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Deep-water and shallow-water limits
of the intermediate long wave equation:
from deterministic and statistical viewpoints

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. This work has not been submitted for any other degree or professional qualification.

December 5, 2022        Guopeng Li
Abstract

In this thesis, we study the convergence problem for the intermediate long wave equation (ILW) from deterministic and statistical viewpoints. ILW models the internal wave propagation of the interface in a two-layer fluid of finite depth, providing a natural connection between the Korteweg-de Vries equation (KdV) in the shallow-water limit and the Benjamin-Ono equation (BO) in the deep-water limit.

In the first part of this thesis, we discuss the convergence problem for ILW in the low regularity setting from a deterministic viewpoint. In particular, by establishing a uniform (in depth) a priori bound, we show that a solution to ILW converges to that to KdV (and to BO) in the shallow-water limit (and the deep-water limit, respectively). The main writing establishes the first convergence result of ILW in the periodic setting. The resolution of the deterministic convergence problem required an intricate harmonic analytic approach, particularly the Fourier restriction norm method. Moreover, the argument works well on the real line and for the analytic nonlinearity.

In the second part of this thesis, we discuss an analogous convergence result from a statistical viewpoint. More precisely, we study convergence of invariant Gibbs dynamics for ILW in the shallow-water and deep-water limits. After a brief review of the construction of the Gibbs measure for ILW, we show that the Gibbs measures for ILW converge in total variation to that for BO in the deep-water limit, while in the shallow-water limit, we can only show weak convergence of corresponding Gibbs measures for ILW to that for KdV. In terms of dynamics, we use a compactness argument to construct invariant Gibbs dynamics for ILW (without uniqueness) and show that they converge to invariant Gibbs dynamics for KdV and BO in the shallow-water and deep-water limits, respectively. Moreover, our results hold for defocusing measures (i.e., we consider the power type nonlinearity $u^k$, for $k \in 2\mathbb{N} + 1$).
Lay Summary

The intermediate long wave (ILW) equation is an interesting physical model. It appears in various oceanic and atmospheric sciences. From a mathematical viewpoint, ILW is an example of a nonlinear dispersive partial differential equation (PDE), which is used to describe the propagation of a long, weakly nonlinear internal wave in a stratified medium (e.g. two-layer flows) of finite total depth.

Broadly speaking, ILW characterise internal waves (such as gravity waves) so that waves are spread out spatially as time evolves. Precisely speaking, ILW is a typical dispersive PDE, where its dispersion nature is depending on the fluid depths. In this thesis, we aim to study the different ILW dynamics due to its varying fluid depths. Then, as the ultimate goal, we study essential convergence questions of ILW dynamics as the fluid depth goes to infinity (deep-water limit) and zero (shallow-water limit), by mathematical approaches. Moreover, we emphasise that the study of limiting behaviours of ILW are physical important, for which ILW form a natural connection between the Korteweg-de Vries equation (KdV) in the shallow-water region, and the Benjamin-Ono equation (BO) in the deep-water region. This thesis is comprised of two parts:

Part I: deterministic theory

1. We start with the uniform (in $\delta$) local well-posedness for ILW. We hope to find the roughest set of initial data for which we can positively construct solutions. In the construction of ILW solution, its dispersion effects vary according to the different fluid depths. Therefore, we must further deduce that the solutions’ existence time is independent of the depth parameter. 2. We could proceed with the limiting argument and verify the ILW solutions converge to the anticipated objects - solutions to the BO in the deep-water limit; and solutions to the KdV in the shallow-water limit.

Part II: probabilistic theory

1. Rather than looking at a single initial data, we start with a family of data which are contained in support of a probability measure. We first construct the Gibbs measure (probability measure) under the ILW flow. 2. Then, we show the convergence of such probability measure of ILW, to the probability measure of BO and KdV, in the deep-water and shallow-water limits. 3. Next, we construct invariant Gibbs dynamics for ILW associated with the above probability measures. 4. Finally, we show that this invariant Gibbs dynamics for ILW converge to those of BO in the deep-water limit and to those of KdV in the shallow-water limit respectively.
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Chapter 1

Introduction

In this thesis, we are interested in the rigorous mathematical study of the limiting behaviour of the intermediate long wave equation (ILW), which is an equation that has captured much interest in both mathematics and physics. ILW is a one-way propagation asymptotic model describing internal wave travels in an appropriate fluid regime. As an example, ILW describes waves travelling in a stratified fluid of finite depth. In particular, see Figure 1.1 below, in the two-layer system (upper layer of $d_1$ and lower layer of $d_2$) of two inviscid incompressible fluids of different densities $\rho_1 < \rho_2$, the amplitude along the interface is characterised by ILW.

![Figure 1.1: Two-layer system](image)

The top boundary of the fluid is now located at $z = 0$, and the function $u(t, x)$ describes the elevation of the internal wave. Therefore, the pycnocline depth is located around $u(t, x)$ and shown as the dotted line in Figure 1.1 which is the cline or fluid layer where the density gradient is greatest within a body of fluids. The interface separates the upper and lower fluids, hindering vertical transport.

The derivation of ILW in the context of internal waves in a two-layer system was originally done by Joseph [72] and Kubota-Ko-Dobbs [92]; see also [2] [109]. There are two ways to rigorously justify ILW: in the sense of consistency [83]; and through the Hamiltonian approach [25] (see...
Consider the ILW equation on the real line $\mathcal{M} = \mathbb{R}$ and on the circle $\mathcal{M} = \mathbb{T}$:

$$\partial_t u - G_\delta \partial_x^2 u = \partial_x (u^2), \quad (t, x) \in \mathbb{R} \times \mathcal{M}. \quad (1.0.1)$$

Again, equation (1.0.1) models the internal wave propagation of the interface in a stratified fluid of finite depth $\delta > 0$, and the unknown $u : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ denotes the amplitude of the internal wave at the interface. See also Remark 1.0.2. The operator $G_\delta$ is defined as the Fourier multiplier

$$G_\delta = -\coth(\delta \partial_x) - \frac{1}{\delta} \partial_x^{-1}, \quad \overline{G_\delta f}(\xi) := -i \left( \coth(\delta \xi) - \frac{1}{\delta \xi} \right) \hat{f}(\xi). \quad (1.0.2)$$

Here, $\coth(\cdot)$ denotes the usual hyperbolic cotangent function:

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}, \quad x \in \mathbb{R} \setminus \{0\};$$

with the convention $\coth(\delta \xi) - \frac{1}{\delta \xi} = 0$ for $\xi = 0$. The frequency $\xi \in \widehat{\mathcal{M}}$ and the frequency domain is defined by

$$\widehat{\mathcal{M}} = \begin{cases} \mathbb{R}, & \text{if } \mathcal{M} = \mathbb{R}, \\ \mathbb{Z}, & \text{if } \mathcal{M} = \mathbb{T}. \end{cases}$$

Physics study indicates ILW is an important physical model, providing a natural connection between the deep-water region (= the Benjamin-Ono regime) and the shallow-water region (= the Korteweg-de Vries equation regime). See also Remark 1.0.1. This leads to one of the most important properties of ILW so that the Korteweg-de Vries equation (KdV) is shallow water limit; and the Benjamin-Ono equation (BO) is the deep water limit. See subsection 1.1.1 for the detailed discussion.

We overview some physical literature where display ILW has played a key role in the context of long internal gravity waves [88, 92, 146, 152]. Also, it is viewed as the crucial model for wave propagation in localised regions of waveguides, such as, ocean pycnocline, atmospheric fronts or regions of temperature inversion [98, 99]. Furthermore, in modelling nonlinear waves propagating on two opposite edges of a quantum Hall system, we refer to [14]. To read about further applications of ILW, we shall refer the interested readers to [37, 104, 110, 127, 144]. Alongside physical applications and its convergence features, ILW displays other interesting features, such as the $N$-soliton solutions, Hamiltonian structure, complete integrability, etc.. See for example [40, 73, 91, 102].

Main interest of this thesis:

- The convergence of ILW to its limiting forms in the shallow and deep water regions.

We recall that the depth parameter of ILW [1.0.1] characterises the dispersion phenomenon of the internal wave propagation. To be specific, by shallow (or deep) region, we are referring to those water regions whose total fluid depth is much smaller (or larger) than the wavelength. The differencing depth parameter $\delta$ leads to the physical observation that the internal waves (characterised by $\delta$) act as an intermediary between waves in shallow-water and deep water regimes. An important consequence is that not only ILW reduce to KdV and BO formally (i.e., at the equation level), it was also shown that the exact stationary wave solution to ILW converges to those solitons of KdV (as $\delta \to 0$) and BO (as $\delta \to \infty$), respectively [3, 72, 109]. For readers’ convenience, we present the stationary wave convergence results of ILW in the following (see [3, Chapter 3]). ILW soliton has the following form:

$$u_{\text{ILW}}(t, x) = \frac{k_1 \sin(\delta k_1)}{\cos(\delta k_1) + \cosh(k_1 x - t\{k_1^2 - \frac{1}{3} k_2^2 \cot(\delta k_1)\}) + k_2^2}.$$  

where $k_1, k_2$ are arbitrary parameters. Then, $u_{\text{ILW}}$ converges to the KdV soliton:

$$u_{\text{KdV}}(t, x) := \frac{k_2^2}{2} \text{sech}^2 \frac{1}{2} (k_1 x - \frac{1}{3} k_2^2 t + k_2),$$
as $\delta \to 0$. Here, a restriction, $0 < k_1 \delta < \pi$ is required, and it is doable as $\delta$ can be taken arbitrarily small. On the other hand, in the case of limit $\delta \to \infty$, we observe that there is no proper limit unless we send $k_1 \to 0$. Then, by taking $\delta k_1 = \pi - \frac{k_1}{2}$ with a positive real constant $C$ (of order 1) yields the BO rational soliton:

$$u_{\text{BO}}(t, x) := \frac{2C}{1 - C^2(x - Ct)^2}.$$ 

This convergence of ILW solitary solutions, together with the convergence of ILW at the equation level give evidence that one can regard the dynamics properties of KdV and BO as the limiting behaviours of ILW, as $\delta \to 0$ and $\delta \to \infty$, respectively.

Back to the physical literature, such convergence problem has attracted much attention around the 1980s [3, 72, 91, 92, 96, 102, 146, 151]. In particular, physicists have tried to understand the relationship between the internal wave propagation and the varying fluid depths. However, the justifications of convergence in the above literature is rather formal (not less general) from the mathematical viewpoint, i.e. the convergence of ILW was shown at the equation level, under specific numerical simulations, by physical experimentation, and the convergence of the solitary waves was just a special form of solution.

Most recently, as such, convergence problems for ILW have been studied extensively from both the applied and theoretical points of view. We refer readers to a recent book [83, Chapter 3] by Klein and Saut for an overview of the subject and the references therein. See also a survey [147]. These two references indicate that the rigorous mathematical study of ILW is still widely open. In particular, one of the fundamental, but challenging questions is the convergence properties of ILW in the deep-water limit (as the depth parameter $\delta$ tending to $\infty$) and in the shallow-water limit (as $\delta \to 0$).

In this thesis, we continue to study the convergence of ILW from mathematical viewpoints. We construct such convergence results from both deterministic and probabilistic viewpoints, and the thesis is devoted to two parts:

**Part I:** deterministic theory (microscopic convergence): we prove the convergence at a single trajectory of ILW solution for a given initial datum.

**Part II:** probabilistic theory (macroscopic convergence): we consider the initial data sampled from some statistical ensemble and then study the convergence properties of the resulting dynamics.

**Remark 1.0.1.** We remark that ILW is a special case of the Whitham equation, which reads the following:

$$\partial_t u + \partial_x \int_{-\infty}^{\infty} K(x-y)u(y,t)dy + \partial_x(u^2) = 0. \tag{1.0.3}$$

Indeed, one can easily see the connection between the Whitham equation and ILW by viewing the ILW operator $G_\delta$ as an integral form (on $\mathbb{R}$):

$$G_\delta f(x) = -\frac{1}{2\delta} \text{p.v.} \int_{\mathbb{R}} \left[ \coth \left( \frac{\pi(x-y)}{2\delta} \right) - \text{sgn}(x-y) \right] f(y)dy, \tag{1.0.4}$$

which is a certain choice of the kernel $K(x-y)$ in (1.0.3). Moreover, the Whitham equation (1.0.3) was developed to understand the breaking of nonlinear dispersive water waves; see for example [167]. In the very first work of Joseph [72], which also observed that (1.0.1) is a special form of (1.0.3), by using the dispersion relation derived in [142].

Moreover, equation (1.0.3) is convenient for seeing the connections between BO, ILW, and KdV, by picking a suitable choice of the Kernel accordingly. All those three equations arise in the propagation of long internal waves in a stratified fluid. See for example [167, Section 6].

**Remark 1.0.2.** In [92], the equation for the motion of the internal wave in a finite-depth fluid was derived with two depth parameters $d_j$, $j = 1, 2$, where $d_1$ and $d_2$ represent the depths of the upper and lower fluids, respectively, and is given by

$$\partial_t u - c_1 G_{d_1} \partial_x^2 u - c_2 G_{d_2} \partial_x^2 u = \partial_x(u^2). \tag{1.0.5}$$
See (25a)-(25b) and (35a)-(35b) in [92]. In [92, VI Summary], the authors proposed a special case when the internal wave is located halfway between the upper and lower fluid boundaries, namely, \( d_1 = d_2 \). In this case, by setting \( \delta = d_1 + d_2 = 2d_1 \), the equation (1.0.5) reduces to the ILW equation (1.0.1) (up to some inessential multiplicative constants). Namely, the physical ILW model with two parameters degenerates into one parameter equation. We also point out that by taking \( d_1 \to 0 \) while keeping \( d_2 \) fixed (or by taking \( d_2 \to 0 \) while keeping \( d_1 \) fixed), we also see that the equation (1.0.5) reduces to the ILW equation (1.0.1).

**Remark 1.0.3.** This thesis focuses on the (deterministic and probabilistic) PDE viewpoints. The rigorous inverse scattering theory for the ILW equation is also of interest. In [146] where it was shown that when \( \delta \to \infty \), the inverse scattering transform scheme of ILW reduces to that of BO. This work provides additional evidence for the convergence properties of ILW to BO.

### 1.1 The intermediate long wave equation

Consider the Cauchy problem of ILW on \( \mathcal{M} = \mathbb{R} \) or \( T \), with the initial condition:

\[
\begin{align*}
\frac{\partial_t u - G_\delta(\partial_x^2 u)}{\partial_x(u^2)} &= 0, \\
u|_{t=0} &= u_0,
\end{align*}
\tag{1.1.1}
\]

where \( \delta > 0 \) is the depth parameter, and \( u \) is a real-valued function. Also, we recall operator \( G_\delta \) from (1.0.4) and (1.0.2):

\[
\bar{G}_\delta \hat{f}(\xi) := -i \left( \coth(\delta \xi) - \frac{1}{\delta \xi} \right) \hat{f}(\xi)
\quad \text{for} \quad \xi \in \hat{\mathcal{M}}.
\tag{1.1.2}
\]

We acknowledge that one of the central motivations for studying the convergence properties of ILW is that physicists have observed that ILW plays the intermediary role between KdV and BO. In what follows, we give a careful analysis (at the equation level) of the correct ILW formulation, which converges to BO in the deep-water limit and KdV in the shallow-water limit. Moreover, we also discuss the results for the generalisation of ILW, in the sense that one can replace quadratic nonlinearity with some other generalised nonlinear terms.

#### 1.1.1 On the limiting formulations

The correct mathematical ILW model in showing the shallow-water and deep-water limits was discussed in [1, 3]. In [1], authors applied a necessary rescaling to ILW (1.1.1) so that the meaningful shallow-water limit was successfully seen as \( \delta \to 0 \). On the other side, [3] introduced a factor \((1 + \frac{1}{\delta})\), see equation (1.1.16) below, which is convenient to see the limiting equations under the same formulation (1.1.16). See further discussion on Remark 1.1.2. Moreover, both [1, 3] pointed out that to keep the meaningful shallow-water limits, we need to magnify the fluid motion by a factor \( \sim \frac{1}{\delta} \); see detailed discussions in the shallow-water limit below.

The following approach we follow from [1], and we notice that two formulations of ILW (1.1.1) and scaled ILW (1.1.14) are reversible via a scaled amplitude (1.1.13). With a slight abuse of notation, we set

\[
\bar{G}_\delta(\xi) = -i \left( \coth(\delta \xi) - \frac{1}{\delta \xi} \right).
\tag{1.1.3}
\]

- **Deep-water limit** \( (\delta \to \infty) \)

  In this case, an elementary computation shows that

  \[
  \lim_{\delta \to \infty} \bar{G}_\delta(\xi) = -i \text{sgn}(\xi)
  \quad \text{for} \quad \xi \in \hat{\mathcal{M}}.
  \tag{1.1.4}
  \]
for any $\xi \in \mathbb{R}$. Indeed, defining $q_\delta(\xi)$ by
\[ q_\delta(\xi) = \frac{1}{\delta} - \xi \coth(\delta \xi) + |\xi|, \quad (1.1.5) \]
one may easily verify that
\[ 0 \leq q_\delta(\xi) = q_\delta(-\xi) \leq \frac{2}{\delta}, \quad (1.1.6) \]
for any $\xi \in \mathbb{R}$; see Lemma 3.1.1 below or Lemma 4.1 in [1]. The limit (1.1.4) indicates that, in the deep-water limit, namely, as $\delta \to \infty$, the ILW equation (1.1.1) converges to the following BO on $\mathcal{M}$:
\[ \partial_t u - \mathcal{H}(\partial_x^2 u) = \partial_x (u^2), \quad (1.1.7) \]
where $\mathcal{H}$ is the spatial Hilbert transform defined by
\[ \mathcal{H}f(n) = -i \text{sgn}(n) b u(n). \]
Formally speaking, by recasting ILW (1.1.1) as
\[ \partial_t u - \mathcal{H}(\partial_x^2 u) + Q_\delta \partial_x u = \partial_x (u^2), \quad (1.1.8) \]
where $Q_\delta = (\mathcal{H} - G_\delta) \partial_x$ is defined as a Fourier multiplier operator with symbol $q_\delta$ in (1.1.5). Then, the bound (1.1.6) shows that $Q_\delta \partial_x$ tends to 0 in a suitable sense, thus yielding the formal convergence of (1.1.8) (and hence of (1.1.1)) to BO (1.1.7) as $\delta \to \infty$. In proving rigorous convergence, one indeed we needs to show that $Q_\delta \partial_x$ tends to 0 in a suitable sense, and thus, in view of the bound (1.1.6), it indicates that in the deep-water regime $\delta \gg 1$, long waves (with relatively small frequencies $|n| \ll \delta$) “well approximate” long waves of infinitely deep water ($\delta = \infty$).

- **Shallow-water limit** ($\delta \to 0$).
  A direct computation shows that, for $\xi \in \mathbb{R}\setminus\{0\}$, we have
  \[ \mathcal{G}_\delta \partial_x^2 u(\xi) = i \left( \coth(\delta \xi) - \frac{1}{\delta \xi} \right) \xi^2 \tilde{u}(\xi) = i \frac{\delta}{3} \xi^3 \tilde{u}(\xi) + o(1), \quad (1.1.9) \]
as $\delta \to 0$. The identity (1.1.9) follows from the following identity with $x = \delta n^2$
\[ \coth(x) - \frac{1}{x} = \frac{x(2^{2x} - 1) - (2^{2x} - 2x - 1)}{x(2^{2x} - 1)} = \frac{x}{3} + o(1), \quad (1.1.10) \]
as $x \to 0$, which can be verified by using the Taylor expansion: $2^{2x} = 1 + 2x + \sum_{k=2}^{\infty} (2x)^k/k!$.

Indeed, we see the power series of $\coth(x)$ is such that
\[ \coth(x) = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{45} + O(x^7), \quad (1.1.11) \]
where $0 < |x| < \pi$. Then, by using (1.1.11), one can consider the operator $\mathcal{G}_\delta$ as a formal power series in $\delta$, which is valid since $\delta$ can be taken arbitrarily small. Therefore, we obtain the following expression:
\[ \mathcal{G}_\delta = \coth(\delta \partial_x) - \frac{1}{\delta} \partial_x^{-1} = \frac{e^{\delta \partial_x} + e^{-\delta \partial_x}}{e^{\delta \partial_x} - e^{-\delta \partial_x}} - \frac{1}{\delta} \partial_x^{-1} = \frac{\delta \partial_x^3}{3} + \frac{\delta^3 \partial_x^5}{45} + O(\delta^5), \quad (1.1.12) \]
which (on the frequency space) agrees to (1.1.9) as $\delta \to 0$.\footnote{While it is not needed in the mean-zero case, we may set $q_\delta(0) = 0$ by continuity.} \footnote{The limiting behavior (1.1.10) also follows from the Taylor expansion of the hyperbolic cotangent function.}
The identity (1.1.9) shows that, the dispersion in (1.1.1) disappears as \( \delta \to 0\), formally yielding the inviscid Burgers equation in the limit. In order to circumvent this issue, we introduce the following scaling transformation for each \( \delta > 0\): \[ v(t, x) = 3\delta^{-1}u(3\delta^{-1}t, x), \] which leads to the following scaled ILW:

\[ \partial_t v - \frac{3}{\delta} G_3 \partial_x^2 v = \partial_x(v^2). \] (1.1.14)

Namely, \( v \) is a solution to the scaled ILW (1.1.14) (with the scaled initial data) if and only if \( u \) is a solution to the original ILW (1.1.1). In view of (1.1.9), the scaled ILW (1.1.14) formally converges to the following KdV equation on \( M \):

\[ \partial_t v + \partial_x^3 v = \partial_x(v^2). \] (1.1.15)

From the physical point of view, the scaling transformation (1.1.13) is a very natural operation to perform. The ILW equation (1.1.1) describes the motion of the fluid interface in a stratified fluid of depth \( \delta > 0\), where \( u \) denotes the amplitude of the internal wave at the interface. As \( \delta \to 0\), the entire fluid depth tends to 0 and, in particular, the amplitude of the internal wave at the interface is \( O(\delta) \), which also tends to 0, in the physical model. Hence, if we want to observe any meaningful limiting behaviour, we need to magnify the fluid motion by a factor \( \sim \frac{1}{\delta} \), which is exactly what the scaling transformation (1.1.13) does. We also point out that studying the convergence problem for the scaled ILW (1.1.14) with \( O(1) \) initial data means that we are indeed studying the original ILW (1.1.1) with \( O(\delta) \) initial data, which is consistent with the physical viewpoint.

Therefore, we conclude from the above deep-water and shallow-water limits discussion that the two formulations are both correct models for internal wave evolution. In the BO-regime \((1 \lesssim \delta \lesssim \infty)\), ILW gives the deep-water limit; in the KdV-regime \((0 \leq \delta \leq 1)\), scaled ILW gives the shallow-water limit.

Remark 1.1.1. There is also a slightly different formulation for the ILW equation; see [3, p. 211]. In this formulation, the ILW equation on \( M \) reads as

\[ \partial_t u - \left(1 + \frac{1}{\delta}\right) G_3 \partial_x^2 u = \partial_x(u^2), \] (1.1.16)

which was introduced for considerations that obscure the difference between the scaling that leads to KdV and BO. In taking \( \delta \to \infty\), we formally have

\[ \partial_t u - G_3(\partial_x^2 u) = \partial_x(u^2) + O(\delta^{-1}), \] (1.1.17)

which indicates that the same convergence result holds for this version (1.1.16) of ILW in the deep-water limit. On the other hand, in the shallow-water regime, in view of (1.1.9), the equation (1.1.16) can be formally written as

\[ \partial_t u - \frac{1}{\delta} G_3(\partial_x^2 u) = \partial_x(u^2) + O(\delta), \] (1.1.18)

which indicates convergence of (1.1.16) to the following KdV:

\[ \partial_t u + \frac{1}{3} \partial_x^3 u = \partial_x(u^2) \] (1.1.19)

without any scaling transformation. Indeed, in the shallow-water limit, a slight modification of our argument shows that an analogue of our main probabilistic result holds for the version (1.1.16) converging to KdV (1.1.19) in the shallow-water limit (i.e. unifies the two limits simultaneously). As for deterministic convergence, \( O(\delta^{-1}) \) include the second derivative loss, and when we consider this term as the nonlinear perturbation, it is unclear if we can handle this term. On the one hand,
the formulation (1.1.16) may seem to be a convenient model since it does not require a scaling transformation in the shallow-water limit. On the other hand, it does not seem to reflect the physical behaviour in the shallow-water regime (where the entire depth and thus the amplitude \( u \) are \( O(\delta) \)).

**Remark 1.1.2.** It is of interest to note from (1.1.6) that the same behaviour occurs in both the short-wave limit (high frequency) and the infinite-depth limit (arbitrary large \( \delta \)). This indicates that waves with relatively short wavelengths compared to the total depth in the water of finite depth behave similarly to long waves of infinitely deep water. On the other hand, long waves, along with the fluid interface in finite-depth water, behave similarly to those on the shallow-water surface.

### 1.1.2 Generalised intermediate long wave equation

The nonlinear perturbation of ILW (1.1.1) is quadratic nonlinearity (often referred to as weakly nonlinearity), which causes the steepening behaviour of one side travelling wave. In particular, the wave characterised by ILW will eventually break unless the steepening (nonlinearity) is counteracted by smoothing (dispersive) effects. While if the dispersive effects are too strong in dominating the propagation, then those waves trend to spread out as they propagate. Hence, we can expect an intermediate state when the nonlinear effects are exactly balanced by the dispersive effects. Those waves carrying such balance features are well-known as solitary waves (the shape-preserving feature of nonlinear waves). Solitary wave solutions have an important physical meaning, which has been studied for the KdV and BO models; see [10, 11, 126]. Such balance feature waves also appear in the intermediate fluid regime. In particular, the existence of a certain class of nonlinear waves of permanent form has been well studied since the 1970s, see [92]. In fact, the convergence problem of ILW dynamics was first observed for solitary wave solutions [72].

We are also interested in studying the convergence problems for the ILW equation with stronger nonlinear steepening effects or some other more generalised nonlinearity. In particular, we study the following equation posed on \( \mathcal{M} \), and it is so-called the generalised intermediate long wave equation (gILW):

\[
\begin{align*}
\partial_t u - G_\delta(\partial^2_x u) &= \partial_x(f(u)), \\
u|_{t=0} &= u_0, \\
(t, x) &\in \mathbb{R} \times \mathcal{M}. \quad (1.1.20)
\end{align*}
\]

Here, the nonlinearity \( f : \mathbb{R} \to \mathbb{R} \) is a real analytic function with an infinite radius of convergence. In particular, this covers all the power-type nonlinearities. See Remark [1.1.3] for more details. When \( f(u) = u^k \) with \( k = 2 \) we cover the ILW equation; \( k = 3 \) is known as modified ILW (mILW); and \( k \geq 4 \) the (usual) gILW.

The derivations of the limiting equations with a different nonlinearity can be done in the same line of subsection 1.1.1; see [63, 66] for the power-type nonlinearity. For completeness, we present the limiting equations in the following.

- **Deep-water limit (\( \delta \to \infty \))**

  The gILW equation (1.1.20) converges to the generalised BO (gBO):

  \[
  \partial_t u - \mathcal{H}(\partial^2_x u) = \partial_x(f(u)), \quad (1.1.21)
  \]

  as \( \delta \to \infty \). When \( f(u) = u^k \) with \( k = 2 \) we cover the BO limit, \( k = 3 \) is known as modified BO (mBO), and \( k \geq 4 \) the (usual) gBO.

- **Shallow-water limit (\( \delta \to 0 \))**

  Let \( v \) be some suitable rescaling amplitude.\(^4\) Then, the scaled gILW

  \[
  \partial_t v - \frac{3}{\delta} G_\delta(\partial^2_x v) = \partial_x(f(v)) \quad (1.1.22)
  \]

  \(^4\text{For example, let } f(u) = u^k. \text{ Then, we take } v(t, x) = (\frac{1}{\delta})^{\frac{1}{k-1}} u(\frac{3}{\delta} t, x). \)
converges to the following generalised KdV (gKdV):
\[
\partial_t v + \partial_x^3 v = \partial_x (f(v)).
\] (1.1.23)

Note that for \( f(v) = v^k \), \( k = 2 \) corresponds to the KdV, \( k = 3 \) is known as modified KdV (mKdV), and for \( k \geq 4 \) the (usual) gKdV.

**Remark 1.1.3.** The hypothesis on \( f \) ensures that \( f \) is of \( C^\infty \) class. In particular,
\[
f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad \forall x \in \mathbb{R}.
\] (1.1.24)

It is clear that any polynomial function is of this form. Moreover, the exponential functions as \( e^x \), \( \sin(x) \), \( \cos(x) \), and their products or compositions are also in this class.

### 1.2 Microscopic limits: deterministic theory

In this section, we overview our deterministic approach. The convergence question of ILW arises naturally from the physical research, see [72, 92, 109], which presented the convergence of solitary-wave solutions to ILW and numerically investigated the convergence of ILW dynamics. At the level of the equations, subsection 1.1.1 showed that BO (deep-water limit, \( \delta \to \infty \)) and KdV (shallow-water limit, \( \delta \to 0 \)) are regarded as the limiting forms of ILW. However, these are not enough to deduce the limiting behaviour of ILW solution for an arbitrary given initial state. In particular, without a priori information on the ILW solutions, such as the existence or how the fluid motions of ILW evolve along with the fluid depth, we cannot pursue the convergence problem of ILW. To answer such a question, the first mathematical study was done by Abdelouhab-Bona-Felland-Saut [1] (1989). They studied the convergence of ILW solutions by using PDE techniques:

- They first showed the well-posedness theory, which guarantees the existence of solutions for a given initial data. Moreover, the existence time must be \( \delta \)-independent.

- Then, they showed the ILW solution is a Cauchy sequence with respect to \( \delta \). Furthermore, they showed that for the same initial data, ILW solution converges to solutions of BO (deep-water limit, \( \delta \to \infty \)) and KdV (shallow-water limit, after scaling, and \( \delta \to 0 \)), respectively.

We will talk about more of [1] in the next subsection 1.2.1 of the literature review. Most recently, Guo-Wang and Han-Wang [63, 66] revisited this type of convergence problem, where they study ILW associated with a higher-order power-type nonlinearity (gILW (1.1.20), and \( f(u) = u^k \) for \( k = 3 \) or \( k \geq 5 \)), see also subsection 1.2.1. These methods in [63, 66] showed that the modern techniques from dispersive PDEs should be widely applicable here. To this end, one of the primary purposes of this thesis is to bring methods developed in the dispersive PDE community and investigate the low regularity convergence problem of ILW dynamics. The physical space we are particularly interested in is the circle \( \mathbb{T} \), and we remark here that the argument applies equally well on the line \( \mathbb{R} \). Moreover, as a byproduct, our method covers the convergence of solutions to gILW (1.1.20).

Our deterministic approach is relatively straightforward (in general, this methodology applies to gILW (1.1.20)): first of all, the discussions of subsection 1.1.1 indicate that ILW (1.1.1) captures the deep-water fluid motions; after a scaling, scaled ILW (1.1.14) captures shallow-water evolution. In practice, we analyse the dispersive effects of internal waves generated at different depths. Then we identify fluid regimes for which dispersion phenomena become most likely to be KdV or BO-like. i.e., we encode the leading order dynamics of ILW solution to have features of BO (in the deep-water regime) and KdV (in the shallow-water regime). Such separation of the fluid regime is a crucial observation in studying the convergence of ILW dynamics. In general, ILW solutions are \( \delta \)-dependent (i.e., the fluid motion depends on the fluid depths and the initial state). However, after we identify the BO-regime and KdV-regime, it appears to be possible to deduce the existence of ILW solutions (\( \delta \)-dependent) are actually uniformly bounded in \( \delta \) (in some sense, it...
is stable in $\delta$); see Propositions 2.3.9 and 2.3.11. In the end, we show that after passing $\delta$ to the limits, those ILW solutions satisfy the limit equations regarded as BO (deep-water limit) and KdV (shallow-water limit), for the same initial condition.

### 1.2.1 Literature review

**The ILW equation convergence:**

The mathematical study of the convergence problem of ILW only appeared in [1]. In [1], Abdelouhab-Bona-Felland-Saut proved the convergence of solutions to ILW:

- (i) Let $s > \frac{3}{2}$ and solutions in $H^s(M)$. Then, the solutions of ILW converge to the solutions of BO (for the same initial data), as $\delta \to \infty$.

- (ii) Let $s \geq 2$ and solutions in $H^s(M)$. Then, the solutions of ILW converge to the solutions of KdV (for the same initial data), as $\delta \to 0$.

We first notice the regularity of the function spaces for these two convergence results is very high, which is due to their method not exploiting any dispersive nature of the equation. Secondly, we notice the regularities are different between the shallow water limit and the infinitely deep water limit. Roughly speaking, after defining the existence of solutions to ILW, they would need to extract the $\delta$-independent estimates for solutions to ILW. Then, with a careful analysis of the $\delta$-independence, they hope to pass the limit on the solutions to ILW, as $\delta \to 0$ and $\delta \to \infty$, respectively. Indeed, in the case of deep-water limit (as $\delta \to \infty$), the $\delta$-independent estimate was obtained automatically from well-posedness theory [1, Theorem 6.1], which concerns ILW (1.1.1) only, and gives that solutions exist in $H^s(M)$ for $s > \frac{3}{2}$. Therefore, they showed that solutions to ILW (1.1.1) converge to those solutions of BO (1.1.7).

On the other hand, they explained that the way they extracted the $\delta$-independent estimates of solutions to ILW (1.1.1) ([1, Theorem 6.1]) does not directly apply to the scaled ILW (1.1.14). Although equations (1.1.14) and (1.1.1) are invariant under some suitable scaling, the $\delta$-uniformity breaks down under this scaling. In particular, they explained that [1, Theorem 6.1] only works for extracting the $\delta$-independent estimates for $\delta$ large. Therefore, the $\delta$-independent estimates for solutions to the scaled ILW (1.1.14) need an independent argument. This was shown by using the infinite many conservation laws of ILW. Hence, by applying the higher-order conserved quantities, the $\delta$-independent estimate for $\delta$ small was obtained with solutions in $H^s(M)$ for $s \geq 2$. For readers’ convenience, we include the convergence of ILW briefly in the following table 1.1; for the details we refer to [1].

**The modified ILW equation convergence:**

In the case $f(u) = u^3$ of (1.1.20), the equation is known as the modified ILW equation (mILW). The convergence of the solutions to mILW was (not for the mKdV-limit) studied by Guo-Wang [63], where they showed that solutions of mILW converges to those of mBO in $H^s(\mathbb{R})$ for $s \geq \frac{3}{2}$, as $\delta \to \infty$. Similar issues occur in the shallow-water region (for $\delta$ small), meaning that the $\delta$-independent estimates cannot be obtained in the same regularity space. Hence, the authors of [63] did not include the convergence of the solutions to the scaled mILW in the shallow water limit.

We remark that the work of [63] showed that modern techniques from dispersive PDEs could be used to push down the regularity thresholds for the convergence problem of the ILW-type equation (the convergence of the mILW in rougher $L^2$-based Sobolev spaces). For readers’ convenience, we include the convergence of mILW briefly in the following table 1.1 for the details we refer to [1].

**The generalised ILW equation convergence:**

In the case of gILW (1.1.20) where $f(u) = u^{k+1}$ for $k \geq 4$, this convergence problem was studied by Han-Wang [66], where they considered the Besov-type spaces on the real line. We shall refer the interested readers to [66].

**Summary table:**

In the following, we summarise the convergence results for ILW, mILW, and gILW. Moreover, for comparison reasons, we include our main deterministic results from Theorems 1.2.3 and 1.2.4.
We make a few comments on Table 1.1:

(i) One of the main goals of this thesis is to treat the low regularity convergence problem of the ILW solution on the circle $\mathbb{T}$. It appears in the literature that from PDE viewpoint, the only ILW convergence problem (with high regularity) on $\mathbb{T}$ was [1]. The other works [63, 66] related to ILW-type equations were only on $\mathbb{R}$.

(ii) In [66], the authors considered gILW (1.1.20) with $f(u) = u^{k+1}$ for $k \geq 4$. The convergence result of the quartic gILW (i.e. $f(u) = u^4$) seems to be missing there. Moreover, the function space in establishing the convergence of solutions in [66] was Besov spaces, we do not go into further details. In this thesis, we are interested in the convergence problem of the ILW-type equation in $L^2$-based Sobolev spaces.

(iii) It is worth emphasising that in [1], the KdV-limit required a higher regularity assumption (unknown for mKdV-limit in [63]).

### On the well-posedness theory:

The well-posedness theory is the first step to getting the convergence results of the solutions to the ILW-type equations. In particular, we need to construct the solution so that its local existence time is independent of the depth parameter $\delta$. The approach to show the uniform in $\delta$ well-posedness theory in our later analysis is closely related to the works of (i) Molinet-Vento [120] (so-called the improved energy method); (ii) and Molinet-Tanaka [119] (by combining the improved energy method with the short-time Strichartz estimates [90]). The arguments of [120, 119] are suitable for dealing with varying dispersion effects. Moreover, [119] study the generalised dispersion effects associated with analytic nonlinearity. Indeed, the arguments of [120, 119] implies that the ILW-type equation (equations (1.1.1) and (1.1.20) with a fixed depth parameter $\delta$) is well-posed in the $L^2$-based Sobolev spaces. See the discussion in the appendix B.

The BO equation (deep-water limit) and the KdV equation (shallow-water limit) are two of the most important dispersive models in the theoretical study. In the appendix B, we decide to include extensive literature on the well-posedness theory for the BO-type equation (in the deep-water limit) and the KdV-type equation (in the shallow-water limit). It also shows the big picture of the recent developments in the dispersive PDEs community. Moreover, the low regularity well-posedness problem of the ILW-type equations (with a fixed depth parameter $\delta$) is widely open; it is rather challenging due to the non-algebraic structure of the dispersion term and is interesting to study in this direction. See also the following Remark 1.2.1.

---

Table 1.1: Summary table

<table>
<thead>
<tr>
<th>Nonlinearity</th>
<th>Space</th>
<th>Limit $\delta \to 0$</th>
<th>Limit $\delta \to \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u^2$</td>
<td>$u \in H^s(\mathcal{M})$</td>
<td>$s \geq 2$ (II)</td>
<td>$s &gt; \frac{3}{4}$ (II)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$s \geq \frac{3}{4}$ (Thm 1.2.4)</td>
<td>$s \geq \frac{3}{4}$ (Thm 1.2.3)</td>
</tr>
<tr>
<td>$u^3$</td>
<td>$u \in H^s(\mathcal{M})$</td>
<td>$s \geq \frac{3}{4}$ (Thm 1.2.4)</td>
<td>$s \geq \frac{1}{2}$ on $\mathbb{R}$ [63]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$s \geq \frac{3}{4}$ (Thm 1.2.3)</td>
<td></td>
</tr>
<tr>
<td>$u^4$</td>
<td>$u \in H^s(\mathcal{M})$</td>
<td>$s \geq \frac{3}{4}$ (Thm 1.2.4)</td>
<td>$s \geq \frac{3}{4}$ (Thm 1.2.3)</td>
</tr>
<tr>
<td>$u^{k+1}$ ($k \geq 4$)</td>
<td>$u \in \dot{B}^{s_k}_{2,1}(\mathbb{R})$</td>
<td>$s_k = \frac{7}{4} - \frac{2}{k}$ [66].</td>
<td>$s_k = \frac{3}{2} - \frac{1}{k}$: [66].</td>
</tr>
<tr>
<td></td>
<td>$u \in H^s(\mathcal{M})$</td>
<td>$s \geq \frac{3}{4}$ (Thm 1.2.4)</td>
<td>$s \geq \frac{3}{4}$ (Thm 1.2.3)</td>
</tr>
</tbody>
</table>

---

In [66], the initial data lies on the Besov-type spaces, and their convergence was taken in some other spaces. We only discuss their results here for the completeness of the table.
Remark 1.2.1. (i) Over the last few years, there have been breakthrough results on the well-posedness of completely integrable PDEs: (i) KdV by Killip-Visan [87]; (ii) and BO by Gérard-Kappeler-Topalov [56]. The ILW equation is known to be completely integrable. Hence, it is of interest to exploit the completely integrable structure of ILW to prove the low regularity well-posedness theory of ILW. We note that the integrable structure of ILW has rarely been studied.

1.2.2 On the convergence of solutions

In the present deterministic section, we are interested in the low regularity convergence of ILW-type equation at the single trajectory. We show that the (scaled) gILW solution emanates from each fixed initial data converges to the gBO solution in the deep-water limit, and to the gKdV solution (after scaling) in the shallow-water limit. This study is motivated by the following:

1. From the physical study, the solitary wave convergent only takes one particular form of the ILW solutions. Hence, it is natural to study the convergence of (scaled) ILW dynamics at a single trajectory.

2. We carry on the deterministic convergence result of (scaled) ILW by Abdelouhab-Bona-Felland-Saut [1] and extend the convergence of (scaled) ILW solution in the low-regularity setting.

3. From a mathematical point of view, it is an interesting and challenging question to study the convergence of (scaled) gILW solutions, in particular due to the higher-order nonlinearity and lack of local smoothing estimates of the periodic data.

4. We continue the studies of Guo-Wang, and Han-Wang [66, 63], and then naturally show the convergence of (scaled) gILW in the periodic setting.

Our strategy for establishing convergence of (scaled) gILW consists of two steps. For simplicity in the explanation, we shall focus on discussing the deep-water limit in the following, where we treat the original gILW (1.1.20) (rather than the scaled gILW (1.1.22) relevant in the shallow-water limit), unless we need to make a specific point in the shallow-water limit.

• Step 1: Construction on the δ-independent solution.

For each finite δ > 0, we first construct a solution $u_δ$ for gILW (1.1.20), for given initial data $u_0$. The energy method plays an essential role in the construction part. More precisely, we implement the so-called improved energy method [120] and the short-time Strichartz estimates [90]. Indeed, a direct application of the argument in [119], implies that for each fixed $0 \leq \delta \leq \infty$ ($\delta = \infty$ correspond to BO and $\delta = 0$ correspond to KdV), the gILW equation is locally well-posed.

Next, we show the local existence time $T$ is independent of the depth parameter $\delta$. To overcome this issue, we need to use the gILW structure and show the uniform (in $\delta$) estimates. By following [119], we then show that gILW is locally well-posed uniformly with respect to the parameter $\delta \gg 1$. We emphasise that the uniformity in $\delta$ is essential in passing to the limit.

• Step 2: Convergence of the (scaled) gILW dynamics at the single trajectories.

The convergence step essentially follows from showing that gILW solutions $\{u_\delta\}_{\delta \geq 1}$ is a Cauchy sequence in $C([0, T]; H^s(T))$. In doing so, we strongly rely on the structure of the gILW equation as well as the uniform difference estimates. In particular, we observe that the linear dispersion of (generalised) ILW behaves like (generalised) BO, uniformly for any $2 \leq \delta \leq \infty$. See Lemma 2.1.3. Moreover, in applying the difference estimate - Proposition 2.3.11, for different $2 \leq \delta, \gamma \leq \infty$, we will need to place one solution, say $u_\delta$ into $M^{s, \gamma}$-norm, and the difference $u_\delta - u_\gamma$ will end up in $M^{s-1, \gamma}$-norm, where we refer to (2.1.17) for the definition of function spaces. Therefore, for different $2 \leq \delta, \gamma \leq \infty$, the norm can be infinity. We can overcome this difficulty by using the equation structure. See further discussion shortly.

Now, let us state the following result, which is a direct consequence from [119].

---

6The convergence results in [1] were obtained in $H^s$ for some $s > \frac{3}{2}$. In our study, our goal is to go (at least) blow $s < 1$.

7Alternatively, we can also directly take the difference between (scaled) gILW and gBO (or gKdV ), since we already know the limiting object at the equation level.
Lemma 1.2.2. Let $0 < \delta \leq \infty$ and $s \geq \frac{3}{4}$. Then, the Cauchy problem associated with (scaled) $gILW$ is locally well-posed in $H^s(M)$. The maximal time of existence $T = T(\|u_0\|_{H^s(M)}, \delta) > 0$ depends on the initial data and the parameter $\delta$.

We remark on the locally well-posedness of the KdV-limit case ($\delta = 0$), the regularity is $s \geq \frac{3}{4}$. However, for any $\delta > 0$ that solution lives in $C([0, T]; H^s(M))$ for $s \geq \frac{3}{4}$ (from Lemma 1.2.2). Due to the restriction of the methods we are applying here, we cannot improve the regularity threshold for the convergence results. The reason is that even in the shallow-water regime ($0 < \delta < 1$), the KdV-like behaviour only occurs when $0 < \delta |n| \lesssim 1$. Namely, we have the leading order of the dispersion relation to scaled $gILW$ is

$$p_3^{(\delta)}(n) \sim |n|^3 \quad \text{if} \quad 0 < \delta |n| \lesssim 1.$$ 

Therefore, BO-like behaviour will dominate on the high-frequencies ($|n| \gg \frac{1}{\delta}$).

As in Sections 1.1.1 we separate our analysis into: (i) deep-water theory; (ii) shallow-water theory. For convenience, we use $D_{T,\delta}$ and $S_{T,BO} = S_{T,\infty}$ to denote the solution maps for the $gILW$ and $gBO$, respectively. For example, both $D_{T,\delta}$ and $S_{T,\infty}$ were constructed in [119]. Now we state our main results:

Theorem 1.2.3 (Deep-water theory). Let $u_0 \in H^s(M)$ for $s \geq \frac{3}{4}$, where $M = \mathbb{R}$ or $\mathbb{T}$.

(i) (uniform local well-posedness). Then, for any $0 < T < 1$, the solution map $D_{T,\delta}$ satisfies that for all $2 \leq \delta \leq \infty$

$$\|D_{T,\delta}u_0\|_{C([0,T];H^s(M))} \leq C(T, \|u_0\|_{H^s(M)}).$$

(1.2.1)

The solution map $D_{T,\delta}: u_0 \to u_\delta$ is continuous from $H^s(M)$ to $C([0,T];H^s(M))$, uniformly on $\delta \in [2, \infty]$. Moreover, the local existence time $T = T(\|u_0\|_{H^s(M)}) > 0$ is independent of $\delta$.

(ii) (deep-water limit of the solution). Let any local existence time $T$ obtained in Part (i), and solutions

$$D_{T,\delta}u_0 = u_\delta \quad \text{for gILW (1.1.20)} \quad \text{and} \quad S_{T,\infty}u_0 = u_\infty \quad \text{for gBO (1.1.21)}.$$ 

Then, we have

$$\lim_{\delta \to \infty} \|u_\delta - u_\infty\|_{C([0,T];H^s(M))} = 0.$$ 

(1.2.2)

Theorem 1.2.3 (and Theorem 1.2.4) established the first low regularity convergence on tori $\mathbb{T}$ for the (generalised) ILW equation to the (generalised) BO equation (deep-water limit, as $\delta \to \infty$) (or to the (generalised) KdV (shallow-water limit, as $\delta \to 0$)). Given a parameter-dependent solution $u_\delta$, it is of physical interest to study the convergence of the single trajectory, which generalised the convergence of the solitary wave solution. Theorem 1.2.3 (and Theorem 1.2.4) is the first such result for the (generalised) ILW equation, in which we implement the Fourier restriction norm method [18, 19], energy method [20], and short-time Strichartz estimate [90].

We also mention a series of recent development on the well-posedness of the ILW-type equations [120, 116, 119]. See these papers for the references therein. While these works establish only the well-posedness theory, we are interested in the convergence of the corresponding dynamics.

For each fixed $1 \leq \delta \leq \infty$, the construction of the $gILW$ solution follows from [119]; in the limiting cases, where $\delta = \infty$ corresponds to $gBO$. Moreover, to establish the uniform local well-posedness theory for all $1 \leq \delta \leq \infty$ (Theorem 1.2.3 (i)), we observe that the dispersion relation of $gILW$ (1.1.20) in the deep-water regime ($1 \leq \delta \leq \infty$) satisfies

$$p_3^{\text{deep}}(n) \sim |n|^2 \quad \text{for} \quad n \in \mathbb{Z},$$

which is uniformly in $\delta$. Then, a direct application of the argument introduced in [119] yields Theorem 1.2.3 (i).

The main purpose is Part (ii) of Theorem 1.2.3. In order to prove the convergence of $u_\delta$ in the deep-water limit, we need to show the sequence of solutions $\{u_\delta\}_{\delta \geq 1}$ is Cauchy, as $\delta \to \infty$. One
The solution map \( S \) satisfies that
\[
\|u_\gamma\|_{M_T^s} \leq C(T)\|u_\gamma\|_{M_T^{s,\delta}},
\]
see Lemma 2.5.2 for details. We point out that this uniform bound on the different parameters in \( \delta \geq 1 \) also plays an important role in the difference estimates, which we refer to Lemma 2.5.3.

Another key ingredient in establishing convergence of the deep-water limit is the uniform local well-posedness. Therefore, it seems that there is no hope we can improve the regularity even after the scaling.

Next, we consider the scaled gILW (1.1.22). Again, for each fixed \( 0 \leq \delta < 1 \), the construction of the scaled gILW solution follows from [119]; in the limiting cases, where \( \delta = 0 \) corresponds to gKdV. In the shallow-water regime, we use \( S_{T,\delta} \) and \( S_{T,\text{KdV}} \) to denote the solution maps for the scaled gILW and gKdV, respectively. Both \( S_{T,\delta} \) and \( S_{T,\text{KdV}} \) were constructed in [119].

**Theorem 1.2.4** (Shallow-water theory). Let \( v_0 \in H^s(M) \) for \( s \geq \frac{3}{4} \), where \( M = \mathbb{R} \) or \( \mathbb{T} \).

(i) (uniform local well-posedness). Then, for any \( 0 < T < 1 \), the solution map \( S_{T,\delta} \) satisfies that for all \( 0 < \delta \leq 1 \)
\[
\|S_{T,\delta}v_0\|_{C([0,T];H^s(M))} \leq C(T, \|v_0\|_{H^s(M)}).
\]
\[
\text{(1.2.3)}
\]

The solution map \( S_{T,\delta} : v_0 \rightarrow v_\delta \) is continuous from \( H^s(M) \) to \( C([0,T];H^s(M)) \), uniformly on \( \delta \in (0,1) \). Moreover, the local existence time \( T = T(\|v_0\|_{H^s(M)}) > 0 \) is independent of \( \delta \).

(ii) (shallow-water limit). Let any local existence time \( T \) obtained in Part (i), and solutions
\[
S_{T,\delta}v_0 = v_\delta \text{ for scaled gILW (1.1.22)} \quad \text{and} \quad S_{T,\text{KdV}}v_0 = v_{\text{KdV}} \text{ for gKdV (1.1.23)}.
\]

Then, we have
\[
\lim_{\delta \to 0} \|v_\delta - v_{\text{KdV}}\|_{C([0,T];H^s(M))} = 0.
\]
\[
\text{(1.2.4)}
\]

Here, in the shallow-water limit, although the limiting equation (gKdV) is locally well-posed in \( H^s \) for \( s \geq \frac{3}{4} \) (see [119]), while for fixed \( \delta > 0 \), scaled gILW is locally well-posed in \( H^s \) for \( s \geq \frac{3}{4} \). In particular, on the high-frequency regime (when \( |n| \gtrsim \frac{1}{\delta} \)), the linear dispersion is dominated by \( |n|^2 \):
\[
p_{\delta}^{(s)}(n) \sim \begin{cases} 
\delta |n|^2 & \text{if } n \gtrsim \frac{1}{\delta}; \\
|n|^3 & \text{if } n \ll \frac{1}{\delta}.
\end{cases}
\]

Therefore, it seems that there is no hope we can improve the regularity even after the scaling. Then, again we can apply the argument introduced in [119] to obtain Theorem 1.2.4 (i).

As compared to the deep-water limit (\( \delta \to \infty \)) studied in Theorem 1.2.4, we observe an interesting phenomenon in the shallow-water limit (\( \delta \to 0 \)). For different parameters, \( \gamma \neq \delta \), if we want to get a bound uniform in \( n \). Namely, see the discussion for getting (2.5.18):
\[
\langle \tau - p_{\delta}^{(s)}(n) \rangle \lesssim \langle \tau - p_{\gamma}^{(s)}(n) \rangle + \langle n \rangle^3,
\]
the difference of the symbols for dispersion is now \( O(n^3) \). In this case, we need to introduce a frequency cutoff on the initial data and consider the smooth solutions \( v_{\delta,N} \) and \( v_{\text{KdV},N} \). Since solutions
\[
v_{\delta,N}, v_{\text{KdV},N} \in C([0,T];C^\infty(\mathbb{T}))
\]
we then yield the estimate in Lemma 2.5.6. Here, the essential point is that the local existence time \( T \) depends only on the \( H^s(\mathbb{T}) \)-norm of the initial data \( u_0 \) and the bound on the \( H^{s+3} \)-norm.
In terms of writing, the main analysis is written on \( T^{1.35} \). The potential energy part, \( E \), where \( S \) follows the conversation laws. The gILW equation \( (1.1.20) \) can be written in the following Hamiltonian formulation:

\[
\partial_t u = \partial_x \frac{dE_\delta(u)}{du},
\]

where \( E_\delta(u) \) is the Hamiltonian (= energy) given by

\[
E_\delta(u) = \frac{1}{2} \int_\mathcal{M} u G_\delta \partial_x u dx + \int_\mathcal{M} F(u) dx,
\]

where the potential energy part, \( F(x) \) is defined by \( F(x) := \int_0^x f(y) dy \). For example, when \( f(u) = u^k \). Then, the potential energy is \( \frac{1}{k+1} \int_\mathcal{M} u^{k+1} dx \). In particular, \( E_\delta(u) \) is conserved under the dynamics of \( (1.1.20) \). Moreover, it is easy to check that the following two quantities are

\[\text{Estimates on } R^N \text{ follow the same type of argument (with a simpler decomposition) we can establish the energy short-time Strichartz estimates for the real-line case. See Lemmas 2.6.2 for details. Then, by Remark 1.2.5, our approach applies well to both } gBO \text{ is locally well-posed in } L^s H^r \text{ such that Theorems 1.2.3 and 1.2.4 improve most of the previous convergence results works in [1] 66 63 8]. Especially, for } f(u) = u^2 \text{ (or } f(v) = v^4) \text{ Theorems 1.2.3 and 1.2.4 improve the convergence results of } f \text{ for (scaled) ILW solution. Moreover, the convergence results for the quartic gILW equation is first studied here, which is covered by letting } f(u) = u^4. \]

Remark 1.2.5. Our solutions are understood as distributional solutions. In particular, for the gILW formulated in \( (1.1.20) \) for any test function \( \phi \in C_c^\infty((-T, T) \times M) \), the following holds

\[
\int_0^\infty \int_M \left[ \phi_x + G_\delta \phi_{xx} \right] u + \phi_x f(u) \right] dx dt + \int_M \phi(0, \cdot) u_0 dx = 0. \tag{1.2.5}
\]

Then, we note that for \( u \in L^\infty((0, T); H^r(M)) \) with \( s > \frac{1}{2} \), \( f(u) \) is well-defined and belongs to \( L^\infty((0, T); H^r(M)) \). Moreover, \( \|u_0\|_{H^s} \leq \|u\|_{L^\infty H^s} \). Finally, we notice that this also ensures that \( u \) satisfies the following Duhamel formulation:

\[
u(t) = S(t)u_0 - i \int_0^t S(t-t') \left[ \partial_x (f(u)) \right](t') dt', \tag{1.2.6}
\]

where \( S(t) := e^{-it \partial_x^2} \) denotes the linear propagator. See also [119, 141] for a similar discussion. Moreover, such observation here is essential in our later unconditional uniqueness part; see Lemma 2.3.7 where we use the fact that the solution satisfies Duhamel formulation (1.2.6). Indeed, we can verify that under our assumption, the solution satisfies Duhamel formulation from [158 Proposition 1.35].

Remark 1.2.6. Theorem 1.2.4 gives an alternative approach to Molinet-Tanaka [119] such that gBO is locally well-posed in \( H^s(M) \) for \( s \geq \frac{1}{2} \).

Remark 1.2.7. (i) Our approach applies well to both \( \mathbb{R} \) and \( T \). As a comparison, the arguments in [63, 66] strongly rely on the local smoothing property, which is not available on the torus. Moreover, our argument handles both the power-type of nonlinearity and trigonometric and exponential nonlinearities.

(ii) In terms of writing, the main analysis is written on \( T \). We shall only provide the crucial short-time Strichartz estimates for the real-line case. See Lemmas 2.6.2 for details. Then, by following the same type of argument (with a simpler decomposition) we can establish the energy estimates on \( \mathbb{R} \). Moreover, the convergence on \( \mathbb{R} \) follows the same type of argument.

• On the global convergence.

We briefly discuss how one can extend our convergence to hold globally-in-time. The essential point follows from the conversation laws. The gILW equation equation \( (1.1.20) \) can be written in the following Hamiltonian formulation:

\[
\partial_t u = \partial_x \frac{dE_\delta(u)}{du},
\]

where \( E_\delta(u) \) is the Hamiltonian (= energy) given by

\[
E_\delta(u) = \frac{1}{2} \int_\mathcal{M} u G_\delta \partial_x u dx + \int_\mathcal{M} F(u) dx,
\]

where the potential energy part, \( F(x) \) is defined by \( F(x) := \int_0^x f(y) dy \). For example, when \( f(u) = u^k \). Then, the potential energy is \( \frac{1}{k+1} \int_\mathcal{M} u^{k+1} dx \). In particular, \( E_\delta(u) \) is conserved under the dynamics of \( (1.1.20) \). Moreover, it is easy to check that the following two quantities are

\[\text{At this moment, we are not able to improve the results [63] in the case of cubic nonlinearity for } \delta \to \infty. \text{ However, we are able to establish convergence results as } \delta \to 0, \text{ which is beyond the scope of the work [63].} \]
then the convergence of solutions can be extended globally-in-time as $\delta$ is bounded uniformly in $L^2$-level of conservation law $M(u)$, such as the $L^2$-conservation: $M(u) = \int u^2 dx$, and the Hamiltonian. For any fixed $\delta$, the Hamiltonian defined in $L^2$-level of conservation law $M(u)$ is at the $H^{\frac{1}{2}}$-level. However, we can easily see the kinetic energy part of $\delta$ is bounded uniformly in $\delta$, for $\delta$ small, by the kinetic energy part of gKdV. Moreover, the Hamiltonian associated with equation (1.1.20) (for large $\delta$) enjoys $H^{\frac{1}{2}}$-level of conservation law uniformly in $\delta$, for large $\delta$. Due to the kinetic energy part of the Hamiltonian associated with equation (1.1.20) is bounded uniformly in $\delta$ by the kinetic energy part of gBO.

Hence, gathering these conservation laws with the above LWP results, let us assume $s \geq 1$, and then we are able to extend the solution constructed in Theorem 1.2.4 for any $T > 0$. This is analogous to [119] Theorems 1.2 & 1.3, so we omit details here and refer interested readers to [119].

We see from the Theorem 1.2.3 that the regularity threshold is $s \geq \frac{1}{2}$. However, according to the Hamiltonian associated with equation (1.1.20), we can only obtain the $H^{\frac{1}{2}}$-level of conservation law in the deep-water region (for $\delta$ large). Therefore, the argument proposed in [119] (in proving GWP) cannot be applied to extend the Theorem 1.2.3 for any $T > 0$. However, if we restrict $f(u) = u^2$, then it is well-known that ILW (1.1.1) enjoys infinite many conservation laws; see for example [102] [151]. In particular, we can use the following $H^1$-type quantity (see p. 368 of [1]):

$$I_2(u) := \int_M \left( \frac{1}{4} u^4 + \frac{3}{2} u^2 T_\delta(u_x) + \frac{1}{2} u_x^2 + \frac{3}{2} \{T_\delta(u_x)_x\}^2 + \frac{1}{\delta} \left( \frac{3}{2} u^3 + \frac{9}{2} u T_\delta(u_x) \right) + \frac{3}{2 \delta^2} u^2 \right) dx,$$

where $T_\delta u(\xi) = -i \coth(2\pi \delta \xi) \tilde{f}(\xi)$. Therefore, this $H^1$-invariant quantity extends Theorem 1.2.3 globally-in-time.
Remark 1.2.10. In [1], the $\delta$-independent estimates for solutions to the equation \([1.1.1]\) were obtained automatically from [1] Theorem 6.1. For the convergence part, with the $\delta$-independent estimates they easily showed the Cauchy property of the family of solutions \(\{v^\delta\}_{\delta \geq 1}\) to \([1.1.1]\). On the other hand, the statement [1] Theorem 6.1 does not directly apply to the formulation \([1.1.14]\). Although we know that \([1.1.14]\) and \([1.1.1]\) are invariant under some suitable scaling, the $\delta$-uniformity breaks down under this scaling. As a result, $\delta$-independent a priori estimate for solution $v_\delta$ to \([1.1.14]\) was established independently. In particular, see [1] Lemma 8.2.2 and it was done through the infinitely many conservation laws of ILW, and we emphasise that an extra regularity assumption has been made on this lemma, which holds for $v_0 \in H^s(\mathbb{R})$ with $s \geq 2$.

In fact, for the gILW case of [66], the uniform (in $\delta$) estimates were also established separately for small and large $\delta$. Moreover, in the study of the gKdV-limit, they also needed a high regularity assumption to handle the derivatives after applying the integration by parts.

Remark 1.2.11. In [63], they could not show the limiting behaviour of shallow-water regions. However, they conjectured that the convergence problem should still be true due to the results in [1]. In their approach [63], they considered (unscaled) mILW \([1.1.20]\). Dispersion effects, in this case characterise the BO-like behaviour uniformly (only if $\delta$ is large). In other words, for small $\delta$, then $\delta$-independent estimates will break down. That is under the same regularity assumption as in [63], solutions cannot exist simultaneously in both the shallow-water and the deep-water regions. In our approach here, we take advantage of a suitable scaling such that we study only the limit behaviour as $\delta \to 0$ to scaled gILW \([1.1.22]\).

Remark 1.2.12. For simplicity of the presentation, we impose the mean-zero condition on the initial condition $u_0$, namely, \(\int_T u_0 dx = 0\).

This assumption can be justified as follows (we note that on $\mathbb{R}$ this idea fails): Let us consider ILW \([1.1.1]\) with $u(x,0) = u_0$. If we integrate the equation in space, we obtain $\partial_t \int_T u(x,t) dx = 0$ and thus \(\int_T u(x,t) dx = \int_T u_0(x) dx\).

Now set \(v(x,t) = u(x-ct,t) - c\) and observe that if $u$ solves ILW with initial data $u(x,0) = u_0$, then $v$ solves \(\partial_t v + 2cv_x - G_\delta(\partial_x^2 v) = \partial_x (v^2)\), with $v(x,0) = u_0(x) - c$. Since if we integrate in time, we still have that $\partial_t \int_T v(x,t) dx = 0$. Then, we conclude that \(\int_T v(x,t) dx = \int_T v_0(x) dx = \int_T u_0(x) dx - 2\pi c\).

But now we can pick the constant $c$ in such a way that $v$ has a mean zero, \(\int_T v(x,t) dx = 0\).

We notice that $v$ doesn’t solve the original ILW anymore but the methods we are developing apply to the new equation step by step. The only difference is that now the multiplier of the linear group is $\sim k^2 - 2ck$ instead of $\sim k^2$. Notice that in all calculations that follow this replacement nothing changes.

Remark 1.2.13. The proofs of Theorems [1.2.3] and [1.2.4] share the same argument. The main strategy of our crucial nonlinear estimates is to use symmetry arguments. In practice, we distribute the derivative loss from the nonlinearity to several functions, and then to handle the derivative loss by applying either improved Strichartz estimates or $X^{s,b}$-type estimates. To decide in which way to recover the derivative loss depends on whether the nonlinear interactions are resonant or non-resonant. Moreover, since the equation for the difference between two solutions enjoys fewer symmetries, this difference will be estimated in a space with lower regularity than that of the
solution itself, and we will use the frequency envelope approach to recover the continuity result with respect to initial data.

1.3 Macroscopic limits: probabilistic theory

In this section, we go over our probabilistic approach. The results are based on joint work with T. Oh and G. Zheng [101].

General background and motivations

The physical dynamics of the microscopic structure are often enormously complex, and any measurement of microscopic quantities is subject to statistical fluctuations, i.e., any small error in the experiment can lead to a very different outcome. The macroscopic behaviour, however, can be described by means of a few parameters such as temperature and pressure. Such macroscopic measurements lead to apparently deterministic results. This contrast between the microscopic and the macroscopic level is the starting point of Classical Statistical Mechanics as developed by Maxwell, Boltzmann, and Gibbs. Roughly speaking, the microscopic complexity can be overcome by a statistical approach, in the sense that the macroscopic determinism then may be regarded as a consequence of a suitable law of large numbers; see [67].

According to the above philosophy, it is not adequate to describe the behaviour of the dynamics by a particular element. Hence, rather than looking into the analytic properties of a single solution, the dynamics should be described by the macrostate properties of viewing a family of microscopic solutions as an ensemble. This is a brand new idea in studying the convergence behaviours of ILW (and gILW) dynamics. The probabilistic approach here is based on the study of the Gibbs measure associated with ILW dynamics, which is a probability measure frequently seen in many problems of probability theory and statistical mechanics. The notion of a Gibbs measure dates back to R.L. Dobrushin (1968-1970) and O.E. Lanford and D. Ruelle (1969) who proposed it as a natural mathematical description of an equilibrium state of a physical system which consists of a very large number of interacting components. In particular, the equilibrium state of a physical system with the Hamiltonian system is described by the probability measure:

\[ \mu(\omega) = \frac{1}{Z} \exp(-\beta H(\omega))d\omega \]  

(1.3.1)

on the probability space \( \Omega \). In this expression, the notation \( d\omega \) refers to a suitable a priori measure on \( \Omega \), \( H(\omega) \) is a function from the space of states to the real numbers; in physical applications, \( H(\omega) \) is interpreted as the Hamiltonian of the configuration \( \omega \), and \( \beta \) is a positive, free parameter; in Physics, is proportional to the inverse of the absolute temperature. Moreover, \( Z > 0 \) is a normalising constant. The above \( \mu \) is called the Gibbs distribution or Gibbs measure relative to the Hamiltonian \( H \).

A Gibbs measure is a mathematical idealisation of an equilibrium state of a physical system which consists of an extensive number of interacting components. From the statistical mechanics viewpoint, which observes thermodynamic properties of systems in terms of the statistics of ensembles of all possible physical states of a system (which is composed of many particles). The statistical ensemble is a probability distribution over all possible dynamics states. Hence, it is rather natural to study the convergence of ILW dynamics via its associated Gibbs measure.

Heuristically, it is a generalisation of the canonical ensemble to infinite systems. The canonical ensemble gives the probability of the dynamics being in state \( \omega \) (equivalently, in terms of the mathematical concept, it is a random variable having value \( \omega \)). The rigorous justification of the ansatz (1.3.1) is a long story which is also the main issue in our probabilistic approach here. This is due to the infinite-dimensional system the formula (1.3.1) may make no sense; see, for example, [67, (0.1)&(0.2)] the Hamiltonian is not well-defined in the infinite-dimensional system, thus the Gibbs measure relative to \( H \) makes no sense. To overcome this obstacle, the traditional approach in statistical physics studied the limit of intensive properties as the size of a finite system approaches infinity. This, however, turns out to be rather difficult in general. Alternatively, we might try to characterise the Gibbs measure (1.3.1) by a property which admits a direct extension to the case

\[ \mu(\omega) = \frac{1}{Z} \exp(-\beta H(\omega))d\omega \]  

(1.3.1)

9The laws of thermodynamics describe how the energy in a system changes and whether the system can perform useful work on its surroundings.
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of an infinite-dimensional system. Such a characterisation can indeed be obtained fairly easily. Moreover, it directly studies infinite systems instead of taking the limit of finite systems.

Problem setup

Our foremost step from the probabilistic approach focuses on the construction of the Gibbs measures for gILW (associated with a power-type nonlinearity) on the circle $T$:

$$\begin{cases}
\partial_t u - G_\delta \partial_x^2 u = \partial_x(u^k) \\
u_{|t=0} = u_0,
\end{cases} \quad (t, x) \in \mathbb{R} \times T, \quad (1.3.2)$$

where $k \geq 2$ is an integer. When $k = 2$, the equation (1.3.2) corresponds to ILW (1.1.1), while, when $k = 3$, it is known as the modified ILW equation (mILW). The equation (1.3.2) can be written in the following Hamiltonian formulation:

$$\partial_t u = \partial_x \frac{dE_\delta(u)}{du},$$

where $E_\delta(u) = E(u; \delta)$ is the Hamiltonian (= energy) given by

$$E_\delta(u) = \frac{1}{2} \int_T u G_\delta \partial_x u dx + \frac{1}{k+1} \int_T u^{k+1} dx. \quad (1.3.3)$$

See, for example, [1, 102] for ILW Hamiltonian. In particular, $E_\delta(u)$ is conserved under the dynamics of (1.3.2). Moreover, it is easy to check that the following two quantities are conserved under the gILW dynamics:

- mean: $\int_T u dx$ and
- mass: $M(u) = \int_T u^2 dx. \quad (1.3.4)$

The mean-zero condition on the initial condition $u_0$ will be carried out in this part as well. In other words, defining the Fourier coefficient $\hat{f}(n)$ by

$$\hat{f}(n) = \mathcal{F}(f)(n) = \int_T f(x)e^{-inx} dx,$$

we will work with real-valued functions of the form

$$f(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \hat{f}(n)e_n(x). \quad (1.3.6)$$

where $e_n(x) = e^{inx}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

Moreover, we recall from Section 1.1.1 the formal derivations of the limiting equations. In particular, we obtained the following

$$-\coth(\delta \partial_x) - \frac{1}{\delta} \partial_x^{-1} \xrightarrow{\delta \to \infty} \mathcal{H} \quad \text{and} \quad -\frac{3}{\delta}(\coth(\delta \partial_x) + \frac{1}{\delta} \partial_x^{-1}) \xrightarrow{\delta \to 0} \partial_x \quad (1.3.7)$$

We also recall the scaled gILW equation:

$$\partial_t v - \frac{3}{\delta} G_\delta \partial_x^2 v = \partial_x(v^k). \quad (1.3.8)$$

Note that the scaled gILW (1.3.8) is also a Hamiltonian PDE with the Hamiltonian:

$$E_\delta(v) = \frac{3}{2\delta} \int_T v G_\delta \partial_x v dx + \frac{1}{k+1} \int_T v^{k+1} dx, \quad (1.3.9)$$

which differs from the Hamiltonian $E(u)$ in (1.3.3) by a divergent multiplicative constant in the kinetic part (= the quadratic part) of the Hamiltonian.

\footnote{Hereafter, we may drop the harmless factor $2\pi$.}
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As mentioned above, in the deterministic setting, the convergence problem of the gILW dynamics (and the scaled gILW dynamics, respectively) to the gBO dynamics (and to the gKdV dynamics, respectively) has been studied in \cite{63,66,64}. These works studied the convergence issue from a microscopic viewpoint in the sense that convergence was established for each fixed initial data \( u_0 \) to gILW \((1.3.2)\) (or each fixed initial data \( v_0 \) to the scaled gILW \((1.3.8)\)). In the probabilistic approach, we study the convergence problem from a macroscopic viewpoint. Namely, rather than considering the limiting behaviour of a single trajectory, we study the limiting behaviour of solutions as a statistical ensemble. Such an approach is of fundamental importance in statistical mechanics, where one replaces “the study of the microscopic dynamical trajectory of an individual macroscopic system by the study of appropriate ensembles or probability measures on the phase space of the system” \cite{97}. In the present work, we, in particular, study the convergence of the dynamics at the Gibbs equilibrium for the gILW equation \((1.3.8)\) in both the deep-water and shallow-water limits. From the physical point of view, it is quite natural to study the fluid motion from the statistical viewpoint, since one is often interested in a prediction of typical behaviour of the fluid. From the theoretical point of view, it is an interesting and challenging question to study convergence of invariant Gibbs dynamics associated with the gILW equation \((1.3.2)\), in particular due to the low regularity of the support of the Gibbs measures.

Our strategy for establishing convergence of invariant Gibbs dynamics for the scaled gILW consists of the following three steps. For simplicity, we only discuss the deep-water limit in the following, where we treat the original gILW \((1.3.2)\) (rather than the scaled gILW \((1.3.8)\) relevant in the shallow-water limit), unless we need to make a specific point in the shallow-water limit.

In the following, we will restrict our attention to (i) \( k = 2 \), corresponding to ILW \((1.1.1)\), and (ii) \( k \in 2\mathbb{N} + 1 \) in \((1.3.2)\), corresponding to the defocusing case. This restriction comes from the Gibbs measure construction. See Remark \((1.3.6)\) for a discussion on the general focusing case, namely, either for (iii) even \( k \geq 4 \) or (iv) \( k \in 2\mathbb{N} + 1 \) with the focusing sign:

\[
\partial_t u - G\partial_x^2 u = -\partial_x(u^k). \tag{1.3.10}
\]

\textbullet \textit{Step 1: Construction and convergence of the Gibbs measures.}

For each finite \( \delta > 0 \), we first construct a Gibbs measure \( \rho_\delta \) for gILW \((1.3.2)\) with the Hamiltonians \( E_\delta(u) \) in \((1.3.3)\), formally written as\(^{12}\)

\[
\rho_\delta(du) = Z_\delta^{-1} e^{-E_\delta(u)} du = Z_\delta^{-1} e^{-\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1} dx} e^{-\frac{1}{2} \int_{\mathbb{T}} u G \partial_x u dx} du. \tag{1.3.11}
\]

The expression \((1.3.11)\) is merely formal and we aim to construct \( \rho_\delta \) as a weighted Gaussian measure with the base Gaussian measure given by

\[
\mu_\delta(du) = Z_\delta^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}} u G \partial_x u dx} du. \tag{1.3.12}
\]

See the next subsection for a precise definition of \( \mu_\delta \). For each \( \delta > 0 \), the Gaussian measure \( \mu_\delta \) is supported on distributions \( D'(\mathbb{T}) \) of negative regularity and thus the potential energy \( \int_{\mathbb{T}} u^{k+1} dx \) is divergent. In order to overcome this issue, we introduce a renormalization on the potential energy, just as in the construction of the \( \Phi^{k+1}_2 \)-measures \cite{151,153,149,139}. When \( k = 2 \), the potential energy is not sign-definite, causing a further problem. By following the work \cite{162}, we overcome this issue by introducing a Wick-ordered \( L^2 \)-cutoff. See Subsection \((1.3.2)\) for a further discussion.

Once the Gibbs measure \( \rho_\delta \) is constructed for each \( \delta > 0 \), we then proceed to prove convergence of the Gibbs measures \( \rho_\delta \) for gILW \((1.3.8)\) to the Gibbs measure \( \rho_{gBO} \) for gBO \((1.1.7)\) in the deep-water limit \( (\delta \to \infty) \). This step involves establishing the \( L^p \)-integrability bound on the densities, \textit{uniformly in} \( \delta \gg 1 \). We point out that, for each \( \delta \gg 1 \), the base Gaussian measure \( \mu_\delta \) is different and thus an extra care is needed in discussing what we mean by the “density”. See Section \((3.2)\) for further details.

\(^{11}\)Strictly speaking, the case (iii) even \( k \geq 4 \) is non-defocusing, not focusing. For simplicity, however, we refer to the non-defocusing case as focusing in the remaining part of the paper.

\(^{12}\)Henceforth, constants such as \( Z_\delta \) denote various normalizing constants, which may be different line by line.
In order to study the shallow-water limit, we need to consider the scaled gILW (1.3.8) with the Hamiltonian $E_\delta(v)$ in (1.3.9). This leads to the construction of the following Gibbs measure:

$$
\tilde{\rho}_\delta (dv) = Z_\delta^{-1} e^{-E_\delta(v)} dv = Z_\delta^{-1} e^{-\frac{1}{\delta} \int_v v^{k+1} dx e^{-\frac{1}{\delta} \int_v G_\delta \partial_x v^k dx} dv.}
$$

(1.3.13)

For each fixed $\delta > 0$, we construct the Gibbs measure $\tilde{\rho}_\delta$ as a weighted Gaussian measure with the base Gaussian measure $\tilde{\rho}_\delta$ given by

$$
\tilde{\rho}_\delta (dv) = Z_\delta^{-1} e^{-\frac{3}{2} \int_v \partial_x v^k} dv.
$$

(1.3.14)

The construction of the Gibbs measure $\tilde{\rho}_\delta$, $\delta > 0$, follows exactly the same lines as that for the Gibbs measure $\rho_{\delta gKdV}$ in (1.3.11). There is, however, a crucial difference in the shallow-water limit in establishing convergence of the Gibbs measures $\tilde{\rho}_\delta$, $\delta \ll 1$, for the scaled gILW (1.3.8) to the Gibbs measure $\rho_{gKdV}$ for gKdV (1.1.15). More precisely, it turns out that the Gibbs measures $\tilde{\rho}_\delta$, $\delta > 0$, for the scaled gILW (1.3.8) and $\rho_{gKdV}$ for gKdV (1.1.15) are singular and the mode of convergence of $\tilde{\rho}_\delta$ to $\rho_{gKdV}$ is weaker (than that in the deep-water limit).

This first step is one of the main novelties of the probabilistic approach, where we establish a uniform bound on the densities (with respect to the underlying probability measure $P$).

- **Step 2: Construction of invariant Gibbs dynamics for the scaled gILW.**

  In this second step, we construct dynamics for gILW (1.3.2) at the Gibbs equilibrium constructed in Step 1. This step follows the compactness argument introduced by Burq, Thomann, Tzvetkov [29] in the context of dispersive PDEs. See [5, 44] for the first instance of this argument in the context of fluid. See also [139, 134]. Due to the use of the compactness argument, the dynamics constructed in this step lacks a uniqueness statement.

- **Step 3: Convergence of the scaled gILW dynamics at the Gibbs equilibrium.**

  This last step essentially follows from the previous two steps together with the triangle inequality. In Step 2, we construct limiting Gibbs dynamics as a limit of the frequency-truncated dynamics (via the compactness argument mentioned above). In this last step, we characterize the convergence established in Step 2 in the Lévy-Prokhorov metric and conclude the desire convergence of the dynamics at the Gibbs equilibrium for the scaled gILW to that for gBO (or to gKdV) via a diagonal argument. The use of the Lévy-Prokhorov metric in this context is new as far as our knowledge is concerned.

### 1.3.1 Literature review

There have been many results on the construction of invariant Gibbs measures for Hamiltonian PDEs. In particular, we briefly mention the works on the construction of invariant Gibbs measures for KdV (mKdV, gKdV) and BO (mBO, gBO).

**Invariant Gibbs measures of gKdV family.**

Consider the following generalized KdV (gKdV) posed on the circle $T$:

$$
\partial_t v + \partial_x^2 v = \pm \partial_x (v^k).
$$

(1.3.15)

In the seminal work [20], Bourgain proved the invariance of the Gibbs measures for KdV ($k = 2$) and mKdV ($k = 3$); see also the work of Oh [129]. For the quartic gKdV ($k = 4$), it was done by Richards [145]. Moreover, it was shown in Lebowitz-Rose-Speer [97] that the non-defocusing case, i.e. either $k$ is even, or we have the $-$ sign in (1.3.15), the Gibbs measure is non-normalizable when $k > 5$, and when $k = 5$, $R \gg 1$, where $R$ denotes the size of a mass cut-off as a suitable renormalization of the measure. Lastly, we point out that the critical value $k = 5$ corresponds to the smallest power of the nonlinearity, where (1.3.15) on $\mathbb{R}$ possesses finite-time blowup solutions [105, 106]. Therefore, Oh-Richards-Thomann in [134] establish the invariance of the Gibbs measure in the following cases:

(i) defocusing gKdV: with the $+$ sign in (1.3.15) and odd $k \geq 3$;
In the case (ii) and when \( k = 5 \), the mass threshold \( R > 0 \) is taken to be sufficiently small. Furthermore, in the work [35], Chapouto-Kishimoto extend the results in [134] to the Fourier-Lebesgue setting. Moreover, we remark that in [134], the invariance property of the Gibbs measure is stated in some mild sense, and it is weaker than the invariance described in [35]. We also refer to Zhidkov’s book [170, Chapter 4] for the construction of infinitely many invariant measures for KdV that correspond to the conservation laws of the equation at different levels of Sobolev regularities.

**Invariant Gibbs measures of gBO family:**

Now, we consider the following generalized BO (gBO) on \( \mathbb{T} \):

\[
\partial_t u - H \partial_{xx}^2 u = \pm \partial_x (u^k).
\]

(1.3.16)

The construction of a Gibbs measure associated to the BO \((k = 2)\) was done by Tzvetkov [162], and the invariance of the constructed Gibbs measure was proved by Deng [46]. The Gibbs measures with log-correlated Gaussian fields were also discussed in the work [136] of Oh-Seong-Tolomeo, including the Gibbs measure associated to the BO \((k = 3)\) and the mBO \((k = 4)\). In particular,

1. focusing Gibbs measure associated to the mBO (i.e. + sign and \( k = 3 \)), and the gBO with \( k \geq 5 \) are non-normalizable;
2. in a comparison of [162], they provided an alternative proof of the construction of the Gibbs measure for the BO, see [136, Remark 1.7];
3. for the defocusing case (i.e. + in (1.3.16) and \( k \geq 3 \) is an odd integer), the construction of Gibbs measure follows from the Wiener chaos estimates (see Lemma 3.1.7) and Nelson’s estimate [123].

### 1.3.2 Construction of Gibbs measures

Consider a finite-dimensional Hamiltonian flow on \( \mathbb{R}^{2n} \):

\[
\partial_t p_j = \frac{\partial H}{\partial q_j} \quad \text{and} \quad \partial_t q_j = -\frac{\partial H}{\partial p_j}
\]

(1.3.17)

with Hamiltonian

\[
H(p, q) = H(p_1, \cdots, p_n, q_1, \cdots, q_n).
\]

The classical Liouville’s theorem states that the Lebesgue measure \( dpdq = \prod_{j=1}^{2n} dp_j dq_j \) on \( \mathbb{R}^{2n} \) is invariant under the dynamics (1.3.17). Then, together with the conservation of the Hamiltonian \( H(p, q) \), we see that the Gibbs measure \( Z^{-1}e^{-H(p,q)}dpdq \) is invariant under the dynamics of (1.3.17). By drawing an analogy, we may hope to construct invariant Gibbs dynamics for Hamiltonian PDEs. This program was initiated by the seminal works by Lebowitz, Rose, and Speer [97] and Bourgain [20, 21], leading to the construction of invariant Gibbs dynamics as well as probabilistic well-posedness. See also [52, 169, 108]. This subject has been increasingly more popular over the fifteen years; see, for example, survey papers [130, 13].

Our first main goal is to construct Gibbs measures for gILW (1.3.2) (and the scaled gILW (1.3.8)). For this purpose, let us first go over the known results in the limiting cases \( \delta = 0 \) and \( \delta = \infty \).

- **Construction of Gibbs measures for gKdV on \( \mathbb{T} \).**

This corresponds to the shallow-water limit \((\delta = 0)\) in our problem. Consider gKdV (1.1.15) posed on the circle with the Hamiltonians \( \mathcal{E}_0(u) \):

\[
\mathcal{E}_0(v) = \frac{1}{2} \int_{\mathbb{T}} (\partial_x v)^2 dx + \frac{1}{k+1} \int_{\mathbb{T}} v^{k+1} dx,
\]

(1.3.18)

which, in view of (1.1.9), is a formal limit of \( \mathcal{E}_\delta(v) \) in (1.3.9) as \( \delta \to 0 \). The Gibbs measure \( \rho_{g\text{KdV}} \)
for gKdV is formally given by
\[
\rho_{gKdV}(dv) = Z_0^{-1} e^{-\varepsilon_0(v)} dv
= Z_0^{-1} e^{-\frac{1}{4} \int_v v^{k+1} dx} e^{-\frac{1}{2} \int_v (\partial_x v)^2 dx} dv.
\] (1.3.19)

The Gibbs measure \( \rho_{gKdV} \) can be constructed as a weighted Gaussian measure with the base Gaussian measure given by the periodic Wiener measure \( \tilde{\mu}_0 \) (restricted to mean-zero functions):
\[
\mu_{gKdV}(dv) = Z_0^{-1} e^{-\frac{1}{2} \int_v (\partial_x v)^2 dx} dv.
\] (1.3.20)

More precisely, the periodic Wiener measure \( \mu_{gKdV} \) is defined as the induced probability measure under the map:
\[
\omega \in \Omega \mapsto X_{gKdV}(\omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} \frac{g_n(\omega)}{|n|} e_n,
\] (1.3.21)
where \( e_n(x) = e^{inz} \) and \( \{g_n\}_{n \in \mathbb{Z}^*} \) is a sequence of independent standard complex-valued Gaussian random variables on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) conditioned that \( g_{-n} = \bar{g}_n \), \( n \in \mathbb{Z} \). Indeed, by Plancherel’s theorem (see (1.3.5) and (1.3.6) for our convention of the Fourier transform), we have
\[
\int_{\mathbb{R}} (\partial_x v)^2 dx = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} n^2 |\hat{v}(n)|^2 = \frac{1}{\pi} \sum_{n \in \mathbb{N}} n^2 |\hat{v}(n)|^2,
\]
where the second equality follows from the fact that \( v \) is real-valued, i.e. \( \hat{v}(-n) = \bar{\hat{v}(n)} \). This shows that we formally have
\[
e^{-\frac{1}{2} \int_v (\partial_x v)^2 dx} dv \sim \prod_{n \in \mathbb{N}} e^{-\frac{1}{2} n^2 |\hat{v}(n)|^2} d\hat{v}(n) \sim \prod_{n \in \mathbb{N}} e^{-\frac{1}{4} |g_n|^2} d g_n
\sim \left( \prod_{n \in \mathbb{N}} e^{-\frac{1}{4} \text{Re} g_n^2} d \text{Re} g_n \right) \left( \prod_{n \in \mathbb{N}} e^{-\frac{1}{4} \text{Im} g_n^2} d \text{Im} g_n \right)
\] (1.3.22)
in the limiting sense with the identification \( \hat{v}(n) = \frac{g_n}{|n|} \). This shows that \( \text{Re} g_n \) and \( \text{Im} g_n \) are given by mean-zero Gaussian random variables with variance \( \pi \). Hence, \( g_n = \text{Re} g_n + i \text{Im} g_n \) has variance \( 2\pi \).

It is easy to show that the support of \( \mu_{gKdV} \) is contained \( H^{\frac{1}{2} - \varepsilon}(\mathbb{T}) \setminus H^{\frac{1}{2}}(\mathbb{T}) \) for any \( \varepsilon > 0 \). By Khintchine’s inequality, one may also show that the support of \( \tilde{\mu}_0 \) is indeed contained in \( W^{\frac{1}{2} - \varepsilon, \infty}(\mathbb{T}) \). See, for example, [12]. Hence, in the defocusing case, namely, when \( k \in 2\mathbb{N} + 1 \), the density \( e^{-\frac{1}{4} \int_v v^{k+1} dx} \) in [1.3.19] with respect to \( \tilde{\mu}_0 \) satisfies \( 0 \leq e^{-\frac{1}{4} \int_v v^{k+1} dx} \leq 1 \), almost surely, which is in particular integrable with respect to \( \tilde{\mu}_0 \). This shows that the Gibbs measure \( \rho_{gKdV} \) can be realized as a weighted \( \tilde{\mu}_0 \):
\[
\rho_{gKdV}(dv) = Z_0^{-1} e^{-\frac{1}{4} \int_v v^{k+1} dx} d\tilde{\mu}_0(v)
\] (1.3.23)
in this case.

In the focusing case, namely, when \( k \in 2\mathbb{N} \) or when the potential energy \( \frac{1}{k+1} \int_v v^{k+1} dx \) in [1.3.3] comes with the + sign, the situation is completely different, since, in this case, the density is no longer integrable with respect to the base Gaussian measure \( \tilde{\mu}_0 \). In the seminal work [97], Lebowitz, Rose, and Speer proposed to consider the Gibbs measure with an \( L^2 \)-cutoff:
\[
\rho_{gKdV}(dv) = Z_0^{-1} 1_{\{|v|^2 dx \leq K\}} e^{-\frac{1}{4} \int_v v^{k+1} dx} d\mu_{gKdV}(v)
\] (1.3.24)
for \( k \in 2\mathbb{N} \) in the non-defocusing case, and more generally in the focusing case:
\[
\rho_{gKdV}(dv) = Z_0^{-1} 1_{\{|v|^2 dx \leq K\}} e^{-\frac{1}{4} \int_v |v|^{k+1} dx} d\mu_{gKdV}(v)
\] (1.3.25)

\footnote{By convention, we assume that \( g_n \) has mean 0 and variance 2\( \pi \), \( n \in \mathbb{Z}^* \). See \( 1.3.22 \) below.}
for any real number \( k > 1 \). In [117, 20], it was shown that, when \( k < 5 \), the Gibbs measures \( \rho_{gKdV} \) in (1.3.24) and (1.3.25) can be constructed as a probability measure for any \( K > 0 \), while it is not normalizable for any cutoff size when \( k > 5 \). The situation at the critical case \( k = 5 \) (for (1.3.25)) is more subtle. Note that the critical value \( k = 5 \) corresponds to the smallest power of the nonlinearity, where the focusing gKdV (namely, (1.1.15) with the \( - \) on the nonlinearity) on the real line possesses finite-time blowup solutions [103, 109]. The Gibbs measure construction when \( k = 5 \) remained a challenging open problem for thirty years and it was completed only recently in the work [157] by Sosoe, Tolomeo, and the second author; when \( k = 5 \), the focusing Gibbs measure in (1.3.25) can be constructed if and only if the cutoff size \( K \) is less than or equal to the mass of the so-called ground state on the real line. See [137] for a further discussion on this issue.

As we see below, in the non-defocusing case, only the \( k = 2 \) case is relevant to us. In this case, the Gibbs measure for KdV relevant to us is given by\(^{14}\)

\[
\rho_{gKdV}(du) = Z_0^{-1} \chi_K \left( \int_\mathbb{T} v^2 dx - \frac{2\pi}{\sqrt{2}} \sigma_{gKdV}(v) \right) e^{-\frac{1}{2} \int_\mathbb{T} v^2 dx} d\mu_{gKdV}(v),
\]  

(1.3.26)

where \( \chi_K : \mathbb{R} \to [0, 1] \) is a continuous function such that \( \chi_K(x) = 1 \) for \( |x| \leq K \) and \( \chi_K(x) = 0 \) for \( |x| \geq 2K \).

See Theorem 1.3.4 below. Here, \( \sigma_{gKdV} \) denotes the variance of \( X_{gKdV}(x) \) in (1.3.21) given by

\[
\sigma_{gKdV} = \mathbb{E}[X_{gKdV}^2] = \frac{1}{2\pi^2} \sum_{n \in \mathbb{Z}} \frac{2\pi}{n^2} = \frac{\pi}{6},
\]  

(1.3.27)

which is independent of \( x \in \mathbb{T} \) due to the translation invariant nature of the problem.

**Construction of Gibbs measures for gBO on \( \mathbb{T} \).**

Next, we go over the (non-)construction of the Gibbs measures associated with gBO (1.1.7), which corresponds to the deep-water limit (\( \delta = \infty \)) in our problem. The Hamiltonian for gBO (1.1.7) is given by

\[
E_\infty(u) = \frac{1}{2} \int_\mathbb{T} u \mathcal{H}u dx + \frac{1}{k+1} \int_\mathbb{T} u^{k+1} dx,
\]  

which, in view of (1.3.7), is a formal limit of \( E_\delta(u) \) in (1.3.3) as \( \delta \to \infty \). Here, \( \mathcal{H} \) denotes the Hilbert transform. Then, the Gibbs measure \( \rho_{gBO} \) for gBO is formally given by

\[
\rho_{gBO}(du) = Z_\infty^{-1} e^{-E_\infty(u)} du
= Z_\infty^{-1} e^{-\frac{1}{\pi} \int_\mathbb{T} \frac{1}{k+1} u^{k+1} dx} e^{-\frac{1}{2} \int_\mathbb{T} u \mathcal{H}u dx} du.
\]

As in the KdV case, we first introduce the base Gaussian measure \( \mu_{gBO} \) by

\[
\mu_{gBO}(du) = Z_\infty^{-1} e^{-\frac{1}{2} \int_\mathbb{T} u \mathcal{H}u dx} du.
\]  

(1.3.28)

More precisely, the Gaussian measure \( \mu_{gBO} \) is defined as the induced probability measure under the map:

\[
\omega \in \Omega \longmapsto X_{gBO}(\omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{|n|^2} e_n,
\]  

(1.3.29)

where \( \{g_n\}_{n \in \mathbb{Z}} \) is as in (1.3.21). In this case, the support of \( \mu_{gBO} \) is contained in \( H^{-\epsilon}(\mathbb{T}) \setminus L^2(\mathbb{T}) \) for any \( \epsilon > 0 \); see (3.2.5) below. Namely, a typical element \( u \) in the support of \( \mu_{gBO} \) is merely a

\(^{14}\)From a PDE point of view, this criticality corresponds to the so-called \( L^2 \)-criticality (or mass-criticality), while, from the viewpoint of mathematical physics, this criticality corresponds to the phase transitions for (non-)normalizability of the focusing Gibbs measure. Here, the phases transitions are two-fold: normalizability for \( k < 5 \) and non-normalizability for \( k \geq 5 \). Also, when \( k = 5 \), normalizability below or at the critical mass and non-normalizability above the critical mass.

\(^{15}\)Hereafter, we use a continuous cutoff function \( \chi_K \) as in [162].
distribution and thus the potential energy is divergent in this case.

Let us first consider the defocusing case \( k \in 2\mathbb{N} + 1 \). Noting that the Gaussian measure \( g_{BO} \) is logarithmically correlated, by introducing a Wick renormalized power \( \mathcal{W}(u^{k+1}) \) (see (1.3.30) below), Nelson’s estimate allows us to define the Gibbs measure \( g_{BO} \):

\[
\rho_{gBO}(du) = Z^{-1}_\infty e^{-\int_T \mathcal{W}(u^{k+1})dx} d\mu_{gBO}(u)
\]
as a limit of the frequency-truncated version, just as in the construction of the \( \Phi_{\frac{k+1}{2}} \)-measure \[135, 139\]; see Theorem 1.3.2 below. See Subsection 3.1.2 for a precise definition of the Wick power \( \mathcal{W}(u^{k+1}) \).

Let us now turn to the focusing case. When \( k = 2 \), Tzvetkov \[162\] constructed the Gibbs measure for the Benjamin-Ono equation (BO) by introducing a Wick-ordered \( L^2 \)-cutoff:

\[
\rho_{gBO}(du) = Z^{-1}_\infty \chi_k \left( \int_T \mathcal{W}(u^2)dx \right) e^{-\frac{1}{2} \int_T W(u)dx} d\mu_{gBO}(u).
\]  

(1.3.30)

See \[136\] for an alternative, concise proof. Note that under the mean-zero assumption, there is no need to introduce a renormalization in this case. See Remark 1.3.3.

In \[136\], Seong, Tolomeo, and the second author showed that the Gibbs measure for the focusing modified BO (with \( k = 3 \)):

\[
\rho_{gBO}(du) = Z^{-1}_\infty \chi_K \left( \int_T \mathcal{W}(u^3)dx \right) e^{-\frac{1}{3} \int_T W(u)dx} d\mu_{gBO}(u)
\]  

(1.3.31)
is not normalizable. Their argument can also be adapted to show that the focusing Gibbs measure is not normalizable for any \( k \geq 3 \). We mention the work \[24\] by Brydges and Slade on a similar non-normalizability result (but with a completely different proof) in the context of the focusing \( \Phi_{\frac{k}{2}} \)-measure. See also Remark 1.3.6 below.

Lastly, we point out that, due to the use of the Wick renormalization, we can only consider integer values for \( k \) in this case (\( \delta = \infty \)) and also in the intermediate case \( 0 < \delta < \infty \) which we will discuss next.

**Construction of Gibbs measures for gILW on \( \mathbb{T} \).**

We finally discuss the construction of the Gibbs measure for the scaled gILW. Let us first consider the unscaled gILW \[1.3.2\] with the Hamiltonian \( E_\delta(u) \) in \[1.3.3\]. Fix \( 0 < \delta < \infty \). Our first goal is to construct the Gibbs measure \( \rho_\delta \) of the form \[1.3.11\]. Let \( \mu_\delta \) be the base Gaussian measure of the form \[1.3.12\], which is nothing but the induced probability measure under the map:

\[
\omega \in \Omega \longrightarrow X_\delta(\omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} \frac{g_n(\omega)}{|K_\delta(n)|} e_n.
\]  

(1.3.32)

Here, \( \{g_n\}_{n \in \mathbb{Z}} \) is as in \[1.3.21\] and \( K_\delta(n) \) is given by

\[
K_\delta(n) := in \hat{\varphi}_\delta(n) = n \coth(\delta n) - \frac{1}{\delta}
\]  

(1.3.33)

with \( \hat{\varphi}_\delta(n) \) as in \[1.3.7\]. For each \( n \in \mathbb{Z}^* \), we have \( K_\delta(n) > 0 \) and moreover, it follows from \[1.1.3\] and \[1.1.6\] that

\[
K_\delta(n) = |n| + O(\frac{1}{\delta})
\]  

(1.3.34)

See also Lemma 3.1.1 This asymptotics allows us to show that, for any given \( 0 < \delta < \infty \), the Gaussian measures \( \mu_\delta \) in \[1.3.12\] and \( \rho_{gBO} \) in \[1.3.28\] are equivalent. See Proposition 3.2.1. In particular, as in the \( \delta = \infty \) case, the Gaussian measure \( \mu_\delta \) is supported on \( H^{-\frac{1}{2}}(\mathbb{T}) \backslash L^2(\mathbb{T}) \) for any \( \varepsilon > 0 \) and thus we need to renormalise the potential energy.

\[16\]Namely, mutually absolutely continuous.
Given $N \in \mathbb{N}$, let $P_N$ be the Dirichlet projection onto the frequencies $\{|n| \leq N\}$ and set $X_{\delta,N} := P_N X_{\delta}$. Note that, for each fixed $\delta > 0$ and $x \in T$, the random variable $X_{\delta,N}(x)$ is a mean-zero real-valued Gaussian random variable with variance
\[
\sigma_{\delta,N} := \mathbb{E}[X_{\delta,N}(x)^2] = \frac{1}{4\pi^2} \sum_{0 < |n| \leq N} \frac{2\pi}{H_{\delta}(n)} = C(\delta) \log(N + 1)
\]
for some $C(\delta) > 0$. Given an integer $k \geq 2$, we define the Wick ordered monomial $W(X_{\delta,N}^k)$ by
\[
W(X_{\delta,N}^k) = H_k(X_{\delta,N}; \sigma_{\delta,N}),
\]
where $H_k(x; \sigma)$ is the Hermite polynomial of degree $k$; see (3.1.16) for the definition. Then, $W(X_{\delta,N}^k)$ converges almost surely to a limit, denoted by $W(X_{\delta}^k)$, in $H^{-\varepsilon}(T)$ for any $\varepsilon > 0$; (Proposition 3.4.5). In particular, the truncated renormalized potential energy $\int_T W(X_{\delta,N}^k) dx$ converges, almost surely and also in $L^p(\Omega)$, to a limit denoted by $\int_T W(X_{\delta}^k) dx$.

With $u_N = P_N u$, we define the truncated Gibbs measure $\rho_{\delta,N}$ by
\[
\rho_{\delta,N}(du) = Z_{\delta,N}^{-1} e^{-\frac{1}{\varepsilon} \int_T W(u_N^{k+1}) dx} d\mu_\delta(u).
\]
We also define the truncated density $G_{\delta,N}(u)$ by
\[
G_{\delta,N}(u) = e^{-\frac{1}{\varepsilon} \int_T W(u_N^{k+1}) dx}.
\]
In view of the convergence of the truncated renormalized potential energy mentioned above, we see that the truncated density $G_{\delta,N}$ converges to the limiting density
\[
G_{\delta}(u) = e^{-\frac{1}{\varepsilon} \int_T W(u^{k+1}) dx}
\]
in measure with respect to $\mu_\delta$.

**Remark 1.3.1.** It is easy to verify from the following that for any fixed $\delta \in (0, \infty)$, both $X_{\delta}$ and $\tilde{X}_{\delta}$ lie almost surely in the space $H^s(T) \setminus L^2(T)$ for any $s < 0$,
\[
\|X_{\delta}\|_{H^s(T)} = \sum_{n \in \mathbb{Z}} \frac{|n|^{2s}}{|K_\delta(n)|} \leq \sum_{n \in \mathbb{Z}} \frac{|n|^{2s}}{|n - c(\delta)|} < \infty,
\]
provided $1 - 2s > 1$ and some constant depends only on $\delta$ (similar computation for $\tilde{X}_{\delta}$).

### 1.3.3 Main results on the convergence of Gibbs measure

We now state the construction of the limiting Gibbs measure $\rho_\delta$ and its convergence property in the deep-water limit.

**Theorem 1.3.2.** Let $k \in 2\mathbb{N} + 1$. Then, the following statements hold.

(i) Let $0 < \delta \leq \infty$. Then, for any finite $p \geq 1$, we have
\[
\lim_{N \to \infty} G_{\delta,N}(u) = G_{\delta}(u) \quad \text{in } L^p(\mu_\delta).
\]

As a consequence, the truncated Gibbs measure $\rho_{\delta,N}$ in (1.3.37) converges, in the sense of (1.3.40), to the limiting Gibbs measure $\rho_\delta$ given by
\[
\rho_\delta(du) = Z_\delta^{-1} G_{\delta}(u) d\mu_\delta(u)
\]
\[
= Z_\delta^{-1} e^{-\frac{1}{\varepsilon} \int_T W(u^{k+1}) dx} d\mu_\delta(u).
\]
In particular, $\rho_{\delta,N}$ converges to $\rho_\delta$ in total variation.
(ii) (deep-water limit of the Gibbs measures). Let $0 < \delta < \infty$. Then, the Gibbs measures $\rho_\delta$ for $gILW$ (1.3.2) and $\rho_{BO} = \rho_\infty$ for $gBO$ (1.1.7) constructed in Part (i) are equivalent. Moreover, $\rho_\delta$ converges to $\rho_{BO}$ in total variation, as $\delta \to \infty$.

Furthermore, when $k = 2$, by replacing the truncated Gibbs measure $\rho_{\delta, N}$ in (1.3.37) by the truncated Gibbs measure with a Wick-ordered $L^2$-cutoff:

$$\rho_{\delta, N}(du) = Z_{\delta, N}^{-1} \chi_K \left( \int_{\Omega} W(u_N^3) dx \right) e^{-\frac{1}{2} \int_{\Omega} u_N^3 dx} d\mu_\delta(u),$$

(1.3.42)

the statements (i) and (ii) hold true for any fixed $K > 0$. Namely, for each $0 < \delta \leq \infty$, the truncated Gibbs measure $\rho_{\delta, N}$ in (1.3.42) converges to the limiting Gibbs measure:

$$\rho_\delta(du) = Z_\delta^{-1} \chi_K \left( \int_{\Omega} W(u^3) dx \right) e^{-\frac{1}{2} \int_{\Omega} u^3 dx} d\mu_\delta(u)$$

(1.3.43)

in the sense of the $L^p(\mu_\delta)$-convergence of the truncated densities as in (1.3.40). Moreover, the resulting Gibbs measure $\rho_\delta$ in (1.3.43) and the base Gaussian measure endowed with the Wick-ordered $L^2$-cutoff $\chi_K \left( \int_{\Omega} W(u^3) dx \right) d\mu_\delta(u)$ are equivalent. For $2 \leq \delta \leq \infty$, the rate of convergence in total variation of $\rho_{\delta, N}$ to $\rho_\delta$ as $N \to \infty$ is uniform for $2 \leq \delta \leq \infty$.

For $0 < \delta < \infty$, the Gibbs measures $\rho_\delta$ in (1.3.43) for ILW (1.3.2) with $k = 2$ and $\rho_{BO}$ in (1.3.30) for BO are equivalent and, as $\delta \to \infty$, the Gibbs measure $\rho_\delta$ converges to $\rho_{BO}$ in total variation.

Given a parameter-dependent Hamiltonian dynamics, it is of significant physical interest to study convergence of the associated Gibbs measures, which may be viewed as the first step toward studying convergence of dynamics at the Gibbs equilibrium. Theorem 1.3.2 and Theorem 1.3.4 is the first such result for the (generalized) ILW equation, appearing in the study of fluids. We also mention a series of recent breakthrough results on the convergence of the Gibbs measures for quantum many-body systems to that for the nonlinear Schrödinger equation, led by two groups (Lewin, Nam, Rougerie) and (Fröhlich, Knowles, Schlein, and Sohinger). See these papers for the references therein. While these works establish only the convergence of the Gibbs measures, we also establish convergence of the corresponding dynamics; see Theorems 1.3.7 and 1.3.9 below.

Fix $k \in 2\mathbb{N} + 1$. For each fixed $0 < \delta \leq \infty$, the construction of the Gibbs measure (Theorem 1.3.2) follows from a standard application of Nelson’s estimate. The main novelty is Part (ii) of Theorem 1.3.2. In order to prove convergence of $\rho_\delta$ in the deep-water limit, we need to estimate the truncated densities $G_{\delta, N}(u)$, uniformly in both $\delta \gg 1$ and $N \in \mathbb{N}$. One subtle point is that for different values of $\delta \gg 1$, the base Gaussian measures $\mu_\delta$ are different. In order to overcome this issue, we indeed estimate $G_{\delta, N}(X_0)$ in $L^p(\Omega)$, uniformly in both $\delta \gg 1$ and $N \in \mathbb{N}$. Namely, we need to directly work with the probability measure $\mathbb{P}$ on $\Omega$. See Section 3.2 for details. We point out that this uniform bound on the truncated densities in $\delta \gg 1$ and $N \in \mathbb{N}$ also plays an important role in the dynamical part, which we discuss in the next subsection.

When $k = 2$, the problem is no longer defocusing and thus Nelson’s argument is not directly applicable. While we could adapt the argument by Tzvetkov [162] for the BO equation, we instead use the variational approach as in the work [136] by Seong, Tolomeo, and the first author, which provides a slightly simpler argument.

**Remark 1.3.3.** We point out that, when $k = 2$, there is no need for a renormalization. Indeed, recalling that $H_\delta(x; \sigma) = x^3 - 3\sigma x$, under the mean-zero condition, we have

$$\int_{\Omega} W(u_N^3) dx = \int_{\Omega} u_N^3 dx - 3\sigma_{\delta, N} \int_{\Omega} u_N dx = \int_{\Omega} u_N^3 dx,$$

showing that a renormalization is not necessary in the $k = 2$ case. The same comment applies to Theorem 1.3.4 in the shallow-water limit.

Next, we consider the scaled gILW (1.3.8) with the Hamiltonian $E_{\delta}(u)$ in (1.3.10). Let $k \in 2\mathbb{N} + 1$. For each fixed finite $\delta > 0$, the construction of the Gibbs measure $\rho_\delta$ in (1.3.13) follows exactly the
same lines as above. Define the base Gaussian measure $\tilde{\mu}_\delta$ in \[1.3.14\] as the induced probability measure under the map:

$$\omega \in \Omega \mapsto \tilde{X}_\delta(\omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} \frac{g_n(\omega)}{|L_\delta(n)|^2} e_n,$$

(1.3.44)

where $\{g_n\}_{n \in \mathbb{Z}^*}$ is as in \[1.3.21\] and $L_\delta(n)$ is given by

$$L_\delta(n) := \frac{3}{\delta} K_\delta(n) = \frac{3n}{\delta} \left( n \coth(\delta n) - \frac{1}{\delta} \right).$$

(1.3.45)

From \[1.3.32\], \[1.3.44\], and \[1.3.45\], we have

$$\tilde{X}_\delta = \sqrt{\frac{2}{3}} X_\delta$$

(1.3.46)

for any $0 < \delta < \infty$. Hence, by setting $\tilde{X}_{\delta,N} = \mathbf{P}_N \tilde{X}_\delta$, it follows from \[1.3.35\] that

$$\tilde{\sigma}_{\delta,N} := \mathbb{E}[\tilde{X}_{\delta,N}^2(x)] = \frac{1}{4\pi^2} \sum_{0 < |n| \leq N} \frac{2\pi}{L_\delta(n)}$$

$$= \frac{\delta}{3} \sigma_{\delta,N} \sim \delta \log(N + 1),$$

(1.3.47)

where $\sigma_{\delta,N}$ is as in \[1.3.35\].

Given $N \in \mathbb{N}$, we define the truncated density $\tilde{G}_{\delta,N}(u)$ by

$$\tilde{G}_{\delta,N}(v) = e^{-\frac{1}{4\pi^2} \int_0^v W(v^{k+1}) dx},$$

where $v_N = \mathbf{P}_N v$ and

$$W(v^{k+1}) = H_{k+1}(v_N; \tilde{\sigma}_{\delta,N}).$$

(1.3.48)

Then, we define the truncated Gibbs measure $\tilde{\rho}_{\delta,N}$ by

$$\tilde{\rho}_{\delta,N}(dv) = \frac{1}{Z_{\delta,N}} e^{-\frac{1}{4\pi^2} \int_0^v W(v^{k+1}) dx} d\tilde{\mu}_\delta(v).$$

(1.3.49)

We now state our main result on convergence of the Gibbs measures in the shallow-water limit. Due to the use of the Wick renormalization for $\delta > 0$, we need to consider a “renormalized” power even in the shallow-water limit ($\delta = 0$):

$$\rho_{gKdV}(dv) = Z_0^{-1} e^{-\frac{1}{4\pi^2} \int_0^v W(v^{k+1}) dx} d\tilde{\mu}_0(v),$$

(1.3.50)

associated with the following gKdV:

$$\partial_t v + \partial_x^3 v = \partial_x W(v^k).$$

(1.3.51)

Here, $W(v^\ell)$ is given by

$$W(v^\ell) = H_\ell(v; \sigma_{gKdV}),$$

(1.3.52)

where $\sigma_{gKdV}$ is as in \[1.3.27\]. In particular, when $\delta = 0$, $W(v^\ell)$ is nothing but the usual Hermite polynomial of degree $\ell$ with the finite variance parameter $\sigma_{gKdV}$, which is well defined without any limiting procedure.

**Theorem 1.3.4.** Let $k \in 2\mathbb{N} + 1$. Then, the following statements hold.
CHAPTER 1. INTRODUCTION

(i) Let $0 < \delta < \infty$. Then, for any finite $p \geq 1$, we have
\[
\lim_{N \to \infty} \tilde{G}_{\delta,N}(v) = \tilde{G}_{\delta}(v) := e^{-\frac{1}{4} \int T \tilde{W}(v^{k+1})dx} \text{ in } L^p(\mu_{\delta}).
\]
(1.3.53)
As a consequence, the truncated Gibbs measure $\tilde{\rho}_{\delta,N}$ in (1.3.49) converges, in the sense of (1.3.53), to the limiting Gibbs measure $\tilde{\rho}_{\delta}$ given by
\[
\tilde{\rho}_{\delta}(dv) = Z_{\delta}^{-1} \tilde{G}_{\delta}(v) d\tilde{\mu}_{\delta}(v) = Z_{\delta}^{-1} e^{-\frac{1}{4} \int T \tilde{W}(v^{k+1})dx} d\tilde{\mu}_{\delta}(v).
\]
(1.3.54)
In particular, $\tilde{\rho}_{\delta,N}$ converges to $\tilde{\rho}_{\delta}$ in total variation. The resulting Gibbs measure $\tilde{\rho}_{\delta}$ and the base Gaussian measure $\mu_{\delta}$ are equivalent.

For $0 < \delta \leq 1$, the rate of convergence (1.3.53) is uniform and thus the rate of convergence in total variation of $\tilde{\rho}_{\delta,N}$ to $\tilde{\rho}_{\delta}$ as $N \to \infty$ is uniform for $0 < \delta \leq 1$.

(ii) (shallow-water limit of the Gibbs measures). Let $0 < \delta < \infty$. Then, the Gibbs measures $\tilde{\rho}_{\delta}$ for the scaled gILW (1.3.8) constructed in Part (i) and $\rho_{\text{gKdV}}$ in (1.3.50) for gKdV (1.3.51) are mutually singular. As $\delta \to 0$, however, $\tilde{\rho}_{\delta}$ converges weakly to $\rho_{\text{gKdV}}$.

Furthermore, when $k = 2$, by replacing the truncated Gibbs measure $\tilde{\rho}_{\delta,N}$ in (1.3.49) by the truncated Gibbs measure with a Wick-ordered $L^2$-cutoff:
\[
\tilde{\rho}_{\delta,N}(dv) = Z_{\delta,N}^{-1} \chi_K \left( \int T \tilde{W}(v^{k})dx \right) e^{-\frac{1}{4} \int T v^{3}dx} d\tilde{\mu}_{\delta}(v),
\]
(1.3.55)
the statements (i) and (ii) hold true for any fixed $K > 0$. Namely, for each $0 < \delta < \infty$, the truncated Gibbs measure $\tilde{\rho}_{\delta,N}$ in (1.3.55) converges to the limiting Gibbs measure:
\[
\tilde{\rho}_{\delta}(dv) = Z_{\delta}^{-1} \chi_K \left( \int T \tilde{W}(v^{2})dx \right) e^{-\frac{1}{4} \int T v^{3}dx} d\tilde{\mu}_{\delta}(v)
\]
(1.3.56)
in the sense of the $L^p(\mu_{\delta})$-convergence of the truncated densities as in (1.3.53). Moreover, the resulting Gibbs measure $\tilde{\rho}_{\delta}$ in (1.3.56) and the base Gaussian measure endowed with the Wick-ordered $L^2$-cutoff $\chi_K \left( \int T \tilde{W}(v^{2})dx \right)$ are equivalent. For $0 < \delta \leq 1$, the rate of convergence in total variation of $\tilde{\rho}_{\delta,N}$ in (1.3.55) to $\tilde{\rho}_{\delta}$ in (1.3.56) as $N \to \infty$ is uniform for $0 < \delta \leq 1$.

For $0 < \delta < \infty$, the Gibbs measures $\tilde{\rho}_{\delta}$ in (1.3.56) for the scaled ILW (1.3.8) (with $k = 2$) and $\rho_{\text{gKdV}}$ in (1.3.50) for KdV (with an $L^2$-cutoff) are mutually singular. As $\delta \to 0$, however, the Gibbs measure $\tilde{\rho}_{\delta}$ converges weakly to $\rho_{\text{gKdV}}$ in (1.3.26).

As compared to the deep-water limit ($\delta \to \infty$) studied in Theorem 1.3.2, we have an interesting phenomenon in this shallow-water limit ($\delta \to 0$). This is due to the fact that, while $L_{\delta}(n) \sim_{\delta} |n|$ for each $\delta > 0$, we have
\[
\lim_{\delta \to 0} L_{\delta}(n) = n^2
\]
for each $n \in \mathbb{Z}^*$. See Lemma 3.1.3. This causes $\tilde{\mu}_{\delta}$, $\delta > 0$, in (1.3.14) and the limiting Gaussian measure $\mu_{\delta}$ in (1.3.20) to be mutually singular. (For each finite $\delta > 0$, the Gaussian measure $\mu_{\delta}$ is supported on $H^{-\varepsilon}(T) \setminus L^2(T)$, $\varepsilon > 0$, whereas $\mu_{0}$ is supported on $H^{\frac{1}{2}-\varepsilon}(T) \setminus H^{\frac{1}{2}}(T)$, $\varepsilon > 0$.) In view of the equivalence of the Gibbs measures and the base Gaussian measures, the first claim in Theorem 1.3.4(ii) essentially follows from this observation. Due to this mutual singularity, the mode of convergence of the Gibbs measures $\tilde{\rho}_{\delta}$ to $\rho_{\text{gKdV}}$ in the shallow-water limit is much weaker as compared to that in the deep-water limit stated in Theorem 1.3.2(i). See Section 3.3 for details.

Remark 1.3.5. Let $k \in 2\mathbb{N} + 1$. Then, the Gibbs measure $\rho_{\text{gKdV}}$ in (1.3.50) for the KdV equation is a well-defined probability measure on $H^{\frac{1}{2}-\varepsilon}(T) \setminus H^{\frac{1}{2}}(T)$, $\varepsilon > 0$. In view of (1.3.52) with (1.3.27), we have $0 < e^{-\frac{1}{4} \int T \tilde{W}(v^{k+1})dx} = e^{-\frac{1}{4} \int T \tilde{W}_{k+1}(v^{k+1})dx} \leq 1$ on $H^{\frac{1}{2}-\varepsilon}(T)$, which is clearly integrable with respect to the base Gaussian measure $\mu_{0}$ in (1.3.20).
Remark 1.3.6. (i) As mentioned above, in [136], Seong, Tolomeo, and the second author proved non-normalizability of the Gibbs measure (1.3.31) (with \( k = 3 \)) for the focusing modified BO (for any cutoff size \( K > 0 \) on the Wick-ordered \( L^2 \)-cutoff). For each fixed \( \delta > 0 \), the same argument allows us to prove non-normalizability of the Gibbs measure (with \( k = 3 \)):

\[
\rho_\delta(du) = Z_\delta^{-1} \chi_K \left( \int_T \mathcal{W}(u^2)dx \right) e^{\frac{i}{\hbar} \int_T \mathcal{W}(u^{k+1})dx} d\mu_\delta(u) \tag{1.3.57}
\]

for the focusing modified ILW equation (1.3.10) with \( k = 3 \). A straightforward modification of the argument in [136] also yields non-normalizability of the focusing\(^{17} \) Gibbs measures (1.3.57) for any \( k \geq 3 \) and \( 0 < \delta \leq \infty \). For any \( k \geq 3 \) and \( \delta > 0 \), the same non-normalizability result also applies to the Gibbs measure:

\[
\tilde{\rho}_\delta(dv) = Z_\delta^{-1} \chi_K \left( \int_T \mathcal{W}(v^2)dx \right) e^{\frac{i}{\hbar} \int_T \mathcal{W}(v^{k+1})dx} d\tilde{\mu}_\delta(v) \tag{1.3.58}
\]

for the focusing scaled gILW (namely, (1.3.8) with the \(-\) sign on the nonlinearity).

(ii) In the shallow-water limit (\( \delta = 0 \)), the Gibbs measure \( \rho_{gKdV} \) for the focusing gKdV (with an appropriate \( L^2 \)-cutoff) exists up to the \( L^2 \)-critical case (\( k = 5 \)). For each \( \delta > 0 \), however, the Gibbs measure for the focusing scaled gILW, \( \delta > 0 \), is not normalizable and thus it is not possible to study the convergence problem for the Gibbs measures (as well as dynamics at the Gibbs equilibrium) in this case. One possible approach may be to study convergence of the truncated Gibbs measure \( \tilde{\rho}_{\delta,N} \) in (1.3.49) (with a Wick-ordered \( L^2 \)-cutoff) for the frequency-truncated scaled gILW to the Gibbs measure \( \rho_{gKdV} \) in (1.3.50) for the focusing gKdV (1.3.51), by taking \( N \to \infty \) and \( \delta \to 0 \) in a related manner. The associated dynamical convergence problem may be of interest as well.

### 1.3.4 Dynamical problem

Our next goal is to study the associated dynamical problems. More precisely, our goal is to construct dynamics for the (scaled) gILW at the Gibbs equilibrium and then to show that the invariant Gibbs dynamics for the (scaled) gILW converges to that for gBO in the deep-water limit (and for gKdV in the shallow-water limit, respectively) in some appropriate sense. In the following, for the sake of the presentation, we refer to the study of the original (unscaled) gILW equation (and the gBO equation) for \( 0 < \delta \leq \infty \) as the deep-water regime, and the study of the scaled gILW equation for \( 0 \leq \delta < \infty \) (and the gKdV equation) as the shallow-water regime.

Let us first consider the deep-water regime. In Theorem 1.3.2, we constructed the Gibbs measure \( \rho_\delta \) in (1.3.41) associated with the following renormalized Hamiltonian:

\[
E_\delta(u) = \frac{1}{2} \int_T u \mathcal{G}_\delta \partial_x u dx + \frac{1}{k+1} \int_T \mathcal{W}(u^{k+1}) dx,
\]

when \( k \in 2\mathbb{N} + 1 \). The corresponding Hamiltonian dynamics is formally given by the following renormalized gILW:

\[
\partial_t u - \mathcal{G}_\delta \partial_x^2 u = \partial_x \mathcal{W}(u^k), \tag{1.3.59}
\]

which needs to be interpreted in a suitable limiting sense. When \( k = 2 \), the measure construction does not require any renormalization (see Remark 1.3.3) and thus we study ILW (1.3.59) with \( k = 2 \) as the corresponding dynamical problem. As mentioned above, our first main goal is to construct dynamics at the Gibbs equilibrium. It is, however, a rather challenging problem to construct strong solutions to these equations with the Gibbsian initial data, even in a probabilistic sense. This is mainly due to the low regularity (namely, \( H^{-\varepsilon(T)} \setminus L^2(T), \varepsilon > 0 \)) of the Gibbsian initial data when \( \delta > 0 \). In fact, for \( 0 < \delta \leq \infty \), the only known case is for the Benjamin-Ono equation (\( k = 2 \) with \( \delta = \infty \)) by Deng [40], where he established deterministic local well-posedness result in a space, containing the support for the Gibbs measure, by a rather intricate argument.

\(^{17}\)Recall our convention that by focusing, we also include the non-defocusing case, namely, (1.3.57) with \( k \in 2\mathbb{N} \) and the \(-\) sign on the potential energy.
and then used Bourgain’s invariant measure argument [20] to construct global-in-time dynamics at the Gibbs equilibrium. By invariance, we mean that (with \( \delta = \infty \) in the BO case)

\[
\rho_\delta(\Phi_\delta(-t)A) = \rho_\delta(A)
\]  

(1.3.60)

for any measurable set \( A \subset H^{-\epsilon}(\mathbb{T}) \) with some small \( \epsilon > 0 \) and any \( t \in \mathbb{R} \), where \( \Phi_\delta(t) \) denotes the solution map:

\[
\Phi_\delta(t) : u_0 \in H^{-\epsilon}(\mathbb{T}) \mapsto u(t) = \Phi_\delta(t)u_0 \in H^{-\epsilon}(\mathbb{T}),
\]

satisfying the flow property

\[
\Phi_\delta(t_1 + t_2) = \Phi_\delta(t_1) \circ \Phi_\delta(t_2)
\]  

(1.3.61)

for any \( t_1, t_2 \in \mathbb{R} \). Here, we used \( H^{-\epsilon}(\mathbb{T}) \) for simplicity but it may be another Banach space, containing the support of the Gibbs measure (as in [40]). We also mention a recent work [59] on sharp global well-posedness of BO in almost critical spaces \( H^s(\mathbb{T}) \), \( s > -\frac{1}{2} \), based on the complete integrability of the equation. When \( 0 < \delta < \infty \), the construction of strong solutions with the Gibbsian initial data is widely open even for \( k = 2 \). When \( k \geq 3 \), the difficulty of the problem increases significantly and nothing is known up to date for the renormalized gBO (with the Gibbs measure initial data):

\[
\partial_t u - \mathcal{H} \partial_x^2 u = \partial_x W(u^k).
\]  

(1.3.62)

For example, when \( k = 3 \) corresponding to the (renormalized) modified BO equation (mBO), the best known (deterministic) well-posedness result for mBO is in \( H^{\frac{5}{2}}(\mathbb{T}) \) [42], while the scaling-critical space is \( L^2(\mathbb{T}) \) and the support of the Gibbs measure is contained in \( H^{-\epsilon}(\mathbb{T}) \setminus L^2(\mathbb{T}) \). When \( 0 < \delta < \infty \), we expect that the problem is much harder due to a rather complicated, non-algebraic nature of the dispersion symbol (see (1.1.2)).

In this paper, we do not aim to construct strong solutions. By a compactness argument, we instead construct global-in-time dynamics of weak solutions at the Gibbs equilibrium (without uniqueness), including the gBO case (\( \delta = \infty \)). In the deep-water limit, we also show that there exists a sequence \( \{\delta_m\}_{m \in \mathbb{N}} \) of the depth parameters, tending to \( \infty \), and solutions, at the BO equilibrium, to the renormalized gILW (1.3.59) with \( \delta = \delta_m \), converging almost surely to solutions, at the Gibbs equilibrium, to the renormalized gBO (1.3.62).

**Theorem 1.3.7** (deep-water regime). Let \( k \in \mathbb{N} + 1 \). Then, the following statements hold.

(i) Let \( 0 < \delta \leq \infty \). Then, there exists a set \( \Sigma_\delta \) of full measure with respect to the Gibbs measure \( \rho_\delta \) in (1.3.41) constructed in Theorem 1.3.2 such that for every \( u_0 \in \Sigma_\delta \), there exists a global-in-time solution \( u \in C(\mathbb{R}; H^s(\mathbb{T})) \), \( s < 0 \), to the renormalized gILW equation (1.3.59) (and to the renormalized gBO equation (1.3.62) when \( \delta = \infty \)) with (mean-zero) initial data \( u|_{t=0} = u_0 \). Moreover, for any \( t \in \mathbb{R} \), the law of the solution \( u(t) \) at time \( t \) is given by the Gibbs measure \( \rho_\delta \).

(ii) There exists an increasing sequence \( \{\delta_m\}_{m \in \mathbb{N}} \subset \mathbb{N} \) tending to \( \infty \) such that the following holds.

- For each \( m \in \mathbb{N} \), there exists a (random) global-in-time solution \( u_{\delta_m} \in C(\mathbb{R}; H^s(\mathbb{T})) \), \( s < 0 \), to the renormalized gILW equation (1.3.59), with the depth parameter \( \delta = \delta_m \), with the Gibbsian initial data distributed by the Gibbs measure \( \rho_{\delta_m} \). Moreover, for any \( t \in \mathbb{R} \), the law of the solution \( u_{\delta_m}(t) \) at time \( t \) is given by the Gibbs measure \( \rho_{\delta_m} \).

- As \( m \to \infty \), \( u_{\delta_m} \) converges almost surely in \( C(\mathbb{R}; H^s(\mathbb{T})) \) to a (random) solution \( u \) to the renormalized gBO equation (1.3.62). Moreover, for any \( t \in \mathbb{R} \), the law of the limiting solution \( u(t) \) at time \( t \) is given by the Gibbs measure \( \rho_{\text{gBO}} = \rho_{\infty} \) in (1.3.41).

When \( k = 2 \), the statements (i) and (ii) hold true without any renormalization (but with the Gibbs measures \( \rho_{\delta_m} \) in (1.3.43) and \( \rho_{\text{BO}} \) in (1.3.30)).

While our construction yields only weak solutions without uniqueness, Theorem 1.3.7 (and Theorem 1.3.9) is the first result on the construction of solutions with the Gibbsian initial data for both the (generalized) ILW equation \( (k \geq 2) \) and the gBO equation \( (k \geq 3) \). Furthermore,
Theorem 1.3.7 presents the first convergence result for the (generalized) ILW equation from a statistical viewpoint. In Theorem 1.3.9 below, we state analogous results in the shallow-water regime.

In proving Theorem 1.3.7 we employ the compactness approach used in [30, 139, 134], which in turn was motivated by the works of Albeverio and Cruzeiro [5] and Da Prato and Debussche [44] in the study of fluids. Our strategy is to start with the frequency-truncated dynamics (say, when $k \in 2\mathbb{N} + 1$):

$$
\partial_t u_{\delta,N} - \mathcal{G}_\delta \partial_x^2 u_{\delta,N} = \partial_x \mathcal{P}_N \mathcal{W}((\mathcal{P}_N u_{\delta,N})^k),
$$

(1.3.63)

which preserves the truncated Gibbs measure $\rho_{\delta,N}$ in (1.3.37). By exploiting the invariance of the truncated Gibbs measures $\rho_{\delta,N}$, we establish tightness (compactness) of the pushforward measures $\nu_{\delta,N}$ (on space-time distributions) of the truncated Gibbs measures under the truncated dynamics (1.3.63). Then, for each fixed $\delta \gg 1$, Skorokhod’s theorem (Lemma 3.1.14) allows us to prove almost sure convergence of the solution $u_{\delta,N}$ to (1.3.63) (after changes of measure spaces) to a limit $u$, which satisfies the Wick-ordered gILW (1.3.59) in the distributional sense. This part follows from exactly the same argument as those in [30, 139, 134]. Due to the use of the compactness, we only obtain global existence of a solution $u$ to (1.3.59) without uniqueness. The main ingredient in this step is the uniform bound on the truncated densities $\{G_{\delta,N}\}_{N \in \mathbb{N}}$ in (1.3.38). Here, we only need the uniformity in $N$ for each fixed $0 < \delta < \infty$, and it is with respect to the base Gaussian measure $\mu_\delta$ in (1.3.12).

A new ingredient in showing convergence of the gILW dynamics (1.3.59) to the gBO dynamics (1.1.7) is the uniform integrability of the truncated densities in both $\delta \gg 1$ and $N \in \mathbb{N}$ established in Theorem 1.3.2. As mentioned above, for different values of $\delta \gg 1$, the base Gaussian measures $\mu_\delta$ are different and thus we need to work directly with the underlying probability measure $P$ on $\Omega$. This shows tightness of the measures $\{\nu_{\delta,N}\}_{\delta \gg 1, N \in \mathbb{N}}$ constructed in the fist step, in both $\delta \gg 1$ and $N \in \mathbb{N}$. Then, by using the Lévy-Prokhorov metric which characterizes weak convergence of probability measures, we conclude the desired convergence from a diagonal argument and the triangle inequality.

**Remark 1.3.8.** Our notion of solutions constructed in Theorem 1.3.7 (and Theorem 1.3.9) basically corresponds to that of martingale solutions studied in the field of stochastic PDEs. See, for example, [10].

Next, we state our dynamical result in the shallow-water regime. In this case, we study following renormalized scaled gILW:

$$
\partial_t v - \frac{3}{\delta} \mathcal{G}_\delta \partial_x^2 v = \partial_x \mathcal{W}(v^k).
$$

(1.3.64)

generated by the renormalized Hamiltonian:

$$
\mathcal{E}_\delta(v) = \frac{3}{2\delta} \int_T v \mathcal{G}_\delta \partial_x v dx + \frac{1}{k + 1} \int_T \mathcal{W}(v^{k+1}) dx.
$$

(1.3.65)

As in the deep-water regime, we construct dynamics for (1.3.64) as a limit of the frequency-truncated dynamics:

$$
\partial_t v_{\delta,N} - \frac{3}{\delta} \mathcal{G}_\delta \partial_x^2 v_{\delta,N} = \partial_x \mathcal{P}_N \mathcal{W}((\mathcal{P}_N v_{\delta,N})^k).
$$

(1.3.66)

**Theorem 1.3.9** (shallow-water regime). Let $k \in 2\mathbb{N} + 1$. Then, the following statements hold.

(i) Let $0 < \delta < \infty$. Then, there exists a set $\tilde{\Sigma}_\delta$ of full measure with respect to the Gibbs measure $\tilde{\rho}_\delta$ in (1.3.54) constructed in Theorem 1.3.4 such that for every $v_0 \in \tilde{\Sigma}_\delta$, there exists a global-in-time solution $v \in C(\mathbb{R}; H^s(\mathbb{T}))$, $s < 0$, to the renormalized scaled gILW equation (1.3.64) with (mean-zero) initial data $v|_{t=0} = v_0$. Moreover, for any $t \in \mathbb{R}$, the law of the solution $v(t)$ at time $t$ is given by the Gibbs measure $\tilde{\rho}_\delta$.

(ii) There exists a decreasing sequence $\{\delta_m\}_{m \in \mathbb{N}} \subset \mathbb{R}_+$ tending to 0 such that the following holds.

- For each $m \in \mathbb{N}$, there exists a (random) global-in-time solution $v_{\delta_m} \in C(\mathbb{R}; H^s(\mathbb{T}))$, $s < 0$, to the renormalized scaled gILW equation (1.3.64), with the depth parameter $\delta = \delta_m$, with the
Gibbsian initial data distributed by the Gibbs measure $\tilde{\rho}_{\delta,m}$. Moreover, for any $t \in \mathbb{R}$, the law of the solution $v_{\delta,m}(t)$ at time $t$ is given by the Gibbs measure $\tilde{\rho}_{\delta,m}$.

- As $m \to \infty$, $v_{\delta,m}$ converges almost surely in $C(\mathbb{R}; H^s(\mathbb{T}))$ to a (random) solution $v$ to the gKdV equation \((1.3.51)\). Moreover, for any $t \in \mathbb{R}$, the law of the limiting solution $v(t)$ at time $t$ is given by the Gibbs measure $\tilde{\rho}_{\delta,KdV}$ in \((1.3.50)\).

When $k = 2$, the statements (i) and (ii) hold true without any renormalization (but with the Gibbs measures $\tilde{\rho}_{\delta}$ in \((1.3.56)\) and $\tilde{\rho}_{\delta,KdV}$ in \((1.3.26)\)).

With the uniform integrability on the truncated densities in both $0 < \delta \leq 1$ and $N \in \mathbb{N}$ (established in Theorem 1.3.4, Theorem 1.3.9 follows from exactly the same argument in the proof of Theorem 1.3.7 and hence we omit details.

**Remark 1.3.10.** Theorems 1.3.7 and 1.3.9 yield the construction and convergence of weak solutions. Due to the use of a compactness argument, we do not have any uniqueness statement. While these solutions are distributional solutions, they do not satisfy the Duhamel formulation, which is the usual notion for strong solutions in the study of dispersive PDEs. Furthermore, due to the lack of uniqueness they remain a weak solution unless they satisfy the invariance property as stated in \((1.3.60)\). This is the reason why we have a weaker invariance property in Theorems 1.3.7 and 1.3.9 which, is, for example, not sufficient to imply the Poincaré recurrence property. See [150] for a further discussion. We also expect that a suitable uniqueness statement would allow us to show convergence of the entire family \{$(u_{\delta})_{\delta>1}$\} in the deep-water limit $(\delta \to \infty)$ and \{$(u_{\delta},v_{\delta})_{0<\delta<1}$\} in the shallow-water limit $(\delta \to 0)$.

Therefore, it would be of significant interest to study probabilistic construction of strong solutions to the (scaled) ILW equation with the Gibbsian initial data. As mentioned above, the $k \geq 3$ case seems to be out of reach at this point. Even as for the $k = 2$ case, the problem is very challenging. For example, in studying low regularity well-posedness of the BO equation, the gauge transform \([158]\) plays a crucial role. For the ILW equation, however, existence of such a gauge transform is unknown; see [33] p. 128.

When $k = 2$, another possible approach would be to exploit the complete integrability of the ILW equation. In the case of the BO equation, there is a recent breakthrough \([56]\) on sharp global well-posedness in $H^s(\mathbb{T})$, $s > -\frac{1}{2}$. Even with the complete integrability, however, the low regularity well-posedness of the ILW equation seems to be very challenging.

Lastly, let us point out that, as for the gKdV equation \((1.1.15)\) (and also \((1.3.51)\)), there is a good well-posedness theory with the Gibbsian initial data; see \([19, 155, 157]\). In particular, in a recent work \([53]\), Chapouto and Kishimoto completed the program initiated by Bourgain \([20]\) on the construction of invariant Gibbs dynamics for the (defocusing) gKdV \((1.1.15)\) for any $k \in 2\mathbb{N} + 1$.

**Remark 1.3.11.** When $k = 2$, the ILW equation is known to be completely integrable with an infinite sequence of conservation laws of increasing regularities. In this work, we study the construction and convergence of the Gibbs measures associated with the Hamiltonians and the corresponding dynamical problem. For the ILW equation, it is also possible to study the construction of invariant measures associated with the higher-order conservation laws. See \([170, 165, 164, 165, 47]\) for such construction of invariant measures associated with the higher order conservation laws in the context of the KdV and BO equations. Once such construction is done, it would be of strong interest to study the related convergence problem. We plan to address this issue in a forthcoming work \([36]\). These invariant measures will be supported on smooth(er) functions and thus this problem is of importance even from the physical point of view.

**Remark 1.3.12.** In this work, we focus our attention to the circle $\mathbb{T}$. From the physical point of view, it seems natural to study the problem on the real line. The difficulty of this problem comes from not only the roughness of the support but also the integrability of typical functions.

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189 The solution map to the frequency-truncated equation such as \((1.3.63)\) enjoys the flow property, and thus a suitable uniqueness statement would imply the flow property for the limiting dynamics.

19 In view of the absolute continuity of the Gibbs measure with respect to the base Gaussian measure, it suffices to study probabilistic local well-posedness with the Gaussian initial data $X_\delta$ in \((1.3.32)\) (or $\tilde{X}_\delta$ in \((1.3.44)\)) in the spirit of \([27, 22, 153]\).
See [23, 133] for the construction of invariant Gibbs dynamics in the context of the nonlinear Schrödinger equations on the real line. See also [89].

**Remark 1.3.13.** There are recent works [54, 171, 172] on convergence of stochastic dynamics at the Gibbs equilibrium. One key difference between our work and these works is that, in [54, 171, 172], a single Gibbs measure remains invariant for the entire one-parameter family of dynamics, whereas, in our work, the Gibbs measure (and even the base Gaussian measure) varies as the depth parameter $\delta$ changes, requiring us to first establish the convergence at the level of the Gibbs measures.
Chapter 2

Microscopic limits: deterministic method

2.1 Preliminaries

In this section, we start by introducing notations. Then, we will explore the behaviours of the dispersion terms to the equations (2.1.1) and (2.1.2). Lastly, we will introduce the function spaces used throughout the paper and their well-known properties.

From now on, we will focus our analysis on the periodic setting, and let us recall the (scaled) gILW equations in the following. In the deep-water region (=BO-regime), we consider the Cauchy problem of gILW:

\[
\begin{aligned}
\partial_t u - G_\delta (\partial_x^2 u) + \partial_x (f(u)) &= 0, \\
{u|}_{t=0} &= u_0,
\end{aligned}
\tag{2.1.1}
\]

In the shallow-water region (=KdV-regime), we consider the Cauchy problem of scaled gILW:

\[
\begin{aligned}
\partial_t v - \frac{3}{2} G_\delta \partial_x^2 v &= \partial_x (f(v)), \\
v|_{t=0} &= v_0,
\end{aligned}
\tag{2.1.2}
\]

The \(G_\delta\) operator is

\[
G_\delta = -\coth(\delta \partial_x) - \frac{1}{\delta} \partial_x^{-1}
\]

and it is understood as the Fourier multiplier defined by,

\[
\hat{G_\delta} f(n) := -i \left( \coth(\delta n) - \frac{1}{\delta n} \right) \hat{f}(n) \quad \text{for } n \in \mathbb{Z} \setminus \{0\}.
\]

Then, we use \(p_\delta^{(d)}\) and \(p_\delta^{(s)}\) to denote the linearised dispersion relation of (scaled) gILW (2.1.1) and (2.1.2). These have the following forms:

\[
p_\delta^{(d)}(n) = n^2 \left( \coth(\delta n) - \frac{1}{\delta n} \right), \quad p_\delta^{(s)}(n) = \frac{3}{\delta} n^2 \left( \coth(\delta n) - \frac{1}{\delta n} \right).
\tag{2.1.3}
\]

2.1.1 Notations

For \(A, B > 0\), we use \(A \lesssim B\) to mean that there exists \(C > 0\) such that \(A \leq CB\). By \(A \sim B\), we mean that \(A \lesssim B\) and \(B \lesssim A\). Moreover, we denote \(A \ll B\), if there is some small \(c > 0\), such that \(A \leq cB\). For two non negative numbers \(a, b\), we denote \(a \vee b := \max\{a, b\}\) and \(a \wedge b := \min\{a, b\}\). We also write \((\cdot) = (1 + | \cdot |^2)^{1/2}\) for the Japanese bracket.

Given a function \(u(t, x)\) on \(\mathbb{R} \times \mathcal{M}\), we use \(\hat{u}\) and \(\mathcal{F}(u)\) to denote the space Fourier transform.
of \( u \) given by
\[
\hat{u}(k) = \int_{\mathcal{M}} e^{-ikx} u(t, x) \, dx \quad \text{for } k \in \widehat{\mathcal{M}}.
\]
We use \( n \in \mathbb{Z} \) to denote the frequency variable on the tori \( \mathbb{T} \), and \( \xi \in \mathbb{R} \) denotes the frequency variable on the real line \( \mathbb{R} \). In the rest of the paper, we shall only restrict our writing on the tori \( (n \in \mathbb{Z}) \), except for Section 2.6 where we give some estimates to support our argument on the real line.

For any \( s \in \mathbb{R} \), we define \( D^s f \) by its Fourier transform:
\[
\hat{D^s f}(n) = |n|^s \hat{f}(n).
\]

Let \( \eta \in C_0^\infty(\mathbb{R}) \) be a even smooth non-negative cutoff function supported on \([-2, 2]\) such that \( \eta \equiv 1 \) on \([-1, 1]\). We define \( \phi \) by \( \phi(n) = \eta(n) - \eta(2n) \), and set \( \phi_{2k}(n) = \phi(2^{-k}n) \) for \( k \in \mathbb{Z} \). Namely, \( \phi_{2k} \) is supported on \( \{2^{k-1} \leq |n| \leq 2^{k+1}\} \). By convention, we denote \( \phi_1(n) = \eta(2n) \).

Let \( \mathbb{Z}_{\geq 0} := \mathbb{Z} \cap [0, \infty) \). Given a (non-homogeneous) dyadic number \( n \in 2^{\mathbb{Z}_{\geq 0}} \), we replace the above definition by \( \phi_N \) for \( N \geq 1 \). Then, we have
\[
\sum_{N \geq 1 \text{ dyadic}} \phi_N = 1.
\]
We notice that \( \text{supp}(\phi_N) \subset \{ \frac{N}{2} \leq |n| \leq 2N \} \) for \( N \geq 2 \) and if \( N = 1 \), \( \text{supp}(\phi_1) \subset \{|n| \leq 1\} \).

Let \( P_N \) be the (non-homogeneous) Littlewood-Paley projector onto the (spatial) frequencies \( \{n \in \mathbb{Z} : |n| \sim N\} \) such that \( \hat{P}_N u = \phi_N \hat{u} \). Then, we have
\[
f = \sum_{N \geq 1} P_N f.
\]
We also set
\[
P_{\geq N} := \sum_{K \geq N \text{ dyadic}} P_K, \quad P_{\leq N} := \sum_{K \leq N \text{ dyadic}} P_K.
\]
We also define an analogous decomposition on the modulation, which we will denote as \( Q_L \) for \( L \) to be dyadic numbers. Then, \( Q_L u = \psi_L(n, \tau) \hat{u} \), where \( \psi_L = \phi_L(t - p_k^{(s)}(n)) \) or \( \psi_L = \phi_L(t - p_k^{(s)}(n)) \), depending on the context. For simplicity, sometimes we also use \( u_N = P_N u \) when there is no confusion.

### 2.1.2 Dispersion relation

In this subsection, we go over the properties of the dispersion relation, which we defined in (2.1.3).

We notice that \( \coth(\cdot) \) plays the essential rule for the expression in (2.1.3). Let us first collect some known results from [1], which expresses the properties of \( \coth(\cdot) \) by using the expansion formula.

**Lemma 2.1.1** ([1] Lemma 8.2.1). Let \( \delta > 0 \) and for all \( n \in \mathbb{Z} \), then we have
\[
n \coth(\delta n) = 1 + \frac{1}{3} \delta n^2 - \frac{1}{3} n^2 h(n, \delta), \tag{2.1.4}
\]
where the remainder \( h(n, \delta) \) satisfies the following conditions:

(i) For any finite \( N \in \mathbb{N} \), we have
\[
\max_{|n| \leq N} \|h(n, \delta)\| \lesssim_N \delta^3.
\]

(ii) There is some absolute constant \( C_0 \) such that for any \( n \in \mathbb{Z} \),
\[
|h(n, \delta)| \leq C_0 \delta. \tag{2.1.5}
\]
Proof. The proof can be seen in [1, Lemma 8.2.1]. We decide to present full details for the reader’s convenience. First of all, we recall the Mittag-Leffler expansion [27] of \( \coth(z) \) such that

\[
z \coth(z) = 1 + \sum_{k=1}^{\infty} \frac{2z^2}{z^2 + (k\pi)^2}
\]

which is valid for \( z \in \mathbb{C}, \ z \neq ik\pi \), and for any integer \( k \). Therefore, for any \( n \in \mathbb{R} \) and \( \delta > 0 \), we compute

\[
n \coth(\delta n) = \frac{1}{\delta} + 2n^2 \delta \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2 + \delta^2 n^2}
\]

\[
= \frac{1}{\delta} + 2n^2 \delta \left[ \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} + \sum_{k=1}^{\infty} \left( \frac{1}{(k^2\pi^2 + \delta^2 n^2)} - \frac{1}{k^2\pi^2} \right) \right]
\]

\[
= \frac{1}{\delta} + \frac{n^2\delta}{3} - n^2 \sum_{k=1}^{\infty} \frac{2\delta^3 n^2}{k^2\pi^2 (k^2\pi^2 + \delta^2 n^2)}.
\]

That is, the remainder term \( h(n, \delta) \) is explicitly given by

\[
h(n, \delta) = \sum_{k=1}^{\infty} \frac{2\delta^3 n^2}{k^2\pi^2 (k^2\pi^2 + \delta^2 n^2)}.
\]

For \( |n| \leq N \), we have

\[
|h(n, \delta)| \lesssim N \delta^3 \sum_{k=1}^{\infty} \frac{1}{k^4} \lesssim N \delta^3.
\]

Moreover, for all \( n \in \mathbb{Z} \), we have

\[
\frac{|h(n, \delta)|}{\delta} \lesssim \sum_{k=1}^{\infty} \frac{1}{k^2} \lesssim 1.
\]

This concludes our proof.

The next result gives a good approximation on \( \coth(\cdot) \). The proof is in [1, Lemma 4.1], and we present an alternative method for its proof.

Lemma 2.1.2 ([1, Lemma 4.1]. For all \( n \in \mathbb{R} \) and every \( \delta > 0 \). Then, we have

\[
-\frac{1}{\delta} + |n| \leq n \coth(\delta n) \leq \frac{1}{\delta} + |n|.
\]  (2.1.6)

Proof. It is equivalent to showing that

\[
|n \coth(\delta n) - |n|| \leq \delta^{-1}.
\]

That is equivalent to showing

\[
|\delta n \coth(\delta n) - \delta |n|| = |\delta n[\coth(\delta n) - \text{sgn}(\delta n)]| \leq 1.
\]

Now, it suffices to show that

\[
\sup_{x \in \mathbb{R}} |x[\coth(x) - \text{sgn}(x)]| \leq 1.
\]

It is trivial when \( x = 0 \). Therefore, we show the case when \( x \neq 0 \). Since for both \( \coth(x) \) and \( \text{sgn}(x) \) are odd function, it is enough to show when \( x > 0 \). Hence, we use the definition of \( \coth(x) \)
yields
\[ x[\coth(x) - \text{sgn}(x)] = x \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{e^x - e^{-x}}{e^x - e^{-x}} \right) = \frac{2xe^{-x}}{e^x - e^{-x}} = \frac{2x}{e^{2x} - 1} \leq 1. \]

We immediately have Corollary 2.1.3 and Lemma 2.1.4. The proof we refer to the later Subsection 3.1.1.

**Corollary 2.1.3.** Let \( K_\delta := n \coth(\delta n) - \frac{1}{\delta} \). Then, for any \( \delta > 0 \), we have
\[
\max \left( 0, |n| - \frac{1}{\delta} \right) \leq K_\delta(n) = n \coth(\delta n) - \frac{1}{\delta} \leq |n|,
\]
where the above inequalities are strict for \( n \neq 0 \). In particular, for \( \delta \geq 2 \) we have
\[ K_\delta(n) \sim |n| \]
for any \( n \in \mathbb{Z}^* \). Furthermore, for each fixed \( n \in \mathbb{Z}^* \), \( K_\delta(n) \) is strictly increasing in \( \delta \geq 1 \) and converges to \( |n| \) as \( \delta \to \infty \).

We now observe that
\[
p_\delta^{(s)}(n) = n \left( n \coth(\delta n) - \frac{1}{\delta} \right) = nK_\delta(n) \in C^4(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})
\]
is a real-valued odd function. Moreover, Lemma 2.1.3 implies that for any \( \delta \geq 2 \)
\[
|p_\delta^{(s)}(n)| = \left| n \left( n \coth(\delta n) - \frac{1}{\delta} \right) \right| \sim |n|^2
\]
for any \( n \in \mathbb{Z}^* \). Therefore, we can deduce that there exists some \( n_0 > 0 \) such that for all \( n \geq n_0 \), we have
\[
|\partial_n p_\delta^{(s)}(n)| \sim n \quad \text{and} \quad |\partial_n^2 p_\delta^{(s)}(n)| \sim 1.
\]

Next, we define \( L_\delta(n) = \frac{3}{2} K_\delta(n) \). Then, we state some basic properties of \( L_\delta(n) \) in the following. Given \( \delta > 0 \), it follows from \( L_\delta(n) = \frac{3}{2} K_\delta(n) \) and Lemma 3.1.1 that
\[ L_\delta(n) \sim_\delta |n| \]
for any \( n \in \mathbb{Z}^* \). We then state the following lemma, which helps us to understand the basic properties of \( L_\delta(n) \). See Lemma 3.1.3 for more details.

**Lemma 2.1.4.** Let \( L_\delta(n) = \frac{3}{2} K_\delta(n) \). The following statements hold.

(i) \( 0 < L_\delta(n) < n^2 \) for any \( \delta > 0 \) and \( n \in \mathbb{Z}^* \).

(ii) For each \( n \in \mathbb{Z}^* \), \( L_\delta(n) \) increases to \( n^2 \) as \( \delta \to 0 \).

(iii) We have
\[
L_\delta(n) \gtrsim \begin{cases} 
 n^2, & \text{if } \delta |n| \lesssim 1, \\
 |n|, & \text{if } \delta |n| \gg 1 \text{ and } \delta \lesssim 1.
\end{cases}
\]

In particular, the following uniform bound holds:
\[ \inf_{0 < \delta \lesssim 1} L_\delta(n) \gtrsim |n| \]
for any \( n \in \mathbb{Z}^* \).

One can see that

\[
p^{(s)}_δ(n) = \frac{3}{δ} n \left( n \coth(δn) - \frac{1}{δ} \right) = \frac{3}{δ} n K_δ(n) = n L_δ(n) ∈ C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})
\]

is a real-valued odd function in \( n \). Moreover, we see from Lemma 2.1.4(iii) that for any \( 0 < δ \ll 1 \) there is a uniform lower bound for all \( n ∈ \mathbb{Z}^* \). Hence, we can deduce that there exists some \( n_0 > 0 \) such that for all \( n ≥ n_0 \), we have

\[
n ≤ δ_0 p^{(s)}_δ(n) ≤ C(δ)n ≤ n^2 \quad \text{and} \quad 1 ≤ δ^2_0 p^{(s)}_δ(n) ≤ C(δ).
\]

**Remark 2.1.5.** Here, the uniform lower bound is crucial in our later counting argument; see Lemma 2.2.4 and the proof of Lemma 2.2.3. Moreover, although the upper bound depends on \( δ \), where the the constant \( C(δ) \) increases as \( δ \) decreases to 0, we have the trivial upper bound that is independent of \( δ \). These are enough to get the uniform estimate in Lemma 2.2.3.

**Lemma 2.1.6** ([157] Lemma 3.1). Let \( δ > 0 \) and \( p^{(s)}_δ(n) = n(n \coth(δn) - \frac{1}{δ}) \). Then, we have the following statements:

\[
\begin{align*}
|p^{(d)}_δ(n)| & ≤ |n|^2, \quad |∂p^{(d)}_δ(n)| = |n|, \quad |∂^2 p^{(d)}_δ(n)| = |n|; \quad \text{if } |n| ≥ \frac{1}{δ}, \\
|p^{(d)}_δ(n)| & ≤ δ|n|^3, \quad |∂p^{(d)}_δ(n)| = δ|n|^2, \quad |∂^2 p^{(d)}_δ(n)| = δ|n|; \quad \text{if } |n| < \frac{1}{δ}.
\end{align*}
\]

### 2.1.3 Function spaces and their basic properties

We finish this section by introducing the function spaces and their properties. Firstly, we introduce a sequence of positive numbers \( \{ω_N\} \), which is an increasing sequence depending on the dyadic number \( N ∈ 2^{Z^2} \). This dyadic sequence of weights \( \{ω_N\} \) is known as a frequency envelope in Section 5. The main purpose of introducing such weight is to be useful in the proof of continuity with respect to initial data. This delicate method was introduced in [90]. Moreover, we collect the following result from [119] Lemma 4.6, which helps us to choose our frequency envelope \( ω_N \).

**Lemma 2.1.7** ([119] Lemma 4.6). Let \( κ > 1 \), suppose the dyadic sequence \( \{ω_N\} \) of positive numbers satisfies

\[
ω_N ≤ ω_{2N} ≤ κω_N \quad \text{for} \quad N ≥ 1,
\]

and \( ω_N → ∞ \) as \( N → ∞ \). Then, for any \( 1 < κ' < κ \), there exists a dyadic sequence \( \{\tilde{ω}_N\} \) such that

\[
\tilde{ω}_N ≤ ω_N, \quad \tilde{ω}_N ≤ \tilde{ω}_{2N} ≤ κ'\tilde{ω}_N \quad \text{for} \quad N ≥ 1
\]

and \( \tilde{ω}_N → ∞ \) as \( N → ∞ \).

For a given dyadic sequence \( \{ω_N\} \) of positive numbers, Lemma 2.1.7 allows us to take \( κ ≤ 2 \). Then, we can define a new dyadic sequence. In practice, see in the proof of Proposition 2.3.9. Let \( N, M \) be dyadic numbers such that \( 1 ≤ M ≤ ℓN \) for some \( ℓ ≥ 2 \). By using \( ω_{2N} ≤ κω_N \), the following holds

\[
\frac{ω_M}{ω_N} ≤ K^{log_2 ℓ} ≤ ℓ
\]

which is uniformly in \( κ \).

Then, we slightly modify the classical Sobolev spaces in the following way: let \( s ≥ 0 \), we define \( H^s_ω(\mathbb{T}) \) associated with the norm

\[
∥u∥_{H^s_ω} := \left( \sum_{N, \text{dyadic}} ω_N^s (1 ∨ N)^{2s} ∥P_N u∥_{L^2}^2 \right)^{\frac{1}{2}}.
\]
One simple observation is that we recover our usual $L^2$-based Sobolev space by choosing $\omega_N \equiv 1$. Namely, $H^s_0(\mathbb{T}) = H^s(\mathbb{T})$ with $\omega_N \equiv 1$.

Let $1 \leq p \leq \infty$ and $T > 0$. Let $B_x$ be any Banach space. Then, we define the following short-hand notation:

$$L^p_x B_x := L^p(\mathbb{R}; B_x) \quad \text{and} \quad L^p_T B_x := L^p([0, T]; B_x)$$

equipped with the norms

$$\|u\|_{L^p_x B_x} = \left( \int_{\mathbb{R}} \|u(t, \cdot)\|_{B_x}^p dt \right)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_{L^p_T B_x} = \left( \int_0^T \|u(t, \cdot)\|_{B_x}^p dt \right)^{\frac{1}{p}},$$

respectively. Note for $p = \infty$ case, and we modify the physical space with the vector space of essentially bounded measurable functions with the essential supremum norm.

As in the usual low-regularity analysis of dispersive PDEs, a crucial ingredient is the Fourier restriction norm method introduced in [18, 19]. Given $s, b, \delta$, with the norm

$$\|u\|_{X^{s,b,\delta}} = \left( \int_{-\infty}^{\infty} (1 + \tau)^{2b} |\hat{u}(\tau, n)|^2 d\tau \right)^{\frac{1}{2}},$$

we can modify the physical space with the vector space of essentially bounded measurable functions with the essential supremum norm.

Moreover, we define the function spaces $M^{s,\delta}$ and $M^{s,\delta}$ in the following way.

$$M^{s,\delta} := L^\infty_t H^s \cap X^{s-1,1,\delta} \quad \text{and} \quad M^{s,\delta} := L^\infty_t H^s \cap X^{s-1,1,\delta},$$

endowed with the natural norm

$$\|u\|_{M^{s,\delta}} = \|u\|_{L^\infty_t H^s} + \|u\|_{X^{s-1,1,\delta}} \quad \text{and} \quad \|u\|_{M^{s,\delta}} = \|u\|_{L^\infty_t H^s} + \|u\|_{X^{s-1,1,\delta}}. \quad (2.1.17)$$

In the same line as above, we define the function spaces $N^{s,\delta}$ and $N^{s,\delta}$:

$$N^{s,\delta} := L^\infty_t H^s \cap Y^{s-1,1,\delta} \quad \text{and} \quad N^{s,\delta} := L^\infty_t H^s \cap Y^{s-1,1,\delta},$$

endowed with the natural norm

$$\|v\|_{N^{s,\delta}} = \|v\|_{L^\infty_t H^s} + \|v\|_{Y^{s-1,1,\delta}} \quad \text{and} \quad \|v\|_{N^{s,\delta}} = \|v\|_{L^\infty_t H^s} + \|v\|_{Y^{s-1,1,\delta}}. \quad (2.1.18)$$

We also use the restriction in time versions of these spaces. Let $T > 0$ be a positive time and $B$ be
a normed space of space-time functions. The restriction space $B_T$ will be the space of functions $u : (0, T) \times \mathbb{T} \to \mathbb{R}$ satisfying

$$\|u\|_{B_T} := \inf\{\|\tilde{u}\|_{\mathbb{R} \times \mathbb{T}} : \tilde{u} = u \text{ on } (0, T) \times \mathbb{T}\} < \infty.$$  

Let us define the linear propagator of (scaled) gILW in the following:

$$S^{(4)}_s(t) = e^{-i\varphi_s \partial_x^2}, \quad S^{(s)}_s(t) = e^{-i\varphi_s \partial_x^2}.$$  

Finally, we introduce a bounded linear operator from $X^{s-1,1,\delta}_\omega,T \cap L^\infty_T H^s_\omega$ into $M^{s,\delta}_\omega$ with a bound which does not depend on $s$, $T$, and $\delta$. Similarly, we have a bounded linear operator from $Y^{s-1,1,\delta}_\omega,T \cap L^\infty_T H^s_\omega$ into $N^{s,\delta}_\omega$. The existence of such an operator guarantees that

$$X^{s-1,1,\delta}_\omega,T \cap L^\infty_T H^s_\omega = M^{s,\delta}_\omega \quad \text{and} \quad Y^{s-1,1,\delta}_\omega,T \cap L^\infty_T H^s_\omega = N^{s,\delta}_\omega.$$  

The following discussion is for the deep-water situation, but it is generally true for both deep and shallow-water settings. We take the same definition from [103] such that $\rho_T$ is of the following form

$$\rho_T(u)(t) := S^{(4)}_s(t)\eta(t)S^{(s)}_s(-\mu_T(t))u(\mu_T(t)), \quad (2.1.20)$$

where $\mu_T$ is the continuous piecewise affine function defined by

$$\mu_T(t) = \begin{cases} 
0 & \text{for } t \notin (0, 2T), \\
t & \text{for } t \in [0, T], \\
2T - t & \text{for } t \in [T, 2T]. 
\end{cases}$$

Then, we have the following existence lemma of $\rho_T$. It also follows from [119], and proof can be seen in [117] Lemma 2.4, where $\omega_N \equiv 1$, but it is obvious that the result does not depend on $\omega_N$.

**Lemma 2.1.8** ([119] Lemma 2.1). Take a dyadic sequence $\{\omega_N\}$ of positive numbers satisfying the assumptions of Lemma 2.1.7. Let $s \in \mathbb{R}$ and $0 < T \leq 1$. Then, we have

$$\rho_T : X^{s-1,1,\delta}_\omega,T \cap L^\infty_T H^s_\omega \to M^{s,\delta}_\omega$$

$u \mapsto \rho_T(u)$

is a bounded linear operator. In particular,

$$\|\rho_T(u)\|_{L^\infty_T H^s_\omega} + \|\rho_T(u)\|_{L^{s-1,1,\delta}_\omega} \lesssim \|u\|_{L^\infty_T H^s_\omega} + \|u\|_{X^{s-1,1,\delta}_\omega},$$  

for all $u \in X^{s-1,1,\delta}_\omega,T \cap L^\infty_T H^s_\omega$. Moreover, it holds that

$$\|\rho_T(u)\|_{L^\infty_T H^s_\omega} \lesssim \|u\|_{L^\infty_T H^s_\omega}$$  

for all $u \in L^\infty T H^s_\omega$. Here, the implicit constants in (2.1.21) and (2.1.22) can be chosen independent of $T$, $\delta$, and $s \in \mathbb{R}$. Furthermore, the same statement holds from $Y^{s-1,1,\delta}_\omega,T \cap L^\infty_T H^s_\omega$ into $N^{s,\delta}_\omega$.

In the rest of this section, we collect some known results, see [119] Lemma 2.2. We remark that Lemma 2.1.11 - commutator-type estimate can be seen as a variant of integration by parts. This type of argument can be seen in [100] Lemma 3.3.

**Lemma 2.1.9** ([119] Lemma 2.2). Let $s > 0$ and take a dyadic sequence $\{\omega_N\}$ of positive numbers satisfying the assumption of Lemma 2.1.7. Then, we have the estimate

$$\|uv\|_{H^s_\omega} \lesssim \|u\|_{H^s_\omega}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^s_\omega}.$$  

In particular, for any fixed real entire function $f$ with $f(0) = 0$, there exists a real entire function
\( G = G[f] \) that is increasing non-negative on \( \mathbb{R}_+ \) such that
\[
\|f(u)\|_{H^s_{\omega}} \lesssim G(\|u\|_{L^\infty}) \|u\|_{H^s_{\omega}}.
\] (2.1.24)

**Lemma 2.1.10** ([119] Lemma 2.3). Let \( s_1 + s_2 \geq 0, \ s_1 \wedge s_2 \geq s_3, \) and \( s_3 < s_1 + s_2 - \frac{1}{2} \). Then,
\[
\|uv\|_{H^{s_3}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}.
\] (2.1.25)

In particular, let \( u, v \in H^s(\mathbb{T}) \) for \( s > \frac{1}{4} \). Then, there exists a increasing and non-negative real entire function \( G = G[f] \) on \( \mathbb{R}_+ \) such that
\[
\|f(u) - f(v)\|_{H^{s-1}} \leq G(\|u\|_{H^s} + \|v\|_{H^s}) \|u - v\|_{H^{s-1}}.
\] (2.1.26)

**Lemma 2.1.11** ([119] Lemma 2.4). Let \( N \in 2^{\mathbb{Z}_+} \) be a dyadic number. Then, there exists a positive constant \( C \) such that for every \( u, v \in L^2(\mathbb{T}) \), and \( \partial_x w \in L^\infty(\mathbb{T}) \), the following holds
\[
\left| \int_{\mathbb{T}} \Pi(u,v)wdx \right| \leq C \|u\|_{L^2_\omega} \|v\|_{L^2_\omega} \|\partial_x w\|_{L^\infty_\omega}.
\]
where
\[
\Pi(u,v) := v\partial_x p_N^2 u + u\partial_x p_N^2 v.
\] (2.2.27)

### 2.2 Uniform linear estimates

The main goal of the section is to establish the following \( \delta \)-independent short-time Strichartz estimates, which was first introduced in [90]. The main strategy follows from [119]. However, the uniform estimates need to be established separately in the shallow-water and deep-water regimes.

**Proposition 2.2.1.** Let \( s > \frac{1}{2} \), \( 0 < T < 1 \), and \( \{\omega_N\} \) satisfies Lemma 2.1.7. Take \( u, v \in C([0,T];H^s_\omega(\mathbb{T})) \) satisfies gLW (2.1.1) and scaled gLW (2.1.2) with initial data \( u_0, v_0 \in H^s_\omega(\mathbb{T}) \) on \([0,T] \), separately. Then, the following statements hold:

(i) for any \( 2 \leq \delta \leq \infty \), we have
\[
\left( \sum_{N, \text{dyadic}} \omega_N^4 \|D_x^{-\delta} p_N u\|_{L^4([0,T];L^4(\mathbb{T}))} \right)^\frac{1}{2} \leq C T^\frac{\delta}{4} \|u\|_{L^\infty([0,T];H^s_{\omega}(\mathbb{T}))};
\] (2.2.1)
\[
\left( \sum_{N, \text{dyadic}} \|D_x^\delta p_N u\|_{L^4([0,T];L^4(\mathbb{T}))} \right)^\frac{1}{4} \leq C T^\frac{8}{\delta} \|u\|_{L^\infty([0,T];H^{s+\frac{3}{2}}_{\omega}(\mathbb{T}))};
\] (2.2.2)

(ii) for any \( 0 < \delta \ll 1 \), we have
\[
\left( \sum_{N, \text{dyadic}} \omega_N^4 \|D_x^{-\delta} p_N v\|_{L^4([0,T];L^4(\mathbb{T}))} \right)^\frac{1}{2} \leq \tilde{C} T^\frac{\delta}{4} \|v\|_{L^\infty([0,T];H^s_{\omega}(\mathbb{T}))};
\] (2.2.3)
\[
\left( \sum_{N, \text{dyadic}} \|D_x^\delta p_N v\|_{L^4([0,T];L^4(\mathbb{T}))} \right)^\frac{1}{4} \leq \tilde{C} T^\frac{8}{\delta} \|v\|_{L^\infty([0,T];H^{s+\frac{3}{2}}_{\omega}(\mathbb{T}))};
\] (2.2.4)

Here, the constant \( C = C(\|u\|_{L^P_{\omega}}) \) and \( \tilde{C} = \tilde{C}(\|v\|_{L^P_{\omega}}) \) is independent of \( \delta \).

Before we can prove Proposition 2.2.1, we need a few ingredients. Let us start with the following \( L^4 \)-Strichartz estimate, originally introduced in [119]. We will adapt it to our dispersion relations (2.1.3).

**Lemma 2.2.2.** Let \( u \in X^0_{\frac{3}{2}}(\mathbb{T} \times \mathbb{R}) \) and \( v \in Y^0_{\frac{3}{2}}(\mathbb{T} \times \mathbb{R}) \). Then, the following estimates hold:
(i) for any \( 2 \leq \delta \leq \infty \), we have
\[
\|u\|_{L^4(\mathbb{R};L^4(\mathbb{T}))} \leq C \|u\|_{X^0_{\frac{3}{2}}(\mathbb{T} \times \mathbb{R})};
\] (2.2.5)
for any $0 < \delta \ll 1$, we have
\[
\|v\|_{L^4(\mathbb{T})} \leq C \|v\|_{Y^{0,\frac{1}{4}}(\mathbb{T} \times \mathbb{R})}.
\] (2.2.6)

Here, the constant $C$ is independent of $\delta$.

**Proof.** The proof follows from the next lemma (see the Appendix in [111] for similar considerations). We briefly explain the strategy for its completeness.

- For the inequality (2.2.6),

We will thus work in the function space $Y^{0,b,\delta}$ endowed with the norm:
\[
\|v\|_{Y^{0,b,\delta}} = \left( \sum_{\tau \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |\tau - \langle \psi^{(s)}(n) \rangle_{2b}|^2 \langle \hat{v}(\tau,n) \rangle^2 \right)^{\frac{1}{2}}
\]

Let $v \in Y^{0,b,\delta}$ for $b \geq \frac{3}{8}$. We use Littlewood-Paley projection (decompose to the modulation) on $v$:
\[
v = \sum_{M, \text{dyadic}} v_M
\]

where $\text{supp} (\hat{v}_M) \subset \{(\tau, n) \in \mathbb{Z}^2 | (\tau - \langle \psi^{(s)}(n) \rangle_{2b}) \sim M\}$. Then, we have
\[
\sum_{M, \text{dyadic}} M^{2b} \|v_M\|_{L^4_{\tau,x}}^2 \sim \|v\|_{X^{0,b}}^2.
\]

By rewriting $M_1$ as $M_1 = 2^k M_2$ with $k \in \mathbb{N}$, we also have
\[
\|v\|_{L^4_{\tau,x}}^2 \leq \sum_{M_1, M_2, \text{dyadic}} \|v_{M_1} u_{M_2}\|_{L^4_{\tau,x}} \lesssim \sum_{k=0}^{\infty} \sum_{M, \text{dyadic}} \|v_M v_{2^k} M\|_{L^4_{\tau,x}}.
\]

Let us now assume Lemma 2.2.3 is true for now. Then, with this lemma in hand, we get the following computation
\[
\sum_{k=0}^{\infty} \sum_{M, \text{dyadic}} \|v_M v_{2^k} M\|_{L^4_{\tau,x}} \lesssim \sum_{k=0}^{\infty} \sum_{M, \text{dyadic}} M^{\frac{1}{2}} (2^k M)^{\frac{1}{2}} \|v_M v_{2^k} M\|_{L^4_{\tau,x}}
\]
\[
\lesssim \sum_{k=0}^{\infty} \sum_{M, \text{dyadic}} M^{\frac{1}{2}} \|v_M\|_{L^4_{\tau,x}} (2^k M)^{\frac{1}{2}} 2^{-\frac{k}{2}} \|v_{2^k} M\|_{L^4_{\tau,x}}
\]
\[
\lesssim \sum_{k=0}^{\infty} 2^{-\frac{k}{2}} \left( \sum_{M, \text{dyadic}} M^{\frac{1}{2}} \|v_M\|_{L^4_{\tau,x}}^2 \right)^{\frac{1}{2}} \left( \sum_{M, \text{dyadic}} (2^k M)^{\frac{1}{2}} \|v_{2^k} M\|_{L^4_{\tau,x}}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \|v\|_{X^{0,\frac{3}{8}}}^2.
\] (2.2.7)

- For the inequality (2.2.5),

We follow exactly the same steps, but this time we work in the function space $X^{0,b}$ endowed with the norm
\[
\|u\|_{X^{0,b}} = \left( \sum_{\tau \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |\tau - \langle \psi^{(d)}(n) \rangle_{2b}|^2 \langle \hat{u}(\tau,n) \rangle^2 \right)^{\frac{1}{2}}
\]

Then, let $v \in X^{0,b,\delta}$ for $b \geq \frac{3}{8}$. The rest of the steps follow same lines of showing (2.2.6).

The following lemma completes the proof of Lemma 2.2.2. Indeed, we see by Plancherel theorem,
\[
\|u_{M_1} u_{M_2}\|_{L^4_{\tau,x}} = \|\hat{u}_{M_1} *_{\tau,n} \hat{u}_{M_2}\|_{L^4_{\tau,x}}
\]
it suffices to show the following lemma.

**Lemma 2.2.3.** Let \( u, v \in L^2(\mathbb{R}; \ell^2(\mathbb{Z})) \) be any real-valued functions and \( N_1, N_2, M, \bar{M} \in 2^{\mathbb{Z}_{>0}} \) be any dyadic numbers. Set \( M = \min\{N_1, N_2\} \) and \( \bar{M} = \max\{N_1, N_2\} \). Then, the following statements hold:

(i) for any \( 2 \leq \delta \leq \infty \), we have

\[
\| (\psi_{N_1}u) \ast_{\tau} (\psi_{N_2}v) \|_{L^2_\delta} \leq CM^{\frac{1}{2}}M^{\frac{1}{4}}\|\psi_{N_1}u\|_{L^2_\delta} \|\psi_{N_2}v\|_{L^2_\delta}.
\]

(ii) for any \( 0 < \delta \ll 1 \), we have

\[
\| (\psi_{N_1}u) \ast_{\tau} (\psi_{N_2}v) \|_{L^2_\delta} \leq CM^{\frac{1}{2}}M^{\frac{1}{4}}\|\psi_{N_1}u\|_{L^2_\delta} \|\psi_{N_2}v\|_{L^2_\delta}.
\]

The constant \( C = C(n_0) > 0 \) is independent of \( \delta \), where \( n_0 \) is a fixed number, and \( \psi_N \) is the projection on the modulation function.

The proof of Proposition 2.2.3 requires the following counting lemma, which follows from [149, Lemma 2].

**Lemma 2.2.4.** Let \( I \) and \( J \) be two intervals on the real line and \( g \in C^1(J; \mathbb{R}) \). Then,

\[
\# \{ x \in J \cap \mathbb{Z}; g(x) \in I \} \leq \frac{|I|}{\inf_{x \in J} |g'(x)|} + 1.
\]

In what follows, we prove Lemma 2.2.3. The argument closely related to [119, Lemma 3.2], see also [153, Lemma 3.1]. The main ingredient is the uniform lower bounds (2.1.11) and (2.1.15). This is the crucial point in applying Lemma 2.2.4 such that our following counting part (cardinality of the set) is independent of \( \delta \).

**Proof of Lemma 2.2.3.** We start with **Step 1** in the following, which will reduce our problems into a counting issue of some sets. We remark that this part of the proof applies to both shallow-water and deep-water regimes. Indeed, we only change \( p^{(d)}_h(n) \) to \( p^{(d)}_b(n) \) in (2.2.8) and (2.2.9).

**• Step 1: reducing the problem into a counting question**

Let \( \psi_N \) be the projection on the modulation function. For example, \( \psi_N u \) is the localisation of \( u \) on \( N \leq \langle \tau - p^{(d)}_h(n) \rangle < 2N \). Then, we may assume that

\[
\sup_n (\psi_N u), \sup_n (\psi_N v) \subset \mathbb{N}
\]

since \( u \) and \( v \) are real-valued. By using the relation

\[
\tau = \tau_1 + \tau_2 \quad \text{and} \quad n = n_1 + n_2,
\]

the Cauchy–Schwarz inequality in \((\tau_1, n_1)\) gives

\[
\| (\psi_{N_1}u) \ast_{\tau} (\psi_{N_2}v) \|_{L^2_\delta}^2 \\
= \sum_{n = 0}^{\infty} \int_{\mathbb{R}} \left( \sum_{n \geq n_1} \int_{\mathbb{R}} \psi_{N_1}(\tau_1, n_1)u(\tau_1, n_1)\psi_{N_2}(\tau - \tau_1, n - n_1)v(\tau - \tau_1, n - n_1) d\tau_1 \right)^2 d\tau \\
\lesssim \sup_{(\tau, n) \in \mathbb{R} \times \mathbb{N}} A(\tau, n) \|\psi_{N_1}u\|_{L^2_\delta}^2 \|\psi_{N_2}v\|_{L^2_\delta}^2,
\]

where \( A(\tau, n) \) is the following

\[
A(\tau, n) = \# \{ (\tau, n) \in \mathbb{R} \times \mathbb{N}; (\tau_1, n_1) \in \sup_n (\psi_N u); (\tau_2, n_2) \in \sup_n (\psi_N v) \} \\
\lesssim \# \{ (\tau_1, n_1) \in \mathbb{R} \times \mathbb{N}; n - n_1 \geq 0, (\tau_1 - p^{(d)}_h(n_1)) \sim N_1, \quad (\tau - \tau_1 - p^{(d)}_b(n - n_1)) \sim N_2 \} \quad \text{(2.2.8)}
\]

\[
\lesssim (N_1 \cap N_2) \# B(\tau, n),
\]

and

\[
\#(\tau, n) \in A, \quad \#(\tau, n) \geq 1,
\]

\[
\sum_{(\tau, n) \in A} \#(\tau, n) \leq \# \{ (\tau, n) \in \mathbb{R} \times \mathbb{N}; (\tau_1, n_1) \in \sup_n (\psi_N u); (\tau_2, n_2) \in \sup_n (\psi_N v) \} \\
\lesssim \# \{ (\tau_1, n_1) \in \mathbb{R} \times \mathbb{N}; n - n_1 \geq 0; (\tau_1 - p^{(d)}_h(n_1)) \sim N_1, \quad (\tau - \tau_1 - p^{(d)}_b(n - n_1)) \sim N_2 \} \quad \text{(2.2.8)}
\]

\[
\lesssim (N_1 \cap N_2) \# B(\tau, n),
\]

\[
\sum_{(\tau, n) \in A} \#(\tau, n) \leq \# \{ (\tau_1, n_1) \in \mathbb{R} \times \mathbb{N}; n - n_1 \geq 0; (\tau_1 - p^{(d)}_h(n_1)) \sim N_1, \quad (\tau - \tau_1 - p^{(d)}_b(n - n_1)) \sim N_2 \} \quad \text{(2.2.8)}
\]

\[
\lesssim (N_1 \cap N_2) \# B(\tau, n),
\]
where \# we denote as the cardinality of the set, and with the following (for simplicity, we set $M := N_1 \lor N_2$)

\[
B(\tau, n) = \{n_1 \geq 0 \mid n - n_1 \geq 0, (\tau - p_\delta^{(d)}(n_1) - p_\delta^{(d)}(n - n_1)) \lesssim N_1 \lor N_2\}
\]

\[
= \{n_1 \geq 0 \mid n - n_1 \geq 0, (\tau - p_\delta^{(d)}(n_1) - p_\delta^{(d)}(n - n_1)) \lesssim M\}
\]  

(2.2.9)

- **Step 2: counting the cardinalities of set $B$.**

  The essential tool in this step is Lemma 2.2.4. For the convenience we define function $g(\cdot)$ to be

  \[
g(n_1) := \tau - p_\delta^{(d)}(n_1) - p_\delta^{(d)}(n - n_1),
\]

  where $p_\delta^{(d)}(\cdot)$ we will replace it to be $p_\delta^{(s)}(\cdot)$ in the shallow-water regime. Moreover, the uniform lower bound on the function $g(\cdot)$ is the key point in this proof.

  **Case I: deep-water region ($2 \leq \delta \leq \infty$).**

  Let us take $u, v$ to be solutions to (2.1.1). Then, we recall from (2.1.10), the definition of $p_\delta^{(d)}(n)$ such that

  \[
  C_1|n|^2 \leq p_\delta^{(d)}(n) = n\left(n \coth(\delta n) - \frac{1}{\delta}\right) = nK_\delta(n) \leq C_2|n|^2,
  \]

  for some constants $C_1, C_2 > 0$ are independent of $\delta$.

  Now, we observe that the cardinality of the set $B$ is divided into low-frequency contribution and high-frequency contribution. When $n \leq M^\frac{1}{2} + 2n_0 + 2$, we start with the following low-frequency case:

  **Case I.a: low-frequency contribution.**

  Let $n \leq M^\frac{1}{2} + 2n_0 + 2$, where $n_0$ is fixed such that we have (2.1.11). Also, we note that we have $0 \leq n_1 \leq n$ and $M \geq 1$ is the dyadic number. Hence, for the fixed $(\tau, n)$. The following bound holds

  \[
  \#B(\tau, n) \leq M^\frac{1}{2} + 2n_0 + 2 \leq C(n_0)M^\frac{1}{2},
  \]

  where the constant depends only on the choice of $n_0$.

  **Case I.b: high-frequency contribution.**

  Now, we consider $n \geq C(n_0)M^\frac{1}{2}$, and recall that $g(n_1) := \tau - p_\delta^{(d)}(n_1) - p_\delta^{(d)}(n - n_1)$. By fixing $(\tau, n)$ and counting $n_1$ on set $B$. Then, we yield the following

  \[
  \#B(\tau, n) \lesssim \#\{n_1 \geq 0 \mid n \geq 2n_1 \text{ and } |g(n_1)| \lesssim M\}
  \]

  \[
  \leq [n_0] + 1 + \#\{n_1 \geq 0 \mid n \geq 2n_1, n_1 \geq n_0, \text{ and } |g(n_1)| \lesssim M\}
  \]

  \[
  \lesssim \#\{n_1 \in [n_0, \frac{n}{2}] \mid n - 2n_1 \geq M, \text{ and } |g(n_1)| \lesssim M\} + [n_0] + M^\frac{3}{2},
  \]

  where $[n_0]$ is an integral part of $n_0$, and the implicit constant is independent of $\delta$. For the first inequality, we used symmetry. Moreover, we notice that since $(n - 1 - \frac{1}{2}L) \geq n_0$ by the assumption, which guarantees the set on the right-hand side is non-empty.

  In order to complete the counting argument, we will apply Lemma 2.2.4. It requires the uniform lower bound on the first derivative of function $g(\cdot)$, which is the following

  \[
  |\partial_{n_1}g(n_1)| = |\partial_{n_1}(p_\delta^{(d)}(n_1) - p_\delta^{(d)}(n - n_1))|
  \]

  \[
  \geq \left|\int_{n_1}^{n - n_1} \partial^2_{xx}p_\delta^{(d)}(x)dx\right| \gtrsim n - 2n_1 \geq M^\frac{1}{2}
  \]

  The above computation makes sense since for any $\delta > 0$ and $x \geq n_1$, the function $\partial^2_{xx}p_\delta^{(d)}(x)$ does not change sign (as $|\partial^2_{xx}p_\delta^{(d)}(x)| \sim 1$). Also, we see $\partial^2_{xx}p_\delta^{(d)}(x)$ is continuous outside 0. Therefore,
Lemma 2.2.4 implies the following:
\[ \# \{ n_1 \in [n_0, \frac{3}{2}] \} \left| n - 2n_1 \right| \geq M, \text{ and } |g(n_1)| \lesssim M \leq M^\frac{3}{2}. \]

Hence, together with Case I.a and Case I.b, we conclude that \( \# \mathcal{B}(\tau, n) \leq C(n_0)M^\frac{3}{2}. \)

Case II: shallow-water region (0 < \( \delta \ll 1 \)).

Let us assume now \( u, v \) are solutions to (2.1.2). Namely, now the projection \( \psi_N u \) is the localisation of \( u \) on \( N \leq (\tau - p^{(s)}_\delta(n)) < 2N \). We recall \( p^{(s)}_\delta(n) \) such that\[ Cn^2 \leq p^{(s)}_\delta(n) = \frac{3n}{\delta} nK_\delta(n) = nL_\delta(n) \leq C_\delta n^2 \]
where \( C \) appears on the lower bound is independent of \( \delta \) and \( C_\delta \) is some constant strictly increasing as \( \delta \) decreases to 0. See Remark 2.1.6.

We follow the same strategy as in Case I such that divide our situations into low-frequency and high-frequency contributions. When \( n \leq M^{\frac{3}{2}} + 2n_0 + 2 \), we have the low-frequency case:

Case II.a: low-frequency contribution.

For the low-frequency contribution, let \[ n \leq M^{\frac{3}{2}} + 2n_0 + 2, \]
where \( n_0 \) is fixed such that we have (2.1.15). Also, we note that we have \( 0 \leq n_1 \leq n \) and \( M \geq 1 \) is a dyadic number. Thus, let us fix \((\tau, n)\). Then, we obtain
\[ \# \mathcal{B}(\tau, n) \leq M^{\frac{3}{2}} + 2n_0 + 2 \leq C(n_0)M^\frac{3}{2}, \]
for some constant depends only on the choice of \( n_0 \).

• High-frequency contribution:

For the high-frequency contribution, we consider \[ n \geq M^{\frac{3}{2}} + 2n_0 + 2. \]

Let us use \( g(\cdot) \) again, to define the following function,
\[ g(n_1) := \tau - p^{(s)}_\delta(n_1) - p^{(s)}_\delta(n - n_1). \]

Let \((\tau, n)\) to be fixed again. Then, we obtain
\[ \# \mathcal{B}(\tau, n) \lesssim \{ n_1 \geq 0 \mid n \geq 2n_1 \text{ and } |g(n_1)| \lesssim M \} \]
\[ \leq [n_0] + \{ n_1 \geq 0 \mid n \geq 2n_1, n_1 \geq n_0, \text{ and } |g(n_1)| \lesssim M \} \]
\[ \lesssim \{ n_1 \in [n_0, \frac{3}{2}] \} \mid n - 2n_1 \mid^2 \gtrsim M, \text{ and } |g(n_1)| \lesssim M \} + [n_0] + M^{\frac{3}{2}}, \]
where \([n_0]\) is an integral part of \( n_0 \), and the implicit constant is independent of \( \delta \). Again, symmetry is applied to the first inequality. Moreover, the high-frequency assumption guarantees the set on the right-hand side is non-empty.

In the following, we will see the uniform lower bound of \( p^{(s)}_\delta(n) \) is now the crucial part to apply Lemma 2.2.4 and again obtain a uniform cardinality on the set \( B \). Hence, follows from the definition of \( g(n) \) and the uniform lower bound of \( p^{(s)}_\delta(n) \):
\[ \left| \partial_{n_1} g(n_1) \right| = \left| \partial_{n_1} (p^{(s)}_\delta(n_1) - p^{(s)}_\delta(n - n_1)) \right| \]
\[ = \left| \int_{n_1}^{n - n_1} \partial^2_{x_2} p^{(s)}_\delta(x) \right| dx \gtrsim n - 2n_1 \geq M^{\frac{3}{2}}. \]

One notice from the above computation is that for any \( \delta > 0 \) and \( x \geq n_1 \), the function \( \partial^2_{x_2} p^{(s)}_\delta(x) \) does not change sign (since \( 1 \lesssim |\partial^2_{x_2} p^{(s)}_\delta(x)| \lesssim 1 \)). Also, we see \( \partial^2_{x_2} p^{(s)}_\delta(x) \) is continuous outside 0.
Therefore, Lemma 2.2.4 implies the following estimate
\[ \# \{ n_1 \in [n_0, \frac{\delta}{2}] \mid n - 2n_1^2 \geq M, \text{ and } |g(n_1)| \lesssim M \} \lesssim M^\frac{1}{2}. \]

This completes the proof.

We recall the linear propagator of (scaled) gILW from (2.1.19). Lemma 2.2.2 allows us to establish the following Strichartz estimate, which deals with the linear solution to (scaled) ILW. The proof follows from Lemma 2.1 in [118], see also [119]. The main idea is to apply the $L^4$-Strichartz estimate, Lemma 2.2.2.

**Lemma 2.2.5.** Let $T > 0$, any $u, v \in L^2(\mathbb{T})$, and $S^{(1)}(t), S^{(2)}(t)$ as defined in (2.1.19). Then, there exists $C > 0$ independent of $\delta$ such that the following statements hold:

(i) for any $2 \leq \delta \leq \infty$, we have
\[ \| S^{(d)}(t)u \|_{L^4([0,T]; L^4(\mathbb{T}))} \leq CT^\frac{1}{2} \| u \|_{L^2(\mathbb{T})}. \]

(ii) for any $0 < \delta \ll 1$, we have
\[ \| S^{(s)}(t)v \|_{L^4([0,T]; L^4(\mathbb{T}))} \leq CT^\frac{1}{2} \| v \|_{L^2(\mathbb{T})}. \]

Now, with Lemma 2.2.5 we are ready to prove the short-time Strichartz estimates Proposition 2.2.1. The proof follows from [119], we decide to present the proof in the following to emphasise the $\delta$-independence of our linear estimates, and the essential tool in our proof is the uniform $\delta$-independence of our linear estimates, and the essential tool in our proof is the uniform Strichartz estimate - Lemma 2.2.2. This short-time Strichartz estimate was originally introduced in [90] to deal with the well-posedness problem of the BO equation. One crucial point of chopping the time interval into pieces (depending on the frequency truncation) is that we can gain a small power of local existence time $T^\theta$ for some $\theta > 0$.

**Proof of Proposition 2.2.1.** It suffices to consider the case $N \gg 1$. We first divide the time interval into small intervals of length $\sim TN^{-1}$, where $N$ is the size of the frequency projection. In doing so, let us define the intervals $\{I_{j,N}\}_{j \in J_N}$ such that
\[ \bigcup_{j \in J_N} I_{j,N} = [0, T], \quad |I_{j,N}| \lesssim TN^{-1}, \quad \text{and} \quad \#J_N \lesssim N. \]

By assumption that $u \in C([0, T]; H^s(\mathbb{T}))$, we see that $\| D_x^s P_N u(t) \|_{L^4(\mathbb{T})}^4 \in C([0, T])$. For $j \in J_N$, we let $c_{j,N} \in I_{j,N}$ to be the point where $\| D_x^s P_N u(t) \|_{L^4(\mathbb{T})}$ reach its minimum value on the interval $I_{j,N}$. We then see the following Duhamel formulation:
\[ P_N u(t) = S^{(k)}_{\delta}(t - c_{j,N}) P_N u(c_{j,N}) + \int_{c_{j,N}}^t S^{(k)}_{\delta}(t - t') P_N \partial_x(f(u))(t') dt', \]
for $t \in I_{j,N}$ and $k = d$ or $s$. We remark here for $k = 2$, we may change the corresponding solution to be $v \in C([0, T]; H^s_x(\mathbb{T}))$. Moreover, the following argument works the same for both Proposition 2.2.1 (i) and (ii). Furthermore, let us recall the following short-hand notation (with the time interval $I_{j,N}$):
\[ L^4(I_{j,N}; L^4(\mathbb{T})) = L^4_{I_{j,N}} L^4_x. \]

We first show the claims (2.2.1) and (2.2.3) in the following.

- **Linear contribution.** Without loss of generality, we consider the estimate for the deep-water regime, which is the following:
\[ S^{(d)}_{\delta}(t - c_{j,N}) P_N u(c_{j,N}) \]
Additionally, with Hölder’s inequality in time, we obtain

\[
\left( \sum_{N \geq 1} \sum_{j} \omega_N^4 \| D_x^{s-\frac{1}{2}} S^{(d)}_\delta (t - c_{j,N}) P_N u(c_{j,N}) \|_{L_{t}^{1} L_{x}^{4}} \right)^{\frac{1}{4}} \lesssim T^{\frac{1}{8}} \left( \sum_{N \geq 1} \sum_{j} \omega_N^4 \| I_{j,N} \|_{L_{t}^{1} L_{x}^{4}} \right)^{\frac{1}{4}}
\]

Therefore, we observe now we can apply the same strategies as in the proof of (2.2.1) and (2.2.3) to obtain the following bound:

\[
\left( \sum_{N \geq 1} \sum_{j} \omega_N^4 \| D_x^{s} P_N u(c_{j,N}) \|_{L_{t}^{1} L_{x}^{4}} \right)^{\frac{1}{4}} \lesssim T^{\frac{1}{8}} \left( \sum_{N \geq 1} \sum_{j} \omega_N^4 \| D_x^{s} P_N u(c_{j,N}) \|_{L_{t}^{1} L_{x}^{4}} \right)^{\frac{1}{4}} \lesssim T^{\frac{1}{8}} \| u \|_{L_{t}^{\infty} H_{x}^{s}}.
\]

Here, the implicit constant is independent of \( \delta \).

\textbf{Duhamel contribution.}

The second contribution comes from Duhamel’s term:

\[
\int_{c_{j,N}}^{t} S^{(d)}_\delta (t - t') P_N \partial_x (f(u))(t') dt'
\]

For simplify we set \( \tilde{f} = f - f(0) \). Then, in the following computation, we first apply Lemma 2.2.5 and use our constrain such that \( \| I_{j,N} \| \sim T N^{-1} \) to obtain the first inequality. Moreover, by using Hölder’s inequality in time and (2.1.24), we have the rest of the computations:

\[
\left( \sum_{N \geq 1} \sum_{j} \omega_N^4 \left\| \int_{c_{j,N}}^{t} S^{(d)}_\delta (t - t') D_x^{s-\frac{1}{2}} P_N \partial_x (f(u))(t') dt' \right\|_{L_{t}^{1} L_{x}^{4}} \right)^{\frac{1}{4}} \lesssim T^{\frac{1}{8}} \left( \sum_{N \geq 1} \sum_{j} \omega_N^4 \left( \int_{c_{j,N}}^{t} \| D_x^{s} P_N \tilde{f}(t') \|_{L_{x}^{4}} dt' \right)^{\frac{1}{4}} \right)^{\frac{1}{4}} \lesssim T^{\frac{1}{8}} \left( \sum_{N \geq 1} \sum_{j} \omega_N^4 \left( \int_{c_{j,N}}^{t} \| D_x^{s} P_N \tilde{f}(t') \|_{L_{x}^{4}} dt' \right)^{\frac{1}{4}} \right)^{\frac{1}{4}} \lesssim C T^{\frac{1}{8}} \| u \|_{L_{t}^{\infty} H_{x}^{s}}.
\]

Here, the constant \( C = C(\| u \|_{L_{t}^{\infty} L_{x}^{4}}) \) is independent of \( \delta \) and satisfies the condition in (2.1.24). This finish the proofs of (2.2.3) and (2.2.4).

Next, we sketch the proof for claims (2.2.2) and (2.2.4). From Bernstein’s inequality in spatial variables, we have the following

\[
\| P_N u \|_{L_{x}^{\infty}} \lesssim N^\frac{1}{2} \| P_N u \|_{L_{x}^{4}}.
\]

Additionally, with Hölder’s inequality in time, we obtain

\[
\| D_x^{s} P_N u \|_{L_{t}^{1} L_{x}^{4}} \lesssim \| I_{j,N} \|_{H_{x}^{s}} \left\| D_x^{s-\frac{1}{2}} P_N u \right\|_{L_{t}^{1} L_{x}^{4}} \lesssim T^{\frac{1}{8}} \left\| D_x^{s} P_N u \right\|_{L_{t}^{1} L_{x}^{4}}.
\]

This implies the following bound:

\[
\left( \sum_{N, \text{dyadic}} \sum_{j} \| D_x^{s} P_N u \|_{L_{t}^{1} L_{x}^{4}} \right)^{\frac{1}{4}} \lesssim T^{\frac{1}{8}} \left( \sum_{N, \text{dyadic}} \sum_{j} \| D_x^{s} P_N u \|_{L_{t}^{1} L_{x}^{4}} \right)^{\frac{1}{4}}. \quad (2.2.10)
\]

Therefore, we observe now we can apply the same strategies as in the proof of (2.2.1) and (2.2.3) with \( \omega_N = 1 \) to the right-hand-side of (2.2.10). Then, we deduce the claims (2.2.2) and (2.2.4).
CHAPTER 2. MICROSCOPIC LIMITS: DETERMINISTIC METHOD

hold.

By following up the Proposition 2.2.1, we obtain the estimates for the difference. In particular, the proof of Corollary 2.2.6 follows the same strategy as in the proof of Proposition 2.2.1. We set \( \omega N = 1 \) and replace all places where we used (2.1.24) to (2.1.26). Hence, we omit the proof. We recall our notation \( a \vee b := \max\{a, b\} \) and \( a \wedge b := \min\{a, b\} \).

**Corollary 2.2.6.** Let \( s > \frac{1}{2} \) and \( 0 < T < 1 \). Then, the following estimates hold:

(i) for any \( 2 \leq \delta < \infty \), let \( u^{(1)}, u^{(2)} \in C([0,T]; H^s(\mathbb{T})) \) satisfy (2.1.1) with \( u_0^{(1)}, u_0^{(2)} \in H^s(\mathbb{T}) \) on \([0,T]\). Then, for \( w := u^{(1)} - u^{(2)} \), we have

\[
\left( \sum_{N, \text{dyadic}} \left[ (1 \vee N)^{s-\frac{3}{2}} \| P_N w \|_{L^s([0,T]; L^4(\mathbb{T}))} \right]^4 \right)^{\frac{1}{4}} \leq C T^\frac{1}{8} \| w \|_{L^\infty([0,T]; H^{s-1}(\mathbb{T}))};
\]

\[
\left( \sum_{N, \text{dyadic}} \left[ (1 \vee N)^{-\frac{s}{2}} \| P_N w \|_{L^s([0,T]; L^4(\mathbb{T}))} \right]^4 \right)^{\frac{1}{4}} \leq C T^\frac{1}{8} \| w \|_{L^\infty([0,T]; H^{s-1}(\mathbb{T}))}.
\]

(ii) for any \( 0 < \delta \ll 1 \), let \( v^{(1)}, v^{(2)} \in C([0,T]; H^s(\mathbb{T})) \) satisfy (2.1.2) with \( v_0^{(1)}, v_0^{(2)} \in H^s(\mathbb{T}) \) on \([0,T]\). Then, for \( \tilde{w} := v^{(1)} - v^{(2)} \), we have

\[
\left( \sum_{N, \text{dyadic}} \left[ (1 \vee N)^{s-\frac{3}{2}} \| P_N \tilde{w} \|_{L^s([0,T]; L^4(\mathbb{T}))} \right]^4 \right)^{\frac{1}{4}} \leq \tilde{C} T^\frac{1}{8} \| \tilde{w} \|_{L^\infty([0,T]; H^{s-1}(\mathbb{T}))};
\]

\[
\left( \sum_{N, \text{dyadic}} \left[ (1 \vee N)^{-\frac{s}{2}} \| P_N \tilde{w} \|_{L^s([0,T]; L^4(\mathbb{T}))} \right]^4 \right)^{\frac{1}{4}} \leq \tilde{C} T^\frac{1}{8} \| \tilde{w} \|_{L^\infty([0,T]; H^{s-1}(\mathbb{T}))}.
\]

Here, the constants \( C = C(u^{(1)}, u^{(2)}) \) and \( \tilde{C} = \tilde{C}(v^{(1)}, v^{(2)}) \) are independent of \( \delta \), and has the following form

\[
C(u, v) := G(\| u \|_{L^p H^s} + \| v \|_{L^p H^s})
\]

such that satisfies the condition in (2.1.26).

2.3 Energy estimates

2.3.1 Preliminary technical estimates

Before we get our main \( \delta \)-independent a priori estimates, there are a few useful estimates we shall state in this subsection.

Let \( 1_T \) be the characteristic function of the time interval \((0,T)\). In \([120]\), it was pointed out that \( 1_T \) does not commute with \( Q_L \). Then, by following \([120]\), we further decompose \( 1_T \) (with respect to some \( R > 0 \) to be fixed later) as

\[
1_T = 1_{low T,R} + 1_{high T,R} \quad \text{such that} \quad F_t(1_{low T,R})(r) = \chi(\frac{r}{R}) F_t(1_{T})(r).
\]

By doing this further decomposition, we avoid the difficulty such that \( 1_T \) and \( Q_L \) are not commute. Moreover, we collect the following three lemmas from \([117]\) Lemma 3.5 - Lemma 3.7]

**Lemma 2.3.1.** Let \( L > 0 \) be an inhomogeneous dyadic number, \( 1 \leq p \leq \infty \), and \( s \in \mathbb{R} \). Then, the operator \( Q_L \) is bounded in \( L^p(\mathbb{R}; H^s(\mathbb{T})) \) uniformly in \( L \). In particular, we have

\[
\| Q_L u \|_{L^p(\mathbb{R}; H^s(\mathbb{T}))} \leq C \| u \|_{L^p(\mathbb{R}; H^s(\mathbb{T}))},
\]

for all \( u \in L^p(\mathbb{R}; H^s(\mathbb{T})) \) and the constant \( C \) is independent of \( L \).
Lemma 2.3.2. Let any $R > 0$ and $T > 0$. Then, the following hold:

\begin{align}
\|1_{T,R}^{\text{high}}\|_{L^1} & \lesssim T \land R^{-1}; \\
\|1_{T,R}^{\text{low}}\|_{L^\infty} + \|1_{T,R}^{\text{low}}\|_{L^\infty} & \lesssim 1.
\end{align}

Lemma 2.3.3. Let $T > 0$, $R > 0$, and $L \gg R$. Then, the following holds

\[ \|Q_L(1_{T,R}^{\text{low}}u)\|_{L^2(R;\mathbb{L}^2(\mathbb{T}))} \lesssim \|Q_{\sim L}u\|_{L^2(R;\mathbb{L}^2(\mathbb{T}))}, \]

for all $u \in \mathbb{L}^2(R;\mathbb{L}^2(\mathbb{T}))$.

In what follows, we study the properties of the resonance function associated with gILW (2.1.1) and scaled gILW (2.1.2). We will study the multi-linear interactions due to the nonlinearity. We say the multi-linear interaction is non-resonance when the resulting frequency of multiple frequencies is large. On the other hand, we say it is the resonance interaction. In the non-resonance case, if the resonance function has a “good” lower bound. Then, in Bourgain’s Fourier restriction norm method, the modulation function provides derivative gain to balance the derivative loss on the nonlinearity.

Let us recall

\begin{align}
p(d) & := n \coth(\delta n) - \frac{1}{\delta} = nK_{\delta}(n), \\
p(s) & := 3n \coth(\delta n) - \frac{1}{\delta} = 3nK_{\delta}(n) = nL_{\delta}(n)
\end{align}

Then, we define the resonance function in the following.

**Definition 2.3.4.** Let $j \in \mathbb{N}$,

(i) for any $2 \leq \delta < \infty$. We define $\Omega_{j,\delta}^{d}(n_1, \ldots, n_{j+1}) : \mathbb{Z}^{j+1} \to \mathbb{R}$ as

\[ \Omega_{j,\delta}^{d}(n_1, \ldots, n_{j+1}) := \sum_{k=1}^{j+1} p_k^{(d)}(n_k) \]

(ii) for any $0 \leq \delta \ll 1$. We define $\Omega_{j,\delta}^{s}(n_1, \ldots, n_{j+1}) : \mathbb{Z}^{j+1} \to \mathbb{R}$ as

\[ \Omega_{j,\delta}^{s}(n_1, \ldots, n_{j+1}) := \sum_{k=1}^{j+1} p_k^{(s)}(n_k) \]

for $(n_1, \ldots, n_{j+1}) \in \mathbb{Z}^{j+1}$.

For simplicity, we may denote a short-hand notation for the resonance function

\[ \Omega_{j,\delta}^{d}(n_1, \ldots, n_{j+1}) = \Omega_{j,\delta}^{d}(\vec{n}) \]

The next two lemmas indicate when the resonance functions $\Omega_{j,\delta}^{d}(\vec{n})$ and $\Omega_{j,\delta}^{s}(\vec{n})$ have a uniform lower bound. Let us recall from (2.1.11) and (2.1.15) that there exists some $n_0 > 0$ such that for all $n \geq n_0$, we have (for simplicity we denote $\partial_n p^{(d)}_\delta(n) = [p^{(d)}_\delta(n)]'$)

\[ [p^{(d)}_\delta(n)]', [p^{(s)}_\delta(n)]' \gtrsim n \]

uniformly in $\delta$.

**Lemma 2.3.5.** Let $k \geq 1$, and $(n_1, \ldots, n_{k+2}) \in \mathbb{Z}^{k+2}$ such that

\[ \sum_{j=1}^{k+2} n_j = 0. \]
Moreover, let us further assume that
\[
\begin{aligned}
|n_1| \sim |n_2| &\geq |n_3|, \\
|n_1| \sim |n_2| &\gg k \max_{j \geq 4} |n_j|, \\
|n_1| \sim |n_2| &\gg k \max_{j \geq 4} |n_j|, & \text{if } k = 1; \\
|n_1| \sim |n_2| &\gg k \max_{j \geq 4} |n_j|, & \text{if } k \geq 2.
\end{aligned}
\]

Then, there exists some \( n_0 > 0 \) such that
(i) for any \( 2 \leq \delta < \infty \), we have
\[|\Omega^\delta_{k+2}(n_1, \ldots, n_{k+2})| \geq |n_3||n_1|, \quad (2.3.5)\]
provided \( |n_1| \geq \max_{0 \leq n \leq n_0} |p^d_\delta(n)|' \).

(ii) for any \( 0 < \delta \ll 1 \), we have
\[|\Omega^\delta_{k+2}(n_1, \ldots, n_{k+2})| \geq |n_3||n_1|, \quad (2.3.6)\]
provided \( |n_1| \geq \max_{0 \leq n \leq n_0} |p^s_\delta(n)|' \).

Proof. Claims of Lemma 2.3.5 (i) and (ii) can be proved similarly, as we have the uniform lower bound
\[|p^d_\delta(n)|' \geq |p^s_\delta(n)|' \geq n.\]

Hence, we focus on the claim Lemma 2.3.5 (i). First, we consider the case \( k \geq 2 \). We separate different cases:

Case 1: \( |n_2| \gg |n_3| \).

Firstly, for some \( n_0 < 0 \) such that \( |n_2| \gg n_0 \). Mean value theorem (MVT) implies that there exists \( k \in \mathbb{R} \) such that \( |k| \sim |n_2| \) and that
\[p^d_\delta(n_2 + \cdots + n_{k+2}) - p^d_\delta(n_2) = |n_3 + \cdots + n_{k+2}| |p^d_\delta(k)'| \geq |n_3||n_1|.\]

Here, we used \( |n_1| \sim |n_2| \) and \( |n_3| \gg k \max_{j \geq 4} |n_j| \). Next, for all \( j \geq 4 \) such that \( |n_j| \leq n_0 \). Then, by MVT again
\[|p^d_\delta(n_j)| \leq |n_j| \max_{0 \leq n \leq n_0} |p^d_\delta(n)|' |n_1| \ll |n_3||n_1|,\]
provided \( |n_1| \geq \max_{0 \leq n \leq n_0} |p^d_\delta(n)|' \) and we have used \( p^d_\delta(0) = 0 \). On the other hand, for all \( j \geq 4 \) such that \( |n_j| \geq n_0 \). Then, we apply MTV twice, we have
\[|p^d_\delta(n_j)| \leq |p^d_\delta(n_j) - p^d_\delta(n_0)| + |p^d_\delta(n_0) - p^d_\delta(0)| \ll |n_0| \max_{0 \leq n \leq n_0} |p^d_\delta(n)|' + |n_j| |n_1| \ll |n_3||n_1|,\]
provided \( |n_1| \geq \max_{0 \leq n \leq n_0} |p^d_\delta(n)|' \). Similarly, we can get \( |p^d_\delta(n_3)| \ll |n_3||n_1| \). Hence, we conclude the following:
\[|\Omega^\delta_{k+2} (\bar{n})| = \left| \sum_{j=0}^{k+1} p^d_\delta(n_j) \right| \geq |n_3||n_1|.\]

Case 2: \( |n_2| \sim |n_3| \).

In this case, we first note that \( |n_3| \gg n_0 \). Then, \( n_1, n_2 \) and \( n_3 \) do not have the same sign. By the symmetry and \( |n_1| \sim |n_2| \sim |n_3| \), we may assume that \( n_2, n_3 > 0 \). Moreover, we observe the
following identity

\[-\Omega_{k+1}^{d,\delta}(\vec{n}) = \int_{n_0}^{n_0} \left( [p_{d}^{(d)}(x + n_3 + \cdots + n_{k+2})'] - [p_{d}^{(d)}(x)]' \right) dx \]

\[+ [p_{d}^{(d)}(n_0 + n_3 + \cdots + n_{k+2}) - p_{d}^{(d)}(n_3)] - p_{d}^{(d)}(n_0) - \sum_{j=4}^{k+2} p_{d}^{(d)}(n_j).\]

Additionally, by the MVT, we have

\[|p_{d}^{(d)}(n_0 + n_3 + \cdots + n_{k+2}) - p_{d}^{(d)}(n_3)| \lesssim (|n_0| + k \max_{j \geq 4} |n_j|)|n_3| \ll |n_3||n_1|,\]

and the fundamental theorem of Calculus implies

\[|p_{d}^{(d)}(x + n_3 + \cdots + n_{k+2})' - [p_{d}^{(d)}(x)]'| = \int_{0}^{n_3+\cdots+n_{k+2}} [p_{d}^{(d)}(x + y)]'' dy.\]

Recall there exists some \(n_0 > 0\) such that for all \(x \geq n_0\), we have \(|p_{d}^{(d)}(x)|''\) does not change sign, since

\[\|p_{d}^{(d)}(x)|''\| \sim 1 \quad \text{and} \quad [p_{d}^{(d)}(x)]'' \text{ is continuous outside } 0.\]

Therefore, for any \(0 \leq x \leq n_2\), there exists some absolute constant \(C\) such that

\[\int_{0}^{n_3+\cdots+n_{k+2}} [p_{d}^{(d)}(x + y)]'' dy \geq \int_{0}^{n_3+\cdots+n_{k+2}} C dy \sim n_3.\]

Here, in the last step, we have used \(n_3 \gg n_j\) for \(j \geq 4\). Therefore, we can deduce that

\[|\Omega_{k+1}^{d,\delta}(\vec{n})| \gtrsim |n_3||n_1|.\]

For the case \(k = 1\), we can argue exactly as above.

\[\square\]

**Lemma 2.3.6.** Let \(k \geq 2\), and \((n_1, \ldots, n_{k+2}) \in \mathbb{Z}^{k+2}\) such that

\[\sum_{j=1}^{k+2} n_j = 0.\]

Moreover, let us further assume that

\[|n_1| \sim |n_2| \gg |n_3| \gtrsim |n_4|, \quad \text{if} \quad k = 2;\]

for \(k \geq 3\) and \(|n_3 + n_4| \gg k \max_{j \geq 5} |n_j|\) we assume that

\[|n_1| \sim |n_2| \gg |n_3| \gtrsim |n_4|.\]

Then, there exists some \(n_0 > 0\) such that

(i) for any \(2 \leq \delta < \infty\), we have

\[|\Omega_{k+2}^{d,\delta}(n_1, \ldots, n_{k+2})| \gtrsim |n_3 + n_4||n_1|, \quad (2.3.7)\]

provided \(|n_1| \gg \max_{0 \leq n \leq n_0} |p_{d}^{(d)}(n)|'\).

(ii) for any \(0 < \delta \ll 1\), we have

\[|\Omega_{k+2}^{d,\delta}(n_1, \ldots, n_{k+2})| \gtrsim |n_3 + n_4||n_1|, \quad (2.3.8)\]
provided \(|n_1| \gg \max_{0 \leq n \leq n_0} |[p_{d}^{(s)}(n)]'|\).

Proof. The proof is almost the same as the proof of Lemma 2.3.5. We only give a sketch. For \(k \geq 3\) case, we first consider (2.3.7) to be case 1, and we separate into the sub-cases when \(|n_3| \gg |n_4|\), and when \(|n_3| \sim |n_4|\).

Case 1.a: \(|n_3| \gg |n_4|\).

We have the following frequency relation

\(|n_3| \gg k \max_{j \geq 5} |n_j| \quad \text{and} \quad |n_3| \sim |n_3 + n_4|\).

Therefore, we argue as in the proof of Lemma 2.3.5 and the following holds

\(|\Omega_{k+1}^{d,s}(\tilde{n})| \gtrsim |n_3||n_1| \sim |n_3 + n_4||n_1|\).

Case 1.b: \(|n_3| \gg |n_4|\).

We need to consider then \(n_3\) and \(n_4\) have the same or different signs. If \(n_3n_4 \geq 0\). Then, we can write \(|n_3 + n_4| = |n_3| + |n_4|\). Moreover, we have

\(|n_3|, |n_4| \gg k \max_{j \geq 5} |n_j|\).

Therefore, the same argument as in the proof of Lemma 2.3.5 implies

\(|\Omega_{k+1}^{d,s}(\tilde{n})| \gtrsim |n_3||n_1| \sim |n_3 + n_4||n_1|\).

In the case when \(n_3n_4 < 0\). By using MVT, there exist \(k_1, k_2 \in \mathbb{R}\) satisfying

\(|k_1| \sim |n_1| \sim |n_2| \quad \text{and} \quad |n_4| \lesssim |k_2| \lesssim |n_3|\)

such that

\[
-\Omega_{k+1}^{d,s}(\tilde{n}) = -(n_1 + n_2) [p_{d}^{(s)}(k_1)]' - (n_3 + n_4) [p_{d}^{(s)}(k_2)]' - \sum_{j=5}^{k+2} p_{d}^{(s)}(n_j)
\]

\[
= (n_3 + n_4 + \cdots + n_{k+2}) [p_{d}^{(s)}(k_1)]' - (n_3 + n_4) [p_{d}^{(s)}(k_2)]' - \sum_{j=5}^{k+2} p_{d}^{(s)}(n_j)
\]

(2.3.9)

where we used the property of \(p_{d}^{(s)}(n)\) being an odd-function. Next, it is enough to show

\(|(n_3 + n_4)[p_{d}^{(s)}(k_2)]'| \quad \text{and} \quad \sum_{j=5}^{k+2} |p_{d}^{(s)}(n_j)|\)

are negligible compared to \(|n_3 + n_4||n_1|\). First of all, as in the proof of Lemma 2.3.5 Case 1, we have

\[
\max_{j \geq 5} |p_{d}^{(s)}(n_j)| \ll \frac{|n_3 + n_4||n_1|}{k}.
\]

Then, for another term, we consider \(|k_2| \leq n_0\), by our constraint we have

\(|[p_{d}^{(s)}(k_2)]'| \leq \max_{0 \leq n \leq n_0} |[p_{d}^{(s)}(n)]'| \ll |n_1|\).

If \(|k_2| \geq n_0\), we have

\(|[p_{d}^{(s)}(k_2)]'| \lesssim |k_2| \lesssim |n_3| \ll |n_1|\)

Therefore, we deduce that \(|[p_{d}^{(s)}(k_2)]'| \ll |n_1|\). Hence, \(\Omega_{k+1}^{d,s}(\tilde{n})| \gtrsim |n_3 + n_4||n_1|\).
Next, we consider (2.3.8) to be case 2, and we follow from case 1. Again, we separate into sub-cases when \(|n_3| \gg |n_4|\), and when \(|n_3| \sim |n_4|\). The same argument as in case 1.a applies to case 2.a (when \(|n_3| \gg |n_4|\)). Let us focus on case 2.a, where (2.3.8) with \(|n_3| \sim |n_4|\).

**Case 2.b:** \(|n_3| \gg |n_4|\).

In this case, if we show the following claim:

\[
|\Omega_{k+2}^\delta(\tilde{n})| \sim |n_3 + n_4| |\partial p^s(k_1)| \quad \text{where} \quad |k_1| \sim |n_1| \sim |n_2|.
\]  

(2.3.10)

Then, Lemma 2.1.6 implies that

\[
\begin{aligned}
|\Omega_{k+2}^\delta(\tilde{n})| &\sim |n_3 + n_4| |n_1|^2 \quad \text{for } |n_1| \lesssim \frac{1}{2}, \\
|\Omega_{k+1}^\delta(\tilde{n})| &\sim \frac{1}{2} |n_3 + n_4| |n_1| \quad \text{for } |n_1| \gtrsim \frac{1}{2}.
\end{aligned}
\]  

(2.3.11)

Since, we are now in the region \(0 < \delta \ll 1\), hence, for any \(n_1\) we have the following lower bound,

\[
|\Omega_{k+1}^\delta(\tilde{n})| \gtrsim |n_3 + n_4| |n_1|,
\]

which conclude (2.3.8).

Now we prove claim (2.3.10). We follow from the argument as in case 1.b, we arrive the same equation as in (2.3.9):

\[
-\Omega_{k+1}^\delta(\tilde{n}) = -(n_1 + n_2) [p^s(k_1)]' - (n_3 + n_4) [p^s(k_2)]' - \sum_{j=5}^{k+2} p_j^s(n_j)
\]

(2.3.12)

where \(k_1\) is between \(-n_1\) and \(n_2\), and \(k_2\) is between \(-n_3\) and \(n_4\). Since, \(|n_1| \sim |n_2| \gg |n_3 + \cdots + n_{k+2}|\) and \(\sum_{j=1}^{k+2} n_j = 0\), we have \(-n_1\) and \(n_2\) must have the same sign. Thus \(|k_1| \sim |n_1| \sim |n_2|\). Moreover, this case, we are under the assumption that \(n_3n_4 < 0\). Therefore, \(|k_2| \sim |n_3| \sim |n_4|\).

Next, according to Lemma 2.1.6, we split into the case when \(|n_1| \lesssim \frac{1}{2}\) and when \(|n_1| \gtrsim \frac{1}{2}\).

**Case 2.b.i:** \(|n_1| \lesssim \frac{1}{2}\).

In this case, we have \(|n_j| \lesssim \frac{1}{2}\) for all \(j \geq 1\). Then, we have the following 4 estimates:

- \(|n_3 + n_4| |\partial p^s(k_1)| \sim |n_3 + n_4| |n_1|^2|
- \(|n_5 + \cdots + n_{k+2}| |\partial p^s(k_1)| \sim |n_3 + n_4| |n_1|^2|
- \(|n_3 + n_4| |\partial p^s(k_2)| \sim |n_3 + n_4| |n_3|^2 \ll |n_3 + n_4| |n_1|^2|
- \(\sum_{j=5}^{k+2} |p_j^s(n_j)| \sim \sum_{j=5}^{k+2} |n_j|^3 \ll |n_3 + n_4| |n_1|^2|

Then, we conclude from (2.3.10) and (2.3.12) that

\[
|\Omega_{k+2}^\delta(\tilde{n})| \sim |n_3 + n_4| |n_1|^2 \gtrsim |n_3 + n_4| |n_1|.
\]

**Case 2.b.ii:** \(|n_1| \gtrsim \frac{1}{2}\).

In this case, we have \(|n_1| \sim |n_2| \gtrsim \frac{1}{2}\). Then, Lemma 2.1.6 implies

- \(|n_3 + n_4| |\partial p^s(k_1)| \sim |n_3 + n_4| |n_1| \gtrsim |n_3 + n_4| |n_1|
- \(|n_5 + \cdots + n_{k+2}| |\partial p^s(k_1)| \sim |n_5 + \cdots + n_{k+2}| |n_1| \gtrsim |n_3 + n_4| |n_1| \frac{1}{2}

Therefore, \(|n_3 + n_4| \gtrsim |n_3 + n_4| |n_1| \frac{1}{2}\).
For the remaining terms, we need to consider cases depending on how big or small these frequencies are when compared to $\frac{1}{\delta}$.

$$|\partial p^{(s)}_{\delta}(k_2)| \sim \begin{cases} \frac{1}{2}|k_2|, & \text{if } |k_2| \gtrsim \frac{1}{\delta}; \\ |k_2|^2, & \text{if } |k_2| \lesssim \frac{1}{\delta}. \end{cases}$$

Since, we have $|k_2| \sim |n_3| \ll |n_1|$ and $|n_1| \gtrsim \frac{1}{\delta}$. Then,

- when $|k_2| \gtrsim \frac{1}{\delta}$, $|\partial p^{(s)}_{\delta}(k_2)| \sim \frac{1}{2}|k_2| \ll |n_1| \sim |\partial p^{(s)}_{\delta}(k_1)|$;
- when $|k_2| \lesssim \frac{1}{\delta}$, $|\partial p^{(s)}_{\delta}(k_2)|^2 \sim \frac{1}{2}|k_2| \ll |n_1| \sim |\partial p^{(s)}_{\delta}(k_1)|$.

Therefore, we always have $|\partial p^{(s)}_{\delta}(k_2)||n_3 + n_4| \ll |n_3 + n_4|^{\frac{1}{3}}|n_1|$.

Next, for each $j \geq 5$, we have

$$|\partial p^{(s)}_{\delta}(n_j)| \sim \begin{cases} \frac{1}{2}|n_j|^2, & \text{if } |n_j| \gtrsim \frac{1}{\delta}; \\ |n_j|^3, & \text{if } |n_j| \lesssim \frac{1}{\delta}. \end{cases}$$

If $|n_j| \lesssim \frac{1}{\delta}$, then

$$|p^{(s)}_{\delta}(n_j)| \sim |n_j|^3 \ll \frac{1}{k^3}|n_3 + n_4||n_j|^2 \ll \frac{1}{k\delta}|n_3 + n_4||n_1|.$$  

If $|n_j| \gtrsim \frac{1}{\delta}$, then

$$|p^{(s)}_{\delta}(n_j)| \sim \frac{1}{\delta}|n_j|^2 \ll \frac{1}{k\delta}|n_3 + n_4||n_1|.$$  

Hence, we can conclude (2.3.10).

For the case $k = 2$, we can argue exactly as above. \hfill \square

### 2.3.2 Uniform energy estimate

In this subsection, we will present the crucial nonlinear estimates. We first notice that our solutions $u_\delta$ and $v_\delta$ to gILW (2.1.1) and scaled gILW (2.1.2) are characterised by $\delta$, but in this construction of solutions part, we do not need to worry about the parameter $\delta$. The main goal of this step is to keep tracking all the estimates that are uniformly in $\delta$ such that we can pass the limit to the solution of (2.1.2) and (2.1.1), separately. Hence, for the simplicity reason, we write $u_\delta = u$ and $v_\delta = v$ in this section and we keep in mind this notation.

Let $s \in \mathbb{R}$. We recall the function spaces $M^{s,\delta}$ and $N^{s,\delta}$ from (2.1.17) and (2.1.18) such that

$$\|u\|_{M^{s,\delta}} = \|u\|_{L^\infty H_2^s} + \|u\|_{X^{s-1,1,\delta}}, \quad \|v\|_{N^{s,\delta}} = \|v\|_{L^\infty H_2^s} + \|v\|_{Y^{s-1,1,\delta}},$$

and set $\omega_N = 1$ we have $M^{s,\delta}$ and $N^{s,\delta}$.

The next two lemmas are the essential tool to obtain unconditional uniqueness, where we used that for $s \gtrsim \frac{1}{2}$, solutions $u$ and $v$ to gILW (2.1.2) and scaled gILW (2.1.1) also satisfy the Duhamel formulation. It follows from (1.19) and is a standard $X^{s,b}$-analysis.

**Lemma 2.3.7.** Take $\{\omega_N\}$ be a dyadic sequence satisfies Lemma 2.1.7 with $1 \leq \kappa \leq 2$. Let $s \gtrsim \frac{1}{2}$, $0 < T < 1$, and $u \in L^\infty([0,T];H^s_2(\mathbb{R}))$ be a solution to (2.1.1). Then, for any $2 \leq \delta \leq \infty$ and $u \in M^{s,\delta}$, we have

$$\|u\|_{M^{s,\delta}} \lesssim \|u\|_{L^\infty_2 H^s_2} + C(\|u\|_{L^\infty_2 H^s_2})\|u\|_{L^\infty_2 H^s_2}. \quad (2.3.13)$$

Moreover, for $j = 1,2$. Let $u^{(j)} \in L^\infty([0,T];H^s(\mathbb{T})$ to be solutions to (2.1.1) with initial data
\( u_0^{(j)} \in H^s(\mathbb{T}) \). Then, the following holds
\[
\|u^{(1)} - u^{(2)}\|_{M_T^{s-1,s}} \lesssim \|u^{(1)} - u^{(2)}\|_{L_T^\infty X_T^{s-1}}
+ C(\|u^{(1)}\|_{L_T^\infty X_T^s} + \|u^{(2)}\|_{L_T^\infty X_T^s})\|u^{(1)} - u^{(2)}\|_{L_T^\infty H_T^{s-1}}. \tag{2.3.14}
\]

Here, the implicit constants are independent of \( \delta \).

**Proof.** For the claim (2.3.13), by following the extension Lemma 2.1.8, it is enough to estimate \( u \in X^{s-1.1.6}_T \). Moreover, we recall from Remark 1.2.5 that \( u \) satisfies the Duhamel formula of (2.1.1) and
\[
\|u_0\|_{H_T^s} \leq \|u\|_{L_T^\infty X_T^s}
\]
for any \( \theta \leq s \). Hence, standard linear estimates in \( X^{s,\delta}_T \)-spaces and (2.1.24) lead to
\[
\|u\|_{X^{s-1,1.6}_T} \lesssim \|u_0\|_{H_T^{s-1}} + \|\partial_x (f(u))\|_{X^{s-1.1.6}_T} \lesssim \|u_0\|_{H_T^{s-1}} + \|f(u) - f(0)\|_{L_T^\infty H_T^s}
\lesssim \|u\|_{L_T^\infty H_T^{s-1}} + C(\|u\|_{L_T^\infty X_T^s})\|u\|_{L_T^\infty X_T^s},
\]
where \( C \) satisfies the condition of (2.1.24). Similarly, by using (2.1.26), we get
\[
\|u^{(1)} - u^{(2)}\|_{X^{s-2,1.6}_T} \lesssim \|u^{(1)} - u^{(2)}\|_{H_T^{s-1}} + \|f(u^{(1)}) - f(u^{(2)})\|_{L_T^\infty H_T^s}
\lesssim \|u^{(1)} - u^{(2)}\|_{L_T^\infty H_T^{s-1}} + C(\|u^{(1)}\|_{L_T^\infty X_T^s} + \|u^{(2)}\|_{L_T^\infty X_T^s})\|u^{(1)} - u^{(2)}\|_{L_T^\infty H_T^{s-1}}
\]
which completes the proof of (2.3.14). \( \square \)

The next lemma is similar to the above. It is for solution \( v \) of scaled gILW (2.1.2). Therefore, if we repeat the same argument, we get the following results.

**Lemma 2.3.8.** Take \( \{\omega_N\} \) be a dyadic sequence satisfies Lemma 2.1.7 with \( 1 \leq \kappa \leq 2 \). Let \( 0 < T < 1 \), and \( v \in L^\infty([0,T];H^s(\mathbb{T})) \) be a solution to (2.1.2). Then, for any \( 0 < \delta \ll 1 \) and \( \omega \in N^{s,\delta}_T \), we have
\[
\|v\|_{N^{s,\delta}_T} \lesssim \|v\|_{L_T^\infty X_T^s} + C(\|v\|_{L_T^\infty X_T^s})\|v\|_{L_T^\infty X_T^s}.
\]
Moreover, for \( j = 1,2 \). Let \( v^{(j)} \in L^\infty([0,T];H^s(\mathbb{T})) \) to be solutions to (2.1.2) with initial data \( v_0^{(j)} \in H^s(\mathbb{T}) \). Then, the following holds
\[
\|v^{(1)} - v^{(2)}\|_{N_T^{s-1,\delta}} \lesssim \|v^{(1)} - v^{(2)}\|_{L_T^\infty H_T^{s-1}}
+ C(\|v^{(1)}\|_{L_T^\infty X_T^s} + \|v^{(2)}\|_{L_T^\infty X_T^s})\|v^{(1)} - v^{(2)}\|_{L_T^\infty H_T^{s-1}}.
\]
Here, the implicit constants are independent of \( \delta \).

In the following, we establish uniform energy estimates for solutions to the gILW (2.1.1) and scaled gILW (2.1.1). This argument is very close in spirit to the improved energy method by Molinet-Vento [120] Proposition 3.4 and Molinet-Tanaka [119] Proposition 4.7. We are analysing nonlinear interactions, which we can observe from the following. Let \( u \in C(\mathbb{R};H^\infty(\mathbb{T})) \) be a smooth solution to (2.1.1). Then, by the Fundamental Theorem of Calculus, we have
\[
\|u(t)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 = -2 \int_0^t \int_\mathbb{T} \partial_x (f(u)) u dx dt'. \tag{2.3.15}
\]
This turn to study the nonlinear interactions on the RHS of (2.3.15). The computation (2.3.15) holds exactly for smooth solution \( v \in C(\mathbb{R};H^\infty(\mathbb{T})) \) to (2.1.2).

This type of argument can also be seen in [119]. Moreover, such an argument works well on the real line situation. Indeed, we only used the integration by parts (IBP) for the following set-up step. The main strategy of our crucial nonlinear estimates is to use symmetry arguments. In practice, we distribute the lost derivative to several functions, and then to recover it by applying either
short-time Strichartz estimates or using $X^{s,b}$-type estimates and gain derivatives from modulation. In practice, to decide which way to recover the loss of derivative depends on whether the nonlinear interactions are resonant. Hence, on $\mathbb{R}$, we shall replace the short-time Strichartz estimates with the real-line setting accordingly.

**Proposition 2.3.9** (uniform energy estimate). Take $\{\omega N\}$ be a dyadic sequence satisfies Lemma 2.1.7. Let $s \geq \frac{3}{4}$, $0 < T < 1$, and $u, v \in L^\infty([0, T]; H^s(\mathbb{T}))$ be a solution to $g(\mathbb{LW})$ and scaled $gILW$ with an initial data $u_0, v_0 \in H^s(\mathbb{T})$, separately. Then, the following statements hold:

(i) for any $0 < \delta \leq \infty$, we have

$$
\|u\|_{L^\infty_T H^s_{L^2}} \leq \|u_0\|_{H^s_{L^2}} + T^\frac{4}{3} C(\|u\|_{M^s_{L^2,T}}) \|u\|_{M^s_{L^2,T}} \|u\|_{L^\infty_T H^s_{L^2}}.
$$

(2.3.16)

(ii) for any $0 < \delta \ll 1$, we have

$$
\|v\|_{L^\infty_T H^s_{L^2}} \leq \|v_0\|_{H^s_{L^2}} + T^\frac{4}{3} C(\|v\|_{N^s_{L^2,T}}) \|v\|_{N^s_{L^2,T}} \|v\|_{L^\infty_T H^s_{L^2}};
$$

(2.3.17)

In particular, the estimates are uniformly in $\delta$.

**Remark 2.3.10.** The restriction on the regularity comes from the resonant interactions of the nonlinearity. In particular, see subcase 2.1 of $A_1$ contribution such that

$$
N \sim N_1 \sim N_2 \sim N_3 \sim kN_4.
$$

In this situation, the nonlinear effects happen to be resonant interactions. To recover the loss of derivative, we share it into 4 terms with high frequencies in $L^4([0, T]; L^4(\mathbb{T}))$, and we apply the short-time Strichartz estimates in Proposition 2.2.1 to control their norms. In particular, we see from Proposition 2.2.1 that by putting 4 terms in $L^4([0, T]; L^4(\mathbb{T}))$ to recover the loss of derivatives, one leads to

$$
4 \left( s - \frac{1}{8} \right) \geq 2s + 1 \iff s \geq \frac{1}{2} + \frac{1}{4} \left( \frac{3}{4} \right).
$$

**Proof of 2.3.9.** The proof for (2.3.16) and (2.3.17) are essentially the same. We present a sketch proof of (2.3.16) for the reader’s convenience.

From Lemma 2.3.7 we know $u \in M^{s,\delta}_{L^2,T}$. Moreover, by taking the Littlewood-Paley projection onto the solution $u$, it is clear that $P_N u \in C([0, T]; H^\infty(\mathbb{T}))$ with $P_N \partial_x u \in L^\infty([0, T]; H^\infty(\mathbb{T}))$. Therefore, (2.3.15) applies, in particular, taking the $L^2$-scalar product of the resulting equation with $P_N u$, multiplying by $\omega^2_N (N)^{2s}$ and integrating over $[0, t]$ with $0 < t < T$, we yield

$$
\omega^2_N (N)^{2s} \|P_N u(t)\|^2_{L^2} = \omega^2_N (N)^{2s} \|u_0\|^2_{L^2} - 2 \omega^2_N (N)^{2s} \int_0^t \int_{\mathbb{T}} \partial_x P_N (f(u)) P_N u \, dx \, dt'.
$$

We use integration by parts, apply Bernstein inequalities, and sum over $N$, we obtain

$$
\|u(t)\|^2_{L^2} = \sum_N \omega^2_N (1 \vee N)^{2s} \left( \left\|P_N u_0\right\|^2_{L^2} - 2 \int_0^t \int_{\mathbb{T}} P_N \partial_x (f(u)) P_N u \, dx \, dt' \right)
\leq \|u_0\|^2_{L^2} + 2 \sum_N \omega^2_N (1 \vee N)^{2s} \int_0^t \int_{\mathbb{T}} P_N (f(u) - f(0)) P_N \partial_x u \, dx \, dt'
\leq \|u_0\|^2_{L^2} + 2 \sum_{N \geq 1} \omega^2_N (N)^{2s} \int_0^t \int_{\mathbb{T}} (f(u) - f(0)) P_N^2 \partial_x u \, dx \, dt',
$$

(2.3.18)

where we recall that $f(0) = 0$ and hence $\partial_x f(u) = \partial_x (f(u) - f(0))$, also we note that $P_0 \partial_x u = 0$.

Let us rewrite the difference of the nonlinearity in the following

$$
f(u) - f(0) = \sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} u^k.
$$
Then, for any fixed $N \in 2\mathbb{N}$ we have the following
\[\int_0^t \int_{\Omega} (f(u) - f(0)) P_N^k \partial_x u \, dx \, dt' = \sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} \int_0^t \int_{\Omega} u^k P_N^k \partial_x u \, dx \, dt'. \tag{2.3.19}\]

Indeed, in order to interchange the summation and the integration in (2.3.19), we need to show the following boundedness. For each fixed $N$,
\[
\sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_0^t \int_{\Omega} |u^k P_N^k \partial_x u| \, dx \, dt' \lesssim N \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_0^t \|u^k\|_{L^2_x} \|u\|_{L^2_t} \, dt' \\
\lesssim N \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_0^t \|u^{k-1}\|_{L^2_x} \|u\|_{L^2_t} \, dt' \\
\lesssim TNC(\|u\|_{L^\infty_{T,x}}) \|u\|_{L^2_{x,T}}^2 < \infty.
\]

Hence, the equation (2.3.19) follows from Fubini-Lebesgue's theorem. Moreover, by (2.3.19) together with Fubini-Tonelli’s theorem, we obtain
\[
\sum_{N \geq 1} \omega_N^2 N^{2s} \left| \int_0^t \int_{\Omega} (f(u) - f(0)) P_N^k \partial_x u \, dx \, dt' \right| \\
= \sum_{N \geq 1} \omega_N^2 N^{2s} \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_0^t \int_{\Omega} u^k P_N^k \partial_x u \, dx \, dt' \\
\leq \sum_{N \geq 1} \sum_{k \geq 1} \omega_N^2 N^{2s} \left[ \frac{|f^{(k)}(0)|}{k!} \right] \int_0^t \int_{\Omega} u^k P_N^k \partial_x u \, dx \, dt' \\
= : \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} I_k^1
\]
where $I_k^1$ is defined by
\[I_k^1 := \sum_{N \geq 1} \omega_N^2 N^{2s} \left| \int_0^t \int_{\Omega} u^k P_N^k \partial_x u \, dx \, dt' \right|.
\]

We notice here, it is easy to check that $I_1^1 = 0$ by IBP. Therefore, we shall prove that for any $k \geq 1$, the following holds
\[I_{k+1}^1 \leq C^k T^4 C_{M} C_k^3 (\|u\|_{X_{T,x}^{1+s}} + \|u\|_{L^2_{x,T} H^s_x}) \|u\|_{L^2_{x,T} H^s_x}. \tag{2.3.21}\]

Here, we define $C_{M} := C(\|u\|_{M_{x,T}^{2s}})$ and $C_1 := \|u\|_{L^2_{x,T} x}$. Since $\sum_{k \geq 1} \frac{|f^{(k+1)}(0)|}{(k+1)!} C^k C_{M} < \infty$, it is clear that (2.3.21) leads to (2.3.17) and (2.3.16) by taking (2.3.18) and (2.3.20) into account. Moreover, the bound on (2.3.21) is independent of $\delta$.

In the following, we fix $k \geq 1$. For simplicity, for any positive numbers $a$ and $b$, the notation $a \lesssim b$ means there exists a positive constant $C > 0$ independent of $k$ such that
\[a \leq C^k b. \tag{2.3.22}\]

Remark that $a \leq k^m b$ for $m \in \mathbb{N}$ can be expressed by $a \lesssim b$ too since an elementary calculation shows $\frac{1}{k} k^m \leq m! e^k$ for $m \in \mathbb{N}$.

**Low-frequency contribution.**

The contribution of the sum over $N \lesssim 1$ in $I_{k+1}^1$ is easily estimated by
\[
\sum_{N \lesssim 1} \omega_N^2 N^{2s} \left| \int_0^t \int_{\Omega} a^{k+1} P_N^k \partial_x u \, dx \, dt' \right| \leq T \sum_{N \lesssim 1} \|u\|_{L^2_{x,T}}^k \|u\|_{L^2_{x,T}} \|P_N^k u\|_{L^2_{x,T}}^k
\]

\(^1\text{Here, } e \text{ is Napier’s constant.}\)
\[ \lesssim_k TC_k \| u \|_{L_T^p H_x^k}^2. \]

- **High-frequency contribution.**

  It thus remains to bound the contribution of the sum over \( N \gg 1 \) in \( I_{k+1}^j \). First, we define the following symbols,

  \[
  A(n_1, \ldots, n_{k+2}) := \sum_{j=1}^{k+2} \phi_N^2 (n_j) n_j, \\
  A_1(n_1, n_2) := \phi_N^2 (n_1) n_1 + \phi_N^2 (n_2) n_2, \\
  A_2(n_4, \ldots, n_{k+2}) := \sum_{j=4}^{k+2} \phi_N^2 (n_j) n_j.
  \]

  Here, \( \phi_N \) is defined in Section 2.1.1. It is clear that

  \[ A(n_1, \ldots, n_{k+2}) = A_1(n_1, n_2) + \phi_N^2 (n_3) n_3 + A_2(n_4, \ldots, n_{k+2}). \]

  Moreover, we see from the symmetry that

  \[
  \int_T u^{k+1} P_N^2 \partial_x u dx = \frac{i}{k+2} \sum_{n_1 + \cdots + n_{k+2} = 0} A(n_1, \ldots, n_{k+2}) \prod_{j=1}^{k+2} \hat{u}(n_j).
  \]

  By symmetry, we can assume that

  \[
  \begin{cases} 
  N_1 \geq N_2 \geq N_3, & \text{if } k = 1; \\
  N_1 \geq N_2 \geq N_3 \geq N_4, & \text{if } k = 2; \\
  N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5 = \max_{j \geq 5} N_j, & \text{if } k \geq 3.
  \end{cases}
  \]

  We notice that the cost of this choice is a constant factor less than \((k+2)^4\). It is also worth seeing that the frequency projection operator \( P_N \) ensures that there is no contribution \footnote{If \( N_1 \leq N/4 \), we thus can assume that \( N_1 \geq N/4 \) and that \( N_2 \geq N_1/k \) with \( N_2 \geq 1 \).}
of any \( N_1 \leq N/4 \).

  **Case 1: \( A_2 \) contribution.**

  We start with the \( A_2 \) contribution, and note that we must have \( k \geq 2 \) since otherwise \( A_2 = 0 \). Also it suffices to consider the contribution of \((\phi_N(n_4))^2 n_4\), since the contributions of \((\phi_N(n_j))^2 n_j\), for \( j \geq 5 \), are simpler. Note that \( N_1 \sim N \) in this case (0 otherwise, due to the observation before).

  First of all, by the Bernstein inequality, we have for \( s' > \frac{1}{2} \)

  \[
  \sum_K \| P_K u \|_{L_T^p H_x^s} \lesssim \sum_K (1 \wedge K^{\frac{1}{2}-s'}) \| u \|_{L_T^p H_x^s} \lesssim \| u \|_{L_T^p H_x^{s'}} \lesssim CM. \tag{2.3.24}
  \]

  This together with Hölder’s, Young’s convolution inequalities (see (2.3.25)), and Proposition 2.2.1 we have

  \[
  \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2s} \left| \int_0^t \int_T (\partial_x P_N^2 P_{N_4} u) \prod_{j=1, j \neq 4}^{k+2} P_{N_j} u dx dt' \right| \lesssim_k \| u \|_{L_T^p H_x^{s'}} \sum_{N_1 \geq N_2 \geq N_3 \geq N_4} \omega_{N_4}^2 N_4^{2s+2} \prod_{j=1}^{k} \| P_{N_j} u \|_{L_T^p H_x^{s'}}.
  \]

  \footnote{If \( N_1 \leq N/4 \), since the \{supp(P_N) \cap supp(P_{N_1}) \} = \emptyset.}
We note that Young’s inequality will be used frequently in our later analyses. To be precise in the application of Young’s convolution inequalities, let us denote 

$$f, g$$

where \( \Pi(\cdot) \) and Proposition 2.2.1 show that \( N \) either

$$N_1 \lesssim N_2 \lesssim k N_4$$

or

$$N_3 \gg k N_4$$

or

$$k = 1$$

Moreover we can also assume that

$$N_3 \geq 1$$

since otherwise the contribution of \( A_1 \) cancelled by integration by parts.

We divide the contribution \( A_1 \) into three cases:

(i) \( N_2 \lesssim N_3 \lesssim k N_4 \),

(ii) \( N_3 \gg k N_4 \) or \( k = 1 \),

(iii) \( N_2 \gg N_3 \).

Moreover, let us define the following notation

$$J_k := \sum_{N \gg 1} \sum_{N_1, \ldots, N_2 \leq N} \omega^{2k} N^2 s \left| \int_0^t \Pi(P_{N_1} u, P_{N_2} u) \prod_{j=3}^{k+2} P_{N_j} u dt \right|,$$

where \( \Pi(f, g) \) is defined by \( \text{(2.1.27)} \). Note that \( N \gg 1 \) ensures that \( N_1 \gg 1 \).

**Subcase 2.1:** \( N_2 \lesssim N_3 \lesssim k N_4 \).

Since \( N \lesssim N_1 \lesssim k N_2 \lesssim k N_3 \lesssim k^2 N_4 \). Then, by Hölder’s, Bernstein’s, Young’s inequality \( \text{(2.3.25)} \), and Proposition 2.2.1, show that

$$J_k \lesssim k^{2(2s+1)} \sum_{N_1, \ldots, N_2 \leq k^2 N_4} \omega^{2k} N^{2s+1} \prod_{j=1}^{k+2} \|P_{N_j} u\|_{L^4} \prod_{j=5}^{k+2} \|P_{N_j} u\|_{L^\infty}$$

$$\lesssim k \sum_{N_1 \geq N_4, N_2 \geq N_4, N_3 \geq N_4} \frac{\omega_{N_1} \omega_{N_2} \omega_{N_3} \omega_{N_4}}{\omega_{N_2}} \left( \frac{N_1}{N_4} \right)^{s-\frac{k}{2}} \left( \frac{N_2}{N_3} \right)^{s-\frac{k}{2}} \left( \frac{N_2}{N_3} \right)^{s-\frac{k}{2}}
The largest two frequencies must be comparable. We must have $w_{\kappa} \lesssim \kappa N$ such that $\kappa \leq 2$ and $N_1 \lesssim \kappa N_2$, we have $\frac{w_{\kappa}}{w_{\kappa_2}} \lesssim k$. Moreover, it is not difficult to see that the last inequality holds when $s \geq \frac{3}{4}$.

**Subcase 2.2:** $N_3 \gg k N_4$ or $k = 1$.

The largest two frequencies must be comparable. We must have $N \sim N_1 \sim N_2$. We take the extensions $\hat{u} = \rho_T(u)$ of $u$ defined in (2.1.20). For simplicity, we define the following functional:

$$J_2(u_1, \ldots, u_{k+2}) := \sum_{N \gg 1} \sum_{N_{1, \ldots, N_{k+2}}} \omega_N^2 N_s^2 \left| \int_T \int T \Pi(u_1, u_2) \prod_{j=3}^{k+2} u_j \, dx \, dt \right|. \tag{2.3.26}$$

By setting $R = N_1^{\frac{1}{3}} N_3^{\frac{2}{3}}$, and then we split $J_1$ into

$$J_1 \leq J_2(\Pi N_{1, t, R}^{\text{high}} \hat{u}, P_n \Pi_{1, t, R}^{\text{low}} \hat{u}, P_n \Pi_{1, t, R}^{\text{high}} \hat{u}, \ldots, P_{N_{k+2}} \Pi_{1, t, R}^{\text{low}} \hat{u}))$$

In particular, we see the following expression

$$J_1 \leq \sum_{N \gg 1} \sum_{N_{1, \ldots, N_{k+2}}} \omega_N^2 N_s^2 \left| \int_T \int T \Pi(\Pi N_{1, t, R}^{\text{high}} \hat{u}, P_n \Pi_{1, t, R}^{\text{low}} \hat{u}) \prod_{j=3}^{k+2} P_n \Pi_{1, t, R}^{\text{low}} \hat{u} \, dx \, dt \right|$$

For $J_2(1, \ldots, 1)$, recall that $N \sim N_1 \sim N_2$. We see from (2.3.2) that

$$\|1_{1, t, R}^{\text{high}}\|_{L^1} \lesssim T^{\frac{1}{4}} N_1^{\frac{1}{4}} N_3^{-1},$$

which gives

$$J_2(1, \ldots, 1) = \sum_{N \gg 1} \sum_{N_{1, \ldots, N_{k+2}}} \omega_N^2 N_s^2 \left| \int_T \int T \Pi(\Pi N_{1, t, R}^{\text{high}} \hat{u}, P_n \Pi_{1, t, R}^{\text{low}} \hat{u}) \prod_{j=3}^{k+2} P_n \Pi_{1, t, R}^{\text{low}} \hat{u} \, dx \, dt \right|$$

In the last inequality, we used (2.1.22). By the strategy as for (2.3.27), we can estimate $J_2(1, \ldots, 1)$ while using (2.3.3).
To estimate $J_{\infty,3}^{(2)}$, we first recall from Lemma 2.3.3 that

$$|\Omega_{\epsilon,3}^{(2)}(\tilde{u})| \gtrsim N_3 N_1 \gg R.$$ 

Then, by defining $L := N_3 N_1$, we further decompose $J_{\infty,3}^{(2)}$ into the following

$$J_{\infty,3}^{(2)} = \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2k} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_N^1 Q_{\geq L}(1_{t,R} \tilde{u}), P_N t R^2 \tilde{u}) \prod_{j=3}^{k+2} P_N \tilde{u} dx dt \right|$$

In particular, we have

$$J_{\infty,3}^{(2)} \leq \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2k} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_N Q_{\geq N_3 N_1}(1_{t,R} \tilde{u}), P_N t R^2 \tilde{u}) \prod_{j=3}^{k+2} P_N \tilde{u} dx dt \right|$$

$$+ \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2k} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_N Q_{\leq L}(1_{t,R} \tilde{u}), P_N t R^2 \tilde{u}) \prod_{j=3}^{k+2} P_N \tilde{u} dx dt \right|$$

$$+ \sum_{n=3}^{k+2} \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2k} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_N Q_{\leq N_3 N_1}(1_{t,R} \tilde{u}), P_N t R^2 \tilde{u}) \prod_{j=3}^{k+2} P_N \tilde{u} dx dt \right|$$

We also see from (2.3.2) and (2.3.3) that $\|1_{t,R}^{\text{high}}\|_{L_x^2} \leq R^{-\frac{1}{2}}$, and then we have

$$\|P_N t R \tilde{u}\|_{L_x^2} \leq \|P_N^1 \tilde{u}\|_{L_x^2} + \|P_N t R^{\text{high}} \tilde{u}\|_{L_x^2} \lesssim \|P_N^1 \tilde{u}\|_{L_x^2},$$

Thus, we can estimate $J_{\infty,3,1}^{(2)}$ by using Lemma 2.1.11, Lemma 2.1.8, Hölder’s inequality, (2.3.4), and (2.3.28). That is the following

$$J_{\infty,3,1}^{(2)} \lesssim \sum_{N_1, \ldots, N_{k+2}} \omega_N^2 N^{2k} \|P_N Q_{\geq L}(1_{t,R} \tilde{u})\|_{L_x^2} \|P_N t R \tilde{u}\|_{L_x^2} \prod_{j=3}^{k+2} \|P_N \tilde{u}\|_{L_x^2}$$

$$\lesssim_k \|u\|_{L_x^2} \sum_{N_1 \geq 1} \omega_N^2 N^{2k-1} \|P_N \tilde{u}\|_{X^{0,1,\ast}} \|P_N^1 \tilde{u}\|_{L_x^2}$$

$$+ T^{\frac{d}{2}} \|u\|_{L_x^2} \sum_{N_1 \geq N_3} \omega_N^2 N^{2k-1} N_3^{\frac{d}{2}} \|P_N^1 \tilde{u}\|_{L_x^2} \|P_N \tilde{u}\|_{L_x^2} \|P_N \tilde{u}\|_{L_x^2}$$

We can immediately estimate $J_{\infty,3,2}^{(2)}$ by the same approach with (2.3.28).

Next, we consider the contribution $J_{\infty,3,3}^{(2)}$. By using lemma 2.1.8, Lemma 2.1.11, Hölder’s...
inequality and (2.3.1) yield

\[ J_{2N_1^2, N_3}^{(2)} \lesssim \sum_{N_1 \geq N_2} \omega_{N_1}^2 N_1^2 \sum_{N_3 \geq 1} \| P_{N_1} Q_{\ll L} (1_{\mathfrak{L}}^L \mathfrak{R} \mathfrak{u}) \|_{L_t^2} \| P_{N_1} Q_{\ll L} (1_{\mathfrak{L}}^L \mathfrak{R} \mathfrak{u}) \|_{L_t^\infty} \]

\[ \times \| P_{N_1} Q_{\ll L} \mathfrak{u} \|_{L_t^2 L_x^\infty} \prod_{j=4}^{k+2} \| P_{N_j} \mathfrak{u} \|_{L_t^\infty} \]

\[ \lesssim_k T^{\frac{1}{2}} \| \mathfrak{u} \|_{L^2_{t,x} H_x^s} \sum_{N_1 \geq N_2} \omega_{N_1}^2 N_1^2 \| P_{N_1} \mathfrak{u} \|_{L_t^2 L_x^\infty} \| D_{N_1}^2 P_{N_1} \mathfrak{u} \|_{X^0, 1.5} \]

\[ \lesssim T^{\frac{1}{2}} \| \mathfrak{u} \|_{L^2_{t,x} H_x^s} \sum_{N_1 \geq N_2} \omega_{N_1}^2 N_1^2 \| P_{N_1} \mathfrak{u} \|_{L_t^2 L_x^\infty} \| D_{N_1}^2 P_{N_1} \mathfrak{u} \|_{X^0, 1.5} \]

For all \( 4 \leq j \leq k+2 \). By a similar argument as above for \( J_{2N_1^2, N_3}^{(2)} \), we can estimate \( J_{2N_1^2, N_3}^{(2)} \) such that

\[ J_{2N_1^2, N_3}^{(2)} \lesssim_k T^{\frac{1}{2}} \| \mathfrak{u} \|_{L^2_{t,x} H_x^s} \]

**Subcase 2.3:** \( N_1 \sim N_2 \gg N_3 \).

In this case, we need to compare the size \( |n_3 + n_4| \) and \( k|n_5| \). By symmetry we can assume \( |n_3| \geq |n_4| \), where \( n_j \) is the \( j \)-th largest frequency. Therefore, we consider the following two cases:

\[ |n_3 + n_4| \gg k|n_5| \quad |n_3 + n_4| \lesssim k|n_5| \]

If \( |n_3 + n_4| \gg k|n_5| \), we have a suitable non-resonance relation (see Lemma 2.3.6). Otherwise we can share the lost derivative between three functions, \( P_{N_1} \mathfrak{u} \), for \( j = 3, 4, 5 \), see in (2.3.29). Hence, let us split \( J_t \) into the following two terms:

\[ J_t \leq \sum_{N \gg 1, N_1, \ldots, N_{k+2}} \omega_N^2 N^2 \left| \int_0^T \int_\mathbb{T} \Pi(P_{N_1} \mathfrak{u}, P_{N_2} \mathfrak{u}) P_{\ll k N_5} \left( \prod_{j=3}^{k+2} P_{N_j} \mathfrak{u} \right) dx dt \right| \]

\[ + \sum_{N \gg 1, N_1, \ldots, N_{k+2}} \omega_N^2 N^2 \left| \int_0^T \int_\mathbb{T} \Pi(P_{N_1} \mathfrak{u}, P_{N_2} \mathfrak{u}) P_{\ll k N_5} \left( \prod_{j=3}^{k+2} P_{N_j} \mathfrak{u} \right) dx dt \right| \]

\[ =: I_t^{(3)} + J_t^{(3)}. \]

One observation from Lemma 2.3.6\(^3\) that if \( k = 2 \). Then, the term \( I_t^{(3)} \) does not appear. The above estimate holds that \( J_t \leq I_t^{(3)} \) with the summation \( \sum_{M \leq N_3} \) instead of \( \sum_{N_5 \ll M \lesssim N_5} \). Note also that the contribution in respectively \( I_t^{(3)} \) and \( J_t^{(3)} \) of respectively \( N_5 = 0 \) and \( N_5 = M = 0 \) does vanish by integration by parts. Therefore we can always assume that \( N_5 \geq 1 \) in \( I_t^{(3)} \) and that \( M \geq 1 \) in \( J_t^{(3)} \). For \( I_t^{(3)} \), since either \( N_1 \sim N \) or \( N_2 \sim N \), we observe that by the Young inequality and the assumption that \( \kappa \leq 2 \)

\[ \sum_{N_1 \geq N_2, k N_2 \geq N_1} (\omega_{N_1}^2 N_1^2 \omega_{N_2}^2 N_2^2) \| P_{N_1} \mathfrak{u} \|_{L_t^2} \| P_{N_2} \mathfrak{u} \|_{L_t^2} \]

\[ \lesssim k^3 \sum_{k N_2 \geq N_1} \omega_{N_1}^2 \omega_{N_2}^2 \left( \frac{N_1}{k N_2} \right) \| D_{N_1}^2 P_{N_1} \mathfrak{u} \|_{L_t^2} \| D_{N_2}^2 P_{N_2} \mathfrak{u} \|_{L_t^2} \lesssim k^3 \| \mathfrak{u} \|_{H_x^s}^2. \]

By using above observation together with Lemma 2.1.11 Hölder’s, Young’s inequality (2.3.25),

\(^3\)In this case if \( k = 2 \), there is a suitable non-resonance.
and Proposition 2.2.1 show

\[ J_1^{(3)} \lesssim \sum_{N_1 \geq N_2, ..., N_{k+2} \geq N_1} (\omega_2^2 N_1^{2a} + \omega_2^2 N_2^{2a})kN_5 \]

\[ \times \int_0^t \|P_{N_1}u\|_L^2 \|P_{N_2}u\|_L^2 \prod_{j=3}^{k+2} \|P_{N_j}u\|_L^2 dt' \]

\[ \lesssim_k \|u\|_{L^p_t H_x^s}^k \|u\|_{L^p_t H_x^s}^k \sum_{N_2 \geq N_2 \geq N_3} \prod_{j=3}^5 \left( \frac{N_5}{N_j} \right)^{\frac{1}{2}} \|D^{\frac{1}{2}}_x P_{N_j}u\|_{L^2_t L^\infty_x} \]

\[ \lesssim_k \|u\|_{L^p_t H_x^s}^k \|u\|_{L^p_t H_x^s}^k \sum_j D^{\frac{1}{2}}_x P_k u \|_{L^2_t L^\infty_x} \lesssim_k T^2 C_M \|u\|_{L^p_t H_x^s}. \]

(2.3.29)

For \( J_1^{(3)} \), we take the extensions \( \tilde{u} = \rho_t(u) \) of \( u \) defined in (2.1.20). Note that we have \( N_1 \sim N_2 \sim N \). We further decompose \( J_1^{(3)} \) as in Case 2. In the same spirit as (2.3.26), we define the functional for the sake of notation:

\[ J_1^{(3)}(u_1, \ldots, u_{k+2}) := \sum_{N_1 \geq N_2, ..., N_{k+2}} \sum_{kN_5 \leq M \leq N_5} \omega_2^2 N_5^{2a} \left( \int_R \int_T \Pi(u_1, u_2) P_M \left( \prod_{j=3}^{k+2} u_j \right) dx dt \right). \]

We let \( R = N_1^2 M^\frac{3}{2} \) and yield

\[ J_1^{(3)} \leq J_1^{(3)}(P_{N_1}1_{t, R}^{\text{high}} u, P_{N_2}1_{t, R}^{\text{high}} \tilde{u}, P_{N_3}u, \ldots, P_{N_{k+2}} u) \]

\[ + J_1^{(3)}(P_{N_1}1_{t, R}^{\text{low}} u, P_{N_2}1_{t, R}^{\text{high}} \tilde{u}, P_{N_3}u, \ldots, P_{N_{k+2}} u) \]

\[ + J_1^{(3)}(P_{N_1}1_{t, R}^{\text{low}} u, P_{N_2}1_{t, R}^{\text{low}} \tilde{u}, P_{N_3}u, \ldots, P_{N_{k+2}} u) \]

\[ =: J_1^{(3)}_{\text{low}, 1} + J_1^{(3)}_{\text{low}, 2} + J_1^{(3)}_{\text{low}, 3}. \]

More precisely, we would have

\[ J_1^{(3)} \leq \sum_{N_1 \geq N_2, ..., N_{k+2}} \sum_{kN_5 \leq M \leq N_5} \omega_2^2 N_5^{2a} \left( \int_R \int_T \Pi(P_{N_1}1_{t, R}^{\text{high}} u, P_{N_2}1_{t, R}^{\text{high}} \tilde{u}) P_M \left( \prod_{j=3}^{k+2} P_{N_j} u \right) dx dt \right). \]

(2.3.29)

Let \( M, N_3 \) be dyadic numbers. Then, for any \( \epsilon > 0 \), we have

\[ \sum_{M \leq N_3} 1 \lesssim N_1^\epsilon, \]

(2.3.30)

where the implicit constant does not depend on \( \epsilon \). By the same argument as we did for \( J_{\text{low}, 1}^{(2)} \), while using (2.3.30), (2.1.22), we obtain

\[ J_1^{(3)}_{\text{low}, 1} + J_1^{(3)}_{\text{low}, 2} \lesssim_k T^\frac{3}{4} \|\tilde{u}\|_{L^p_t H_x^s}^k \|\tilde{u}\|_{L^p_t H_x^s}^k \lesssim_k T^\frac{3}{4} \|u\|_{L^p_t H_x^s}. \]

\[ \text{For instance, } \epsilon = \frac{3}{8} \wedge (s'/2 - \frac{1}{4}) \]

\[ (s'/2) \]
CHAPTER 2. MICROSCOPIC LIMITS: DETERMINISTIC METHOD

Now, to estimate $J_{\infty,3}^{(3)}$, we recall from Lemma 2.3.6 that

$$|\Omega_{k+1}^{\delta,\delta}(\tilde{n})| \gtrsim n_3 + n_4 |N_1 | \sim MN_1$$

Hence, by letting $L := MN_1$, we further decompose as the following way:

$$J_{\infty,3}^{(3)} \leq J_{\infty,3}^{(3)}(P_{N_1}Q_{\geq L}(1_{l,R}^{\text{low}}), P_{N_2}Q_{\geq L}(1_{l,R}^{\text{low}}), P_{N_3}u, \ldots, P_{N_{k+2}}u)$$

$$+ J_{\infty,3}^{(3)}(P_{N_1}Q_{\leq L}(1_{l,R}^{\text{low}}), P_{N_2}Q_{\leq L}(1_{l,R}^{\text{low}}), P_{N_3}u, \ldots, P_{N_{k+2}}u)$$

$$+ J_{\infty,3}^{(3)}(P_{N_1}Q_{\leq L}(1_{l,R}^{\text{low}}), P_{N_2}Q_{\leq L}(1_{l,R}^{\text{low}}), P_{N_3}u, \ldots, P_{N_{k+2}}u)$$

$$+ \ldots + J_{\infty,3}^{(3)}(P_{N_1}Q_{\leq L}(1_{l,R}^{\text{low}}), P_{N_2}Q_{\leq L}(1_{l,R}^{\text{low}}), P_{N_3}u, \ldots, P_{N_{k+2}}Q_{\geq L}u)$$

$$=: J_{\infty,3,1}^{(3)} + \ldots + J_{\infty,3,k+2}^{(3)}.$$

In particular,

$$J_{\infty,3,1}^{(3)} \leq \sum \omega^2 N^2 \left| \int_{\mathbb{R}} \int_{\mathbb{S}} \Pi(P_{N_1}Q_{\geq MN_1}(1_{l,R}^{\text{low}}), P_{N_2}(1_{l,R}^{\text{low}}))M \left( \prod_{j=3}^{k+2} P_{N_j}u \right) dx dt' \right|$$

$$+ \sum \omega^2 N^2 \left| \int_{\mathbb{R}} \int_{\mathbb{S}} \Pi(P_{N_1}Q_{\leq MN_1}(1_{l,R}^{\text{low}}), P_{N_2}(1_{l,R}^{\text{low}}))M \left( \prod_{j=3}^{k+2} P_{N_j}u \right) dx dt' \right|$$

$$+ \sum_{l=3}^{k+2} \sum \omega^2 N^2 \left| \int_{\mathbb{R}} \int_{\mathbb{S}} \Pi(P_{N_1}Q_{\leq MN_1}(1_{l,R}^{\text{low}}), P_{N_2}(1_{l,R}^{\text{low}})) \times M \left( \prod_{j=3}^{k+2} P_{N_j}u \right) dx dt' \right|$$

where $\sum$ denotes $\sum_{N>1} \sum_{N_1 \sim N_2 \sim N} \sum_{k_3 \in \mathbb{S}}$ for simplicity.

Here, we observe the argument of $J_{\infty,3,1}^{(2)}$ applies to $J_{\infty,3,1}^{(3)}$. Then, Lemmas 2.1.1 and 2.1.8 Hölder’s inequality, (2.3.4), (2.3.28), and (2.3.30) give the following estimate:

$$J_{\infty,3,1}^{(3)} = \sum \omega^2 N^2 \left| \int_{\mathbb{R}} \int_{\mathbb{S}} \Pi(P_{N_1}Q_{\geq MN_1}(1_{l,R}^{\text{low}}), P_{N_2}(1_{l,R}^{\text{low}}))M \left( \prod_{j=3}^{k+2} P_{N_j}u \right) dx dt' \right|$$

$$\lesssim \sum_{N_1, \ldots, N_{k+2}, M \leq N_0} \sum \omega^2 N^2 \left| \prod_{j=3}^{k+2} \|P_{N_j}u\|_{H_x^s} \right|$$

Moreover, $J_{\infty,3,2}$ immediately follows from the above computation.

Now, we apply the same strategy as we did for $J_{\infty,3,3}^{(2)}$ to estimate $J_{\infty,3,3}^{(3)}$. Then, from (2.3.1), (2.3.30), and Lemma 2.1.8, we have

$$J_{\infty,3,3}^{(3)} = \sum \omega^2 N^2 \left| \int_{\mathbb{R}} \int_{\mathbb{S}} \Pi(P_{N_1}Q_{\leq MN_1}(1_{l,R}^{\text{low}}), P_{N_2}Q_{\leq MN_1}(1_{l,R}^{\text{low}})) \right.$$
Lastly, we conclude for all 4 \( \leq j \leq k + 2 \), the same strategy as above, we have
\[
J_{\infty, 3, j}^{(3)} \lesssim_k T_j^2 \| u \|^2_{L^2 T H_j^s}
\]

**Case 3: \( \phi_3^2 (n_3) n_3 \) contribution.**
Finally, we consider the contribution of \( \phi_3^2 (n_3) n_3 \). We may assume that \( N_3 \gg k N_4 \). Otherwise, the proof is the same as the contribution of \( A_2 \). When \( N_3 \gg k N_4 \), we can obtain the desired estimate as in Case 2. This completes the proof.

### 2.3.3 Uniform difference estimates

In the following, let us establish an a priori estimate at the regularity level \( s - 1 \) on the difference between two solutions. Then, the symmetrisation process, as in the Proof of 2.3.9, fails when we establish an energy estimate for the difference between two solutions. We artificially introduce one derivative gain to overcome such difficulty. Moreover, we do not use the frequency envelope, and we set \( \omega_N = 1 \) (always argue in the standard Sobolev space \( H^s (\mathbb{T}) \)). The main strategy follows from [120 Proposition 3.5] and [119 Proposition 5.1].

Let us take two solutions \((u^{(1)}, u^{(2)}) \) in \( M_T^s \) of gILW \((2.1.1)\) with the initial data \((u_0^{(1)}, u_0^{(2)}) \) in \( (H^s (\mathbb{T}))^2 \). Then, we denote the difference by \( w = u^{(1)} - u^{(2)} \), which satisfies
\[
\partial_t w - G_\delta \partial_x^2 w = -\partial_x (f(u^{(1)}) - f(u^{(2)})).
\] (2.3.31)

The idea is now to establish analogous estimates as in Proposition 2.3.9 to the equation (2.3.34). The following argument can be seen in [119], and again it applies well to real-line settings. One should notice here that the equation (2.3.34) enjoys fewer symmetries. Then, we will estimate the difference in a space with lower regularity than that of the solution itself. Such difference estimates help to show the uniqueness of the solutions. Furthermore, in proving the local well-posedness part, we will use the frequency envelope approach to recover the continuity result with respect to initial data.

**Proposition 2.3.11.** Let \( s \geq \frac{\delta}{2} \). For \( j = 1, 2 \), we take \( w^{(j)} \in M_T^s \) be solution of gILW \((2.1.1)\) with the initial data \( u_0^{(j)} \in H^s (\mathbb{T}) \), and \( w^{(j)} \in N_T^s \) be solution of scaled gILW \((2.1.2)\) with the initial data \( v_0^{(j)} \in H^s (\mathbb{T}) \). Then, the following statements hold:

(i) for any \( 2 \leq \delta \leq \infty \), we have
\[
\| u^{(1)} - u^{(2)} \|^2_{L^\infty T H_j^{s-1}} \leq \| u_0^{(1)} - u_0^{(2)} \|^2_{H_j^{s-1}} + T_\delta^j C \left( \| u^{(1)} \|_{M_j^s, \ell}, \| u^{(2)} \|_{M_j^s, \ell} \right) \times \| u^{(1)} - u^{(2)} \|_{M_j^{s-1, \ell}} \| u^{(1)} - u^{(2)} \|_{L^\infty T H_j^{s-1}}.
\] (2.3.32)

(ii) for any \( 0 < \delta \ll 1 \), we have
\[
\| u^{(1)} - u^{(2)} \|^2_{L^\infty T H_j^{s-1}} \leq \| u_0^{(1)} - u_0^{(2)} \|^2_{H_j^{s-1}} + T_j^4 C \left( \| v^{(1)} \|_{N_j^s, \ell}, \| v^{(2)} \|_{N_j^s, \ell} \right) \times \| u^{(1)} - u^{(2)} \|_{N_j^{s-1, \ell}} \| u^{(1)} - u^{(2)} \|_{L^\infty T H_j^{s-1}}.
\] (2.3.33)

Moreover, the estimate is independent of \( \delta \).

**Proof.** The proof of (2.3.32) and (2.3.33) share the same argument. We only present the proof of (2.3.32) in the following.

For simplicity, let us denote the two solutions are \((u, v) \) in \((Z_T^s)^2 \) associated with the initial data \((u_0, v_0) \) in \((H^s (\mathbb{T}))^2 \). The difference \( w = u - v \) satisfies
\[
\partial_t w - G_\delta \partial_x^2 w = -\partial_x (f(u) - f(v)).
\] (2.3.34)
By rewriting \( f(u) - f(v) \) as the following
\[
f(u) - f(v) = \sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} (u^k - v^k) = \sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} w \sum_{i=0}^{k-1} u^i v^{k-1-i}. \tag{2.3.35}
\]

We proceed as in the proof of Proposition 2.3.9, we see from (2.3.34) that for \( t \in [0,T] \) and we obtain
\[
\|w(t)\|^2_{H^s_x} \leq \|w_0 - v_0\|^2_{H^s_x} + 2 \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{(k-1)!} \max_{i \in \{0, \ldots, k-1\}} I^I_{k,i}, \tag{2.3.36}
\]
where \( I^I_{k,i} \) is defined by
\[
I^I_{k,i} := \sum_{N \geq 1} N^{2(s-1)} \left| \int_0^t \int_{\mathbb{T}} w^i u^{k-1-i} w P_N^k \partial_x w \, dx \, dt \right|.
\]

It is clear that \( I^I_{1,i} = 0 \) from IBP. Therefore we are reduced to estimating the contribution of
\[
I^I_{k+1} \leq \sum_{N \geq 1} N^{2(s-1)} \left| \int_0^t \int_{\mathbb{T}} z^k w P_N^k \partial_x w \, dx \, dt \right| \tag{2.3.37}
\]
where \( z^k \) stands for \( w^i u^{k-i} \) for some \( i \in \{0, \ldots, k\} \). We set \( C_M := C(||u||_{M^s_T} + ||v||_{M^s_T}) \), and we claim that for any \( k \geq 1 \) the following bound holds
\[
I^I_{k+1} \leq C^k T M^k C_M^k ||w||_{M^s_T} ||w||_{L^\infty_T H^{s-1}_x}. \tag{2.3.38}
\]

Hence, we conclude this leads to (2.3.32) and (2.3.33) by following the same sort of discussion after (2.3.21). Now, let us show claim (2.3.38). Let \( k \geq 1 \) be fixed, and we estimate \( I^I_k \). Also, we will use the notation \( a \lesssim_b \) as defined in (2.3.22).

- **Low-frequency contribution.**

  The contribution of the sum over \( 1 \leq N \lesssim 1 \) in (2.3.37) is easily estimated. For the situation \( 1 - s < 0 \), we notice that \( L^2(\mathbb{T}) \subset H^{0-s}(\mathbb{T}) \). Then, for \( 1 - s > 0 \) we apply (2.1.23), and obtain the following
\[
\sum_{N \leq 1} (1 \vee N)^{2(s-1)} \left| \int_0^t \int_{\mathbb{T}} z^k w P_N^k \partial_x w \, dx \, dt \right| \lesssim T \sum_{N \leq 1} ||w||_{L^\infty_T H^{s-1}_x} \||z^k P_N^k \partial_x w||_{L^\infty_T H^{s-1}_x} \leq T \left\{ ||z^k||_{L^\infty_T H^{s-1}_x} \||z^k P_N^k \partial_x w||_{L^\infty_T H^{s-1}_x} + ||\partial_x w||_{L^\infty_T H^{s-1}_x} \right\} \lesssim \|z^k\||_{L^\infty_T H^{s-1}_x}^{N^s-2s} \||z^k P_N^2 w||_{L^\infty_T H^{s-1}_x} + ||z^k||_{L^\infty_T H^{s-1}_x} \||z^k P_N^2 w||_{L^\infty_T H^{s-1}_x} \lesssim_k T \||z^k||_{L^\infty_T H^{s-1}_x}^2.
\]

Our assumption is that \( s > \frac{1}{2} \). Then, we have the Sobolev inequality such that \( H^{\frac{s}{2}+}(\mathbb{T}) \subset L^\infty(\mathbb{T}) \). In the last inequality, we used \( 0 < 1 - s < \frac{1}{2} \) and iteratively applied (2.1.23) to obtain
\[
||z^k||_{H^{1-s}_x} \lesssim ||z||_{H^s_x}.
\]

- **High-frequency contribution.**

  Let \( N \gg 1 \). A similar argument to (2.3.23) we see from the symmetry that
\[
\left| \int_T (w P_N^k \partial_x w) z^k \, dx \right| \lesssim \sum_{n_1 + \cdots + n_{k+2} = 0} A(n_1, n_2) w_1 w_2 \prod_{j=3}^{k+2} \tilde{e}(n_j)
\]
We observe that the frequency projectors in $\Pi(n, n_2) = \phi_N^2(n_1) + \phi_N^2(n_2)$. Therefore, we see the following expression immediately

$$
\sum_{N \gg 1} N^{2(s-1)} \left| \int_0^t \int_{\mathbb{R}} z^k \partial_x w P_N \hat{\partial}_x w dxdt \right|
\leq \sum_{N \gg 1} \left( \sum_{n_{1}, \ldots, n_{k+2}} N^{2(s-1)} \left| \int_0^t \Pi(P_{N_1} w, P_{N_2} w) \prod_{j=3}^{k+2} P_{N_j} z_j dxdt \right| \right)
=: J_t,
$$

where $\Pi(f, g)$ is defined by (2.1.27). By symmetry, we may assume that

$$
\begin{cases}
N_1 \geq N_2; \\
N_3 \geq N_4, \text{ for } k = 2; \\
N_3 \geq N_4 \geq N_5 = \max_{j \geq 5} N_j, \text{ for } k \geq 3.
\end{cases}
$$

We observe that the frequency projectors in $\Pi(\cdot, \cdot)$ implies that

$$
N_1 \sim N \quad \text{ or } \quad N_2 \sim N.
$$

In particular, we have $N_1 \gtrsim N$. Moreover, we can assume that $N_3 \geq 1$, since one application of IBP shows that the contribution of $N_3 = 0$ vanish. Finally, we can also assume that $N_2 \geq 1$.

This is due to the case when $N_2 = 0$, we must have $k N_3 \gtrsim N_1$, and it easy to see the following computation

$$
J_t = \sum_{N \gg 1} \left( \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_0^t \Pi(P_{N_1} w, P_{N_2} w) \prod_{j=3}^{k+2} P_{N_j} z_j dxdt \right| \right)
\leq kT \|w\|_{L^{\infty}_x H^{-1}_t} \|z_1\|_{L^{\infty}_x H^1_t} \|P_0 w\|_{L^{\infty}_x H^1_t} \prod_{j=4}^{k+2} \|z_j\|_{L^{\infty}_x H^1_t} \lesssim k T \|w\|_{L^2_x H^{-1}_t}^2.
$$

Here, we have used that $P_0 \partial_x w = 0$. In the last inequity, we use the following observation

$$
|P_0 w(x)| = |F^{-1}_x \{ \phi_0(n) \hat{\omega}(n) \}(x)| = |\hat{\omega}(0)| \leq \|n\|^{-1} \|\hat{\omega}(n)\|_{L^2_x},
$$

for $n \in \mathbb{Z}$. Therefore, to show claim (2.3.38), let us divide $J_t$ into the following three cases:

1. $N_1 \lesssim k N_4$ (or $k \geq 2$);
2. $N_1 \gg k N_4$ and $N_2 \gtrsim N_3$ (or $k = 1$ and $N_2 \gtrsim N_3$);
3. $N_1 \gg k N_4$ and $N_2 \ll N_3$ (or $k = 1$ and $N_2 \ll N_3$).

**Case 1:** $N_1 \lesssim k N_4$ (or $k \geq 2$).

We apply Hölder’s inequality, Young’s convolution inequalities (see (2.3.25, 2.3.24), Proposition...
Therefore, in this case, the contribution of $J_{3.1}$:

$$J_t = \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_0^t \int_T \Pi(P_{N_1} w, P_{N_2} w) \prod_{j=3}^{k+2} P_{N_j} z_j dx dt \right|$$

$$\leq \sum_{N_1, \ldots, N_{k+2}} (N_1^{2s-1} + N_2^{2s-1}) \prod_{j=1}^2 \|P_{N_j} w\|_{L_{T,x}^s} \prod_{l=3}^4 \|P_{N_l} z_l\|_{L_{T,x}^s} \prod_{j=5}^{k+2} \|P_{N_j} z_j\|_{L_{T,x}^s}$$

$$\lesssim_k \sum_{N_1, N_2 \geq k^{-1} N_1 \geq k^{-1} N_2} N_1^{2s-1} \|P_{N_1} w\|_{L_{T,x}^s} \|P_{N_2} w\|_{L_{T,x}^s} \|P_{N_3} z_3\|_{L_{T,x}^s} \|P_{N_4} z_4\|_{L_{T,x}^s}$$

$$\lesssim_k \sum_{N \geq k = 1} k^{2s-2} \left( \frac{N_1}{k N_4} \right)^{\frac{2s-1}{2}} \left( \frac{N_2}{k N_4} \right)^{\frac{2s-1}{2}} \left( \frac{N_3}{N_4} \right)^{s-\frac{1}{2}} \times \prod_{j=1}^2 \|D_{T,x}^{\frac{1}{2}} P_{N_j} w\|_{L_{T,x}^s} \prod_{l=3}^4 \|D_{T,x}^{\frac{1}{2}} P_{N_l} z_l\|_{L_{T,x}^s}$$

$$\lesssim_k T^{\frac{s}{2}} C_M \|w\|^2_{L_T^2 H_x^{s-\frac{1}{2}}}.$$ 

**Case 2:** $N_1 \gg k N_4$ and $N_2 \gtrsim N_3$ (or $k = 1$ and $N_2 \gtrsim N_4$).

Firstly, let us recall that $N_1 \sim N$ or $N_2 \sim N$, which implies that $N \sim N_1 \sim N_2 \gtrsim N_3$.

Therefore, in this case, the contribution of $J_t$ can be estimated by the same way as the $A_1$ contribution in Proposition 2.3.9. In particular, we shall replace

$$N^{2s}, P_{N_j} w, P_{N_j} u, P_{N_j} u, \quad \text{for} \quad j = 3, \ldots, k+2$$

by the following

$$N^{2s-2}, P_{N_j} w, P_{N_j} w, P_{N_j} z_j, \quad \text{for} \quad j = 3, \ldots, k+2,$$

respectively. Then, as the result, we can obtain

$$J_t = \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_0^t \int_T \Pi(P_{N_1} w, P_{N_2} w) \prod_{j=3}^{k+2} P_{N_j} z_j dx dt \right|$$

$$\leq C T^{\frac{s}{2}} \left( \|w\|_{X^{s-2,1}} + \|w\|_{L_T^\infty H_x^{s-1}} \right) \|w\|_{L_T^p H_x^{s-1}}$$

for $s \geq \frac{3}{4}$.

**Case 3:** $N_1 \gg k N_4$ and $N_2 \ll N_3$ (or $k = 1$ and $N_2 \ll N_3$).

Observe that in this case $N_1 \sim N_3 \sim N \gg N_2 \lor N_4$. We further divide the contribution of $J_t$ into the following three subcases:

3.1. $N_2 \gg k N_4$ or $k = 1$, 

3.2. $k N_4 \gtrsim N_2 \gtrsim N_5$ (or $k = 2$ and $N_4 \gtrsim N_2$), 

3.3. $N_2 \ll N_5$ (or $k = 3$).

**Subcase 3.1:** $N_2 \gg k N_4$ or $k = 1$.

Recall that we have $N_2 \geq 1$ and thus this subcase contains the case $N_4 = 0$. We take the extensions

$$\begin{cases} 
\hat{w} = \rho_T(w) & \text{of } w \\
\hat{z}_j = \rho_T(z_j) & \text{of } z_j 
\end{cases}$$
for \( j \geq 3 \), see definition (2.1.20). For simplicity, we shall use the following notation:

\[
J_{\infty}^{(3,1)}(u_{1}, \cdots, u_{k+2}) := \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(u_1, u_2) \prod_{j=3}^{k+2} u_j \, dx dt \right|
\]

By setting \( R = N_{1}^{1/2}N_{2}^{1/2} \), we divide \( J_t \) as

\[
J_t \leq J_{\infty}^{(3,1)}(P_{N_{1}}1_{t,R}^{\text{high}} \hat{w}, P_{N_{2}} \hat{w}, P_{N_{3}}1_{t,R}^{\text{high}} \hat{z}_3, P_{N_{4}} \hat{z}_4, \cdots, P_{N_{k+2}} \hat{z}_{k+2})
+ J_{\infty}^{(3,1)}(P_{N_{1}}1_{t,R}^{\text{low}} \hat{w}, P_{N_{2}} \hat{w}, P_{N_{3}}1_{t,R}^{\text{high}} \hat{z}_3, P_{N_{4}} \hat{z}_4, \cdots, P_{N_{k+2}} \hat{z}_{k+2})
+ J_{\infty}^{(3,1)}(P_{N_{1}}1_{t,R}^{\text{low}} \hat{w}, P_{N_{2}} \hat{w}, P_{N_{3}}1_{t,R}^{\text{low}} \hat{z}_3, P_{N_{4}} \hat{z}_4, \cdots, P_{N_{k+2}} \hat{z}_{k+2})
= J_{\infty,1}^{(3,1)} + J_{\infty,2}^{(3,1)} + J_{\infty,3}^{(3,1)}.
\]

In more details, the above can be expressed as the following

\[
J_t \leq \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_{1}}1_{t,R}^{\text{high}} \hat{w}, P_{N_{2}} \hat{w}) \prod_{j=3}^{k+2} P_{N_{j}} \hat{z}_j \, dx dt \right|
+ \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_{1}}1_{t,R}^{\text{low}} \hat{w}, P_{N_{2}} \hat{w}) \prod_{j=3}^{k+2} P_{N_{j}} \hat{z}_j \, dx dt \right|
+ \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_{1}}1_{t,R}^{\text{low}} \hat{w}, P_{N_{2}} \hat{w}) \prod_{j=3}^{k+2} P_{N_{j}} \hat{z}_j \, dx dt \right|
= : J_{\infty,1}^{(3,1)} + J_{\infty,2}^{(3,1)} + J_{\infty,3}^{(3,1)}.
\]

For \( J_{\infty,1}^{(3,1)} \), we see from (2.3.3) that

\[
\|1_{t,R}^{\text{high}}\|_{L^1} \lesssim T^{1/2} N_{1}^{-1} N_{2}^{-1},
\]

which gives

\[
J_{\infty,1}^{(3,1)} = \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(P_{N_{1}}1_{t,R}^{\text{high}} \hat{w}, P_{N_{2}} \hat{w}) \prod_{j=3}^{k+2} P_{N_{j}} \hat{z}_j \, dx dt \right|
\lesssim \sum_{N_1, \ldots, N_{k+2}} N_{1}^{2s-1} \|1_{t,R}^{\text{high}}\|_{L^1} \|P_{N_{1}} \hat{w}\|_{L^p} \|P_{N_{2}} \hat{w}\|_{L^p} \|P_{N_{j}} \hat{z}_j\|_{L^p} \prod_{j=4}^{k+2} \|P_{N_{j}} \hat{z}_j\|_{L^p}
\lesssim_k T^{1/2} \sum_{N_1, N_2} N_{1}^{2s-1} \|P_{N_{1}} \hat{w}\|_{L^p} \|P_{N_{2}} \hat{w}\|_{L^p} \|P_{N_{j}} \hat{z}_j\|_{L^p}
\lesssim_k T \|\hat{w}\|^2_{L^p H^{s-1}} \lesssim_k T \|\hat{w}\|^2_{L^p H^{s-1}}.
\]

Here, in the first inequality, we used (2.1.22) and the fact that \( N_{2}^{2s-1} \leq N_{1}^{2s-1} \) for \( s \geq \frac{3}{4} > \frac{1}{2} \).

This strategy for \( J_{\infty,1}^{(3,1)} \) applies to \( J_{\infty,2}^{(3,1)} \) and we have a same bound.

For \( J_{\infty,3}^{(3,1)} \), we recall from Lemma 2.3.5 we have

\[
|\Omega_{k+1}^{d,4}(\hat{v})| \gtrsim N_2 N_1 =: L.
\]

This allows us to decompose \( J_{\infty,3}^{(3,1)} \) as the following

\[
J_{\infty,3}^{(3,1)} \leq J_{\infty}^{(3,1)}(P_{N_{1}} Q_{\mathbb{Z}L}(1_{t,R}^{\text{low}} \hat{w}), P_{N_{2}} \hat{w}, P_{N_{3}}1_{t,R}^{\text{low}} \hat{z}_3, \cdots, P_{N_{k+2}} \hat{z}_{k+2})
\]
where \( J^{(3, 1)}_{\infty, 3, n} \) for \( 4 \leq n \leq k + 2 \) corresponds to the term in which \( Q_{\geq L} \) lands on \( P_{N_n^k, n} \). More precisely, we have the following expression:

\[
J^{(3, 1)}_{\infty, 3} \leq \sum_{n=4}^{k+2} N^{2(s-1)} \left| \int_R \int_T \Pi(P_{N_n^k, n}Q_{\geq N_n^k, n}(1_{L,R}w), P_{N_n^k, n}1_{L,R}z) \prod_{j=4}^{k+2} P_{N_n^k, n}^j \frac{dx}{dt} \right| \\
+ \sum_{n=4}^{k+2} N^{2(s-1)} \left| \int_R \int_T \Pi(P_{N_n^k, n}Q_{\geq N_n^k, n}(1_{L,R}w), P_{N_n^k, n}Q_{\geq N_n^k, n}(1_{L,R}z)) \prod_{j=4}^{k+2} P_{N_n^k, n}^j \frac{dx}{dt} \right| \\
\times (P_{N_n^k, n}Q_{\geq N_n^k, n}(1_{L,R}z)) \prod_{j=4}^{k+2} P_{N_n^k, n}^j \frac{dx}{dt} \\
+ \sum_{n=4}^{k+2} N^{2(s-1)} \left| \int_R \int_T \Pi(P_{N_n^k, n}Q_{\geq N_n^k, n}(1_{L,R}w), P_{N_n^k, n}Q_{\geq N_n^k, n}(1_{L,R}z)) \prod_{j=4}^{k+2} P_{N_n^k, n}^j \frac{dx}{dt} \right| \\
= \sum_{j=1}^{k+2} J^{(3, 1)}_{\infty, 3, j},
\]

where \( \sum \) denotes \( \sum_{n=4}^{k+2} \sum_{n_1, \ldots, n_{k+2}} \) for simplicity. Also, we have

\[
R = N^4 \ll N_2^4 = L.
\]

Then, by the Hölder’s inequality, (2.3.4), (2.3.28) and Lemma 2.1.8 imply that

\[
J^{(3, 1)}_{\infty, 3, 1} = \sum_{n_1, \ldots, n_{k+2}} N^{2(s-1)} \left| \int_R \int_T \Pi(P_{N_n^k, n}Q_{\geq N_n^k, n}(1_{L,R}w), P_{N_n^k, n}Q_{\geq N_n^k, n}(1_{L,R}z)) \prod_{j=4}^{k+2} P_{N_n^k, n}^j \frac{dx}{dt} \right| \\
\leq \sum_{n_1, \ldots, n_{k+2}} N^{2s-1} \|P_{N_n^k, n}Q_{\geq L}(1_{L,R}w)\|_{L^\infty} \|P_{N_n^k, n}1_{L,R}z\|_{L^\infty} \prod_{j=4}^{k+2} \|P_{N_n^k, n}^j\|_{L^\infty} \\
\lesssim \sum_{n_1, n_2} N^{2s-1} \|P_{N_1, n_2}w\|_{X^{0, 1, s}} \|P_{N_2, n_2}w\|_{L^\infty H_x^{s-\frac{1}{2}}} \|P_{N_2, n_2}1_{L,R}z\|_{L^\infty} \\
\lesssim k \sum_{n_1, n_2} N^{2s-1} \|P_{N_1, n_2}w\|_{X^{0, 1, s}} \|P_{N_2, n_2}w\|_{L^\infty H_x^{s-\frac{1}{2}}} \|P_{N_2, n_2}1_{L,R}z\|_{L^\infty} \\
+ T^{\frac{s}{2}} \sum_{n_1, n_2} N^{2s-1} \|P_{N_1, n_2}w\|_{X^{0, 1, s}} \|P_{N_2, n_2}w\|_{L^\infty H_x^{s-\frac{1}{2}}} \|P_{N_2, n_2}1_{L,R}z\|_{L^\infty} \\
\lesssim k T^{s} \|w\|_{L^\infty H_x^{s-1}} \|w\|_{X^{2, 1, s}} \lesssim k T^{s} \|w\|_{M^{s-1, 1}} \|w\|_{L^\infty H_x^{s-1}}.
\]
Similar computation and (2.3.1) show
\[
J_{3,2}^{(3)} = \sum N^{2(s-1)} \left| \int \mathbb{I}(P_N Q_{<N} N_{1,1} (1_{T} \hat{w}_J), P_N \hat{w}) P_N Q_{>N} N_{1,1} (1_{T} \hat{w}_J) \prod_{j=4}^{k+2} P_N \hat{z}_j \, dx dt \right|
\]
\[
\leq_k T^2 C_M^{k-1} \| \hat{w} \|_{L_T^\infty H_x^\infty} \| \hat{z}_3 \|_{X^{s+1,5}} \leq_k T^2 \| w \|_{L_T^\infty H_x^\infty}^2.
\]

Next, we consider the contribution of \(J_{3,3}^{(3)}\). By using Lemma 2.1.8, the Hölder’s inequality, and (2.3.1) show
\[
J_{3,3}^{(3)} = \sum N^{2(s-1)} \left| \int \mathbb{I}(P_N Q_{<N} N_{1,1} (1_{T} \hat{w}_J), P_N \hat{w}) \prod_{j=4}^{k+2} P_N \hat{z}_j \, dx dt \right|
\]
\[
\leq_k \sum_{N_1, \ldots, N_{k+2}} N^{2s-1} \| P_N 1_{T} \hat{w}_J \|_{L_T^\infty} \| P_N Q_{>N} \hat{w} \|_{L_T^\infty} \| P_N \hat{z}_j \|_{L_T^\infty} \prod_{j=4}^{k+2} \| P_N \hat{z}_j \|_{L_T^\infty}
\]
\[
\leq_k T^2 C_M^{k-1} \sum_{N_1 \geq N_2} N^{2s-1} \| P_N \hat{w} \|_{L_T^\infty} \| P_N \hat{z}_j \|_{L_T^\infty} \sum_{N_1 \geq N_2} N^{s-1} \| P_N \hat{w} \|_{X^{s+1,5}} \leq_k T^2 \| w \|_{M_T^{s+1,5}} \| w \|_{L_T^\infty H_x^\infty}
\]
since \(s > \frac{3}{4} > \frac{1}{2}\). Similarly, we obtain the bound on \(J_{3,4}^{(3)}\), i.e. when the case
\[
\begin{align*}
Q_{>L} & \text{ hits } P_N \hat{z}_4; \\
Q_{<L} & \text{ hit } P_N \hat{w}, P_N \hat{w}_J, \text{ and } P_N \hat{z}_3.
\end{align*}
\]

Therefore, we have
\[
J_{3,4}^{(3)} \leq_k C_M^{k-2} \sum_{N_1, N_2, N_4} N^{2s-1} \| P_N 1_{T} \hat{w}_J \|_{L_T^\infty} \| P_N \hat{w} \|_{L_T^\infty} \| P_N \hat{z}_3 \|_{L_T^\infty} \| P_N Q_{>N} \hat{w} \|_{L_T^\infty} \| P_N \hat{z}_4 \|_{L_T^\infty} \| P_N \hat{z}_4 \|_{L_T^\infty}
\]
\[
\leq_k T^{1/2} \sum_{N_1, N_2, N_4} N^{2s-1} \| P_N \hat{w} \|_{L_T^\infty} \| P_N \hat{z}_3 \|_{L_T^\infty} \| P_N \hat{w} \|_{L_T^\infty} \| P_N \hat{z}_4 \|_{X^{s+1,5}}
\]
\[
\leq T^2 C_M^{k-2} \| \hat{w} \|_{L_T^\infty H_x^\infty} \| \hat{z}_3 \|_{L_T^\infty} \| P_N \hat{z}_4 \|_{X^{s+1,5}} \leq_k T^2 C_M^{k} \| w \|_{L_T^\infty H_x^\infty}^2.
\]

We conclude this subcase by noticing from the above argument that \(J_{3,3}^{(3)}\) for \(5 \leq j \leq k + 2\) can be estimated by the same bound as that of \(J_{3,4}^{(3)}\). The same argument yields for \(5 \leq j \leq k + 2\),
\[
J_{3,5}^{(3)} \leq_k T^2 C_M^{k} \| w \|_{L_T^\infty H_x^\infty}^2.
\]

**Subcase 3.2:** \(N_5 \lesssim N_2 \lesssim k N_4\) (or \(k = 2\) and \(N_4 \gtrsim N_2\)).

Note that the cases \(N_2 = 0\) or \(N_4 = 0\) have already treated so that we can assume that \(N_2 \geq 1\) and \(N_4 \geq 1\). Let us recall the following relation:
\[
N \sim N_1 \sim N_3 \Rightarrow N_2 \vee N_4.
\]

It suffices to consider the case \(N_5 \lesssim N_2 \lesssim N_4\) since \(k \geq 1\). First, we observe the following, which separates the case into when the derivative hits the low-frequency part \(P_N \hat{w}\) and when the
where $P$ of $P_I$.

Furthermore, for the term $I_{(3,2)}$, we further divide it as in Case 3 in Proposition 2.3.9. We will compare the size of $|n_2 + n_4|$ and $k|n_5|$, and we arrive

$$I_{(3,2)} \leq \sum N^{2(s-1)} \left| \int_0^t \int_\mathbb{T} \partial_x P_N^2 P_N w P_N z_3 P_{kN+1} \left( P_{N_7} w \prod_{j=4}^{k+2} P_{N_j} z_j \right) dxdt' \right|$$

$$+ \sum_{kN+1 \leq M \leq N} N^{2(s-1)} \left| \int_0^t \int_\mathbb{T} \partial_x P_N^2 P_N w P_N z_3 P_M \left( P_{N_7} w \prod_{j=4}^{k+2} P_{N_j} z_j \right) dxdt' \right|$$

$$=: I_{(3,2)}^{(1)} + I_{(3,2)}^{(2)},$$

where $\Sigma$ denotes $\sum_{N \gg 1} \sum_{N_1 \leq \ldots \leq N_{k+2}}$ for the simplicity. Let us remark that if $k = 2$, then the term $I_{(3,2)}^{(1)}$ does not appear, and it holds that $I_{(3,2)} \leq I_{(3,2)}^{(1)}$ with the summation $\Sigma_{M \leq N}$ instead of $\Sigma_{kN+1 \leq M \leq N}$.

To estimate $I_{(3,1)}^{(3,2)}$, we use Hölder’s, Bernstein’s, Minkowski’s inequalities, Proposition 2.2.1 and Corollary 2.2.6 to obtain the following

$$I_{(3,2)}^{(1)} \lesssim \sum_{N_1 \ldots N_{k+2}} N^{2(s-1)} \left( k^3 N_0^2 \right) \left( \frac{N_7}{N_4} \right)^{-1} \left( \frac{N_5 + 1}{N_4} \right)^\frac{1}{2}$$

$$\times \left( \prod_{j=6}^{k+2} \left\| P_{N_j} z_j \right\|_L^2 \right)$$

$$\lesssim \sum_{N_1 \leq \ldots \leq N_{k+2}} \left( \frac{N_2}{N_4} \right)^\frac{1}{2} \left( \frac{N_5 + 1}{N_4} \right)^\frac{1}{2}$$

$$\times \left( \prod_{j=6}^{k+2} \left\| P_{N_j} z_j \right\|_L^2 \right)$$

$$\lesssim \left( \prod_{j=6}^{k+2} \left\| P_{N_j} z_j \right\|_L^2 \right)$$

$$\lesssim \left( \prod_{j=6}^{k+2} \left\| P_{N_j} z_j \right\|_L^2 \right)$$

$$\lesssim \left( \prod_{j=6}^{k+2} \left\| P_{N_j} z_j \right\|_L^2 \right)$$

since $\frac{1}{2} \geq \frac{2}{3} + \frac{1}{6}$.

In order to estimate $I_{(3,2)}^{(1,2)}$, we shall further decompose it. Again, we take the extensions

$$\left\{ \begin{array}{l}
\bar{w} = \rho_T(w) \quad \text{of} \quad w \\
\bar{z}_j = \rho_T(z_j) \quad \text{of} \quad z_j
\end{array} \right.$$
Putting $R = N^{\frac{1}{1}} M^{4}$, we see that

$$I(t,2)^{(3,2)} \leq I_{\infty,2}^{(3,2)}(P_{N}^{1}_{\text{high}}(t,R) \tilde{u}, P_{N}^{1}_{\text{low}}(t,R) \tilde{u}, P_{N_{1}}^{1}(t,R) \tilde{z}_{3}, P_{N_{2}}^{1}(t,R) \tilde{z}_{4}, \ldots, P_{N_{k+2}}^{1}(t,R) \tilde{z}_{k+2})$$

$$+ I_{\infty,2}^{(3,2)}(P_{N}^{1}_{\text{low}}(t,R) \tilde{u}, P_{N}^{1}_{\text{high}}(t,R) \tilde{z}_{3}, P_{N_{1}}^{1}(t,R) \tilde{z}_{4}, \ldots, P_{N_{k+2}}^{1}(t,R) \tilde{z}_{k+2})$$

$$+ I_{\infty,2}^{(3,2)}(P_{N}^{1}_{\text{low}}(t,R) \tilde{u}, P_{N}^{1}(t,R) \tilde{z}_{3}, P_{N_{1}}^{1}(t,R) \tilde{z}_{4}, \ldots, P_{N_{k+2}}^{1}(t,R) \tilde{z}_{k+2})$$

$$=: I_{\infty,2}^{(3,2)} + I_{\infty,2}^{(3,2)} + I_{\infty,2}^{(3,2)}.$$

In particular, the above inequality can be expressed as

$$I(t,2)^{(3,2)} \leq \sum N^{2(s-1)} \int_{R}^{T} \int \partial_{x} P_{N}^{2} P_{N}^{1}_{\text{high}}(t,R) \tilde{u} P_{N_{1}}^{1}(t,R) \tilde{z}_{3} P_{M}^{1}(t,R) \tilde{z}_{j} \int dxdt'$$

$$+ \sum N^{2(s-1)} \int_{R}^{T} \int \partial_{x} P_{N}^{2} P_{N}^{1}_{\text{low}}(t,R) \tilde{u} P_{N}^{1}_{\text{high}}(t,R) \tilde{z}_{3} P_{M}^{1}(t,R) \tilde{z}_{j} \int dxdt'$$

$$+ \sum N^{2(s-1)} \int_{R}^{T} \int \partial_{x} P_{N}^{2} P_{N}^{1}_{\text{low}}(t,R) \tilde{u} P_{N}^{1}(t,R) \tilde{z}_{3} P_{M}^{1}(t,R) \tilde{z}_{j} \int dxdt'$$

$$=: I_{\infty,2}^{(3,2)} + I_{\infty,2}^{(3,2)} + I_{\infty,2}^{(3,2)}$$

where $\sum$ denotes $\sum N^{\geq 1} \sum_{N_{1}, \ldots, N_{k+2}} \sum_{k N_{\text{low}} \leq M \leq N_{k}}$ for simplicity. For $I_{\infty,2,1}$, we see from $(2.3.3)$ that

$$\|1_{\text{high}}^{l}(t,R) \|_{L^{1}} \lesssim T^{\frac{1}{4}} N_{1}^{-\frac{1}{4}} M^{-1}.$$

Then, we have

$$I_{\infty,2,1}^{(3,2)} \lesssim \sum N^{1}_{\geq 1} \sum_{N_{1}, \ldots, N_{k+2}} \sum_{k N_{\text{low}} \leq M \leq N_{k}} N^{2(s-1)} \|1_{\text{high}}^{l}(t,R) \|_{L^{1}} \|P_{N}^{1}_{\text{low}}(t,R) \tilde{u} \|_{L^\infty_{t,r} L^2_{z}} \|P_{N_{1}}^{1}(t,R) \tilde{z}_{3} \|_{L^\infty_{t} L^2_{z}}$$

$$\times \left\| P_{M}^{1}(t,R) \tilde{z}_{j} \right\|_{L^\infty_{t,r}}$$

$$\lesssim T^{\frac{1}{4}} \sum \sum N^{2(s-1)} \|P_{N_{1}}^{1}(t,R) \tilde{u} \|_{L^\infty_{t} L^2_{z}} \|P_{N_{1}}^{1}(t,R) \tilde{z}_{3} \|_{L^\infty_{t} L^2_{z}}$$

$$\times \left\| P_{N_{j}}^{1}(t,R) \tilde{z}_{j} \right\|_{L^\infty_{t} H^{s'}_{z}} \cdot$$

Note that $(2.1.25)$ leads to

$$\sum N_{1}, \ldots, N_{k+2} \left\| P_{N_{j}}^{1}(t,R) \tilde{z}_{j} \right\|_{H^{s'}_{z}} \lesssim k \|P_{N_{1}}^{1}(t,R) \tilde{z}_{4} \|_{H^{s'}_{z}} \|P_{N_{j}}^{1}(t,R) \tilde{z}_{j} \|_{H^{s'}_{z}} \left(2.3.40\right)$$

with $\frac{1}{2} < s' < s$. This together with $(2.3.30)$ and $(2.1.22)$ shows that

$$I_{\infty,2,1}^{(3,2)} \lesssim k T^{\frac{1}{4}} \|\tilde{u}\|_{L^\infty_{t} H^{s'}_{z}} \|\tilde{u}\|_{L^\infty_{t} H^{s'}_{z}} \|\tilde{z}_{3}\|_{L^\infty_{t} H^{s'}_{z}} \|\tilde{z}_{4}\|_{L^\infty_{t} H^{s'}_{z}} \sum N_{1}^{-\frac{4}{4}}$$

$$\lesssim k T^{\frac{1}{4}} \|\tilde{u}\|_{L^\infty_{t} H^{s'}_{z}}^{2}.$$

Similarly, $I_{\infty,2,2}^{(3,2)}$ can be estimated by the same bound as above.

$$I_{\infty,2,2}^{(3,2)} \lesssim k T^{\frac{1}{4}} \|\tilde{u}\|_{L^\infty_{t} H^{s'}_{z}}^{2}.$$
For $I_{\infty, 2, 3}^{(3.2)}$, we see from Lemma 2.3.6 that we have a non-resonant function such that

$$|x_{k+1}(\tilde{u})| \gtrsim MN_1 \sim |n_2 + n_4|N_1.$$  

Therefore, by setting $L := MN_1$, we further decompose as

$$I_{\infty, 2, 3}^{(3.2)} \leq I_{\infty, 2, 3}^{(3.2)}(P_{N_1}Q_{\lesssim L}(1_{L,R}^w \tilde{u}), P_{N_2} \tilde{w}, P_{N_3}1_{L,R}^{\tilde{z}_2} \tilde{x}, \ldots, P_{N_{k+2}} \tilde{z}_{k+2}) + I_{\infty, 2, 3}^{(3.2)}(P_{N_1}Q_{\lesssim L}(1_{L,R}^w \tilde{u}), P_{N_2} \tilde{w}, P_{N_3}Q_{\lesssim L}(1_{L,R}^{\tilde{z}_3} \tilde{x}), \ldots, P_{N_{k+2}} \tilde{z}_{k+2}) + \cdots + I_{\infty, 2, 3}^{(3.2)}(P_{N_1}Q_{\lesssim L}(1_{L,R}^w \tilde{u}), P_{N_2}Q_{\lesssim L}(1_{L,R}^{\tilde{z}_3} \tilde{x}), \ldots, P_{N_{k+2}}Q_{\lesssim L}(1_{L,R}^{\tilde{z}_k} \tilde{x}), P_{N_{k+2}}Q_{\lesssim L}(1_{L,R}^{\tilde{z}_{k+2}}) = I_{\infty, 2, 3, 1}^{(3.2)} + \cdots + I_{\infty, 2, 3, k+2}^{(3.2)},$$

where $I_{\infty, 2, 3, n}^{(3.2)}$ for $4 \leq n \leq k + 2$ corresponds to the term in which $Q_{\lesssim L}$ lands on $P_{N_n} \tilde{z}_n$. We see a more explicit explanation blow:

$$I_{\infty, 2, 3}^{(3.2)} \leq \sum N^{2(s-1)} \left| \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_x P_N^2 P_{N_1}Q_{\lesssim MN_1}(1_{L,R}^w \tilde{u}) P_{N_3}1_{L,R}^{\tilde{z}_3} P_{N_4} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j dx dt \right|$$

$$+ \sum N^{2(s-1)} \left| \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_x P_N^2 P_{N_1}Q_{\lesssim MN_1}(1_{L,R}^w \tilde{u}) P_{N_3}Q_{\lesssim MN_1}(1_{L,R}^{\tilde{z}_3}) P_{N_4} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j dx dt \right|$$

$$+ \sum N^{2(s-1)} \left| \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_x P_N^2 P_{N_1}Q_{\lesssim MN_1}(1_{L,R}^w \tilde{u}) P_{N_3}Q_{\lesssim MN_1}(1_{L,R}^{\tilde{z}_3}) \times P_M \left( P_{N_2}Q_{\lesssim MN_1} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right) dx dt \right|$$

$$+ \sum N^{2(s-1)} \left| \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_x P_N^2 P_{N_1}Q_{\lesssim MN_1}(1_{L,R}^w \tilde{u}) P_{N_3}Q_{\lesssim MN_1}(1_{L,R}^{\tilde{z}_3}) \times \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right|$$

$$= \sum_{j=1}^{k+2} I_{\infty, 2, 3, j}^{(3.2)},$$

where $\sum$ denotes $\sum_{N_1, \ldots, N_{k+2}} \sum_{1 \leq k, N_2, \ldots, N_4 \leq N_4}$ for simplicity.

The contribution of $I_{\infty, 2, 3, 1}^{(3.2)}$ is estimated, thanks to Lemma 2.1.8 (2.3.3), (2.3.4), (2.3.28), and (2.3.40), by

$$I_{\infty, 2, 3, 1}^{(3.2)} \lesssim \sum_{N_1, \ldots, N_{k+2}} \sum_{1 \leq k, N_2, \ldots, N_4 \leq N_4} N_1^{2s-1-1} M^{-1} \left\| \hat{P}_{N_1} \tilde{w} \right\|_{X^{0.1, s}} \left\| \hat{P}_{N_3}1_{L,R}^{\tilde{z}_3} \right\|_{L_T^{\infty}} \times \left\| \hat{P}_{N_2} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right\|_{L_T^{\infty}}$$

$$\lesssim \sum_{N_1, \ldots, N_{k+2}} \sum_{M \leq N_4} N_1^{2s-2} \left\| \hat{P}_{N_1} \tilde{u} \right\|_{X^{0.1, s}} \left\| \hat{P}_{N_3}1_{L,R}^{\tilde{z}_3} \right\|_{L_T^{\infty}} \left\| \hat{P}_{N_2} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right\|_{L_T^{\infty}}$$

$$\lesssim \sum_{N_1 \geq N_4} \sum_{M \leq N_4} (N_1^{2s-2} \left\| \hat{P}_{N_1} \tilde{w} \right\|_{X^{0.1, s}} \left\| \hat{P}_{N_1} \tilde{z}_3 \right\|_{L_T^{\infty}} \left\| \hat{P}_{N_2} \tilde{z}_4 \right\|_{L_T^{\infty}} \left\| \hat{P}_{N_2} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right\| \left\| \hat{P}_{N_2} \tilde{z}_4 \right\|_{L_T^{\infty}}$$

$$+ T^{\frac{1}{2}} N_1^{2s-2} M^{-1} \left\| \hat{P}_{N_1} \tilde{w} \right\|_{X^{0.1, s}} \left\| \hat{P}_{N_1} \tilde{z}_3 \right\|_{L_T^{\infty}} \left\| \hat{P}_{N_2} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right\| \left\| \hat{P}_{N_2} \tilde{z}_4 \right\|_{L_T^{\infty}}.$$
CHAPTER 2. MICROSCOPIC LIMITS: DETERMINISTIC METHOD

Next, we estimate $I^{(3,2)}$, by

$$I^{(3,2)}_{\infty,2,3,2} \lesssim_k T^{\frac{1}{2}} \|w\|_{L^p_t H^{\alpha-1}_x}^2.$$

Next, we estimate $I^{(3,2)}_{\infty,2,3,3}$. By using Lemma 2.1.8, (2.3.3), (2.3.30), (2.3.1) and (2.3.40), we have

$$I^{(3,2)}_{\infty,2,3,3} \lesssim \sum_{N_{1,\ldots,N_{k+2}} M \leq N_4} N_4^{2s-1} M \|P_{N_{1}} w\|_{L^p_t L^q_x} \|P_{N_{2}} z_3\|_{L^p_t L^q_x} \|P_{N_{3}} z_4\|_{L^p_t L^q_x} \|P_M \{(P_{N_{2}} Q_{\geq L} w)\} \prod_{j=4}^{k+2} P_{N_{j}} \tilde{z}\|_{L^2_t L^p_x}$$

$$\leq k T^{\frac{1}{2}} \sum_{N_{1,\ldots,N_{k+2}} M \leq N_4} N_4^{2s-1} M \|P_{N_{1}} w\|_{L^p_t L^q_x} \|P_{N_{2}} z_3\|_{L^p_t L^q_x} \|P_{N_{3}} z_4\|_{L^p_t L^q_x} \|P_{N_{4}} z_2\|_{L^p_t H^{\alpha-1}_x}$$

$$\lesssim k T^{\frac{1}{2}} \|\tilde{z}_3\|_{L^p_t H^{\alpha-1}_x} \|\tilde{z}_4\|_{L^p_t H^{\alpha-1}_x} \|\tilde{w}\|_{L^p_t H^{\alpha-1}_x} \|\tilde{w}\|_{L^p_t H^{\alpha-1}_x}$$

with $\frac{1}{2} < s' < s$. In the first inequality, we have inserted $MM^{-1}$, which guarantees that we can apply (2.3.40). The second inequality we notice

$$\|P_{N_{1}} Q_{\leq L} \tilde{z}_3\|_{L^p_t L^q_x} \leq \|P_{N_{1}} Q_{\leq L} \|_{L^p_t L^q_x} + \|P_{N_{1}} Q_{\leq L} \tilde{z}_3\|_{L^p_t L^q_x}$$

$$\leq k T^{\frac{1}{2}} R^{-\frac{1}{2}} \|P_{N_{1}} Q_{\leq L} \|_{L^p_t L^q_x} + \|P_{N_{1}} Q_{\leq L} \tilde{z}_3\|_{L^p_t L^q_x}$$

$$\lesssim k T^{\frac{1}{2}} R^{-\frac{1}{2}} \|P_{N_{1}} \tilde{z}_3\|_{L^p_t L^q_x} + k T^{\frac{1}{2}} \|P_{N_{2}} \tilde{z}_3\|_{L^p_t L^q_x}$$

where $R = N_4^{\frac{3}{2}} M^\frac{1}{2}$. Moreover, in this case we have $N_3 \sim N_1 \gg N_2$, and we notice the following observation

$$N_4^{2} N_2^{2s-2} N_1^{2s-1} \lesssim N_1^{2s-1} N_2^{2s-1},$$

which allows us to sum over all the dyadic pieces of $N_j$. By a similar argument, we also have

$$I^{(3,2)}_{\infty,2,3,4} \lesssim T^{\frac{1}{2}} \sum_{N_{1,\ldots,N_{k+2}} M \leq N_4} N_4^{2s-1} M \|P_{N_{1}} w\|_{L^p_t L^q_x} \|P_{N_{2}} z_3\|_{L^p_t L^q_x}$$

$$\times \|P_{N_{3}} Q_{\leq MN_4} \tilde{w}\|_{L^2_t H^{\alpha-1}_x} \|P_{N_{4}} Q_{\geq MN_4} \tilde{z}_4\|_{L^p_t L^q_x} \prod_{j=5}^{k+2} P_{N_{j}} \tilde{z}\|_{L^2_t H^{\alpha-1}_x}$$

$$\lesssim k T^{\frac{1}{2}} \|\tilde{w}\|_{L^p_t H^{\alpha-1}_x} \|\tilde{z}_4\|_{X^{-\frac{1}{2},1}\|L^p_t H^{\alpha-1}_x} \|\tilde{w}\|_{L^2_t H^{\alpha-1}_x} \|\tilde{w}\|_{L^2_t H^{\alpha-1}_x}$$

Here, $\frac{1}{2} < s' < \frac{3}{2}$. Similarly, we can estimate $I^{(3,2)}_{\infty,2,3,n}$ for $5 \leq n \leq k+2$ by the same bound as above

$$\sum_{n=5}^{k+2} I^{(3,2)}_{\infty,2,3,n} \lesssim k T^{\frac{1}{2}} \|w\|_{L^p_t H^{\alpha-1}_x}^2.$$
which completes the estimate of the contribution of \( I^{(3,2)}_t \).

On the other hand, the contribution of \( J^{(3,2)}_t \) can be controlled by \( I^{(3,2)}_t \) since \( s > \frac{1}{2} \) and \( N_2^{2s-1} \leq N_1^{2s-1} \). Indeed, we see the following expression

\[
J^{(3,2)}_t \leq \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_0^t \int_0^T \partial_x P^2_N P_{N_1} w P_{N_2} z P_N \left( \partial_x P^2_{N_2} P_{N_1} w \prod_{j=4}^{k+2} P_{N_j} z \right) dx \right| dt' \\
+ \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \sum_{N_5 \leq M \leq N_2} N^{2(s-1)} \left| \int_0^t \int_0^T \partial_x P^2_N P_{N_1} w P_{N_2} z P_M \left( \partial_x P^2_{N_2} P_{N_1} w \prod_{j=4}^{k+2} P_{N_j} z \right) dx \right| dt' \\
=: J^{(3,2)}_{t,1} + J^{(3,2)}_{t,2}.
\]

Observe that \( J^{(3,2)}_t \) is bounded by the right-hand side of the first inequality in (2.3.39).

**Subcase 3.3:** \( N_2 \ll N_5 \) (\( k \geq 3 \)).

Recall that we can assume \( N_2 \geq 1 \) and note that we can also assume that \( N_5 \geq 1 \) since \( N_5 = 0 \) is included in Subcase 3.2: \( N_2 \geq N_5 \). Also, recall that we assumed symmetry

\[ N_1 \geq N_2 \quad \text{and} \quad N_3 \geq N_4 \geq N_5. \]

As well as in this case, \( N \sim N_1 \sim N_3 \). As in Subcase 3.2, we evaluate \( I^{(3,2)}_{t,1} \) and \( J^{(3,2)}_{t,1} \). Define \( L_k \) for \( k \geq 3 \) so that

\[ L_3 := N_2 \quad \text{and} \quad L_k := N_2 \lor k N_6 \quad \text{for} \quad k \geq 4. \]

Therefore, we compare the size of \( |n_5 + n_4| \) and \( |n_2| \lor k|n_6| \) to see a good resonance function as we have seen before. Moreover, we decompose

\[
I^{(3,2)}_t \leq \sum_{N \gg 1} N^{2(s-1)} \left| \int_0^t \int_0^T \partial_x P^2_N P_{N_1} w P_{N_2} z P_{N_5} \left( \partial_x P^2_{N_2} P_{N_1} w \prod_{j=4}^{k+2} P_{N_j} z \right) dx \right| dt' \\
+ \sum_{L_k \ll M \leq N_4} N^{2(s-1)} \left| \int_0^t \int_0^T \partial_x P^2_N P_{N_1} w P_{N_2} z P_M \left( \partial_x P^2_{N_2} P_{N_1} w \prod_{j=4}^{k+2} P_{N_j} z \right) dx \right| dt' \\
=: I^{(3,3)}_{t,1} + I^{(3,3)}_{t,2},
\]

where \( \sum \) denotes \( \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \) for simplicity. In what follows, we may assume that \( k \geq 4 \) since the case \( k = 3 \) can be treated by the same way as Subcase 3.2. For \( I^{(3,3)}_{t,1} \), Hölder’s, Bernstein’s and Minkowski’s inequalities, Proposition 2.2.1 and Corollary 2.2.6 give

\[
I^{(3,3)}_{t,1} \lesssim \left( \sum_{N \gg 1} N^{2s-1} N_1^2 + \sum_{N \gg 1} N^{2s-1}(k N_6)^2 \right) \\
\times \left( \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \prod_{j=4}^{k+2} P_{N_2} w \prod_{j=4}^{k+2} P_{N_5} z \prod_{j=4}^{k+2} P_{N_6} z \prod_{j=4}^{k+2} P_{N_6} z \right) \\
\lesssim_k W_{L^\infty_{H^s} H^{s-1}_t} \sum_{N_4 \geq N_5 \gg N_2} \left( \frac{N_2}{N_4} \right)^2 \left( \frac{N_2}{N_5} \right)^2 \left( \frac{N_4}{N_6} \right)^2 \left( \frac{N_2}{N_6} \right)^2 \left( \frac{N_2}{N_6} \right)^2 \left( \frac{N_2}{N_6} \right)^2 \\
\lesssim_k T C M \left\| w \right\|_{L^\infty_{H^s} H^{s-1}_t}.
\]
since $\frac{3}{4} \geq \frac{7}{12} + \frac{1}{5}$. On the other hand, $I^{(3,3)}_{t,3}$ can be estimated by the same way as $I^{(3,2)}_{t,2}$. We also notice that $I^{(3,2)}_{t,3}$ is controlled by $I^{(3,2)}_{t,2}$. This concludes the proof. \hfill \Box

### 2.4 Uniform local well-posedness

In this section, we show that the gILW equation (2.1.1) and the scaled gILW equation (2.1.2) are locally well-posed in $H^s(\mathbb{T})$ for $s \geq \frac{3}{4}$. Moreover, the local existence time are uniformly in $\delta$.

The main idea is that we still follow the energy method. First of all, we will show the uniqueness of the solution by considering the difference between the two solutions. Then, Proposition 2.3.11 plays a key role in this step. In the second part, we will show the existence of solutions to our desire equations. This is done by considering a smooth approximation and using our a priori estimates Proposition 2.3.9. In the final step, we will complete our proof by showing the continuous dependence on the initial data. This is done by using the frequency envelope, which is analogue to [90]. The argument in general is applicable to both Theorems 1.2.4 and 1.2.3. We present the proof of the well-posedness part of Theorem 1.2.4 in terms of the solution $u$ to gILW (2.1.1) in the following. For the proof of Theorem 1.2.3, we replace $u$ by $v$ and restrict the regularity assumption to $s \geq \frac{3}{4}$.

#### Uniqueness

Let $s \geq \frac{3}{4}$, $u_0 \in H^s(\mathbb{T})$, and $0 < T < 1$. Consider $(u^{(1)}, u^{(2)}) \in (L^\infty([0,T]; H^s(\mathbb{T})))^2$ be two solutions to the gILW equation (2.1.1) that are generated from $u_0$. Then, Lemma 2.3.7 implies that $(u^{(1)}, u^{(2)}) \in (M^{\delta}_T)^2$. Moreover, set the difference $w := u^{(1)} - u^{(2)}$, we have

$$
\|w\|_{M^{\delta}_T} \leq \|w\|_{L^\infty_T H^s} + C(\|u^{(1)}\|_{L^\infty_T H^s}, \|u^{(2)}\|_{L^\infty_T H^s}) \|w\|_{L^\infty_T H^{s-1}}.
$$

From Proposition 2.3.11 and $w(0) = 0$, we have

$$
\|w\|^2_{L^\infty_T H^{s-1}} \leq T^\frac{1}{2} C(\|u^{(1)}\|_{M^{\delta}_T}, \|u^{(2)}\|_{M^{\delta}_T}) \|w\|_{M^{\delta}_T} \|w\|_{L^\infty_T H^{s-1}},
$$

which implies that

$$
u^{(1)}(t) \equiv u^{(2)}(t) \quad \text{on} \quad [0, T_0]
$$

with $0 < T_0 \leq T$. We notice here the equivalency only depends on $\|u^{(1)}\|_{M^{\delta}_T}$ and $\|u^{(2)}\|_{M^{\delta}_T}$. Therefore, we conclude $u^{(1)}(T_0) = u^{(2)}(T_0)$ and we can iterate the same argument on $[T_0, T]$. Finally, this proves that $u^{(1)} \equiv u^{(2)}$ on $[0, T]$ after a finite number of iteration.

#### Local existence

First, we recall the well-known results (see [1]) that the Cauchy problem (2.1.1) is locally well-posed in $H^s(\mathbb{T})$ for $s > \frac{3}{4}$ with a minimal time of existence $T = T(\|u_0\|_{H^s})$. Indeed, for any $s \in \mathbb{R}$ and any smooth test function $\phi$, the $H^s$-scalar product $\langle G_\phi, \phi \rangle_{H^s}$ vanishes. The classical energy method (which consists of the parabolic regularisation, Kato–Ponce commutator estimate, and Bona–Smith estimate (23)) imply the local well-posedness in $C([0, T]; H^s(\mathbb{T}))$ for $s > \frac{3}{4}$.

- **Existence of smooth approximation**

Let $u \in C([0, T_0]; H^\infty(\mathbb{T}))$ be a smooth solution to (2.1.1) emanating from a smooth initial data $u_0 \in H^\infty(\mathbb{T})$ with $\|u_0\|_{H^s} < \infty$. Let $v_N \equiv 1$. Then, Lemma 2.3.7 gives the following

$$
\|u\|_{M^{\delta}_T} \leq \|u\|_{L^\infty_T H^s} + C(\|u\|_{L^\infty_T H^{s-1}}) \|u\|_{L^\infty_T H^s}.
$$

Then, together with Proposition 2.3.9, there exists a constant $C = C(\|u\|_{L^\infty_T H^s})$ such that

$$
\|u\|^2_{L^\infty_T H^s} \leq \|u\|^2_{H^s} + T^\frac{1}{2} C \|u\|^2_{L^\infty_T H^s}.
$$

(2.4.1)

for any $0 < T \leq \min(1, T_0)$ and any $\frac{3}{4} \leq s \leq 2$.

Let $T_* \geq T_0$ denote the maximal time of existence of $u(t) \in H^\infty(\mathbb{T})$. We recall from [1] that the well-posedness result ensures that $\lim_{T \nearrow T_*} \|u\|_{L^\infty_T H^s} = \infty$ whenever $T_* < \infty$. We note
\[ \|u(0)\|_{H^s} \leq \|u(0)\|_{H^r} < \infty \] and let \( T_1 = C(2\|u_0\|_{H^s})^{-4} \). From a continuity argument that for all \( 0 < T < \min(T_0, T_1) \), we have
\[ \|u\|^2_{L^p_t H^s_x} \leq 2\|u_0\|^2_{H^s} < \infty. \]

For any \( \frac{3}{4} \leq s \leq 2 \), (2.4.1) leads to the following
\[ \|u\|^2_{L^p_t H^s_x} \leq 2\|u_0\|^2_{H^s} < \infty. \]

Therefore, it ensures that \( T_1 < T_* \), i.e., \( u \) does exist on \([0, T_1]\) with \( u \in C([0, T_1]; H^\infty(T)) \).
Moreover, without loss of generality, we take \( T_1 = 1 \) and \( T_* \geq 1 \).

**Compactness**

We are now in the right place to perform compactness argument. Let
\[ u_0 \in H^s(\mathbb{T}) \quad \text{with} \quad \frac{3}{4} \leq s \leq 2. \]

Consider that the smooth sequence \( \{u_{0,n}\}_{n \geq 1} \) which converges to \( u_0 \) in \( H^s(\mathbb{T}) \). We deduce from previous step that the sequence of solutions \( \{u_n\}_{n \geq 1} \) generated by \( \{u_{0,n}\}_{n \geq 1} \) is included in \( C([0, T]; H^\infty(\mathbb{T})) \) and satisfies
\[ \|u_n\|^2_{L^p_t H^s_x} \leq 2\|u_0\|^2_{H^s} < \infty, \quad (2.4.2) \]
where \( 0 < T < 1 \). This proves that the sequence \( \{u_n\}_{n \geq 1} \) is bounded in \( L^\infty([0, T]; H^s(\mathbb{T})) \) and thus \( f(u_n) \) is bounded in \( L^\infty([0, T]; H^s(\mathbb{T})) \).

The boundedness of \( \{u_n\}_{n \geq 1} \) together with the Banach-Alaoglu Theorem imply that \( \{u_n\}_{n \geq 1} \) converges in the weak\(^*\) topology of \( L^\infty([0, T]; H^s(\mathbb{T})) \) to some limit, say \( u \).

Moreover, in the following, we show the Cauchy property of \( \{u_n\}_{n \geq 1} \) in \( L^\infty([0, T]; H^{s-1}(\mathbb{T})) \). Let \((u_{n_1}, u_{n_2}) \in (L^\infty([0, T]; H^s(\mathbb{T})))^2 \) to be the solutions generated by \( \{u_{0,n_1}, u_{0,n_2}\} \), where \( 1 \leq n_1 \leq n_2 \). For simplicity, let us set
\[ w = u_{n_1} - u_{n_2} \quad \text{with} \quad w(0) = u_{0,n_1} - u_{0,n_2}. \]

We first note from Proposition 2.3.11 that
\[ \|w\|^2_{L^p_t H^{s-1}_x} \leq \|w(0)\|^2_{H^{s-1}} + T^{\frac{s}{4}}C(\|u_{n_1}\|_{M^{s-1}_t} + \|u_{n_2}\|_{M^{s-1}_t})\|w\|_{M^{s-1}_t} \|w\|_{L^p_t H^{s-1}_x}. \]

Lemma 2.3.7 second part (2.3.14) provide the following estimate
\[ \|w\|_{M^{s-1}_t} \leq \|w\|_{L^p_t H^{s-1}_x} + T^{\frac{s}{4}}C(\|u_{n_1}\|_{L^p_t H^s_x} + \|u_{n_2}\|_{L^p_t H^s_x})\|w\|_{L^p_t H^{s-1}_x}. \]

Let \( j = 1, 2 \). From (2.4.2) and Lemma 2.3.7 with \( w_N = 1 \) give the following
\[ \|u_n\|_{Z^j} \leq \|u_n\|_{L^p_t H^s_x} + C(\|u_{0,n}\|_{L^p_t H^s_x})\|u_n\|_{L^p_t H^s_x} < C(\|u_{0,n}\|_{H^r}), \]
where the constant only depends on \( \|u_{0,n}\|_{H^r} \). Thus, together with (2.4.2), we can conclude that there exist constants \( C(\|u_{0,n}\|_{H^r}, \|u_{0,n}\|_{H^r}) \) and \( C'(\|u_{0,n}\|_{H^r}, \|u_{0,n}\|_{H^r}) \) depend only on \( \|u_{0,n}\|_{H^r} \) such that the following holds
\[ \|w\|^2_{L^p_t H^{s-1}_x} \leq \|w(0)\|^2_{H^{s-1}} + T^{\frac{s}{4}}C(\|w\|^2_{L^p_t H^{s-1}_x} + T^{\frac{s}{4}}C'\|w\|^2_{L^p_t H^{s-1}_x} \leq \|w(0)\|^2_{H^{s-1}}. \]

Since \( \{u_{0,n}\}_{n \geq 1} \) is a convergent sequence in \( H^s(\mathbb{T}) \) by assumption, therefore it is convergent in \( H^{s-1}(\mathbb{T}) \). For any \( \varepsilon > 0 \), there exist \( n_\varepsilon \), large enough such that for all \( n_1 \geq n_\varepsilon \), we have
\[ \|w\|^2_{L^p_t H^{s-1}_x} < \varepsilon. \]

\(^5\)For example, we may take \( u_{0,n} = P_{\leq n} * u_0 \), it is clear that the sequence \( \{u_{0,n}\}_{n \geq 1} \) belongs to \( H^\infty(\mathbb{T}) \) and converges to \( u_0 \) in \( H^s(\mathbb{T}) \).
Namely, we showed that
\[ \{u_n\}_{n \geq 1} \] is a Cauchy sequence in \( C([0, T]; H^{s-1}(T)) \).
Thus, the weak* convergence of \( \{u_n\}_{n \geq 1} \) in \( L^\infty([0, T]; H^s(T)) \) and the Cauchy property of \( \{u_n\}_{n \geq 1} \) in \( C([0, T]; H^{s-1}(T)) \), we can deduce that \( \{u_n\}_{n \geq 1} \) converges to some function \( u \in C([0, T]; H^s(T)) \cap L^\infty([0, T]; H^s(T)) \)
for any \( s' < s \). Thus, in view of (1.2.5), we can pass to the limit on the nonlinear term \( f(u_n) \) that
\[ \partial_x f(u_n) \xrightarrow{D'} \partial_x f(u), \]
i.e., convergence in the distributional sense. Moreover, the linear part converges in the distribution sense, which we can easily pass to the limit.

• Strong continuity in \( H^s(T) \)

Let \( u \) be a solution of (2.1.1) generated by the initial data \( u_0 \in H^s(T) \). Due to the time translation invariance, reversibility in time (invariance by the change of variables \( (t, x) \mapsto (-t, -x) \) of (2.1.1)) and the uniqueness of the solutions, it suffices to check that the continuity of \( u(t) \in H^s(T) \) at \( t = 0 \).

Since the limiting object we obtained is
\[ u \in C([0, T]; H^{s'}(T)) \cap L^\infty((0, T); H^s(T)) \]
for some \( s' < s \). By the following theorem.

**Lemma 2.4.1** ([155]). Let \( V \) and \( Y \) be Banach spaces, \( V \) reflexive, \( V \) a dense subset of \( Y \) and the inclusion map of \( V \) into \( Y \) continuous. Then,
\[ C_w([0, T]; Y) \cap L^\infty([0, T]; V) = C_w([0, T]; V). \]
\( C_w([0, T]; V) \) denotes the the subspace of \( L^\infty([0, T]; V) \) consisting of those functions which are a.e. equal to weakly continuous functions with values in \( V \).

It yields that \( u(t) \) is is weakly continuous (w.r.t time) in \( H^s(T) \) on \([0, T]\). Namely,
\[ u \in C_w([0, T]; H^s(T)), \]
and the following holds
\[ \| u_0 \|_{H^s} \leq \liminf_{t \to 0} \| u(t) \|_{H^s}. \] (2.4.3)
On the other hand, we see from (3.2.73) with \( \omega_N \equiv 1 \) that
\[ \limsup_{t \to 0} \| u(t) \|_{H^s} \leq \| u_0 \|_{H^s}. \] (2.4.4)
From (2.4.3) and (2.4.4) we conclude that
\[ \lim_{t \to 0} \| u(t) \|_{H^s} = \| u_0 \|_{H^s}. \] (2.4.5)
Finally, weak continuity of \( u(t) \) in \( H^s(T) \) on \([0, T]\) and continuous of norms (2.4.5) imply:
\[ [0, T] \ni t \mapsto u(t) \in H^s(T) \]
is continuous. This finishes the proof that \( u \in C([0, T]; H^s(T)) \).

**Continuity with respect to initial data**
In this subsection, we show the continuity of the flow map. The proof of continuous dependence on the data in the following is different from the classical Bona–Smith approximation argument (see [21]). This approach is known as the frequency envelope argument, which relies on the following lemma.

**Lemma 2.4.2** ([20]). Suppose that the sequence of spacial functions \( g_n \to g \) in \( H^s(\mathbb{T}) \). Then there exists a sequence \( \{\omega_M\} \) for \( M = 2^{6n} \) of positive dyadic numbers which satisfies

\[
2^s\omega_M \leq \omega_{2M} \leq 2^{s+1}\omega_M \quad \text{and} \quad \frac{\omega_M}{M^s} \to \infty
\]

such that

\[
\sup_n \sum_M \omega_M^2 \|P_M g_n\|^2_{L^2} < \infty.
\]

**Remark 2.4.3.** In the application of Lemma 2.4.2, the choice of \( \omega_N \) depends only on the initial data and the approximation sequence of the initial condition.

We make use of the frequency envelope \( \omega_N \). Let \( \{u_n\}_{n \geq 1} \) be a sequence of solutions in \( C([0,T]; H^s(\mathbb{T})) \) with

\[
u_{0,n} \to u_0 \quad \text{in} \quad H^s(\mathbb{T}).
\]

We want to prove that the emanating solution \( u_n \) tends to \( u \) in \( C([0,T]; H^s(\mathbb{T})) \). Let \( P_{\leq K} u = \sum_{N \leq K} P_N u \). Then, by the triangle inequality, we have the following

\[
\|u_n - u\|_{L^P H^s} \leq \|u_n - P_{\leq K} u_n\|_{L^P H^s} + \|P_{\leq K} (u_n - u)\|_{L^P H^s} + \|P_{\leq K} u - u\|_{L^P H^s}. \tag{2.4.6}
\]

First, for any fixed \( K \), a direct consequence of Proposition 2.3.11 gives us a Lipschitz bound of the difference, which tells us that there exists \( n_0 \) such that for \( n \geq n_0 \), the following holds:

\[
\|P_{\leq K} (u_n - u)\|_{L^P H^s} \leq (2K)\|P_{\leq K} (u_n - u)\|_{L^P H^{s-1}} \leq K\|u_{n_0} - u_0\|_{H^{s-1}} < \varepsilon. \tag{2.4.7}
\]

Secondly, for \( K \) large enough, we have

\[
\|P_{\leq K} u - u\|_{L^P H^s} < \varepsilon. \tag{2.4.8}
\]

Finally, by Lemma 2.4.2, there exists a dyadic sequence \( \{\omega_N\} \) of positive numbers satisfies \( \omega_N \leq \omega_{2N} \leq \kappa \omega_N \) for \( N \geq 1 \) (from Lemma 2.1.7) we may assume that \( 1 \leq \kappa \leq 2 \) and this is equivalent to Lemma 2.4.2 such that

\[
\|u_0\|_{H^s} < \infty, \quad \sup_{n \geq 1} \|u_{0,n}\|_{H^s} < \infty, \quad \text{and} \quad \omega_N \not\to \infty.
\]

For \( K > 0 \) large enough, by applying Proposition 2.3.9, we have the following

\[
\|P_{\leq K} u_n - u_n\|^2_{L^P H^s} = \sum_{N > K} \|P_N u_n\|^2_{L^P H^s} \leq \sum_{N > K} \omega_N^2 \|P_N u_n\|^2_{L^P H^s} \leq \omega_K^{-2} \sum_{n \geq 1} \|u_n\|^2_{L^P H^s} \leq \omega_K^{-2} \sum_{n \geq 1} \|u_{0,n}\|^2_{H^s} < \varepsilon. \tag{2.4.9}
\]

Therefore, we plug \( 2.4.7, 2.4.8 \), and \( 2.4.9 \) into \( 2.4.6 \), we obtain the continuity of the flow map.

### 2.5 On limiting behaviour of \( gILW \)

In this section, we will explore the limiting behaviour of the \( gILW \) equation. For this section, we will keep track of the depth parameter \( \delta \). In particular, we write the \( \delta \) parameter in the subscripts such that \( u_0 \) and \( v_0 \). The main idea is to show the family of solutions \( \{v_0\}_{\delta > 0} \) and \( \{u_0\}_{\delta \geq 1} \) are Cauchy sequences in \( C_T H^s \). Then, we know that in the complete metric space, they admit some

---

\( ^6 \)take \( \omega_M = \omega_N N^s \) in view of Lemma 2.4.2
Moreover, the following holds:

\[ v_\delta - v_{KdV} \quad \text{or} \quad u_\delta - u_{BO}. \]

Then, show the limit is zero. In what follows, we show \( \{v_\delta\}_{\delta > 0} \) and \( \{u_\delta\}_{\delta \geq 1} \) are Cauchy sequence.

### 2.5.1 Deep-water limit on \( \mathbb{T} \)

In this subsection, we finish the proof of Theorem 1.2.3. For any \( 0 \leq \delta, \gamma \leq 1 \), let us use the following subscripts notation

\[ u_\delta \in M_{\gamma}^{s, \delta} \]

denotes a solution to \( gILW_\delta \).

Then, \( u_\gamma \) and \( u_\delta \) are two solutions to \( gILW_\gamma \) and \( gILW_\delta \) (of (2.1.1)) with same initial data. Then, set the difference \( w = u_\gamma - u_\delta \), which solves the following equation:

\[
\begin{aligned}
\partial_t w - \mathcal{H}(\partial_x^2 w) + K_\delta(\partial_x w) &= (K_\gamma - K_\delta)\partial_x u_\gamma - \partial_x(f(u_\gamma) - f(u_\delta)) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad…\end{aligned}\]

Here, \( K_\delta = (\mathcal{H} - \mathcal{G}_\delta)\partial_x \) is defined as the Fourier symbol

\[ q_\delta(n) = \frac{1}{\delta} - n \coth(\delta n) + |n|. \]

It is convenient to use the notation

\[ T_{\delta, \gamma}(u) = K_\gamma(u) - K_\delta(u). \]

The goal here is to show that solutions \( \{u_\delta\}_{\delta \geq 1} \) is Cauchy in \( C([0,T];H^s(\mathbb{T})) \) for \( s \geq \frac{3}{4} \).

**Proposition 2.5.1.** Let \( s \geq \frac{3}{4} \), and any \( \delta \geq 2 \). Then, the one-parameter family \( \{u_\delta\}_{\delta \geq 1} \) is Cauchy in \( C([0,T];H^s(\mathbb{T})) \) as \( \delta \to \infty \), where \( T \) is the local existence time.

Before we proceed with the proof of Proposition 2.5.1, we will need the following lemma, which studies the properties of \( M_{\gamma}^{s, \delta} \) for different \( \delta \).

**Lemma 2.5.2.** Let \( s > \frac{1}{2} \), \( 0 < T < 1 \) be the local existence time, and \( u_\gamma \in M_{\gamma}^{s, \delta} \) is a solution to \( gILW_\gamma \). Then, for any \( 2 \leq \delta, \gamma \leq \infty \), we have

\[ \|u_\gamma\|_{M_{\gamma}^{s, \delta}} \leq C\|u_\gamma\|_{M_{\gamma}^{s, \gamma}}, \]

Moreover, the following holds:

\[ \|u_\gamma\|_{M_{\gamma}^{s, \delta}} \leq C\|u_\gamma\|_{L_T^2 H_x^s} + C(\|u_\gamma\|_{L_T^\infty H_x^s})\|u_\gamma\|_{L_T^2 H_x^s}. \]

**Proof.** Let us recall the definition of \( M_{\gamma}^{s, \gamma} = L_T^2 H_x^s \cap X_T^{s-1, \gamma} \). Moreover, we recall the symbol for the deep-water formulation \( p_\delta^{(d)}(n) \) from (2.1.3), and notice that

\[ \tau - p_\delta^{(d)}(n) = \tau - p_\delta^{(d)}(n) - (p_\delta^{(d)} - p_\gamma^{(d)}(n)) \]

\[ = \tau - p_\gamma^{(d)}(n) - in(q_\delta(n) - q_\gamma(n)), \]

where \( q_\delta(n) \) is defined in (2.5.2). Therefore, we obtain the following

\[ \langle \tau - p_\delta^{(d)}(n) \rangle \lesssim \langle \tau - p_\gamma^{(d)}(n) \rangle + (\delta^{-1} + \gamma^{-1})(n), \]

(2.5.3)
where we used the fact that $0 \leq g_\delta(n) \leq \frac{2}{T}$. Now by using (2.5.3) we have
\[
\|u_\gamma\|_{X_T^{s-1,1.\gamma}} \lesssim \|u_\gamma\|_{X_T^{s-1,1.\gamma}} + (\delta^{-1} + \gamma^{-1})\|u_\gamma\|_{X_T^{s,0}}
\]
\[
= \|u_\gamma\|_{X_T^{s-1,1.\gamma}} + (\delta^{-1} + \gamma^{-1})\|u_\gamma\|_{L_T^2 H_x^s}
\]
\[
\lesssim \|u_\gamma\|_{X_T^{s-1,1.\gamma}} + C\|u_\gamma\|_{L_T^2 H_x^s},
\]
for any $\gamma, \delta \geq 2$. In particular, claim
\[
\|u_\gamma\|_{M_T^{s,\delta}} \leq C\|u_\gamma\|_{M_T^{s,\gamma}}
\]
holds for any $\gamma, \delta \geq 2$. Moreover, we follow the same sort of approach as in Lemma 2.3.7. In particular, $u_\gamma$ satisfies the Duhamel formulation and we have the following
\[
\|u_\gamma\|_{X_T^{s-1,1.\gamma}} \lesssim \|u_\gamma\|_{L_T^2 H_x^s} + G(\|u_\gamma\|_{L_T^2 H_x^s})\|u_\gamma\|_{L_T^2 H_x^s}
\]
We can now conclude that for any $\gamma, \delta \geq 2$, we have
\[
\|u_\gamma\|_{M_T^{s,\delta}} \leq C\|u_\gamma\|_{M_T^{s,\gamma}} \lesssim_T \|u_\gamma\|_{L_T^2 H_x^s} + G(\|u_\gamma\|_{L_T^2 H_x^s})\|u_\gamma\|_{L_T^2 H_x^s}
\]
where the implicit constant only depends on the local existence time.

\[\square\]

Lemma 2.5.3. Let $s > \frac{1}{2}$, $0 < T < 1$ be the local existence time, and $u_\delta, u_\gamma \in C([0, T]; H^s(T))$ such that $u_\gamma \in M_T^{s,\gamma}$ is a solution to gILW_\gamma with initial data $u_0, \gamma$; and $u_\delta \in M_T^{s,\delta}$ is a solution to gILW_\delta with initial data $u_0, \delta$. Then, for any $2 \leq \gamma, \delta < \infty$, we have
\[
\|w\|_{M_T^{s-1,\delta}} \lesssim \|w\|_{L_T^2 H_x^{s-1}} + C(\|u_\gamma\|_{L_T^2 H_x^s} + \|u_\delta\|_{L_T^2 H_x^s})\|w\|_{L_T^2 H_x^{s-1}} + C(\delta^{-1} + \gamma^{-1})\|u_\gamma\|_{L_T^2 H_x^{s-1}}
\]
where $w = u_\gamma - u_\delta$.

Proof. We first take the difference of gILW_\gamma and gILW_\delta. Then, $w = u_\gamma - u_\delta$ satisfies the following equation
\[
\partial_t w - G_\delta \partial_x^2 w + (G_\delta - G_\gamma)\partial_x \cdot \partial_x u_\delta = \partial_x f(u_\gamma) - \partial_x f(u_\delta).
\]
Recall that
\[
M_T^{s-1,\gamma} = L_T^2 H_x^{s-1} \cap X_T^{s-2,1.\gamma}.
\]
Then, by following the same lines as in the proof of (2.3.14), it is enough to estimate $w$ in $X_T^{s-2,1.\delta}$ norm. Therefore, by using the Duhamel formulation, for any $\delta, \gamma \geq 2$
\[
\|w\|_{X_T^{s-2,1.\delta}} \lesssim \|u_0, \gamma - u_0, \delta\|_{H_x^{s-1}} + \|f(u_\gamma) - f(u_\delta)\|_{L_T^2 H_x^{s-1}} + \|(G_\delta - G_\gamma)\partial_x \cdot \partial_x u_\delta\|_{X_T^{s-2,0.\delta}}
\]
\[
\lesssim \|w\|_{L_T^p H_x^{s-1}} + G(\|u_\delta\|_{L_T^p H_x^s} + \|u_\gamma\|_{L_T^p H_x^s})\|w\|_{L_T^p H_x^{s-1}} + \|(G_\delta - G_\gamma)\partial_x u_\delta\|_{L_T^p H_x^{s-1}}
\]
\[
\lesssim \|w\|_{L_T^p H_x^{s-1}} + G(\|u_\delta\|_{L_T^p H_x^s} + \|u_\gamma\|_{L_T^p H_x^s})\|w\|_{L_T^p H_x^{s-1}} + (\delta^{-1} + \gamma^{-1})\|u_\gamma\|_{L_T^p H_x^{s-1}},
\]
where we used that $g_\delta(n) \leq \frac{2}{T}$ and the implicit constant only depends on $T$.

To approach Lemma 2.5.1 let us show that $\{u_\delta\}_{\delta \geq 2}$ is Cauchy sequence in $C([0, T]; H^{s+1}(T))$ for $s \geq \frac{1}{2}$.

Lemma 2.5.4. Let $s \geq \frac{3}{4}$, $2 \leq \gamma, \delta \leq \infty$, and $0 < T < 1$ be the local existence time. Then, there exists $C = C(T, \|u_0\|_{H^s(T)}) > 0$ independent of $\delta, \gamma$ such that
\[
\|w(t)\|_{C([0, T]; H^{s-1}(T))} \leq C \left(\frac{1}{\delta} + \frac{1}{\gamma}\right),
\]
(2.5.4)
where \( w = u_\gamma - u_\delta \).

**Proof.** In the following proof, we follow the same sprites of proof of Proposition \[2.3.11\]. We first use the equation \[2.5.1\] with \( w(0) = 0 \) and apply the same strategy as we did from \[2.3.34\] to \[2.3.36\] to obtain:

\[
\|w(t)\|_{L^2_{T,x}}^2 \lesssim \sum_{N \geq 1} N^{2s-2} \|T_{\delta,\gamma} P_N \partial_x u_\gamma\|_{L^2_{T,x}} \|P_N w\|_{L^2_{T,x}} + 2 \sum_{k \geq 1} \left| \frac{f(k)}{(k-1)!} \right| \max_{i \in \{0, \ldots, k-1\}} I_{k,i}^{\delta,\gamma},
\]

where \( I_{k,i}^{\delta,\gamma} \) is defined by

\[
I_{k,i}^{\delta,\gamma} := \sum_{N \geq 1} N^{2(s-1)} \left| \int_0^t \int_T u_\delta^i u_\gamma^{k-1-i} \partial_x w p_N^2 \partial_x w dx dt \right|.
\]

Recall \( T_{\delta,\gamma}(u) = K_\gamma(u) - K_\delta(u) \) and \( \tilde{K}_\delta(u) = q(u) \leq \frac{2}{3} \) (Lemma \[2.1.2\]). Then, we apply Cauchy’s inequality to obtain

\[
\sum_{N \geq 1} N^{2s-2} \|T_{\delta,\gamma} P_N \partial_x u_\gamma\|_{L^2_{T,x}} \|P_N w\|_{L^2_{T,x}} \leq C \left( \frac{1}{\delta} + \frac{1}{\gamma} \right)^2 \|u_\gamma\|_{L^\infty_T H^s_x}^2 + c(T) \|w\|_{L^2_T H^{s-1}_x},
\]

for some \( c(T) \ll 1 \). Therefore, Proposition \[2.3.11\] leads to

\[
\|w(t)\|_{L^2_T H^{s-1}_x}^2 \leq C \left( \frac{1}{\delta} + \frac{1}{\gamma} \right)^2 \|u_\gamma\|_{L^\infty_T H^s_x}^2 + c(T) \|w(t)\|_{L^2_T H^{s-1}_x}^2 + T^k c \left( \|u_\gamma\|_{M^{s,k}_T} + \|u_\delta\|_{M^{s,k}_T} \right) \|w\|_{M^{s-1,k}_T} \|w\|_{L^2_T H^{s-1}_x}.
\]

From Lemma \[2.3.7\] and Lemma \[2.5.2\] for any \( 2 \leq \delta, \gamma \leq \infty \), we have

\[
\|u_\gamma\|_{M^{s,k}_T} + \|u_\delta\|_{M^{s,k}_T} \lesssim T \|u_\gamma\|_{L^\infty_T H^s_x} + C(\|u_\gamma\|_{L^\infty_T H^s_x}) \|u_\gamma\|_{L^\infty_T H^s_x} + \|u_\delta\|_{L^\infty_T H^s_x} + C(\|u_\delta\|_{L^\infty_T H^s_x}) \|u_\delta\|_{L^\infty_T H^s_x}.
\]

Next, we need to estimate \( \|w\|_{M^{s-1,k}_T} \) in \(2.5.6\), which follows from \(2.5.3\) and by noticing \( w(0) = 0 \)

\[
\|w\|_{M^{s-1,k}_T} \lesssim C(\|u_\gamma\|_{L^\infty_T H^s_x} + \|u_\gamma\|_{L^\infty_T H^s_x}) \|w\|_{L^\infty_T H^{s-1}_x} + C(\delta^{-1} + \gamma^{-1}) \|u_\gamma\|_{L^\infty_T H^{s-1}_x}.
\]

Moreover, we recall that local well-posedness implies that the family \( \{u_\delta\}_{\delta \geq 1} \) is bounded in \( C([0,T]; H^s(T)) \) for any \( s \geq \frac{1}{2} \). In particular, there exists a \( \delta \)-independent constant \( M \) such that

\[
\|u_\delta\|_{L^\infty_T H^s_x} \lesssim \|u_0\|_{H^s} < M.
\]

Therefore, from \(2.5.6\) we have

\[
\|w(t)\|_{L^2_T H^{s-1}_x}^2 \leq C \left( \frac{1}{\delta} + \frac{1}{\gamma} \right)^2 M + c(T) \|w(t)\|_{L^2_T H^{s-1}_x}^2 + C T^{\frac{k}{2}} \|w(t)\|_{L^2_T H^{s-1}_x}^2 + T^2 \left( \frac{1}{\delta} + \frac{1}{\gamma} \right)^2 + \frac{1}{2} \|w(t)\|_{L^2_T H^{s-1}_x}^2 \lesssim T \left( \frac{1}{\delta} + \frac{1}{\gamma} \right)^2.
\]

\[\Box\]
Proof of Proposition 2.5.1. By the triangle inequality, we first write the following:

\[ \|u_\delta - u_\gamma\|_{CT; H^s} < \|u_\delta - P_{\leq K} u_\delta\|_{CT; H^s} + \|P_{\leq K} u_\delta - P_{\leq K} u_\gamma\|_{CT; H^s} + \|P_{\leq K} u_\gamma - u_\gamma\|_{CT; H^s} \]

Let \( \eta > 0 \), then there exists \( K_0 \) such that for \( K \geq K_0 \) we have the following:

\[ \|u_\delta - P_{\leq K} u_\delta\|_{CT; H^s} < \frac{\eta}{3}; \quad \|P_{\leq K} u_\gamma - u_\gamma\|_{CT; H^s} < \frac{\eta}{3}. \]

Notice that Lemma 2.5.4 implies that for all \( \delta, \gamma \) such that \( 2 \leq \delta \leq \gamma \leq \infty \), there exists a constant \( C = C(T, \|u_0\|_{H^r}) \) independent of \( \delta \) and \( \gamma \) such that for any \( K \),

\[ \|P_{\leq K}(u_\delta) - P_{\leq K}(u_\gamma)\|_{CT; H^s} \leq 2K\|u_\delta - u_\gamma\|_{CT; H^s} \leq \frac{CK}{\delta} \]

provided that same initial data such that \( w(0) = 0 \). Now, we choose \( K = \delta^{\frac{1}{2}} \), so that as \( \delta \to \infty \)

\[ \|u_\delta - u_\gamma\|_{CT; H^s} < \eta. \]

As \( \eta \) is arbitrary, hence we finish the proof.

To complete the proof of Theorem 1.2.3, we let \( s \geq \frac{3}{2} \). Then, from Lemma 2.5.1 there exists \( u \in C([0, T]; H^s(\mathbb{T})) \) such that \( u_\delta \) converges to \( u \) in \( C([0, T]; H^r(\mathbb{T})) \), as \( \delta \to \infty \). We now show \( u \) is indeed a solution to gBO. We notice for any \( \delta \geq 1 \), the following bound is true

\[ \|K_\delta(\partial_x u_\delta)\|_{H^{s-1}} \leq \frac{2}{\delta} \|u_\delta\|_{H^s} \leq \frac{c}{\delta}, \quad (2.5.7) \]

for some absolute constant \( c \). Then, it is readily seen that \( u \) is actually the solution of the gBO corresponding to the initial data \( u_0 \). Indeed, we have seen that we can write gILW as perturbed gBO:

\[ \partial_t u_\delta + \mathcal{H}(\partial_x^2 u_\delta) + \partial_x(f(u_\delta)) + K_\delta(\partial_x u_\delta) = 0. \]

We have the almost everywhere convergence of the linear part:

\[ \partial_t u_\delta + \mathcal{H}(\partial_x^2 u_\delta) \xrightarrow{P} \partial_t u + \mathcal{H}(\partial_x^2 u), \]

i.e. convergent in the distributional sense, as \( \delta \to \infty \). Also, by the analyticity of \( f(\cdot) \), we have \( f(u_\delta) \) converges to \( f(u) \) in \( C([0, T]; H^s(\mathbb{T})) \) as \( \delta \to \infty \). Moreover, by \( (2.5.7) \), \( K_\delta(\partial_x u_\delta) \) vanishes as \( \delta \to \infty \). Therefore, it is enough to conclude our claim.

2.5.2 Shallow-water limit on \( \mathbb{T} \)

In this subsection, we study the limiting behaviour of (scaled) gILW (2.1.2) as \( \delta \to 0 \). For any \( 0 < \delta, \gamma \ll 1 \), we take \( v_\delta \) and \( v_\gamma \) are two solutions to scaled gILW \( \delta \) and scaled gILW \( \gamma \) (of (2.1.2)) with same initial data. Then, set difference \( w = v_\delta - v_\gamma \), which solves the following equation:

\[ \begin{cases} 
\partial_t w + \partial_x w + H_\delta(\partial_x w) = (H_\gamma - H_\delta)\partial_x v_\gamma - \partial_x (f(v_\delta) - f(v_\gamma)) \\
w(0) = 0.
\end{cases} \]

(2.5.8)

Here, \( H_\delta \) is defined by the Fourier multiplier:

\[ \hat{H}_\delta(n) := -n^2 \frac{h(n, \delta)}{\delta} \]

(2.5.9)

and \( h(n, \delta) \) is defined in Lemma 2.1.1 (similar we definition \( H_\gamma \)). Moreover, for convenience, we denote

\[ L_{\delta, \gamma} = H_\gamma - H_\delta. \]

(2.5.10)
We notice that in the shallow-water formulation, the symbol \( \sigma(2.5.9) \) is not uniformly (of \( n \)) decay in \( \delta \). Therefore, the argument for the deep-water limit is not fully available. In particular, this causes difficulty in showing the Cauchy properties of \( \{v_\delta\}_{\delta > 0} \). To overcome this difficulty, we start with the following frequency truncated equation, where we apply a frequency truncation on the nonlinearity as well as the initial data:

\[
\begin{aligned}
  &\partial_t P_K v - \frac{3}{2} G_0 \partial_x^2 P_K v = P_K \partial_x(f(P_K v)), \\
  &P_K v|_{t=0} = P_K v_0.
\end{aligned}
\] (2.5.11)

Then, the corresponding solution \( P_K v_\delta \) is supported on frequency \( |n| \leq K \).

**Proposition 2.5.5.** Let \( s \geq \frac{3}{4}, K \in \mathbb{N}, \) and local existence time \( 0 < T < 1 \). Assume that

\[
P_K v_\delta \in C([0,T]; C^\infty(\mathbb{T}))
\]

is a solution to the scaled gILW [2.5.11].

Then, the one-parameter family \( \{P_K v_\delta\}_{\delta > 0} \) is Cauchy in \( C([0,T]; H^{s-1}(\mathbb{T})) \) as \( \delta \to 0 \).

To prove Proposition 2.5.5, we use the following technique lemmas.

**Lemma 2.5.6.** Let \( s > \frac{1}{2}, 0 \leq \gamma, \delta < 1, K \in \mathbb{N}, \) and local existence time \( 0 < T < 1 \). Assume that

\[
P_K v_\delta \in C([0,T]; C^\infty(\mathbb{T}))
\]

is a solution to the scaled gILW [2.5.11].

Then, there exists \( \delta_0 \) such that for any \( 0 < \delta, \gamma < \delta_0 \) we have

\[
\|P_K v_\gamma\|_{N_T^{s,\delta}} \lesssim \|P_K v_\gamma\|_{N_T^{s,\gamma}},
\] (2.5.12)

where the implicit constant only depends on \( T \), for all \( 0 < \delta, \gamma < \delta_0 \). Moreover, the following estimate holds:

\[
\|P_K v_\gamma\|_{N_T^{s,\delta}} \lesssim \|P_K v_\gamma\|_{L_T^\infty X_T^s} + G(\|P_K v_\gamma\|_{L_T^{\infty, s}})\|P_K v_\gamma\|_{L_T^\infty X_T^s}.
\] (2.5.13)

**Proof.** Let us recall the definition of \( N_T^{s,\gamma} \)-space and symbol for scaled gILW [2.1.2] (Lemma 2.1.1):

\[
N_T^{s,\gamma} = L_T^\infty \cap Y_T^{s-1,1,\gamma} \quad \text{and} \quad p_\delta^{(s)}(n) = n^3 + n^3 \frac{h(n, \delta)}{\gamma}
\] (2.1.2)

Moreover, we notice that

\[
\tau - p_\delta^{(s)}(n) = \tau - p_\gamma^{(s)}(n) - \left( p_\delta^{(s)} - p_\gamma^{(s)}(n) \right)
\]

\[
= \tau - p_\gamma^{(s)}(n) - \left( n^3 \frac{h(n, \delta)}{\gamma} - n^3 \frac{h(n, \gamma)}{\gamma} \right).
\] (2.1.5)

Here, we recall the definition of \( h(n, \delta) \) from Lemma 2.1.1

\[
h(n, \delta) = \sum_{k=1}^{\infty} \frac{2\delta^3 n^2}{k^2 \pi^2 (k^2 \pi^2 + \delta^2 n^2)}.
\] (2.1.6)

In particular, under the assumption that \( n \leq K \), we have we have

\[
\frac{h(n, \delta)}{\delta} = O(\delta^2) \quad \text{as} \quad \delta \to 0
\] (2.1.7)

Then, from \( 2.5.15 \), \( 2.5.16 \), and \( 2.5.17 \), we have

\[
\langle \tau - p_\delta^{(s)}(n) \rangle \lesssim \langle \tau - p_\gamma^{(s)}(n) \rangle + O(\delta^2 K^3) \quad \text{as} \quad \delta \to 0.
\] (2.1.8)
Now by using (2.5.18) we have the following control in $Y^{s, \delta, \gamma}$-norm: for any $0 < \gamma, \delta < 1$,

$$
\|P_K v_\gamma\|_{Y_T^{s-1, \delta, \gamma}} \lesssim \|P_K v_\gamma\|_{Y_T^{s-1, \gamma}} + O(\delta^2)K^2 \|P_K v_\gamma\|_{Y_T^{s, \gamma}}
$$

as $\delta \to 0$. Moreover, take $\delta_0$ is such that for any $0 < \gamma, \delta < \delta_0$, we have

$$
\sup_{|n| \leq K} K^2 \frac{h(n, \delta)}{\delta} = O(\delta^2)K^2 < 1 \quad \text{as} \quad \delta \to 0.
$$

Then, (2.5.19), (2.5.14), and (2.5.20) give

$$
\|P_K v_\gamma\|_{N_T^{s, \delta, \gamma}} = \|P_K v_\gamma\|_{Y_T^{s-1, \delta, \gamma}} + \|P_K v_\gamma\|_{L_T^\infty H_{T, \delta}^s}
$$

for any $0 < \gamma, \delta < \delta_0$. Moreover, we follow the same sort of approach as in Lemma 2.5.7. In particular, $P_K v_\gamma$ satisfies the Duhamel formulation of the scaled gILW (2.1.2) and we have the following

$$
\|P_K v_\gamma\|_{Y_T^{s-1, \gamma}} \lesssim \|P_K v_\gamma\|_{L_T^\infty H_{T, \delta}^s} + G(\|P_K v_\gamma\|_{L_T^\infty})\|P_K v_\gamma\|_{L_T^\infty H_{T, \delta}^s}
$$

Now, from (2.5.21) and (2.5.22), we can conclude that for any $0 < \gamma, \delta < \delta_0$, the following is true

$$
\|P_K v_\gamma\|_{N_T^{s, \gamma}} \lesssim \|P_K v_\gamma\|_{N_T^{s, \delta}}
$$

We also need the following lemma.

**Lemma 2.5.7.** Let $s > \frac{1}{2}$, $0 \leq \gamma, \delta < 1$, $K \in \mathbb{N}$, and $0 < T < 1$ be the local existence time. Assume $P_K v_\gamma$ and $P_K v_\delta$ solve the scaled gILW (2.5.11). Then, for any $0 < \delta, \gamma < 1$ we have

$$
\|P_K w\|_{N_T^{s-1, \delta}} \lesssim \|P_K w\|_{L_T^\infty H_{T, \delta}^{s-1}} + O(\delta^2)K^2 \|P_K v_\gamma\|_{L_T^\infty H_{T, \delta}^s}
$$

as $\delta \to 0$, and where $P_K w = P_K(v_\gamma - v_\delta)$.

**Proof.** We first take the difference of the frequency truncated equation scaled gILW $\gamma$ and scaled gILW $\delta$. Then, set $P_K w = P_K(v_\gamma - v_\delta)$, which satisfies the following equation

$$
\partial_t P_K w - \frac{3}{\delta} \mathcal{G}_\delta \partial_x^2 P_K w + (H_\gamma - H_\delta) \partial_x P_K v_\gamma = P_K(\partial_x f(P_K v_\gamma) - \partial_x f(P_K v_\delta)).
$$

By following the same proof as in [120] Lemma 3.1 and recall that

$$
N_T^{s-1, \gamma} = L_T^\infty H_{T, \delta}^{s-1} \cap Y_T^{s-2, \gamma},
$$

it is enough to estimate $w_N$ in $Y^{s-2, \gamma}$-norm. Therefore, by using the Duhamel formulation, we
We recall that \( H_\delta \) and \( L_{\delta,\gamma} \) from (2.5.9) and (2.5.10). Then, from (2.5.16) under the assumption that \( n \leq K \), we have (2.5.17). Therefore, we have

\[
\| L_{\delta,\gamma} P K v_\gamma \|_{L^p_T H^s_x} \lesssim O(\delta^2) K^2 \| P K v_\gamma \|_{L^p_T H^s_x}, \tag{2.5.25}
\]

as \( \delta \to 0 \). Then, we continue from (2.5.24) to have

\[
\| P K w \|_{Y^{s,2,\delta}_T} \lesssim \| P K w \|_{L^p_T H^s_x} + O(\delta^2) K^2 \| P K v_\gamma \|_{L^p_T H^s_x} + C(\| P K v_\gamma \|_{L^p_T H^s_x} + \| P K v_\gamma \|_{L^p_T H^s_x}) \| P K w \|_{L^p_T H^s_x} \tag{2.5.26}
\]

as \( \delta \to 0 \). This finished the proof of (2.5.23).

\[ \Box \]

Now, we ready to prove Proposition 2.5.5. Here, for any \( 0 < \gamma < \delta < 1 \), let us set \( w = v_\gamma - v_\delta \). Then, we are considering the difference between the two frequency truncated equations:

\[
\partial_t P K w + \delta^2 P K w + H_\delta(\partial_x P K w) = (H_\delta - H_\delta)\partial_x P K v_\gamma - P K(\partial_x f(P K v_\delta) - f(P K v_\gamma)). \tag{2.5.27}
\]

**Proof of Proposition 2.5.5.** By using (2.5.27) and the same initial data condition \( (w(0) = 0) \), we apply the same strategy as in (2.5.5) to obtain:

\[
\| P K w(t) \|_{H^s} \lesssim \| L_{\delta,\gamma} P K \partial_x v_\gamma \|_{L^2_{T,x}} \| P K w \|_{L^2_{T,x}} + 2 \left| \max_{(k-1)! \in \{0, \ldots, k-1\}} I_{k,i}^{\delta,\gamma} \right|, \tag{2.5.28}
\]

where \( I_{k,i}^{\delta,\gamma} \) is now defined by

\[
I_{k,i}^{\delta,\gamma} := K^{2(s+1)} \left| \int_0^T \int_0^T \int_0^T \int_0^T v_\gamma^{k-1-i}(t') w P K \partial_x v \partial_x w dxdt' \right|.
\]

For the first term on the right-hand-side of (2.5.28), we apply Cauchy’s inequality to have

\[
K^{2s-2} \| L_{\delta,\gamma} P K \partial_x v_\gamma \|_{L^2_{T,x}} \| P K w \|_{L^2_{T,x}} \lesssim T c_1 \| L_{\delta,\gamma} P K v_\gamma \|_{L^2_T H^s_x} + c_2 \| w \|_{L^p_T H^s_x}, \tag{2.5.29}
\]

with \( c_2 \ll 1 \) coming from the Cauchy’s inequality. Let \( s \geq \frac{3}{4} \), from local well-posedness, we have

\[
\| P K v_\delta \|_{L^p_T H^s_x} \leq \| v_\delta \|_{L^p_T H^s_x} \leq C(\| v_\delta \|_{H^s}), \tag{2.5.29}
\]

where the constant is independent of \( 0 < \delta < 1 \). Then, under condition \( |n| < K \) that for any \( 0 < \delta, \gamma < 1 \) we have

\[
\sup_{|n| \leq K} K^2 \frac{h(n, \delta)}{\delta} = O(\delta^2) K^2 \quad \text{as} \quad \delta \to 0. \tag{2.5.30}
\]

By (2.5.29) and (2.5.30), given \( \varepsilon > 0 \) there exists \( 0 < \delta_0 < 1 \) such that for any \( 0 < \delta, \gamma < \delta_0 \), we have

\[
c_1 \| L_{\delta,\gamma} P K v_\gamma \|_{L^2_T H^s_x} \lesssim \left\| K^2 \frac{h(\delta, n)}{\delta} P K v_\delta \right\|_{L^2_T H^s_x}^2 < \varepsilon, \tag{2.5.31}
\]
Next, for the second term of the right-hand-side of (2.5.28), we apply Proposition 2.3.11. Then, we arrive at the following
\[
\|P_Kw\|^2_{L^\infty_t H^{-1}_x} \leq \varepsilon + TC_2 \|P_Kw\|^2_{L^\infty_t H^{-1}_x} + T^4 C(\|P_Kv_\gamma\|_{N^{-1}_t} + \|P_Kv_\delta\|_{N^{-1}_t})\|P_Kw\|_{N^{-1}_t} \cdot \|P_Kw\|_{L^\infty_t H^{-1}_x}. \tag{2.5.32}
\]

From Lemma 2.3.7 and Lemma 2.5.6, there exists \(\delta_0\) such that for any \(0 < \delta, \gamma < \delta_0 < 1\),
\[
\|P_Kv_\gamma\|_{N^{-1}_t} + \|P_Kv_\delta\|_{N^{-1}_t} \leq C_1 \|P_Kv_\gamma\|_{L^\infty_t H^s_x} + \|P_Kv_\gamma\|_{L^\infty_t H^s_x} + C_2 = C_1 \|P_Kv_\gamma\|_{L^\infty_t H^s_x} + C_2 \|P_Kv_\gamma\|_{L^\infty_t H^s_x} \tag{2.5.33}
\]

Here, \(C_1\) is coming from (2.5.13) and \(C_2\) is coming from (2.5.12). Next, we need to estimate \(\|w\|_{N^{-1}_t} \) in (2.5.32), which follows from Lemma 2.5.7 with \(w(0) = 0\). In particular, by equation (2.5.23) we have
\[
\|P_Kw\|_{N^{-1}_t} \leq C(\|P_Kv_\gamma\|_{L^\infty_t H^s_x} + \|P_Kv_\delta\|_{L^\infty_t H^s_x}) \|P_Kw\|_{L^\infty_t H^s_x} + \|P_Kw\|_{L^\infty_t H^s_x} + O(\delta^2)K^2 \|P_Kv_\gamma\|_{L^\infty_t H^s_x} \tag{2.5.34}
\]

Therefore, we plug (2.5.33), (2.5.34), into (2.5.32), and we obtain
\[
\|P_Kw\|^2_{L^\infty_t H^{-1}_x} \leq \varepsilon + TC_2 \|P_Kw\|^2_{L^\infty_t H^{-1}_x} + T^4 C_3(\|P_Kw\|_{L^\infty_t H^{-1}_x} + \|P_Kw\|_{L^\infty_t H^{-1}_x} + \varepsilon)\|P_Kw\|_{L^\infty_t H^{-1}_x}. \tag{2.5.35}
\]

By applying Cauchy’s inequality on the last term of the right-hand-side of (2.5.35), we conclude that given \(\varepsilon > 0\), there exists \(\delta_0\) such that for any \(0 < \delta, \gamma < \delta_0 < 1\) we have
\[
\|P_Kw\|^2_{L^\infty_t H^{-1}_x} \leq \varepsilon, \tag{2.5.36}
\]

as \(\varepsilon\) is arbitrary, and therefore we finished the proof of Proposition 2.5.5.

To finish the proof of Theorem 1.2.4, the following proposition is the final result regarding the convergence of gILW solutions to those of gKdV.

**Proposition 2.5.8.** Let \(s \geq \frac{3}{4}\), \(v_0 \in H^s(\mathbb{T})\), and \(v_\delta\) be solution of scaled gILW (2.1.2) with initial data \(v_0\). Then, for any \(0 < T < 1\), we have \(v_\delta \to v\) in \(C([0, T]; H^s(\mathbb{T}))\) as \(\delta \to 0\), where \(v\) is the solution of the gKdV (1.1.23) with the same initial data \(v_0\).

**Proof.** Let \(v_{0,N}\) be the specific smooth approximation of \(v_0\) as defined in Section 2.4. Let \(v_{\delta,N}\) denote the solution of (2.1.2) with initial data \(v_{0,N}\) and let \(v_N\) denote the solution to the Cauchy problem for the gKdV (1.1.23) with \(v_{0,N}\) as initial data. Moreover, let \(P_{\leq N}v_{\delta,N}\) Then, triangle inequality implies
\[
v_\delta - v = (v_\delta - v_{\delta,N}) + (v_{\delta,N} - P_{\leq K}v_{\delta,N}) + P_{\leq K}(v_{\delta,N} - v_N) + (v_N - v).
\]

Let us fix \(v_0 \in H^s(\mathbb{T})\) for \(s \geq \frac{3}{4}\), and let \(v_{0,N}\) be the smooth approximations to \(v_0\) defined in Section 2.4. Let \(v_N\) denote the solution of the gKdV (1.1.23) with initial data \(v_{0,N}\) and let \(v\)}
denote the solution with the initial data $v_0$. The theory developed in [119] implies that given $\varepsilon > 0$ and fix $N = N(\varepsilon)$, for any $s \geq \frac{3}{4}$ and $0 < T < 1$, we have
\[
\|v_N - v\|_{C^T H^s_x} < \frac{\varepsilon}{5}.
\] (2.5.36)
Let $v_{\delta,N}, v_\delta$ denote the solution of the gILW (2.1.2) with initial data $v_{0,N}, v_{0,N}$, separately. Then, it is also known from the theory developed in Section 2.4 that given $\varepsilon > 0$ and fix $N = N(\varepsilon)$, for any $s \geq \frac{3}{4}$ and $0 < T < 1$, we have
\[
\|v_{\delta,N} - v_{\delta}\|_{C^T H^s_x} < \frac{\varepsilon}{5}.
\] (2.5.37)
uniformly in $0 < \delta < 1$. Moreover, given $\varepsilon > 0$ and fix $K = K(\varepsilon)$ large enough such that
\[
\|P_{\leq K}(v_{\delta,N} - v_N)\|_{C^T H^s_x} \leq 2K\|P_{\leq K}(v_{\delta,N} - v_N)\|_{C^T H^{s-1}_x} < \frac{\varepsilon}{5},
\] (2.5.38)
Lastly, given $\varepsilon > 0$, fix $N(\varepsilon), K(\varepsilon) \in \mathbb{N}$ so that (2.5.36), (2.5.37), and (2.5.38) hold. Then, by Proposition 2.5.5 we have
\[
\|P_{\leq K}(v_{\delta,N} - v_N)\|_{C^T H^s_x} \leq 2K\|P_{K}(v_{\delta,N} - v_N)\|_{C^T H^{s-1}_x} < \frac{\varepsilon}{5},
\] Since $\varepsilon > 0$ was arbitrary, we finished the proof of Proposition 2.5.8.

2.6 On the discussion of limiting behaviour on $\mathbb{R}$

In the following section, we will sketch the convergence of gILW (2.1.1) and scaled gILW (2.1.2) on real line settings. For construction on the uniform local well-posedness, we have seen from the previous discussion that the crucial part is to obtain the short-time Strichartz estimates on smooth solutions to the gILW (2.1.1) and scaled gILW (2.1.1).

2.6.1 Short-time Strichartz estimates on $\mathbb{R}$

In this subsection, our strategy genuinely follows from [90]. We first establish the Short-time Strichartz estimate for the linearised equation with some general nonlinearity. This type of argument can also be seen in [116, 117].

In the real line setting, although we can use the classical smoothing effect derived in [79], our dispersion of (scaled) gILW is too weak, so there is no derivative gain. In particular, we observe the following: let $\mathcal{L}$ be a general dispersion operator and from [79], we have
\[
\|e^{-t\mathcal{L}[\mathcal{L}''(t)]^{\frac{1}{2}}u_0}\|_{L^4(\mathbb{R};L^\infty(\mathbb{R}))} \lesssim \|u_0\|_{L^2(\mathbb{R})}.
\] (2.6.1)
Then, in the view of gILW (2.1.1) and scaled gILW (2.1.2), we would have
\[
\mathcal{L} = \mathcal{G}_{d}\partial_x^2, \quad \text{or} \quad \mathcal{L} = \frac{3}{\delta}\mathcal{G}_{d}\partial_x^2.
\]
Hence, (2.6.1) gives no derivative gain. We recall the linear propagators in the deep-water and shallow-water regions
\[
S^{(d)}_\delta(t) = e^{-t\mathcal{G}_{d}\partial_x^2}, \quad S^{(s)}_\delta(t) = e^{-t\frac{3}{\delta}\mathcal{G}_{d}\partial_x^2}.
\]
Then, we shall recall the standard Strichartz estimates from [63, Lemma 3.5].

**Lemma 2.6.1.** Let $(q,r)$ be admissible pair such that
\[
2 \leq q, r \leq \infty, \quad \text{and} \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{r}.
\]
CHAPTER 2. MICROSCOPIC LIMITS: DETERMINISTIC METHOD

Then, for any $\phi \in L^2(\mathbb{R})$, the following statements hold:

(i) for any $2 \leq \delta \leq \infty$, we have

$$\|S_\delta^{(d)}(t)\phi\|_{L^q_tL^r_x} \lesssim \|\phi\|_{L^2}.$$  \hfill (2.6.2)

(ii) for any $0 < \delta \ll 1$, we have

$$\|S_\delta^{(s)}(t)\phi\|_{L^q_tL^r_x} \lesssim \|\phi\|_{L^2}.$$ \hfill (2.6.3)

Here, the implicit constant is independent of $\delta$.

Next, we follow the same strategy as in [90]. For any $2 \leq \delta \leq \infty$, let $u(t, x)$ to be any solution defined on $[0, T]$ to the following linear equation

$$\partial_t u + G_\delta \partial_x^2 u = F.$$ \hfill (2.6.4)

For any $0 < \delta \ll 1$, let $v(t, x)$ to be any solution defined on $[0, T]$ to the following linear equation

$$\partial_t v + \frac{3}{\delta} G_\delta \partial_x^2 v = F.$$ \hfill (2.6.5)

Then, we have the following short-time Strichartz estimates. The following argument can also be seen in [141, Lemma 2.9].

Lemma 2.6.2. Let $T > 0$, $u(t, x)$ to be any solution satisfies (2.6.4) on $[0, T]$, and $v(t, x)$ to be any solution satisfies (2.6.5) on $[0, T]$. Moreover, suppose that $(q, r) \in \mathbb{R}^+$ is admissible and $2 \leq p \leq q$.

Then, for any $\theta > 0$, the following statements hold:

(i) for any $2 \leq \delta \leq \infty$, we have

$$\|u\|_{L^p_tL^r_x} \lesssim N^{\frac{1}{q}} \|D_x^{\frac{1}{2} + \theta} u\|_{L^p_tL^2_x} + \|D_x^{\frac{1}{2} - \theta} F\|_{L^p_tL^2_x}.$$ \hfill (2.6.6)

(ii) for any $0 < \delta \ll 1$, we have

$$\|v\|_{L^p_tL^r_x} \lesssim N^{\frac{1}{q}} \|D_x^{\frac{1}{2} + \theta} v\|_{L^p_tL^2_x} + \|D_x^{\frac{1}{2} - \theta} F\|_{L^p_tL^2_x}.$$ \hfill (2.6.7)

Here, the implicit constant is independent of $\delta$.

Proof. This proof essentially follows from the tori setting. See also [141, Lemma 2.9]. We only sketch the proof.

It is enough to show that, for any dyadic $N > 1$, the following holds

$$\|u_N\|_{L^p_tL^r_x} \lesssim N^{\frac{1}{q}} \|u_N\|_{L^p_tL^2_x} + N^{\frac{1}{q} - 1} \|F_N\|_{L^p_tL^2_x}.$$ \hfill (2.6.8)

For each $N$, it is enough to have

$$[0, T] = \bigcup_{j \in J_N} I_{j,N}, \quad \text{where} \quad I_{j,N} := [a_j, b_j], \quad \|I_{j,N}\| \sim N^{-1}, \quad \text{and} \quad \#J \sim N.$$

Moreover, $u_N(t)$ solves the Duhamel formulation

$$u_N(t) = e^{-(t-a_j)\partial_x^2} u_N(a_j) + \int_{a_j}^t e^{-(t-t')\partial_x^2} F_N(t')dt'.$$
for any \( t \in I_{j,N} \). Then, we have

\[
\| u_N \|_{L_t^p L_x^q} \lesssim \left( \sum_j \| e^{-(t-a_j)G_{ij} \partial_x^2} u_N(a_j) \|_{L^p(I_{j,N} ; L_x)}^p \right)^{\frac{1}{p}} + \left( \sum_j \left\| \int_{a_j}^t e^{-(t-t')G_{ij} \partial_x^2} F_N(t') \, dt' \right\|_{L^p(I_{j,N} ; L_x)}^p \right)^{\frac{1}{p}} := I + II.
\]

For the linear term we use Hölder’s inequality in time and \( (2.6.2) \),

\[
I \lesssim \left( \sum_j |I_{j,N}|^{1-\frac{p}{q}} \| u_N(a_j) \|_{L^q(I_{j,N} ; L_x)}^p \right)^{\frac{1}{p}} \sim N^{\frac{1}{q}} \left( \sum_j |I_{j,N}| \| u_N(a_j) \|_{L^2_x}^p \right)^{\frac{1}{p}} \lesssim N^{\frac{1}{q}} \left( \sum_j \int_{I_{j,N}} \| u_N(t') \|_{L^p_x}^p \, dt' \right)^{\frac{1}{p}} = N^{\frac{1}{q}} \| u_N \|_{L_t^p L_x^q}.
\]

As for or the Duhamel term, we use Hölder’s inequality in time, Minkowski, and \( (2.6.2) \),

\[
II \lesssim \left( \sum_j |I_{j,N}|^{1-\frac{p}{q}} \left\| \int_{a_j}^t e^{-(t-t')G_{ij} \partial_x^2} F_N(t') \, dt' \right\|_{L^q(I_{j,N} ; L_x)}^p \right)^{\frac{1}{p}} \lesssim \left( \sum_j |I_{j,N}|^{1-\frac{p}{q}} \left\{ \int_{I_{j,N}} \| e^{-(t-t')G_{ij} \partial_x^2} F_N(t') \|_{L^q(I_{j,N} ; L_x)} \, dt' \right\}^p \right)^{\frac{1}{p}} \sim \left( \sum_j |I_{j,N}|^{1-\frac{p}{q}} \| F_N(t') \|_{L^p(I_{j,N} ; L_x^q)}^p \right)^{\frac{1}{p}} \lesssim \left( \sum_j |I_{j,N}|^{1-\frac{p}{q}+p-1} \| F_N(t') \|_{L^p(I_{j,N} ; L_x^q)}^p \right)^{\frac{1}{p}} \sim N^{\frac{1}{q} - 1} \left( \sum_j \| F_N(t') \|_{L^p(I_{j,N} ; L_x^q)}^p \right)^{\frac{1}{p}} = N^{\frac{1}{q} - 1} \| F_N(t') \|_{L_t^p L_x^q}.
\]

Following a similar argument as in Proposition \( 2.3.9 \) we can obtain \( \delta \)-independent energy estimates. Together with energy estimates and the estimates for the difference between two solutions, we can construct our solution on \( \mathbb{R} \) similarly. Moreover, we can conclude our convergence results on \( \mathbb{R} \) by running the same convergent argument.
Chapter 3

Macroscopic limits: probabilistic method

We focus on the power-type nonlinearity in the probabilistic study of the convergence of gILW dynamics. Let us recall in the deep-water region (BO-regime), we consider the Cauchy problem of gILW on $\mathbb{T}$:

$$
\begin{align*}
\partial_t u - G_\delta (\partial_x^2 u) &= \partial_x (u^k), \\
 u|_{t=0} &= u_0,
\end{align*}
$$

$t, x) \in \mathbb{R} \times \mathbb{T}$.

(3.0.1)

The associated Hamiltonian

$$
E(u) := \frac{1}{2} \int_\mathbb{T} u G_\delta \partial_x u dx + \frac{1}{k+1} \int_\mathbb{T} u^{k+1} dx.
$$

In the shallow-water region (KdV-regime), we consider the Cauchy problem of scaled gILW on $\mathbb{T}$:

$$
\begin{align*}
\partial_t v - \frac{3}{2} G_\delta (\partial_x^2 v) &= \partial_x (v^k), \\
v|_{t=0} &= v_0,
\end{align*}
$$

$t, x) \in \mathbb{R} \times \mathbb{T}$.

(3.0.2)

The associated Hamiltonian

$$
H(v) := \frac{3}{2\delta} \int_\mathbb{T} v G_\delta \partial_x v dx + \frac{1}{k+1} \int_\mathbb{T} v^{k+1} dx.
$$

Again, recall that the operator

$$
\delta f(n) := -i \left( \coth(\delta n) - \frac{1}{\delta n} \right) \tilde{f}(n) \quad \text{for } n \in \mathbb{Z}
$$

The deep-water limit equation ($\delta = \infty$) is

$$
\partial_t u - \mathcal{H}(\partial_x^2 u) = \partial_x (u^k).
$$

(3.0.3)

The associated Hamiltonian is

$$
E_\infty(u) := \frac{1}{2} \int_\mathbb{T} u \mathcal{H} \partial_x u dx + \frac{1}{k+1} \int_\mathbb{T} u^{k+1} dx
\quad = \frac{1}{2} \int_\mathbb{T} (|D_x^\perp u|^2 dx + \frac{1}{k+1} \int_\mathbb{T} u^{k+1} dx.
$$

The shallow-water limit equation ($\delta = 0$) is

$$
\partial_t v + \partial_x^2 v = \partial_x (v^k).
$$

(3.0.4)
The associated Hamiltonian is given by
\[ E_0(v) := -\frac{1}{2} \int \varphi_x^2 v dx + \frac{1}{k+1} \int v^{k+1} dx \]
\[ = \frac{1}{2} \int (\varphi_x v)^2 dx + \frac{1}{k+1} \int v^{k+1} dx \]

3.1 Preliminaries

In this section, we present several basic lemmas.

3.1.1 On the variance parameters

In this subsection, we establish elementary lemmas on the variance parameters \( K_\delta(n) \) in (1.3.33) and \( L_\delta(n) \) in (1.3.45) for the Gaussian Fourier series \( X_\delta \) in (1.3.32) and \( e^{X_\delta} \) in (1.3.44), respectively.

Lemma 3.1.1. Let \( K_\delta(n) \) be as in (1.3.33). Then, for any \( \delta > 0 \), we have
\[ \max \left( 0, |n| - \frac{1}{\delta} \right) \leq K_\delta(n) = n \coth(\delta n) - \frac{1}{\delta} \leq |n|, \]
\[ (3.1.1) \]
where the above inequalities are strict for \( n \neq 0 \). In particular, we have
\[ K_\delta(n) \sim \delta |n| \]
\[ (3.1.2) \]
for any \( n \in \mathbb{Z}^* \). Furthermore, for each fixed \( n \in \mathbb{Z}^* \), \( K_\delta(n) \) is strictly increasing in \( \delta \geq 1 \) and converges to \( |n| \) as \( \delta \to \infty \).

The bound (1.1.6) implies that, for \( \delta \geq 2 \), we have
\[ K_\delta(n) \geq |n| - \frac{1}{2} \sim |n| \]
\[ (3.1.3) \]
for any \( n \in \mathbb{Z}^* \).

Proof. For \( x \in \mathbb{R} \setminus \{0\} \), define \( h \) by
\[ h(x) = 1 - x \coth(x) + |x| = 1 + |x| - x \frac{e^x + e^{-x}}{e^x - e^{-x}} \]
such that
\[ K_\delta(n) = |n| - \frac{1}{\delta} h(\delta n). \]
\[ (3.1.4) \]
In view of (1.1.10), we set \( h(0) = 0 \) such that \( h \) is continuous. We claim that
\[ 0 < h(x) < \min(1, |x|) \]
\[ (3.1.5) \]
for any \( x \in \mathbb{R} \setminus \{0\} \). Indeed, we first note that \( h \) is an even function. For \( x > 0 \), a direct computation shows
\[ h(x) = 1 + x - x \frac{e^{2x} + 1}{e^{2x} - 1} = 1 - \frac{2x}{e^{2x} - 1} \in (0, 1), \]
\[ (3.1.6) \]
from which the claim (3.1.5) follows. Then, the bound (3.1.1) follows from (3.1.4) and (3.1.5). The equivalence (3.1.2) is a direct consequence of (3.1.1) and the fact that \( K_\delta(n) > 0 \) for \( n \in \mathbb{Z}^* \).
Fix \( n \in \mathbb{N} \). By writing \( K_\delta(n) = |n| - \frac{h(\delta n)}{\delta n} n \), the claimed strict monotonicity of \( K_\delta(n) \) in \( \delta \geq 1 \) follows once we show that \( \frac{h(x)}{x} \) is strictly decreasing and its limit as \( x \to \infty \) is 0. A direct computation shows that

\[
\frac{d}{dx} \left( \frac{h(x)}{x} \right) = -\frac{e^{4x} - 2e^{2x} - 4xe^{2x} + 1}{x^2(e^{2x} - 1)^2} < 0
\]

for \( x \geq 1 \). Namely, \( K_\delta(n) \) is increasing for \( \delta \geq 1 \). From (3.1.6), we have

\[
\frac{h(x)}{x} = \frac{1}{x} - \frac{2}{e^{2x} - 1},
\]

from which we conclude \( \lim_{x \to \infty} \frac{h(x)}{x} = 0 \). This concludes the proof of Lemma 3.1.1.

Remark 3.1.2. Note that we have \( q_\delta(n) = \frac{1}{\delta} h(\delta n) \), where \( q_\delta(n) \) is as in (1.1.5). Then, (3.1.5) in Lemma 3.1.1 yields (1.1.6) with the right-hand side replaced by \( \frac{1}{\delta} \).

Next, we state basic properties of \( L_\delta(n) \) defined in (1.3.45). Given \( \delta > 0 \), it follows from \( L_\delta(n) = \frac{3}{\delta} K_\delta(n) \) and Lemma 3.1.1 that

\[
L_\delta(n) \sim \delta |n|
\]

for any \( n \in \mathbb{Z}^* \).

Lemma 3.1.3. The following statements hold.

(i) \( 0 < L_\delta(n) < n^2 \) for any \( \delta > 0 \) and \( n \in \mathbb{Z}^* \).

(ii) For each \( n \in \mathbb{Z}^* \), \( L_\delta(n) \) increases to \( n^2 \) as \( \delta \to 0 \).

(iii) We have

\[
L_\delta(n) \gtrsim \begin{cases} n^2, & \text{if } \delta |n| \lesssim 1, \\ |n|, & \text{if } \delta |n| \gtrsim 1 \text{ and } \delta \lesssim 1. \end{cases}
\]

In particular, the following uniform bound holds:

\[
\inf_{0 < \delta \lesssim 1} L_\delta(n) \gtrsim |n|
\]

for any \( n \in \mathbb{Z}^* \).

(iv) Define \( h(n, \delta) \) by

\[
L_\delta(n) = n^2 - h(n, \delta)n^2.
\]

Then, we have

\[
\sum_{n \in \mathbb{Z}} h^2(n, \delta) = \infty
\]

for any \( \delta > 0 \).

Proof. From (1.3.45), we have \( L_\delta(n) = \frac{3}{\delta} K_\delta(n) \). Hence, from Lemma 3.1.1 we have \( L_\delta(n) > 0 \) for any \( n \in \mathbb{Z}^* \). On the other hand, from the Mittag-Leffler expansion \[11\) on p. 189\], we have

\[
\pi z \coth(\pi z) = \pi z \cot \left( \frac{\pi z}{i} \right) = 1 + \sum_{k=1}^{\infty} \frac{2z^2}{k^2 + z^2}
\]

(3.1.11)
for \( z \in \mathbb{C} \setminus i\mathbb{Z} \). Then, from (1.3.35) and (3.1.11), we have

\[
L_\delta(n) = 6n^2 \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2 + \delta^2 n^2} = 6n^2 \left( \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2} - \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2 + \delta^2 n^2} \right) = n^2 - n^2 \sum_{k=1}^{\infty} \frac{6\delta^2 n^2}{k^2 \pi^2 (k^2 \pi^2 + \delta^2 n^2)}
\]

(3.1.12)

for any \( \delta > 0 \) and \( n \in \mathbb{Z} \). Hence, we conclude that \( L_\delta(n) < n^2 \) for any \( n \in \mathbb{Z}^* \). This proves the claim (i).

From (3.1.9) and (3.1.12), we have

\[
h(n, \delta) = 6\delta^2 \sum_{k=1}^{\infty} \frac{n^2}{k^2 \pi^2 (k^2 \pi^2 + \delta^2 n^2)}.
\]

(3.1.13)

which tends to 0 as \( \delta \to 0 \). We also note that the expression after the first equality in (3.1.12) shows that \( L_\delta(n) \) is monotonic in \( \delta \). This yields the claim (ii). From (3.1.13), we see that, as \( n \to \infty \), \( h(n, \delta) \to 0 \) (for each fixed \( \delta > 0 \)), which yields (3.1.10). This proves the claim (iv).

Lastly, we prove (iii). Suppose \( \delta |n| \lesssim 1 \). Then, from (3.1.12) we have

\[
L_\delta(n) = 6n^2 \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2 + \delta^2 n^2} \gtrsim n^2 \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} \gtrsim n^2.
\]

(3.1.14)

Now, suppose \( \delta |n| \gg 1 \) and \( \delta \lesssim 1 \). Then, from (3.1.12) and a Riemann sum approximation, we have

\[
\frac{L_\delta(n)}{|n|} \gtrsim \sum_{k=1}^{\infty} \frac{|n|}{k^2 \pi^2 + \delta^2 n^2} \gtrsim \frac{1}{\delta} \sum_{k=1}^{\infty} \frac{1}{\pi^2 (\frac{k}{|n|})^2 + 1} \gtrsim \frac{1}{\delta |n|} \gtrsim \int_0^{\infty} \frac{dx}{\pi^2 x^2 + 1} \approx \frac{1}{\delta |n|}.
\]

(3.1.15)

Note that the implicit constants in (3.1.14) and (3.1.15) are independent of \( \delta \). This proves the claim (iii).

### 3.1.2 Tools from stochastic analysis

In the following, we review some basic facts on the Hermite polynomials and the Wiener chaos estimate. See, for example, [94][123].

We define the \( k \)th Hermite polynomials \( H_k(x; \sigma) \) with variance \( \sigma \) via the following generating function:

\[
e^{tx - \frac{1}{2} \sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x; \sigma)
\]

(3.1.16)

for \( t, x \in \mathbb{R} \) and \( \sigma > 0 \). When \( \sigma = 1 \), we set \( H_k(x) = H_k(x; 1) \). Then, we have

\[
H_k(x; \sigma) = \sigma^{\frac{k}{2}} H_k(\sigma^{-\frac{1}{2}} x).
\]

(3.1.17)

It is well known that \( \{H_k(\sqrt{\sigma})\}_{k \in \mathbb{N}_0} \) form an orthonormal basis of \( L^2(\mathbb{R}; \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx) \). In the following, we list the first few Hermite polynomials for readers’ convenience:

\[
H_0(x; \sigma) = 1, \quad H_1(x; \sigma) = x, \quad H_2(x; \sigma) = x^2 - \sigma, \quad H_3(x; \sigma) = x^3 - 3\sigma x, \quad H_4(x; \sigma) = x^4 - 6\sigma x^2 + 3\sigma^2.
\]
From (3.1.16), we obtain the following recursion relation:

$$\partial_x H_k(x; \sigma) = k H_{k-1}(x; \sigma)$$

for any \( k \in \mathbb{N} \), and the following identity:

$$H_k(x + y) = \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} H_\ell(y),$$

which, together with (3.1.17), yields

$$H_k(x + y; \sigma) = \sigma^k \sum_{\ell=0}^{k} \binom{k}{\ell} \sigma^{-\frac{k-\ell}{2}} x^{k-\ell} H_\ell(\sigma^{-\frac{1}{2}} y)$$

(3.1.18)

Let \( \{g_n\}_{n \in \mathbb{Z}} \) be an independent family of standard complex-valued Gaussian random variables conditioned that \( g_n = \overline{g_{-n}} \). We first recall the following bound:

$$\sup_{n \in \mathbb{Z}} |g_n| \leq C_{\varepsilon, \omega} < \infty$$

(3.1.19)

almost surely for some random constant \( C_{\varepsilon, \omega} > 0 \); see Lemma 3.4 in [42]. See also Appendix in [128].

We define a real-valued, mean-zero Gaussian white noise \( W \) on \( \mathbb{T} \) by

$$W(x; \omega) = \sum_{n \in \mathbb{Z}} g_n(\omega)e^{-in x}.$$  

(3.1.20)

Next, we introduce the isonormal Gaussian process \( \{W_f : f \in L^2(\mathbb{T})\} \) associated to the Gaussian white noise \( W \).

**Definition 3.1.4.** The isonormal Gaussian process \( \{W_f : f \in L^2(\mathbb{T})\} \) is a real-valued, mean-zero Gaussian process indexed by the real separable Hilbert space \( L^2(\mathbb{T}) \) such that

$$\mathbb{E}[W_f W_g] = \langle f, g \rangle_{L^2_\mathbb{T}}$$

for \( f, g \in L^2(\mathbb{T}) \). Moreover, we can realize \( W_f \) as follows:

$$f \in L^2(\mathbb{T}) \mapsto W_f = \langle f, W \rangle_{L^2_\mathbb{T}} = \sum_{n \in \mathbb{Z}} \hat{f}(n) g_n(\omega),$$  

(3.1.21)

where \( W \) is as in (3.1.20).

**Remark 3.1.5.** The action (3.1.21) on \( f \) by the white noise is referred to as the white noise functional in [139, 134]. Note that \( W_f \) is basically the ‘periodic’ Wiener integral on \( \mathbb{T} \).

In the following, we denote \( L^2(\Omega, \sigma, \{W\}, \mathbb{P}) \) the space of real-valued, square-integrable random variables that are measurable with respect to \( W \). We present below a fundamental result in Gaussian analysis, providing us an orthogonal decomposition of this \( L^2 \)-probability space. Let \( \Gamma^W_k \) be the \( L^2(\Omega) \)-completion of the linear span of the set \( \{H_k(W_f) : f \in L^2(\mathbb{T}); \|f\|_{L^2} = 1\} \). We call
\( \Gamma_k^W \) the kth Wiener chaos associated to \( W \). The following Wiener-Itô decomposition holds:

\[
L^2(\Omega, \sigma\{W\}, \mathbb{P}) = \bigoplus_{k=0}^{\infty} \Gamma_k^W. \tag{3.1.22}
\]

The orthogonal decomposition \( \tag{3.1.22} \) indicates that random variables belonging to Wiener chaoses of different orders are uncorrelated (namely, \( L^2(\Omega) \)-orthogonal). See also the following particular case that we will often use in our computations.

**Lemma 3.1.6.** Let \( Y_1, Y_2 \) be two real-valued, mean-zero, and jointly Gaussian random variables with variances \( \sigma_1 = \mathbb{E}[Y_1^2] > 0 \) and \( \sigma_2 = \mathbb{E}[Y_2^2] > 0 \). Then, for \( k, m \in \mathbb{N} \cup \{0\} \), we have

\[
\mathbb{E}[H_k(Y_1; \sigma_1)H_m(Y_2; \sigma_2)] = 1_{k=m} \cdot k!(\mathbb{E}[Y_1Y_2])^k. \tag{3.1.23}
\]

For example, with \( f, h \in L^2(\mathbb{T}) \), the random variables \( Y_1 = W_f \) and \( Y_2 = W_h \), with \( \sigma_1 = \|f\|_{L^2}^2 \) and \( \sigma_2 = \|h\|_{L^2}^2 \) satisfy the identity \( \tag{3.1.23} \).

Next, we state the Wiener chaos estimate, which is a consequence of Nelson’s hypercontractivity \( \tag{123} \). See, for example, \( \tag{154} \) Theorem I.22. See also \( \tag{159} \) Proposition 2.4.

**Lemma 3.1.7** (Wiener chaos estimate). Let \( \{g_n\}_{n \in \mathbb{Z}} \) be an independent family of standard complex-valued Gaussian random variables conditioned that \( g_n = g_{-n} \). Given \( k \in \mathbb{N} \), let \( \{Q_j\}_{j \in \mathbb{N}} \) be a sequence of polynomials in \( g = \{g_n\}_{n \in \mathbb{Z}} \) of degrees at most \( k \) such that \( \sum_{j \in \mathbb{N}} Q_j(g) \in \mathbb{R} \), almost surely. Then, for any finite \( p \geq 2 \), we have

\[
\left\| \sum_{j \in \mathbb{N}} Q_j(g) \right\|_{L^p(\Omega)} \leq (p - 1)^{\frac{k}{2}} \left\| \sum_{j \in \mathbb{N}} Q_j(g) \right\|_{L^2(\Omega)}. \]

Lastly, we provide a brief discussion on the Wick renormalisation.

**Wick renormalization.** Let \( \{\beta_k, k \in \mathbb{N}\} \) be i.i.d. real-valued standard Gaussian random variables, which can be built from the Gaussian white noise \( W \) in \( \tag{3.1.20} \). Consider the polynomial \( Q(x_1, \ldots, x_n) \) with \( n \) variables. We denote its degree by \( \text{deg}(Q) \). Then, the random variable \( Q(\beta_1, \ldots, \beta_n) \) belongs to the sum of the first \( \text{deg}(Q) \) Wiener chaoses, that is,

\[
Q(\beta_1, \ldots, \beta_n) \in \bigoplus_{k \leq \text{deg}(Q)} \Gamma_k^W.
\]

One can find a unique polynomial \( P \) with the same degree and the same coefficient on the leading order term such that \( P(\beta_1, \ldots, \beta_n) \in \Gamma_k^W \), that is, \( P(\beta_1, \ldots, \beta_n) \) is the projection of \( Q(\beta_1, \ldots, \beta_n) \) onto \( \Gamma_{\text{deg}(Q)}^W \). We call such a polynomial \( P \) as the Wick-ordered version of \( Q \), and we write \( P = W(Q) \).

**Example 1.** (i) Consider the polynomial \( Q(x_1, \ldots, x_n) = x_1^{k_1} \cdots x_n^{k_n} \). Then, we have

\[
W(Q)(x_1, \ldots, x_n) = \prod_{j=1}^{n} H_{k_j}(x_j),
\]

where \( H_k \) is the kth Hermite polynomial with variance \( \sigma = 1 \).

(ii) Given \( N \in \mathbb{N} \), consider the following truncated random Fourier series \( X_{\delta,N} \):

\[
X_{\delta,N}(x, \omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} \frac{g_n(\omega)}{|K_\delta(n)|^{\frac{1}{2}}} e_n. \tag{3.1.24}
\]

Note that \( X_{\delta,N} = P_N X_\delta \), where \( X_\delta \) is as in \( \tag{1.3.32} \). For each \( x \in \mathbb{T} \), \( X_{\delta,N}(x) \) is a real-valued, mean-zero Gaussian random variable with variance \( \sigma_{\delta,N} \) in \( \tag{1.3.35} \). Then, the Wick-ordered version of \( X_{\delta,N}^k, k \in \mathbb{N} \), is given by \( W(X_{\delta,N}^k) = H_k(X_{\delta,N}; \sigma_{\delta,N}) \). Compare this with \( \tag{1.3.36} \).

In the following, we will use the notion of the Ky-Fan metric on \( L^0(\Omega) \), which characterizes the convergence in probability; see \( \tag{18} \). Let \( X, Y \) be two real-valued random variables defined on a
common probability space $\Omega$, then the Ky-Fan distance between $X$ and $Y$ is given by

$$d_{KF}(X, Y) = E[1 \land |X - Y|],$$

where $a \land b := \min\{a, b\}$.

3.1.3 Various modes of convergence for probability measures and random variables

We conclude this section by going over various modes of convergence for probability measures and random variables. See, for example, [143, Chapter 3] and [160, Chapter 2] for further discussions. See also [48].

- Convergence in probability and the Ky-Fan distance.

Let $X$ and $Y$ be two real-valued random variables defined on a common probability space $\Omega$. Then, the Ky-Fan distance between $X$ and $Y$ is defined by

$$d_{KF}(X, Y) = E[1 \land |X - Y|],$$

where $a \land b := \min\{a, b\}$. It is known that the Ky-Fan distance characterizes convergence in probability. Namely, a sequence $\{Z_n\}_{n \in \mathbb{N}}$ of random variables converges in probability to some limit $Z$ if and only if $d_{KF}(Z_n, Z) \to 0$ as $n \to \infty$.

The usual continuous mapping theorem [15, Problem 5.17 on p. 83] states that if a sequence $\{Z_n\}_{n \in \mathbb{N}}$ converges to a limit $Z$ in probability, then, given a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, $\{\phi(Z_n)\}_{n \in \mathbb{N}}$ converges to $\phi(Z)$ in probability. For our purpose, we need to extend this continuous mapping theorem for uniform convergence in probability.

Lemma 3.1.8 (uniform continuous mapping theorem). Let $J \subset [0, \infty]$ be an index set. Suppose that $\{Z_{\delta,n}\}_{n \in \mathbb{N}}$ converges in probability to a limit $Z_\delta$ uniformly in $\delta \in J$, as $n \to \infty$ in the following sense:

$$\lim_{n \to \infty} \sup_{\delta \in J} d_{KF}(Z_{\delta,n}, Z_\delta) = 0 \quad (3.1.25)$$

or equivalently, for any $\eta > 0$,

$$\lim_{n \to \infty} \sup_{\delta \in J} P(|Z_{\delta,n} - Z_\delta| > \eta) = 0. \quad (3.1.26)$$

Suppose that the family of random variables $\{Z_\delta\}_{\delta \in J}$ is tight, meaning that for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathbb{R}$ such that

$$\sup_{\delta \in J} P(Z_\delta \in K_\varepsilon^c) \leq \varepsilon. \quad (3.1.27)$$

Then, given any continuous function $\phi : \mathbb{R} \to \mathbb{R}$, we have

$$\lim_{n \to \infty} \sup_{\delta \in J} d_{KF}(\phi(Z_{\delta,n}), \phi(Z_\delta)) = 0. \quad (3.1.28)$$

Note that the tightness assumption on $\{Z_\delta\}_{\delta \in J}$ is crucial.

Proof. Let us first show the equivalence of (3.1.25) and (3.1.26). Let $0 < \eta < 1$. Then, by Markov’s inequality and (3.1.25), we have

$$P(|Z_{\delta,n} - Z_\delta| > \eta) \leq E\left[1_{(|Z_{\delta,n} - Z_\delta| > \eta)} \frac{|Z_{\delta,n} - Z_\delta|}{\eta}\right] \leq \frac{1}{\eta} d_{KF}(Z_{\delta,n}, Z_\delta)$$

and

$$d_{KF}(Z_{\delta,n}, Z_\delta) = E[|Z_{\delta,n} - Z_\delta| \land 1] \leq \eta + P(|Z_{\delta,n} - Z_\delta| \geq \eta).$$
This proves the equivalence of (3.1.25) and (3.1.26).

We now prove (3.1.28). Fix $\beta > 0$. In view of (3.1.27), there exists $\eta = \eta(\beta) \geq 1$ such that

$$
sup_{\delta \in J} \mathbb{P}(|Z_\delta| > \eta) \leq \beta. \tag{3.1.29}
$$

Since $\phi$ is continuous, it is uniformly continuous on $[-2\eta, 2\eta]$. In particular, there exists small $\varepsilon = \varepsilon(\phi, \beta) > 0$ such that

$$
|\phi(x) - \phi(y)| \leq \beta, \quad \text{whenever } x, y \in [-2\eta, 2\eta] \text{ with } |x - y| \leq \varepsilon. \tag{3.1.30}
$$

Without loss of generality, we assume that $\varepsilon \leq \eta$. Note that these parameters $\varepsilon, \beta$, and $\eta$ do not depend on $n \in \mathbb{N}$.

From (3.1.26), we have

$$
\lim_{n \to \infty} \sup_{\delta \in J} \mathbb{E}\left[\left(|\phi(Z_{\delta,n}) - \phi(Z_\delta)| \land 1\right)1_{|Z_{\delta,n} - Z_\delta| > \varepsilon}\right] \\
\leq \lim_{n \to \infty} \sup_{\delta \in J} \mathbb{P}( |Z_{\delta,n} - Z_\delta| > \varepsilon) = 0. \tag{3.1.31}
$$

On the other hand, from (3.1.30) and (3.1.29), we have

$$
\mathbb{E}\left[\left(|\phi(Z_{\delta,n}) - \phi(Z_\delta)| \land 1\right)1_{|Z_{\delta,n} - Z_\delta| \leq \varepsilon}\right] \\
\leq \mathbb{E}\left[\left(|\phi(Z_{\delta,n}) - \phi(Z_\delta)| \land 1\right)1_{|Z_{\delta,n} - Z_\delta| \leq \varepsilon, |Z_\delta| \leq \eta}\right] + \mathbb{E}\left[\left(|\phi(Z_{\delta,n}) - \phi(Z_\delta)| \land 1\right)1_{|Z_\delta| > \eta}\right] \leq 2\beta. \tag{3.1.32}
$$

Since $\beta > 0$ is arbitrary, (3.1.28) follows from (3.1.31) and (3.1.32).

\textbf{• Convergence in total variation and the Hellinger distance.}

Let $\nu$ and $\mu$ be two probability measures on a measurable space $(E, \mathcal{E})$, the \textit{total variation distance} $d_{TV}$ of $\mu$ and $\nu$ is given by

$$
d_{TV}(\mu, \nu) := \sup \{ |\mu(A) - \nu(A)| : A \in \mathcal{E} \}. \tag{3.1.33}
$$

This metric induces a much stronger topology than the one induced by the weak convergence.

Next, we recall the notion of the Hellinger integral [13 \textbf{19}]. Let $\mu$ and $\nu$ be two probability measures on a measurable space $(E, \mathcal{E})$. Note that both $\mu$ and $\nu$ are absolutely continuous with respect to the probability measure $\lambda = \frac{1}{2}(\mu + \nu)$. Then, the Hellinger integral of $\mu$ and $\nu$ is defined by

$$
H(\mu, \nu) = \int_E \sqrt{\frac{d\mu}{d\lambda} \frac{d\nu}{d\lambda}} d\lambda. \tag{3.1.34}
$$

In fact, the definition (3.1.34) is independent of the choice of a probability measure $\lambda$ such that $\mu, \nu \ll \lambda$. When $\mu$ and $\nu$ are equivalent (i.e. mutually absolutely continuous), we can write $H(\mu, \nu)$ as

$$
H(\mu, \nu) = \int_E \sqrt{\frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}} d\mu. \tag{3.1.35}
$$

Note that $0 \leq H(\mu, \nu) \leq 1$. The Hellinger integral provides a criterion for singularity (and equivalence) of two probability measures. It is known [19] Proposition 2.20] that $H(\mu, \nu) = 0$ if

\footnote{For example, let $\mu_N$ denote the law of the random variable $\frac{1}{\sqrt{N}}(Y_1 + \ldots + Y_N)$, where $Y_i, i \in \mathbb{N}$, are i.i.d. random variables with $P(Y_1 = 1) = P(Y_1 = -1) = \frac{1}{2}$. Then, the classical central limit theorem asserts that $\mu_N$ converges weakly to the standard Gaussian measure on $\mathbb{R}$, while due to the discrete nature of $\mu_N$, its total variation distance from the standard Gaussian measure is always one.}
and only if $\mu$ and $\nu$ are mutually singular. Thus, for $\mu$ and $\nu$ to be equivalent, we must have $H(\mu, \nu) > 0$. In fact, when $\mu$ and $\nu$ are product measures on $(\mathbb{R}^\infty, \mathcal{B}_{\mathbb{R}^\infty})$, the condition $H(\mu, \nu) > 0$ is also sufficient (Kakutani’s theorem). See Theorem 2.7 in [13].

**Lemma 3.1.9.** Let $\{\mu_n\}_{n \in \mathbb{N}}$ and $\{\nu_n\}_{n \in \mathbb{N}}$ be two sequences of probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $\mu_n$ and $\nu_n$ are equivalent for any $n \in \mathbb{N}$. Let $\mu = \otimes_{n \in \mathbb{N}} \mu_n$ and $\nu = \otimes_{n \in \mathbb{N}} \nu_n$. Then, we have $H(\mu, \nu) = \prod_{n \in \mathbb{N}} H(\mu_n, \nu_n)$ and

- $H(\mu, \nu) > 0$ if and only if $\mu$ and $\nu$ are equivalent. In this case, we have

$$\frac{d\mu}{d\nu} = \prod_{n \in \mathbb{N}} \frac{d\mu_n}{d\nu_n},$$

(3.1.36)

- $H(\mu, \nu) = 0$ if and only if $\mu$ and $\nu$ are mutually singular.

With the notations as above, we introduce the *Hellinger distance* $d_H$ of $\mu$ and $\nu$ by setting

$$d_H(\mu, \nu) = \left( \frac{1}{2} \int_E \left( \sqrt{\frac{d\mu}{d\lambda}} - \sqrt{\frac{d\nu}{d\lambda}} \right)^2 d\lambda \right)^{\frac{1}{2}},$$

(3.1.37)

where $H(\mu, \nu)$ is the Hellinger integral defined in (3.1.35). It is clear that $0 \leq d_H(\mu, \nu) \leq 1$. We state Le Cam’s inequality, relating the total variation distance and Hellinger distance; see Lemma 2.3 in [160]².

**Lemma 3.1.10.** Let $d_{TV}$ and $d_H$ be as in (3.1.33) and (3.1.37), respectively. Then, we have

$$(d_H(\mu, \nu))^2 \leq d_{TV}(\mu, \nu) \leq \sqrt{2} \cdot d_H(\mu, \nu)$$

for any probability measures $\mu$ and $\nu$ on a measurable space $(E, \mathcal{E})$. In particular, a sequence $\{\mu_k\}_{k \in \mathbb{N}}$ of probability measures on $(E, \mathcal{E})$ converges to some limit $\mu$ in total variation if and only if it converges to the same limit in the Hellinger distance.

In the remaining part of the paper, we do not make use of the Hellinger distance. We, however, decided to introduce it here due to its connection to the total variation distance and also to the fact that Hellinger integral plays an important role in the proof of Lemma 3.2.2. See also Remark 3.2.3 (iii).

**• Kullback-Leibler divergence (= relative entropy).**

We now define the Kullback-Leibler divergence $d_{KL}(\mu, \nu)$ between $\mu$ and $\nu$ by setting

$$d_{KL}(\mu, \nu) = \begin{cases} \int_E \log \frac{d\mu}{d\nu} \, d\mu, & \text{if } \mu \ll \nu, \\ \infty, & \text{otherwise}, \end{cases}$$

(3.1.38)

which is nothing but the relative entropy of $\mu$ with respect to $\nu$. While the total variation distances and the Hellinger distance are metrics, the Kullback-Leibler divergence is not a metric. For example, $d_{KL}(\cdot, \cdot)$ is not symmetric, and moreover, the symmetrized version $d_{KL}(\mu, \nu) + d_{KL}(\nu, \mu)$ is not a metric, either. If $\mu$ and $\nu$ are product measures of the form $\mu = \otimes_{n \in \mathbb{N}} \mu_n$ and $\nu = \otimes_{n \in \mathbb{N}} \nu_n$, then we have

$$d_{KL}(\mu, \nu) = \sum_{n \in \mathbb{N}} d_{KL}(\mu_n, \nu_n).$$

(3.1.39)

The following lemma shows that convergence in the Kullback-Leibler divergence (or in relative entropy) implies convergence in total variation and in the Hellinger distance. See Lemmas 2.4 and 2.5 in [160] for the proof.

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²Note a slightly different multiplicative constant in the definition of the Hellinger distance in [160].
Lemma 3.1.11. Let $d_{TV}$, $d_H$, and $d_{KL}$ be as in (3.1.33), (3.1.37), and (3.1.38), respectively. Then, we have

\[ d_H(\mu, \nu) \leq \frac{\sqrt{d_{KL}(\mu, \nu)}}{\sqrt{2}} \]  

(3.1.40)

and

\[ d_{TV}(\mu, \nu) \leq \frac{\sqrt{d_{KL}(\mu, \nu)}}{\sqrt{2}}. \]  

(3.1.41)

The second inequality (3.1.41) is known as Pinsker’s inequality and it is slightly stronger than $d_{TV}(\mu, \nu) \leq \sqrt{d_{KL}(\mu, \nu)}$, which follows from Lemma 3.1.10 and (3.1.40).

- Weak convergence the Lévy-Prokhorov metric.

Finally, let us introduce the Lévy-Prokhorov metric for probability measures on a separable metric space $(\mathcal{M}, d)$. Given $\varepsilon > 0$, we define an $\varepsilon$-neighborhood of a measurable subset $A \subset \mathcal{M}$ by

\[ A^\varepsilon := \{ z \in \mathcal{M} : d(z, x) < \varepsilon \text{ for some } x \in \mathcal{M} \}. \]

Given two probability measures $\mu$ and $\nu$ on $\mathcal{M}$, their Lévy-Prokhorov distance $d_{LP}(\mu, \nu)$ is defined by

\[ d_{LP}(\mu, \nu) := \inf \{ \varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \} \text{ for all measurable } A \subset \mathcal{M}. \]  

(3.1.42)

Note that the Lévy-Prokhorov metric is indeed a metric on the space of probability measures on $\mathcal{M}$. It is known that the Lévy-Prokhorov metric induces the same topology as the topology for weak convergence. Together with this property, we only need one additional property of the Lévy-Prokhorov metric in this paper, that is, the triangle inequality; see (3.4.33) below. See [16, 48] and [8, Section 30.3] for a further discussion.

Lastly, we recall the Prokhorov theorem and the Skorokhod representation theorem.

Definition 3.1.12. Let $\mathcal{J}$ be any nonempty index set. A family $\{\rho_i\}_{i \in \mathcal{J}}$ of probability measures on a metric space $\mathcal{M}$ is said to be tight if, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathcal{M}$ such that $\sup_{i \in \mathcal{J}} \rho_i(K_\varepsilon) \leq \varepsilon$. We say that $\{\rho_i\}_{i \in \mathcal{J}}$ is relatively compact, if every sequence in $\{\rho_i\}_{i \in \mathcal{J}}$ contains a weakly convergent subsequence.

Note that the index set $\mathcal{J}$ does not need to be countable. We now recall the following Prokhorov theorem from [16, 8].

Lemma 3.1.13 (Prokhorov theorem). If a sequence of probability measures on a metric space $\mathcal{M}$ is tight, then it is relatively compact. If in addition, $\mathcal{M}$ is separable and complete, then relative compactness is equivalent to tightness.

Lastly, we recall the following Skorokhod representation theorem from [8, Chapter 31].

Lemma 3.1.14 (Skorokhod representation theorem). Let $\mathcal{M}$ be a complete separable metric space (i.e., a Polish space). Suppose that probability measures $\{\rho_n\}_{n \in \mathbb{N}}$ on $\mathcal{M}$ converges weakly to a probability measure $\rho$ as $n \to \infty$. Then, there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and random variables $X_n, X : \Omega \to \mathcal{M}$ such that

\[ \mathcal{L}(X_n) = \rho_n \quad \text{and} \quad \mathcal{L}(X) = \rho, \]

and $X_n$ converges $\mathbb{P}$-almost surely to $X$ as $n \to \infty$.

3.2 Gibbs measures in the deep-water regime

In this section, we go over the construction of the Gibbs measures for the gILW equation (1.3.2), including the gBO case ($\delta = \infty$), and prove convergence of the Gibbs measures in the deep-water
limit (as \( \delta \to \infty \)). Recall from the discussions that in deep-water regime, we work with the formulation (3.0.1) of gILW:

\[
\partial_t u - G_\delta(\partial_x^2 u) + \partial_x(u^k) = 0.
\]

Recall the base Gaussian measure for gBO dynamics (3.0.3): The Gaussian measure \( \mu_{gBO} \), formally given by

\[
\mu_{gBO}(du) = Z^{-1} e^{-\frac{1}{2} \int du u \mathcal{H}_0 u} du,
\]

can be realized as the probability measure induced by the random Fourier series

\[
X_{gBO} = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{|n|^\frac{1}{2}} e^{inx},
\]

where \( \{g_n : n \in \mathbb{Z}\} \) is the Gaussian sequence as before. Also, let us recall from (1.3.32) that Gaussian measure for gILW dynamics in the deep water region

\[
\mu_\delta(du) = Z^{-1} e^{-\frac{1}{2} \int du \mathcal{H}_\delta u} du
\]

is the probability measure induced by

\[
X_\delta(\omega) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega) e_n}{|K_\delta(n)|^\frac{1}{2}},
\]

where \( K_\delta(n) = n \coth(\delta n) - \frac{1}{\delta} \) is strictly positive for \( n \in \mathbb{Z}^* \).

As mentioned in the introduction, we construct the Gibbs measure as a weighted Gaussian measure, where the base Gaussian measure is given by \( \mu_\delta \) in (1.3.12) with the understanding that it is given by \( \mu_\infty \) in (1.3.28) when \( \delta = \infty \). For \( 0 < \delta < \infty \), let \( K_\delta(n) \) be as in (3.1.24). We extend the definition of \( K_\delta(n) \) to \( \delta = \infty \) by setting

\[
K_\infty(n) = |n|,
\]

which is consistent with Lemma 3.1.1. Then, a typical element under the Gaussian measure \( \mu_\delta \) in (1.3.12) (and \( \mu_\infty \) in (1.3.28)) is given by \( X_\delta \) in (1.3.32) when \( 0 < \delta < \infty \) and \( X_\infty := X_{gBO} \) in (1.3.29) when \( \delta = \infty \). It is easy to see that, given \( 0 < \delta \leq \infty \), \( X_\delta \in H^{-\varepsilon}(\mathbb{T}) \setminus L^2(\mathbb{T}) \) for any \( \varepsilon > 0 \), almost surely. Indeed, from Lemma 3.1.1 we have \( K_\delta(n) \sim |n| \). Hence, with \( X_{\delta,N} = P_N X_\delta \) in (3.1.24), it follows from Lemma 3.1.7 that there exists \( C_\delta > 0 \) such that, for any finite \( p \geq 1 \),

\[
\|X_{\delta,N}\|_{L_p^\infty H^{-\varepsilon}_p} \leq p^{\frac{1}{2}} \|\nabla\|^{-\varepsilon} X_{\delta,N}(x)\|_{L_p^2 L_2^2} \leq C_\delta p^{\frac{1}{2}} \left( \sum_{0 < |n| \leq N} \frac{1}{|n|^{1+2\varepsilon}} \right)^{\frac{1}{2}} \sim C_\delta p^{\frac{1}{2}},
\]

uniformly in \( N \in \mathbb{N} \), provided that \( \varepsilon > 0 \). A similar computation together with the Borel-Cantelli lemma shows that \( X_{\delta,N} \) converges, in \( L^p(\Omega) \) and almost surely, to the limit \( X_\delta \) in \( H^{-\varepsilon}(\mathbb{T}) \) for any \( \varepsilon > 0 \). The fact that \( X_\delta \notin L^2(\mathbb{T}) \) almost surely follows from Lemma B.1 in [31].

In Subsection 3.2.1 we first study various properties of the base Gaussian measures \( \mu_\delta \). See Proposition 3.2.1. By restricting our attention to the defocusing case \( (k \in 2\mathbb{N} + 1) \), we then go over the construction of the Gibbs measures in Subsection 3.2.2. In Subsection 3.2.3, we continue to study the defocusing case and establish convergence in total variation of the Gibbs measure \( \rho_\delta \) to \( \rho_{BO} \) in the deep-water limit \( (\delta \to \infty) \). Finally, in Subsection 3.2.4, we present the proof of Theorem 1.3.2 when \( k = 2 \).

First, we establish the fact that the base Gaussian measure \( \mu_\delta \) for gILW of the deep water formulation is equivalent to the base Gaussian measure \( \mu_{gBO} \) for gBO (3.0.3). In particular, these two Gaussian measures (hence the associated Gibbs measures) share the same support.
3.2.1 Equivalence of the base Gaussian measures

**Proposition 3.2.1.** (i) Let $X_1$ and $X_{gBO}$ be as in (1.3.32) and (1.3.29), respectively. Then, given any $\varepsilon > 0$ and finite $p \geq 1$, $X_1$ converges to $X_{gBO}$ in $L^p(\Omega; H^{-\varepsilon}(\mathbb{T}))$ and in $H^{-\varepsilon}(\mathbb{T})$ almost surely, as $\delta \to \infty$. In particular, the Gaussian measure $\mu_\delta$ in (1.3.32) converges weakly to the Gaussian measure $\mu_{gBO}$ in (1.3.28), as $\delta \to \infty$.

(ii) For any $0 < \delta < \infty$, the Gaussian measures $\mu_\delta$ and $\mu_{gBO}$ are equivalent.

(iii) As $\delta \to \infty$, the Gaussian measure $\mu_\delta$ converges to $\mu_{gBO}$ in the Kullback-Leibler divergence defined in (3.1.38). In particular, $\mu_\delta$ converges to $\mu_{gBO}$ in total variation.

Part (iii) of Proposition 3.2.1 plays an essential role in establishing convergence in total variation of the Gibbs measure $\rho_\delta$ to $\rho_{gBO}$ in the deep-water limit ($\delta \to \infty$).

In proving Part (ii) of Proposition 3.2.1 we resort to the following Kakutani’s theorem [75] in the Gaussian setting (or the Feldman-Hajek theorem [51, 65]; see also [78, Theorem 2.9]). See, for example, [32, 138, 140, 61], where Kakutani’s theorem was used in the study of dispersive PDEs.

In particular, see also Proposition B.1 in [32].

**Lemma 3.2.2.** Let $\{A_n\}_{n \in \mathbb{Z}^*}$ and $\{B_n\}_{n \in \mathbb{Z}^*}$ be two sequences of independent, real-valued, mean-zero Gaussian random variables with $\mathbb{E}[A_n^2] = a_n > 0$ and $\mathbb{E}[B_n^2] = b_n > 0$ for all $n \in \mathbb{N}$. Then, the laws of the sequences $\{A_n\}_{n \in \mathbb{Z}^*}$ and $\{B_n\}_{n \in \mathbb{Z}^*}$ are equivalent if and only if

$$
\sum_{n \in \mathbb{Z}^*} \left( \frac{a_n}{b_n} - 1 \right)^2 < \infty.
$$

(3.2.6)

If they are not equivalent, then they are singular.

We first present a short proof of Lemma 3.2.2 based on Lemma 3.1.9. See also the proof of Theorem 2.9 in [75].

**Proof of Lemma 3.2.2.** Given $n \in \mathbb{Z}^*$, let $\mu_n$ and $\nu_n$ denote the laws of $A_n$ and $B_n$, respectively, and set $\mu = \bigotimes_{n \in \mathbb{Z}^*} \mu_n$ and $\nu = \bigotimes_{n \in \mathbb{Z}^*} \nu_n$. Namely, $\mu$ and $\nu$ are the laws of the sequences $\{A_n\}_{n \in \mathbb{Z}^*}$ and $\{B_n\}_{n \in \mathbb{Z}^*}$, respectively. The Hellinger integral $H(\mu, \nu)$ defined in (3.1.35) is given by an infinite product:

$$
H(\mu, \nu) = \prod_{n \in \mathbb{Z}^*} H(\mu_n, \nu_n) = \prod_{n \in \mathbb{Z}^*} \int_{\mathbb{R}} \sqrt{\frac{d\mu_n}{d\nu_n}} \frac{1}{\sqrt{2\pi(a_n b_n)^{\frac{3}{2}}}} e^{-\frac{1}{2}(\frac{x}{a_n} + \frac{x}{b_n})^2} dx
$$

$$
= \prod_{n \in \mathbb{Z}^*} \frac{\sqrt{2} (a_n b_n)^{\frac{3}{4}}}{\sqrt{a_n + b_n}}.
$$

Thus, we have

$$
(H(\mu, \nu))^4 = \prod_{n \in \mathbb{Z}^*} \frac{4a_n b_n}{(a_n + b_n)^2} = \prod_{n \in \mathbb{Z}^*} \left( 1 - \frac{(a_n - b_n)^2}{(a_n + b_n)^2} \right) = \prod_{n \in \mathbb{Z}^*} \left( 1 - \left( \frac{a_n}{b_n} - 1 \right)^2 \right).
$$

Hence, $H(\mu, \nu) > 0$ if and only if

$$
\sum_{n \in \mathbb{Z}^*} \frac{(a_n - b_n)^2}{(a_n + b_n)^2} = \sum_{n \in \mathbb{Z}^*} \left( \frac{a_n}{b_n} - 1 \right)^2 \left( \frac{a_n}{b_n} + 1 \right)^2 < \infty.
$$

(3.2.7)

Note that the condition (3.2.7) is equivalent to the condition (3.2.6), since if one of the sums in (3.2.6) or (3.2.7) converges, then $\frac{a_n}{b_n}$ must tend to 1 as $n \to \infty$, which implies the other sum also converges. Then, the desired conclusion follows from Lemma 3.1.9.

We now present the proof of Proposition 3.2.1.
Proof of Proposition 3.2.1. (i) Let \( \varepsilon > 0 \) and fix finite \( p \geq 1 \). Then, it follows from Lemma 3.1.7 (1.3.29), (1.3.32), and \( \sqrt{a} - \sqrt{b} \leq \sqrt{a - b} \) for any \( a \geq b \geq 0 \) together with (3.1.11) in Lemma 3.1.11 that

\[
\|X_\delta - X_{\delta BO}\|_{L^p_H} \lesssim \|\langle \nabla \rangle^{-\varepsilon}(X_\delta - X_{\delta BO})(x)\|_{L^q_L^q} \sim \left( \sum_{n \in \mathbb{Z}^*} \frac{1}{|n|^{2\varepsilon}} \left| \frac{n}{K_\delta(n)} \right|^2 \right)^{\frac{1}{2}} \leq \frac{C_{\varepsilon,\omega}}{\delta} \sum_{n \in \mathbb{Z}^*} \frac{(n)^{2\varepsilon}}{|n|^{1+2\varepsilon}(|n| - \frac{1}{2})} \to 0,
\]

as \( \delta \to \infty \). See also (3.1.3) for the penultimate step in (3.2.8).

As for the almost sure convergence, we repeat a computation analogous to (3.2.8) but with (3.1.19) in place of \( E[|g_n|^2] \sim 1 \). Then, together with Lemma 3.1.1 (for \( \delta \geq 2 \)), we have

\[
\|X_\delta(\omega) - X_{\delta BO}(\omega)\|^2 \lesssim \frac{C_{\varepsilon,\omega}}{\delta} \sum_{n \in \mathbb{Z}^*} \frac{(n)^{2\varepsilon}}{|n|^{1+2\varepsilon}(|n| - \frac{1}{2})} \to 0,
\]

as \( \delta \to \infty \), provided that \( 0 < \varepsilon_0 < \varepsilon \). Recalling that \( \mu_\delta \) and \( \mu_{\delta BO} \) are the laws of \( X_\delta \) and \( X_{\delta BO} \), we conclude weak convergence of \( \mu_\delta \) to \( \mu_{\delta BO} \). This proves (i).

(ii) Rewrite \( X_\delta \) in (1.3.32) (and in 1.3.29) when \( \delta = \infty \) with the understanding (3.2.4) as

\[
X_\delta(\omega) = \sum_{n \in \mathbb{N}} \left( \frac{\text{Re } g_n}{\pi K_\delta^2(n)} \cos(nx) - \frac{\text{Im } g_n}{\pi K_\delta^2(n)} \sin(nx) \right).
\]

For \( n \in \mathbb{Z}^* \), set

\[
A_n = \frac{\text{Re } g_n}{\pi K_\delta^2(n)} \quad \text{and} \quad A_{-n} = -\frac{\text{Im } g_n}{\pi K_\delta^2(n)},
\]

and

\[
B_n = \frac{\text{Re } g_n}{\pi |n|^{\frac{3}{2}}} \quad \text{and} \quad B_{-n} = -\frac{\text{Im } g_n}{\pi |n|^{\frac{3}{2}}} \quad \text{for } \delta = \infty
\]

with \( a_{\pm n} = E[A_n^2] = \frac{1}{\pi K_\delta(n)} \) and \( b_n = E[B_n^2] = \frac{1}{\pi |n|} \). Then, from Lemma 3.1.1 we have

\[
\sum_{n \in \mathbb{Z}^*} \left( \frac{a_n}{b_n} - 1 \right)^2 \leq \sum_{n \in \mathbb{Z}^*} \frac{(|n| - K_\delta(n))^2}{K_\delta^2(n)} \leq C_\delta \sum_{n \in \mathbb{Z}^*} \frac{1}{n^2} < \infty.
\]

Therefore, the claimed equivalence of \( \mu_\delta \) and \( \mu_\infty \) follows from Kakutani's theorem (Lemma 3.2.2).

(iii) In this part, we prove that \( \mu_\delta \) converges to \( \mu_{\delta BO} \) in the Kullback-Leibler divergence defined in (3.1.38). Once this is achieved, convergence in total variation follows from Pinsker's inequality (3.1.41) in Lemma 3.1.11.

Let us first write \( \mu_\delta \), \( 0 < \delta \leq \infty \), as the product of Gaussian measures on \( \mathbb{R} \) (see also 1.3.22):

\[
d\mu_\delta = \left( \bigotimes_{n \in \mathbb{N}} K_\delta^2(n) e^{-\frac{i}{2} K_\delta(n)(\text{Re } \bar{u}(n))^2} d\text{Re } \bar{u}(n) \right) \times \left( \bigotimes_{n \in \mathbb{N}} K_\delta^2(n) e^{-\frac{i}{2} K_\delta(n)(\text{Im } \bar{u}(n))^2} d\text{Im } \bar{u}(n) \right).
\]
with the identification \((3.2.4)\) when \(\delta = \infty\). With \(x = (x_1, x_2) \in \mathbb{R}^2\), we then have
\[
d\mu_\delta = \bigotimes_{n \in \mathbb{N}} K_\delta(n) \frac{e^{-\frac{1}{2\pi}K_\delta(n)|x|^2}}{2\pi} dx.
\]
(3.2.10)

In Part (ii), we proved the equivalence of \(\mu_\delta\) and \(\mu_\infty\). From \((3.2.8)\), we see that the Radon-Nikodym derivative \(\frac{d\mu_\delta}{d\mu_\infty}\) is given by
\[
d\mu_\delta = \bigotimes_{n \in \mathbb{N}} K_\delta(n) \frac{e^{-\frac{1}{2\pi}K_\delta(n)|x|^2}}{2\pi} dx.
\]
(3.2.11)

See \((3.1.36)\). Then, from \((3.1.38)\) and \((3.1.39)\) with \((3.2.10)\) and \((3.2.11)\), we have
\[
d_{KL}(\mu_\delta, \mu_\infty) = \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^2} \left( \log \frac{K_\delta(n)}{n} + \frac{1}{2\pi} (n - K_\delta(n))^2 \right) \frac{K_\delta(n)}{2\pi} e^{-\frac{1}{2\pi}K_\delta(n)|x|^2} dx
\]
\[
= \sum_{n \in \mathbb{N}} \left( \log \frac{K_\delta(n)}{n} + \frac{1}{2\pi} (n - K_\delta(n))^2 \int_{\mathbb{R}^2} \frac{K_\delta(n)}{2\pi} e^{-\frac{1}{2\pi}K_\delta(n)|x|^2} dx \right)
\]
\[
= \sum_{n \in \mathbb{N}} \phi \left( \frac{n}{K_\delta(n)} \right),
\]
(3.2.12)

where \(\phi(t) := t - 1 - \log t\). Note that \(\phi(1) = 0\) and \(\phi'(t) > 0\) for \(t > 1\). Then, it follows from Lemma \(3.1.1\) that for each fixed \(n \in \mathbb{N}\), we have
\[
\phi \left( \frac{n}{K_\delta(n)} \right) \text{ decreases to } \phi(1) = 0,
\]
(3.2.13)
as \(\delta \to \infty\), since \(\frac{n}{K_\delta(n)}\) decreases to \(1\) as \(\delta \to \infty\). Hence, if the right-hand side of \((3.2.12)\) is finite for some \(\delta \gg 1\), then the observation \((3.2.13)\) allows us to apply the dominated convergence theorem and conclude
\[
\lim_{\delta \to \infty} d_{KL}(\mu_\delta, \mu_\infty) = \lim_{\delta \to \infty} \sum_{n \in \mathbb{N}} \phi \left( \frac{n}{K_\delta(n)} \right) = \sum_{n \in \mathbb{N}} \lim_{\delta \to \infty} \phi \left( \frac{n}{K_\delta(n)} \right) = 0,
\]
yielding the desired convergence in the Kullback-Leibler divergence.

It remains to check that the right-hand side of \((3.2.12)\) is finite for some \(\delta \gg 1\). In fact, we show that the right-hand side of \((3.2.12)\) is finite for any \(\delta > 0\). By a direct computation, we have \(\phi(t) \leq (t - 1)^2\) for \(t \geq 1\). Then, from Lemma \(3.1.1\) we have
\[
\sum_{n \in \mathbb{N}} \phi \left( \frac{n}{K_\delta(n)} \right) \leq \sum_{n \in \mathbb{N}} \frac{(n - K_\delta(n))^2}{K_\delta^2(n)} \leq C_\delta \sum_{n \in \mathbb{N}} \frac{1}{n^2} < \infty
\]
for any \(\delta > 0\). This concludes the proof of Proposition \(3.2.1\). \(\square\)

**Remark 3.2.3.** (i) By using the Wiener chaos estimate (Lemma \(3.1.7\), Chebyshev’s inequality, and the Borel-Cantelli lemma, one can easily upgrade the convergence of \(X_\delta\) to \(X_{\delta BO}\) to that in \(L^2(\Omega; W^{-\varepsilon, \infty}(T))\) and in \(W^{-\varepsilon, \infty}(T)\) almost surely.

(ii) From \((3.2.8)\), we see that the difference \(X_\delta - X_{\delta BO}\) lives in \(H^{\frac{1}{2} - \varepsilon}(T)\) although neither \(X_\delta\) nor \(X_{\delta BO}\) belongs to \(L^2(T)\).

(iii) In order to prove convergence of \(\mu_\delta\) to \(\mu_\infty\) in total variation, it is indeed possible to directly show that \(\mu_\delta\) converges to \(\mu_\infty\) in the Hellinger distance \(d_H\) defined in \((3.1.37)\) and invoke Lemma \(3.1.10\). \footnote{In fact, in \(W^{\frac{1}{2} - \varepsilon, \infty}(T)\) if we use the Wiener chaos estimate (Lemma \(3.1.7\).}
3.2.2 Construction of the Gibbs measure for the defocusing gILW equation

In this subsection, we present the construction of the Gibbs measure for the gILW equation (1.3.2), 0 ≤ δ ≤ ∞ with the understanding that the δ = ∞ case corresponds to the gBO (1.1.1), in the defocusing case: k ∈ 2N + 1. We treat the k = 2 case, corresponding to the ILW equation (1.1.1), in Subsection 3.2.7 Our basic strategy is to follow the argument presented in [139] on the construction of the complex Ψ_{k+1} measures, by utilizing the Wiener chaos estimate (Lemma 3.1.7) and Nelson’s estimate. In order to establish convergence of the Gibbs measures in the deep-water limit (δ → ∞), however, we need to establish an L^p(Ω)-integrability of the (truncated) densities, uniformly in both the frequency-truncation parameter N ∈ N and the depth parameter δ ≫ 1. See Proposition 3.2.6. This uniform bound also plays a crucial role in the dynamical part presented in Section 3.4.

Fix the depth parameter 0 ≤ δ ≤ ∞. Given N ∈ N, let X_{δ,N} = P_N X_δ, where X_δ is defined in (1.3.32):

\[ X_{δ,N}(ω) := P_N X_δ(ω) = \frac{1}{2\pi} \sum_{0 < |n| \leq N} \frac{g_n(ω)}{K_δ^N(n)} e_n \]

with the identification (3.2.4) when δ = ∞. When δ = ∞, we also set

\[ X_δ \in BO := X_{∞,N} = P_N X_∞ = P_N X_δ \in BO, \]

where X_δ is as in (1.3.29). Given k ∈ N, let \( W(X_{δ,N}^k) = H_k(X_{δ,N}; \sigma_{δ,N}) \) denotes the Wick power defined in (1.3.36), where \( \sigma_{δ,N} \) is as in (1.3.35). Then, the truncated Gibbs measure \( ρ_{δ,N} \) in (3.2.37) can be written as

\[ \rho_{δ,N}(A) = Z_{δ,N}^{-1} \int_{H^∞} 1_{\{u ∈ A\}} e^{\frac{1}{\delta} \frac{1}{2} \int_{T} W(u_N^{k+1}) dx} dμ_δ(u) \]

\[ = Z_{δ,N}^{-1} \int_{H^∞} 1_{\{X_δ(ω) ∈ A\}} e^{\frac{1}{\delta} \frac{1}{2} \int_{T} W(X_{δ,N}^{k+1}(ω)) dx} dμ(ω) \]  

(3.2.14)

for any measurable set \( A \subset H^∞(T) \) with some small \( ε > 0 \), where \( u_N = P_N u_∞ \). In the following, we freely interchange the representations in terms of \( X_δ \) and in terms of \( u_∞ \) distributed by \( μ_δ \), when there is no confusion. Let us first construct the limiting Wick power \( W(X_{δ,N}^k) \) and the related stochastic objects.

Proposition 3.2.4. Let \( k ∈ N \) and 0 ≤ δ ≤ ∞. Given \( N ∈ N \), let \( W(X_{δ,N}^k) \) be as in (1.3.36). Then, given any finite \( p ≥ 1 \), the sequence \( \{W(X_{δ,N}^k)\}_{N ∈ N} \) is Cauchy in \( \text{L}^p(Ω; W^{s,∞}(T)) \), \( s < 0 \), thus converging to a limit, denoted by \( W(X_{δ,N}^k) \). This convergence of \( W(X_{δ,N}^k) \) to \( W(X_{δ,N}^k) \) also holds almost surely in \( W^{s,∞}(T) \). Furthermore, given any finite \( p ≥ 1 \), we have

\[ \sup_{N ∈ N} \sup_{2 ≤ δ ≤ ∞} |||W(X_{δ,N}^k)|||_{L^p(Ω)} < ∞ \]

(3.2.15)

and

\[ \sup_{2 ≤ δ ≤ ∞} |||W(X_{δ,N}^k) - W(X_{δ,N}^k)|||_{L^p(Ω)} \rightarrow 0 \]

(3.2.16)

for any \( M ≥ N \), tending to ∞. In particular, the rate of convergence is uniform in \( 2 ≤ δ ≤ ∞ \).

As a corollary, the following two statements hold.

(i) Let 0 < δ ≤ ∞. Given \( N ∈ N \), let \( R_{δ,N}(u) = R_{δ,N}(u; k + 1) \) denotes the truncated potential energy defined by

\[ R_{δ,N}(u) := \frac{1}{k + 1} \int_{T} W((P_N u)^{k+1}) dx = \frac{1}{k + 1} \int_{T} H_{k+1}(P_N u; \sigma_{δ,N}) dx, \]

(3.2.17)

where \( \sigma_{δ,N} \) is as in (1.3.35) with the identification (3.2.4) when δ = ∞; see \( \sigma_{∞,N} \) in (3.2.52).
Then, given any finite \( p \geq 1 \), the sequence \( \{R_{\delta,N}(u)\}_{N \in \mathbb{N}} \) converges to the limit:

\[
R_{\delta}(u) = \frac{1}{k+1} \int_{\mathbb{T}} W(u^{k+1})dx = \lim_{N \to \infty} \frac{1}{k+1} \int_{\mathbb{T}} W(P_Nu^{k+1})dx
\]

in \( L^p(d\mu_{\delta}) \), as \( N \to \infty \). Furthermore, there exists \( \theta > 0 \) such that given any finite \( p \geq 1 \), we have

\[
\sup_{N \in \mathbb{N}, |(u)| = N} \sup_{2 \leq \delta \leq \infty} \|R_{\delta,N}(u)\|_{L^p(d\mu_{\delta})} < \infty,
\]

with \( R_{\delta,\infty}(u) = R_{\delta}(u) \), and

\[
\|R_{\delta,M}(u) - R_{\delta,N}(u)\|_{L^p(d\mu_{\delta})} \leq \frac{C_{k,\delta} p^{k+1}}{N^\theta}
\]

for any \( M \geq N \geq 1 \). For \( 2 \leq \delta \leq \infty \), we can choose the constant \( C_{k,\delta} \) in \( (3.2.20) \) to be independent of \( \delta \) and hence the rate of convergence of \( R_{\delta,N}(u) \) to the limit \( R_{\delta}(u) \) is uniform in \( 2 \leq \delta \leq \infty \).

(ii) Let \( 0 < \delta \leq \infty \). Given \( N \in \mathbb{N} \), let \( F_N(u) = F_N(u; k) \) be the truncated renormalized nonlinearity in \( (1.3.63) \) given by

\[
F_N(u) := \partial_s P_N W((P_Nu)^k) = \partial_s P_N H_k(P_Nu; \sigma_N),
\]

where \( \sigma_N \) is as in \( (1.3.35) \) with the identification \( (3.2.4) \) when \( \sigma = \infty \); see \( \sigma_{\infty,N} \) in \( (3.2.52) \). Then, given any finite \( p \geq 1 \), the sequence \( \{F_N(u)\}_{N \in \mathbb{N}} \) is Cauchy in \( L^p(d\mu_{\delta}; H^s(\mathbb{T})) \), \( s < -1 \), thus converging to a limit denoted by \( F(u) = \partial_s W(u^k) \). Furthermore, given any finite \( p \geq 1 \), we have

\[
\sup_{N \in \mathbb{N}, |(u)| = N} \sup_{2 \leq \delta \leq \infty} \|F_N(u)\|_{H^s_{L^p}} < \infty,
\]

with \( F_{\infty}(u) = F(u) \), and

\[
\sup_{2 \leq \delta \leq \infty} \|F_M(u) - F_N(u)\|_{H^s_{L^p}} \to 0
\]

for any \( M \geq N \), tending to \( \infty \). In particular, the rate of convergence of \( F_N(u) \) to the limit \( F(u) \) is uniform in \( 2 \leq \delta \leq \infty \).

Remark 3.2.5. In the proof of Proposition \( 3.2.4 \) we use \( 3.1.3 \) to obtain a lower bound on \( K_\delta(n) \), uniformly in \( 2 \leq \delta \leq \infty \), for any fixed \( n \in \mathbb{Z}^* \). The lower bound \( \delta = 2 \) is by no means sharp. For example, in view of the strict monotonicity of \( K_\delta(n) \) in \( \delta \geq 1 \) (for fixed \( n \in \mathbb{Z}^* \)) and the fact that \( K_\delta(n) \neq 0 \) for \( n \in \mathbb{Z}^* \) as stated in Lemma \( 3.1.1 \), a slight modification of the proof of Proposition \( 3.2.4 \) yields the uniform (in \( \delta \)) bounds for \( 1 \leq \delta \leq \infty \). Since our main interest is to take the limit \( \delta \to \infty \), we do not attempt to optimize a lower bound for \( \delta \). The same comment applies to the subsequent results presented in this section and hence to Theorem \( 1.3.2 \).

Proof of Proposition \( 3.2.4 \). Given \( N \in \mathbb{N} \) and \( x, y \in \mathbb{T} \), we set

\[
\gamma_N(x - y) := E[X_{\delta,N}(x)X_{\delta,N}(y)] = \frac{1}{4\pi^2} \sum_{0 < |n| \leq N} \mathbb{E}\left[ \frac{g_n g_m}{(K_{\delta}(n)K_{\delta}(m))^{\frac{3}{2}}} \right] e_n(x)e_m(y)
\]

\[
= \frac{1}{2\pi} \sum_{0 < |n| \leq N} \frac{e_n(x - y)}{K_{\delta}(n)}.
\]

Here, \( X_{\delta,N}(x) \) is a random variable, which is a real-valued function defined on a sample space, and \( g_n \) is an independent standard \( \mathbb{C} \)-valued Gaussian random variable with \( g_{-n} = \overline{g_n} \). Then, the last
equality of (3.2.24) follows from the mean-zero of \( g_n \),

\[
\mathbb{E}[g_n g_m] = \begin{cases} 0, & \text{if } n \neq m; \\ \mathbb{E}[g_n g_m] = \mathbb{E}[|g_n|^2] = 2\pi & \text{if } n = -m. 
\end{cases}
\]

Note that we have

\[
\gamma_N(x - y) = \mathbb{E}[X_{\delta,N}(x)X_{\delta,M}(y)]
\]

for any \( M \geq N \geq 1 \). In the following, we set \( u_N = P_N u \).

Let us first make a preliminary computation. Given \( n, m \in \mathbb{Z}^* \), we have

\[
\mathbb{E} \left[ F(H_k (X_{\delta,N}; \delta_N, \delta_M))(n) \right] F(H_k (X_{\delta,N}; \delta_N, \delta_M))(m) 
= \int_{\mathbb{T}^2} E[H_k(X_{\delta,N}(x); \delta_N, \delta_M)H_k(X_{\delta,N}(y); \delta_N, \delta_M)] e_{-n+m}(x) e_{-m}(x - y) dy dx.
\]

From Lemma 3.1.6 with (3.2.24) we have

\[
\mathbb{E}[H_k(X_{\delta,N}(x); \delta_N, \delta_M)H_k(X_{\delta,N}(y); \delta_N, \delta_M)] = k! \gamma_N^k(y - x).
\]

Then, by integrating first in \( y \) and then in \( x \) and applying Lemma 3.1.6 with (3.2.24), we have

\[
\mathbb{E} \left[ F(H_k (X_{\delta,N}; \delta_N, \delta_M))(n) \right] F(H_k (X_{\delta,N}; \delta_N, \delta_M))(m) 
= k! \int_{\mathbb{T}^2} \gamma_N^k(y - x) e_{-m}(x - y) dy e_{-n+m}(x) dx 
= k! \mathbf{1}_{n = m} \cdot \int_{\mathbb{T}^2} \gamma_N^k(y - x) e_n(y - x) dy dxdy.
\]

Fix small \( \varepsilon > 0 \). Then, by Sobolev’s inequality with finite \( r > 1 \) such that \( r \varepsilon > 1 \), we have

\[
\|W(u_k^N)\|_{W^{s,\infty}} \lesssim \|W(u_k^N)\|_{W^{s+r,\infty}}.
\]

Let \( p \geq r \). Then, by (3.2.26), Minkowski’s inequality (with \( p \geq r \gg 1 \)), the Wiener chaos estimate (Lemma 3.1.7), (3.2.25), and the boundedness of the torus \( \mathbb{T} \), we have

\[
\|W(u_k^N)\|_{W^{s,\infty}} \lesssim \|W(u_k^N)\|_{L^p(\mu)} \overset{\text{p \geq r \gg 1}}{\lesssim} \|\mathbb{E}[\nabla]W(u_k^N)\|_{L^p(\mu)} \lesssim \|\mathbb{E}[\nabla]W(u_k^N)\|_{L^p(\mu)} \lesssim \frac{p \varepsilon}{2\pi} \left( \int_{\mathbb{T}^2} (\mathbb{E}[H_k(X_{\delta,N}; \delta_N, \delta_M))(n) e_n(x) \right)_{L^p(\mu)} \right)^{\frac{p \varepsilon}{2\pi}}.
\]

Note that

\[
\gamma_N^k(y - x) e_n(y - x) dy dx = \frac{1}{(2\pi)^{k-2}} \sum_{0 < |n_j| \leq N} \frac{1_{n_1 + \cdots + n_k}}{\prod_{j=1}^{k} K_{\delta}(n_j)},
\]
Hence, from (3.2.27) and (3.2.28), and Lemma 3.1.1 we obtain

\[
||| \mathcal{W}(u_N^k) |||_{L^p(\mu)} \leq C_{k, \delta} p^{\frac{\varepsilon}{2}} \left( \sum_{0 < |n_j| \leq N} \frac{1}{\prod_{j=1}^{k} \langle n_j \rangle} (n_1 + \ldots + n_k)^{2(s+\varepsilon)} \right)^{\frac{1}{2}} \\
\leq C_{k, \delta} p^{\frac{\varepsilon}{2}} \left( \sum_{n_1, \ldots, n_k \in \mathbb{Z}} \frac{1}{\prod_{j=1}^{k} \langle n_j \rangle} (n_1 + \ldots + n_k)^{2(s+\varepsilon)} \right)^{\frac{1}{2}} < \infty,
\]

(uniformly in \( N \in \mathbb{N} \), provided that \( s + \varepsilon < 0 \). This last condition can be guaranteed for \( s < 0 \) by taking \( \varepsilon > 0 \) sufficiently small. In view of (3.1.3), the bound (3.2.29) holds uniformly in \( 2 \leq \delta \leq \infty \) (namely, the constant \( C_{k, \delta} \) can be chosen to be independent of \( 2 \leq \delta \leq \infty \)). This proves (3.2.15).

Let \( M \geq N \geq 1 \) and \( p \geq 2 \). Proceeding as above, we have

\[
||| \mathcal{W}(u_N^k) - \mathcal{W}(u_N^k) |||_{L^p(\mu)} \leq C_{k, \delta} p^{\frac{\varepsilon}{2}} \left( \sum_{0 < |n_j| \leq M} \frac{1}{\prod_{j=1}^{k} \langle n_j \rangle} (n_1 + \ldots + n_k)^{2(s+\varepsilon)} \right)^{\frac{1}{2}} - \sum_{0 < |n_j| \leq N} \frac{1}{\prod_{j=1}^{k} \langle n_j \rangle} (n_1 + \ldots + n_k)^{2(s+\varepsilon)} \right)^{\frac{1}{2}} \\
\leq C_{k, \delta} p^{\frac{\varepsilon}{2}} \left( \sum_{0 < |n_j| \leq M} \frac{1}{\prod_{j=1}^{k} \langle n_j \rangle} (n_1 + \ldots + n_k)^{2(s+\varepsilon)} \right)^{\frac{1}{2}} \\
\leq C_{k, \delta} p^{\frac{\varepsilon}{2}} N^{\max(s,-\frac{1}{2})+2\varepsilon}
\]

for any \( \varepsilon > 0 \), provided that \( s < 0 \). By choosing \( 0 < 2\varepsilon < \min \left( -s, \frac{1}{2} \right) \), we then obtain

\[
||| \mathcal{W}(u_N^k) - \mathcal{W}(u_N^k) |||_{L^p(\mu)} |||_{L^p(\mu)} \longrightarrow 0,
\]

as \( N \to \infty \). In view of (3.1.3), the bound (3.2.30) holds uniformly in \( 2 \leq \delta \leq \infty \) and thus the convergence in (3.2.31) holds uniformly in \( 2 \leq \delta \leq \infty \), yielding (3.2.16).

By applying Chebyshev’s inequality (see also Lemma 4.5 in [162]), to (3.2.30) (with \( M = \infty \)) and summing over \( N \in \mathbb{N} \) we have

\[
\sum_{N=1}^{\infty} P\left( ||| \mathcal{W}(u_N^k) - \mathcal{W}(u_N^k) |||_{L^p(\mu)} > \frac{1}{j} \right) \leq \sum_{N=1}^{\infty} e^{-cN^{\frac{2}{\max(s,-\frac{1}{2})+2\varepsilon)}} j^{-\frac{2}{\varepsilon}} \\
\leq e^{-c''j^{-\frac{2}{\varepsilon}}} < \infty.
\]

Therefore, we conclude from the Borel-Cantelli lemma that there exists \( \Omega_j \) with \( P(\Omega_j) = 1 \) such that for each \( \omega \in \Omega_j \), there exists \( N_j = N_j(\omega) \in \mathbb{N} \) such that

\[
||| \mathcal{W}(u_N^k)(\omega) - \mathcal{W}(u_N^k)(\omega) |||_{L^p(\mu)} < \frac{1}{j}
\]

for any \( N \geq N_j \). By setting \( \Sigma = \bigcap_{j=1}^{\infty} \Omega_j \), we have \( P(\Sigma) = 1 \). Hence, we conclude that \( \mathcal{W}(u_N^k) \) converges almost surely to \( \mathcal{W}(u_N^k) \) in \( W^{s,\infty}(\mathbb{T}) \).

Let us briefly discuss how to obtain the corollaries (i) and (ii). We only discuss the difference
estimates \(3.2.21\) and \(3.2.23\). The first corollary on \(R_{\delta,N}(u)\) (Part (i)) easily follows from the discussion above (in particular \(3.2.30\) with \(k\) replaced by \(k+1\)) by noting that

\[ |R_{\delta,M}(u) - R_{\delta,N}(u)| \leq C_k \| \mathcal{W}(u_{M}^{k+1}) - \mathcal{W}(u_{N}^{k+1}) \|_H, \]

for any \(s < 0\). We can take \(s = -\frac{1}{2}\) for example.

As for the second corollary on \(F_N(u)\), we just need to note that

\[
\begin{aligned}
&\|F_M(u) - F_N(u)\|_H^p \leq \| (P_M - P_N) \mathcal{W}(u_M^k) \|_H^{k+1} \|_L^p(d\mu_u) \\
&\quad + \| \mathcal{W}(u_M^k) - \mathcal{W}(u_N^k) \|_H^{k+1} \|_L^p(d\mu_u) \\
&=: \infty + \Pi.
\end{aligned}
\]

For \(s < -1\), we can estimate \(\Pi\) in \(3.2.32\) just as in \(3.2.30\). As for the first term \(\infty\) in \(3.2.32\), we note that due to the projection \(P_M - P_N\), we have \(|u| = |n_1 + \cdots + n_k| > N\) in a computation analogous to \(3.2.29\), which in particular implies \(\max_{j=1,\ldots,k} |n_j| \gtrsim k\). Hence, a slight modification of \(3.2.31\) yields the desired bound \(3.2.23\).

Next, we study the densities for the truncated Gibbs measures \(\rho_{\delta,N}\) in \(3.2.14\). As mentioned above, we restrict our attention to the defocusing case in this subsection. Namely, we fix \(k \in 2\mathbb{N}+1\). See Subsection \(3.2.4\) for the \((k=2)\) case. Given \(0 < \delta \leq \infty\) and \(N \in \mathbb{N}\), let \(G_{\delta,N}(u)\) be the truncated density defined in \((1.3.38)\). Our main goal is to establish an \(L^p\)-integrability of the truncated density \(G_{\delta,N}(u)\) for the following two purposes:

- In order to construct the limiting Gibbs measure \(\rho_{\delta}\) for each fixed \(0 < \delta \leq \infty\) (Theorem \(1.3.2\) (i)), we establish such an \(L^p\)-integrability of the truncated density, uniformly in \(N \in \mathbb{N}\) but for each fixed \(0 < \delta \leq \infty\).

- In order to prove convergence of the Gibbs measures in the deep-water limit (Theorem \(1.3.2\) (ii)), we establish an \(L^p\)-integrability of the truncated density, uniformly in both \(N \in \mathbb{N}\) and \(\delta \gg 1\).

Here, we need to study the \(L^p\)-integrability of \(G_{\delta,N}(u)\) with respect to the Gaussian measure \(\mu_{\delta}\) in \((1.3.12)\), which is different for different values of \(\delta\). In order to establish a uniform (in \(\delta\)) bound, it is therefore more convenient to work with the Gaussian process \(X_{\delta}\) and the underlying probability measure \(\mathbb{P}\) on \(\Omega\).

Given \(0 < \delta \leq \infty\) and \(N \in \mathbb{N}\), we define \(G_{\delta,N}(X_{\delta}) = G_{\delta,N}(X_{\delta}; k+1)\) by

\[
G_{\delta,N}(X_{\delta}) = e^{-R_{\delta,N}(X_{\delta})} = e^{-\int_0^{\infty} \mathcal{W}(X_{\delta}^{k+1}) dx},
\]

where \(R_{\delta,N}(X_{\delta}) = R_{\delta,N}(X_{\delta}; k+1)\) is the truncated potential energy defined in \((3.2.17)\).

**Proposition 3.2.6.** Let \(k \in 2\mathbb{N}+1\) and fix finite \(p \geq 1\). Given any \(0 < \delta \leq \infty\), we have

\[
\sup_{N \in \mathbb{N}} \| G_{\delta,N}(X_{\delta}) \|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \| G_{\delta,N}(u) \|_{L^p(d\mu_u)} \leq C_{p,k,\delta} < \infty.
\]

In addition, the following uniform bound holds for \(2 \leq \delta \leq \infty\):

\[
\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \| G_{\delta,N}(X_{\delta}) \|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \| G_{\delta,N}(u) \|_{L^p(d\mu_u)} \leq C_{p,k} < \infty.
\]

Define \(G_{\delta}(X_{\delta}) = G_{\delta,\infty}(X_{\delta})\) by

\[
G_{\delta}(X_{\delta}) = e^{-R_{\delta}(X_{\delta})}
\]

with \(R_{\delta}(X_{\delta})\) as in \((3.2.18)\). Then, \(G_{\delta,N}(X_{\delta})\) converges to \(G_{\delta}(X_{\delta})\) in \(L^p(\Omega)\). Namely, we have

\[
\lim_{N \to \infty} \| G_{\delta,N}(X_{\delta}) - G_{\delta}(X_{\delta}) \|_{L^p(\Omega)} = 0.
\]
As a consequence, the uniform bounds (3.2.33) and (3.2.34) hold even if we replace supremum in $N \in \mathbb{N}$ by the supremum in $N \in \mathbb{N} \cup \{\infty\}$.

Theorem 1.3.2(i) follows as a direct corollary to Proposition 3.2.6, allowing us to define the limiting Gibbs measure $\rho_\delta$ in (1.3.41). Fix $0 < \delta \leq \infty$. Then, (3.2.35) with $p = 1$ implies that the partition function $Z_{\delta,N} = \|G_{\delta,N}(u)\|_{L^1(\mu_A)}$ of the truncated Gibbs measure $\rho_{\delta,N}$ in (1.3.37) converges to the partition function $Z_{\delta} = \|G_{\delta}(u)\|_{L^1(\mu_A)}$ of the Gibbs measure $\rho_{\delta}$ in (1.3.41). Let $\mathcal{B}_{H^{-\varepsilon}}$ denote the collection of Borel sets in $H^{-\varepsilon}(\mathbb{T})$. Then, once again from (3.2.35), we have

$$
\begin{align*}
\lim_{N \to \infty} \sup_{A \in \mathcal{B}_{H^{-\varepsilon}}} |\rho_{\delta,N}(A) - \rho_{\delta}(A)| & = \lim_{N \to \infty} \sup_{A \in \mathcal{B}_{H^{-\varepsilon}}} \left| \frac{Z_{\delta,N}}{Z_{\delta}} \rho_{\delta,N}(A) - \rho_{\delta}(A) \right| \\
& \leq Z_{\delta}^{-1} \lim_{N \to \infty} \sup_{A \in \mathcal{B}_{H^{-\varepsilon}}} \int_{H^{-\varepsilon}} 1_A(u) |G_{\delta,N}(u) - G_{\delta}(u)| \mu_{\delta}(u) \\
& \leq Z_{\delta}^{-1} \lim_{N \to \infty} \|G_{\delta,N}(X_{\delta}) - G_{\delta}(X_{\delta})\|_{L^1(\mathcal{B})} \\
& = 0.
\end{align*}
$$

This proves convergence in total variation of $\rho_{\delta,N}$ to $\rho_{\delta}$. By using (3.2.36) in place of (3.2.35), a slight modification of the argument above yields uniform convergence in total variation of $\rho_{\delta,N}$ to $\rho_{\delta}$ for $2 \leq \delta \leq \infty$. See (3.2.35) below. We omit details.

We now present the proof of Proposition 3.2.6.

**Proof of Proposition 3.2.6.** We break the proof into two steps.

- **Step 1:** We first prove the uniform $L^p$-bounds (3.2.33) and (3.2.34). Given $k \in 2\mathbb{N} + 1$, the Hermite polynomial $H_{k+1}$ has a global minimum; there exists finite $a_{k+1} > 0$ such that $H_{k+1}(x) \geq -a_{k+1}$ for any $x \in \mathbb{R}$. It follows from (3.2.17) that

$$
H_{k+1}(x; \sigma) \geq -\sigma^{\frac{k+1}{2}} a_{k+1}
$$

for any $x \in \mathbb{R}$ and $\sigma > 0$. Hence, from (3.2.17) and (3.2.38) with (1.3.35), we have

$$
\begin{align*}
-R_{\delta,N}(X_{\delta}) & = -\frac{1}{k+1} \int_{-\theta}^{\theta} H_{k+1}(X_{\delta,N}; \sigma_{\delta,N}) \, dx \\
& \leq \frac{2\pi}{k+1} \sigma_{\delta,N}^{\frac{k}{2}} a_{k+1} \leq A_{k,\delta}(\log(N+1))^{\frac{k+1}{2}}
\end{align*}
$$

for some $A_{k,\delta} > 0$, uniformly in $N \in \mathbb{N}$. The bound (3.2.39) is exactly where the defocusing nature of the equation plays a crucial role.

**Remark 3.2.7.** Recall the uniform lower bound (3.1.3) for $2 \leq \delta \leq \infty$ (with the identification (3.2.4) when $\delta = \infty$). In view of (1.3.35), we can then choose $A_{k,\delta}$ to be independent of $2 \leq \delta \leq \infty$ (and $N \in \mathbb{N}$) as in the proof of Proposition 3.2.4. Similarly, by restricting our attention to $2 \leq \delta \leq \infty$, we can choose the constant $c_{k,\delta}$ in (3.2.40) below to be independent of $2 \leq \delta \leq \infty$ since the constant $C_{k,\delta}$ in (3.2.20) is independent of $2 \leq \delta \leq \infty$. As a result, the constants in $B_{k,\delta,p}$ and $C_{2}(k,\delta,p)$ in (3.2.44) below can be chosen to be independent of $2 \leq \delta \leq \infty$.

By applying Proposition 3.2.4(i) and Chebyshev’s inequality (see also Lemma 4.5 in [162]), we have, for some $C_1 > 0$ and $c_{k,\delta} > 0$,

$$
P\left(p| R_{\delta,M}(X_{\delta}) - R_{\delta,N}(X_{\delta})| > \lambda \right) \leq C_1 e^{-c_{k,\delta} p^{-1} \lambda^{\frac{2k}{2k+1}}} \lambda^{\frac{2k}{2k+1}}
$$

(3.2.40)
Therefore, we obtain
\[ \|G_{\delta,N}(X_\delta)\|_{L^p(\Omega)} = \int_0^\infty \mathbb{P}\left(e^{-pR_{\delta,N}(X_\delta)} > \alpha\right) d\alpha \leq 1 + \int_1^\infty \mathbb{P}\left(-pR_{\delta,N}(X_\delta) > \log \alpha\right) d\alpha, \]
we see that the desired bound (3.2.33) follows once we show that there exist \( C_2 = C_2(k, \delta, p) > 0 \) and \( \beta > 0 \) such that
\[ \mathbb{P}\left(-pR_{\delta,N}(X_\delta) > \log \alpha\right) \leq C_2\alpha^{-(1+\beta)} \] (3.2.41)
for any \( \alpha > 0 \) and \( N \in \mathbb{N} \). We prove (3.2.41) via a standard application of the so-called Nelson’s estimate. Namely, given \( \alpha > 0 \), we choose \( N_0 = N_0(\alpha) > 0 \) and establish (3.2.41) for \( N \geq N_0 \) and \( N < N_0 \) in two different ways.

Given \( \lambda := \log \alpha > 0 \), we choose \( N_0 > 0 \) by setting
\[ \lambda = 2pA_{k,\delta}(\log(N_0 + 1))^{\frac{k}{k+1}}. \] (3.2.42)
Then, from (3.2.39) and (3.2.42), we have
\[ -pR_{\delta,N_0}(X_\delta) \leq pA_{k,\delta}(\log(N_0 + 1))^{\frac{k}{k+1}} = \frac{1}{2} \lambda. \] (3.2.43)
Hence, from (3.2.42) and (3.2.40), we have
\[ \mathbb{P}\left(-pR_{\delta,N}(X_\delta) > \lambda\right) \leq \mathbb{P}\left(-p(R_{\delta,N}(X_\delta) - R_{\delta,N_0}(X_\delta)) > \frac{1}{2} \lambda\right) \leq \mathbb{P}\left|R_{\delta,N}(X_\delta) - R_{\delta,N_0}(X_\delta)\right| > \frac{1}{2} \lambda \leq C_1 e^{-c_{k,\delta,p} \frac{\log \alpha}{\lambda^{\frac{2}{k+1}}}} \leq C_2 e^{-c_{k,\delta,p} \frac{\log \alpha}{\lambda^{\frac{2}{k+1}}}}, \] (3.2.44)
for any \( N \geq N_0 \). On the other hand, for \( N < N_0 \), it follows from (3.2.39) and (3.2.42) that
\[ -pR_{\delta,N}(X_\delta) \leq pA_{k,\delta}(\log(N + 1))^{\frac{k}{k+1}} < \frac{1}{2} \lambda \]
and thus we have
\[ \mathbb{P}\left(-pR_{\delta,N}(X_\delta) > \lambda\right) = 0. \] (3.2.45)
Putting (3.2.44) and (3.2.45) together, we conclude that (3.2.41) holds for any \( \alpha > 1 \) and \( N \in \mathbb{N} \). Therefore, we obtain
\[ \|G_{\delta,N}(X_\delta)\|_{L^p(\Omega)} \leq C_3(k, \delta, p) < \infty \] (3.2.46)
for any \( N \in \mathbb{N} \).

For \( 2 \leq \delta \leq \infty \), it follows from Remark 3.2.7 that the constant \( C_3(k, \delta, p) \) in (3.2.46) can be chosen to be independent of \( 2 \leq \delta \leq \infty \), thus yielding (3.2.34).

- **Step 2**: Next, we show the (uniform) \( L^p \)-convergence of the truncated densities.

Fix \( 0 < \delta \leq \infty \). The \( L^p \)-convergence (3.2.35) of the truncated density \( G_{\delta,N}(X_\delta) \) follows from the uniform bound (3.2.33) and a standard argument (see [161, Remark 3.8]). More precisely, as a consequence of Proposition 3.2.4(i) and the continuous mapping theorem, we see that \( G_{\delta,N}(X_\delta) = e^{-R_{\delta,N}(X_\delta)} \) converges in probability to the limit \( G_{\delta}(X_\delta) = e^{-R_{\delta}(X_\delta)} \). Then, the
\[ L^p\text{-convergence } (3.2.35) \text{ follows from the uniform bound } (3.2.33) \text{ and this softer convergence in probability. While we omit details of the argument in this case, we present details of an analogous argument in establishing the uniform } L^p\text{-convergence } (3.2.36) \text{ in the following.} \]

In the following, we present the proof of (3.2.36) and thus restrict our attention to \( 2 \leq \delta \leq \infty \). Proposition 3.2.4 (i), the continuity of the exponential function, and the uniform continuous mapping theorem (Lemma 3.1.8)\footnote{In this case, we use the tightness of \( \{R_\delta(X_\delta)\}_{2 \leq \delta \leq \infty} \), coming from (3.2.19), to verify the hypothesis (3.1.27) in Lemma 3.1.8.} we see that \( G_{\delta,N}(X_{\delta}) \) converges in probability to \( G_{\delta}(X_{\delta}) \) as \( N \to \infty \), uniformly in \( 2 \leq \delta \leq \infty \). Then, by setting

\[ A_{\delta,N,\varepsilon} = \{|G_{\delta,N}(X_{\delta}) - G_{\delta}(X_{\delta})| \leq \varepsilon\}, \]

we have

\[ \sup_{2 \leq \delta \leq \infty} P(A_{\delta,N,\varepsilon}) \to 0, \]

as \( N \to \infty \). Then, from (3.2.47), Cauchy-Schwarz’s inequality, the uniform (in \( \delta \) and \( N \), including \( N = \infty \)) bound (3.2.34), and (3.2.48), we obtain

\[ \sup_{2 \leq \delta \leq \infty} \|G_\delta(X_{\delta}) - G_{\delta,N}(X_{\delta})\|_{L^p(\Omega)} \leq \epsilon + \sup_{2 \leq \delta \leq \infty} \|G_\delta(X_{\delta}) - G_{\delta,N}(X_{\delta})\|_{L^p(\Omega)} \cdot \sup_{2 \leq \delta \leq \infty} P(A_{\delta,N,\varepsilon})^{\frac{1}{p}} \leq 2\epsilon \]

for sufficiently large \( N \gg 1 \). This proves the uniform (in \( \delta \)) \( L^p\)-convergence \( (3.2.36) \). This concludes the proof of Proposition 3.2.6.

### 3.2.3 Convergence of the Gibbs measures in the deep-water limit

In this subsection, we present the proof of Theorem 1.3.2 (ii). Once again, we restrict our attention to the defocusing case: \( k \in 2\mathbb{N} + 1 \). The construction of the Gibbs measures in the previous subsection shows that, for each \( 0 < \delta < \infty \), the Gibbs measure \( \rho_{\delta} \) and the base Gaussian measure \( \mu_{\delta} \) are equivalent. On the other hand, from Proposition 3.2.1, we know that the Gaussian measures \( \mu_{\delta} \) are all equivalent for \( 0 < \delta < \infty \). Therefore, we conclude that the Gibbs measure \( \rho_{\delta} \), \( 0 < \delta < \infty \), for the defocusing gILW equation (1.3.2) and the Gibbs measure \( \rho_{gBO} = \rho_{\infty} \) for the defocusing gBO equation (1.1.7) are equivalent. This proves the first claim in Theorem 1.3.2 (ii). Hence, it remains to show that the Gibbs measure \( \rho_{\delta} \) converges to \( \rho_{gBO} \) in total variation, as \( \delta \to \infty \).

Before proceeding to the proof of the uniform convergence of \( \rho_{\delta} \) to \( \rho_{gBO} \), let us first present the following \( L^p \)-convergence of the (truncated) densities. For \( 0 < \delta \leq \infty \), let \( X_\delta \) and \( X_{\delta,BO} = X_{\infty} \) be as in (1.3.32) and (1.3.29), respectively, and let \( R_\delta(X_\delta) \) (and \( G_\delta(X_\delta) \), respectively) be the limit of \( R_{\delta,N}(X_\delta) \) constructed in Proposition 3.2.4 (and of \( G_{\delta,N}(X_{\delta}) \) constructed in Proposition 3.2.6, respectively).

**Lemma 3.2.8.** Let \( k \in 2\mathbb{N} + 1 \). Then, given \( N \in \mathbb{N} \), we have

\[ \lim_{\delta \to \infty} \|G_{\delta,N}(X_{\delta}) - G_{\infty,N}(X_{\delta,BO})\|_{L^p(\Omega)} = 0. \]

As a corollary, we have

\[ \lim_{\delta \to \infty} \|G_\delta(X_{\delta}) - G_\infty(X_{\delta,BO})\|_{L^p(\Omega)} = 0. \]
In particular, the partition function $Z_{\delta}$ of the Gibbs measure $\mu_{\delta}$ in (1.3.41) converges to the partition function $Z_{\rho_{BO}} = Z_{\infty}$ of the Gibbs measure $\mu_{BO} = \mu_{\infty}$, as $\delta \to \infty$.

Remark 3.2.9. In view of the argument presented in (3.2.37), one may be tempted to conclude directly from (3.2.50) in Lemma 3.2.8 that $\mu_{\delta}$ converges to $\mu_{BO} = \mu_{\infty}$ in total variation as $\delta \to \infty$. However, this is not possible. This is due to the fact that the base Gaussian measures $\mu_{\delta}$ and $\mu_{\infty}$ are different. If we were to mimic the argument in (3.2.37), the integral in the third step of (3.2.54) (namely, 5) would be replaced by

$$\int_{\Omega} \left( \mathbf{1}_{\{X_{\delta} \in A\}} G_{\delta,N}(X_{\delta}) - \mathbf{1}_{\{X_{BO} \in A\}} G_{BO}(X_{BO}) \right) dP = \int_{\Omega} \mathbf{1}_{\{X_{\delta} \in A\}} \left( G_{\delta,N}(X_{\delta}) - G_{BO}(X_{BO}) \right) dP$$

(3.2.51)

While we can apply (3.2.50) in Lemma 3.2.8 to control the first term on the right-hand side of (3.2.51), we can not handle the second term as it is. Note that the difference $1_{\{X_{\delta} \in A\}} - 1_{\{X_{BO} \in A\}}$ with respect to the $P$-integration (and taking the supremum in $A \in B_{\mu_{\infty}}$) is closely related to the convergence in total variation of $\mu_{\delta}$ to $\mu_{\infty}$ proven in Proposition 3.2.1 (iii), which plays a crucial role in the proof of convergence in total variation of $\rho_{\delta}$ to $\rho_{BO}$ presented below.

Proof of Lemma 3.2.8. Fix $N \in \mathbb{N}$. From (1.3.35) and Lemma 3.1.1 we have

$$\sigma_{\delta,N} = \frac{1}{2\pi} \sum_{0 < |n| \leq N} \frac{1}{K_{\delta}(n)} \to \frac{1}{2\pi} \sum_{|n| \leq N} \frac{1}{|n|} =: \sigma_{\infty,N}, \quad (3.2.52)$$

as $\delta \to \infty$. It also follows from the definitions (1.3.29), (1.3.32), and Lemma 3.1.1 that, for any $x \in \mathbb{T}$ and $\omega \in \Omega$, $X_{\delta,N}(x)$ converges to $X_{BO,N}(x)$ as $\delta \to \infty$. Moreover, from (1.3.32) and Lemma 3.1.1, we have

$$|X_{\delta,N}(x; \omega)| \leq \sum_{0 < |n| \leq N} \frac{|g_n(\omega)|}{K_\delta^2(n)} \leq C_{N,\omega} < \infty$$

for any $2 \leq \delta \leq \infty$, $x \in \mathbb{T}$, and $\omega \in \Omega$. Then, by the dominated convergence theorem applied to the integration in $x \in \mathbb{T}$, we have

$$R_{\delta,N}(X_{\delta}(\omega)) = \frac{1}{k+1} \int_{\mathbb{T}} H_k(X_{\delta,N}(x; \omega); \sigma_{\delta,N}) dx \to R_{\infty,N}(X_{BO}(\omega)) \quad (3.2.53)$$

as $\delta \to \infty$, for any $\omega \in \Omega$. As a consequence, we see that $G_{\delta,N}(X_{\delta}(\omega))$ converges to $G_{\infty,N}(X_{BO}(\omega))$ as $\delta \to \infty$, for any $\omega \in \Omega$. Moreover, from the uniform (in $\omega$) bound (3.2.39), we conclude that $G_{\delta,N}(X_{\delta})$ converges to $G_{\infty,N}(X_{BO}(\omega))$ in $L^p(\Omega)$, as $\delta \to \infty$. This proves (3.2.49).

By the triangle inequality, we have

$$\|G_{\delta}(X_{\delta}) - G_{\infty}(X_{BO})\|_{L^p(\Omega)} \leq \|G_{\delta}(X_{\delta}) - G_{\delta,N}(X_{\delta})\|_{L^p(\Omega)} + \|G_{\delta,N}(X_{\delta}) - G_{\infty,N}(X_{BO})\|_{L^p(\Omega)}$$

(3.2.54)

Then, by first applying (3.2.49) above and then (3.2.36) in Proposition 3.2.6 to (3.2.54) (namely, 5)
we first take $\delta \to \infty$ and then $N \to \infty$), we obtain
\[
\lim_{\delta \to \infty} \|G_\delta(X_\delta) - G_\infty(X_{BO})\|_{L^p(\Omega)} \leq 2 \lim_{N \to \infty} \left( \sup_{2 \leq \delta \leq N} \|G_\delta(X_\delta) - G_{\delta,N}(X_\delta)\|_{L^p(\Omega)} + \lim_{\delta \to \infty} \|G_{\delta,N}(X_\delta) - G_{\infty,N}(X_{BO})\|_{L^p(\Omega)} \right).
\]

This proves (3.2.50). \qed

We are now ready to show that the Gibbs measure $\rho_\delta$ in (1.3.37) converges to $\rho_{BO} = \rho_\infty$ in total variation as $\delta \to \infty$. By the triangle inequality, we have
\[
d_{TV}(\rho_{\delta,N}, \rho_{BO}) \leq d_{TV}(\rho_{\delta,N}, \rho_{\delta}) + d_{TV}(\rho_{\delta,N}, \rho_{BO}) + d_{TV}(\rho_{\delta}, \rho_{\delta,N}) \tag{3.2.55}
\]
for any $N \in \mathbb{N}$. From Theorem 1.3.2 (i) (see also Proposition 3.2.6), we have
\[
\lim_{N \to \infty} \sup_{2 \leq \delta \leq N} d_{TV}(\rho_{\delta,N}, \rho_\delta) = 0. \tag{3.2.56}
\]
Hence, it suffices to prove
\[
\lim_{\delta \to \infty} d_{TV}(\rho_{\delta,N}, \rho_{\delta,N}) = 0 \tag{3.2.57}
\]
for any $N \in \mathbb{N}$. Indeed, by applying (3.2.56) and (3.2.57) to (3.2.55) (namely, by first taking $\delta \to \infty$ and then $N \to \infty$), we obtain
\[
\lim_{\delta \to \infty} d_{TV}(\rho_{\delta,N}, \rho_{BO}) \leq \lim_{N \to \infty} \left( \sup_{2 \leq \delta \leq N} d_{TV}(\rho_{\delta,N}, \rho_\delta) + \lim_{\delta \to \infty} d_{TV}(\rho_{\delta,N}, \rho_{\delta,N}) \right) = 0.
\]

In the following, we prove (3.2.57) for any fixed $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$. Then, Lemma 3.2.8 with $p = 1$ implies that the partition function $Z_{\delta,N} = \|G_{\delta,N}(u)\|_{L^1(d\mu)}$ of the truncated Gibbs measure $\rho_{\delta,N}$ in (1.3.37) converges to the partition function $Z_{\infty,N} = \|G_{\infty,N}(u)\|_{L^1(d\mu)}$ of the truncated Gibbs measure $\rho_{\infty,N}$ for the gBO equation as $\delta \to \infty$. Then, from Proposition 3.2.1 (ii), we have
\[
\lim_{\delta \to \infty} \sup_{A \in \mathcal{B}_{H^{-\varepsilon}}} |\rho_{\delta,N}(A) - \rho_{\infty,N}(A)| = \lim_{\delta \to \infty} \sup_{A \in \mathcal{B}_{H^{-\varepsilon}}} \left| \frac{Z_{\delta,N}}{Z_{\infty,N}} \rho_{\delta,N}(A) - \rho_{\infty,N}(A) \right| \leq Z_{\infty,N}^{-1} \lim_{\delta \to \infty} \sup_{A \in \mathcal{B}_{H^{-\varepsilon}}} \left| \int_{H^{-\varepsilon}} 1_A(u) \left( G_{\delta,N}(u) \frac{d\mu_\delta}{d\mu_\infty}(u) - G_{\infty,N}(u) \right) d\mu_\infty(u) \right| \tag{3.2.58}
\]
\[
\leq Z_{\infty,N}^{-1} \lim_{\delta \to \infty} \int_{H^{-\varepsilon}} |G_{\delta,N}(u) - G_{\infty,N}(u)| d\mu_\infty(u) + Z_{\infty,N}^{-1} \lim_{\delta \to \infty} \int_{H^{-\varepsilon}} |G_{\delta,N}(u)| \frac{d\mu_\delta}{d\mu_\infty}(u) - 1 | d\mu_\infty(u). \tag{3.2.59}
\]

From (1.3.38) and (3.2.52), we see that $G_{\delta,N}(u)$ converges to $G_{\infty,N}(u)$ $\mu_\infty$-almost surely, as $\delta \to \infty$. Moreover, it follows from (3.2.39) and Remark 3.2.7 that
\[
|G_{\delta,N}(u) - G_{\infty,N}(u)| \leq C_\delta < \infty \tag{3.2.59}
\]
for any $2 \leq \delta \leq \infty$. Hence, by the dominated convergence theorem, we obtain
\[
\lim_{\delta \to \infty} \int_{H^{-\varepsilon}} |G_{\delta,N}(u) - G_{\infty,N}(u)| \, d\mu_\infty(u) = 0.
\]
(3.2.60)

By Scheffé's theorem (Lemma 2.1 in [160]; see also Proposition 1.2.7 in [95]), we have
\[
d_{TV}(\mu_\delta, \mu_\infty) = \frac{1}{2} \int_{H^{-\varepsilon}} \left| \frac{d\mu_\delta}{d\mu_\infty}(u) - 1 \right| \, d\mu_\infty(u).
\]
(3.2.61)

Then, it follows from the convergence in total variation of $\mu_\delta$ to $\mu_\infty$ as $\delta \to \infty$ (Proposition 3.2.1(iii), (3.2.61)), and the uniform (in $\delta$) bound (3.2.39) (and Remark 3.2.7) for $2 \leq \delta \leq \infty$ that
\[
\lim_{\delta \to \infty} \int_{H^{-\varepsilon}} G_{\delta,N}(u) \left| \frac{d\mu_\delta}{d\mu_\infty}(u) - 1 \right| \, d\mu_\infty(u)
\leq C_N \lim_{\delta \to \infty} \int_{H^{-\varepsilon}} \left| \frac{d\mu_\delta}{d\mu_\infty}(u) - 1 \right| \, d\mu_\infty(u)
= 2C_N \lim_{\delta \to \infty} d_{TV}(\mu_\delta, \mu_\infty)
= 0.
\]
(3.2.62)

Therefore, from (3.2.58), (3.2.60), and (3.2.62), we conclude (3.2.57) and hence convergence in total variation of $\rho_\delta$ to $\rho_{\mathrm{BO}}$ as $\delta \to \infty$. This concludes the proof of Theorem 1.3.2 when $k \in 2\mathbb{N} + 1$.

### 3.2.4 Gibbs measures for the ILW equation: variational approach

We conclude this section by presenting the proof of Theorem 1.3.2 for the $k = 2$ case, corresponding to the ILW equation (1.1.1). In this case, the problem is no longer defocusing and thus we need to consider the (truncated) Gibbs measures with a Wick-ordered $L^2$-cutoff of the form (1.3.42) and (1.3.43). As pointed out in Remark 1.3.3, there is no need for a renormalization on the potential energy under the current (spatial) mean-zero condition.

Fix $K > 0$ in the remaining part of this section. Given $0 < \delta \leq \infty$ and $N \in \mathbb{N}$, define the truncated density $G_{\delta,N}^K(u)$ by
\[
G_{\delta,N}^K(u) = \chi_K \left( \int_{\mathbb{T}} W(u_N^2) \, dx \right) e^{-\frac{1}{2} \int_{\mathbb{T}} u_N^2 \, dx}
= \chi_K \left( \int_{\mathbb{T}} H_2(u_N; \sigma_{\delta,N}) \, dx \right) e^{-\frac{1}{2} \int_{\mathbb{T}} u_N^2 \, dx},
\]
(3.2.63)
where $u_N = P_N u$ and $\chi_K : \mathbb{R} \to [0, 1]$ is a continuous function such that $\chi_K(x) = 1$ for $|x| \leq K$ and $\chi_K(x) = 0$ for $|x| \geq 2K$.

In view of the discussion in Subsections 3.2.2 and 3.2.3 Theorem 1.3.2 for $k = 2$ follows once we prove the following uniform bounds.

**Proposition 3.2.10.** Fix finite $p \geq 1$ and $K > 0$. Then, given any $0 < \delta \leq \infty$, we have
\[
\sup_{N \in \mathbb{N}} \| G_{\delta,N}^K(X_\delta) \|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \| G_{\delta,N}^K(u) \|_{L^p(\mu_\delta)} \leq C_{p,K} < \infty.
\]

In addition, the following uniform bound holds for $2 \leq \delta \leq \infty$:
\[
\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \| G_{\delta,N}^K(X_\delta) \|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \| G_{\delta,N}^K(u) \|_{L^p(\mu_\delta)}
\leq C_{p,K} < \infty.
\]

Let us first discuss how to conclude Theorem 1.3.2 when $k = 2$, by assuming Proposition 3.2.10.
Define the limiting density $G^K_\delta(u)$ by

$$G^K_\delta(u) = \chi_K \left( \int_T W(u^2) dx \right) e^{-\frac{1}{2} \int_T u^3 dx}. \tag{3.2.64}$$

Note that Proposition 3.2.4(i) guarantees that $\int_T W(u^2) dx$ and $\int_T u^3 dx = \int_T W(u^3) dx$ in (3.2.64) exist as the limits in $L^p(\mu_\delta)$ of the truncated versions. Hence, the truncated density $G^K_{\delta,N}(u)$ converges in measure to the limiting density $G^K_\delta(u)$ in (3.2.64). Hence, once we prove Proposition 3.2.10 we can repeat the argument in Step 2 in the proof of Proposition 3.2.6 to show the following convergence results.

**Corollary 3.2.11.** Let $0 < \delta \leq \infty$. Then, $G^K_{\delta,N}(X_\delta)$ converges to $G^K_\delta(X_\delta)$ in $L^p(\Omega)$. Namely, we have

$$\lim_{N \to \infty} \|G^K_{\delta,N}(X_\delta) - G^K_\delta(X_\delta)\|_{L^p(\Omega)} = 0.$$

Furthermore, the convergence is uniform in $2 \leq \delta \leq \infty$:

$$\lim_{N \to \infty} \sup_{2 \leq \delta \leq \infty} \|G^K_{\delta,N}(X_\delta) - G^K_\delta(X_\delta)\|_{L^p(\Omega)} = 0.$$

This proves an analogue of Theorem 1.3.2(i) when $k = 2$. The equivalence of the Gibbs measure $\rho_\delta$ in (1.3.43), $0 < \delta < \infty$, and $\rho_{\text{gBO}}$ in (1.3.30) follows from (i) the equivalence of the Gibbs measure $\rho_\delta$ and the Gaussian measure with the Wick-ordered $L^2$-cutoff:

$$\chi_K \left( \int_T W(u^2) dx \right) d\mu_\delta(u),$$

(including $\delta = \infty$ with the understanding that $\rho_\infty = \rho_{\text{gBO}}$), and (ii) the equivalence of the base Gaussian measure $\mu_\delta$, $0 < \delta \leq \infty$ (Proposition 3.2.1(ii)).

Finally, we discuss convergence of the Gibbs measure $\rho_\delta$ to $\rho_{\text{gBO}}$ in the deep-water limit ($\delta \to \infty$). In the defocusing case discussed in the previous subsection, the bound (3.2.39) provided the uniform (in $2 \leq \delta \leq \infty$ and $\Omega \in \Omega$) bound on the truncated density $G_{\delta,N}(u)$; see the discussion after (3.2.53). See also (3.2.59) and (3.2.62). In the current non-defocusing case, however, the bound (3.2.39) is not available to us. Nonetheless, in view of (1.3.36) and (1.3.35) with Lemma 3.1.1 the Wick-ordered $L^2$-cutoff in (3.2.63) with (1.3.36) implies

$$\left| \int_T u_N^2 dx \right| \leq \sigma_{\delta,N} + 2K \leq C_{N,K} < \infty, \tag{3.2.65}$$

for any $2 \leq \delta \leq \infty$ and $N \in \mathbb{N}$, where $C_{N,K}$ is independent of $2 \leq \delta \leq \infty$. Then, by Sobolev’s inequality with (3.2.65), we have

$$\left| \int_T u_N dx \right| \leq \|u_N\|^2_{H^{\frac{3}{2}}} \leq N^\frac{3}{2} C_{N,K}^\frac{3}{2}, \tag{3.2.66}$$

which provides a bound on the truncated density $G^K_{\delta,N}(u)$ in (3.2.63), uniformly in $2 \leq \delta \leq \infty$. With this bound on $G^K_{\delta,N}(u)$, we can repeat the argument presented in Subsection 3.2.3 to conclude the desired convergence in total variation of $\rho_\delta$ to $\rho_{\text{gBO}}$ as $\delta \to \infty$.

In the remaining part of this section, we present the proof of Proposition 3.2.10. Given $0 < \delta \leq \infty$ and $N \in \mathbb{N}$, set

$$R_{\delta,N}(u) = \frac{1}{3} \int_T u_N^3 dx + A \left( \int_T W(u_N^2) dx \right)^2, \tag{3.2.67}$$

where $W(u_N^2) = W_{\delta,N}(u_N^2) = H_2(u_N; \sigma_{\delta,N})$. Then, as in [130], we consider the following truncated
density:
\[ G^K_{\delta,N}(u) = e^{-R_{\delta,N}(u)} = e^{-\frac{1}{2} \int_0^1 u_N^2 dx - A|\int_0^1 W(u_N) dx|^2} \quad (3.2.68) \]
for some suitable \( A > 0 \). Noting that
\[ \chi_K(x) \leq \exp \left( -A|x|\gamma \right) \exp(A^2 K\gamma) \quad (3.2.69) \]
for any \( K, A, \gamma > 0 \), we have
\[ G^K_{\delta,N}(u) \leq C_{A,K} \cdot G^K_{\delta}(u). \]

Hence, Proposition 3.2.10 follows once we prove the following uniform bounds on \( G^K_{\delta,N}(u) \).

**Proposition 3.2.12.** Fix finite \( p \geq 1 \). Then, there exists \( A_0 = A_0(p) > 0 \) such that
\[ \sup_{N \in \mathbb{N}} \|G^K_{\delta,N}(X_{\delta})\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \|G^K_{\delta,N}(u)\|_{L^p(d\mu_\delta)} \leq C_{p,K,A} < \infty \quad (3.2.70) \]
for any \( 0 < \delta \leq \infty \), \( K > 0 \), and \( A \geq A_0 \). In addition, the following uniform bound holds for \( 2 \leq \delta \leq \infty \):
\[ \sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|G^K_{\delta,N}(X_{\delta})\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \|G^K_{\delta,N}(u)\|_{L^p(d\mu_\delta)} \leq C_{p,K,A} < \infty \quad (3.2.71) \]
for any \( K > 0 \) and \( A \geq A_0 \).

As mentioned in the introduction, we employ the variational approach, introduced by Barashkov and Gubinelli [9], to prove Proposition 3.2.12. In particular, we follow closely to the argument in [136], where the \( \delta = \infty \) case was treated via the variational approach. See also [61] [133] [131] [26] [132] for recent works on dispersive PDEs, where the variational approach played a crucial role.

Let us first introduce some notations. Let \( W(t) \) be a cylindrical Brownian motion in \( L^2(\Omega) = P_{\neq 0}L^2(\mathbb{T}) \) of mean-zero functions on \( \mathbb{T} \), where \( P_{\neq 0} \) denotes the projection onto the non-zero frequencies. Namely, we have
\[ W(t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} B_n(t)e_n, \quad (3.2.72) \]
where \( \{B_n\}_{n \in \mathbb{Z}^*} \) is a sequence of mutually independent complex-valued Brownian motions such that \( B_n = B_{-n}, n \in \mathbb{Z}^* \). Then, we define a centered Gaussian process \( Y_\delta(t) \) by
\[ Y_\delta(t) = (G_\delta \partial_x)^{-\frac{1}{2}}W(t), \quad (3.2.73) \]
where \((G_\delta \partial_x)^{-\frac{1}{2}}\) is the Fourier multiplier operator with the multiplier \( (K_\delta(n))^{-\frac{1}{2}} \) with \( K_\delta(n) \) as in (1.3.33). In view of (1.3.32), we have \( \mathcal{L}(Y_\delta(1)) = \mu_\delta \). Given \( N \in \mathbb{N} \), we set \( Y_{\delta,N} = P_N Y_\delta \). Then, from (1.3.35), we have
\[ \mathbb{E}[\|Y_{\delta,N}(1)\|] = \sigma_{\delta,N} \sim \delta \log(N + 1). \]

Next, we recall the Boué-Dupuis variational formula. Let \( \mathcal{H}_\delta \) denote the collection of drifts, which are progressively measurable processes belonging to \( L^2([0,1]; L^2(\mathbb{T})) \), \( \mathbb{P} \)-almost surely. We now state the Boué-Dupuis variational formula [4] [166]. See, in particular, Theorem 7 in [166].

**Lemma 3.2.13.** Given \( 0 < \delta \leq \infty \), let \( Y_\delta \) be as in (3.2.73). Fix \( N \in \mathbb{N} \). Suppose that \( F : C^\infty(\mathbb{T}) \to \mathbb{R} \) is measurable such that \( \mathbb{E}[|F(Y_{\delta,N}(1))|^p] < \infty \) and \( \mathbb{E}[|e^{-F(Y_{\delta,N}(1))}|^q] < \infty \) for some

---

*By convention, we normalize \( B_n \) such that \( \text{Var}(B_n(t)) = 2\pi t \).*
1 < p, q < ∞ with \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, we have

\[
- \log E\left[ e^{-F(\delta,N(1))} \right] = \inf_{\theta \in H_\delta} E\left[ F(Y_{\delta,N}(1)) + \mathbf{P}_N I_\delta(\theta)(1) \right] + \frac{1}{2} \int_0^1 \| \theta(t) \|_{L^2}^2 \, dt,
\]

where \( I_\delta(\theta) \) is defined by

\[
I_\delta(\theta)(t) = \int_0^t (\mathcal{G}_\delta \partial_x)^{\frac{1}{2}} \theta(t') \, dt'
\]

and the expectation \( E = E_\mathcal{P} \) is an expectation with respect to the underlying probability measure \( \mathcal{P} \).

**Remark 3.2.14.** (i) As far as the proof of Proposition 3.2.12 is concerned, we only need to work with \( Y_{\delta,N} \) evaluated at time \( t = 1 \). As such, we could have stated Lemma 3.2.13 with \( Y_{\delta,N} \) in place of \( Y_{\delta,N}(1) \), thus allowing us to avoid introducing \( W(t) \) in (3.2.72) and \( Y(t) \) in (3.2.73). We, however, did not do so since the natural setting of the Boué-Dupuis formula is as stated above. For example, (3.2.74) allows us to choose a \( Y_\delta \)-dependent drift \( \theta \), which is crucial in showing non-normalizability of the focusing Gibbs measures \( \rho_\theta \). See [136].

(ii) In view of the discussion above, in order to prove Proposition 3.2.12, it is possible to work with a slightly different and weaker variational formula stated in [61, Proposition 4.4], where an expectation is taken with respect to a shifted measure.

In the following, we prove Proposition 3.2.12 by applying Lemma 3.2.13 to \( G_{\delta,N}^K(\nu) \) in (3.2.68). Before proceeding to the proof of Proposition 3.2.12, let us state a preliminary lemma on the pathwise regularity bounds of \( Y_{\delta,N}(1) \) and \( I_\delta(\theta)(1) \).

**Lemma 3.2.15.** (i) Let \( \varepsilon > 0 \) and fix finite \( p \geq 1 \). Then, given any \( 0 < \delta \leq \varepsilon \), we have

\[
E\left[ \| Y_{\delta,N}(1) \|_{W^{-\varepsilon,\infty}}^p + \| W(Y_{\delta,N}^2(1)) \|_{W^{-\varepsilon,\infty}}^p + \| W(Y_{\delta,N}^3(1)) \|_{W^{-\varepsilon,\infty}}^p \right] \leq C_{\varepsilon,p,\delta} < \infty,
\]

uniformly in \( N \in \mathbb{N} \). Furthermore, by restricting our attention to \( 2 \leq \delta \leq \infty \), we can choose the constant \( C_{\varepsilon,p,\delta} \) in (3.2.75) to be independent of \( \delta \).

(ii) Let \( 0 < \delta \leq \varepsilon \). For any \( \theta \in H_\delta \), we have

\[
\| I_\delta(\theta)(1) \|_{H^\frac{1}{2}}^2 \leq C_\delta \int_0^1 \| \theta(t) \|_{L^2}^2 \, dt,
\]

where the constant \( C_\delta > 0 \) can be chosen to be independent of \( 2 \leq \delta \leq \infty \).

**Proof.** By noting that \( \mathcal{L}(Y_{\delta,N}(1)) = \mathcal{L}(X_{\delta,N}) \), we see that Part (i) follows from Proposition 3.2.4. As for the bound (3.2.76), it follows from Minkowski’s and Cauchy-Schwarz’s inequalities and the lower bound (3.1.2) of \( K_\delta(\nu) \) that

\[
\| I_\delta(\theta)(1) \|_{H^\frac{1}{2}} = \left\| \left( \nabla \right)^{\frac{1}{2}} \left( \mathcal{G}_\delta \partial_x \right) \right\|_{L^2} \leq C_\delta \int_0^1 \| \theta(t) \|_{L^2} \, dt \leq C_\delta \int_0^1 \| \theta(t) \|_{L^2}^2 \, dt \leq \frac{1}{2} \theta(t) \|_{L^2}^2 \, dt.
\]

When \( 2 \leq \delta \leq \infty \), the lower bound (3.1.3) allows us to choose the constant \( C_\delta \) to be independent of \( 2 \leq \delta \leq \infty \). □

Fix \( 0 < \delta \leq \varepsilon \) and finite \( p \geq 1 \). We first prove the bound (3.2.70). In view of the Boué-Dupuis formula (Lemma 3.2.13), it suffices to establish a lower bound on

\[
\mathcal{M}_{\delta,N}(\theta) = E\left[ p \mathbf{R}_{\delta,N}(Y_{\delta}(1) + I_\delta(\theta)(1)) + \frac{1}{2} \int_0^1 \| \theta(t) \|_{L^2}^2 \, dt \right],
\]

with a slightly different and weaker variational formula stated in [61, Proposition 4.4], where an expectation is taken with respect to a shifted measure.
uniformly in $N \in \mathbb{N}$ and $\theta \in \mathbb{R}_+$. We set

$$Y_{\delta,N} = P_N Y_{\delta} = P_N Y_{\delta}(1) \quad \text{and} \quad \Theta_{\delta,N} = P_N \Theta_{\delta} = P_N I_\delta(\theta)(1).$$

From (3.2.67) and (3.1.18), we have

$$\text{From (3.2.67) and (3.1.18), we have} \quad \text{coming from (3.2.79) and (3.2.80). From (3.2.79) and (3.2.81) together with Lemmas 3.2.16 and}$$

(hence, from (3.2.77) and (3.2.15), we obtain

$$\text{where the first term on the right-hand side vanishes under the expectation. Hence, from (3.2.77)}$$

$$\text{and (3.2.78), we have}$$

$$\text{we establish a pathwise lower bound on} \quad \text{We now recall the following lemma from [136, Lemma 4.1], where the}$$

(i) \text{There exist small $\varepsilon > 0$ and a constant $c = c(p) > 0$ and $C_0 > 0$ such that}

$$p \left| \int_T W(Y_{\delta,N}^2) \Theta_{\delta,N} dx \right| \leq c \| W(Y_{\delta,N}^2) \|_{H^{\varepsilon,\infty}}^2 + \frac{1}{100} \| \Theta_{\delta,N} \|_{H^\frac{1}{2}}^2,$$

$$p \left| \int_T Y_{\delta,N} \Theta_{\delta,N}^2 dx \right| \leq c \| Y_{\delta,N} \|_{H^{\varepsilon,\infty}}^2 + \frac{1}{100} \left( \| \Theta_{\delta,N} \|_{H^\frac{1}{2}}^2 + \| \Theta_{\delta,N} \|_{L^2}^4 \right),$$

$$p \left| \int_T \Theta_{\delta,N}^3 dx \right| \leq \frac{1}{100} \| \Theta_{\delta,N} \|_{H^\frac{1}{2}}^2 + C_0p^2 \| \Theta_{\delta,N} \|_{L^2}^4,$$

uniformly in $N \in \mathbb{N}$ and $0 < \delta \leq \infty$.

(ii) \text{Let $A > 0$. Given any small $\varepsilon > 0$, there exists $c = c(\varepsilon, p, A) > 0$ such that}

$$A_p \left\{ \int_T \left( W(Y_{\delta,N}^2) + 2Y_{\delta,N} \Theta_{\delta,N} + \Theta_{\delta,N}^2 \right) dx \right\}^2 \geq \frac{A_p}{4} \| \Theta_{\delta,N} \|_{L^2}^4 - \frac{1}{100} \| \Theta_{\delta,N} \|_{H^\frac{1}{2}}^2 - c \left( \| Y_{\delta,N} \|_{H^{\varepsilon,\infty}}^2 + \left( \int_T W(Y_{\delta,N}^2) dx \right)^2 \right),$$

uniformly in $N \in \mathbb{N}$ and $0 < \delta \leq \infty$.

As in [136], we establish a pathwise lower bound on $\mathcal{M}_{\delta,N}(\theta)$ in (3.2.79), uniformly in $N \in \mathbb{N}$ and $\theta \in \mathbb{R}_+$, by making use of the positive terms:

$$\mathcal{U}_{\delta,N}(\theta) = E \left[ \frac{A_p}{4} \| \Theta_{\delta,N} \|_{L^2}^4 + \frac{1}{100} \left( \int_0^1 \| \theta(t) \|_{L^2}^2 dt \right) \right].$$

coming from (3.2.79) and (3.2.80). From (3.2.79) and (3.2.81) together with Lemmas 3.2.16 and 3.2.15 we obtain

$$\inf_{N \in \mathbb{N}} \inf_{\theta \in \mathbb{R}_+} \mathcal{M}_{\delta,N}(\theta) \geq \inf_{N \in \mathbb{N}} \inf_{\theta \in \mathbb{R}_+} \left\{ -C_{p,\delta,A} + \frac{1}{100} \mathcal{U}_{\delta,N}(\theta) \right\} \geq -C_{p,\delta,A} > -\infty,$$

(3.2.82)
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provided that \( A = A(p) \gg 1 \) is sufficiently large. Hence, the uniform (in \( N \)) bound (3.2.70) follows from Lemma 3.2.13 with (3.2.68) and (3.2.82).

Next, we restrict our attention to \( 2 \leq \delta \leq \infty \). In this case, the constant \( C_{\varepsilon,p,\delta} \) in (3.2.75) of Lemma 3.2.15 is independent of \( \delta \) and, as a result, we see that the constant \( C_{p,\delta,A} \) in (3.2.82) is also independent of \( 2 \leq \delta \leq \infty \). Therefore, the second bound (3.2.71) follows from Lemma 3.2.13 with (3.2.68) and (3.2.82). This concludes the proof of Proposition 3.2.12 and hence of Theorem 1.3.2 when \( k = 2 \).

3.3 Gibbs measures in the shallow-water regime

In this section we study the \( gKdV \)-limit, and let us concentrate on the regime \( 0 < \delta \ll 1 \). Consider the following (reformulated) \( gILW \):

\[
\partial_t v - \frac{3}{\delta} G_\delta(\partial_x^2 v) = \partial_x(v^3).
\] (3.3.1)

We present the proof of Theorem 1.3.4. Namely, we go over the construction and convergence in the shallow-water limit (\( \delta \to 0 \)) of the Gibbs measure \( \tilde{\mu}_\delta \) associated with the scaled \( gILW \) equation (1.3.8). For each fixed \( 0 < \delta < \infty \), the scaling transformation (1.1.13) simply introduces a constant factor, depending on \( \delta \). Hence, the regularity properties of the support of the base Gaussian measures \( \mu_\delta \) in (1.3.12) for the unscaled problem and \( \tilde{\mu}_\delta \) in (3.3.1) for the scaled problem are the same for each fixed \( 0 < \delta < \infty \), and thus we can repeat the argument in Section 3.2 to construct the Gibbs measure \( \tilde{\mu}_\delta \) supported on \( H^{-\varepsilon}(\mathbb{T}) \setminus L^2(\mathbb{T}) \), \( \varepsilon > 0 \), yielding Theorem 1.3.4(i) for each fixed \( 0 < \delta < \infty \). The main difference in this shallow-water regime appears in establishing uniform (in \( \delta \)) bounds and convergence as \( \delta \to 0 \). This is due to the singularity of the base Gaussian measures \( \tilde{\mu}_\delta, \delta \ll 1 \), supported on \( H^{-\varepsilon}(\mathbb{T}) \setminus L^2(\mathbb{T}) \), and \( \tilde{\mu}_0 \) in (3.3.2) supported on \( H^{\varepsilon}(\mathbb{T}) \setminus H^2(\mathbb{T}) \); see Proposition 3.3.1.

In Subsection 3.3.1, we first study the singularity and convergence properties of the base Gaussian measures. Then, we briefly go over the construction and convergence of the Gibbs measure \( \tilde{\mu}_\delta \) for the defocusing case \( (2N + 1) \) in Subsections 3.3.2 and 3.3.3. In Subsection 3.3.4 we discuss the variational approach to treat the \( k = 2 \) case.

3.3.1 Singularity of the base Gaussian measures

Given \( 0 < \delta < \infty \), let \( \tilde{\mu}_\delta \) be as in (1.3.14) and let \( \tilde{\mu}_0 \) be as in (1.3.20). Then, a typical element under \( \tilde{\mu}_\delta \) (and under \( \tilde{\mu}_0 \), respectively) is given by the Gaussian Fourier series \( \tilde{X}_\delta \) in (1.3.44) and by \( X_{gKdV} \) in (1.3.21), respectively. Given \( N \in \mathbb{N} \), set

\[
\tilde{X}_{\delta,N} = P_N \tilde{X}_{\delta,N} \quad \text{and} \quad X_{gKdV,N} = P_N X_{gKdV}.
\] (3.3.2)

Then, in view of (1.3.46), we see that, for each \( 0 < \delta < \infty \), \( \tilde{X}_{\delta,N} \) converges in \( L^p(\Omega) \) for any finite \( p \geq 1 \) and almost surely to the limit \( \tilde{X}_\delta \) in \( H^{-\varepsilon}(\mathbb{T}) \setminus L^2(\mathbb{T}) \), \( \varepsilon > 0 \), as \( N \to \infty \). On the other hand, it is well known [20] [33] that \( X_{gKdV,N} \) converges, in \( L^p(\Omega) \) and almost surely, to the limit \( X_{gKdV} \) in \( H^{\varepsilon}(\mathbb{T}) \setminus H^2(\mathbb{T}) \), \( \varepsilon > 0 \), as \( N \to \infty \).

**Proposition 3.3.1.** (i) Given any \( \varepsilon > 0 \) and finite \( p \geq 1 \), \( \tilde{X}_\delta \) converges to \( X_{gKdV} \) in \( L^p(\Omega; H^{-\varepsilon}(\mathbb{T})) \) and almost surely in \( H^{-\varepsilon}(\mathbb{T}) \), as \( \delta \to 0 \). In particular, the Gaussian measure \( \tilde{\mu}_\delta \) converges weakly to the Gaussian measure \( \mu_{gKdV} \), as \( \delta \to 0 \).

(ii) Let \( \varepsilon > 0 \). Then, for any \( 0 < \delta < \infty \), the Gaussian measures \( \tilde{\mu}_\delta \) and \( \mu_{gKdV} \) are singular as probability measures \( H^{-\varepsilon}(\mathbb{T}) \).

In Section 3.2 the convergence in total variation of \( \mu_\delta \) to \( \mu_\infty \) played an essential role in establishing the convergence in total variation of \( \rho_\delta \) to \( \rho_{gRO} \). Proposition 3.3.1 only provides weak convergence of the base Gaussian measures \( \tilde{\mu}_\delta \) to \( \mu_0 \), and the singularity between the base Gaussian measures suggests that we do not expect any stronger mode of convergence (such as convergence in total variation). As a result, we only expect weak convergence of the associated Gibbs measures \( \tilde{\rho}_\delta \) to \( \rho_{gKdV} \) in (1.3.50) in the shallow-water limit \( (\delta \to 0) \).
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Proof of Proposition 3.3.1. Let \( \varepsilon > 0 \). From (1.3.21), (1.3.44), and Lemma 3.1.7 we have

\[
\| \tilde{X}_\delta - X_{\text{gKdV}} \|_{L^2 H^{-\varepsilon}} \lesssim_p \| (\nabla)^{-\varepsilon} (\tilde{X}_\delta - X_{\text{gKdV}}) (x) \|_{L^2 L^2_x} \sim \left( \sum_{n \in \mathbb{Z}^*} \left( \frac{1}{(n)^{2\varepsilon}} \left( \frac{1}{L^2_{\delta}} (n) - \frac{1}{|n|} \right)^2 \right)^{\frac{1}{2}} \right).
\]  

(3.3.3)

It follows from (3.1.8) in Lemma 3.1.3 that the summand is bounded by \( (n)^{-1-2\varepsilon} \) uniformly in \( 0 < \delta \lesssim 1 \), which is summable in \( n \in \mathbb{Z}^* \). Moreover, from Lemma 3.1.3(ii), we see that, for each \( \delta > 0 \), the summand tends to 0 as \( \delta \to 0 \). Hence, by the dominated convergence theorem, we conclude that \( \tilde{X}_\delta \) converges to \( X_{\text{gKdV}} \) in \( L^p(\Omega; H^{-\varepsilon}(\mathbb{T})) \). As for almost sure convergence, we repeat a computation analogous to (3.3.3) but with (3.1.19) in place of \( E[|g_n|^2] \sim 1 \). We omit details. See (3.2.9) for an analogous argument in the unscaled case.

Next we prove part (ii) by using Lemma 3.2.2. From (1.3.44) and (1.3.21), we have

\[
\tilde{X}_\delta (\omega) = \sum_{n \in \mathbb{N}} \left( \frac{\text{Re} g_n}{\pi L^2_{\delta}} \cos (nx) - \frac{\text{Im} g_n}{\pi L^2_{\delta}} \sin (nx) \right)
\]

for \( 0 \leq \delta < \infty \) with the understanding that \( \tilde{X}_0 = X_{\text{gKdV}} \) and \( L_0 (n) = n^2 \). For \( n \in \mathbb{Z}^* \), set

\[
A_n = \frac{\text{Re} g_n}{\pi L^2_{\delta}(n)} \quad \text{and} \quad A_{-n} = -\frac{\text{Im} g_n}{\pi L^2_{\delta}(n)}
\]

and

\[
B_n = \frac{\text{Re} g_n}{\pi |n|} \quad \text{and} \quad B_{-n} = -\frac{\text{Im} g_n}{\pi |n|},
\]

with \( a_{\pm n} = E[A_n^2] = \frac{1}{\pi L_0(n)} \) and \( b_n = E[B_n^2] = \frac{1}{\pi n^2} \). Then, from Lemma 3.1.3(iv), we have

\[
\sum_{n \in \mathbb{Z}^*} \left( \frac{b_n}{a_n} - 1 \right)^2 = \sum_{n \in \mathbb{Z}^*} \frac{(n^2 - L_0(n))^2}{n^4} = \sum_{n \in \mathbb{Z}^*} h^2(n, \delta) = \infty
\]

for any \( \delta > 0 \), where \( h(n, \delta) \) is as in (3.1.9). Therefore, we conclude from Kakutani’s theorem (Lemma 3.2.2) that, for any \( 0 < \delta < \infty \), the Gaussian measures \( \tilde{\mu}_\delta \) and \( \tilde{\rho}_0 \) are mutually singular. \( \square \)

3.3.2 Construction of the Gibbs measures for the defocusing scaled gILW equation

In this subsection, we briefly go over the construction of the Gibbs measure \( \tilde{\mu}_\delta \), \( 0 < \delta < \infty \), for the scaled gILW equation (1.3.8) in the defocusing case \( k \in 2\mathbb{N} + 1 \). (Theorem 1.3.4(i)). We treat the \( k = 2 \) case in Subsection 3.3.4.

Fix the depth parameter \( 0 < \delta < \infty \). Given \( N \in \mathbb{N} \), let \( \tilde{X}_{\delta, N} = P_N \tilde{X}_\delta \), where \( \tilde{X}_\delta \) is defined in (1.3.44). Given \( k \in \mathbb{N} \), let

\[
\mathcal{W}(\tilde{X}^k_{\delta, N}) = H_k(\tilde{X}_{\delta, N}; \tilde{\sigma}_{\delta, N})
\]

(3.3.4)

denote the Wick power defined in (1.3.48) where \( \tilde{\sigma}_{\delta, N} \) is as in (1.3.47). Then, the truncated Gibbs measure \( \tilde{\rho}_{\delta, N} \) in (1.3.49) can be written as

\[
\tilde{\rho}_{\delta, N}(A) = Z_{\delta, N}^{-1} \int_{H^{-\varepsilon}} \mathbf{1}_{\{v \in \Delta A\}} e^{\frac{-1}{2} \int_{\Omega} \mathcal{W}(\tilde{X}^k_{\delta, N} (\omega)) d\tilde{\mu}_\delta (\omega)}
\]

\[
= Z_{\delta, N}^{-1} \int_{\Omega} \mathbf{1}_{\{X(\omega) \in \Delta \}} e^{\frac{-1}{2} \int_{\Omega} \mathcal{W}(\tilde{X}^k_{\delta, N} (\omega)) d\tilde{\mu}_\delta (\omega)},
\]

\( ^7 \)As in Section 3.2, we freely interchange the representations in terms of \( \tilde{X}_\delta \) and in terms of \( v \) distributed by \( \tilde{\mu}_\delta \), when there is no confusion.
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Then, given any finite \( p \geq 1 \), the sequence \( \{W(X_{\delta,N}^k)\}_{N \in \mathbb{N}} \) is Cauchy in \( L^p(\Omega; W^{s,\infty}(T)) \), \( s < 0 \), thus converging to a limit denoted by \( W(X_{\delta}^k) \). This convergence of \( W(X_{\delta,N}^k) \) to \( W(X_{\delta}^k) \) also holds almost surely in \( W^{s,\infty}(T) \). Furthermore, given any finite \( p \geq 1 \), we have

\[
\sup_{N \in \mathbb{N}} \sup_{0 < \delta \leq 1} \|W(X_{\delta,N}^k)\|_{W^{s,\infty}(\Omega)} < \infty
\]

and

\[
\sup_{0 < \delta \leq 1} \|W(X_{\delta,M}^k) - W(X_{\delta,N}^k)\|_{L^p(\Omega)} \rightarrow 0
\]

for any \( M \geq N \), tending to \( \infty \). In particular, the rate of convergence is uniform in \( 0 < \delta \leq 1 \).

As a corollary, the following two statements hold.

(i) Let \( 0 < \delta < \infty \). Given \( N \in \mathbb{N} \), let \( \tilde{\hat{R}}_{\delta,N}(v) = R_{\delta,N}(v;k+1) \) denote the truncated potential energy defined by

\[
\tilde{\hat{R}}_{\delta,N}(v) := \frac{1}{k+1} \int_T W((\text{P}_N v)^{k+1}) dx = \frac{1}{k+1} \int_{\mathbb{T}} H_{k+1}(\text{P}_N v; \tilde{\sigma}_{\delta,N}) dx,
\]

where \( \tilde{\sigma}_{\delta,N} \) is as in (3.3.47). Then, given any finite \( p \geq 1 \), the sequence \( \{\tilde{\hat{R}}_{\delta,N}(v)\}_{N \in \mathbb{N}} \) converges to the limit:

\[
\tilde{\hat{R}}_{\delta}(v) = \frac{1}{k+1} \int_T W(v^{k+1}) dx = \lim_{N \rightarrow \infty} \frac{1}{k+1} \int_{\mathbb{T}} W((\text{P}_N v)^{k+1}) dx
\]

in \( L^p(d\mu_{\delta}) \), as \( N \rightarrow \infty \). Furthermore, there exists \( \theta > 0 \) such that given any finite \( p \geq 1 \), we have

\[
\sup_{N \in \mathbb{N}} \sup_{0 < \delta \leq 1} \|\tilde{\hat{R}}_{\delta,N}(v)\|_{L^p(d\mu_{\delta})} < \infty,
\]

with \( \tilde{\hat{R}}_{\delta,\infty}(v) = \tilde{\hat{R}}_{\delta}(v) \), and

\[
\|\tilde{\hat{R}}_{\delta,M}(v) - \tilde{\hat{R}}_{\delta,N}(v)\|_{L^p(d\mu_{\delta})} \leq \frac{C_{k,\delta} \theta^{k+1}}{N^\theta}
\]

for any \( M \geq N \geq 1 \). For \( 0 < \delta \leq 1 \), we can choose the constant \( C_{k,\delta} \) in (3.3.8) to be independent of \( \delta \) and hence the rate of convergence of \( \tilde{\hat{R}}_{\delta,N}(v) \) to the limit \( \tilde{\hat{R}}_{\delta}(v) \) is uniform in \( 0 < \delta \leq 1 \).

(ii) Let \( 0 < \delta < \infty \). Given \( N \in \mathbb{N} \), let \( \tilde{\hat{F}}(u) = \tilde{\hat{F}}(u;k) \) be the truncated renormalized nonlinearity in (3.3.6) given by

\[
\tilde{\hat{F}}(v) := \partial_x \text{P}_N W((\text{P}_N v)^k) = \partial_x \text{P}_N H_k(\text{P}_N v; \tilde{\sigma}_{\delta,N}),
\]

where \( \tilde{\sigma}_{\delta,N} \) is as in (3.3.47). Then, given any finite \( p \geq 1 \), the sequence \( \{\tilde{\hat{F}}_{\delta,N}(v)\}_{N \in \mathbb{N}} \) is Cauchy in \( L^p(d\mu_{\delta}; H^s(T)) \), \( s < -1 \), thus converging to a limit denoted by \( \tilde{\hat{F}}(v) = \partial_x W(v^k) \). Furthermore, given any finite \( p \geq 1 \), we have

\[
\sup_{N \in \mathbb{N}} \sup_{0 < \delta \leq 1} \|\tilde{\hat{F}}_{\delta,N}(v)\|_{L^p(d\mu_{\delta})} < \infty
\]

and

\[
\sup_{0 < \delta \leq 1} \|\tilde{\hat{F}}_{M}(v) - \tilde{\hat{F}}_{N}(v)\|_{L^p(d\mu_{\delta})} \rightarrow 0
\]
for any $M \geq N$, tending to $\infty$. In particular, the rate of convergence of $\tilde{F}_N(v)$ to the limit $\tilde{F}(v)$ is uniform in $0 < \delta \leq 1$.

Proof. Proposition 3.3.2 follows from a straightforward modification of the proof of Proposition 3.2.4. The only notable difference is that instead of using the bounds (3.1.2) and (3.1.3) for $K_\delta(n)$, we need to use the bounds (3.1.7) and (3.1.8) for $L_\delta(n)$. We omit details.

Given $0 < \delta < \infty$ and $N \in \mathbb{N}$, we define $\tilde{G}_{\delta,N}(\tilde{X}_\delta) = \tilde{G}_{\delta,N}(\tilde{X}_\delta; k + 1)$ by

$$\tilde{G}_{\delta,N}(\tilde{X}_\delta) = e^{-\tilde{R}_{\delta,N}(\tilde{X}_\delta)} = e^{-\frac{1}{\delta} \int_{\mathbb{T}} \omega(\tilde{X}_{\delta,N}) \, dx},$$

where $\tilde{R}_{\delta,N}(\tilde{X}_\delta) = \tilde{R}_{\delta,N}(\tilde{X}_\delta; k + 1)$ is the truncated potential energy defined in (3.3.5). Then, a slight modification of the proof of Proposition 3.2.6 yields the following proposition.

Proposition 3.3.3. Let $k \in 2\mathbb{N} + 1$ and fix finite $p \geq 1$. Given any $0 < \delta < \infty$, we have

$$\sup_{N \in \mathbb{N}} \|G_{\delta,N}(\tilde{X}_\delta)\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \|\tilde{G}_{\delta,N}(v)\|_{L^p(\mu_\delta)} \leq C_{p,k,\delta} < \infty.$$  

(3.3.9)

In addition, the following uniform bound holds for $0 < \delta \leq 1$:  

$$\sup_{N \in \mathbb{N}} \|G_{\delta,N}(\tilde{X}_\delta)\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \|\tilde{G}_{\delta,N}(v)\|_{L^p(\mu_\delta)} \leq C_{p,k} < \infty.$$  

(3.3.9)

Define $\tilde{G}_\delta(\tilde{X}_\delta)$ by

$$\tilde{G}_\delta(\tilde{X}_\delta) = e^{-\tilde{R}_\delta(\tilde{X}_\delta)}$$

with $\tilde{R}_\delta(\tilde{X}_\delta)$ as in (3.3.6). Then, $G_{\delta,N}(\tilde{X}_\delta)$ converges to $\tilde{G}_\delta(\tilde{X}_\delta)$ in $L^p(\Omega)$. Namely, we have

$$\lim_{N \to \infty} \|G_{\delta,N}(\tilde{X}_\delta) - \tilde{G}_\delta(\tilde{X}_\delta)\|_{L^p(\Omega)} = 0.$$  

Furthermore, the convergence is uniform in $0 < \delta \leq 1$:

$$\lim_{N \to \infty} \sup_{0 < \delta \leq 1} \|G_{\delta,N}(\tilde{X}_\delta) - \tilde{G}_\delta(\tilde{X}_\delta)\|_{L^p(\Omega)} = 0.$$  

(3.3.10)

As a consequence, the uniform bounds (3.3.9) and (3.3.10) hold even if we replace supremum in $N \in \mathbb{N}$ by the supremum in $N \in \mathbb{N} \cup \{\infty\}$.

Theorem 3.3.4(i) follows as a direct corollary to Proposition 3.3.3, allowing us to define the limiting Gibbs measure $\tilde{\mu}_\delta$ in (3.5.4). See the discussion right after Proposition 3.2.6.

For $0 < \delta < \infty$, the Gibbs measure $\tilde{\mu}_\delta$ is equivalent to the base Gaussian measure $\mu_\delta$. Similarly, the Gibbs measure $\rho_{KdV}$ in (3.5.0) equivalent to the base Gaussian measure $\mu_0$. Recalling from Proposition 3.3.1 that the base Gaussian measures $\tilde{\mu}_\delta$, $0 < \delta < \infty$, and $\mu_0$ are mutually singular, we conclude that the Gibbs measures $\tilde{\mu}_\delta$ in (3.5.4) and $\rho_{KdV}$ in (3.5.0) are mutually singular. This proves the first claim in Theorem 3.3.4(iii).

Proof of Proposition 3.3.2 From (3.2.38) with (3.47), we have

$$-\tilde{R}_{\delta,N}(\tilde{X}_\delta) = -\frac{1}{k + 1} \int_{\mathbb{T}} H_{k+1}(\tilde{X}_{\delta,N}; \tilde{\sigma}_{\delta,N}) \, dx$$

$$\leq \frac{2\pi}{k + 1} \tilde{\sigma}_{\delta,N} a_{k+1} \leq \tilde{A}_{k,\delta} (\log(N + 1))^{\frac{1}{k+1}}$$  

(3.3.11)

for some $\tilde{A}_{k,\delta} > 0$, uniformly in $N \in \mathbb{N}$. Then, we can simply repeat the proof of Proposition 3.2.6 using Proposition 3.3.2 in place of Proposition 3.2.4. For $0 < \delta \leq 1$, it follows from (1.3.47) and Lemma 3.1.3 that the constant $\tilde{A}_{k,\delta}$ in (3.3.11) can be chosen to be independent of $0 < \delta \leq 1$. Similarly, by restricting our attention to $0 < \delta \leq 1$, we...
can choose the constant $c_{k,\delta}$ in an analogue of \((3.2.40)\) in the current setting to be independent of \(0 < \delta \leq 1\) since the constant $C_{k,\delta}$ in \((3.3.8)\) is independent of \(0 < \delta \leq 1\). Moreover, in applying Lemma \(3.1.8\) in Step 2 of the proof of Proposition \(3.2.6\) we need the uniform bound \((3.3.7)\), replacing \((3.2.10)\). This observation yields the uniform bounds \((3.3.9)\) and \((3.3.10)\).

**Remark 3.3.4.** Given $N$, define $\sigma_{gKdV,N}$ by

$$
\sigma_{gKdV,N} = \mathbb{E}[X_{gKdV,N}^2(x)] = \frac{1}{4\pi^2} \sum_{0 < |n| \leq N} \frac{2\pi}{n^2},
$$

which is uniformly bounded in $N \in \mathbb{N}$. Here, $X_{gKdV,N}$ is as in \((3.3.2)\). We then extend the definition of $L_\delta(n)$ and $\tilde{G}_{\delta,N}$ to the $\delta = 0$ case by setting $L_0(n) = n^2$ and

$$
\tilde{G}_{0,N}(X_{gKdV}) = e^{-\frac{1}{4\pi^2} \int \omega(X_{gKdV,N})dx},
$$

where $\omega(X_{gKdV,N}) = H_{k+1}(X_{gKdV,N}; \sigma_{gKdV,N})$. We also set

$$
\tilde{G}_{0}(X_{gKdV}) = e^{-\frac{1}{4\pi^2} \int \omega(X_{gKdV})dx},
$$

where $\omega(X_{gKdV}) = H_{k+1}(X_{gKdV}; \sigma_{gKdV})$ as in \((1.3.52)\). Then, by setting $\tilde{X}_0 = X_{gKdV}$, Proposition \(3.3.3\) extends to $\delta = 0$. In particular, the uniform bounds \((3.3.9)\) and \((3.3.10)\) hold for $0 \leq \delta \leq 1$.

### 3.3.3 Convergence of the Gibbs measures in the shallow-water limit

It remains to prove that the Gibbs measure $\tilde{\rho}_\delta$ converges weakly to $\rho_{gKdV}$ as $\delta \to 0$. We first state an analogue of Lemma \(3.2.8\)

**Lemma 3.3.5.** Let $k \in 2\mathbb{N} + 1$. Then, given $N \in \mathbb{N}$, we have

$$
\lim_{\delta \to 0} \|\tilde{G}_{\delta,N}(\tilde{X}_\delta) - \tilde{G}_{0,N}(X_{gKdV})\|_{L^p(\Omega)} = 0.
$$

As a corollary, we have

$$
\lim_{\delta \to 0} \|\tilde{G}_{\delta}(\tilde{X}_\delta) - \tilde{G}_{0}(X_{gKdV})\|_{L^p(\Omega)} = 0.
$$

In particular, the partition function $Z_\delta$ of the Gibbs measure $\tilde{\rho}_\delta$ in \((1.3.54)\) converges to the partition function $Z_{gKdV} = Z_0$ of the Gibbs measure $\rho_{gKdV} = \tilde{\rho}_0$ in \((1.3.50)\), as $\delta \to 0$.

**Proof.** From Lemma \(3.1.3\) we see that $\tilde{\sigma}_{\delta,N}$ in \((1.3.47)\) converges to $\sigma_{gKdV,N}$ in \((3.3.12)\) as $\delta \to 0$. With this observation, we can simply repeat the proof of Lemma \(3.2.8\). We omit details. \(\square\)

We are now ready to prove weak convergence of $\tilde{\rho}_\delta$ to $\rho_{gKdV}$ in the shallow-water limit ($\delta \to 0$). Fix small $\varepsilon > 0$. Let $A$ be any Borel subset of $H^{-1}(\mathbb{T})$ with $\tilde{\mu}_0(\partial A) = 0$, where $\partial A$ denotes the boundary of the set $A$. Our goal is to show that

$$
\tilde{\rho}_\delta(A) - \rho_{gKdV}(A) \to 0
$$

as $\delta \to 0$, which, together with the portmanteau theorem, yields the desired weak convergence.

By the triangle inequality, we have

$$
|\tilde{\rho}_\delta(A) - \rho_{gKdV}(A)| \leq |\tilde{\rho}_\delta(A) - \tilde{\rho}_\delta(N)(A)| + |\tilde{\rho}_\delta,N(A) - \rho_{gKdV,N}(A)| + |\rho_{gKdV,N}(A) - \rho_{gKdV}(A)|,
$$

\(\text{(3.3.15)}\)
where \( \rho_{gKdV,N} \) denotes the truncated Gibbs measure for \( \delta = 0 \) given by
\[
\rho_{gKdV,N}(A) = Z_{0,N}^{-1} \int_{H^{-\varepsilon}} 1_{\{v \in A\}} e^{-\frac{1}{\varepsilon} \int_{\mathbb{T}} W(v_N^{(n+1)}) dv_0(v)}
\]
\[
= Z_{0,N}^{-1} \int_{\Omega} 1_{\{X_{gKdV}(\omega) \in A\}} \tilde{G}_{0,N}(X_{gKdV}) dP(\omega)
\]
for any measurable set \( A \subset H^{-\varepsilon}(\mathbb{T}) \). From Proposition 3.3.3 and Remark 3.3.4, we have
\[
\lim_{N \to \infty} \sup_{0 \leq \delta \leq 1} |\rho_\delta(A) - \rho_{\delta,N}(A)|
\]
\[
= \lim_{N \to \infty} \sup_{0 \leq \delta \leq 1} \|\tilde{G}_{\delta,N}(\tilde{X}_\delta) - \tilde{G}_{\delta}(\tilde{X}_\delta)\|_{L^1(\Omega)}
\]
\[
= 0,
\]
with the identification \( \tilde{X}_0 = X_{gKdV}, \tilde{\rho}_0 = \rho_{gKdV} \), and \( \tilde{\rho}_{0,N} = \rho_{gKdV,N} \), where \( \tilde{G}_{0,N}(\tilde{X}_0) \) and \( \tilde{G}_{\delta}(\tilde{X}_\delta) \) are as in (3.3.13) and (3.3.14), respectively. Hence, in view of (3.3.15), (3.3.16), and (3.3.17), it suffices to prove
\[
\lim_{\delta \to 0} |\tilde{\rho}_{\delta,N}(A) - \rho_{gKdV,N}(A)|
\]
\[
= \lim_{\delta \to 0} \|\tilde{G}_{\delta,N}(\tilde{X}_\delta) 1_A(\tilde{X}_\delta)\| - Z_{0,N} \|E[\tilde{G}_{\delta,N}(X_{gKdV}) 1_A(X_{gKdV})]\|
\]
\[
= 0
\]
for some \( N \in \mathbb{N} \).

First, note that it suffices to show that
\[
E[\tilde{G}_{\delta,N}(\tilde{X}_\delta) 1_A(\tilde{X}_\delta)] - E[\tilde{G}_{0,N}(X_{gKdV}) 1_A(X_{gKdV})] \to 0
\]
as \( \delta \to 0 \) since, by taking \( A = H^{-\varepsilon}(\mathbb{T}) \), (3.3.19) implies \( Z_{\delta,N} \to Z_{0,N} \) as \( \delta \to 0 \).

By the triangle inequality, we have
\[
\|E[\tilde{G}_{\delta,N}(\tilde{X}_\delta) 1_A(\tilde{X}_\delta)] - E[\tilde{G}_{0,N}(X_{gKdV}) 1_A(X_{gKdV})]\|
\]
\[
\leq E[|\tilde{G}_{\delta,N}(\tilde{X}_\delta) - \tilde{G}_{0,N}(X_{gKdV})|]
\]
\[
+ E[\tilde{G}_{0,N}(X_{gKdV}) 1_A(\tilde{X}_\delta) - 1_A(X_{gKdV})].
\]

From Lemma 3.3.5, we have
\[
E[|\tilde{G}_{\delta,N}(\tilde{X}_\delta) - \tilde{G}_{0,N}(X_{gKdV})|] \to 0,
\]
as \( \delta \to \infty \). As for the second term on the right-hand side of (3.3.20), we first note that \( \sigma_{gKdV,N} \) defined in (3.3.12) is uniformly bounded in \( N \in \mathbb{N} \). Then, together with (3.2.38) and (3.3.13), we conclude that
\[
0 < \tilde{G}_{0,N}(X_{gKdV}(\omega)) \lesssim 1,
\]
uniformly in \( \omega \in \Omega \) and \( N \in \mathbb{N} \). Hence, from (3.3.22) and \( \tilde{\mu}_0(\partial A) = 0 \) (which implies \( E[1_{\partial A}(X_{gKdV})] = 0 \)), we have
\[
E[\tilde{G}_{0,N}(X_{gKdV}) 1_A(\tilde{X}_\delta) - 1_A(X_{gKdV})]
\]
\[
\lesssim E[1_A(\tilde{X}_\delta) - 1_A(X_{gKdV})]
\]
\[
= E[1_{\text{int}A}(X_{gKdV}) \cdot |1_A(\tilde{X}_\delta) - 1_A(X_{gKdV})|]
\]
\[
+ E[1_{\text{int}A}(X_{gKdV}) \cdot |1_A(\tilde{X}_\delta) - 1_A(X_{gKdV})|],
\]
where \( \text{int}A \) denotes the interior of \( A \) given by \( \text{int}A = A \setminus \partial A \). From Proposition 3.3.1(i) and the
openness of $\text{int} A$ and $\text{int} A^c$, the integrands of the terms on the right-hand side of (3.3.23) tend to 0 as $\delta \to 0$. Hence, by the bounded convergence theorem, we conclude that

$$E[\tilde{G}_{0,N}(X_{\delta,KdV})|1_A(\tilde{X}_\delta) - 1_A(X_{\delta,KdV})] \longrightarrow 0,$$

(3.3.24)

as $\delta \to 0$. Therefore, putting (3.3.20), (3.3.21), and (3.3.24) together, we conclude (3.3.19), which in turn implies (3.3.18). Finally, from (3.3.16), (3.3.17), and (3.3.18), we conclude (3.3.15), namely, weak convergence of $\tilde{\rho}_s$ to $\rho_{\delta,KdV}$ as $\delta \to 0$. This concludes the proof of Theorem 1.3.4 when $k \in 2N + 1$.

### 3.3.4 Gibbs measures for the scaled ILW equation: variational approach

We conclude this section by briefly going over the proof of Theorem 1.3.4 when $k = 2$, based on the variational approach as in Subsection 3.2.4 The major part of the argument follows exactly as in Subsection 3.2.4 and thus we only describe necessary definitions and steps.

Fix $K > 0$ in the remaining part of this section. Given $0 \leq \delta < \infty$ and $N \in \mathbb{N}$, define the truncated density $\tilde{G}_{\delta,N}^K(v)$ by

$$\tilde{G}_{\delta,N}^K(v) = \chi_K \left( \int_{\mathbb{T}} W(v_N^2)dx \right) e^{-\frac{1}{2} \int_{\mathbb{T}} v_N^2 dx}$$

$$= \chi_K \left( \int_{\mathbb{T}} H_2(v_N; \tilde{\sigma}_{\delta,N})dx \right) e^{-\frac{1}{2} \int_{\mathbb{T}} v_N^2 dx},$$

where $v_N = P_{N} v$, $\tilde{\sigma}_{\delta,N}$ is as in (1.3.47) when $0 < \delta < \infty$, and $\sigma_{0,N} = \sigma_{\delta,KdV,N}$. As in the unscaled case discussed in Subsection 3.2.4 Theorem 1.3.4 for $k = 2$ follows once we prove the following uniform bounds.

**Proposition 3.3.6.** Fix finite $p \geq 1$ and $K > 0$. Then, given any $0 \leq \delta < \infty$, we have

$$\sup_{N \in \mathbb{N}} \|\tilde{G}_{\delta,N}^K(\tilde{X}_\delta)\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \|\tilde{G}_{\delta,N}^K(v)\|_{L^p(d\tilde{\mu})} \leq C_{p,\delta,K} < \infty.$$

In addition, the following uniform bound holds for $0 \leq \delta \leq 1$:

$$\sup_{N \in \mathbb{N}} \sup_{0 \leq \delta \leq 1} \|\tilde{G}_{\delta,N}^K(\tilde{X}_\delta)\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \sup_{0 \leq \delta \leq 1} \|\tilde{G}_{\delta,N}^K(v)\|_{L^p(d\tilde{\mu})} \leq C_{p,K} < \infty.$$

Once we have Proposition 3.3.6 we can argue exactly as in Subsection 3.2.4 to conclude Theorem 1.3.4 In particular, (3.2.66) and (3.3.66) provide a bound on the truncated density $\tilde{G}_{\delta,N}$, uniformly in $0 \leq \delta \leq 1$, replacing the defocusing bound (3.3.11). We omit details.

In order to prove Proposition 3.3.6 we consider the truncated density with a taming by a power of the Wick-ordered $L^2$-norm as in Subsection 3.2.4 Given $0 \leq \delta < \infty$ and $N \in \mathbb{N}$, set

$$\tilde{R}_{\delta,N}(v) = \frac{1}{3} \int_{\mathbb{T}} v_N^3 dx + A \left| \int_{\mathbb{T}} W(v_N^2) dx \right|^2,$$

where $W(v_N^2) = W_{\delta,N}(v_N^2) = H_2(v_N; \tilde{\sigma}_{\delta,N})$. Then, we also define the truncated density with a taming by a power of the Wick-ordered $L^2$-norm:

$$\tilde{G}_{\delta,N}^K(v) = e^{-\tilde{R}_{\delta,N}(v)} = e^{-\frac{1}{2} \int_{\mathbb{T}} v_N^2 dx - A \left| \int_{\mathbb{T}} W(v_N^2) dx \right|^2}$$

for some suitable $A > 0$. Then, from (3.2.69), we have

$$\tilde{G}_{\delta,N}^K(v) \leq C_{A,K} \cdot \tilde{G}_{\delta,N}^K(v)$$

and, hence, Proposition 3.3.6 follows once we prove the following uniform bounds.
Proposition 3.3.7. Fix finite $p \geq 1$. Then, there exists $A_0 = A_0(p) > 0$ such that
\[
\sup_{N \in \mathbb{N}} \|G_{a,N}^K(X_\delta)\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \|G_{a,N}^K(v)\|_{L^p(d\mu)} \leq C_{p,\delta,K,A} < \infty
\]
for any $0 \leq \delta < \infty$, $K > 0$, and $A \geq A_0$. In addition, the following uniform bound holds for $0 \leq \delta \leq 1$:
\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq \delta \leq 1} \|G_{a,N}^K(X_\delta)\|_{L^p(\Omega)} = \sup_{N \in \mathbb{N}} \|G_{a,N}^K(v)\|_{L^p(d\mu)} \leq C_{p,K,A} < \infty
\]
for any $K > 0$ and $A \geq A_0$.

In order to set up the variational formulation, let us introduce some notations. Define $\tilde{Y}_\delta(t)$ by
\[
\tilde{Y}_\delta(t) = \left(\frac{3}{2}G_{a}\partial_x\right)^{-\frac{1}{2}}W(t),
\]
where $W(t)$ is as in (3.2.72) and $(\frac{3}{2}G_{a}\partial_x)^{-\frac{1}{2}}$ is the Fourier multiplier operator with the multiplier $(L_{a}(n))^{-\frac{1}{2}}$ with $L_{a}(n)$ as in (1.3.45). In view of (1.3.44), we have $L(\tilde{Y}_\delta(1)) = \tilde{\mu}_\delta$. Given $N \in \mathbb{N}$, we set $\tilde{Y}_{\delta,N} = P_N\tilde{Y}_\delta$. The variational formulation in the current problem is given by the following lemma.

Lemma 3.3.8. Given $0 \leq \delta < \infty$, let $\tilde{Y}_\delta$ be as in (3.3.25). Fix $N \in \mathbb{N}$. Suppose that $F : C^\infty(\mathbb{T}) \to \mathbb{R}$ is measurable such that $E[F(\tilde{Y}_{\delta,N}(1))] < \infty$ and $E[|e^{-F(\tilde{Y}_{\delta,N}(1))}|^q] < \infty$ for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have
\[
-\log E[e^{-F(\tilde{Y}_{\delta,N}(1))}] = \inf_{\theta \in \mathcal{H}_a} E\left[F(\tilde{Y}_{\delta,N}(1) + P_N\tilde{I}_\delta(\theta)(1)) + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L_2}^2 dt\right],
\]
where $\tilde{I}_\delta(\theta)$ is defined by
\[
\tilde{I}_\delta(\theta)(t) = \int_0^t (\frac{3}{2}G_{a}\partial_x)^{-\frac{1}{2}}\theta(t')dt'.
\]

With Lemma 3.3.8 in hand, we can proceed as in Subsection 3.2.4 to prove Proposition 3.3.7 by using Lemma 3.2.16 and the following lemma.

Lemma 3.3.9. (i) Let $\varepsilon > 0$ and fix finite $p \geq 1$. Then, given any $0 \leq \delta < \infty$, we have
\[
E\left[\|\tilde{Y}_{\delta,N}(1)\|_{W_{-\varepsilon,\infty}}^p + \|W(\tilde{Y}_{\delta,N}^2(1))\|_{W_{-\varepsilon,\infty}}^p + \|W(\tilde{Y}_{\delta,N}^3(1))\|_{W_{-\varepsilon,\infty}}^p\right] \leq C_{\varepsilon,\delta,p} < \infty,
\]
uniformly in $N \in \mathbb{N}$. Furthermore, by restricting our attention to $0 \leq \delta \leq 1$, we can choose the constant $C_{\varepsilon,\delta,p}$ in (3.3.26) to be independent of $\delta$.

(ii) Let $0 \leq \delta < \infty$. For any $\theta \in \mathcal{H}_a$, we have
\[
\|\tilde{I}_\delta(\theta)(1)\|_{H_2}^2 \leq \int_0^1 \|\theta(t)\|_{L_2}^2 dt,
\]
where $\mathcal{H}_a$ denotes the collection of drifts, which are progressively measurable processes belonging to $L^2([0,1];L_{a,p}(\mathbb{T}))$, $\mathbb{P}$-almost surely, as in Subsection 3.2.4.

The proof of Lemma 3.3.9 follows exactly as in the proof of Lemma 3.2.16 using the lower bounds (3.1.7) and (3.1.8) of $L_{a}(n)$ (in place of (3.1.12) and (3.1.13)). We omit details.

We conclude this section by recalling Proposition 3.3.7 implies Proposition 3.3.6 which in turn implies Theorem 1.3.4 for $k = 2$. 

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3.4 Dynamical problem

In this section, we study the dynamical problem associated with the Gibbs measures constructions in the previous sections. In the following, we only consider the deep-water regime \(0 < \delta \leq \infty\) (namely, we work on the unsealed problem (1.1.1)) and present the proof of Theorem 1.3.7 since Theorem 3.3.9 in the shallow-water regime (\(0 \leq \delta < \infty\)) follows from a similar argument. Our main strategy is to use a compactness argument as in \([30, 139, 134]\). In fact, as mentioned in the introduction, the proof of Theorem 1.3.7(i) follows from exactly the same argument as that presented in [139 Section 5]. As for the dynamical convergence result in Theorem 1.3.7(ii), we can repeat the same argument but with one additional key ingredient: the uniform (in \(\delta\) and \(N\)) integrability of the (truncated) densities (Proposition 3.2.6). For conciseness of the presentation, we restrict our attention to \(2 \leq \delta \leq \infty\) in the following and discuss the proof of Theorem 1.3.7. For each fixed \(0 < \delta < 2\), the same argument (without uniformity in \(\delta\)) applies to yield Theorem 1.3.7(i).

In the remaining part of this section, fix \(k \in 2\mathbb{N} + 1\) and \(s < 0\). The \(k = 2\) case follows from exactly the same argument by replacing the truncated Gibbs measure \(\rho_{\delta,N}\) in (1.3.37) and the Gibbs measure \(\rho_{\delta}\) in (1.3.41) by \(\rho_{\delta,N}\) in (1.3.42) and \(\rho_{\delta}\) in (1.3.43), respectively, and thus we omit details. In Subsection 3.4.1, we first study the truncated gILW equation (1.3.63) and construct global-in-time invariant Gibbs dynamics associated with the truncated Gibbs measure \(\rho_{\delta,N}\) in (1.3.37) for each \(N \in \mathbb{N}\) and \(2 \leq \delta \leq \infty\); see Lemma 3.4.1 below. This allows us to construct a probability measure \(\nu_{\delta,N} = \rho_{\delta,N} \circ \Phi_{\delta,N}^{-1}\) on space-time functions as the pushforward of the truncated Gibbs measure \(\rho_{\delta,N}\) under the solution map \(\Phi_{\delta,N}\) for the truncated gILW equation (1.3.63). Then, by using the uniform (in \(\delta\) and \(N\)) bound on the (truncated) densities (Proposition 3.2.6), we prove that \(\{\nu_{\delta,N}\}_{2 \leq \delta \leq \infty, N \in \mathbb{N}}\) is tight (Proposition 3.4.2). The main new point in this work is that we prove tightness not only in the frequency cutoff parameter \(N \in \mathbb{N}\) but also in the depth parameter \(2 \leq \delta \leq \infty\). In Subsection 3.4.2, we then present the proof of Theorem 1.3.7 by constructing the limiting dynamics. For each fixed \(2 \leq \delta \leq \infty\), we can simply repeat the argument in \([30, 139, 134]\), based on the Skorokhod representation theorem (Lemma 3.1.14), and construct the limiting invariant Gibbs dynamics (without uniqueness) as \(N \to \infty\), yielding Theorem 1.3.7(i).

As for proving Theorem 1.3.7(ii), by exploiting the tightness of \(\{\nu_{\delta,N}\}_{2 \leq \delta \leq \infty, N \in \mathbb{N}}\), we use a diagonal argument together with the triangle inequality for the Lévy-Prokhorov metric, characterizing weak convergence, to show that there exists a sequence \(\{\delta_m\}_{m \in \mathbb{N}}\), tending to \(\infty\), such that \(u_{\delta_m}\) converges almost surely to some limit \(u\) in \(C(\mathbb{R}; H^s(\mathbb{T}))\). Here, in order to have the claimed almost sure convergence of \(u_{\delta_m}\) to \(u\), we apply the Skorokhod representation theorem (Lemma 3.1.14). Furthermore, in order to show that \(u_{\delta_m}, m \in \mathbb{N}\), satisfies the renormalized gILW equation (1.3.59), we need to apply the Skorokhod representation theorem (Lemma 3.1.14) infinitely many times (i.e. once for each \(m \in \mathbb{N}\)).

3.4.1 Pushforward of the truncated Gibbs measure

Given \(2 \leq \delta \leq \infty\) and \(N \in \mathbb{N}\), consider the truncated gILW equation (1.3.63):

\[
\partial_t u_{\delta,N} - \mathcal{G}_\delta \partial_x^2 u_{\delta,N} = F_N(u_{\delta,N}) = \partial_x P_N \mathcal{W}(P_N u_{\delta,N})^k = \partial_x P_N H_k(P_N u_{\delta,N}; \sigma_{\delta,N}),
\]

where \(\sigma_{\delta,N}\) is as in (1.3.35) and \(F_N\) is as in (3.2.21). We first prove global well-posedness of (3.4.1) and invariance of the truncated Gibbs measure \(\rho_{\delta,N}\) defined in (1.3.37).

**Lemma 3.4.1.** Let \(2 \leq \delta \leq \infty, N \in \mathbb{N}, \) and \(s < 0\). Then, the truncated gILW equation (3.4.1) is globally well-posed in \(H^s(\mathbb{T})\). Moreover, the truncated Gibbs measure \(\rho_{\delta,N}\) is invariant under the dynamics of (3.4.1).

**Proof.** The proof of this lemma follows from that of Lemma 5.1 in [139] and thus we will be brief...
here. We first decompose (3.4.1) into two parts:
\[
 u_{\delta,N} = u_{\delta,N}^{\text{low}} + u_{\delta,N}^{\text{high}} = P_N u_{\delta,N} + P_N^\perp u_{\delta,N},
\]  
(3.4.2)
where \( P_N = \text{Id} - P_N \). Then, \( u_{\delta,N}^{\text{low}} \) and \( u_{\delta,N}^{\text{high}} \) satisfy the following equations:

(i) nonlinear dynamics on the low-frequency part \( \{0 < |n| \leq N\} \):
\[
 \partial_t u_{\delta,N}^{\text{low}} - \delta_\sigma \partial^2_x u_{\delta,N}^{\text{low}} = \partial_x P_N H_k(u_{\delta,N}^{\text{low}}; \sigma_{\delta,N}).
\]  
(3.4.3.4)

(ii) linear dynamics on the high frequency part \( \{|n| > N\} \):
\[
 \partial_t u_{\delta,N}^{\text{high}} = -G \partial^2_x u_{\delta,N}^{\text{high}} = 0.
\]  
(3.4.4)

We now view the equations (3.4.3) and (3.4.4) on the Fourier side. As a decoupled system of linear equation (for each frequency \(|n| > N\), (3.4.4) is globally well-posed. As for (3.4.3), it is a system of finitely many ODEs with a Lipschitz vector field and thus by the Cauchy-Lipschitz theorem, it is locally well-posed. Furthermore, a direct computation shows that the \( L^2 \)-norm of \( u_{\delta,N}^{\text{low}} \) is conserved under the flow of (3.4.3), which yields global well-posedness of (3.4.3). Putting together, we conclude that (3.4.1) is globally well-posed.

Next, we prove invariance of the truncated Gibbs measure \( \rho_{\delta,N} \). We first write \( \rho_{\delta,N} \) in (1.3.37) as
\[
 \rho_{\delta,N} = \rho_{\delta,N}^{\text{low}} \otimes \rho_{\delta,N}^{\text{high}},
\]  
(3.4.5)
where \( \rho_{\delta,N}^{\text{low}} \) and \( \rho_{\delta,N}^{\text{high}} \) are given as follows:

(i) the low-frequency component \( \rho_{\delta,N}^{\text{low}} \) is the finite-dimensional Gibbs measure on \( P_N H^s(\mathbb{T}) \), defined by
\[
d\rho_{\delta,N}^{\text{low}}(u) = Z_{\delta,N}^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}} H_k(u; \sigma_{\delta,N}) ds} d\mu_{\delta,N}^{\text{low}}(u),
\]
where \( \mu_{\delta,N}^{\text{low}} = (P_N)_{*} \mu_\delta \) is the pushforward image measure under \( P_N \) of the base Gaussian measure \( \mu_\delta \) in (1.3.12). Namely, \( \mu_{\delta,N}^{\text{low}} \) is the induced probability measure under the map \( \omega \in \Omega \mapsto X_{\delta,N}(\omega) = P_N X_\delta(\omega) \), where \( X_\delta \) is as in (1.3.32).

(ii) the high-frequency component \( \rho_{\delta,N}^{\text{high}} \) is nothing but the Gaussian measure \( (P_N^\perp)_{*} \mu_\delta \) given as the (infinite) product of Gaussian measures at each frequency \(|n| > N\):
\[
 (Z_{\delta,N}^{-1})^{-1} \bigotimes_{|n| > N} e^{-\frac{1}{2} \|u_n\|_{H^s(\mathbb{T})}^2} du_n(n).
\]  
(3.4.6)

By the classical Liouville theorem and the conservation of the (truncated) Hamiltonian for (3.4.3), we see that the Gibbs measure \( \rho_{\delta,N}^{\text{low}} \) is invariant under the flow of (3.4.3). On the other hand, the linear dynamics (3.4.4) acts as a rotation on the Fourier coefficient at each frequency \(|n| > N\), preserving the Gaussian measure at each frequency \(|n| > N\) in (3.4.6). As a result, the Gaussian measure \( \rho_{\delta,N}^{\text{high}} = (P_N^\perp)_{*} \mu_\delta \) is invariant under the linear dynamics (3.4.4). In view of (3.4.2) and (3.4.5), we conclude invariance of the truncated Gibbs measure \( \rho_{\delta,N} \) under the flow of the truncated gIWin equation (3.4.1). 

As a consequence of Lemma 3.4.1, we can define the solution map \( \Phi_{\delta,N} : H^s(\mathbb{T}) \to C(\mathbb{R}; H^s(\mathbb{T})) \) associated to (3.4.1). More precisely, for \( t \in \mathbb{R} \), we define \( \Phi_{\delta,N}(t) : H^s(\mathbb{T}) \to H^s(\mathbb{T}) \) by
\[
 \phi \in H^s(\mathbb{T}) \mapsto \Phi_{\delta,N}(t)(\phi) = u_{\delta,N}(t),
\]  
(3.4.7)
where $u_{\delta,N}$ is the global-in-time solution to the truncated gILW equation (3.4.1) with initial data $u_{\delta,N}(0) = \phi$.

Next, we introduce the pushforward image measure $\nu_{\delta,N}$ of the truncated Gibbs measure $\rho_{\delta,N}$ under the solution map $\Phi_{\delta,N}$:

$$\nu_{\delta,N} = \rho_{\delta,N} \circ \Phi_{\delta,N}^{-1}.$$  \hspace{1cm} (3.4.8)

Here, we view $\nu_{\delta,N}$ as a probability measure on $C(\mathbb{R}; H^{s}(\mathbb{T}))$ endowed with the compact-open topology, induced by the following metric:

$$\text{dist}(u, v) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|u - v\|_{C([-j, j]; H^{s})}}{1 + \|u - v\|_{C([-j, j]; H^{s})}}.$$  \hspace{1cm} (3.4.9)

Recall that, under this topology, a sequence $\{u_n\}_{n \in \mathbb{N}} \subset C(\mathbb{R}; H^{s}(\mathbb{T}))$ converges if and only if it converges uniformly on $[-K, K]$ for each finite $K > 0$. We also recall that the metric space $(C(\mathbb{R}; H^{s}(\mathbb{T})), \text{dist})$ is complete and separable.

Then, it follows from the local Lipschitz continuity of $\Phi_{\delta,N}$ that $\Phi_{\delta,N}$ is continuous from $H^{s}(\mathbb{T})$ into $C(\mathbb{R}; H^{s}(\mathbb{T}))$, which shows that $\nu_{\delta,N}$ is a well-defined probability measure on $C(\mathbb{R}; H^{s}(\mathbb{T}))$ endowed with the compact-open topology. Note that we have

$$\int_{C(\mathbb{R}; H^{s})} F(u) d\nu_{\delta,N}(u) = \int_{H^{s}} F(\Phi_{\delta,N}(\phi)) d\rho_{\delta,N}(\phi)$$

for any bounded measurable function $F : C(\mathbb{R}; H^{s}(\mathbb{T})) \to \mathbb{R}$.

Our main goal in this subsection is to prove the following tightness result on $\{\nu_{\delta,N}\}_{2 \leq \delta \leq \infty, N \in \mathbb{N}}$. We point out that tightness holds not only over $N \in \mathbb{N}$ but also over $2 \leq \delta \leq \infty$, which is the key new feature of this proposition.

**Proposition 3.4.2.** Let $s < 0$. Then, the family $\{\nu_{\delta,N}\}_{2 \leq \delta \leq \infty, N \in \mathbb{N}}$ of probability measures on $C(\mathbb{R}; H^{s}(\mathbb{T}))$ is tight, and hence is relatively compact.

Before proceeding to the proof of Proposition 3.4.2, we state two auxiliary lemmas. The first lemma establishes uniform (in $\delta$ and $N$) space-time bounds on the solutions to the truncated gILW equation (3.4.1). We postpone its proof to the end of this subsection.

Given $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, we define the space $W^{1,p}_{T}H^{s}_2 = W^{1,p}([-T, T]; H^{s}(\mathbb{T}))$ by the norm:

$$\|u\|_{W^{1,p}_{T}H^{s}_2} = \|u\|_{L^{p}_{T}H^{s}} + \|\partial_t u\|_{L^{p}_{T}H^{s}}.$$  \hspace{1cm} (3.4.10)

**Lemma 3.4.3.** Let $s < 0$, and fix finite $p \geq 1$. Then, there exists $C_p > 0$ such that

$$\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|u\|_{L^{p}_{T}H^{s}} \|u\|_{L^{p}_{T}(d\nu_{\delta,N})} \leq C_p T^{\frac{s}{2}},$$

$$\sup_{N \in \mathbb{N}} \sup_{2 \leq \delta \leq \infty} \|u\|_{W^{1,p}_{T}H^{s}_2} \|u\|_{L^{p}_{T}(d\nu_{\delta,N})} \leq C_p T^{\frac{s}{2}}.$$  \hspace{1cm} (3.4.11)

The following interpolation lemma allows us to control the Hölder regularity (in time) by the two quantities controlled in Lemma 3.4.3 above. For $\alpha \in (0, 1)$ and $s \in \mathbb{R}$, define the space $C^{\alpha}_{T}H^{s}_2 = C^{\alpha}([-T, T]; H^{s}(\mathbb{T}))$ by the norm

$$\|u\|_{C^{\alpha}_{T}H^{s}_2} = \sup_{t_1, t_2 \in [-T, T] \atop t_1 \neq t_2} \frac{\|u(t_1) - u(t_2)\|_{H^{s}}}{|t_1 - t_2|^\alpha} + \|u\|_{L^{\infty}_{T}H^{s}}.$$  \hspace{1cm} (3.4.12)

**Lemma 3.4.4 (\cite{29} Lemma 3.3).** Let $T > 0$ and $1 \leq p \leq \infty$. Suppose that $u \in L^{p}_{T}H^{s}_2$ and

\footnote{Recall that the space of continuous functions from a separable metric space $X$ to another separable metric space $Y$ with the compact-open topology is separable; see \cite{107}. See also the paper \cite{27} Corollary 3.3.}
\[ \partial_t u \in L^p_T H^{s_2}_x \text{ for some } s_2 \leq s_1. \text{ Then, for } \delta > p^{-1}(s_1 - s_2), \text{ we have} \]
\[ \|u\|_{L^p_T H^{s_2}_x} \leq \|u\|_{L^p_T H^{s_1}_x} + \|u\|_{W^{1,p}_T H^{s_2}_x}. \quad (3.4.13) \]

Moreover, there exist \( \alpha > 0 \) and \( \theta \in [0, 1] \) such that for all \( t_1, t_2 \in [-T, T] \), we have
\[ \|u(t_2) - u(t_1)\|_{H^{s_2-\theta}_x} \leq \|t_2 - t_1\|^\theta \|u\|_{L^p_T H^{s_1}_x} + \|u\|_{W^{1,p}_T H^{s_2}_x}. \quad (3.4.14) \]

As a consequence, we have
\[ \|u\|_{C_T^\alpha H^{s_2-\theta}_x} \leq \|u\|_{L^p_T H^{s_1}_x} + \|u\|_{W^{1,p}_T H^{s_2}_x}. \quad (3.4.15) \]

**Proof.** As for \((3.4.13)\) and \((3.4.14)\), see the proof of Lemma 3.3 in [30]. The bound \((3.4.15)\) follows from \((3.4.12)\), \((3.4.13)\), and \((3.4.14)\) with Young’s inequality.

We present the proof of Proposition 3.4.2.

**Proof of Proposition 3.4.2.** Let \( s < s_1 < s_2 < 0 \) and \( \alpha (0, 1) \). By the Arzelà-Ascoli theorem, the embedding \( C^0([-T, T]; H^s(\mathbb{T})) \subseteq C([-T, T]; H^s(\mathbb{T})) \) is compact for each \( T > 0 \). From Lemma 3.4.4 (with large \( p \gg 1 \)) and Lemma 3.4.3, we have
\[ \sup_{N \in \mathbb{N}} \sup_{2 \leq s \leq \infty} \left\{ \|u\|_{C_T^\alpha H^{s_1}_x} \right\}_{L^p(dv_{s,N})} \leq \sup_{N \in \mathbb{N}} \sup_{2 \leq s \leq \infty} \left\{ \|u\|_{L^p_T H^s_x} \right\}_{L^p(dv_{s,N})} + \sup_{N \in \mathbb{N}} \sup_{2 \leq s \leq \infty} \left\{ \|u\|_{W^{1,p}_T H^{s_2}_x} \right\}_{L^p(dv_{s,N})} \quad (3.4.16) \]
\[ \leq C_T \delta^{\frac{1}{2}}. \]

Given \( j \in \mathbb{N} \) and \( \varepsilon (0, 1) \), define \( K_\varepsilon \) by setting
\[ K_\varepsilon := \{ u \in C(\mathbb{R}; H^s(\mathbb{T})) : \|u\|_{C_T^\alpha H^{s_1}_x} \leq C_{0 \varepsilon^{-\frac{1}{2}} T_j^{1+\frac{1}{2}}} \text{ for all } j \in \mathbb{N} \}, \quad (3.4.17) \]
where \( T_j = 2^j \). Then, by Chebyshev’s inequality and \((3.4.16)\), we have
\[ \sup_{N \in \mathbb{N}} \sup_{2 \leq s \leq \infty} \nu_{N, \varepsilon}(K_\varepsilon) \leq \sum_{j=1}^{\infty} \nu_{N, \varepsilon} \left( \|u\|_{C_T^\alpha H^{s_1}_x} > C_{0 \varepsilon^{-\frac{1}{2}} T_j^{1+\frac{1}{2}}} \right) \]
\[ \leq C_0 \|\|u\|_{L^p_T H^s_x}\|_{L^p(dv_{s,N})} \]
\[ \leq \left( C_0^{-p} C_0^p \sum_{j=1}^{\infty} T_j^{-p} \right) \varepsilon < \varepsilon, \]
where the last step follows from choosing \( C_0 > 0 \) sufficiently large in the definition \((3.4.17)\) of \( K_\varepsilon \).

It remains to show that \( K_\varepsilon \) is compact in \((C(\mathbb{R}; H^s(\mathbb{T})), \text{dist})\), namely, endowed with the compact-open topology. While the proof of this fact was presented in the proof of Proposition 5.4 in [139], we present the argument for readers’ convenience. Let \( \{u_n\}_{n \in \mathbb{N}} \subseteq K_\varepsilon \). It follows from \((3.4.17)\) that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( C^0([-T_j, T_j]; H^s(\mathbb{T})) \) for each \( j \in \mathbb{N} \) and hence is compact in \( C([-T_j, T_j]; H^s(\mathbb{T})) \) for each \( j \in \mathbb{N} \). Then, by a diagonal argument, we can extract a subsequence \( \{u_{n_{\ell}}\}_{\ell \in \mathbb{N}} \) that is convergent in \( C([-T_j, T_j]; H^s(\mathbb{T})) \) for each \( j \in \mathbb{N} \). Hence, \( \{u_{n_{\ell}}\}_{\ell \in \mathbb{N}} \) is convergent in \((C(\mathbb{R}; H^s(\mathbb{T})), \text{dist})\). This proves that \( K_\varepsilon \) is relatively compact in \((C(\mathbb{R}; H^s(\mathbb{T})), \text{dist})\). It is clear that \( K_\varepsilon \) is closed as well, and hence we conclude the proof.

We conclude this subsection by presenting the proof of Lemma 3.4.3.

**Proof of Lemma 3.4.3.** The proof essentially follows the same lines in the proof of Lemma 5.5 in [139]. From \((3.4.9)\), the invariance of \( \mu_{N,\varepsilon} \) under the truncated gLLW dynamics \((3.4.1)\), Cauchy-Schwarz’s inequality, Proposition 3.2.4 (see \((3.2.15)\) with \( k = 1 \)), and Proposition 3.2.6 (see
(3.2.34), we have
\[
\|u\|_{L^p_T H^*_{L^p(d\nu_{1,N})}} = \|\Phi_{\delta,N}(t)\|_{L^p_T H^*_{L^p(d\nu_{1,N})}}
\]
\[
= \|\Phi_{\delta,N}(t)\|_{L^p_T H^*_{L^p}\|_{L^p_T}}
\]
\[
\leq T^\frac{1}{p}\|\|\Phi_{\delta,N}(t)\|_{L^p_T H^*_{L^p}(d\nu_{1,N})}|\|_{\chi}
\]
\[
\leq T^\frac{1}{p}\|\|u\|_{L^p_T H^*_{L^p(d\nu_{1,N})}}|\|_{\chi}
\]
\[
\leq T^\frac{1}{p},
\]
uniformly in \(N \in \mathbb{N}\) and \(2 \leq \delta \leq \infty\). This proves (3.4.10).

Next, we prove the second bound (3.4.11). By writing \(G_\delta \partial_x^2 = (G_\delta \partial_x)\partial_x\), it follows from (1.3.33) and Lemma 3.3.1 that
\[
\sup_{2 \leq \delta \leq \infty} \|G_\delta \partial_x^2 f\|_{H^{*,-2}} \leq \|f\|_{H^{*}}.
\]

Then, from (3.4.1) and (3.4.19), we have
\[
\|\|u\|_{W^{1,\infty}_T H^*_{L^p(d\nu_{1,N})}} = \|\|\partial_t u\|_{L^p_T H^*_{L^p(d\nu_{1,N})}}\|_{L^p_T H^*_{L^p(d\nu_{1,N})}}
\]
\[
\leq \|\|G_\delta \partial_x^2 u\|_{L^p_T H^*_{L^p(d\nu_{1,N})}} + \|\|F_N(u)\|_{L^p_T H^*_{L^p(d\nu_{1,N})}}\|_{L^p_T H^*_{L^p(d\nu_{1,N})}}
\]
\[
\leq \|\|u\|_{L^p_T H^*_{L^p(d\nu_{1,N})}} + \|\|F_N(u)\|_{L^p_T H^*_{L^p(d\nu_{1,N})}}\|_{L^p_T H^*_{L^p(d\nu_{1,N})}},
\]
uniformly in \(2 \leq \delta \leq \infty\) and \(N \in \mathbb{N}\), where \(F_N(u)\) is as in (3.2.21). Then, the rest follows as in (3.4.18) from Cauchy-Schwarz’s inequality, Proposition 3.2.6 and Proposition 3.2.4 (see (3.2.15) and (3.2.22)).

### 3.4.2 Proof of Theorem 1.3.7

In this subsection, we present the proof of Theorem 1.3.7. We first work with fixed \(2 \leq \delta \leq \infty\) and construct invariant Gibbs dynamics to the renormalized gILW equation (1.3.59):

\[
\partial_t u_\delta - G_\delta \partial_x^2 u_\delta = F(u_\delta)
\]

\[
= \partial_x W(u_\delta^{(k)})
\]

with the understanding that it corresponds to the renormalized gBO equation (1.3.62) when \(\delta = \infty\), where \(F(u)\) is the limit of \(F_N(u)\) in (3.2.21) constructed in Proposition 3.2.4 (ii). In view of Proposition 3.4.2 the family \(\{u_{\delta,N}\}_{N \in \mathbb{N}}\) is tight. Hence, by the Prokhorov theorem (Lemma 3.1.13), there exists a subsequence \(\{u_{\delta,N_j}\}_{j \in \mathbb{N}}\) converging weakly to some limit \(\nu_\delta\) denoted by \(\nu_\delta\). Namely, we have

\[
d_{LP}(u_{\delta,N_j}, u_\delta) \to 0
\]

as \(j \to \infty\), where \(d_{LP}\) denotes the Lévy-Prokhorov metric defined in (3.1.42).

By the Skorokhod representation theorem (Lemma 3.1.14), there exist some probability space \((\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta)\) and \(C(\mathbb{R}; H^{*}\mathbb{(T)})\)-valued random variables \(u_{\delta,N_j}\) and \(u_\delta\), such that

\[
\mathcal{L}(u_{\delta,N_j}) = \nu_{\delta,N_j} \quad \text{and} \quad \mathcal{L}(u_\delta) = \nu_\delta,
\]

and \(u_{\delta,N_j}\) converges \(\mathbb{P}_\delta\)-almost surely to \(u_\delta\) in \(C(\mathbb{R}; H^{*}(\mathbb{T}))\) as \(j \to \infty\). By repeating the argument in [50] [139] [134] (see, in particular, Subsection 5.3 in [139]), we obtain the following global existence result for the gILW equation (3.4.20) with the Gibbsian initial data (Theorem 1.3.7 (i)).

\(\nu_\delta\)The space \(\mathcal{M} = C(\mathbb{R}; H^{*}\mathbb{(T)})\) endowed with the compact-open topology is complete and separable, and thus \(\mathcal{P}(\mathcal{M}) = \{\text{all the probability measures on } \mathcal{M}\}\) is complete; see, for example, [16] Theorem 6.8 on p. 73.
Proposition 3.4.5. Let $u_{δ,N_j}$, $j \in \mathbb{N}$, and $u_δ$ be as above. Then, $u_{δ,N_j}$ and $u_δ$ are global-in-time distributional solutions to the truncated gILW equation (3.4.1) and the renormalized gILW equation (3.4.20), respectively. Moreover, we have
\[ \mathcal{L}(u_{δ,N_j}(t)) = \rho_{δ,N_j} \quad \text{and} \quad \mathcal{L}(u_δ(t)) = \rho_δ \] (3.4.23)
for any $t \in \mathbb{R}$.

Proof. While the proof of Proposition 3.4.5 follows exactly the same lines in Subsection 5.3 of [139], we present details (with some modifications from [139]) for readers’ convenience. We also point out that Proposition 3.4.5 will be applied iteratively in the proof of Theorem 1.3.7 (ii) presented below.

Fix $t \in \mathbb{R}$. Let $R_t : C(\mathbb{R}; H^s(\mathbb{T})) \to H^s(\mathbb{T})$ be the evaluation map defined by $R_t(φ) = φ(t)$. Note that $R_t$ is a continuous function. Then, from (3.4.8) and the invariance of the truncated Gibbs measure $ρ_δ$ (Lemma 3.4.1), we have
\[ \nu_{δ,N} = \rho_{δ,N} \circ \Phi_{δ,N}^{-1} = \rho_{δ,N} \circ \left( R_t \circ \Phi_{δ,N} \right)^{-1} = \left( R_t \circ \Phi_{δ,N} \right) \ast \rho_{δ,N} = \left( \Phi_{δ,N}(t) \right) \ast \rho_{δ,N} \] (3.4.24)
Then, it follows from (3.4.22) and (3.4.24) that
\[ \mathcal{L}(u_{δ,N_j}(t)) = \nu_{δ,N_j} \circ R_t^{-1} = \rho_{δ,N_j}. \] (3.4.25)
By the construction, $u_{δ,N_j}$ converges to $u_δ$ in $C(\mathbb{R}; H^s(\mathbb{T}))$ almost surely with respect to $\tilde{P}_δ$. Thus, we have
\[ u_{δ,N_j}(t) = R_t(u_{δ,N_j}) \to u_δ(t) = R_t(u_δ) \] almost surely as $j \to \infty$, which in particular implies $u_{δ,N_j}(t)$ converges in law to $u_δ(t)$ as $j \to \infty$. Namely, $\mathcal{L}(u_{δ,N_j}(t))$ converges weakly to $\mathcal{L}(u_δ(t))$ as $j \to \infty$. On the other hand, recall from Theorem 1.3.2 (i) that $ρ_{δ,N_j}$ converges to $ρ_δ$ in total variation as $j \to \infty$, which in particular implies that $ρ_{δ,N_j}$ converges weakly to $ρ_δ$. Hence, in view of (3.4.25) and the uniqueness of the limit, we conclude $\mathcal{L}(u_δ(t)) = ρ_δ$. This proves (3.4.23).

Next, we show that the random variable $u_{δ,N_j}$ is indeed a global-in-time distributional solution to (3.4.1). Given a test function $φ \in \mathcal{D}(\mathbb{R} \times \mathbb{T}) = C_c^∞(\mathbb{R} \times \mathbb{T})$, define $V_{φ,j} : C(\mathbb{R}; H^s(\mathbb{R})) \to \mathbb{R}$ by
\[ V_{φ,j}(u) = \langle (φ, \partial_t u - G_δ \partial_x^2 u - F_{N_j}(u)) \rangle, \] (3.4.26)
where $(\cdot, \cdot)$ denotes the $D_{1,x}D'_{1,x}$ pairing. It is easy to see that $V_{φ,j}$ is continuous. In view of the separability of $\mathcal{D}(\mathbb{R} \times \mathbb{T})$, let $\{ ϕ_m \}_{m \in \mathbb{N}}$ be a countable dense subset of $\mathcal{D}(\mathbb{R} \times \mathbb{T})$. Then, in view of (3.4.9), (3.4.26), and the definition (3.4.7) of $Φ_{δ,N_j}$, we have
\[ \| V_{φ,m,j} \|_{L^1(υ_{δ,N_j})} = \int_{H^s} |V_{φ,m,j}(Φ_{δ,N_j}(φ))| dυ_{δ,N_j}(φ) = 0 \] (3.4.27)
for any $m \in \mathbb{N}$. Namely, there exists a set $Σ_m \subset C(\mathbb{R}; H^s(\mathbb{T}))$ such that $υ_{δ,N_j}(Σ_m) = 1$ and $V_{φ,m,j}(u) = 0$ for any $u \in Σ_m$. Now, set $Σ = \bigcap_{m \in \mathbb{N}} Σ_m$. Then, we have $υ_{δ,N_j}(Σ) = 1$ and, moreover, $V_{φ,j}(u) = 0$ for any $u \in Σ$ and $φ \in \mathcal{D}(\mathbb{R} \times \mathbb{T})$, where the latter claim follows from (3.4.27) and the density of $\{ ϕ_m \}_{m \in \mathbb{N}}$.

Finally, we prove that the random variable $u_δ$ is a global-in-time distributional solution to (3.4.20). It follows from the almost sure convergence of $u_{δ,N_j}$ to $u_δ$ in $C(\mathbb{R}; H^s(\mathbb{T}))$ that
\[ \partial_t u_{δ,N_j} - G_δ \partial_x^2 u_{δ,N_j} \to \partial_t u_δ - G_δ \partial_x^2 u_δ \] (3.4.28)
in $\mathcal{D}'(\mathbb{R} \times \mathbb{T})$, $\tilde{P}_δ$-almost surely, as $j \to \infty$.

Next, we show almost sure convergence of the truncated nonlinearity $F_{N_j}(u_{δ,N_j})$ to $F(u_δ) = ...$
\[ \partial_t \mathcal{W}(u_\delta). \] Given \( M \in \mathbb{N} \), write
\[
F_{N_j}(u_{\delta,N_j}) - F(u_\delta) = (F_{N_j}(u_{\delta,N_j}) - F_M(u_{\delta,N_j})) + (F_M(u_{\delta,N_j}) - F_M(u_\delta)) + (F_M(u_\delta) - F(u_\delta)). \tag{3.4.29}
\]
Noting that \( u \in C(\mathbb{R}; H^s(\mathbb{T})) \mapsto F_M(u) \in C(\mathbb{R}; H^{s-1}(\mathbb{T})) \) is continuous, it follows from the almost sure convergence of \( u_{\delta,N_j} \) to \( u_\delta \) in \( C(\mathbb{R}; H^s(\mathbb{T})) \) that
\[
F_M(u_{\delta,N_j}) \to F_M(u_\delta)
\]
in \( C(\mathbb{R}; H^{s-1}(\mathbb{T})) \), \( \bar{\mathbb{P}}_\delta \)-almost surely, as \( j \to \infty \). As for the first term on the right-hand side of (3.4.29), for fixed \( T > 0 \), it follows from (3.4.9), the invariance of the truncated Gibbs measure \( \rho_{\delta,N} \), and Proposition 3.2.6 that
\[
\| F_{N_j}(u_{\delta,N_j}) - F_M(u_{\delta,N_j}) \|_{L^2 H^{s-1}_\delta} \leq \| F_{N_j}(u_{\delta,N_j}) - F_M(u_{\delta,N_j}) \|_{L^2(\mathcal{Q}_\delta)} + \| F_M(u_{\delta,N_j}) - F_M(u_\delta) \|_{L^2(\mathcal{Q}_\delta)} + \| F_M(u_\delta) - F(u_\delta) \|_{L^2(\mathcal{Q}_\delta)}.
\]
where the implicit constants are independent of \( N_j \). By applying Proposition 3.2.4(ii), we conclude that the first term on the right-hand side of (3.4.29) converges to 0 in \( L^2(\mathcal{Q}_\delta; L^2([-T,T]; H^{s-1}(\mathbb{T}))) \) as \( j,M \to \infty \). Hence, by extracting a subsequence, the first term on the right-hand side of (3.4.29) converges to 0 in \( L^2([-T,T]; H^{s-1}(\mathbb{T})) \), \( \bar{\mathbb{P}}_\delta \)-almost surely, as \( j,M \to \infty \). A similar argument shows that, by extracting a subsequence, the third term on the right-hand side of (3.4.29) converges to 0 in \( L^2([-T,T]; H^{s-1}(\mathbb{T})) \), \( \bar{\mathbb{P}}_\delta \)-almost surely, as \( M \to \infty \).

Putting all together with (3.4.29), we conclude that, up to a subsequence, \( F_{N_j}(u_{\delta,N_j}) \) converges to \( F(u_\delta) \) in \( L^2([-T,T]; H^{s-1}(\mathbb{T})) \), \( \bar{\mathbb{P}}_\delta \)-almost surely, as \( j \to \infty \). Since the choice of \( T > 0 \) was arbitrary, we can apply this argument for \( T_m = 2^m, m \in \mathbb{N} \). Thus, with \( m = 1 \), there exists a subsequence \( F_{N_{j_1}}(u_{\delta,N_{j_1}}) \) and a set \( \Sigma_1 \) of full \( \bar{\mathbb{P}}_\delta \)-probability such that \( F_{N_{j_1}}(u_{\delta,N_{j_1}})(\omega) \) converges to \( F(u_\delta)(\omega) \) in \( L^2([-T_1,T_1]; H^{s-1}(\mathbb{T})) \) for each \( \omega \in \Sigma_1 \) as \( j_1 \to \infty \). For each \( m \geq 2 \), we can extract a further subsequence \( F_{N_{j_m}}(u_{\delta,N_{j_m}}) \) of \( F_{N_{j_{m-1}}}(u_{\delta,N_{j_{m-1}}}) \) and a subset \( \Sigma_m \subseteq \Sigma_{m-1} \) of full \( \bar{\mathbb{P}}_\delta \)-probability such that \( F_{N_{j_m}}(u_{\delta,N_{j_m}})(\omega) \) converges to \( F(u_\delta)(\omega) \) in \( L^2([-T_m,T_m]; H^{s-1}(\mathbb{T})) \) for each \( \omega \in \Sigma_m \) as \( j_m \to \infty \). By a diagonal argument, we conclude that, passing to a subsequence, we have \( F_{N_j}(u_{\delta,N_j}) \) converges to \( F(u_\delta) \) in \( L^2_{loc} H^{s-1}(\mathbb{T}) \), \( \bar{\mathbb{P}}_\delta \)-almost surely, which in particular implies that this subsequence converges to \( F(u_\delta) \) in \( \mathcal{D}'(\mathbb{R} \times \mathbb{T}) \), \( \bar{\mathbb{P}}_\delta \) almost surely. Therefore, together with (3.4.29), we conclude that \( u_\delta \) is a global-in-time distributional solution to (3.4.20).

Finally, we present the proof of Theorem 3.7(ii). In the discussion at the beginning of this subsection, we used Proposition 3.4.2 and the Prokhorov theorem (Lemma 3.1.13) to conclude that, for each fixed \( 2 \leq \delta \leq \infty \), there exists a sequence \( N_j \to \infty \) such that (3.4.21) holds. In the following, we iteratively apply this argument for integers \( \delta \geq 2 \) and apply a diagonal argument.

(i) Let \( \delta = 2 \). Then, it follows from Proposition 3.4.2 that the family \( \{\nu_{2,N}\}_{N \in \mathbb{N}} \) is tight. Hence, by the Prokhorov theorem (Lemma 3.1.13), there exists a weakly convergent subsequence \( \{\nu_{2,N_j}\}_{j \in \mathbb{N}} \). Namely, there exists a probability measure \( \nu_2 \) on \( C(\mathbb{R}; H^s(\mathbb{T})) \) such that \( d_{\mathcal{L}^p}(\nu_{2,N_j}, \nu_2) \to 0 \) as \( j \to \infty \).

(ii) For \( \delta = 3 \), we apply the same argument to \( \{\nu_{3,N_j}\}_{j \in \mathbb{N}} \) to conclude that there exists a weakly convergent subsequence \( \{\nu_{3,N_j}\}_{j \in \mathbb{N}} \) with \( \{N_j(3)\}_{j \in \mathbb{N}} \subseteq \{N_j(2)\}_{j \in \mathbb{N}} \). Namely, there exists a probability measure \( \nu_3 \) on \( C(\mathbb{R}; H^s(\mathbb{T})) \) such that \( d_{\mathcal{L}^p}(\nu_{3,N_j}, \nu_3) \to 0 \) as \( j \to \infty \).
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(iii) We iterate this procedure for each integer \( \delta \geq 4 \) and construct a weakly convergent subsequence \( \{ \nu_{\delta, N_j^{(\delta)}} \}_{j \in \mathbb{N}} \) with \( \{ N_j^{(\delta)} \}_{j \in \mathbb{N}} \subset \{ N_j^{(\delta-1)} \}_{j \in \mathbb{N}} \). Namely, there exists a probability measure \( \nu_{\delta} \) on \( C(\mathbb{R}; H^s(\mathbb{T})) \) such that

\[
d_{LP}(\nu_{\delta, N_j^{(\delta)}}, \nu_{\delta}) \rightarrow 0, \tag{3.4.30}
\]

as \( j \rightarrow \infty \).

(iv) Let \( \mathbb{N}_{\geq 2} = \mathbb{N} \cap [2, \infty) \). We take a diagonal sequence \( \{ \nu_{\delta, N_j^{(\delta)}} \}_{j \in \mathbb{N}_{\geq 2}} \), where \( j(\delta) \) is chosen such that \( j(\delta) \) is increasing in \( \delta \) and

\[
d_{LP}(\nu_{\delta, N_j^{(\delta)}}, \nu_{\delta}) \leq \frac{1}{\delta}, \tag{3.4.31}
\]

By Proposition 3.4.2 and the Prokhorov theorem (Lemma 3.1.13), the family \( \{ \nu_{\delta, N_j^{(\delta)}} \}_{\delta \in \mathbb{N}_{\geq 2}} \) is tight and thus admits a weakly convergent subsequence \( \{ \nu_{\delta_m, N_j^{(\delta_m)}} \}_{m \in \mathbb{N}} \) to some limit, which we denote by \( \nu_{\infty} \). Namely, we have

\[
d_{LP}(\nu_{\delta_m, N_j^{(\delta_m)}}, \nu_{\infty}) \rightarrow 0, \tag{3.4.32}
\]
as \( m \rightarrow \infty \). By the triangle inequality for the Lévy-Prokhorov metric \( d_{LP} \) with (3.4.31) and (3.4.32), we have

\[
d_{LP}(\nu_{\delta_m}, \nu_{\infty}) \leq d_{LP}(\nu_{\delta_m}, \nu_{\delta_m, N_j^{(\delta_m)}}) + d_{LP}(\nu_{\delta_m, N_j^{(\delta_m)}}, \nu_{\infty})
\]

\[
\leq \frac{1}{\delta_m} + d_{LP}(\nu_{\delta_m, N_j^{(\delta_m)}}, \nu_{\infty}) \rightarrow 0,
\]
as \( m \rightarrow \infty \) (and hence \( \delta_m \rightarrow \infty \)). Hence, \( \nu_{\delta_m} \) converges weakly to \( \nu_{\infty} \) as \( m \rightarrow \infty \).

By the Skorokhod representation theorem (Lemma 3.1.14), there exist a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and \( C(\mathbb{R}; H^s(\mathbb{T})) \)-valued random variables \( u_{\delta_m} \) and \( u \) such that

\[
\mathcal{L}(u_{\delta_m}) = \nu_{\delta_m} \quad \text{and} \quad \mathcal{L}(u) = \nu_{\infty}, \tag{3.4.34}
\]

and \( u_{\delta_m} \) converges \( \mathbb{P} \)-almost surely to \( u \) in \( C(\mathbb{R}; H^s(\mathbb{T})) \) as \( m \rightarrow \infty \).

Next, we show that \( u_{\delta_m} \) is a global-in-time distributional solution to the renormalized gILW equation (3.4.20) (with \( \delta = \delta_m \)). It follows from (3.4.30) and the Skorokhod representation theorem (Lemma 3.1.14) that there exist a probability space \( (\Omega_m, \mathcal{F}_m, \mathbb{P}_m) \) and \( C(\mathbb{R}; H^s(\mathbb{T})) \)-valued random variables \( \tilde{u}_{\delta_m, N_j^{(\delta_m)}} \) and \( \hat{u}_{\delta_m} \) such that

\[
\mathcal{L}(\tilde{u}_{\delta_m, N_j^{(\delta_m)}}) = \nu_{\delta_m, N_j^{(\delta_m)}} \quad \text{and} \quad \mathcal{L}(\tilde{u}_{\delta_m}) = \nu_{\delta_m}, \tag{3.4.35}
\]

and \( \tilde{u}_{\delta_m, N_j^{(\delta_m)}} \) converges \( \mathbb{P}_m \)-almost surely to \( \hat{u}_{\delta_m} \) in \( C(\mathbb{R}; H^s(\mathbb{T})) \) as \( j \rightarrow \infty \). Arguing as in the proof of Proposition 3.4.3, we see that \( \tilde{u}_{\delta_m} \) is a global-in-time distributional solution to the renormalized gILW equation (3.4.20). Hence, from (3.4.34) and (3.4.35), we conclude that \( u_{\delta_m} \) is a global-in-time distributional solution to the renormalized gILW equation (3.4.20).

It remains to show that \( u \) satisfies the renormalized gBO equation (1.3.62) in the distributional sense. The almost sure convergence of \( u_{\delta_m} \) to \( u \) implies that

\[
\partial_t u_{\delta_m} - G_\delta \partial_x^2 u_{\delta_m} \rightarrow \partial_t u - \mathcal{H} \partial_x^2 u \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times \mathbb{T}) \quad \text{as} \quad m \rightarrow \infty.
\]

Next, we discuss convergence of the nonlinearity. Let \( F(u_\delta) = \partial_x W(u_\delta^2) \)
be as in Proposition \[3.2.4\](ii). Given \(M \in \mathbb{N}\), write
\[
F(u_{\delta_m}) - F(u) = (F(u_{\delta_m}) - F_M(u_{\delta_m})) + (F_M(u_{\delta_m}) - F_M(u)) + (F_M(u) - F(u)),
\]
(3.4.37)
From the continuity of \(F_M\) and the almost sure convergence of \(u_{\delta_m}\) to \(u\), we see that the second term on the right-hand side of (3.4.37) tends to 0 in \(C(\mathbb{R}; H^{s-1}(\mathbb{T}))\), \(\mathbb{P}\)-almost surely, as \(m \to \infty\). As for the first and third terms on the right-hand side of (3.4.37), we need to exploit the uniform (in \(\delta\) and \(N\)) bounds, which is the main difference from the proof of Proposition 3.4.5 presented above. Let \(T > 0\). Then, from (3.4.38) and the invariance of the truncated Gibbs measure \(\rho_{\delta,N}\), we have
\[
\begin{align*}
\|F(u_{\delta_m}) - F_M(u_{\delta_m})\|_{L^2_H} & = \|F(u) - F_M(u)\|_{L^2_H} \quad \|F_M(\phi) - F_M(\phi)\|_{L^2_H} \\
& \leq T^{\frac{1}{2}} Z_{\delta_m}^{\frac{1}{2}} \|G_{\delta_m}\|_{L^1(\Omega)} \|F(\phi) - F_M(\phi)\|_{L^1(\delta m)H^{s-1}_e}
\end{align*}
\]
with the understanding that \(u_{\delta_m} = u\) when \(m = \infty\). From Proposition \[3.2.6\] we have
\[
\sup_{m \in \mathbb{N}\cup\{\infty\}} \sup_{m \in \mathbb{N}\cup\{\infty\}} \|G_{\delta_m}\|_{L^1(\Omega)} \lesssim 1.
\]
(3.4.39)
Then, from (3.4.38), (3.4.39), and Proposition 3.2.4(ii) (see (3.2.23) with \((M, N) = (\infty, N)\)), we conclude that the first and third terms on the right-hand side of (3.4.37) converge to 0 in \(L^2(\Omega; L^2([-T, T]; H^s(\mathbb{T})))\) as \(M \to \infty\). Then, by first taking \(m \to \infty\) and then \(M \to \infty\) in (3.4.37), we conclude that, by extracting a subsequence, \(F(u_{\delta_m})\) converges to \(F(u)\) in \(L^2([-T, T]; H^s(\mathbb{T}))\), \(\mathbb{P}\)-almost surely, as \(m \to \infty\). By repeating the argument at the end of the proof of Proposition 3.4.5 we see that up to a further subsequence, \(F(u_{\delta_m})\) converges to \(F(u)\) in \(\mathcal{D}'(\mathbb{R} \times \mathbb{T})\), \(\mathbb{P}\)-almost surely, as \(m \to \infty\). Therefore, together with (3.4.36), we conclude that \(u\) is a global-in-time distributional solution to the renormalized gBO equation (3.4.20). This concludes the proof of Theorem 1.3.7(ii).

Remark 3.4.6. In this paper, we considered probability measures on \(H^{-\xi}(\mathbb{T})\) for fixed small \(\xi > 0\). In the following, we briefly explain how to remove the dependence on \(\xi\). First, we set
\[
H^{0^-}(\mathbb{T}) := \bigcap_{s > 0} H^{-s}(\mathbb{T}) = \bigcap_{j \in \mathbb{N}} H^{-s_j}(\mathbb{T}),
\]
with \(s_j = \frac{1}{j}\). Then, we equip \(H^{0^-}(\mathbb{T})\) with the following distance:
\[
\|f - g\|_{H^{-s_j}} = \sum_{j=1}^{\infty} 2^{-j} \frac{|f - g|_{H^{-s_j}}}{1 + |f - g|_{H^{-s_j}}}
\]
By definition, we have \(d(f_n, f) \to 0\) if and only if \(f_n\) converges to \(f\) in \(H^{-s_j}(\mathbb{T})\) for each \(j \in \mathbb{N}\). Let \(D\) be the set of smooth functions \(Q \in C^\infty(\mathbb{T})\) of the form
\[
Q(x) = \sum_{|n| \leq N} q_n e_n(x),
\]
with \(q_n \in \mathbb{Q}\) and \(N \in \mathbb{N}\). Then, \(D\) is a countable dense subset of \(H^{-s_j}(\mathbb{T})\) for any \(j \in \mathbb{N}\). Let \(f \in H^{0^-}(\mathbb{T})\). Then, for each \(j \in \mathbb{N}\), there exists \(Q_{j,N} \in D\) such that
\[
\|Q_{j,N} - f\|_{H^{-s_j}} \leq 2^{-N}.
\]
Now, set \( Q_N = Q_{N,N} \in D, \ N \in \mathbb{N} \). Then, given \( \varepsilon > 0 \), by choosing \( N \geq \frac{1}{\varepsilon} \), we have
\[
\|Q_N - f\|_{H^{-\varepsilon}} \leq \|Q_N - f\|_{H^{-\frac{1}{2}}} = \|Q_{N,N} - f\|_{H^{-\frac{1}{2}}} \leq 2^{-N}.
\]
Hence, we have
\[
d(Q_N, f) \leq \sum_{j=1}^{N} 2^{-j} \|Q_N - f\|_{H^{-\frac{1}{2}}} + \sum_{j=N+1}^{\infty} 2^{-j} \leq 2^{-N} + 2^{-N} \rightarrow 0,
\]
as \( N \rightarrow \infty \). In other words, we just proved that \( D \) is also a countable dense subset of \( H^{0-}(\mathbb{T}) \) with respect to the metric \( d \). Hence, from [77], we see that \( C(\mathbb{R}; H^{0-}(\mathbb{T})) \) is separable.\(^{10}\) This allows us to repeat the entire paper by replacing \( C(\mathbb{R}; H^{-\varepsilon}(\mathbb{T})) \) with \( C(\mathbb{R}; H^{0-}(\mathbb{T})) \).

\(^{10}\) Note that we have
\[
C(\mathbb{R}; H^{0-}(\mathbb{T})) = \bigcap_{j=1}^{\infty} C(\mathbb{R}; H^{-\frac{1}{2}}(\mathbb{T})).
\]
Appendix A

Equation derivation

In this section, we present the discussion of the derivation of the ILW equation. A rigorous derivation (in the sense of consistency) is given in [38] using a two-layer system, that is, a system of two layers of fluids of different densities, the density of the total fluid being discontinuous though, see also [41]. In the following discussion, the spacial domain is $\mathbb{R}$. All the boldface notations denote the vector-valued functions.

system setup for internal equatorial waves

The internal wave along the intermediate region is modelled by two layers of fluids in two dimensions since the motion in the vertical direction is neglected. The layers are separated by a common interface (the thermocline/ pycnocline), see the following figure.

The two-fluid domains are

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -h < y < \eta(x, t)\} \quad \text{and} \quad \Omega_1 = \{(x, y) \in \mathbb{R}^2 : \eta(x, t) < y < h_1\}.$$

The functions and parameters associated with the upper layer will be marked with subscript 1. Also, subscript $c$ (implying a common interface) will be used to denote the evaluation of the internal wave. Propagation of the internal wave is assumed to be in the positive $x$-direction. The direction of the gravity force is in the negative $y$-axis.
APPENDIX A. EQUATION DERIVATION

The function \( \eta(x, t) \) describes the evolution of the internal wave with the mean zero assumption,

\[
\int_\mathbb{R} \eta(x, t) \, dx = 0.
\]

In fact, it is sufficient to have \( \int_\mathbb{R} \eta(x, t) \, dx \leq \infty \), because the mean value involves division of the length of the corresponding interval, which in this case is \( \int_\mathbb{R} \, dx = \infty \). The fluids are incompressible with densities \( \rho \) and \( \rho_1 \). The stable stratification is given by the immiscibility condition \( \rho > \rho_1 \).

For the incompressible fluid, we have the divergence free vector field:

\[
\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u}_1 = 0.
\]

The stream functions, \( \psi(x, y, t) \) and \( \psi_1(x, y, t) \), are related to the velocity fields \( \mathbf{u} = (u, v) \) and \( \mathbf{u}_1 = (u_1, v_1) \) from the relations

\[
u = \partial_y \psi, \quad u_1 = \partial_y \psi_1, \quad v = -\partial_x \psi \quad \text{and} \quad v_1 = -\partial_x \psi_1
\]

Let \( \phi \) and \( \phi_1 \) to be the velocity potentials. A piecewise linear current profile can be represented by the velocity fields of the following form

\[
u = \partial_x \phi + \gamma y + \kappa, \quad u_1 = \partial_x \phi_1 + \gamma_1 y + \kappa_1, \quad v = \partial_y \phi \quad \text{and} \quad v_1 = \partial_y \phi_1.
\]

where \( \gamma = -\partial_y v + \partial_x u \) and \( \gamma_1 = -\partial_y v_1 + \partial_x u_1 \) are the constant vorticities, \( \kappa \) and \( \kappa_1 \) are constants representing the current horizontal velocities at \( y = 0 \).

We assume that the functions \( \eta(x, t), \phi(x, y, t) \), and \( \phi_1(x, y, t) \) are smooth functions, i.e., they belong to the Schwartz class \( S(\mathbb{R}) \) with respect to \( x \) (for any \( y \) and \( t \)). The assumption implies that for large values of \( |x| \) the internal wave attenuates, and is vanishing at infinity,

\[
\lim_{|x| \to \infty} \eta(x, t) = \lim_{|x| \to \infty} \phi(x, y, t) = \lim_{|x| \to \infty} \phi_1(x, y, t) = 0.
\]

Governing equations

The Euler equations for the two layers are

\[
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \mathbf{F},
\]

\[
\partial_t \mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_1 = -\frac{1}{\rho_1} \nabla p_1 + \mathbf{g} + \mathbf{F}_1.
\]

where \( \mathbf{F} = 2\omega \nabla \psi \) and \( \mathbf{F}_1 = 2\omega \nabla \psi_1 \). are the Coriolis forces per unit mass at the equator with \( \omega \) being the rotational speed of the Earth, \( \mathbf{g} = (0, 0, -g) \) is the Earth acceleration, \( \rho \) and \( \rho_1 \) (due to the assumption of incompressibility) are the constant densities, and \( \rho \) and \( \rho_1 \) are the corresponding pressures. The pressure gradients can be expressed as

\[
\nabla p = -\rho \nabla (\partial_t \phi + \frac{1}{2} |\nabla \psi|^2 - (\gamma + 2\omega) \psi + gy),
\]

\[
\nabla p_1 = -\rho_1 \nabla (\partial_t \phi_1 + \frac{1}{2} |\nabla \psi_1|^2 - (\gamma_1 + 2\omega) \psi_1 + gy).
\]

The dynamic boundary condition at the interface \( p = p_1 \) gives the Bernoulli equation

\[
\rho(\partial_t \phi_c + 2|\nabla \psi_c|c - (\gamma + 2\omega) \theta + gn) = \rho_1(\partial_t \phi_{1,c} + 2|\nabla \psi_{1,c}|c - (\gamma_1 + 2\omega) \theta_1 + gn)
\]

(1.0.2) where the subscript \( c \) is a notation for an evaluation at the common interface \( y = \eta(x, t) \). The stream function replace \( y \) by \( \eta = \psi(x, \eta, t) \) and \( \theta_1 = \psi(x, \eta, t) \). The Bernoulli equation \( (1.0.2) \) gives the time evolution of \( \xi := \rho\phi_c - \rho_1\phi_{1,c} \) which is the momentum type variable in the Hamiltonian formulation of the problem. The coordinate variable is \( \eta(x, t) \) and it evolves according
to the kinematic boundary condition at the interface
\[ \partial_t \eta = v - uw_x = v_1 - u_1 x. \]
This can be expressed in terms of the velocity potentials, using (1.0.1), as
\[ \partial_t \eta = (\partial_y \phi)_c - (\partial_x \phi)_c + \gamma \eta + \kappa \partial_x \eta. \] (1.0.3)
The kinematic boundary condition at the bottom, requiring that there is no velocity component in the y-direction on the flat bed, is given by
\[ \partial_y \phi(x, -h, t) = 0 \quad \text{and} \quad \partial_x \psi(x, -h, t) = 0, \]
where \(-h\) is the depth of the bottom boundary. Additionally, there is a kinematic boundary condition at the top, requiring that there is no velocity component in the y-direction on the surface, given by
\[ \partial_y \phi_1(x, h_1, t) = 0 \quad \text{and} \quad \partial_x \psi_1(x, h_1, t) = 0, \]
where \(h_1\) is the top boundary.

**Hamiltonian formulation**

The wave motion on the interface (thermocline/pycnocline) admits a (nearly) Hamiltonian representation. Let us introduce the following notations for the velocity potentials values at the interface
\[ \varphi := \phi(x, \eta(x, t), t) \quad \varphi_1 := \phi_1(x, \eta(x, t), t). \]
The variable
\[ \xi := \rho \varphi - \rho_1 \varphi_1 \] (1.0.4)
plays the role of a generalised momentum. We assume that \(\xi \in S(\mathbb{R})\) for all \(t\). The equations of motion are
\[ \begin{cases} 
\partial_t \eta = \delta \xi H \\
\partial_t \xi = -\delta \eta H + \Gamma \theta 
\end{cases} \] (1.0.5)
where \(H = H(\eta, \xi)\) is the total energy of the system, \(\Gamma\) is defined by
\[ \Gamma := \rho \gamma - \rho_1 \gamma_1 + 2 \omega (\rho - \rho_1), \]
and the stream function \(\theta\) is evaluated at \(y = \eta(x, t)\) such that
\[ \theta(x, t) = -\int_{\infty}^{x} \partial_t \eta(x', t)dx' = -\partial_x^{-1} \partial_t \eta. \]
Under the transformation
\[ \xi \rightarrow \zeta = \xi - \frac{\Gamma}{2} \int_{\infty}^{x} \eta(x', t)dx', \]
the dynamics (1.0.5) can be formally written in the canonical form
\[ \begin{cases} 
\partial_t \eta = \delta \xi H \\
\partial_t \xi = -\delta \eta H .
\end{cases} \]
Moreover, the dynamics (1.0.5) could be further expressed in terms of the variable \(\vartheta = \partial_x \xi\) for a
Hamiltonian written in terms of $u$ and $\eta$:

$$
\partial_t \eta = -\partial_x \left( \frac{\delta H}{\delta u} \right), \quad \partial_t u = -\partial_x \left( \frac{\delta H}{\delta \eta} \right) - \Gamma \partial_\eta \eta.
$$

(1.0.6)

In order to obtain the Hamiltonian in terms of variables defined at the interface we need to introduce the so-called Dirichlet–Neumann (DN) operators

$$
\begin{aligned}
G(\eta)\varphi &= \phi_n(1 + \partial_x \eta^2)^{\frac{1}{2}} \quad \text{for } \Omega, \\
G_1(\eta)\varphi_1 &= \phi_{1,n}(1 + \partial_x \eta^2)^{\frac{1}{2}} \quad \text{for } \Omega_1
\end{aligned}
$$

where $\phi_n$ and $\phi_{1,n}$ are the normal derivatives of the velocity potentials, at the interface, and $n$ and $n_1$ are the outward normals to the corresponding domains, with $n = -n_1$. We need also the operator $B$:

$$
B := \rho G_1(\eta) + \rho_1 G(\eta).
$$

(1.0.7)

Using the boundary conditions (1.0.3), we have

$$
\begin{aligned}
G(\eta)\varphi &= -\partial_\eta \eta \partial_\eta \phi_c + \partial_\gamma \phi_c = \partial_\eta \eta + (\gamma \eta + \kappa)\partial_\gamma \eta, \\
G_1(\eta)\varphi_1 &= -\partial_\gamma \eta \partial_\gamma \phi_{1,c} - \partial_\gamma \phi_{1,c} = -\partial_\gamma \eta - (\gamma_1 \eta + \kappa_1)\partial_\gamma \eta.
\end{aligned}
$$

Hence, we obtain $G(\eta)\varphi + G_1(\eta)\varphi_1 = \mu$ where $\mu$ is defined by

$$
\mu := (\gamma - \gamma_1)\eta + (\kappa - \kappa_1)\partial_\gamma \eta.
$$

Together with (1.0.4) we obtain

$$
\rho G_1(\eta) + \rho_1 G(\eta) = \rho_1 \mu + G_1(\eta)\xi.
$$

Moreover, $\varphi$ and $\varphi_1$ can be solved by:

$$
\varphi = B^{-1}(\rho_1 \mu + G_1(\eta)\xi) \quad \varphi_1 = B^{-1}(\rho \mu + G(\eta)\xi).
$$

The expression for the Hamiltonian in terms of $\xi, \eta \in \mathcal{S}^\prime(\mathbb{R})$ has the form

$$
H(\eta, \xi) = \frac{1}{2} \int_\mathbb{R} \xi G(\eta) B^{-1} G_1(\eta) \xi dx - \frac{\rho p_1}{2} \int_\mathbb{R} \mu B^{-1} \mu dx - \int_\mathbb{R} (\kappa + \gamma \eta) \xi \partial_\gamma \eta dx \\
+ \rho_1 \int_\mathbb{R} \mu B^{-1} G(\eta) \xi dx + \frac{\rho}{6\gamma} \int_\mathbb{R} (\gamma \eta + \kappa)^3 - 3 \xi^3 dx \\
- \frac{\rho_1}{6\gamma} \int_\mathbb{R} (\gamma_1 \eta + \kappa_1)^3 - 3 \xi^3 dx + \frac{1}{2} g(\rho - \rho_1) \int_\mathbb{R} \eta^2 dx.
$$

In the special case when $\gamma_1 = \gamma, \kappa_1 = \kappa$ and $\mu = 0$ the Hamiltonian acquires the form

$$
H(\eta, \xi) = \frac{1}{2} \int_\mathbb{R} \xi G(\eta) B^{-1} G_1(\eta) \xi dx - \int_\mathbb{R} (\kappa + \gamma \eta) \xi \partial_\gamma \eta dx \\
+ \frac{\rho - \rho_1}{6\gamma} \int_\mathbb{R} (\gamma \eta + \kappa)^3 - 3 \xi^3 dx + \frac{1}{2} g(\rho - \rho_1) \int_\mathbb{R} \eta^2 dx.
$$

When $\gamma_1 \neq \gamma$ and $\kappa_1 = \kappa = 0$ then $\mu = (\gamma - \gamma_1)\eta \partial_\gamma \eta$ and the Hamiltonian becomes

$$
H(\eta, \xi) = \frac{1}{2} \int_\mathbb{R} \xi G(\eta) B^{-1} G_1(\eta) \xi dx + \rho_1 (\gamma - \gamma_1) \int_\mathbb{R} \eta \partial_\gamma \eta B^{-1} G(\eta) \xi dx \\
- \frac{1}{2} \rho \rho_1 (\gamma - \gamma_1)^2 \int_\mathbb{R} \eta \partial_\gamma \eta B^{-1} \eta \partial_\gamma \eta dx - \gamma \int_\mathbb{R} \xi \eta \partial_\gamma \eta dx
$$
In the situation with $\kappa_1 = \kappa$ which is physically realistic because the unperturbed currents at the two layers have the same speed at the interface (i.e. absence of a vortex sheet), we have also

$$
\mu = \frac{(\gamma - \gamma_1)\eta\partial_x \eta}{\eta} \quad \text{and} \quad H(\eta, \xi) = \frac{1}{2} \int_{\mathbb{R}} G(\eta)B^{-1}G_1(\eta)\xi d\eta - \frac{1}{2} \int_{\mathbb{R}} \eta \partial_x \eta B^{-1} \eta \partial_x \eta d\eta - \gamma \int_{\mathbb{R}} \xi \partial_x \eta d\eta
$$

(1.0.8)

where

$$
\frac{\rho \gamma^2 - \rho_1 \gamma_1^2}{6} \int_{\mathbb{R}} \eta^3 d\eta + \frac{g(\rho - \rho_1)}{2} \int_{\mathbb{R}} \eta^2 d\eta.
$$

The stability of the flow under disturbances of arbitrary wavenumber is assumed. In the next section, we will introduce appropriate scales and provide the corresponding expansions of the DN operators in terms of the small-order scale parameters.

**Scales and approximations**

Let us introduce a small parameter which will be used to separate the order of the terms in the model:

$$
\varepsilon = \frac{a}{h_1}
$$

where the constant $a$ represents the average amplitude of the waves $\eta(x, t)$ under consideration, and $\varepsilon \ll 1$.

The Dirichlet–Neumann (DN) operators have the following structure

$$
G = G^{(0)} + G^{(1)} + G^{(2)} + \cdots
$$

where $G^{(n)}(\eta)$ is an operator, such that $G^{(n)}(\nu \eta) = \nu^n G^{(n)}(\eta)$ for any constant $\nu$, i.e., $G^{(n)} \sim \varepsilon^n \sim \eta^n$, since $\eta \sim h_1 \varepsilon$ and similarly for $G_1$. Let $D = -i\partial_x$. Then, the corresponding expansions are:

$$
G(\eta) = D \tanh(hD) + D\eta D - D \tanh(hD) \eta D \tanh(hD) + O(\eta^2)
$$

$$
G_1(\eta) = D \tanh(h_1D) + D\eta D - D \tanh(h_1D) \eta D \tanh(h_1D) + O(\eta^2).
$$

We will study the equations of motion under the additional approximation that the wavelengths $L$ are much bigger than $h_1$. In particular,

$$
\delta = \frac{h_1}{L} \ll 1.
$$

Here, we notice that the wave number $k = \frac{2\pi}{L}$ is an eigenvalue or a Fourier multiplier for the operator $D = i\partial_x$ (when acting on waves of form $e^{ikx}$) we make the following further assumptions about the scales

- $\delta = O(\varepsilon)$;
- $hk = O(1)$ and $h_1k = O(\delta)$ (i.e. $\frac{h_1}{k} \sim \delta \ll 1$). This corresponds to a deep lower layer;
- $\xi \sim O(1)$;
- The physical constants $h_1, \rho, \rho_1, \gamma, \gamma_1 \sim O(1)$.

Since the operator $D = -i\partial_x$ has an eigenvalue $k$, thus we keep in mind that $hD = O(1)$ and...
$h_1 D = O(\delta)$. By writing the DN–operators in the form

$$G(\eta) = \frac{1}{h_1} \left( (h_1 D) \tanh(hD) + (h_1 D) \frac{\eta}{h_1} (h_1 D) - (h_1 D) \tanh(hD) \frac{\eta}{h_1} (h_1 D) \tanh(hD) + \cdots \right)$$

$$G_1(\eta) = \frac{1}{h_1} \left( (h_1 D) \tanh(hD) - (h_1 D) \frac{\eta}{h_1} (h_1 D) + (h_1 D) \tanh(hD) \frac{\eta}{h_1} (h_1 D) \tanh(hD) + \cdots \right)$$

Let us denote $D', \eta'$ to be such that together with $h$ and $h_1$ are assumed to be of order 1. Then, we can determine explicitly the scale factors

$$G(\eta) = \frac{1}{h_1} \left( \delta(h_1 D') \tanh(hD') + \delta^3(h_1 D') \frac{\eta'}{h_1} (h_1 D') - \delta^3(h_1 D') \tanh(hD') \frac{\eta'}{h_1} (h_1 D') \tanh(hD') + O(\delta^4) \right)$$

$$G_1(\eta) = \frac{1}{h_1} \left( \delta(h_1 D') \left[ \delta(h_1 D') - \delta^3 \frac{(h_1 D')^3}{3} \right] - \delta^3(h_1 D') \frac{\eta'}{h_1} (h_1 D') + \delta^3(h_1 D')(h_1 D') \frac{\eta'}{h_1} (h_1 D')(h_1 D') + O(\delta^4) \right)$$

Let us introduce the short-hand notation $t_h := \tanh(hD)$, $D_1 := h_1 D$ and omit the $'$ notation for convenience we truncate the DN–expansions as follows:

$$G(\eta) = \frac{1}{h_1} \left( \delta D_1 t_h + \delta^3 D_1 \frac{\eta}{h_1} D_1 - \delta^3 D_1 t_h \frac{\eta}{h_1} D_1 t_h \right) + O(\delta^4)$$

$$G_1(\eta) = \frac{\delta^2}{h_1} D_1 \left( 1 - \delta \frac{\eta}{h_1} - \delta^2 \frac{D_1^2}{3} \right) D_1 + O(\delta^5) \tag{1.0.9}$$

Note that $h$ appears only in the definition of the operator $t_h$, which is of order 1. Since $h_1$ is assumed of order 1, then formally the order of the differentiation $\partial_x$ is $\delta$. Hence the order of the integration measure $dx$ is $\frac{1}{\delta}$.

We see now that the leading order terms in $G$ are $O(\delta)$ and the leading order terms in $G_1$ are $\delta^2$. Hence, $G_1 G^{-1} \sim \delta \ll 1$ and moreover with the $B$ defined in (1.0.7) we can expand as follows:

$$GB^{-1} = G \frac{1}{\rho_1 G + \rho G_1} = \frac{1}{\rho_1 (1 + \frac{\rho}{\rho_1} G G^{-1})} \tag{1.0.10}$$

Both $G$ and $G_1$ are self-adjoint, implies that $GB^{-1} G_1$ is self-adjoint. Let us define the notation $T_h := -i \coth(hD) = (it_h)^{-1} = (i \tanh(hD))^{-1}$. Then, the substitution of (1.0.9) in (1.0.10) gives

$$B^{-1} = \frac{1}{\rho_1} \left( G^{-1} G G^{-1} G_1 + \rho_1^2 G^{-1} G_1 G^{-1} G_1 - \cdots \right) = \frac{1}{\delta D_1} t_h^{-1} T_h + O(1),$$

$$B^{-1} G = \frac{1}{\rho_1} \left( 1 - \frac{\rho}{\rho_1} G G^{-1} G_1 + \rho_1^2 G^{-1} G_1 G^{-1} G_1 - \cdots \right) = \frac{1}{\rho_1} \left( 1 - \frac{\rho}{\rho_1} \right) T_h \delta D + O(\delta^2).$$

The quantity $\eta \partial_x \eta = \delta^3 \left( \rho \frac{\partial}{\partial h_1} \right) D_1 (D_1)^{-1} \sim \delta^3$. The contribution of the integral density $\eta \partial_x \eta B^{-1} \eta \partial_x \eta$ in the Hamiltonian is therefore of order $\delta^5$. Recall that $dx \sim \frac{1}{\delta}$ and $\psi = \partial_x \xi$. Hence, by keeping terms up to $\delta^3$ in the Hamiltonian (1.0.8) we have

$$H(\eta, \vartheta) = \delta \frac{h_1}{2 \rho_1} \int_\mathbb{R} \vartheta^2 dx + \delta^2 \frac{A}{2} \int_\mathbb{R} \eta^2 dx + \delta \int_\mathbb{R} \eta \vartheta dx - \delta^2 \frac{1}{2 \rho_1} \int_\mathbb{R} \eta^2 \vartheta dx - \delta^2 \frac{L^2}{2 \rho_1} \int_\mathbb{R} \vartheta T_h \partial_x \vartheta dx$$

$$+ \delta^2 \frac{\gamma_1}{2} \int_\mathbb{R} \eta^2 \vartheta dx + \delta^2 \frac{\partial^2}{\partial \vartheta^2} - \rho_1 \gamma_1 \int_\mathbb{R} \eta^2 dx - \delta^3 \frac{h_1^2}{2 \rho_1} \int_\mathbb{R} \vartheta^2 \partial_x \vartheta dx$$

$$+ \delta^3 \frac{h_1 L}{\rho_1} \int_\mathbb{R} \vartheta T_h \partial_x \vartheta dx + \delta^3 \frac{(7 - \gamma_1) h_1 L}{2 \rho_1} \int_\mathbb{R} \eta^2 T_h \partial_x \vartheta dx.$$
where the constant $A = g(\rho - \rho_1) + \kappa(\rho_\gamma - \rho_1 \gamma_1)$. We notice that $H(\eta, \vartheta)$ is of order $\delta$. This gives the proper scaling of $\partial_x$ which should be also of order $\delta$, same as the order of $\partial_x$. The variation $\delta \vartheta$ bears a scale factor $\delta$ as well. The Hamiltonian relation (1.0.6) with the scaling written explicitly therefore are
\[
\partial_t \eta + \frac{1}{\rho_1} \partial_x \left( \frac{\delta H}{\delta \vartheta} \right) = 0 \quad \text{and} \quad \partial_t \vartheta + \Gamma \partial_t \eta + \frac{1}{\delta} \partial_x \left( \frac{\delta H}{\delta \eta} \right) = 0,
\]
which produces the coupled system
\[
\partial_t \eta + \kappa \partial_x \eta + \frac{h_1}{\rho_1} \partial_x \vartheta - \delta \frac{1}{\rho_1} \partial_x (\eta \vartheta) - \delta \frac{\rho h_1^2}{\rho_1^2} L_0 \partial_x^2 \vartheta + \delta \gamma_1 \eta \partial_x \eta + \delta^2 \frac{h_1^3}{\rho_1^4} \left( \frac{1}{3} + \frac{\rho^2 T_0^2}{2 \rho_1^2} \right) \partial_x^2 \vartheta
+ \delta^2 \frac{\rho h_1}{\rho_1^2} \left[ \partial_x (\eta L_0 \partial_x \vartheta) + T_0 \partial_x^2 \vartheta (\eta \vartheta) \right] + \delta^2 \frac{\rho h_1 (\gamma - \gamma_1)}{\rho_1} \partial_x^2 \vartheta (\eta \vartheta) = 0
\]
and
\[
\partial_t \vartheta + \kappa \partial_x \vartheta + \Gamma \partial_t \eta + A \partial_x \eta - \delta \frac{1}{\rho_1} \partial_x \vartheta + \delta \gamma_1 \partial_x (\eta \vartheta) + \delta \gamma_1 \partial_x (\eta \vartheta) + \delta (\rho_\gamma^2 - \rho_1 \gamma_1^2) \eta \partial_x \eta
+ \delta^2 \frac{\rho h_1 (\gamma - \gamma_1)}{\rho_1} \partial_x (\eta L_0 \partial_x \vartheta) + \delta^2 \frac{\rho h_1 (\gamma - \gamma_1)}{\rho_1} \partial_x (\eta \partial_x \vartheta) = 0.
\]

The ILW equation

From equations (1.0.11) and (1.0.12) by neglecting the terms of order $\delta^2$, we obtain
\[
\partial_t \eta + \kappa \partial_x \eta + \frac{h_1}{\rho_1} \partial_x \vartheta - \delta \frac{1}{\rho_1} \partial_x (\eta \vartheta) - \delta \frac{\rho h_1^2}{\rho_1^2} L_0 \partial_x^2 \vartheta + \delta \gamma_1 \eta \partial_x \eta = 0
\]
\[
\partial_t \vartheta + \kappa \partial_x \vartheta + \Gamma \partial_t \eta + A \partial_x \eta - \delta \frac{1}{\rho_1} \partial_x \vartheta + \delta \gamma_1 \partial_x (\eta \vartheta) + \delta \gamma_1 \partial_x (\eta \vartheta) + \delta (\rho_\gamma^2 - \rho_1 \gamma_1^2) \eta \partial_x \eta = 0
\]
Let us use a Galilean transformation of coordinates as follows:
\[
X = x - \kappa t, \quad T = t, \quad \partial_X = \partial_x, \quad D \rightarrow -i \partial_X, \quad \text{and} \quad \partial_T = \partial_t + \kappa \partial_x.
\]

Then, the equations of motion can be written as
\[
\partial_T \eta + \frac{h_1}{\rho_1} \partial_X \vartheta - \delta \frac{1}{\rho_1} \partial_X (\eta \vartheta) - \delta \frac{\rho h_1^2}{\rho_1^2} L_0 \partial_X^2 \vartheta + \delta \gamma_1 \eta \partial_X \eta = 0
\]
\[
\partial_T \vartheta + \Gamma \partial_T \eta + (A - \Gamma \kappa) \partial_X \eta - \delta \frac{1}{\rho_1} \partial_X \vartheta + \delta \gamma_1 \partial_X (\eta \vartheta) + \delta (\rho_\gamma^2 - \rho_1 \gamma_1^2) \eta \partial_X \eta = 0.
\]

By noticing $\kappa$ is the physical constant and $g \gg 2\omega \kappa$. We have that
\[
A - \Gamma \kappa = g(\rho - \rho_1) + \kappa(\rho_\gamma - \rho_1 \gamma_1) - \kappa(\rho_\gamma - \rho_1 \gamma_1) - \kappa(\rho_\gamma - \rho_1 \gamma_1) + 2\kappa \omega (\rho - \rho_1)
= (g - 2\kappa \omega)(\rho - \rho_1) \approx g(\rho - \rho_1).
\]

The equations of motion are reduced to the following:
\[
\partial_T \eta + \frac{h_1}{\rho_1} \partial_X \vartheta - \delta \frac{1}{\rho_1} \partial_X (\eta \vartheta) - \delta \frac{\rho h_1^2}{\rho_1^2} L_0 \partial_X^2 \vartheta + \delta \gamma_1 \eta \partial_X \eta = 0
\]
\[
\partial_T \vartheta + \Gamma \partial_T \eta + g(\rho - \rho_1) \partial_X \eta - \delta \frac{1}{\rho_1} \partial_X \vartheta + \delta \gamma_1 \partial_X (\eta \vartheta) + \delta (\rho \gamma^2 - \rho_1 \gamma_1^2) \eta \partial_X \eta = 0.
\]

The leading order terms (i.e., neglecting the terms with $\delta$ above) produce a system of linear equations with constant coefficients from where the speed(s) of the travelling waves (in the leading
order) could be determined:

\[ c = -\frac{h_1}{2\rho_1} \Gamma - \left( \frac{h_1}{4\rho_1} \Gamma^2 + \frac{h_1}{\rho_1} g(\rho - \rho_1), \right) \]  

(1.0.14)

where the + sign is for the right-running waves and the − sign is for the left-running waves. These speeds coincide with the speeds in the case of infinitely deep lower layer, indeed, the \( h \)-dependence comes only from the term \( T_h \) which is of order \( \delta \).

In what follows, \( c \) in (1.0.14) could be either of the two solutions of the dispersion equation, the choice of solution determines if the results are relevant for the left or the right-running waves.

For the travelling wave, which depends on the characteristic variable \( X - cT \), we also have \( \vartheta = \rho_1c\eta \) and in order to obtain a single nonlinear equation for \( \eta \) we expect a relation which involves terms of order \( \delta \). In other words, we consider an expansion of the form

\[ \vartheta = \frac{\rho_1}{h_1} c\eta + \delta \alpha \eta^2 + \delta \beta T_h \partial_X \eta, \quad (1.0.15) \]

for some constants \( \alpha \) and \( \beta \) we determine the following. This type of relationship is known also as the Johnson transformation. The substitution of \( \vartheta \) from (1.0.15) in (1.0.13) when keeping only the terms up to order \( \delta \) leads to two equations for \( \eta \). Therefore these two equations must coincide. This leads to equality of the coefficients in front of the terms of the same type, which further allows determining the previously unknown \( \alpha \) and \( \beta \):

\[ \alpha = \rho_1(\rho_1 C^2 + 2h_1 \Gamma c - \gamma_1 h_1^2 \Gamma + \rho_1 \gamma_1 h_1 c + h_1^2) \frac{2h_1}{2h_1(2\rho_1 c + h_1 \Gamma)}, \quad \beta = \rho(\rho_1 c^2 + h_1 \Gamma c) \frac{2h_1}{2\rho_1 c + h_1 \Gamma}. \]

We arrive at the ILW equation for \( \eta \) is

\[ \partial_T \eta + c\partial_X \eta - \delta \frac{\rho h_1 c^2}{2\rho_1 c + h_1 \Gamma} T_h \partial^2_X \eta - \delta = \frac{-3\rho_1 c^2 + 3\rho_1 \gamma_1 h_1 c + h_1^2 (\rho_1^2 - \rho_1 \gamma_1^2)}{h_1(2\rho_1 c + h_1 \Gamma)} \eta \partial_X \eta = 0 \]  

(1.0.16)

The ILW in the irrotational case (when \( \gamma = \gamma_1 = \omega = 0 \), \( \kappa = \Gamma = 0 \), in this case \( \kappa = 0 \) and hence \( (X, T) \equiv (x, t) \)) becomes

\[ \partial_T \eta + c\partial_X \eta - \delta \frac{\rho h_1 c}{2\rho_1} T_h \partial^2_X \eta - \delta = \frac{3c}{2h_1} \eta \partial_X \eta = 0. \]

Let us write the ILW equation (1.0.16) in the form

\[ \partial_T \eta + c\partial_X \eta + \delta \mathcal{A} \eta \partial_X \eta - \delta \mathcal{B} T_h \partial^2_X \eta = 0 \quad (1.0.17) \]

where

\[ \mathcal{A} := \frac{-3\rho_1 c^2 + 3\rho_1 \gamma_1 h_1 c + h_1^2 (\rho_1^2 - \rho_1 \gamma_1^2)}{h_1(2\rho_1 c + h_1 \Gamma)}, \quad \mathcal{B} := \frac{\rho h_1 c^2}{2\rho_1 c + h_1 \Gamma} \]

are physical parameters. Moreover, for \( 0 < k_0 < \frac{2\pi}{\kappa} \), let us mention that the one-soliton solution of (1.0.17) has the form

\[ \eta(X, T) = \frac{2\mathcal{B}}{\mathcal{A} \cos(k_0 h) + \cosh \left[k_0(X - X_0 - (c - \delta Bk_0 \cot(k_0 h))T)\right]} \frac{k_0 \sin(k_0 h)}{k_0 \sin(k_0 h) + k_0 \sin(k_0 h)} \]

In the above formula, \( X_0 \) and \( k_0 \) are the soliton parameters (i.e. arbitrary constants within their range of allowed values). \( X_0 \) is the initial position of the crest of the soliton and \( k_0 \) is related to its amplitude. The wave speed of the soliton is \( c = -\delta B k_0 \cot(k_0 h) \) and the correction of order \( \delta \) depends on the coefficient \( \mathcal{B} \) and the dispersion law related to the dispersive term and also on the spectral parameter \( k_0 \).
APPENDIX A. EQUATION DERIVATION

Connection to the Benjamin–Ono equation

In this section, we demonstrate that the Benjamin–Ono equation could be obtained as a special kind of a long-wave limit from the ILW equation.

In the limit $h \to \infty$ which corresponds to an infinitely deep lower layer we have

$$ T_h = -i \coth(hD) \to -i \text{sign}(D), \quad T_h \partial_X \to |D|. $$

The ILW equation (1.0.17) becomes the Benjamin–Ono equation

$$ \partial_T \eta + c \partial_X \eta + \delta A \eta \partial_X \eta - \delta B |D| \partial_X \eta = 0 $$

The limit to the one-soliton solution of the BO equation (with $h \to \infty$ but $k_0 h \lesssim 1$ finite, $k_0 h = \pi - \frac{h}{q}$ where $q$ is a constant) can be written in the form

$$ \eta(X, T) = \frac{\eta_0}{1 + (\frac{\rho A}{2B})^2 [X - X_0 - (c + \frac{4}{3} \delta A k_0) T]^2} $$

where $X_0$ is the initial position of the soliton and $\eta_0$ is its amplitude. The relation to the constant $q$ (and $k_0$) is

$$ \eta_0 = \frac{4Bq}{A} = \frac{4Bk_0}{A(\pi - k_0 h)}. $$

Connection to the KdV equation

In this section, we provided a discussion showing that KdV equation could be obtained as a special kind of a long-wave limit from the ILW.

The KdV limit could be obtained by assuming $|hD| \ll 1$ (or $hk \ll 1$). The operator

$$ T_h = -i \coth(hD) \approx -i \left( \frac{1}{hD} + \frac{1}{3} hD \right) = \frac{1}{h} \partial_X^{-1} - \frac{1}{3} h \partial_X. $$

Then, from the ILW equation (1.0.17), we have

$$ \partial_T \eta + \left( c - \delta A 2 \rho c + h_1 \Gamma \frac{h_1}{h} \right) \partial_X \eta + \delta A \eta \partial_X \eta + \delta B \frac{h \Gamma}{3} \partial_X^3 \eta = 0. $$

We notice that in this case $\frac{h_1}{h} \approx \delta \ll 1$ the correction to $c$ in the second term is of order $\delta^2$ and to be neglected. Hence, we obtain the KdV equation in the form

$$ \partial_T \eta + c \partial_X \eta + \delta A \eta \partial_X \eta + \delta B \frac{h \Gamma}{3} \partial_X^3 \eta = 0. $$

The one-soliton solution of KdV can also be obtained from ILW case. For $k_0 h \ll 1$ we have $\sin(k_0 h) \approx k_0 h$, $\cos(k_0 h) \approx 1$, and also the identity $1 + \cosh(z) = 2 \cosh^2 \left( \frac{z}{2} \right)$. Moreover, we observe

$$ B k_0 \cot(k_0 h) \approx B k_0 \left( \frac{1}{h} - \frac{k_0 h}{3} \right) = B \frac{h}{3} - B \frac{h}{3} k_0 = \frac{\rho c^2}{2 \epsilon \rho_1 + \Gamma h_1} \frac{h_1}{h} - B \frac{h}{3} k_0. $$

Here, we notice

$$ \frac{\rho c^2}{2 \epsilon \rho_1 + \Gamma h_1} \frac{h_1}{h} = \frac{\rho c^2}{2 \epsilon \rho_1 + \Gamma h_1} \delta \ll 1, $$

which does not depend on $k_0$. It represents the small $O(\delta^2)$ correction to the constant wave speed.
APPENDIX A. EQUATION DERIVATION

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c. Moreover, $B \frac{h}{A} k_0^2 \sim k_0^2$. Then, the approximation leads to

$$\eta(X, T) = \frac{3B}{A} \frac{hk_0^2}{1 + \cosh [k_0(X - X_0 - (c + \delta \frac{2B}{A} h k_0^2)T)]}$$

$$= \frac{B}{A} \frac{hk_0^2}{\cosh^2 \left[ \frac{k_0}{2} (X - X_0 - (c + \delta \frac{2B}{A} h k_0^2)T) \right]}.$$  

Let a new constant $K = \frac{k_0}{2}$, we have KdV soliton

$$\eta(X, T) = \frac{4B}{A} \frac{hk^2}{\cosh^2 [K(X - X_0 - (c + 4\delta \frac{2B}{A} K^2)T)].}$$

Study on the dispersion term

The dispersion operator $G_\delta$ is defined as the Fourier multiplier

$$G_\delta f(x) = -\coth(\delta \partial_x) + \frac{1}{\delta} \partial_x^{-1}, \quad \hat{G_\delta f}(\xi) := -i \left( \coth(\delta \xi) - \frac{1}{\delta \xi} \right) \hat{f}(\xi).$$

Thus, we use $G_\delta f(x) = -i \mathcal{F}\{(\coth(\delta \xi) - (\delta \xi)^{-1})\hat{f}(\xi)\}(x)$ and we have

$$G_\delta f(x) = -\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \left[ \coth(\delta \xi) - \frac{1}{\delta \xi} \right] \hat{f}(\xi) d\xi$$

$$= -\frac{i}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \left[ \coth(\delta \xi) - \frac{1}{\delta \xi} \right] \left( \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi y} f(y) dy \right) d\xi$$

$$= -\frac{1}{2\delta} \text{p.v.} \int_{\mathbb{R}} \coth \left( \frac{\pi(x - y)}{2\delta} \right) - \frac{1}{\delta} \text{sgn}(x - y) f(y) dy.$$  

Hence, we see that $G_\delta$ has an integral form.
Appendix B

On the development of the (deterministic) well-posedness theory

The KdV and the BO equations are two fundamental equations in the field of dispersive PDEs. They are one of the main equations of study in the area of dispersive PDEs, with extensive literature regarding the well-posedness theory. In what follows, without causing any confusion, we denote $v(t,x)$ as the solution to the KdV (or mKdV, gKdV), and $u(t,x)$ as the solution to the BO (or mBO, gBO).

We first look into the Cauchy problem of the generalised KdV:

$$\begin{align*}
\partial_t v + \partial_x^3 v + \partial_x (f(v)) &= 0, \\
v|_{t=0} &= v_0, \\
(t, x) &\in \mathbb{R} \times \mathcal{M},
\end{align*}$$

where $f(v) = v^k$. This corresponds to KdV with $k = 2$, mKdV with $k = 3$, and gKdV with $k \geq 4$.

• KdV well-posedness theory:

The Cauchy problem of KdV has been studied extensively. In [81], Kenig-Ponce-Vega showed the local well-posedness (LWP) in $H^s(\mathbb{R})$ for $s \geq -3/4$. This result is sharp, in the sense that the flow map fails to be uniformly continuous in $H^s(\mathbb{R})$ for $s < -3/4$; see [33]. Then, in [39], Colliander-Keel-Staffilani-Takaoka-Tao established global well-posedness (GWP) for $s > -3/4$ (via the so-called method of almost conservation law). Moreover, a stunning result of Killip-Visan [87] showed GWP in $H^{-1}(\mathcal{M})$ by using the complete integrability of the equation. This result is sharp, as the equation is known to be ill-posed below $H^{-1}(\mathbb{R})$ due to Molinet [112]. In the periodic domain, in [81], they showed the LWP in $H^s(\mathbb{T})$ for $s \geq -1/2$, and see [39] for the GWP. Lastly, the work of Kappeler-Topalov [84] by the so-called inverse method, which is based on the complete integrability of the KdV equation, they showed GWP in $H^{-1}(\mathbb{T})$. For further related work, see [7, 168].

• mKdV and gKdV well-posedness theories:

Let us now turn to the mKdV. In [80], Kenig-Ponce-Vega proved the LWP in $H^s(\mathbb{R})$ for $s \geq 1/4$. It has been shown that this result is sharp in the sense that the flow map fails to be uniformly continuous below $H^{1/4}(\mathbb{R})$, see [32] [33]. Then, GWP was shown for $s > 1/4$ in [39] (see also [53]). See also in [115] [98] in the context of unconditional uniqueness.

With periodic boundary conditions, Bourgain [19] proved LWP of mKdV in $H^s(\mathbb{T})$ for $s \geq 1/4$. In [32], Kenig-Ponce-Vega showed that when $s < 1/2$, the trilinear estimate breaks down. In [22], Bourgain also proved that the flow map corresponding to mKdV is not in $C^3$ for $s < 1/7$. However, these negative results do not necessarily imply that mKdV is ill-posed in $H^s(\mathbb{T})$ for $s < 1/2$. Indeed, by exploring the nonlinear smoothing effect, Takaoka-Tsutsumi [156] showed the LWP in $H^s(\mathbb{T})$ for $s > 3/8$. Then, this was improved by Nakanishi-Takaoka-Tsutsumi [121] to $H^s(\mathbb{T})$ for $s > 1/5$. 149
Moreover, in [117], Molinet-Pilod-Vento covered the endpoint case $s = \frac{1}{3}$, by using the improved energy method developed in [120] together with the construction of modified energies. See also [39, 34] for the related results.

Lastly, for the results related to the gKdV, we refer to [57, 80, 111, 119]. Here, we only emphasise the work of Molinet-Tanaka [119], which showed LWP in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$ and $H^s(\mathbb{T})$ for $s > \frac{2}{3}$ (similar results in the real line setting see [141]).

Next, we go over the literature on the Cauchy problem of the generalised BO:

$$
\begin{align*}
\partial_t u + H\partial^2_x u + \partial_x(f(u)) &= 0, \\
|_{t=0} &= u_0, \quad (t,x) \in \mathbb{R} \times \mathcal{M},
\end{align*}
$$

where $f(u) = u^k$. This corresponds to BO with $k = 2$, mBO with $k = 3$, and gBO with $k \geq 4$.

- **BO well-posedness theories:**
  First of all, without using any dispersion Iorio Jr showed GWP in $H^s(\mathcal{M})$ for $s > \frac{3}{2}$, see [1, 70]. Then, the first significant result was done by Koch-Tzvetkov [90] who proved LWP in $H^s(\mathbb{R})$ for $s > \frac{5}{4}$. The main idea of the proof is to improve the dispersive estimates (Strichartz estimates) by localising them in space frequency-dependent time intervals together with classical energy estimates. This method was improved by Kenig-Koenig [78] who obtained LWP in $H^s(\mathbb{R})$ for $s > \frac{9}{8}$. A breakthrough was made by Tao [157] who obtained the LWP (due to the conserved quantities, thus the GWP) in $H^s(\mathbb{R})$. The new ingredient applies a gauge transformation to eliminate the high-low interaction (terms involving the interaction of very low and very high frequencies, where the derivative falls on the very high frequencies). For the further developments see [28, 68, 113].

  On the other hand, without applying a gauge transformation, in [120], Molinet-Vento showed LWP in $H^s(\mathcal{M})$ for $s \geq \frac{1}{2}$. Here, they proposed an improved energy method, which is less powerful to get low regularity results but allows us to deal with perturbations of BO. See the further development in [116], where Molinet-Pilod-Vento extended their result to the regularity of $s > \frac{1}{4}$ (see also [86] in the periodic setting). See also Ifrim-Tataru [69] for the GWP in $L^2(\mathbb{R})$ by a normal form approach. Moreover, Gérard-Kappeler-Topalov [56] via the complete integrability showed that BO is GWP in $H^s(\mathbb{T})$ for $s > \frac{1}{2}$, which is a sharp result in the sense that BO is ill-posed for $s \leq \frac{1}{2}$.

- **mBO and gBO well-posedness theories:**
  For mBO we mention the work of Kenig-Takaoka [84], where they proved that mBO is GWP in $H^s(\mathbb{R})$ for $s \geq \frac{1}{2}$. The Sobolev exponent $\frac{1}{2}$ is optimal in the sense that the solution map is not uniformly continuous for $s < \frac{1}{2}$. See also the work of Kishimoto [85] and Guo [59].

  Lastly, we mention the results on gBO [60, 113, 95, 119]. We only focus on the work of Molinet-Tanaka [119], which showed that gBO is LWP in $H^s(\mathcal{M})$ for $s > \frac{1}{2}$ (see in [95] for a similar result on $\mathbb{T}$ but with the short-time Fourier restriction method).
Bibliography


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