of a Butterworth filter of order 7 with various level of low-pass filter cutoffs. From Figure 6.16 it can be seen that there is a slight corruption introduced into the RDSs for the filtered data. To allow comparison of the RDSs, all RDSs have been normalized to give an amplitude of unity at \(t = 0.1\) s.

![Figure 6.16: RDSs for two-way spanning concrete slab acceleration data, collected with a level-crossing range of 0 to \(+\infty\). All RDSs normalized to have an amplitude of unity at \(t = 0.1\) s for comparison.](image)

By subtracting the normalized RDS of the unfiltered data from that of the filtered data, shown in Figure 6.17a, it can be seen that the cause of this corruption is the autocorrelation of the filtering artefacts. A close approximation of these filtering artefacts can be recreated as shown in Figure 6.17b through filtering an impulse function using the same sampling and filtering parameters.

However, there are key differences between the isolated filtering artefacts and the filtered impulses caused primarily by the non-white nature of the sensor noise. This non-white noise can be identified by applying the RDT to an INP, a section of data collected when the slab is not excited. The RDSs for the filtered and unfiltered INP are shown in Figure 6.18. The presence of a noise impulse lasting
(a) Isolated filtering artefacts phenomena obtained by subtraction of RDS of unfiltered data from RDSs in Figure 6.16.

Figure 6.17: Isolated filtering artefacts in data collected from two-way spanning concrete slab subject to footfall excitation. Data filtered with a Butterworth low-pass filter of order 7 prior to analysis with the RDT.

several samples indicates that the noise has a non-uniform power spectrum and contains significant low-frequency components. This is a common feature of the noise profiles of both piezoelectric and MEMS accelerometers [407].

Figure 6.18: RDSs for two-way spanning concrete slab indicative noise profile calculated using the RDT with a level-crossing range of 0 to +∞.

The impact of the extended noise impulse is that the fitted parameters of the system changes depending on which part of the RDS is fitted, as shown in Figures 6.19a and 6.19b. As the majority of the noise impulse is excluded after the trimming of the first three data-points from the unfiltered RDS, the fitted modal parameters stabilize at a constant value. However, as the filtering artefacts have significant amplitude past three data-points the modal parameters do not converge to a constant result.

The removal of the filtering artefacts from the 2-way spanning slab data using an INP is demonstrated in Figure 6.20a. Five minutes of data where there
was no excitation of the slab were isolated, filtered and analysed through the range-crossing RDT with limits of 0 and $+\infty$ to create the filtered INP RDS, $\hat{D}_{\text{INP, filtered}}$, which was scaled using Equation 6.4. By scaling and subtracting the filtered INP RDS from the RDS of the filtered data collected when the slab was excited ($\hat{D}_{\text{filtered}}$), the majority of the filtering artefacts are removed, as seen in Figure 6.20a and the biasing of modal estimates greatly reduced, as shown in Figure 6.20b. This allows analysis of the slabs dynamic response with minimal corruption of the modal estimates by filtering artefacts.

Figure 6.19: Estimated modal parameters for RDSs of concrete slab data with various levels of trimming (exclusion of data points from start of RDS) and low-pass filter cutoffs.
(a) Random decrement signatures. All RDSs normalized to have an amplitude of unity at $t = 0.1$ s for comparison. (b) Estimated damping of dominant mode for unfiltered, low-pass filtered, and low-pass filtered and INP corrected RDSs.

Figure 6.20: Estimates of modal parameters using random decrement signatures, with filtering artefacts removed prior to analysis of RDSs through use of an indicative noise profile.

6.6 Summary

Due primarily to the presence of noise within the autocorrelation function, or estimates of a systems free-response, filtering of data for use in time-domain operational modal analysis induces filtering artefacts. Within this chapter it has been demonstrated how this results in higher errors in damping estimates for systems with higher damping ratios, or data which have a lower signal-to-noise ratio, and that these errors may be on the order of hundreds of percent of the true damping value. Using acceleration data collected from a two-way spanning concrete slab subject to footfall excitation, the filtering artefacts have been isolated. A range of techniques for minimizing the effects of filtering artefacts in time-domain operational modal analysis have been discussed and two novel practical solutions, in the form of either excluding sections of the autocorrelation functions corrupted by noise impulses/filtering artefacts from the modal analysis, or by generating estimates of the filtering artefacts using indicative noise profiles, have been presented and their application demonstrated using numerical and real-world data.
Chapter 7

Short-time random decrement technique (ST-RDT)

This chapter introduces the short-time random decrement technique, a time-domain operational modal analysis (OMA) method which provides a basis for addressing research objective 3:

RO 3. Quantify uncertainty in OMA modal parameter estimates.

The short-time random decrement technique (ST-RDT) is a novel extension to the random decrement technique (RDT), discussed in Chapter 2. It combines aspects of subsampling and moving average filters to allow the uncertainty in modal parameter estimates to be quantified for both stationary and non-stationary dynamic behaviour. Through the use of overlapping windows of data it allows for the dependence of the uncertainty in modal parameters induced by the correlation of structural responses to be accounted for. In this chapter the application of the ST-RDT is demonstrated using real-world data, and key statistical properties of modal estimates from the ST-RDT are derived based on subsampling theory and the theory of moving average filters. Use of the ST-RDT for quantifying uncertainty in modal parameters for weakly non-linear dynamical systems is demonstrated using real-world case studies.

The chapter is organized as follows. A new derivation of the random decrement signature (RDS) as a conditional correlation function is presented in Section 7.1.2. It is shown that due to the correlated behaviour of dynamic systems there are a narrow range of conditions under which the RDS is an estimate of the free-response of a dynamic system. The components of the RDS which cannot be directly modelled through modal analysis as a finite sum of damped sinusoids are identified. These non-free response components result in errors in the modal
parameter estimates for the structural modes and the appearance of spurious or noise modes within the modal parameter estimates.

The ST-RDT is introduced in Section 7.3 and its application is demonstrated using acceleration data collected from the MX3D Bridge, the world’s first 3D printed steel bridge. Mode-shape generation using the ST-RDT modal estimates is presented in Section 7.3.2 with an overview of the selection of appropriate parameters for use within the ST-RDT presented in Section 7.3.3.

The relationship between modal estimates generated using the ST-RDT, subsampling theory and the quantification of uncertainty in estimated modal parameters is derived in Section 7.4 for both linear and weak non-linear modal behaviour. Subsampling theory, introduced in Chapter 2, provides a means of quantifying statistical distributions and confidence intervals from a finite sample of data whilst maintaining any correlations or non-stationarity within the observed data. In subsampling, data is divided into overlapping windows or subsamples and analysed separately on the basis that the probability mechanism used to generate each subsample is the same as that of the system as a whole [312]. The use of overlapped windows is discussed in the context of the additional information which it may impart when analysing weak non-linear modal behaviour as well as the presence of serially correlated errors within the modal analysis estimates which the use of overlapped windows introduces.

The latter sections of the chapter present real-world applications of the ST-RDT for: assessment of a historic glass-fibre reinforced footbridge (Section 7.5), analysis of one week of data from a multistory timber building under wind excitation (Section 7.6), modal analysis of low-frequency standing surface waves on lakes (Section 7.7), and prediction of the vibration response of the world’s first 3D printed steel footbridge (Section 7.8).

### 7.1 The Random Decrement Technique

As discussed in the literature review, OMA can be split into two broad categories: frequency-domain OMA, where the modal parameters are estimated based on the frequency power spectrum of the ambient vibration data, and time-domain OMA whereby the estimates of free-response or correlation function of the data is used to estimate the modal parameters. The random decrement technique (RDT) combined with modal analysis methods such as least-squares curve fitting, the Ibrahim time-domain method [237], or the matrix pencil method [254], is a popular time-domain OMA method which has been shown to be effective in
the identification of non-linear structural behaviour under broadband-random excitation [17].

### 7.1.1 Discrete time approximation

As the majority of OMA uses digital measurements of structural response which are recorded at discrete time intervals, all derivations are presented in this chapter relative to uniform discrete time intervals, demarcated with subscript $i$. In reality, the forcing applied to dynamic systems and the system response itself is continuous. The transformation of the force applied to the system from continuous time $t$ to discrete uniform time intervals $\Delta t$ is given by approximating the continuous force $p(t)$ across the time span $t$ with a series of discrete impulse forces $F$ at times $t_i$ [281] using Equation 7.1.

$$p(t) \approx \sum_{i<t} F(t_i) \Delta t \delta(t - t_i) \quad (7.1)$$

The continuous response of the system $x(t)$ is given by Duhamel’s integral [15], presented in Equation 7.2 for a viscously damped single degree of freedom system, with mass $m$, natural frequency $\omega$, damping $\zeta$ and damped natural frequency $\omega_d$.

$$x(t) = \frac{1}{m \omega_d} \int_0^t F(t_i) e^{-\zeta \omega(t-t_i)} \sin(\omega_d(t-t)) \, dt \quad (7.2)$$

This can be approximated across discrete time steps $\Delta t$ as the superposition of the unit-impulse response functions $h(t - t_i)$, described in Chapter 6, due to the discrete impulse forces $F(t_i)$ using Equation 7.3.

$$x(t) \approx \sum_{i<t} F(t_i) h(t - t_i) \Delta t \quad (7.3)$$

While the new definition of the RDS is presented in this section relative to the discrete time approximation given by Equation 7.3, an equivalent continuous time definition of the RDS can be derived through replacing the summation of discrete time-lagged responses with the equivalent time-domain integral.

### 7.1.2 Relationship between random decrement signatures and correlation functions

Previous definitions of the RDS, such as that provided by Asmussen et al. [257], have focused on the use of the RDT under the assumptions of white-noise stochas-
tic forcing and with white-noise corruption of the measured structural response, under which conditions the RDS is an estimate of the free-response of the system. Further to this, previous work has focused on the use of vibration data as both the triggering and analysis data, potentially limiting the applicability of the RDT for quantification of weak non-linear modal behaviour.

In this section the RDS is redefined as a conditional correlation. In the context of dynamic systems this conditional correlation is the sum of the expected value of the analysis data, conditional on the trigger data, due to: i) the initial amplitude response, ii) the initial first derivative response, iii) the forced response, and iv) the noise within the data. The new definition of the RDS accounts for the correlation between the forcing applied to a structure and its dynamic response, as well as the correlations induced by the use of non-independent segments of data within the RDS, such as where segments of data overlap.

This revised definition of the RDS is crucial for accurate estimation of the modal parameters, as some parts of the RDS cannot be directly modelled as a finite sum of damped sinusoids within the modal analysis and some parts of the RDS are associated with non-white forcing of the system, or non-white measurement noise. For numerical examples of non-white forcing and noise components see Chapter 6 and Section 7.1.3. A specific real-world example of where non-white forcing is encountered is given in Section 7.7. These non-white components of the RDS manifest in the modal parameter estimates as both errors in the estimated modal parameters and non-real modes of vibration, also referred to as spurious or noise modes [17].

As described within Chapter 2, the RDS $\hat{D}_{Xt}(\tau)$ for an analysis data set $X$ from which the modal parameters are to be estimated, formed for $N_{rd}$ instances of the triggering condition $T_i = \text{True}$, at a time-lag $\tau$ before or after the triggering condition is met is given by Equation 7.4.

$$\hat{D}_{Xt}(\tau) = \frac{1}{N_{rd}} \sum_{n=0}^{N_{rd}} (X(t_i + \tau)|T_i = \text{True}) \quad (7.4)$$

In terms of probability theory, the RDS is an ensemble average and can be considered as the expected value of the data set $X$ at a time lag $\tau$ conditional on the trigger condition being true, described fully through Equation 7.5.

$$\hat{D}_{Xt}(\tau) = E(X_{t_i + \tau}|T_i = \text{True}) \quad (7.5)$$

To understand the relationship between the expected value $E(X_{t_i + \tau}|T_i = \text{True})$
and the response of a dynamic system, the components which make up each segment of data at a time-lag $\tau$, $X_{t_i+\tau}$, must be understood. Consider a series of segments $x_a$, $x_b$, and $x_c$, collected from vibration data $X$ for a single-degree of freedom dynamic system excited through white noise excitation. Segments extend between time-lags $\tau = 0$ and $\tau = 2$ seconds, and are collected using a level-crossing triggering condition with a trigger level of $1 \text{ m/s}^2$, hence $T_i = \text{True}$ if $X_i = 1 \text{ m/s}^2$. As shown in Figure 7.1, each segment comprises of the summation of four separate components:

i. $x_0$ - The response of the system due to the initial amplitude of the response at $\tau = 0$.

ii. $x_d$ - The response of the system due to the initial first derivative of the response at $\tau = 0$.

iii. $x_F$ - The response of the system due to forcing applied at $\tau > 0$.

iv. $x_n$ - The noise in the sensor used to measure the data.

Figure 7.1: Components of segments of vibration data collected from a single degree of freedom system
The definition of the RDS as the expected value of $X$ can be expanded as the summation of these four components given in Equation 7.6:

$$\hat{D}_{XT}(\tau) = E(X_{0,t_i+\tau} + X_{d,t_i+\tau} + X_{F,t_i+\tau} + X_{n,t_i+\tau}|T_i = \text{True})$$  

(7.6)

Equation 7.6 may be expanded to give Equation 7.7:

$$\hat{D}_{XT}(\tau) = E(X_{0,t_i+\tau}|T_i = \text{True}) + E(X_{d,t_i+\tau}|T_i = \text{True}) + E(X_{F,t_i+\tau}|T_i = \text{True}) + E(X_{n,t_i+\tau}|T_i = \text{True})$$

(7.7)

After separating the RDS into its constituent parts, each can be considered separately. Note that through considering each component of the segments separately it is not assumed that there is no correlation between the components, for example it is evident that the amplitude of the free-response is likely to be correlated with the amplitude of the forced response in each segment. However, this correlation between components will only appear within the RDS if there exists a correlation between each component of the data and the triggering condition selected.

The following sections will consider each of the components of the RDS individually in order to derive a description of the RDS which accounts for the correlated components of dynamic systems exhibiting stationary or linear behaviour, presented in Section 7.1.2.5, or weakly non-linear behaviour, as presented in Section 7.1.2.6.

### 7.1.2.1 Initial amplitude component of the RDS segments

The component of each segment of data due to the initial amplitude of the system’s response, $x_0$, taken from a $j$-degree of freedom linear dynamic system at any time-lag $\tau$ is described by Equation 7.8:

$$x_0(\tau) = \sum_{j=1}^{j} A_{0,j} e^{-\zeta_j \omega_j \tau} \cos(\omega_j \tau)$$

(7.8)

In Equation 7.8, $\zeta_j$ is the damping ratio of mode $j$, $\omega_j$ is the angular natural frequency of mode $j$, and $A_{0,j}$ is the amplitude of mode $j$. This response is only dependant on the modal parameters of the system, assumed to be constant for a linear-dynamic system, and the initial amplitude of the response of each of the $j$ modes, $A_{0,j}$, at $\tau = 0$. By inspection of Equation 7.8 it is clear that $x_0(\tau)$ can be exactly modelled as sum of damped sinusoids during the modal analysis.

If there is no correlation between the vibration response $X$ and the trigger condition $T$, then they are independent variables $X \perp T$. As they are indepen-
dent variables, the expected value of the amplitude of the signal $E(X_i|T_i)$ at time lag $\tau = 0$ approaches the expected value of $X$ given by Equation 7.9

$$E(X_i|T_i = \text{True}) = E(X|X \perp \perp T)$$ (7.9)

$E(X)$ is given by the summation of the expected amplitude of the $j$ modes of vibration at $\tau = 0$ given by Equation 7.10

$$E(X) = \sum_{j=1}^{j} E(A_{0,j})$$ (7.10)

Assuming the oscillations are a zero-mean process, the expected amplitude of each mode of vibration is zero where $X \perp \perp T$. The variance of the expected amplitude of the $j$ modes of vibration at $\tau = 0$ is given by the central limit theorem through Equation 7.11

$$\text{Var}(E(A_{0,j})) = \frac{\sigma^2_{A_{0,j}}}{N_{rd}}$$ (7.11)

In Equation 7.11, $\sigma^2_{A_{0,j}}$ is the variance in the amplitude of mode $j$, and $N_{rd}$ is the number of instances of the triggering condition $T_i = \text{True}$.

If the vibration response $X$ is correlated with $T$, $X \not\perp \not\perp T$, the expected value of $E(X_i|T_i)$ at time lag $\tau = 0$ is the summation of the expected amplitude of each individual mode of vibration conditional on the trigger condition being true given by Equation 7.12

$$E(X_i|T_i = \text{True}) = E(X_i|T_i = \text{True}, X \not\perp \not\perp T) = \sum_{j=1}^{j} E(A_{0,j}|T_i = \text{True}, X \not\perp \not\perp T)$$ (7.12)

As might be expected, the equation for the variance of the expected amplitudes of each mode must also be revised to account for correlation between the initial amplitude and the triggering condition, as given in Equation 7.13

$$\text{Var}(E(A_{0,j}|T_i = \text{True})) = \frac{\sigma^2_{A_{0,j}|T_i=\text{True}}}{N_{rd}}$$ (7.13)

### 7.1.2.2 Initial first derivative component of the RDS segments

The component of each segment associated with the first derivative of the system, $x_d$, at any time lag $\tau$ is described by Equation 7.14 the impulse response function
for a \( j \)-degree of freedom linear dynamic system.

\[
x_d = \sum_{j=1}^{j} A_{d,j}e^{-\zeta_j \omega_j \tau} \sin(\omega_j \tau)
\]

(7.14)

In Equation 7.14, \( A_{d,j} \) is the amplitude of the response due to the initial first derivative response of each of the \( j \) modes of vibration. As with the initial amplitude response, the first derivative response is only dependant on the modal parameters of the system, assumed to be constant for a linear-dynamic system, and the initial first derivative of the response of each of the \( j \) modes, \( A_{d,j} \), at \( \tau = 0 \). Again by inspection it is clear from Equation 7.14 that the response due to the initial first derivative response can be directly modelled as sum of damped sinusoids during the modal analysis.

Similar to the initial amplitude component \( x_0 \), if the first derivative response of the system, \( \dot{X} \), is independent of the triggering condition, \( \dot{X} \perp T \), the expected value of the first derivative response given the triggering condition is the summation of the expected first derivative responses of the individual modes of vibration at time lag \( \tau = 0 \). This summation is given by Equation 7.15.

\[
E(\dot{X}) = \sum_{j=1}^{j} E(A_{d,j})
\]

(7.15)

The variance of the expected value of the initial first derivative response is given by Equation 7.16.

\[
V ar(E(A_{d,j})) = \frac{\sigma_{A_{d,j}}^2}{N_{rd}}
\]

(7.16)

Again, assuming the oscillations are a zero-mean process, the expected value for the first derivative response of each mode of vibration is zero where \( \dot{X} \perp T \). If the first derivative response of the system is correlated with the triggering condition, \( \dot{X} \not\perp T \), the expected value of the first derivative response given the triggering condition is the summation of the expected first derivative responses of the individual modes of vibration at time lag \( \tau = 0 \) conditional on \( T_i = \text{True} \), as described by Equation 7.17.

\[
E(\dot{X}_i | T_i = \text{True}) = E(\dot{X}_i | T_i = \text{True}, \dot{X} \not\perp T) = \sum_{j=1}^{j} E \left( A_{d,j} | T_i = \text{True}, \dot{X} \not\perp T \right)
\]

(7.17)
The variance of $E(A_{d,j}|T_i = \text{True})$ is given by Equation 7.18

$$
\text{Var}(E(A_{d,j}|T_i = \text{True})) = \frac{\sigma_{A_{d,j}|T_i=\text{True}}^2}{N_{rd}} \quad (7.18)
$$

### 7.1.2.3 Forced component of the RDS segments

The component of each segment of data associated with the forcing applied to the system after $\tau = 0$, $x_F(\tau)$, is given by Equation 7.19

$$
x_F(\tau) = \sum_{\delta_t = 0}^{\delta_t = \tau} \sum_{j=1}^{j} A_{F,j,\delta_t} e^{-\zeta_j \omega_j (\delta_t + \tau)} \sin(\omega_j (\delta_t + \tau)) \quad (7.19)
$$

In Equation 7.19, $A_{F,j,\delta_t}$ is the amplitude of the forced response of mode $j$ due to the application of force at time-lag $\delta_t$ after $\tau = 0$. While the isolated forced response of the system due to a single impulse could be directly modelled as a sum of damped sinusoids, the summation of the $\delta_t$ time-lagged forced responses cannot due to the presence of a time-lag term. Therefore a goal of selecting a triggering condition is to minimize the absolute amplitude of the expected value of the forced response, discussed further in Section 7.3.3.2.

Unlike the components of each segment associated with the initial amplitude of the response ($x_0$) and the initial first derivative of the response ($x_d$), where the expected value of the RDS was purely dependant on whether $X \perp T$ and $\dot{X} \perp T$ respectively, when considering the forced component of the segments attention must be given to both whether the forcing $F$ is independent of the trigger condition, $F \perp T$, and whether the forcing at time-lag $\delta_t$ is independent of the forcing applied at each of the previous $n$ time lags, $F_{\delta_t} \perp F_{\delta_t-n}$.

Considering first the independence of the forcing and the triggering condition. If $F \perp T$ the expected value of $F$ is given by Equation 7.20 and the variance of $F$ is given by Equation 7.21

$$
E(F_i) = \sum_{j=1}^{j} E(A_{F,j}) \quad (7.20)
$$

$$
\text{Var}(E(F_j)) = \frac{\sigma_{A_{F,j}}^2}{N_{rd}} \quad (7.21)
$$

If $F \not\perp T$ the expected value of $F$ is given by Equation 7.22 and the variance of

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\( F \) is given by Equation 7.23:

\[
E(F_i|T_i = \text{True}) = \sum_{j=1}^{j} E(A_{F,j}|T_i = \text{True})
\] (7.22)

\[
\text{Var}(E(F_i|T_i = \text{True})) = \frac{\sigma_{A_{F,j}|T_i=\text{True}}^2}{N_{rd}}
\] (7.23)

However, under either condition special attention should be paid to the possibility of correlation between the forcing between subsequent time steps. If the forcing is a true white noise process it is both temporally and spatially stochastic. Spatially stochastic is here defined as that the force applied to mode of vibration \( j \) is independent of the forcing applied to mode \( k \) at the same step \( i \): \( E(F_{i,j}) \perp \perp E(F_{i,k}) \). Temporally stochastic is here defined as the force applied at time \( \delta_i \) being independent of the force applied at \( \delta_{i-n} \) for all values of \( n \): \( E(F_{i}) \perp \perp E(F_{i-n}) \). For civil structures it is rare that either of these are true. For example the force applied at one point at a structure is likely to be broadly similar to that applied elsewhere. Similarly the force applied to a structure at a given point in time is likely to be broadly similar to the force applied a short time earlier and that applied a short time later. With regards to the weak temporal correlation of the forcing, the forcing applied to structures is usually considered broadband random for civil structures, that is it has no dominant frequency component within the frequency range of interest. However, artefacts of the broadband forcing may still appear in the RDS, as discussed within Chapter 6 and illustrated in Sections 7.1.3 and 7.7.

### 7.1.2.4 Noise component of RDS

The measurement noise, \( x_n \), within the data set \( X \) is an inherent part of analysing data from real-world structures. The component of each segment of data associated with the measurement noise, \( x_n(\tau) \), is given by Equation 7.24:

\[
x_n(\tau) = \varepsilon(\tau)
\] (7.24)

In Equation 7.24 \( \varepsilon \) is a function defining the noise. It is unlikely that the noise can be directly modelled as a sum of damped sinusoids. Therefore, as with the forced response, the trigger condition should be selected so as to minimize the absolute amplitude of the expected value of the noise response, discussed further in Section 7.3.3.2.

If the noise is independent of the triggering condition, \( \varepsilon \perp \perp T \), the expected
value of the noise component of the signal given the triggering condition equals
the expected value of the noise, \( \mathbb{E}(\varepsilon(\tau)|T_i = \text{True}) = \mathbb{E}(\varepsilon(\tau)) \). The variance of
the noise is given by \( \text{Var}(\mathbb{E}(\varepsilon(\tau))) = (\sigma_\varepsilon(\tau)^2)/N_{rd} \). If \( \varepsilon(\tau) \not\perp T \) the variance
of the expected value of the noise is conditional on the basis that the triggering
condition is true, described through Equation 7.25.

\[
\text{Var}(\mathbb{E}(\varepsilon(\tau)|T_i = \text{True})) = \frac{\sigma_\varepsilon^2(\tau)T_i=\text{True}}{N_{rd}} \tag{7.25}
\]

Again consideration must be given to the possibility of both spatial and temporal
correlation within the noise. An example of spatial correlation within the noise
would be electrical interference induced by the environment of the sensors, such as
sensor cabling running alongside high voltage cables, while temporal correlation
of the noise would occur where the noise is not a white-noise process and therefore
has a non-uniform frequency spectra. Temporal correlation of noise and its impact
on RDSs is discussed at length in Chapter 6 and illustrated in Section 7.1.3.

### 7.1.2.5 Definition of the RDS for linear dynamic systems

Combining the components of vibration data discussed in the previous sections,
the RDS at a time lag \( \tau \) for a \( j \)-degree of freedom linear dynamic system analysed
with a generalized triggering condition can be fully described by Equation 7.26.

\[
\hat{D}_{XT}(\tau) = \mathbb{E}(X_{0,t_i+\tau}|T_i = \text{True}) + \mathbb{E}(X_{d,t_i+\tau}|T_i = \text{True}) + \mathbb{E}(X_{F,t_i+\tau}|T_i = \text{True}) + \mathbb{E}(\varepsilon_t|T_i = \text{True}) \tag{7.26}
\]

Equation 7.26 may expanded as Equation 7.27.

\[
\hat{D}_{XT}(\tau) = \sum_{j=1}^{j} E(A_{0,j}|T_i = \text{True})e^{-\zeta_j\omega_j \tau} \cos(\omega_j \tau)
+ \sum_{j=1}^{j} E(A_{d,j}|T_i = \text{True})e^{-\zeta_j\omega_j \tau} \sin(\omega_j \tau)
+ \sum_{\delta_i=0}^{\delta_i=\tau} \sum_{j=1}^{j} E(A_{F,j,\delta_i}|T_i = \text{True})e^{-\zeta_j\omega_j(\delta_i + \tau)} \sin(\omega_j(\delta_i + \tau))
+ E(\varepsilon_t|T_i = \text{True}) \tag{7.27}
\]

Using the trigonometric identity \( a \cos(t) + b \sin(t) = \sqrt{a^2 + b^2} \sin(t + \tan^{-1}(\frac{a}{b})) \),
parts of Equation 7.27 may be simplified using the redefined quantities in Equa-
tion 7.28 and Equation 7.29.

\[
\sqrt{E(A_{0,j}|T_i = \text{True})^2 + E(A_{d,j}|T_i = \text{True})^2} = E(A_{R,j}|T_i = \text{True}) \quad (7.28)
\]

\[
\tan^{-1} \left( \frac{E(A_{0,j}|T_i = \text{True})}{E(A_{d,j}|T_i = \text{True})} \right) = \phi_{j,\text{lag}} \quad (7.29)
\]

Using the definitions in Equations 7.28 and 7.29, the generalized equation for the RDS given in Equation 7.30 can be derived.

\[
\hat{D}_{XT}(\tau) = \sum_{j=1}^{j} E(A_{R,j}|T_i = \text{True}) e^{-\zeta_j \omega_j \tau} \sin(\omega_j \tau + \phi_{j,\text{lag}})
\]

\[
+ \sum_{\delta_i=0}^{\delta_i=\tau} \sum_{j=1}^{j} E(A_{F,j}(\tau)|T_i = \text{True}) e^{-\zeta_j \omega_j (\delta_i + \tau)} \sin(\omega_j (\delta_i + \tau))
\]

\[
+ E(\varepsilon_i|T_i = \text{True}) \quad (7.30)
\]

An equivalent variance of the RDS is therefore given by Equation 7.31.

\[
Var(\hat{D}_{XT}(\tau)) = \frac{\sigma^2_{A_{R,j}|T_i = \text{True}}}{N_{rd}} + \frac{\sigma^2_{A_{F,j}|T_i = \text{True}}}{N_{rd}} + \frac{\sigma^2_{\varepsilon(\tau)}}{N_{rd}} \quad (7.31)
\]

Equation 7.30 can be split into two constituent parts. The first of these, that part of the RDS, \(\hat{D}_{XT,0}(\tau)\), which is related to the free-response of the system and can be directly modelled as a finite sum of damped sinusoids, is given in Equation 7.32.

\[
\hat{D}_{XT,0}(\tau) = \sum_{j=1}^{j} E(A_{R,j}|T_i = \text{True}) e^{-\zeta_j \omega_j \tau} \sin(\omega_j \tau + \phi_{j,\text{lag}}) \quad (7.32)
\]

The second part of the RDS, \(\hat{D}_{XT,n}(\tau)\), which is associated with the time-lagged forced response of the system and noise within the measurements is given in Equation 7.33.

\[
\hat{D}_{XT,n}(\tau) = \sum_{\delta_i=0}^{\delta_i=\tau} \sum_{j=1}^{j} E(A_{F,j}(\tau)|T_i = \text{True}) e^{-\zeta_j \omega_j (\delta_i + \tau)} \sin(\omega_j (\delta_i + \tau))
\]

\[
+ E(\varepsilon_i|T_i = \text{True}) \quad (7.33)
\]

\(\hat{D}_{XT,n}(\tau)\) contributes nothing to the understanding of the free-response of the system and cannot be directly modelled as finite sum of damped sinusoids.

The objective when selecting an appropriate triggering condition is to maxi-
mize the ratio of $\hat{D}_{XT,0}(\tau)$ to $\hat{D}_{XT,n}(\tau)$ for the time lag of interest, $\tau$. However, a key issue faced in modal analysis through the RDT is that at present there are no methods for estimating the errors in the modal estimates introduced by a non-zero value of $\hat{D}_{XT,n}(\tau)$, or for estimating the magnitude of $\hat{D}_{XT,n}(\tau)$. The modal analysis represents a highly non-linear transformation from the full RDS, $\hat{D}_{XT}(\tau)$, to an approximation of the RDS as a sum of damped sinusoids $\tilde{D}_{XT}(\tau)$. If $\hat{D}_{XT,n}(\tau)$ tends to zero as the number of segments included within the RDS increases, as would occur for white noise forcing and measurement noise when the RDS is formed through a range-crossing triggering condition [259], the solution is to maximize the number of segments included within the RDS such that $\hat{D}_{XT,n}(\tau) \ll \hat{D}_{XT,0}(\tau)$ for all $\tau$.

Note that the issues posed by non-independence or correlated errors within the segments, highlighted by Asmussen et al. [257], is encapsulated within the component of the RDS given by $\hat{D}_{XT,n}(\tau)$. The correlation of errors due to overlapping segments, is purely a function of the correlation of the forced and noise components of the segments with the triggering condition.

### 7.1.2.6 Definition of the RDS for non-linear dynamic systems

As discussed in earlier chapters, real-world structures are very rarely linear dynamic systems, instead exhibiting a wide range of long- and short-term weak non-linear modal behaviour. To expand the definition of the RDS to include this description of non-linearity the subscript $\gamma$ is introduced to the modal parameters, where $\gamma$ refers to the modal parameters of the system at a specific point in time. The equation for an RDS including non-linear behaviour is given in Equation 7.34.

$$
\hat{D}_{XT}(\tau) = \sum_{j=1}^{\delta_\tau=\tau} E(A_{R,j,\gamma}|T_i = \text{True}) e^{-\zeta_{j,\gamma,\omega_j,\gamma}\tau} \sin(\omega_{j,\gamma,\tau} + \phi_{j,\text{lag},\gamma}) \\
+ \sum_{\delta_t=0}^{\delta_t=\tau} \sum_{j=1}^{\delta_t} E(A_{F,j,\gamma+\delta_t}(\tau)|T_i = \text{True}) e^{-\zeta_{j,\gamma+\delta_t,\omega_j,\gamma+\delta_t(\delta_t+\tau)} \sin(\omega_{j,\gamma+\delta_t(\delta_t+\tau)})} \\
+ E(\varepsilon_t|T_i = \text{True})
$$

(7.34)

As might be expected, the introduction of non-linear modal behaviour into the model of the structural system greatly complicates the definition of the RDS. Considering the part of the RDS which could previously be exactly described as a finite sum of damped sinusoids, $\hat{D}_{XT,0}(\tau)$, if the modal parameters are now
non-constant the likely distribution of these modal parameters will increase as the
number of segments included within the RDS increases. Therefore to accurately
estimate the modal parameters from the RDS it is necessary to minimize the
number of segments included within it. This directly contradicts the need to
increase the number of segments such that \( \hat{D}_{XT,n}(\tau) \ll \hat{D}_{XT,0}(\tau) \).

The study of non-linear dynamical systems requires a careful balance to be
struck between increasing the number of segments within the RDS so as to mini-
mize \( \hat{D}_{XT,n}(\tau) \), and minimizing the range of weak non-linear behaviour exhibited
within \( \hat{D}_{XT,0}(\tau) \) so as to accurately approximate the dynamic behaviour.

7.1.2.7 RDS summary

To summarize the key points from the previous subsections:

- Each segment of vibration data \( X \) is made up of four components: i) the
  response due to the initial amplitude at the start of the segment, \( x_0 \), ii) the
  response due to the initial first derivative of the response at the start of the
  segment, \( x_d \), iii) the response due to the forcing applied to the system after
  that start of the segment, \( x_F \), and iv) the noise within the data, \( x_n \).

- The RDS is the expected value of the vibration response at a time lag \( \tau \)
given the trigger condition \( T \), \( \hat{D}_{XT}(\tau) = E(X_{t+\tau}|T_{t+\tau}) \).

- The components of the RDS comprising of the expected values of the ini-
tial amplitude and first derivative responses of the segments \( x_0 \) and \( x_d \),
\( \hat{D}_{XT,0}(\tau) \), for a linear dynamic system can be exactly modelled as a finite
sum of damped sinusoids through the modal analysis.

- The components of the RDS comprising the expected values due to the
  forcing applied after the start of the segment and the noise in the data \( x_F \)
and \( x_n \), \( \hat{D}_{XT,n}(\tau) \), cannot be exactly modelled as a finite sum of damped
sinusoids.

- The components of the RDS comprising of the expected values of the ini-
tial amplitude and first derivative responses of the segments \( x_0 \) and \( x_d \),
\( \hat{D}_{XT,0}(\tau) \), for a non-linear dynamic system cannot be exactly modelled as a
finite sum of damped sinusoids as the modal parameters of the system are
changing.

- A triggering condition should be chosen so as to maximize the amplitude of
\( \hat{D}_{XT,0}(\tau) \) relative to \( \hat{D}_{XT,n}(\tau) \). For a linear dynamic system excited through
white-noise forcing and with white-noise corruption of the data by additive
noise this can be achieved by maximizing the number of segments included
within the RDS.

- If the structures response is non-linear the number of segments included
  within the RDS should be minimized so as to reduce the variation in the
  modal parameters.

### 7.1.3 Graphical description of correlated errors within RDSs

In this section a graphical description of the different types of correlated errors
which might be observed in RDSs for a linear-dynamic system is presented. These
errors are broadly categorized into two intersecting categories:

- Errors associated with a non-zero noise component of the RDS.
- Errors associated with a non-zero forcing component of the RDS.

Correlated errors can only be avoided if both the forcing and noise components of
the RDS are completely independent of the triggering condition. To demonstrate
the pervasiveness of correlated errors, the vector random decrement technique,
collecting simultaneous segments from multiple analysis data sets, is used along-
side 30-minutes of numerical data, sampled at 100Hz, generated for a single-degree
of freedom oscillator. The system has a natural frequency of 10 Hz and a damping
ratio of 2% of critical damping.

The vector RDT is applied using a range-crossing triggering condition with segments of data collected where the noise-corrupted acceleration data falls in the range of zero to $\infty$. At time steps where the triggering condition is met, segments of data are collected from the acceleration data, the data used to apply forcing to the SDOF oscillator (the force data), and the data used to corrupt the acceleration data (the noise data). These segments are used to form an acceleration RDS, a force RDS, and a noise RDS, using Equation 7.4.

Presented in Figure 7.2 are the vector RDSs generated when the system is
excited by a zero-mean Gaussian stochastic (white-noise) process, with the data
corrupted with additive white-noise prior to analysis with the vector RDT. In the
top row of subplots, the individual vector RDSs are presented, with the Welch
power spectrum density (PSD) of the data used to form the RDS (black) and the
RDS itself (red) presented in the lower row of subplots.
Figure 7.2: Comparison of RDSs generated from acceleration data, applied force, and additive noise for a single degree of freedom oscillator excited by white noise forcing, and corrupted with additive white noise.

It can be seen from the reduction of the PSD of the force RDS, that as the force is largely uncorrelated with the triggering condition the force-component of the RDS is approaching zero. However, the use of the corrupted acceleration data as a triggering condition has still introduced a correlation. This correlation in the force RDS will also appear as a correlation within the forced components of the acceleration RDS, which was shown by Chauhan et al. [416] to introduce bias into damping estimation. Segments of force data are collected when the amplitude of the noise-corrupted acceleration data is positive. Where the acceleration data is positive is weakly correlated to where a positive force was applied previously. If the acceleration signal has a positive first derivative, there is a higher likelihood of this applied forcing resulting in the triggering condition being met at a later time step than if the first derivative of the signal is negative, that is the system is approaching rest. This marginal likelihood causes the peak in the PSD of the force RDS at the same frequency as the natural frequency of the SDOF system, and corresponding small amplitude oscillations observed in the force RDS.

This marginal likelihood depending on the sign of the first derivative of the
system is also present in the noise RDS. However a more significant component of the noise RDS which can also be visually identified in the acceleration RDS, is the large positive amplitude of the noise RDS at time-lag zero, and the spike in the amplitude at time-lag zero in the acceleration RDS. As discussed at length in [Chapter 6](#), this is the autocorrelation of the noise within the data.

In [Figure 7.3](#), the system is excited with white-noise forcing but the noise used to corrupt the acceleration measurements is now non-white, having a significant low-frequency component which is commonly observed in many micro-electromechanical sensors (MEMS) and piezoelectric sensors [407]. This increased low-frequency component in the signal noise results in increased correlation between the triggering condition and the noise, at the expense of a reduction in the correlation between the systems uncorrupted acceleration response and the triggering conditions, resulting in the acceleration RDS having a significant low frequency component corresponding to the autocorrelation function for the non-white noise.

![Comparison of RDSs generated from acceleration data, applied force, and additive noise for a single degree of freedom oscillator excited by white noise forcing, and corrupted with additive white noise.](image)

**Figure 7.3:** Comparison of RDSs generated from acceleration data, applied force, and additive noise for a single degree of freedom oscillator excited by white noise forcing, and corrupted with additive white noise.

In [Figure 7.4](#), the RDSs are presented for where the forcing of the system
is non-white, having a broad frequency component centered on 2.98 Hz. As in Figure 7.2, white noise is used to corrupt the acceleration response prior to analysis. As the acceleration response is correlated with the forcing which has a significant frequency component, and the corrupted acceleration response is used as a triggering condition, there exists significant correlation between the triggering condition and the force applied to the system, as shown by the significant frequency content of the force RDS. This translates to correlation of the forced components within the acceleration RDS, making the acceleration RDS a mixture of the free- and forced-response of the SDOF system. A real-world example of non-white forcing is discussed in Section 7.7.

![Figure 7.4: Comparison of RDSs generated from acceleration data, applied force, and additive noise for a single degree of freedom oscillator excited by non-white noise forcing, and corrupted with additive white noise.](image)

7.1.4 Relationship between modal parameters and the RDS

Most applications of the RDT use the RDS to estimate the modal parameters of the system through techniques such as the matrix pencil method [254], the
Ibrahim time domain method [253], or least-squares curve fitting of the RDS [17]. This is here represented as a non-linear transformation of the RDS, \( \hat{D}_{XT} \), given in Equation 7.35 to the estimated modal parameters, summarized through the variable \( \theta = [\omega, \zeta, A, \phi_{lag}] \).

\[
\theta = f \left( \hat{D}_{XT} \right) \quad (7.35)
\]

If the components of the RDS which can be directly modelled as a finite sum of damped sinusoids \( \hat{D}_{XT,0} \) was equal to a non-zero value, and the components of the RDS \( \hat{D}_{XT,n} \) associated with the time-lagged forced response of the system and noise within the measurements was equal to zero, the modal parameters will be exactly equal to the true modal parameters of the system \( \theta_0 \) for a linear-dynamic system, as described by Equation 7.36.

\[
\theta_0 = \theta = f \left( \hat{D}_{XT,0} \right) \quad (7.36)
\]

Note that \( \theta_0 \) is unlikely to include all the modal parameters for the system, only those with non-zero amplitude within the RDS. However, if \( \hat{D}_{XT,n} \) is not equal to zero, the true modal parameters are corrupted. This manifests in \( \theta \) as both noise modes, \( \theta_n \), spurious sets of modal estimates associated with fitting of \( \hat{D}_{XT,n} \) within the modal analysis, and corruption of the modal estimates, \( \theta_\varepsilon \), due to variation in the fitting of \( \hat{D}_{XT,0} \) due to the presence of \( \hat{D}_{XT,n} \). Therefore \( \theta \) is fully described by Equation 7.37.

\[
\theta = [(\theta_0 + \theta_\varepsilon), \theta_n] \quad (7.37)
\]

For a non-linear system, the modal parameters of the dynamic system in the RDS are a weighted average of the responses or the expected value of the modal parameters given that the triggering condition is true, as described by Equation 7.38.

\[
\theta = [(E(\theta_0|T = \text{True}) + \theta_\varepsilon), \theta_n] \quad (7.38)
\]

The objective of the following sections is to present subsampling and the ST-RDT as methods for quantifying the uncertainty in the modal parameters \( \theta_\varepsilon \) in the presence of weak non-linear modal behaviour and for identifying noise modes, \( \theta_n \) within the data.
7.2 Estimating the uncertainty and weak non-linearity of OMA modal parameters

To accurately quantify the non-linear dynamic behaviour of civil structures it is necessary to:

i. Understand the relationship between the expected value of the modal parameters and the triggering condition, \( E(\theta_0 | T = \text{True}) \).

ii. Understand the distribution of the errors in the modal estimates, \( \theta_e \).

iii. Distinguish modal parameters of the system \( \theta_0 \) from noise modes \( \theta_n \).

To understand the true distribution of \( \theta \) in the context of estimating the modal parameters of civil structures the process must be divided into two parts. The vibration data collected from the structure is first analysed with the short-time random decrement technique, a novel OMA method described in Section 7.3 which incorporates subsampling theory to estimate the distribution of modal parameters exhibited by the structure for a fixed window length. The results from the ST-RDT must then be interpreted in the context of subsampling theory to understand the relationship between the distribution of estimated parameters, \( \theta \), the window length \( L_w \), and the true distribution of modal parameters \( \theta_0 \), as discussed in Section 7.4.

7.3 The Short-time Random Decrement Technique (ST-RDT)

The fundamental objective of modal analysis is to characterise the dynamic behaviour of a system through its modal parameters; the natural frequencies, damping ratios and mode shapes. For an idealized linear system with finite modes of vibration, the full range of dynamic behaviour could be fully explained with a finite set of modal parameters. However, real systems are not linear. At longer time scales the dynamic behaviour of a system is dependant on the condition of the system; any degradation of materials or damage which may have occurred. One level down from this are the environmentally induced sources of non-linearity such as thermally induced changes in stiffness or changes in mass of a structure. At the most extreme end of the spectrum are the near-instantaneous change in dynamic behaviour caused by hysteresis or amplitude-dependant behaviour.
Therefore it is no longer possible to describe the dynamic behaviour of a system with a finite set of discrete modal parameters. Instead the modes of a dynamical system must be described by distributions of possible values and models of how the dynamic behaviour changes in response to other factors.

The short-time random decrement technique (ST-RDT) seeks to build approximations of the modal parameters of the system using subsampling theory through dividing the data into short overlapping windows or subsamples, each of which are analysed independently using a variation on the RDT. The theoretical basis of subsampling theory is that any given subsample of the data is generated using the same probability mechanism as that which describes the full population. Similarly, the theoretical basis of the ST-RDT is that the modal parameters estimated for a short window of acceleration data are directly related to the modal parameters which could be used to describe an infinite sample of acceleration data from the system. Similarly, the errors observed in the modal parameter estimates for a short window of data are directly related to the distribution of errors in the modal parameter estimates which would occur if an infinite sample of acceleration data was observed. In Section 7.3.1 the application of the ST-RDT is demonstrated using real-world acceleration data, with the process for generating mode shapes using modal estimates from the ST-RDT presented in Section 7.3.2. The selection of appropriate parameters for ST-RDT analysis are discussed in Section 7.3.3.

### 7.3.1 Application of ST-RDT

An overview of the steps for application of the ST-RDT are presented in Figure 7.5 and described in greater detail in Section 7.3.1.1 to 7.3.1.7.
In this section, the application of the ST-RDT to achieve optimal modal parameter estimates is demonstrated using acceleration data collected from the MX3D Bridge, discussed in detail in Section 7.8.
7.3.1.1 Data collection for ST-RDT

In order to accurately identify modal parameters from the vibration response of structures in-service, careful attention must be paid to the location of accelerometers and the sample rate selected.

If there is prior knowledge of the mode shapes of the structure, either from finite element modelling or previous testing, accelerometers should be located so as to maximize the normalized amplitude of as many mode shapes as possible. If an accelerometer is located at close to the node point (point of zero amplitude) of a mode shape, the signal-to-noise ratio for that mode will be lower and detection of it within the acceleration data will be compromised. Where the location of multiple accelerometers must be selected, a balance should be sought between: detecting the maximum number of modes across the structure, avoiding the detection of localized modes which do not significantly contribute to the dynamic behaviour of the structure as a whole, and maximizing the normalized amplitude of the measured modes. For structures which are likely to have close modes, such as those with strong structural symmetry, accelerometers should be orientated so as to maximize the relative difference between the directions of oscillation in order to allow greater separation of the modal parameters in later steps.

The sample rate should be selected to be a minimum of three times greater than the highest frequency of interest. While it is possible to detect modes of vibration up to the Nyquist frequency (half the sampling rate), the frequency roll-off induced by filtering prior to the analogue-to-digital conversion, alongside any weak non-linear modal behaviour which may cause significant variation in the natural frequencies, may result in corruption in modal estimates for modes above the one third sampling frequency threshold [15].

The acceleration data collected from three accelerometer channels fixed to the underside of the MX3D Bridge deck, discussed in detail in Section 7.8, is presented in Figure 7.6 and used to demonstrate the application of the ST-RDT within this section. For consistency with later notation, these data sets are designated $X$, $Y$, and $Z$. All data was sampled at 100 Hz, with no filtering applied to the data after the analogue-to-digital conversion.
Figure 7.6: Three channels of accelerometer data for the MX3D Bridge used to demonstrate the ST-RDT. All data sampled at 100 Hz. The data consists of walked crossings of the bridge by an individual pedestrian, interspersed with the pedestrian conducting heel-drops at a variety of locations on the bridge deck. Heel-drops appear as the peak acceleration responses within the plotted data. For further information on the pedestrian excitation present in the data set including the crossing routes and heel-drop locations refer to Appendix F.

7.3.1.2 Division of data into overlapping windows

The data from all channels is divided into overlapping windows of length $L_w$, with an overlap of $L_b$ between windows, with the condition that the overlap length must be less than the window length, $L_b < L_w$, and that the window length must be less than the total length of the data set ($N_{total}$), $L_w < N_{total}$. The window length must be long enough to contain multiple oscillations of the lowest frequency mode within the data, but short enough so as to prevent any weak non-linear modal behaviour which might be present within the data from being obscured. A greater overlap length results in a higher temporal resolution of modal estimates, but has an associated computational and storage cost. These trade-offs in the selection of window and overlap length are discussed in detail in Section 7.4 and may require iteration for selection of appropriate values.

For demonstration of the ST-RDT, a window length of three-minutes, with
an overlap length of two-minutes, is used. The first six overlapping windows are highlighted in Figure 7.7.

Figure 7.7: Example of subdivision of data into overlapping windows. Start and end points of the first six overlapping windows, with a window length of three-minutes and overlap length of two-minutes, highlighted.

7.3.1.3 Formation of random decrement signatures

Within a window of data, an adapted version of the vector random decrement technique [257, 258, 259] is used to form a random decrement signature (RDS), an ensemble average of segments of data where a user-specified triggering condition is met. To apply the vector RDT, a triggering condition is first defined. There are a wide range of different triggering conditions which have been proposed, a selection of which are discussed in Section 7.3.3.2. Each of the three channels of acceleration data \(X, Y, Z\) are used as both the triggering channel, the channel of data which is used to define where the triggering condition is met, and the analysis channel, the channel of data from which the modal parameters are to be estimated. Note that a single channel of data can be used as both the analysis and triggering channel, as shown in Table 7.1.
Table 7.1: Combinations of analysis channels and trigger channels for application of the vector RDT with three unique data sets \([X, Y, Z]\). In each letter pair in the table, the first letter refers to the analysis channel and the second letter refers to the trigger channel.

<table>
<thead>
<tr>
<th>Trigger channel</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis channel</td>
<td>X</td>
<td>XX</td>
<td>XY</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>YY</td>
<td>YX</td>
</tr>
<tr>
<td></td>
<td>Z</td>
<td>ZX</td>
<td>ZY</td>
</tr>
</tbody>
</table>

Where a trigger condition \(T_t\) is met within the triggering channel, a segment of data of length \(L_{rd}\) is collected from the analysis channel of data.

Unlike the original implementation of the vector RDT \([257, 258, 259]\), a weighting channel is introduced into the ST-RDT. The weighting channel is used to apply a weighting factor \(W_t\) to the segments of data where the triggering condition is met within the trigger channel. The weighting factor increases the flexibility of the RDT through allowing segments of data which may be of greater interest, such as those that better fit a non-binary triggering condition, to be given greater weight within the RDS. The weighting factor is the value of the weighting channel when the trigger condition is met within the triggering channel. As discussed in Section 7.3.3.2, when used with an all-points triggering condition (collecting a segment of data starting at every point within the data) a weighting factor equal to the amplitude of the trigger-channel allows the RDS to be exactly equal to the cross-correlation of the trigger and analysis channels of data, unifying the definition of the RDT with classical correlation theory.

The RDS \(\hat{D}_{A,T,W}\) for each analysis \(A_x\), trigger \(T_x\) and weighting \(W_x\) set of channels of length \(N_{total}\) time steps is generated through a weighted ensemble average of all segments of data of length \(L_{rd}\) where the triggering condition \(T_t = \text{True}\) using the generalized formula given in Equation 7.39.

\[
\hat{D}_{A,T,W} = \frac{1}{\sum_{t=1}^{N_{total}} W_{x,t} \left| A_{x,t} \right| T_{x,t} = \text{True}} \sum_{t=1}^{N_{total}} W_{x,t} \left| A_{x,t} \right| T_{x,t} = \text{True} \quad (7.39)
\]

For demonstrating the ST-RDT in this section, a range-crossing triggering condition of where the absolute amplitude of the acceleration falls within the range of \(0.1 \text{ m/s}^2 \pm 0.05 \text{ m/s}^2\) is used alongside a weighting condition of the amplitude of the trigger channel. An RDS length \(L_{rd}\) of two seconds is used. As an example, the formula for the RDS for the use of data from channel \(X\) as the analysis channel, and data from channel \(Y\) as the trigger and weighting channels.
\( \hat{D}_{XY} \) is presented in Equation 7.40 below alongside the corresponding trigger condition given in Equation 7.41.

\[
\hat{D}_{XY} = \frac{1}{\sum_{t=1}^{N_{total}} (|Y_t| \quad T_t = \text{True})} \sum_{t=1}^{N_{total}} Y_t \left( X_{t,t+L_{rd}} \quad T_t = \text{True} \right)
\]

(7.40)

\[
T_t = \begin{cases} 
\text{True}, & \text{if } 0.05m/s^2 < |Y_t| < 0.15m/s^2. \\
\text{False}, & \text{otherwise.} 
\end{cases}
\]

(7.41)

The RDSs for the first three-minute window of the example data presented previously in Figure 7.7 for each of the nine unique pairs of trigger and analysis channels described in Table 7.1 are presented in Figure 7.8.

Figure 7.8: Example of RDSs (\( \hat{D} \)) generated with the triggering condition given in Equation 7.41 for the first three-minute window of data presented previously in Figure 7.7. RDSs generated using the generalized equation given in Equation 7.39. In the notation for the RDSs, the first letter corresponds to the analysis channel of data, and the second letter corresponds to the trigger channel of data.
7.3.1.4 Trimming of RDSs

As discussed at length in Chapter 6, the RDSs must be trimmed to remove the correlation function of the noise or forcing within the data. Guidance on identifying the section of the RDS to be trimmed is given in Chapter 6 and summarized in Section 7.3.3.3.

For the example data, the first two time steps of each RDS ($\tau = 0 \text{s}$ to $\tau = 0.02 \text{s}$) is excluded from the modal analysis based on the correlation functions for a section of the MX3D acceleration data containing only noise, as highlighted in Figure 7.9.

Figure 7.9: Trimming of RDSs ($\hat{D}$) to excluded part of RDSs corrupted with the correlation of noise within the data. RDSs between $\tau = 0 \text{s}$ and $\tau = 0.02 \text{s}$ (highlighted in red) excluded from modal analysis.

7.3.1.5 Estimating modal parameters from RDSs

To estimate the modal parameters from each RDS, the matrix mencil method (MPM) [238, 417, 278, 251, 279], described in Chapter 2, is used.

Each ST-RDT RDS is analysed separately in order to avoid the corruption of modal parameters which may be induced by individual RDSs which are heavily...
corrupted by noise or non-structural oscillations and to aid the separation of close modes of vibration.

When applying the MPM, \( L_{rd} \) is the length of the RDS and the time array \( t \) must be adjusted to account for trimming of the RDS. For example, in the case of the example trimmed RDSs shown in Figure 7.9, the time array would start at \( t = 0.02 \) seconds.

In the ST-RDT, the value of \( M_k \), the number of fitted damped sinusoids, is incremented by one from \( M_k = 1 \) until the explained variance reaches a user specified target, as discussed in detail in Chapter 5 and summarized in Section 7.3.3.4. The proportion of explained variance is quantified through the coefficient of determination, \( R^2 \), given by Equation 7.42 in which \( \hat{D}_{M_k} \) is the order \( M_k \) approximation of the RDS.

\[
R^2 = 1 - \frac{(\hat{D} - \hat{D}_{M_k})^2}{\hat{D} - \frac{1}{n} \sum_{i=1}^{n} D_i} \quad (7.42)
\]

For the example data, the target explained variance is set at \( R^2 = 0.999 \). The fitted RDSs obtained through the MPM analysis of each individual RDS are presented in Figure 7.10.
Figure 7.10: Comparison of measured trimmed $\hat{D}$ and matrix pencil method (MPM) fitted RDSs ($\tilde{D}$) for the first three-minute window of data presented previously in Figure 7.7. RDSs fitted to achieve a target explained variance of $R^2 = 0.999$.

The residuals: the difference between the measured RDS, $\hat{D}$, and the MPM fitted RDS to achieve the target explained variance or $R^2_{\text{targ}}$, $\tilde{D}$, are presented in Figure 7.11. The oscillations within the residuals with an initial amplitude of zero suggest that the component of the RDS associated with forcing after time lag $\tau = 0$ and correlation of the noise with the triggering condition are the likely dominant sources of errors within the RDS approximation of the systems weighted free-response.
7.3.1.6 Repetition of ST-RDT analysis for new windows of data

The steps laid out in Section 7.3.1.3 to 7.3.1.5 are then repeated for the subsequent windows of data, with the estimated modal parameters which achieve the target explained variance stored for further analysis.

A comparison of the RDSs of the example data for where channel $X$ is used as the analysis channel, and channel $Y$ is used as the triggering and weighting channels, for the first six windows of data is presented in Figure 7.12. It can be seen that as the window of data is incremented the relative magnitude of different modal components changes. The small variation in the RDS between window three ($\hat{D}_{XY,3}$) and window six ($\hat{D}_{XY,6}$) visible at close to 1.8 seconds in Figure 7.12 suggests a potential shift in frequency of the low frequency component of the mode, or shifts in the relative magnitude of close modes of vibration, as supported by the power spectral density plots of the RDSs presented in Figure 7.13.
Figure 7.12: Comparison of trimmed RDSs $\hat{D}_{XY}$ for where channel $X$ is used as the analysis channel, and channel $Y$ is used as the triggering and weighting channels. The numerical subscript refers to the window of data from which the RDS was generated.

Figure 7.13: Comparison of power spectral densities of trimmed RDSs $\hat{D}_{XY}$ for where channel $X$ is used as the analysis channel, and channel $Y$ is used as the triggering and weighting channels. The numerical subscript refers to the window of data from which the RDS was generated.

7.3.1.7 Plotting and interpreting modal estimates from the ST-RDT

As introduced earlier in this chapter, the ST-RDT allows for the variability and uncertainty within modal estimates to be quantified. This section includes several examples of ways in which the ST-RDT estimates may be plotted to identify modes of vibration, detect indications of weak non-linear modal behaviour, and understand the uncertainty or volatility of modal estimates from multiple time steps. More detailed methods for automatic identification of modes of vibration within the ST-RDT modal estimates, and quantification of weak non-linear modal behaviour, are provided in Chapters 8 and 9.
The ST-RDT frequency estimates for the three channels of example data across the 90 minutes of data presented in Figure 7.6 are presented as a histogram in Figure 7.14. It can be seen that the results are dominated by a peak in the results at close to 5 Hz, indicating one or more modes of vibration. The key issue with the histogram of frequency estimates is that it is difficult to distinguish close modes of vibration, and cannot aid with identification of weak non-linear modal behaviour.

![Figure 7.14: Histogram of ST-RDT frequency estimates for example data plotted previously in Figure 7.6. Histogram bin width of 0.02 Hz.](image)

One method for distinguishing close modes of vibration within ST-RDT results is through consideration of the 2D histogram of frequency versus damping, as shown for the example ST-RDT modal estimates in Figure 7.15. In the case of the MX3D Bridge, all modes exhibit typical damping values of between 0.5% and 2%, preventing visual separation of the modes of vibration.

![Figure 7.15: 2D Histogram of ST-RDT frequency and damping estimates for example data plotted previously in Figure 7.6. Histogram bin dimensions of 0.04 Hz by 0.3% of critical damping. Darker colors show regions with high densities of modal estimates.](image)

The results can also be viewed in the context of the windows of time used to generate the RDS, as shown in Figure 7.16. This gives indications of when
particular modes dominate the results and can allow the identification of close modes of vibration and weak non-linear modal behaviour.

Figure 7.16: 2D Histogram of ST-RDT frequency estimates versus time (lower subplot) for example data plotted previously in Figure 7.6, one channel of which is shown in the upper subplot. Histogram bin dimensions of 30 seconds by 0.04 Hz. Darker colors show regions with high densities of frequency estimates. Frequency estimates plotted relative to the centre of the time window from which the RDS was calculated.

Considering the mode at close to 5.0 Hz which dominates the ST-RDT results for the example data, plotted in greater detail in Figure 7.17, an increase in the variability of the modal estimates during periods of lower excitation of the bridge is observed, likely due to a lower signal-to-noise ratio. Significant shifts in the frequency over the course of the data set appear to be correlated with high amplitude oscillations are also observed, as well as longer-term shifts in frequency likely to be associated with changes in mass on the bridge, and the temperature of the bridge, discussed in greater detail in Chapter 9.
For the mode close to 11.1 Hz, plotted in Figure 7.18, which may have incorrectly been identified as a pair of close modes in conventional RDT analysis due to the substantial frequency shift which occurs, similar patterns of long-term variation in the natural frequency are observed. The mode exhibits less pronounced variation in the uncertainty of the frequency estimates, as given by the spread of frequency estimates at a single time step corresponding to differing results from the different trigger and analysis channels, than were observed for the mode close to 5.0 Hz.
The most extreme indicators of weak non-linear modal behaviour are observed in the results for the mode of vibration close to 6.8 Hz, shown in Figure 7.19, where the frequency of the mode appears to bifurcate during periods of high excitation. This may indicate that the frequency of this mode is dependant on the amplitude of the oscillation, a potential signifier of amplitude dependant modal stiffness, or may indicate the presence of close modes of vibration which are only excited during high amplitude events. Again, through conventional OMA the behaviour of this mode may have been incorrectly classified as a set of distinct close modes of vibration, rather than a single mode exhibiting pronounced weak non-linear behaviour.
7.3.2 Mode shape generation through ST-RDT

An advantage of the ST-RDT over conventional OMA methods is that it allows mode shapes to be generated for multiple time steps and then combined into a more representative mode shape whilst quantifying the uncertainty in the mode shape estimate.

The generation of mode shapes using the ST-RDT is demonstrated using the data from the vertical channels of data for the seven accelerometers fixed to the underside of the MX3D Bridge, discussed in detail in Section 7.8. The data is analysed using the same parameters and selection of triggering and weighting conditions as were used in the example application of the ST-RDT presented in Section 7.3.1.

7.3.2.1 Identifying a mode of vibration

The first step for generating mode shapes is identifying and isolating a single mode of vibration within the ST-RDT modal estimates. In the case of well-separated modes this can be achieved by identifying an upper and lower frequency bound
in which the mode’s natural frequency varies. For close modes, or modes which have natural frequencies which vary significantly over the course of the data set, more complex methods, such as those discussed in Chapter 8, are required.

The mode used to demonstrate the generation of mode shapes is identified as falling between 10.8 Hz and 11.4 Hz, the frequency estimates for which are plotted in Figure 7.20. The lack of multiple peaks within the histograms of frequency estimate for a single window of data, translating to a single line in the 2D histogram of frequency versus time, strongly implies that this is a single mode of vibration.

![Figure 7.20: Frequency estimates for a mode of vibration between 10.8 Hz and 11.4 Hz. Modal estimates obtained through the ST-RDT analysis of seven channels of vertical accelerometer data of the MX3D Bridge, discussed in detail in Section 7.8, using the ST-RDT parameters discussed in Section 7.3.1.](image)

7.3.2.2 Select mode shape trigger channel

As in the vector RDT [258], mode shapes are generated using a single accelerometer as a consistent trigger channel for the ST-RDT. The combining of mode shapes for multiple trigger channels is discussed in Section 7.3.2.5.
For the example data, channel one of seven is selected as the trigger channel for the initial mode shape generation.

### 7.3.2.3 Combining amplitude and phase information

To simplify later steps, the amplitude of each modal estimate, a scalar value giving the amplitude of the oscillation, is combined with the phase direction, a binary value indicating whether the oscillation is in or out of phase with the trigger channel value. Where the absolute value of a phase angle for the modal estimates is greater than \( \pi/2 \) is designated as out of phase, and assigned a phase direction of -1. Where the absolute value of the phase angle is less than \( \pi/2 \), the phase direction is +1. The assignment of phase angles is illustrated in Figure 7.21.

![Figure 7.21: Phase angle direction classification for all ST-RDT modal estimates generated with channel 1 as the triggering channel, and with frequencies falling between 10.8 Hz and 11.4 Hz.](image)

Where the phase direction is negative, the modal amplitude is multiplied by minus one, combining the phase and amplitude information within a single variable referred to as the modal vector amplitude.

The binary phase direction is used in the generation of the mode shapes, rather than the phase value itself, to account for any biasing within the phase angles of modal estimates associated with noise or forced components of the RDS. While the amplitude of such spurious components is likely to be small, biasing within the phase angles can lead to mode shapes in locations on the structure which are phase lagged from their true oscillation. This becomes especially apparent when the modal estimates from many separate time steps, each of which include spurious modal estimates with a consistently biased small non-zero or non-\( \pi \) phase angle are used in the generation of mode shapes.
7.3.2.4 Generating mode shapes distributions

Rather than generating a single mode shape for the mode of vibration, mode shape distributions are used, such as those plotted in Figure 7.22 for the example acceleration data with accelerometer 1 as the trigger channel. Mode shape histograms combine all variability in the ST-RDT modal vector amplitudes, such as may be caused by physical changes to the dynamic system, or the presence of spurious modal estimates within the ST-RDT.

Figure 7.22: Distribution of modal vector amplitudes for the seven accelerometers in the data set for the mode identified as having frequencies falling in the range 10.8 Hz to 11.4 Hz. Accelerometer one used as triggering channel.

Where the mode shapes are to be used for further analysis, such as the updating of finite element models, or estimation of a structure’s vibration response, the distributions can be used to generate independent confidence intervals, for generation of upper- and lower-bound mode shapes, as plotted in Figure 7.23. These confidence interval mode shapes do not include the information of the correlation between the different channels, only the amplitude given through generating confidence intervals of the distributions of values given in Figure 7.22. However, these could easily be adapted to maintain this correlated information, for example by generating a distribution of the modal vector amplitudes from all accelerometers.
eters where the trigger accelerometer amplitude falls outside its calculated 95% confidence interval.

Figure 7.23: Mode shapes generated from modal vector amplitude distributions for the seven accelerometers in the data set for the mode identified as having frequencies falling in the range 10.8 Hz to 11.4 Hz. Accelerometer one used as triggering channel. Mode shapes cubically interpolated between accelerometer locations.

As might be expected, the greatest range of amplitudes is observed where the
amplitude of the median mode shape is greatest, as shown in Figure 7.24.

Figure 7.24: Absolute difference between 2.5% bound and 97.5% bound mode shapes shown in Figure 7.23. Differences cubically interpolated between accelerometer locations.

7.3.2.5 Combining mode shape distributions from multiple trigger channels

To combine the mode shape estimates from multiple different trigger channels a common baseline analysis channel must be selected for scaling of all modal vector amplitude distributions. It is recommended that this channel, referred to as the scaling location, is that which gives the largest sum absolute median modal vector amplitude across the distributions for all trigger channels.

For each set of distributions for a given trigger channel, all distributions are scaled such that the distribution for the scaling location has a new median of unity. The distributions of all estimates, which now share a common baseline of the median modal vector amplitude for the scaling location having an amplitude of unity, can now be combined. This scaling of all values by a scaled baseline acts to scale the distributions, ensuring that modal vector amplitudes which are outliers in the unscaled distributions, remain as outliers in the scaled distributions.

The unscaled modal vector amplitudes distributions for the seven trigger channels (accelerometer locations) are plotted in Figure 7.25 and the scaled distributions are plotted in Figure 7.26 with the combined distributions are plotted in Figure 7.27.
Figure 7.25: Unscaled distributions of modal vectors amplitudes for seven triggering locations for mode with natural frequency falling between 10.8 Hz and 11.4 Hz.
Figure 7.26: Distributions of modal vectors amplitudes for seven triggering locations for mode with natural frequency falling between 10.8 Hz and 11.4 Hz, scaled to give a median modal vector amplitude of unity for accelerometer seven across each individual distribution.
Figure 7.27: Combined distributions of modal vectors amplitudes for seven triggering locations for mode with natural frequency falling between 10.8 Hz and 11.4 Hz, scaled to give a median modal vector amplitude of unity for accelerometer seven across each individual distribution.

The median mode shape generated from the combined distributions of modal vector amplitudes is presented in Figure 7.28. The advantage with the ST-RDT approach to generating mode shapes compared to conventional OMA methods is that as the amount of data increases, the confidence in the generated mode shape also increases. As mode shapes are generated from scaled distributions of modal parameters, mode shape estimates can be updated with any new data that becomes available through appending new modal estimates to the modal vector amplitude distributions.
Figure 7.28: Median mode shape for mode with natural frequency falling between 10.8 Hz and 11.4 Hz, generated from combined distributions of modal vectors amplitudes for seven triggering locations and cubically interpolated between measurement locations.

7.3.3 RDT parameter selection

The selection of the input parameters for the RDT used within the ST-RDT is highly dependant on the characteristics of the system to be analyzed. As the relationship between errors in the RDS and errors in the modal parameter estimates is still poorly understood, it is difficult to derive a theoretical basis for selection of these parameters. Instead, this section presents general guidance on the selection of the RDT parameters based on the analysis of the case studies presented elsewhere in this thesis as well as existing literature on the RDT. The window length and overlap length used within the ST-RDT is addressed separately in Section 7.4.

For all parameters the objective can be summarized as maximizing the power of the free-response component of the RDS, $\hat{D}_{XT,0}$, which can be exactly described as a sum of damped sinusoids, whilst minimizing the power of the non-free response component of the RDS, $\hat{D}_{XT,n}$ comprising the forced and noise components of the RDS.

7.3.3.1 RDS length

Selecting an appropriate length of RDS is a multi-faceted and system specific problem. It is unlikely that a global optimum RDS length exists which results in the most accurate parameter estimates for all modes of vibration. As the length of the RDS, $L_{rd}$, increases the amplitude of free-oscillation component of the RDS, $\hat{D}_{XT,0}$ comprising the $j$ modes of vibration, approaches zero. The rate at which the amplitude of each of the $j$ components approach zero is defined by their
damping envelope \( e^{-\zeta_j \omega_j \tau} \), where \( \zeta_j \) is the damping ratio for mode \( j \) as a fraction of the critical damping, \( \omega_j \) is the angular natural frequency of mode \( j \), and \( \tau \) is the time-lag since the start of the free-oscillation. As the oscillation of mode \( j \) approaches zero, there is a higher ratio of the noise component of the RDS to the amplitude of mode \( j \). Therefore for each additional time-lag considered as part of the RDS length, the information about the oscillation of mode \( j \) which is imparted relative to the amplitude of the noise component, decreases. As might be expected, the rate of this decrease is directly proportional to the ratio of the damping envelope to the variance of the noise component of the RDS. This ratio, here denoted as \( \eta_j(\tau) \) for mode \( j \) may be expressed numerically at time-lag \( \tau \) through combining the damping enveloped component of Equation 7.32 with the expected value of the force and noise components of the RDS, as given by Equation 7.33.

\[
\eta_j(\tau) = \frac{E(A_{R,j}|T_i = \text{True}) e^{-\zeta_j \omega_j \tau}}{\hat{D}_{XT,n}(\tau)} (7.43)
\]

By inspection of Equation 7.43, if the component of the RDS \( \hat{D}_{XT,n}(\tau) \) associated with the time-lagged forced response of the system and noise within the measurements is independent of the trigger condition, the optimal value of the RDS length to maximize the ratio \( \eta_j(\tau) \) is theoretically a single time-lag. However, the accuracy of the modal estimates is also dependant on the number of time lags used within the modal analysis. Observing an oscillation over a longer RDS length increases the number of points available for analysis within the matrix pencil method modal analysis. This must be balanced with the diminishing returns from each additional point contained within the RDS due to the reduction in the amplitudes of oscillation.

A further issue is the unknown number of modes present within the RDS. As the number of modes increases, a longer RDS length is required so that the modes may be separated. As a minimum, least-squares curve fitting approaches would require \( j \) time-lags so as to separate \( j \) modes of vibration, with each additional time-lag allowing for greater separation of noise and free-oscillation components within the RDS [33]. The matrix pencil method, and by extension the Ibrahim time domain method, both of which rely on analysis of time-lagged Hankel matrices, require a minimum RDS length equal to the number of modes of vibration multiplied by the number of time-lags used within the Hankel matrices. For the implementation of the matrix pencil method as was presented in Section 7.3.1.5, a minimum RDS length of \( 2j + 1 \) would be required to distinguish \( j \) modes of vibration. Once again the accuracy of the modal estimates is expected to increase
as the number of time-lags within the RDS increases.

If the behaviour of the system is non-linear, either due to short-term non-linearity such as amplitude dependence, or if the RDS contains modal components with modal parameters which change over the length of the data set, as might be observed due to changes in modal stiffness with temperature, a longer RDS length may allow these behaviours to be approximated as multiple close modes of vibration. However in practice it is beneficial to limit the RDS length so that the fitted modes of vibration are strongly correlated with the triggering condition, allowing for greater separation of non-linearity in future analysis of the ST-RDT results, as discussed further in Chapter 9.

### 7.3.3.2 Trigger and weighting condition

As discussed in previous sections, the selection of the triggering and weighting conditions should be guided by:

i. Maximizing the ratio of the free-response component of the RDS, $\hat{D}_{XT,0}$, to the non-free response component of the RDS, $\hat{D}_{XT,n}$.

ii. Minimizing the range of weak non-linear modal behaviour present within the segments of data used to generate the RDS.

The ratio of $\hat{D}_{XT,0}$ to $\hat{D}_{XN,n}$, when $\hat{D}_{XN,n}$ is a summation of zero-mean white-noise stochastic processes which are independent of the trigger condition, is entirely dependent on the number of segments included within the RDS. As shown by Asmussen et al. [259], the expected value of $\hat{D}_{XN,n}$ at any time lag is zero and its expected variance $\sigma^2_{\hat{D}_{XN,n}}$ can be calculated using the central limit theorem through Equation 7.44:

$$\sigma^2_{\hat{D}_{XN,n}} = \frac{\sigma^2_n}{N_{rd}}$$  \hspace{1cm} (7.44)

In Equation 7.44, $\sigma^2_n$ is the variance of the white noise stochastic process, and $N_{rd}$ is the number of segments contained within the RDS. This provides a simple metric for calculating the number of segments required to achieve a specified expected reduction in $\hat{D}_{XN,n}$. This is demonstrated in Figure 7.29, in which the reduction in the expected variance of the noise and forced components of the RDS is plotted based on an initial expected variance of 100 when the RDS comprises a single segment.
As shown in Section 7.1.3 the assumption of uncorrelated errors is unlikely to be true. Despite the presence of correlated errors in data for real-life systems, the findings presented in Figure 7.29 serve as useful baseline values during calibration of RDT parameters.

If there is prior knowledge of the type of weak non-linear modal behaviour which may be encountered this could be used to guide the selection of the weighting and triggering conditions. For example if it is known that the structure exhibits an amplitude dependency, it may be worth exploring the use of range- or level-crossing triggering as has been explored in depth by Xu et al. [270], Huang and Gu [267], Li et al. [278], Tamura and Suganuma [264], Li et al. [419], as well as the early RDT work by Cole Jr [235], Cole Jr [236] and Chang [251]. In order to ensure the minimum number of segments are met, segments could then be weighted to give greater weight to segments which fulfill pre-specified amplitude and first derivative response criteria. However, in selection of a triggering and weighting condition, great care must be taken to avoid projection of expected behaviour on to the observed behaviour. As demonstrated in Chapter 6 the presence of filtering artefacts within data analysed with the range-crossing RDT may cause biasing of the modal estimates, with this biasing being correlated with the amplitude of the modal response. This correlated biasing may be mistaken for amplitude-dependant behaviour. Similarly, correlation in the forced response may be mistakenly identified as amplitude dependant behaviour, a real-world example of which is given in Section 7.7.

A more robust starting position when selecting the weighting and triggering conditions is to assume no knowledge of the structural behaviour prior to analysis. This can help to avoid the issues of projecting false prior beliefs on to the data. The suggested starting point for all analyses is the use of an all-points
triggering condition, in which a segment of data is collected starting at every time step within the ST-RDT window, with a sign-weighting condition, where each segment is weighted according to the $+ve/-ve$ sign of the amplitude in the triggering channel, or an amplitude-weighting condition, where each segment is weighted using the amplitude of the triggering channel (equivalent to a correlation function). It may be assumed that it is unlikely that any non-linear behaviour is identical across all overlapping windows used within the ST-RDT. Therefore through post-analysis of the ST-RDT modal estimates, the relationship between the weak non-linear modal behaviour and external variables, such as the temperature, wind direction or amplitude of oscillation, may be robustly modelled, as discussed in detail in Chapters 8 and 9, and Sections 7.6 and 7.7.

### 7.3.3.3 Trimming of RDS

Trimming of the RDS is carried out to remove components associated with strong correlations between the noise or force components of the RDS, $\hat{D}_{XT,n}$, and the triggering condition. This is discussed in detail within the context of removing filtering artefacts from RDSs in Chapter 6, as well as within the case studies presented at the end of this chapter.

### 7.3.3.4 Target explained variance

Within the ST-RDT as presented in Section 7.3.1, the model order to be used in the matrix pencil method modal analysis is systematically increased until a target explained variance (TEV) is met. The TEV approach is discussed in detail in Chapter 5 and summarized briefly here. The explained variance between the measured RDS ($\hat{D}$) and the order-$M$ approximation of the RDS obtained through the matrix pencil method ($\hat{D}_M$) is quantified with the coefficient of determination, $R^2$, calculated using Equation 7.45

$$ R^2 = 1 - \frac{(\hat{D} - \hat{D}_M)^2}{\hat{D} - \frac{1}{n} \sum_{i=1}^{n} \hat{D}_i} $$  \hspace{1cm} (7.45)

Selecting an appropriate TEV is an iterative process. If the TEV is too low, modes of vibration with smaller amplitudes of oscillation relative to the amplitude of the RDS may not be fitted. This not only leads to modes not being identified, but may cause biasing of the modal estimates for other modes of vibration. Selecting a very high TEV may lead to the fitting of noise- and forced-response components within the RDS, leading to spurious modal estimates and biasing of the estimated
modal parameters. Despite the complication of the ST-RDT modal estimates which may result from selecting a higher TEV, in many cases the failure to capture and characterize all modes of vibration will dominate the choice of TEV value.

7.4 Window length, overlap length and use of the ST-RDT for estimation of uncertainty and weak non-linearity

The selection of the window length, $L_w$, and the overlap between windows, $L_b$, used within the ST-RDT is strongly linked to subsampling theory and the theory of moving averages for quantifying non-linear behaviour and estimating uncertainty in the presence of correlated and non-identically distributed data. As introduced in Sections 7.1.2 and 7.3, the ST-RDT is a subsampling methodology which provides a weighted estimates of the modal parameters of the system conditional on the triggering condition. To understand the dynamic behaviour of real-world structures, the modal estimates $\theta$ would ideally be separated into two components, introduced in Section 7.1.4: i) $\theta_0$, the distribution of true modal parameters of the structure, and ii) $\theta_\varepsilon$ the errors or uncertainty in the modal estimates. Unfortunately as demonstrated in previous sections these values are often correlated. In the example using data from the MX3D Bridge, the signal to noise ratio of the signal, and by extension the uncertainty in $\theta$, was correlated with the amplitude of the excitation, which in turn was correlated with the temperature of the bridge. The objective of the ST-RDT is to simultaneously quantify both the uncertainty in the modal estimates and the weak non-linear behaviour so they may be used to form probabilistic models of the structure’s dynamic behaviour.

As with the RDT parameters discussed in Section 7.3.3, the choice of the window length and overlap length in the ST-RDT is highly system specific and is dependant on:

i. the modal parameters volatility - The magnitude of the change in modal parameters relative to small changes in external parameters, conditional on the triggering condition,

ii. the uncertainty volatility - The magnitude of the change in modal parameters due to a small change in the noise- and forced-components of the RDS, $\hat{D}_{XT,n}$
iii. the strength of the correlation of weak non-linear modal behaviour, and the correlation of errors, between overlapping and adjacent windows of data.

When selecting window and overlap lengths the objective is to ensure the uncertainty volatility and the modal parameter volatility are representative of the behaviour of the structure. That is, the uncertainty estimated for the modal parameters is indicative of what might be observed in future modal parameter estimates, and that any weak non-linear behavior may be correctly accounted for when modelling the system. As discussed in Section 7.1.2.6, this often requires a trade-off in which a shorter window length and fewer segments of observed data would result in lower levels of variation in the non-linear modal parameters, while a longer window length would allow for reduction of the unwanted force and noise components of the RDS and more accurate modal parameters if the systems behaviour is assumed to be linear.

As highlighted in previous sections, the use of overlapping windows within the ST-RDT introduces correlated errors between the modal estimates for overlapped windows. While this is less than ideal, the use of overlapping windows is essential if non-linear behaviour is to be accurately observed and quantified. This can be illustrated through the example data given in Figure 7.30 in which a step in the amplitude is observed between $t = 135$ and $t = 185$.

![Example data set containing stepped output response.](image)

If a non-overlapped moving average is calculated the magnitude of this step is underestimated and the relative magnitudes and location of the step is dependant on the time step used as the start of the first non-overlapped window, as shown in Figure 7.31.

![Figure 7.31: Example data set containing stepped output response.](image)
Figure 7.31: Non-overlapped moving average of data given in Figure 7.30. Averages plotted at centre of window of data from which they were generated.

A moving average ensures that the maximum average value is identified by considering all possible window start points, as shown in Figure 7.32, reflects the symmetry of the step function in the original data and therefore maximizes the information available in later modelling of the system.

Figure 7.32: Overlapped moving average of data given in Figure 7.30. Averages plotted at centre of window of data from which they were generated.

The optimal value of overlap to be used in the ST-RDT would be a single time step, however this can prove computationally expensive and drastically increase the storage required by the estimated modal parameters. When selecting the overlap length consideration should be given to the volatility of possible causes of non-linearity. Based on the case studies presented at the end of this chapter, a minimum overlap length of one fifth the window length is recommended to ensure that the results obtained are not strongly dependant on the window start time selected.

Selecting the window length to be used within the ST-RDT is more complicated, as it requires volatility of the modal parameters to be balanced with the volatility of the uncertainty. Some guidance for the selection of a minimum window length may be adapted from guidance on the minimum data set length for accurate modal parameter estimates found in the OMA literature, however as discussed in Chapter 2, the applicability of these minimum data set lengths to real-world systems exhibiting correlated errors and non-linear modal behaviour is largely untested.
More generalized guidance on the selection of window length may be adapted from the subsampling and moving average literature. Politis et al. [309] presents two generalized methods for selecting the window and overlap length when subsampling, referred to as the calibration method and the minimum volatility method.

The calibration method, based on work by Loh [420], requires approximating a parametric model of the data and generating pseudo-samples for analysis. This model may then be used to relate the confidence intervals of the subsampled data, denoted $\alpha_c$, to the confidence intervals of the parameterized model, $\lambda_c$. In the context of OMA this could be achieved through the use of simplified numerical or finite element (FE) models of the structure utilizing distributions of modal parameters, referred to by Haukaas and Gardoni [421] as Bayesian FE models. Pseudo-samples of numerically generated random vibration data from this model could then be analysed with the ST-RDT and a function derived to relate the distribution of estimated modal parameters $\alpha_c$ to the true distribution of modal parameters $\lambda_c$ for a given window length. The practicality of such an approach is likely to be limited as it requires an accurate FE model of the structure and broad assumptions about the independence of the applied forcing and noise within the data.

The minimum volatility method [309] is a more practical method for selecting an appropriate window length for the ST-RDT and does not require prior knowledge of the systems expected behaviour. Minimum volatility is a purely heuristic method in which the window length is iterated to minimize the volatility (change) in parameters between adjacent, non-overlapping, windows of data. Consider the data presented in the left subplot of Figure 7.33, comprising of a linear trend corrupted with additive white noise. If the objective is to select a window length for calculation of the mean average, $\mu_{c,i}$, of the data for each window $i$, the mean absolute volatility of the data, $VI$, for a given window length $L_w$ and with the full data set divided into $N_{\text{windows}}$ non-overlapping windows is given by

$$VI = \frac{1}{N_{\text{windows}}} \sum_{i=2}^{i=N_{\text{windows}}} |\mu_{c,i} - \mu_{c,i-1}|$$

(7.46)

The mean absolute volatility of the average for the data is presented in the right subplot of Figure 7.33, indicating a window length of approximately 250 samples to minimize the mean absolute volatility calculated using Equation 7.46.
Figure 7.33: Left subplot) Noisy data exhibiting a positive linear trend with time. Right subplot) Mean absolute volatility of subsampled averages based on Equation 7.46.

The minimum volatility method could be expanded to the ST-RDT by considering a range of window lengths and quantifying the change in modal parameters observed for non-overlapping windows. More advanced applications of this approach could use a fitted probabilistic model, of the type described in Chapters 8 and 9, as a means of quantifying the degree of volatility which can be explained by the fitted model, with the window length then iterated until a reasonable approximation of the uncertainty in the modal parameters under the probabilistic model was achieved. This approach can be of particular use when there are sources of weak non-linear modal behaviour which are not quantified by a given model.

Further guidance on the selection of appropriate block sizes and overlap lengths can be found if the ST-RDT is considered to be a novel version of a moving average or boxcar filter. The moving average filter is the optimal solution for reducing noise in a signal whilst retaining the sharpest step-function response [422]. The sample frequency, $f_s$, of the moving average filter is the sample rate of the raw data prior to application of the ST-RDT, and the corresponding frequency roll-off, $H[f]$, of the moving average filter is described by Equation 7.47.

$$H[f] = \begin{cases} 
\frac{\sin(\pi f/f_s) L_w}{L_w \sin(\pi f/f_s)} & \text{if } 0 < f < 0.5f_s \\
1 & \text{otherwise} 
\end{cases} \quad (7.47)$$

Using the frequency roll-off of the moving average as a guideline, the ST-RDT window length can be selected relative to the dominant frequency component ($f$) of the weak non-linear behaviour of interest. For example, when analysing diurnal temperature induced changes in modal parameters, the ST-RDT window
length should be increased to remove higher frequency sources of weak non-linear
behaviour. When analysing rapidly changing sources of weak non-linear modal
behaviour, such as might be induced by vortex shedding of wind [423], a far
shorter window length is required to maximize the relative differences in modal
behaviour observed within each window of the ST-RDT analysis.

If the computational and storage requirements are available, a further ap-
proach which might be considered is summing the RDSs for adjacent non-
overlapping windows as part of a multi-stage analysis. This provides a minimally
expensive method of forming RDSs for any window length which is an integer
value of the minimum ST-RDT window length (\(L_w\)) selected, whilst maintaining
the equivalent temporal resolution given by the overlapped windows.

As discussed above, the range of weak non-linear behaviour, as well as the re-
lationship between the uncertainty in modal estimates and the true distribution of
modal parameters exhibited by the structure, is dependant on the window length
selected for the ST-RDT and is likely to be system dependant. For many practi-
cal applications of the ST-RDT, such as damage detection, quantifying thermal
variation in behaviour, or estimating the loading on the structure, as long as the
window length is fixed, accurate estimates of the structures behaviour may be ob-
tained through probabilistic modelling, discussed at length in Chapters 8 and 9.
The benefit of this approach is that it does not require careful iteration of the
window length to identify the optimal length for minimizing volatility. All that
is required is for the window length selected to provide a reasonably consistent
relationship between the fitted modal parameters and the true parameters of the
system. Any uncertainty in the modal parameters can be incorporated into the
modelling of the distributions of parameters observed under the weak non-linear
modal behaviour of interest, as demonstrated in the case studies presented in

Chapter 8

7.5 Case study - The Aberfeldy footbridge

This section is adapted from earlier work published by Wynne et al. [3] as a
chapter in Civil Structural Health Monitoring.

A key advantage of the ST-RDT compared to conventional time-domain OMA
methods is that it allows the weak non-linear modal behaviour of the structure
due to changes in the mass or stiffness of the structure to be identified. This
can allow changes in modal parameters due to damage to be distinguished from
those associated with thermally- or environmentally-driven changes in stiffness,
or changes in mass due to changes in the static loading applied to the structure.

An example of this benefit can be seen in the analysis of acceleration data collected from the Aberfeldy Footbridge, the world’s first major advanced composite footbridge, presented in detail in Appendix C. A comparison of the modal parameters of the structure was previously conducted by Stratford [424] based on the parameters estimated through forced excitation measurements by Pavic et al. [425] and a set of modal estimates based on ambient excitation measurements in 2011. Stratford [424] concluded that there had been a meaningful decrease in natural frequencies of the structure which was likely to have been induced by damage to the bridge.

However, analysis of a new acceleration data set collected from the bridge in 2019 suggests that the likely cause of the decrease in natural frequencies of the bridge observed by Stratford [424] was due to the additional mass provided by pedestrians during the ambient excitation measurements. As highlighted in Figure 7.34, the additional mass provided by pedestrians leads to systematic and significant changes in the natural frequencies of the bridge. These changes in natural frequency due to the added mass of a single pedestrian are of the order of 10% of the natural frequency for the 1st horizontal mode of the bridge. The additional mass provided by pedestrians was not present in analysis conducted by Pavic et al. [425] as forced excitation of the bridge deck was used.

![Figure 7.34: ST-RDT frequency estimates (black dots) for 1st horizontal mode. Vertical mid-span acceleration data shown in grey on second y-axis.](image)

When the variation in natural frequencies associated the additional mass and damping provided by pedestrians is considered within the analysis, there is little evidence of a meaningful decrease in natural frequencies associated with damage to the bridge. This is highlighted within Figure 7.34 as the natural frequency of the bridge appears to return to a baseline value close to that observed by Pavic et al. [425], as evidenced further in Figure 7.35.

The ability to quantify the variation in modal parameters due to variations in
the mass and stiffness of a structure in-service represents a key step towards accurate in-situ damage detection. Alongside this, automated tracking and quantification of weak non-linear modal behaviour in-situ provides a basis for estimating other structural parameters, as discussed further in Chapter 9.

### 7.6 Case study - Analysis of the Whitmore timber building

An advantage of the ST-RDT, as demonstrated through the analysis of accelerometer data from the Whitmore timber building discussed in detail in Appendix D, is that the large number of modal estimates produced through analysis of overlapping windows greatly simplifies the identification of real modes of vibration within the modal estimates. This is demonstrated in Figure 7.36. The modal estimates for a single window of ST-RDT analysis contain high number of spurious modal estimates associated with noise, a result of the low signal-to-noise ratio of the acceleration data. However, as these spurious modal estimates are not correlated across separate ST-RDT analysis windows, the modes of vibration of the system, which appear as dense regions of modal estimates which are correlated across time in Figure 7.36, can be easily identified. These dense clusters of modal estimates associated with real modes of vibration of the structure form the basis for the automated identification of modes presented in Chapter 8.
Figure 7.36: Natural frequencies extracted from 52 Whitmore Road building acceleration data with the ST-RDT, plotted alongside acceleration data used in analysis. Frequency estimates where a channel was used as both the analysis and triggering data set are highlighted as black scatter markers.

An additional benefit of the independent analysis of subsamples of the data through the ST-RDT is that it allows the correlations of modal parameters with other factors to be quantified without presupposition of a relationship. For example, conventional RDT analysis of amplitude dependant dynamic behaviour
is typically based on analysing the data with a variety of level-crossing triggers. The key issue with this approach is that if there are correlations between the triggering conditions and other components of the signal, such as the forcing applied to the system or the signal-to-noise ratio, this may result in biasing of the modal estimates.

An example use of the ST-RDT results for exploring amplitude dependant behaviour of the Whitmore Building is presented in Figure 7.37. It can be seen that there is evidence of small increases in natural frequency and decreases in damping of the fundamental mode when the amplitude of the RDS, or the amplitude of the mode, exceeds a small threshold value. However, while there is evidence of amplitude dependant behaviour, changes in the modal parameters with amplitude are largely insignificant when compared to the errors in the modal parameters, as evidenced by the wide distributions of frequency and damping estimates. Given the wide range of uncertainty within the modal parameter estimates, further data from the structure would be required to assess whether there is a meaningful change in dynamic behaviour with amplitude.

![Figure 7.37: Modal and RDS envelope amplitude dependant behaviour of natural frequency and damping of the fundamental mode of vibration of 52 Whitmore Road.](image)

7.7 Case study - Lake seiches

The use of the ST-RDT is not limited to the analysis of civil structures but can be used to estimate the modal parameters of any dynamic system. This is demonstrated in Appendix E and summarized in this section through the application
of the ST-RDT to estimate the natural frequencies and damping ratios of standing waves in lakes using water level data collected from Lake Tahoe and Lake Geneva, previously analysed using the conventional RDT by Wynne et al. [4]. These standing waves, known as barotropic or surface seiches, are the oscillation of the water surface under a gravitational restoring force which occur after an upwelling event, where the water within the lake is forced towards one end of a lake basin by wind stress on the surface of water, earthquakes or landslides. There are several features of lake seiches which make them challenging to analyse. Seiche periods are typically very low, on the order of minutes to days. Due to these low natural periods, the low frequency components of the wind forcing driving the seiches, associated with diurnal fluctuations or typical storm duration, become significant. The low frequency components of the RDS associated with the broadband forced response of the system is highlighted in Figure 7.38. As discussed further in Appendix E, the failure to account for the correlation of the forced response of the RDS within the RDT analysis presented by Wynne et al. [4] may explain the broad range of damping estimates previously reported for Lake Geneva.

![Figure 7.38: Example of random decrement signature for Lake Geneva water level data.](image)

While many modes of vibration of a civil structure may be simultaneously excited by wind forcing, a sustained wind force and direction is required to excite each seiche mode. Therefore, the data for both lakes are dominated by oscillations associated with a single seiche mode corresponding to the prevailing wind direction in the lake basins. When the data is analysed as a monolithic block, this makes it challenging to obtain accurate modal parameters for other, weakly excited modes. However, these modes are well extracted using the ST-RDT, as demonstrated in Figures 7.39 and 7.40, as periods of time with different wind directions are temporally separated.
Figure 7.39: ST-RDT estimates of natural period of Lake Tahoe surface seiches verses time. Likely modes of vibration for the surface seiche appear as dense areas of points within the results, indicating higher densities of modal estimates.

Figure 7.40: Natural periods of Lake Geneva surface seiches modes

While estimates of the modal parameters of the Lake Tahoe seiche with a period of between 11.3 and 11.8 minutes had previously been extracted by Wynne et al. [4] using the RDT, the other fundamental seiche periods had not previously been extracted. Similarly, while the RDT analysis presented by Wynne et al. [4] had provided modal parameter estimates for the Lake Geneva seiche with a period of approximately 73.9 minutes, modal parameters for the accompanying mode with a natural period of 35.5 minutes had not previously been extracted. Identifying and quantifying these other seiches greatly expands the broader un-
derstanding of lake energy dynamics and how changes in atmospheric conditions or water levels may impact on mixing of the water column within the lakes [4]. The ST-RDT provides a robust method for accurately quantifying weakly excited modes through leveraging the temporally correlated nature of specific oscillatory behaviour. Further examples of this can be seen in the analysis of wind speed and direction dependant behaviour of the seiche periods and damping presented in Appendix E.

### 7.8 Case study - Vibration response of the MX3D footbridge

An area where in-situ structural monitoring of existing structures may offer significant benefit is in ensuring compliance with serviceability requirements, such as meeting the required levels of user-comfort. These performance criteria often guide the design of lightweight structures, or those with novel, efficient forms and construction techniques, due to their lower stiffness-to-strength ratio when compared with traditional designs. A key serviceability criterion for the design of floors in buildings and bridges is ensuring that vibrations do not cause discomfort to users or impair the use of the structure [426]. This requirement has led to the development of guidelines on the design of structures for footfall induced vibration [427, 428, 426, 189, 429, 430].

The ST-RDT allows for variability of modal parameters in-situ, due to the additional mass and damping provided by pedestrians, to be incorporated into the assessment of footfall induced vibration. An example of this is the assessment of the vibration response of the MX3D Bridge, detailed in Appendix F and summarized here. This work is currently under review at *Case Studies in Construction Materials* under the title of *Dynamic testing and analysis of the world’s first metal 3D printed bridge*.

Within this assessment, the vibration response of the MX3D bridge was predicted using the methodology set out by Willford and Young [426] based on modal parameters measured using the ST-RDT, those measured through impact hammer experimental modal analysis (EMA), and the modal parameters predicted using a finite element (FE) model of the structure. These predictions were then compared to the measured vibration response of the bridge during a series of crossings by a single pedestrian. As the ST-RDT produces a range of estimates of the modal parameters of the structure, the median natural frequencies and damping ratios were used when predicting the bridge’s vibration responses, as
highlighted in Figure 7.41. The median modal parameters were used as these are believed to offer a good approximation of pedestrians experience of the bridge response, as due to the differing mode shapes of the bridge it is not possible for the minimum frequency and damping values to occur for all modes of vibration simultaneously.

Using the median natural frequencies, damping and mode shapes from the ST-RDT results in a significantly higher peak vibration response $R$ when compared to those predicted using the EMA modal estimates or FE modal predictions, as shown in Figure 7.42. The FE model produces significantly lower estimates of the modal mass of the structure, resulting in significantly lower estimates of the vibration response factor. Differences in the response factor predictions based on the impact hammer and ST-RDT measurements of the modal parameters are associated with differences in the natural frequencies and damping ratios. The impact hammer measurements produce larger measurements of damping than the ST-RDT results for most modes suggesting some degree of amplitude-dependant damping behaviour [264]. Alongside differences in damping, the modes which are well detected by the ST-RDT are different from those captured by the impact hammer excitation, which struggles to detect some modes due to low mode shape amplitudes at the impact location. This results in greater uncertainty in both the scaled mode shapes and modal masses for the impact hammer modal estimates, which, combined with the higher damping ratios, lead to lower predictions of the response factor.

When the vibration predictions based on the FE predictions of modal param-
Figure 7.42: Comparison of the maximum vertical response-factor curves predicted based on the modal parameters from the FE model, impact hammer analysis and ST-RDT analysis.

Figure 7.43: Vertical response factors during bridge crossings by a single pedestrian walking at 1.70 Hz.

Parameters and ST-RDT and EMA measurements of modal parameters are compared to the measured vibration response at two walking frequencies, as plotted in Figures 7.43 and 7.44, it can be seen that the predicted response factor based on the ST-RDT modal estimates provides excellent agreement with the measured peak response factor for both walking frequencies.

The accurate predictions of the vibration response of the MX3D Bridge highlights several of the benefits of the ST-RDT over both EMA and conventional OMA, such as the ability to accurately estimate modal parameters for a wide number of modes and accounting for changes in modal behaviour associated with the added mass and damping provided by pedestrians, and amplitude-dependant behaviour of the structure.
7.9 Summary

This chapter has introduced the short-time random decrement technique (ST-RDT), a time-domain OMA method that allows for the quantification of uncertainty and non-linear behaviour in modal estimates.

The ST-RDT is based on the new definition of the random decrement signature (RDS) for dynamic systems as a conditional correlation formed from the expected value of the analysis data, conditional on the triggering condition, of four distinct components of the data:

- the initial amplitude response of the analysis data,
- the initial first derivative response of the analysis data,
- the forced response of the analysis data,
- the noise within the data.

Through the derivations presented in this chapter it has been shown that the first two components listed above can be directly modelled as a sum of damped sinusoids, while the other two components cannot. The inability to directly model the forced response and noise as a sum of damped sinusoids results in errors in the modal estimates. Further to this, where the system exhibits weak non-linear modal behaviour, changes in the modal parameters associated with changes to the structural system, the modal parameters estimated from the RDS are a weighted estimate of the true time-varying modal parameters. This new definition of the RDS removes the assumptions of independent segments of data, white noise forcing, and white noise corruption of the measured response by noise, imposed in the definitions of the RDS derived by Asmussen et al. [257].
The application of the ST-RDT for time-domain OMA has been demonstrated and linked to the iteration of model orders to achieve a target explained variance, introduced in Chapter 5, and the trimming of correlation functions to remove the autocorrelation of noise discussed in Chapter 6. Alongside this, the relationship between modal estimates generated using the ST-RDT and subsampling theory has been described, providing a basis for further expansion of the relationship between ST-RDT modal estimates and the uncertainty associated with in-service ambient vibration monitoring. Initial guidance on the selection of suitable ST-RDT parameters, such as window length and overlap length, has been developed based on numerical simulations, subsampling theory and established practice for moving-average filters. Through the numerical simulations and real-world case studies presented in this chapter, the ST-RDT has been demonstrated as a robust and accurate OMA method which allows spurious modal estimates to be distinguished, uncertainty in the modal parameters to be quantified, and weak non-linear dynamic behaviour identified.

A key difference between the ST-RDT and other time-domain OMA methods is that the number of modal estimates produced for a single data set may be many orders of magnitude greater. This opens new possibilities for the automated identification of modes of vibration within the modal estimates, as developed in the next chapter. The use of the ST-RDT for identifying and quantifying weak non-linear modal behaviour also allows for the estimation of structural parameters usually measured by more expensive, delicate or complex in-situ monitoring technology, as explored in Chapter 9.
Chapter 8

Automated identification of modes of vibration in operational modal analysis

The statistical tools in this chapter provide methods for addressing research objective 4:

RO 4. Automate the detection and separation of modes of vibration within OMA modal estimates.

As highlighted in the literature review and by the survey of industry perceptions, automated methods of identifying modes of vibration within modal estimates produced through operational modal analysis (OMA) are required to aid the adoption of long-term monitoring to inform structural engineering design. The challenges of automated identification of modes of vibration using existing techniques from the machine learning literature are presented in this chapter. A method for automated identification of modes in modal estimates from the short-time random decrement technique (ST-RDT) which overcomes the limitations of existing methods of automated modal identification and allows the uncertainty or volatility within the modal estimates to be quantified is introduced. In Chapter 9, this method is expanded for the identification and quantification of weak non-linear modal behaviour, where the natural frequencies, damping ratios and mode shapes of a structure may change over time.

As discussed in previous chapters, OMA allows the modal parameters of dynamical systems to be estimated using measurements made under in-service or ambient loadings [17]. However, interpreting OMA estimates is challenging due to the noise and uncertainty in the modal estimates. For time-domain OMA
methods such as stochastic subspace identification (SSI) or the random decrement technique (RDT), each modal estimate corresponds to either: i) a mode of vibration of the system, or ii) a spurious estimate associated with the fitting of noise within the correlation function. The spurious modal estimates, also referred to in the literature as noise modes [431, 313], have no value for understanding the dynamic behaviour of the structure.

Within the modal estimates for a mode of vibration, there are two sources of uncertainty. The first source of uncertainty is the errors in the modal estimates associated with the approximation of a free-response of the dynamic system, referred to in this chapter as the modal errors. The second source of uncertainty is the weak non-linear modal and/or dynamic behaviour of the mode; the change in the modal parameters which occurs due to changes in the effective mass, stiffness or damping of a structure. These changes can be induced by a broad range of factors, such as changes in temperature [432, 433, 330, 215], changes in the static loading [359, 360], or changes in the amplitude of structural response [264, 276].

The identification and separation of modal errors and weak non-linear modal behaviour is further complicated by close modes, modes that have similar natural frequencies, as are often encountered when analysing structures with structural symmetry such as buildings with bi-directional floor plan symmetry.

A set of five key objectives were identified for the automated identification of modes of vibration within OMA modal estimates, based on the analysis of data from real-world structures presented in this research and the findings of the literature review:

1. Automated identification of modes of vibration within the modal estimates.
2. Automated separation of modes of vibration within modal estimates.
3. Automated separation of spurious modal estimates from modal estimates associated with modes of vibration.
4. Automated quantification of the modal errors for each mode of vibration.
5. Automated identification, quantification and/or removal of weak non-linear modal behaviour from the modal estimates.

Fulfilling these objectives would enable automated structural health monitoring [321], the integration of in-situ vibration monitoring into digital shadows and digital twins [170], and remove the subjectivity associated with the manual interpretation of modal estimates. While this chapter is largely focused on the
analysis of modal estimates generated with the ST-RDT, the methods demonstrated and the observations made are applicable to a wide range of time- and frequency-domain OMA methods and may be adapted for use with data from other in-situ monitoring technology.

The aim of this chapter is not to summarise all possible methods for automated identification of modes of vibration, nor to provide a detailed account of the development and refinement of the methods presented. Instead, this chapter highlights the limitations of existing methods for automated modal identification, and introduces a new method which allows for robust and efficient identification of modes of vibration within OMA modal estimates.

8.1 Artificial generation of modal estimates

Numerically generated data designed to mimic the distributions of frequency and damping estimates observed within ST-RDT modal estimates for real structures are used to demonstrate the methods presented in this chapter. These artificial data sets include the following key characteristics:

- Each mode of vibration has a “true” natural frequency and damping ratio.
- The modal errors in the frequency and damping estimates for each mode of vibration are normally distributed.
- The error in the frequency and damping estimates are not correlated between time steps.
- The number of modal estimates for each mode of vibration varies between time steps.
- The modal estimates for each time step include frequency and damping estimates from one or more modes of vibration alongside spurious modal estimates associated with noise.
- The modal estimates may include close modes of vibration.
- Modes of vibration may exhibit systematic weak non-linear modal behaviour.

Examples of these characteristics can be seen in the ST-RDT modal estimates from real-world structures, such as in the modal estimates for the timber buildings discussed in Section 8.4, an example of which is plotted in Figure 8.1. For
comparison, an artificial data set of modal estimates generated using the characteristics listed above is presented in Figure 8.2.

Figure 8.1: ST-RDT modal estimates from the Treet timber building, discussed in Section 8.4, displaying the characteristics which form the basis for the artificial data generation.

Figure 8.2: Example of artificial data generated for a system with six modes of vibration, with characteristics broadly modelled on the Treet timber building data.

The advantage of testing, developing and demonstrating the methods using artificial modal estimates is that the true parameters of the generated modes and the true assignment of each modal estimate to a mode of vibration or spurious noise are known. However, some characteristics of the real modal estimates are
not captured in the artificial data, such as the correlation of errors in the modal estimates which may exist between adjacent time steps and between modes of vibration, discussed in Chapter 7. While this could be included within the artificial data, it was found that the methods demonstrated in this chapter are robust to correlated errors if there is a reasonably large number of independent modal estimates available for analysis. Correlated errors may impact the analysis of weak non-linear modal behaviour, as discussed in detail in Chapter 9.

8.2 Limitations of existing clustering techniques for automated detection of modes of vibration

Identifying modes of vibration within OMA modal estimates may be viewed as a clustering problem. There is a data set of modal estimates and the objective is to assign each data object, a set of modal estimates such as a pair of frequency and damping estimates, to a larger set or cluster. For automated OMA each cluster should contain either modal estimates which relate to a single mode of vibration, or should contain only spurious modal estimates which can be excluded from further analysis. As the “correct” assignment of modal estimates to clusters is typically unknown, automated identification of modes is an unsupervised machine learning problem.

This section provides an overview of the limitations of some existing clustering methods for automated identification of modes with OMA modal estimates. As there has been relatively little research into the application of these methods within the OMA literature, details of the development of the clustering methods, their application, examples of when the methods may allow for accurate identification of modes within OMA modal estimates, and details of further clustering techniques explored as part of this research are presented in Appendix G.

Clustering methods are split into two categories: hard-clustering methods, where each data object is assigned to a single cluster, and soft-clustering methods, where each data object is assigned a probability of being assigned to each cluster.

A widely used hard-clustering technique is k-means clustering. Each “object” is assigned to one of $k_{clusters}$ clusters which minimizes the Euclidean or squared distance between each of the objects $x_{data}$ and the cluster points $\mu_k$, where cluster point $\mu_k$ is the mean of the objects $x_{data}$ assigned to cluster $k_{cluster}$. Each object is only assigned to the cluster which results in the smallest
Euclidean distance between $x_{data}$ and $\mu_k$. After each assignment, the cluster points $\mu_k$ are updated, and the process is repeated until the assignment of objects to clusters does not change.

As discussed in Chapter 2, while there has been some past success using k-means clustering for automated identification of modes of vibration in stability diagrams [313, 314, 315, 316, 317], little work has explored the use of k-means clustering for identifying modes of vibration when the modal estimates come from a windowed analysis of data, in which far greater numbers of modal estimates are collected and weak non-linear behaviour may be observed.

There are several limitations of k-means clustering for automated identification of modes, as illustrated in Figure 8.3. In this figure, the assignment of pairs of frequency $f_i$ and damping $\zeta_i$ estimates to clusters with cluster centers $\mu_{f,k}$ and $\mu_{\zeta,k}$ are shown based on three different assignment criteria:

- One-dimensional (1D) frequency k-means: Assignment based on minimizing $\sqrt{(\mu_{f,k} - f_i)^2}$

- Two-dimensional (2D) frequency-damping k-means: Assignment based on minimizing $\sqrt{(\mu_{f,k} - f_i)^2 + (\mu_{\zeta,k} - \zeta_i)^2}$

- 2D scaled frequency-damping k-means: Assignment based on minimizing $\sqrt{(\mu_{f,k} - f_i)^2 + (\mu_{\zeta,k} - \zeta_i)^2}$ with all $f_i$ and $\zeta_i$ scaled to fall in the range $(0,1)$ prior to clustering.

The number of clusters is set equal to the true number of modes within the data. 1D frequency k-means reasonably identifies modes if the mode is well separated from other modes. However, a large number of spurious modal estimates are still assigned to these clusters. The 1D frequency k-means performs poorly where the frequency estimates from different modes overlap, even when the damping estimates of the mode are well distinguished, such as for the modes between 2 Hz and 3 Hz. Where the frequency and damping estimates for multiple modes overlap, such as the two modes between 5 Hz and 6 Hz, k-means clustering cannot separate the modes. 2D frequency-damping k-means results in all modes being split across multiple clusters, due to the higher variance of the damping estimates compared to the variance of the frequency estimates for each mode. This can be partially resolved by scaling all frequency and damping estimates to fall in the range $(0,1)$. However, where there is a large disparity between the variance in the frequency and damping for a single mode, such as the mode at 3.6 Hz in Figure 8.3, the mode is still split between multiple clusters. As assignment of modal estimates to clusters is through minimizing the Euclidean distance, it
is challenging to separate spurious modal estimates using k-means clustering. If further clusters were introduced, some clusters would contain only spurious modal estimates. However, introducing more clusters is also likely to result in modal estimates from each mode of vibration being split over multiple adjacent clusters.

![Comparison of 1D, 2D, and 2D scaled k-means clustering of frequency and damping estimates. K-means clustering performed using Scikit-learn Python library \[435\]. Top plot shows true assignment of artificial modal estimates.](image)

Figure 8.3: Comparison of 1D, 2D, and 2D scaled k-means clustering of frequency and damping estimates. K-means clustering performed using Scikit-learn Python library \[435\]. Top plot shows true assignment of artificial modal estimates.
To overcome some of the limitations of k-means clustering, Ester et al. [326] developed density-based spatial clustering of applications with noise (DBSCAN). DBSCAN does not require pre-specification of the number of clusters in the data set, can separate noise in the data set, and can be used with arbitrarily shaped clusters [326]. Two parameters are used within DBSCAN, \( \varepsilon_D \) which defines a distance around the boundary of a cluster in which points are searched for, and \( \text{MinPts} \), the minimum number of points required to define a valid cluster.

The use of DBSCAN is demonstrated in Figure 8.4 based on the artificial data introduced in Figure 8.3. DBSCAN is applied using the frequency and damping estimates, with all estimates scaled to fall in the range (0,1). All non-overlapping modes are well separated with \( \varepsilon_D = 0.02 \) and \( \text{MinPts} = 20 \), with a number of the modal estimates correctly identified as noise. The overlapping modes of vibration between 5 Hz and 6 Hz are not separated. Alongside this, extreme outliers associated with the modes of vibration are excluded. The exclusion of outlier values may result in the uncertainty of the modal estimates for a mode being underestimated. This issue is more pronounced for modes with higher variance in the modal estimates. While DBSCAN removes the need to specify the number of clusters, a challenge in k-means clustering [436], in DBSCAN \( \varepsilon_D \) and \( \text{MinPts} \) require careful selection and the choice of appropriate values for these parameters is largely subjective and data set dependant.

DBSCAN may also be used for identifying modes in the time-frequency domain of the ST-RDT modal estimates, as demonstrated in Figure 8.4. However, if the detection of a mode within the modal estimates is intermittent and the mode is not detected for some time steps, DBSCAN will split this mode across separate clusters. Similarly, if the mode exhibits variations in the natural frequency due to weak non-linear modal behaviour, the cluster may split the mode across separate clusters or map multiple modes to the same cluster. Identifying modes of vibration in the presence of weak non-linear modal behaviour is discussed further in Section 8.3.3 and addressed in Chapter 9.

Several of the issues which limited the use of hard-clustering methods for automated mode detection can be overcome through mixture models. Unlike the hard clustering methods discussed above, where each data object is assigned to a single cluster, mixture models or soft-clustering are probabilistic models of the data from which the probability of each data object belonging to a specific cluster may be calculated. This can resolve the issues posed by noise in the modal estimates or where the modal estimates contain close or overlapping modes of vibration.

Gaussian mixture models (GMMs) are probabilistic models in which each of
Figure 8.4: DBSCAN of artificial data introduced in Figure 8.3. DBSCAN applied using frequency and damping estimates scaled to fall in the range (0,1) with $\varepsilon_D = 0.01$ and MinPts = 20. Top plot shows true assignment of artificial modal estimates, lower plot show cluster assignment using DBSCAN based on frequency estimates only. DBSCAN implemented with the Scikit-learn Python library [435].

The primary use of GMMs in the OMA literature has been for identification of modes of vibration within stability diagrams [323], identification of damage detection without separation of modes of vibration [320, 321, 324] and as a standalone method of frequency-domain OMA [322]. No examples of the use of GMMs for modal identification using modal estimates from multiple windows of data were found in the literature.

Presented in Figure 8.6 are two GMMs with varying number of Gaussian components fitted to the artificial data presented previously in Figures 8.3 and 8.4.
Figure 8.5: Example of where DBSCAN may fail to identify modes in the time-frequency domain of ST-RDT modal estimates due to intermittent appearance of modes within the modal estimates and overlap of close modes of vibration induced by weak non-linear modal behaviour. Top plot shows true assignment of artificial modal estimates, lower plots show cluster assignment using DBSCAN clustering based on frequency estimates and time step. DBSCAN implemented with the Scikit-learn Python library [435].

When the number of Gaussian components is equal to the number of modes within the data, the overlapping modes between 5 Hz and 6 Hz are not separated. Instead, one of the Gaussian components approximates the uniform background noise within the modal estimates. These overlapping modes are identified if the number of Gaussians components is increased to two greater than the number of modes in the data set. However, as shown in the comparison of fitted and true parameters in Figure 8.7, the background noise within the modal estimates causes an overestimation of the variance of the uncertainty in the damping estimates and a slight underestimation of the uncertainty in the frequency estimates.

Like DBSCAN, GMMs may perform poorly in the presence of weak non-linear modal behaviour as the distributions of modal estimates associated with a single mode of vibration are likely to be non-Gaussian. This is examined further in Appendix G.

The limitations of existing clustering methods for identifying modes of vibration within OMA modal estimates have been demonstrated in this section. The next section introduces a mixture model of the modal estimates which accounts for the spurious modal estimates, and allows for the separation of close or overlapping modes of vibration. In Chapter 9, this model is expanded for the quantification of weak non-linear modal behaviour.
8.3 Non-Gaussian mixture models for automated identification of modes of vibration

The performance of mixture models for automated detection of modes in OMA modal estimates can be improved if the mixture models are modified to better represent the expected distribution of modal estimates. These modified mixture models are referred to here as non-Gaussian mixture models. In this chapter, the linear non-Gaussian mixture model is introduced. In Chapter 9, this non-Gaussian mixture model is expanded further to incorporate the expected distributions of modal estimates when a structure exhibits weak non-linear modal behaviour.

A modification to the mixture model that was found to improve the automated detection of modes of vibration is the inclusion of a uniform noise component within the mixture model. The rationale for the inclusion of this component
Figure 8.7: Fitted GMMs parameters for GMM with nine Gaussian components compared to true parameters used in generation of data. Fitting of GMMs implemented with the Scikit-learn Python library [435].

is that if the data is a white noise process and contains no dominant modal components it would be expected that the distribution of modal parameters fitted through OMA would be uniform between 0 Hz and the Nyquist frequency. In reality, the choice of windowing within OMA will introduce some degree of correlation and corresponding non-uniform distribution, but this may still be reasonably approximated as a uniform distribution within the frequency and damping ranges of interest. The probability of observing the measured modal estimates $X_{data}$ under the revised mixture model with fitted parameters $\Delta$ is then given by [Equation 8.2], the product of the probabilities of observing each of the $N$ individual modal estimates ($x_{data}$) summed across each of the $k_{clusters}$ Gaussian distributions and the uniform distribution $U(L_{low}, L_{high})$.

$$P(X|\Delta, m_f, L_{low}, L_{high}) = \prod_{n=1}^{N} \left( \sum_{k_{clusters}=1}^{K} m_{f,k}P(x_{data}|\Delta_k) + m_{f,K+1}U(L_{low}, L_{high}) \right)$$

(8.2)

If the mixture model is a 1D mixture model using only the frequency estimates, the values of $X$ are the frequency estimates, $\Delta$ comprises of the mean frequencies of the modes $f_0$ and their accompanying variance $\sigma^2_f$, and $L_{low}$ and $L_{high}$ are the minimum and maximum observed frequencies in the modal estimates. If the
mixture model is a 2D mixture model using the frequency and damping estimates, the values of $X$ are the pairs of frequency and damping estimates, $\Delta$ comprises of the mean frequencies and damping ratios of the modes $f_0$ and $\zeta_0$, alongside their accompanying variances $\sigma_f^2$ and $\sigma_\zeta^2$. $L_{low}$ and $L_{high}$ are the minimum and maximum observed frequency and damping values in the modal estimates such that $L_{low} = \{f_{min}, \zeta_{min}\}$ and $L_{high} = \{f_{max}, \zeta_{max}\}$.

The unweighted probability of a modal estimate under the 1D uniform component $U(L_{low}, L_{high})$ is given by $P(f_i) = 1/(f_{high} - f_{low})$. For the 2D uniform component $U(L_{low}, L_{high})$, the unweighted probability is given by $P(f_i) = 1/((f_{max} - f_{min})(\zeta_{max} - \zeta_{min}))$.

Inclusion of the uniform component within the mixture model improves the separation of noise from the detected modes of vibration, as demonstrated in Figure 8.8 using the artificial data presented previously in Figure 8.3. The uniform component removes the need to approximate the distribution of spurious modal estimates through broad Gaussian distributions and reduces inaccuracies in the estimated variance of the modes previously discussed, as demonstrated in the comparison of fitted and true parameters presented in Figure 8.9. The greatest deviation in the true and fitted parameters are observed for the overlapping modes between 5 Hz and 6 Hz, with a slight overestimation of the variance of the damping for the two modes and the relative probability of observing the modes within the data, associated with the uncertainty in the assignment of modal estimates to the mixture model components for these modes.

The revised mixture model has been shown to improve the automated detection of modes of vibration within artificial modal estimates. However, there are additional complexities in the implementation of the model described by Equation 8.2 for automated modal identification which must be addressed, including maximizing the probability of the modal estimates given the model parameters and reparameterization of the solution space for efficient modal detection.
Figure 8.8: Identification of modes of vibration through a mixture model with seven 2D Gaussian components and a 2D uniform distribution. The true distributions of modal estimates are presented in the top subplot, while the fitted distributions are presented in the lower plot.
Figure 8.9: Comparison of true parameters of artificial data with seven modes of vibration, presented in Figure 8.3, and parameters estimated through fitting of a mixture model with seven 2D Gaussian components and a 2D uniform distribution.
8.3.1 Log-probability for optimizing mixture model parameters

The objective when fitting the parameters of the mixture model defined by Equation 8.2 is to find the set of parameters which maximizes the likelihood of the model parameters given the observed modal estimates. These maximum likelihood parameters offer the best description of the modal estimates using the mixture model. Therefore, the parameters describe the most likely modes of vibration given the observed modal estimates.

In practice, there are a variety of optimizers which may be used for this problem such as Newton’s method [437], expectation maximization [438], or quasi-Newton methods such as the L-BFGS algorithm [439]. In this chapter, adaptive moment estimation (Adam), a stochastic gradient descent (SGD) method, has been used. SGD and Adam have some advantages for the automated detection of modes of vibration, including escaping local minima within the negative log-likelihood space. Full details on the implementation of Adam and discussion of the benefits of SGD for optimization are presented in Appendix H.

When finding the set of model parameters which maximize the likelihood of the model parameters (the product of probabilities of the observed modal estimates) it is recommended to use the sum log-probabilities. Using the log-probabilities has several advantages when compared to optimization using the probabilities themselves. Firstly, as the logarithmic transform is monotonic, the location of local maxima and minima are identical for the likelihood and the log-likelihood [437]. Summing the log probabilities is typically more computationally efficient than calculating the product of all probabilities [437]. Alongside this, the log-probabilities also resolves the issues posed by the small probabilities assigned to modal estimates which are far from a mode of vibration. These small probabilities may lead to computational underflow, in which the small values are less than the floating point accuracy used in the computation. Finally, the gradient of the log-likelihood function is better scaled for optimization [437]. The gradient of the probability density functions of the normal distribution (the probability of the discrete modal estimates given the model parameters) used within Equation 8.2 is rapidly changing, which can result in repeated overshooting of parameters during optimization. By using the log-likelihood, the change in gradient of the probability density function is much smaller, allowing for more stable and rapid convergence.
8.3.2 Reparameterizing constrained quantities

The mixture model defined by Equation 8.2 contains parameters which should be reparameterized to ensure stability when optimizing the sum log-likelihoods of the mixture model parameters.

The variance of the frequency and damping within the Gaussian components of the mixture model must be positive valued. To ensure that updating of these parameters does not result in negative values, the logarithm of the variances are used when updating the mixture model parameters. The exponential of these log values are taken prior to calculating the probabilities of the modal estimates under the model parameters. As the exponential of any number is positive, this ensures that the variances of the frequency and damping cannot be negative.

To ensure that the mixing fractions $m_f$ sum to unity, the softmax function is used. This transforms the arbitrary values of $z_j$ which are updated during fitting of the mixture model into the mixing fraction $m_{f,j}$ for mixture model component $j$ as $m_{f,j} = e^{z_j} / \sum_{i=1}^{K} e^{z_i}$.

It is beneficial to constrain the mean values of frequency and damping to within a predefined range in order to ensure that during updating of the model parameters components do not stray outside of the range of modal estimates and be assigned a mixing fraction of zero. This can be achieved through a shifted and scaled logistic sigmoid function. For an arbitrary value $a$ which is to be reparameterized as $a_s$, where $a_s$ is constrained to the range $a_{min}$ to $a_{max}$, this is defined using Equation 8.3:

$$a_s = (a_{max} - a_{min}) \left( \frac{1}{1 + e^{-a}} \right) + a_{min} \quad (8.3)$$

8.3.3 Automated identification of modes of vibration in the presence of weak non-linear modal behaviour

A key limitation of automated identification of modes of vibration using the mixture model described by Equation 8.2 is that while it fulfills the majority of the criteria set out at the start of this chapter: it automatically identified and separates modes of vibration, quantifies the modal errors in the modes, and is robust to noise and close modes of vibration; it does not account for non-linear modal behaviour, the changes in the natural frequencies and damping ratios which may occur due to changes in the external environment or the loading applied to the system.

The impact of non-linear behaviour on the fitted modal parameters are evident
from the results presented in Figure 8.10. In this artificial data set the natural frequencies of the six modes of vibration vary systematically with time, as can be seen in the time-frequency subplots. However, as this mixture model given by Equation 8.2 does not account for the systematic variation in the frequency, the modal errors (the variance of the modes of vibration within a single time step) are overestimated. Despite this, the total variance of the modes, the combined variance associated with the modal errors and the non-linear modal behaviour, as may be interpreted from the frequency-damping plots, is still well approximated by the fitted model.

Figure 8.10: Fitting of artificial modal data exhibiting systematic variations in natural frequencies with the mixture model given in Equation 8.2.

Weak non-linear modal behaviour can lead to the same systematic variation in the modal parameters as those presented in the artificial data in Figure 8.10. To maximize the information which can be extracted from OMA, causes of weak non-linear modal behaviour must be understood and robust methods developed to account for them within the automated detection of modes of vibration. Quantifying weak non-linear modal behaviour within automated OMA forms the basis of the next chapter, which provides simple methods for incorporating weak non-linear modal behaviour into the mixture model developed and demonstrated in this chapter.
8.4 Case study - Timber buildings

To demonstrate the robustness and adaptability of the probabilistic mixture model for automated detection of modes of vibration, selected results are presented from a larger ST-RDT analysis of ten timber structures presented in full in Appendix I.

The data from the ten timber structures were collected under a range of measurement conditions, including different accelerometers types, numbers of accelerometer channels, sensor layouts, and excitation levels or signal-to-noise ratios. Full details of the data used in the analysis, alongside details of the timber structures, methods used and further results are presented in Appendix I.

The data from all structures were analysed with a consistent set of ST-RDT parameters to facilitate comparison of the modal parameters. This methodology was extended to the automated detection of modes using the probabilistic mixture model, with the number of modal components included in the mixture model defined through a pre-specified criteria based on peak locations within the histogram of ST-RDT frequency estimates.

Prior to fitting of the mixture model it was necessary to discard a fraction of modal estimates associated with spurious or noise components of the estimated system response. These modal estimates are typically characterized by negative damping ratios and having amplitudes which are small when compared to the summed amplitude of modal components fitted to the RDS, referred to here as the normalized amplitude of the modal component. The normalized amplitude of the modal component was used as threshold for excluding modal estimates which were likely to correspond to noise components of the fitted system response. The exact value of this amplitude threshold was independently determined for each structure based on finding the logistic regression coefficient, which maximized the separation of modal components with positive damping, which were likely to correspond to physical modes of the system, and negative damping, which were likely to correspond to spurious estimates. While this approach does not fully remove all modal estimates with negative damping, as some of these have accompanying normalized amplitudes falling above the fitted threshold, it largely maintains the Gaussianity of the results, ensuring robust fitting with the probabilistic mixture model. An example of separation of modal estimate which are likely to be spurious from physical modal estimates based on logistic regression is presented in Figure 8.11.

The means and three standard deviation intervals for the modes of vibration, identified within the modal estimates for the timber structures using the auto-
Figure 8.11: Example of separation of spurious and physical modal estimates through use of a normalized amplitude threshold, fitted to maximize the separation of positive and negative damping estimates through logistic regression.

The estimated identification procedure and non-Gaussian mixture model, are plotted in Figure 8.12. In this plot modes which have a fitted standard deviation of the frequency greater than 0.25 Hz are separated as noise modes (plotted in grey) which are likely to correspond to fitting of noise within the modal estimates and not physical modes of vibration. Tight clusters of modes of vibration are clearly distinguishable in Figure 8.12 and have been well identified through fitting of the mixture model. Despite the presence of close modes and high levels of variance in the modal estimates, the mixture model has identified a range of overlapping modes of vibration. Specific examples of the separation of close modes of vibration can be seen in the results for Trento-CLT and Holz8. It should be noted that based on previous work it is likely that the two clusters of modal estimates with the lowest frequencies identified for BRE do not correspond to the fundamental (lowest) mode of the structure, but to components of the response associated with the oscillation of a rooftop pagoda.
Figure 8.12: Frequency and damping estimates from ST-RDT analysis (black) plotted alongside means and three standard deviation boundaries of fitted modes of vibration. Noise modes are defined as where the standard deviation of the frequency for a fitted mode is greater than 0.25 Hz. Modal estimates which are likely to correspond to noise and which were excluded prior to fitting of mixture model are shown in grey.
It should be noted that all data sets used within the analysis of timber buildings were relatively short, ranging from under 30 minutes to 160 minutes. Longer data sets would allow larger number of independent modal parameters to be estimated using the ST-RDT. This is likely to improve the efficacy of the mode detection, allowing for more robust quantification of the uncertainty in the modal parameters, greater separation of spurious modal estimates, and greater confidence in the accuracy of the mean natural frequencies and damping ratios of the modes. Despite the limitations imposed by the short data lengths, the results presented in Figure 8.12 demonstrate the effectiveness of the automated modal identification using the probabilistic mixture model developed in this chapter.

8.5 Summary

In this section a robust technique using non-Gaussian mixture models has been developed for the automated detection of modes of vibration within OMA results. The technique has been demonstrated using artificial data designed to mimic the modal estimates collected from structures in-service, and data collected from ten timber structures. This technique fulfills the criteria introduced at the beginning of the chapter for addressing research objective 4 by allowing:

1. Automated identification of modes of vibration within the modal estimates.
2. Automated separation of modes of vibration within modal estimates.
3. Automated separation of spurious modal estimates from modal estimates associated with modes of vibration.
4. Automated quantification of the modal errors for each mode of vibration.

The limitations of other methods of automated mode detection, such as k-means clustering, DBSCAN and Gaussian Mixture Models have been demonstrated, alongside discussion of the practical reasons for optimization of mixture model parameters using log-probabilities and the importance of reparameterizing model parameters. The method developed for automated identification of modes of vibration requires minimal user selected parameters, which is likely to aid in its application to other data sets and data types.

In the next chapter the automated identification of modes of vibration is expanded to account for weak non-linear modal behaviour when fitting the mixture model. Removal of weak non-linear modal behaviour is a key requirement if damage induced changes in dynamic behaviour are to be identified. Alongside
this, it is shown that through quantifying weak non-linear modal behaviour other structural parameters may be estimated using modal estimates collected from structures under in-service or ambient loadings.
Chapter 9

Understanding and quantifying structural behaviour through long-term monitoring

Building on the work presented in the previous chapters, this chapter presents a series of methods and case studies which address research objective 5:

RO 5. Relate changes in dynamic behaviour measured using ambient vibration monitoring to the physical changes of a structure and its environment.

As has been touched on in previous chapters, the behaviour of civil structures in-service may change over time due to changes in the loading applied to the structure, damage to the structure and changes in the external environment. To understand in-service structural behaviour it is essential to identify the causes of these variations, quantify the changes in behaviour they induce and relate the changes in measured data to the physical behaviour of the structural system.

In this chapter the primary drivers of weak non-linear modal behaviour (WNLMB), the changes in the dynamic in-situ behaviour of civil structures due to changes in the mass or stiffness of the structure, are discussed in detail. This discussion is focused on two primary drivers of WNLMB:

1. changes in temperature,
2. changes in static or gravity loading.

Equations based on the linear dynamic theory of structural behaviour are derived in Section 9.1 which relate these drivers of WNLMB to the changes in natural frequency of the structure. Subsequent sections focus on methods by which these
relationships may be used to quantify and isolate the causes of WNLMB, and how they may be applied for estimating a range of structural parameters in-service. Section 9.3 presents an overview of linear regression and quantile regression for estimating the parameters of the WNLMB equations derived in Section 9.1. These methods are utilized as a part of a two-step process. First, the modal estimates from a single mode of vibration must be isolated, either through defining bounds within which modal estimates correspond to a single mode of vibration or through the automated procedures presented in the previous chapter. The modal estimates from this mode are then used to estimate the parameters which relate changes in the frequency of the mode, to changes in temperature or static loading. The strengths and limitations of these regression methods are discussed in the context of analysing WNLMB, and guidance is given on the application of the methods for quantifying and removing WNLMB. The performance of the two regression methods for the analysis of thermally-induced changes in dynamic behaviour is compared in Section 9.3.3 using data collected as part of an experimental case study of the dynamic behaviour of beam structures at elevated temperatures. While this chapter is primarily focused on the analysis of modal estimates collected through operational modal analysis (OMA), the findings and methods are broadly applicable to a wide range of structural monitoring technologies.

A more robust and efficient one-step method for quantifying WNLMB is presented in Section 9.4. This method incorporates the equations of WNLMB from Section 9.1 into the non-Gaussian mixture model introduced in Section 8.3 of the previous chapter. This allows for the automated identification of modes of vibration in the presence of WNLMB. The physical basis of the WNLMB equations derived in Section 9.1 allows the automated identification of modes of vibration to be extrapolated to previously unseen environmental conditions, overcoming a key limitation of the mixture model developed in Section 8.3. The robustness of the method for automated mode detection in the presence of WNLMB is demonstrated in Section 9.4.3 using ST-RDT frequency estimates from analysis of accelerometer data from the MX3D Bridge, introduced previously in Section 7.8.

The equations relating changes in natural frequencies to changes in static loading are used as the basis for two load-estimation methods presented in Section 9.5. The first of the methods, detailed in Section 9.5.1, utilizes an empirical approach to load estimation through random sampling of distributions of frequency estimates from different modes of vibration. This method is demonstrated using data collected from a beam structure under both individual and multiple static loads. The second load-estimation method uses the probabilistic model of WNLMB
developed in Section 9.4 alongside Markov Chain Monte Carlo to estimate the posterior distribution of likely loading applied to a structure. The application of this method for load estimation is presented in Section 9.5.3 alongside a case study using data collected from the MX3D Bridge. Both load-estimation methods incorporate the uncertainty in the modal estimates generated through the ST-RDT, as was discussed in detail in Chapter 7 and propagate the uncertainty to the estimated load, allowing the confidence in the load predictions under variable levels of excitation and behaviour volatility to be assessed.

As discussed in Section 9.1.4, the methods developed for predicting the magnitude and location of static load applied to a structure may be directly used for predicting the magnitude and location of damage applied to a structure due to the relationship between changes in frequency induced by changes in mass and those induced by changes in stiffness.

9.1 Equations for weak non-linear modal behaviour

Changes in mass and stiffness of structures in-service cause all civil structures to exhibit changes in dynamic parameters due to WNLMB. Changes in mass or the static load of a structure may be induced by the additional mass associated with users of the structures, such as pedestrians or vehicles, non-permanent structural fixings and furniture, or the applied mass of ponding water or snow. Changes in stiffness will be induced by thermal expansion and contraction of the structure, thermally-induced changes in the behaviour of structural supports, as well as damage to structural elements and the degradation of construction materials.

A failure to account for the real-world variability of structural dynamic behaviour associated with WNLMB will inevitably impede the use of OMA for structural health monitoring (SHM), as any changes in structural behaviour associated with damage to the structure may be masked by thermally-induced or static load-induced changes in dynamic behaviour. Similarly, a failure to account for non-damage related causes of WNLMB within a SHM system is likely to result in a large number of false-positives, as a non-damage induced change in behaviour may be mistakenly identified as being associated with structural damage.

Alongside the need to account for WNLMB in SHM, understanding the causes of WNLMB may allow OMA to be used as a basis for estimating a range of other structural parameters usually measured by more expensive or delicate sensors, such as the in-service stiffness of structural elements, the thermal strains which
structures experience in-situ, or the loading applied to structures. This would greatly increase the information which might be extracted through OMA which may be used to guide the design of future structures.

In this section, the relationship between WNLMB induced changes in natural frequencies and the corresponding changes in mass and/or stiffness matrices of linear dynamic systems are presented. This work builds on work by Parloo et al. [350], who presented methods for mode shape scaling, discussed further in Section 9.1.1. The equations derived by Parloo et al. [350] are expanded to provide specific equations for analysing two common drivers of WNLMB in civil structures:

1. changes in temperature,
2. changes in static or gravity loading.

In the context of the OMA and SHM literature, these relationships form part of a physics-based approach to in-service structural monitoring [208]. The advantages of the physics-based approach is that it allows the behaviour of the structure to be predicted for unseen structural and environmental conditions, and all parameters used within the analysis have a direct relationship to the fundamental structural parameters of the system. Later sections of this chapter merge these relationships with methods more traditionally considered as part of a data-based approach to SHM, where machine learning and statistical pattern recognition are used as the basis for quantifying structural behaviours such as damage [207]. The advantage of the data-based approach is that it can provide more efficient methods for analysing the in-service monitoring data, and reduce some of the subjectivity and complexity which may be introduced through physics-based modelling of the structural system. By combining both approaches into physics-based machine learning methods [442], robust and efficient methods for analysing in-situ structural monitoring data which maintain the interpretability of the results may be developed.

This section is focused on WNLMB induced changes in the change in natural frequencies. WNLMB will also cause changes in the damping and mode shapes of a structure in-service, such as the change in mode shapes induced by damage discussed by Kim et al. [443]. However, as discussed in Chapter 7, the mode shape and damping estimates generated using ambient vibration data are far more sensitive to noise within the data and non-zero forced components of the estimated free-response of the structure. This sensitivity results in the variation in damping ratios and mode shapes being primarily dominated by modal errors, associated
with the approximation of the structural response within the modal analysis, which may obscure WNLMB induced changes in the parameters. Despite the focus on WNLMB induced changes in natural frequencies within this chapter, many of the methods discussed in later sections may be adapted for the analysis of changes in damping and mode shapes.

9.1.1 Modal mass and modal stiffness

To relate the changes in stiffness and mass of a structure to real-valued structural parameters, the relationship between natural frequencies, the stiffness and mass matrices, and the mode shapes of a structure must be understood. The natural frequencies of a structure are the frequencies at which a structure resonates when excited. Each natural frequency has a characteristic vector mode shape \( \{ \psi \} \).

The shape of a mode is dictated by the stiffness matrix of the structure \([K]\): the resistance to deformation due to the stiffness, and the mass matrix of a structure \([M]\): the inertial resistance to deformation [15]. The characteristic equation of motion for the free oscillation of an undamped multi-degree of freedom system can be derived from these quantities as given by Equation 9.1 where \( \{ \ddot{X} \} \) is the vector of acceleration responses of the system, \( \{ X \} \) is the vector of displacement responses of the system and \( \{ 0 \} \) is a vector of zeros.

\[
[M]\{ \ddot{X} \} + [K]\{ X \} = \{ 0 \} \tag{9.1}
\]

For a linear dynamic system, the displacement response at any time \( t \) is harmonic such that \( \{ X(t) \} = \{ \psi \} e^{i\omega t} \), where \( \{ \psi \} \) is the mode shape: a vector of constants giving the amplitude of the response at each point on the system, and \( \omega \) is a natural frequency of the system response. Substituting \( \{ X(t) \} = \{ \psi \} e^{i\omega t} \) and \( \{ \dot{X}(t) \} = -\omega^2 \{ \psi \} e^{i\omega t} \) into Equation 9.1 gives Equation 9.2.

\[
-\omega^2 [M]\{ \psi \} e^{i\omega t} + [K]\{ \psi \} e^{i\omega t} = \{ 0 \} \tag{9.2}
\]

Dividing Equation 9.2 by the shared constant term \( e^{i\omega t} \) gives Equation 9.3.

\[
-\omega^2 [M]\{ \psi \} + [K]\{ \psi \} = \{ 0 \} \tag{9.3}
\]

Equation 9.3 may be rearranged to give Equation 9.4.

\[
[K]\{ \psi \} = \omega^2 [M]\{ \psi \} \tag{9.4}
\]
Equation 9.4 is of the form of a generalized eigenvalue problem: \( Ax = \lambda I x \). The squared natural frequencies \( \omega^2 \) and mode shape vectors \( \{ \psi \} \) which fulfill the equivalence given by Equation 9.4 are respectively the eigenvalues and eigenvectors of the equation.

Throughout this chapter, the equations of WNLMB will be presented with the natural frequencies \( \omega \) expressed in units of radians per second. However, to aid the interpretability of the results, frequency results \( f \) will be presented in units of hertz (Hz) with the conversion between \( f \) and \( \omega \) given by Equation 9.5.

\[
f = \frac{\omega}{2\pi} \quad (9.5)
\]

In Equation 9.4 the matrices of mass \( [M] \) and stiffness \( [K] \) are both real-valued, with respective units of kg and N/m, or a scaled equivalent. The mode shape \( \{ \psi \} \) is unitless with arbitrary scaling. The relationship between the amplitude of structural response for mode \( j \), \( \{ \psi_j \} \), and the real-valued mass and stiffness of the structure is given by the modal mass of mode \( j \), \( m_{m,j} \), and modal stiffness of mode \( j \), \( m_{k,j} \). The modal mass and modal stiffness for a mode \( j \) are defined by Brownjohn and Pavic [353] through Equations 9.6 and 9.7.

\[
m_{m,j} = \{ \psi_j \}^T [M] \{ \psi_j \} \quad (9.6)
\]
\[
k_{m,j} = \{ \psi_j \}^T [K] \{ \psi_j \} \quad (9.7)
\]

As discussed by Brownjohn and Pavic [353], two methods for scaling the mode shapes are commonly used in the analysis of civil structures: unity-scaling, where the mode shapes are scaled to have a maximum absolute amplitude of unity, and mass-normalization, where mode shapes are scaled such that the modal mass has a value of unity.

In mass-normalization, the mode shape \( \{ \psi_j \} \) is scaled such that \( m_{m,j} = 1 \). This is achieved using a factor \( \alpha_{M,j} = \sqrt{m_{m,j}} \), with the scaled mode shape \( \{ \phi_{M,j} \} \) defined as \( \{ \psi_j \} = \alpha_{M,j} \{ \phi_{M,j} \} \). Substituting the modal mass normalized mode shape into Equation 9.6 gives Equation 9.8.

\[
\alpha_{M,j} \{ \phi_{M,j} \}^T [M] \alpha_{M,j} \{ \phi_{M,j} \} = m_{m,j} \quad (9.8)
\]

Using the definition of \( \alpha_{M,j} \) given above, Equation 9.8 can be expanded as Equation 9.9.

\[
\sqrt{m_{m,j}} \{ \phi_{M,j} \}^T [M] \sqrt{m_{m,j}} \{ \phi_{M,j} \} = m_{m,j} \quad (9.9)
\]

Equation 9.9 can then be simplified to give the definition defined above through
Equation 9.10 and Equation 9.11

\[ m_{m,j}{\phi_{M,j}}^T [M] {\phi_{M,j}} = m_{m,j} \]  
\[ {\phi_{M,j}}^T [M] {\phi_{M,j}} = 1 \]  

(9.10)  
(9.11)

In practice, unity scaling of mode shapes is more commonly used, as mass-normalization will result in very small mode shape amplitudes due to the high mass of civil structures. For this reason the unity-scaled mode shapes \{\phi\} are used throughout this chapter. A new factor, \( \alpha \) is introduced, defined for mode \( j \) through Equation 9.12 to relate the mass-normalized mode shape, \{\phi_{M}\}, and unity-scaled mode shape, \{\phi\}.

\[ {\phi_{M,j}} = \alpha_j {\phi_j} \]  

(9.12)

The primary use of mode shape scaling factors in OMA is for scaling of mode shapes such that the relative amplitude of the modes for a given applied force may be calculated [350, 352, 353, 351]. However, mode shape scaling factors are also crucial in the analysis of WNLMB as it allows changes in the natural frequencies of a system to be related to the real-valued mass and stiffness matrices of the structure through the derivation presented in Section 9.1.2.

By understanding the relationship between the mode shape scaling and the drivers of WNLMB introduced at the start of this section, the change in natural frequencies associated with different forms of WNLMB may be quantified, isolated and predicted. This can allow the real-valued changes in stiffness and mass of the structure to be estimated based on the changes in natural frequencies and allow automated identification of modes of vibration in the presence of WNLMB.

9.1.2 Mass-change, stiffness-change and the mode shape scaling factor

This section prevents the derivation of the mass-change and stiffness-change approaches for estimating the mode shape scaling factor, \( \alpha \), for unity-scaled mode shapes, \( \phi \), adapted from Parloo et al. [350] and Khatibi et al. [351].

As discussed above, the natural frequency of a mode \( \omega_0 \) is given by the generalized eigenvalue problem in Equation 9.13 which relates the mass ([M]) and stiffness ([K]) matrices, scaled by the corresponding mode shape \{\psi_0\}.

\[ [K]\{\psi_0\} = \omega_0^2 [M]\{\psi_0\} \]  

(9.13)
If a small change is made to the stiffness matrix \([\Delta K]\) and the mass matrix \([\Delta M]\), it is expected that there will be a corresponding change in the natural frequency. The natural frequency of the mode after the change in the stiffness-matrix and mass-matrix is denoted as \(\omega_1\), with a corresponding mode shape \(\{\psi_1\}\) and is given by Equation 9.14

\[
([K] + [\Delta K]) \{\psi_1\} = \omega_1^2 ([M] + [\Delta M]) \{\psi_1\}
\]  

(9.14)

Rearranging and equating Equation 9.13 and Equation 9.14 gives Equation 9.15

\[
[M] \{\psi_0\} \omega_0^2 - [K] \{\psi_0\} = ([M] + [\Delta M]) \{\psi_1\} \omega_1^2 - ([K] + [\Delta K]) \{\psi_1\}
\]  

(9.15)

Rearranging Equation 9.15 further so that the mass- and stiffness- terms are grouped gives Equation 9.16

\[
[M] \{\psi_0\} \omega_0^2 - ([M] + [\Delta M]) \{\psi_1\} \omega_1^2 = [K] \{\psi_0\} - ([K] + [\Delta K]) \{\psi_1\}
\]  

(9.16)

Equation 9.16 may then be expanded to give Equation 9.17

\[
[M] \{\psi_0\} \omega_0^2 - [M] \{\psi_1\} \omega_1^2 - [\Delta M] \{\psi_1\} \omega_1^2 = [K] \{\psi_0\} - [K] \{\psi_1\} - [\Delta K] \{\psi_1\}
\]  

(9.17)

Rearranging and factoring Equation 9.17 gives Equation 9.18

\[
[M] (\{\psi_0\} \omega_0^2 - \{\psi_1\} \omega_1^2) - [\Delta M] \{\psi_1\} \omega_1^2 = [K] (\{\psi_0\} - \{\psi_1\}) - [\Delta K] \{\psi_1\}
\]  

(9.18)

Assuming that the relative change in the mass and stiffness matrices are small, it will be assumed that the change in the mode shape will also be small. Therefore it is assumed that \(\{\psi_0\} \cong \{\psi_1\} = \{\psi\}\). Substituting this result into Equation 9.18 gives Equation 9.19

\[
[M] (\{\psi\} \omega_0^2 - \{\psi\} \omega_1^2) - [\Delta M] \{\psi\} \omega_1^2 = [K] (\{\psi\} - \{\psi\}) - [\Delta K] \{\psi\}
\]  

(9.19)

Equation 9.19 can be rearranged and simplified as Equation 9.20

\[
[M] \{\psi\} (\omega_0^2 - \omega_1^2) = [\Delta M] \{\psi\} \omega_1^2 - [\Delta K] \{\psi\}
\]  

(9.20)

Pre-multiplying both sides of Equation 9.20 by the transposed mode shape matrix, \(\{\psi\}^T\) gives Equation 9.21

\[
\{\psi\}^T [M] \{\psi\} (\omega_0^2 - \omega_1^2) = \{\psi\}^T ([\Delta M] \{\psi\} \omega_1^2 - [\Delta K] \{\psi\})
\]  

(9.21)
Substituting the mass-normalized mode shapes, \( \{ \psi \} = \alpha_M \{ \phi_M \} \), into Equation 9.21 gives Equation 9.22.

\[
\alpha_M \{ \phi_M \}^T [M] \alpha_M \{ \phi_M \} \left( \omega_0^2 - \omega_1^2 \right) = \alpha_M \{ \phi_M \}^T \left( [\Delta M] \alpha_M \{ \phi_M \} \omega_1^2 - [\Delta K] \alpha_M \{ \phi_M \} \right)
\]

(9.22)

Gathering the terms in Equation 9.22 gives Equation 9.23.

\[
\alpha_M^2 \{ \phi_M \}^T [M] \{ \phi_M \} \left( \omega_0^2 - \omega_1^2 \right) = \alpha_M^2 \{ \phi_M \}^T \left( [\Delta M] \{ \phi_M \} \omega_1^2 - [\Delta K] \{ \phi_M \} \right)
\]

(9.23)

As defined in the previous section, \( \{ \phi_M \}^T [M] \{ \phi_M \} = 1 \), therefore Equation 9.23 may be reduced to Equation 9.24.

\[
\alpha_M^2 \left( \omega_0^2 - \omega_1^2 \right) = \alpha_M^2 \{ \phi_M \}^T \left( [\Delta M] \{ \phi_M \} \omega_1^2 - [\Delta K] \{ \phi_M \} \right)
\]

(9.24)

Equation 9.24 can be simplified further by dividing through by \( \alpha_M^2 \) to give Equation 9.25.

\[
\left( \omega_0^2 - \omega_1^2 \right) = \{ \phi_M \}^T \left( [\Delta M] \{ \phi_M \} \omega_1^2 - [\Delta K] \{ \phi_M \} \right)
\]

(9.25)


\[
\left( \omega_0^2 - \omega_1^2 \right) = \{ \phi \}^T \left( [\Delta M] \{ \phi \} \omega_1^2 - [\Delta K] \{ \phi \} \right)
\]

(9.26)

Factoring the right hand side of Equation 9.26 gives Equation 9.27.

\[
\left( \omega_0^2 - \omega_1^2 \right) = \alpha^2 \{ \phi \}^T \left( [\Delta M] \{ \phi \} \omega_1^2 - [\Delta K] \{ \phi \} \right)
\]

(9.27)

Therefore, the value of \( \alpha \) may be calculated through the application of a known change to the real-valued mass and/or stiffness matrix of the system through solving Equation 9.28.

\[
\alpha = \sqrt{\frac{\omega_0^2 - \omega_1^2}{\{ \phi \}^T \left( [\Delta M] \{ \phi \} \omega_1^2 - [\Delta K] \{ \phi \} \right) \{ \phi \}}}
\]

(9.28)

Alternatively, if the value of \( \alpha \) is known, the frequency of the system due to some change in the mass or stiffness matrix of the system is given by Equation 9.29.

\[
\omega_1 = \sqrt{\frac{\omega_0^2 + \alpha^2 \{ \phi \}^T [\Delta K] \{ \phi \}}{1 + \alpha^2 \{ \phi \}^T [\Delta M] \{ \phi \}}}
\]

(9.29)
9.1.3 Models of WNLMB

Equation 9.29 provides a generalized relationship between changes in the mass- and/or stiffness-matrices of a structure, and the corresponding change in natural frequency of the system. This relationship can be used to identify, quantify and estimate a range of causes of WNLMB. In this section, Equation 9.29 is used to derive specific relationships for thermally-induced and static-load induced changes in natural frequency.

9.1.3.1 Thermally-induced WNLMB

As the temperature of a structure changes there are corresponding changes in the stiffness of the structural materials, joints, and boundary conditions \[241\]. Different thermally-induced changes in effective stiffness are likely observed for each individual element and connection, as well as the physical processes which govern their interaction such as contact area or friction. Therefore the change in stiffness of a given mode \[\Delta K\] is formed from a matrix of functions relating the change in temperature of an element of the stiffness matrix \(j\) to its corresponding change in stiffness as given by Equation 9.30.

\[
\Delta k_j = f(\Delta T_j) \quad (9.30)
\]

For typical construction materials subjected to relatively small variations in temperature, this change in real stiffness is approximately linear \[338\] such that Equation 9.30 may reasonably be approximated as a proportional relationship \(\Delta k_j \propto \Delta T_j\).

The relationship between changes in stiffness and changes in temperature can be simplified further if it is assumed that the spatial variation of the temperature, and by extension the spatial variation of thermally-induced changes in stiffness, is negligible. Under this assumptions the change in stiffness of a mode can be approximated as \(\{\phi\}^T[\Delta K]\{\phi\} \approx \Delta k\), where \(\Delta k\) is proportional to the real-valued average change in effective modal stiffness. Incorporating the assumptions of linear proportionality with the assumptions of small spatial variations in temperature and effective modal stiffness variation, the generalized relationship between stiffness and temperature given by Equation 9.31 can be formed where \(\Delta T\) is the average change in temperature of the structure.

\[
\{\phi\}^T[\Delta K]\{\phi\} \approx \Delta k \propto \Delta T \quad (9.31)
\]
A temperature-change factor $\alpha_K$ is defined to relate the change in temperature to the average change in the real-valued stiffness of the mode scaled by the squared mode shape scaling factor $\alpha^2$ through Equation 9.32:

$$\alpha_K = \frac{1}{\omega_0^2} \left( \frac{\alpha^2 \Delta k}{\Delta T} \right)$$

(9.32)

Note that while the inclusion of $\omega_0$, the natural frequency of the mode before the temperature change, in this definition is largely unneeded, it can have practical benefits when estimating $\alpha_K$ as higher frequency modes have been observed to be more sensitive to changes in temperature. Therefore when estimating $\alpha_K$ simultaneously for multiple modes, as discussed in Section 9.4, the scaling can help constrain the possible solution space to be explored.

Substituting the definition given in Equation 9.32 into Equation 9.29 and assuming the change in mass of the structure is zero gives Equation 9.33.

$$\omega_1 = \sqrt{\omega_0^2 + \alpha_K \Delta T \omega_0^2}$$

(9.33)

Equation 9.33 may be factored to give Equation 9.34.

$$\omega_1 = \sqrt{\omega_0^2(1 + \alpha_K \Delta T)}$$

(9.34)

9.1.3.2 Static load-induced WNLMB

Civil structures are subjected to changes in the effective mass matrix $[\Delta M]$ due to changes in the matrix of static loading $[\Delta L]$ applied to the structure. These changes can arise from the self-weight of pedestrians or vehicles, as well as temporary fittings, fixtures and furnishings added to a structure. Note, throughout this chapter, it is assumed that the change in the mass matrix of the structure is proportional to the change in the matrix of static loading applied to the structure, $[\Delta M] \propto [\Delta L]$.

If the change in static load can be well approximated by a single change in the mass matrix at element $j$, the mass-change matrix only contains a single non-zero element. Therefore the unity-scaled mode shape vector $\{\phi\}$ can be replaced by the amplitude of the unity-scaled mode shape, $A_j$, at the location $j$ of the static load. Therefore the corresponding change in frequency under the assumption of no change in the stiffness matrix of the structure is given by Equation 9.35, where
\[ \Delta L \text{ is the sum change in static loading.} \]

\[ \omega_1 = \sqrt{\frac{\omega_0^2}{1 + \alpha^2 \Delta L A_j^2}} \quad (9.35) \]

If the change in static loading cannot be approximated as having occurred at a single location, an alternative approach is to define an equivalent change in mass of the structure using Equation 9.36.

\[ \{\phi\} [\Delta M] \{\phi\} = \alpha^2 \beta_L \Delta L \quad (9.36) \]

In Equation 9.36, \( \beta_L \) is a weighting factor between zero and one which relates the distribution of changes in static loading to the positions on the mode shape at which the change in static loading is applied. If the change in static loading is primarily located in the vicinity of the nodes of a mode, where \( A_j \rightarrow 0 \), the value of \( \beta_L \) is close to zero, and the observed change in frequency will also approach zero. If the change in static loading is primarily located at the peak of a mode, where \( A_j \rightarrow 1 \), then \( \beta_L \rightarrow 1 \) and the observed change in frequency approaches that which would be observed if the loading was concentrated at the peak of the mode. The relationship between the value of \( \beta_L \) and the amplitude of the mode shape is a squared relationship, as \( \beta_L = A_j^2 \) if the change in static loading is localized to a single location \( j \) of the mass matrix. Combining Equation 9.36 with Equation 9.29 under the assumption that \([\Delta K] = 0\) gives Equation 9.37.

\[ \omega_1 = \sqrt{\frac{\omega_0^2}{1 + \alpha^2 \beta_L \Delta L}} \quad (9.37) \]

If it is assumed that \( \beta_L = 1 \), Equation 9.37 provides a lower bound method for estimating the magnitude of changes in static loading in the absence of knowledge of its location. Similarly, it can be seen from inspection of Equation 9.37 that if the change in static loading is located at the node point of a mode shape such that \( \beta_L = 0 \), no observable change in frequency will be observed, giving an upper bound on the change in static loading of +/-\( \infty \).

For clarity a new mode shape scaling factor to be used for changes of static loading \( \alpha_L \) is introduced, defined as \( \alpha_L = \alpha^2 \). Therefore, Equation 9.36 becomes Equation 9.38.

\[ \omega_1 = \sqrt{\frac{\omega_0^2}{1 + \alpha_L \beta_L \Delta L}} \quad (9.38) \]
9.1.4 Relationship between damage and static load in modal behaviour

Later sections of this chapter will primarily focus on the use of WNLMB for static load prediction, as it is simpler to apply a known static load to experimental and real-world structures than it is to apply known damage induced changes in stiffness to the same structure. However, all findings and methods presented in this chapter for static load prediction are directly transferable to quantifying the magnitude and location of damage to a structure. This can be demonstrated through considering the relationship between changes in the mass and/or stiffness matrices of a structure presented previously in Equation 9.29 and reproduced in Equation 9.39.

\[
\omega_1 = \sqrt{\frac{\omega_0^2 + \alpha^2\{\phi\}^T[\Delta K]\{\phi\}}{1 + \alpha^2\{\phi\}^T[\Delta M]\{\phi\}}} \tag{9.39}
\]

Through inspection of Equation 9.39, it is evident that the change in frequency associated with any decrease in the stiffness matrix \([\Delta K]\) might also be associated with an increase in the mass change matrix \([\Delta M]\).

In practice, the only way in which changes in mass and stiffness may be distinguished is the context in which they occur. Changes in static loading may be transient; associated with the temporary mass from pedestrians or vehicles, or semi-permanent; associated with the introduction of furniture or fixings onto a structure. For civil infrastructure such as bridges, the changes in static loading are also significantly more likely to be an increase in the mass of the structure from a baseline level, due to the application of static loads, rather than a reduction in the mass of a structure from the original baseline mass. In contrast, changes in stiffness are likely to manifest within the modal behaviour as a permanent step-change in the natural frequencies, due to the sudden appearance of damage, or an irreversible steady decrease in stiffness associated with the slow propagation of damage within the structure. If these events were viewed in the context of having been changes in mass they would correspond to either a sudden and permanent change in the mass of a structure, or a steady increase or decrease in the mass of a structure, both of which seem unlikely given the typical static loads applied to civil structures in-service.

Therefore, it is more prudent to devise models for predicting the static loading applied to structures in-service, as these provide real-valued estimates of the static loading alongside providing a basis for damage detection as described above. It should be stressed that damage detection based on the identification of changes in frequency is unlikely to be robust if there is a failure to account for changes in mass.
in static loading, as changes in the natural frequencies associated with changes in static loading may mask damage-induced changes in natural frequencies and prevent early detection of damage to the structure.

9.1.5 Incorporating uncertainty into the mode shape scaling factor

The conventional method of estimating the mode shape scaling factor ($\alpha$) in OMA is through the mass change method, discussed in detail in Parloo et al. [350], where a known mass is applied to a structure at a known location. The natural frequency of a mode $j$ is measured before ($\omega_{0,j}$) and after ($\omega_{1,j}$) application of the known mass $\Delta m$. With prior knowledge of the unity-scaled mode shapes, $\phi$, the value of $\alpha$ for mode $j$ may then be calculated using Equation 9.29.

If the mode shape scale factor is known, the associated factors relating changes in frequency to changes in static loading, $\alpha_L$, may be calculated using Equation 9.38. The temperature-induced frequency change factor $\alpha_K$ cannot be calculated through this method, as by definition it includes an unknown factor relating the change in temperature to the weighted average of the change in stiffness of the structure. Alongside this limitation, the mass change method requires prior knowledge of the mode shapes of the structure and does not typically account for the uncertainty inherent within the estimates of the natural frequencies $\omega_{0,j}$ and $\omega_{1,j}$. As the natural frequencies are squared within the calculation of $\alpha$, small errors in the frequency estimates can translate to large errors in the mode shape scale factor.

To incorporate the uncertainty in the frequency estimates it is necessary to use multiple independent or semi-independent estimates of the frequencies of the system under a range of different changes in mass (for calculation of $\alpha_L$) or change in temperature (for calculation of $\alpha_K$). Subsequent sections of this chapter will examine regression methods and probabilistic methods through which frequency estimates from different applied static loadings or temperatures, denoted as a vector $\{\omega_i\}$, with accompanying vectors of temperature $\{T_i\}$ and/or static loading $\{L_i\}$, may be used to estimate the parameters of the equations given in the previous sections and summarized in Table 9.1. Note that previous definitions given in this section were based on the change in temperature $\Delta T_i$ and change in $\Delta L_i$. In subsequent sections, it is assumed that all changes in temperature and static loading are measured from a baseline of zero such that $\Delta T_i = T_i$ and $\Delta L_i = L_i$.
9.1.6 Equation summary

In this section the equations for mode shape scaling have been used as the basis for deriving relationships between thermally-induced WNLMB and static-load induced WNLMB. A summary of the key equations derived in this section is presented in Table 9.1. As discussed in Section 9.1.4, the equation for static-load induced WNLMB may also be reformulated to relate damage-induced changes in stiffness of a structure to changes in natural frequency.

Table 9.1: Equations for simple models for relating changes in frequency to drivers of WNLMB

<table>
<thead>
<tr>
<th>Driver of WNLMB</th>
<th>Isolated location ( j )</th>
<th>Distributed location</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal</td>
<td>N/A</td>
<td>( \omega_1 = \sqrt{\omega_0^2(1 + \alpha_K \Delta T)} )</td>
</tr>
<tr>
<td>Static load</td>
<td>( \omega_1 = \sqrt{\frac{\omega_0^2}{1 + \alpha_L \Delta L \lambda_j^2}} )</td>
<td>( \omega_1 = \sqrt{\frac{\omega_0^2}{1 + \alpha_L \beta_L \Delta L}} )</td>
</tr>
</tbody>
</table>

9.2 Artificial generation of modal estimates exhibiting WNLMB

Within this chapter, the numerically generated modal estimates of frequency introduced in Chapter 8 will be used to demonstrate various methods for estimating the parameters of the WNLMB equations derived in Section 9.1. These artificial frequency estimates have the characteristics listed below and represent an increased complexity when compared to those used in Chapter 8:

- Each mode of vibration \( j \) has a “true” natural frequency \( \omega_j \).
- The modal errors in the frequency estimates for each mode of vibration are Gaussian and independent of the WNLMB.
- The errors in the frequency estimates are not correlated between time steps.
- The number of modal estimates for each mode of vibration varies between time steps.
- The modal estimates for each time step include frequency estimates from one or more modes of vibration alongside spurious modal estimates associated with noise in the data.
• The modal estimates may include close or overlapping modes of vibration.

• Modes of vibration exhibit systematic WNLMB based on the equations for WNLMB given in Table 9.1.

• Within the equations of WNLMB for changes in static loading, it is assumed that the value of $\beta_L$ is constant for a time step, representing a constant distribution of load on the structure at that time, and is independent between time steps, representing a change in the load distribution with time. This assumption is interrogated further in Section 9.4.

9.3 Regression methods for quantifying WNLMB

The parameters of the equations of WNLMB derived in Section 9.1 can be estimated through regression methods or regression analysis, statistical methods for estimating the relationships between dependent or response variables, such as the natural frequency of a mode, and independent or feature variables, such as the variation in temperature of a structure. The regression methods presented in this section form part of a two-step analysis process for quantifying WNLMB as individual modes of vibration must first be isolated within the OMA modal estimates prior to the regression analysis, as highlighted in the block diagram presented in Figure 9.1. A one-step analysis process based on the use of probabilistic mixture models is presented in Section 9.4.

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**Figure 9.1**: Block diagram for application of regression methods for analysis of WNLMB.

In this section two regression methods, linear regression and quantile regression, are presented. For both methods it is assumed that the feature variable

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$x$ is related to the response variable $y$ through $y = a + Bx + \varepsilon$. In ordinary least squares (OLS) linear regression, the parameters $a$ and $B$ are estimated through minimizing the sum squared error term $\sum_i \varepsilon_i^2$. In quantile regression, $a$ and $B$ are estimated through minimizing the pinball loss function $\tau_Q \left( \sum_{y_i > (a + Bx_i)} |y_i - (a + Bx_i)| + (1 - \tau_Q) \sum_{y_i < (a + Bx_i)} |y_i - (a + Bx_i)| \right)$, a weighted sum of the positive and negative error terms for a given quantile level $\tau_Q$.

As discussed in Chapter 2, linear regression has been the primary method for analysis of WNLMB and analysis of in-situ structural monitoring data more broadly. However, in this section it is demonstrated that quantile regression provides a more robust alternative for quantifying variations in structural behaviour induced by changes in temperature or loading. Technical details relating to the application of linear and quantile regression, alongside an introduction to other regression methods for the analysis of WNLMB, are presented in Appendix J.

### 9.3.1 Regression analysis of thermally-induced WNLMB

Figure 9.2 shows artificial data which exhibits thermally-induced WNLMB which may be encountered when undertaking long-term monitoring of civil structures in-service. Alongside the thermally-induced variation of the true natural frequency of the mode, the data includes normally-distributed noise about the true frequency of the mode, an approximation of the modal errors observed for modal estimates from real-structures, and a uniformly distributed background noise component, simulating modal estimates associated with fitting of the spurious or noise components of the structural response, both discussed in Chapter 7.

![Figure 9.2: Example of artificially generated frequency estimates exhibiting thermally induced weak non-linear modal behaviour.](image)

For the thermally-induced WNLMB presented in Figure 9.2, the relationship between frequency $\omega_j$ and temperature $T_j$ is given by $\omega_j = \sqrt{\omega_0^2(1 + \alpha K T_j)}$. This
relationship can be rearranged to the form \( y = a + Bx \) as \( \omega_j^2 = \omega_0^2 + \alpha_K T_j \omega_0^2 \), where \( a = \omega_0^2 \) and \( B = \alpha_K \omega_0^2 \).

A comparison of the true and fitted relationships for linear and quantile regression analysis of the artificial data with thermally-induced WNLMB introduced in Figure 9.2 is presented in Figure 9.3. For the linear regression, \( \omega_0 \) is overestimated by approximately 0.1%, and \( \alpha_K \) is underestimated by approximately 9.4%. The 50% quantile regression results provide a closer estimate of \( \omega_0 \) and \( \alpha_K \), with \( \omega_0 \) being overestimate by approximately 0.03% and \( \alpha_K \) underestimated by approximately 2.93%.

![Figure 9.3: Comparison of performance of OLS linear regression and quantile regression for quantification of thermally-induced WNLMB in artificially generated frequency estimates.](image)

These errors in the parameter estimates occur due to the non-symmetrical distribution of spurious modal estimates or background noise relative to the mean frequency of the mode. While this noise is assumed to be uniformly distributed within the plotted frequency range, the variation in the mean frequency at each time step due to the thermal WNLMB ensures that the noise is non-symmetrically distributed relative to the mean frequency of the mode across all time steps.

An advantage of quantile regression is that the fitted 5% and 95% quantiles provide an effective means of separating spurious or noise estimates from frequency estimates associated with the mode of vibration, allowing the variance in the estimated parameters of the mode of vibration to be quantified. As discussed
in detail in Section 9.4, this is an important requirement for the application of OMA as a method of damage detection, as it allows the probability of damage to be assessed in the presence of uncertainty in the modal estimates.

The degree of over- or under-estimation of the WNLMB parameters estimated using linear and quantile regression is dependent on the bounds used when isolating the modes of vibration of interest, as these control the distribution of spurious frequency estimates relative to the distribution of modal frequency estimates. The error in the estimated parameters will also increase if the noise within the estimates is non-uniform over time, particularly where the quantity of noise estimates is correlated with temperature. An example of when this might occur is the decreased signal-to-noise ratio of the measured response that is likely to be associated with periods of low excitation of bridges during nighttime hours, where it might be expected that the temperature of the structure will be lower.

9.3.2 Regression analysis of load-induced WNLMB

If the magnitude of loading, \( L \), and the weighted squared-amplitude of the unity scaled mode shape at the locations of the applied loading, \( \beta_L \), are known the parameters of the load-induced WNLMB equation \( \omega_i = \frac{\sqrt{\omega_0^2}}{1 + \alpha_L \beta_L L} \) can be estimated through linear or quantile regression. For load-induced WNLMB, the regression coefficients \( a = 1/\omega_0^2 \) and \( B = \alpha_L/\omega_0^2 \) may be estimated using the inverse squared frequency \( 1/\omega_j^2 \) as the response variable, as described for thermally-induced WNLMB.

The challenge when analysing load-induced WNLMB through regression methods is that the location of the loading, and by extension the location factor \( \beta_L \), is typically not known. Alongside this, there is an inherent uncertainty within any estimate of the mode shapes of a structure which may result in errors and biasing of the location factor \( \beta_L \).

As shown in the artificial data presented in Figure 9.4, for a given applied loading, there is a range of changes in natural frequency of a mode which may be observed alongside the variation in frequency associated with modal errors. The upper bound of this variation in frequency correspond to when the load is concentrated at the node point of a mode such that \( \beta_L = 0 \), where there will be no detectable change in natural frequency. The lower bound of the variation in frequency occurs when the load is located at the peak of a mode shape such that \( \beta_L = 1 \), where the maximum possible change in natural frequency will be observed.

If the artificially generated data presented in Figure 9.4 is analysed with OLS
linear regression assuming $\beta_L L = L$, that is neglecting the value of $\beta_L$, it can be seen from the results presented in Figure 9.5 that the linear regression coefficient approximates the value given by $\beta_L = 0.5$, as illustrated in Figure 9.5. This approximation of $\beta_L = 0.5$ occurs within the linear regression analysis of the artificial data as the values of $\beta_L$ used to generate the artificial frequency estimates were drawn from the uniform distribution $U(0, 1)$. The expected mean of this distribution is 0.5. Theoretically, if $\beta_L$ was known to be this uniform distribution, the true value of $\alpha_L$ might be inferred as twice the regression coefficient $\alpha_L = 2B\omega_0^2$. However, if the true distribution of $\beta_L$ was different, the fitted regression coefficient will approach that given by the mean value of $\beta_L$ and any relationship between the true value of $\alpha_L$ and the regression coefficient would be lost without prior knowledge of the distribution of $\beta_L$. As discussed for the thermally-induced WNLMB, the lack of symmetry of the background noise relative to the mean frequency of the mode results in a biased estimation of the regression coefficients which is clearly visible at low loads.

As can be seen from the quantile regression results for artificially generated frequency estimates presented in Figure 9.5, the fitted median quantile provides an excellent approximation of the variation in load when $\beta_L = 0.5$. The fitted 5% and 95% quantiles also allow for approximation of the expected variation in frequency given by $\beta_L = 0$ and $\beta_L = 1$. These quantiles are not an exact measure of the boundaries given by $\beta_L = 0$ and $\beta_L = 1$ as they are biased by the combined impact of modal errors and the distribution of $\beta_L$.

When the load is exactly zero, modal errors lead to a normally distributed spread of frequency estimates centered on the unloaded frequency of the system, $\omega_0$. This distribution of frequency estimates due to modal errors is reflected in the quantile regression results by the intercept of the 5% and 95% fitted quantiles not being equal to the unloaded frequency of the system.
Figure 9.5: Comparison of performance of OLS linear regression and quantile regression for quantification of load-induced WNLMB in artificially generated frequency estimates.

The biasing associated with these modal errors is compounded by the way in which modal errors interact with the distribution of static load-induced changes in natural frequency. Under the model of distributed static-load induced changes in natural frequency, the frequency estimates for a load $L$ are expected to fall in the range given by $\beta_L L = 0 \ (\beta_L = 0)$ and $\beta_L L = L \ (\beta_L = 1)$. The results for a specific time step remain normally distributed due to the modal errors, as it was assumed within the artificial data generation that there was a fixed value of $\beta_L$ for each time step. However, when the frequency estimates for multiple time steps which have the same load value but different values of $\beta_L$ are plotted as a histogram, the distribution of frequency estimates approaches that given by summing of values sampled from a normal distribution, due to the modal errors, and an inverse uniform distribution, given by the uniform probability of observing any $\beta_L$ value in the range zero to one. An approximation of this distribution is presented in Figure 9.6 generated through random sampling of the distribution $\omega_i = N(\sqrt{(\omega_0^2)/(1 + \alpha_L \beta_L L)}, \sigma_0^2)$, where $\sigma_0^2$ is the variance of the normally distributed errors in the modal estimates, referred to as the modal errors, and the value of $\beta_L$ is generated from a uniform distribution $U(0,1)$.

The effect of the combined modal errors and distribution of $\beta_L$ is that as the load increases, a greater proportion of the frequency estimates are expected to
fall within the boundaries given by $\beta_L = 0$ and $\beta_L = 1$, with the modal errors having a smaller impact on the range of frequency variations when compared to frequency variations associated with different values of $\beta_L$. This provides an explanation for the closer approximation of the $\beta_L L$ boundaries at higher loads visible in Figure 9.5.

9.3.3 Case study - Temperature-change experiment

This section is expanded and adapted from work originally published by Wynne et al. as a chapter in *Civil Structural Health Monitoring as Mass and Temperature Changes in Operational Modal Analysis* [5].

As discussed in the introduction to this chapter, a challenge in identifying the dynamic properties of large structures through OMA is the changes in temperature which may occur during data collection. The effect of these changes may mask any damage to the structure or lead the investigator to incorrectly conclude that damage has occurred. Quantification of such effects requires the use of short-time OMA methods, such as the ST-RDT introduced in Chapter 7 where the structure’s response is assumed to be linear across short periods of time. Once the variation in the frequency of individual modes of vibration due to the thermally-induced WNLMB has been approximated through the short-time OMA methods, regression methods may be used to relate the changes in
frequency to changes in the real-valued stiffness of the structure.

An experiment, referred to as the temperature-change experiment, was carried out to assess the robustness of the ST-RDT as a means for quantifying thermally-induced WNLMB, and to allow comparison of linear regression and quantile regression for estimating the parameters of the models of thermally-induced WNLMB introduced in Section 9.1. A flat plate was heated in laboratory conditions using radiant panels with the changes in dynamic properties estimated during heating, steady temperature and cooling of the plate through the ST-RDT.

A schematic of the experimental setup is presented in Figure 9.7. Modes of vibration were visually identified within the ST-RDT results, with modal estimates separated into alternating blocks of training data, used for fitting of the regression models, and hold-back data, used for assessing the quality of the fitted regression parameters. Further details of the experimental methodology, modal analysis and further results are presented in Appendix K.

For all modes of vibration, the ST-RDT frequency estimates were related to changes in temperature through \( \omega_i = \sqrt{\omega_0^2(1 + \alpha\Delta T)} \). The parameters of this equation were independently fitted to the six modes of vibration using OLS linear regression and quantile regression. For the quantile regression analyses, the 10%, 25%, 50%, 75% and 90% quantiles were fitted. The regression analyses presented in this section used the average temperature recorded across the six thermistors fixed to the beam during testing. The results of multivariable regression, where multiple features are used to predict a single response variable, are discussed in Appendix K. These multivariable regression results exhibit issues with overfitting of the data discussed in detail in Appendix J.

The fitted linear and quantile regression models for six modes of vibration are compared with the ST-RDT modal estimates in Figure 9.8. The linear regres-
sion results, presented in the left column of subplots, display a consistent biasing in the fitted parameters due to the non-uniform distribution of spurious modal estimates relative to the mean frequencies of the modes, as discussed previously for the artificial data in Section 9.3.1. This biasing is most evident for modes 5 and 6, with the fitted parameters resulting in a significant overestimation of the frequencies at the highest temperatures, where the natural frequencies are at their lowest. This biasing is sensitive to the frequency bounds selected, with a lower value for the lower frequency bound likely to lead to a reduction in the biasing at higher temperatures. However, this may result in an underestimation of frequencies at lower temperatures, and may introduce greater levels of noise between the frequency bounds. In comparison, the median fitted quantile estimated using quantile regression closely follows the densest region of frequency estimates which is indicative of the true natural frequency of the system at these temperatures. Alongside this, the other fitted quantiles allow the differences in behaviour during heating and cooling of the plate to be extracted, as discussed further in Appendix K.

An advantage of the physics-based models of WNLMB derived in Section 9.1 is that the fitted model parameters can be related to other structural parameters. For example, the thermal WNLMB scaling coefficients, $\alpha_K$, provides an indication of the change in stiffness of the structure with temperature. For the free-free beam, this change in global stiffness is primarily driven by the change in the beams elastic modulus. The values of $\alpha_K$ fitted in the regression analysis, presented in Figure 9.9, provide excellent agreement with the approximate percentage change in elastic modulus of carbon steel at the peak temperature of the test as given in Meyer et al. [44].

This section has presented analyses of a simple structure under varying temperatures and demonstrated some challenges of extracting dynamic properties from systems exhibiting WNLMB. The results of the regression analyses indicate that quantile regression, rather than the more broadly used linear regression, is a more robust method for estimating the parameters of the thermally-induced WNLMB relationships, as it allows for noise within the frequency estimates to be accounted for within the analysis. The advantage of the physics-based models of WNLMB derived in Section 9.1, that they allow real-valued parameters to be estimated, has been demonstrated.
9.4 Incorporating non-linear modal behaviour into automated mode detection

The regression methods presented in Section 9.3 formed the second stage in a two-step process for analysing WNLMB, requiring modes of vibration to have first
been identified within the modal estimates. There are several issues with this approach for the analysis of long-term monitoring data collected from civil structures in-service. First, isolating modal estimates corresponding to a single mode of vibration may require significant user input for the selection of appropriate frequency bounds or other metrics for separating modes of vibration. This becomes more apparent in the analysis of close modes of vibration, where the natural frequencies may intersect due to WNLMB associated with changes in temperature, loading or damage. There is the added complication that the bounds or metrics selected for isolating a single mode of vibration may not be applicable under all plausible loading and environmental conditions. For example, if hard frequency bounds are used to isolate a mode of vibration there is the possibility that large changes in loading or temperature may cause WNLMB which results in the frequency of the mode exceeding these bounds. The regression methods also fail to explicitly account for either the variability in the frequency estimates at each time step due to modal errors, or the background noise which is inherent in modal estimates from OMA, although as discussed in the previous section, quantile regression allows this variability to be approximated.
These limitations can be overcome if a one-step approach is used which incorporates the models of WNLMB derived in Section 9.1 into the non-Gaussian mixture model introduced in Section 8.3. A block diagram for the one-step approach to automated identification of modes of vibration and quantification of WNLMB is presented in Figure 9.10.

The mixture model introduced in Section 8.3 for automated mode detection was defined through \( n \) Gaussian distributions, which correspond to dense regions of modal estimates associated with individual modes of vibration, and a uniform distribution which is used to account for the background noise in the modal estimates. The probability of a single frequency estimate \( \omega_{i=1} \) given the fitted parameters of the mixture model \( \Delta \), with \( K \) modes of vibration and without consideration of WNLMB, is given by Equation 9.40

\[
P(\omega_{i=1}|\Delta) = \sum_{k=1}^{K} (m_{f,k}P(\omega_{i=1}|N(\omega_k, \sigma_k^2))) + m_{f,K+1}P(\omega_{i=1}|U(f_{low}, f_{high}))
\]

In Equation 9.40, \( P(\omega_{i=1}|N(\omega_k, \sigma_k^2)) \) is the probability of observing the frequency estimate \( \omega_i \) given the probability density function (PDF) of mode \( k \), assumed to be a normal distribution with mean \( \omega_k \) and variance \( \sigma_k^2 \). Similarly, \( P(\omega_{i=1}|U(f_{low}, f_{high})) \) is the probability of observing the frequency estimate \( \omega_i \) given the PDF for the uniform distribution of background noise between frequencies \( f_{low} \) and \( f_{high} \). The relative probability of the frequency estimate having come from one of the \( K + 1 \) distributions, independent of the value of the frequency estimate \( \omega_i \), is given by the mixing fractions \( m_{f,1}, m_{f,2}, \ldots, m_{f,K+1} \), defined such that \( \sum_{k=1}^{K+1}(m_{f,k}) = 1 \).

Within the mixture model defined in Equation 9.40, it was assumed that the mean natural frequency of a mode of vibration is constant over time and
that the distribution of modal estimates from that mode is given by the normal distribution \( N(\omega_k, \sigma_k^2) \).

If there are measurements of the temperature \( T_i \) of the structure and the location and magnitude of any loading \( L_i \), this information may be incorporated into the mean of the normal distribution such that the mean frequency \( \omega_k \) will vary over time according to Equation 9.41

\[
\omega_k = \sqrt{\frac{\omega_{0,k}^2 (1 + \alpha_{K,k} T_i)}{1 + \alpha_{L,k} \beta_{L,k} L_i}}
\]

In Equation 9.41 if \( T_i, \beta_{L,k} \) (the location of loading relative to the mode shapes of the structure) and \( L_i \) are known, the unknown parameters \( \alpha_{K,k} \) and \( \alpha_{L,k} \) can be found through maximizing the likelihood, the product of probabilities of the observed frequency estimates, each a sum given by the probabilistic mixture model defined by Equation 9.42

\[
P(\omega_i | \Delta, T_i, \beta_L, L_i) = \prod_{n=1}^{N} \left[ \sum_{k=1}^{K} m_{f,k} P \left( \omega_{i=1} \left| N \left( \sqrt{\frac{\omega_{0,k}^2 (1 + \alpha_{K,k} T_i)}{1 + \alpha_{L,k} \beta_{L,k} L_i}, \sigma_k^2} \right) \right) + m_{f,K+1} P \left( \omega_{i=1} | U(f_{low}, f_{high}) \right) \right] \]

This maximization can be carried out using an optimizer, as discussed in Section 8.3 as the gradient of the likelihood function may be automatically calculated using automatic differentiation through Python packages such as Autograd, used throughout this chapter.

As has been discussed in previous sections, it is rare to have exact values for \( \beta_L \) for any modes of vibration, as this requires knowledge of both where loading is located on the structure and accurate estimates of all mode shapes. To account for the lack of knowledge of \( \beta_L \), a distribution of \( \beta_L \) may be assumed. The simplest assumption of \( \beta_L \) which may be made is that there is an equal probability of observing all values of \( \beta_L \). The accompanying probabilistic distribution for this assumption is that the value of \( \beta_L \) is generated from a uniform distribution \( \beta_L = U(0, 1) \).

In the physical context of some real-world structures, where the locations of loading may be constrained, it may be necessary to refine this assumption. However, it is important to stress that the distribution of \( \beta_L \) is used to model the distribution of static loading relative to the mode shapes of the structure and not
the distribution relative to the structure itself. While a physical relationship will exist between the distributions of loading relative to the structure and relative to the mode shapes, restraints on the distribution of loading relative to the structure may lead to only marginal differences in the distribution of loading relative to the mode shapes. To take an extreme example of a constraint on the physical distribution of static loading, it is unlikely that static loading will be applied to the underside of a bridge deck. However, despite this extreme constraint on the physical distribution of loading on the structure, the distribution of loading relative to the vertical mode shapes of the bridge deck is unchanged, as the loading may still occur at any location on the top surface of the bridge deck.

Introducing the assumed uniform distribution of $\beta_L$ into Equation 9.42 gives Equation 9.43 for the probability of the frequency estimates given the fitted model parameters $\Delta = \{\omega_0, \alpha_K, \alpha_L, \sigma_0, m\}$, temperature $T_i$ and static loading $L_i$.

$$P(\omega_i | \Delta, T_i, L_i) = \prod_{n=1}^{N} \left( \sum_{k=1}^{K} m_{f,k} P(\omega_{i=1} N \left( \frac{\omega_0^2 \alpha_K T_i}{1 + \alpha_K U(0, 1)L_i}, \sigma_k^2 \right) \right) + m_{f,K+1} P(\omega_i=1|U(f_{low}, f_{high})) \right) \quad (9.43)$$

For the purposes of maximizing the likelihood of the parameters of the mixture model through optimization, Equation 9.43 is problematic as it requires the joint sampling of a normal distribution $N(\mu, \sigma^2)$ and an inverse uniform distribution $1/U(0, 1)$. To form a PDF from Equation 9.43 it would be necessary to integrate the PDF of the normal distribution with the PDF of the inverse uniform distribution. However, there is no closed form solution for this integration, so a PDF cannot be formed from Equation 9.43.

To overcome the lack of a closed-form solution for Equation 9.43 it is necessary to approximate the PDF for modal estimates associated with a single mode of vibration. One method for approximating this distribution is to separate the PDF of frequency at a given load value $L_i$ into three parts as described by Equation 9.43.
\[ P(\omega_i | \Delta, T_i, L_i) = \begin{cases} 
H_a P\left(\omega_i \left| N \left( \frac{\omega_i^2 (1 + \alpha_{K,k} T_i)}{1 + \alpha_{L,K} L_i}, \sigma_k^2 \right) \right. \right) & \text{if } \omega_i < \sqrt{\frac{\omega_{0,k}^2 (1 + \alpha_{K,k} T_i)}{1 + \alpha_{L,K} L_i}} \\
H_b P\left(\omega_i \left| \frac{\omega_{0,k}}{U(0,1)} \left( \frac{1 + \alpha_{K,k} T_i}{1 + \alpha_{L,K} L_i} \right) \right. \right) & \text{if } \sqrt{\frac{\omega_{0,k}^2 (1 + \alpha_{K,k} T_i)}{1 + \alpha_{L,K} L_i}} \geq \omega_i \leq \omega_{0,k} \\
H_c P(\omega_i \mid N(0, \sigma_k^2)) & \text{if } \omega_i > \omega_{0,k} 
\end{cases} \tag{9.44} \]

Within Equation 9.44, \( H_a \), \( H_b \) and \( H_c \) are scale factors to ensure continuity of the piecewise probability distribution at the boundaries given by \( \beta_L = 0 \) and \( \beta_L = 1 \) and to ensure the sum probability over all possible frequency values is unity.

A simpler approximation of Equation 9.43 is to change the variables with which the modal errors are associated. Previously it has been assumed that these normally distributed modal errors were associated with the measured frequency estimates through Equation 9.45.

\[ P(\omega_i | \Delta, T_i, L_i) = P\left(\omega_i \left| N \left( \frac{\omega_i^2 (1 + \alpha_{K,k} T_i)}{1 + \alpha_{L,K} L_i}, \sigma_k^2 \right) \right. \right) \tag{9.45} \]

However, if instead the modal errors are assumed to be associated with uncertainty in the load values the expression for the likelihood of the model parameters given the frequency estimates is described by Equation 9.46 which has the practical advantage of having a closed form solution.

\[ P(\omega_i | \Delta, T_i, L_i) = P\left(\omega_i \left| \sqrt{\frac{\omega_{0,k}^2 (1 + \alpha_{K,k} T_i)}{1 + \alpha_{L,K} U(0,1)L_i} + N(0, \sigma_k^2)} \right. \right) \tag{9.46} \]

A comparison of the histograms of frequency estimates generated with a constant value of \( T_i \) and \( L_i \) through random sampling of the probability distributions given by Equations 9.45 and 9.46 is presented in Figure 9.11. It can be seen that the difference between the two distributions of frequency estimates is minor, suggesting that Equation 9.45 may be well approximated through Equation 9.46.
Figure 9.11: Comparison of probability density functions approximated using one million frequency estimates randomly sampled using Equations 9.45 and 9.46. In both equations a constant value of $\Delta T = 0$, $\Delta L = 10$, and $\alpha_L = 0.01$ is used.

The full mixture model for the frequency estimates in the presence of WNLMB with known temperature $T_i$ and change in applied static load $L_i$ incorporating the approximation from Equation 9.46 is given by Equation 9.47.

$$P (\omega_i | \Delta, T_i, L_i) = \prod_{n=1}^{N} \sum_{k=1}^{K} m_{f,k} P \left( \omega_{i=1} \left| \frac{\omega^2_0 k (1 + \alpha_{K,k} T_i)}{1 + \alpha_{L,k} U(0,1) L_i + N(0, \sigma^2_k)} \right) + m_{f,K+1} P \left( \omega_{i=1} | U(f_{low}, f_{high}) \right) \right) \tag{9.47}$$

The parameters of the mixture model ($\Delta$) described by Equation 9.47 can be estimated to maximize the probability of the observed modal estimates under the fitted mixture model parameters through a wide range of different optimization algorithms. Within this section the Adam optimizer, detailed in Appendix H, is used. To aid the stability of optimization, the natural frequencies $\omega_0$ and variances $\sigma^2_k$ are reparameterized as described in Section 8.3.2. Using the shifted and scaled logistic sigmoid function, described in Section 8.3.2, $\alpha_K$ is reparameterized to fall in the range -0.01 to 0.01. The same method is used to reparametrize $\alpha_L$ to fall in the range 0 to 0.01, as an increase in loading must result in a decrease in natural frequency. To ensure the value of $\alpha_L$ does not exceed these bounds, all load values are scaled to fall in the range 0 to 10.
9.4.1 Application of automated mode detection in presence of WNLMB

The PDF for a frequency estimate \( f_i = \omega_i / (2\pi) \) given a mode \( k \) with model parameters \( \Delta_k \), derived from Equation 9.46, is given in Equation 9.48.

\[
P(f_i | \Delta_k, T_i, L_i) = \frac{\omega_{0,k}^2 (1 + \alpha_{K,k} T_i) (\text{erf}(G_1) - \text{erf}(G_2))}{4\alpha_{L,k} L_i \pi^2 f_i^3}
\]  

(9.48)

In Equation 9.48, \( \text{erf} \) is the error function \[446\], which may be approximated through Horner’s method \[447\] to allow automatic differentiation, and \( G_1 \) and \( G_2 \) are given by Equations 9.49 and 9.50.

\[
G_1 = \frac{4(1 + \alpha_{L,k} L_i) \pi^2 f_i^2 - \omega_{0,k}^2 (1 + \alpha_{K,k} T_i)}{4\sqrt{2\pi^2 \sigma_{f_i}^2}}
\]  

(9.49)

\[
G_2 = \frac{4\pi^2 f_i^2 - \omega_{0,k}^2 (1 + \alpha_{K,k} T_i)}{4\sqrt{2\pi^2 \sigma_{0,k} f_i^2}}
\]  

(9.50)

The full probability function for a frequency estimate \( f_i \) given the \( K \) modal model components and the uniform noise component is given by Equation 9.51.

\[
P(f_i | \Delta, T_i, L_i) = \sum_{k=1}^{K} m_{f,k} P(f_i | \Delta_k, T_i, L_i) + m_{f,K+1} \left( \frac{1}{f_{\text{high}} - f_{\text{low}}} \right)
\]  

(9.51)

In Equation 9.51, \( m_f \) is the mixing fraction, as described in Chapter 8. The likelihood of the model parameters given the full set of \( N \) frequency estimates \( f \) is given by Equation 9.52.

\[
P(f | \Delta, T_i, L_i) = \prod_{n=1}^{N} \left( \sum_{k=1}^{K} m_{f,k} P(f_i | \Delta_k, T_i, L_i) + m_{f,K+1} \left( \frac{1}{f_{\text{high}} - f_{\text{low}}} \right) \right)
\]  

(9.52)

The set of model parameters \( \Delta \) which maximize the combined probability of the frequency estimates given by Equation 9.52 may then be found through optimizing the log-probability of the frequency estimates, as described in Section 8.3.1.

9.4.2 Numerical demonstration of automated mode detection in presence of WNLMB

The use of mixture model described in Equation 9.47 for automated mode detection in the presence of WNLMB is demonstrated using the artificially generated
frequency estimates, presented in Figure 9.12, containing two modes of vibration and background noise. In these artificially generated frequency estimates, the natural frequency of a mode \( j \) at time step \( i \) is related to the temperature \( T \) through 
\[
\omega_i = \sqrt{\omega_{0,j}^2 (1 + \alpha_{K,j} T_i)}.
\]
The natural frequency of the same mode is related to the loading applied to the structure through 
\[
\omega_i = \sqrt{(\omega_{0,j}^2) / (1 + \alpha_{L,j} \beta_{L,j,i} L_i)}.
\]
The value of \( \beta_{L,j,i} \) at a time step \( i \) is assumed to be constant. Each mode of vibration has normally distributed errors with variance \( \sigma_j^2 \), with the distribution of errors applied to the variation in frequency associated with the applied loading. Therefore, the artificial frequency estimates for a mode of vibration \( j \) are generated from the equation 
\[
\omega_i = \sqrt{(\omega_{0,j}^2 (1 + \alpha_{K,j} T_i)) / (1 + \alpha_{L,j} \beta_{L,j,i} L + N(0, \sigma_j^2))}.
\]

Figure 9.12: Artificially generated frequency estimates for data containing two modes of vibration which exhibit thermally-induced and static-load induced WNLMB, alongside spurious background noise.

The parameters of the mixture model described by Equation 9.47 were optimized using the Adam optimizer, introduced in Chapter 8 and described in Appendix H. The model was fitted with two modal components, initialized with natural frequencies equal to the two maximum peaks of a histogram of the frequency estimates with a bin width of 0.1 Hz. The boundaries of the uniform noise component of the mixture model were set as equal to the maximum and
minimum frequency values recorded in the full data set. The initialization values for $\alpha_K$ were zero, for $\alpha_L$ were $1 \times 10^{-4}$, and for $\sigma$ was equivalent to a standard deviation of 2 Hz. The mixing fractions were initialized with equal values. The Adam optimizer had a subset size of eight samples and an initial step size of 0.001 for all parameters. The step size was reduced by a factor of ten after five epochs of analysis.

After analysis of each 50 subsets of the training data, the likelihood of the fitted model parameters, based on Adam optimization using the training data, were evaluated using the holdback data, highlighted in Figure 9.12. After each epoch, the model parameters were reinitialised using the set of model parameters that maximized the probability of the frequency estimates within the holdback data.

The average log-likelihood given the frequency estimates in the training and holdback data under the mixture model parameters at each evaluation of stage of the analysis is plotted in Figure 9.13. It can be seen that after approximately four epochs the average log-likelihood plateaus, suggesting convergence of the optimizer to an optimal set of model parameters. Further initializations would be required to assess the stability of the fitted parameters and whether other local optima may exist within the solution space. The log-likelihood of the fitted parameters of the mixture model given the frequency estimates are higher than the log-likelihood under the true parameters of the mixture model; the parameters used in the generation of the frequency estimates. This is a consequence of analysing a finite set of data, where the randomly generated distributions of frequency estimates provide only an approximation of the true parameters of frequency estimates. This translates to the artificial data being better described (having a higher probability) under a set of parameters that are slightly different from the true parameters used in the data generation.
Figure 9.13: Log-likelihood of the model parameters given the artificially generated frequency estimates in the training and holdback data at every fiftieth step of the optimization process.

Despite this difference in parameters, it can be seen from the comparison of the fitted and true model parameters, presented in Table 9.2, that the fitted mixture model accurately describes the physical behaviour of the system. The largest deviation in values is observed in the parameters for $\alpha_K$ of mode 2. As demonstrated by the comparison of the modal boundaries presented in Figure 9.14, the deviation due to small differences between the fitted and true model parameters is very small when contextualized within the WNLMB induced changes in frequencies of the modes.

Table 9.2: Comparison of fitted and true parameters for artificial data containing two modes of vibration.

<table>
<thead>
<tr>
<th></th>
<th>Mode 1 parameters</th>
<th>Mode 2 parameters</th>
<th>Noise parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>True</td>
<td>Fitted</td>
<td>True</td>
</tr>
<tr>
<td>$f_0$</td>
<td>15.00</td>
<td>15.00</td>
<td>20.00</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>0.0015</td>
<td>0.0016</td>
<td>0.0010</td>
</tr>
<tr>
<td>$\alpha_K$</td>
<td>$-2.73 \times 10^{-3}$</td>
<td>$-2.70 \times 10^{-3}$</td>
<td>$-5.03 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\alpha_L$</td>
<td>$6.03 \times 10^{-3}$</td>
<td>$5.99 \times 10^{-3}$</td>
<td>$6.57 \times 10^{-5}$</td>
</tr>
<tr>
<td>$m$</td>
<td>0.67</td>
<td>0.66</td>
<td>0.20</td>
</tr>
</tbody>
</table>
Figure 9.14: Comparison of boundaries given by fitted and true parameters of the mixture model. Boundaries plotted as where $\beta_L = 0$ minus two standard deviations $\sigma$ of Gaussian noise, and where $\beta_L = 1$ plus two standard deviations $\sigma$ of modal errors.

An advantage of the use of the physics-based mixture model is that the behaviour of the system can be predicted for thermal and loading conditions which were not included within the training data if this behaviour broadly conforms to the behaviour of the structure within the physical models derived in Section 9.1 as demonstrated in Figure 9.15. Despite the wide distribution of frequency estimates associated with the high loading values, the parameters which were fitted using a narrower range of temperature and loading variations allow the frequencies which are likely to be associated with modes of vibration to be isolated within the boundaries given by $\beta_L = 0$ and $\beta_L = 1$. 
Figure 9.15: Comparison of predicted boundaries of frequency estimates from the same model used in Figure 9.14 under unseen temperature and loading conditions. Boundaries plotted as where $\beta_L = 0$ minus two standard deviations $\sigma$ of Gaussian noise, and where $\beta_L = 1$ plus two standard deviations $\sigma$ of modal errors.

Alongside predicting frequency changes associated with WNLMB under unseen environmental and loading conditions, the mixture model remains robust for the analysis of larger numbers of modes of vibration and the analysis of close modes of vibration, as demonstrated through the optimization of model parameters for artificial data containing modal estimates associated with seven modes of vibration presented in Figures 9.16 and 9.17. Despite the presence of overlapping and close modes within the data, the model parameters describing the WNLMB and the variance within the modal estimates are accurately estimated for all modes of vibration.
Figure 9.16: Identification of seven modes of vibration within artificial frequency estimates in which each mode exhibits WNLMB. Boundaries plotted as where $\beta_L = 0$ minus two standard deviations $\sigma$ of Gaussian noise, and where $\beta_L = 1$ plus two standard deviations $\sigma$ of modal errors.

Figure 9.17: Comparison of true and fitted model coefficients for a model containing seven modes of vibration which exhibit WNLMB.
9.4.3 Case study - Analysis of non-linear modal behaviour of the MX3D Bridge

This case study demonstrates the automated mode detection using a probabilistic mixture model incorporating WNLMB with acceleration, temperature, and load cell data collected from the MX3D Bridge while it was installed at the University of Twente.

9.4.3.1 Methods

The data was collected during a series of repeated controlled loadings of the bridge between the 22\textsuperscript{nd} of June 2020 and the 25\textsuperscript{th} of June 2020, the data for which has previously been introduced in Section 7.8.

The data set used for the automated mode detection comprised four cycles of known static loading applied to the bridge deck alongside a known pattern of excitation of the bridge by a single pedestrian. The locations of the static loading applied to the structure, alongside the crossing routes of the pedestrians, are shown in Figure 9.18. Key features of the pattern of applied loading are presented in Figure 9.19. Data from a fifth cycle of loading, alongside a shorter sample of pedestrian crossing only loading, was withheld during the analysis for validating the fitted mixture model. The pattern of static loading was repeated at different times of day, with the repetitions of the testing spread over four consecutive days. All data were sampled at 100 Hz for the duration of the tests and was analysed without pre-processing or filtering. All data sets were concatenated for analysis, with the boundaries between data sets highlighted in Figure 9.20.
Figure 9.18: Schematic of static loading applied to the MX3D Bridge at the University of Twente alongside pedestrian crossing routes.

Figure 9.19: One cycle of pattern of static loading applied to the MX3D Bridge at the University of Twente by a static mass, located at mass locations shown in Figure 9.18 and a single pedestrian (mass approximately 87.5 kg) performing either bridge crossings along routes shown in Figure 9.18 or heel drops at mass locations.

The acceleration data from the vertical accelerometer channels were analysed...
using the ST-RDT with an amplitude trigger and weighting condition, and a window length of 30 seconds. This window length was selected as the primary interest was accurately estimating the variation in natural frequencies of the structure due to the combined weight of the applied static loads and the pedestrian, rather than accurately estimating the damping and mode shapes of the structure, for which a longer ST-RDT window length is advised. An overlap of 15 seconds between adjacent ST-RDT windows was used in the analysis, alongside an RDS length of 2 seconds, approximately ten times the period of the fundamental mode of the structure.

All RDSs were analysed with the matrix pencil method with the model order iterated by one starting from unity to achieve a target explained variance of the fitted RDSs of $R^2 = 0.99$. The natural frequency estimates of the bridge in the range 0 Hz to 30 Hz estimated through the ST-RDT are presented in Figure 9.20 alongside the average temperature recorded across all thermistors during the tests, and the sum output of the four load cells (the total static load applied to the bridge). The average temperature and sum load have been processed with a moving average with a window length of 30 seconds, to correspond to the ST-RDT frequency estimates used as the basis for the automated mode detection. Note, that as the moving average load is used, the loading used in the analysis is an average of the load over the 30 second window. As shown in Figure 9.19, this translates to a higher average load during the heel drop events when the pedestrian is stationary at the static mass locations.
Figure 9.20: Natural frequency estimates for the MX3D Bridge, estimated using the ST-RDT with a window length of 30 seconds, plotted alongside raw acceleration (acc.) data from accelerometer A01X, the sum load measured by the load cells, and the average temperature (temp.) recorded across all bridge thermistors.

Examining the frequency estimates which are likely to be related to a single mode of vibration, such as the frequency estimates plotted in Figure 9.21, it can be seen that there is evidence of WNLMB as the changes in frequency appear correlated with changes in the average temperature and sum static loading. Note that while the location of the static loading and the pedestrian was recorded during the tests, to demonstrate the automated mode detection it is assumed that the location of loading on the structure is unknown. Similarly, while the mode shapes of the structure have previously been extracted, as used in Section 7.8, these are also assumed to be unknown in this section.
To split the data into training data and holdback data, the frequency estimates were divided into five-minute blocks, with every third five-minute block forming part of the holdback data. This ensured that both training and holdback data contained a representative sample of temperatures. The set of frequency estimates generated from data collected during the fifth application of the pattern of known static loading alongside a section of data containing only pedestrian loading, both collected on the 25th of June 2020, was withheld during the automated mode detection analysis for use in demonstrating the probabilistic load prediction presented in Section 9.5.4 and for assessing the quality of the automated modal detection using unseen data. Training, holdback and unseen data are highlighted in Figure 9.22.

The number of modal components to be included within the mixture model was selected based on the number of peaks within a histogram with bin width 0.05 Hz of all frequency estimates collected across the data set, as plotted in Figure 9.22 alongside the identified peaks. A peak was here defined as being a minimum of 0.5 Hz from the nearest peak and having an amplitude greater than 300 frequency estimates within the histogram bin. A total of 14 peaks were identified in the histogram based on these criteria.
The log-likelihood of the model parameters given the frequency estimates, as described by Equation 9.52, was maximized using the Adam optimizer, described in Section 8.3.1. All standard deviations of the modal components were initialized with a value equivalent to a standard deviation of 0.1 Hz, with the mixture model components initialized with a mixing fraction proportional to the peak value of the histogram presented in Figure 9.22. The uniform noise component of the mixture model was initialized with a mixing fraction of 0.1%.

The initial values of $\alpha_K$ and $\alpha_L$ were selected through a grid-search optimization, assuming fixed values of the mean natural frequency, standard deviation, and mixing fraction for each mode. The load was converted to a change in static load through subtracting the minimum sum load in the data set, with the load scaled to fall in the range 0 to 10 for the automated mode detection to ensure computational stability.

The Adam optimizer used a subset size of eight ST-RDT frequency estimates and an initial step size of 0.01. Given the large number of frequency estimates available (approximately one million modal estimates across the concatenated data sets), an epoch size of 10% of the data set length was used in the analysis to reduce computation time, with the data analysed each epoch randomly selected without repetition. The step size was reduced by a factor of ten after three epochs, and by a further factor of ten after a further three epochs. A total of nine epochs were used in the optimization.
9.4.3.2 Results

The average log-likelihood of the model parameters at every 50th subset of the Adam optimization is plotted in Figure 9.23. There is a clear plateau in the average log-likelihood after four epochs indicating convergence of the model parameters on a maximum of the solution space, a set of model parameters which maximize the probability of the observed frequency estimates.

![Figure 9.23](image)

Figure 9.23: Average negative log-likelihood of model parameters given data at every 50th subset of the Adam optimization procedure.

The assignment of frequency estimates to the components of the modes are plotted in Figure 9.24 with the intensity of the color indicating the strength of the assignment to a given mode. Between 0 Hz and 8 Hz, a number of overlapping modes of vibration are fitted to the data. The broad fitted components in this range are likely to be associated with weakly excited modes which exhibit high variance in the ST-RDT frequency estimates. Mixed with these are the more clearly defined modes of the structure, such as mode 3 in Figure 9.24 which correspond to higher density regions of frequency estimates.

Well defined and separated modes are clearly identified between 8 Hz and 25 Hz, corresponding to the higher fundamental modes of the structure. It can be seen that intermittently detected modes in this range, such as the mode close to 20 Hz, fall below the noise floor defined through the uniform noise component of the model and are not separated. Inclusion of further components of the mixture model are likely to allow these components to be separated and quantified. Alongside this, modes which are not observed in the training data, such as the mode close to 28 Hz, cannot be identified using the fitted mixture model. This highlights the need for a large data set with various types and levels of excitation from which modal parameters are estimated to ensure that the fitted mixture model is representative of the full range of dynamic behavior a structure may
Figure 9.24: Bounds of modal components identified through automated modal identification with data from the MX3D Bridge. Bounds plotted to show variation in frequency due to temperature and load, plus the one standard deviation of noise.

A comparison between the histogram of frequency estimates of the structure, as previously presented in Figure 9.22 and a histogram of frequency estimates generated through random sampling of the probability density function given by
Equation 9.52 using the parameters of the fitted mixture model and the measured sum loading and average temperature is presented in Figure 9.25. There is excellent agreement between the distribution of measured frequency estimates and those sampled from the fitted mixture model. This agreement suggests that the fitted model well approximated the observed frequency estimates across the frequency range, despite the presence of overlapping close modes and the presence of modes which were approximated as part of the uniform noise component.

Figure 9.25: Comparison between histogram of ST-RDT frequency estimates for the MX3D Bridge and a histogram of frequency estimates generated through random sampling of the probability density function given by Equation 9.52 and the fitted mixture model parameters.

Four modes isolated through the mixture model are plotted against time in Figure 9.26 and against load after removal of the temperature induced variation in frequency in Figure 9.27. The fitted model bounds show that alongside the modes being well identified by the mixture model, changes in natural frequencies due to changes in temperature and static loading are also well quantified across a range of modes of vibration. The thermally-induced and static load induced WNLMB identified through the mixture model illustrate the importance of accounting for WNLMB in the analysis of real structural behaviour. As demonstrated in the previous chapter, a failure to account for WNLMB is likely to lead to frequency estimates being split across multiple components within any procedure for the automated identification of modes of vibration. The magnitude of the WNLMB plotted in Figure 9.25 demonstrate the importance of WNLMB when analysing modal behaviour for damage detection, so that damage induced WNLMB may be separated from other drivers of WNLMB such as changes in loading or temperature.
Figure 9.26: Four isolated modes of vibration from the MX3D Bridge plotted against time.
Figure 9.27: Four isolated modes of vibration from the MX3D Bridge plotted against load after removal of the temperature induced variation in frequency.
To evaluate the efficacy of the fitted mixture model, the mixture model components previously plotted in Figures 9.26 and 9.27 are plotted using only the unseen data withheld when estimating the mixture model parameters in Figures 9.28 and 9.29. While the accuracy of the automated mode detection is high, these plots highlight some of the limitations of the fitted model parameters, notably that the distribution of modal parameters for the modes close to 12 Hz and 14.7 Hz is centered at a slightly higher natural frequency than expected. This is likely due to the limited size of the data set which encompassed only a small range of temperatures. This might be avoided through fitting of the mixture model parameters using a data set which included a wider range of temperatures.

Figure 9.28: Four isolated modes of vibration from the MX3D Bridge plotted against time for unseen data not used when estimating mixture model parameters.
A further implication of the limited availability of data from the MX3D Bridge while it was installed at the University of Twente was that it was not possible to create a fully unseen data set. It was necessary to refine the methods presented in this chapter using the full set of modal parameters collected at Twente, including analysing the unseen data to identify causes of deviations between measured and predicted modal behaviour. Therefore, the unseen data was not truly unseen. Future work should seek to validate the methods presented in this section through assessing the probability of unseen data given a fitted mixture model without plotting of the data. This would address potential biasing in the initialization of the mixture model parameters, or the characteristics of the mixture model itself, which may result from visual inspection of the data.
9.4.3.3 Summary

This case study has demonstrated the use of a probabilistic mixture model incorporating WNLMB as a method for automated mode detection. The method is robust to spurious modal estimates, requires minimal selection of parameters, and can allow for interpolation of the expected WNLMB to previously unseen loading and thermal conditions.

9.5 Load prediction using WNLMB

The relationship for thermally-induced WNLMB, derived in Section 9.1, has previously been demonstrated as a means of estimating the change in elastic modulus with temperature for simple structures in Section 9.3.3. The accompanying relationships for static load-induced WNLMB also has a physical interpretation, corresponding to the static-loading applied to the structure. Therefore, if there is an an estimate of the parameters for the WNLMB equations and frequency measurements from the same structure, they may be used as a basis for quantifying the magnitude and location of static loading applied to a structure. A block diagram for the use of WNLMB as a basis for load prediction is presented in Figure 9.30 highlighting how the two methods of load prediction presented in Sections 9.5.1 and 9.5.3 respectively relate to the wider context of analysing WNLMB and in-service ambient vibration monitoring.

As discussed within the literature review, while there has been limited work on load prediction using OMA, this has primarily focused on the use of finite element model simulations for estimating added mass [359]. However, a key limitation of this approach, and the use of finite element models within SHM more generally, is the need for an accurate finite model of the structure and the selection of likely damage/loading scenarios [360]. The objective of this section is to introduce two methods by which the static loading applied to a structure may be estimated without a finite element model of the structure and whilst incorporating the uncertainty inherent in modal estimates from OMA.

9.5.1 Random sampling mass estimation

This section is adapted from work originally published by Wynne et al. in Engineering Structures as Quantifying mass changes with ambient vibration measurements [6].

Static loading is one of the key unknowns in the structural design process.
Figure 9.30: Block diagram for application of methods for load prediction using WNLMB.

While weigh-in-motion stations and load cells allow the load to be measured over small areas or small structures, they are not viable for more complex structures such as long-span bridges or larger buildings and can be prohibitively expensive. OMA can reveal changes in the dynamic behaviour of civil structures measured with accelerometers, potentially offering a low-cost, durable technique for estimating changes of mass in-service. To do this, the dynamic parameters of the structure must be estimated with sufficient accuracy and then processed using a mass change estimation method robust enough to accommodate noise and measurement error. In this section a new technique for estimating changes in mass, random sampling mass estimation (RSME), is introduced, with its application to a laboratory-scale beam structure demonstrated in Section 9.5.2.

RSME allows the mass added to or removed from a structure to be estimated based on the structure’s WNLMB whilst incorporating the uncertainty in the modal parameter estimates. This method relies on accurate tracking of the modal parameters of the structure using the ST-RDT, introduced in Chapter 7. As previously discussed, the advantages of the ST-RDT over conventional OMA methods is that it allows the dynamic parameters of structures to be estimated over a short time frame, and it produces multiple estimates for each modal parameter, indicating the uncertainty in the parameter measured. This uncertainty can be projected to the mass change estimation allowing for more accurate results and the quantification of uncertainty within the mass change predictions.
RSME, an overview of which is shown in Figure 9.31, builds on the mode shape scaling research by Parloo et al. [350] to estimate the magnitude and location on the structure of an added mass based on known modal masses, scaled mode shapes, and unloaded natural frequencies, whilst incorporating the uncertainties within the modal estimates. RSME may be reformulated for detecting changes in stiffness of a structure such as may be induced by damage, as described in the previous section. As discussed in Section 9.1, the change in a mode’s natural frequency due to the addition of a small mass is related to both the magnitude of the added mass and the location on the mass in relation to the normalized mode shape of that mode.

9.5.1.1 Estimating the mode shape scaling factor

To estimate the mode shape scaling factor, \( \alpha_{L,n} \), for the \( n \)th mode of vibration, the frequency of the mode must be measured in both the unloaded state and when some known change is made to the mass matrix \( [\Delta M] \). The relationship between the unloaded natural frequency of the \( n \)th mode, \( \omega_{n,1} \), and the frequency of the mode after the change to the mass matrix, \( \omega_{n,1} \), is given by Equation 9.53.

\[
\alpha_{L,n} = \sqrt{\frac{\omega_{n,0}^2 - \omega_{n,1}^2}{\{\phi_n\}^T ([\Delta M] \omega_{n,1}^2) \{\phi_n\}}} \quad (9.53)
\]

The ST-RDT produces multiple estimates for the natural frequency of mode \( n \) within each window of data, due to the use of different triggering channels and a range of model orders. These distributions of modal estimates represent the uncertainty within our estimates of the natural frequency of the mode. To accurately capture this uncertainty in our estimates of the mode shape scaling factor \( \alpha_{L,n} \), Monte Carlo sampling is used to select combinations of \( \omega_{n,0} \) and \( \omega_{n,1} \) for use with Equation 9.53. As the physics of structural systems prevent the natural frequency of the system from increasing when the amount of added mass is increased, a limitation is placed such that \( \omega_{n,0} > \omega_{n,1} \). An example of how combinations of \( \omega_{n,0} \) and \( \omega_{n,1} \) are combined to estimate \( \alpha_{L,n} \) is shown in Figure 9.32 based on modal estimates from the first vertical mode of vibration of the simple beam structure discussed in Section 9.5.2.1.

9.5.1.2 RSME - One or more added masses, known mass locations

If there is information describing the location of each of the added masses on the structure (for example, from closed-circuit television (CCTV) or site pho-
Collect data from unloaded structure

Use ST-RDT to obtain estimates of the $n$ unloaded natural frequencies $\omega_{n,u}$ and amplitude normalized mode shapes $\phi_n$.

Collect data from the structure with a known applied mass.

Use ST-RDT to obtain estimates of the $n$ loaded natural frequencies $\omega_{n,l}$.

Use Monte Carlo sampling to collect $k$ estimates of $\omega_{n,u}$ and $\omega_{n,l}$ from ST-RDT results. Calculate $k$ values of the scaling factor $\alpha_L$.

Use ST-RDT and Monte Carlo sampling to obtain $k$ estimates of the $n$ loaded natural frequencies $\omega_{n,l}$.

**Unknown static load applied to structure**

**Single added mass**

For each of the $k$ sets of modal estimates:

- Calculate mass required to cause change in frequency of each mode $n$ at each point on structure using Equation (3).
- Identify intersection points on structure where required mass change and location is the same in two or more modes of vibration.

- Form 2D histogram of mass magnitude and location intersection points
- Filter histogram using a Gaussian kernel filter
- Peak of histogram corresponds to the most likely mass magnitude and location

**Multiple added masses**

Known added mass magnitudes
Obtain mass magnitudes $m_j$ at points on $\phi_n$ with unknown amplitude $A_{j,n}$

- Cauchy least-squares minimization of the $k \times n$ modal estimates using Equation (5) to find the unknown variables.

Known added mass locations
Calculate amplitude ($A_{j,n}$) of $\phi_n$ at location of each of the $j$ masses ($m_j$).

If the location of the $j$ discrete added masses ($\Delta m_j$) applied to a structure are known then the equations of static-load induced WNLMB derived in **Section 9.1** may be reformulated in terms of the amplitude ($A_{j,n}$) of the unity-scaled mode
shapes \( \{ \phi_n \} \) at those mass locations. For small changes in frequency, this gives Equation 9.54

\[
\sum_j \Delta m_j A_{j,n}^2 = (\omega_{n,0}^2 - \omega_{n,1}^2) / (\alpha_{L,n}^2 \omega_{n,1}^2)
\] (9.54)

Each mode of vibration may have the same amplitude at multiple different locations on the mode shape. However, with the exclusion of structural symmetry, it is unlikely that there are multiple locations on a structure where the amplitude of all modes of vibration is simultaneously repeated. Therefore Equation 9.54 may be reformulated as a set of simultaneous equations to be solved, as given in Equation 9.55. This has the effect of limiting the solution space of mass values to those which would cause the observed change in system frequency across all modes of vibration, reducing the variance in the total added mass estimates.

\[
\begin{bmatrix}
\sum_j \Delta m_j A_{j,0}^2 \\
\sum_j \Delta m_j A_{j,1}^2 \\
\vdots \\
\sum_j \Delta m_j A_{j,n}^2
\end{bmatrix}
= 
\begin{bmatrix}
(\omega_{0,0}^2 - \omega_{0,1}^2) / (\alpha_{L,0}^2 \omega_{0,1}^2) \\
(\omega_{1,0}^2 - \omega_{1,1}^2) / (\alpha_{L,1}^2 \omega_{1,1}^2) \\
\vdots \\
(\omega_{n,0}^2 - \omega_{n,1}^2) / (\alpha_{L,n}^2 \omega_{n,1}^2)
\end{bmatrix}
\] (9.55)

The procedure for estimating the total added mass on a structure is as follows. The acceleration data for the unloaded system and the data from the system with some known added mass applied is analysed separately using the ST-RDT. As discussed in Chapter 7, the use of multiple different accelerometer channels as the analysis, trigger and weighting data, and the use of a range of model orders in the matrix pencil method, results in multiple estimates of the modal parameters for each mode of vibration. As discussed in Section 9.5.1.1, Monte-Carlo sampling is used to select \( k \) estimates of the unloaded \( (\omega_{n,0}) \) natural frequencies of the system.

Figure 9.32: One thousand combinations of unloaded natural frequency of first vertical mode of vibration, and natural frequency of same mode subjected to an added mass of 200 g at 1.7 m along length of span.
and $k$ estimates of the natural frequency when loaded with a known change in mass ($\omega_{n,1}$). These are combined to create $k$ different estimates of the mode shape scaling factor ($\alpha_{L,n}$) for each mode of vibration.

The acceleration data for the system with unknown changes in mass is now analysed with the ST-RDT, generating multiple estimates of the natural frequencies ($\omega_{n,1}$) of the system for each mode of vibration. Monte-Carlo sampling is used to select a set of $k$ estimates for each of the $n$ natural frequencies of the system subjected to the unknown added mass. For the $n$ modes of vibration, these $k$ estimates of the natural frequency of each mode subjected to unknown added mass ($\omega_{n,1}$) are combined with $k$ estimates of the unloaded natural frequency of each mode ($\omega_{n,0}$) collected previously and the $k$ estimates of the mode shape scaling factor ($\alpha_{L,n}$) calculated earlier for each mode. This forms a $k$-by-$n$ set of modal parameter estimates. Linear regression is then used to find the optimal values of $\Delta m_j$ for the $k$-by-$n$ set of modal estimates. The total added mass applied to the system is then $\sum_j \Delta m_j$. As Equation 9.54 is highly non-linear it is prone to extreme outliers, most notably in the case where $\alpha_{L,n}$ approaches zero. Therefore a Cauchy loss function $\rho(z) = \ln(1 + z)$ which minimizes the impact of extreme outliers, is used in the regression.

In this section, the value of $k$ is set at 1,000, with the random sampling procedure repeated 100 times (100 different $k$-by-$n$ sets of modal estimates are fitted separately) to estimate the variability within the total added mass estimates due to the random sampling procedure.

9.5.1.3 RSME - Multiple added masses, known added mass magnitudes

Theoretically, if the magnitude of multiple added masses $\Delta m_j$ in Equation 9.55 were known it would be possible to estimate their location on a structure through finding the set of concurrent mode shape amplitudes ($A_{n,j}$) which minimized the error between the left and right-hand sides of Equation 9.55. However, the solution space for this equation is highly non-linear, due to the non-linearity of the mode shape equations, and gradient descent methods have a high likelihood of becoming trapped in local minima. While there may be practical applications for this problem, its solution is beyond the scope of this study but is discussed within Section 9.5.3.
9.5.1.4 RSME - Single added mass

If there is only a single concentrated added mass applied to the structure, the mass distribution matrix \([\Delta M]\) contains only a single non-zero element. Therefore both the magnitude of the added mass \((\Delta m)\) and its location on the structure may be estimated through a graphical or grid-search implementation of RSME, despite the highly non-linear solution space. The graphical approach has the advantage of avoiding the issues caused by the non-smooth minimization functions discussed in Section 9.5.1.3. The accuracy of these estimates is defined by the sensitivity of the modes to mass change (the magnitude of \(\alpha_{L,n}\)), the number of modes observable within the data, and the symmetry or asymmetry of the modes, while the computational intensity of such an approach is dependant on the resolution of the grid-search.

To estimate the magnitude and location of a single added mass it is necessary to first obtain estimates of the unloaded frequency \(\left(\omega_0\right)\), the normalized mode shapes \((\phi)\) and the mode shape scaling factors \((\alpha_L)\) for each mode by applying a known mass to the structure. As discussed previously, when these parameters are estimated through the ST-RDT there are multiple estimates of each parameter, each associated with a different set of ST-RDT modal estimates. Then, for a new frequency estimate made when the added mass magnitude and location are unknown, the added mass required to cause the observed change in frequency is calculated for each mode of vibration using Equation 9.54 as shown in Figure 9.33.

The estimates of the change in added mass required for the observed change in frequency of each mode are then combined to find the locations along the length of the span where they intersect, as plotted in Figure 9.34. These locations correspond to points on the span where the calculated change in mass would cause the observed change in natural frequency of two different modes of vibration, a partial solution to the set of simultaneous equations in Equation 9.55.

This process is repeated using 1,000 randomly sampled values of the current frequency and 1,000 randomly sampled values of the previous estimates of the \(\alpha\) values to build up a 2D histogram of the likely added mass magnitude and location, an example of which is shown in Figure 9.35.

To prevent a single, potentially outlier, frequency or \(\alpha\) estimate from dominating the estimates of the change in mass the 2D-histogram is filtered with a symmetrical Gaussian kernel filter \([\text{422}]\) as shown in Figure 9.36. This filtering benefits areas where there are large numbers of intersections in close proximity, corresponding to locations and magnitudes of the change in mass which lead to
similar partial solutions to the set of simultaneous equations presented in Equation 9.55. The area of greatest density of partial solutions is the peak of the filtered histogram and corresponds to the most likely magnitude and location of the change in mass.

Note, the results presented above model the simple beam as a 1-dimensional structure. To expand the results to 2-dimensions the change in mass required to induce the observed frequency change must be considered over both the length and breadth of the structure. Intersections of mass change and location are then found in 3-dimensions, with the most likely magnitude and location of the change in mass being the densest region of the 3-dimensional filtered histogram.
Figure 9.35: Gaussian filtered histogram of likely added mass locations. Symmetrical Gaussian kernel filter with a standard deviation of 4 bins used for filtering.

Figure 9.36: Gaussian filtered histogram of likely magnitude and location of the change in mass. Symmetrical Gaussian kernel filter with a standard deviation of 4 bins used for filtering.

of magnitude and location estimates.

9.5.2 Case study - Random sampling mass estimation

In this section, ST-RDT and RSME are demonstrated using experimental data for a laboratory-scale beam structure supporting concentrated added masses. It is shown that where multiple masses are present on the structure, the location of the masses, such as that gathered through site photos or CCTV, is required for estimating the total change in mass by finding the least-squares solution to the RSME implementation of the mass-change equations. The challenges posed by the complex solution space and non-linearity of the mass-change equations when solving for the location of the changes in mass with known magnitudes are discussed in the context of RSME. The graphical implementation of RSME which overcomes these challenges, introduced in the previous section, is used to accurately identify the magnitude and location of individual changes in mass.
without external knowledge of the mass’s location or magnitude.

As discussed in the previous section, RSME incorporates the uncertainty in the estimates of the dynamic parameters and allows least-squares estimation of the total change in mass applied to a structure if the locations of the changes in mass are known, or prediction of both the magnitude and location of a single concentrated change in mass through the novel graphical implementation. For individual masses added to the beam, 92.0% of mass location predictions have errors smaller than 2% of the length of the span and 69.4% of added mass magnitude predictions are within $\pm 10$ g, approximately 0.05% of the mass of the beam. For concurrently added mass locations, the mean error of the estimates for the total change in mass where two or more added masses are present on the beam is $-2$ g, approximately 0.011% of the mass of the beam.

9.5.2.1 Experimental Set-up

To validate RSME, a series of experiments were conducted to test its prediction of the change in mass applied to a simple beam structure, shown schematically in Figure 9.37. A 3.0 m length of IPE 80 steel beam, steel grade S235 and weight 6.11 kg/m, was suspended from spring supports to give an approximation of free-free support conditions. The spring supports were located at 0.3 m from each end of the span and were shaped to maximize freedom of movement. 50 g and/or 100 g and/or 200 g masses, approximately 0.25%, 0.5% or 1% of the mass of the beam, were fixed at a variety of locations along the top flange.

At each added mass location a minimum of two minutes of vertical acceleration data were collected from five uni-axial accelerometers fixed to the top flange of the beam as shown in Figure 9.37. All data were sampled at 3516.14 Hz, with random excitation of the beam approximated through light brushing of the beam with a stiff-bristled brush [272]. All acceleration data were high-pass filtered to
10 Hz using a Butterworth filter (filter order= 7) to remove components of the acceleration response associated with the movement of the beam on the spring supports.

Only the vertical acceleration response of the beam was measured in this experiment. All acceleration data sets for the single or multiple added mass experiments were concatenated before analysis with ST-RDT and RSME. For both the single and multiple added mass experiments, the mode shape scaling factor $\alpha_L$ was estimated based on the ST-RDT results for two minutes of data from the unloaded structure, and two minutes of data when a mass of 200 g was added at 1.7 m along the length of the span.

### 9.5.2.2 Modal analysis

An example of the results from the ST-RDT analysis of the acceleration data is presented in Figure 9.38. This plot is a histogram, with the number of estimates in each frequency bin at each time step indicated by the colour of that area. This extract shows how one of the natural frequencies varies as added masses are applied at different points along the simple beam structure discussed in Section 9.5.2.1. In this plot, the natural frequency is seen as the dark lines in the histogram, with changes in the location of the added mass distinguishable by the sudden changes in natural frequency.

![Figure 9.38: Histogram of ST-RDT frequency estimates of second vertical mode of vibration. Bin dimensions of 5 seconds by 0.025 Hz.](image)

Throughout the ST-RDT analyses presented in this section, an amplitude weighting condition in conjunction with an all-points triggering condition was used. Therefore each RDS represents either the auto-correlation of the window of data (where the same accelerometer channel is used as both the trigger and analysis data) or cross-correlation of the window of data (where different accelerometer channels are used for the trigger and analysis data). Each RDS was 0.15 seconds in length and was independently fitted with a model order of 1 to
20. All accelerometer data are divided into 30-second windows, with an overlap of 25 seconds between adjacent windows of data for the ST-RDT analysis.

**9.5.2.3 Results - Multiple mass change locations - Magnitude estimation**

Figure 9.39 shows the estimates of the total change in mass applied to the beam where first a 200 g mass, and then both a 100 g and a 200 g mass are placed on the beam. These estimates are based on 100 separate Cauchy least-squares minimization’s of Equation 9.55 at each time step with known locations of the added masses and 1,000 estimates of the unloaded frequency, mode shape scale factor, and current frequency of each mode. The least-squares minimization was initialized with an estimate of the change in mass at each location drawn from a uniform random distribution between 0 g and +500 g.

![Figure 9.39: Estimates of the total change in mass applied to beam based on Cauchy least-squares minimization of Equation 9.55 for data where the number of added masses was less than or equal to two.](image)

Between 3.2 minutes and 6.5 minutes, where each window of ST-RDT results contains only a single added mass applied to the beam, excellent estimates of the total change in mass are achieved with a mean absolute error of less than 0.68 g, and a mean error of -0.03 g, less than 0.0001% of the mass of the beam. Greater variability is observed between 7 minutes and 165 minutes where two added masses are present on the beam, with a mean absolute error across all time steps and initialisations of 29.42 g. However, the lack of bias within the estimates leads to a mean error of -4.05 g, approximately 0.02% of the mass of the beam.
Figure 9.40: Estimates of total change in mass applied to beam based on Cauchy least-squares minimization of Equation 9.55 for data where the number of additional masses was more than two.

The results for more than two additional masses applied to the structure are plotted in Figure 9.40. In this plot, a slightly greater variation in the results is observed, resulting in a mean absolute error of 54.44 g and a mean error of 11.96 g (0.07% of the mass of the beam). This increased mean error is primarily driven by larger errors and biasing where multiple added masses are located outside of the span support locations (between 142 minutes and 150 minutes), likely due to the increased curvature of the mode shapes in these locations. This higher curvature results in greater uncertainty in the estimated change in mass, as small errors in the assumed mass location or mode shape lead to large changes in frequency. Therefore small errors in the predicted amplitude of the normalized mode shapes translate into larger errors in the predicted magnitude of the added mass. Greater variability is observed after 166 minutes when three or more added masses are present on the structure, with notable biasing of the estimated sum change in mass where added masses are located at close to centre span. However, no corresponding increase in Cauchy least-squares error is observed for the estimated total change in mass, as shown in Figure 9.41. This suggests that the solutions obtained are alternative sets of mass change values (with a different sum change in mass) which would cause the same observed change in frequency of the modes. This is a limitation of RSME for larger numbers of concurrent changes in mass.

Figure 9.41 shows an increase in the minimum Cauchy least-squares error for each time step through the data set. This might suggest that a change in
temperature across the course of the experiments, or slight shifting of the support locations, resulted in changes in the unloaded frequency of all/some of the modes of vibration. For long-term measurements, this effect could be mitigated by updating the estimates of the mode shape factors $\alpha_L$, either by re-calibration with a known mass change or possibly by continuous updating using the mass estimates generated by RSME. Alternatively, if the frequency shifts were due to a known and measurable cause, such as changes in temperature, the frequency estimates could be corrected to account for this influence before the prediction of the magnitude of the change in mass.

9.5.2.4 Results - Single mass change location

The estimated mass change magnitudes and locations obtained through the graphical implementation of RSME are presented in Figures 9.42 and 9.43 for when a single added mass is applied to the beam.

As shown in Figure 9.44 the overall accuracy of the graphical implementation of RSME is high, with 69.4% of the mass change magnitudes predicted within $\pm 10$ g (0.055% of the mass of the beam) of their true value, and 82.9% of mass change magnitudes predicted within $\pm 20$ g (0.109% of the mass of the beam) of their true value. 93.9% of mass change locations were predicted within $\pm 0.1$ m
Figure 9.43: Estimated location of change in mass for a single added mass applied to the beam.

of their true location, with the error of 92.0% of location predictions being less than 2% of the length of the span. The results show a slight bias towards underestimating the change in mass believed to be driven either by over-estimates of the $\alpha_L$ factor for the experiment or by the uncertainty in the mode shape for the beam.

As for when the beam was loaded with multiple added masses, the highest errors in the estimates of magnitude and location for a single mass change location occur when the added mass is located close to the free end of the beam, as shown between 35 minutes and 40 minutes in Figures 9.42 and 9.43, likely due to the increased curvature of the mode shapes at these locations.

When the change in mass is located close to the node point of a given mode, where the amplitude of the mode shape is zero, there is higher uncertainty in the estimated change in mass as one or more of the modes of vibration exhibit no change in frequency due to the zero amplitude. However, as RSME relies on combining the results for multiple modes of vibration, the peaks of the histograms used to identify the most likely mass change magnitudes and locations are not materially affected by the presence of node points.

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9.5.2.5 Notes on real-world application

The experimental testing discussed in this section has shown that RSME and the ST-RDT show promise for estimating changes in static loading by incorporating the uncertainty in the modal estimates. A key difference between real-world structures and the experiment presented in the Section 9.5.2 is that it is rare for the excitation force applied to real-world structures to be continuous. A non-continuous excitation force will lower the signal-to-noise ratio of the signal, which for the conventional RDT has been shown to increase the error in the modal estimates [449]. This increased error in the modal estimates is likely to occur for the ST-RDT and may necessitate the use of a longer window of data. However, as RSME incorporates the variability of the modal estimates when predicting the magnitude and location of changes in static loading it will be more robust to noise than if a single set of modal parameter estimates were used. Instead, the increased variability of modal parameters will be reflected in the results as a broader range of likely static load locations and magnitudes.

A limitation of RSME is that it is not possible to distinguish between changes in stiffness and changes in static loading. However, it is often possible to extrapolate the causes of changes in natural frequencies from the practical context in which they are observed as discussed in detail in Section 9.1.4. For example, thermally induced changes in stiffness are likely to be highly correlated with the external temperature and the diurnal cycle, whereas changes in static-loading are likely to be correlated with the use of the structure such as traffic flows. Conversely, damage induced changes in stiffness are likely to be irreversible and represent a new baseline set of modal parameters for the structure; a step-change in dynamic behaviour. As has been demonstrated elsewhere in this chapter, it is possible to distinguish different sources of WNLMB by the context in which they occur. Through methods such as the probabilistic mixture model presented in Section 9.4 it would be possible to account for these other causes of WNLMB when applying RSME to real-world structures.

9.5.2.6 Summary

OMA offers a promising method of estimating in-situ static loading on a structure. It has been shown that by tracking and quantifying WNLMB and accounting for the uncertainty within the modal parameter estimates, accurate estimates can be made of the total changes in the mass of a simple 2-dimensional structure if the mass change locations are known through a novel procedure: random sampling mass estimation (RSME). If only a single localised mass is applied to the
structure, both its location and magnitude may be estimated through a graphical implementation of the mass-change equations based on partial solutions to the set of mass-change simultaneous equations.

The ST-RDT offers a simple and effective method of quantifying and tracking WNLMB and has the advantage that the uncertainty within the modal estimates is captured within the results. This uncertainty can be propagated forward through random sampling to allow the uncertainty in the mass change predictions to be estimated and allow more accurate mass change estimation.

For the simple beam structure tested under individual concentrated added masses, 92.0% of mass change location predictions have errors smaller than 2% of the length of the span and 69.4% of mass change magnitude predictions are within ±10 g, approximately 0.05% of the mass of the beam. For the same structure under variably distributed added masses with known locations, the average error in the estimated total change in mass for two additional masses applied to the beam was –4.05 g, approximately 0.02% of the mass of the beam, and for three or four additional masses was 11.96 g, approximately 0.07% of the mass of the beam. Beyond estimating changes in mass, the method can be adapted for changes in stiffness, such as may be induced by damage to a structure.

### 9.5.3 Ambient vibration based probabilistic load prediction

The previous sections presented RSME, an empirical approach for estimating the static load applied to a structure using WNLMB. Key limitations of RSME are that it required accurate mode shapes for the structure, and was not robust for the prediction of distributed loading without prior knowledge of the load locations. This section introduces an alternative method for load prediction based on the use of the probabilistic mixture model incorporating WNLMB introduced in [Section 9.4](#). The advantage of this approach is that it requires only frequency estimates from the structure alongside previous estimates of the mixture model parameters. No estimates of the mode shapes of the structure or knowledge of the load location is required at any stage of the probabilistic approach to load prediction using ambient vibration data.

The combined probability of an observed set of frequency estimates $f_i$, based on the probabilistic mixture model defined in [Section 9.4](#) is given by Equa-
\[ P (f_i | \Delta, T_i, L_i) = \sum_{k=1}^{K} m_{f,k} P (f_i | \Delta_k, T_i, L_i) + m_{f,K+1} \frac{1}{f_{high} - f_{low}} \]  \hspace{1cm} (9.56) \]

Assuming that the parameters of the mixture model have previously been estimated for known loadings, Equation 9.56 also allows the likelihood of any value of \( \beta_{L,k} L_i \) given a new set of frequency estimates to be calculated. Note that in Equation 9.56, the uniform distribution used to model noise within the modal estimates conveys no direct information about the static loading applied to the structure as it is associated with noise within the data. As such under this noise all loadings between \(-\infty\) and \(\infty\) are of equal likelihood. However, it is necessary to include within the load estimation so that noise within the frequency estimates is accounted for when calculating the likelihood of \( \beta_{L,k} L_i \).

If the mixture model described by Equation 9.56 contains only a single modal component \((k = 1)\), the value of \(L_i\) cannot be calculated without knowledge of \(\beta_{L,k=1}\). At best, the lower bound solution for the static loading applied to the structure may found through assuming \(\beta_{L,k=1} = 1\), this corresponds to the minimum applied loading which would cause the observed change in frequency. The upper bound solution is given as \(L_i \rightarrow -\infty\) where \(\beta_L \rightarrow 0\).

If the mixture model contains more than one modal component \((k \geq 2)\), the shared value of \(L_i\) can be used as a basis for finding the values of \(\beta_{L,k}\) which maximize the combined probability of the observed vector of frequency estimates \(f_i\) at the time step. However, an identifiability problem still exists, as all values of \(\beta_{L,k}\) within \(\beta_L L_i\) may be multiplied by a scale factor if the value of \(L_i\) was scaled by the inverse of the same factor. To overcome the challenges posed by the lack of identifiability, a new \(k\) length vector \(\beta_m\) is defined using Equation 9.57.

\[ \beta_m = \frac{\beta_L}{\max(\beta_L)} \]  \hspace{1cm} (9.57) \]

Scaling of the vector of \(\beta_L\) values by the maximum value of \(\beta_L\) ensures that at least one of the values of the new vector \(\beta_m\) must be equal to one. This resolves the identifiability issue discussed above by constraining the space of possible solutions.

There is a range of possible justifications for the suitability of the scaling of \(\beta_L\) that should be considered within the load prediction analysis. The first is how likely it is that the static load applied to a structure is applied at or close to the peak of a mode shape. For some structures under spatially varying load, such
as bridges, it is likely that for a portion of a crossing the load will be located close to the node point of a mode shape. Similarly, if the modes being analysed correspond to modes with multiple peaks and troughs, the likelihood of the static loading being close to the peak of a mode for a portion of the time increases. If the static loading is associated with transient loads which are imparting driving forces to the structure, such as vehicles or pedestrians, it is also likely that modal estimates may be slightly biased towards larger values of $\beta_L$, as the observability of a mode within the ambient vibration data is directly related to the amplitude of the response of that mode. This amplitude is greatest when the driving force is located at the peak of a mode, suggesting that the most likely value of $\beta_L$ to be observed from some structures may approach $\beta_L = 1$. As more modes of vibration with well distinguished mode shapes are included within the analysis, the likelihood of the static load being close to the peak of one of the mode shapes increases. However, these observations may not be true for all structures and should be interrogated as part of the probabilistic load-prediction procedure.

The vector of values of $\beta_m L_i$ which maximize the combined probability of the observed frequency estimates using Equation 9.56 may be found through an optimization routine, such as expectation maximization or Adam, introduced in Section 8.3.1. However, this will provide only a single set of values for $\beta_m L_i$. In order to understand the sensitivity of this optima to the values of $\beta_m$ and $L_i$ it is necessary to calculate the likelihood of $\beta_m$ and $L_i$ given the frequency estimates over a range of realistic values.

This problem can be framed within the context of Bayes theorem, the formula for which is given in Equation 9.58.

$$P(\Theta|y) = \frac{P(y|\Theta)P(\Theta)}{P(y)}$$  \hspace{1cm} (9.58)

The objective is to calculate the posterior $P(\Theta|y)$ [450]. Within the context of load-prediction using WNLMB, $\Theta$ is a vector of the combined static load and scaled static load location factors $\beta_m L_i$, and $y$ is the data formed from the vector of frequency estimates $\omega_i$, the vector of temperature measurements $T_i$, and the parameters $\Delta$ of the mixture model: $P(\Theta|y) = P(\beta_m L_i|\Delta, \omega_i, T_i)$.

The prior distribution $P(\Theta)$ are plausible values of $\beta_m L_i$. In this section it will be assumed that all loading values $L_i$ within the limits $L_{\text{low}}$ and $L_{\text{high}}$, the range of loading which are likely to be applied to the structure, have uniform likelihood and that all values of $\beta_m$ in the range zero to one have uniform likelihood, as the values of $\beta_L$ are constrained to this range within the definitions given in Section 9.1. The distribution $P(y|\Theta)$ is the probability of the observed frequencies, temperatures
and model parameters given the values of $\beta_m L_i$, $P(y|\Theta) = P(\Delta, \omega_i, T_i|\beta_m L_i)$ which may be calculated using Equation 9.56. The final term in Equation 9.58 is the marginal likelihood $P(y)$, the likelihood of $\beta_m L_i$ over all estimated frequencies, temperatures and model parameters, and is found through the integrating over all possible values of $\beta_m L_i$ using Equation 9.59

$$P(y) = P(\Delta, \omega_i, T_i) = \int (P(\Delta, \omega_i, T_i|\beta_m L_i)P(\beta_m L_i)) \, d(\beta_m L_i) \quad (9.59)$$

Equation 9.59 is intractable where the frequency estimates contain multiple modes of vibration, each with a unique value of $\beta_m$. Therefore it is necessary to approximate the posterior distribution $P(\beta_m L_i|\Delta, \omega_i, T_i)$

One method for approximating this distribution is through a grid-search method, whereby the values of each of the $k$ values of $\beta_m$ and $L_i$ are evaluated over a range of plausible values. However, when $k$ becomes large such an approach may become infeasible. Furthermore, it also requires careful selection of a range of plausible values of $L_i$, which introduces a degree of subjectivity into the approximated distribution equivalent to the assumption of a non-uniform prior over load.

An alternative approach for approximating the posterior distribution $P(\beta_m L_i|\Delta, \omega_i, T_i)$ is to use a Markov Chain Monte Carlo (MCMC) method. MCMC methods approximate the posterior distribution through sampling. A “walker” is created with an initial estimate of the parameters for which the posterior distribution is to be estimated. For load prediction with MCMC, these parameters are the $k$ values of $\beta_m$ and $L_i$. The likelihood of these initial parameters given the observed frequency and temperature data, alongside the previously fitted mixture model parameters, are then calculated using Equation 9.56. The parameters are then updated through a pre-defined algorithm, such as a small perturbation through a randomly sampled step-size relative to the current parameters. The likelihood of the new parameters are then calculated. If the likelihood of the new parameters meets some acceptance criteria, such as the ratio of likelihoods of the new and current parameters being greater than a randomly sampled number between zero and unity, the current parameters are updated to the new parameters. If the acceptance criteria is not met, the new parameters are rejected and the current parameters remain unchanged. The current parameters are then appended to a record of all parameters used, known as an MCMC “chain”. The process is repeated for a set number of iterations.

Given enough iterations, the MCMC chain will effectively approximate the posterior likelihood distribution given by different values of $\beta_m$ and $L_i$, as the
number of iterations which the walker spends in a given region of the parameter space is a function of the likelihood of the parameters \[\beta_m, L_i\]. MCMC can use individual walkers, initialized independently multiple times, such as in the Metropolis-Hastings algorithm [451], the broad steps of which were described above. The chains from the multiple initializations can then be combined to improve the approximation of the posterior [450].

More advanced MCMC methods have been proposed which allow for a more rapid and thorough approximation of the posterior distribution. Hamiltonian Monte Carlo (HMC) [452] introduces a momentum coefficient in combination with a step-size for each walker. This allows walkers to better escape local minima and more rapidly traverse the solution space than random-walk methods such as the Metropolis-Hastings algorithm [452]. While this is similar to the momentum coefficient as described for the Adam optimizer discussed in Appendix H, a key difference is that the momentum coefficient of HMC is also randomly sampled and is independent of the value of the MCMC chain at each iteration [452].

An expansion of HMC, the no u-turn sampler (NUTS) was introduced by Hoffman and Gelman [453]. This removes the need to select the number of iterations or steps which each walker takes. Instead, walkers are automatically terminated when they begin to revisit areas of the solution space which have already been traversed [453]. This can allow for efficient traversal of the approximated posterior density without the need to tune step sizes or run unneeded further iterations [453]. Further improvements in the efficiency of MCMC samples were proposed by Goodman and Weare [454] who introduced affine invariant MCMC which makes use of the simultaneous results from many different MCMC walkers, with the parameters of each walker updated based on the relative location of the other walkers within the solution space [454].

An issue encountered in MCMC sampling is the initial selection of the parameters. It is possible that the random perturbations of the initial values of \(\beta_m\) and \(L_i\) result in an unlikely solution. In this scenario it will take several iterations for a walker to move to a more likely set of parameters, creating a string of strongly correlated outlier results referred to in the MCMC literature as the “burn-in” [455]. To prevent biasing or skew of the approximated posterior distribution, it is necessary to discard the initial portion of the chains from each walker containing the correlated outlier results.

A further issue is auto-correlation within the results, as discussed at length by Link and Eaton [456] and Foreman-Mackey et al. [457]. An MCMC walker can remain at a single location in the solution space for several iterations leading to non-independent samples. This is closely related to the issues associated with
the overlap of windows within the ST-RDT discussed at length in Chapter 7. To accurately approximate the posterior likelihood it is necessary to ensure that there are a large number of semi-independent samples. To check the independence of the samples, the autocorrelation of the MCMC chains may be plotted, as discussed further in Section 9.5.4. Foreman-Mackey et al. [457] present the integrated autocorrelation time as a metric for quantifying the degree of autocorrelation within the MCMC chains. As discussed by Foreman-Mackey et al. [458] a minimum integrated autocorrelation time of 50 is recommended to ensure that the number of iterations used in the MCMC simulation is sufficient to well approximate the posterior likelihood.

9.5.3.1 Numerically generated frequency data example

In this section, the use of the probabilistic mixture model and the MCMC algorithms for load prediction is demonstrated using the two-mode and seven-mode artificial data introduced in Section 9.4. For both artificial data sets, new artificial frequency estimates were generated, presented in Figures 9.45 and 9.46. These data sets used the same model parameters as for those data sets used to fit the probabilistic mixture model in Section 9.4 but included previously unseen values of temperature and loading. For comparison, the two-mode and seven-mode artificial data sets use the same variations in temperature and loading.
Figure 9.45: Artificially generated frequency estimates for data containing two modes of vibration which exhibit thermally-induced and static-load induced WNLMB, and spurious background noise. The static loading applied to the system is assumed to be unknown but the model parameters are assumed to have been estimated using frequency estimates collected under known loadings.
Figure 9.46: Artificially generated frequency estimates for data containing seven modes of vibration which exhibit thermally-induced and static-load induced WNLMB, and spurious background noise. The static loading applied to the system is assumed to be unknown but the model parameters are assumed to have been estimated using frequency estimates collected under known loadings.
The posterior likelihood distribution of $\beta_m L_i$ was approximated using the affine invariant MCMC algorithm [454], discussed in the previous section, as implemented within the Emcee Python library [458]. This posterior was independently approximated for each time step within the modal estimates, with the initialisation values of the chains based on minimization of Equation 9.56 through the L-BFGS algorithm [439] as implemented in the Scipy Python library [459]. The initialization values for the chains were perturbed with a uniform distribution $U(-0.1, 0.1)$ for values of $\beta_m$ (whilst ensuring all values of $\beta_m$ fell in the range zero to one) and with a normal distribution $N(0, 0.1^2)$ for values of $L_i$. A uniform prior of $U(-30, 30)$ was used to constrain the possible loadings $L_i$ applied to the structure. 5,000 iterations of the affine invariant MCMC were used to ensure that the integrated autocorrelation time of the chains was less than 50, with each affine invariant MCMC analysis consisting of 20 walkers.

The MCMC chains for the analysis of frequency estimates of time step fifty of the artificial data containing two modes of vibration are presented in Figure 9.47. There is a burn-in period of several hundred iterations which appear in the results as the individual walkers converging towards a stable value. Results from this burn-in period, conservatively estimated at 1,000 iterations, are excluded in further plots of the approximated posterior distribution. All values of $\beta_m$ for mode 2 are unity for all time steps after the first 1,000 iterations, indicating that the load prediction has correctly identified that the greatest change in frequency relative to the values of $\alpha_L$, and by extension the maximum value of $\beta_L$, has occurred in mode 2. Presented alongside the results from the individual chains are the true values of $\beta_L$, the scaled value $\beta_m$ as given by Equation 9.57, the true value of $L_i$, and the equivalent loading if $L_i$ is scaled based on the maximum value of $\beta_L$. The most likely value does not coincide with the true loading on the system, as neither mode has a value of $\beta_L = 1$. Instead, the load prediction is a lower bound estimate under the assumption that $\beta_L$ equals unity for one of the modes. When the scaling of the $\beta_L$ values to $\beta_m$ is considered the load prediction is excellent, with all values of the MCMC chains being tightly clustered around the true value of the scaled loading.
Figure 9.47: MCMC chains for analysis of frequency estimates from time step 50 of data presented in Figure 9.45. Results presented alongside true values of $\beta_m$ and $L_i$ for comparison.

Histograms of the values of $\beta_m$ and $L_i$ from the MCMC chains plotted in Figure 9.47 are presented in Figure 9.48. It can be seen that the values of $\beta_m$ for mode 1 are approximately normally distributed about the true value, as might be expected given the assumption of normally distributed errors within the mixture model of frequencies, with the values of $L_i$ normally distributed but with an offset from the true scaled load value, suggesting a small bias in the distribution of noise within the limited number of frequency estimates at the time step.
Figure 9.48: Corner plots of MCMC chains presented in Figure 9.47. A burn-in period of 1,000 iterations excluded before plotting. Results presented alongside true and scaled values of $\beta_m$ and $L_i$ for comparison.

A 2D histogram of the MCMC load predictions across the full data set for the two-mode of vibration data is presented in Figure 9.49 in comparison to both the true loading, and the loading scaled based on the scaling required between $\beta_L$ and $\beta_m$. In all cases the posterior likelihood of the load is tightly clustered about the scaled load values. As previously discussed, the scaling of $\beta_L$ to $\beta_m$ ensures that the posterior load distribution is a lower bound load prediction, as evidenced by the majority of the posterior densities falling below the absolute true loading applied to the structure at all time steps.
As discussed in the previous section, as more modes of vibration are included within the load prediction, the likelihood of one of the modes having a true $\beta_L$ value close to unity increases. However, the interpretability of the results is more challenging, as the presence of close modes within the data can lead to potential trade-offs within the posterior distribution of $\beta_L L_i$. This can be seen in the corner-plot of the MCMC chains from time step 50 of the artificial data containing seven modes of vibration plotted in Figure 9.50. Due to the increased number of modes there is a far wider distribution of values of $\beta_m$ for each mode of vibration but an increased likelihood that one of the modes has a true value of $\beta_L$ close to one, as evidenced by the small difference between the true value of $L_i$ and the scaled value of $L_i$. 

Figure 9.49: Posterior likelihood of load estimated through affine invariant MCMC from artificial frequency estimates containing two modes of vibration.
Figure 9.50: Corner plots of MCMC chains for analysis of artificial data containing seven modes of vibration at time step 50 of the data presented in Figure 9.46. A burn-in period of 1,000 iterations excluded before plotting. Results presented alongside true and scaled values of $L_i$ for comparison.

This trade-off is reflected in the posterior distributions of static loading for the seven modes of vibration artificial data across all time steps, plotted in Figure 9.51. The posterior distributions are far wider, due to the greater uncertainty in the value of $\beta_L$ associated with each mode of vibration. However, as discussed above, as one of the modes is likely to have a value of $\beta_L$ close to unity, the me-
dian of these distributions are more closely centred about the true static loading applied to the system.

![Figure 9.51: Posterior likelihood of load estimated through affine invariant MCMC from artificial frequency estimates containing seven modes of vibration.](image)

To compare the load predictions from the two and seven modes vibration data, the difference from the empirical cumulative distribution function (ECDF) is plotted in Figure 9.52. This plot shows the fraction of MCMC estimates for each time step which are less than the true load (black lines) and the max($\beta L$)-scaled load (red lines), minus the fraction of estimates which would be expected if the percentage of MCMC load values less than the true/scaled load were uniformly distributed between 0% and 100%. This gives a metric to compare the bias in the results and whether the calculated posterior is narrower or wider than would be expected, as discussed in depth by Talts et al. [460].

In Figure 9.52, the results are plotted as the MCMC load values which are less than the true and max($\beta L$)-scaled load applied to the structure (for positive true load values) or which are greater than the true or max($\beta L$)-scaled load applied to the system (for negative true load values). Approximately 62% of the MCMC load predictions across the time steps are less than the true load values for the two modes data, indicating a strong bias in the results as for the majority of the time steps the maximum value of $\beta L$ is far less than unity. However, when compared to the max($\beta L$)-scaled load, the MCMC values for the two modes artificial data are approximately uniformly distributed, indicating a lack of bias and accurate approximation of the posterior. For the seven modes artificial data, the increased number of modes increase the likelihood that one of the independently sampled $\beta L$ values is close to unity, so the MCMC load predictions within a time step are less
biased towards under predicting the true load applied to the system. Therefore
the difference between the distributions of the true load $L_i$ and the $\max(\beta L)$-scaled load are broadly similar, with both approaching a uniform distribution which indicates a lack of bias and accurate approximation of the posterior of the load using MCMC.

![Figure 9.52: Comparison of difference from empirical cumulative distribution function (ECDF) for MCMC load predictions using two modes (top subplot) and seven modes (lower subplot).](image)

9.5.4 Case study - Probabilistic load-prediction using frequency estimates from the MX3D Bridge

The MCMC method of approximating the posterior distribution of static loading based on WNLMB is demonstrated in this section using the frequency estimates collected from the MX3D Bridge while it was installed at the University of Twente, previously introduced in Section 9.4.3.

9.5.4.1 Methods

The ST-RDT frequency estimates used in the case-study are from 10 hours of data which correspond to four repetitions of the controlled loading of the structure described in Section 9.4.3 and previously presented in Figure 9.20. The previously fitted mixture model parameters were fitted using frequency estimates when the bridge was subjected to a known static loading regime. These model parameters are used as the basis for approximating the posterior distribution of static loading on the full data set including both the previously seen training and holdback data, and data for which it is assumed the loading on the structure is unknown and which was not included in the fitting of the mixture model parameters. Note
that as the mixture model was fitted using the 30 second moving-average load applied to the bridge, the posterior distribution are predictions of the 30 second moving-average load applied to the structure.

The affine invariant MCMC, initialized through the method presented in the previous section, and implemented with 30 walkers for 10,000 iterations, was used for the load prediction. The first 7,500 iterations of each chain were discarded as a burn-in period based on visual inspection of the chains and the autocorrelation of the chains, examples of which are presented in Figure 9.53.

Figure 9.53: Top plot) Autocorrelation of MCMC chains from time step 100 of MX3D frequency estimates presented in Figure 9.20, lower left plot) MCMC chains of $\beta$-normalized load predictions for analysis of frequency estimates from time step 100 of MX3D frequency estimates presented in Figure 9.20, lower right plot) histogram of $\beta$-normalized load predictions after burn-in period.

9.5.4.2 Results

The posterior distributions of loading are compared to the true loading applied to the structure in Figure 9.54 for the full data set, and for the previously unseen data only in Figure 9.55. The MCMC load predictions offer a good approximation of the loading applied to the structure, with the majority of the true loading values falling within the 50% density range of the approximated posterior. This extends to the unseen data sets, with a strong correlation observed between the 50% density range of load predictions and the true loading applied to the structure.
As has been discussed in the previous section, the probabilistic approach to
load prediction provides a lower bound load estimate. Evidence that the posterior distributions are a lower bound solution can be seen in Figure 9.54, where the posterior distributions of loading are plotted against the true loading applied to the structure, as the true loading consistently falls at the upper bounds of the posterior likelihood distribution. For the most likely static load prediction to match the true load applied to the structure one of the values of $\beta_L$ must equal unity. However, the static loading applied to the bridge came from the mass of the pedestrian and the concentrated mass of slabs applied to the structure at individual, discrete locations. Therefore, it is unlikely that the mass of the slabs was located such that $\beta_L = 1$. This contributes to the difference between the predicted posterior distribution of $\beta_L$-normalized loading magnitudes and the true sum static load applied to the structure.

As shown in Figure 9.56, the results are robust as the predictions for the unseen loading data set are broadly similar to that from the training data set, indicating that the results are not due to the fitting of the noise within the frequency estimates. The difference from the ECDF illustrates that despite the method giving a lower-bound estimate of the applied loading, the biasing of the load predictions towards underestimating the true applied static loading remains small, likely due to the large number of fitted components within the mixture model. As discussed in Section 9.4.3, future work should seek to validate these results using truly unseen data.

![Figure 9.56: Comparison of difference from empirical cumulative distribution function (ECDF) for MCMC load predictions generated for data collected from the MX3D Bridge.](image)

If the uncertainty within the load predictions was underestimated there would be disproportionate numbers of time steps where all load predictions would fall above or below the true loading applied to the structure. If the uncertainty in the load predictions was overestimated, there would be very small numbers of time steps where all load predictions were above or below the true loading applied to
the structure. The near uniformity of the percentage of MCMC estimates which fall below the true load values for a given time step, as plotted in Figure 9.56, indicates that the uncertainty within the load predictions are well calibrated.

An issue which is encountered in the MCMC load prediction is the correlation of the frequency estimates generated from overlapping windows of data, alongside the inherent correlation within dynamic behaviour as discussed in Chapter 7. This correlation extends to the MCMC load predictions such that the results from any time step are correlated with those from adjacent time steps. The practical implications of this are that there may be multiple time steps where the load is consistently over or under predicted. Therefore, it is inadvisable to consider the accuracy of the load predictions at the level of individual time steps, but to consider only the range of load predictions over the full data set. Practically, this may limit the MCMC method for load prediction for estimating the short-term variations in loading, which may be better predicted using RSME.

A further issue that must be resolved is the intermittent detection of some modes of vibration within the modal estimates. If a mode of vibration is not observable within the data for a time step, correlated spurious modal estimates may lead to corruption of the load predictions, as it may appear that there is a cluster of frequency estimates which may only be observed by a larger than reality change in static loading. These errors contributes to outlier chains within the MCMC predictions, which appear in Figure 9.54 as section of the posterior density for an individual time step which are separated from the majority of the posterior density.

One potential solution to the problems posed by correlation and intermittently detected modes of vibration is to combine the frequency estimates from multiple time steps when predicting the posterior distribution of load. An example of this approach based on the use of five time steps of frequency estimates is presented in Figures 9.57 and 9.58. The results show a significant reduction in the number of outlier chains of load predictions and a more consistent predicted load. However, the time-domain fidelity of the load predictions suffer, with shorter term variations in load less clearly detected.
Figure 9.57: Approximated posterior distribution of static loading on the MX3D Bridge, estimated using the frequency estimates presented in Figure 9.20 and affine invariant MCMC using modal estimates from a window length of five time steps.

Figure 9.58: Approximated posterior distribution of static loading on the MX3D Bridge, estimated using the unseen frequency estimates presented in Figure 9.20 and affine invariant MCMC using modal estimates from a window length of five time steps.
A further issue with combining the frequency estimates from multiple time steps in the load prediction is shown in the difference from the ECDF, plotted in Figure 9.59. This plot suggests that while the load predictions are well centered, with the median load prediction being close to the true load applied to the bridge, there is an overestimation of the uncertainty within the load predictions. Therefore, the likelihood of extreme loading events is under estimated. This is to be expected as the variance of the frequency estimates has been artificially increased when combining the frequency estimates from multiple time steps with the mixture model which was fitted using individual time steps. While this is disadvantageous for the load prediction, it does indicate that the uncertainty estimated through the fitted mixture model was well calibrated with that observed within the data for individual time steps.

The results presented in Figures 9.58 and 9.59 suggest that for the purpose of load prediction it may be advantageous to fit a mixture model to observed frequency estimates from multiple time steps. This would address the issues posed by temporally correlated errors and dynamic behaviour as frequency estimates which are likely to share a common set of dynamic behaviour would be evaluated as a combined set. Alongside this, combining the data from multiple time steps may help to ensure that intermittently detected modes are well described by the fitted mixture model. However, combining multiple time steps will reduce the temporal resolution of the load predictions which may result in the magnitude of short term loading events being under predicted.

9.5.4.3 Summary

This case study has presented a demonstration of load prediction based on WNLMB of natural frequencies through the use of a probabilistic mixture model.
The method has shown to be robust to noise within the data and thermally-induced WNLMB. The results indicate that despite the method providing a lower-bound approximation of the static loading applied to a structure, as there is a biasing towards large \( \beta_L \) values for bridges under footfall excitation, the measured and predicted loads are tightly correlated. Unlike RSME, the probabilistic approach utilizing MCMC does not require estimates of the mode shapes of a structure, or any knowledge of the locations of loads on the structure.

The limitations of the probabilistic mixture model, such as errors induced by correlated behaviour and measurements, and intermittently detected modes have been discussed, and a potential solution based on combining frequency estimates from multiple time steps has been introduced.

### 9.5.5 Case study - In-service behaviour of the MX3D Bridge

This section presents a comparison of the frequency estimates collected from the MX3D Bridge while it was installed at the University of Twente, presented previously in Sections 9.4.3 and 9.5.4, and data collected from the bridge in Spring 2022, after its installation in Amsterdam in August 2021. The objective of this case study is to highlight some of the issues encountered when analysing complex structural systems in-service, the importance of quantifying uncertainty in OMA, and how the ST-RDT may be used as a diagnostic tool for separating WNLMB from other causes of variation in modal estimates. The challenges of load prediction using MCMC in the presence of high levels of uncertainty in the modal parameters are demonstrated.

#### 9.5.5.1 Methods

The data used in the analysis was collected in five minute blocks, once per hour, between the 23rd of March 2022 and the 31st of March 2022, with a second series of data sets collected between the 21st of April 2022 and the 1st of May 2022. All data sets were sampled at 200Hz.

For the ST-RDT analysis of the Amsterdam data, each five minute block of data was analysed independently with the same analysis parameters as were used in the analysis of the data collected at the University of Twente described in Section 9.4.3. The mixture model was fitted using the methodology described in Section 9.4.3 with the last three days of data withheld during fitting of the mixture model as an unseen data set. The load prediction used the fitted mixture
model, documented in Section 9.5.5.4, alongside affine invariant MCMC initialized using the method described in Section 9.5.3. Thirty MCMC walkers were ran for 5,000 iterations, with the initial 4,000 iterations of the chains discarded to remove the burn-in period from the predictions of loading.

9.5.5.2 Results - Comparison of dynamic behaviour

A comparison of the ST-RDT frequency estimates from data collected at the University of Twente, and after installation of the bridge in Amsterdam is presented in Figure 9.60. For the purpose of plotting the data collected in Amsterdam throughout this section, time intervals where no data were collected are removed from the plots.

While there are visually identifiable modes of vibration within the ST-RDT frequency estimates for the Amsterdam data, the modal estimates show a far higher variance than those estimated using the data collected at Twente. However, the modal estimates do not exhibit significant heteroskedasticity as this variance is consistent over time, suggesting that the higher variance is not a result of the wider range of excitation of the bridge. For example, the data collected in Amsterdam in the early hours of the morning should be broadly similar to that collected for the majority of the time in Twente, as in both cases the excitation of the structure is through crossings by individual pedestrians. Instead, the higher variance of modal estimates from the Amsterdam data is likely the result of the addition of concrete within the swirls after its installation in Amsterdam. This concrete was placed to aid with cleaning detritus from inside the swirls. Unfortunately, as the concrete was a late addition to the design, it negatively affects the operation of the permanent sensor network, as it provides an alternative load path to the foundations which does not pass through the load cells.
The addition of the concrete makes comparison of modal parameters estimated in Amsterdam and Twente challenging. The additional mass leads to changes in the dominant modes of the structure, the natural frequencies of all modes, and which modes are excited during crossings by pedestrians. Inspection of the bridge suggests the concrete within the swirls has cracked since installation, likely due to thermal expansion of the structure. The additional mass provided by the concrete, alongside the additional friction surfaces which the cracked concrete provides, may explain the increased damping ratio for the bridge in Amsterdam. As shown in the comparison of damping ratios plotted in Figure 9.61, the damping of all modes has more than doubled, further altering the dynamic behaviour of the structure.
Figure 9.61: Comparison of ST-RDT damping estimates from the MX3D Bridge at the University of Twente and after installation in Amsterdam.

The likely cause of this increased damping is the concrete shifting within the swirls and the bridge deck making contact with other parts of the foundations, as well as an electrical box installed adjacent to one of the swirls. This is based on visual observations of the bridge during pedestrian crossings made in May 2022. Evidence for these contact points can be seen in the raw acceleration data collected from the bridge at Amsterdam, an example of which is plotted in Figure 9.62. The impact of the shifting concrete and the complicated boundary condition is to cut oscillations short. However, the point at which oscillations are cut short is dependant on the combined oscillation of all modes of vibration and the amplitude of the structural response. As the discontinuities cannot be well approximated as a sum of damped sinusoids, this is likely to translate to greater variance in the modal parameters estimated using the ST-RDT.
Figure 9.62: Raw acceleration data, collected in Amsterdam, from an MX3D Bridge deck accelerometer during crossing by a single pedestrian. Left subplot) Acceleration response during pedestrian crossing, right subplot) free oscillation of bridge deck after pedestrian has left the bridge. Examples of discontinuities in the oscillation which are likely due to the bridge deck making contact with concrete in the swirls and/or the foundations or street furniture highlighted in red.

9.5.5.3 Results - Fitted mixture model

Despite the high variance of the modal parameters, the distribution of frequency estimates can still be well approximated through the WNLMB mixture model introduced in Section 9.4, as demonstrated in the likely assignment of modal estimates to the modes of vibration presented in Figure 9.63, and a comparison of the histogram of ST-RDT frequency estimates from Amsterdam and those generated through random sampling of the fitted PDF for the mixture model presented in Figure 9.64.
Figure 9.63: Automatically identified modes of the MX3D Bridge using data collected in Amsterdam, plotted against time. Red lines show the WNLMB induced variation in frequencies due to changes in temperature and static loading, including two standard deviations (std.) of the variance of the fitted modes.
Figure 9.64: Comparison between histogram of ST-RDT frequency estimates for the MX3D Bridge in Amsterdam and a histogram of frequency estimates generated through random sampling of the probability density function given by Equation 9.52 and the fitted mixture model parameters.
The thermally-induced WNLMB is clearly distinguished in the some of the fitted mixture model components presented in Figure 9.63. Alongside this, there is some evidence of load-induced WNLMB. The magnitude of load-induced WNLMB may be inferred from Figure 9.63 by the magnitude of the deviation from parallel of the upper and lower plotted bounds.

The key challenge when analysing the modal estimates collected in Amsterdam is the higher variance within the modal estimates. Unlike the estimates of the dynamic behaviour of the bridge while it was installed at the University of Twente, the magnitude of changes in natural frequencies induced by WNLMB is orders of magnitude smaller than the variation induced by spurious components within the RDSs used in the ST-RDT.

9.5.5.4 Results - In-service load prediction

The high variance of the estimated modal parameters negatively impacts the predictions of the posterior distributions of change in static loading applied to the bridge in Amsterdam, estimated through the method presented in the previous section and plotted in Figure 9.65. However, the performance of the load prediction for unseen data, plotted separately in Figure 9.66, is broadly similar to that of the training data.

Figure 9.65: Approximated posterior distribution of static loading on the MX3D Bridge in Amsterdam, estimated using the frequency estimates presented in Figure 9.63 and affine invariant MCMC.
Figure 9.66: Approximated posterior distribution of static loading on the MX3D Bridge in Amsterdam, estimated using the unseen frequency estimates presented in Figure 9.63 and affine invariant MCMC.

The similarity in the load predictions for the training and unseen data is illustrated in the difference from the ECDF, presented in Figure 9.67. There is a strong bias within the predicted load posteriors towards under prediction of the load on the bridge. Possible causes for this include other causes of non-linear modal behaviour masking load-induced WNLMB or bias within the fitted mixture model. Alternatively, the approximated posterior may still be accurate for the loading on the bridge when the distribution of loading is considered. At times of peak loading it is known that there are static pedestrians on the bridge. Therefore the assumption that load is concentrated such that one of the values of $\beta_L$ approaches unity may not be true. If none of the values of $\beta_L$ approached unity, the posterior distribution would be centered at a lower value that the total static load applied to the bridge. Similarly, the assumption of a uniform distribution of $\beta_L$ when fitting the mixture model may not be correct. Further work is required to validate how the distribution of loading impacts the efficacy of the load predictions using the probabilistic mixture model.
Figure 9.67: Comparison of difference from empirical cumulative distribution function (ECDF) for MCMC load predictions generated for data collected from the MX3D Bridge in Amsterdam.

9.5.5.5 Summary

ST-RDT frequency estimates generated using acceleration data collected from the MX3D Bridge have highlighted the additional challenges posed by real-world structures due to complicated boundary conditions. Despite the challenges associated with analysing the acceleration data collected from the MX3D Bridge in Amsterdam, it has been shown that the proposed method for automated modal identification introduced in Section 9.4 remains effective. Alongside this, the ST-RDT has been demonstrated as a robust method that allows WNLMB to be separated from variations in modal estimates induced by spurious components of the acceleration response. Through considering the lack of heteroskedasticity of the results, the cause of the spurious modal components has been identified as issues associated with the foundations and boundary conditions of the bridge.

The load prediction using the probabilistic mixture model and MCMC has been demonstrated using data collected from the bridge in-service, with promising early results presented. Further data from the bridge in-service is needed to validate the assumed uniformity of the distribution of $\beta_L$, and the assumption that the value of $\beta_L$ approaches unity for at least one of the modes of vibration within the mixture model.

9.6 Summary

In this chapter, methods have been presented which allow WNLMB of civil structures in-service to be robustly and accurately quantified, allowing changes in dynamic behaviour measured using ambient vibration monitoring to be related to physical changes of a structure and its environment.

In Section 9.1 the relationships between changes in frequency and two com-
mon drivers of WNLMB; temperature changes and changes in applied static loading, were derived based on the physical behaviour of linear dynamical systems. These derivations included a relationship for quantifying frequency changes due to local changes of the mass matrix and weighted average relationships for distributed or global changes to the stiffness or the mass matrix.

The use of regression methods for estimating the parameters of these relationships were presented in Section 9.3. These methods formed part of a two-step process, requiring that the natural frequencies associated with a single mode of vibration first be isolated before the regression analysis. It was shown that while linear regression has been the predominant method of analysing environmentally induced WNLMB, quantile regression offers a more robust method of quantifying WNLMB which is relatively insensitive to noise within the data, or variations in WNLMB due to factors such as the location of static loading on a structure. This was demonstrated in Section 9.3.3 where the change in elastic modulus of a steel beam due to temperature was quantified using regression methods.

An alternative approach for quantifying and analysing WNLMB through a probabilistic mixture model of WNLMB was introduced in Section 9.4. The advantage of such an approach is that it is a one-step process with the automated modal detection carried out in parallel to estimation of the parameters of the WNLMB relationships derived in Section 9.1. As a result of this, the method may be robustly used for the accurate analysis of close and overlapping modes of vibration, as demonstrated using ST-RDT frequency estimates generated for accelerometer data from the MX3D Bridge. A further advantage of the WNLMB probabilistic mixture model is that it allows the likelihood of new frequency estimates under previously unseen conditions to be evaluated, a metric that may form the basis for damage detection in-service.

Two methods for predicting the static loading applied to structures in-service were introduced in Section 9.5 and the relationship between damage detection and estimating applied static loading discussed in detail. An empirical approach, random sampling mass estimation (RSME), based on resampling of modal estimates and non-linear regression, was presented in Section 9.5.1 and demonstrated using an experimental case study in Section 9.5.2. A probabilistic approach for load prediction using the mixture model derived in Section 9.4 was introduced in Section 9.5.3. Through the use of Markov Chain Monte Carlo, this method allows the posterior distribution of load applied to a structure to be approximated, as demonstrated using artificially generated frequency estimates and ST-RDT frequency estimates collected from the MX3D Bridge under controlled and in-service loadings.
All methods presented in this chapter provide robust and interpretable solutions to the challenges of analysing in-service monitoring data and allow WNLMB to be used for estimating a wide range of structural parameters. These parameters can be used to guide the design of future structures through understanding the in-service stiffness of structures, the quantities and locations of loadings applied to structures in-service, or where and when damage to structures may have occurred. The techniques presented in this chapter are efficient and may be implemented using the continuous output of data provided by long-term ambient vibration monitoring. In line with the findings of the survey of the structural engineering industry presented in Chapter 3, these methods present direct indications of the confidence and uncertainty within the parameter estimates which build on the use of the ST-RDT as a method for quantifying uncertainty in modal parameter estimates. The outputs from the WNLMB analysis presented in this chapter can allow for the real-time integration of structural monitoring data with digital shadows and digital twins, introduced in Chapter 2, or may be used as a basis to guide the design, operation and maintenance of future civil structures through performance-based design.
Chapter 10

Conclusions

This study has addressed how long-term in-service monitoring may be used to guide the design and operation of civil structures. It has primarily examined the use of ambient vibration monitoring and operational modal analysis (OMA) as a low-cost and robust method for quantifying the global behaviour of structures in-situ. Through assumptions about the statistical properties of the applied forcing and the dynamic response of the system, OMA allows the modal parameters of a structure to be estimated without knowledge of the forcing applied to the system.

The current civil engineering design process may be characterized as reactive and prescriptive, with catastrophic failures driving the development of codified design guidance which specifies the types and quantities of materials to be used or the loadings that a structure must be designed to resist. Recently, the limitations of this approach have been highlighted, including the barrier they present to the adoption of new materials and construction techniques, and the need for a holistic design approach that balances the required safety of a structure with the need for efficient material use. These issues are especially pertinent given the climate emergency and the need to maintain, repair and replace the large number of infrastructure assets approaching the end of their design life.

“Smart structures”, structures instrumented with sensors in-situ, offer an established technology for collecting data from civil structures in-service. Digital shadows, virtual models of assets automatically updated with real-world measurements, and digital twins, where the control and operation of the real-world assets are controlled by the virtual model, offer a design paradigm for integrating measurements from smart structures in a way that can be understood and interrogated by engineers.

Current limitations of OMA for the long-term monitoring of civil structures have been addressed in this study so that ambient vibration monitoring may form
a viable basis for integrating data collected from smart structures with digital shadows and digital twins.

10.1 Fulfillment of research objectives

The five research objectives, developed based on gaps in past research identified in the literature review, presented in Chapter 2, and the findings from a survey of industry perceptions of long-term monitoring, presented in Chapter 3, were:

RO 1. Establish a standardized method for selection of model order in OMA.

RO 2. Identify and remove biasing of OMA modal parameters estimates introduced by pre-processing of ambient vibration data.

RO 3. Quantify uncertainty in OMA modal parameter estimates.

RO 4. Automate the detection and separation of modes of vibration within OMA modal estimates.

RO 5. Relate changes in dynamic behaviour measured using ambient vibration monitoring to the physical changes of a structure and its environment.

Fulfillment of these objectives has demonstrated the potential of long-term in-service monitoring of civil structures through ambient vibration monitoring to improve the design and operation of civil structures.

10.1.1 Fulfillment of RO 1: Standardized model order selection

In Chapter 5, the coefficient of determination combined with target explained variance was introduced as a robust metric that removes the subjectivity of selecting a model order when approximating the response of dynamic systems.

The model order is a key parameter within OMA, as it represents the number of modes of vibration which are assumed to be identifiable within the measured dynamic response. Existing OMA methods often calculate modal parameters over a wide range of model orders. However, for long-term in-service vibration monitoring this can be computationally intense and require large numbers of spurious modal estimates to be stored and processed.

Alongside reducing the number of modal estimates to be calculated and stored, an advantage of the proposed approach is that it provides an indication of the
degree of uncertainty within the approximated dynamic response, as measured responses including a larger number of modes of vibration or a lower signal-to-noise ratio require a higher model order to achieve a pre-specified target explained variance. The use of a common baseline target explained variance allows the comparison of modal estimates collected from the same structure under different levels of excitation, the comparison of modal parameters collected from separate structures and the impact of different analysis parameters to be objectively quantified.

10.1.2 Fulfillment of RO 2: Removing bias due to pre-processing of vibration data

Filtering of ambient vibration data is commonly used in OMA to reduce unwanted noise, remove signals outside of the frequency band of interest, or for the analysis of close modes of vibration.

However, within Chapter 6 it has been shown that Gibbs filtering artefacts induces biasing within modal parameters estimated through time-domain OMA. Numerical and real-world case studies have demonstrated how this systematic biasing may lead to incorrect conclusions about structural behaviour or might be mistaken as amplitude-dependent behaviour. Trimming of the correlation functions used within time-domain OMA to remove the auto- or cross-correlation of noise within the data was demonstrated as a robust and efficient alternative to filtering of the data. Where filtering of the data is required, such as to remove oscillations outside of the frequency range of interest, it is necessary to remove filtering artefacts from the correlation functions to ensure accurate and unbiased modal parameters estimates. This can be achieved through trimming of the correlation functions to remove the filtering artefacts, or through a modal analysis procedure which includes an approximation of the filtering artefacts produced using an indicative noise profile, a section of sensor data containing only noise.

10.1.3 Fulfillment of RO 3: Quantifying uncertainty in modal parameter estimates

In order to build confidence in long-term in-situ monitoring it is necessary to quantify and understand causes of uncertainty within the estimates of in-service structural behaviour. To achieve this, the short-time random decrement technique (ST-RDT) was developed.

As described in Chapter 6 the ST-RDT is a novel extension to the random decrement technique, an established time-domain OMA method, whereby
weighted free-responses of the structural response given a predefined triggering condition, referred to as random decrement signatures (RDSs), are approximated for overlapping windows of data. The ST-RDT is based on subsampling theory, through which information about the statistics of any population may be gained by evaluating the statistic on a subsample of the same dataset. Unlike other sampling approaches such as bootstrapping, subsampling maintains the original structure of the data and is therefore robust to correlated errors, non-linear behaviour, and heteroskedasticity, all of which are commonly encountered in the analysis of ambient vibration data.

To understand the causes of error in modal parameters, a new description of the RDS as a conditional correlation function was derived. For a linear-dynamic system subjected to unmeasured forcing and corruption by additive sensor noise, it was shown that the RDS is comprised of two components. The first component includes the initial amplitude and first derivative response of the system and can be exactly modelled as a finite sum of damped sinusoids. The second component of the RDS includes the time-lagged correlation functions of the forced response of the system and the correlation function of noise within the analysis data. The noise and forced response components of the RDS provide no information about the dynamic response of the system and cannot be exactly modelled as a finite sum of damped sinusoids. This new derivation removes the assumptions and limitations presented by previous definitions of the RDS within the literature, as it does not assume independence of the segments of data used to form the RDS and explicitly accounts for the correlation between the system response, applied forcing, and noise within the data. The new definition of the RDS is expanded to include systems exhibiting weak non-linear behaviour, where the RDS is comprised of segments of data with varying modal parameters. For non-linear systems, the initial amplitude and first derivative response of the system cannot be exactly modelled as a finite sum of damped sinusoids, but may still be approximated as such.

The application of the ST-RDT for estimating the natural frequencies and damping ratios of civil structures was demonstrated using real-world acceleration data. A probabilistic approach for generating mode shapes using the ST-RDT was developed which allows for error bars to be introduced to the mode shapes. Alongside this, the new definition of the RDS and subsampling theory was used as a basis for describing the relationship between the modal parameters produced using the ST-RDT and the uncertainty in the modal parameters. Guidance was provided on the selection of suitable parameters used within the ST-RDT, with a specific focus on accurate quantification of the uncertainty in the modal parame-
ters and accurately identifying and quantifying weak non-linear modal behaviour.

A series of case studies of real-world structures demonstrated some of the advantages of the ST-RDT for in-service monitoring of civil structures. This included identifying that the change in modal parameters of the Aberfeldy footbridge previously identified within the literature was likely associated with weak non-linear modal behaviour of the bridge induced by the mass of pedestrians, and not related to damage to the structure.

The analysis of the Whitmore timber building illustrated the ease with which modes of vibration may be identified within ST-RDT modal estimates when compared to conventional methods such as stability diagrams. Analysis of the Whitmore timber building further demonstrated how correlations between modal parameters and other variables such as the amplitude of structural response can be quantified using the ST-RDT without the presupposition of a relationship.

The improved efficacy of the ST-RDT when compared to the conventional RDT for detecting weakly or intermittently excited modes of vibration was demonstrated by estimating the modal parameters of surface waves or seiches for two lakes. This case study also illustrates the application of the ST-RDT in areas other than the analysis of civil structures, greatly expanding the potential impact of the method.

The integration of the ST-RDT with existing methodologies for assessing the serviceability of civil structures was demonstrated through the prediction of the vibration response of the MX3D Bridge. It was shown that the ST-RDT allowed for more accurate predictions of the bridge’s vibration response due to the higher accuracy of the modal estimates that it produced and as it allows the weak non-linear modal behaviour of the structure under pedestrian loading to be accounted for within the analysis.

10.1.4 Fulfillment of RO 4: Automated mode identification

The ST-RDT has allowed for a fundamental shift in the automated detection of modes of vibration within OMA due to the large number of modal estimates it produces and the temporal structure inherent within ST-RDT modal estimates.

A probabilistic mixture model was derived in Chapter 8 which allowed for robust and accurate identification of modes of vibration based on estimates of the natural frequencies and damping ratios of a system. This method separates spurious modal estimates associated with noise in the data and allows for the analysis of modes of vibration with close or overlapping distributions of modal
parameters. The outputs from this mixture model provide direct metrics for quantifying the uncertainty of modal parameters associated with each mode of vibration detected within the data.

The use of the mixture model was demonstrated using acceleration data collected from ten timber structures, analysed with a common set of modal analysis parameters. It was shown that the identification of modes was robust across a broad range of monitoring conditions and that the use of an automated procedure greatly enhanced the accessibility of ambient vibration monitoring.

10.1.5 Fulfillment of RO 5: Modal identification in the presence of weak non-linear modal behaviour

To enhance the efficacy of the automated identification of modes of vibration, physics-based models were developed in Chapter 9 for the changes in natural frequencies induced by two different causes of weak non-linear modal behaviour; thermally-induced changes in stiffness and static-load induced changes in mass. The models were integrated into the probabilistic mixture model to allow for the automated identification, quantification and tracking of modes of vibration exhibiting weak non-linear modal behaviour. As demonstrated using data from the MX3D Bridge, this allows the uncertainty associated with the approximated dynamic response in the ST-RDT to be separated from uncertainty in the modal parameters due to the applied static loading or temperature of the structure.

As these models of weak non-linear modal behaviour are based on the physical behaviour of linear dynamic systems, it was shown that they allow ambient vibration monitoring to be used for estimating structural parameters usually measured by more expensive, delicate or complicated sensors. This was demonstrated in the ST-RDT analysis of the beam structure under varying temperatures where it was shown that the change in natural frequencies of the beam may be used to estimate the factor relating the variation in elastic modulus to temperature through quantile regression.

These models of weak non-linear modal behaviour allow the static loading applied to structures in-service, a key unknown within the civil engineering design process, to be estimated using measurements of a structure’s dynamic response. Two methods were developed for this purpose. Random sampling mass estimation (RSME), provides an empirical approach for estimating the location and magnitude of concentrated static loading applied to a structure based on the simultaneous change in natural frequencies of multiple modes. RSME allows the uncertainty in the modal estimates to be projected to the predictions of mass lo-
cation and magnitude through a novel graphical sampling implementation. The second approach for estimating the static loading applied to structures in-service utilizes Markov-chain Monte Carlo and the probabilistic mixture model developed for automated identification of modes of vibration within the ST-RDT modal estimates. This method allows the posterior likelihood of the static loading applied to the structure to be approximated based on the simultaneous change of natural frequencies for one or more modes of vibration. The method is simple to apply in practice, requiring only a section of data during which known static loadings are applied. Furthermore, as the method does not require a numerical model of the structure or estimates of the mode shapes of the structure, it may be readily adapted for use with any civil structure.

The methods developed for predicting the static loading applied to civil structures based on weak non-linear modal behaviour may also be used for detecting, locating and quantifying any damage to the structure, as the damage-induced change in stiffness results in an equivalent change in natural frequency as applied static loading.

The methods developed within this study provide a foundation for the integration of long-term monitoring into the structural engineering design process. The methods utilize relatively low-cost sensors, are easily accessible by non-specialist operators, may be applied to a broad range of different structures and can be adapted for use with a range of structural sensing technology.

10.2 Potential further work

This study has highlighted that a key barrier to the further theoretical development of OMA methods is understanding the relationship between spurious components of the time-domain correlation functions or frequency-domain power spectra and errors within the modal estimates. As was commented on in this study, the separation of such errors is a complex and multifaceted problem that is dependent on factors such as the number of modes identifiable within the data, specific time-lagged correlations within the forcing applied to the system and noise within the data, and any non-linearity within the system response. The ST-RDT resolves these issues through subsampling. However, a greater understanding of the causes of errors within modal estimates might be used to guide the selection of optimal ST-RDT parameters.

The models of non-linear behaviour presented in this study could be expanded to include more complex structural systems, where the relationship between pa-
Parameters such as stiffness and natural frequencies might be impacted by factors such as the boundary conditions of the system. Similarly, the models of non-linear modal behaviour could be expanded to include the damping and mode shapes of the structure. This may allow for more accurate estimation of parameters such as applied static loading or damage location and might facilitate more robust and accurate automated detection of modes of vibration.

Further work should also consider methods for separating changes in natural frequencies associated with applied static loading from those associated with damage to a structure. As was discussed in this study, these manifest in different ways within the time-history of the structural response. This should allow for the development of methods that can automatically distinguish damage from other causes of weak non-linear modal behaviour.

It is expected that there will be long-term changes in the dynamic behaviour of civil structures in-service. Over time these small changes will reduce the efficacy of the automated procedure for identifying modes of vibration in the presence of non-linear modal behaviour. Therefore, methods are needed for re-calibration of the probabilistic models. This might be achieved in practice through the application of known static loads to a structure, allowing the weak non-linear modal behaviour parameters to be re-calibrated.

Attention should also be given to the application of the methods developed in this study to emerging monitoring technologies such as drone-based vibration monitoring, distributed monitoring through vehicles and mobile phones, or sub-Nyquist monitoring. Key challenges to overcome in this area include incorporating the uncertainty of the spatial and temporal distribution of sensors within the predicted structural parameters.

Finally, as highlighted in the study of industry perceptions used to guide this research, if long-term monitoring is to be widely adopted as a tool to support the design and maintenance of civil structures, the issues surrounding data ownership and the legal liabilities associated with the use of long-term monitoring in engineering design must be resolved. Further consideration should be given to how to incentivise the uptake of long-term monitoring, as this study has demonstrated that in-service monitoring and design feedback loops have the potential to radically alter the structural engineering design process.
Publications

Published papers


Papers under review


Papers in preparation

Bibliography


349


376


Appendices
Appendix A

Questionnaire design and respondent characteristics

This appendix provides additional information regarding the design of the questionnaire used for the survey of industry perceptions of long-term monitoring detailed in Chapter 3 alongside details of the ethical considerations, data privacy and survey respondent characteristics. This appendix is adapted from work by Wynne et al. originally published as *Perceptions of long-term monitoring for civil and structural engineering* in *Structures* [1].

A.1 Questionnaire development and dissemination

The questionnaire, a copy of which is presented in Appendix B, was designed to gather quantitative and qualitative information from those involved in civil/structural engineering across three areas:

- Perceptions of unknowns and uncertainties in the existing civil/structural engineering design process.
- Perceptions and current uses of long-term monitoring within industry.
- Future potential and barriers to adoption of long-term monitoring within civil and structural engineering.

To ensure that the respondents constituted a representative sample of the engineering design community, information about the respondents engineering experience and current/most recent engineering employment was also collected as part of the questionnaire.

To gather information across the areas listed above an online questionnaire, delivered via Google Forms [461], was developed. The questionnaire, developed
in line with guidance provided in Brace [462] and Saris and Gallhofer [463], contained a mixture of open- and closed-ended questions. Open-ended questions, also referred to as open-requests for answers [463], were used to ensure that the respondents was given ample opportunity to convey their beliefs, free from the confines which may be introduced by categorical selections or scales. A key aspect of the questionnaire was that it was designed to gather information on both perceptions and beliefs. This was reinforced throughout the question wording, making explicit reference to the individuals own views and experiences. The qualifier “why?” was introduced to many of the open-ended questions so as to encourage the respondents to expand upon their answers [462]. The closed-ended questions were predominantly categorical and sought to gather quantitative information on the respondents beliefs which could be unambiguously compared across the sample set. Careful consideration was given to the choice of categorical questions over numerical scales, which have been shown to be highly subjective and more open to misinterpretation [463]. The questionnaire was divided up into clearly defined sections to both indicate to respondents their progression through the process, and remove ambiguity as to the focus of each question, for example by clearly demarcating which questions referred to current use of long-term monitoring as opposed to the potential future uses. Each section was introduced with a short (one or two line) preamble to provide context for the questions within the section and further reduced ambiguity [463].

Hyperlinks to the questionnaire were disseminated between June 2020 and August 2021, primarily via correspondence with regional, national and international professional institutions who shared the questionnaire with their members through direct correspondence, newsletters and via their websites. Additional dissemination was carried out via posts on online forums and professional networks, such as LinkedIn and GeoWorld. It is should be noted that no incentive was given for completion of the questionnaire, and that the questionnaire was only available in English.

A.2 Ethical considerations and data privacy
In line with European General Data Protection Regulation [464], explicit consent was obtained at the start of the questionnaire for collection, processing and storage of the data. All data was anonymised prior to processing through assignment of a unique numerical identifier. The purpose of the study, and how data would be used and stored, was communicated to the participants through a data collection consent form at the start of the questionnaire, adapted from that provided by the University of Edinburgh [465]. Demographic information about
the participants was limited to that strictly applicable to the research, with all potentially sensitive or non-relevant information, such as the gender and race of the participants, not included as part of the data collection.

### A.3 Respondent characteristics

In total, 146 responses to the questionnaire were collected from participants spread across 31 countries, with all continents represented in the data apart from Antarctica. The geographical distribution of the respondents are provided in Figure A.1 highlighting that while respondents were geographically diverse, respondents current/most recent employment were predominantly in English-speaking countries. Participants were also disproportionately likely to be from countries with high gross-domestic product (GDP) per capita. This should be considered when interpreting the results as the current use of long-term monitoring in lower-income countries, alongside the challenges they faced for integration and implementation of long-term monitoring within design, are likely to be different.

![Figure A.1: Predominant country of current/most recent employment of survey respondents.](image-url)

The respondents represented a cross-section of levels of engineering experience, as illustrated within Figure A.2 with a skew towards more senior positions, with 59.6% of respondents coming from senior or leadership positions, compared to 30% of respondents indicating lower levels of experience. This may indicate a greater interest in technology such as long-term monitoring from senior members within the field, but may also indicate differences in the engagement with professional institution literature and communications (the primary method for disseminating the questionnaire) or differing time-constraints.
47.6% of respondent most recent/current employer were large companies (more than 200 employees), 28.3% were medium sized companies (100 to 200 employees), 14.5% were companies with fewer than 100 employees, and 9.6% identified as self-employed. Within these companies, the respondents represented an extensive range of self-identified areas of expertise, as shown in Figure A.3. As may be expected, more general classifiers, such as “design” and “structural”, were self-identified as classifiers by a wider range of respondents than more technically specific areas such as ”environmental” or ”digital”. However, when respondents were asked to describe in their own words areas in which they have specialized, a much broader range of, in some cases highly specific, areas were elicited, illustrating both the intersectionality of civil and structural engineering and the inter-disciplinary nature of engineering design, findings further reflected in a similar question which asked respondents for a one sentence description of their employer.
Appendix B

Long Term Monitoring in Civil & Structural Design Questionnaire

Presented over the following pages is the online questionnaire used to understand the perception of long-term monitoring in civil and structural design, discussed in detail in Chapter 3.
Long Term Monitoring in Civil & Structural Design

This survey forms part of a research project looking at whether long-term monitoring data from structures can be used to improve the civil and structural engineering design process.

The survey is split into four sections:
1. Information about your engineering experience and current/most recent engineering employment.
2. Perceptions of unknowns and uncertainties in the existing civil/structural engineering design process.
3. Perceptions of current use of long-term monitoring in the civil/structural engineering design process.
4. Thoughts on future use of long-term monitoring in the civil/structural engineering design process.

All questions are optional and the survey may be discarded at any time. The survey will take approximately 10 to 20 minutes to complete.

This research is being undertaken by Zachariah Wynne as part of his PhD study. The research is overseen by the Institute of Infrastructure and Environment at the University of Edinburgh. For further information related to the project please contact Zachariah Wynne at Z.Wynne@sms.ed.ac.uk. Funding for this research is provided by EPSRC Doctoral Training Partnership Studentship (EP/R513209/1).

If you have any concerns about the conduct or content of this research please contact the Director of Research for the School of Engineering, University of Edinburgh, at Dob@eng.ed.ac.uk.

* Required

You are being invited to take part in research on long-term monitoring in the civil and structural engineering design process. Zachariah Wynne at the University of Edinburgh is leading this research, with funding provided by EPSRC Doctoral Training Partnership Studentship (EP/R513209/1).

Before you decide to take part it is important you understand why the research is being conducted and what it will involve. Please take time to read the following information carefully.

WHAT IS THE PURPOSE OF THE STUDY?
The purpose of the study is to understand perceptions of long-term monitoring in the civil and structural engineering design process.

DO I HAVE TO TAKE PART?
No – it is entirely up to you. If you do decide to take part you are still free to withdraw at any time and without giving a reason. All questions are optional and the survey may be discarded at any time.

WHAT WILL HAPPEN IF I DECIDE TO TAKE PART?
The survey is split into four sections and will take 10 -20 minutes to complete:
1. Information about your engineering experience and current/most recent engineering employment.
2. Perceptions of unknowns and uncertainties in the existing civil/structural engineering design process.
3. Perceptions of current use of long-term monitoring in the civil/structural engineering design process.
4. Thoughts on future use of long-term monitoring in the civil/structural engineering design process.

The survey consists of a mixture of multiple-choice and written questions.

WHAT ARE THE POSSIBLE BENEFITS OF TAKING PART?
By sharing your experiences with us, you will be helping Zachariah Wynne and the University to better understand whether long-term monitoring of structures can be used to improve the engineering design process.

ARE THERE ANY RISKS ASSOCIATED WITH TAKING PART?
This research involves the transmission of data over the internet. During online communication security of information cannot be guaranteed.

WHAT IF I WANT TO WITHDRAW FROM THE STUDY?
Agreeing to participate in this project does not oblige you to remain in the study nor have any further obligation to this study. If, at any stage, you no longer want to be part of the study, please inform the project administrator Zachariah Wynne at Z.Wynne@sms.ed.ac.uk. You should note that your data may be used in the production of formal research outputs (e.g. journal articles, conference papers, theses and reports) prior to your withdrawal and so you are advised to contact the research team at the earliest opportunity should you wish to withdraw from the study. On specific request we will destroy all your identifiable answers, but we will need to use the data collected prior to your withdrawal, and to maintain our records of your consenting participation.

DATA PROTECTION AND CONFIDENTIALITY
Your data will be processed in accordance with Data Protection Law. All information collected about you will be kept strictly confidential. Your data will be referred to by a unique participant number rather than by name. Raw data will only be viewed by the researcher/research team. All electronic data will be stored on a password-protected computer file and all paper records will be stored in a locked filing cabinet. Your consent information will be kept separately from your responses in order to minimise risk. Anonymised data will be publicly available for a minimum of 10 years after the completion of the study, in line with the Engineering and Physical Sciences Research Council (EPSRC) requirement for open access data.

WHAT WILL HAPPEN WITH THE RESULTS OF THIS STUDY?
The results of this study may be summarised in published articles, reports and presentations. Quotes or key findings will always be made anonymous in any formal outputs unless we have your prior and explicit written permission to attribute them to
1. I confirm that I have read and understood the Data Collection Consent for the above study. *
   
   *Mark only one oval.*
   
   - Agree
   - Disagree

2. I understand that my participation is voluntary and that I can ask to withdraw at any time without giving a reason and without legal rights being affected. *
   
   *Mark only one oval.*
   
   - Agree
   - Disagree

3. I understand that my anonymised data will be stored for a minimum of 10 years and may be used in future ethically approved research. *
   
   *Mark only one oval.*
   
   - Agree
   - Disagree

4. I confirm that I am above 18 years of Age.*
   
   *Mark only one oval.*
   
   - Agree
   - Disagree

5. I agree to take part in this study *
   
   *Mark only one oval.*
   
   - Agree
   - Disagree

6. Engineering Experience
   
   *Mark only one oval.*
   
   - Student
   - Intern/Apprentice
   - Engineer
   - Chartered/Senior Engineer
   - Project Manager
   - Team Leader
   - Senior Management
   - Other: ________________________________

The Respondent

This section is to understand your role and background in civil/structural engineering.
7. What was the size of your current/most recent employer in the engineering sector?

Mark only one oval.

- [ ] Self-employed
- [ ] Small (2 - 20 employees)
- [ ] Medium (20 - 200 employees)
- [ ] Large (200+ employees)

8. What is the primary country in which you work?


9. Area of Expertise (Tick all that apply)

Check all that apply.

- [ ] Construction
- [ ] Design
- [ ] Digital
- [ ] Civil
- [ ] Structural
- [ ] Infrastructure
- [ ] Environmental
- [ ] Geotechnical
- [ ] Project Management
- [ ] Water
- [ ] CAD/Technician
- [ ] Site Engineer
- [ ] Other: __________________________

10. Please provide a one sentence description of your current/most recent employer (e.g. timber engineering consultant, infrastructure consultant, temporary works designer).


11. Please provide a one sentence description covering any areas in which you specialised (e.g. concrete structural design, people flow analysis, hydraulic modelling).

Unknowns in the Structural Design Process

This section of the survey is to understand your perceptions of where uncertainty lies within the civil/structural design process.
12. Please rate your confidence that the characteristic design loads (given or calculated) available in codes and standards reflect reality based on your personal experience.

*Mark only one oval per row.*

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<th>Excessive overestimate of true value</th>
<th>Reasonable overestimate of true value</th>
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<tr>
<td>Impaired loading - Thermal</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
</tr>
</tbody>
</table>

13. Please rate your confidence that the characteristic dynamic design loads (given or calculated) available in codes and standards reflect reality based on your personal experience.

*Mark only one oval per row.*

| Dynamic loading - Footfall                        | ☐                                   | ☐                                    | ☐                  | ☐                          | ☐                                       | ☐   |
| Dynamic loading - Vehicle                         | ☐                                   | ☐                                    | ☐                  | ☐                          | ☐                                       | ☐   |
| Dynamic loading - Wind                            | ☐                                   | ☐                                    | ☐                  | ☐                          | ☐                                       | ☐   |
| Dynamic loading - Seismic                         | ☐                                   | ☐                                    | ☐                  | ☐                          | ☐                                       | ☐   |
14. Please rate your confidence that the design parameters (given or calculated) available in codes and standards reflect reality based on your personal experience.

Mark only one oval per row:

<table>
<thead>
<tr>
<th>Substantial overestimate of true value</th>
<th>Overestimate of true value</th>
<th>Close to true value</th>
<th>Acceptable underestimate of true value</th>
<th>Excessive underestimate of true value</th>
<th>N/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material strengths</td>
<td></td>
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</tr>
<tr>
<td>Material stiffnesses</td>
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<td></td>
</tr>
<tr>
<td>Structural joint strength</td>
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<td>Structural joint stiffness</td>
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<td>Geotechnical /foundation strength</td>
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<tr>
<td>Geotechnical /foundation stiffness</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

15. Please rate your confidence that the performance limits available in codes, standards or other guidance documents are appropriate based on your personal experience.

Mark only one oval per row:

<table>
<thead>
<tr>
<th>Substantial overestimate of safe limit</th>
<th>Acceptable overestimate of safe limit</th>
<th>Safe limit close to true value</th>
<th>Underestimate of safe limit</th>
<th>Excessive underestimate of safe limit</th>
<th>N/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static deflection limits</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Acceptable acceleration limits</td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

16. If you had access to a single set of perfect information about a structure you designed in the past to inform your future designs, what would be of most use to you?

______________________________________________________________________________
______________________________________________________________________________
______________________________________________________________________________
______________________________________________________________________________

17. In which area of civil/structural design do you think there is the greatest uncertainty about design assumptions? Why?

______________________________________________________________________________
______________________________________________________________________________
______________________________________________________________________________
______________________________________________________________________________
18. Have projects you have worked on been evaluated by the designers following construction?

Mark only one oval.

☐ Yes
☐ No
☐ N/A

19. If projects are assessed, how and when does this assessment take place? Is this common practice or is it limited to specific projects?

________________________________________________________________________
________________________________________________________________________
________________________________________________________________________

20. Is your current/most recent employer actively involved in monitoring of structures following construction?

Mark only one oval.

☐ Yes
☐ No
☐ N/A

21. If long-term monitoring data was available from previous projects, where could it be of use within the design process?

________________________________________________________________________
________________________________________________________________________
________________________________________________________________________

22. Are you familiar with the concept of a Digital Twin?

Mark only one oval.

☐ Yes
☐ No

23. Is your current/most recent employer actively using Digital Twins?

Mark only one oval.

☐ Yes
☐ No
☐ Don't know

24. If you are familiar with the concept of digital twins, please define it in your own words.

________________________________________________________________________
________________________________________________________________________
________________________________________________________________________
________________________________________________________________________
Integration of long-term monitoring into the future structural design process

25. Do you think long-term monitoring data could be of use to you in future projects? Why/Why not?


26. What barriers do you see for the use of long-term monitoring data in the structural design process?


27. If there was an opportunity to engage in long-term monitoring of a structure do you think your current/most recent employer should do it? Why/Why not?


28. Do you think your current/most recent employer would engage in long-term monitoring of structure? Why/Why not?


29. If your employer had access to long-term monitoring data from a project designed by another company do you think your company should use it? Why/why not?


30. Do you think your employer would use long-term monitoring data from another company? Why/Why not?


31. What concerns would you have about the use of long-term monitoring data to inform the engineering design of future structures?
Thank you for your help.

Thank you for taking part in this survey.

32. If you have any other thoughts on the use of long-term monitoring within the design process, please feel free to note them here.

________________________________________________________________________________________________________________________________________________________________________________________________________________________

33. If you are interested in taking part in an additional interview on the use of long-term monitoring within the design process please provide your name and email address in the space below.

________________________________________________________________________________________________________________________________________________________________________________________________________________________

34. If you are interested in receiving a summary of the survey responses please add your email address in the space below.

________________________________________________________________________________________________________________________________________________________________________________________________________________________
Appendix C

Case study - The Aberfeldy footbridge

This section is adapted from earlier work published by Wynne et al. in a chapter of *Civil Structural Health Monitoring as Operational Modal Analysis of a Historic GRP Structure*.

The Aberfeldy Footbridge was the world’s first major advanced composite footbridge. Constructed from glass-fibre reinforced plastic (GFRP) in 1991, the lightweight cable-stayed footbridge provided the client with a low-cost access route which needed minimal site equipment for erection. Since opening to the public, the resin used within the structure has degraded exposing the glass fibres and there have been issues with the connections between various components of the bridge. Following damage to the structure in 1997 by the crossing of a light vehicle, the bridge deck was strengthened with bonded GFRP panels. It has been reported that there has been a gradual change in dynamic properties. Between 1995 and 2000 the damping of the first vertical mode is reported to have reduced by 52% with a reduction in natural frequency of 2%. Additional testing carried out in 2011 suggests that the natural frequency of the first vertical mode had reduced by a further 5%.

This case-study presents a new analysis of data from previous dynamic tests carried out by other researchers in 2011 and 2013, alongside data from tests carried out by the authors in 2019. A variety of time and frequency domain operational modal analysis techniques are used to quantify the uncertainty within the results. The reported changes in dynamic properties over the past 14 years are compared to the known degradation and repair work carried out on the structure. These results indicate that the previously reported changes in dynamic properties may be due to weakly non-linear behaviour of the structure, close modes and
non-damage induced changes in dynamic behaviour. This illuminates the information which can be gained from in-depth dynamic analysis of civil structures, the challenges of damage detection and the need to quantify the range of dynamic behaviour a structure may exhibit.

C.1 Introduction

The Aberfeldy footbridge is a pedestrian cable-stayed bridge spanning the River Tay in Scotland. The primary structure is glass-fibre reinforced polymer (GFRP), a lightweight and flexible composite material. The bridge is easily excited by footfall loading due to its low mass. Within this document operational modal analysis (OMA), where the dynamic characteristics of a structure (natural frequencies, damping ratios and mode shapes) are estimated from an unknown input loading, is applied to data collected from the bridge under footfall excitation in 2011 [424], 2013 and 2019. These results are compared to the results of ambient testing of the structure in 1995 [466] and forced excitation testing of the structure carried out in 2000 [425]. The results show the challenges of analysing data from lightweight and degraded structures due to weak non-linearity introduced by changes in the dynamic behaviour caused by human-structure interaction, damage to the structure and loose connections.

C.2 Description and condition of the bridge

Opened in 1992, the bridge’s primary structure is formed of pultruded glass-fibre reinforced polymer (GFRP) sections and aramid suspension cables. With a main span of 63m, symmetric approach spans of 25m and a deck width of 2.12m, the bridge was the largest advanced composite bridge at the time of construction. At a total weight per metre span of less than 200kg [425], the bridge was assembled on site with minimal heavy equipment, reducing disruption and cost for the bridge owner, the Aberfeldy Golf Club [467]. An overview of the materials used within the structure and their masses is shown in Figure C.1.

To reduce uplift against wind, improve footfall behaviour and reduce flutter instability, the central cells of the deck at mid-span are filled with concrete. Despite this additional mass the structure experiences large amplitudes of oscillation under light footfall loading.

Considerable degradation of the GFRP has occurred since construction, most notably in the parapets of the structure where glass fibres have been exposed in
numerous places. Debonding of the GFRP is visible on the edge & cross beams and has been exacerbated by the growth of moss and algae on the structure. Separation of the components and loosening of joints is visible in numerous places including both parapets and the aluminium columns which act to tie down and support the approach-spans. Friction joints were used to connect the handrails to the upright to stiffen the parapet. However, due to a lack of rigidity between the upright and the deck, these frictional joints have loosened leading to separation of the components. This results in audible rattles as the bridge oscillates. The parapets have suffered extensive damage, likely due to vandalism, with numerous railings broken or missing on the northern approach span. Apart from surface degradation caused by moss and algae, the pylons do not exhibit signs of damage.

The bridge deck was damaged in 1999 by the passage of a small vehicle towing a trailer of sand. Following this damage, the deck was reinforced with pultruded GFRP plates, adhesively bonded to the top of the deck, and strengthening of the deck edge beams through the addition of Carbon Fibre Reinforced Polymer (CFRP) sheets. The strengthening increased the weight of the structure by 0.17kN/m span.

There is visible sagging of the bridge deck between the edge beams at mid-span of the structure. This sagging has not been referred to in previous reports on the condition of the structure and was anecdotally reported by local residents to have worsened in the past decade. There is extensive mould and algae growth.
on the deck structure but few signs of wear or erosion of the GFRP plates or CFRP.

Presented in Table C.1 are the Natural Frequencies and damping ratios for two sets of ambient excitation measurements collected in 1995 and 2011 [466, 424], and a single set of forced excitation measurements collected in 2000 [425]. Frequency results are based on inspection of the power spectral density and frequency response functions (FRFs) for the ambient and forced excitation respectively. The damping results for 1995 are based on the free decay following a jumping test by a single pedestrian. The damping results for 2000 are based upon direct-curve fitting of a FRF.

Table C.1: Natural frequency and damping values for 1995 [466], 2000 [425] and 2011 [424]. H = Horizontal mode, V = Vertical mode, T = Torsional mode.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (Hz)</th>
<th>Damping (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>1.00</td>
<td>0.98</td>
</tr>
<tr>
<td>V1</td>
<td>1.59</td>
<td>1.52</td>
</tr>
<tr>
<td>V2</td>
<td>1.92</td>
<td>1.86</td>
</tr>
<tr>
<td>V3</td>
<td>2.59</td>
<td>2.49</td>
</tr>
<tr>
<td>H2</td>
<td>2.81</td>
<td>2.73</td>
</tr>
<tr>
<td>V4</td>
<td>3.14</td>
<td>3.01</td>
</tr>
<tr>
<td>T1</td>
<td>3.44</td>
<td>3.48</td>
</tr>
<tr>
<td>V5</td>
<td>3.63</td>
<td>3.50</td>
</tr>
<tr>
<td>V6</td>
<td>4.00</td>
<td>3.91</td>
</tr>
<tr>
<td>T2</td>
<td>4.31</td>
<td>4.29</td>
</tr>
<tr>
<td>V7</td>
<td>4.60</td>
<td>4.40</td>
</tr>
<tr>
<td>V8</td>
<td>5.10</td>
<td>4.93</td>
</tr>
</tbody>
</table>

A new analysis of the 2011 data sets; triaxial data collected from a mobile phone accelerometer at 50 Hz, alongside data collected in 2013 using a mobile phone accelerometer at a variable sample rate between 20 Hz and 100 Hz, is included in later sections. The 2013 data has been resampled through linear interpolation to achieve a constant sample rate of 20 Hz. The loading on the structure over the course of these data sets includes jumping, heel drops and jogging of a single pedestrian.

Alongside this, 95 minutes of data were collected in December 2019 at a sample rate of 1024 Hz from 3 triaxial accelerometers, located at mid-span and 10.5 m either side of mid-span on the west side of the bridge deck, and 2 bi-axial accelerometers, located 10.5 m either side of mid-span on the east side of the bridge deck. The accelerometers used had a sensitivity of 1067 mV/g. This data constitutes both light wind and footfall excitation of the structure by individuals and
groups of pedestrians.

Where possible the data from 2011 and 2013 has been analysed in Section 4. However due to the low-quality of these data sets, much of the analysis has been focused on the use of the high-quality data collected in 2019.

C.3 Analysis methods

As the input force to the structure is unmeasured, assumptions about both the force and the structure must be made to allow the application of operational modal analysis (OMA). The structure is assumed to behave linearly; for a given input force the bridges behaviour is constant. The loading is assumed to be a Gaussian, broadband stochastic force; it has no dominant frequency component which falls within the same range as the natural frequencies of the structure and the force is uncorrelated with previous forces applied to the structure. By collecting a large data set under a range of operational conditions the likelihood that the loading approximates this assumption due to the variability in footfall and wind loading can be increased. The challenge with light-weight structures under footfall excitation is that the changes in dynamic properties with the number of people interacting with different parts of the structure is significant, violating the assumption of linear dynamic behaviour.

Within this document four OMA methods are used to identify the dynamic properties of the structure; Welch’s method [221], Covariance-driven Stochastic Subspace Identification (SSI-Cov) [17], the Random Decrement Technique (RDT) [246] and the short-time Random Decrement Technique (ST-RDT).

The ST-RDT is similar to conventional RDT except that Random Decrement Signatures (RDSs) are formed for short overlapping windows of data and correlation between data segments is allowed. This has the advantage that the behaviour of the structure is assumed to be linear only over the length of the window, not over the full data set. This allows weakly non-linear behaviour; where a structure has identifiable modes of vibration but the response of a mode exhibits variations in behaviour [15], to be observed and quantified.

C.4 Frequency domain analyses

In Figure C.2 the Welch PSDs for the 2011, 2013, and 2019 data are plotted based on a Hanning window, a weighted tapered cosine, of length 3 minutes with 1.5 minutes overlap. Modification of the type of window and the window length was
found to have little impact on the peaks of the frequency spectra. The 2019 PSDs are noisy, despite the strength of the measured acceleration signal, preventing accurate natural frequency and damping estimates. These noisy peaks are believed to be due to both the loose joints and connection of the bridge, leading to step changes in vibration behaviour, and the human-structure interaction leading to changes in the frequency and damping behaviour of the structure over the length of the data set. The 2011 data set shows peaks at lower frequencies than the 2013 or 2019 data set, the reason for this is unknown. The 2013 data set, collected using a mobile phone accelerometer, shows numerous windowing artefacts, believed to be induced by the re-sampling of the data. These results indicate that whilst the sensitivity of mobile phone accelerometers may be sufficient to monitor civil structures, the lower sample rates and inconsistent sample rates introduces significant artefacts into modal estimates.

![Figure C.2: Welch power spectral density of 2011, 2013 and 2019 acceleration data. Top plot: Transverse acceleration. Lower plot: Vertical acceleration. Modes identified by Pavic et al. [425] shown for comparison.](image)

**C.5 Time domain analyses**

**C.5.1 Covariance-driven Stochastic Subspace Identification (SSI-Cov)**

SSI-Cov was applied to the data using a maximum Toeplitz block length of 1.6 seconds with a maximum model order of 100. All channels of acceleration data
were used, with the data down sampled from 1024 Hz to 102.4 Hz prior to application of SSI-Cov. The results of the SSI-Cov analysis are shown as a stability diagram in Figure C.3. The natural frequency (frq.) and damping (dmp.) are deemed to be stable if they have varied by less than 0.5% and 5% respectively from the previous model order. The Modal Assurance Criteria (MAC) is used to identify stable modes. A mode is identified as stable if the MAC value is greater than 98%. For reference, the average of the 2019 Welch PSD results presented previously in Figure C.2 are plotted on the second y-axis.

![Figure C.3: SSI-Cov results (Toeplitz block length of 1.6 seconds) and Welch Power Spectral Density (PSD) for 2019 data.](image)

### C.5.2 Random Decrement Technique (RDT)

Each of the 2019 vertical and transverse acceleration data sets were analysed individually using the RDT with a range-crossing trigger level with upper and lower bounds of 1 and 1.1 multiples the standard deviation of the data set. An equivalent negative trigger level was applied simultaneously and all segments meeting it multiplied by -1. The segments were averaged together to form a random decrement signature (RDS) of 1.5 seconds length.

Each RDS ($\hat{D}$) was fitted individually on the assumption that they were the free response of a system with $N$ degrees of freedom system:

$$\hat{D} \approx \sum_{i=1}^{N} A_i e^{-\zeta_i \omega_i t} \cos(\omega_i t + \phi_{i,lag})$$  \hspace{1cm} (C.1)
Where \( N \) is the number of modes, \( A_i \) is the initial amplitude of the free-decay, \( \zeta_i \) is the damping of the mode as a percentage of critical damping, \( \omega_i \) is the natural frequency of the mode, \( \phi_{i,\text{lag}} \) is the the phase lag of the response and \( t \) is a time array corresponding to the length of the RDS. Fitting of the RDSs was carried out using the Levenberg-Marquardt algorithm non-linear least-squares curve fitting of Equation \( C.1 \) with the value of \( N \) varying from 1 to 20.

Each RDS contained between 8,726 and 12,030 segments of data. The RDSs for all transverse acceleration data sets are well described by two modes of vibration, an example is shown in Figure \( C.4 \) with average natural frequencies of 0.94 Hz and 2.63 Hz, and damping ratios of 1.05\% and 2.00\% respectively. The vertical RDSs are poorly fitted, as shown in Figure \( C.5 \) This poor fitting is believed to be caused primarily by the human-structure interaction. As pedestrians cross the bridge, the mass concentrated within each mode changes, changing its natural frequency. In tandem to this, the damping of each mode changes due to the additional energy dissipation introduced by the pedestrians. This results in the distinctive sharp peaks and troughs of the RDS in Figure \( C.5 \) due to the presence of multiple close and overlapping modes of vibrations within the different segments.

![Figure C.4: Transverse RDS for accelerometer at mid-span. Fitted signal based on two modes of vibration.](image)

C.5.3 **Short-time Random Decrement Technique (ST-RDT)**

The ST-RDT was applied to the 2019 vertical, horizontal and lateral acceleration data using an all-points triggering and sign-weighting conditions, with a window length of 60 s, and an overlap between neighbouring windows of 55 s. An RDS length of 1.5 seconds was used, with the first two data points excluded from the
Figure C.5: Vertical RDS for accelerometer at mid-span. Fitted signal based on five modes of vibration.

analysis to remove the correlation of noise and forced components from the RDS, alongside a target explained variance of $R^2 = 0.999$. Data was down sampled from 1024 Hz to 128 Hz prior to analysis. The results for the first vertical and horizontal modes are presented in Figure C.6 and Figure C.7 respectively.

Figure C.6: ST-RDT frequency estimates (black dots) for 1st horizontal mode. Vertical mid-span acceleration data shown in grey on second y-axis.

Figure C.7: ST-RDT frequency estimates (black dots) for 1st vertical mode. Vertical mid-span acceleration data shown in grey on second y-axis.

There is a clear variation in the behaviour of the structure with relation to
the amplitude of the oscillation due to the human-structure interaction, with this behaviour remaining consistent across all accelerometers. This is characterised for each mode by the reduction in the frequency as a pedestrian approaches the anti-node of the mode, as annotated in Figure C.6 and Figure C.7. The damping behaviour exhibits much larger ranges of weak non-linearity but remains consistent between separate sections of data. This weak non-linearity may be due to the changes in the number of people on the structure and their relationship to one another as well as the anti-nodes of the mode shapes. Despite the weak non-linearities across the data sets, the results for each mode of vibration may be pooled to create distributions of likely values based on the data collected, as shown in Figure C.8 and Figure C.9 for the 1st horizontal and vertical modes respectively.

Figure C.8: Histograms of frequency and damping values for 1st horizontal mode. Histograms plotted to show 95% of estimated values. Red lines indicate results of 2000 testing of bridge [425].

Figure C.9: Histograms of frequency and damping values for 1st vertical mode. Histograms plotted to show 95% of estimated values. Red lines indicate results of 2000 testing of bridge [425].
C.6 Comparison and discussion

The results of analysing the data en-masse, either through Welch’s method, SSI-Cov or conventional RDT do not identify clear linear modes. The analysis using the ST-RDT suggests that this is because the properties of those modes vary with time, likely because of the additional mass from people crossing the bridge.

Modes of vibration which could be clearly identified from the ST-RDT are summarised in Table C.2. These results are based on the bootstrapped mean values of the distributions of frequency and damping, examples of which were given in Figure C.8 and Figure C.9. Frequency and damping values are treated as pairs of values; a given bootstrap sample contains a fixed pair of frequency and damping values which must be re-sampled as a set.

Table C.2: Natural frequency and damping values obtained from ST-RDT analysis. Values shown in brackets represent the 95% confidence intervals of mean values obtained from bootstrapping of distributions. \( n \) value represents the number of occurrences of the mode in ST-RDT results.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (Hz)</th>
<th>Damping (%)</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>1.00</td>
<td>0.98</td>
<td>0.945 (0.0003)</td>
</tr>
<tr>
<td>V1</td>
<td>1.59</td>
<td>1.52</td>
<td>1.53 (0.0009)</td>
</tr>
<tr>
<td>V2</td>
<td>1.92</td>
<td>1.86</td>
<td>1.84 (0.0013)</td>
</tr>
<tr>
<td>V3</td>
<td>2.59</td>
<td>2.49</td>
<td>2.48 (0.0009)</td>
</tr>
<tr>
<td>H2</td>
<td>2.81</td>
<td>2.73</td>
<td>2.63 (0.0010)</td>
</tr>
<tr>
<td>V4</td>
<td>3.14</td>
<td>3.01</td>
<td>-</td>
</tr>
<tr>
<td>T1</td>
<td>3.44</td>
<td>3.48</td>
<td>-</td>
</tr>
<tr>
<td>V5</td>
<td>3.63</td>
<td>3.50</td>
<td>3.46 (0.0012)</td>
</tr>
<tr>
<td>V6</td>
<td>4.00</td>
<td>3.91</td>
<td>3.84 (0.0023)</td>
</tr>
<tr>
<td>T2</td>
<td>4.31</td>
<td>4.29</td>
<td>-</td>
</tr>
<tr>
<td>V7</td>
<td>4.60</td>
<td>4.40</td>
<td>-</td>
</tr>
<tr>
<td>V8</td>
<td>5.10</td>
<td>4.93</td>
<td>4.91 (0.0016)</td>
</tr>
</tbody>
</table>

The damping results show a clear increase for the values obtained from the analysis of the 2019 ambient data. Part of this is likely due to the increased damping imparted by pedestrians, however an over estimation of damping values may occur due to the rapid shifts in natural frequency which exist over the 60s window length.

The bootstrapped mean natural frequency is lower for most modes. However, this lower mean value is likely due to the reduction in natural frequencies due to additional mass from pedestrians. Examining the ST-RDT results for the lowest horizontal mode in the time domain, as shown in Figure C.6, it can be seen that
the measured natural frequency, with a bootstrapped mean of 0.945 Hz, returns to a value of around 0.98 Hz or 1.00 Hz as the amplitude of the signal dies away. This could indicate that once a pedestrian moves off the bridge, its unloaded natural frequency remains close to the values recorded in 1995 and 2000. This behaviour is repeated across the majority of modes of the structure.

The exception to this behaviour is the 1\textsuperscript{st} vertical mode. This appears to revert to a baseline natural frequency of between 1.60 Hz and 1.65 Hz, higher than the value recorded in 2000. As noted earlier, there is some non-linear behaviour induced by rattling of connections and tie-downs, which may lead to the recorded increase in natural frequency. Alternatively, it may be that the sagging of the bridge at mid-span has led to a change in the natural frequency. There is the possibility that differing temperatures between the 2000 and 2019 data collection regimes could result in the variation in results, however it would be expected that this would have a blanket effect on all natural frequencies, something which the results for other modes of vibration provide no evidence to support.

C.7 Summary

The analyses have shown that the natural frequencies of the Aberfeldy footbridge are lower for the 2019 dataset than previous measurements but there is evidence that this is due to the differences in pedestrian loading of the structure between the different data sets. Little evidence has been found that any significant change in dynamic properties of the Aberfeldy footbridge has occurred in the past twenty years, despite continued degradation and damage to the structure.

Understanding the weak non-linear behaviour of structures in-service could allow damage induced changes in the dynamic behaviour to be identified. However, dynamics based structural health monitoring is limited to types of degradation and damage which affect the global dynamic properties. Some of the damage to the Aberfeldy footbridge, such as the sagging of the bridge deck, is likely to fall outside of this scope. The lack of significant changes in the dynamic behaviour of the bridge would suggest limited damage of the primary load resisting systems.

A key challenge for long-term monitoring of structures, particularly lightweight structures like the Aberfeldy footbridge, is separating changes in dynamic behaviour due to damage from those changes in external factors such as loading and temperature. Short-time OMA methods, such as the short-time random decrement technique, provide a simple method through which changes in structural behaviour may be visually identified and assessed.
Appendix D

Case study - Analysis of the Whitmore timber building

This case study presents the use of the ST-RDT to extract the modal parameters from a seven day programme of vibration monitoring of 52 Whitmore Road, a composite timber and concrete building.

D.1 The building

52 Whitmore Road, Hackney, UK is a six-storey, composite cross-laminated timber (CLT) and concrete mixed-use building. Designed by Waugh Thistleton architects, with engineering by KHL UK, at the time of completion it was the world’s largest CLT structure [470]. A photograph of the completed structure presented in Figure D.1 with an axonometric view of the building is presented in Figure D.2.

Figure D.1: Photo of 52 Whitmore Road. Photo credit: Will Price [470].

Figure D.2: Axonometric view of 52 Whitmore Road. Image credit: Waugh Thistleton [471].
D.2 Instrumentation and data collection

Acceleration data was collected from three single-channel accelerometers fixed to a block and arranged to measure transverse (channel 1), longitudinal (channel 2) and vertical (channel 3) oscillations. The accelerometers were located at the north-east corner of the structure on the sixth floor, between 2014-01-24 23:59:48 and 2014-02-03 11:26:15. All data was sampled at 1652 Hz, with data sets split into 30 minute intervals for storage.

D.3 Modal analysis

The data was analysed with the short-time random decrement technique (ST-RDT), a short-time operational modal analysis method. Prior to analysis all data was highpass filtered with a Butterworth filter (filter order=1) to reduce components of the signal below the cutoff frequency of 0.1 Hz. Due to the relatively small accelerations of the building, the acceleration data was dominated by high frequency noise. To reduce this noise all data was lowpass filtered with a Butterworth filter (filter order=5) with a cutoff frequency of 25 Hz to reduce components of the signal outside of the frequency range of interest. After filtering the acceleration data was downsampled by a factor of six, from the initial sample rate of 1652 Hz to 275.3 Hz to prevent unnecessary computational effort resulting from analysis of the filtered frequency components of the data which are outside of the frequency range of interest. A comparison of 30 minutes of the raw data, and the data after filtering and down-sampling is presented in Figure D.3.

![Comparison of raw acceleration data and acceleration data after bandpass filtering and down-sampling.](image)

The ST-RDT was applied with an amplitude weighting condition and an all points triggering condition such that each random decrement signature (RDS)
formed was either: a time-lagged autocorrelation function, where the analysis
data channel was the same as the triggering/weighting channel of data, or a
time-lagged cross-correlation function, where the analysis data channel was a
different accelerometer channel than that used in the triggering/weighting channel
of data. A window length of 10 minutes, an overlap between adjacent windows
of 9 minutes and 30 seconds, and an RDS length of 1 second were used in the
ST-RDT analysis. The first 0.15 seconds of each RDS was trimmed in order to
exclude the majority of the filtering artefacts, noise impulses and the dominant
part of the autocorrelation of the forced response of the structure from the modal
analysis.

Each RDS was analysed with an increasing model order between 1 and 40 until
a target explained variance, as measured through the coefficient of determination
($R^2$), of 0.99 was achieved.

D.4 Results

The natural frequencies estimated through ST-RDT are plotted against the times-
tamps corresponding to the centre of the ST-RDT windows in Figure D.4 along-
side the filtered and downsampled acceleration data from one of the accelerom-
eters. In this plot the results are separated based on the triggering/ weighting
channel of data used, with results where the triggering channel is the same as the
analysis channel plotted as black scatter points. Modes of vibration are the dense
regions of frequency estimates in the results which extend across multiple time
steps. It can be seen that while the amplitude of the motion of the building was
very small, a number of noisy modes of vibration are detected across the length
of the data set, and there are distinct transverse, longitudinal, and vertical modes
of vibration.
Figure D.4: Natural frequencies extracted from 52 Whitmore acceleration data with the ST-RDT, plotted alongside acceleration data used in analysis.

The frequency and damping estimates, separated according to the trigger/weighting channel used in the ST-RDT, are plotted in Figure D.5, with likely modes of vibration corresponding to regions with high densities of modal estimates. Where the damping of the mode is negative it is likely to correspond to either spurious components in the RDSs, resulting from noise within the data, or be due to the non-zero forced component of the RDS which may be induced by a
correlation between the motion of the building and the forcing applied to it. The very high damping values between 3 Hz and 5 Hz are likely to result from similar sources, or may be associated with localized modes of vibration induced by the relative flexibility of the CLT structure compared to steel and concrete buildings. The most clearly defined modes, as visible in Figure D.4, have damping ratios in the range of 2% to 5%, broadly in line with what might be expected for a timber structure of this height.

Figure D.5: Natural frequencies and damping estimates extracted from 52 Whitmore acceleration data with the ST-RDT

The mean and standard deviations for the natural frequencies and damping of modes which were visually identified from Figures D.1 and D.2 are presented in Table D.1.

The frequency and damping of the fundamental mode of vibration, plotted
Table D.1: Natural frequencies and damping ratios of visually identified modes of vibration of 52 Whitmore Road. Standard deviations of mean values shown in brackets. Dominant channel is the ST-RDT trigger channels which resulted in a majority of the modal estimates in the frequency range used.

<table>
<thead>
<tr>
<th>Assumed frequency limits [Hz]</th>
<th>Number of estimates</th>
<th>Frequency [Hz]</th>
<th>Damping [%]</th>
<th>Dominant channel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower</td>
<td>Higher</td>
<td>30,882</td>
<td>2.78 (0.01)</td>
<td>2.73 (0.76)</td>
</tr>
<tr>
<td>2.70</td>
<td>2.85</td>
<td>23,910</td>
<td>3.58 (0.05)</td>
<td>4.89 (1.84)</td>
</tr>
<tr>
<td>3.45</td>
<td>3.70</td>
<td>20,527</td>
<td>5.71 (0.22)</td>
<td>1.50 (1.76)</td>
</tr>
<tr>
<td>5.00</td>
<td>6.50</td>
<td>124,594</td>
<td>8.41 (0.16)</td>
<td>3.89 (1.30)</td>
</tr>
<tr>
<td>8.00</td>
<td>9.00</td>
<td>94,315</td>
<td>11.40 (0.40)</td>
<td>2.70 (1.84)</td>
</tr>
<tr>
<td>10.50</td>
<td>12.50</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

against time in Figure D.6 is primarily in the direction of Channel 2, and has a mean natural frequency of 2.78Hz, and a mean damping ratio of 2.54%. The ST-RDT modal estimates for this mode of vibration are largely consistent across the length of the data, suggesting that the behaviour of the mode is only weakly affected by changes in external conditions.

Figure D.6: Natural frequency and damping estimates of for the fundamental mode of vibration of 52 Whitmore Road, based on ST-RDT estimates from where channel 2 was used as triggering, weighting and analysis data sets.

There is some evidence for amplitude dependant behaviour of both the frequency and damping of the mode, as shown in Figure D.7, with the ST-RDT estimates displaying lower frequency and damping values at higher modal amplitudes and envelope amplitudes, as measured through the summed amplitudes of the fitted modal components of the RDS. The results suggest a slight increase
in damping from 2.5% to 3% as the modal/envelope RDS amplitude increases to 0.005 m/s², with the damping then decreasing by approximately 0.5% as the modal/RDS envelope amplitude approaches 0.0008 m/s². The frequency shows a more linear decrease with increasing amplitude, from a mean value of approximately 2.785 Hz at a modal/RDS envelope amplitude of 0.0001 m/s² to 2.773 Hz at a modal/RDS envelope amplitude of 0.0008 m/s². The amplitude of the fundamental mode is strongly correlated with the envelope amplitude of the RDS. This suggests that the oscillation of the fundamental mode dominates the movement of the building at the accelerometer location. Due to this strong correlation between modal amplitude and envelope amplitude, it is not possible to separate the relationship between damping and modal amplitude from that between damping and the RDS envelope amplitude.

Figure D.7: Modal and RDS envelope amplitude dependant behaviour of natural frequency and damping of the fundamental mode of vibration of 52 Whitmore Road.

Similar relationships are observed for the four other modes between 0 Hz and 20 Hz which can be visually identified from Figures D.4 and D.5, as shown in Figures D.8 and D.9. The results suggest a stronger correlation with the modal amplitude, rather than the envelope amplitude of the RDS.
Figure D.8: Comparison of RDS envelope amplitude with natural and frequencies and damping ratios of four modes of vibration of 52 Whitmore Road.
Figure D.9: Comparison of mode amplitude with natural and frequencies and damping ratios of four modes of vibration of 52 Whitmore Road.
D.5 Summary

The natural frequencies and damping ratios of 52 Whitmore Road, a six-storey CLT building, display indications of amplitude dependant behaviour, but are largely insensitive to other changes in the external environment. The fundamental mode of the building over the course of seven days of monitoring has a mean natural frequency of 2.79 Hz with a standard deviation of 0.02 Hz, and a mean damping ratio of 2.56%, with a standard deviation of 0.95%. The movement of the mode at 2.79 Hz dominates the structural response. There are indications that all modes of vibration detected in the range 0 Hz to 20 Hz exhibit some degree of amplitude dependant behaviour, with this behaviour more strongly correlated with the amplitude of each mode of vibration, rather than the amplitude of the building movement.
Appendix E

Case study - Lake seiches

The use of the ST-RDT is not limited to the analysis of civil structures, but can be used to estimate the modal parameters of any dynamic system. This is demonstrated in this section through the application of the ST-RDT to estimate the natural frequencies and damping ratios of standing waves in lakes using water level data collected from Lake Tahoe and Lake Geneva, previously analysed using the conventional RDT by Wynne et al. These standing waves, known as barotropic or surface seiches, are the oscillation of the water surface under a gravitational restoring force which occur after an upwelling event, where the water within the lake is forced towards one end of a lake basin by wind stress on the surface of water, earthquakes or landslides. Accurate quantification of the periods and damping of surface seiches is a key factor in understanding lake energy dynamics, their interactions with ecosystems, modelling shoreline erosion, and assessing the risk of damaging seiche events. The ST-RDT allows the variation in the natural frequencies and damping ratios across the data from the two lakes to be quantified, and this variation related to the lakes annual cycle, wind speed and water level, allowing for a greater understanding of the frequency with which seiche events occur, the atmospheric conditions required for their formation, and how changes in the lakes water level might impact on the dynamic characteristics of seiche behaviour.

E.1 Lake Tahoe

Lake Tahoe, located on the border between California and Nevada in the USA, is the largest alpine lake in North America. The natural frequencies and damping ratios of the surface seiches were estimated through the ST-RDT using water level data collected across three monitoring periods: i) 30 July 2013 to 6 December
2013, ii) 1 January 2014 to 10 May 2014, iii) 6 January 2015 to 15 May 2015. All data was collected using a water pressure transducer located offshore from Homewood, shown in Figure E.1 with a sample rate of 30 seconds.

Data was high-pass filtered with a 30 minute Butterworth filter (filter order \(= 1\)) prior to analysis to remove the daily and seasonal fluctuations in the water level. Given the relative rarity of surface seiche events, the data was analysed with a window-length of 120-hours with an overlap-length of 117-hours. The RDS was formed with an RDS length of 45-minutes, approximately twice the lowest expected seiche period based on inspection of the frequency spectrum of the data, with the first two data-points of the RDS excluded to remove the autocorrelation of the noise within the data and accompanying filtering artefacts. All ST-RDT analyses used an all-points triggering condition with amplitude weighting, with the water level data from the Homewood location used simultaneously as the analysis, triggering and weighting channels in the ST-RDT analysis. The RDS is therefore exactly equal to the autocorrelation of the data, as discussed in Section 7.3.3. The model order used for modal analysis of each RDS was iterated to achieve a minimum target explained variance \( (R^2) \) of 0.95. All modal estimates with a negative damping ratio, which are indicative of modal estimates associated with fitting of a non-zero forcing components within the RDS, were discarded.

The ST-RDT estimates of the natural periods of oscillation in minutes are presented in Figure E.2 for the three data sets. Given the low natural frequencies typically observed for surface seiches, all results in this section are presented in terms of the equivalent period of oscillation \( T_j \) in minutes, given for mode \( j \) by
\[ T_j = \frac{1}{f_j}/60 \] where \( f_j \) is the natural frequency in Hz.

Figure E.2: ST-RDT estimates of natural period of Lake Tahoe surface seiches versus time. Likely modes of vibration for the surface seiche appear as dense areas of points within the results, indicating higher densities of modal estimates.

In Figure E.3, the natural period estimates presented previously in Figure E.2 are plotted against the accompanying damping ratio. Also presented in this figure are the natural periods and damping ratios estimated using conventional RDT analysis by Wynne et al. [4] and natural periods predicted through finite element modelling of the Lake Tahoe surface seiche carried out by Roberts et al. [472].
Figure E.3: Comparison of ST-RDT estimates of natural periods and damping ratio of Lake Tahoe surface seiches, with natural periods and damping ratio estimated through conventional RDT analysis \[4\] (white crosses) and natural periods predicted using FE modelling \[472\] (white dashed lines). Top plot) 2013 Lake Tahoe data, middle plot) 2014 Lake Tahoe data, lower plot) 2015 Lake Tahoe data. Bright regions of the 2D histogram correspond to higher densities of ST-RDT modal estimates. Histogram bin dimensions of 0.02 minutes period by 0.05% damping.

Figure E.3 suggests that the conventional RDT analysis \[4\] resulted in the median period of the dominant surface seiche, the seiche with a period of between 11.3 minutes and 11.8 minutes in the ST-RDT results, being underestimated by between 2% and 6%. Similarly the ST-RDT results suggest the median damping ratio of the dominant seiche was underestimated in the conventional RDT analysis by approximately 60% for the 2013 data, overestimated by approximately 70% for the 2014 data, and underestimated by 25% for the 2015 data. However in all
cases the conventional RDT estimates of damping fell within the 95% confidence interval of the observed damping ratio for the dominant seiche modes. This suggests that the damping ratios previously obtained may have been accurate, but failure to account for weak non-linear behaviour biased these estimates towards higher amplitude seiche events.

Several other clusters of natural period and damping estimates indicating other modes of vibration are visible in the ST-RDT results with periods of approximately 9 minutes (1% damping), 4.8 minutes (0.5% damping), 3.6 minutes (0.5% damping) and 2.5 minutes (0.5% damping). These periods are substantially shorter than any previously observed seiche periods for the lake, but Welch frequency analysis of the raw data, plotted in Figure E.4 show evidence of these modes are also present in the frequency power spectrum, suggesting that these oscillations may be real and consistent fluctuations in the water level. Whether these fluctuations correspond to surface seiches or other wind generated waves cannot be extrapolated from the raw data or ST-RDT results but bears further investigation. The ST-RDT results presented previously in Figure E.2 would suggest that the close peaks in the Welch frequency spectra between 11.3 minutes and 11.8 minutes both correspond to a single mode of vibration, as no corresponding close modes are observed in the ST-RDT results. Instead it may be that there is a single mode of vibration which is sensitive to the wind speed or direction, with slight changes in this forcing resulting in a corresponding change in mode shape and period.

![Welch power spectral density of Lake Tahoe water elevation data compared to median periods of seiches identified through the ST-RDT analysis. Welch frequency spectrum calculated for a window length of 48 hours.](image)

**Figure E.4:** Welch power spectral density of Lake Tahoe water elevation data compared to median periods of seiches identified through the ST-RDT analysis. Welch frequency spectrum calculated for a window length of 48 hours.

Inspection of the periods presented in Figure E.2 suggests the presence of systematic variation in the seiche periods, indicated by trends within the natural periods which span multiple non-overlapping windows of data. The most promi-
nent of these are the significant changes in the periods of the first and second
dominant modes of up to 10% of the natural period. Darbyshire and Darbyshire
[473] has previously suggested that the period of surface seiche oscillation may
be dependant on the wind speed, with higher wind speeds being correlated with
larger upwelling events accompanied by a change in seiche period. Similarly it
may be that small changes in the dominant wind direction causes oscillation of
the surface seiches around slightly different nodal lines.

To explore this behaviour the natural period estimates of the modes centred
on 11.4 minutes and 9 minutes, the two modes of oscillation which dominate the
ST-RDT estimates, are compared with the average wind-speed and direction
over the ST-RDT windows in Figures E.5 and E.6. The first dominant surface
seiche is negatively correlated with the wind speed, whilst higher wind speeds are
positively correlated with higher damping estimates. There are weak indications
of potential correlations between the period and damping and the wind direction
for the first dominant seiche mode. There is a strong positive correlation between
the period of the second dominant seiche mode and the wind direction and speed,
whilst the damping ratio of the second dominant seiche mode appears to be
positively correlated with the wind speed, with a weak negative correlation with
wind direction. These findings would fit with the results presented by Parsmar
and Stigebrandt [474] who suggested that the damping of baroclinic or internal
seiches, where it is the transition zone of warm surface water to colder water
at greater depths which oscillates, was heavily dependant on the bathymetry
(underwater topography) of the lake basin. The results for Lake Tahoe suggests
that a similar phenomena might exist for surface seiches, with both the period
and damping of the surface seiche being strongly sensitive to small changes in the
dominant wind speed and direction.
Figure E.5: Lake Tahoe first dominant surface seiche periods compared to weighted average wind-speed and direction. Linear line of best fit shown in red.
Figure E.6: Lake Tahoe second dominant surface seiche periods compared to weighted average wind-speed and direction. Linear line of best fit shown in red.
E.2 Lake Geneva

Lake Geneva straddles the border of France and Switzerland, and with a surface area of 580 km\(^2\) is one of the largest freshwater lakes in Europe. The data used in the ST-RDT analysis was water level data, sampled at 10 minute intervals, collected continuously at the three locations shown in Figure E.7 between the 1\(^{st}\) of January 1974 and the 7\(^{th}\) of January 2013.

![Bathymetric map of Lake Geneva showing locations of data collection](image)

The data was analysed with an all-points triggering condition with amplitude weighting, using a window length of 30-days, an overlap length of 29-days, and an RDS length of 6-hours. All data was filtered with a highpass Butterworth filter (filter order=2) with a filter period of 4.5 hours to remove diurnal and seasonal components from the water level data prior to the ST-RDT analysis. The water level data from each of three-locations was used as both the analysis, and a combined triggering and weighting channel in the ST-RDT analysis, resulting in nine unique RDSs for analysis for each window of data. An amplitude weighting of the segments was used alongside the model order being iterated to achieve a minimum target explained variance (\(R^2\)) of 0.95.

Presented in Figure E.8 are the ST-RDT estimates of the natural periods of the dominant seiches detected across the 30-year dataset. Only two modes of oscillation are consistently identifiable across the nearly forty years of data, with median periods of 73.9 minutes and 35.5 minutes minutes, both with an average damping ratio of approximately 0.5% as shown in Figure E.9.
The results obtained with the ST-RDT represent a substantial improvement over the conventional random decrement technique [4] which failed to accurately identify the second dominant mode of vibration at 35.5 minutes. While the natural period of the first dominant seiche, estimated at between 73.63 minutes and 73.8 minutes using the conventional random decrement technique [4], is broadly similar to the ST-RDT results, the average damping of the seiche estimated using the ST-RDT is approximately ten times lower than that estimated using the conventional RDT (4.72% [4]). This difference is believed to be due to a failure to account for the autocorrelation of the wind forcing in the modal analysis of the
RDS by Wynne et al. [4], which caused a strong biasing of the modal estimates. Failure to account for the correlation of the forced response may also explain the broad range of damping estimates across the three measurement locations reported in that work. In the ST-RDT analysis the first six data points of the RDS were excluded to account for this broadband forced response, an example of which is shown in Figure E.10.

![Example of random decrement signature for Lake Geneva water level data.](image)

The reason why the second dominant mode of vibration is not detected in the water level data prior to 1980 is unknown. During this period there are no meaningful changes in the water level of the lake, which is artificially controlled [475], nor were there notable changes in the wider atmospheric conditions in the lake basin or in the equipment used to collect the water level data.

The median mode shapes for the dominant Lake Geneva surface seiches, generated using the methodology presented in Section 7.3.2, are presented in Figures E.11 and E.12. At Location 2028, located in the shallow eastern end of the lake basin, the average amplitude of the seiche is more than five times higher than at the other locations, perhaps indicating an upwelling of the seiche as it enters the shallower part of the basin.
Figure E.11: Mode shape for Lake Geneva surface seiche with natural period between 72 minutes and 76 minutes generated using the ST-RDT modal estimates for water level data collected between 1970 and 2013 at the locations shown in Figure E.7.
Figure E.12: Mode shape for Lake Geneva surface seiche with natural period between 34 minutes and 37 minutes generated using the ST-RDT modal estimates for water level data collected between 1970 and 2013 at the locations shown in Figure E.7.
E.3 Summary

In this case study the cross-applicability of the short-time random decrement technique (ST-RDT) to other sensor measurements has been demonstrated through the analysis of lake surface seiches. Through direct comparison with the conventional random decrement technique (RDT), the additional insights arising from short-time operational modal analysis methods, such as detection of a wider range of modes and non-biased estimation of weak non-linear behaviour, have been illustrated. This work has highlighted the importance of excluding components of correlation functions arising from correlation between the system response and applied forcing to avoid biasing and corruption of modal estimates.
Case study - Vibration response of the MX3D footbridge

This case study is adapted from Wynne et al. [8], currently under review at Case Studies in Construction Materials under the title of Dynamic testing and analysis of the world’s first metal 3D printed bridge.

The MX3D Bridge is the world’s first additively manufactured metal bridge. It is a 10.5 m-span footbridge, and its dynamic response is a key serviceability consideration. The bridge has a flowing, sculptural form and its response to footfall was initially studied using a 3D finite element (FE) model featuring the designed geometry and material properties obtained from coupon tests. The bridge was tested using experimental modal analysis (EMA) and operational modal analysis (OMA) during commissioning prior to installation. The results have shown that the measured vibration response of the bridge under footfall excitation is 200% greater than predictions based on the FE model and contemporary design guidance. The difference between predicted and measured behaviour is attributed to the complexity of the structure, underestimation of the modal mass in the FE model, and the time-variant modal behaviour of the structure under pedestrian footfall. Both OMA and EMA give a dominant natural frequency for the bridge of between 5.19 Hz and 5.32 Hz, higher than the FE model prediction of 4.31 Hz, and average damping estimates across all modes of vibration below 15 Hz of 0.61% and 0.74% respectively, higher than the 0.5% assumed within the design guidance, slightly reducing the peak response factor predicted for the bridge.
F.1 Introduction

F.1.1 Data-driven design for vibration

Data-driven design, where in-situ structural monitoring is used to refine design assumptions, has the potential to reduce i) the uncertainty in the structural design process, ii) material usage, and iii) costs, while increasing the user comfort and lifetime of the structure [476]. The key to this approach is the blending of in-situ observations of similar structural forms with predictions of behaviour made through engineering models, such as finite element (FE) modelling [477]. While there is a wide range of available methods for: a) in-situ structural sensing [362, 363, 361], b) updating of numerical structural models [478, 479, 480], and c) fusing in-situ measurements with numerical models [167, 366], little attention has been given to date in how in-situ structural monitoring may provide the greatest benefit for improving structural design [481].

An area where in-situ structural monitoring of existing structures may offer significant benefit is in ensuring compliance with serviceability requirements, such as meeting the required levels of user-comfort. These performance criteria often guide the design of lightweight structures, or those with novel, efficient forms and construction techniques, due to their lower stiffness-to-strength ratio when compared with traditional designs. A key serviceability criterion for the design of floors in buildings and bridges is ensuring that vibrations do not cause discomfort to users or impair the use of the structure [426]. This requirement has led to the development of guidelines on the design of structures for footfall induced vibration [427, 428, 429, 189, 429, 430].

The structural system parameters need to be accurately predicted in order to estimate the vibration response. For regular structural systems, modal parameters can be determined using simplified methods (e.g. [426]). However, for more complex and novel structural designs, numerical models are generally required. Once the modal parameters have been determined, equations based on the theory of linear structural dynamics can be used to estimate the vibration response of the structure to the impulsive and harmonic pedestrian forces [482]. The efficacy and accuracy of this process for structurally complex additively manufactured structures such as the MX3D Bridge is, however, largely untested, preventing benchmarking for use in future designs.

Recent work to characterize the vibration response of steel footbridges under footfall excitation has primarily focused on the analysis of conventionally designed and manufactured structures (e.g. [483, 484, 485]). Work on the impact of novel
structural materials on dynamic behaviour include Ivorra et al. [486] who highlighted the impact of retrofitting of glass fiber-reinforced polymer on the modal parameters of a steel footbridge. Alongside this, Soria et al. [487] highlighted the importance of external factors such as temperature in the dynamic behaviour of a steel stress-ribbon footbridge and how changes in modal parameters may impact vibration response.

F.1.2 The MX3D Bridge

An example of a structure which is pushing the boundaries of modern construction technology is the MX3D Bridge, which is the world’s first metal additively manufactured (AM) bridge [488]. The process of additive manufacturing is also commonly referred to as 3D printing. The MX3D Bridge (the bridge), shown in Figure F.1, was constructed by the Dutch start-up company MX3D using wire arc AM (WAAM), a type of directed energy deposition [489]. The bridge has a span of 10.5 m, and comprises a substructure, handrails, deck plate, end beams and non-structural decorative swirls at the ends of the handrails. All parts except for the deck plate and end beams were built using WAAM. The bridge is supported by two pinned supports and two roller supports, with load distributed about the diagonal axis of the bridge deck, in line with the spanning direction of the primary bridge substructure pictured in Figure F.1. To understand the behaviour of the bridge and inform future designs, comprehensive material, cross-section and structural testing has been undertaken [488, 490, 491], and the bridge has been extensively instrumented with structural and environmental sensors including load cells, strain gauges, displacement transducers, inclinometers, accelerometers and thermistors, as part of the Smarter Bridge Sensor Network [492].

![Handrails, Deck plate, Non-structural decorative swirls](image1)

(a) The MX3D Bridge during temporary installation at Dutch Design Week 2018.

![Substructure, End beam](image2)

(b) Beam elements of the MX3D bridge, prior to installation of bridge deck. Image credit: MX3D.

Figure F.1: The MX3D Bridge.

The permanently installed instrumentation provides a unique opportunity to
understand the interaction of pedestrians with the bridge, and to demonstrate how data from structural sensors can be used to feed back critical information into the structural design process, such as the accuracy of design stage predictions of structural behaviour and the in-situ performance of additively manufactured structures.

The motivation of this research is to investigate the dynamic behaviour of a novel structure, and to compare a design approach which takes advantage of in-situ sensing to characterize the structure’s dynamic parameters to conventional approaches for predicting the vibration response of structures under footfall excitation. The vibration response of the bridge, when subjected to footfall excitation by a single pedestrian walking at two different frequencies, is measured and the response is compared to predictions obtained using an established method for footfall-induced vibration. The input parameters for the predictions are derived from one of the following three methods: i) a finite element (FE) model (this is representative of the design-stage estimate of the response), ii) measurements of the dynamic parameters by experimental modal analysis of the bridge excited by an impact hammer, and iii) measurements of the dynamic parameters from operational modal analysis of the bridge under footfall loading.

F.2 Background

F.2.1 Methods of modal analysis

In-situ modal analysis allows the dynamic behaviour of a structure to be quantified and is divided into experimental and operational modal analyses.

Experimental modal analysis (EMA) is the conventional method of estimating the modal parameters of a completed structure. A known force is applied to the structure and the dynamic response is measured using accelerometers, which is then used to estimate the natural frequencies, damping ratios, mode shapes and modal masses of the structure.

Operational modal analysis (OMA) uses measurements of the dynamic response of a structure under ambient excitation from wind, footfall or vehicles, to estimate the modal parameters. Unlike EMA, the applied force is unknown and is usually assumed to be a broadband stochastic force. However, a key advantage of OMA over EMA is that it can provide a better understanding of the behaviour of a structure as all measurements are made under forces and conditions more closely approximating the real in-service conditions.

One aspect of dynamic behaviour which both EMA and OMA may fail to
capture is time-variant modal behaviour, where a linear dynamic model based on certain modal parameters describes the behaviour of a system well, but those parameters change over time [15]. While there are still clearly defined modes of vibration, the frequencies, damping and mode shapes of these modes change due to changes in the external environment (additional mass, changes in temperature) or to the material (hysteresis, damage) [15]. These changes in dynamic behaviour can have significant implications for understanding civil structures in-situ, and may affect the vibration serviceability of floors and footbridges if the modal parameters used in design do not correspond to the appropriate environment for the design scenario in question.

In this study, time-variant modal behaviour was induced by pedestrians crossing the bridge, as they provide additional mass and damping to the system. To accurately quantify this time-variant modal behaviour, a new time-domain OMA method, the short-time random decrement technique (ST-RDT), has been developed. This technique allows the modal parameters of civil structures to be estimated using ambient vibration measurements over short time intervals, allowing changes in dynamic parameters to be tracked over time. By tracking changes in the dynamic parameters, the impact of time-variant modal behaviour can be quantified, allowing for a more accurate assessment of serviceability performance and diagnosis of the causes of differences in behaviour between the design and the structure [6, 5, 3].

F.3 Methods

To predict the vibration response of the bridge using published design guidance, three sets of modal parameters were used, and the results compared: the modal parameters predicted using the FE model of the structure, those estimated through experimental modal analysis, and those estimated through the ST-RDT. Note that all EMA and OMA is limited to analysis of the vertical vibration only, since the peak vertical vibrations observed in the measured acceleration response of the bridge are approximately 30 times greater than the transverse accelerations caused by crossings by a single pedestrian. Modes that were primarily transverse were included in the analysis of the FE model results. However, only the vertical components of these mode shapes were used within the analysis.
F.3.1 Predicting the vibration response

The vibration response of the bridge was first predicted using the resonant response analysis methodology set out by Willford and Young [426]. An extensive review of other design guidance for footfall induced vibration of footbridges is provided by Maraveas et al. [494]. In Willford and Young [426], it is assumed that the harmonic force $F_h$ applied to the bridge for a given footfall frequency $f_w$, in hertz, is represented using the first four harmonics, $h$, of the walking frequency $f_h$ as $f_h = hf_w$, where $h$ is an integer between one and four. The real ($a_{\text{real},h,m}$) and imaginary ($a_{\text{imag},h,m}$) components of the acceleration response at each mode $m$ of the structure below 15 Hz are then calculated through Equations F.1 and F.2.

$$a_{\text{real},h,m} = \left( \frac{f_h}{f_m} \right)^2 \frac{F_h \mu_{r,m} \mu_{e,m} \rho_{h,m}}{m_m^2} \frac{A_m}{A_m^2 + B_m^2} \quad (F.1)$$

$$a_{\text{imag},h,m} = \left( \frac{f_h}{f_m} \right)^2 \frac{F_h \mu_{r,m} \mu_{e,m} \rho_{h,m}}{m_m^2} \frac{B_m}{A_m^2 + B_m^2} \quad (F.2)$$

Within Equations F.1 and F.2, $f_m$ is the natural frequency of mode $m$ in Hertz, $\mu_{r,m}$ and $\mu_{e,m}$ are respectively the amplitudes of the unity-scaled mode shape at the response measurement and excitation locations, $\rho_{h,m}$ is a factor related to the ratio of the stride length to the span of the bridge deck, and $m_m$ is the modal mass of mode $m$ associated with the unity-scaled mode shape. The factors $A_m$ and $B_m$ are calculated using the ratio of walking and natural frequencies, $A_m = 1 - (f_h/f_m)^2$ and $B_m = 2\zeta_m(f_h/f_m)$, where $\zeta_m$ is the modal damping of mode $m$. The harmonic footfall force, $F_h$, is calculated using the dynamic load factors, derived from footfall time history measurements made by Kerr [495]. The unity-scaled mode shape is scaled to have a peak amplitude of one.

The magnitude of the acceleration response at a given walking harmonic $|a_h|$ is then calculated using Equation F.3.

$$|a_h| = \sqrt{\left( \sum_m a_{\text{real},h,m} \right)^2 + \left( \sum_m a_{\text{imag},h,m} \right)^2} \quad (F.3)$$

This acceleration magnitude is then scaled according to the root-mean squared (RMS) acceleration level of the average threshold of perception [426] to calculate the response factor for each walking frequency harmonic $R_{1-4}$. The total response factor $R$ is calculated as $R = \sqrt{R_1^2 + R_2^2 + R_3^2 + R_4^2}$.

This process is repeated across a range of likely walking frequencies and all plausible combinations of measurement point $\mu_r$ and excitation point $\mu_e$ on the
structure. The range of plausible walking frequencies are here conservatively taken as 0 Hz to 3 Hz. The extreme value of 3 Hz is considered within the analysis as the bridge’s permanent installation location in Amsterdam suggests it is plausible that a pedestrian may cross the bridge rapidly while other stationary pedestrians are on the structure.

F.3.2 Experimental modal analysis

Roving impact hammer excitation [15] was used to estimate the modal parameters of the bridge, as shown in Figure F.2a. As the impact hammer testing occurred prior to the installation of the permanent bridge sensor network, the bridge response was measured using four tri-axial accelerometers temporarily installed on the underside of the bridge substructure; the impact locations and temporary accelerometer locations are shown in Figure F.3. The measured acceleration response of the bridge, sampled at 1706.76 Hz, and impact hammer force were used to generate accelerance impulse response functions (IRFs) with a duration of seven seconds. IRFs were fitted using the matrix pencil method [254], a form of singular value decomposition, to estimate the natural frequencies, damping ratios and scaled mode shapes of the bridge. To calculate the modal mass of the structure, the accelerance mode shapes, calculated using the multiple-input multiple-output methodology as described by Catbas et al. [496], were converted to mobility mode shapes by dividing the accelerance mode shape amplitude by the circular frequency of the mode. From the mobility mode shape, the modal mass \( m_m \) was calculated through Equation F.4

\[
m_m = \left( \frac{A_{m,hd}}{A_{m,max}} \right)^2 \frac{A_{m,hd}}{A_{m,hd}} \tag{F.4}
\]

In Equation F.4, \( A_{m,hd} \) is the amplitude of the mobility mode shape at the accelerometer location, and \( A_{m,max} \) is the maximum absolute amplitude of the mobility mode shape [353].
F.3.3 Operational modal analysis

For OMA of the bridge, vertical acceleration data sampled at 100 Hz was collected from seven accelerometers on the underside of the substructure, installed as part of the permanent bridge sensor network and shown in Figure F.3, during crossings of the bridge by a single pedestrian walking at guided average step frequencies of 1.70 Hz and 2.79 Hz, shown in Figure F.2b. The pedestrians pace frequency was guided via a metronome, with the average footfall rate validated using data collected from the load cells and strain gauges installed on the bridge as part of
the permanent bridge sensor network. The acceleration data sets collected were analysed with the short-time random decrement technique (ST-RDT).

The majority of OMA methods assume that the dynamic response of the system to a known input is constant over time, neglecting the impact that time-variant modal behaviour may have on estimates of dynamic parameters. To overcome the shortcomings of existing OMA techniques, the short-time random decrement technique (ST-RDT), a time-domain OMA method, has been developed by the authors [6, 5, 3]. The ST-RDT is an expansion on the random decrement technique [287], and allows variations in a structure’s dynamic properties to be tracked over time by dividing the measured acceleration of a structure into short windows of data. The system’s behaviour is assumed to be linear within the window of data, but not across separate windows.

Conventional time-domain OMA methods may skew towards periods of high excitation, due to the use of correlation functions or approximated free-responses within the modal analysis. The response of the structure during these periods of high excitation may not be representative of the typical dynamic response of the structure under footfall excitation. The ST-RDT allows the median response of the structure to be estimated, which provides a more representative measure of the structural response of the structure which would be experienced by pedestrians. The ST-RDT accounts for autocorrelation of noise within the measurements through the trimming of the weighted correlation functions [2] and allows the uncertainty in the modal parameters to be quantified across separate windows of analysis [3]. Unlike other proposed methods of quantifying the uncertainty or variation in OMA modal parameters, such as the bootstrapped random decrement technique [303], the ST-RDT is robust to heteroscedasticity of the approximated structural response under varying excitation levels, and makes no assumptions of the distribution of errors within the modal estimates. Therefore it provides an unbiased approximation of the variability within the modal estimates from which representative samples, such as the median modal parameters, may be generated.

The ST-RDT relies on three sources of data: i) the analysis data: the data from (often multiple) sensors from which modal parameters are estimated; ii) the trigger data, which define which sections of the analysis data are of interest; and iii) the weighting data, which are used to apply a weighting to the analysis data to further focus on the behaviour of interest. These sources of data do not have to be unique, for example a simple application of the ST-RDT may be to collect a single set of acceleration data which can then be used as the analysis, trigger and weighting data sets. For the analysis of the bridge, all vertical deck accelerometer channels from the permanent sensor network were used in turn as the analysis,
triggering and weighting data sets.

To apply the ST-RDT to a single analysis data set, the analysis, trigger and weighting data sets are first divided into short time windows with an overlap between neighbouring windows. Within each window, a short segment of data is collected where the trigger condition is met. Triggering conditions include: all-points (collecting a segment of data starting at every time-step), range-crossing (collecting a segment of data when the trigger channel falls in a given range) and peak-triggering (where the value in the trigger channel is larger than the two adjacent values). The segments of data are weighted using the value of the weighting data at the time at which the segments are collected. These segments are then averaged together to form a random decrement signature (RDS). During this averaging, stochastic components of the segments, such as forcing after the start of a segment and system noise, approach zero. Therefore each RDS is a weighted estimate of a structure’s response to some initial perturbation under the conditions defined by the triggering and weighting data sets. Modal parameters are estimated from the RDS using the matrix pencil method [254].

A key advantage of the ST-RDT over conventional OMA methods is the ability to track changes in modal parameters with a far shorter time resolution. An example of the frequency and damping variations of the first vertical mode of vibration of the bridge is shown in the histograms in Figure F.4. The frequency and damping estimates appear as dark lines in Figure F.4 with the relative darkness of these lines giving an indirect indication of the overall uncertainty in the frequency and damping estimates at each time. This method of analysis also illustrates that the variations in natural frequency and damping due to human-structure interaction can be clearly distinguished from the uncertainty in the frequency estimate at each time-step. For example, in the data presented in Figure F.4 which reflects the behaviour of the bridge during various crossings by a single pedestrian following the routes shown in Figure F.3, the natural frequency and damping varies depending on the location of the pedestrian on the structure. For the purpose of the response factor calculations, the median frequency, damping and mode shapes of each mode of vibration are used.
For the ST-RDT analysis of the bridge, all data were divided into windows of 120 seconds, with an overlap of 115 seconds between windows, selected to balance the time resolution with which variation in modal parameters can be measured, and the accuracy of the estimates. Each permanent sensor network accelerometer was taken as an individual analysis data set and analysed with an all-points trigger condition. The analysis of each accelerometer data set was repeated multiple times using each of the vertical accelerometer data sets as the weighting data. This allowed the mode shapes of the bridge to be estimated for each window of data in a similar manner to the vector random decrement technique [259]. The five second RDS from each weighting and analysis data set combination was analysed using the matrix pencil method, with model orders (the number of damped sinusoids fitted to the RDS) from 5 to 30. Prior to the matrix pencil analysis, the first two data points of each RDS were removed to exclude the autocorrelation of noise within the data from the analysis, as described by Wynne et al. [2]. The use of seven analysis data sets (the seven accelerometer channels), and seven triggering data sets (the same seven accelerometer channels), in combination with analysis at model orders from 5 to 30 results in a maximum of $7 \times 7 \times \sum_{k=5}^{30} k = 22,295$ modal estimates per time step. Results where the fitting of the RDS resulted in a coefficient of determination of less than 0.99, suggesting under-fitting of the RDS, were excluded from further analysis.

A drawback of OMA is that, as the force is unmeasured, it is not possible to estimate the scaled mode shapes and modal mass of the structure. Therefore the modal masses were calculated using a series of five heel-drops carried out at the
third points along the deck span. Heel-drops provide a simple way of applying a
known force to a structure without the need for extensive instrumentation [493].
The force from these heel-drops was approximated as a triangular impulse, as
given by Wyatt [497], and is shown in Figure F.5.

![Figure F.5: Approximation of the heel-drop force time history [497] at 100 Hz sample rate.](image)

The heel-drop data were used in combination with the unity scaled mode
shapes, determined using the ST-RDT, to estimate the modal masses of the bridge
according to Equation (F.4), with the heel-drop and ST-RDT mode shapes paired
using the modal assurance criterion [346].

### F.3.4 Finite element model predictions

An estimate of the vibration response of the bridge was made based on infor-
mation that would be available to an engineer at the design stage. This serves
as a baseline to show the additional information that could be revealed through
in-situ measurements of the bridge’s response. At the design stage, the vibration
response must be estimated based on predictions of the structure’s modal param-
eters. For simple structures this can be achieved through analytical methods,
such as those described by Willford and Young [426], while for more complex
structures, such as the MX3D Bridge, finite element (FE) modelling is required
[488, 498].

The FE model of the bridge, shown in Figure F.6 and described in detail in
Kyvelou et al. [498], was developed in the FE software package ABAQUS (2016
[499]) and was validated against the results of the in-situ structural tests, under-
taken as part of the safety verification of the bridge – see Figure F.2 [488, 491].
The nominal geometry of the bridge was imported into ABAQUS from the CAD
file from which the bridge was printed. The conventionally-manufactured stainless
steel metallic deck plate and end beams were modelled using 4-node quadrilateral S4R shell elements with reduced integration and hourglass control [499] while a combination of linear triangular S3 shell elements [499] and S4R shell elements were employed to accurately replicate the more intricate geometry of the AM bridge components (i.e. the substructure, handrails and swirls). The interconnections between the conventional and WAAM bridge components, which, in reality, were achieved by manual welds, were simulated using TIE [500] constraints.

Figure F.6: Finite element model of the MX3D Bridge.

The two-stage Ramberg-Osgood expression [501] was employed for the representation of the material response of all bridge components. The mechanical properties of the conventionally manufactured end beams and deck plate were taken from Afshan and Gardner [500], while their material response was modelled as isotropic. The mechanical properties of the WAAM bridge components were determined by means of tensile testing on coupons printed using the same material and printing parameters as the bridge [502], while their material response was modelled as orthotropic [503] through use of the *LAMINA and *POTENTIAL keywords [499] for the elastic and plastic material responses, respectively. More details of the FE model used herein can be found in Kyvelou et al. [498].

The general static solver [499] was first employed for the application of gravity load on the bridge; this was followed by an eigenvalue analysis to determine the natural frequencies and corresponding mode shapes. The Lanczos eigensolver [499] was used, including the effect of structural coupling during the natural frequency extraction procedure. The design-stage estimate of damping for all modes was taken as 0.5%, based on the value given for ‘Welded steel bridges with little or no services, fixtures or fittings’ in Table A2 by Willford and Young [426].

**F.3.5 Mode shape and modal mass scaling**

All FE, EMA and OMA, mode shapes used in the study were interpolated with a third-order polynomial across the bridge deck, with interpolation carried out
using a uniform discretized grid of 0.01 m.

The FE model suggests that the maximum amplitudes of vertical movement ($U_3$) in all modes of vibration will occur in the decorative swirls attached to the ends of the handrails, as shown in Figure F.7. However, since the swirls are not accessible, no user of the structure would feel their vibration and no excitation would be applied to the swirls. As a result, accelerometers were not placed on the swirls for either EMA or OMA using the temporary or permanent accelerometer networks. For consistency, therefore, the unity-scaled mode shapes and modal masses from the FE model, EMA, and OMA are scaled to a maximum deck movement amplitude of one. The modal mass calculated using this scaling is referred to herein as the *apparent* modal mass. This may be greater than the static mass of the bridge [353] since the presence of the swirls, and their contribution to the apparent modal mass, appears to reduce the amplitude of the deck vibration.

![Swirl movement.](image1) ![Deck movement, swirls hidden for clarity.](image2)

Figure F.7: Examples of FE model modal amplitudes of the MX3D Bridge.

### F.4 Results

Response factors were calculated for the bridge using i) the FE predictions of the modal properties, ii) the EMA measurements of the modal properties and iii) the ST-RDT measurements of the modal properties. The bridge span was taken as 10.5 m, the stride length (i.e. two steps) of the pedestrian was estimated as 1.43 m, and the pedestrian weight, as measured during testing, was 826 N (84.2 kg). The response factor was calculated for all walking frequencies between 0.01 Hz and 3.00 Hz at 0.01 Hz intervals and for all possible combinations of excitation and measurement locations.

Predictions of the response factor were made at the locations of the permanent
sensor network bridge deck accelerometers for the FE model and ST-RDT results, and for the hammer impact locations for the EMA results. These results were then interpolated with a third-order polynomial across the bridge deck as it is assumed that the bridge deck behaves as a single membrane. A comparison between linear interpolation and the third-order polynomial interpolation of the response factor highlight that the differences in peak response factor due to the choice of interpolation used is of the order of 5%, with cubic-interpolation producing consistently higher peak response factor predictions. The choice of interpolation does not meaningfully impact the conclusions drawn. In the interpolation it was assumed that there was zero vertical movement at the bridge bearings. As the acceleration response was not measured on the end-beams of the bridge, this results in an interpolated response factor of zero across the ends of the bridge deck.

F.4.1 Design-stage predictions

The maximum vertical response factor $R$ for each possible bridge deck location based on the FE model predictions of the modal parameters is shown in Figure F.8 alongside the walking frequency that induces the peak response factor at each position on the bridge deck.

The sharp changes in peak $R$ walking frequency in Figures F.8 to F.10 are a result of the different mode shapes of the bridge. Due to the distinct natural frequencies, the peak response factor at a given location on the bridge deck is induced by specific walking frequencies. The boundaries between these specific walking frequencies correspond to the transitions between different modes of vibration dominating the vertical structural oscillation.
F.4.2 Modal parameters estimated by EMA

The maximum vertical response factor $R$ and associated walking frequency for each possible location on the bridge deck, based on the impact hammer measured modal parameters is shown in Figure F.9. As a result of the cubic interpolation of the mode shapes, the response factor predicted using modal parameters measured with EMA marginally exceeds the performance target for external bridges of $R < 64$ recommended by Willford and Young [426] at one location on the bridge deck.
F.4.3 Modal parameters estimated by OMA

The maximum vertical response factor $R$ and associated walking frequency for each possible bridge deck location based on the median ST-RDT measured modal parameters is shown in Figure F.10. As with EMA modal parameter predictions, the response factor predicted using modal parameters measured with OMA marginally exceeds the performance target for external bridges at some locations on the bridge deck.

![Figure F.10: Maximum vertical response factor $R$ based on the ST-RDT measures of the modal parameters.](image)

F.4.4 Comparison of response factor predictions

The maximum response curves based on the FE model predictions, and the impact hammer and ST-RDT measurements of the modal parameters, are presented in Figure F.11. The FE model predictions result in substantially lower estimates of the peak response factor, more than three times lower than the response factors predicted using the impact hammer and ST-RDT measurements of the modal properties. While the impact hammer and ST-RDT results are broadly similar, there are differences in the location of some response factor peaks, with the ST-RDT results showing a larger number of distinguishable peaks and a higher average response factor. The ST-RDT and impact hammer modal parameter estimates result in a prediction that the performance target of $R < 64$, suggested by Willford and Young [426], will be marginally exceeded at walking frequencies close to 2.8 Hz.
F.4.5 Comparison of the modal parameters

The variations in the predicted response factors are due to differences in the modal parameters used in the response factor calculation, namely the natural frequencies, damping ratios and modal masses, compared in Figure F.12. Alongside the modal parameters, the modal assurance criterion (MAC) scores are plotted as connections between parameters, illustrating where the mode shapes generated from the finite element model, impact hammer analysis and ST-RDT analysis share high consistency.

The FE model results contain a higher number of vibration modes than the measurements recorded on the bridge. The impact hammer and ST-RDT modes of vibration are based on analysis of the vertical acceleration data only, therefore modes that are primarily transverse, or localised within the handrails or swirls, are not included. Modes that are primarily transverse have very low vertical amplitudes, and therefore do not contribute significantly to the response factor calculations. The lowest-frequency vertical mode in the FE model, with a natural frequency of 4.22 Hz, is dominated by movement of the swirls (hence its high apparent modal mass), and is thus not sufficiently recorded by the substructure sensors.

The relatively flexible connection of the swirls to the main structure during the temporary installation of the bridge at the University of Twente, induces several close modes of vibration which are captured within the FE model. In the impact hammer and ST-RDT results, traces of these close modes can be seen due to small differences in natural frequencies and overlapping peaks in the frequency spectra. The extent of the overlap between these close modes of vibration, and the lack of
accelerometers on the swirls of the bridge that would allow their distinct mode shapes to be identified however, prevent the separation of the close modes for the impact hammer results. The similarities in the mode shapes of the FE model modes at low frequencies can be seen in the high MAC scores when compared with the mode at 5.18 Hz in the ST-RDT modal estimates. While close modes of vibration can be clearly distinguished and separated in the ST-RDT results, the heel-drop results used for estimating the modal mass are not sufficiently separated to allow distinct mode shapes to be calculated.

It is supposed that at higher frequencies the bridge does not act monolithically, with the flexible connection between the swirls and the main structure resulting in greater variations between the FE model predictions of the mode shapes and the mode shapes quantified through the impact hammer and ST-RDT analysis. This is further complicated by the complex geometry of the structure which result in mode shapes which are dominated by torsional oscillation about the primary load bearing diagonal of the bridge deck.

For the majority of modes measured in the impact hammer and ST-RDT re-
sults, the damping is greater than the 0.5% assumed in the FE model. An average damping across all modes of vibration below 15 Hz of 0.61% for the ST-RDT results, and 0.74% for the impact hammer results is observed. If the response factor is predicted using the impact hammer measurements of modal mass and natural frequencies and 0.5% damping, as was assumed in the FE model, the peak predicted response factor for the impact hammer measurements increases by 13.16% from 64.19 to 72.65. However if the response factor is predicted using the ST-RDT measurements of modal mass and natural frequencies and 0.5% damping, the peak predicted response factor for the ST-RDT measurements reduces by 4.91%, from 93.60 to 89.00. This is due to the variation in damping across the different modes of vibration and the contribution of higher frequency modes with lower measured damping ratios to the peak estimated response factor.

The differences in the FE predicted and EMA and OMA observed natural frequencies of the structure, and how these align with the harmonics of the walking frequencies, cause differences in which walking frequency induce the peak response factor and small variations in the peak response factor. Where harmonics of a walking frequency coincide with multiple modes of vibrations, the predicted response factor is higher. However, the primary reason for the lower response factors predicted based on the FE model parameters is the higher apparent modal mass predictions when compared to the impact hammer and ST-RDT results, shown in comparison to the static mass of the bridge (8 tonnes) in Figure F.12. As discussed in Section F.3.5, the apparent modal mass is higher than the static mass of the bridge due to the scaling of the mode shapes to have a maximum amplitude of unity at the point of largest bridge deck oscillation. This results in the amplitude of the mode shapes being greater than unity at the ornamental swirls. The inertial effect of modal mass is to reduce the amplitude of vibration for a given input force; therefore, the higher modal mass predicted using the FE model suggests that the bridge experiences greater accelerations across nearly all modes of vibration than would be predicted using the FE model. The lower EMA observed modal masses may be due to a difference in stiffness of the connection between the handrails and/or the ornamental swirls to the main bridge structure, or greater flexibility of the bridge bearings that allow greater deck movement. It is unclear from the results whether the true stiffness of the connection between the handrails and the swirls is higher or lower than what was assumed in the FE model, as the dynamic coupling between connected components is highly sensitive to the ratios and distributions of mass and stiffness within the coupled elements.

Differences in the response factor predictions based on the impact hammer and
ST-RDT measurements of the modal parameters are associated with differences in the natural frequencies and damping ratios. The impact hammer measurements produce larger measurements of damping that the ST-RDT results for most modes suggesting some degree of amplitude-dependant damping behaviour [264]. Alongside differences in damping, the modes which are well detected by the ST-RDT are different from those captured by the impact hammer excitation, which struggles to detect some modes due to low mode shape amplitudes at the impact location. Evidence for the lack of excitation of some modes within the impact hammer analysis can be seen in the lack of distinct pairs of modes in the MAC score connections plotted in Figure F.14. This results in greater uncertainty in both the scaled mode shapes and modal masses for the impact hammer modal estimates, which, combined with the higher damping ratios, lead to lower predictions of the response factor.

**F.4.6 Comparison with the measured bridge response**

The peak response factor of the bridge, recorded across all permanent sensor network accelerometers for a series of crossings by a pedestrian, is presented in Figures F.13 and F.14 for walking frequencies of 1.70 Hz and 2.79 Hz respectively. These are the same data sets used for calculating the modal parameters using the ST-RDT. Also presented in these plots are the maximum predicted response factors presented previously in Figure F.12 from the FE, impact hammer and ST-RDT analyses at the exact walking frequency and ±0.1 Hz to account for variability in walking frequencies, as quantified using load cell and strain gauge data collected from the permanent bridge sensor network during the pedestrian crossings.

![Graph showing response factors during bridge crossings by a single pedestrian walking at 1.70 Hz.](image)

Figure F.13: Vertical response factors during bridge crossings by a single pedestrian walking at 1.70 Hz.
The predicted response factor based on the ST-RDT modal estimates provides excellent agreement with the measured peak response factor for both walking frequencies, with the slight variations most likely due to variability of the walking frequency of less than 0.1 Hz. While the impact hammer modal parameters also provide a reasonable prediction of the response factor, the over-estimation of the damping ratio and natural frequencies combine to result in an under-prediction of approximately 30% of the bridge’s response factor at the most responsive walking frequencies. These results suggest that, for a responsive structure such as the MX3D Bridge, human-structure interaction is important. Since OMA measures the true performance of the person-bridge system, it gives the most relevant parameters for estimating the performance of the structure.

The crossing route over the bridge, shown in Figure F.3, was systematically varied for both walking frequencies. The first ten Centre, five South and five North crossings (as per Figure F.3) were performed from the pinned supports to the roller supports, with the direction then reversed. This change in the crossing route explains some of the variability of the measured response factors. A further possible cause of the deviation in measured and predicted response factors is the indeterminacy of the load distribution within the structure, which is known to vary with the temperature of the bridge, as well as discrepancies arising from the interpolation of predicted response factors between measurement locations.

The performance target for external bridges suggested by Willford and Young [426] (Section 3.3.3.) of $R < 64$ is met for all crossings at a walking frequency of 1.7 Hz. However, as predicted by the modal parameters from the ST-RDT, this target is exceeded at a walking frequency of 2.79 Hz. The performance target is exceeded only for short periods of time and is quickly damped out, suggesting any discomfort to users would be minimal.
It can be seen from Figures F.13 and F.14 that the prediction based on the modal parameters from the FE model under-predicts the response factor by approximately 50% for a walking frequency of 1.70 Hz, and approximately 200% for the 2.79 Hz walking test. Both the impact hammer and ST-RDT predictions of the response factor are far closer to the measured response of the bridge, with both slightly underestimating the true peak and a higher deviation for the 2.79 Hz walking frequency.

F.5 Summary

Measurements and analysis of the MX3D Bridge, the world’s first additively manufactured bridge, have demonstrated the challenges in ensuring accurate design-stage predictions of the vibration response for complex or novel structures. The measured vibration response slightly exceeds the performance target ($R < 64$) at specific walking frequencies, and is larger than that predicted using modal parameter predictions from the finite element modelling due to an overestimation of the modal masses. The impact hammer experimental modal analysis failed to accurately capture all modes of vibration that are excited by footfall excitation and overestimated structural damping. Accurate predictions of the vibration response could be made using dynamic parameters estimated through operational modal analysis with the ST-RDT on the bridge, which allows for more accurate predictions of the modal parameters and the time-variant modal behaviour induced by the human-structure interaction to be quantified.

This work highlights the potential for smart structures and in-situ measurements of dynamic parameters to provide a benchmark for similar structures and inform whether predictions of dynamic behaviour are likely to be unduly optimistic, hence mitigating the risk of exceeding serviceability requirements in future designs.
Appendix G

Clustering methods for automated detection of modes of vibration

This appendix provides additional details on existing clustering methods explored for automated identification of modes of vibration within operational modal analysis (OMA) modal estimates.

G.1 K-means clustering

K-means clustering is an unsupervised machine learning technique \[325\] through which each “object” is assigned to one of \(k\) clusters. Assignment of objects to clusters is typically carried out through minimizing the Euclidean or squared distance between each of the objects \(x_n\) and the cluster points \(\mu_k\), where cluster point \(\mu_k\) is the mean of the objects \(x_n\) assigned to cluster \(k\) \[434\]. Each object is only assigned to the cluster which results in the smallest Euclidean distance between \(x_n\) and \(\mu_k\) \[325\]. The procedure for application of k-means clustering with a Euclidean error criteria \[325\] is:

1. Initialize values for each of the \(k\) cluster points, \(\mu_k\). Within this thesis, \(k\)-means++, as implemented within Scikit-learn is used \[505\] \[435\].

2. Find the assignment \(k\) which minimizes the Euclidean distance of each point from the cluster centre, \(\sqrt{(x_n - \mu_k)^2}\), for each of the \(n\) data objects.

3. Recalculate the cluster points based on the mean of the point assignment:
\[
\mu_k = \frac{\sum_{n=1}^{N}(z_{nk}x_n)}{\sum_{n=1}^{N}(z_{nk})},
\]
where \(z_{nk}\) is a binary indicator representing the assignment of point \(x_n\) to cluster \(k\).
4. Repeat steps 2) and 3) until the assignment of points to clusters does not change from the previous iteration.

The simplest application of k-means clustering for automated modal identification is to use only the frequency estimates from the modal analysis, referred to here as one-dimensional (1D) k-means clustering. The steps for k-means clustering of frequency estimates are as follows:

1. Initialize $k$ mean cluster frequencies, $f_k$ using k-means++ initialization.

2. Assign each frequency estimate $f_i$ to the cluster which minimizes $\sqrt{(f_i - f_k)^2}$.

3. Recalculate the mean cluster frequencies based on the mean of all frequency estimates assigned to the cluster $f_k = \frac{\sum_{i=1}^{N} (z_{ik} f_i)}{\sum_{i=1}^{N} (z_{ik})}$, where $z_{ik}$ is a binary operation representing the assignment of a frequency estimate $f_i$ to cluster $k$.

4. Repeat steps 2. and 3. until the assignment of frequency estimates has not changed.

If the data has low numbers of spurious modal estimates and well separated modes of vibration, such as the artificial data shown in Figure G.1, 1D k-means clustering using the Euclidean distance can be very successful, with modes quickly identified and separated for further analysis.

![Figure G.1](image-url) 1D k-means clustering of frequency estimates for well-distinguished clusters. K-means clustering performed using Scikit-learn Python library [435].
As the assignment for the 1D k-means clustering is based on the Euclidean distance of the frequency estimates, close modes of vibration cannot be well separated through 1D k-means, as demonstrated in Figure G.2. However, if the modes have distinct damping ratios, as in the artificial data presented in Figure G.2, this issue can be resolved through two-dimensional (2D) k-means clustering method based on the Euclidean distance of the frequency and damping estimates from the cluster center. The method for application of 2D k-means clustering is:

1. Initialize $k$ mean cluster frequencies, $f_k$, and $k$ mean cluster damping ratios, $d_k$, using k-means++ initialization.

2. Assign each frequency-damping estimate pair $\{f_i, d_i\}$ to the cluster which minimizes $\sqrt{(f_i - f_k)^2 + (d_i - d_k)^2}$.

3. Recalculate the mean cluster frequencies and damping ratios based on the mean of all estimates assigned to the cluster $f_k = \frac{\sum_{i=1}^{N}(z_{ik}f_i)}{\sum_{i=1}^{N}(z_{ik})}$ and $d_k = \frac{\sum_{i=1}^{N}(z_{ik}d_i)}{\sum_{i=1}^{N}(z_{ik})}$, where $z_{ik}$ is a binary operation representing the assignment of the frequency-damping estimate set $\{f_i, d_i\}$ to cluster $k$.

4. Repeat steps 2. and 3. until the assignment of all modal estimates has not changed.
The 2D k-means clustering has the ability to separate close modes with distinct frequency and damping ratios, as shown in Figure G.3.

![Figure G.3: 2D k-means clustering of frequency and damping estimates for close modes of vibration. Top plot shows true assignment of artificial modal estimates to modes, lower plot shows cluster assignment using k-means clustering based on frequency estimates only. K-means clustering performed using Scikit-learn Python library [435].](image)

However, this approach can cause further problems, as demonstrated in Figure G.4. Due to the increased variance in the damping estimates compared to the frequency estimates, the use of the 2D Euclidean distance leads to poor separation of the modes of vibration. This is partly resolved by scaling all frequency and damping estimates to fall in the range (0,1), referred to as min-max standardization [506]. However, scaling of the frequency and damping estimates still results in the poorer assignment of modal estimates to distinct clusters than was achieved with the 1D frequency-only k-means clustering approach.

As the number of modes within the data increases additional clusters may be required as the optimal cluster centres which minimize the Euclidean distance may be dictated by spurious modal estimates. An example of this is shown in Figure G.5. While modes may be clearly visually identified within the data for selecting a value of $k$, the lack of a mode between 0 Hz and 1 Hz leads to the presence of a noise cluster, a cluster containing only spurious modal estimates, while the modes at 3.57 Hz and 3.85 Hz are both assigned to the same cluster. This issue is closely related to the selection of the number of clusters, $k$, to be used in the clustering. If $k$ is smaller than the number of real modes present within the data, some or all of the clusters will contain multiple modes of vi-
Figure G.4: Comparison of 1D, 2D, and 2D scaled k-means clustering of frequency and damping estimates. K-means clustering performed using Scikit-learn Python library [435].

However, if \( k \) is too large modes may be split across multiple adjacent clusters. Several approaches have been put forward for selecting an appropriate value of \( k \), discussed at greater length in Pham et al. [436] such as the incremental introduction of clusters [507]. One method for selecting \( k \) is to examine how the inertia or sum of Euclidean distances of the samples to their closest cluster centre varies. A plot of the variation in inertia with \( k \) for the artificial data presented in Figure G.5 is shown in Figure G.6. It can be seen that there is a small change in the rate of decrease of inertia with the increased number of clusters once the true number of modes is exceeded, referred to colloquially in the literature as an “elbow” [508]. A generalized rule of thumb, as discussed by Sugar [509], is to select \( k \) as one greater than the value at the elbow if the data includes background noise which may distort the clustering. However, the applicability of this method and the identification of the elbow is open to interpretation.

The results in this section highlight some of the limitations of k-means clustering for the separation of modes of vibration in OMA results. The selection of a value of \( k \) is subjective and requires careful tuning to best separate the
Figure G.5: 2D k-means clustering of frequency and damping estimates for larger numbers of modes of vibration. Top plot shows true assignment of artificial modal estimates to modes, lower plots show cluster assignment using k-means clustering based on frequency and damping estimates. K-means clustering performed using Scikit-learn Python library [435].

Figure G.6: Variation in inertia with number of clusters $k$ used in 1D k-means clustering of artificial data presented in Figure G.4

variables of interest (modes of vibration). Alongside this, the background noise associated with spurious modal estimates cannot be accounted for in k-means clustering, as all estimates are assigned to the cluster which minimizes the Euclidean distance. While a further step could be introduced in which outliers are excluded from clusters, or clusters which fail to meet some minimum density or modal parameter uniformity criteria are excluded, such as in the work of Neu
et al. [314], this would introduce further parameters to be tuned, complicating the automated OMA process. K-means clustering is also poorly suited for the identification and separation of close modes. If the modal estimates for these modes overlap, as frequently occurs for close modes, the Euclidean distance of some of these modal estimates to the “correct” cluster centre (the true mode of vibration) will be greater than that to the cluster centre corresponding to the close mode. This inability to deal with overlapping clusters becomes a greater issue when the modal estimates display weak non-linear modal or dynamic behaviour, which may result in arbitrarily shaped or overlapping modal estimates from multiple modes of vibration.

The assignment of points to the clusters is also sensitive to the initial cluster centres selected. In practice this may be resolved through multiple random initializations of the initial cluster centres, allowing the stability of the assignment of any given point to a given cluster, as well as the stability of the cluster centres themselves, to be quantified. The various solutions to the clustering represent local minima within the solutions space [510], points at which the assignment of points has become fixed. These local minima may not correspond to the global minimum of the solution space, as discussed in greater detail in Appendix H.

The final issue with k-means clustering to note is that the clustering tells us relatively little about the modes of vibration. The clustering gives us the mean value of all points assigned to the cluster, the Euclidean distance of points from the cluster centre, and the number of points assigned to each cluster. Subsequent steps would be required to assess the shape of the cluster distributions and the stability of the clusters in the presence of new data.

G.2 Density-based spatial clustering of applications with noise (DBSCAN)

To overcome some of the limitations of k-means clustering, Ester et al. [326] developed density-based spatial clustering of applications with noise (DBSCAN). DBSCAN does not require pre-specification of the number of clusters in the dataset, can separate noise in the data set, and can be used with arbitrarily shaped clusters [326].

Two parameters are used within DBSCAN, $\varepsilon_D$ which defines a distance around the boundary of a cluster in which points are searched for, and MinPts, the minimum number of points required to define a valid cluster. The application of DBSCAN is as follows:
1. Randomly select a point in the data set as the start of a new cluster.

2. If there are MinPts with a Euclidean distance less than $\varepsilon_D$ assign those points to the cluster.

3. For the points which are newly assigned to the cluster, assign all points within Euclidean distance less than $\varepsilon_D$ to the cluster. Repeat for each batch of newly assigned points.

4. Repeat step 3 until all points within the cluster have been evaluated.

5. Randomly select a point that has not been assigned to a cluster.

6. Repeat steps 2 to 4 for the new cluster, evaluating only points which have yet to be assigned to a cluster.

7. Once all points have been evaluated, assign all points which have not been assigned to a cluster as noise.

The use of DBSCAN is demonstrated in Figure G.7 based on the artificial data introduced in Figure G.4. It can be seen that all modes are well separated with $\varepsilon_D = 0.05$Hz and MinPts = 100, with a number of the modal estimates correctly identified as noise. As might be expected, if the modal estimates from two or more modes of vibration overlap or are with a distance of $\varepsilon_D$, the modal estimates from these modes will inevitably be assigned to the same cluster. Where there is a significant degree of overlap in the modal estimates, it will often not be possible to select a value of $\varepsilon_D$ which separates these modes without splitting each mode of vibration across multiple clusters. The performance of DBSCAN may decrease if both the frequency and damping are used as clustering parameters, due to the difference in the variance in the damping and frequency estimates, as discussed with k-means clustering.

As discussed by Li et al. [318] and Ye and Zhao [319], DBSCAN offers promise as a clustering method for automating the identification of stable modes in stability diagrams. A key benefit in this use is that the criteria for defining a stable mode traditionally used in stability diagrams, such as ensuring a specified variation in the frequency between adjacent model orders, may be used as a basis for the value of $\varepsilon_D$ in DBSCAN. This is demonstrated in Figure G.8 where a frequency stability only criteria is used to identify likely stable modes within the stability diagram of acceleration data from the Aberfeldy Bridge, discussed previously in Chapter 7 and Appendix C.
Figure G.7: DBSCAN of artificial data introduced in Figure G.4. DBSCAN applied using frequency estimates only with $\varepsilon_D = 0.05\,Hz$ and MinPts = 100. Top plot shows true assignment of artificial modal estimates, lower plots show cluster assignment using DBSCAN based on frequency estimates only. DBSCAN implemented with the Scikit-learn Python library [435].

Figure G.8: Example of use of DBSCAN to identify stable modes in an SSI-Cov stability diagram of acceleration data from the Aberfeldy Bridge $\varepsilon_{D,f} = 0.01\,Hz$, $\varepsilon_{D_{,model-order}} = 1$ and MinPts = 5. DBSCAN implemented with the Scikit-learn Python library [435].
DBSCAN may also be used for identifying modes in the time-frequency domain of the ST-RDT modal estimates, as demonstrated in Figure G.9. However, as illustrated in Figure G.10 if the detection of a mode within the modal estimates is intermittent and it is not detected for some time steps, DBSCAN will split this mode across separate clusters. Similarly, if the mode exhibits variations in the natural frequency due to weak non-linear modal behaviour, the cluster may split the mode across separate clusters or map multiple modes to the same cluster.

Figure G.9: Example of use of DBSCAN to identify modes in the time-frequency domain of artificial modal estimates. Top plot shows true assignment of artificial modal estimates, lower plots show cluster assignment using DBSCAN clustering based on frequency estimates and time step. DBSCAN implemented with the Scikit-learn Python library [435].

G.3 Other hard-clustering methods

Aside from DBSCAN, a wide range of other clustering methods have been developed which seek to overcome some of the limitations of k-means clustering. These typically seek to remove the need to pre-specify the number of clusters within the data and to separate noise from the clusters [511].

One family of these alternatives to k-means clustering are hierarchical clustering methods, which are split into agglomerative approaches [511], where each data point starts as its own cluster and clusters are merged, and divisive approaches [511], where all observations are assigned to a single cluster which is then recursively split into smaller clusters. Whether clusters are merged/split is
Figure G.10: Example of where DBSCAN may fail to identify modes in the time-frequency domain of ST-RDT modal estimates due to intermittent appearance of modes within the modal estimates and overlap of close modes of vibration induced by weak non-linear modal behaviour. Top plot shows true assignment of artificial modal estimates, lower plots show cluster assignment using DBSCAN clustering based on frequency estimates and time step. DBSCAN implemented with the Scikit-learn Python library [435].

decided by predefined metric and linkage criteria. A metric criterion is a measure of the distance between observations and is used to assign points to clusters, such as the Euclidean distance, while a linkage criterion is used to define how clusters are merged/split, such as the simple-linkage criteria where the clusters which have the minimum Euclidean distance between cluster centers are merged. Hierarchical approaches remove the need to specify the number of clusters, instead allowing the hierarchy of clustering steps to be partitioned at a specified level. Methods for selecting the level of the partition are discussed in Kaufman and Rousseeuw [511].

Where the clusters are arbitrarily shaped, mean-shift clustering can be used. In mean-shift clustering, points are assigned a weight as a function of their distance from the cluster centre. A weighted mean of the points is then used to update the cluster centre (the mean shift of the cluster). An advantage of this approach over k-means clustering is that the bandwidth or shape of the weighting function has a physical meaning, which can be of use if the expected distribution of points within a cluster is known [512]. The weighting function can also allow noise within the data to be separated as points outside the bandwidth of the weighting function will have very small weights in the fitted clusters. However, as discussed by Cheng [512], the success of the clustering is strongly dependent
on the selection of an appropriate weighting function. If the weighting function
used is a Gaussian kernel, mean-shift clustering is equivalent to Gaussian mix-
ture models, discussed in the next section, with each Gaussian component having
equal weight.

A variation on DBSCAN that can resolve some of the issues when the data
contains high levels of noise or differing densities of clusters is Ordering Points
To Identify Cluster Structure (OPTICS) [513]. OPTICS introduces two addi-
tional metrics for the assignment of points to existing clusters, a core distance;
the distance of a data point from a core-point defined as any point with a mini-
mum of MinPts within $\varepsilon_D$, and a reachability distance; the distance of any point
from a core-point. The results are presented as reachability plots, in which the
reachability-distances are plotted with points ordered based on their cluster as-
signment and reachability distance. Within these plots, the clusters appear as
valleys: areas with larger numbers of points with smaller reachability distances.

Memory and computational issues may be encountered when clustering OMA
modal estimates due to the number of data points to be analysed. Two methods
that can be useful in this scenario are mini-batch k-means [514], in which k-means
clustering is performed on randomly selected subsets of the full data set until the
change in cluster centres between subsets converge towards zero, and Balanced
Iterative Reducing and Clustering using Hierarchies (BIRCH), which allows real-
time clustering of data through the use of summary statistics, removing the need
for recalculation of all Euclidean distances and cluster centres for each new batch
of data [515].

G.4 Gaussian-mixture models

Gaussian mixture models (GMMs) are probabilistic models in which each of the
$k$ clusters are represented as a Gaussian distribution with a mean value $\mu_k$, a
variance $\sigma^2_k$, and a mixing weight $m_f$. The probability of observing the measured
modal estimates $X$ under the GMM with fitted parameters $\Delta$ is then given by the
product of the probability of observing each of the $N$ individual modal estimates
$(x_n)$ summed across each of the $K$ Gaussian distributions [325]:

$$P(X|\Delta, m) = \prod_{n=1}^{N} \sum_{k=1}^{K} m_f P(x_n|\Delta_k) \quad (G.1)$$

To fit the GMM to the data, the parameters are iteratively optimized to find the
set of parameters that maximize the probability of the observed data under the
fitted parameters. This maximization can be carried out through methods such as expectation-maximization (EM) \[438\] or gradient descent methods, where the objective is to minimize some cost function for the observed data under the model parameters \[437\]. In the context of GMMs, the objective is to maximize the probability of the observed data under the fitted model, with the corresponding cost function being to minimize the negative-sum log-likelihood of the model parameters given the observed data. Here, the negative transforms the maximization problem into a minimization problem, while the use of the sum logarithm of the data is a more efficient method of calculating the product of all probabilities of the individual data points under the fitted model \[437\].

As with the clustering methods discussed in the previous section, the GMMs may be used with a single array of 1D input data, referred to as univariate GMMs, or with two or more dimensional data, referred to as multivariate GMMs \[437\]. The results of fitting a univariate-GMM through the EM algorithm utilizing the frequency estimates of the artificial data introduced previously in Figure G.5 are presented as a histogram in Figure G.11. While the means of the GMM are close to the true natural frequencies of the eight modes of vibration present within the artificial data, the variance of these distributions have been overestimated due to the presence of the background noise within the frequency estimates.

If the same artificial data is analysed with a multivariate GMM, using both frequency and damping estimates, this issue becomes more apparent, as demonstrated in Figure G.12. The noise within the modal estimates has led to several wide Gaussians which approximate the uniform noise in the artificially generated modal estimates. As a result of this, while the number of components is equal to the number of modes of vibration in the artificial data, some modes are grouped within a single fitted Gaussian. As with the k-means clustering, to account for the increased variance in the damping estimates the frequency and damping estimates were scaled to fall in the range \(0,1\) \[506\] before fitting the GMM.

If the number of Gaussians included within the GMM is increased by one, so there is the flexibility to allow the uniform noise in the modal estimates to be modelled as a wide Gaussian distribution, better results may be achieved, as demonstrated in Figure G.13. As with k-means clustering and DBSCAN, the fitted parameters of GMM are sensitive to the initialization of the parameters due to the presence of local minima. It is therefore recommended that the fitting be repeated multiple times so that the volatility in the fitted results may be quantified, and the set of fitted parameters which maximize the log-likelihood of the data under the fitted model identified.
Figure G.11: Expected distributions of modes of vibration in artificial frequency data presented in Figure G.5 estimated through 1D GMM, fitted with the EM algorithm. Frequency estimates fitted with eight mixture components, with fitting initialized with the results of 1D k-means clustering. The true distributions of modal estimates are presented in the top subplot, while the fitted distributions are presented in the lower subplot. 1D GMM implemented with the Scikit-learn Python library [435].
Figure G.12: Expected distributions of modes of vibration in artificial frequency data presented in Figure G.5 estimated through 2D GMM, fitted with expectation maximization. Frequency and damping estimates fitted with eight mixture components, with fitting initialized with the results of 2D k-means clustering. The true distributions of modal estimates are presented in the top subplot, while the fitted distributions are presented in the lower subplot. 2D GMM implemented with the Scikit-learn Python library [435].
Figure G.13: Expected distributions of modes of vibration in artificial frequency data presented in Figure G.5 estimated through 2D GMM, fitted with the EM algorithm. Frequency and damping estimates fitted with nine GMM components, with fitting initialized with the results of 2D k-means clustering. The true distributions of modal estimates are presented in the top subplot, while the fitted distributions are presented in the lower subplot. 2D GMM implemented with the Scikit-learn Python library [435].
The advantage of GMMs over hard clustering is that each frequency estimate is assigned a probability of belonging to each of the GMM components. Therefore, unlike k-means clustering, it is robust when analysing close modes and provides indications when modal estimates are likely to be noise, as demonstrated in Figure G.14. Despite the overlap in the modal estimates and the presence of noise within the data, the identified mean frequencies and damping values are close to the true mean values for the modes, as summarized in Table G.1. As previously noted in Chapter 8, the presence of noise within the data results in an overestimation of the variance of the fitted GMMs compared to the true variances of the distributions from which the modal estimates are generated, as can be seen in Table G.1.

Figure G.14: 2D GMM of two close modes. Coloring of second subplot corresponds to the likelihood of each frequency estimate belonging to either Mode 1 (red) or Mode 2 (blue). The true distributions of modal estimates are presented in the top subplot, while the fitted distributions are presented in the lower subplot. 2D GMM implemented with the Scikit-learn Python library [435].

As with k-means clustering, the selection of the number of components for the GMM is a key parameter. One method of selecting the number of components is to examine the variation in the sum log-likelihood of the fitted model given the data for varying numbers of components and identify an elbow in the same way as discussed for k-means clustering. An alternative approach is to examine the variation in the Akaike information criterion (AIC) [516] and the Bayesian information criterion (BIC) [517]. The AIC is defined as $AIC = -2 \log(\hat{L}) + 2d$. 

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Table G.1: Comparison of true and fitted means and standard deviations (std.) of close modes of vibration plotted in Figure G.14.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mean frequency [Hz]</th>
<th>Mean damping [%]</th>
<th>Std. frequency [Hz]</th>
<th>Std. damping [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - True</td>
<td>4.30</td>
<td>1.50</td>
<td>0.050</td>
<td>0.500</td>
</tr>
<tr>
<td>1 - Fitted</td>
<td>4.30</td>
<td>1.50</td>
<td>0.046</td>
<td>0.470</td>
</tr>
<tr>
<td>2 - True</td>
<td>4.50</td>
<td>3.00</td>
<td>0.075</td>
<td>1.000</td>
</tr>
<tr>
<td>2 - Fitted</td>
<td>4.49</td>
<td>2.82</td>
<td>0.130</td>
<td>1.730</td>
</tr>
</tbody>
</table>

and the BIC is defined as $BIC = -2 \log(\hat{L}) + \log(N)d$, where $\hat{L}$ is the maximum likelihood of the model given the data and fitted parameters, $d$ is the number of components included in the GMM, and $N$ is the number of samples available for fitting. As might be interpreted through inspection of the equations, due to the negative log-likelihood, the optimal number of components occurs when the AIC or BIC reaches their minimum value. The key difference between the AIC and BIC in the context of GMMs is that the BIC more heavily penalizes the introduction of new components into the GMM, with the magnitude of this penalization dependent on the data available during the fitting \[517\]. The variation in the AIC and the BIC for 2D GMMs with a varying number of components, fitted to the artificial data presented in Figure G.5, is plotted in Figure G.15 with a clear minimum when the data is fitted with 2D GMMs with 10 components. While this minimum is greater than the true number of modes in the data, the presence of the uniform noise within the modal estimates, which is not explicitly accounted for in the GMM, results in the need for larger numbers of components so that the uniform noise may be approximated within the GMM.

Figure G.15: Variation in AIC and BIC for fitting of 2D GMM using artificial data plotted in Figure G.5.

Despite GMMs robustness to close modes of vibration, automated modal anal-
ysis with GMMs can suffer when the modal estimates display weak non-linear modal behaviour, as the distribution of modal estimates across a finite time-span is unlikely to be Gaussian, as demonstrated in Figure G.16. To account for such weak non-linear behaviour, the data could be fitted with a higher number of components, which could then be merged based on the time-frequency variation in modal parameters, or the weak non-linearity can be accounted for when modelling the data, as discussed in Chapter 9.

Figure G.16: Poor performance of 2D GMMs in presence of weak non-linear modal behaviour due to non-Gaussianity of distribution of modal estimates. The true distributions of modal estimates are presented in the top subplot, while the fitted distributions are presented in the lower subplot. 2D GMM implemented with the Scikit-learn Python library [435].
Appendix H

Methods for optimisation of mixture model parameters

The fitting of the parameters of the non-Gaussian mixture model introduced in Chapter 8 and expanded for non-linear modal behaviour in Chapter 9 was carried out with the Adaptive moment estimation (Adam) optimization algorithm, a stochastic gradient descent (SGD) based optimizer introduced by Kingma and Ba [518].

Conventional gradient descent algorithms suffer from two key sets of problems. The first of these are the problems posed by local solutions, which occur where the minimization reaches a non-optimal solution such that any small change in parameters will lead to a higher cost of the function under the fitted model, referred to as local minima [518]. This cost function may be anything, but in the context of this chapter, it specifically refers to minimizing the negative log-likelihood of the fitted mixture model parameters given the observed set of modal estimates. At a local minimum the gradient is zero and hence the updating of the model parameters using the gradient, such as in the expectation maximization algorithm, will result in zero change in the model parameters. The second problem in conventional gradient descent algorithms is that the calculation of the gradient of the solution space becomes computationally intensive for large data sets and/or numbers of model parameters [518].

H.1 Stochastic gradient descent

SGD attempts to resolve these issues by using a small subset or mini-batch of the full data set. The gradient of the function is calculated using this mini-batch, and the values of the parameters are iterated in line with the gradient descent [518].
The intuitive explanation for SGD can be interpreted by considering the arbitrary 1D cost function plotted in Figure H.1. In this function, the value of $Y$ reaches a global minimum when $X = 7.38$. However, the cost function contains many local minima with a gradient of zero. If the cost function is approximated using a mini-batch of eight of the values of $Y$, it can be seen that these local minima are “smoothed”, such that the minima of the function approximated with the mini-batch are close to the true global minima of the function. Sampling of a different mini-batch will result in different approximated minima, allowing the gradient descent to converge on the true global minima of the function without becoming trapped within a local minimum.

![Figure H.1: Approximation of complex function using a mini-batch of the data](image)

**H.2 Adaptive moment estimation**

In Adaptive moment estimation (Adam), the updating of the model parameters is based on running averages of the gradient ($m_t$) and the second moment of the gradient ($v_t$) [518], which are similar to the momentum term as introduced by Rumelhart et al. [519]. To provide intuitive physical reasoning to the use of the average gradient ($m_t$) and the second moment of the gradient ($v_t$), the optimizer is often described as a marble, with the function to be minimized being the surface on which the marble sits. In this analogy the velocity of the marble is a function of the weighted average gradient of the surface over which the marble has previously traversed, while its acceleration is dictated by the second moment of the gradient of the surface over which the marble has previously traversed. The inclusion of $m_t$ and $v_t$ in the SGD can further aid the gradient descent to escape local minima, as the Adam optimizer will overshoot any minima it encounters. If these minima are local minima, the Adam optimizer will, hopefully, escape this
region of the solution space due to the non-zero values of $m_t$ and $v_t$. However, if the minima is a global optimal minimum, the sign of the gradient descent will then change after the overshoot, with the Adam optimizer once again beginning to converge towards the global minima.

The steps for application of Adam over a single epoch are presented below. Before application of the Adam optimizer, the data set must be split into a training and hold-back data set, such that the fitted model parameters may be assessed using unseen-data. In the context of SGD, one epoch is typically taken as equal to the length of the training data set.

1. Initialize the parameters $c$ of the function to be minimized.

2. Set the step size $S_z$ for the analysis. This is how much the parameters will be iterated by when the ratio of the gradient and the second moment of gradient equals 1.

3. Initialize Adam with $m_t = 0$ and $v_t = 0$.

4. Randomly shuffle the hold-back data and split into mini batches of equal length.

5. Evaluate the gradient $g$ of the function to be minimized across all model parameters using the first mini-batch of data. In this research, the gradient of functions were calculated using the Autograd Python package [415].

6. Update the values of $m_t = (1 − \beta_1)g + \beta_1 m_t$ and $v_t = (1 − \beta_2)g^2 + \beta_2 v_t$ for all model parameters, where $\beta_1$ and $\beta_2$ are weighting factors, with suggested values of $\beta_1 = 0.9$ and $\beta_2 = 0.999$ [518].

7. Apply the bias correction to the values of $m_t$ and $v_t$ as $\hat{m}_t = (m_t)/(1 − \beta_1^i)$ and $\hat{v}_t = (v_t)(1 − \beta_2^i)$, where $i$ is the number of mini-batches previously analysed.

8. Increment the parameters $c$ such that $c = c + S_t$, where $S_t = S_z(\hat{m}_t/(\sqrt{\hat{v}_t} + \epsilon))$ in which $\epsilon$ is a small value (typically $1 \times 10^{-8}$) to prevent division by zero.

9. Repeat steps 5 to 8 for all subsequent mini-batches of data.

At predefined intervals, the negative log-likelihood of the model parameters given the hold-back data should be evaluated using the current fitted model parameters, $c$. The Adam optimizer is reinitialized at the start of each new epoch using the
values of $c$ from previous epochs which resulted in the minimum negative log-likelihood of the hold-back data under the fitted model. This helps to ensure the fitted parameters are robust and are not adversely affected by the overfitting of the data.

As the Adam optimizer will always overshoot a minimum, it is best practice to reduce the step size $S_z$ after a pre-specified number of epochs to allow the optimizer to converge, with this convergence appearing as a stabilization of the cost for the fitted model under the training data. An example of the convergence seen in the average negative log-likelihood during the fitting of the artificial data presented in Chapter 8, Figure 8.3, is presented in Figure H.2. In this analysis, a mini-batch of eight samples of the modal estimates were used, with the cost function evaluated every 50 mini-batches and the step size reducing by a factor of ten every five epochs. It can be seen that after approximately ten epochs, the sum negative log-likelihood of the data under the fitted parameters is stable with little variation between mini-batches, suggesting convergence of the optimizer has occurred.

![Figure H.2: Example of convergence of cost function during Adam optimization.](image)

As with other gradient descent based optimization routines, Adam is sensitive to the initialization values selected. Therefore it is necessary to repeat the optimization using a range of initialization values to assess the stability of the fitted model parameters.
Appendix I

Case study - Timber buildings

This case study presents the modal parameters of ten timber structures under wind excitation, estimated using two novel variations on the random decrement technique (RDT), the short-time random decrement technique (ST-RDT) and amplitude-ranked random decrement technique (AR-RDT). The results support previous observations that there is a correlation between the amplitude of movement of tall structures and the damping of its fundamental mode. However, the results indicate that this is likely due to the presence of spurious components within the modal estimates collected from structures under low levels of excitation. A strong positive correlation is also identified between the fundamental natural frequency of the structures and their heights, as well as a weak correlation between the damping of the building and the slenderness, with the average damping decreasing slightly as slenderness increases.

Alongside presenting the results for the ten structures analysed, an automated methodology is presented for the analysis of ambient vibration data from timber structures such that future data sets may be compared to the results presented. This methodology encompasses both the modal analysis of the ambient vibration data, the removal of spurious modal estimates associated with fitting of noise and the forced response within the estimates of the structural free response, and the automatic detection of modes of vibration in the operational modal analysis (OMA) modal estimates through a novel probabilistic model, removing the biasing and interpretability introduced through many OMA techniques.

I.1 Background

The data used in this study was originally presented in Reynolds et al. [262], and comprises of data collected from ten multi-storey timber buildings. A brief
description of the buildings is presented below:

- BRE Innovation building - A two-storey timber structure located at the BRE innovation park, England.
- Holz8 - An eight-storey composite timber-concrete building located in Bad Aibling, Germany.
- Kampa - A seven-story timber structure located in Waldhausen, Germany.
- LCT One - The life-cycle tower, an eight-storey composite timber-concrete structure located in Dornbirn, Austria.
- Treet - A fourteen-storey timber structure located in Bergen, Norway.
- Trento CLT - A cross-laminated composite timber-concrete structure located at the University of Trento, Italy.
- Trento TF - A timber frame composite timber-concrete structure located at the University of Trento, Italy.
- UEA Enterprise Centre - A two-storey timber structure located at the University of East Anglia, England.

I.2 Methods

I.2.1 Instrumentation and data collection

For each building ambient vibration data was collected using accelerometers. The key details of each of the data sets collected are presented in Table I.1 including the number of accelerometer channels used, the original sample rate, and the length of the data set. For some structures multiple separate data sets were collected at different times of day or with different accelerometer orientations. These sub-data sets are differentiated with numerical suffixes.
<table>
<thead>
<tr>
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<th>Number of channels</th>
<th>Data set length [minutes]</th>
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</tr>
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</tr>
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</table>

Table I.1: Details of data sets used in ambient vibration analyses of ten timber structures.

### I.2.2 Modal analysis

The acceleration data sets were individually analysed using two OMA techniques; the short-time random decrement technique (ST-RDT) and the amplitude-ranked random decrement technique (AR-RDT). This allowed for the identification of weak non-linear modal and dynamic behaviour, alongside quantification of the uncertainty, volatility and any sources of bias within the modal estimates with respect to both time and amplitude of structural response.
Prior to the ST-RDT and AR-RDT analyses, all data sets were filtered with a Butterworth bandpass filter with a highpass filter cutoff of 0.1 Hz (filter order of 1) and an lowpass filter cutoff of 30 Hz (filter order of 5) to remove components of the acceleration signal outside of the frequency range of interest. After filtering of the data, the data was downsampled to the analysis sample rates shown in Table I.2, which were selected to minimize the computational intensity of the analyses, aid comparably of the modal parameter estimates, and ensure that all frequency components of interest were captured in the analysis.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Original sample rate (HZ)</th>
<th>Analysis sample rate (Hz)</th>
</tr>
</thead>
<tbody>
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<td>BRE</td>
<td>500</td>
<td>100</td>
</tr>
<tr>
<td>Holz8</td>
<td>200</td>
<td>100</td>
</tr>
<tr>
<td>Kampa</td>
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<td>100</td>
</tr>
<tr>
<td>LCT</td>
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<td>100</td>
</tr>
<tr>
<td>Limnologen</td>
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<td>100</td>
</tr>
<tr>
<td>Stadhaus</td>
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<td>100</td>
</tr>
<tr>
<td>Treet</td>
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<td>170</td>
</tr>
<tr>
<td>Trento-CLT</td>
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<td>Trento-TF</td>
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<td>100</td>
</tr>
<tr>
<td>UEA</td>
<td>500</td>
<td>100</td>
</tr>
</tbody>
</table>

Table I.2: Down sampling of data sets prior to ST-RDT and AR-RDT analysis.

I.2.2.1 ST-RDT Modal analysis

The short-time random decrement technique (ST-RDT) is a short-time operational modal analysis method. In the ST-RDT, the data is divided into overlapping windows, with segments of the data meeting a predefined triggering condition within the data used to form an ensemble weighted average referred to as a random decrement signature (RDS). The ST-RDT analyses of the data used an all-points triggering condition alongside an amplitude weighting condition, a window length of five minutes, and an overlap between adjacent windows of four minutes and 45 seconds.

In order to remove the filtering artefacts associated with the filtering of the data, as well as components of the RDS associate with potential non-white forced response of the structures under wind excitation, the first 0.1 seconds of all RDSs were excluded prior to modal analysis with the matrix pencil method [6]. The model order of the fitted matrix pencil was incremented by one, starting from unity, until the fitted RDS achieved a target explained variance ($R^2$ score) of 0.99 or a maximum model order of 40 was reached. A range of random decrement
signature (RDS) lengths between one second and five seconds were analysed in intervals of 0.5 seconds to identify a suitable RDS length. Post-processing of the results identified that an RDS length of 2.5 seconds was broadly suitable across all data sets. For brevity, only the results for an RDS length of 2.5 seconds are presented in this case-study.

I.2.2.2 Amplitude-ranked RDT

The amplitude-ranked random decrement technique (AR-RDT) allows any variation in modal parameters and modal uncertainty or biasing with amplitude of dynamic structural response to be quantified.

In the AR-RDT approach, a segment of the analysis data is taken starting at every data point, with each segment of data weighted using the sign of the analysis at the start of the data (i.e. all segments are weighted to have a positive initial amplitude). The segments are then sorted in descending amplitude based on the sign-weighted amplitude at the start of the segment. A list of starting amplitudes is defined based on dividing the list of initial segment amplitudes into 1,000 lists of as close to equal length as is possible. An RDS is then formed through an ensemble average of the 2,000 segments with an initial amplitude less than or equal to the the starting amplitude defined above. Where the data length is less than $2,000 \times 1,000 = 2,000,000$ data points, there will be overlap between the amplitude-ranked RDSs formed. The degree of this overlap may be inferred from the dataset lengths presented in Table I.1 and downsampled sample rates presented in Table I.2.

The AR-RDT results in 1,000 RDSs for analysis per data channel, with modal analysis of the RDSs carried out using the same methodology as described for the ST-RDT.

I.2.3 Automated modal identification

To identify modes of vibration in the ST-RDT and AR-RDT modal estimates, a probabilistic mixture model was fitted to the modal estimates with an estimated natural frequency between 0 Hz and 10 Hz, and an estimated damping ratio between -10% and 30%.

The mixture model described the natural frequency estimates and damping ratios through $n$ two-dimensional (2D) Gaussian distributions representing the modes of vibration, each with a mean natural frequency $f_n$, a mean damping ratio $\zeta_n$ and a diagonal covariance matrix formed from the independent variance in the frequency estimates, $\sigma^2_{f,n}$, and damping estimates $\sigma^2_{\zeta,n}$. Alongside the $n$
Gaussian distributions representing the modes of vibration, the mixture model included a 2D uniform distribution between 0 Hz and 10 Hz, and -10% damping and 30% damping, which was used to model the spurious modal estimates which result from analysing ambient vibration data. In the context of the ST-RDT and AR-RDT frequency estimates, the variance of the $n$ normal distributions can be considered as the uncertainty in the modal estimates, with modes that are poorly detected or which are sensitive to changes in the external environment exhibiting a wider distribution of natural frequency and damping estimates. The full probabilistic mixture model of the ST-RDT and AR-RDT frequency ($f_i$) and damping ($\zeta_i$) estimates for a structure within the ranges of interest is given by

\[
[f_i, \zeta_i] = \sum_{j=1}^{n} (m_{f,j}N(\mu, \Sigma_c)) + m_{f,n+1}U(0, 10)U(-10, 30) \tag{I.1}
\]

In Equation I.1, $\mu$ are the means of the distributions $\mu = [f_j, \zeta_j]$, $\Sigma_c$ is the diagonal covariance matrix $\Sigma_c = [[\sigma_{f,j}^2, 0], [0, \sigma_{\zeta,j}^2]]$, and $m_f$ is the vector of mixing fractions such that $\sum_{j=1}^{n+1} (m_{f,j}) = 1$.

Fitting of the parameters of the probabilistic mixture model was carried out through minimization of the negative log-likelihood of the model parameters given the observed frequency and damping estimates, implemented with the adaptive moment estimation (Adam) stochastic gradient descent (SGD) method \cite{518}. Adam was selected due to the large number of modal estimates within the analysis and the highly non-linear solution space of the minimization problem, which features a large number of local minima, making optimization through direct gradient descent methods or algorithms such as the Expectation-Maximization algorithm \cite{518} computationally expensive and sensitive to the initial parameters selected.

The value of $n$, the number of Gaussian components to be included within the mixture model (the number of assumed modes), was set as equal to the number of peaks in the histogram of frequency estimates between 0 Hz and 10 Hz with a bin width of 0.04 Hz. A peak was defined as where the number of estimates in a bin was greater than 10% of the maximum number of estimates in a histogram bin (10% of the peak value of the histogram), and as being a minimum of +/- 0.1 Hz from an adjacent peak. Peaks where the number of modal estimates in the histogram bin was less than ten were excluded.

In the Adam optimization a mini-batch size of eight frequency estimates was used, with the error of the fitted model parameters evaluated using the full set of
frequency estimates every 50 mini-batches. The initial values of $f_i$ for each of the $n$ modal components were initialized in the Adam optimization with mean values equal to the peaks identified from the histogram of frequency estimates. An initial value of $\zeta_n$ of 1% was used for all modes of vibration. The initial standard deviations for all components of the probabilistic model used to model modes of vibration were 0.15 Hz and 1.5% damping. All model components including the noise component were initialized with mixing fractions equal to $m = 1/(n + 1)$. Adam used an initial step size of 0.001 for all parameters, which was reduced to 0.0001 after ten iterations of analysis of the full set of modal estimates (ten epochs). After each epoch the model was reinitialized using the set of model parameters which resulted in the lowest negative log-likelihood in the previous epoch. The softmax function \[440\] was used to ensure the fitted mean frequencies were in the range 0 Hz to 10 Hz, and the fitted mean damping values were in the range -10% to 30%. The logarithm of the variances was used in the SGD to ensure a positive variance value. The mixing fractions were normalized using the softmax function to ensure they summed to unity.

Prior to fitting of the mixture model it was necessary to discard a fraction of modal estimate associated with spurious or noise components of the estimated system response. These modal estimates are typically characterized by negative damping ratios and having amplitudes which are small when compared to the summed amplitude of modal components fitted to the RDS, referred to here as the normalized amplitude of the modal component. The normalized amplitude of the modal component was used as threshold for excluding modal estimates which were likely to correspond to noise components of the fitted system response. The exact value of this amplitude threshold was independently determined for each structure based on finding the logistic regression coefficient which maximized the separation of modal components with positive damping, which were likely to correspond to physical modes of the system, and negative damping, which were likely to correspond to spurious estimates. While this does not fully remove all modal estimates with negative damping, as some of these have accompanying normalized amplitudes falling above the fitted threshold, it largely maintains the Gaussianity of the results, ensuring robust fitting with the mixture model described above. An example of separation of modal estimate which are likely to be spurious from physical modal estimates based on logistic regression is presented in [Figure 1.1](#).
I.3 Results and discussion

I.3.1 ST-RDT results

The time-frequency results from the ST-RDT analyses of each of the timber structures are presented in Figures I.2 and I.3. For all structures multiple modes of vibration with a natural frequency of less than 10 Hz were detected, with these modes appearing as dense regions of frequency estimates which are stable over time.
Figure I.2: Time versus ST-RDT estimates of natural frequency for BRE, Holz8, Kampa, LCT, and Limnologen data sets. For brevity, all sub-data sets for a given structure are plotted in the same subplots. Red lines indicate the boundaries between the consecutive sub-data sets analysed, with a five minute gap left between consecutive sub-data sets in the plots.
Figure I.3: Time versus ST-RDT estimates of natural frequency for Stadhaus, Treet, Trento-CLT, Trento-TF, and UEA data sets. For brevity, all sub-data sets for a given structure are plotted in the same subplots. Red lines indicate the boundaries between the consecutive sub-data sets analysed, with a five minute gap left between consecutive sub-data sets in the plots.
The means and three standard deviation intervals for the modes of vibration identified within the modal estimates for the timber structures using the automated identification procedure are plotted in Figure I.4. In this plot modes which have a fitted standard deviation of the frequency greater than 0.25 Hz are separated as noise modes (plotted in grey) which are likely to correspond to fitting of noise within the modal estimates and not physical modes of vibration. Tight clusters of modes of vibration are clearly distinguishable in Figure I.4 and have been well identified through fitting of the mixture model. Despite the presence of close modes in the results and high levels of variance in the modal estimates, the mixture model has identified a range of overlapping modes of vibration. Specific examples of the separation of close modes of vibration can be seen in the results for Trento-CLT and Holz8. It should be noted that based on previous work it is likely that the two clusters of modal estimates with the lowest frequencies identified for BRE do not correspond to the fundamental (lowest) mode of the structure, but to components of the response associated with the oscillation of a rooftop pagoda.
Figure I.4: Frequency and damping estimates from ST-RDT analysis (black) plotted alongside means and three standard deviation boundaries of fitted modes of vibration. Noise modes defined as where the standard deviation of the frequency for a fitted mode is greater than 0.25 Hz. Modal estimates which are likely to correspond to noise and which were excluded prior to fitting of mixture model are shown in grey.
A comparison of the identified mean frequencies and mean damping ratios and the height (h), along-wind dimension (L), and slenderness (h/L) of the buildings is presented in Figure I.5, with the mode of vibration with the lowest natural frequency highlighted in Figure I.5. The results suggest the frequency of the fundamental mode is linearly dependent on the height of the structure, while there is some weak correlation between the damping of the fundamental mode and the slenderness of the building.

Figure I.5: Comparison of ST-RDT estimates of mean natural frequency and damping ratios of modes with height, along wind dimension, and slenderness of the timber buildings.
I.3.2 Amplitude-ranked RDT results

The frequencies estimated using the AR-RDT are plotted against the amplitude at the start of the RDS in Figures I.6 and I.7. As might be expected, the highest variance in the frequencies occur at the lowest amplitudes where the noise and forced components of the RDS form a larger component of the fitted dynamic response compared to the free-response of the system. This increased variance leads to the distinctive tapered shape of the frequency-amplitude plots. The stability of the mean frequency of each mode with increasing amplitude suggests that the natural frequencies of the system are not dependant on the amplitude of the structural response, and that the frequency estimates are not biased by the amplitude at which the structural response is measured.
Figure I.6: Frequency estimates from amplitude-ranked RDT plotted against amplitude at start of RDS for BRE, Holz8, Kampa, LCT and Limnologen data sets.
Figure I.7: Frequency estimates from amplitude-ranked RDT plotted against amplitude at start of RDS for Stadhaus, Treet, Trento-CLT, Trento-TF and UEA data sets.
The natural frequency and damping estimates from the AR-RDT are plotted in Figure I.8 with the modes of vibration identified in the fitted probabilistic model highlighted based on the same methodology used in the plotting of modes for the ST-RDT modal estimates. There are several key differences between the AR-RDT identified modes and the ST-RDT identified modes. The fundamental mode for Trento-TF, located at approximately 3.2 Hz, is not identified in the AR-RDT, due to the RDS being dominated by the structural response to the modes at 4.5 Hz and 5.5 Hz. The dominance of small numbers of modes also explains the reduction in the number of modes detected through the automated procedure. At low amplitudes, the variance in all modes is larger. However, as the amplitude increases this variance reduces resulting in lower numbers of modes dominating the structural response of the system at high amplitudes. The lower numbers of modes detected is not necessarily an issue, as it can allow the dominant modes to be separated from spurious modes, such as modal estimates associated with oscillation of the pagoda within the BRE results.

A comparison of the mean natural frequencies and damping ratios identified through fitting of the mixture model to the AR-RDT results with the height (h), along-wind dimension (L), and slenderness of the buildings (h/L) is presented in Figure I.9. The mean fundamental frequencies identified from the mixture model fitted with AR-RDT modal estimates also exhibit a strong correlation with the height of the buildings, while the damping exhibits the weak correlation with the buildings slenderness ratio, as previously observed for the ST-RDT modal estimates.
Figure I.8: Confidence ellipses for the amplitude ranked RDT frequency and damping estimates based on modes identified through fitted probabilistic mixture model.
Figure I.9: Comparison of AR-RDT estimates of mean natural frequency and damping ratios of modes with height, along wind dimension, and slenderness of the timber buildings.
With the exception of the failure to detect the fundamental mode of vibration for Trento-TF, discussed above, the mean frequency and damping estimates from fitting of the mixture model to the AR-RDT modal estimates are broadly similar to those identified using the ST-RDT modal estimates, as demonstrated in Figure I.10. The largest deviations are seen in the mean damping ratios for UEA and Stadhaus, with the ST-RDT producing substantially lower estimates for the mean damping ratio.

Figure I.10: Comparison of ST-RDT and AR-RDT estimates of mean natural frequency of fundamental modes with building height, and mean damping of fundamental mode with building slenderness.

Part of this deviation may be explained by the differences in amplitudes of the RDSs produced using the ST-RDT, which will consistently produce RDSs with lower amplitudes than the AR-RDT as segments of data are not ranked based on the sign-weighted amplitude at the start of the segment but based on the order in which they are observed in the time-ordered data. These lower amplitudes can be seen in Figures I.11 and I.12 in which the damping ratio of all modal estimates falling within two standard deviations of the limits identified for the fundamental mode of vibration is compared to the amplitude of the RDS from which the modal estimate was generated. It can be seen that as the amplitude of the RDS increases, there are small increases in the fitted damping ratios. This might be an indication of amplitude-dependant damping behaviour, as previously observed by Reynolds et al. [260, 261, 262], Marukawa et al. [263], Tamura and Suganuma [264], Quan et al. [265] and Kim et al. [266], among others. However, as the increase in damping ratio is most apparent for very small amplitudes of structural response it would seem more likely that the variation in damping ratio
observed is due to biasing of the damping estimates at low amplitudes due to higher levels of noise and the forced response within the RDS when compared to the free-response of the system.

Figure I.11: Comparison of RDS amplitude and damping ratio for ST-RDT and AR-RDT modal estimates of the fundamental mode for BRE, Holz8, Kampa, LCT, and Limnologen data sets. Results limited to non-spurious modal estimates falling within two standard deviations of the mean values given by the fitted mixture models.
Figure I.12: Comparison of RDS amplitude and damping ratio for ST-RDT and AR-RDT modal estimates of the fundamental mode for Stadhaus, Treet, Trento-CLT, Trento-TF, and UEA data sets. Results limited to non-spurious modal estimates falling within two standard deviations of the mean values given by the fitted mixture models.

Stronger evidence for the biasing due to the presence of the forced and noise components within the RDSs at low amplitudes can be seen when the amplitude of the modal response is compared to the damping of the fundamental mode, as presented in Figure I.13 and I.14. Sorting of the damping estimates by the modal amplitude results in tighter clusters of results and smoother variations in
damping than was previously observed when the damping is compared to the amplitude of the RDS, as plotted in Figures I.11 and I.12.

Figure I.13: Comparison of modal amplitude and damping ratio for ST-RDT and AR-RDT modal estimates of the fundamental mode for BRE, Holz8, Kampa, LCT, and Limnologen data sets. Results limited to non-spurious modal estimates falling within two standard deviations of the mean values given by the fitted mixture models.
Figure I.14: Comparison of modal amplitude and damping ratio for ST-RDT and AR-RDT modal estimates of the fundamental mode for Stadhaus, Treet, Trento-CLT, Trento-TF, and UEA data sets. Results limited to non-spurious modal estimates falling within two standard deviations of the mean values given by the fitted mixture models.

The RDS amplitude ranked and modal amplitude ranked moving averages for the AR-RDT and ST-RDT damping results from the fundamental modes of vibration, presented previously in Figures I.11 to I.14 are compared in Figure I.15. It can be seen that the damping across most modes of vibration is stable, with only small variations in the estimated average damping ratio. The damping ratio
for most of the fundamental modes of the timber structures is approximately 2%. Where the modal or RDS amplitudes are very small the mean damping ratios are typically slightly lower. However, as discussed above this may be due to the noise and forced components within the estimated free-response of the structures. At higher amplitudes the results appear to be less stable, exhibiting larger variations in damping. This may be due to the small number of modal estimates collected at higher amplitudes, or may indicate real variation in damping behaviour associated with factors such as wind-induced vortex shedding [264]. The exceptions to the typical behaviour are Trento-CLT and Trento-TF, both of which exhibit unusually large damping ratios that exhibit a strong correlation with the modal amplitude. This may be due to the increased noise and the number of close modes within the modal estimates collected for these structures.

![Graph](image)

Figure I.15: RDS amplitude ranked and modal amplitude ranked moving averages for the AR-RDT and ST-RDT damping results from the fundamental modes of vibrations.

### I.4 Summary

The modal parameters of ten timber structures under ambient wind excitation have been quantified and compared. The estimated modal parameters indicate a strong correlation between the natural frequency of the fundamental mode of vibration of a timber structure and the structures height. The results also suggest
that the estimated damping of this fundamental mode is positively correlated with the amplitude of oscillation of the fundamental mode, and that this correlation is stronger and more consistent than any correlation between the damping and the amplitude of oscillation of the structure as a whole. However, the results would indicate that this may be due to the smaller amplitude of the free-response component within the RDSs when compared to the amplitude of noise and forced components within the RDSs. These results highlight the need for long-term monitoring of structural behaviour in-situ so that real and spurious modal estimates may be better separated and any variation in dynamic behaviour with changes in the external environment quantified.

Alongside these findings, this case study has demonstrated the use of an automated procedure for the analysis of ambient vibration data using novel variations of the random decrement technique and a probabilistic mixture model for identification of modes of vibration. The procedure removes issues with the biasing of modal estimates and the challenges associated with interpreting operational modal analysis modal estimates and has been shown as an effective method through which the modal parameters of civil structures may be estimated and the uncertainty in those parameters automatically quantified. This automated procedure allows the modal estimates from different data sets or structures to be compared and provides a baseline for future dynamic monitoring of timber structures.
Appendix J

Regression methods for quantifying weak non-linear modal behaviour

In this section additional details are provided for the application of linear regression and quantile regression for estimating the parameters of the WNLMB equations derived in Section 9.1.

Regression methods, or regression analysis, are statistical methods for estimating the relationships between dependant or response variables, and independent or feature variables. These methods attempt to minimize a cost function to estimate the parameters of a function that relates the response and feature variables. A brief introduction to linear regression and quantile regression is presented in Sections J.1 and J.2.

As discussed in the literature review, the primary use of regression methods for the analysis of WNLMB in OMA modal estimates has been for the quantification of environmental variation in natural frequencies through linear regression. Unlike the derivation presented in Section 9.1, in which it was assumed that the stiffness of a structure was linearly related to temperature, resulting in the temperature being related to the squared frequency of the mode, in the work by Peeters and De Roeck [241] and Ramos et al. [330], and within the use of regression analysis for analysis of thermally-induced WNLMB in the OMA literature more generally, it has been assumed that the natural frequency of a mode was linearly related to the changes in temperature. Perhaps as a result of this, within both Peeters and De Roeck [241] and Ramos et al. [330] the regression analysis is split into separate temperature ranges. This may represent a linear approximation of the expected squared relationship of the frequency and temperature described in Section 9.1.
As highlighted by Tee et al. [339] little work has explored the use of quantile regression for OMA or SHM, as discussed in detail in Section J.2.

### J.1 Linear regression

Simple or univariable linear regression is used where the relationship between a single response \( y \) and a single feature variable \( x \) may be approximated as a linear relationship of the form \( y \approx a + Bx \), where \( a \) is the intercept; the value of the response \( y \) when the feature \( x = 0 \), and \( B \) is a constant value [336]. As this relationship is an approximation, it is expected that there is some error term \( \varepsilon \) within the relationship [336] so that the relationship between \( x \) and \( y \) may be fully described by \( y = a + Bx + \varepsilon \).

There are a variety of different simple linear regression methods, but the most commonly used is ordinary least squares (OLS) regression [336] where the objective when estimating parameters \( a \) and \( B \) from a dataset of size \( n \) is to minimize the sum squared error term as given by Equation J.1.

\[
\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (y_i - (a + Bx_i))^2 \tag{J.1}
\]

By inspection of Equation J.1 it can be seen that this is the same cost function as was discussed for k-means clustering in Section 8.2, where the Euclidean distance between the points and the cluster centres was minimized as part of the k-means clustering.

If multiple feature variables \( (x_0, x_1, \ldots, x_n) \) are used to predict the response variables, the regression method is known as multivariable linear regression, with the function having the form \( y = a + B_0x_0 + B_1x_1 + \ldots + B_nx_n + \varepsilon \).

Multivariable linear regression or multiple linear regression should not be confused with multivariate regression, in which multiple responses \( (y_0, y_1, \ldots, y_n) \) are predicted, either from a single feature (multivariate univariable regression) or multiple features (multivariate multivariable regression) [520]. Within this chapter, only individual responses are to be predicted. More fully, the two methods used in this section are OLS univariate univariable regression (a single feature is used to predict a single response) and OLS univariate multivariable regression (multiple features are used to predict a single response). For brevity, these will be referred to as univariable linear regression and multivariable linear regression.

Minimizing the OLS cost function to find the intercept and regression coefficients of the linear model can be achieved through a variety of different optimizers.
In this section, the LAPACK routine, based on finding the optimal solution to the least-squares problem through singular-value decomposition \cite{521}, as implemented within the Scikit-Learn Python library \cite{435} is used.

In the context of analysing WNLMB, linear regression can be an effective tool for quantifying and removing a variety of different effects. It can allow systematic variations in behaviour associated with thermal expansion and contraction of the structure, or the static load applied from the structure, to be separated from changes in the natural frequency which may be associated with damage. The first stage in all of the analyses presented in this section is the identification of frequency estimates from a single mode of vibration, either through establishing frequency bounds containing a single mode of vibration visual inspection of the results or through the methods presented in Chapter 8. Once frequency estimates for a single mode of vibration have been isolated, it is necessary to reformulate the equations of common causes of WNLMB summarized in Table 9.1 to be of the form given in Equation J.1. Consider the relationship between changes in temperature and changes in natural frequency, introduced in Section 9.1 and reproduced in Equation J.2

\[
\omega_i = \sqrt{\omega_0^2(1 + \alpha_K T_i)}
\]  

(J.2)

In Equation J.2 the feature variable \(x\) is \(T_i\) and the response variable \(y\) is the observed frequencies of the system \(\omega_i\). Equation J.2 can be rearrange to the form \(y = a + Bx\) as \(\omega_i^2 = \omega_0^2 + \alpha_K T_i \omega_0^2\), where \(a = \omega_0^2\) and \(B = \alpha_K \omega_0^2\). As can be seen from the artificial data presented in Figure J.1 consisting of frequency estimates for a single mode of vibration subjected to thermally induced WNLMB as given by Equation J.2 estimating the parameter \(\alpha_K\) through linear regression analysis allows for the effective removal of the thermal effects which induce variations in the natural frequency. Through quantifying the thermally-induced variations in modal parameters, the wider thermal behaviour of the structure can be understood, as covered in detail in Section 9.3.3 or may be removed such that damage-induced changes in behaviour may be identified.

A weakness of linear regression for analysis of WNLMB is that the results are sensitive to outliers or noise within the data, as demonstrated in Figure J.2. This plot presents the same artificial data as presented in Figure J.1 but where 10% of the artificially generated frequency estimates correspond to spurious frequency estimates or noise within the data. While the majority of thermal effects have been removed, the presence of noise within the data results in an overestimation of \(\omega_0\) by approximately 0.1%, and an underestimation of \(\alpha_K\) approximately 9.4%.
Figure J.1: Example of removal of thermally induced variations in natural frequencies through linear univariable regression, demonstrated using artificially generated frequency estimates.

While there was noise in the frequency estimates presented in Figure J.1, this noise was normally distributed about the true natural frequency of the system at each time-step, an approximation of the modal errors observed for modal estimates from real-structures, as discussed in Chapter 7. These normally distributed modal errors translate to a symmetrical distribution of noise within the linear regression. The key difference in the data presented in Figure J.2 is the presence of the background noise or spurious frequency estimate. While this noise is assumed to be uniformly distributed within the plotted frequency range, the variation in the mean frequency at each time-step due to the thermal WNLMB ensures that the noise is non-symmetrically distributed relative to the mean frequency of the mode across all time-steps.
Figure J.2: Example of removal of thermally induced variations in natural frequencies through linear univariable regression, demonstrated using artificially generated frequency estimates containing 10% spurious frequency estimates.

The degree of over- or under-estimation of the WNLMB parameters is dependent on the bounds used when isolating the modes of vibration of interest, as these control the distribution of spurious frequency estimates relative to the distribution of modal frequency estimates. The error in the estimated parameters will also increase if the noise within the estimates is non-uniform over time, particularly where the quantity of noise estimates is correlated with temperature. An example of when this might occur is the increased noise that is likely to be associated with periods of low excitation of bridges during nighttime hours, where it might be expected that the temperature of the structure will be lower. Similarly, errors will increase if there is evidence of other modes within the isolated frequency bounds, and OLS linear regression cannot be used for the analysis of close modes of vibration, which will be subjected to the same fluctuations in temperature but may exhibit different thermally-induced WNLMB. While weighted least-squares regression, discussed further in Section J.3 could be used for the analysis of close modes or to reduce the impact of noise on the fitted regression parameters, it was found that the parameters for thermally-induced WNLMB were highly sensitive to the weighting criteria selected and offered little to no improvement in the accuracy of parameter estimation when compared to OLS regression.
For some structures there may be multiple temperature measurements from
thermistors located on different parts of the structure. In this scenario, an aver-
age temperature may be used within the univariable linear regression. However,
a technique that may account for the sensitivity of the thermally induced weak
non-modal behaviour to the spatial variability of the temperature changes is to
use multivariable linear regression. In this application, a unique coefficient $\alpha_{K,j}$
is introduced which relates the change in frequency of the structure to the tem-
perature measured at each of the $j$ thermistors. The multivariable linear regres-
sion equation for the analysis of thermally-induced WNLMB therefore becomes

$$\omega_i^2 = \omega_0^2 + \alpha_{K,0} T_0 \omega_0^2 + \alpha_{K,1} T_1 \omega_0^2 + \ldots + \alpha_{K,j} T_j \omega_0^2$$

However, the multivariable linear regression may negatively impact the inter-
pretability of the regression coefficients due to multicollinearity: the correlation
of features within the analysis. It is expected that the temperature recorded at
thermistors on different parts of a structure is likely to be strongly correlated. In
the extreme situation of perfect multicollinearity, such that the temperature at
two thermistors is proportional $T_0 \propto T_1$, it is evident that no unique value of the
coefficients $\alpha_{K,0}$ and $\alpha_{K,1}$ will exist, as any increase in $\alpha_{K,0}$ may be offset by a
decrease in $\alpha_{K,1}$. Hence, the physical meaning of $\alpha_K$ as a means of relating the
change in stiffness of a structure to the change in temperature is lost. Perfect
multicollinearity may also result in the failure of some OLS regression techniques,
as discussed in greater detail by Alin [522] and Fahrmeir [336].

Aside from limiting the interpretability of the regression coefficients for the
intercorrelated features, the increased number of features increases the risk of
overfitting the data. Overfitting occurs when the regression coefficients are scaled
in such a way as to model the noise within the response data. In this context noise
is any feature of the data which is not correlated with the explanatory variables
over an infinite data set. Overfitting limits the applicability of the fitted regression
coefficients to other data sets. An example of overfitting has been seen previously
in this section in Figure J.2, where the fitted model parameters differed from the
ture model parameters due to the presence of noise within the data. Where there
are large numbers of features, or the amount of data used when fitting the model
is small, or there are correlations between the noise and the response parameters
(as were induced by the selection of frequency bounds in Figure J.2), overfitting
is more likely to occur. Methods for reducing the impact of multicollinearity and
overfitting are discussed in Section J.3.

A challenge faced when applying linear regression for estimating the param-
ters of the models summarized in Table 9.1, which relate the static load applied
to a structure, $L$, to the natural frequency of the structure $\omega_i$ at a time-step
through Equation J.3, is that the values of $\beta_L$ and $L$ are coupled.

$$\omega_i = \sqrt{\frac{\omega_0^2}{1 + \alpha_L \beta_L L}} \quad (J.3)$$

If the value of $\beta_L$; the squared amplitude of the unity scaled mode shape at the locations of the applied loading, and the applied static load $L$ is known, the feature to be used in the regression analysis $x$ is $x = \beta_L \Delta L$, with the function to be fitted to the data given in Equation J.4

$$\frac{1}{\omega_i^2} = \frac{1}{\omega_0^2} + \frac{\alpha_L}{\omega_0^2} x \quad (J.4)$$

In Equation J.4, the response vector is $1/\omega_i^2$, the intercept $a$ is $1/\omega_0^2$ and the regression coefficient $B$ is $\alpha_L/\omega_0^2$. Univariable linear regression with known values of $\beta_L L$, with the value of $\beta_L$ used to generate the artificial frequency estimates having been randomly sampled from a uniform distribution $U(0, 1)$, is demonstrated in Figure J.3. As with the removal of thermal effects, these results are sensitive to the presence of noise and/or other modes within the window of data selected for use within the regression.

![Figure J.3](image)

Figure J.3: Example of removal of load induced variations in natural frequencies through linear univariable regression with known $\beta_L \Delta L$, demonstrated using artificially generated frequency estimates.

In reality, it is rare to know the value of $\beta_L$ when analysing OMA data, as the
location of static loads on a structure relative to the mode shapes is not typically known and there is inherent uncertainty when estimating the mode shapes of a structure using ambient vibration data. If the data presented in Figure J.3 is analysed with univariable linear regression assuming $\beta L = L$, that is neglecting the value of $\beta_L$, it can be seen that the values fitted through OLS linear regression approximate the values given by $\beta_L = 0.5$, as illustrated in Figure J.4. This approximation of $\beta_L = 0.5$ occurs within the linear regression analysis of the artificial data as the values of $\beta_L$ used to generate the artificial frequency estimates were drawn from the uniform distribution $U(0, 1)$. The expected mean of this distribution is 0.5. Theoretically, if $\beta_L$ was known to be this uniform distribution, the true value of $\alpha_L$ might be inferred as twice the regression coefficient $\alpha_L = 2B\omega_0^2$. However, if the true distribution of $\beta_L$ was different, the fitted regression coefficient will approach that given by the mean value of $\beta_L$ and any relationship between the true value of $\alpha_L$ and the regression coefficient would be lost without prior knowledge of the distribution of $\beta_L$.

Figure J.4: Example of removal of load induced variations in natural frequencies through linear univariable regression with known $L$ and unknown $\beta_L$, demonstrated using artificially generated frequency estimates.

As the values of $\beta_L$ at each time-step are unknown, the static load component of the natural frequency presented in Figure J.4 is not fully removed. Instead, the average static load component of the WNLMB, associated with the mean value of $\beta_L$, is removed. This is a key limitation of OLS linear regression for analysis.
of structural behaviour: it is based on minimization of the squared error term between the response and some linearly scaled feature. As has been demonstrated in this section, the presence of noise within the frequency estimates may skew this relationship. Alongside this, it may require the use of large simplifying assumptions about the structural behaviour, such as the assumption of a constant value of $\beta_L$ as presented in Figure J.4.

### J.2 Quantile regression

Some of the issues due to noise within the data and unknown factors within the response to be estimated can be resolved using quantile regression. In this context, a quantile is the fraction of a dataset that falls above a given partition. In OLS linear regression, the objective was to minimize the OLS error term of the data as given by [Equation J.1](#). In quantile regression, the objective is to minimize a weighted sum of the positive and negative error terms for a given quantile level $\tau$ [336], often referred to as the pinball loss function, as given by [Equation J.5](#).

$$
\tau \left( \sum_{y_i > (a + Bx_i)} \left| y_i - (a + Bx_i) \right| \right) + (1 - \tau) \left( \sum_{y_i < (a + Bx_i)} \left| y_i - (a + Bx_i) \right| \right)
$$

(J.5)

The use of quantile regression for analysis of the artificial data with thermally induced weak non-linear data and noise introduced previously in Figure J.2 is presented in Figure J.5. Whereas the relationship between the temperature and squared frequency fitted through linear regression was heavily skewed due to the presence of noise within the data, it can be seen that the median (50%) quantile fitted through quantile regression offers a closer approximation of the true relationship in the data than the linear regression analysis presented previously in Figure J.2. Alongside this, the fitted 5% and 95% quantiles provide an effective means of separating spurious or noise estimates from frequency estimates associated with the mode of vibration, allowing the variance in the estimated parameters of the mode of vibration to be quantified. As discussed in detail in Section 9.4, this is an important requirement for the application of OMA as a method of damage detection, as it allows the likelihood of damage to be assessed in the presence of uncertainty in the modal estimates. Despite the success of quantile regression for quantifying and removing thermally induced WNLMB, the results are still sensitive to the presence of non-uniform noise within the modal estimates and the presence of close modes of vibration as these may bias the estimated distribution of quantiles for a specific mode of vibration. However,
through evaluation of a range of quantiles the biasing induced by such features may be assessed.

Figure J.5: Example of quantification of thermally induced variations in natural frequencies through quantile univariable regression, demonstrated using artificially generated frequency estimates containing 10% spurious frequency estimates.

Quantile regression analysis of well-separated modes of vibration is also an effective method for estimating the expected variation in frequency induced by distributed changes in static loading, as it allows the boundaries given by $\beta_L = 0$ (load is primarily located at the node point of a mode) and $\beta_L = 1$ (load is primarily located at the peak of a mode) to be estimated. This is demonstrated in Figure J.6 using the artificial frequency estimates previously introduced in Figure J.4 with 10% of the frequency estimates replaced with spurious noise. The fitted median quantile closely corresponds with the expected change in frequency under application of load when $\beta_L = 0.5$, as the median of the uniform distribution of $\beta_L$, is approximately 0.5. Alongside this, the fitted values of the 5% and 95% quantiles provide an approximation of the relationships given when $\beta = 0$ and $\beta = 1$ respectively. Through comparison of the fitted quantiles and the exact results of the mode of vibration given by $\beta_L = 0$ and $\beta_L = 1$ in the time-domain, presented in Figure J.7, it can be seen that the quantiles are not an exact measure of the boundaries given by $\beta_L = 0$ and $\beta_L = 1$ as they are biased by two factors.

The first of these factors is the modal errors, the errors in the frequency estimates for a mode of vibration at each time step. When the load is exactly zero,
Figure J.6: Example of quantification of distributed load-induced variations in natural frequencies through quantile univariable regression, demonstrated using artificially generated frequency estimates containing 10% spurious frequency estimates.

Figure J.7: Comparison of boundaries of expected frequencies given by $\beta = 0$ and $\beta = 1$ for load induced variations in natural frequencies, and 5% and 95% quantiles of the data fitted through quantile regression.

Modal errors lead to a normally distributed spread of frequency estimates centered on the unloaded frequency of the system, $\omega_0$. This distribution of frequency estimates due to modal errors is reflected in the quantile regression results by the intercept of the 5% and 95% fitted quantiles not being equal to the unloaded frequency of the system. The second factor which biases the results is that under the model of distributed static-load induced changes in natural frequency,
the frequency estimates for a load $L$ are expected to fall in the range given by $\beta_L L = 0$ ($\beta_L = 0$) and $\beta_L L = L$ ($\beta_L = 1$). The results for a specific time-step remain normally distributed due to the modal errors, as it was assumed within the artificial data generation that there was a fixed value of $\beta_L$ for each time-step. However, when the frequency estimates for multiple time-steps which have the same load value but different values of $\beta_L$ are plotted as a histogram, the distribution of frequency estimates approaches that given by a convolution of a normal distribution, due to the modal errors, and a uniform distribution, given by the uniform probability of observing any $\beta_L$ value in the range zero to one. An approximation of this distribution is presented in Figure J.8, generated through random sampling of the distribution given by Equation J.6.

$$\omega_i = N\left(\sqrt{\frac{\omega_0^2}{1 + \alpha L \beta_L L}}, \sigma^2_0\right)$$  \hspace{1cm} (J.6)

In Equation J.6 $\sigma_0^2$ is the variance of the normally distributed errors in the modal estimates, referred to as the modal errors, and the value of $\beta_L$ is generated from a uniform distribution $U(0, 1)$. The effect of this combined inverse uniform distribution and normally distributed modal errors in the context of the quantile analysis is that as the load increases, a greater proportion of the frequency estimates are expected to fall within the boundaries given by $\beta_L = 0$ and $\beta_L = 1$, with the modal errors having a smaller impact on the range of frequency variations when compared to frequency variations associated with different values of $\beta_L$. This provides an explanation for the closer approximation of the $\beta_L L$ boundaries at higher loads visible in Figures J.6 and J.7. In light of this, for the estimation of parameters associated with load-induced WNLMB through quantile regression, the ideal training data set would consist of data from when large static loads were applied at a wide range of locations on the structure. This is in contrast to the ideal training data set for estimating the model parameters through linear regression, where it is beneficial to have data from a much wider range of applied static loadings to ensure a more uniform distribution of euclidean errors across the loading range.

### J.3 Other regression methods

The issues associated with outlier noise measurements and non-uniformity of noise is discussed extensively within the regression analysis literature such as in Fahrmeir [336] and Draper [520]. One method for reducing the impact of these
outlier measurements in linear regression is through weighted least-squares (WLS) linear regression where a vector of weights \( w_i \) is introduced into the error term using Equation J.7:

\[
\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} w_i (y_i - (a + Bx_i))^2 \tag{J.7}
\]

This vector of weights may be any set of values. In the context of analysis of WNLMB, the use of the inverse distance of points from the cluster centres in k-means clustering and the likelihood of the modal estimates under the probabilistic mixture model, both described in the Chapter 8, were explored as weighting criteria. It was found that for well-separated modes of vibration this offered meaningful improvements in the fitted model parameters as it reduced the bias introduced by spurious noise. However the results were heavily skewed to the average temperature or loading on the structure, as this coincided with the centre of the fitted cluster for k-means clustering or the mean frequency value within the probabilistic mixture model.

Alongside the least-squares error and weighted least-squares error functions, there is a range of other error functions which have been developed for reducing the impact of outlier results. An overview of some of these error functions for linear regression can be found in Fahrmeir [336]. An example of one of these error functions, the Cauchy loss function, is discussed in Section 9.5.1.

As discussed in previous sections, multivariable regression using in-situ monitoring data is often plagued by multicollinearity in the data due to the high
intercorrelation between features used in the regression. Common examples of this include the strong correlation between the simultaneous temperature measurements from different parts of a structure. As previously discussed, multicollinearity does not limit the ability to fit a regression model to the data but does limit the interpretability of the regression coefficients for the intercorrelated features and the degree of redundancy within the set of features used to predict the response.

Approaches for examining the results from multicollinear regression analyses including repeating the analysis using individual features or subsets of the features, or increasing the amount of data available for the regression analysis to reduce the intercorrelation between features. Two specific methods which may be used when analysing multicollinear in-situ structural monitoring data are ridge regression and lasso regression.

Ridge regression, as used within work by Zhang et al. for removing environmental variability from OMA estimates, introduces a term $\alpha_{ridge}$ to the least-squares regression equation which is used to penalize large regression coefficients $\{\beta\} = \{B_0, B_1, \ldots, B_n\}$. The cost function to be minimized in ridge regression is given in Equation J.8:

$$\left( \sum_{i=1}^{n} (y_i - (\alpha + Bx_i))^2 \right) + \alpha_{ridge} \sum B^2 \quad (J.8)$$

In this cost function, the penalization is based on the squared regression coefficients, referred to as the L2 norm or L2 penalty. As the value of $\alpha_{ridge}$ increases, the penalization applied to large coefficients increases. This can help to reduce the overfitting of the data through the application of large coefficients with different signs (+/-) to intercorrelated features as well as encouraging the assignment of small regression coefficients to redundant features.

An expansion on ridge regression is lasso regression which uses the L1 norm penalty or the absolute value of regression coefficients within the cost function given by Equation J.9:

$$\left( \sum_{i=1}^{n} (y_i - (\alpha + Bx_i))^2 \right) + \alpha_{ridge} \sum |B| \quad (J.9)$$

Extensive discussions of ridge and lasso regression can be found in Fahrmeir and Draper.
J.4 Summary

This appendix has provided additional details on the use of regression for analysis of weak non-linear modal behaviour. The issues posed by multicollinearity and overfitting have been highlighted and alternative regression methods which may be used to address them have been identified. Quantile regression has been demonstrated as a more robust method for analysis of in-situ structural monitoring data, rather than the more widely used linear regression methods.
Appendix K

Case study - Temperature-change experiment

This section is expanded and adapted from work originally published by Wynne et al. as a chapter in Civil Structural Health Monitoring as Mass and Temperature Changes in Operational Modal Analysis [5].

As discussed in Chapter 9, a challenge in identifying the dynamic properties of large structures through OMA is the changes in temperature which may occur during data collection. The effect of these changes may mask any damage to the structure or lead the investigator to incorrectly conclude that damage has occurred. Quantification of such effects requires the use of short-time OMA methods, such as the ST-RDT introduced in Chapter 7, where the structure’s response is assumed to be linear across short periods of time. Once the variation in the frequency of individual modes of vibration due to the thermally-induced WNLMB has been approximated through the short-time OMA methods, regression methods may be used to relate the changes in frequency to changes in the real-valued stiffness of the structure.

An experiment, referred to as the temperature-change experiment, was carried out to assess the robustness of the ST-RDT as a means for quantifying thermally-induced WNLMB, and to allow comparison of linear regression and quantile regression for estimating the parameters of the models of thermally-induced WNLMB introduced in Section 9.1. A flat plate was heated in laboratory conditions using radiant panels with the changes in dynamic properties estimated during heating, steady temperature and cooling of the plate through the ST-RDT.
K.1 Experimental setup and modal analysis

A 600 mm length of 50 x 5 mm flat plate, steel grade S235, was suspended 0.59 m above two 400 mm by 300 mm radiant heating panels. Suspended supports, located 25 mm from either end of the plate, were shaped to minimize the contact area and provide an approximation of a free-free support condition. The plate was excited through random impulse excitation, with the acceleration response measured using a high-temperature accelerometer bolted to the plate at 80 mm from mid-span. The thickness of the plate was chosen with the intent of minimising the temperature gradients both along the length of the plate and through the cross-section (i.e. to ensure that the imposed heat flux was relatively uniform over the exposed surface, and to ensure that the Biot number was much less than 1.0). To verify (and quantify) this, the temperature of the plate was measured using six type K thermocouples welded directly to the plate at equal intervals along the top and bottom face of the plate. Once the radiant panels were ignited, the plate heated to a maximum temperature of 290 °C in approximately 13 minutes before the radiant panels were shut-off and the plate was allowed to cool to ambient temperature.

Triaxial acceleration data was collected from the accelerometer with a sample rate of 4098.36 Hz. This data was analysed using the ST-RDT with an amplitude triggering and weighting condition, a window length of 10 seconds, and an overlap between adjacent analysis windows of 5 seconds. RDSs were analysed using the matrix pencil method, with the model order incremented by one from unity until a target explained variance of the fitted RDS of greater than 0.99 was achieved. An RDS length of 0.04 seconds, approximately three times the fundamental frequency of the plate, was used in the ST-RDT analysis. Results from RDSs which required a model order greater than 20 to achieve the target explained variance, indicating high levels of noise within the data and low levels of modal oscillations, were
K.2 Temperature variation

As shown in Figure K.2, the temperature differential across the depth of the plate reaches a maximum value of 37°C, with an average difference in temperature between the bottom and top of the plate of +23.8°C during heating and +11.3°C during cooling of the plate.

There is a significant temperature differential between the right-hand side (RHS) and the centre/left-hand side (LHS) of the plate, illustrated in the right plot of Figure K.2. This temperature differential is supported by data from an infrared camera recorder. The discrepancy can be attributed to both a slightly increased elevation of the RHS and an asymmetry of the heated air plume due to the location of extraction fans and draughts during the experiment. These factors result in a lower incident heat flux on the RHS, and therefore a decreased temperature, which is most pronounced during heating of the plate.

Figure K.2: Left plot: Measured temperature at thermocouples. Right plot: Variation in temperature across length of plate, interpolated at times shown on left plot.

K.3 Identification of modes of vibration

Modes of vibration were visually identified within the ST-RDT results, with frequency bounds selected to allow a minimum separation of 2.5 Hz between the likely minimum/maximum natural frequency for the mode and the edge of the frequency bound. The natural frequency estimates of the six modes to be analysed are plotted against time in Figure K.3 and the average temperature of the plate in Figure K.4. All modes show some evidence of differences in behaviour
during the heating and cooling phases, with this being most apparent in the separation of frequencies at high temperatures for mode 4. This is likely due to the differential temperature gradients across the plate discussed above. To account for these differences in behaviour during the assessment of the regression methods, the data was split into five-minute intervals, with every third five-minute block used as holdback data for assessing the quality of the fitted regression models.

The equation relating changes in temperature to changes in natural frequencies used in all regression analyses was \( \omega_i = \sqrt{\omega_{20}^2 (1 + \alpha_K \Delta T)} \).

The parameters of this equation were independently fitted to the six modes of vibration using OLS linear regression and quantile regression, introduced in Section 9.3. For the quantile regression analyses, the 10%, 25%, 50%, 75% and 90% quantiles were fitted. Both regression analyses were conducted using the average temperature recorded across the six thermistors, referred to here as the univariable case, and all thermistor measurements, referred to as the multivariable case.

**K.4 Results - Linear regression**

The univariable (red line) and multivariable (blue line) linear regression results are presented in Figure K.5. In both cases, there are only minor differences between the univariable and multivariable results, with the multivariable regression results showing evidence of overfitting of the data, characterized by a non-smooth variation in the expected frequency with temperature. It would be expected that the variation in natural frequency with temperature will be smooth, as for the free-free condition the change in frequency is driven only by changes in the elastic modulus of the plate with temperature. The results for modes 5 and 6 show a significant overestimation of the frequencies at the highest temperatures, where the natural frequencies are at their lowest. This is likely due to the presence of spurious frequency estimates within the frequency bounds leading to biasing of the fitted results as discussed in Section J.1. This biasing is sensitive to the frequency bounds selected, with a lower value for the lower frequency bound likely to lead to a reduction in the biasing at higher temperatures. However, this may result in an underestimation of frequencies at lower temperatures, and may introduce greater levels of noise between the frequency bounds. The coefficient of determination (\( R^2 \) scores) for the training and holdback data are presented in Figure K.6. The lack of improvement between the \( R^2 \) scores for the multivariable regression compared to the univariable regression suggests that using the data
Figure K.3: ST-RDT natural frequency estimates for six modes of vibration during heating and cooling of a flat plate. Modes identified through visual inspection of the frequency results in the time-domain.

from all thermistors offers very little additional information about the expected WNLMB of the system. The strong correlation between the $R^2$ scores for the training and holdback data indicate that the fitted regression parameters are robust, with the non-unity $R^2$ scores likely due to the presence of background noise.
Figure K.4: ST-RDT natural frequency estimates for six modes of vibration during heating and cooling of a flat plate compared to average temperature of plate recorded across the six thermistors.

A comparison of the expected frequencies of the six modes at a temperature of $T = 0^\circ C$, which is equal to the square root of the linear regression intercept, is presented in Figure K.7. Also plotted is the difference between the univariable and
Figure K.5: Comparison of ST-RDT natural frequency estimates and linear regression predictions of natural frequency for univariable and multivariable linear regression analysis.

multivariable regression values, illustrating that the differences in the intercept between the approaches are minor, being of the order of +/- 1% to 2%.
Figure K.6: Comparison of $R^2$ scores between training and holdback data for univariable and multivariable linear regression analyses.

Figure K.7: Square root of the linear regression intercepts, which are equal to the expected natural frequency of the system at a temperature of $0^\circ C$.

The temperature mode shape scaling coefficients ($\alpha_K$), which are equal to the linear regression coefficients divided by the regression intercept, are presented in Figure K.8. These results demonstrate the issues posed by multicollinearity of the thermistor data, with the fitted values of $\alpha_K$ for the multivariable regression having a wide range of signs and values, indicative of overfitting of the data. However, the univariable regression coefficients fall within a tight range of values between -0.00025 and -0.00034. This tight range of values is to be expected for analysis of the thermal behaviour of the free-free beam structure, as the change in stiffness of the structure is dependant only on the beam's elastic modulus. Therefore the $\alpha_K$ coefficient gives the percentage change in elastic modulus of the beam for a one-degree increase in temperature. This highlights the interpretability of the univariable regression coefficients and how WNLMB may be used as a basis for quantifying a range of structural parameters.
Figure K.8: Linear regression coefficients scaled by linear regression intercepts. Equivalent to the percentage change in elastic modulus of the beam structure for a one degree increase in temperature.

K.5 Results - Quantile regression

The multivariable quantile regression results indicate the same overfitting of the data as was discussed for the multivariable linear regression results. Therefore, for brevity, only the results of the univariable quantile regression analysis are presented in this section.

The fitted quantiles are presented in Figure K.9. It can be seen that the quantile regression offers a robust approach for accounting for the non-uniform background noise within the ST-RDT frequency estimates. This robustness to noise is most apparent for modes 5 and 6, where linear regression significantly over-predicted the natural frequency of the system at high temperatures due to the background noise. In comparison, the median fitted quantile closely follows the densest region of frequency estimates which is indicative of the true natural frequency of the system at these temperatures. A further advantage of the quantile regression approach is that the use of the other quantiles allows the differences in behaviour during heating and cooling to be approximated, as seen in the right-hand column of plots in Figure K.9.

The quantile regression intercepts, plotted in Figure K.10 provide some indication of these differences in heating and cooling behaviour, highlighting that the widest range of intercepts corresponds to the highest modes of vibration. This occurs even though visual inspection of these modes would suggest they have lower variances in the frequency estimates than mode 1. This would suggest that the wide variation of intercepts for higher modes is primarily driven by differential behaviour during the heating and cooling phases.

The median (50%) quantile scaled regression coefficients, plotted in Figure K.8.
Figure K.9: Comparison of ST-RDT natural frequency estimates and predicted quantiles of the data based on univariable quantile regression analysis.

Figure K.11 show a tighter range of values than was observed for the linear regression results, with a mean value of approximately -0.00033. This is very close to the approximate percentage change in elastic modulus of carbon steel at the peak temperature of the test as given in Meyer et al. [444], plotted for reference in Figure K.11.
Figure K.10: Square root of the quantile regression intercepts, which are equal to the expected quantile of the natural frequency of the system at a temperature of 0°C.

Figure K.11: Quantile regression coefficients scaled by quantile regression intercepts. Equivalent to the real-valued percentage change in elastic modulus of the beam structure with a one degree increase in temperature. Also plotted are the average percentage change in elastic modulus of carbon steel between 200°C and 250°C, as given in Meyer et al. [444].

K.6 Summary

This section has presented analyses of a simple structure under varying temperatures and demonstrated some challenges of extracting dynamic properties from systems exhibiting WNLMB.

The ST-RDT has been demonstrated as a robust method for extracting dynamic parameters from rapidly time-varying systems with a high degree of accuracy. The results of the regression analyses indicate that quantile regression, rather than the more broadly used linear regression, is a more robust method for estimating the parameters of the thermally-induced WNLMB relationships as it
allows for noise within the frequency estimates to be accounted for within the analysis. The physical meaning of the univariable regression coefficients for simple beam structures has been discussed, with this physical interpretation being a key benefit when analysing WNLMB which is lost when multivariable regression is used.

Future work would look to explore the use of non-linear relationships between temperature and elastic modulus to more accurately model the WNLMB which occurs at elevated temperatures.