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NON-LORENTZIAN GEOMETRY OF FLUIDS AND STRINGS

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Doctor of Philosophy
University of Edinburgh
2022
Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. This work has not been submitted for any other degree or processional qualification.

(Emil Have)
Til mor og far
Abstract

Non-Lorentzian geometry is a branch of geometry where, roughly speaking, the notion of a metric is replaced by something else. We begin by providing an overview of non-Lorentzian geometries, and we describe how they as Cartan geometries which emphasises their connection to kinematical symmetries. We also review their construction as G-structures, and the formalism of $1/c^2$ expansions. Examples of non-Lorentzian geometries include Carrollian, Newton–Cartan and Aristotelian geometries. After reviewing relativistic fluid dynamics, we develop a theory of boost-agnostic fluids using Lagrangian methods. These fluids couple to Aristotelian backgrounds, and their non-dissipative transport is described by an action coupled to Aristotelian geometry. In the final part of the thesis, we describe nonrelativistic approximations of string theory using $1/c^2$ expansions, and we show that this generalises the Gomis–Ooguri nonrelativistic string. The target space geometry of these string theories is a string Newton–Cartan-like geometry that arises from a stringy $1/c^2$ expansion of Lorentzian geometry.

Lay summary

The interplay between geometry and physics has flourished since Einstein's theory of General Relativity beautifully combined pseudo-Riemannian, or Lorentzian, geometry with gravity and showed that the gravitational force is encoded in the curvature of spacetime. General Relativity is predicated on Einstein's principle of relativity, which states that the laws of physics are the same in all inertial frames (i.e., unaccelerated reference frames), and that all observers, regardless of their relative motion, measure the same speed of light. This last
postulate is what truly distinguishes Einstein’s notion of relativity, and it implies that reference frames that move relative to each other with some velocity are related by Lorentz boosts. It also implies that the speed of light is the speed limit of the universe. This directly leads to Einstein’s special relativity, which plays out in four-dimensional (three space dimensions and one time) spacetime known as Minkowski space. The symmetries, i.e., all the things we can do to an observer in Minkowski space while still having them observe the same laws of physics, form a group called the Poincaré group, and consists, for example, of the Lorentz boost. In other words, we can put our observer in an inertial frame that moves with a particular velocity relative to the original reference frame using a Lorentz boost, and the laws of physics for the observer will be unchanged. Including the effects of gravity requires us to curve up the spacetime in such a way that it still locally looks like Minkowski spacetime, and doing so leads to Lorentzian geometry which is described by a metric. This is analogous to how a sphere (like the Earth) looks flat if we zoom in enough, but over large distances, the curvature becomes noticeable. Now, imagine we replace Einstein’s principle of relativity with that of Galilei: the first part remains unchanged, but the second part is replaced by the requirement that observers moving with any constant relative velocity see the same laws of physics. We can then go through exactly the same steps as above to construct first a flat Galilean spacetime which can be curved up, which gives rise to a Newton–Cartan geometry. This geometry is no longer described by a metric and is an example of a non-Lorentzian geometry. Such geometries have many practical applications in a physics context: since many important physical systems, notably condensed matter systems, are to a very good approximation described by Galilean symmetries, non-Lorentzian geometry is the appropriate geometric framework for such systems. While the real-world geometry does not curve, the knowledge of how to couple such systems to general geometries allows us to extract a lot of information and is particularly useful in the context of fluid dynamics, which is one of the few tools in a theorist’s toolbox that allows us to deal with so-called strongly coupled systems.

There are many examples of non-Lorentzian geometries, and their mathematical structure as well as their applications in physics ranging from fluid dynamics to quantum gravity is a thriving field of research.
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List of publications

   The results of this work are not included in the thesis.

   Chapter 4 and parts of Chapter 3 are based on this publication.

   Chapter 5 contains the results presented in this paper.

   This publication is not included in the thesis.

   Parts of Chapter 2 are based on results obtained in this paper.

   Parts of Chapter 2 are based on results obtained in this paper.

   The results of this work are included in Chapter 5.
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Chapter 1

Introduction

The Lorentzian paradigm has long reigned supreme in the world of theoretical (high energy) physics, and rightfully so; after all, the quantum field theories that describe the experiments at CERN are relativistic, and the framework of General Relativity is Lorentzian geometry. Nevertheless, there are length scales at which, to a good approximation, fundamental processes are governed by other symmetries. Atomic physics and condensed matter, for example, tend to take place in settings where things are not moving too fast and where the relevant relativity principle is not that of Einstein, but that of Galilei. In other words, the symmetries of the system are no longer described by the Poincaré group, but by an appropriate Galilean-like symmetry group. Harnessing the power of symmetries has remained a core principle of modern theoretical physics. In the realm of fluids, such symmetries dictate the form of the hydrodynamic relations [8, 9]. More generally, we can extract knowledge about field theories by coupling them to curved backgrounds that locally realise the appropriate symmetry group, which, for example, allows us to study anomalies; notably the Weyl anomaly [10]. In an approach pioneered by Dam Than Son in the context of the quantum Hall effect in [11–13], the coupling to curved space allows us to extract information about a strongly coupled quantum system, where the usual toolkit of perturbation theory fails.

In addition to these concrete scenarios where the Lorentzian description is not particularly useful, recent years have seen a surge in the interest of going beyond the Lorentzian paradigm. This is because even after almost a 100 years, there is still no quantum theory of gravity, which remains the holy grail of theoretical physics. One proposal, which originated with Hořava in [14, 15], involves a non-Lorentz invariant but renormalisable and UV complete theory of gravity, which is now known as Hořava–Lifshitz gravity (although it is related to Einstein–aether theory (or \(\mathcal{E}\)-theory) as developed by Jacobson et al. [16–18]). Just as General Relativity is a dynamical theory of Lorentzian geometry, Hořava–Lifshitz gravity is a dynamical theory of a particular non-Lorentzian geometry known as Aristotelian geometry (see, e.g., [4, 19]). While the experimental evidence is stacked against Hořava–Lifshitz gravity [20, 21], there are other uses for non-Lorentzian geometry in the context of quantum gravity. To illustrate this, consider the Bronstein cube of physical theories in Figure 1.1. This cube is a cartoon of physical theories, which depicts how various theories, such as General
Relativity, can be obtained from Classical Mechanics by “turning” on fundamental constants of nature: $\hbar$, $G$ and $1/c$. One can perturbatively expand a given theory in an appropriately small parameter; for example, one can perturb around Nonrelativistic Gravity to obtain corrections in $1/c$ as indicated on the diagram. Nonrelativistic Gravity is described by a slightly modified version of Newton–Cartan geometry, another example of a non-Lorentzian geometry [22–24], and could provide a more tractable starting point for the construction of a putative theory Nonrelativistic Quantum Gravity, which could pave the way for a similar construction for a full theory of Quantum Gravity.

Let us highlight a number of concrete settings where non-Lorentzian geometry plays an important rôle.

**Non-AdS holography & “AdS/CMT”** The holographic principle [25–28] stipulates an equivalence between a quantum theory of gravity in $(d + 1)$ dimensions (in the “bulk”) and a non-gravitational quantum field theory in $d$ dimensions, heuristically said to live on the “boundary”. The first concrete realisation of this principle originated in
string theory, where Maldacena “derived” the duality by considering a stack of D3-branes [29, 30]. This stack has a description both in terms of open and closed strings, and by taking a suitable low-energy limit, the open string description reduces to four-dimensional $N = 4$ superconformal Yang–Mills (SYM) theory with gauge group $SU(N)$ in the “planar limit” (i.e., large-$N$), while the closed string description reduces to type IIB (super)gravity on $AdS_5 \times S^5$. The conjecture, then, is that the full type IIB string theory on $AdS_5 \times S^5$ is dual to $N = 4$ SYM with gauge group $SU(N)$ for any $N$. While this AdS/CFT correspondence remains the gold standard of holography, there has been a concerted effort to find other manifestations of the holographic principle. In particular, motivated by potential applications to strongly coupled condensed matter systems [31, 32], attempts at developing nonrelativistic holographic theories have been made. It was in this context that Newton–Cartan geometry was rediscovered as the boundary geometry in Lifshitz holography [33, 34] (see also [35]).

**Celestial (or flat space) holography** A particularly interesting realisation of the holographic principle involves a gravity in asymptotically flat spacetimes [36–40]. This is because our universe, to a good approximation, is flat, and so having a quantum description of gravity in our universe in terms of some much more (non-gravitational) tractable quantum field theory is highly desirable. In the AdS/CFT correspondence, one of the basis entries in the “dictionary” that forms the basis of the duality is the matching between the global symmetries of the two sides: on the gravitational side, the global symmetries are the isometries of $AdS_5$, which form the group $SO(4, 2)$, while on the field theory side, the same symmetry group arises as conformal spacetime symmetries (disregarding supersymmetry). In asymptotically flat spacetimes, the asymptotic symmetries are enhanced from the Poincaré group to the Bondi–Metzner–Sachs (BMS) group [41–43]. Hence, quantum gravity in four-dimensional asymptotically flat space should be dual a BMS field theory living on the three-dimensional null boundary [44–47]. The BMS group in four dimensions is isomorphic to the conformal Carroll group [48, 49], so a BMS field theory is the same a conformal Carrollian field theory. There are some unique challenges facing the flat holography program: the null nature of the conformal boundary means that it, like in Lifshitz holography, admits a non-Lorentzian geometric description: it is an example of a Carrollian geometry. Moreover, unlike in AdS spacetimes, the boundary is *leaky* and allows radiative flux to leak through the boundary. To take this into account, the dual conformal Carrollian field theory should include source terms [50]. There is another perspective one may take on flat space holography. This proposal is known as celestial holography, where the holographic dual of quantum gravity in four-dimensional asymptotically flat spacetimes is a two-dimensional *celestial* conformal field theory that lives on the conformal sphere at infinity [51–54] (see [55, 56] for reviews and additional references). Celestial holography is based on the observation that the gravitational S-matrix expressed in a basis of boost eigenstates assumes the same form as two-dimensional conformal correlation functions, and so brings the powerful machinery of two-dimensional conformal theory into play. More recently, it was realised the *ambitwistor* string theory dynamically generates celestial operator product expansions [57]. Ambitwistor strings
were originally developed to provide a dynamical theory for the Cachazo–He–Yuan formulae for scattering amplitudes [58, 59]. Ambitwistor strings are related the tensionless (or null) limit of string theory [60, 61], which in turn are related to Carrollian worldsheet limits. Even more recently, a concrete top-down construction of flat space holography was put forward in [62].

**String theory**  String theory is a marvellous mathematical framework that describes the dynamics of extended objects. Originally, and ambitiously, billed as a candidate “theory of everything” to describe every force of nature, string theory should be considered a framework in the same sense as quantum field theory (QFT). QFT can be made to describe many facets of nature: notably, the standard model of particle physics is a Lorentz invariant QFT that very accurately describes three of the four fundamental forces of nature. In the realm of condensed matter physics, QFT can be used to describe phenomena such as superconductivity and, more generally, phase transitions, but these QFTs are not Lorentz invariant: rather, they are Galilei invariant. If we extend this line of thinking to string theory, a natural question to ask is: what happens to string theory if we replace Lorentz invariance with another type of kinematical invariance?

One of the first non-Lorentzian string theories was developed by Gomis and Ooguri as well as Danielsson et al. in [63, 64]. This nonrelativistic string theory, which was also called *wound string theory*, was inspired by techniques developed in the context of non-commutative open strings and open membrane (⃗Ω) theory [65–68]. This string theory exhibits a Galilean spectrum, but requires a compact direction in the target space around which the strings can wind. This winding roughly corresponds to mass, and it was recognised early on that states with zero winding play the rôle of “Newtonian gravitons” and mediate an instantaneous gravitational force [69]. More than a decade later, these nonrelativistic string theories were generalised to a curved target space in [70–76], which involves so-called *string Newton–Cartan geometry*, which is a type of non-Lorentzian geometry.

While nonrelativistic string theory is certainly the most studied non-Lorentzian string theory, this points to the existence of a landscape of non-Lorentzian string theories where the target space (and even the worldsheet) is non-Lorentzian. One concrete way to begin the study of such non-Lorentzian string theories, which has already yielded many results, is via a very interesting link with double field theory [77–83].

**Exotic states of matter**  Non-Lorentzian symmetries are relevant for condensed matter systems. A particularly interesting example involves a phase of matter involving exotic quasiparticles known as *fractons* [84, 85]. Fracton phases of matter include excitations with restricted mobility [86, 87], and thus constitute an unfamiliar and fundamentally new (though still purely theoretical) phase of matter. Interestingly, fracton theories do not fit into the Wilsonian framework for QFTs: low-energy observables depend on

---

1While fracton phases of matter have not yet been observed experimentally, there are various concrete proposals to engineer such phases in ultra-cold atoms or in antiferromagnets [88, 89], and it may be that vortices in superfluids and the lowest Landau level of fractional quantum Hall states admit fractonic excitations [90–92].
the short-distance physics (see [93] and, in particular, references therein). This UV/IR mixing is one the chief obstacles in the theoretical description of these phases of matter.

In the absence of a standard quantum field theory description,\(^2\) one of the most powerful tools at our disposal is that of hydrodynamics, which is an almost universally applicable long-range description a given system at finite temperature. Hydrodynamics essentially describes the conserved quantities of the theory while taking into account the laws of thermodynamics such as positivity of entropy production. These conserved quantities are in turn dictated by the symmetries of theory, so, to a large extent, the only input of the hydrodynamic theory is the symmetries. In the case of theories with a conserved dipole moment, the spacetime symmetries are Aristotelian, i.e., there is no boost symmetry, so the full symmetry algebra takes the form

\[
\begin{align*}
\{M_{ij}, M_{kl}\} &= -4\delta_{[k[i}M_{j]l]} \\
\{M_{ij}, P_{k}\} &= -2\delta_{[i |j}P_{k]} \\
\{P_{i}, Q_{j}^{(2)}\} &= \delta_{ij}Q^{(0)} \\
\{M_{ij}, Q_{k}^{(2)}\} &= -2\delta_{ij}Q_{k}^{(2)},
\end{align*}
\]

(1.0.1)

where \(M_{ij}\) are the spatial rotations, \(P_{i}\) spatial translations, while \(Q^{(0)}\) is the \(U(1)\) charge, and \(Q^{(2)}_{i}\) is the dipole moment. In addition to these generators, the algebra contains a central Hamiltonian \(H\), which generates time translations.

Recently, it has been shown that fracton theories (or, more precisely, field theories with global or local dipole symmetries) generically couple to Aristotelian geometries [4] (see also [95]). Hydrodynamics without boost symmetry was developed using the technology of hydrostatic partition functions in [2] (see also [96]). To extend these results to theories with a conserved dipole moment, we need to include the symmetric tensor gauge field that arises when gauging the dipole symmetry. However, maintaining dipole gauge invariance on generic curved spaces is tricky (see [4]), which makes writing down the hydrostatic partition function tricky. More generally, the restricted mobility makes it unclear whether fractons admit a hydrodynamic description at all.\(^3\) This is similar to the situation for systems with Carroll symmetry\(^4\), see, e.g., [97], where it is also unclear whether a hydrodynamical description exists (although notions of Carroll fluids do exist, see, e.g., [98,99], although they do not use variational methods).

Some aspects of fracton fluids have nevertheless appeared in the recent literature [100–102], but it is fair to say that a full description of fracton fluid dynamics remains an open problem.

The bizarre trait of not being able to freely move offers a novel window to widen our understanding of physical (quantum field) theories, gravitational physics [103, 

\(^2\)Another line of attack involves using large-N techniques, which was recently explored in [94].

\(^3\)Attempts at writing down a hydrostatic partition function for fractons yields strange results, lending credence to the possibility that “conventional” fracton fluids do not exist.

\(^4\)And, as was remarked in [4], Gaussian fracton theories are Carrollian. In particular, Carroll particles are immobile just like isolated fractons.
holography [105], and might even have applications in the context of quantum information storage [85, 106–108]. For further details and references we refer to the reviews [109, 110].

Finally, let us mention that there are other tantalising indications that Carroll symmetry may be of relevance for the theoretical description of some condensed matter systems. We will elaborate on some of the above points in what follows.

1.1 What are the possible kinematics of space and time?

First asked in a groundbreaking paper by Levi-Leblond and Bacry in 1967 [111], the question of which kinematics are possible was highlighted by Dyson as a missed opportunity in his 1972 Gibbs lecture [112]. By *kinematics* we mean the symmetries under which the laws of nature are invariant: for example, the fundamental laws of physics as we know them are invariant under spatial rotations, translations in space and time, and inertial transformations *a.k.a.* boosts. In a “relativistic” theory, the transformation between inertial frames is a Lorentz boost, while in a “nonrelativistic” theory it is a Galilean boost. Of course, the terminology “nonrelativistic” used to describe theories where the speed of light is infinite is unfortunate, although convention makes it hard to avoid. Such theories should instead be referred to as *Galilean*: it is *not* the case that there is no principle of relativity; rather, the Galilean principle of relativity applies. As Galilei himself put it in 1632 [113]:

“Shut yourself up with some friend in the main cabin below decks on some large ship […] have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. […] The cause of all these correspondences of effects is the fact that the ship’s motion is common to all the things contained in it, and to the air also. That is why I said you should be below decks; for if this took place above in the open air, which would not follow the course of the ship, more or less noticeable differences would be seen in some of the effects noted.”

In other words, there exist special Galilean coordinate systems where the equations of motion for mechanical systems take the same form. These are the *inertial systems*. Any coordinate system that moves uniformly relative to an inertial system is itself an inertial system.

Similarly, for theories with a vanishing speed of light, what we might call the “the Carrollian principle of relativity” applies. Here, an observer with finite energy must remain
stationary as space becomes absolute. The Carrollian inertial frames are related by uniform motion in time: a Carrollian boost changes time according to \( t' = t + \vec{\beta} \cdot \vec{x} \), where \( \vec{\beta} \) is the Carrollian boost parameter that behaves like an inverse velocity.

Another class of symmetries are those that are Aristotelian: these are the symmetries of absolute time and space, and as such describe truly nonrelativistic spacetimes in the sense that a principle of relativity is absent; i.e., inertial frames are not related by any boost transformation.

Classifying these symmetries, i.e., classifying kinematical Lie algebras up to isomorphism, is just the first part of the story. This is because the Lie algebra does not uniquely determine the geometric realisation of the Lie algebra. As a case in point, the Poincaré algebra acts transitively on both Minkowski spacetime and a Carrollian version of anti-de Sitter space. To classify the kinematics, then, we need to also classify the geometric realisations of the kinematic symmetries on spacetimes. Concretely, these spacetimes are constructed as homogeneous spaces of the Lie groups corresponding to the kinematical Lie algebras, e.g., Minkowski space in \((d + 1)\) dimensions corresponds to the homogenous space \( \mathbb{R}^{d+1} = \text{ISO}(d, 1)/\text{SO}(d, 1) \) of the Poincaré Lie group \( \text{ISO}(d, 1) \).

1.2 Fluid dynamics

Fluid dynamics provide a universal description of the long-wavelength behaviour of physical systems. As such, fluid dynamics plays a crucial rôle as one of only a handful of tools capable of describing the physics of strongly coupled systems. Once a hydrodynamic description is established for a system, its evolution is governed by the hydrodynamic equations of motion, which express the conservation of currents such as the energy-momentum tensor. The relevant currents are parametrised in terms of fluid variables such as temperature, fluid velocity and chemical potential via the constitutive relations—see, e.g., [8, 9].

In the standard treatment of hydrodynamical frameworks some type of boost symmetry is assumed, namely Galilean boost symmetry for non-relativistic hydrodynamics and Lorentz boost symmetry for its relativistic counterpart. Consequently, these symmetries are present in the well-known Navier-Stokes equations [8] and relativistic hydrodynamics [9] or magneto-hydrodynamics [114–117] respectively. While these boost symmetries are conventionally assumed, there is no a priori reason to require any type of boost symmetry in the formulation of hydrodynamics. From a theoretical point of view they are not necessary as the hydrodynamic equations generally follow from conservation of energy/momentum and other charges, which in turn are connected to time/space translations and possible extra global symmetries. Boost symmetries, on the other hand, provide relations between components of the various currents, and are as such not an essential ingredient, though they are reflected

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5 There exists another possibility: a particle with zero energy can move in an unrestricted fashion, and can be viewed as an ultralocal limit of a tachyon.

6 Assuming spatial rotational symmetry is not necessary either, though often taken as an extra symmetry as is also the case in the present work. Hydrodynamics for anisotropic systems has been studied in several places, e.g., in [118, 119].
as extra symmetries of the resulting hydrodynamic equations. In fact, as we will return to in more detail shortly, there exist many physical systems that do not exhibit boost symmetry. In particular, as soon as there is a preferred reference frame, i.e., a medium with respect to which the fluid moves, this symmetry will be broken. Moreover, breaking of boost symmetry also occurs in critical systems with scaling symmetry characterised by a generic dynamical exponent $z$, i.e., systems with Lifshitz symmetry. Importantly, the no-go theorem of [120] says that, when $z \neq 1, 2$, such systems cannot exhibit boost symmetry if they allow for a fluid description. A similar no-go theorem was found in [121] from a field theoretic point of view.

A systematic treatment of perfect fluids with translation and rotation symmetries, applicable in the absence of any type of boost symmetry was first given in [120]. In a subsequent work [122] the first-order hydrodynamics and transport for these perfect fluids were studied when linearising around a zero velocity background. Furthermore, hydrodynamics of systems without boosts has been addressed previously (see e.g. [123, 124] and [125–127]) but our starting point, the perfect fluid thermodynamics introduced in [120], differs from these works.

Our hydrodynamic description starts from the perfect fluid energy-momentum tensor for non-boost invariant systems obtained in [120]. This includes a new thermodynamic variable, the kinetic mass density $\rho$, which is the thermodynamic dual of the magnitude of the fluid velocity squared, $v^2$. Furthermore, as an immediate consequence of the absence of boost symmetry, all extensive thermodynamic quantities now also depend on the extra intrinsic thermodynamic quantity $v$. The analysis of [120] shows that this more general class of perfect fluids leads to corrections to the Euler equations, which might be observable in hydrodynamic fluid experiments. One also finds new expressions for the speed of sound in perfect fluids, reducing to known results when boost symmetry is present. A concrete realisation of this framework can be obtained by considering an ideal gas of Lifshitz particles, enabling for example to obtain expressions for the speed of sound for corresponding classical and quantum Lifshitz gases [120]. Furthermore, the linearised first-order analysis in [122] has provided novel expressions for the linearised Navier–Stokes equation including new dissipative and non-dissipative first order transport coefficients.

As is well known, in order to account for dissipative effects, the conserved currents entering the effective description are expanded to a given order in derivatives of the hydrodynamic fields under the assumption that these derivative corrections are small compared to some intrinsic length scale of the microscopic system (e.g., the mean free path). In the currents, and therefore also in the resulting hydrodynamic equations of motion, each independent derivative correction term is multiplied by a transport coefficient, such as viscosity and conductivity. The values of these coefficients are constrained by the requirement that the divergence of the entropy current is non-negative and, additionally, by the Onsager relations (or, rather, their appropriate generalisation: absence of anti-symmetric transport) in systems with time reversal symmetry. For systems with a microscopic description, the specific form of certain transport coefficients can be determined via Kubo formulae. If a system admits a gravitational dual, further relations abound: notably, it has been shown via the AdS/CFT correspondence that shear viscosity divided by entropy density is equal to $\frac{1}{4\pi}$ for a strongly coupled plasma in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [128].
To obtain the first-order hydrodynamics for non-boost invariant systems following the paradigm reviewed above, our analysis makes crucially use of the geometry that is connected to non-boost invariant fluids along with the power of hydrostatic partition functions and the entropy current formalism. The geometry on which non-boost invariant fluids live is the geometry that realises (locally) only translational (space and time) and rotational symmetries. These symmetries are precisely the Aristotelian symmetries that we discussed above, and the corresponding geometry is that of curved absolute spacetime, known as Aristotelian geometry.

An interesting feature of non-boost invariant fluids is the appearance of non-dissipative transport coefficients at first order, alongside dissipative transport coefficients. By applying the entropy current constraint to the full non-linear constitutive relations we will show that there are 10 dissipative transport coefficients and 6 non-dissipative ones. We also show that the number of transport coefficients is unaffected by the introduction of background curvature to first order in derivatives. Following, one can further separate the non-dissipative ones into two types, hydrostatic and non-hydrostatic, which in the present case turns out to be 2 and 4 transport coefficients, respectively. For the case of Lifshitz fluids, these numbers become 7, 1 and 2, respectively. We will show that both the hydrostatic and non-hydrostatic transport coefficients can be obtained using Lagrangian methods. The hydrostatic transport coefficients feature in the non-canonical part of the entropy current and coincide with contributions that can be computed using an action principle obtained by allowing for time dependence in the hydrostatic partition function, see, e.g., and the earlier works. Furthermore, we find that when restricting to linearised perturbations all the non-hydrostatic transport coefficients vanish due to the Onsager relations.

We now return to a brief discussion of the physical relevance of non-boost invariant hydrodynamics.

Relevance of non-boost invariant hydrodynamics

As remarked above, for many systems in nature one does not have the luxury of assuming boost symmetry. In condensed matter systems, for example, one of the most prominent example where boost symmetries are absent are fracton phases of matter, which we discussed above. Their spacetime symmetries are Aristotelian, and they couple to curved Aristotelian geometries. One can also study critical points where boost symmetries are absent and viscous electron fluids. With this application in mind, it is shown in the original Refs. how the framework of non-boost invariant hydrodynamics can be adapted to (non-relativistic) scale invariant fluids with critical exponent $\zeta$. This includes particular expressions for the speed of sound in generic $\zeta$ Lifshitz fluids as well as specific results for

\footnote{In the way we will later set up our calculations, we consider fluids which could have relativistic or Galilean boosts. The Carrollian boost invariant fluid as realised in, e.g., will not be considered in this thesis.}

\footnote{Similarly, non-relativistic (Galilean) fluids live on the geometry that locally realises these symmetries and in addition Galilean boosts, i.e., Newton–Cartan geometry. This was used in, e.g., [1,129–132].}
the first-order transport coefficients in the linearised case. In particular, it was shown that the sound attenuation constant depends on both shear viscosity and thermal conductivity. The framework was also recently used in [146] to study out-of-equilibrium energy transport in a quantum critical fluid with Lifshitz scaling symmetry following a local quench between two semi-infinite fluid reservoirs. It is also interesting to note that Lifshitz hydrodynamics is relevant in connection with non-AdS holographic realisations of systems with Lifshitz thermodynamics [132, 147–150], see also [151–155].

More generally non-boost invariant hydrodynamics is of relevance to any system with a reference frame, such as æther theories or in various active matter systems exhibiting, e.g., flocking behavior (cf., Figure 1.2), see, e.g., [156], and [157] for a more recent example. General active matter systems typically do not have conserved energy or momentum as the divergence of the energy and momentum currents are equal to “driving” terms. Hence, non-boost invariant hydrodynamics is only an approximate description for configurations that are close to equilibrium configurations at “cruising speed” where the driving terms vanish.

Assuming that the fundamental laws of physics are Lorentz invariant, the necessity of some type of reference frame to obtain a system with broken boosts is obvious. Dispersion relations which are non-analytic and incompatible with boost invariance, such as those of capillary waves and domain-wall fluctuations in superfluid interfaces (ripplons), see for example [158], do indeed describe the propagation of particular fluctuations with respect to
a medium. But in order to have a hydrodynamic description of excitations with respect to a medium, we also need that energy and momentum of these excitations are approximately conserved. This is a non-trivial requirement, and requires a relatively weak coupling of the excitations to the medium. In addition, in order to be in the hydrodynamic regime, the interaction times and length scales of the excitations with themselves must be much smaller than those of the excitations with the medium. For example, for electrons in a crystal, the electron-electron scattering rate must be much higher than the rate for scattering of impurities and phonons in order to possibly have hydrodynamic flow [144].

It is of separate interest to understand systems with broken boost symmetry from an effective field theory point of view. Here, the boost breaking arises because we integrate out the degrees of freedom of the medium in a state which breaks the boost symmetry, for example because the medium has a fixed density or particle number. In [159] a classification of condensed matter systems that break boost symmetry but preserve rotation and translation invariance was presented. The simplest possibility presented there was a so-called type I framid, which is a system where the unbroken spacetime translation and rotation generators are unmodified in the presence of a boost-breaking state. Curiously, this possibility does not seem to be realised in nature, which was also seen in a recent analysis of the structure of Goldstone bosons associated to boost breaking [160]. In such a putative type I framid the expectation value of the energy-momentum tensor must be proportional to the metric with the sum of the energy density and the pressure equal to zero. This is similar to the effective energy-momentum tensor one can associate to a cosmological constant, and also the form of the energy-momentum tensor in the presence of additional Carroll boost invariance [120]. It is unclear whether any systems in nature properly realise Carroll symmetry and this observation may be in fact equivalent to the non-existence of type I framids. For a more detailed discussion of systems with (approximate) Carroll symmetry, we refer the reader to [161].

A simple example which gives rise to a system without boost invariance and which does appear in nature is a superfluid with a spontaneously broken $U(1)$ symmetry of the type considered in [162]. These systems contain a coupling $\int \text{d}^{d+1}x A_\mu J^\mu$ where $J^\mu$ is the current associated to the global $U(1)$ symmetry and $A_\mu$ acquires an expectation value $A_\mu = \lambda \delta_\mu^0$. This is reminiscent of a chemical potential and we assume that at finite $\lambda$ the ground state breaks the global $U(1)$ symmetry. If $\lambda$ is constant the system remains invariant under translations and rotations. Our hydrodynamical description will still be a good approximation as long as $|\partial_\mu \lambda| \ll |\lambda|$ so that energy and momentum are approximately conserved. To find the energy-momentum tensor of the theory, it is convenient to use vielbeins to convert curved into flat indices, and to assume that $A_\mu = \lambda \delta_\mu^0$. This is now a scalar and not a tensor from the spacetime point of view, and one can obtain a conserved stress-tensor by varying with respect to the vielbeins as described in e.g. [163]. The result of this computation is a new stress tensor of the form $T_{\mu\nu}^{\text{new}} \sim T_{\mu\nu}^{\text{old}} + J_\mu A_\nu$ which is clearly not symmetric and therefore incompatible with boost invariance. Moreover it shows that the new generator of time translations is a linear combination of the old generator plus the $U(1)$ current. As described in detail in [159], other breaking patterns are also possible.
1.3 Nonrelativistic string theory

We have already discussed the prospect of considering string theories with general non-Lorentzian symmetries. In this thesis, we will focus on nonrelativistic string theory, which remains the best understood non-Lorentzian string theory, and we develop a formulation based on $1/c^2$ expansions. Following the pioneering work of [63, 64], nonrelativistic (NR) string theory has developed into an active field of research. There is a growing class of string theories whose worldsheet or target space geometry is non-Lorentzian. Such settings are relevant for non-Lorentzian versions of holographic dualities, to better understand certain limits of relativistic string/M-theory, and to study non-Lorentzian theories of quantum gravity. The nonrelativistic string studied here belongs to this larger class of string theories.

The generalisation to curved target space geometries was considered in [70–76], and these geometries were further studied in [164–168]. The beta functions were studied in [74, 169–171], while a Hamiltonian perspective was taken in [71, 172–177]. Open NR strings and DBI-like actions for NR Dp-branes were described in [178–187], and the connection to limits of the AdS/CFT correspondence was investigated in [70, 73, 75, 188–194]. Supersymmetric NR strings were considered in [188, 195–197], and the relation between double field theory and NR strings was established in [77–83]. For a recent review of NR strings, see [198].

The idea of perturbatively expanding in an appropriately small parameter is a central tenet of physics: in perturbative QFT, the expansion parameter is $\hbar$, while post-Newtonian expansions assume both weak fields and small velocities and thus can be considered expansions in both $G$ and $1/c$. Further afield, the gradient expansion of hydrodynamics corresponds to an expansion in small values of the wave number, or, in other words, a long wavelength expansion.

One advantage of performing a $1/c^2$ expansion, as opposed to taking the $c \to \infty$ limit, is that we can, in principle, go to any order in the expansion that we like. We can illustrate this principle using the Bronstein cube, where the novelty of the systematic $1/c$ expansion lies in the fact that it permits us to probe the edges of the cube (along the $1/c$ axis). This is exemplified above in Figure 1.1 for the case of gravity.

The string $1/c^2$ expansion introduced in [3] singles out both the time direction and a distinguished space direction, which form the so-called longitudinal directions; the remaining directions are called transverse. This stands in contrast to the particle $1/c^2$ expansion of [22–24], where only time is singled out. As was shown in these works, the particle $1/c^2$ expansion up to $O(c^{-2})$ leads to type II torsional Newton–Cartan geometry, contains Newton–Cartan geometry when the torsion is zero. As we will show, the string $1/c^2$ expansion of a Lorentzian geometry with metric $G_{MN}$ gives rise to what we call type II string Newton–Cartan (SNC) geometry. The expansion of the metric $G_{MN}$ up to order $c^{-2}$ is

$$G_{MN} = c^2(-\tau_M^0\tau_N^0 + \tau_M^1\tau_N^1) + H_{MN} + O(c^{-2}),$$

and this reduces to SNC geometry if the strong foliation constraint

$$d\tau^A = \epsilon^A_{\ B} \omega^B \wedge \tau^B, \tag{1.3.2}$$

Type II SNC involves an additional field that appears in the expansion of the transverse part of the metric.
is satisfied for the 1-forms $\tau^A$ that appear at order $c^2$ in the expansion of the metric (1.3.1). This is in complete analogy with the particle case. In this expression, $\omega$ is a 1-form determined by this equation, and $A, B = 0, 1$ are longitudinal tangent space index. The reason we add the qualifier “strong” to the condition (1.3.2) is that the $1/c^2$ expansion of string theory comes with its own foliation constraint

$$d\tau^A = \alpha^A B \wedge \tau^B,$$

(1.3.3)

for arbitrary 1-forms $\alpha^A B$. This condition (1.3.3) is nothing but the Frobenius integrability condition, which guarantees that the $\tau^A$ define a codimension-2 foliation, and arises as the leading order (LO) Einstein equation, which in turn comes from the beta functions of relativistic string theory. Ignoring the dilaton and the Kalb–Ramond field, these are $R_{MN} = 0$ to leading order in $\alpha'$, and the $1/c^2$ expansion of this equation leads to (1.3.3). Choosing $\alpha^A B$ to be proportional to $\epsilon^A B$ reduces (1.3.3) to the strong foliation constraint (1.3.2).

At this point, one might wonder why we must use the string $1/c^2$ expansion rather than the particle $1/c^2$ expansion. Had we performed a particle $1/c^2$ expansion, the resulting leading order string theory would be the Galilean string [199]. These strings suffer from the problem that they do not admit oscillations, which follows from the fact that the particle $1/c^2$ expansion places the kinetic and potential terms at different orders in $1/c^2$. This precludes exchange of energy between the potential and kinetic terms and thus does not lead to oscillations, and as such these “string theories” describe rigid extended objects with only center-of-mass motion.

It was already observed by Gomis and Ooguri that the longitudinal spatial direction that we single out in the string $1/c^2$ expansion must be compact for the resulting string theories to have a non-trivial spectrum [63]. One way to understand this is that the compact direction provides an additional length scale, say $R$, which due to the presence of the intrinsic string length scale $\ell_s$ is required to form a dimensionless parameter in terms of which we can perform the $1/c^2$ expansion. This is because, at the end of the day, the $1/c^2$ expansion must be an expansion in terms of a dimensionless parameter formed by quantities already present in the theory (e.g., $c, \hbar$ etc. as well as other intrinsic characteristic quantities) and which is small when $c$ is large. For string theory, the parameter is $\ell_s/R$, which, as we will show, corresponds to an expansion in $1/c^2$ in the sense that the centre of mass velocity in the compact direction is much smaller than the speed of light. Since $\ell_s/R$ is small when $R$ is large, this also has the interpretation as an expansion around a decompactification limit.

We show that the LO string theory only probes the longitudinal target space and is “topological” in flat space. The equation of motion for the embedding scalar string imposes the Frobenius constraint (1.3.3).

At the next order in $1/c^2$—the next-to-leading order (NLO)—the theory becomes much richer. If the target space is such that the otherwise arbitrary 1-forms in (1.3.3) are traceless, $\alpha^A A = 0$, the Lagrangian of the theory reduces to that of the SNC string [72, 73]. This is still more general than the strong foliation constraint (1.3.2), where $\alpha^A B \sim \epsilon^A B$.\(^{10}\) When the

\(^{10}\)Recently, other studies have also proposed relaxing the strong foliation constraint motivated by the quantum theory [166, 167].
target space is flat, the NLO theory is the Gomis–Ooguri string [63].

The string theory at next-to-next-to-leading order (NNLO) is more complicated due to the proliferation of terms that the $1/c^2$ expansion gives rise to. We show that the gauge fixed NNLO theory on flat space has a spectrum that corresponds to the expansion of the spectrum of the relativistic string.

The string $1/c^2$ expansion in particular involves expanding the embedding scalars $X^M = x^M + c^{-2} y^M + \cdots$. The subleading components of these make sure that the string theory at any given order “remembers” the string theories at all previous orders. For example, the dynamics of the LO theory are imposed in the NLO theory by $y^M$.

An important ingredient of the Gomis–Ooguri string is the Kalb–Ramond field $B_M$, which is included in the theory via a Wess–Zumino (WZ) term. By tuning the Kalb–Ramond field to a critical value, we can lift the dynamical foliation constraint (1.3.2). This, together with a Stückelberg symmetry between $m_M$ and $B_M$, is a crucial ingredient in the matching between the SNC and TNC strings strings in [75], which in our formulation plays out at NLO. At NNLO, there is again a Stückelberg symmetry, but it is not possible to match type II TSNC and type II TNC without forcing certain fields to be zero by hand in a procedure analogous to that employed in [73] to match SNC and TNC in the absence of a Kalb–Ramond field.

In parallel to the string $1/c^2$ expansion of the geometry, the underlying symmetry algebra, the Poincaré algebra, also gets expanded [75] (see [24] for the particle equivalent). In the string theory, this plays out at the level of the Noether charges of the string theory, which have Poisson brackets that match order-by-order with the string $1/c^2$ expansion of the Poincaré algebra.

A natural next step to consider is the quantisation of the $1/c^2$ expanded string theories. It will turn out that the Poisson brackets change at each order, so one cannot just $1/c^2$ expand the relativistic brackets. Instead, we work out the phase space versions of the $1/c^2$ expanded string theories and perform the Dirac procedure to find the gauge-fixed Poisson brackets, which allow us to pass to the quantum theory. In the quantum theory, the same normal ordering constant as in the relativistic closed bosonic string theory appears at each order.

**Outline of thesis**

This thesis is structured as follows. In Chapter 2, we give an account of non-Lorentzian geometry. This chapter is based partly on [5,6]. We start by considering Minkowski spacetime and its limits in Section 2.1. By taking the speed of light, $c$, to infinity and zero, respectively, we will uncover Galilean and Carrollian spacetimes, and we will see how their geometric and algebraic structures are inherited from their Lorentzian antecedent. We then explore how Minkowski, Galilei and Carroll spacetimes arise as homogeneous spaces in Section 2.1.1, while providing a prescription that allows us to identify the geometric structure of the spacetimes from an algebraic perspective. We then discuss the more general concept of a *kinematical Lie algebra* in Section 2.2, which culminates in an (incomplete) classification of kinematical spacetimes. After a quick discussion of fiber bundles in Section 2.3, we showcase how the homogeneous spaces of the Poincaré group, in addition to featuring Minkowski
space, also capture the asymptotic structure of Minkowski space. More precisely, the blow-ups of timelike and spacelike infinity turn out to be homogeneous spaces of the Poincaré group equipped with Carrollian structures. In Section 2.5, we discuss Cartan geometries modelled on homogeneous spaces and their Klein pairs, which allows us to construct general geometries that locally look like the homogeneous space on which they are modelled. These Cartan geometries are examples of $G$-structures, which we discuss in Section 2.6, where we also point out that these in general have intrinsic torsion which plays an important rôle in non-Lorentzian geometry. We then go on to construct various geometries as Cartan geometries, starting with Lorentzian geometry in Section 2.7, and then continuing with Carrollian and Newton–Cartan geometries in Sections 2.8 and 2.9, respectively. In Section 2.10, we construct Aristotelian geometry as a Cartan geometry. Then, in Section 2.11, we develop the notion of type II Newton–Cartan geometry using the technology of Lie algebra expansions, and we show how this geometry may be obtained directly from a $1/c^2$ expansion of a Lorentzian structure in Section 2.11.2. These techniques can be generalised to string $1/c^2$ expansions, which leads to what is called string Newton–Cartan geometry, which we explore in Section 2.12.

In Chapter 3, we provide a review of relativistic hydrodynamics. We begin, gently, in Section 3.1 with a survey of the traditional approach to fluid dynamics that involves writing down constitutive relations and checking that the transport coefficients they feature obey the second law of thermodynamics. Then, in Section 3.2, we discuss the Lagrangian description of non-dissipative transport, which involves putting the fluid on a curved background.

Chapter 4, to which Chapter 3 serves as a warm-up, is based on [2] and is about fluids without boost symmetries. In Section 4.1, we recap and establish facts about perfect boost-agnostic fluids and their coupling to Aristotelian geometry as defined in Section 2.10. Next, in Section 4.2, we generalise the considerations of Section 3.1.4 by formulating the entropy current on curved spacetime and define three sectors of transport: hydrostatic non-dissipative, non-hydrostatic non-dissipative and dissipative transport. Subsequently, in Section 4.3, we present a Lagrangian description of both hydrostatic and non-hydrostatic transport using the methods of Section 3.2. We also connect the Lagrangian formulation to the non-canonical contributions to the entropy current. Finally, in Section 4.4 we obtain and combine all first order transport coefficients using constitutive relations.

In Chapter 5, we investigate the string $1/c^2$ expansion of closed bosonic string theory based on [3, 7]. In Section 5.1, we describe the string $1/c^2$ expansion that lies at the heart of the expansion of string theory that this chapter investigates, and we discuss the interpretation of the string $1/c^2$ expansion as an expansion around a decompactification limit. We expand the spectrum of a closed relativistic bosonic string on a background with a compact circle that is wound by the string. In Section 5.2, we discuss the gauge structure of the string $1/c^2$ expansion of Lorentzian geometry that we met in Section 2.12. Then, in Section 5.3, we expand both the Nambu–Goto and Polyakov actions up to NNLO. Furthermore, we employ the string $1/c^2$ expansion to gravity and demonstrate that the LO part of Einstein's equations imposes a two-dimensional foliation structure in the sense of Frobenius on the longitudinal target space. The also discuss the LO equation of motion for the embedding fields and show that when $\alpha^{AB}$ in (1.3.3) is traceless, this equation of motion is automatically satisfied. In
Section 5.4, we explicitly demonstrate the equivalence between the Gomis–Ooguri and the NLO theory when the background satisfies (1.3.3) with $\alpha^A_A = 0$. Following this, we discuss the rôle of the WZ term in Section 5.5 and show that we can lift the foliation constraint by fine-tuning the Kalb–Ramond field. We also discuss Stückelberg symmetries between the subleading longitudinal geometric fields and the Kalb–Ramond field. We then go on to consider the spectrum on flat space in Section 5.6, which involves fixing the residual gauge redundancies. At NLO, this reproduces the spectrum of the Gomis–Ooguri string, while the spectrum of the NNLO theory matches the result obtained by $1/c^2$ expanding the relativistic spectrum in Section 5.1.1. In Section 5.7, we consider the target space symmetries of the $1/c^2$ expanded string theories and show that the symmetry algebra corresponds to the string $1/c^2$ expansion of the Poincaré algebra that was considered in [75]. In Section 5.8, we develop the phase space formulation of the LO, NLO and NNLO string theories. Concretely, this is achieved by $1/c^2$ expanding the relativistic phase space action, and we go through the Dirac procedure and find the Dirac brackets at each order in Section 5.8.2. We then quantise the theories in Section 5.8.3 by deriving the commutators and writing down the normal ordering constant.

Finally, in Chapter 6, we provide a conclusion and an outlook.
Chapter 2

Non-Lorentzian geometry

In this chapter, we lay the foundations for the remainder of this thesis by developing the mathematical framework of non-Lorentzian geometry; or, perhaps, "non-(pseudo)-Riemannian geometry" in the sense that such geometries do not possess a metric.¹ We realise these various non-Lorentzian geometries as Cartan geometries, for which we provide an overview in Section 2.5. We then illustrate how Lorentzian geometry arises as a Cartan geometry in Section 2.7 and show how this acquires the usual geometric interpretation in local coordinates, where we may write down a metric $g_{\mu\nu}$ and construct a torsion-free metric compatible connection $\nabla$: the Levi–Civita connection.

This chapter contains, in addition to a general review of non-Lorentzian geometry, results that first appeared in [5, 6] and [7]; these appear primarily in Sections 2.4, 2.11 and 2.12.

2.1 Minkowski spacetime & its limits

We begin by an exposition of the properties of Minkowski space: the geometrical arena of special relativity. Minkowski space in $(d + 1)$ dimensions $\mathbb{M}^{d+1}$ is a pair $\left(\mathbb{R}^{d+1}, \eta\right)$, where $\eta : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}$ is the Minkowski inner product, which defines the proper distance $\Delta : \mathbb{R}^{d+1} \to \mathbb{R}$ between two points (or events) $a, b \in \mathbb{M}^{d+1}$ with $a = (t_1, x_1)$ and $b = (t_2, x_2)$

$$\Delta(b - a) = \eta(b - a, b - a) = -c^2(t_2 - t_1)^2 + ||x_2 - x_1||^2,$$  

(2.1.1)

where $c$ is the speed of light. The proper distance, or its negative, the proper time, combines both time and space. In the words of Hermann Minkowski [203]:

¹Interestingly, some non-Lorentzian geometries belong to the class of sub-Riemannian geometries (see, e.g., [200] for an overview of sub-Riemannian geometry). Exploring this connection and its consequences is under active investigation.

²To be precise, the additional structure of the vector space $\mathbb{R}^{d+1}$ is not necessary, and should be replaced by $(d + 1)$-dimensional affine space $d+1$, although we will ignore this subtlety. See, for example, [201, 202] for additional details.
The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

This mixing of space and time means that there is no invariant way to characterise events as simultaneous. Rather, the notion of simultaneity is superseded by that of causality: two events \( a, b \in \mathbb{M}^{d+1} \) are causally connected if they lie within their respective lightcones; to every point in Minkowski space is associated a lightcone defined as those points that are zero proper distance away from that point

\[
\mathbb{L}_a = \{ b \in \mathbb{R}^{d+1} \mid \Delta(b - a) = 0 \}.
\] (2.1.2)

![Figure 2.1: The lightcone \( \mathbb{L}_a \) of an event \( a \). Two events are causally related if one lies within the lightcone of the other.](image)

The isometry group of Minkowski space consists of those transformations that leave invariant the proper distance (2.1.1). This group is called the Poincaré group, and has the form of a semidirect product \( \text{ISO}(d, 1) \cong O(d, 1) \rtimes \mathbb{R}^{d+1} \), where \( O(d, 1) \subset \text{GL}(d+1, \mathbb{R}) \) is the \((d+1)\)-dimensional Lorentz group consisting of those \( L \in \text{GL}(d+1, \mathbb{R}) \) satisfying \( L^T \eta L = \eta \).

It is useful to embed the Poincaré group into \( \text{GL}(d+2, \mathbb{R}) \) as the matrices

\[
\text{ISO}(d, 1) \cong \left\{ \begin{pmatrix} L & v \\ 0 & 1 \end{pmatrix} \middle| L \in O(d, 1) \text{ & } v \in \mathbb{R}^{d+1} \right\},
\] (2.1.3)
which acts on events \((a, 1) \in \mathbb{R}^{d+2}\) as \(\begin{pmatrix} L & v' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} La + v' \\ 1 \end{pmatrix}\), which is a combination of Lorentz transformation and a spacetime translation. The associated Lie algebra \(\text{iso}(d, 1)\) similarly embeds into \(\mathfrak{gl}(d + 2, \mathbb{R})\) as

\[
is\text{iso}(d, 1) \cong \left\{ \begin{pmatrix} A & v' \\ 0 & 0 \end{pmatrix} \mid L \in AT \eta + \eta A = 0 \ \& \ v \in \mathbb{R}^{d+1} \right\},
\]

(2.1.4)

obtained by setting \(L = 1 + A\), where \(A\) is “small”. To determine the brackets of \(\text{iso}(d, 1)\), it is useful to introduce a basis \((L_{mn}, P_m)\) in the following way

\[
\begin{pmatrix} A \\ v' \\ 0 \\ 0 \end{pmatrix} = v^m P_m + \frac{1}{2} A^{mn} L_{mn},
\]

(2.1.5)

where \(L_{mn} = -L_{nm}\), and \(m, n, \cdots = 0, \ldots, d\). The \(L_{mn}\) generate the Lorentz algebra \(\mathfrak{so}(d, 1)\), consisting of boosts and rotations, while \(P_m\) generates spacetime translations. Explicitly, the brackets of the Poincaré algebra in this basis take the form

\[
\begin{align*}
[L_{ab}, L_{cd}] &= \eta_{bc} L_{ad} - \delta_{bd} L_{ac} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc}, \\
[L_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b, \\
[L_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b, \\
[B_a, P_b] &= \delta_{ab} H, \\
[B_a, B_b] &= c^2 L_{ab}, \\
[B_a, H] &= c^2 P_a,
\end{align*}
\]

(2.1.6)

where we used that \(\eta_{mn} = \text{diag}(-c^2, 1, \ldots, 1)\). The rotations \(L_{ab}\) generate the rotation subalgebra \(\mathfrak{so}(d)\), while \(P_a\) and \(B_a\) — the spatial momenta and the boosts — transform as \(\mathfrak{so}(d)\) vectors. The generator of time translations \(H\) transform in the scalar representation.

---

\(^2\)This Lie algebra is also generated by those vectors \(\xi \in \mathfrak{X}(\mathbb{R}^{d+1})\) along which the Lie derivative of \(\eta\) vanishes, \(\xi \lhd \eta = 0\).
Setting $c = 0$ gives rise to the algebra

\[
\begin{aligned}
\{L_{ab}, L_{cd}\} &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc}, \\
\{L_{ab}, B_c\} &= \delta_{bc} B_a - \delta_{ac} B_b, \\
\{L_{ab}, P_c\} &= \delta_{bc} P_a - \delta_{ac} P_b, \\
\{B_a, P_b\} &= \delta_{ab} H,
\end{aligned}
\]

(2.1.8)

where we omitted brackets that give zero. This algebra was first studied by Lévy-Leblond [204] (and, slightly later, by Sen Gupta [205]), who named it after Lewis Carroll, the pen name of Charles Dodgson who wrote *Alice’s Adventures in Wonderland* and the sequel *Through the Looking-Glass*. Sending the speed of light to zero means that motion is impossible, poetically captured by Carroll in a passage involving the Red Queen’s Race [206]:

““Well, in our country,” said Alice, still panting a little, “you’d generally get to somewhere else—if you run very fast for a long time, as we’ve been doing.”

“A slow sort of country!” said the Queen. “Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!””

The Carrollian limit is sometimes called the “ultrarelativistic” limit, but since this would correspond to characteristic velocities close to $c$, $v_{ch}/c \approx 1$, this is slightly misleading. A better name, which is also used in the literature, is the “ultralocal” limit, since no material body can move in the Carrollian regime.

The opposite Galilean limit where $c \to \infty$ requires us to first redefine the boosts as $\tilde{B}_a = c^2 B_a$, so that (2.1.7) becomes

\[
\begin{aligned}
\{L_{ab}, L_{cd}\} &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc}, \\
\{L_{ab}, \tilde{B}_c\} &= \delta_{bc} \tilde{B}_a - \delta_{ac} \tilde{B}_b, \\
\{L_{ab}, P_c\} &= \delta_{bc} P_a - \delta_{ac} P_b, \\
\{\tilde{B}_a, P_b\} &= c^{-2} \delta_{ab} H, \\
\{\tilde{B}_a, \tilde{B}_b\} &= c^{-2} L_{ab}, \\
\{\tilde{B}_a, H\} &= P_a,
\end{aligned}
\]

(2.1.9)
so with $c^{-2} = 0$, this becomes the Galilean algebra (where we redefined $\tilde{B}_a = B_a$)

\[
\begin{align*}
[L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc}, \\
[L_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b, \\
[L_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b, \\
[B_a, H] &= P_a .
\end{align*}
\tag{2.1.10}
\]

This process is also known as a Inönü–Wigner contraction [207].

The Carrollian and Galilean limits drastically affect the causal structure as captured by the lightcone (cf., Figure 2.1). In the first limit, $c \to 0$, the lightcone closes and forms a line, so there is no causal relationship between any two points not on the same line. This is a reflection of the fact that material bodies move only in time. In the Galilean limit, the lightcone opens up to become a hyperplane of simultaneity: any two events in this plane are simultaneous, and any two events are causally connected.

\[
\begin{align*}
&\, c = 0 & &\, c \ll 1 & &\, c = 1 & &\, c \gg 1 & &\, c = \infty \\
&\text{c = 0} & &\text{c \ll 1} & &\text{c = 1} & &\text{c \gg 1} & &\text{c = \infty}
\end{align*}
\]

Figure 2.2: Lightcones opening and closing as $c \to \infty$ and $c \to 0$, respectively. In the intermediate cases, where $c \ll 1$ and $c \gg 1$, the algebras (and the spacetimes) are described by expansions in either $c$ or $1/c$. This gives rise to “type II” structures, which we discuss in Section 2.1.1.

### 2.1.1 Minkowski, Carroll and Galilei as homogeneous spaces

A useful description of Minkowski space $\mathbb{M}^{d+1}$ is as a homogeneous space of the Poincaré group $\text{ISO}(d, 1)$. Since the group action $\text{ISO}(d, 1) \times \mathbb{M}^{d+1} \to \mathbb{M}^{d+1}$, under which $x \mapsto g \cdot x$ for $g \in \text{ISO}(d, 1)$, is transitive (i.e., any point $x \in \mathbb{M}^{d+1}$ can be reached from any other point $y \in \mathbb{M}^{d+1}$ by the action $g \cdot y = x$ for some $g \in \text{ISO}(d, 1)$), while the origin is left invariant by all $\text{ISO}(d, 1)$ transformations except for the translations (cf., (2.1.3)), i.e., the Lorentz group $\text{SO}(d, 1)$. In the context of homogeneous spaces this group that leaves a distinguished point invariant is called the stabiliser subgroup. These considerations imply that Minkowski space is a coset space of the Poincaré group

\[
\mathbb{M}^{d+1} \cong \text{ISO}(d, 1)/\text{SO}(d, 1).
\tag{2.1.11}
\]
By introducing exponential coordinates, one may explicitly verify that that this has the structure of Minkowski space [201]. A convenient way to capture this information is in terms of a Klein pair \((g, h)\), which consists of a Lie algebra \(g\) and a subalgebra \(h\ subset g\) (see [208] for more details). For Minkowski space, the Klein pair consists of the isometry Lie algebra \(g \cong \text{iso}(d, 1)\) and the Lie algebra \(h \cong \text{so}(d, 1)\) of the stabiliser subgroup. In this lingo, Minkowski space is the geometric realisation of the Klein pair \((\text{iso}(d, 1), \text{so}(d, 1))\).

In the same way, given the Carroll and Galilei algebras, which are the Lie algebras of the Carroll and Galilei groups, we can write down geometrically realisable Klein pairs that give rise to Carroll and Galilei spacetimes. For the Carroll algebra \(\text{carr}(d, 1)\), whose brackets are given in (2.1.8), the Klein pair that gives rise to Carrollian spacetime is \((\text{carr}(d, 1), \text{iso}(d))\), where \(\text{iso}(d) \cong \langle L_{ab}, B_a \rangle\). Galilei spacetime corresponds to the Klein pair \((\text{gal}(d, 1), \text{iso}(d))\), where \(\text{gal}(d, 1)\) is the Galilean algebra (2.1.10) and, as was the case for the Carrollian spacetime, \(\text{iso}(d) \cong \langle L_{ab}, B_a \rangle\). Note that all these Klein pairs are reductive, which means that there exists a split
\[ g = m \oplus h \quad \text{such that} \quad [h, m] \subset m. \] (2.1.12)
There exist interesting spaces for which the Klein is not reductive: for example, the lightcone \(LC\) is not reductive.

### 2.1.2 Invariant spacetime tensors

While it might seem odd at first that both the Galilean and Carrollian spacetimes share the same stabiliser, the point is that the \(\text{iso}(d)\)-invariant structures on the respective spacetimes will not be the same, which means that their geometric properties are not the same. For a geometrically realisable and reductive Klein pair \((g, h)\), there is a one-to-one correspondence between \(h\)-invariant tensors on \(m\) and \(G\)-invariant tensor fields on the associated homogeneous space \(M \cong G/H\) [208].

Assuming that the Klein pair is reductive\(^4\) (2.1.12), we are interested in \(h\)-invariant tensors in \(m, m^*, \otimes^2 m\) and \(\otimes^2 m^*\). The action of \(h\) on \(m\) is given be the usual adjoint action, i.e., the bracket, while \(h\) acts on \(m^*\) by the coadjoint action, which is defined as follows: if \(\alpha \in m^*\) and \(X \in h\), then \(X \cdot \alpha = -\alpha \circ \text{ad}_X\). Let’s illustrate how this works for the Carroll algebra (2.1.8). Introduce a dual basis \((\pi^a, \eta)\) for \(m^*\) such that
\[ \pi^a(P_b) = \delta^a_b, \quad \eta(H) = 1, \] (2.1.13)
we find that \(H \in m\) is invariant, corresponding to a vector field on \(M\). The coadjoint action

\(^4\)Similar methods works even in the non-reductive case, in which case we need the so-called linear isotropy representation. See [208] for details.
From this, we see that there is an invariant symmetric spatial tensor \( \pi^2 = \delta_{ab} \pi_a \pi_b \in \otimes^2 m^* \), and that there are no other invariant of low rank. The \( \mathfrak{h} \)-invariants \( H \in m \) and \( \pi^2 \in \otimes^2 m^* \) give rise to an invariant vector field and a spatial symmetric contravariant tensor on the Carrolian spacetime, which together form a Carollian structure. This Carollian structure is what replaces the Minkowski metric for Carollian spacetimes.

Indeed, if we were to repeat the calculation above for the Klein pair of Minkowski space \( (\text{iso}(d, 1), \text{so}(d, 1)) \), we would find two invariant tensors in \( \otimes^2 m^* \) of Lorentzian signature, corresponding to the Minkowski metric and its inverse.

For Galilean spacetime, identical calculations show that the \( \mathfrak{h} \)-invariant structures of low rank consist of \( \eta \in m^* \) and \( P^2 \in \otimes^2 m \), corresponding to an invariant 1-form and a spatial symmetric covariant tensor on the Galilei spacetime: these together are known as a Galilean structure.

### 2.2 Kinematical Lie algebras & homogeneous spacetimes

Having explored how Carrollian and Galilean geometries arise as limits of Minkowski geometry above, we now describe how to systematise the construction of non-Lorentzian Klein geometries. Following the pioneering work of Bacry and Lévy-Leblond [111], a more exhaustive classification of possible kinematics was put forward in [208]. In this context, the "possible kinematics" refers to the isometries of an underlying "kinematical spacetime", which in the case of Riemannian or Lorentzian geometry would be some maximally symmetric space. For example, the isometries of \((d + 1)\)-dimensional Minkowski spacetime \( \mathbb{M}^{d+1} \) generate the Poincaré Lie algebra \( \text{iso}(d, 1) \), which is a prime example of a kinematical Lie algebra. The Poincaré algebra consists of spatial rotations that generate \( \text{so}(d) \) as well as boosts and space-time translations. Based on this, one might be led to the conclusion that the Poincaré algebra is uniquely associated to Minkowski spacetime, but as was emphasised already in [208], one must specify the action of the kinematical Lie algebra on the spacetime itself in order to make precise which generators are boosts and which are translations. As we elaborate below, this requires us to refine the discussion to homogeneous spaces rather than just Lie algebras. The spacetimes are such that the stabiliser of any point contains the rotation subalgebra \( \text{so}(d) \) of that point, which implies that all invariant tensors will be rotationally invariant; this property is also known as "spatial isotropy".
More precisely, a kinematical Lie algebra is defined as follows [208].

**Definition 1.** A \((d + 1)\)-dimensional kinematical Lie algebra with with \(d\)-dimensional spatial isotropy is real Lie algebra \(k\) with the properties that it (1) contains a Lie subalgebra \(r \cong so(d)\); and (2): as a representation of \(r\), the kinematical Lie algebra \(k\) admits a decomposition of the form \(k = r \oplus 2V \oplus S\), where \(2V\) are two copies of the vector irreducible representation of \(so(d)\) and \(S\) is trivial scalar representation of \(so(d)\). A Lie group whose Lie algebra is a kinematical Lie algebra is called a kinematical Lie group.

Put differently, this definition says that a kinematical Lie algebra contains something that can be identified with rotations, and that under the action of these rotations \(k\) decomposes into something that can be identified with boosts and momenta (the “2\(V\)”) and time translations (the “\(S\)”). Note also that these algebras are kinematical in the “particle sense”: it is possible to also classify kinematical Lie algebras a string sense, in which case, or, indeed, the \(p\)-brane sense. Work in this direction is ongoing.

Definition 1 implies the existence of a basis for any kinematical Lie algebra of the form \((L_{ab}, B_a, P_a, H)\), where \(a, b, \cdots = 1, \ldots, d\) are spatial \(so(d)\) indices. The generators \(L_{ab} = -L_{ba}\) generate the rotation subalgebra \(r\), while \(H\) generates \(S\) and \(B_a, P_a\) generate \(2V\). Relative to this basis, the Lie brackets that follow from Definition 1 are

\[
\begin{align*}
[L_{ab}, L_{cd}] &= \delta_{bc}L_{ad} - \delta_{ac}L_{bd} - \delta_{ba}L_{ac} + \delta_{ad}L_{bc}, \\
[L_{ab}, B_c] &= \delta_{bc}B_a - \delta_{ac}B_b, \\
[L_{ab}, P_c] &= \delta_{bc}P_a - \delta_{ac}P_b, \\
[L_{ab}, H] &= 0.
\end{align*}
\]

(2.2.1)

We may impose additional brackets as long as they are consistent with the Jacobi identity, but all kinematical Lie algebras are subject to the brackets (2.2.1). In this sense, one can think of the brackets (2.2.1) as forming the “skeleton” of kinematical Lie algebras. Now, while the notation employed above suggests \(B_a\) and \(P_a\) admit interpretations as boosts and momenta, one cannot assign any geometrical interpretation to these generators without specifying the action of the kinematical Lie group on a spacetime. This is something we will explicitly demonstrate in the context of the Poincaré algebra below, but the upshot is that what acts as boosts in one spacetime may act as momenta in another.

We now describe the process that allows us to pass from a kinematical Lie group to a homogeneous spacetime, which uses the language of Klein geometry that we already met in Section 2.1.1 for the Minkowski, Carroll and Galilei spacetimes. In general, a Klein geometry of a Lie group \(G\) is a homogeneous space of \(G\); that is, a smooth manifold \(M\) on which \(G\) acts smoothly and transitively. The intuition is that every point of \(M\) “looks the same” through the optics of \(G\); cf., the discussion of Minkowski space as a homogenous space in Section 2.1. If we pick an arbitrary point \(o \in M\) and designate it as the origin, then the subgroup \(H\) of \(G\) consisting of elements which fix the origin is called the stabiliser. \(H\) is a closed subgroup of \(G\) and \(M\) is (locally) isomorphic to the space \(G/H\) consisting of left cosets \(gH\), for \(g \in G\), where the Lie group \(G\) acts on \(G/H\) via left multiplication. Let \(g\) and \(h\) denote the Lie algebras...
of $G$ and $H$, respectively. Then, as we saw above, to a homogeneous space of $\mathcal{G}$ we may assign a Klein pair $(g, h)$. Conversely a Klein pair $(g, h)$ is said to be **geometrically realisable** if there exists a Lie group $\mathcal{G}$ with Lie algebra $g$ such that the connected subgroup $\mathcal{H}$ of $\mathcal{G}$ corresponding to $h$ is closed. Not every Lie pair is geometrically realisable, but it is possible to show that there is a one-to-one correspondence between simply-connected homogeneous spaces and geometrically realisable (effective) Klein pairs. We summarise the above in the following definition of homogeneous kinematical spacetimes [208].

**Definition 2.** A **homogeneous kinematical spacetime** is a Klein geometry $M$ of a kinematical Lie group $\mathcal{K}$ with the following properties:

1. $M$ is a connected smooth connected manifold
2. The action of $\mathcal{K}$ on $M$ is transitive and locally effective, and has stabiliser $\mathcal{H}$
3. The stabiliser $\mathcal{H}$ is a closed subgroup of $\mathcal{K}$, and its Lie algebra $h$ contains $\mathfrak{r} \cong \mathfrak{so}(d)$ and admits the $\mathfrak{r}$-decomposition $h = \mathfrak{r} \oplus V$.

The upshot of this definition is that the spacetime $M$ corresponds to the homogeneous space $\mathcal{K}/\mathcal{H}$ and as such has the same dimension as the number of generators in $g/h$. Property (2) guarantees that $\mathcal{K}/\mathcal{H}$ is well defined as a homogeneous space (see [208] for more details). The third property is a statement about the form of the stabiliser: the stabiliser of a point in $M$ must contain the rotational subalgebra and one copy of the vector representation of $\mathfrak{so}(d)$, and by being part of the stabiliser that $V$ acquires the interpretation of boosts. Note that Definition 2 does not lead to Aristotelian spacetimes, which we will discuss in Section 2.10.

The classification of kinematical Lie algebras and their associated homogeneous spacetimes is partially summarised in Table 2.1 below. Following our discussion of the Minkowski, Carroll and Galilean structures in Section 2.1.2, we have classified each spacetime according to whether they come equipped with a Lorentzian, Riemannian, Galilean or Carrollian. We remark that other structures exist, some of which we explore in the next section.

### 2.3 Bundles and all that

In this section, we collect a few facts about fibre bundles that we will use in the following, based on [5, 209–212]. In particular, it will turn out that certain homogeneous spacetimes of the Poincaré group naturally acquire the structure of line bundles, which are examples of vector bundles. A (real) vector bundle of rank $k$ with “base” manifold $M$ is a manifold $E$, called the total space, along with a surjective map $\pi : E \to M$, called the projection, which satisfies the following properties

1. For all points $p \in M$, the fibre $\pi^{-1}(p)$ is a $k$-dimensional vector space;
2. Each point $p \in M$ has an open neighbourhood $U$ and comes equipped with a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$, which is called a local trivialisation, and it maps the vector space $\pi^{-1}(p)$ isomorphically to the vector space $(p) \times \mathbb{R}^k$;
Table 2.1: An incomplete list of simply-connected homogeneous \((d + 1)\)-dimensional kinematical spacetimes as defined in Def. 2. Absent are some one-dimensional spaces without any particular structure, as well as a number of torsional Galilean spaces. See \[208\] for the full list and additional details.

(iii) If \((U, \varphi_U)\) and \((V, \varphi_V)\) are two local trivialisations with non-empty intersection, \(U \cap V \neq \emptyset\), then the composition

\[ \varphi_U \circ \varphi_V^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k, \tag{2.3.1} \]

maps \((p, v) \mapsto (p, t_{UV}(p)v)\), where the matrix-valued transition function \(t_{UV} : U \cap V \to GL(k, \mathbb{R})\) is smooth.

If the neighbourhood in (ii) can be taken to be all of \(M\), the local trivialisation becomes global, and the bundle is to be trivial. A trivial vector bundle of rank \(k\) thus has the form \(M \times \mathbb{R}^k\). A line bundle is a vector bundle of rank 1, and all the line bundles that we will encounter in Section 2.4 are trivial. A section of a vector bundle is a smooth map \(s : M \to E\) such that \(\pi \circ s = id_M\). A section smoothly assigns to each point in the base \(M\) a vector in the fibre above it. Vector bundles always admit sections.

The prime example of a vector bundle is the tangent bundle \(TM\) of an \(n\)-dimensional manifold \(M\): this is the disjoint union of tangent spaces over each point \(p \in M\)

\[ TM = \bigcup_{p \in M} T_p M, \tag{2.3.2} \]

and comes with projection \(\pi : TM \to M\) sending \(v \in T_p M\) to \(p \in M\). The fibre of each point is \(\pi^{-1}(p) \cong \mathbb{R}^n\), making the tangent bundle into a real vector bundle of rank \(n\). Sections of \(TM\) are vector fields.

Another type of bundle that we will encounter is the principal bundle. Like a vector bundle, a principal bundle consists of a manifold \(E\), the total space, and a projection \(\pi : P \to M\) to the base \(M\), with fibres \(\pi^{-1}(m)\), for \(m \in M\), that have the structure of a Lie group \(G\).
In addition, a principal bundle comes equipped with a right $G$-action $P \times G \to P$ sending $(p, g) \mapsto p \cdot g$. We may then identify the base $M$ with the orbit space $M = P/G$. A principal bundle is subject to suitable generalisations of conditions (ii) and (iii) for vector bundles: now each point $m \in M$ has an open neighbourhood $U$ with a $G$-equivariant diffeomorphism $\psi_U : \pi^{-1}(U) \to U \times G$, which is the local trivialisation, such that $\psi_U(p) = (\pi(p), g_U(p))$ for a $g_U : \pi^{-1}(U) \to G$ satisfying $G$-equivariance

$$g_U(p \cdot g) = g_U(p)g.$$ \hfill (2.3.3)

As above, if the local trivialisation $\psi_U$ extends to all of $P$, the bundle is trivial, $P \cong M \times G$, and one can show that principal bundles only admit sections if they are trivial (however, since they are still locally trivial, they do admit local sections).

An important example of a principal bundle is the frame bundle, which is a principal $GL(n, \mathbb{R})$-bundle over a $d$-dimensional manifold $M$. It is constructed as follows: a frame at a point $p \in M$ is a vector space isomorphism $u : \mathbb{R}^n \to T_p M$, and the image under $u$ of the canonical basis $e_i$ of $\mathbb{R}^n$ is a basis $u(e_i)$ for $T_p M$. Two bases $u', u$ at $p$ are related by a $GL(n, \mathbb{R})$ transformation: more precisely, since $g := u^{-1} \circ u \in GL(n, \mathbb{R})$, we have that $u' = u \circ g$, which defines a right $GL(n, \mathbb{R})$-action on the set $F_p(M)$ of frames at $p$. This makes the disjoint union

$$F(M) = \bigsqcup_{p \in M} F_p(M),$$ \hfill (2.3.4)

into the total space of a principal $GL(n, \mathbb{R})$-bundle: the frame bundle. Local sections of the frame bundle $s : U \to F(M)$ define frames. We will have more to say about the frame bundle in Section 2.6.

To a principal $G$-bundle $\pi : P \to M$ we may associate a vector bundle, called an associated vector bundle, in the following way: if $G$ acts on a vector space $V$ via automorphisms, the corresponding representation is $\rho : G \to GL(V)$. We can then define a vector bundle over $M$ by

$$P \times_G V := (P \times V) / G,$$ \hfill (2.3.5)

Figure 2.3: A vector bundle $\pi : E \to M$ with a section $s : M \to E$. 

[Diagram of a vector bundle with a section]
where we quotient by the right \( G \)-action on \( P \times V \), \((p, f)g = (p \cdot g, \rho(g^{-1})f)\), and the induced projection \( \pi_V : P \times_G V \to M \) given by \( \pi_F(p, v) := \pi(p) \), which, even though \((p, v)\) is only defined up to a \( G \)-transformation, is well defined since \( \pi(p \cdot g) = \pi(p) \), which follows from the identification \( M = P/G \). In general, one can consider associated fibre bundles, where \( V \) is a more general space than a vector space.

As an example of an associated vector bundle, consider the adjoint representation \( \text{ad} : G \to \text{GL}(g) \) mapping \( G \ni g \mapsto d(\text{Ad}(g))|_e \), where \( \text{Ad}_g(h) = ghg^{-1} \) for \( h \in G \). The adjoint vector bundle is then \( \text{ad} P := P \times_G g \).

### 2.4 Homogeneous spacetimes of the Poincaré group & asymptotic infinity

![Penrose Diagram](image)

**Figure 2.4:** The Penrose diagram of Minkowski spacetime \( \mathbb{M} \) with its hyperbolic slicing. We have also illustrated how \( T_i \) and \( S_i \) arise as the blow-ups of, respectively, timelike and spacelike infinities, while \( N_i \) fibers over \( \mathcal{I} \) and can be understood as the bundle of scales of the conformal Carrollian structure of \( \mathcal{I} \).

To emphasise the point made above that the same kinematical Lie algebra can give rise to many different homogeneous spaces, this section provides a discussion of the homogeneous spaces of the \((d + 1)\)-dimensional Poincaré group following [5]. These include several non-
Lorentzian spaces, which, interestingly, turn out to capture the asymptotic structure of asymptotic infinity [5]: the \((d + 1)\)-dimensional Carrollian spaces \(S_{\pi, Ti}\), as well as a novel \((d + 1)\)-dimensional “doubly Carrollian space” \(N_i\), which is the bundle of scales of \(\mathcal{F}\). The identification of these spaces with the blow-ups of space- and timelike infinity in the sense of Ashtekar–Hansen is discussed further in Section 2.4.2.

In general, the homogeneous spaces of the \((d + 1)\)-dimensional Poincaré group are determined locally by Klein pairs of the form \((\text{iso}(d, 1), \mathfrak{h})\) with \(\mathfrak{h}\) a subalgebra of \(\text{iso}(d, 1)\). The most obvious example is, of course, Minkowski spacetime \(M\) with Klein pair \((\text{iso}(d, 1), \text{so}(d, 1))\), with \(\text{so}(d, 1)\) the Lorentz subalgebra. A slightly less obvious example is obtained by instead considering the Klein pair \((\text{iso}(d, 1), \text{iso}(d - 1, 1))\), i.e., by replacing the Lorentz algebra by the \(d\)-dimensional Poincaré algebra \(\text{iso}(d - 1, 1)\) which is clearly also a subalgebra of \(\text{iso}(d, 1)\). In contrast to Minkowski space, the Poincaré group acts on the resulting space in a way that does not allow for the construction of a nondegenerate invariant metric. Instead, one finds a pseudo-Carrollian structure consisting of a degenerate Lorentzian metric and a distinguished vector field. As we will explain in more detail below, the resulting \((d + 1)\)-dimensional spacetime fibers over \(d\)-dimensional de Sitter space \(dS_d\) and the degenerate metric is the pull-back by the projection of the constant positive curvature metric on \(dS_d\).

Although the physical significance of this construction appears rather opaque at first sight, it was observed by Gibbons in [213] that this is precisely the universal structure at spatial infinity \(S_{\pi}\) of Ashtekar and Hansen’s (AH) [214]. In a generic asymptotically flat spacetime various physical fields acquire direction-dependent limits at the point \(i^0\). One therefore considers a blow-up of \(i^0\), such that fields at \(i^0\) can be regarded as smooth fields on the blow-up. The blow-up is constructed as the space of space-like geodesics approaching \(i^0\) with unit tangent vector. The set of all such curves turns out to be parametrised by the homogeneous space discussed above where the \(dS_d\)-slices parametrise the choices of tangent vectors and the coordinate along the fibre correspond to the tangential acceleration which is not fixed by the construction of [214]. We will therefore refer to the homogeneous space of the Poincaré group with Klein pair \((\text{iso}(d, 1), \text{iso}(d - 1, 1))\) as \(S_{\pi}\).

The above construction immediately suggests the existence of another homogeneous space of the Poincaré group corresponding to the universal structure at (either future or past) timelike infinity that we will refer to as \(T_i\). In this case the subgroup is isomorphic to the euclidean group in one lower dimension. The homogeneous space is now equipped with a Carrollian structure and fibers over \(d\)-dimensional hyperbolic space \(H^d\) instead of de Sitter space. In fact, the existence of this space was already revealed in the classification of spatially isotropic homogeneous spacetimes of [208] (see also [215]) where it was called the anti-de Sitter–Carroll spacetime (henceforth AdSC) and identified with the Carrollian limit of AdS.

Looking at the Penrose diagram (cf., Figure 2.4) of an asymptotically flat spacetime, the appearance of the universal structure at timelike and spacelike infinities as \((d + 1)\)-dimensional homogeneous spaces of the Poincaré group further suggests the existence of

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6That \(N_i\) is the “bundle of scales” of \(\mathcal{F}\) means that \(N_i \cong R^+ \times \mathcal{F}\), i.e., it is a trivial principal \(R^+\)-bundle over \(\mathcal{F}\). One can view the extra direction as a “geometrification” of the conformal structure on \(\mathcal{F}\). In the same way, the lightcone \(LC_d\) is the bundle of scales of the celestial sphere, \(LC_d \cong R^+ \times CS^{d-1}\); see [5] for more details.
another homogeneous space related to the universal structure at null infinity. While the latter is indeed described by a homogeneous space of the Poincaré group, namely $\mathcal{S}$, it is only of dimension $d$. The above picture is nevertheless completed by an additional $(d + 1)$-dimensional space $\mathcal{N}_i$ fibering over $\mathcal{S}$. It turns out that $\mathcal{N}_i$ also fibers over the lightcone, and that both the lightcone and $\mathcal{S}$ fiber over the celestial sphere, resulting in the commuting square of fibrations displayed below together with all the other homogeneous spaces under consideration:

$$
\begin{array}{cccc}
\mathbb{M} & \text{Spi} & \text{Ti} & \text{Ni} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
dS & \mathcal{H} & \mathcal{S} & \text{LC} \\
\end{array}
$$

where LC is either the future or past lightcone (without the apex, cf., Figure 2.5) and $\mathcal{C}_S$ is the celestial sphere. To the right of the square we have denoted their dimensions, which shows that Spi and Ti do not have the conventional interpretation as a boundary of one lower dimension. As we will see, all manifolds in (2.4.1) admit transitive actions of the Poincaré group; although the action is not effective for LC and $\mathcal{C}_S$, where the translations act trivially. While the Poincaré-invariant structures of Ti and Spi are (pseudo-)Carrollian, that of $\mathcal{N}_i$ is a novel carrollian-like structure which involves two invariant vectors and a corank-two degenerate metric. We tentatively dub this structure a “doubly-Carrollian” structure\(^7\), by analogy with the fibration $\text{LC} \rightarrow \mathcal{C}_S$. Concretely, one observes that the Carrollian structure on LC arises naturally from interpreting LC as the total space of the bundle of scales of the conformal structure of $\mathcal{C}_S$. In the same way, the doubly-Carrollian structure of $\mathcal{N}_i$ arises naturally from interpreting $\mathcal{N}_i$ as the total space of the bundle of scales of the conformal Carrollian structure of $\mathcal{S}$. Consistent with this interpretation is the fact that the symmetries of the doubly-carrollian structure of $\mathcal{N}_i$ agree with the BMS symmetries [41,42], which are the symmetries of the conformal Carrollian structure on $\mathcal{S}$.

\section*{2.4.1 Embeddings à la Penrose–Rindler}

Although the spaces under consideration were motivated as homogeneous spaces of the Poincaré group, their simplest description turns out to involve their embedding as codimension-2 submanifolds in a pseudo-euclidean space. The use of an auxiliary six-dimensional pseudo-euclidean space to study four-dimensional physics has a long and illustrious pedigree. It was perhaps first used by Dirac [217] in order to discuss conformally invariant wave equations and later by Kasner [218] and Fronsdal [219] in order to embed the Schwarzschild black hole.

\footnote{The doubly-Carrollian structure bears a superficial resemblance to so-called stringy carrollian structures encountered in a “string Carroll geometry” [216], which is the (much less studied) Carrollian counterpart of string Newton–Cartan geometry that we describe in Section 2.12. We stress, however, that they are not the same.}
We start by describing the pseudo-euclidean space $E$ where a signature-$Q$ positive curvature, so that $\rho Q$ curvature, making $E$ metries of $E$ call $Q$ quadric hypersurface cut out by the equation $\eta x$ and $\frac{1}{\sqrt{\epsilon}}(x^{d+1} \pm x^{d+2})$ are null ($x^{d+1}$ and $x^{d+2}$ are spacelike and timelike, respectively). Relative to these coordinates, the metric on $x$ or $x$ is timelike and $A = (x^{d+1} + x^{d+2})$ are null ($x^{d+1}$ and $x^{d+2}$ are spacelike and timelike, respectively). Relative to these coordinates, the metric on $E$ is expressed as

$$g_E = \eta_{AB} dx^A dx^B = -(dx^0)^2 + \sum_{a=1}^{d} (dx^a)^2 + 2dx^+ dx^-.$$  \hfill (2.4.2)

It clearly has signature $(d+1, 2)$. We will let $\mathbb{R}^{d,1}$ denote the lorentzian vector space $(\mathbb{R}^{d+1, \bar{\eta}})$, where $\bar{\eta} = \text{diag}(-1, 1, \ldots, 1)$. A typical point in $E^{d+1,2}$ is denoted by $(x, x^+, x^-)$ with $x^\pm \in \mathbb{R}$ and $x \in \mathbb{R}^{d,1}$.

We now introduce some algebraic subspaces of $E^{d+1,2}$. Let $\epsilon \in \mathbb{R}$ and let $\mathcal{D}_\epsilon$ denote the quadric hypersurface cut out by the equation $\eta_{AB} x^A x^B = \epsilon$. In particular, if $\epsilon = 0$, we shall call $\mathcal{D}_0$ the null quadric. These quadrics are preserved by a subgroup $O(d+1,2)$ of the isometries of $E^{d+1,2}$, which acts transitively on every $\mathcal{D}_\epsilon \neq 0$. The null quadric contains a singular point (namely, the origin in $E^{d+1,2}$) and $O(d+1,2)$ acts transitively on the complement.

If $\epsilon = -\rho^2 < 0$, then the induced metric on $\mathcal{D}_\epsilon$ is lorentzian of constant negative curvature, making $\mathcal{D}_{\epsilon < 0}$ into the hyperboloid model of AdS$_d$ with radius of curvature $\rho$. If $\epsilon = \rho^2 > 0$, then the induced metric on $\mathcal{D}_\epsilon$ has signature $(d, 2)$ and has constant positive curvature, so that $\mathcal{D}_{\epsilon > 0}$ is a pseudo-sphere of radius of curvature $\rho$, or, equivalently a signature-$(d, 2)$ version of de Sitter space.

Let $\sigma \in \mathbb{R}$ and let $\mathcal{M}_\sigma$ denote the null hypersurface with equation $x^- = \sigma$. For $\sigma \neq 0$, the

![Figure 2.5: Future lightcone without apex](image-url)
subgroup of $O(d+1,2)$ which preserves $\mathcal{M}_0$ is isomorphic to the Poincaré group $O(d,1)\ltimes \mathbb{R}^{d,1}$.

It is given explicitly by the following matrices

$$
\begin{pmatrix}
A & 0 & v \\
-v^T \eta A & 1 & -\frac{1}{2} \eta(v,v) \\
0^T & 0 & 1
\end{pmatrix}
\begin{aligned}
A^T \eta A &= \eta \\
v \in \mathbb{R}^{d,1}
\end{aligned}
\subset O(d+1,2). 

\tag{2.4.3}
$$

The subgroup of $O(d+1,2)$ which preserves $\mathcal{M}_0$ is larger and it includes also “dilatations”. Every matrix in the Poincaré group (2.4.3) decomposes into a product

$$
\begin{pmatrix}
A & 0 & v \\
-v^T \eta A & 1 & -\frac{1}{2} \eta(v,v) \\
0^T & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & v \\
-v^T \eta & 1 & -\frac{1}{2} \eta(v,v) \\
0^T & 0 & 1
\end{pmatrix}
\begin{pmatrix}
A & 0 & 0 \\
0^T & 1 & 0 \\
0^T & 0 & 1
\end{pmatrix}
\tag{2.4.4}
$$

of a Lorentz transformation $A$ fixing the points $(0, x^+, x^-) \in \mathbb{E}^{d+1,2}$ and a translation $v$.

At the level of the Lie algebra, $so(d+1,2)$ is spanned by the vector fields

$$
M_{AB} := \eta_{AC} x^C \partial_B - \eta_{BC} x^C \partial_A \in \mathcal{D}(E^{d+1,2}),
$$

with Lie brackets

$$
[M_{AB}, M_{CD}] = \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{AD} M_{BC}.
$$

The Poincaré algebra $\mathfrak{g}$ is the subalgebra of $so(d+1,2)$ whose vector fields are tangent to the null hypersurfaces $\mathcal{M}_0$ for any $\sigma$. It is spanned by

$$
\begin{align*}
L_{ab} &:= M_{ab} = x^a \partial_b - x^b \partial_a \\
P_a &:= M_{a+} = x^- \partial_a - x^a \partial_+ \\
B_a &:= M_{0a} = -x^0 \partial_a - x^a \partial_0 \\
H &:= M_{0+} = -x^0 \partial_+ - x^- \partial_0,
\end{align*}
$$

where $a, b = 1, \ldots, d$. Its Lie brackets are

$$
\begin{align*}
[L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc} \\
[L_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b \\
[L_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \\
[B_a, B_b] &= L_{ab} \\
[B_a, P_b] &= \delta_{ab} H.
\end{align*}
$$

\tag{2.4.8}

If $\sigma = 0$, there is an enhancement of symmetry and the subalgebra of $so(d+1,2)$ tangent to $\mathcal{M}_0$ has an additional generator: namely, $D := M_{-+} = x^+ \partial_+ - x^- \partial_-$. This enhances the Poincaré group to the subgroup of $O(d+1,2)$ consisting of matrices of the form

$$
\begin{pmatrix}
A & 0 & v \\
-a^T \eta A & \eta & -\frac{1}{2} \eta(v,v) \\
0^T & 0 & a^{-1}
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
0^T & 1 & 0 \\
0^T & 0 & 1
\end{pmatrix}
\begin{pmatrix}
A & 0 & 0 \\
0^T & 1 & 0 \\
0^T & 0 & 1
\end{pmatrix},
$$

\tag{2.4.9}

where the additional symmetry is given by nonzero $a \in \mathbb{R}$.
**Poincaré orbits in \( \mathbb{E}^{d+1,2} \)**

In discussing the orbits of the Poincaré group on \( \mathbb{E}^{d+1,2} \) we find it convenient to restrict ourselves to the identity component of the Poincaré group, denoted \( G \), and given by

\[
G = \left\{ \begin{pmatrix} A & 0 & v \\ v^T \tilde{\eta} A & 1 & -\frac{1}{2} \tilde{\eta}(v, v) \\ 0^T & 0 & 1 \end{pmatrix} \middle| A \in \text{SO}(d, 1)_0, \quad v \in \mathbb{R}^{d,1} \right\},
\]

where \( \text{SO}(d, 1)_0 \) is the identity component of the Lorentz group.

Since \( G \subset \text{O}(d + 1, 2) \), it preserves the quadrics \( Q_\epsilon \) for any \( \epsilon \in \mathbb{R} \), and by definition it also preserves the null hyperplanes \( \mathcal{N}_\sigma \) for any \( \sigma \in \mathbb{R} \). Therefore it preserves their intersections

\[
\mathcal{M}_{\epsilon, \sigma} := Q_\epsilon \cap \mathcal{N}_\sigma.
\]

**Embedding Minkowski**

Figure 2.6: An embedding of \((d + 1)\)-dimensional Minkowski spacetime \( \mathcal{M}_{d+1} \) as the intersection \( \mathcal{D}_0 \cap \mathcal{N}_1 \) in the ambient space \( \mathbb{E}^{d+1,2} \)

Our first observation is that for any \( \epsilon \), provided that \( \sigma \neq 0 \), \( \mathcal{M}_{\epsilon, \sigma} \) is an embedding of Minkowski spacetime \( \mathcal{M}_{d+1} \) in \( \mathbb{E}^{d+1,2} \). Let us first show that \( \mathcal{M}_{\epsilon, \sigma} \) is an orbit of \( G \). Suppose that \( (x, x^+, \sigma) \) is a point in \( \mathcal{M}_{\epsilon, \sigma} \). Because it lies in the quadric \( \mathcal{D}_\epsilon \) and \( \sigma \neq 0 \), we may solve for \( x^+ \) in terms of \( x \):

\[
x^+(x) = \frac{\epsilon - \tilde{\eta}(x, x)}{2\sigma},
\]

so that \( \mathcal{M}_{\epsilon, \sigma} \) is the image of \( \mathbb{R}^{d,1} \) under the embedding \( x \mapsto (x, x^+(x), \sigma) \). The resulting paraboloid is illustrated in Figure 2.6 for \( \epsilon = 0 \) and \( \sigma = 1 \).
The action of the Poincaré group on \((x, x^+, \sigma)\) can be read off from equation (2.4.10) and we see that it corresponds to \(x \mapsto \lambda x + \sigma v\). This action is transitive on \(\mathbb{R}^{d,1}\) and hence transitive on \(\mathcal{M}_{\epsilon, \sigma}\). Since \(x^- = \sigma\) is a constant, the pull-back to \(\mathcal{M}_{\epsilon, \sigma}\) of the pseudo-euclidean metric \(g_{\mathbb{E}}\) in equation (2.4.2) agrees with the Minkowski metric, proving that for any \(\epsilon \in \mathbb{R}\) and \(\sigma \neq 0\), \(\mathcal{M}_{\epsilon, \sigma}\) is isometric to \(\mathbb{M}^{d+1}_{\epsilon}\).

We pick an origin \((0, \frac{\epsilon}{2\sigma}, \sigma) \in \mathcal{M}_{\epsilon, \sigma}\). The subgroup \(H \subset G\) fixing the origin is the proper orthochronous Lorentz subgroup consisting of matrices of the form

\[
H = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0^T & 1 & 0 \\ 0^T & 0 & 1 \end{pmatrix} \right| A \in \text{SO}(d,1)_0 \right\}. \tag{2.4.14}
\]

Its Lie algebra \(\mathfrak{h}\) consists of the span of \(L_{ab}, B_a\), defined in equation (2.4.7). We write this as \(\mathfrak{h} = \langle L_{ab}, B_a \rangle\).

If \(\sigma = 0\), the quadric condition does not fix \(x^+\). The Poincaré orbits in \(\mathcal{M}_{\epsilon, 0}\) depend on the sign of \(\epsilon\), so we must distinguish between three cases, depending on whether \(\epsilon > 0\), \(\epsilon < 0\) or \(\epsilon = 0\).

**Embedding Spi**

Let \(\epsilon = \rho^2 > 0\). Then \(\mathcal{M}_{\rho^2, 0}\) consists of those points \((x, x^+, 0)\) where \(\tilde{\eta}(x, x) = \rho^2\) and \(x^+ \in \mathbb{R}\) is otherwise arbitrary. The condition \(\tilde{\eta}(x, x) = \rho^2\) cuts out a one-sheeted hyperboloid in \(\mathbb{R}^{d,1}\), i.e., a \(d\)-dimensional de Sitter space \(dS_d\). The proper orthochronous Lorentz group \(\text{SO}(d,1)_0\) acts transitively on this hyperboloid. The translation \(v\) in \(G\) acts via \((x, x^+, 0) \mapsto (x, x^+ - \tilde{\eta}(v, x), 0)\) and hence we see that \(G\) acts transitively on \(\mathcal{M}_{\rho^2, 0}\). Let \(e_d := (0, 0, \ldots, 0, 1) \in \mathbb{R}^{d,1}\) be an elementary spacelike vector and let us choose an origin \((\rho e_d, 0, 0)\) for \(\mathcal{M}_{\rho^2, 0}\). The subgroup \(H \subset G\) which fixes the origin consists of matrices

\[
H = \begin{pmatrix} A & 0 & v \\ -v^T \tilde{\eta} A & 1 & -\frac{1}{2} \tilde{\eta}(v, v) \\ 0^T & 0 & 1 \end{pmatrix} \tag{2.4.15}
\]

where \(A \in \text{SO}(d,1)_0\) is such that \(A e_d = e_d\) and \(v \in \mathbb{R}^{d,1}\) is such that \(\tilde{\eta}(v, e_d) = 0\). This subgroup \(H\) is isomorphic to the Poincaré group \(\text{SO}(d-1,1)_0 \times \mathbb{R}^{d-1,1}\) in one lower dimension. Its Lie algebra \(\mathfrak{h}\) can be determined as those vector fields in (2.4.7) which vanish at the origin and we can see that \(\mathfrak{h} = \langle L_{ij}, B_i, P_i, H \rangle\), where \(i, j = 1, \ldots, d - 1\). For any \(\epsilon > 0\), \(\mathcal{M}_{\epsilon, 0}\) is an

\[\text{We can make contact with the hyperboloid picture of } dS_{d+2} \text{ in the following way. Setting } \epsilon = -\rho^2 \text{ and parametrizing the hyperboloid as } x^0 = y^0 \sigma^{-1}, x^a = \sigma y^a, x^- = \sigma \text{ and } x^+ \text{ as in (2.4.12), the induced metric on the hyperboloid becomes}
\]

\[
ds^2 = \rho^2 \sigma^{-2} d\sigma^2 + \sigma^2 \rho^{-2} \left( -(dy^0)^2 + (dy^a)(dy_a) \right).
\]

This is the usual Poincaré patch of \(dS_{d+2}\). For \(\epsilon < 0\) and \(\sigma \neq 0\) the Minkowski spaces \(\mathcal{M}_{\epsilon, \sigma}\) described above therefore correspond to this slicing of \(dS_{d+2}\). The conformal boundary of \(dS_{d+2}\) is reached for \(\sigma \to \infty\). The light-like hypersurface \(\sigma = 0\) is not covered by the Poincaré patch coordinates. As we will see below, the corresponding space \(\mathcal{M}_{\epsilon, \rho^2}\) is not Minkowski space but \(T^{1,1}\).
embedding in $\mathbb{R}^{d+1,2}$ of the blow-up $\text{Spi}_{d+1}$ of the spatial infinity $i^0$ of Minkowski spacetime.

This embedding shows that $\text{Spi}_{d+1}$ fibers over $dS_d$, identifying $dS_d$ with any one of the one-sheeted hyperboloids in $\mathbb{R}^{d,1}$. The projection $\text{Spi}_{d+1} \to dS_d$ sends $(x, x^+, 0) \mapsto x$. This is a trivial bundle and hence $\text{Spi}_{d+1} \cong dS_d \times \mathbb{R}$. Every smooth function $f$ on $dS_d$ defines a section $dS_d \to \text{Spi}_{d+1}$ by $x \mapsto (x, f(x), 0)$. These sections are in one-to-one correspondence with the $\text{Spi}$-supertranslations, an infinite-dimensional abelian ideal of the Lie symmetries of the pseudo-carrollian structure [213] on $\text{Spi}_{d+1}$.

**Embedding $\text{Ti}^\pm$**

Let $\epsilon = -\rho^2 < 0$. Now $\mathcal{M}_{\rho^2, 0}$ consists of points $(x, x^+, 0)$ where $\bar{\eta}(x, x) = -\rho^2$ and $x^+ \in \mathbb{R}$. The condition $\bar{\eta}(x, x) = -\rho^2$ defines a two-sheeted hyperboloid in $\mathbb{R}^{d,1}$ which is acted on transitively by $\text{SO}(d, 1)$. Under the identity component $\text{SO}(d, 1)_0$, each sheet is an orbit. The translation $v$ in $G$ acts via $(x, x^+, 0) \mapsto (x, x^+ - \bar{\eta}(v, x), 0)$ and hence $G$ acts with two orbits on $\mathcal{M}_{\rho^2, 0}$:

$$\mathcal{M}_{\rho^2, 0} = \mathcal{M}_{\rho^2, 0}^+ \cup \mathcal{M}_{\rho^2, 0}^-,$$

where

$$\mathcal{M}_{\rho^2, 0}^\pm = \left\{ \begin{pmatrix} x \\ x^+ \\ 0 \end{pmatrix} \bigg| \bar{\eta}(x, x) = -\rho^2, \quad \pm x^0 > 0, \quad \text{and} \quad x^+ \in \mathbb{R} \right\}. \quad (2.4.16)$$

Let $e_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{d,1}$ be an elementary timelike vector and let us fix the origin $\pm \rho e_0, 0, 0 \in \mathcal{M}_{\rho^2, 0}^\pm$. The subgroup $H \subset G$ which fixes the origin is common to both orbits and consists of matrices

$$H = \begin{pmatrix} A & 0 & v \\ -v^T \bar{\eta} A & 1 & -\frac{1}{2} \bar{\eta}(v, v) \\ 0^T & 0 & 1 \end{pmatrix} \quad (2.4.18)$$

where $A \in \text{SO}(d, 1)_0$ fixes $e_0$ and $v \in \mathbb{R}^{d,1}$ is perpendicular to $e_0$. This subgroup is isomorphic to the euclidean group $\text{SO}(d) \ltimes \mathbb{R}^d$ in one lower dimension, corresponding to the hyperplane of $\mathbb{R}^{d,1}$ perpendicular to $e_0$. Its Lie algebra $\mathfrak{h}$ is spanned by those vector fields in equation (2.4.7) which vanish at the origin: namely, $\mathfrak{h} = \{ l_{ab}, p_a \}$. As shown in [208] (see also [215]), $\mathcal{M}_{\rho^2, 0}^\pm$ define embeddings of $\text{AdS}_{d+1}$, the carrollian limit of $\text{AdS}_{d+1}$. As discussed in Section 2.4.2, $\text{AdS}_{d+1}$ is isomorphic (as a homogeneous space of the Poincaré group) to the blow-ups $\text{Ti}^\pm_{d+1}$ of the timelike infinites $i^\pm$ of Minkowski spacetime. We shall therefore refer to $\text{AdS}_{d+1}$ simply as $\text{Ti}_{d+1}$.

Just as with $\text{Spi}_{d+1}$, this embedding shows that $\text{Ti}_{d+1}$ fibers over projective space $\mathcal{H}^d$, where we identify $\mathcal{H}^d$ with any one of the sheets of the two-sheeted hyperboloids $\bar{\eta}(x, x) = -\rho^2$. The fibration $\text{Ti}_{d+1} \to \mathcal{H}^d$ sends $(x, x^+, 0) \mapsto x$. Again this is a trivial bundle and hence $\text{Ti}_{d+1} \cong \mathcal{H}^d \times \mathbb{R}$. The smooth sections $\mathcal{H}^d \to \text{Ti}_{d+1}$ can be identified with the smooth functions on $\mathcal{H}^d$ and correspond to the $\text{Ti}$-supertranslations, the infinite-dimensional abelian ideal of the Lie symmetries of the carrollian structure of $\text{Ti}_{d+1}(= \text{AdS})$, which were determined in [222].
Finally we let $\epsilon = 0$ and consider $\mathcal{M}_{0,0} = \mathcal{D}_0 \cap \mathcal{N}_0$. The point $(x, x^+, 0)$ lies in $\mathcal{M}_{0,0}$ if and only if $\bar{\eta}(x, x) = 0$, so that $x$ lies on the lightcone in $\mathbb{R}^{d,1}$. Under $\text{SO}(d, 1)_0$, the lightcone $\mathcal{L}C \subset \mathbb{R}^{d,1}$ breaks up into three orbits:

\[ \mathcal{L}C = \mathcal{L}C^- \cup \{0\} \cup \mathcal{L}C^+, \tag{2.4.19} \]

where $\mathcal{L}C^\pm$ are the future/past lightcones with the apex removed. Provided that $x \in \mathcal{L}C^\pm$, the translations in $G$ can relate any two values of $x^+$, but if $x = 0$, then each of the points $(0, x^+, 0)$ is fixed by $G$. In summary, $\mathcal{M}_{0,0}$ breaks up into two $(d + 1)$-dimensional orbits $\mathcal{M}_{0,0}^\pm$ and a line $\ell = \{(0, x^+, 0) \mid x^+ \in \mathbb{R}\}$ of fixed points under the Poincaré group $G$; that is,

\[ \mathcal{M}_{0,0} = \mathcal{M}_{0,0}^- \cup \ell \cup \mathcal{M}_{0,0}^+ \quad \text{with} \quad \ell = \bigcup_{x^+ \in \mathbb{R}} \{(0, x^+, 0)\}, \tag{2.4.20} \]

where

\[ \mathcal{M}_{0,0}^\pm = \left\{ \begin{pmatrix} x \\ x^+ \\ 0 \end{pmatrix} \bigg| \bar{\eta}(x, x) = 0, \quad \pm x^0 > 0, \quad \text{and} \quad x^+ \in \mathbb{R} \right\}. \tag{2.4.21} \]

Let $e_- := \frac{1}{\sqrt{2}}(e_d - e_0) = (-\frac{1}{\sqrt{2}}, 0, \ldots, 0, \frac{1}{\sqrt{2}}) \in \mathbb{R}^{d,1}$ and let us fix the origin $(\pm e_-, x^+, 0) \in \mathcal{M}_{0,0}^\pm$. The subgroup $H \subset G$ which fixes the origin is common to both $\mathcal{M}_{0,0}^\pm$ and consists of matrices

\[ H = \begin{pmatrix} A & 0 & v \\ -v^T \bar{\eta}A & 1 & -\frac{1}{2} \bar{\eta}(v, v) \\ 0^T & 0 & 1 \end{pmatrix} \tag{2.4.22} \]

where $A \in \text{SO}(d, 1)_0$ fixes $e_-$ and $v \in \mathbb{R}^{d,1}$ is perpendicular to $e_-$. This subgroup is isomorphic to the $d$-dimensional Carroll group. Its Lie algebra $\mathfrak{h}$ is spanned by those vector fields in equation (2.4.7) which vanish at the origin: namely, $\mathfrak{h} = \langle L_{ij}, P_i, L_{id} + B_1, H - P_d \rangle$, for $i, j = 1, \ldots, d - 1$. Again, since the stabiliser subgroup is common to both $\mathcal{N}_i^\pm$, they are isomorphic as homogeneous spaces of $G$. We will therefore refer to either one of these two spaces simply as $\mathcal{N}_i$. 

As shown in [6], we will identify $\mathcal{M}_{0,0}^\pm$ with $\mathcal{N}_i^\pm$, the bundle of scales of the conformal carrollian structure of $\mathcal{G}^\pm$. This embedding of $\mathcal{N}_i^\pm$ shows that it fibers over the future/past lightcone $\mathcal{L}C^\pm$, with the fibration $\mathcal{N}_i^\pm \to \mathcal{L}C^\pm$ given simply by $(x, x^+, 0) \mapsto x$. Together with
the identification of \( \mathcal{N}_d^\pm \) as a bundle over \( \mathcal{I}^\pm \), we can see that there is a double fibration

\[
\begin{array}{ccc}
\mathcal{N}_d^+ & \xleftarrow{\mathcal{I}^+} & \mathcal{I}^+ \\
\xrightarrow{\mathcal{I}^-} & \mathcal{I}^- & \xrightarrow{\mathcal{I}^-} \\
\mathcal{L}^\pm & \xrightarrow{\mathcal{C}^\pm} & \mathcal{C}^\pm
\end{array}
\]

which allows us to view \( \mathcal{N}_d^\pm \) as sitting inside \( \mathcal{L}^\pm \times \mathcal{I}^\pm \) as their fibred product over the celestial sphere \( \mathcal{C}^\pm \). Said differently, the fibration \( \mathcal{N}_d^\pm \to \mathcal{I}^\pm \) is the pull-back fibre bundle of the fibration \( \mathcal{L}^\pm \to \mathcal{C}^\pm \) via the fibration \( \mathcal{I}^\pm \to \mathcal{C}^\pm \).

The fibration \( \mathcal{N}_d^\pm \to \mathcal{I}^\pm \) can also be understood from the embedding picture. Let \( \mathbb{P}^{d+2} \) be the projective space of \( \mathbb{E}^{d+1,2} \). It is the quotient of \( \mathbb{E}^{d+1,2} \setminus \{ 0 \} \) by the action of the nonzero reals \( \mathbb{R}^\times \) which rescales the nonzero vectors: \( x \mapsto \lambda x \) for \( x \in \mathbb{E}^{d+1,2} \setminus \{ 0 \} \) and \( \lambda \in \mathbb{R}^\times \). As explained in [220, Section 9.2], the image of the null quadric \( Q_0 \) in \( \mathbb{P}^{d+2} \) is a conformal compactification \( M^\#_{d+1} \) of Minkowski spacetime \( M^{d+1} \). The image of points in \( Q_0 \) with \( x^- \neq 0 \) correspond to the interior points of \( M^\#_{d+1} \) (corresponding to Minkowski spacetime itself), whereas the image of points with \( x^- = 0 \) correspond to the conformal boundary of Minkowski spacetime in this compactification. The points in \( \mathcal{N}^\pm_{d+1} \) map to \( \mathcal{I}^\pm \), which is the identification of \( \mathcal{I}^+ \) and \( \mathcal{I}^- \), which are after all indistinguishable as homogeneous spaces of \( G \), whereas the points in the singular line \( \ell \) (except for the origin) get mapped to the same point \( I_0 \in \mathbb{P}^{d+2} \) which is the identification of \( i^0, i^+ \) and \( i^- \).

**Summary**

We may summarise the above discussion by explicitly decomposing \( \mathbb{E}^{d+1,2} \) in terms of orbits of the connected Poincaré group:

\[
\mathbb{E}^{d+1,2} = \left( \bigsqcup_{\epsilon \in \mathbb{R}, \epsilon \neq 0} M_{0,\epsilon,0} \cong \mathbb{M} \right) \sqcup \left( \bigsqcup_{\epsilon > 0} M_{+,\epsilon,0} \cong \mathbb{S}^i \right) \sqcup \left( \bigsqcup_{\epsilon < 0} M_{-,\epsilon,0} \cong \mathbb{T}^i \right) \sqcup \left( \bigsqcup_{\epsilon \neq 0} M_{0,\epsilon,0} \cong \mathbb{S}^i \right) \sqcup \left( \bigsqcup_{x^+ \in \mathbb{R}} \left\{ \left( x^+, 0 \right) \right\} \right) \quad (2.4.24)
\]

We may now pass to the projective space \( \mathbb{P}^{d+2} = (\mathbb{E}^{d+1,2} \setminus \{ 0 \}) / \mathbb{R}^\times \) to obtain

\[
\mathbb{P}^{d+2} = \left( \bigsqcup_{\tau \in \mathbb{R}} M \right) \sqcup \mathbb{S}^i \sqcup \mathbb{T}^i \sqcup \mathcal{I} \sqcup \{ I \}, \quad (2.4.25)
\]

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where \( \tau = \varepsilon/\sigma^2 \) is a projective invariant. Restricting to the projectivised null quadric we obtain the conformal compactification

\[ M^\# = M \cup I \cup \{ I \} \quad (2.4.26) \]

of Penrose and Rindler [220, Section 9.2]. Although they treat the four-dimensional case \((d = 3\) here), their results are dimension agnostic. Here \( I \) is the identification of \( I^+ \) and \( I^- \) and \( \{ I \} \) is the singleton set obtained by identifying \( i^0 \) and \( i^\pm \). However we see that if we do not restrict to the null quadric, we actually obtain \( \text{Spi} \) and \( Ti^\pm \) as limits of a family of embedded Minkowski spacetimes.

### 2.4.2 Spi and Ti as blow-ups of spatial and timelike infinity

The following discussion is based on the original work of Ashtekar–Hansen (AH) [214]. Although a detailed discussion of the AH construction lies beyond the scope of this work, we will summarise the salient features in the following. In a conformal compactification of Minkowski spacetime the conformal boundary at spatial infinity is given by a single point \( i^0 \). This remains true for more general asymptotically flat spacetimes in the definition of AH. However, various physical fields, e.g., the connection coefficients, admit only direction-dependent limits at \( i^0 \). One therefore constructs a blow-up of \( i^0 \), such that fields at \( i^0 \) can be regarded as smooth fields on a blow-up manifold \( \text{Spi} \).

The blow-up manifold is constructed using the behaviour of certain inextensible spacelike curves approaching \( i^0 \). The AH definition gives rise to a universal lorentzian metric at \( i^0 \) that is used to demand that these curves have unit tangent vector at \( i^0 \). Such tangent vectors form a hyperboloid in the tangent space of \( i^0 \), with induced metric being \( dS_d \). This defines the asymptotic geometry at spatial infinity in the sense of [223]. However, the differentiability conditions in the AH definition allow also to define (direction-dependent) connection coefficients at \( i^0 \). Using these one demands that the spacelike curves be geodesics of the original asymptotically flat manifold. This requirement leaves undefined the component of the acceleration along the tangent vector of the curve at \( i^0 \). This additional parameter, taking values in the real numbers, can therefore be used to distinguish between asymptotic spacelike curves and thus becomes an additional coordinate on \( \text{Spi} \).

From the above construction it is apparent that \( \text{Spi} \) has the structure of a fibre bundle. The base space is the (one-sheeted) unit hyperboloid with fibre \( \mathbb{R} \). There are two natural tensor fields defined on \( \text{Spi} \): a nowhere vanishing vector field \( n \in \mathcal{X}(\text{Spi}) \) that generates diffeomorphism of the fibre and a corank-one \( \gamma \in \Gamma(\odot^2 T^*\text{Spi}) \) of lorentzian signature with constant positive curvature (the pullback of \( \gamma \) to the base space \( dS \) is the metric on \( dS \)), which furthermore satisfies \( \gamma(n, -) = 0 \). This is exactly the invariant structure that the Klein pair of \( \text{Spi} \) gives rise to, and we therefore recognise the AH construction of \( \text{Spi} \) as the (simply-connected) homogeneous space of this Klein pair. This observation was first made in [213].

This construction is applicable, mutatis mutandis, to future/past timelike infinity. Here, the AH construction leads a universal riemannian metric at \( i^\pm \), which is now used to demand that timelike curves approaching (or emanating from) \( i^\pm \) have unit tangent vector at \( i^\pm \), with
those tangent vectors now giving the tangent space of $i^{±}$ the structure of hyperbolic space $H^{d}$. Exactly as for $S^{p}$, the component of the acceleration along the tangent vector of a curve at $i^{±}$ can be used to distinguish between asymptotic timelike curves and is taken to be an additional coordinate on $T_i$. Hence $T_i$ is a (trivial) line bundle over hyperbolic space, whose invariant structure is a carrollian structure, consisting of a nowhere vanishing $\xi \in \mathcal{T}(T_i)$ and a corank-one positive semi-definite $\mathcal{h} \in \Gamma(\odot^2T^*T_i)$ of constant negative curvature whose kernel is spanned by $\xi$: $\mathcal{h}(\xi, -) = 0$. This precisely our space $T_i \cong \text{AdSC}$.

2.5 Cartan geometry

Having described non-Lorentzian Klein geometries, the next question we ask is: what are the differential geometries associated to these spaces which are such that they locally look like the non-Lorentzian Klein geometry? This is analogous to the familiar construction of a Lorentzian manifold $M$, which locally looks like Minkowski space in these sense that the tangent space at each point $p \in M$ is isomorphic to Minkowski space $M$.

Recall from our discussion above that given a kinematical Lie group $G$, we can construct an associated homogeneous space $M$ by choosing a stabiliser subgroup $H \subset G$ which leaves any point in the space invariant. The homogeneous space $M$ is (locally isomorphic) to the coset manifold $G/H$. Infinitesimally, the coset manifold corresponds to what is called the Klein pair $(g, h)$, where $g$ is the Lie algebra of $G$ and $h$ the Lie algebra of $H$. In this section, following [6], we describe the construction of Cartan geometries associated to a Klein pair. In the literature, this is sometimes known as “gauging” the algebra $g$ [224, 225], but as demonstrated in [6], the gauging procedure depends on the choice of stabiliser $h$: in particular, starting from the Poincaré group, choosing different subalgebras $h$ leads to very different geometries as we saw in Section 2.4. From a mathematical point of view, the gauging procedure is nothing but the construction of a Cartan geometry,9 as we now explain. Note that the gauging procedure usually also involves assigning dynamics to the geometric fields (see, e.g., [228]), something that was considered from the perspective of Cartan geometry in [6].

Let us fix a Klein pair $(g, h)$ with $\dim g - \dim h = n$, which we furthermore assume to be reductive (2.1.12), and let us fix a connected Lie group $\mathcal{G}$ with Lie algebra $h$.

Let $M$ be a smooth manifold of dimension $n = \dim g - \dim h$. The gauging procedure starts with a one-form $\Lambda \in \Omega^1(M, g)$ in $M$ with values in $g$, i.e., $\Lambda$ is a Lie algebra $g$ valued one-form. For $X \in g$, we let $\overline{X} \in m$ denote its image under the canonical surjection $g \to m$. Let $e := \overline{\Lambda} \in \Omega^1(M, m)$ denote the projection of $\Lambda$ to $m$. The main assumption in the gauging procedure is that $e$ is an inverse vielbein, or a “coframe field”. On an open subset $U \subset M$ on which the cotangent bundle admits a trivialisation, we can choose an open cover $\{U_\alpha\}$ (note that $\alpha$ labels the open cover and is not a tensor index) of $M$ such that the cotangent bundle is trivialisable over each $U_\alpha$. The gauging procedure then starts with a one-form $A_\alpha \in \Omega^1(U_\alpha, g)$ over each $U_\alpha$, with the assumption that its projection $e_\alpha \in \Omega^1(U_\alpha, m)$ is

---

9See [226] for a more comprehensive treatment of Cartan geometry, and [227] for a less comprehensive (but decidedly delightful) treatment.
a local coframe, i.e., if we introduce local coordinates \( \{x^\mu\} \) on \( U_\alpha \), where \( \mu = 1, \ldots, n \) is a “spacetime” index, we can write

\[
e_\alpha = e_\mu \, dx^\mu,
\]

(2.5.1)

where \( e_\mu \) are the \( m \)-valued components of a one-form on \( U_\alpha \). In other words, if we choose a vector space complement \( m \) of \( \mathfrak{h} \) in \( \mathfrak{g} \) and a basis \( X_i \) for \( m \), where \( i = 1, \ldots, n \) is a “tangent space index”, we can write \( e_\alpha = e_\alpha^i X_i = e_\mu^i X_i \, dx^\mu \), where we used (2.5.1), where \( e_\mu^i \) are the components of the coframe field, i.e., an isomorphism \( T_p M \rightarrow m \), sending \( \delta_\mu \rightarrow e_\mu^i X_i \). The fields \( e_\mu^i \) are nothing but the familiar inverse vielbeins as employed in the physics literature. Each such pair \( (U_\alpha, A_\alpha) \) is called a Cartan gauge in [226, §5.1].

This then prompts the question of how the one-forms \( A_\alpha \) and \( A_\beta \) are related in a non-empty overlap \( U_\alpha \cap U_\beta \). This is typically not discussed in the literature on the gauging procedure, but based on the examples at our disposal, it is reasonable to demand that on a non-empty overlap \( U_\alpha \cap U_\beta \), the one-forms \( A_\alpha \) and \( A_\beta \) should be related by an \( \mathcal{H} \)-gauge transformation: namely,

\[
A_\beta = A_\alpha + h_\alpha^\beta \delta H,
\]

(2.5.2)

for some smooth \( h_\alpha^\beta : U_\alpha \cap U_\beta \rightarrow \mathcal{H} \) and where \( \delta H \) is the left-invariant Maurer–Cartan form on the group \( H \). In the case of matrix groups, which we may assume since \( \mathcal{H} \) is a subgroup of \( GL(m) \), the above relation says, explicitly, that for all \( p \in U_\alpha \cap U_\beta \),

\[
A_\beta(p) = h_\alpha^\beta(p)^{-1} A_\alpha(p) h_\alpha^\beta(p) + h_\alpha^\beta(p)^{-1} \, dh_\alpha^\beta(p).
\]

(2.5.3)

Infinitesimally, we can write \( h_\alpha^\beta = 1 + \Lambda_{\alpha\beta}, \) where \( \Lambda_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathfrak{h} \), and where \( \mathfrak{h} \) is a subalgebra of \( \mathfrak{gl}(m) \). The infinitesimal difference at the point \( p \) (which we now suppress) is

\[
\delta A := A_\beta - A_\alpha \text{ is then}
\]

\[
\delta A = dA + [A, \Lambda],
\]

(2.5.4)

where we suppressed the labels \( \alpha, \beta \).

Formally, Cartan geometry on \( M \) modelled on \( (\mathfrak{g}, \mathfrak{h}) \) with group \( \mathcal{H} \) is a \( \mathcal{H} \)-principal bundle \( P \rightarrow M \) equipped with a non-degenerate one-form \( A \in \Omega^1(P, \mathfrak{g}) \) called the Cartan connection. A Cartan geometry is said to be of the first order if \( \mathcal{H} \) acts faithfully on \( m \). This holds by construction in our set-up, since \( \mathcal{H} \) is defined as a subgroup of \( GL(m) \). As shown in [226, §3.3], a first-order Cartan geometry is an \( \mathcal{H} \)-structure; that is, \( P \) is a sub-bundle of the frame bundle of \( M \) where frames transform under local \( \mathcal{H} \)-transformations on overlaps (cf., the transformation law (2.5.3)). Since we assume the Cartan geometry to be reductive, the Cartan connection \( A = \omega + e \) splits into an Ehresmann \( \mathfrak{h} \)-connection \( \omega \) on \( P \) and a soldering form \( e \) taking values in \( m \) (see Section 2.6 for more details). We can pass from a first-order formulation to a second-order formulation by expressing the \( \mathfrak{h} \)-component of \( A \) in terms of the soldering forms \( e \). The conditions that allow us to do this typically involve setting (components) of the curvature (to be discussed below) to zero; these are sometimes called “conventional constraints” in terminology borrowed from supergravity [229].

The Klein model of a Cartan geometry modelled on \( (\mathfrak{g}, \mathfrak{h}) \) with group \( H \) is precisely the case where \( P = \mathcal{G} \), so that \( M = \mathcal{G}/\mathcal{H} \) and \( A \) is the left-invariant Maurer–Cartan form, so that it
obeys the structure equation \( dA + \frac{1}{2}[A, A] = 0 \). For a general Cartan geometry, the curvature \( F := dA + \frac{1}{2}[A, A] \in \Omega^2(P, \mathfrak{g}) \) is not necessarily zero. Indeed, flat Cartan geometries are locally isomorphic to \( \mathcal{G}/\mathcal{H} \). Thus we may understand the curvature of a Cartan connection as the failure of the Cartan geometry to be locally isomorphic to the Klein model on which it is modelled.

The curvature \( F \) splits as \( F = \Theta + \Omega \), where \( \Theta \in \Omega^2(P, \mathfrak{h}) \) and \( \Theta \in \Omega^2(P, \mathfrak{m}) \) are given by

\[
\Theta = d\omega e + \frac{1}{2}[e, e]_m \quad \text{and} \quad \Omega = F_\omega + \frac{1}{2}[e, e]_h,
\]

where we used \( \mathfrak{h} \)-invariance of \( m \), and where \( F_\omega := d\omega + \frac{1}{2}[\omega, \omega] \) and \( d\omega e := de + [\omega, e] \). By \( [e, e]_m \) and \( [e, e]_h \) we mean the projection of \( [e, e] \) onto \( m \) and \( h \), respectively.

The curvature satisfies the **Bianchi identity** \( dA F = dF + [A, F] = 0 \). This is an immediate consequence of \( d^2 = 0 \) and the Jacobi identity of \( \mathfrak{g} \). Since we are in the reductive case, the Bianchi identity splits into two identities:

\[
d\omega \Theta + [e, \Theta]_h = 0 \quad \text{and} \quad d\omega \Theta + [e, \Theta]_m + [e, \Theta]_m = 0,
\]

where we in general define the action of \( d\omega \) as \( d\omega \alpha := d\alpha + [\omega, \alpha] \) and where the Lie bracket hides a wedge.

### 2.6 G-structures & intrinsic torsion

As we already mentioned above, first-order Cartan geometries are \( G \)-structures with \( G = \mathcal{H} \). Formally, a \( G \)-structure on an \( n \)-dimensional manifold \( M \) is a principal \( G \)-subbundle \( P \subset F(M) \) of the frame bundle [202, 209]. For example, if \( M \) is equipped with a Riemannian metric, we can restrict ourselves to orthonormal frames which are related on overlaps by a local \( O(n) \) transformation, i.e., \( \Lambda \in O(n) \) acts on the set of orthonormal frames at \( p \in P \subset F(M) \) by sending the orthonormal frame \( u \) to another orthonormal frame \( u' = u \circ \Lambda \), which in components and considered infinitesimally becomes the perhaps more familiar transformation of a Riemannian vielbein \( E_i \)

\[
\delta E_i = \lambda_i^j E_j,
\]

for \( \lambda_i^j \in \mathfrak{o}(n) \), and \( O(n) \) is the group that leaves the “tangent space metric” \( \delta_{ij} \) invariant. Such a \( G \)-invariant tensor field \( \delta \) on \( M \) is generally known as a **characteristic tensor field**. In this way, a Riemannian metric on \( M \) leads to an \( O(n) \)-structure on \( M \).

As we saw above, a convenient way to get both a coframe (a.k.a. soldering form or inverse vielbein) and an Ehresmann connection on a \( G \)-structure is from a reductive first-order Cartan geometry: in this case, as we saw, the \( m \)-component of the Cartan connection defines a soldering form, while the \( h \)-component defines an Ehresmann connection. A soldering form \( e \in \Omega^1(P; m) \) defines a bundle isomorphism between the tangent bundle \( TM \) and the so-called fake tangent bundle \( P \times_G m \), which is an associated vector bundle as defined in Section 2.3. As a vector space, \( m \cong \mathbb{R}^n \), and \( G \subset GL(n, \mathbb{R}) \) acts on \( m \) via the defining
representation. In this way, the soldering form provides an identification between tensor bundles over \(M\) with associated vector bundles to the principal bundle (and \(G\)-structure) \(P \to M\). An Ehresmann connection \(\omega \in \Omega^1(P, h)\) defines a Koszul connection on a given associated vector bundle (see, e.g. [209] for more details). A Koszul connection \(\nabla\) on a vector bundle \(E \to M\) is a bilinear map
\[
\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E),
\]
where \(\mathcal{X}(M)\) is the space of vector fields on \(M\), and \(\Gamma(E)\) is the space of sections on the vector bundle. This map sends \((X, s) \mapsto \nabla_X s\), and satisfies the relations \(\nabla f X s = f \nabla X s\) and \(\nabla X fs = X(f)s + f \nabla X s\) for all \(f \in C^\infty(M)\), \(X \in \mathcal{X}(M)\) and \(s \in \Gamma(E)\). In particular, the Ehresmann connection determines a Koszul connection on the fake tangent bundle \(P \times_G m\), and using the soldering form \(e\), this maps to an adapted affine connection on \(TM\). In Section 2.7, we will explicitly go through the procedure outlined above in the case of Lorentzian geometry.

In general, a \(G\)-structure has **intrinsic torsion**. This is the part of the torsion which is independent of the adapted connection. Formally, the intrinsic torsion is given by the cokernel of a so-called Spencer differential. While we do not need the detailed construction (the interested reader should consult [209]), classifying non-Lorentzian geometries in terms of their intrinsic torsion leads to a useful “taxonomy”, and, in the case of Newton–Cartan geometry, leads to the familiar three classes: torsionless, twistless torsionless and torsional Newton–Cartan geometry.

### 2.7 Lorentzian geometry

Lorentzian, or pseudo-Riemannian, geometry is the geometric framework that underlies Einstein’s theory of general relativity. It is immensely rich and accommodates a myriad of spectacular phenomena, chief among which are black holes. The purpose of this section is to go through the construction of Lorentzian geometry as a Cartan geometry; we refer to this as the **gauging procedure**.

The starting datum for the gauging procedure is the Klein pair \((g, h)\) of \((n\)-dimensional) Minkowski spacetime, where \(g\) is the Poincaré algebra and \(h\) is a Lorentz subalgebra.\(^{10}\) Although often described as “gauging the Poincaré algebra”, this is, as illustrated in Section 2.4, imprecise: we are gauging a Klein pair \((g, h)\); although it should be clear from the description above, if anything is being gauged, it is actually the subalgebra \(h\).

In the basis \(L_{mn} = -L_{nm}\) and \(P_m\), the brackets of the Poincaré algebra are given in (2.1.6). This is basis for \(g\) is adapted to the reductive split \(g = h \oplus m\), with \(h = \langle L_{mn}\rangle\) and \(m = \langle P_m\rangle\).

Let \(M\) be a four-dimensional smooth manifold. We shall work locally in an open subset \(U \subset M\) with the tacit assumption that we are repeating these calculations on each member of an open cover for \(M\) with gluing conditions as explained in Section 2.5, particularly

\(^{10}\)Note that we could have started with any of the Lorentzian Klein pairs of Table 2.1: these spaces are model mutants and give rise to the same Cartan geometries. See [6] for more details.
equation (2.5.2). We shall let
\[ A = \frac{1}{2} \omega^{mn} L_{mn} + e^m P_m \in \Omega^1(U, g), \]  
with its associated curvature
\[ F = \frac{1}{2} \Omega^{mn} L_{mn} + \Theta^m P_m \in \Omega^2(U, g) \]  
Explicitly, we have
\[ \Omega^{mn} = d\omega^{mn} + \omega^m_p \wedge \omega^{pn} \]
\[ \Theta^m = d\nabla e^m := de^m + \omega^m_n \wedge e^n, \]  
where indices are lowered with \( \eta_{mn} \). The Bianchi identities take the form
\[ d\nabla \Omega^{mn} = 0 \quad \text{and} \quad d\nabla \Theta^m = \Omega^n_m \wedge e^n. \]  
where the action of \( d\nabla \) is defined by
\[ d\nabla \alpha^{mn} := d\alpha^{mn} + \omega^m_p \wedge \alpha^{pn} + \omega^n_p \wedge \alpha^{mp}. \]  

By construction, the Cartan geometry under discussion, being of the first order, is a \( G \)-structure with \( G = K = O(d, 1) \), as discussed in Section 2.6. The characteristic tensor fields are a Lorentzian metric and its inverse, which reside in \( \odot^2 m \) and \( \odot^2 m \), respectively. The intrinsic torsion of a Lorentzian structure is zero: this follows from the fundamental theorem of Riemannian geometry.

The Lorentzian metric on \( M \) has the local expression \( g = \eta_{mn} e^m e^n \) which allows us to identify the bundle \( P \to M \) as the bundle of oriented orthonormal frames. Being reductive, the connection \( \omega \) takes values in \( h \) and hence it is metric-compatible and if we furthermore impose that the torsion vanishes, \( \Theta^m = 0 \) (sometimes such conditions are called curvature constraints, see, e.g., [230], the connection essentially becomes the Levi-Civita connection. These curvature constraints allow us to solve for the Ehresmann connection in terms of the vielbeine, which from the point of the Cartan connection are independent objects. These can be imposed by hand, but, as demonstrated from the perspective of Cartan geometry in [6], one may construct gauge invariant Lagrangians which will allow us to obtain \( \Theta^m = 0 \) dynamically by varying the Lagrangian.

In a Cartan gauge \( (U, \Lambda) \) coordinatised by \( \{ x^\mu \} \), we can write the Cartan connection \( A \in \Omega^1(U, g) \) as
\[ A_\mu = \frac{1}{2} \omega^m_{\mu n} L_{mn} + e^m_{\mu} P_m, \]  
where the \( m \)-component \( e^m_{\mu} \) are the components of a coframe as argued above. The coframe \( e^m \) transform infinitesimally under \( h \)-gauge transformations according to (2.5.4). Writing \( \Lambda = -\Lambda^{mn} L_{ab} \), the \( m \)-component of (2.5.4) tells us that
\[ \delta e^m = \Lambda^{mn} e^n, \]  
43
which is the familiar statement that “vielbeine transform under local Lorentz transformations”.

Let $E_{\mu m}$ denote the components of the canonically dual frame, i.e., the vielbein, satisfying

$$e_{\mu m} E_{\nu n} = \delta^m_n, \quad e_{\mu m} E^m_{\nu} = \delta^\nu_{\mu}.$$  \hfill (2.7.8)

The $h$-gauge transformation of $E_m$ can be inferred from these relations by using that $\delta_m^n$ is $h$-gauge invariant; this gives us that $\delta E_{\mu m} = \Lambda_m \delta^m_{\nu}$. From the vielbeine, we can construct both a metric and its inverse

$$g_{\mu \nu} = \eta_{mn} e_{\mu m} e_{\nu n}, \quad g^{\mu \nu} = \eta_{mn} E_{\mu m} E_{\nu n},$$  \hfill (2.7.9)

which are invariant under $h$-gauge transformations.

The $h$-component $A^h$ of $A$ defines an Ehresmann connection on the principal $\mathcal{H}$-bundle $P \to M$ of the Cartan geometry. As we saw, this connection induces a Koszul connection on the fake tangent bundle $P \times H^m \to M$, which is the vector bundle associated to $P$ via the $\mathcal{H}$-representation $m$. Locally, sections of $P \times H^m$ are $m$-valued functions on $U$. The coframe $e^m$ gives a bundle isomorphism $\mathcal{T}M \to P \times H^m$ sending $\partial_\mu \mapsto e^m \partial_\mu$ and allowing us to transport the Koszul connection to an affine connection $\nabla$ on $\mathcal{T}M$. This affine connection has connection coefficients $\Gamma_{\mu \nu}^\rho$ defined by

$$\nabla_\mu \partial_\nu = \Gamma_{\mu \nu}^\rho \partial_\rho.$$  \hfill (2.7.10)

They can be determined in terms of the Cartan connection via the so-called vielbein postulate, which says that we obtain the same result if we differentiate $\partial_\mu$ with the affine connection $\nabla$ and then map to $P \times H^m$ or first map to $P \times H^m$ and differentiate with the $h$-part of the Cartan connection. Explicitly, in the former operation we obtain

$$\partial_\nu \Gamma_{\mu \nu}^\rho \partial_\rho \implies \Gamma_{\mu \nu}^\rho e_{\mu m} E_{\nu n} P_m,$$  \hfill (2.7.11)

whereas in the latter we obtain

$$\partial_\nu e_{\nu m} P_m \implies \partial_\nu e_{\nu m} P_m + e_{\nu m} [A^h_{\mu m}, P_m],$$  \hfill (2.7.12)

where $A^h$ is the $h$-component of the Cartan connection. Equating the two expressions, we obtain

$$\Gamma_{\mu \nu}^\rho e_{\mu m} P_m = \partial_\mu e_{\nu m} P_m + e_{\nu m} [A^h_{\mu m}, P_m] = \partial_\mu e_{\nu m} P_m - e_{\nu n} \omega_{\mu n m} P_m,$$  \hfill (2.7.13)

from where we read off the following expression for the adapted affine connection

$$\Gamma_{\mu \nu}^\rho = E_{\mu m} \partial_\mu e_{\nu m} + \omega_{\mu n m} e_{\nu n} E_{\rho m}.$$  \hfill (2.7.14)

\hfill (i)

If we introduce a covariant derivative $D_\mu$ that acts as $D_\mu e_{\mu m} = \partial_\mu e_{\mu m} - \Gamma_{\mu \nu}^\rho e_{\mu m} + \omega_{\mu n m} e_{\nu n}$, the vielbein postulate (2.7.13) can be expressed as $D_\mu e_{\mu m} = 0$, which is the form in which it appears in for example [19, 230].
The torsion of $\Gamma$ is given by $\Theta^m$, i.e.,

$$\Theta^m_{\mu \nu} = 2 \partial_{[\mu} e_{\nu]}^m + 2 \omega^m_{[\mu} e_{\nu]}^n = 2 \Gamma^\rho_{[\mu \nu]} e^m_\rho,$$

(2.7.15)

and the curvature constraint $\Theta^m = 0$ allows us to solve for $\omega$ in terms of the vielbeine:

$$\omega_{\mu}^{\ m n} = 2 E^\rho_{\lambda} \partial_{[\mu} e^m_{\nu]} - e_{\mu p} E^\rho_{\lambda n} \partial_{[\lambda} e^p_{\nu]},$$

(2.7.16)

which allows us to pass from a first-order formulation to a second-order formulation. Plugging the above into (2.7.14), we get the Levi–Civita connection:

$$\Gamma^\rho_{\mu \nu} = \frac{1}{2} g_{\rho \lambda} \left( \partial_\mu g_{\nu \lambda} + \partial_\nu g_{\mu \lambda} - \partial_\lambda g_{\mu \nu} \right).$$

(2.7.17)

The other $h$-component of $F$, namely the one associated with $L_{mn}$, becomes the Riemann tensor and is expressible in terms of the Levi–Civita connection in the usual way. For more details, see [6, 19, 230].

### 2.8 Carrollian geometry

We repeat the process described in the section above for a Cartan geometry modelled on the Klein pair $(g, h)$, where $g$ is the Carroll algebra in the basis (2.1.8) and $h = (L_{ab}, B_a) \cong \text{iso}(d)$ the stabiliser subalgebra. This section is based on [6, 202] (see also [228, 231, 232]).

Consider an open subset $U \subset M$ of a four-dimensional smooth manifold $M$ and introduce a one-form $A \in \Omega^1(U, g)$ with values in $g$ which, relative to the above basis, can be expanded as

$$A = \frac{1}{2} \omega^{ab} L_{ab} + \psi^a B_a + e^a P_a + \tau H.$$

(2.8.1)

Its curvature $F \in \Omega^2(U, g)$ is given by

$$F = dA + \frac{1}{2} [A, A] = \frac{1}{2} \Omega^{ab} L_{ab} + \Psi^a B_a + \Theta^a P_a + \Xi H,$$

(2.8.2)

where

$$\Omega^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega_c^b = F^a_{ab},$$

$$\Psi^a = d\psi^a + \omega^a_b \wedge \psi^b = d^\nabla \psi^a,$$

$$\Theta^a = de^a + \omega^a_b \wedge e^b = d^\nabla e^a,$$

$$\Xi = d\tau + \psi^a \wedge \tau_a,$$

(2.8.3)

where $d^\nabla$ is the $\tau$-covariant exterior derivative and $F^a_{ab}$ its curvature two-form. Indices have
been lowered with \( \delta_{ab} \). The Bianchi identity \( dF + [A, F] = 0 \) also splits as follows:

\[
\begin{align*}
d^\nabla \Omega^{ab} &= 0, \\
d^\nabla \psi^a &= \Omega^a_{\ b} \wedge \psi^b, \\
d^\nabla \Theta^a &= \Omega^a_{\ b} \wedge e^b, \\
de^\nabla &= e^a \wedge \Theta^a = 0,
\end{align*}
\]  

(2.8.4)

where \( d^\nabla \Omega^{ab} = d\Omega^{ab} + \omega^a_{\ c} \wedge \Omega^{cb} - \omega^b_{\ c} \wedge \Omega^{ca} \). Just like Lorentzian geometry that we considered above in Section 2.7, a Carrollian geometry is a \( G \)-structure with \( G = \mathcal{H} \) given by the stabiliser subgroup that defines the Carrollian Cartan geometry. To explicitly exhibit this group as a subgroup of \( \text{GL}(d + 1, \mathbb{R}) \), it is useful to recall from Section 2.1.2 that the \( h \)-invariant tensors in \( m, m^* \) of a Carrollian geometry consists of \( H \in m \) and \( \pi^2 \in \mathcal{O} \). As a vector space, \( m \cong \mathbb{R}^{d+1} \), and letting \( e_0, \ldots, e_d \) denote the canonical basis for \( \mathbb{R}^{d+1} \). Letting \( \alpha^0, \ldots, \alpha^d \) denote the canonically dual basis, we are looking for the subgroup \( \mathcal{H} \) of \( \text{GL}(d + 1, \mathbb{R}) \) that preserves \( \alpha^0 \) and \( \sum_{a=1}^d e_a e_a \). By writing everything explicitly as matrices, one may explicitly verify that a Carrollian geometry is a \( G \)-structure with

\[
G = \left\{ \begin{pmatrix} 1 & \nu^T \\ 0 & A \end{pmatrix} \bigg| A \in \text{O}(d) \ & & \nu \in \mathbb{R}^d \right\} \subset \text{GL}(d + 1, \mathbb{R}).
\]  

(2.8.5)

We can recast our results above in terms of the Carrollian structure on the manifold \( M \). We work in a Cartan gauge \((U, A)\) where, as above, \((U, x^\mu)\) is also a chart with local coordinates \( x^\mu \), relative to which the Cartan connection \( A \in \Omega^1(U; g) \) given in (2.8.1) has components

\[
A_{\mu} = \frac{1}{2} \omega_{\mu}^{\ ab} L_{ab} + \psi^a_{\ b} + e_{\mu}^a p_a + \tau_{\mu} H.
\]  

(2.8.6)

Like \( \mathcal{H}, \) \( C \) is a reductive Klein geometry, so the \( m \)-component of \( A \) defines a coframe. Let \( (\nu^\mu, E_{\mu \ a}) \) denote the components of the canonically dual frame. The Carrollian structure is then given by \( (\nu^\mu, h_{\mu \nu}) \), where \( h_{\mu \nu} = \delta_{ab} e^a_{\mu} e_{\nu}^b \) satisfying the following completeness relations:

\[
\tau_{\mu} \nu^\mu = -1, \quad e^a_{\mu} \nu^\mu = 0, \quad \tau_{\mu} E_{\mu \ a} = 0, \quad e_{\mu}^a E_{\mu \ b} = \delta^a_b \quad \text{and} \quad -\tau_{\mu} \nu^\nu + e_{\mu}^a E_{\mu \ a} = \delta^\nu_\nu.
\]  

(2.8.7)

It is convenient to define \( \gamma \in \Gamma(\mathcal{O}^2 T U) \) via \( \gamma = \delta_{\mu \nu} E_{\mu \ a} E_{\nu \ b} \) with components \( \gamma^\mu_{\ \nu} = \delta_{\mu \nu} E_{\mu \ a} E_{\nu \ b} \). It follows from the completeness relations that

\[
\gamma^\mu_{\ \nu} h_{\rho \nu} - \nu^\mu \tau_{\nu} = \delta^\mu_{\nu}.
\]  

(2.8.8)

The \( h \)-gauge transformations are again found by applying (2.5.4). Writing \( \Lambda = \frac{1}{2} \Lambda_{ab} L_{ab} + \lambda^a B_a \) and using the completeness relations, (2.5.4) implies that

\[
\delta \tau_{\mu} = \lambda_a e_{\mu}^a, \quad \delta e_{\mu}^a = \Lambda_{\ a \ b} e_{\mu}^b, \quad \delta \nu^\mu = 0, \quad \delta E_{\mu \ a} = \nu^\mu \lambda_a + \Lambda_{\ a \ b} E_{\mu \ b},
\]  

(2.8.9)

which imply that

\[
\delta h_{\mu \nu} = 0, \quad \delta \gamma^\mu_{\ \nu} = 2 \lambda^a E_{(\mu \ a)\nu}.
\]  

(2.8.10)
The Carroll structure \((v, h)\) is \(h\)-invariant, and is the curved space generalisation of the Carroll structure we met in Section 2.1.2.

To determine the form of the affine connection, we again impose the vielbein postulate, which now gives

\[
\Gamma^\rho_{\mu\nu} (\tau_\rho H + e_\rho^a P_a) = (\partial_\mu \tau_\nu + e_\nu^a \psi_\mu A^a) H + \left( \partial_\mu e_\nu^a + \omega_\mu A^a b e_\nu^b \right) P_a ,
\]

which upon application of the completeness relations gives rise to

\[
\Gamma^\rho_{\mu\nu} = -v^\rho \partial_\mu \tau_\nu - v^\rho e_\nu^a \psi_\mu A^a + E^\rho A^a \partial_\mu e_\nu^a + E^\rho A^a \omega_\mu A^b e_\nu^b .
\]

(2.8.12)

The torsion of the affine connection (2.8.12) satisfies

\[
\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = -v^\lambda \Xi_{\mu\nu} + E^\lambda A^a \Theta_{\mu\nu}^a ,
\]

(2.8.13)

where \(\Xi_{\mu\nu}\) and \(\Theta_{\mu\nu}^a\) are the components of the \(m\)-component of the curvature (2.8.2).

Imposing the curvature constraints\(^{12}\) \(\Xi = \Theta^a = 0\) allows us to solve for the \(h\)-components of the Cartan connection in terms of the Carrollian structure, and leads to torsion-free affine connections.

Since the affine connection \(\nabla\) comes from an Ehresmann connection \(\omega\), it is adapted to the carrollian structure; that is, the tensors \((v, h)\) defining the carrollian structure are parallel (or, in other words, the affine connection is “Carroll metric compatible”)

\[
\nabla_\mu v_\nu = 0 \quad \text{and} \quad \nabla_\mu h_{\nu\lambda} = 0 .
\]

(2.8.14)

As shown in [228,233], the most general torsion-free connection adapted to the carrollian structure \((v, h)\) has coefficients given by

\[
\Gamma^\lambda_{\mu\nu} = -v^\lambda \partial_\mu \tau_\nu + \frac{1}{2} \gamma^\lambda (\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}) + v^\lambda \Sigma_{\mu\nu} ,
\]

(2.8.15)

where \(\Sigma_{\mu\nu} = \Sigma_{\rho\mu\nu}\) is an arbitrary symmetric tensor subject to the condition

\[
v^\mu \Sigma_{\mu\nu} = 0 ,
\]

(2.8.16)

arising from the fact that \(v\) is parallel.

Unlike a Lorentzian geometry, a Carrollian geometry has intrinsic torsion. It was shown in [209] that the intrinsic torsion is given by \(\mathcal{L}_v h\), an object which admits an interpretation as the extrinsic curvature of the spatial slices of the Carrollian geometry. The classification in terms of intrinsic torsion leads to four classes of Carrollian geometries, and we refer to [209] for more details.

\(^{12}\)These constraints also arise dynamically as equations of motion for the Lagrangian obtained by gauging the Klein pair upon which the Carroll Cartan geometry is modelled; see [6] for details.
2.9 Newton–Cartan geometry & the Bargmann algebra

The Galilean algebra (2.1.10) admits a central extension, which is known as the Bargmann algebra. It involves augmenting (2.1.10) with the bracket

\[ [P_a, B_b] = \delta_{ab} Z, \]

(2.9.1)

where \( Z \) is a central charge, which, as we now demonstrate, is related to mass [230]. Consider a nonrelativistic free particle with mass \( M \), which is described by the action

\[ S = \int_{t_1}^{t_2} dt L, \quad L = \frac{1}{2} M \dot{x}^a \dot{x}^a. \]

(2.9.2)

The action is invariant under Galilean boosts \( x^a \rightarrow x^a + v^a t \), whereas the Lagrangian is not: it transforms into a total derivative

\[ \delta v_i L = \frac{d}{dt} (Mx^a v^a), \]

(2.9.3)

and promoting \( v^a \rightarrow v^a(t) \) to a function of \( t \), we find that

\[ \delta v^a(t) L = \frac{d}{dt} (Mx^a v^a) + \dot{v}^a (M\dot{x}^a t - Mx^a), \]

(2.9.4)

which allows us to read off the conserved Noether charge corresponding to boosts

\[ \mathcal{B}^a = M\dot{x}^a t - Mx^a = \mathcal{P}^a t - Mx^a, \]

(2.9.5)

where \( \mathcal{P}^a \) is the conserved charge corresponding to translations \( x^a \rightarrow x^a + d^a \). The basic Poisson brackets are \([x^a, \mathcal{P}^b] = \delta^{ab} \), and so we find that

\[ [\mathcal{P}_a, \mathcal{B}_b] = \delta_{ab} M, \]

(2.9.6)

thereby reproducing the bracket (2.9.1).

In the basis of (2.1.10) augmented with the central charge \( Z \), the Klein pair corresponding to the homogeneous Bargmann space \((\mathfrak{g}, \mathfrak{h})\) with \( \mathfrak{g} \) the Bargmann algebra and \( \mathfrak{h} = (L_{ab}, B_a, Z) \cong \text{iso}(d) \oplus \mathfrak{u}(1) \). Since (2.9.1) implies that \([\mathfrak{h}, \mathfrak{m}] \not\subset \mathfrak{m}\), this, like the lightcone LC in Table 2.1, is not a reductive. As such, the \( \mathfrak{h} \)-component of the Cartan connection no longer defines an Ehresmann connection, which in turn does not lead to an \( \mathfrak{h} \)-invariant affine connection. Nevertheless, the gauging procedure still works [6, 19, 201, 230]. As before, we consider an open \( U \subset M \) and introduce the Cartan connection \( \Lambda \in \Omega^1(U, \mathfrak{g}) \) given by

\[ \Lambda = \frac{1}{2} \omega^{ab} L_{ab} + \psi^a B_a + m Z + e^a P_a + \tau H, \]

(2.9.7)
with its associated curvature

\[ F = dA + \frac{1}{2} [A, A] = \frac{1}{2} \Omega^{ab} L_{ab} + \Psi^a B_a + \mathcal{M} Z + \Theta^a P_a + \Xi H , \]  

(2.9.8)

where now

\[ \Omega^{ab} = \Gamma^{ab} , \]
\[ \psi^a = d\psi^a + \omega^a_b \wedge \psi^b = d^a \psi^a , \]
\[ \mathcal{M} = dm + e^a \wedge \psi^a , \]
\[ \Theta^a = d^a e^a + \psi^a \wedge \tau , \]
\[ \Xi = d\tau . \]  

(2.9.9)

The intrinsic torsion of a Galilean structure is given by \(\Xi = d\tau\) [209]. Depending on the properties of \(d\tau\), there are three types of Newton–Cartan geometries as first identified in [33, 34]:

1. **Torsionless Newton–Cartan geometry** (NC): has vanishing intrinsic torsion, \(d\tau = 0\);

2. **Twistless torsional Newton–Cartan geometry** (TTNC): here, \(\tau\) is hypersurface ortho-

gonal, \(\tau \wedge d\tau = 0\);

3. **Torsional Newton–Cartan geometry** (TNC): the intrinsic torsion is unconstrained, and, in particular, satisfies \(\tau \wedge d\tau \neq 0\).

In a Cartan gauge \((U, A)\), the Cartan connection (2.9.7) becomes

\[ A_\mu = \frac{1}{2} \omega_\mu^{ab} L_{ab} + \psi_\mu^a B_a + m_\mu Z + e_\mu^a P_a + \tau_\mu H , \]  

(2.9.10)

where we assume that the \(g/h\)-component forms a coframe. As before, we write the dual frame as \((\nu^\mu, E^a_\mu)\), which satisfies the same relations as in (2.8.7). The invariant spacetime tensors for Newton–Cartan geometry are the same as those for Galilean geometries we worked out in Section 2.1.2: the \(h\)-invariant objects are here \(\eta \in (g/h)^*\) and \(b^2 \in \odot^2 m\), corresponding to the timelike coframe field \(\tau \in \Omega^1 (U)\) and \(h = \delta^{ab} E_a E_b \in \Gamma (\odot^2 T U)\) with components \(h^{\mu \gamma}\). Together, \((\tau, h)\) form a Galilean structure, but a Newton–Cartan geometry also includes the \(h\)-valued field \(m\).

Writing \(\Lambda = \frac{1}{2} \Lambda^{ab} L_{ab} + \lambda^a B_a + \sigma Z\), the \(h\)-gauge transformations of the (co)frame follow from (2.5.4) and the completeness relations (2.8.7)\(^{13}\)

\[ \delta \nu^\mu = \lambda^a E^\mu_a , \quad \delta e_\mu^a = \Lambda^a_b e_\mu^b - \tau_\mu \lambda^a , \quad \delta E^\mu_a = \Lambda^a_b E^\mu_b , \quad \delta m_\mu = \partial_\mu \sigma + e_\mu^a \lambda_a , \]  

(2.9.11)

\(^{13}\)Note that the boost transformations look off compared to works such as [19,234]. This boils down to the fact that the generators are defined slightly differently: if we redefine \(H \mapsto -H\) and \(L_{ab} \mapsto -L_{ab}\), we obtain their results. The same goes for the boost-invariant combinations in (2.9.12).
where we omitted those fields that are inert. It is useful to define the boost-invariant (but not $Z$-invariant) combinations

$$\hat{v}^\mu = v^\mu - h^{\mu\nu} v_\nu, \quad \bar{h}_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu + \tau_\mu v_\nu + \tau_\nu v_\mu, \quad (2.9.12)$$

which satisfy the relations

$$\hat{v}^\mu \tau_\mu = -1, \quad \hat{v}^\mu \bar{h}_{\mu\nu} = -2m_\nu + h^{\mu\rho} m_\mu m_\rho \tau_\nu. \quad (2.9.13)$$

Although the Bargmann Klein pair is non-reductive, one can still write down an affine “pseudo-connection”,\(^{14}\) which, while transforming correctly under general coordinate transformations, fails to be $h$-invariant. This connection and its $Z$-transformation are given by

$$\bar{\Gamma}_\rho^\mu_{\mu\nu} = -v^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\lambda} (\partial_\mu \bar{h}_{\nu\lambda} + \partial_\nu \bar{h}_{\mu\lambda} - \partial_\lambda \bar{h}_{\mu\nu}), \quad (2.9.14)$$

$$\delta_Z \bar{\Gamma}_\rho^\mu_{\mu\nu} = \frac{1}{2} (\partial_\lambda \nu \sigma + (\partial_\lambda)_{\mu\nu} \partial_\mu \sigma + (\partial_\lambda)_{\mu\nu} \partial_\mu \sigma).$$

Only when $d\tau = 0$ is $\bar{\Gamma}_\rho^\mu_{\mu\nu} h$-invariant, and in this case one can also solve $\omega^{ab}$ and $\psi^a$ in terms of $(\tau, e^a, m)$ by imposing the curvature constraints $\Theta^a = \mathcal{M} = 0$.\(^{201}\)

The conundrum of $Z$-non-invariance may, as proposed in \([236]\), be dealt with by just accepting the fact that there is no $Z$-invariant connection, and then organise the coupling to these geometries such that the full theory is $Z$-invariant.

It is interesting to note that formally cancelling the non-invariance by introducing a Stückelberg field $\chi$ which transforms as $\delta_Z \chi = \sigma$, as was done in \([19]\), leads to Aristotelian geometry, which we discuss in the next section.

Finally, we remark that one may obtain Newton–Cartan geometries by performing a null reduction \([201, 237, 238]\). Consider a $(d+2)$-dimensional Lorentzian geometry $(N, g)$ with a nowhere-vanishing null Killing vector field $\xi$. In adapted coordinates where $\xi = \partial_\mu$, we can write the metric on $N$ as

$$g = 2\tau(du + m) + h, \quad (2.9.15)$$

for $\tau, m \in \Omega^1(N)$ and $h \in \Gamma(\otimes^2 TN)$ satisfying $h(\xi, -) = 0$. Assuming that $\xi$ integrates to a one-parameter subgroup of isometries $\Gamma$ that acts smoothly on $N$, the quotient $M := N/\Gamma$ defines a projection $\pi: N \rightarrow M$. The $(d + 1)$-dimensional manifold $M$ then comes equipped with a Newton–Cartan structure $(\tau, m, h)$ inherited from the Lorentzian structure $g$ on $N$. The pair $(g, \xi)$ is known as a Bargmann structure and is in this sense equivalent to a Newton–Cartan structure.

### 2.10 Aristotelian geometry

There exists a class of Lie algebras that does not fit into the definition of a kinematical Lie algebra given in Definition 1: these are the Aristotelian Lie algebras, which do not include

\(^{14}\)One can, of course, write down an infinite family. These were classified in \([235]\), but (2.9.14) is “canonical” in that it arises from the Noether procedure \([234]\). It is also distinguished by only being linear in $m_\mu$.\(^{201}\)
boost symmetries. As such, an Aristotelian Lie algebra $\mathfrak{g}$ with $d$-dimensional spatial isotropy is defined to contain a Lie subalgebra $\mathfrak{r} \cong \mathfrak{so}(d)$, as a representation of which $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{V} \oplus \mathfrak{S}, \quad (2.10.1)$$

which, compared to Definition 1, contains only a single copy of the vector representation, corresponding to spatial translations. Aristotelian Lie algebras were classified in [208]. In the following, we will construct a Cartan geometry modelled on the static Aristotelian Lie algebra, for which we choose a basis $(L_{ab}, P_a, H)$, relative to which the Lie brackets read

\[
\begin{align*}
[L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc}, \\
[L_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b, \\
\end{align*}
\]

(2.10.2)

with all other brackets zero. We consider the Klein pair $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h} = \mathfrak{r} \cong \mathfrak{so}(3)$. This leads to a reductive split

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad (2.10.3)$$

where $\mathfrak{h} = \langle L_{ab} \rangle$ and the $(d + 1)$-dimensional Abelian ideal $\mathfrak{m}$ is generated by $(P_a, H)$. To gauge the Aristotelian algebra, we start by writing down the Cartan connection $A \in \Omega^1(U, \mathfrak{g})$

$$A = \frac{1}{2} \omega^{ab} L_{ab} + e^a P_a + \tau H. \quad (2.10.4)$$
The associated curvature $F = dA + \frac{1}{2}[A, A] \in \Omega^2(U, g)$ is

$$F = \frac{1}{2} \Omega^{ab} L_{ab} + \Theta^a P_a + \Xi H,$$

(2.10.5)

where, explicitly,

$$\Omega^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^{cb},$$
$$\Theta^a = de^a + \omega^{ab} \wedge e^b =: d\nabla e^a,$$
$$\Xi = d\tau,$$

(2.10.6)

where indices are lowered using $\delta_{ab}$. The Bianchi identity $dF + [A, F] = 0$ translates into the relations\(^{15}\)

$$d\nabla \Omega^{ab} = 0,$$
$$d\nabla \Theta^a = \Omega^c_{\ ab} \wedge e^c,$$

(2.10.7)

where $d\nabla \Omega^{ab} = d\Omega^{ab} + \omega^{ac} \wedge \Omega^{cb} + \omega^{bc} \wedge \Omega^{ac}$. Ongoing work aims to gauge the Aristotelian algebra using the prescription developed in [6] and identify the resulting gravity theory with Hořava–Lifshitz gravity [15].

In a Cartan gauge, the objects $(\tau_\mu, e_\mu^a)$ form a coframe, and, as above, we denote the corresponding frame by $(v_\mu, E_\mu^a)$, which again satisfy the relations (2.8.7).

The invariants, which are found using the methods of Section 2.1.2, comprise the fields\(^{16}\)

$$\tau_\mu, \quad h_{\mu\nu} = \delta_{ab} e_\mu^a e_\nu^b, \quad \nabla^\mu, \quad h^{\mu\nu} = \delta^{ab} E_\mu^a E_\nu^b.$$

(2.10.8)

In this way, an Aristotelian structure simultaneously admits a Galilean and a Carrollian structure, and we remark that Aristotelian geometries also admit Lorentzian and Riemannian invariants; explicitly, $\mp \tau_\mu \tau_\nu + h_{\mu\nu}$.

As a $G$-structure, an Aristotelian geometry has a structure group that preserves all the characteristic tensor fields, which corresponds to simultaneously preserving a Carrollian and a Galilean structure. This leads directly to the following explicit form of $G$ as a subgroup of $GL(d + 1, \mathbb{R})$:

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \middle| A \in O(d) \right\} \subset GL(d + 1, \mathbb{R}).$$

(2.10.9)

The intrinsic torsion of an Aristotelian $G$-structure is given by $d\tau + \epsilon_\nu h_{\mu\nu} [209]$; and, in this sense, combines the intrinsic torsions of Galilean and Carrollian $G$-structures. One can write down an affine connection whose torsion is given only by the intrinsic torsion and with respect to which all the geometric fields in (2.10.8) are parallel. This connection is given

---

\(^{15}\)The Bianchi identity $d\Xi = 0$ is trivial since $\Xi$ is an exact form.

\(^{16}\)Note that there are two $h$’s here, which are not inverses of each other: one $h \in \Gamma(\odot^2 T U)$ and one $h \in \Gamma(\odot^2 T^* U)$. In the literature that uses Aristotelian geometry, e.g., [2,4], these are distinguished only by the position of the spacetime indices. To facilitate comparison with these works, we will adopt the same convention.
by [4]

\[ \Gamma_{\mu\nu}^\rho = -v^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} (\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \frac{1}{2} h^{\rho\sigma} \tau_\nu \xi_\nu h_{\mu\sigma}, \] (2.10.10)

and its torsion is

\[ 2\Gamma_{[\mu,\lambda]}^\rho = -v^\rho (d\tau)^{\mu\lambda} - h^{\rho\sigma} \tau^{[\lambda} \xi^{\sigma]} h_{\mu]} - \frac{1}{2} c^2 J_{\mu\nu}, \] (2.10.11)

It would be interesting to investigate the rôle of this particular connection in a theory of dynamical Aristotelian geometry obtained using the methods of [6].

2.11 Type II Newton–Cartan geometry

In this section, we introduce the notion of type II torsional Newton–Cartan (TNC) geometry as developed in [22, 24, 239] using a systematic expansion in inverse powers of the speed of light \( c \) as developed in [240] (see also [241, 242]). Type II TNC geometry is the relevant geometry for nonrelativistic gravity as developed in [22–24, 239], and when the intrinsic torsion of the Galilean structure vanishes, type II TNC geometry reduces to Newton–Cartan geometry as developed in Section 2.9 when a certain field (which decouples from the action) is taken out of the description. In this context, Newton–Cartan geometry is sometimes referred to as “type I Newton–Cartan geometry” to distinguish it more clearly from “type II”.

2.11.1 Particle \( 1/c^2 \) expansions of the Poincaré algebra & their gauging

To match with the literature on type II TNC geometry, it is useful to change the basis for the Poincaré Lie algebra (2.1.7) by defining

\[ J_{ab} := -L_{ab}, \] and \( H := -H \), in terms of which the brackets of the Poincaré algebra become

\[
\begin{align*}
[J_{ab}, J_{cd}] & = \delta_{ac} J_{bd} - \delta_{bc} J_{ad} - \delta_{ad} J_{bc} + \delta_{bd} J_{ac}, \\
[J_{ab}, B_c] & = \delta_{ac} B_b - \delta_{bc} B_a, \\
[J_{ab}, P_c] & = \delta_{ac} P_b - \delta_{bc} P_a, \\
[P_a, B_b] & = \delta_{ab} H, \\
[B_a, B_b] & = -c^2 J_{ab}, \\
[H, B_a] & = c^2 P_a.
\end{align*}
\] (2.11.1)
Letting $X$ collectively denote the generators $(J_{ab}, P_a, B_a)$, we first formally expand the generators in powers of $1/c^2$ according to

$$X = \sum_{k \in \mathbb{N}_0} c^{-2k}X^{(2k)},$$

$$H = \sum_{k \in \mathbb{N}_0} c^{-2k}H^{(2k-2)}.$$  \hspace{1cm} (2.11.2)

We define the *level* of a generator to be the (even) integer in parentheses in the superscript; for example $P_1^{(10)}$ is a “level-10 generator”. Thus, in this way of labelling the generators, $H$ begins at level $-2$, while $X$ begins at level 0. This means that we obtain the following brackets (where $m, n \in \mathbb{N}_0$)

$$[J^{(2m)}_{ab}, P^{(2n)}_a] = 2\delta_{c[a}P^{(2m+2n)}_{b]} , \quad [H^{(2m-2)}, B^{(2n)}_a] = P^{(2m+2n)}_a , \quad (2.11.3a)$$

$$[P^{(2m)}_a, B^{(2n)}_b] = \delta_{ab}H^{(2m+2n)} , \quad [B^{(2m)}_a, J^{(2n)}_{bc}] = 2\delta_{a[b}B^{(2m+2n)}_{c]} , \quad (2.11.3b)$$

$$[B^{(2m)}_a, B^{(2n)}_b] = -J^{(2m+2n)}_{ab} , \quad [J^{(2m)}_{ab}, J^{(2n)}_{cd}] = 4\delta_{[b[c}J^{(2m+2n)}_{d]a]} . \quad (2.11.3c)$$

We will denote the $1/c^2$ expanded Poincaré algebra, which contains infinitely many generators, by $\text{iso}_{1/c^2}(d + 1, 1)$. Now, notice that for a given, fixed integer $n \geq 0$, the set of all generators of level $2k \geq 2n$ forms an (infinite dimensional) ideal in $\text{iso}_{1/c^2}(d + 1, 1)$ that we call $i_n$. The biggest ideal is $i_0$, and in general these ideals “sit inside each other” according to

$$i_0 \supset i_1 \supset i_2 \supset \ldots \quad (2.11.4)$$

This means that for each integer $n \geq 0$, we can form the quotient algebra

$$q_n = \text{iso}_{1/c^2}(d + 1, 1)/i_n , \quad (2.11.5)$$

which for each $n$ is an example of a kinematical Lie algebra in the sense of Definition 1. For $n = 0$, the only level $-2$ generator is $H^{(-2)}$, which generates an Abelian one-dimensional algebra, i.e., $q_0 \cong \mathbb{R}$. For $n = 1$, the algebra $q_1$ has the following nonzero brackets

$$[J^{(0)}_{ab}, P^{(0)}_c] = 2\delta_{c[a}P^{(0)}_{b]} , \quad [H^{(-2)}, B^{(0)}_a] = P^{(0)}_a , \quad (2.11.6a)$$

$$[P^{(0)}_a, B^{(0)}_b] = \delta_{ab}H^{(0)} , \quad [B^{(0)}_a, J^{(0)}_{bc}] = -2\delta_{a[b}B^{(0)}_{c]} , \quad (2.11.6b)$$

$$[J^{(0)}_{ab}, J^{(0)}_{cd}] = 4\delta_{[b[c}J^{(0)}_{d]a]} \quad (2.11.6c)$$

which we recognise as the Bargmann algebra with central charge $H$. This labelling is natural from the perspective of the Noether charges, as shown for the case of string theory in [7].

There is another way of assigning levels to the generators, introduced in [22, 24], which is

\[\text{...}^{17}\text{Although we leave them out, one could also include odd powers of }1/c.\text{ See [243] for a survey of such terms in the context of gravity.}\]
more natural from the perspective of the geometry. Writing instead

\[
X = \sum_{k \in \mathbb{N}_0} c^{-2k} X^{(2k)}, \\
H = \sum_{k \in \mathbb{N}_0} c^{-2k} H^{(2k)}.
\] (2.11.7)

which differs from (2.11.2) in that now all generators begin at level 0. With this assignment of levels, the brackets of the algebra iso\(_{1/c^2}(d + 1, 1)\) become

\[
[J_{ab}^{(2m)}, P_c^{(2n)}] = 2\delta_{c[a} P_{b]}^{(2m+2n)}, \quad [H^{(2m)}, B_a^{(2n)}] = P_a^{(2m+2n)},
\] (2.11.8a)

\[
[P_a^{(2m)}, B_b^{(2n)}] = \delta_{ab} H^{(2m+2n+2)}, \quad [B_a^{(2m)}, J_{bc}^{(2n)}] = -2\delta_{a[b} B_{c]}^{(2m+2n)},
\] (2.11.8b)

\[
[B_a^{(2m)}, B_b^{(2n)}] = -J_{ab}^{(2m+2n+2)}, \quad [J_{ab}^{(2m)}, J_{cd}^{(2n)}] = 4\delta_{[b|c} J_{d]a]}^{(2m+2n)}.
\] (2.11.8c)

Once again, all generators of a levels 2k > 2n (note the strict inequality) for a fixed integer \(n \geq 0\) form an ideal that we call \(J_n\), in terms of which we can form quotient algebras of the form

\[
w_n = \text{iso}_{1/c^2}(d + 1, 1)/ J_n,
\] (2.11.9)

which, like \(q_n\), is a kinematical Lie algebra (cf., Definition 1) for each \(n\). The first algebra \(w_0\) has the nonzero brackets

\[
[J_{ab}^{(0)}, P_c^{(0)}] = 2\delta_{c[a} P_{b]}^{(0)}, \quad [H^{(0)}, B_a^{(0)}] = P_a^{(0)},
\] (2.11.10a)

\[
[B_a^{(0)}, J_{bc}^{(0)}] = -2\delta_{a[b} B_{c]}^{(0)}, \quad [J_{ab}^{(0)}, J_{cd}^{(0)}] = 4\delta_{[b|c} J_{d]a]}^{(0)}
\] (2.11.10b)

which we recognise as the Galilean algebra (2.1.10). The algebra \(w_1\) has the following nonzero brackets

\[
[J_{ab}^{(0)}, P_c^{(0)}] = 2\delta_{c[a} P_{b]}^{(0)}, \quad [H^{(0)}, B_a^{(0)}] = P_a^{(0)},
\] (2.11.11a)

\[
[B_a^{(0)}, J_{bc}^{(0)}] = -2\delta_{a[b} B_{c]}^{(0)}, \quad [J_{ab}^{(0)}, J_{cd}^{(0)}] = 4\delta_{[b|c} J_{d]a]}^{(0)},
\] (2.11.11b)

\[
[P_a^{(0)}, B_b^{(0)}] = \delta_{ab} H^{(2)}, \quad [B_a^{(0)}, B_b^{(0)}] = -J_{ab}^{(2)},
\] (2.11.11c)

\[
[J_{ab}^{(2)}, P_c^{(0)}] = 2\delta_{c[a} P_{b]}^{(2)}, \quad [J_{ab}^{(0)}, P_c^{(2)}] = 2\delta_{c[a} P_{b]}^{(2)},
\] (2.11.11d)

\[
[B_a^{(2)}, J_{bc}^{(0)}] = -2\delta_{a[b} B_{c]}^{(2)}, \quad B_a^{(0)}, J_{bc}^{(2)} = -2\delta_{a[b} B_{c]}^{(2)},
\] (2.11.11e)

\[
[H^{(2)}, B_a^{(0)}] = P_a^{(2)}, \quad [H^{(0)}, B_a^{(2)}] = P_a^{(2)},
\] (2.11.11f)

\[
[J_{ab}^{(2)}, J_{cd}^{(2)}] = 4\delta_{[b|c} J_{d]a]}^{(2)}.
\] (2.11.11g)

This algebra is the “type II TNC algebra”, and we can obtain type II TNC geometry by gauging the Klein pair \(\langle w_1, h \rangle\), where the subalgebra \(h\) is

\[
h = \langle J_{ab}^{(0)}, J_{ab}^{(2)}, B_a^{(0)}, B_a^{(2)}, P_a^{(2)}, H^{(2)} \rangle.
\] (2.11.12)
Like the Bargmann Klein pair that leads to Newton–Cartan geometry, this is not a reductive split: for example, the bracket between \( P_\alpha^{(0)} \) and \( B_\beta^{(0)} \) returns \( H^{(2)} \in \mathfrak{h} \), which, in contrast to the Bargmann generator \( Z \), is not central. We can compute the invariants of low rank using the linear isotropy representation as in Section 2.1.2. These invariants live \( \mathfrak{w}_1/\mathfrak{h} \), \( (\mathfrak{w}_1/\mathfrak{h})^* \), \( \circ^2 \mathfrak{w}_1/\mathfrak{h} \) and \( \circ^2 (\mathfrak{w}_1/\mathfrak{h})^* \), and, introducing a dual basis \( (\pi^a, \eta^1) \) for \( (\mathfrak{w}_1/\mathfrak{h})^* \) as usual, it can be checked that the invariants form a Galilean structure: \( \eta \in (\mathfrak{w}_1/\mathfrak{h})^* \) and \( P^2 \in \circ^2 \mathfrak{w}_1/\mathfrak{h} \). We can write down a \( \mathfrak{w}_1 \)-valued Cartan connection

\[
A = \frac{1}{2} \Omega^{ab}_i (0) f^{i(0)}_{ab} + \frac{1}{2} \Omega^{ab}_i (2) f^{i(2)}_{ab} + \frac{1}{2} \Phi^a_i (0) B^a_i + \frac{1}{2} \Phi^a_i (2) B^a_i + 2 \pi^0 P^0_a + \frac{mH^{(2)}}{c^2} + e^a P^0_a + \tau H^{(0)},
\]

(2.11.13)

where we assume that the pair \((\tau, e^a)\) forms a coframe. In type II TNC geometry, the subleading fields that correspond to generators that appear in the expansion of \( H \) and \( P_a \), which form \( m \) for the Poincaré algebra, become gauge fields defined on the manifold in the same way as the field \( m_\mu \) in Newton–Cartan geometry. To pass to a second order formulation, one should find suitable curvature constraints that allow us to write the fields \((\omega^{ab}_0, \omega^{ab}_2, \psi^a_0, \psi^a_2)\) in terms of \((e^a, \tau, m, \pi^a)\) and the frame fields that form an inverse to \((\tau, e^a)\).

There is another way to describe type II TNC geometry, and, indeed, more general geometries, that starts from a \( 1/c^2 \) expansion of the Lorentzian metric. This is the topic of the next section.

### 2.11.2 Particle \( 1/c^2 \) expansion in the second order formulation

Consider a \((d + 1)\)-dimensional manifold \( M \) equipped with a Lorentzian metric \( g \). In local coordinates, we assume that the metric \( g_{\mu\nu} (\mu, \nu = 0, 1, \ldots, d) \) admits a split into timelike and spacelike components as follows

\[
g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu} ,
\]

(2.11.14)

with a similar relation holding for the inverse metric \( g^{\mu\nu} \),

\[
g^{\mu\nu} = -\frac{1}{c^2} T^\mu T^\nu + \Pi^{\mu\nu} .
\]

(2.11.15)

The relation \( g_{\mu\rho} g^{\rho\gamma} = \delta^\gamma_\nu \) implies that the geometric objects involved in the splits above satisfy

\[
T_\mu \Pi^{\mu\nu} = 0 = T^\mu \Pi_{\mu\nu} , \quad T_\mu T^\mu = -1 , \quad \Pi_{\mu\rho} \Pi^{\rho\nu} = \delta^\gamma_\nu + T^\gamma T_\mu .
\]

(2.11.16)

To turn this into a “non-relativistic” (NR) geometry, we formally expand in powers of \( 1/c^2 \). Note that concrete applications of this scheme requires the existence of a suitable characteristic velocity \( v_{\text{ch}} \ll c \) such that the formal \( 1/c^2 \) expansion turns into an expansion in the dimensionless parameter \( \epsilon = v_{\text{ch}}^2/c^2 \). In the case of non-relativistic approximations to gravity, the characteristic velocity in a fixed background is \( v_{\text{ch}} = \sqrt{Gm_{\text{ch}}/\ell_{\text{ch}}} \), where \( G \) is Newton’s constant and \( m_{\text{ch}} \) and \( \ell_{\text{ch}} \) denote a characteristic mass and length, respectively. In string theory, the characteristic velocity corresponds to the center of mass motion of a string.
winding a circle direction [3].

The geometric fields $T_\mu$ and $\Pi_{\mu\nu}$ are expanded in powers of $1/c^2$ as

\[
T_\mu = \tau_\mu + c^{-2}m_\mu + c^{-4}B_\mu + O(c^{-6}) ,
\]
\[
\Pi_{\mu\nu} = h_{\mu\nu} + c^{-2}\Phi_{\mu\nu} + c^{-4}Y_{\mu\nu} + O(c^{-6}) ,
\]  

(2.11.17)

where $\tau_\mu$ and $e^a_\mu$ form the coframe ($\tau_\mu$, $e^a_\mu$) that also appears in the $w_1$-valued Cartan connection (2.11.13), and where $h_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu$ and $\Phi_{\mu\nu} = \delta_{ab} (e^a_\mu \pi^b_\nu + e^b_\nu \pi^a_\mu)$. Note that the fields $B_\mu$ and $Y_{\mu\nu}$ do not form part of type II TNC geometry.

This means that the metric expands according to

\[
g_{\mu\nu} = -c^2 \tau_\mu \tau_\nu + \tilde{h}_{\mu\nu} + c^{-2} \tilde{\Phi}_{\mu\nu} ,
\]  

(2.11.18)

where

\[
\tilde{h}_{\mu\nu} = h_{\mu\nu} - 2\tau_{(\mu} m_{\nu)} ,
\]
\[
\tilde{\Phi}_{\mu\nu} = \Phi_{\mu\nu} - m_\mu m_\nu - 2B_{(\mu} \tau_{\nu)} .
\]  

(2.11.19a, 2.11.19b)

For the inverse structures, we have similar expansions

\[
T^\mu = v^\mu + c^{-2}X^\mu + c^{-4}Z^\mu + O(c^{-6}) ,
\]
\[
\Pi^{\mu\nu} = h^{\mu\nu} + c^{-2}p^{\mu\nu} + c^{-4}Q^{\mu\nu} + c^{-6}W^{\mu\nu} + O(c^{-8}) .
\]  

(2.11.20)

The relations (2.11.16) imply that the leading order (LO) fields satisfy

\[
v^\mu \tau_\mu = -1 ,
\]
\[
v^\mu h_{\mu\nu} = \tau_\mu h^{\mu\nu} = 0 ,
\]
\[
\delta^\mu = -v^\mu \tau_\nu + h^{\mu\rho} h_{\rho\nu} .
\]  

(2.11.21)

These relations furthermore imply that the subleading fields that appear in $T^\mu$ and $\Pi^{\mu\nu}$ are entirely determined by the subleading fields that appear in $T_\mu$ and $\Pi_{\mu\nu}$:

\[
X^\mu = -v^\mu \Phi - h^{\mu\rho} Y^{\nu} \Phi_{\nu\rho} ,
\]
\[
Z^\mu = v^\mu \Phi^2 - v^\mu m_\nu h^{\nu\rho} \Phi_{\rho\sigma} + v^\mu v^\nu B_\nu - h^{\mu\rho} v^{\nu} Y_{\nu\rho} + h^{\mu\rho} v^\nu \Phi \Phi_{\nu\rho} + h^{\mu\rho} h^{\nu\sigma} \Phi_{\sigma\lambda} \Phi_{\nu\rho} .
\]
\[
p^{\mu\nu} = 2v^{[\mu} h^{\nu]} \rho m_\rho - h^{\mu\rho} h^{\nu\sigma} \Phi_{\rho\sigma} ,
\]
\[
Q^{\mu\nu} = v^\mu v^\nu h^{\rho\sigma} m_\rho m_\sigma + h^{\mu\rho} h^{\nu\sigma} \Phi_{\sigma\lambda} h^{\rho\kappa} \Phi_{\rho\kappa} - h^{\mu\rho} h^{\nu\sigma} Y_{\rho\sigma} - 2h^{\mu\rho} v^\rho v^\nu \Phi_{\lambda (\rho \sigma)} .
\]  

(2.11.22)

\[
W^{\mu\nu} = v^\mu v^\nu \left[ 2h^{\rho\sigma} B_\rho m_\sigma - 2v_{(\mu} h^{\nu)} \rho m_\rho - 2v^{(\mu} h^{\nu)} \sigma h^{\rho\lambda} m_\rho \Phi_{\lambda \sigma} \right] + \text{terms we don't need} ,
\]

where we defined

\[
\Phi = -v^\mu m_\mu .
\]  

(2.11.23)

In deriving these expressions, we have used the fact that the LO relations (2.11.21) imply that
any contravariant 2-tensor $X^{\mu\nu}$ may be decomposed as

$$X^{\mu\nu} = \tau_\rho \tau_\lambda \nu^\mu \nu^\nu X^{\rho\lambda} + h^\mu_\sigma h^\nu_\rho h^\nu_\kappa h^\rho_\lambda X^{\rho\lambda} - 2\nu^{(\mu} h^{\nu)}\sigma h^\rho_\sigma \tau_\lambda X^{\rho\lambda}. \quad (2.11.24)$$

For the square root of the determinant of the metric, we write

$$\sqrt{-\det(g)} = cE, \quad (2.11.25)$$

where $E$ expands in powers of $1/c^2$ as

$$E = e \left( 1 + c^{-2} \left[ \Phi + \frac{1}{2} h^{\mu\nu} \Phi_{\mu\nu} \right] \right) + O(c^{-4}), \quad (2.11.26)$$

where $e = \sqrt{\det(\tau_\mu \tau_\nu + h_{\mu\nu})}$.

The subleading fields that appear in the metric $g_{\mu\nu}$ up to (and including) next-to-leading order (NLO) are $\tau_\mu$, $h_{\mu\nu}$, $m_\mu$, $\Phi_{\mu\nu}$, $B_\mu$. It is sometimes useful to decompose the spatial part of the metric $\Pi_{\mu\nu}$ in terms of vielbeins

$$\Pi_{\mu\nu} = \delta_{ab} \epsilon^a_\mu \epsilon^b_\nu, \quad (2.11.27)$$

where $a, b = 1, \ldots, d$ are spatial tangent space indices. The vielbeins have a $1/c^2$ expansion of the form

$$\epsilon^a_\mu = e^a_\mu + c^{-2} \pi^a_\mu + O(c^{-4}), \quad (2.11.28)$$

where the subleading components satisfy

$$h_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu, \quad \Phi_{\mu\nu} = 2\delta_{ab} e^a_\mu \pi^b_\nu, \quad (2.11.29)$$

which implies $\nu^{\mu\nu} \Phi_{\mu\nu} = 0$. We already met these relations below Eq. (2.11.17). The metric transforms under general coordinate transformations infinitesimally generated by a vector $\Xi^\mu$ as $\delta g_{\mu\nu} = \Xi^\mu g_{\nu} - g_{\mu} \Xi^\nu$. The generator $\Xi^\mu$ has a $1/c^2$ expansion of the form [24]

$$\Xi^\mu = \xi^\mu + c^{-2} \zeta^\mu + c^{-4} \kappa^\mu + O(c^{-6}). \quad (2.11.30)$$

In addition to general coordinate transformations, the vielbeins $\tau_\mu$ and $\epsilon^a_\mu$ transform under local Lorentz transformations ($\Lambda^a_\mu, c^{-1} \Lambda^a_\mu$) (see [24] for details), which expand as

$$\Lambda^a = \lambda^a + c^{-2} \eta^a + O(c^{-4}), \quad (2.11.31)$$

$$\Lambda^a_\mu = \lambda^a_\mu + c^{-2} \sigma^a_\mu + O(c^{-4}).$$
Combining all these transformations, we get
\[
\begin{align*}
\delta \tau_\mu &= \xi_\mu \tau_\mu, \\
\delta h_{\mu\nu} &= \xi_\mu h_{\mu\nu} + 2\lambda_\alpha e_\mu{}^{a} \tau_\nu, \\
\delta m_\mu &= \xi_\mu m_\mu + \xi_\alpha \tau_\mu + \lambda_\alpha e_\mu{}^{a}, \\
\delta \Phi_{\mu\nu} &= \xi_\mu \Phi_{\mu\nu} + \xi_\alpha h_{\mu\nu} + 2\lambda_\alpha e_\mu{}^{a} m_\nu + 2\lambda_\alpha \pi_\mu{}^{a} \tau_\nu + 2\eta_\alpha e_\mu{}^{a} \tau_\nu, \\
\delta B_\mu &= \xi_\mu B_\mu + \xi_\alpha m_\mu + \xi_\kappa \tau_\mu + \eta_\alpha e_\mu{}^{a} + \lambda_\alpha \pi_\mu{}^{a}.
\end{align*}
\]
(2.11.32)

Using the completeness relation
\[
e^\nu_a e_\mu{}^{a} - \nu^\nu \tau_\mu = \delta^\nu_\mu \text{ we can write } \lambda_\alpha \pi_\mu{}^{a} = \lambda_\alpha \pi_\nu{}^{a} e_\nu{}^{b} e_\mu{}^{b} - \lambda_\alpha \pi_\nu{}^{a} \nu^\nu \tau_\mu. \text{ This means that we can write for the combination}
\[
\eta_\alpha e_\mu{}^{a} + \lambda_\alpha \pi_\mu{}^{a} = \eta_\alpha e_\mu{}^{a} + \lambda_\alpha \pi_\nu{}^{a} e_\nu{}^{b} e_\mu{}^{b} - \lambda_\alpha \pi_\nu{}^{a} \nu^\nu \tau_\mu = \tilde{\eta}_\alpha e_\mu{}^{a} - \lambda_\rho h^{\rho\nu} \nu^\kappa \Phi_{\sigma \kappa} \tau_\mu,
\]
(2.11.33)

where \( \tilde{\eta}_\alpha = \eta_\alpha + \lambda_\beta \pi_\nu{}^{b} e_\nu{}^{a} \) and where we used
\[
\lambda_\rho h^{\rho\nu} \nu^\kappa \Phi_{\sigma \kappa} = \lambda_\alpha \pi_\rho{}^{a} \nu^\rho,
\]
(2.11.34)

which follows from (2.11.29) and the definition \( \lambda_\mu = \lambda_\alpha e_\mu{}^{a} \).

The field \( \Phi_{\mu\nu} \) decouples from the Lagrangian of nonrelativistic gravity when \( d\tau = 0 \) [22, 24]. In this case, the transformation of \( m_\mu \) above reduces to
\[
\delta m_\mu = \xi_\mu m_\mu + \partial_\mu (\tau_\nu \zeta^\nu) + \lambda_\alpha e_\mu{}^{a},
\]
(2.11.35)

which is the transformation of \( m_\mu \) in “type I TNC” that we found in (2.9.11) with gauge parameter \( \sigma = \tau_\nu \zeta^\nu \). Hence, omitting the field \( \Phi_{\mu\nu} \) we find the same field with the same transformation properties as in the Newton–Cartan geometry developed in Section 2.9. In this sense, type II TNC reduces to type I TNC when \( d\tau = 0 \).

### 2.12 String Newton–Cartan geometry

In Section 2.11 above, we considered the particle \( 1/c^2 \) expansions of Lorentzian geometries. At the level of the kinematical Lie algebras, we already observed below Definition 1 that there exist string and \( p \)-brane kinematical Lie algebras, where the rotation subalgebra is not \( so(d) \) but instead \( so(p, 1) \times so(d - p) \), where \( p = 1 \) for the string kinematical Lie algebras. At the level of the \( 1/c^2 \) expansions of the target space geometry, one can get string and, more generally, \( p \)-brane Newton–Cartan geometries by singling out a larger longitudinal subspace. In this Section, we perform a string \( 1/c^2 \) expansion of Lorentzian geometry following [3], which leads to geometries of string Newton–Cartan type. When truncating the expansion properly, this procedure produces the string Newton–Cartan geometry developed in [164] and used in the context of non-relativistic string theory in [72, 73, 75, 169]. We will put this geometry to good use in Section 5 when we discuss the \( 1/c^2 \) expansion of closed bosonic string theory.
Using the methods first developed in [22–24], we will now discuss how the string $1/c^2$ expansion of a $(d+2)$-dimensional Lorentzian geometry on a manifold $M$ leads to various notions of SNC geometry on $M$. First, in complete analogy with (2.11.14), we assume that the Lorentzian metric and its inverse can be written as

\[
G_{MN} = c^2 (-T^M_0 T^0_N + T^M_1 T^1_N) + \Pi^\perp_{MN} = c^2 \eta_{AB} T^A_M T^B_N + \Pi^\perp_{MN}, \\
G^{MN} = c^{-2} (-T^M_0 T^0_N + T^M_1 T^1_N) + \Pi^{\perp MN} = c^{-2} \eta^{AB} T^A_M T^B_N + \Pi^{\perp MN},
\]

(2.12.1)

where $M, N = 0, 1, \ldots, d+1$ are spacetime indices and $A, B = 0, 1$ are longitudinal two-dimensional tangent space indices. Here, $\eta_{AB}$ denotes the two-dimensional longitudinal Minkowski metric, $\eta_{AB} = \text{diag}(-1, 1)$. Thus, comparing this to what we did above in Section 2.11.2, this just singles out a two-dimensional Lorentzian subspace rather than just the time direction. The fields in the decompositions in (2.12.1) satisfy the relations

\[
T^A_M \Pi^\perp_{MN} = 0, \quad T^A_M T^B_N = \delta^B_A, \quad T^A_M \Pi^{\perp MN} = 0,
\]

(2.12.2)

which imply the completeness relation

\[
\delta^N_M = \Pi^\perp_{MN} \Pi^{\perp LN} + T^A_M T^N_A.
\]

(2.12.3)

The fields in the decompositions in (2.12.1) still depend on $c$, and we assume that they admit a Taylor expansion in $1/c^2$ of the form

\[
T^A_M = \tau^A_M + c^{-2} m^A_M + c^{-4} b^A_M + \mathcal{O}(c^{-6}), \\
\Pi^\perp_{MN} = H^\perp_{MN} + c^{-2} \phi^\perp_{MN} + \mathcal{O}(c^{-4}).
\]

(2.12.4)

Plugging these into the expression for $G_{MN}$ above, we get

\[
G_{MN} = c^2 \tau_{MN} + H_{MN} + c^{-2} \phi_{MN} + \mathcal{O}(c^{-4}),
\]

(2.12.5)

where

\[
\tau_{MN} = \eta_{AB} T^A_M T^B_N, \quad H_{MN} = H^\perp_{MN} + 2 \eta_{AB} \tau(M^A m^B_N), \\
\phi_{MN} = \phi^\perp_{MN} + \eta_{AB} m^A_M m^B_N + 2 \eta_{AB} \tau(M^A B^B_N).
\]

(2.12.6)

Taking the limit $c \to \infty$ leads to the SNC geometry of [164] and only leaves the fields $\tau^A_M$ and $H_{MN}$. The causal structure of SNC geometry can be found using the same considerations as we employed in Section 2.1. Coordinatising Minkowski space $\mathbb{M}^{d+2}$ with $x^M = (x^A, x^A')$ where $A = 0, 1$ label the longitudinal directions and $A' = 2, \ldots, d+1$ label the transverse directions, the lightcone is defined by $c^2(-x^0 x^0 + x^1 x^1) + \delta_{A'B'} x^A x^{B'} = 0$. In the limit $c \to \infty$, the lightcone opens up in the transverse directions leading to two $d$-dimensional “lightwedges” defined by $x^0 = x^1$ and $x^0 = -x^1$; see Figure 2.8.

In Section 5.2 we discuss additional details about the string $1/c^2$ expansion of Lorentzian geometry, including the gauge transformations of the fields. The set of fields $\{\tau^A_M, m^A_M, H^\perp_{MN}, \phi^\perp_{MN}\}$ that arises by truncating the expansions in (2.12.4) at order $c^{-2}$ gives rise a geometry that
forms the direct generalisation of “type II Newton–Cartan geometry” in [22–24], where a “particle $1/c^2$ expansion is performed which only singles out the time direction. For this reason, we dub the geometry defined by the set of fields $\{\tau_M^A, m_M^A, H_{MN}^\perp, \phi_{MN}^\perp\}$ “type II string Newton–Cartan geometry”, or type II SNC geometry for short. As will see in Section 5, the string theory that emerges at NLO couples to type II SNC geometry, but it does not couple to $\phi_{MN}^\perp$. The string theory at NNLO couples to the geometry defined by truncating the expansion (2.12.4) at order $c^{-4}$, but again it does not couple to the field that arises at this order in the expansion of $\Pi_{MN}^\perp$, and so we have refrained from writing it in (2.12.4). The NNLO string does, however, couple to $B_M^A$.

A natural condition to impose on the geometries that arise from the $1/c^2$ expansion is to demand that the LO longitudinal 1-forms $\tau_M^A$ give rise to a codimension-2 foliation, which by Frobenius’ theorem means that they satisfy
\[
d\tau^A = \alpha^A_B \wedge \tau^B,
\]
where the $\alpha^A_B$ are arbitrary 1-forms. As we will see in Section 5.3.4, this condition arises from the string $1/c^2$ expansion of Einstein’s equations.

If we remove the field $\phi_{MN}^\perp$ from the description, type II SNC geometry as constructed above reduces to (type I) SNC geometry if we impose the strong foliation constraint of [165,244]
\[
d\tau^A = \epsilon^A_B \omega \wedge \tau^B,
\]
corresponding to the special case $\alpha^A_B = \omega \epsilon^A_B$ for some 1-form defined by (2.12.8). When (2.12.8) holds, the transformation properties of the field $m_M^A$ reduce to those of type I SNC geometry (see Section 5.2 for details). In other words, when the strong foliation constraint is
Table 2.2: Overview of geometries of SNC type and how they arise from the string $1/c^2$ expansion of Lorentzian geometry. All of these may be subjected to foliation constraints: the most general such constraint is that $\alpha^A$ defines a co-dimensions-2 foliation, i.e., $d\alpha^A = \alpha^A_B \wedge \alpha^B$ for arbitrary 1-forms $\alpha^A_B$. Important special cases include $\alpha^A_A = 0$ and $\alpha^A_B = \epsilon^A_B \omega$ for some 1-form $\omega$. The latter is known as the strong foliation constraint. When the strong foliation constraint holds the gauge transformations of $\tau^A_M, H^A_{MN}, m^A_M$ for type II SNC agree with those of type I SNC.

satisfied, the data $(\tau^A_M, H^A_{MN}, m^A_M)$ describes SNC geometry. This is entirely analogous to the situation in torsional Newton–Cartan geometry [22–24], where type II TNC reduces to Newton–Cartan geometry when the clock form is exact and the field $\Phi$ is removed from the description. We summarise this in Table 2.2.
Chapter 3

A primer on fluid dynamics

Fluid dynamics is an effective long-range description of small fluctuations around thermo-
dynamic equilibrium of a given quantum many body system at finite temperature. Fluid
dynamics, or hydrodynamics, is a theory of locally conserved currents. As such, the first
step in setting up a hydrodynamic theory is to classify all the conserved Noether currents,
corresponding to whatever global symmetries the system under scrutiny is in possession of.
These symmetries fall into two classes: spacetime symmetries, like the ones captured by the
kinematical algebras listed in 2.1, and internal symmetries like a $U(1)$ symmetry, or, more
exotically, a dipole symmetry. A fluid is described by a small set of hydrodynamic variables,
and the conserved currents are expressed in terms of these hydrodynamic variables via the
constitutive relations, which are ordered in a derivative expansion. At any given order in
this derivative expansion, the constitutive relations include all possible tensor structures
allowed by the global symmetries, which are multiplied by undetermined functions of the
hydrodynamic variables; these are known as transport coefficients. Their precise form
must be derived from an underlying microscopic theory and cannot be determined within
the hydrodynamic framework. They are, however, constrained by the local second law of
thermodynamics and by the Onsager relations, which impose microscopic time-reversal
invariance.

In this chapter, we illustrate these concepts by developing the theory of relativistic fluids
up to first order in the gradient expansion, based on [8, 9, 134–138].

3.1 Relativistic fluids

3.1.1 Hydrodynamical variables & constitutive relations

Consider a $(d + 1)$-dimensional relativistic fluid that fills Minkowski space $\mathbb{M}^{d+1}$ (cf., Section
2.1). The spacetime symmetries of a relativistic fluid form the Poincaré group, and the
resulting conserved Noether current is captured by the energy-momentum tensor (EMT)
$T^\mu{}_{\nu} = T^{\mu\rho} \eta_{\rho\nu}$, where $T^{\mu\nu} = T^{\nu\mu}$. Its conservation amounts to the vanishing of the divergence

$$\partial_\mu T^\mu{}_{\nu} = 0.$$  

(3.1.1)
More precisely, $T_{\mu\nu}$ is the current associated with spacetime translations: $\nu = 0$ corresponds to the Noether current for time translations, while $\nu = i = 1, \ldots, d$ corresponds to the Noether current for a spatial translation in the $x^k$-direction. The conserved currents for Lorentz boosts and rotations are expressible in terms of the EMT as $x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}$. If there is also an “internal” global $U(1)$ symmetry, the fluid will have a conserved vector current $J^\mu$

$$\partial_\mu J^\mu = 0.$$  \hspace{1cm} (3.1.2)

The EMT $T_{\mu\nu}$ and the $U(1)$ current $J^\mu$ are expressed in terms of the hydrodynamic variables: these are the local temperature $T(x)$, local fluid velocity $u^\mu(x)$, satisfying $u^\mu u_\mu = -1$, and the local chemical potential $\mu(x)$. The origin of these hydrodynamic variables can be understood in the following way: consider a statistical mechanics system (either classical or quantum) in the grand canonical ensemble, which describes the possible states of the system in thermal equilibrium with a reservoir. The thermal equilibrium state around which we consider fluctuations is characterised by a density operator proportional to the exponential of the Noether charges

$$\rho = \frac{1}{Z} \exp(\beta^\mu P_\mu + \gamma Q), \hspace{1cm} (3.1.3)$$

where $P_\mu$ is the Noether charge for spacetime translations, and $Q$ the $U(1)$ Noether charge, and $\beta_\mu$ and $\gamma$ are constants associated to the equilibrium configuration. In this expression, $Z$ is the grand canonical partition function

$$Z = \text{Tr} \exp(\beta^\mu P_\mu + \gamma Q). \hspace{1cm} (3.1.4)$$

It is conventional to separate out the temperature dependence of the parameters $\beta_\mu$ and $\gamma$ as

$$\beta^\mu = \frac{1}{T} u^\mu, \quad \gamma = \frac{1}{T} \mu, \hspace{1cm} (3.1.5)$$

where $u^\mu u_\mu = -1$ is the fluid velocity, $T$ the temperature and $\mu$ the chemical potential. Thus, we may equivalently describe the equilibrium configuration in terms of $T$, $u^\mu$ and $\mu$. To consider fluctuations around the given equilibrium state, we consider slowly varying functions $T(x)$, $u^\mu(x)$ and $\mu(x)$ that perturb around the equilibrium configuration. These are the hydrodynamic (or fluid) variables. The fact that the hydrodynamic variables vary slowly by assumption implies that we can arrange the expression of the currents in a gradient expansion, where higher orders in derivatives are less important.

We now introduce the fluid projector

$$\Pi^{\mu\nu} := \eta^{\mu\nu} + u^\mu u^\nu, \hspace{1cm} (3.1.6)$$

which projects orthogonally to the fluid velocity, $\Pi^{\mu\nu} u_\nu = 0$. A decomposition originally due to Carl Eckhart [245] arranges the currents (without loss of generality) into components that are, respectively, transverse and parallel to the fluid flow

$$T^{\mu\nu} = \delta u^\mu u^\nu + \mathcal{P}^{\mu\nu} + 2q^{(\mu} u^{\nu)} + t^{\mu\nu},$$

$$J^\mu = N u^\mu + j^\mu,$$  \hspace{1cm} (3.1.7)
where
\[
\mathcal{E} = u_\mu u_\nu T^{\mu\nu}, \quad \mathcal{P} = \frac{1}{d}\Pi_{\mu\nu} T^{\mu\nu}, \\
\mathcal{N} = -u_\mu J^\mu, \quad q^\mu = -\Pi^{\mu}_{\nu\rho} T^{\nu\rho}, \\
t^{\mu\nu} = \frac{1}{2} \left( \Pi^{\mu}_{\rho\sigma} \Pi^{\nu\sigma} + \Pi^{\nu}_{\rho\sigma} \Pi^{\mu\sigma} - 2 \frac{d}{d} \Pi^{\mu\nu} \Pi_{\rho\sigma} \right) T^{\rho\sigma}, \quad j^\mu = \Pi^{\mu}_{\sigma} J^\sigma.
\]

Note that the tensorial objects \( j^\mu, q^\mu \) and \( t^{\mu\nu} \) are all transverse to the fluid velocity. The name of the game in hydrodynamics is then to express these quantities in terms of the hydrodynamic variables in a derivative expansion, where, e.g., quantities like \( \mathcal{E} \) will be functions of scalars like \( T, u_\mu \partial_\mu T \) etc.

### 3.1.2 Ideal hydrodynamics

At zeroth order in derivatives, which corresponds to ideal (or perfect) hydrodynamics, all the transverse tensors \( j^\mu, q^\mu \) and \( t^{\mu\nu} \) must vanish. This is because \( u^\mu \) is the only hydrodynamical variable with an index, and since these tensors are transverse to \( u^\mu \), they must necessarily be at least \( \mathcal{O}(\partial) \). This means that the constitutive relations at ideal order become
\[
T^{\mu\nu}_{(0)} = (\mathcal{E}(T, \mu) + P(T, \mu)) u^\mu u^\nu + P(T, \mu) \eta^{\mu\nu}, \\
J^\mu_{(0)} = \eta(T, \mu) u^\mu,
\]
where \( \mathcal{E}(T, \mu), P(T, \mu) \) and \( \eta(T, \mu) \) are slowly varying functions of the temperature and the chemical potential, but not their derivatives. The subscript "\((0)\)" is there to remind us that these expressions are at ideal order, i.e., zeroth order in derivatives.

To better understand the form of the constitutive relations of the ideal charged fluid above, which also illustrates the rationale behind Eckhart’s decomposition (3.1.7), we consider a relativistic fluid in equilibrium and at rest with (constant) energy density \( \mathcal{E} \), pressure \( P \) and charge density \( \eta \). Such a fluid has an EMT and charge current given by
\[
T^{\mu\nu}_{(0)} = \text{diag}(\mathcal{E}, P, \ldots, P), \quad J^\mu_{(0)} = (\eta, 0),
\]
which we can also express covariantly in terms of the rest frame fluid velocity \( u^\mu = (1, \vec{0}) \)
\[
T^{\mu\nu}_{(0)} = \mathcal{E} u^\mu u^\nu + P(\eta^{\mu\nu} + u^\mu u^\nu), \quad J^\mu = \eta u^\mu,
\]
which we can straightforwardly generalise to arbitrary \( u^\mu \). Thus, the constitutive relations (3.1.9) for the ideal charged hydrodynamics take the same form as the EMT and charge current for a perfect fluid, where the energy-density, pressure and charge density are slowly varying functions of the hydrodynamic variables \( T \) and \( \mu \). In thermodynamic equilibrium, they simply take their equilibrium values.

The expression \( P(T, \mu) \) for the pressure in terms of the temperature and the chemical potential is known as the equation of state. From this equation of state, one may determine
the entropy density $s$ and the charge density $n$ as

$$s = \left( \frac{\partial P}{\partial T} \right)_\mu, \quad n = \left( \frac{\partial P}{\partial \mu} \right)_T,$$

(3.1.12)

and these thermodynamic quantities satisfy the **Euler relation**

$$\mathcal{E} = -P + sT + \mu n,$$

(3.1.13)

and the variation of the equation of state gives rise to the **Gibbs–Duhem relation**

$$dP = sdT + nd\mu.$$

(3.1.14)

The ideal EMT and $U(1)$ current satisfy the hydrodynamical equations of motion

$$\partial_{\mu} T^{\mu\nu}_{(0)} = \partial_{\mu} J^{\mu}_{(0)} = 0,$$

(3.1.15)

and these relations will be used to simplify expressions that appear in first order hydrodynamics.

### 3.1.3 First order hydrodynamics & frame choices

First order hydrodynamics involves including first order derivatives of the hydrodynamic variables in the constitutive relations. At first order, the hydrodynamical variables are no longer uniquely defined: one may redefine $T(x)$, $u^\mu(x)$ and $\mu(x)$ by adding arbitrary gradient terms of the hydrodynamic variables, since these give rise to the same constant equilibrium value when the gradients vanish. This means that, up to first order in derivatives, the quantities $\mathcal{E}$, $P$ and $N$ that appear (3.1.7) have the form

$$\mathcal{E} = \mathcal{E}(T, \mu) + F_\mathcal{E}(\partial T, \partial u, \partial \mu),$$

$$P = P(T, \mu) + F_P(\partial T, \partial u, \partial \mu),$$

$$N = n(T, \mu) + F_N(\partial T, \partial u, \partial \mu),$$

(3.1.16)

where, as above, $\mathcal{E}$, $P$ and $n$ are determined by the equilibrium equation of state. The functions $F_\mathcal{E}$, $F_P$ and $F_N$ depend on the specific choice hydrodynamic variables: such a choice is known as a **choice of frame**, and changes between two sets of hydrodynamical variables are known as **frame transformations**. Unlike the hydrodynamic variables, the EMT $T^{\mu\nu}$ and the current $J^\mu$ originate (in principle) from a microscopic theory, and so their form (3.1.7) must remain invariant under frame transformations. A frame transformation can be written as

$$T \rightarrow T' = T + \delta T,$$

$$u^\mu \rightarrow u'^\mu = u^\mu + \delta u^\mu,$$

$$\mu \rightarrow \mu' = \mu + \delta \mu,$$

(3.1.17)
where $\delta T$, $\delta u^\mu$, and $\delta \mu$ are all first order in derivatives of the hydrodynamic variables, which justifies the notation “$\delta$”, since they behave as infinitesimal transformations. To preserve the normalisation $u^{\mu\nu}u^\nu_\mu = -1$, we must have that $u^{\mu\nu}\delta u^\mu = 0$. The requirement that the EMT and the current take the same form when expressed in the original frame and in the primed frame imposes additional constraints on $[\delta T, \delta u^\mu, \delta \mu]$. Working up first derivatives, we find that the conditions
\begin{equation}
\delta T^{\mu\nu} = T^{\prime\mu\nu} - T^{\mu\nu} = 0, \quad \delta J^\mu = J^{\prime\mu} - J^\mu = 0, \quad (3.1.18)
\end{equation}

imply that
\begin{equation}
\delta \mathcal{E} = \delta \mathcal{P} = \delta \mathcal{N} = \delta T^{\mu\nu} = 0, \quad (3.1.19a)
\end{equation}
\begin{equation}
\delta q^\mu = - (\mathcal{E} + \mathcal{P}) \delta u^\mu, \quad \delta j^\mu = - \mathcal{N} \delta u^\mu. \quad (3.1.19b)
\end{equation}

This suggests two special frames: we can either choose $\delta u^\mu$ such that $j^\mu = 0$, known as Eckart frame, or such that $q^\mu = 0$, which is known as Landau frame. In Eckart frame, there is no charge flow in the rest frame of the fluid, while in Landau frame, there is no energy flow in the fluid rest frame. We will mostly be using the Landau frame. Now, the freedom to fix $\delta T$ and $\delta \mu$ remains. The relations in (3.1.19a) imply, for example, that
\begin{equation}
\mathcal{E}(T, \mu) + F_\mathcal{E}(\partial T, \partial u, \partial \mu) = \mathcal{E}(T', \mu') + F_\mathcal{E}'(\partial T', \partial u', \partial \mu'), \quad (3.1.20)
\end{equation}
and so the transformed first-order functions that appear in (3.1.16) are independent of $\delta u^\mu$, i.e.,
\begin{align*}
F_\mathcal{E}' &= F_\mathcal{E} - \left(\frac{\partial \mathcal{E}}{\partial T}\right)_\mu \delta T - \left(\frac{\partial \mathcal{E}}{\partial \mu}\right)_T \delta \mu, \\
F_P' &= F_P - \left(\frac{\partial P}{\partial T}\right)_\mu \delta T - \left(\frac{\partial P}{\partial \mu}\right)_T \delta \mu, \\
F_N' &= F_N - \left(\frac{\partial n}{\partial T}\right)_\mu \delta T - \left(\frac{\partial n}{\partial \mu}\right)_T \delta \mu.
\end{align*}

Thus, we may choose $\delta T$ and $\delta \mu$ such that two of the functions that appear in (3.1.16) are zero, and one conventionally chooses
\begin{equation}
F_\mathcal{E}' = F_N' = 0, \quad (3.1.22)
\end{equation}
such that
\begin{equation}
\mathcal{E} = \mathcal{E}, \quad \mathcal{N} = n. \quad (3.1.23)
\end{equation}

We emphasise that there are infinitely many other hydrodynamical frames that one might choose to work with instead, and sometimes it is more natural to not work in, say, Landau frame. In particular, the geometric approach to fluid dynamics, to be discussed in Section (3.2), comes with its own frame choice at first order, something we will explicitly encounter in Section 4.3.3.
We now work out the constitutive relations in Landau frame for the charged relativistic fluid at first order in derivatives. In Landau frame, we have that $\mathcal{E} = \mathcal{E}$ and $N = n$, as well as $q^\mu = 0$. Hence, according to Eckart’s decomposition (3.1.7), we first need to classify the scalars, transverse vectors and transverse traceless symmetric tensors that we may build from first derivatives of the hydrodynamic variables. These are then used to construct the expressions up to first order in derivatives of the quantities $\mathcal{P}$, $t^{\mu\nu}$ and $j^\mu$. One can construct the following scalars at $O(\delta)$ (note that $\Pi^{\mu\nu} \partial_\mu u_\nu = \partial_\mu u^\mu$)

$$u^\rho \partial_\rho T, \quad u^\rho \partial_\mu u_\rho, \quad \partial_\rho u^\rho, \quad (3.1.24)$$

and the following transverse vectors

$$\Pi^{\mu\nu} \partial_\nu T, \quad \Pi^{\mu\nu} \partial_\mu \mu, \quad \Pi^{\mu}_\nu u^\rho \partial_\rho u^\nu \quad (3.1.25)$$

and a single transverse traceless symmetric tensor

$$\sigma^{\mu\nu} = \Pi^{\mu\rho} \Pi^{\nu\sigma} \left( \partial_\rho u_\sigma + \partial_\sigma u_\rho - \frac{2}{d} \eta_{\rho\sigma} \partial_\lambda u^\lambda \right). \quad (3.1.26)$$

However, this data is not independent: the zeroth order hydrodynamic equations of motion (3.1.1) and (3.1.2) relate some of these quantities. For the scalars, the equations $u_\mu \partial_\nu T^\mu_{(0)} = \partial_\mu J^\mu_{(0)}$, where the subscript $(0)$ denotes the ideal EMT and $U(1)$ current, imply that only one of the three scalars in (3.1.24) is independent, and we choose the scalar to be $\partial_\rho u^\rho$. Similarly, the transverse equation of motion $\Pi_{\mu\nu} \partial_\rho T^{\rho\nu}$ tells us that there are only two independent vectors in (3.1.25), and we choose to eliminate $\Pi_{\mu}^\nu u^\rho \partial_\rho u^\nu$. This means that the expressions for $\mathcal{P}$, $j^\mu$ and $t^{\mu\nu}$ to $O(\delta)$ can be written as

$$\mathcal{P} = P(T, \mu) - \xi(T, \mu) \partial_\rho u^\rho + O(\delta^2),$$

$$j^\mu = -\sigma(T, \mu) T^{\mu\nu} \partial_\nu (\mu/T) + \chi_T(T, \mu) \Pi^{\mu\nu} \partial_\nu T + O(\delta^2),$$

$$t^{\mu\nu} = -\eta(T, \mu) \sigma^{\mu\nu} + O(\delta^2), \quad (3.1.27)$$

where we introduced a number of transport coefficients: $\xi$ is known as the bulk viscosity, $\sigma$ is the charge conductivity, $\chi_T$ turns out, as we shall see, to be zero, and $\eta$ is the shear viscosity. These cannot be determined within the hydrodynamic framework and must be derived from an underlying microscopic theory. Note also that it is conventional to consider derivatives of $\mu/T$ rather than just $\mu$. Thus, we have found the constitutive relations of the relativistic fluid up to first order in Landau frame to be

$$T^{\mu\nu} = \xi(T, \mu) u^\mu u^\nu + P(T, \mu) \Pi^{\mu\nu} - \xi(T, \mu) \Pi^{\mu\nu} \partial_\rho u^\rho - \eta(T, \mu) \sigma^{\mu\nu} + O(\delta^2),$$

$$J^\mu = n(T, \mu) u^\mu - \sigma(T, \mu) T \Pi^{\mu\nu} \partial_\nu (\mu/T) + \chi_T(T, \mu) \Pi^{\mu\nu} \partial_\nu T + O(\delta^2). \quad (3.1.28)$$

The transport coefficients appearing in the constitutive relations are subject to additional constraints: they must be such that the local second law of thermodynamics is satisfied, which we will discuss in the next section. In addition, they must satisfy the Onsager relations,
which implement time-reversal symmetry at the level of the transport coefficients [246, 247]. The Onsager relations will be discussed in Section 4.4.2.

### 3.1.4 The entropy current and the second law of thermodynamics

A very important ingredient in hydrodynamics is the second law of thermodynamics, which states that there exists an entropy current \( S^\mu \) such that

\[
\partial_\mu S^\mu \geq 0 .
\] (3.1.29)

Note that this equation is *on-shell*; i.e., this only holds when the hydrodynamic equations (3.1.1) and (3.1.2) are satisfied. This will be important later. Up to first order, this entropy current takes the form

\[
S^\mu = su^\mu + S^\mu_{(1)} + \mathcal{O}(\partial^2) ,
\] (3.1.30)

where \( s \) is the entropy density of the ideal fluid we met in Section 3.1.2, and \( S^\mu_{(1)} \) is the most general first order current that is constructed from the fluid variables. The perfect fluid is *non-dissipative* (or isentropic): it does not produce entropy

\[
\partial_\mu (su^\mu) = 0 .
\] (3.1.31)

To derive this, we use the ideal hydrodynamic equations of motion. With \( T^{\mu\nu} \) and \( J^\mu \) given by (3.1.9), the projection \( u_\nu \partial_\mu T^{\mu\nu} = 0 \) tells us that

\[
u^\mu \partial_\mu P = \partial_\mu ((E + P) u^\mu) = \partial_\mu ((sT + \mu n) u^\mu),
\] (3.1.32)

where we used the Euler relation (3.1.13). Using the Gibbs–Duhem relation (3.1.14), we can rewrite the left-hand side as

\[
u^\mu \partial_\mu P = su^\mu \partial_\mu T + nu^\mu \partial_\mu \mu,
\] (3.1.33)

and so, using the equation of motion \( \partial_\mu (nu^\mu) = 0 \), the parallel projection of the conservation of the EMT (3.1.32) reduces to \( \partial_\mu (su^\mu) = 0 \).

In general, the entropy current \( S^\mu \) consists of two pieces

\[
S^\mu = S_{\text{can}}^\mu + S_{\text{non}}^\mu ,
\] (3.1.34)

where \( S_{\text{can}}^\mu \) is the **canonical entropy current**, whose form is obtained by covariantising the

\footnote{More generally, this has to be true when coupling the fluid to arbitrary curved backgrounds. This imposes additional constraints on the transport coefficients; see [138]. As a concrete example, this allows us one to show that the non-canonical entropy current that appears in (3.1.34) vanishes at first order.}
Euler relation (3.1.13)

\[ S^\mu_{\text{can}} = \frac{1}{T}P^\mu - \frac{1}{T}T^\mu\nu u_\nu - \frac{1}{T}J^\mu = su^\mu + (E' - \mathcal{E})\beta^\mu - \mu(N - n)\beta^\mu + \frac{1}{T}q^\mu - \frac{1}{T}J^\mu = su^\mu + (T^\mu\nu - T^\mu_\nu(0))\beta_\nu - \frac{1}{T}(J^\mu - J^\mu(0)), \]  

(3.1.35)

where we used the Eckart decomposition (3.1.7) as well as the perfect fluid EMT and \( u^\mu \) current. We also used the Euler relation (3.1.13) to rewrite \( \rho + \mathcal{E} = sT + \mu n \). We also defined the thermal vector

\[ \beta^\mu := \frac{u^\mu}{T}. \]  

(3.1.36)

The canonical entropy current \( S^\mu_{\text{can}} \) as written above is frame invariant, but simplifies when going to, e.g., Landau frame. The non-canonical entropy current \( S^\mu_{\text{non}} \) contains “the rest”, something we will discuss further in Section 4.2. For the charged relativistic fluid we consider here, it turns out that the non-canonical entropy current at first order is zero [138].

Using the expression for the canonical entropy current in (3.1.35), we may use the equations of motion (3.1.1) and (3.1.2) to write

\[ \partial_\mu S^\mu = \partial_\mu (P\beta^\mu) - T^\mu\nu \partial_\mu \beta_\nu - J^\mu \partial_\mu (\mu/T) = (T^\mu_\nu(0) - T^\mu_\nu)\partial_\mu \beta_\nu + (J^\mu - J^\mu(0))\partial_\mu (\mu/T), \]  

(3.1.37)

where we used that the perfect fluid is non-dissipative. It is amusing to note that this can also be written as

\[ \partial_\mu S^\mu = \frac{1}{2}(T^\mu_\nu(0) - T^\mu_\nu)\xi_{\mu\nu} + (J^\mu - J^\mu(0))\partial_\mu (\mu/T), \]  

(3.1.38)

which is relevant for our considerations in Section 4.2. In Landau frame and to first order, the right-hand side of (3.1.37) becomes a sum of quadratic forms (recall that the perfect fluid has an identically conserved entropy current) — one quadratic form for each type of structure, scalar, vector and tensor — in the first derivative data, which in each case must be positive semidefinite

\[ \partial_\mu S^\mu = -T^\mu_\nu(1)\partial_\mu \beta_\nu - J^\mu(1)\partial_\mu (\mu/T) = -\frac{1}{T}T^\mu_\nu(1)\partial_\mu u_\nu - J^\mu(1)\partial_\mu (\mu/T) = \xi_{\mu\rho}(s)T^\rho(3x3)^{3x3}_A(3x3)B(s) + \eta_{\rho\sigma}(v)\partial_{\rho T}A^{(v)}(3x3)B(\nu)(3x3) + \eta_TA^{(t)} \geq 0, \]  

(3.1.39)

where we used that in Landau frame, \( T^\mu_\nu(1)u_\nu = 0 \), and where, in order to explicitly write the divergence of the entropy current as a quadratic form, we introduced the following basis vectors of scalars and vectors (cf., (3.1.24) and (3.1.25))

\[ b^{(s)} = \begin{pmatrix} \partial_\mu u^\mu \\ u^\mu \partial_\mu (\mu/T) \\ u^\mu D_\mu T \end{pmatrix}, \quad b^{(v)} = \begin{pmatrix} u^\nu \partial_\nu u^\mu \\ \Pi^\nu_\mu \partial_\nu T \\ \Pi^\nu_\nu \partial_\nu (\mu/T) \end{pmatrix}, \]  

(3.1.40)
where the matrices that appear in (3.1.39) take the form
\[
A^{(s)}_{(3 \times 3)} = \frac{1}{T} \begin{pmatrix}
\zeta & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A^{(v)}_{(3 \times 3)} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\chi_T/2 \\
0 & T\sigma & 0
\end{pmatrix},
\]
while the tensor structure can be written as
\[
A^{(t)} = \frac{1}{4} \left( \Pi^{\mu \nu} \Pi^{\nu \sigma} + \Pi^{\mu \sigma} \Pi^{\nu \rho} - \frac{2}{d} \Pi^{\mu \nu} \Pi^{\rho \sigma} \right) \left( \partial_{\mu} u_{\nu} + \partial_{\nu} u_{\mu} \right) \left( \partial_{\rho} u_{\sigma} + \partial_{\sigma} u_{\rho} \right).
\]

Having exhibited the divergence of the entropy current (3.1.39) as a sum of quadratic forms, we may now apply Sylvester’s criterion to determine the constraints imposed on the transport coefficients by demanding that each of the quadratic forms is positive semidefinite. Sylvester’s criterion tells us that

**Theorem 1 (Sylvester’s criterion).** A matrix \( A \in \text{Mat}(n, \mathbb{R}) \) is positive (resp. negative) semidefinite if and only if all its principal minors have nonnegative (resp. nonpositive) determinants. A principal minor is obtained by removing the same columns and rows; for example, a principle minor of an \( n \times n \) matrix may be obtained by removing, say, columns 1 and 2 and rows 1 and 2.

Using Sylvester’s criterion for the quadratic forms (3.1.41) that appear in (3.1.39), we find the following conditions on the transport coefficients
\[
\eta \geq 0, \quad \zeta \geq 0, \quad \sigma \geq 0, \quad \chi_T = 0.
\]

We will discuss the entropy current further in Section 4.2.

### 3.2 Lagrangian description of non-dissipative transport

Until now, we’ve been describing the approach to fluids that has been around for nearly a hundred years; baby’s first fluids, if you like. Building on earlier works on the “hydrostatic partition function” [136, 137], it was shown in [134, 135] that (most of) the non-dissipative sector of transport generically admits a Lagrangian description, in a way to be made precise below. Moreover, a remarkable result by Sayantani Bhattacharyya [248, 249] shows that, at least for relativistic fluids, a hydrostatic analysis constrains all potentially entropy-destroying terms to vanish, and there are no further constraints beyond those that arise from hydrostatics. Of the terms that do produce entropy, i.e., the dissipative terms, only the ones that appear at leading order in the gradient expansion are constrained to be sign-definite. This Lagrangian approach provides nifty way of capturing both the hydrodynamical equations of motion and the non-dissipative transport properties of a fluid system using variational techniques. In this section, we develop the general theory behind this Lagrangian approach, we illustrate its construction for an ideal charged relativistic fluid. This section will serve as a warm-up for Section 4.3, where we apply this formalism to fluids without boost symmetry.
3.2.1 The hydrostatic partition function

The power of the Lagrangian approach comes from geometrising the hydrodynamic variables. A hydrostatic (or equilibrium) configuration on a background \((g_{\mu\nu}, A_\mu)\), where \(g_{\mu\nu}\) is a pseudo-Riemannian metric and \(A_\mu\) a 1-form gauge field, is characterised by a timelike Killing vector \(k^\mu\) and a \(U(1)\) parameter such that the Killing equations are satisfied [250]. To write down a hydrostatic partition function on a Lorentzian background consisting of a metric \(g_{\mu\nu}\) and an Abelian gauge field \(A_\mu\), we assume the existence of a set consisting of a Killing vector \(k^\mu\) and a “Killing gauge transformation” \(\Lambda^\mu\) such that

\[
\begin{align*}
\delta_k g_{\mu\nu} &= \xi_k g_{\mu\nu} = 0, \\
\delta_k A_\mu &= \xi_k A_\mu + \partial_\mu \Lambda^\mu = 0, \\
\end{align*}
\]  
(3.2.1)

where we collectively denoted the symmetry parameters as \(\mathfrak{K} = (k^\mu, \Lambda^\mu)\). These symmetry parameters are part of the background data, and, as we will see, defines the hydrodynamic variables. If we change our description of the background by performing either a general coordinate transformation infinitesimally parametrised by \(\xi^\mu\) or a \(U(1)\) gauge transformation \(\lambda\), the symmetry parameters themselves must change in such a way that (3.2.1) remains true in the new description. In other words, denoting an infinitesimal gauge transformation (consisting of \(\xi^\mu\) and \(\lambda\)) by “\(\delta\)”, we must have that

\[
\delta(\delta_k g_{\mu\nu}) = \delta(\delta_k A_\mu) = 0. 
\]  
(3.2.2)

For the metric, which only transforms under general coordinate transformations, this means that

\[
\begin{align*}
\delta(\delta_k g_{\mu\nu}) &= \xi_{\delta k} g_{\mu\nu} + \xi_k (\xi_{\delta} g_{\mu\nu}) \\
&= \xi_{\delta k} g_{\mu\nu} + [\xi_k, [\xi_{\delta} g_{\mu\nu} \\
&= \xi_{\delta k} g_{\mu\nu} - \xi_{[\xi_{\delta}, \xi_k]} g_{\mu\nu} \\
&= \xi_{(\delta k - [\xi_{\delta}, \xi_k])} g_{\mu\nu} \\
&= 0 ,
\end{align*}
\]  
(3.2.3)

where in the second equality, we used (3.2.1). This allows us to conclude that

\[
\delta k^\mu = ([\xi_{\delta}, k])^\mu = \xi_{\delta} k^\mu . 
\]  
(3.2.4)

In writing the above, we have used the Killing conditions (3.2.1) and the identity

\[
[\xi_X, \xi_Y]T = \xi_{[X, Y]} T ,
\]  
(3.2.5)

which is true for any tensor \(T\) and any vectors \(X\) and \(Y\). Repeating this exercise for \(A_\mu\), we find that

\[
\delta \Lambda^\mu = \xi_{\delta} \Lambda^\mu - \xi_k \lambda .
\]  
(3.2.6)
On such backgrounds, we can write down the Euclideanised grand-canonical partition function \( Z \), the logarithm of which gives an expression for the free energy. What we call the hydrostatic partition function is the quantity which, when Wick rotated, gives rise to this free energy, and it contains all gauge and diffeomorphism invariant quantities that we can build from the data \( \{ g_{\mu \nu}, A_\mu, k^\mu, \Lambda^k \} \).

The next step in the construction of the hydrostatic partition function involves identifying the fluid velocity and the local temperature with the geometric data. This is done as follows: we first identify the affine parameter \( \lambda \) along integral curves of \( k^\mu \) with “time”, which we analytically continue to “Euclidean time” \( \lambda \to -i \lambda^E \), which we then compactify on the “thermal circle” by identifying \( \lambda^E \sim \lambda^E + \frac{1}{T} \), where \( T \) is the temperature.

The invariant length \( L \) of the time circle is then given
\[
L = \frac{1}{T} \int_\gamma \left| \dot{\gamma}(\lambda^E) \right| d\lambda^E, \tag{3.2.7}
\]
where \( \gamma \) is an integral curve of the timelike Killing vector \( k \), i.e., \( \dot{\gamma}(\lambda^E) = k(\gamma(\lambda^E)) \). But since \( k \) is Killing, the integrand is constant along integral curves of \( k^\mu \), which means that we can write
\[
L = \frac{1}{T} \sqrt{-k^2}, \tag{3.2.8}
\]
where \( L = T_0 \) is the “global temperature”. In addition to the temperature, the Killing vector allows us to define the fluid velocity \( u^\mu \), which satisfies \( g_{\mu \nu} u^\mu u^\nu = -1 \), as
\[
u^\mu = \frac{k^\mu}{\sqrt{-k^2}}. \tag{3.2.9}
\]

Other gauge-invariant quantities that we can construct with the data at hand involves the chemical potential, which can be written as
\[
\mu = T(\Lambda^k + k^\mu A_\mu). \tag{3.2.10}
\]
Note that the identification of the gauge-invariant quantity above with the chemical potential is not immediately obvious, but will follow a posteriori. We will elaborate further on the hydrostatic partition function when we discuss its construction for boost-agnostic fluids in Section 4.3.1.

### 3.2.2 Action for non-dissipative transport

To construct an action that works off-shell and off-equilibrium, we no longer require the symmetry parameters to be Killing. To emphasise this, we denote them by \( \chi = (\beta^\mu, \Lambda^k) \),

---

2This follows from the fact that the Lie derivative, which we can identify with the directional derivative, along \( k \) of the scalar integrand vanishes.
which generate reparameterisations and gauge transformations, respectively

\[
\begin{align*}
\delta_X g_{\mu\nu} &= \xi_\beta g_{\mu\nu}, \\
\delta_X A_\mu &= \xi_\beta A_\mu + \partial_\mu \Lambda^\beta,
\end{align*}
\]

(3.2.11)

where the symmetry parameters transform in the same way as \( k^\mu \) and \( \Lambda^\beta \) under arbitrary reparameterisations \( \xi^\mu \) and gauge transformations \( \lambda \), which we collectively denote \( \delta_\Xi \)

\[
\begin{align*}
\delta_\Xi \beta^\mu &= \xi_\xi \beta^\mu, \\
\delta_\Xi \Lambda^\beta &= \xi_\xi \Lambda^\beta - \xi_\beta \lambda.
\end{align*}
\]

(3.2.12)

In the language of [134,135], \( \beta^\mu \) is known as the **thermal vector**, and \( \Lambda^\beta \) is the **thermal twist**.

The hydrostatic partition function in the more general Lagrangian sense can be written as

\[
S = \int d^{d+1}x \sqrt{-g} \mathcal{L}(g_{\mu\nu}, A_\mu, \beta^\mu, \Lambda^\beta),
\]

(3.2.13)

where the “Lagrangian” \( \mathcal{L} \) has a derivative expansion of the form

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \cdots,
\]

(3.2.14)

where the subscript denotes the order in derivatives. In this way, \( \mathcal{L}_0 \) consists of an arbitrary function of the zero derivative scalars \( T \) and \( \mu \), which are defined in terms of the background geometry and \( \mathcal{X} \). This arbitrary function may, a posteriori, be identified with the equation of state, so that \( \mathcal{L}_0 = P(T, \mu) \), something we will explicitly see below in Section 3.2.3. At first order in derivatives, we must identify all independent\(^3\) scalars that are first order in derivatives. Each of these scalars is multiplied by an arbitrary function of the hydrodynamic variables \( T \) and \( \mu \), and these functions will be identified with the set of non-dissipative transport coefficients. When \( \mathcal{X} \) satisfies the Killing equations (3.2.1), some of these scalars will be related leaving a smaller set of independent scalars: the independent scalars when the Killing equations hold give rise to **hydrostatic non-dissipative transport**. The remaining scalars are responsible for **non-hydrostatic non-dissipative transport**. We will explicitly see this in Section 4.3.

The variation of the fluid functional (3.2.13) can be written as

\[
\delta S = \int d^{d+1}x \sqrt{-g} \left( \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \delta A_\mu + G_\mu \delta \beta^\mu + H \delta \Lambda^\beta \right).
\]

(3.2.15)

This gives rise to the following reparameterisation and \( U(1) \) Ward identities

\[
\begin{align*}
\nabla_\mu T^{\mu\nu} &= F_{\nu\mu} J^\mu - A_\mu (\nabla_\nu J^\mu) + G_\mu \nabla_\nu \beta^\mu + \nabla_\mu (\beta^\mu G_\nu) + H \partial_\nu \Lambda^\beta, \\
\nabla_\mu J^\mu &= \nabla_\mu (H \beta^\mu).
\end{align*}
\]

(3.2.16)

These are true off-shell, but they are not the hydrodynamic equations of motion, which would

\(^3\)Some scalars may be related via integration by parts.
\[ \nabla_\mu T^{\mu\nu} = F_{\nu\mu} J^\mu, \]
\[ \nabla_\mu J^\mu = 0. \]
(3.2.17)

As demonstrated in [135], we must introduce a \textit{constrained} variational principle to get these hydrodynamic equations of motion. Such a constrained variational principle is constructed by defining a variation \( \delta \) that acts as
\[ \delta \beta^\mu = \delta \Xi \beta^\mu, \]
\[ \delta \Lambda^\beta = \delta \Xi \Lambda^\beta, \]
\[ \delta g_{\mu\nu} = \delta A_\mu = 0, \]
(3.2.18)
and then define on-shell hydrodynamic configurations as those that satisfy \( \delta S = 0 \).

In practice, this means the following: we obtain the energy-momentum tensor and the \( U(1) \) current by varying \( g_{\mu\nu} \) and \( A_\mu \), respectively, while keeping \( \beta^\mu \) and \( \Lambda^\beta \) fixed. Furthermore, we obtain the hydrodynamic equations of motion, i.e., the on-shell conservation equations, by varying \( g_{\mu\nu} \) and \( A_\mu \) under diffeomorphisms and \( U(1) \) gauge transformations, while, again, keeping \( \beta^\mu \) and \( \Lambda^\beta \) fixed (this follows from the fact that on-shell configurations must satisfy \( \delta S = 0 \)). This formulation will be put to good use in Section 4.3, where we use it to construct the non-dissipative sector of boost-agnostic fluids.

### 3.2.3 Lagrangian description of the ideal fluid

To see the Lagrangian approach in action in the simplest case, let us now reproduce the results for the charged ideal fluid we obtained in Section 3.1.2. As discussed above, the zeroth order fluid action is
\[ S = \int d^{d+1}x \sqrt{-g} \ (T, \mu). \]
(3.2.19)

The variation of \( P \) with respect to \( T \) and \( \mu \) gives rise to the following Gibbs–Duhem relation
\[ dP = \left( \frac{\partial P}{\partial T} \right)_\mu dT + \left( \frac{\partial P}{\partial \mu} \right)_T d\mu \]
\[ := s dT + \rho d\mu, \]
(3.2.20)
where we defined
\[ s = \left( \frac{\partial P}{\partial T} \right)_\mu, \quad \rho = \left( \frac{\partial P}{\partial \mu} \right)_T. \]
(3.2.21)

If we vary the action (3.2.19) with respect to the background fields \( g_{\mu\nu} \) and \( A_\mu \) \textit{while keeping} \( \beta^\mu \) \textit{and} \( \Lambda^\beta \) \textit{fixed}, we obtain the energy-momentum tensor and the \( U(1) \) current
\[ \delta_{\text{bkg}} S = \int d^{d+1}x \sqrt{-g} \left( \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \delta A_\mu \right). \]
(3.2.22)

The background variations are
\[ \delta T = \frac{T}{2} u^\mu u^\nu \delta g_{\mu\nu}, \quad \delta u^\mu = \frac{1}{2} u^\mu u^\nu u^\rho \delta g_{\nu\rho}, \quad \delta \mu = \frac{\mu}{2} u^\mu u^\nu \delta g_{\mu\nu} + u^\mu \delta A_\mu. \]
(3.2.23)
Using these, and the relation \( \delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu \nu} \delta g_{\mu \nu} \), the energy-momentum tensor and \( U(1) \) current become

\[
T^{\mu \nu} = \mathcal{E} u^\mu u^\nu + \Pi^{\mu \nu},
\]

\[
J^\mu = \rho u^\mu,
\]

where we used the Euler relation \( \mathcal{E} = sT + \mu \rho - P \). These are exactly the constitutive relations for an ideal charged fluid we found in (3.1.11).
Chapter 4

Boost-agnostic fluids

Having whetted our appetite (or, perhaps, thirst) for fluids in the previous chapter, this chapter develops a hydrodynamical theory with Aristotelian spacetime symmetries based on \cite{2}. Since relativistic symmetries arise from their Aristotelian counterpart by the addition of Lorentz boosts, the fluids that we develop here will contain the relativistic fluids we constructed above as a special case, and we provide the explicit identification in Section 4.4.5. This formulation also contains Galilean fluids and, at least in principle, Carrollian fluids, although their precise nature remains elusive (see, e.g., \cite{97–99}). For this reason, we refer to fluids with Aristotelian spacetime symmetries as boost-agnostic fluids.

4.1 Non-boost invariant perfect fluids and geometry

4.1.1 Perfect fluids on flat spacetime

Consider a charged perfect fluid in \((d+1)\)-dimensions, which has spatial rotational invariance and translational invariance in both time and space, as was studied in \cite{120, 122}. One can choose the fluid variables to be chemical potential \(\mu\), temperature \(T\) and fluid velocity \(v^i\). These variables are allowed to depend on space and time. It is assumed, however, that locally there exists a thermodynamic equilibrium. We furthermore assume the pressure \(P\) to be a function of \(\mu, T\) and \(v^2\). In other words, we assume the equation of state to be of the form \(P = P(T, \mu, v^2)\). Through the Gibbs-Duhem relation,

\[
\text{d}P = s\text{d}T + n\text{d}\mu + \frac{1}{2}\rho\text{d}v^2,
\]

we can express entropy density \(s\), charge density \(n\) and kinetic mass density \(\rho\) in terms of the fluid variables. Finally, we have the Euler relation

\[
\mathcal{E} = -P + sT + \mu n + \rho v^2,
\]

(4.1.2)
which expresses the total energy density $\mathcal{E}$ in terms of the fluid variables. In the presence of a boost symmetry, the corresponding Ward identity implies that the term containing kinetic mass density in (4.1.1) and (4.1.2) can be absorbed into the other fluid variables [120, 122]. This leads to velocity independent thermodynamic relations; a hallmark of boost invariant systems.

Just as for the relativistic fluid, the conserved currents, i.e., the EMT $T^{\mu\nu}$ and the charge current $J^\mu$, can be expressed as functions of the fluid variables in a derivative expansion, the form of which is dictated by the symmetry of the problem. The divergence of these conserved currents gives rise to the dynamics of the fluid. The most general homogeneous and isotropic perfect fluid without boost symmetries, i.e., at zeroth order in derivatives, is characterised by [120, 122]

$$
T^0_0 = -\mathcal{E}, \quad T^i_0 = -(\mathcal{E} + P)v^i, \quad T^0_j = \rho v^j, \quad T^i_j = P\delta^i_j + \rho v^i v^j,
$$

(4.1.3)

$$
J^0 = n, \quad J^i = n v^i.
$$

(4.1.4)

Here we presented the fluid written in the LAB frame, in which the observer is at rest. For fluids without boost symmetries, it is useful to define the internal energy

$$
\mathcal{\tilde{E}} = \mathcal{E} - \rho v^2,
$$

(4.1.5)

in terms of which the Euler and Gibbs-Duhem relations read

$$
\mathcal{\tilde{E}} + P = T s + \mu n, \quad \frac{d\mathcal{\tilde{E}}}{\tau} + \frac{1}{2} \rho d v^2 = T d s + \mu d n, \quad dP = s dT + n d\mu + \frac{1}{2} \rho d v^2.
$$

(4.1.6)

In the subsequent analysis, we will drop the charge current to simplify the presentation. The inclusion of the charge current was considered in [96], using the Schwinger–Keldysh effective field theory approach for non–equilibrium systems developed in [251–254] and adapted to non-relativistic systems in [255].

### 4.1.2 Recovering Lorentzian fluids

In this section, following [120], we show how the boost-agnostic framework above recovers the ideal Lorentzian fluid of Section 3.1.2. Lorentz boost invariance implies the Ward identity

$$
T^i_0 + T^i_0 = 0,
$$

(4.1.7)

which, using (4.1.3), tells us that the kinetic mass density is no longer an independent hydrodynamic quantity

$$
\rho v^i = (\mathcal{E} + P)v^i \Rightarrow \rho v^2 = (\mathcal{E} + P)v^2,
$$

(4.1.8)

and hence the Euler relation (4.1.2) becomes

$$
(\mathcal{E} + P)(1 - v^2) = sT + n\mu = \mathcal{\tilde{E}} + P,
$$

(4.1.9)
where, as above, the internal energy $\tilde{E}$ is
\[
\tilde{E} = \mathcal{E} - \rho v^2 = \mathcal{E}(1 - v^2) - P v^2,
\] (4.1.10)
which means that we can write $\rho$ as
\[
\rho = \frac{\mathcal{E} + P}{1 - v^2}.
\] (4.1.11)
At this stage, we already recognise the appearance of the ubiquitous $\gamma$ factor:
\[
\gamma = \sqrt{1 - v^2}.
\] (4.1.12)
The Gibbs–Duhem relation for pressure in (4.1.6) then takes the form
\[
dP = s d\mathcal{T} + n d\mu + \frac{\gamma^2}{2} (\tilde{E} + P) dv^2.
\] (4.1.13)
By performing the following redefinitions
\[
\mathcal{T} = \gamma^{-1} \tilde{\mathcal{T}}, \quad s = \gamma \tilde{s}, \quad \mu = \gamma^{-1} \tilde{\mu}, \quad n = \gamma \tilde{n},
\] (4.1.14)
we can remove the term with $dv^2$ in (4.1.13) to get
\[
dP = \tilde{s} \tilde{\mathcal{T}} + \tilde{n} d\tilde{\mu},
\] (4.1.15)
where we used the Euler relation in the form $\tilde{\mathcal{E}} + P = \tilde{s} \tilde{\mathcal{T}} + \tilde{n} \tilde{\mu}$ and the relation $dv^2 = 2 \gamma^{-3} d\gamma$. Thus, for Lorentzian fluids, the pressure (and, by extension, $\tilde{E}$) is independent of $v^2$. Using the relation for $\rho$ in (4.1.11), we can write the EMT as
\[
T^{00} = -(\mathcal{E} + P) \gamma^2 + P,
\]
\[
T^{i0} = -(\tilde{\mathcal{E}} + P) v^i \gamma^2,
\]
\[
T^{ij} = -(\tilde{\mathcal{E}} + P) v^i v^j \gamma^2 + P \delta^i_j,
\]
\[
J^0 = \gamma \tilde{n},
\]
\[
J^i = \tilde{n} \gamma v^i.
\] (4.1.16)
If we identify
\[
u^\mu = \gamma (1, v^i), \quad u_\mu = \gamma (-1, v^i),
\] (4.1.17)
this precisely recovers the Lorentzian EMT in (3.1.11).

### 4.1.3 Curved geometry for boost-agnostic fluids

Boost-agnostic fluids live on Aristotelian spacetime as introduced in Section 2.10. Since we will have occasion to integrate over Aristotelian spacetimes when writing down actions for Lagrangian transport, we need an appropriate integration measure. Such a measure is given
by the determinant of \((\tau_\mu, e_\mu^a)\), which we denote by \(e\), i.e.,
\[
e = \det(\tau_\mu, e_\mu^a).
\] (4.1.18)

The intrinsic torsion of an Aristotelian structure appears in (2.10.11) and is given by \(d\tau\) and \(\xi_v h\). The first of these, the torsion 2-form, has components
\[
(d\tau)_{\mu\nu} =: \tau_{\mu\nu} = \partial_\mu \tau_\nu - \partial_\nu \tau_\mu.
\] (4.1.19)

The other part of the intrinsic torsion is sometimes called the extrinsic curvature, and it has components
\[
K_{\mu\nu} = -\frac{1}{2} \xi_v h_{\mu\nu}.
\] (4.1.20)

The extrinsic curvature is purely spatial,
\[
\nu^\mu K_{\mu\nu} = 0,
\] (4.1.21)
and its trace satisfies
\[
K := h^{\mu\nu} K_{\mu\nu} = -e^{-1} \partial_\mu (e v^\mu).
\] (4.1.22)

Up to first order in derivatives we can make do with Lie derivatives, exterior derivatives and divergences. At second order in the derivative expansion, however, a choice of connection (for example (2.10.10)) must be made in order to write down curvatures.

Finally, we remark that flat Aristotelian spacetime in Cartesian coordinates corresponds to
\[
\tau_\mu = \delta_\mu^0, \quad h_{\mu\nu} = \delta_\mu^i \delta_\nu^i, \quad \nu^\mu = -\delta_\mu^0, \quad h^{\mu\nu} = \delta^i_\mu \delta^i_\nu,
\] (4.1.23)
where we split the spacetime index \(\mu = (0, i)\) into temporal and spatial directions.

4.1.4 Perfect fluids on a curved background

Consider a perfect fluid living on an Aristotelian geometry described by \(\{\tau_\mu, h_{\mu\nu}\}\) with fluid velocity \(u^\mu\) satisfying
\[
u^\mu \tau_\mu = 1.
\] (4.1.24)

On flat space (4.1.23), we can write the fluid velocity as \(u^\mu = (1, v^i)\), and the curved space analogue of \(v^2\) is \(u^2 = h_{\mu\nu} u^\mu u^\nu\). When combined with the completeness relation \(h^{\mu\rho} h_{\rho\nu} = \delta_\nu^\mu + v^\mu \tau_\nu\), this implies the following decomposition of the fluid velocity \(u^\mu\),
\[
u^\mu = -\nu^\mu + h^{\mu\rho} h_{\rho\nu} u^\nu,
\] (4.1.25)
in terms of timelike and spacelike components, respectively. The perfect fluid energy-momentum tensor (4.1.3) generalised to a curved background reads \[ T_{\mu\nu} = - \left( \tilde{E} + P + \rho u^2 \right) u^\mu \tau_\nu + \rho u^\mu u^\rho h_{\rho\nu} + P \delta^\mu_\nu. \] (4.1.26)

Using the property (4.1.24), this can also be written as

\[ T_{\mu\nu} = u^\mu u^\rho \left[ - \left( \tilde{E} + P + \rho u^2 \right) \tau_\rho \tau_\nu + \rho h_{\rho\nu} \right] + P \delta^\mu_\nu. \] (4.1.27)

It is furthermore useful to express the energy-momentum tensor \( T_{\mu\nu} \) as

\[ T_{\mu\nu} = - T_{\mu\tau} \tau_\nu + T_{\mu\rho} h^{\rho\nu}. \] (4.1.28)

where

\[ T^\mu = \tilde{E} u^\mu + P h^{\mu\rho} h_{\rho\nu} u^\nu, \] (4.1.29a)

\[ T^{\mu\nu} = P h^{\mu\nu} + \rho u^\mu u^\nu, \] (4.1.29b)

are the energy current and momentum-stress tensor, respectively.

If we are dealing with a theory on a generic curved Aristotelian background for which there is an action principle, then diffeomorphism invariance implies the following conservation equation for the energy-momentum tensor

\[ e^{-1} \partial_\mu (e T^\mu_{\ \rho}) + T^\mu_{\ \rho} \tau_\mu - \frac{1}{2} T^{\mu\nu} \partial_\rho h_{\mu\nu} = 0. \] (4.1.30)

This will be shown in Section 4.3.3. When we are dealing with an on-shell theory (as we are in the case of dissipative fluids) then we simply impose (4.1.30) as the correct conservation equation. This is the analogue of declaring \( \nabla_\mu T^{\mu\nu} = 0 \) to be the conservation equation in the relativistic case. Notice that in flat space (in Cartesian coordinates), Eq. (4.1.30) reproduces the usual divergence of the energy-momentum tensor. The first term expresses the usual divergence of the energy-momentum tensor, while the remaining terms are currents contracted with their sources, on which a derivative acts. It thus takes the standard form of a divergence of a current being equal to the sum of the responses times the derivative of the sources. For completeness, we remark that the equation of motion (4.1.30) admits the covariantisation

\[ \nabla_\mu T^\mu_{\ \rho} + T^\mu_{\ \nu} \nabla_\rho \tau_\mu - \frac{1}{2} T^{\mu\sigma} \nabla_\rho h_{\mu\sigma} - (\Gamma^\mu_{\mu\sigma} - e^{-1} \partial_\sigma (e) T^\sigma_{\ \rho}) T^\rho_{\ \rho} + 2(\Gamma^\lambda_{\mu\rho}) T^\mu_{\ \lambda} = 0, \] (4.1.31)

for any choice of connection \( \Gamma^\rho_{\mu\nu} \).

---

1Contracting (4.1.30) with a vector \( k^\rho \), we get

\[ 0 = e^{-1} \partial_\mu (ek^\rho T^\mu_{\ \rho}) + T^\mu_{\ \nu} \tau_k - \frac{1}{2} T^{\mu\nu} \psi_k h_{\mu\nu}, \]

so if \( k^\rho \) is Killing (cf., Eqs. (4.3.2a) and (4.3.2b)), we find that \( 0 = e^{-1} \partial_\mu (ek^\rho T^\mu_{\ \rho}) \), that is to say, \( k^\rho T^\mu_{\ \rho} \) is a conserved current.
It is useful to recast the equation of motion (4.1.30) for the perfect fluid (4.1.29a)–(4.1.29b), using the thermodynamic relations (4.1.6), in the following form

\[ 0 = \frac{s}{2} h_{\mu\nu} \left( \xi_\beta h_{\mu\nu} - h_{\mu\sigma} u^\sigma \xi_\beta \tau_\nu - h_{\nu\sigma} u^\sigma \xi_\beta \tau_\mu \right), \]  
(4.1.32a)

\[ 0 = \Pi^\rho_{\mu} h_{\rho\lambda} \left[ -s T h^{\mu\nu} \xi_\beta \tau_\mu + u^\lambda \xi_\beta p \right. 
\left. + \frac{p}{2} \left( u^\mu h^{\kappa\lambda} + u^\kappa h^{\mu\lambda} + u^\lambda h^{\mu\kappa} \right) \left( \xi_\beta h_{\kappa\mu} - h_{\kappa\sigma} u^\sigma \xi_\beta \tau_\mu - h_{\mu\sigma} u^\sigma \xi_\beta \tau_\kappa \right) \right], \]  
(4.1.32b)

where we introduced the spatial projector

\[ \Pi^\rho_{\mu} = \delta^\rho_{\mu} - u^\rho \tau_{\mu}, \]  
(4.1.33)

which satisfies

\[ \Pi^\rho_{\mu} u^\mu = 0 = \Pi^\rho_{\mu} \tau_{\rho}, \quad \Pi^\mu_{\alpha} \Pi^\alpha_{\nu} = \Pi^\mu_{\nu}. \]  
(4.1.34)

Furthermore, the Lie-derivative \( \xi_\beta \) in (4.1.32a) and (4.1.32b) is defined with respect to the vector

\[ \beta^\mu = u^\mu / T. \]  
(4.1.35)

In the remainder of this section we establish some identities that will be crucial later on. Equation (4.1.32a) expresses entropy conservation and (4.1.32b) represents conservation of momentum. Using that \( s \) and \( \rho \) can both be thought of as functions of \( T \) and \( u^2 \) along with the identities

\[ u^\mu \xi_\beta \tau_\mu = -\frac{1}{T^2} u^\mu \partial_\mu T, \]
\[ u^\mu u^\nu \xi_\beta h_{\mu\nu} = \frac{1}{T} u^\mu \partial_\mu u^2 - 2 \frac{u^2}{T^2} u^\mu \partial_\mu T, \]

where \( u^2 = h_{\mu\nu} u^\mu u^\nu \), we can view the perfect fluid equation as providing \( \xi_\beta \tau_{\rho} \) in terms of \( \xi_\beta h_{\mu\nu} - h_{\mu\sigma} u^\sigma \xi_\beta \tau_\nu - h_{\nu\sigma} u^\sigma \xi_\beta \tau_\mu \), i.e.,

\[ \xi_\beta \tau_{\rho} = \frac{1}{2} X_\rho^{\mu\nu} \left( \xi_\beta h_{\mu\nu} - h_{\mu\sigma} u^\sigma \xi_\beta \tau_\nu - h_{\nu\sigma} u^\sigma \xi_\beta \tau_\mu \right), \]  
(4.1.37)

where \( X_\rho^{\mu\nu} \) is given by

\[ X_\rho^{\mu\nu} = \frac{1}{s} \Pi^\sigma_{\rho} \left\{ 2 \left( \frac{\partial P}{\partial u^2} \right)_s h_{\sigma\lambda} u^\lambda u^\mu u^\nu + 2 \left( \frac{\partial P}{\partial u^2} \right)_s h_{\sigma\lambda} u^\lambda u^\mu u^\nu + p \left( u^\mu \delta^\nu_{\sigma} + u^\nu \delta^\mu_{\sigma} \right) \right\} 
\quad + \tau_{\rho} \left[ u^\mu u^\nu \left( \frac{\partial P}{\partial s} \right) u^2 \right], \]  
(4.1.38)

The expression in parentheses on the RHS of (4.1.37) has the nice property that

\[ \xi_\beta h_{\mu\nu} - h_{\mu\sigma} u^\sigma \xi_\beta \tau_\nu - h_{\nu\sigma} u^\sigma \xi_\beta \tau_\mu = \frac{1}{T} \left( \xi_\mu h_{\nu\nu} - h_{\mu\sigma} u^\sigma \xi_\mu \tau_\nu - h_{\nu\sigma} u^\sigma \xi_\mu \tau_\mu \right), \]  
(4.1.39)
so that these are Lie derivatives along velocity with, notably, an absence of $T$ derivatives. Finally, we remark that on flat space (cf., Eq. (4.1.23)), this special combination reduces to

$$\mathcal{L}_{\beta} h_{\mu\nu} - h_{\mu\sigma} u^\sigma \mathcal{L}_{\beta} \tau_{\nu} - h_{\nu\sigma} u^\sigma \mathcal{L}_{\beta} \tau_{\mu} \overset{\text{flat}}{=} \frac{1}{T} \left( h_{\mu\sigma} \partial_\nu u^\sigma + h_{\nu\sigma} \partial_\mu u^\sigma \right),$$

which will be useful in Section 4.4, where, after setting up the general problem of first-order corrections to non-boost invariant fluids, we specialise to flat space.

### 4.2 Entropy current

Going beyond perfect fluids means moving away from local thermodynamic equilibrium and requires derivative corrections to be added to the energy-momentum tensor. We will work up to first order in derivatives. The goal will be to identify all allowed tensorial structures whose coefficients are known as transport coefficients. By “allowed” we mean “allowed by symmetry” and furthermore “allowed by entropy considerations”, just as in Section 3. We will impose that the fluid locally obeys the second law of thermodynamics, and hence there must exist an entropy current $S^\mu$ with non-negative divergence

$$e^{-1} \partial_\mu (eS^\mu) \geq 0.$$  

(4.2.1)

By the entropy current we mean the most general current, constructed from the fluid variables, up to first order in derivatives such that it reduces to $su^\mu$ for a perfect fluid and such that its divergence is non-negative for all fluid configurations. The requirement that the divergence of the entropy current is non-negative constrains the transport coefficients appearing in the expansion of the energy-momentum tensor. In this section, we elucidate the structure of the entropy current beyond perfect fluid order and, in particular, show that the appropriate fluid variables on curved space involve Lie derivatives of the geometric objects $\tau_{\mu}$ and $h_{\mu\nu}$ along $\beta_{\mu} = u_{\mu} T$.

As in Eq. (3.1.35), the canonical part of the entropy current is obtained by covariantising the thermodynamic Euler relation (4.1.6)

$$s = \frac{1}{T} \xi + \frac{1}{T} P,$$  

(4.2.2)

leading to

$$S^\mu_{\text{can}} = -T^\mu_{\nu} \beta^\nu + P \beta^\mu,$$  

(4.2.3)

such that $\tau_{\mu} S^\mu_{\text{can}} = s$ for a perfect fluid. However, as we already discussed, there are generically terms present in the entropy current that do not arise in this way: such terms form the non-canonical piece of the entropy current, $S^\mu_{\text{non}}$, and we may in general write

$$S^\mu = S^\mu_{\text{can}} + S^\mu_{\text{non}} = -T^\mu_{\nu} \beta^\nu + P \beta^\mu + S^\mu_{\text{non}}.$$  

(4.2.4)

Using the decomposition of the entropy current (4.2.4), we can recast the LHS of the second
law (4.2.1) as
\begin{equation}
e^{-1} \partial_{\mu} (eS^\mu) = \left( T^\mu - T^\mu_{(0)} \right) \xi_\beta \tau_\mu - \frac{1}{2} \left( T^{\mu \nu} - T^{\mu \nu}_{(0)} \right) \xi_\beta h_{\mu \nu} + e^{-1} \partial_{\mu} (eS^\mu_{\text{non}}) ,
\end{equation}

where we used the Gibbs–Duhem relation (4.1.1) as well as
\begin{equation}
-T_{(0)}^\mu \xi_\beta \tau_\mu + \frac{1}{2} T_{(0)}^{\mu \nu} \xi_\beta h_{\mu \nu} = e^{-1} \partial_{\mu} (eP^\beta) ,
\end{equation}

with the perfect fluid energy-momentum tensors $T^\mu_{(0)}$ and $T^{\mu \nu}_{(0)}$ given in Eqs. (4.1.29a) and (4.1.29b). Here, the subscript $(0)$ indicates that these terms are zeroth order in derivatives and thus correspond to the perfect fluid contributions.

Following the classification in refs. [134,135], we split the currents $T^\mu - T^\mu_{(0)}$ and $T^{\mu \nu} - T^{\mu \nu}_{(0)}$, which are at least first order in derivatives, into dissipative and non-dissipative parts. The latter are further subdivided into hydrostatic (HS) terms and non-hydrostatic (NHS) terms, to be defined shortly, so that we find
\begin{align}
T^\mu - T^\mu_{(0)} &= T^\mu_D + T^\mu_{\text{HS}} + T^\mu_{\text{NHS}} , \tag{4.2.7a} \\
T^{\mu \nu} - T^{\mu \nu}_{(0)} &= T^{\mu \nu}_D + T^{\mu \nu}_{\text{HS}} + T^{\mu \nu}_{\text{NHS}} . \tag{4.2.7b}
\end{align}

We will now define these three contributions separately. They should be thought of as independent contributions to the energy-momentum tensor and they all have the same symmetry properties with respect to rotations (and boosts or scale symmetries if these are present). For example spatial rotational symmetries dictate that $T^\mu_D$, $T^{\mu \nu}_{\text{HS}}$ and $T^{\mu \nu}_{\text{NHS}}$ are all separately symmetric under the interchange of $\mu$ and $\nu$.

The dissipative terms produce entropy
\begin{equation}
e^{-1} \partial_{\mu} (eS^\mu) = T^\mu_D \xi_\beta \tau_\mu - \frac{1}{2} T^{\mu \nu}_D \xi_\beta h_{\mu \nu} \geq 0 ,
\end{equation}

with equality holding if and only if $T^\mu_D = T^{\mu \nu}_D = 0$, while the non-dissipative NHS terms by definition obey
\begin{equation}
T^\mu_{\text{NHS}} \xi_\beta \tau_\mu - \frac{1}{2} T^{\mu \nu}_{\text{NHS}} \xi_\beta h_{\mu \nu} = 0 .
\end{equation}

These terms thus make a vanishing contribution to the divergence of the canonical entropy current. Finally, the non-dissipative HS terms are defined to cancel the divergence of the non-canonical entropy current
\begin{equation}
e^{-1} \partial_{\mu} (eS^\mu_{\text{non}}) = - T^\mu_{\text{HS}} \xi_\beta \tau_\mu + \frac{1}{2} T^{\mu \nu}_{\text{HS}} \xi_\beta h_{\mu \nu} ,
\end{equation}

and will play a major rôle in the next section. Note that (4.2.9) implies that the HS energy-momentum tensor is only defined up the addition of NHS terms, since they will leave (4.2.10) invariant. This is analogous to what happens when solving inhomogeneous differential equations, where any solution of the homogeneous equation can be added to a particular
solution of the inhomogeneous equation to obtain a new solution of the inhomogeneous equation. Because of this, there is an inherent ambiguity in HS transport. We will make a specific choice that fixes this ambiguity as will be detailed in Sec. 4.3, where we construct actions for HS and NHS transport respectively.

The goal of this chapter will be to classify the allowed terms appearing in the three parts: D, HS and NHS of the energy-momentum tensor. We will start this analysis with a detailed study of the HS and NHS terms for which it is possible to write down a Lagrangian.

4.3 Non-dissipative transport

In this section we study the Lagrangian description (cf., Section 3.2) of non-dissipative boost-agnostic transport. We will see in Section 4.4 that at first order in derivatives all non-dissipative transport coefficients can be obtained from an action. We start with the hydrostatic partition function for fluids in thermal equilibrium. This requires curved backgrounds admitting a Killing vector that generates time translations. Following the framework developed in Section 3.2, we then relax the condition that there is a Killing vector, moving away from thermal equilibrium. This leads to an action for HS transport. The relaxation of the presence of a Killing vector allows for more terms to be added to the action. These extra terms all correspond to NHS transport as we will show in the last subsection of this section.

4.3.1 Hydrostatic partition function

If we assume a stationary curved background $\mathcal{M}_S$ with a time-translation symmetry generated by $\mathcal{H}$, we can write down the thermal partition function

$$Z = \text{Tr}[e^{-\mathcal{H}/T}], \quad (4.3.1)$$

which, provided the Aristotelian background curves sufficiently weakly compared to the mean free path, is known as the hydrostatic partition function or the equilibrium partition function [136, 137] (see also [130] for the construction of the hydrostatic partition function in the context of Newton–Cartan backgrounds). The time-translation symmetry implies that (4.3.1) gives rise to static responses. Phrased in the language of geometry, we take the time-translation symmetry of the background, which is described by $\tau_\mu$ and $h_{\mu\nu}$, to be generated by a timelike Killing vector $\beta^\mu$, where $\beta^\mu = (1, 0, \ldots, 0)$ in suitable coordinates. Timelike (and future pointing) means that $\tau_\mu \beta^\mu > 0$. The Killing equations for an Aristotelian
geometry are\(^2\)

\[
\mathcal{L}_\beta \tau_\mu = 0, \quad (4.3.2a)
\]

\[
\mathcal{L}_\beta h_{\mu\nu} = 0. \quad (4.3.2b)
\]

These conditions imply equilibrium via (4.1.37) as they trivially solve the leading order equations of motion. The vector \(\beta^\mu\) leads to a preferred choice of local temperature and velocity given by

\[
T = 1/(\tau_\mu \beta^\mu), \quad (4.3.3)
\]

where the velocity \(u^\mu\) satisfies \(\tau_\mu u^\mu = 1\), cf., (4.1.24). We see that the requirement of positive temperature is equivalent to the requirement that \(\beta^\mu\) is future-pointing timelike, i.e., \(\tau_\mu \beta^\mu > 0\).

To make contact with the thermal partition function (4.3.1), we need to analytically continue “time”,\(^3\) which we identify with the affine parameter \(\lambda_t\) along integral curves of \(\beta^\mu\), \(\lambda_t \rightarrow -i\lambda_t^E\). We then compactify this to the “thermal circle” by identifying \(\lambda_t^E \sim \lambda_t^E + 1/T\). In this way, a functional integral over the complex-time manifold will return the partition function \(\mathcal{Z}\) in (4.3.1). Now, \(\mathcal{Z}\) itself can be written in a derivative expansion, and by writing

\[
S_{HPF} = -i \log \mathcal{Z}, \quad (4.3.4)
\]

we can now also expand \(S_{HPF}\) in derivatives

\[
S_{HPF} = \sum_n S_{(n)}_{HPF}, \quad (4.3.5)
\]

where \(S_{(n)}_{HPF}\) takes the form of an integral over \(M_S\) built from objects with \(n\) derivatives. With a slight abuse of terminology, we will refer to \(S_{HPF}\) as the hydrostatic partition function.

In order to construct the hydrostatic partition function explicitly, we need to identify the allowed terms up to first order in derivatives taking into consideration the conditions of thermal equilibrium on a curved background imposed by the Killing equation. The first Killing equation (4.3.2a) can be written as

\[
T^{-1} \partial_\mu T - u^\nu (\partial_\nu \tau_\mu - \partial_\mu \tau_\nu) = 0, \quad (4.3.6)
\]

\(^2\)The field \(h_{\mu\nu}\) is constrained to have one zero eigenvalue, so one can write it as \(h_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu\) in terms of unconstrained spatial vielbeins. Since the latter transform under local rotations, the Killing vector equation (4.3.2b) can equivalently be written as

\[
\mathcal{L}_\beta e^a_\mu = \lambda^a_\nu e^a_\nu,
\]

where \(\lambda^a_\nu = -\lambda^\mu_a\) is an infinitesimal local rotation.

\(^3\)This uses the Matsubara formalism, where the field theory acquires a finite temperature upon complexifying the time, whose periodicity is identified with the inverse temperature; see [256] for more details. In the special case of Lorentzian geometry, this “Wick rotation” turns the Lorentzian geometry into a Euclidean geometry, but the Matsubara formalism can be used regardless of the relativity principle that is in force.
while the second Killing equation \((4.3.2b)\) can be written as

\[
\mathcal{L}_u h_{\mu\nu} - \frac{u^\rho}{T} h_{\rho\nu} \partial_\mu T - \frac{u^\rho}{T} h_{\rho\mu} \partial_\nu T = 0. \tag{4.3.7}
\]

By contracting \((4.3.6)\) with \(u^\mu\) and \(v^\mu\) we obtain

\[
0 = u^\mu \partial_\mu T, \quad 0 = T^{-1} u^\mu \partial_\mu T + u^\mu \mathcal{L}_v \tau_\mu. \tag{4.3.8a}
\]

Contracting \((4.3.7)\) with \(v^\mu v^\nu\) gives nothing as a result of the fact that \(h_{\mu\nu}\) has one zero eigenvalue with eigenvector \(v^\mu\). Contracting with \(u^\mu u^\nu\), \(v^\mu v^\nu\) and \(h_{\mu\nu}\) leads to

\[
0 = u^\mu \partial_\mu u^2, \quad 0 = \frac{1}{2} v^\mu \partial_\mu u^2 - \frac{1}{2} u^\mu v^\nu \mathcal{L}_\nu h_{\mu\nu} + u^2 u^\mu \mathcal{L}_\nu \tau_\mu, \quad 0 = e^{-1} \partial_\mu (e u^\mu), \tag{4.3.9c}
\]

where we defined \(u^2 = h_{\mu\nu} u^\mu u^\nu\).

In order to construct the hydrostatic partition function \(S_{\text{HPF}}\), we write down the most general expansion in derivatives of the background fields \(\tau_\mu\) and \(h_{\mu\nu}\) under the assumption that the Killing equations \((4.3.2a)\) and \((4.3.2b)\) are obeyed. At zeroth order in derivatives, we can build two scalars,

\[
T, \quad u^2. \tag{4.3.10}
\]

At first order, taking into account the relations \((4.3.8a)\)–\((4.3.9c)\), there are two independent one-derivative scalars, which we can take to be

\[
\nu^\mu \partial_\mu T, \quad \nu^\mu \partial_\mu u^2. \tag{4.3.11}
\]

Scalars such as \(u^\mu \partial_\mu T, u^\mu \partial_\mu u^2, e^{-1} \partial_\mu (e u^\mu), u^\mu \mathcal{L}_\nu \tau_\mu, u^\mu u^\nu \mathcal{L}_\nu h_{\mu\nu}\) are either zero or related to \((4.3.11)\) via the relations \((4.3.8a)\)–\((4.3.9c)\), possibly using partial integration. Furthermore, scalars such as \(h_{\mu\nu} u^\mu \mathcal{L}_\nu u^\nu, h^{\mu\nu} \mathcal{L}_\mu h_{\mu\nu}\) and \(h^{\mu\nu} \mathcal{L}_\mu h_{\mu\nu}\) do not lead to anything new as they can be rewritten in terms of \((4.3.11)\). Up to first order, we therefore obtain

\[
S_{\text{HPF}} = \int d^{d+1}x \left( P(T, u^2) + F_1(T, u^2) \nu^\mu \partial_\mu T + F_2(T, u^2) \nu^\mu \partial_\mu u^2 + O(\partial^2) \right), \tag{4.3.12}
\]

where the functions \(P, F_1\) and \(F_2\) are all arbitrary functions of \(T\) and \(u^2\).

Since the background is stationary we can use adapted coordinates (known as static gauge in \([130]\)), where we choose a time direction \(t\) such that the Killing vector \(\beta^\mu\) is given by \(\beta^\mu = \delta^\mu_t\). As \(\beta^\mu\) is Killing, the tensors \(\tau_\mu\) and \(h_{\mu\nu}\) are independent of \(t\). Thus, in these coordinates the fluid velocity \(u^\mu\) is given by \(^4 u^\mu = T(x) \delta^\mu_i\), where the temperature \(T\) only depends on \(x^i\) with \(i = 1, \ldots, d\) and not on \(t\).

The complex-time background has the structure of a fiber bundle, where the thermal

\[^4\text{Note that since } u^\mu \propto \delta^\mu_i, \text{ we are in a comoving frame.}\]
We thus obtain the hydrostatic partition function

\[ \tau_\mu \, d\mu = N(dt - A_t \, dx^1), \]
\[ h_{\mu \nu} \, d\mu \, d\nu = \sigma_{ij} \left( dx^i + X^i(\mu - A_i) \right) \left( dx^j + X^j(\mu - A_j) \right). \] (4.3.13)

The metric \( \sigma_{ij} \) is invertible and has signature \((1, \ldots, 1)\). This parameterisation makes manifest that \( h_{\mu \nu} \) has one zero eigenvalue. The integration measure \( e \) is \( e = N\sqrt{\sigma} \) where \( \sigma = \det \sigma_{ij} \). Further, the 1-form \( A = A_i \, dx^i \) is a Kaluza–Klein type gauge connection in that \( \delta t = \Lambda(x) \) and \( \delta A_i = \delta \mu \Lambda \) leave the parameterisation invariant. The other fields \( N, X^i \) and \( \sigma_{ij} \), which depend on \( x^1 \) but not on \( t \), are thus all gauge invariant. Since \( \tau_\mu \, u^\mu = 1 \) with \( u^\mu = T(x) \delta^\mu_1 \) it must be that \( T = N^{-1} \). Finally, the vector \( \nu^\mu \) satisfying \( \tau_\mu \, \nu^\mu = -1 \) is given by

\[ \nu^\mu = -N^{-1} \delta^\mu_1 + N^{-1} X^i \left( \delta^\mu_i + A_i \delta^\mu_1 \right). \] (4.3.14)

We can now ask again what are the invariant scalars up to first order in derivatives. These have to be gauge invariant under the Kaluza–Klein gauge transformation \( \delta A_i = \delta \mu \Lambda \). At zeroth order in derivatives, we find \( N \) and \( X^2 = \sigma_{ij}X^iX^j \), so that at first order in derivatives we can build two scalars

\[ X^i \partial_i N, \quad X^i \partial_i X^2. \] (4.3.15)

We thus obtain the hydrostatic partition function

\[ S = \int_\Sigma d^4x \sqrt{\sigma} \left( \hat{P}(N, X^2) + G_1(N, X^2)X^i \partial_i N + G_2(N, X^2)X^i \partial_i X^2 \right). \] (4.3.16)

A term such as \( G_3(N, X^2) \partial_i \left( \sqrt{\sigma} X^i \right) \) can be absorbed into the \( G_1 \) and \( G_2 \) terms after partial integration. Likewise, a term such as \( G_4(N, X^2)\sqrt{\sigma} \sigma^{ij} \xi_X \sigma_{ij} \), where \( \sigma^{ij} \) is the inverse of \( \sigma_{ij} \), can be written as \( 2G_4(N, X^2) \partial_i \left( \sqrt{\sigma} X^i \right) \) so that this, too, is nothing new. Thus we see that this line of reasoning leads to the same hydrostatic partition function as in (4.3.12). For ease of comparing (4.3.16) and (4.3.12) we note that the vanishing of \( u^\mu \partial_\mu T, u^\mu \partial_\mu u^2 \) and \( \partial_\mu \left( eu^\mu \right) \) follows immediately from the \( t \)-independence of the fields involved in the Kaluza–Klein reduction. Furthermore one observes that \( e \nu^\mu \partial_\mu F = \sqrt{\sigma} X^i \partial_i F \) where \( F \) is any function of \( T = N^{-1} \) and \( u^2 = N^{-2}X^2 \). We can now set (4.3.16) and (4.3.12) equal to each other and in principle read off the bijective relation between the sets of functions \( \{ P, F_1, F_2 \} \) and \( \{ \hat{P}, G_1, G_2 \} \).

### 4.3.2 Hydrodynamic frame transformations \& Landau frame

This section briefly discusses the general features of hydrodynamic frame transformations for boost-agnostic fluids (cf., Sec. 3.1.3 for a discussion of hydrodynamics frame transformations for the relativistic fluid). Consider the energy-momentum tensor to first order, which in a
generic frame takes the form,
\[ T^\mu_\nu = -\left( \tilde{\varepsilon} + P + \rho u^2 \right) u^\mu \tau_\nu + \rho u^\mu u^\rho h_{\rho \nu} + P \delta^\mu_\nu - \tilde{T}^\mu_\nu + \tilde{T}^{\mu \rho}_\nu h_{\rho \nu}. \]  
(4.3.17)

Redefining \( T \) and \( u^\mu \) as
\[ T = T' + \delta T, \quad u^\mu = u'^\mu + \delta u^\mu, \]  
(4.3.18)
with \( \tau_\mu \delta u^\mu = 0 \) (in order to preserve the normalisation \( \tau_\mu u^\mu = 1 \)), leads to an energy-momentum tensor of the form
\[ T^\mu_\nu = -\left( \tilde{\varepsilon}' + P' + \rho' u'^2 \right) u'^\mu \tau'_\nu + \rho' u'^\mu u'^\rho h_{\rho' \nu} + P' \delta^\mu_\nu - \tilde{T}^\mu_\nu + \tilde{T}^{\mu \rho}_\nu h_{\rho' \nu}, \]  
(4.3.19)
where \( \tilde{\varepsilon}' = \tilde{\varepsilon}(T', u'^2) \) etc., and where
\[ \tilde{T}^\mu_\nu = T^\mu_\nu + \left( \delta \tilde{\varepsilon} + \delta P + \rho' \delta u^2 + u'^2 \delta \rho \right) u'^\mu + \left( \tilde{\varepsilon}' + P' + \rho' u'^2 \right) \delta u^\mu + \delta P u^\mu, \]  
\[ \tilde{T}^{\mu \rho}_\nu = T^{\mu \rho}_\nu + \rho' u'^\mu \delta u^\rho + \rho' u'^\rho \delta u^\mu + u'^\mu \delta \rho + \delta P h^{\mu \rho}. \]  
(4.3.20)

The defining condition for the transformed energy-momentum tensor (4.3.19) to be in Landau frame is
\[ u'^\nu T^\mu_\nu = -\tilde{\varepsilon}' u'^\mu \iff \tilde{T}^\mu_\nu = \tilde{T}^{\mu \rho}_\nu h_{\rho \nu} u'^\nu \iff \tilde{T}^{\mu \gamma}_\nu u'^\gamma = 0, \]  
(4.3.21)
where \( \tilde{T}^{\mu \gamma}_\nu = -\tilde{T}^\mu_\nu \tau_\gamma + \tilde{T}^{\mu \rho}_\nu h_{\rho \nu} \) which will be the case provided we have
\[ T^{\mu \gamma}_\nu u'^\gamma = \left( \delta \tilde{\varepsilon} + \frac{1}{2} \delta \rho \delta u^2 \right) u'^\mu + \left( \tilde{\varepsilon}' + P' \right) \delta u^\mu. \]  
(4.3.22)

This can be solved for \( \delta s \) and \( \delta u^\mu \) (by contracting with \( \tau_\mu \) and \( h_{\mu \rho} u'^\rho \)) to give
\[ u'^\mu \delta s + s' \delta u^\mu = T^{\mu \gamma}_\nu u'^\gamma, \]  
\[ s' \delta u^\mu = T^{\mu \gamma}_\nu \Pi^{\nu \sigma}_\rho u'^\gamma, \]  
(4.3.23a)
(4.3.23b)
where \( \Pi^{\nu \sigma}_\rho = \delta^{\nu \sigma}_\rho - u'^\mu \tau_\sigma \), and where we used \( \delta \tilde{\varepsilon} + \frac{1}{2} \delta \rho \delta u^2 = \tilde{T}' \delta s \) and \( \tilde{\varepsilon}' + P' = \tilde{T}' s' \).

In Landau frame, it follows from (4.3.21) that the divergence of the entropy current (4.2.5) reads
\[ e^{-1} \delta_{\mu} (eS^\mu) = \frac{1}{T} T^{\mu \gamma}_\nu h_{\gamma \rho} u^\rho \xi_{\mu} \tau_\nu - \frac{1}{2T} T^{\mu \gamma}_\nu \xi_{\mu} h_{\gamma \nu} + e^{-1} \delta_{\mu} \left( eS^\mu_{(1)\text{non}} \right), \]  
(4.3.24)
which we can equivalently express as
\[ e^{-1} \delta_{\mu} (eS^\mu) = -\frac{1}{2T} T^{\mu \gamma}_\nu \left( \xi_{\mu} h_{\gamma \nu} - h_{\rho \nu} u^\rho \xi_{\mu} \tau_\nu - h_{\mu \rho} u^\rho \xi_{\mu} \tau_\nu \right) + e^{-1} \delta_{\mu} \left( eS^\mu_{(1)\text{non}} \right). \]  
(4.3.25)
We will use this to rewrite the divergence of the entropy current in Section 4.4.

### 4.3.3 Action for hydrostatic non-dissipative transport

In order to compute transport coefficients using (4.3.12), we will drop the restriction to stationary configurations; that is to say, we relax the requirement that \( \beta^\mu \) is Killing. This will lead to an action for the hydrostatic non-dissipative transport coefficients\(^5\) This is related to the discussion below equation (4.2.7a) in the following way. As we will show, the energy-momentum tensor obtained by varying the geometric variables in the action that follows from the hydrostatic partition function without the condition that \( \beta^\mu \) is Killing, is equal to the HS part of the energy-momentum tensor as defined in equation (4.2.10).

We now have geometric variables \( \tau^\mu \) and \( h^{\mu\nu} \) and fluid variables \( \beta^\mu \). To construct a variational principle for non-dissipative boost-agnostic transport we follow the prescription developed in Section 3.2.2. Using this procedure, we first work out the diffeomorphism Ward identity

\[
\delta_\xi S_{\text{HS}} = \int_M d^{d+1}x \, e \left( -T^\mu \delta_\xi \tau^\mu + \frac{1}{2} T^{\mu\nu} \delta_\xi h_{\mu\nu} + F_\mu \delta_\xi \beta^\mu \right) = 0 . \tag{4.3.26}
\]

Here \( S_{\text{HS}} \) is the same action as in (4.3.12) except that now \( \beta^\mu \) is no longer a Killing vector, and \( F_\mu \) is the response to varying \( \beta^\mu \) under diffeomorphisms. Setting the diffeomorphism variation to zero for any \( \xi^\mu \) leads to the off-shell diffeomorphism Ward identity,

\[
e^{-1} \partial_\mu \left( e T^\mu_\rho \right) + T^\mu \partial_\rho \tau^\mu - \frac{1}{2} T^{\mu\nu} \partial_\rho h_{\mu\nu} = F_\mu \partial_\rho \beta^\mu + e^{-1} \partial_\mu \left( e F_\rho \beta^\mu \right) , \tag{4.3.27}
\]

where we recall \( T^\mu_\nu = -T^{\mu}_\tau \tau^\nu + T^{\mu}_\rho h_{\rho\nu} \). Using that the fluid equations of motion follow from a diffeomorphism transformation of \( \beta^\mu \) we see that on shell the left- and right-hand sides vanish separately. We conclude that in order to compute the energy-momentum tensor we vary \( \tau^\mu \) and \( h^{\mu\nu} \) keeping \( \beta^\mu \) fixed, and furthermore that the on-shell energy-momentum conservation equation is given by

\[
e^{-1} \partial_\mu \left( e T^\mu_\rho \right) + T^\mu \partial_\rho \tau^\mu - \frac{1}{2} T^{\mu\nu} \partial_\rho h_{\mu\nu} = 0 , \tag{4.3.28}
\]

as stated before in Eq. (4.1.30).

This procedure reproduces the perfect fluid equations of motion on an arbitrary curved background as discussed in Section 4.1.4. To see this we consider the action up to zeroth order in derivatives, i.e.,

\[
S_{(0)} = \int_M d^{d+1}x \, e P(T, u^2) , \tag{4.3.29}
\]

with \( P \) the pressure as we will see a posteriori. Since we vary the background sources keeping \( \beta^\mu \) fixed, we have \( \delta T = -T u^\mu \delta \tau^\mu \) and \( \delta u^\mu = -u^\mu u^\rho \delta \tau^\rho \). Using further that

\(^5\)In terms of the classification of [134, 135], the non-dissipative transport coefficients considered in this section are class \( L = H_S \cup \bar{H}_S \), i.e., those that have a Lagrangian description.
\[ \delta e = e \left( -\nu^\mu \delta \tau_\mu + \frac{1}{2} h^{\mu \nu} \delta h_{\mu \nu} \right), \]

we find that

\[ T_{(0)}^\mu = P T^\mu + \left( \frac{\delta P}{\delta T} \right) u^2 T u^\mu, \quad (4.3.30a) \]

\[ T_{(1)}^{\mu \nu} = P h^{\mu \nu} + 2 \left( \frac{\delta P}{\delta u^2} \right) T u^\mu u^\nu. \quad (4.3.30b) \]

Using the thermodynamic relations

\[ \left( \frac{\delta P}{\delta T} \right) u^2 = s, \quad \left( \frac{\delta P}{\delta u^2} \right) T = \frac{1}{2} \rho, \quad s T = \tilde{\varepsilon} + P \]

as well as the relation (4.1.25) between \( v^\mu \) and \( u^\mu \), we recover the perfect fluid energy-momentum tensor (4.1.29a) and (4.1.29b).

Let us next consider the first order derivative terms in (4.3.12). We will denote the first order part of the action by \( S_{(1)} \), i.e.

\[ S_{(1)} = \int d^{d+1}x \left( F_1(T, u^2) \nu^\mu \partial_\mu T + F_2(T, u^2) \nu^\mu \partial_\mu u^2 \right). \quad (4.3.31) \]

We thus have \( S_{HS} = S_{(0)} + S_{(1)} \).

As we discussed in the context of the relativistic fluid in Section 3.1.3, when we introduce derivative corrections, the notion of temperature and velocity can undergo field redefinitions whereby two equally valid definitions of temperature and velocity can differ by derivatives of the fluid variables. We will present our final results in Landau frame, which is defined by declaring that the full (all order in derivatives) energy-momentum tensor is such that

\[ T_{\mu \nu}(0) + T_{\mu \nu}(1) = T_{\mu \nu}(0) + T_{\mu \nu}(1)_{HS}, \quad (4.3.34a) \]

\[ T_{\mu \nu}(0) + T_{\mu \nu}(1) = T_{\mu \nu}(0) + T_{\mu \nu}(1)_{HS}, \quad (4.3.34b) \]

where the left hand side is in Lagrangian frame and is computed by variation of the action,
while the right-hand side is in any frame – for example in Landau frame. The right-hand side is computed by applying a frame transformation to the left-hand side and arranging the result according to the number of derivatives. At perfect fluid order, the expressions look the same, but they are written with respect to different choices of $T$ and $u^\mu$.

Let us next compute the variation of the first derivative terms in the action (4.3.31). Using $\delta v^\lambda = v^\lambda u^\mu \delta \tau^\mu - h^\lambda{}(\mu\nu\sigma) \delta h_{\mu\nu\sigma}$, we obtain

\[
\mathcal{T}^\mu (1) = u^\mu \left[ \left( \frac{\partial F_1}{\partial u^\lambda_T} - \frac{\partial F_2}{\partial T} \right) u^2 \right] (2u^\mu v^\lambda \partial_\lambda T - T v^\lambda \partial_\lambda u^2) + u^\mu K(T F_1 + 2u^2 F_2),
\]

(4.3.35a)

\[
\mathcal{T}^\mu (1)_\nu = (h^\mu{}\nu \lambda - h^\lambda{}\mu\nu) \left( F_1 \partial_\lambda T + F_2 \partial_\lambda u^2 \right)
+ 2 \left[ \left( \frac{\partial F_1}{\partial u^\mu} - \frac{\partial F_2}{\partial T} \right) u^2 \right] u^\mu u^\nu \lambda \partial_\lambda T + 2F_2 K u^\mu u^\nu,
\]

(4.3.35b)

where $K$ is the trace of the extrinsic curvature defined in equation (4.1.22). Combining these according to $\mathcal{T}^\mu (1)_\nu = -\mathcal{T}^\mu (1)_{\tau \nu} + \mathcal{T}^\mu (1)_{h^\mu{}\nu}$ yields the first order part of the energy-momentum tensor in Lagrangian frame,

\[
\mathcal{T}^\mu (1)_{\tau \nu} = 2v^\lambda u^\mu h_{\rho \sigma} \Pi^{\rho}{}_{\nu} u^\rho \left[ \left( \frac{\partial F_1}{\partial u^\lambda_T} - \frac{\partial F_2}{\partial T} \right) u^2 \right] + 2F_1 v^\lambda h_{\mu}{}^{\mu} \rho h_{\rho \nu} \partial_\lambda T
+ \left[ v^\lambda u^\mu \tau^\nu T \left( \frac{\partial F_1}{\partial u^\mu} - \frac{\partial F_2}{\partial T} \right) u^2 \right] + 2F_2 v^\lambda h_{\mu}{}^{\mu} \rho h_{\rho \nu} \partial_\lambda u^2
- T u^\mu \tau^\nu F_1 K + 2F_2 K u^\mu \Pi^{\rho}{}_{\nu} h_{\rho \sigma} u^\rho.
\]

This should be added to the perfect fluid energy-momentum tensor

\[
\mathcal{T}^\mu (0)_{\tau \nu} = -(\tilde{\mathcal{E}} + P + \rho u^2) u^\mu \tau^\nu + \rho u^\mu u^\rho h_{\rho \nu} + P \delta^\mu^\nu,
\]

(4.3.37)

coming from the variation of $S_{(0)}$.

In Section 4.3.2 we worked out the transformation from any frame to Landau frame, indicated by primed variables. Using those results we obtain the Landau frame expression

\[
\mathcal{T}^\mu (1)_{HS \nu} = \mathcal{T}^\mu (1)_{HS} h_{\rho \sigma} \Pi^{\rho}{}_{\sigma}{}_{\nu},
\]

(4.3.38)

where we remind the reader that $\mathcal{T}^\mu (1)_{HS}$ is computed using (4.3.34b). This gives

\[
\mathcal{T}^\mu (1)_{HS} = \mathcal{T}^\mu (1)_{\lambda \Pi^{\rho}{}_{\sigma}} \frac{u^\lambda}{\sqrt{T}} \left[ \rho' \left( u^\mu \delta^\rho_{\sigma} + u^\mu \rho \delta^\mu_{\sigma} \right) + 2u^\rho' \left( h^{\mu \rho} \left( \frac{\partial P'}{\partial u^2} \right) u^2 + u'^\mu u'^\rho \left( \frac{\partial P'}{\partial u^2} \right) u^2 \right) \right]
+ \mathcal{T}^\mu (1) + \mathcal{T}^\mu (1)_{\tau \nu} \frac{u^\tau}{\sqrt{T}} \left[ u^\mu u^\nu \left( \frac{\partial P'}{\partial s^2} \right) u^2 + h^{\mu \rho} \left( \frac{\partial P'}{\partial s^2} \right) u^2 \right],
\]

(4.3.39)

with the prime denoting Landau frame fluid variables. We defined $u^\rho' = h_{\rho \kappa} u^\kappa$. Terms such as $\mathcal{T}^\mu (1)$ are given in (4.3.35b), but where we must replace the $T$ and $u^\mu$ by $T'$ and $u'^\mu$.  

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We will drop the primes and use the relations
\[
\begin{align*}
\partial_\mu T &= -T^\beta_\mu \tau_\mu - Tu^\rho \tau_{\mu \rho}, \\
\partial_\mu u^2 &= Tu^\nu (\xi_\beta h_{\mu \nu} - u_\mu \xi_\beta \tau_\nu - u_\nu \xi_\beta \tau_\mu) - u^2 u^\rho \tau_{\mu \rho} - u^\rho \omega_{\rho \mu},
\end{align*}
\] (4.3.40a)
(4.3.40b)
where \( \tau_{\mu \nu} \) is the torsion 2-form defined in (4.1.19) and where \( \omega_{\rho \mu} = \partial_\rho u_\mu - \partial_\mu u_\rho \). Using the equations of motion (4.1.38) to eliminate \( \xi_\beta \tau_\mu \) derivatives, allows us to write
\[
T^\mu_\nu = \frac{1}{2} \eta^\mu_\nu \alpha_\beta (\xi_\beta h_{\alpha \beta} - u_\alpha \xi_\beta \tau_\beta - u_\beta \xi_\alpha \tau_\beta) + \frac{1}{2} \eta^\mu_\nu \alpha_\beta \omega_{\alpha \beta} + \frac{1}{2} \eta^\mu_\nu \alpha_\beta K_{\alpha \beta},
\] (4.3.41)
where \( K_{\alpha \beta} \) is the extrinsic curvature introduced in (4.1.20). In obtaining this result we have also used the following relation
\[
T^\mu_\nu \omega_{\mu \nu} = T^\mu_\nu (\xi_\beta h_{\mu \nu} - u_\mu \xi_\beta \tau_\nu - u_\nu \xi_\beta \tau_\mu) - u_\nu u^\rho \tau_{\mu \rho} - 2u^\rho K_{\nu \rho}.
\] (4.3.42)
This was done in order to make \( \eta^\mu_\nu \alpha_\beta = -\eta^\nu_\mu \alpha_\beta \) spatial in its last two indices (to avoid ambiguities among some of the \( \eta \) tensors as a result of (4.3.42)), i.e., \( \tau_{\alpha \beta} \eta^\mu_\nu \alpha_\beta = 0 \).

The \( \eta^\mu_\nu \alpha_\beta \) tensor that features in (4.3.41) can be written as
\[
\eta^\mu_\nu \alpha_\beta = J_1 h^{\mu \nu} h^{\alpha \beta} + J_2 u^\mu u^{\nu} u^\alpha \beta + J_3 h^{\mu \nu} h^{(\alpha} \beta) + J_4 (h^{\mu \nu} u^\alpha u^\beta + h^\alpha \beta u^\mu u^\nu) + J_5 (h^{\mu \nu} u^{(\alpha} \beta) + h^\alpha \beta u^{\mu} \nu) + J_6 (h^{\mu} u^\nu u^\alpha \beta + u^\alpha \beta u^{\mu} \nu) + 2J_7 (v^{\mu} h^{\nu} (\alpha \beta) + v^{(\alpha} h^{\beta)} (\mu \nu) - \frac{1}{2} A_1 (h^{\mu \nu} u^\alpha \beta - h^\alpha \beta u^{\mu} \nu) + \frac{1}{2} A_2 (h^{\mu} u^\nu u^{\alpha \beta} - h^\alpha \beta u^{\mu} \nu) + \frac{1}{2} A_3 (u^{\mu} u^{\nu} h^{\alpha \beta} - u^\alpha \beta u^\mu u^{\nu}) + 2A_4 (v^{\mu} h^{\nu} (\alpha \beta) - v^{(\alpha} h^{\beta)} (\mu \nu)),
\] (4.3.43)

\[\text{Note that } T^\mu_\nu \text{ in (4.3.41) is only defined up to terms proportional to } v^{\mu} v^{\nu}, \text{ since } v^{\mu} v^{\nu} \delta h_{\mu \nu} = 0.\]
where the eleven scalars $J_{1,...,7}, A_{1,2,3,4}$ are given by

\[
J_1 = \frac{T}{s} F_1 \left[ s \left( \frac{\partial P}{\partial s} \right)_{u^2} - 2u^2 \left( \frac{\partial P}{\partial u^2} \right)_s \right], \tag{4.3.44}
\]

\[
J_2 = \frac{2}{s} F_2 \left[ s \left( \frac{\partial P}{\partial s} \right)_{u^2} - 2u^2 \left( \frac{\partial P}{\partial u^2} \right)_s - 2\rho \right] + \frac{4u^2}{s} f_A \left( \frac{\partial P}{\partial u^2} \right)_s - \frac{2}{s} f_B \left[ s \left( \frac{\partial P}{\partial s} \right)_{u^2} - 2\rho \right], \tag{4.3.45}
\]

\[
J_3 = -TF_2, \tag{4.3.46}
\]

\[
J_4 = \frac{2}{s} F_1 \left[ s \left( \frac{\partial P}{\partial s} \right)_{u^2} - 2u^2 \left( \frac{\partial P}{\partial u^2} \right)_s - 2\rho \right], \tag{4.3.47}
\]

\[
J_5 = \frac{T}{s} F_1 \left[ 2 \left( \frac{\partial P}{\partial u^2} \right)_s - \rho \right] + 2F_2 \left[ \left( \frac{\partial P}{\partial s} \right)_{u^2} + T \right] - 2f_A \left( \frac{\partial P}{\partial s} \right)_{u^2}, \tag{4.3.48}
\]

\[
J_6 = \frac{2F_1}{s} \left[ s \left( \frac{\partial P}{\partial s} \right)_{u^2} - \rho \right] - 2f_A \left( \frac{\partial P}{\partial s} \right)_{u^2} + \frac{2\rho}{s} f_B, \tag{4.3.49}
\]

\[
J_7 = \frac{P}{s} \left[ \frac{\partial P}{\partial s} \right] - TF_2, \tag{4.3.50}
\]

\[
A_1 = -\frac{4}{s} F_1 \left( \frac{\partial P}{\partial s} \right)_{u^2} + \frac{TF_1}{s} \left[ s \left( \frac{\partial P}{\partial s} \right)_{u^2} - 2u^2 \left( \frac{\partial P}{\partial u^2} \right)_s - 2\rho \right] + \frac{4u^2}{s} F_1 \left[ \left( \frac{\partial P}{\partial u^2} \right)_s \left( \frac{\partial P}{\partial s} \right)_{u^2} - \left( \frac{\partial P}{\partial s} \right)_s \left( \frac{\partial P}{\partial u^2} \right)_{u^2} \right] + \frac{2}{s} (F_2 + f_B) \left[ s \left( \frac{\partial P}{\partial s} \right)_{u^2} - 2u^2 \left( \frac{\partial P}{\partial u^2} \right)_s \right], \tag{4.3.51}
\]

\[
A_2 = \frac{F_1}{s} \left[ \rho + 2 \left( \frac{\partial P}{\partial s} \right)_{u^2} + 2T \left( \frac{\partial P}{\partial u^2} \right)_s \right] + 2F_2 \left( \frac{\partial P}{\partial s} \right)_{u^2} + 2f_A \left( \frac{\partial P}{\partial s} \right)_{u^2} - 2f_A \left( \frac{\partial P}{\partial s} \right)_{u^2}, \tag{4.3.52}
\]

\[
A_3 = -\frac{F_1}{s} \left[ \rho \left( \frac{\partial P}{\partial s} \right)_{u^2} + T \left( \frac{\partial P}{\partial u^2} \right)_s \right] + 2F_2 \left( \frac{\partial P}{\partial s} \right)_{u^2} + 2\rho \frac{F_2 - 2f_A}{s} \left( \frac{\partial P}{\partial s} \right)_{u^2}, \tag{4.3.53}
\]

\[
A_4 = J_7, \tag{4.3.54}
\]

where we defined the recurring combinations

\[
f_A := T \left[ \left( \frac{\partial F_2}{\partial T} \right)_{u^2} - \left( \frac{\partial F_1}{\partial u^2} \right)_T \right], \quad f_B := T \left[ \left( \frac{\partial F_2}{\partial T} \right)_{u^2} - \left( \frac{\partial F_1}{\partial T} \right)_{u^2} \right]. \tag{4.3.55}
\]

The coefficients $J_i$ make up the symmetric part of $\eta_{\mu \nu \alpha \beta}^{HS}$ under the interchange of $\mu \nu$ and $\alpha \beta$ while the coefficients $A_i$ make up the anti-symmetric part of $\eta_{\mu \nu \alpha \beta}^{HS}$. The remaining $\eta$
tensors in (4.3.41) are given by
\[
\eta_{\mu
u}^{\text{tor}} = 2 \left[ \frac{1}{T} (TF_1 + 2u^2 F_2) \left( \frac{\partial P}{\partial s} \right) u_2 + T \right] - \frac{2u^2}{T} \left( \frac{\partial P}{\partial s} \right) u_2 f_A \right] h^{\mu\nu} u^{[\alpha\beta]} \\
+ 2 \left[ (TF_1 + 2u^2 F_2) \left( \frac{\partial P}{\partial s} \right) u_2 \right] - 2u^2 \left( \frac{\partial P}{\partial s} \right) u_2 f_A - 2TF_B \right] u^{\mu} u^{\nu} u^{[\alpha\beta]},
\]
\[
\eta_{\mu
u}^{\text{rot}} = \frac{4}{T} \left( T \right) (TF_1 + u^2 F_2) v^[\nu h_\nu [\alpha\beta]} - 4 F_2 v^[\mu u^{\nu}] u^{[\alpha\beta]} ,
\]
\[
\eta_{\mu
u}^{\text{ext}} = -2 \left[ F_1 \left( \frac{\partial P}{\partial s} \right) u_2 - 2F_2 \right] u^{\mu} u^{\nu} h^{\alpha\beta} - 2F_1 \left( \frac{\partial P}{\partial s} \right) u_2 h^{\mu\nu} h^{\alpha\beta} + 8F_2 v^[\mu h_\nu [\alpha\beta]} \]
\[
- \frac{4}{T} \left[ F_2 \left( \frac{\partial P}{\partial s} \right) u_2 - f_A \left( \frac{\partial P}{\partial s} \right) u_2 \right] h^{\mu\nu} u^{\alpha\beta} \]
\[
- \frac{4}{T} (F_2 - f_A) \left( \frac{\partial P}{\partial s} \right) u_2 u^{\mu} u^{\nu} u^{\alpha\beta}.
\]

An important consistency check for these results is that these expressions recover the results of [122] in the limit of linearised perturbations around a fluid at rest (see [2] for more details). Equation (4.3.41) is the main result of this subsection. We will next discuss how this is related to the non-canonical entropy current as it should via the frame-independent definition (4.2.10).

### 4.3.4 Non-canonical entropy current

In the previous subsection, we obtained explicit expressions for the contributions to the energy-momentum tensor that arise from the action \( S_{\text{HS}} \). As discussed in Section 4.2, the HS part of the energy-momentum tensor is related to the divergence of the non-canonical part of the entropy current, cf., (4.2.10). The goal of this subsection is to show that there exists a non-canonical entropy current whose divergence obeys (4.2.10) where the energy-momentum tensor is the one we just obtained.

It can be shown that the converse is also true, i.e., starting from the most general non-canonical entropy current and demanding that its divergence obeys (4.2.10), where the energy-momentum tensor is the most general one allowed by symmetries, we find (using only on-shell relations) that the same result for the non-canonical entropy current

In the Lagrangian frame, the divergence of the non-canonical entropy current (4.2.10) must obey
\[
e^{-1} \partial_\mu \left( e S^\mu_{(1)\text{non}} \right) = - \mathcal{T}^\mu_{(1)} \mathcal{E}_\mu + \frac{1}{2} \mathcal{E}^\nu \mathcal{E}_\nu h_{\mu\nu}, \quad (4.3.59)
\]
where \( \mathcal{T}^\mu_{(1)} \) and \( \mathcal{E}^\nu \) are given in (4.3.35a) and (4.3.35b). This can be solved for \( S^\mu_{(1)\text{non}} \) up to identically conserved currents, leading to
\[
S^\mu_{(1)\text{non}} = - \frac{1}{T} v^\mu F_1 u^\rho \partial_\rho T + \frac{1}{T} u^\mu F_1 v^\rho \partial_\rho T - \frac{1}{T} v^\mu F_2 u^\rho \partial_\rho u^2 + \frac{1}{T} u^\mu F_2 v^\rho \partial_\rho u^2 . \quad (4.3.60)
\]
The total entropy current (4.2.4) in the Lagrangian frame can be written as \( S^\mu = su^\mu - 95 \).
where we have split the contributions from the zeroth order derivative terms, which is just the perfect fluid result $s u^\mu$, from the terms containing first order derivatives. Substituting the result obtained in the previous subsection for $T_{(1)\nu}^\mu$ (see equation (4.3.36)) and using (4.3.60) we obtain for the full entropy current in Lagrangian frame,

$$S^\mu = s u^\mu - u^\mu F_1 e^{-1} \partial_\rho (e v^\rho) + u^\mu \left( \frac{\partial F_2}{\partial T} - \frac{\partial F_1}{\partial u^2} \right) v^\rho \partial_\rho u^2,$$

(4.3.61)

where we draw the reader’s attention to the fact that a wealth of miraculous cancellations have taken place. One may at this point object that the full entropy current could also receive contributions from the dissipative sector of transport. We will show next that in the Lagrangian frame only the HS sector contributes to the entropy current.

To show this we first observe that in Landau frame (denoted here by a prime just like in Section 4.3.2) the total entropy current (4.2.4) is given by

$$S^\mu = s' u'^\mu + S_{(1)\text{non}}^\mu,$$

(4.3.62)

simply because Landau frame is equivalent to demanding that the canonical entropy current is that of a perfect fluid, i.e., $T_{(1)\nu}^\mu \beta^\nu \sim T_{(1)\nu}^\mu u^\nu = 0$ by definition. If we next take the Lagrangian frame result (4.3.61) and we transform it to Landau frame using (4.3.23a) and (4.3.23b) we obtain (4.3.62) with $S_{(1)\text{non}}^\mu$ as given in (4.3.60) (written in terms of primed variables). In other words the Lagrangian frame entropy current is the same as the total entropy current (4.3.62) and so since the total entropy current is frame independent it must be that (4.3.61) equals the total entropy current.

In Landau frame the right hand side of equation (4.2.10) can be written as

$$e^{-1} \partial_\mu \left( e S_{(1)\text{non}}^\mu \right) = \frac{1}{2} T_{(1)\text{HS}}^{\mu \nu} \left( \xi_\mu h_{\nu \nu} - u_\nu \xi_\mu \tau_\mu - u_\mu \xi_\nu \tau_\nu \right),$$

(4.3.63)

where $u_\mu = h_{\mu \nu} u^\nu$ and where $T_{(1)\text{HS}}^{\mu \nu}$ is the Landau frame expression for the HS contributions to $T_{(1)\nu}^\mu$. This was computed in the previous subsection in (4.3.39). As a consistency check we will explicitly verify that this is indeed the case.

To first order in derivatives, we can rewrite the right hand side of (4.3.59) in terms of $u'$ rather than $u$, and using the relation

$$-T_{(1)\nu}^\mu = T_{(1)\nu}^\mu u'^\nu - T_{(1)\text{HS}}^{\mu \rho} u'^\rho u'^\nu,$$

(4.3.64)

we find that

$$e^{-1} \partial_\mu \left( e S_{(1)\text{non}}^\mu \right) = T_{(1)\rho}^{\mu \rho} u'^\rho \xi_\beta \tau_\mu + \frac{1}{2} T_{(1)}^{\mu \nu} \left( \xi_\beta \xi_\rho h_{\nu \nu} - h_{\nu \rho} u'^\rho \xi_\beta \tau_\mu - h_{\mu \rho} u'^\rho \xi_\beta \tau_\nu \right).$$

(4.3.65)

Dropping the prime and using the perfect fluid equations of motion in the form (4.1.37), we can replace the $\xi_\beta \tau_\mu$ by $\xi_\beta h_{\mu \nu} - h_{\nu \rho} u'^\rho \xi_\beta \tau_\mu - h_{\mu \rho} u'^\rho \xi_\beta \tau_\nu$ terms, so that we obtain (4.3.63)
with

\[
T^{\mu\nu}_{(1)HS} = T^{\mu\nu}_{(1)} + T^{\rho}_{(1)\sigma} u^{\rho} X_{\mu}^{\nu},
\]

(4.3.66)

where \(X_{\rho}^{\mu\nu}\) is given in (4.1.38) and where \(T^{\mu\nu}_{(1)HS}\) is in Landau frame as can be seen by comparing to (4.3.39).

### 4.3.5 Action for non-hydrostatic non-dissipative transport

In Section 4.3.3 we dropped the condition that \(\beta^{\mu}\) is a Killing vector and used the hydrostatic partition function to find an action for hydrostatic non-dissipative transport. Once we drop the condition that \(\beta^{\mu}\) is Killing we can add more terms to the action at first order in derivatives because we can no longer use the Killing equations to relate various derivatives. These extra terms can be obtained by looking at all scalars one can construct from the Lie derivatives of \(\tau_{\mu}\) and \(h_{\mu\nu}\). These are listed in equations (4.3.8a)–(4.3.9c). We can multiply each of these scalars by an arbitrary function forming new scalar terms that can be added to the action \(S_{HS}\). Using the freedom to perform partial integrations we can drop the last term of the form \(\tilde{F}T^{-1}\delta_{\mu} (\epsilon u^{\mu})\). This leads to 4 additional terms each multiplied by one of the functions \(F_3\) to \(F_6\). The full first order action becomes

\[
S_{(1)} = \int d^{d+1}x e \left( F_1 \tau_{\mu}\partial_{\mu} T + F_2 u^{\mu}\partial_{\mu} u^2 + F_3 u^{\mu}\partial_{\mu} T + F_4 u^{\mu}\partial_{\mu} u^2 + F_5 u^{\mu}\xi_{\nu}\partial_{\mu} h_{\mu\nu} \right) + F_6 u^{\mu}\xi_{\nu}\partial_{\nu} h_{\mu\nu} \right) .
\]

(4.3.67)

The additional contributions to the energy current and stress tensor (in Lagrangian
frame) due to the novel $F_3$ to $F_6$ contributions to the action are

$$T^\mu_{F_1} = \left( \frac{\partial F_1}{\partial u^z} \right)_T u^\mu (2u^2\nu^\lambda \partial_\lambda T - T\nu^\lambda \partial_\lambda u^2) + TF_1 u^\mu K ,$$  \hspace{1cm} (4.3.68a)

$$T^\mu_{F_2} = -\left( \frac{\partial F_2}{\partial u^2} \right)_u u^\mu (2u^2\nu^\lambda \partial_\lambda T - T\nu^\lambda \partial_\lambda u^2) + 2u^2 F_2 u^\mu K ,$$  \hspace{1cm} (4.3.68b)

$$T^\mu_{F_3} = F_3 (u^\mu + v^\mu) u^\rho \partial_\rho T - TF_3 u^\mu e^{-1} \partial_\rho (e\mu^\rho)$$

$$- \left( \frac{\partial F_3}{\partial u^2} \right)_T u^\mu (T u^\rho \partial_\rho u^2 - 2u^2 u^\rho \partial_\rho T) ,$$  \hspace{1cm} (4.3.68c)

$$T^\mu_{F_4} = F_4 (u^\mu + v^\mu) u^\rho \partial_\rho u^2 - 2u^2 F_4 u^\mu e^{-1} \partial_\rho (e\mu^\rho)$$

$$+ \left( \frac{\partial F_4}{\partial T} \right)_u u^\mu (T u^\rho \partial_\rho u^2 - 2u^2 u^\rho \partial_\rho T) ,$$  \hspace{1cm} (4.3.68d)

$$T^\mu_{F_5} = u^\mu u^\nu \nu_\rho \partial_\rho \left[ \frac{T}{2} \left( \frac{\partial F_5}{\partial u^2} \right)_T u^2 + 2u^2 \left( \frac{\partial F_5}{\partial u^2} \right)_T + F_5 \right] - F_5 u^\mu$$

$$- F_5 e^{-1} \partial_\rho (e\mu^\rho) - F_5 L u^\mu + u^\mu \rho^\rho \partial_\rho F_5 - u^\mu \rho^\rho \partial_\rho F_5 ,$$  \hspace{1cm} (4.3.68e)

$$T^\mu_{F_6} = -2u^\mu u^\rho u^\sigma \kappa_{\rho\sigma} \left[ \frac{T}{2} \left( \frac{\partial F_6}{\partial u^2} \right)_T u^2 + 2u^2 \left( \frac{\partial F_6}{\partial u^2} \right)_T + 2F_6 \right] ,$$  \hspace{1cm} (4.3.68f)

$$T^\mu_{F_1} = F_1 (h^{\mu\nu}\lambda - h^{\lambda\mu\nu} - h^{\lambda\nu\mu}) \partial_\lambda T + 2 \left( \frac{\partial F_1}{\partial u^2} \right)_T v^\lambda \partial_\lambda T u^\mu v^\nu ,$$  \hspace{1cm} (4.3.68g)

$$T^\mu_{F_2} = F_2 (h^{\mu\nu}\lambda - h^{\lambda\mu\nu} - h^{\lambda\nu\mu}) \partial_\lambda u^2 + 2F_2 u^\mu v^\nu - 2 \left( \frac{\partial F_2}{\partial T} \right)_u v^\lambda \partial_\lambda T u^\mu v^\nu ,$$  \hspace{1cm} (4.3.68h)

$$T^\mu_{F_3} = F_3 (h^{\mu\nu} + 2 \left( \frac{\partial F_3}{\partial u^2} \right)_T u^\mu v^\nu ) u^\rho \partial_\rho T ,$$  \hspace{1cm} (4.3.68i)

$$T^\mu_{F_4} = F_4 (h^{\mu\nu} u^\rho \partial_\rho u^2 - 2 \left( \frac{\partial F_4}{\partial u^2} \right)_u u^\mu u^\nu u^\rho \partial_\rho T - 2F_4 u^\mu u^\nu e^{-1} \partial_\rho (e\mu^\rho) ,$$  \hspace{1cm} (4.3.68j)

$$T^\mu_{F_5} = F_5 u^\rho (h^{\mu\nu} v^\rho - h^{\rho\mu} v^\nu - h^{\rho\nu} v^\mu) \tau_{\rho\sigma} + 2 \left( \frac{\partial F_5}{\partial u^2} \right)_T u^\mu u^\nu u^\rho \tau_{\rho\sigma} ,$$  \hspace{1cm} (4.3.68k)

$$T^\mu_{F_6} = -2F_6 h^{\mu\nu} u^\rho u^\sigma \kappa_{\rho\sigma} - 4 \left( \frac{\partial F_6}{\partial u^2} \right)_T u^\mu u^\nu u^\rho u^\sigma \kappa_{\rho\sigma} - 2F_6 h^{\lambda(\mu\nu)} (\partial_\lambda u^2 - 2u^\rho \partial_\rho T)$$

$$+ 4F_6 u^{(\mu\nu)} L u^{(\nu\lambda)} + 2 \left( 2u^{(\mu\nu)} v^{(\nu\lambda)} - u^{(\mu\nu)} v^{(\lambda\nu)} \right) \left[ \left( \frac{\partial F_6}{\partial T} \right)_u \partial_\lambda T + \left( \frac{\partial F_6}{\partial u^2} \right)_T \partial_\lambda u^2 \right]$$

$$+ 4F_6 (u^{\mu\nu}) e^{-1} \partial_\lambda (e\mu^\lambda) + 2F_6 u^\mu u^\nu K ,$$  \hspace{1cm} (4.3.68l)

where for completeness we have included the $F_1$ and $F_2$ parts as well. These were already derived earlier in equations (4.3.35a) and (4.3.35b). In writing the $F_6$ part of $T^\mu\nu$ we used the freedom to remove a term proportional to $v^{\mu\nu}$.

We will next rewrite these expressions by writing them in terms of $\Sigma_\beta h_{\mu\nu}$ and $\Sigma_\beta \tau_\mu$. Using equations (4.3.40a), (4.3.40b), (4.3.42) and
\[ e^{-1} \partial_\mu (e u^\mu) = \frac{T}{2} h^{\rho \sigma} (\xi_\beta h_{\rho \sigma} - 2 u_\rho \xi_\beta \tau_\sigma), \]  
\[ \xi_\mu v^\mu = u^\mu u^\nu v^\tau v_\rho - Th^{\mu \nu v^\rho} (\xi_\beta h_{\nu v} - u_\nu \xi_\beta \tau_\rho), \]  
where we remind the reader that \( u_\mu = h_\mu \psi u^\gamma \) and \( \omega_\mu = \partial_\mu u_\nu - \partial_\nu u_\mu \). In general, \( T^\mu \) and \( T^{\mu \nu} \) take the following form

\[ T^\mu = \chi^{\mu \nu} \xi_\nu \tau_\nu + \frac{1}{2} \Sigma^{\mu \nu \rho} \xi_\rho h_{\nu \rho} + \frac{1}{2} \Sigma_{\text{ext}}^{\mu \nu \rho} \kappa_{\nu \rho}, \]  
\[ T^{\mu \nu} = \Delta^{\mu \nu \rho} \xi_\rho \tau_\rho + \frac{1}{2} \tilde{\eta}^{\mu \nu \rho \sigma} \xi_\rho h_{\nu \rho} + \frac{1}{2} \tilde{\eta}_{\text{ext}}^{\mu \nu \rho \sigma} \kappa_{\rho \sigma} + \frac{1}{2} \tilde{\eta}_{\text{rot}}^{\mu \nu \rho \sigma} \tau_{\rho \sigma}, \]

where \( \Sigma^{\mu \nu \rho} = \Sigma^{\mu \rho \nu} \), \( \Delta^{\mu \nu \rho} = \Delta^{\nu \rho \mu} \), \( \tilde{\eta}^{\mu \nu \rho \sigma} = \tilde{\eta}^{\nu \mu \rho \sigma} = \tilde{\eta}^{\nu \rho \mu \sigma} = \tilde{\eta}^{\nu \rho \sigma \mu} \) and similarly for the other tensors. We find that

\[ \Sigma_{\text{ext}}^{\mu \nu \rho} = 2 (TF_1 - F_5 + 2u^2F_2) u^\mu h^{\nu \rho} \]
\[ - 4 \left[ \frac{\partial}{\partial T} (F_2 + F_6) - \frac{\partial}{\partial u^2} (TF_1 - F_5 - 2u^2F_6) \right] u^\mu u^\nu u^\rho, \]  
\[ \tilde{\eta}_{\text{rot}}^{\mu \nu \rho \sigma} = 4(F_2 + F_6)v^{(\mu \nu \gamma)}(\rho u^{\rho} - h_{\rho \sigma} u^{\sigma} - h_{\nu \rho} u^{\sigma}) \]  
\[ - 2 (TF_1 - F_5 + 2u^2F_2) h^{\mu \nu \rho \sigma} u^{\rho} u^{\sigma} = 4(TF_1 - F_5 + u^2(F_2 - F_6)) v^{(\mu \nu \gamma)}(\rho u^{\rho} - h_{\rho \sigma} u^{\sigma}), \]  
where, as above, we dropped \( v^{\mu \nu \gamma} \) terms and where we defined \( \tilde{\eta}_{\text{rot}}^{\mu \nu \rho \sigma} = - \tilde{\eta}_{\text{rot}}^{\mu \nu \sigma \rho} \) such that \( \tau_\rho \tilde{\eta}_{\text{rot}}^{\mu \nu \rho \sigma} = 0 \) in order that (4.3.42) does not lead to any ambiguities among the various
tensors. Furthermore we find that
\[
\chi^{\mu\nu} = 2T \left( F_5 + T \left( \frac{\partial F_5}{\partial u^2} \right) u^2 + 2u^2 \left( \frac{\partial F_5}{\partial u^2} \right)_T \right) v^{\mu} u^{\nu}, \quad (4.3.72a)
\]
\[
\tilde{\eta}^{\mu\nu\rho\sigma} - \tilde{\eta}^{\rho\sigma\mu\nu} = 4T(F_2 - F_6) \left( h^{\mu\nu} u^{(\rho\sigma)} - h^{\rho\sigma} u^{(\mu\nu)} \right) + 4T \left( 3F_6 - F_2 \right) \left( \epsilon^{\mu\nu}(\rho\sigma)v^{\rho\sigma} - \epsilon^{\rho\sigma}(\mu\nu)v^{\rho\sigma} \right) + 4TF_4 \left( h^{\mu\nu} u^{\rho\sigma} - h^{\rho\sigma} u^{\mu\nu} \right) + 16T \left( \frac{\partial F_6}{\partial u^2} \right)_T \left( u^{\rho\sigma} u^{(\mu\nu)} - u^{\mu\nu} u^{(\rho\sigma)} \right), \quad (4.3.72b)
\]
\[
\tilde{\eta}^{\mu\nu\rho\sigma} + \tilde{\eta}^{\rho\sigma\mu\nu} = 4T(F_2 + F_6) \left[ -2v^{\mu} u^{(\rho\sigma)} \right] + \left( h^{\mu\nu} v^{(\rho\sigma)} + h^{\rho\sigma} v^{(\mu\nu)} \right) - \left( v^{(\rho\sigma)} v^{(\rho\sigma)} + v^{(\rho\sigma)} v^{(\mu\nu)} \right)], \quad (4.3.72c)
\]
\[
\Delta^{\mu\nu} - \Sigma^{\mu\nu} = 2T \left( TF_1 - F_5 + u^2(F_2 - F_6) \right) h^{\rho(\mu\nu)} - T \left( TF_1 - F_5 + 2u^2F_2 \right) v^{\rho} h^{\mu\nu} + 2T(F_2 + F_6)(v^{\rho} + u^{\rho}) u^{(\mu\nu)} + 2T^2 \left( \frac{\partial}{\partial u^2} \right)_T \left( v^{\rho} u^{\mu\nu} - 2u^{\rho} u^{(\mu\nu)} \right) - 2T \left( \frac{\partial}{\partial u^2} \right)_T \left( TF_1 - F_5 - 2u^2F_6 \right) \left( v^{\rho} u^{\mu\nu} - 2u^{\rho} u^{(\mu\nu)} \right), \quad (4.3.72d)
\]
\[
\Delta^{\mu\nu} + \Sigma^{\mu\nu} = 2T \left( TF_1 + F_5 + 2u^2F_2 - u^2(F_2 + F_6) \right) h^{\rho(\mu\nu)} - T \left( TF_1 + F_5 + 2u^2F_2 \right) v^{\rho} h^{\mu\nu} + 2T(F_2 + F_6)v^{\rho} u^{(\mu\nu)} - 2T(2TF_3 + 2u^2F_4) u^{\rho} h^{\mu\nu} + 4T \left( F_4 + T \left( \frac{\partial F_4}{\partial u^2} \right)_T \right) u^{\rho} u^{\mu\nu} + 2T \left( -\frac{\partial}{\partial u^2} \right)_T \left( TF_1 - F_5 - 2u^2F_6 \right) + T \left( \frac{\partial}{\partial u^2} \right)_T(F_2 + F_6) + 4F_4) v^{\rho} u^{\mu\nu} + 2T \left( -2\frac{\partial}{\partial u^2} \right)_T(2F_1 - F_5 + 2u^2F_6) + 2T \left( \frac{\partial}{\partial u^2} \right)_T(F_2 - F_6) + F_2 + F_6) u^{\rho} u^{(\mu\nu)}, \quad (4.3.72e)
\]

where we have discarded terms in $T^{\mu\nu}$ proportional to $v^{\mu} v^{\nu}$.

As we have seen in Section 4.2 the NHS terms are defined as those contributions to the energy-momentum tensor for which
\[
-T^{\mu\nu} \epsilon_{\beta} \tau_{\mu} + \frac{1}{2} T^{\mu\nu} \epsilon_{\beta} h^{\mu\nu} = 0. \quad (4.3.73)
\]

Using equations (4.3.70a) and (4.3.70b) we can see that for this to be the case it is necessary that $\Sigma^{\mu\nu}_{\text{ext}}, \tilde{\eta}^{\mu\nu\rho\sigma}_{\text{int}}, \tilde{\eta}^{\mu\nu\rho\sigma}_{\text{ext}}$ and $\tilde{\eta}^{\mu\nu\rho\sigma}_{\text{int}}$ all vanish. The $\chi^{\mu\nu}$ is anti-symmetric under the interchange of the first pair of symmetric indices with the second pair of symmetric indices ensuring that there are no $L_{\beta} h^{\mu\nu}$ squared contributions to (4.3.73). The condition $\tilde{\eta}^{\mu\nu\rho\sigma} + \tilde{\eta}^{\rho\sigma\mu\nu} = 0$ guarantees that the $\tilde{\eta}^{\mu\nu\rho\sigma}$ is anti-symmetric under the interchange of the first pair of symmetric indices with the second pair of symmetric indices ensuring that there are no $L_{\beta} h^{\mu\nu}$ squared contributions to (4.3.73). Finally, cancellation of the cross terms $L_{\beta} h_{\mu\nu} L_{\beta} \tau_{\rho}$ requires that we set $\Delta^{\mu\nu\rho}$ —
We thus conclude that when \((4.3.74)\) holds, equation \((4.3.73)\) holds off shell. Setting \(F_1\) and \(F_2\) equal to their NHS values we obtain the following action for pure NHS transport at first order

\[
S_{\text{NHS}} = \int d^{d+1}x \left( F_3 u^\mu \partial_\mu T + F_4 u^\mu \partial_\mu u^2 - T F_5 v^\mu \partial_\mu \tau^\mu - 2 T F_6 u^\mu v^\nu \partial_\mu h_{\nu \nu} \right). \tag{4.3.75}
\]

The NHS currents are then schematically

\[
\mathcal{J}_{\text{NHS}}^\mu = \sum_{i=1}^6 \mathcal{J}_{\text{F}_i}^\mu|_{F_1 = T^{-1} F_5 + 2 T^{-1} u^2 F_6; \ F_2 = - F_6}, \tag{4.3.76}
\]

On shell and in Landau frame we have for \(\mathcal{J}^\mu = \sum_{i=1}^6 \mathcal{J}_{\text{F}_i}^\mu\) and \(\mathcal{J}^\nu = \sum_{i=1}^6 \mathcal{J}_{\text{F}_i}^\nu\) that

\[
\mathcal{T}_{\text{NHS}}^\mu + \mathcal{T}_{\text{NHS}}^\nu = \mathcal{T}^\mu + \mathcal{T}^\nu + \mathcal{T}^\sigma u_\sigma X_\rho^{\mu \nu} - \mathcal{T}^\rho X_\rho^{\mu \nu} = \frac{1}{2} \eta^\mu_{{\text{HS}}} \left( \frac{\xi_\alpha \partial_\alpha \tau_\beta - u_\alpha \xi_\beta \tau_\alpha}{2} \right) + \frac{1}{2} \eta^\nu_{{\text{HS}}} \left( \frac{\xi_\alpha \partial_\alpha \tau_\beta - u_\alpha \xi_\beta \tau_\alpha}{2} \right), \tag{4.3.77}
\]

where we remind the reader that \(\mathcal{T}^\mu\) is defined in equation \((4.3.34b)\). In here the tensors are given by

\[
\eta^\mu_{{\text{HS}}} = \eta^\mu_{{\text{HS}}} + \left( - \Sigma^\rho \delta_\beta + \eta^\rho_{{\text{HS}}} \delta_\beta \right) X_\rho^{\mu \nu} + \left( \Delta^\rho \delta_\beta - \Sigma^\rho \delta_\beta \right) u_\sigma X_\rho^{\mu \nu} \quad \frac{1}{2} \eta^\nu_{{\text{HS}}} \left( \frac{\xi_\alpha \partial_\alpha \tau_\beta - u_\alpha \xi_\beta \tau_\alpha}{2} \right), \tag{4.3.78a}
\]

The pure NHS part in Landau frame is given by

\[
\mathcal{T}_{\text{NHS}}^{\mu \nu} = \frac{1}{2} \eta^\mu_{{\text{HS}}} \left( \frac{\xi_\alpha \partial_\alpha \tau_\beta - u_\alpha \xi_\beta \tau_\alpha - u_\beta \xi_\alpha \tau_\beta}{2} \right), \tag{4.3.79}
\]

where \(\eta^\mu_{{\text{HS}}} = -\eta^\mu_{{\text{HS}}} \eta^\nu_{{\text{HS}}}\) is obtained by substituting \((4.3.74)\) into \((4.3.78a)\).

One might wonder what the expression for the pure HS part is. However, for the HS sector it is only the symmetric part of \(\eta^\mu_{{\text{HS}}} \eta^\nu_{{\text{HS}}}\) as well as the objects \(\eta^\mu_{{\text{HS}}} \eta^\nu_{{\text{HS}}}\) and \(\eta^\mu_{{\text{HS}}} \eta^\nu_{{\text{HS}}}\) that are uniquely determined. These all depend on two functions \(F_2 + F_5\) and \(TF_1 - F_5 + 2u^2 F_2\). There is no unique HS expression for the remaining four functions in the action. The reason
behind this is that we know that
\[-T^\mu_\beta e_\beta r_\mu + \frac{1}{2} T^\mu_\nu e_\nu h_\mu_\nu = e^{-1} \partial_\mu (e^{S^\mu}_{\non}) , \quad (4.3.80)\]

but this only uniquely fixes the symmetric part of $\eta^{\mu\nu\rho\sigma}$ as well as the extrinsic, torsion and rotation $\eta$-tensors. The anti-symmetric part cannot be fixed. This freedom is precisely encoded by the NHS terms. In a sense the HS coefficients belong to the “quotient space” of non-dissipative transport coefficients modulo the NHS ones. Hence two HS transport coefficients are equivalent if they differ by an NHS term. In Section 4.3.3 we picked a representative of the HS sector by setting $F_3 = F_4 = F_5 = F_6 = 0$.

The second line in $(4.3.78a)$ is anti-symmetric under interchanging the pair $\mu \nu$ with $\alpha \beta$ as can be seen from the fact that
\[-\chi^{\rho\sigma} + (\Delta^{\rho\lambda\sigma} - \Sigma^{\rho\lambda\sigma}) u_\lambda + \tilde{\eta}^{\rho\kappa\sigma\lambda} u_\kappa u_\lambda = 2T \left[ 2T u^2 \left( \frac{\partial F_6}{\partial T} \right) u^2 - 2T F_1 - 2u^2 F_6 + 2F_5 + 4u^2 \left( \frac{\partial F_5}{\partial u^2} \right) \right] u^{[\rho\sigma]} . \quad (4.3.81)\]

This means that the symmetric part of $\eta^{\mu\nu\alpha\beta}$ does not contain terms that are quadratic in $X_\rho^{\mu\nu}$. This explains why the $J$ coefficients in $(4.3.43)$ do not contain product of $\rho$ and $P$ and/or derivatives thereof, while the $A$ coefficients do admit such terms.

## 4.4 First order corrections

This section can be viewed as a continuation of Section 4.2, in which we use constitutive relations and non-negativity of entropy production to find all the allowed first order corrections to the boost-agnostic perfect fluid energy-momentum tensor. We also show how to recover Lifshitz fluids as well as Lorentz boost invariant fluids from our general framework.

### 4.4.1 Constitutive relations

Using our result $(4.3.63)$, the relation $(4.3.25)$ tells us that to second order in derivatives and in Landau frame,
\[ e^{-1} \partial_\mu (e^{S^\mu}) = -\frac{1}{2T} \left( T^{\mu\nu}_{(1)} - T^{\mu\nu}_{(1)HS} \right) (\xi_\mu h_\mu_\nu - h_\rho_\nu u^\rho \xi_\mu \tau_\mu - h_\mu_\rho u^\rho \xi_\mu \tau_\nu) , \quad (4.4.1) \]

where $T^{\mu\nu}_{(1)HS}$ is the Landau frame hydrostatic contribution as defined in $(4.2.7b)$, and $T^{\mu\nu}_{(1)}$ is the full energy-momentum tensor in Landau frame.

Since the divergence of the entropy current is a quadratic form in the derivatives of the fluid variables, equation $(4.4.1)$ tells us which derivatives we should use to write the constitutive relations for the energy-momentum tensor. The fluid variables are $\xi_\mu \tau_\mu$ and $\xi_\mu h_\mu_\nu$, and we may thus write the following constitutive relation for the part of the energy-
momentum tensor that is not of hydrostatic origin\(^7\)

\[
T^{\mu\nu}_{(1)} - T^{\mu\nu}_{(1)\text{HS}} = \frac{1}{2} \eta^{\mu\nu\rho\sigma} \xi_{\mu} h_{\rho \sigma} + \tilde{\xi}^{\mu\nu} \xi_{\mu} \tau_{\rho} . \tag{4.4.2}
\]

By redefining \(\tilde{\xi}^{\mu\nu}\), this can be written equivalently as

\[
T^{\mu\nu}_{(1)} - T^{\mu\nu}_{(1)\text{HS}} = \frac{1}{2} \eta^{\mu\nu\rho\sigma} (\xi_{\mu} h_{\rho \sigma} - h_{\kappa \sigma} u^{\kappa} \xi_{\mu} \tau_{\rho} - h_{\rho \kappa} u^{\kappa} \xi_{\mu} \tau_{\sigma}) + \zeta^{\mu\nu} \xi_{\mu} \tau_{\rho} . \tag{4.4.3}
\]

Upon substituting the constitutive relations into the right hand side of (4.4.1) we obtain a quadratic form. Non-negative entropy production will restrict the form of the \(\eta^{\mu\nu\rho\sigma}\) tensor, and it tells us that \(\zeta^{\mu\nu}\) must vanish. This is because there are no \(\xi_{\mu} \tau_{\rho}\) squared terms in (4.4.1). Furthermore, terms involving the anti-symmetric combination of velocity derivatives (analogous to the \(\eta^{\mu\nu\rho\sigma}_{\text{rot}}\) term in (4.3.41)) are also explicitly forbidden by the requirement that the divergence of the entropy current is a quadratic form. We thus conclude that

\[
T^{\mu\nu}_{(1)} - T^{\mu\nu}_{(1)\text{HS}} = \frac{1}{2} \eta^{\mu\nu\rho\sigma} (\xi_{\mu} h_{\rho \sigma} - h_{\kappa \sigma} u^{\kappa} \xi_{\mu} \tau_{\rho} - h_{\rho \kappa} u^{\kappa} \xi_{\mu} \tau_{\sigma}) , \tag{4.4.4}
\]

so that all that is left is to classify all the allowed terms that make up \(\eta^{\mu\nu\rho\sigma}\). This can be achieved by looking at the symmetries of the fluid.

In addition to the SO\((d)\) Ward identity (which is manifest in the symmetry of \(T^{\mu\nu}\)) the energy-momentum tensor must respect the symmetries of the thermal state around which we expand. In the absence of boost symmetries the thermal state spontaneously breaks the SO\((d)\) symmetry down to the SO\((d - 1)\) subgroup that preserves the velocity \(h^{\mu\rho} h_{\rho \nu} u^{\nu}\). In flat space these are the rotations preserving \(v^{\nu}\). In other words, different absolute values of velocities correspond to different thermodynamic states of the theory.

Therefore, the natural tensor structures are the SO\((d - 1)\) invariant tensors \(v^{\mu}\) as well as

\[
P^{\mu\nu} = h^{\mu\nu} - n^{\mu} n^{\nu} , \quad n^{\mu} = \frac{h^{\mu\nu} h_{\nu \rho} u^{\rho}}{\sqrt{u^{2}}} , \tag{4.4.5}
\]

where \(n^{\mu} n^{\nu} h_{\mu \nu} = 1\). The tensor \(P^{\mu\nu}\) is a projector onto the space orthogonal to the unit

\(^7\)The \(\eta\)-tensor in (4.4.2) should not be confused with the \(\eta\)-tensor that appears in the context of Lagrangian transport in (4.3.77).
vector \( n^\mu \). In terms of these tensor structures, the constitutive relation takes the form\(^8,9\)

\[
\eta^{\mu\nu\rho\sigma} = t \left( p^{\mu\rho}v^{\nu\sigma} + p^{\mu\sigma}v^{\nu\rho} - \frac{2}{d-1} p^{\mu\gamma}p^{\rho\sigma} \right) + \frac{4s_1}{u^2} v(\mu\nu)\eta(\rho\sigma) + s_2 n^\mu n^\nu n^\rho n^\sigma + s_3 p^{\mu\gamma}p^{\rho\sigma}
\]

leading to a total of 14 transport coefficients. The \( f_1 \) term in was also observed in \[123\].

We see that the \( \eta \)-tensor has a part that is anti-symmetric under interchanging the pairs of symmetric indices. This is related to non-hydrostatic non-dissipative transport and is the topic of Section 4.4.2. This leaves 10 coefficients that could contribute to dissipative transport. The normalisation of the 14 coefficients has been chosen such that all coefficients have the same scaling dimension which is \( d \), the number of spatial dimensions. We note that \( h^{\mu\nu} \) will be assigned a scaling dimension of 2 while \( v^\mu \) and \( u^\mu \) will have scaling dimension 3.

A unique feature of Landau frame is that the derivative corrections to the energy current is given entirely in terms of the \((2,0)\) momentum-stress tensor (cf., the second relation in \(4.3.21\)), which in turn means that the \((1,1)\) energy-momentum tensor \(4.1.28\) at first order can be constructed from the \((2,0)\) momentum-stress tensor. More precisely, defining \( T^{\mu\nu}_{(1)D,NHS} := T^{\mu\nu}_{(1)D} + T^{\mu\nu}_{(1)NHS} = T^{\mu\nu}_{(1)} - T^{\mu\nu}_{(1)HS} \), where each term is a symmetric tensor, we have

\[
( T^{\mu\nu}_{(1)D,NHS} )^\mu = \frac{1}{2} \left[ -\eta^{\mu\rho\kappa\lambda} u^\rho \tau_\nu + \eta^{\mu\rho\kappa\lambda} h^\rho \nu \right] ( \xi^\nu \xi^\nu \lambda - \xi^\nu \xi^\nu \lambda - \xi^\nu \xi^\nu \lambda + \xi^\nu \xi^\nu \lambda ),
\]

where we have used the relation \(4.1.28\). On flat space \(4.1.23\), where \( u^\mu = (1, \nu) \), the energy-momentum tensor, and by extension the tensor \( \eta^{\mu\nu\rho\sigma} \), may be further decomposed as

\[
( T^{(1)D,NHS}_0 )^\mu = \frac{1}{2} \eta_{ijk} ( \partial_k v^i + \partial_i v^k ) + \eta_{ijk} \partial_i v^k, \quad (4.4.8a)
\]

\[
( T^{(1)D,NHS}_j )^\mu = \frac{1}{2} \eta^{ijkl} ( \partial_k v^l + \partial_l v^k ) + \eta^{ijkl} \partial_i v^k, \quad (4.4.8b)
\]

\(^8\)We remark again that this object has two redundancies: we can add to \( \eta^{\mu\nu\rho\sigma} \) any term of the form \( v^\mu v^\nu Y^{\rho\sigma} \) or \( v^\mu v^\nu Z^{\rho\sigma} \) for arbitrary \( Y^{\mu\nu} \) and \( Z^{\mu\nu} \) without changing \( T^{\mu\nu} \).

\(^9\)The rationale behind the naming scheme we have adopted for the transport coefficients will become apparent in the next section (see in particular Eq. \(4.4.16\)), where we show that in the expression for the divergence of the entropy current, the coefficients \( \{ s_1, s_2, \ldots \} \) multiply scalar structures, the coefficients \( \{ f_1, f_2, \ldots \} \) multiply vector structures, while, finally, the coefficient \( t \) multiplies a single tensor structure.
where the flat space tensors $\kappa_{jk}$, $\eta_{jk1}$, $\kappa^{ijk}$ are given by
\[
\kappa_{jk} = \eta^{ijkl} \cdot \kappa_{l}^{jk}, \quad \eta_{jk1} = \eta^{ijkl} \cdot \kappa_{l}^{jk1}, \quad \kappa^{ijk} = \eta^{ijkl} \cdot \kappa_{l}^{jk}, \quad (4.4.9)
\]
which means that
\[
\kappa_{jk} = \frac{f_1}{\sqrt{v^2}} \cdot p_{jk} + \frac{s_1}{\sqrt{v^2}} \cdot n_{j} \cdot n_{k}, \quad (4.4.10a)
\]
\[
\eta_{jk1} = \frac{f_3 + f_{\text{NHS}}}{\sqrt{v^2}} \cdot (p_{jk} \cdot n_{l} + p_{jl} \cdot n_{k}) + \frac{s_3}{\sqrt{v^2}} \cdot n_{j} \cdot n_{l} \cdot n_{k}, \quad (4.4.10b)
\]
\[
\kappa^{ijk} = \frac{f_3 - f_{\text{NHS}}}{\sqrt{v^2}} \cdot (p_{ik} \cdot n_{j} + p_{ij} \cdot n_{k}) + \frac{s_3}{\sqrt{v^2}} \cdot p_{ij} \cdot n_{k} + \frac{s_2}{\sqrt{v^2}} \cdot n_{i} \cdot n_{j} \cdot n_{k}, \quad (4.4.10c)
\]
\[
\eta^{ijk1} = \frac{f_3}{\sqrt{v^2}} \cdot (p_{ik} \cdot p_{jl} + p_{il} \cdot p_{jk} - \frac{2}{d-1} p_{ij} \cdot p_{kl}) + s_3 \cdot p_{ij} \cdot p_{kl} + f_3 \cdot (p_{ik} \cdot n_{j} \cdot n_{l} + p_{jl} \cdot n_{i} \cdot n_{k} + p_{ij} \cdot n_{l} \cdot n_{k}) + s_2 \cdot n_{i} \cdot n_{j} \cdot n_{k}, \quad (4.4.10d)
\]
where the result has been written in terms of
\[
n^{i} = \frac{v^{i}}{\sqrt{v^2}}, \quad p^{ij} = \delta^{ij} - \frac{v^{i}v^{j}}{v^2} = \delta^{ij} - n^{i}n^{j}, \quad (4.4.11)
\]
which are the flat space versions of (4.4.5).

### 4.4.2 Non-hydrostatic non-dissipative transport & Onsager relations

The subsector of transport obtained by isolating the anti-symmetric part of $\eta$, i.e., $\eta^\mu_{\nu\rho\sigma} \subset\n\eta^\mu_{\nu\rho\sigma} with \eta^\mu_{\nu\rho\sigma} = -\eta^\rho_{\mu\nu\sigma}$, corresponds to the non-hydrostatic (NHS) non-dissipative transport. By using (4.4.7) and (4.4.1), such terms trivially produce no entropy. The constitutive relations tell us that there are at most 4 transport coefficients of this type. In Section 4.3.5 we found precisely 4 terms in the action that corresponded to the NHS sector. Extracting the anti-symmetric part of (4.4.6), we get
\[
\eta^\mu_{\nu\rho\sigma}_{\text{A}} = \frac{4f_{\text{NHS}}}{\sqrt{u^2}} \cdot (\eta^\mu_{\nu\rho\sigma}(\rho_{\nu\sigma}) - \eta(\rho_{\nu\sigma})^\mu_{\nu\sigma}) + \frac{2s_1}{\sqrt{u^2}} \cdot (p_{\mu\nu\rho}(\rho_{\nu\sigma}) - p_{\rho\sigma}n^\mu_{\nu\sigma}) + \frac{2s_2}{\sqrt{u^2}} \cdot (n^\mu_{\nu\rho}n^\rho_{\nu\sigma} - n^\rho_{\nu\sigma}n^\mu_{\nu\sigma}) + s_3 \cdot (p_{\mu\nu\rho}n^\rho_{\nu\sigma} - p_{\rho\sigma}n^\mu_{\nu\sigma}) \cdot (4.4.12)
\]

Demanding the absence of NHS transport is, at linear order, equivalent to the Onsager relations [246,247], which express the fact that there are no anti-symmetric contributions to the $\eta$-tensor in systems with time-reversal symmetry. More explicitly, consider linearised perturbations around global thermal equilibrium $(T_0, v_0^i)$ in flat space,
\[
v^i = v_0^i + \delta v^i, \quad T = T_0 + \delta T,
\]

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where $\delta v^i$ and $\delta T$ are the fluctuations. The resulting change in the energy-momentum tensor to first order in fluctuations is

$$\delta T^{\mu\nu}_{(1)} = \frac{1}{2} \eta^{\mu\nu\rho\sigma} (\partial_\rho \delta u_\sigma + \partial_\sigma \delta u_\rho),$$

where $\delta u^\mu = (0, \delta v^i)$ and indices are lowered by $h_{\mu\nu} = \delta_{i\mu} \delta_{i\nu}$. The Onsager relations then tell us that

$$\eta^{\mu\nu\rho\sigma}_{(1)} = \eta^{\rho\sigma\mu\nu}_{(1)},$$

which is the linearised version of the general requirement of symmetry, $\eta^{\mu\nu\rho\sigma} = \eta^{\rho\sigma\mu\nu}$.

Imposing symmetry on the $\eta$-tensor is equivalent to the vanishing of all NHS coefficients,

$$f^{\text{NHS}} = s_1^{\text{NHS}} = s_2^{\text{NHS}} = s_3^{\text{NHS}} = 0. \quad (4.4.13)$$

Hence, ignoring the NHS sector, we obtain the following constitutive relations for the dissipative sector

$$\kappa_{jk} = \frac{f_1}{v^2} p_{jk} + \frac{s_1}{v^2} n_j n_k, \quad (4.4.14a)$$

$$\kappa^{ijk} = \frac{f_3}{v^2} (p^{ik} n^j + p^{jk} n^i) + \frac{1}{\sqrt{v^2}} s_3 p^{ij} n^k + \frac{1}{\sqrt{v^2}} s_4 n^j n^i n^k, \quad (4.4.14b)$$

$$\eta^{ijkl} = t \left( p^{ik} p^{jl} + p^{il} p^{jk} - \frac{2}{d-1} p^{ij} p^{kl} \right)$$

$$+ f_2 (p^{ik} n^j n^l + p^{jk} n^i n^l + p^{il} n^j n^k + p^{ij} n^k n^l)$$

$$+ s_3 p^{ij} p^{kl} + s_6 (p^{kl} n^i n^j + p^{ij} n^k n^l) + s_2 n^i n^j n^k n^l, \quad (4.4.14c)$$

leaving us with 10 candidate coefficients for dissipative transport. Note that in the absence of NHS terms we have the identity $\eta_{ijkl} = \eta_{ij0kl} = \eta_{kj0il} = \kappa_{ijkl}$ (cf., (4.4.9) and (4.4.10a)–(4.4.10d)). For the remainder of this section we will be working in flat space. In the next subsection we will show that all these coefficients contribute to dissipation provided they obey suitable inequalities.

### 4.4.3 Dissipative transport

In this section, we derive additional constraints on the dissipative transport coefficients from the requirement of positivity of entropy production. Using our results (4.4.14a)–(4.4.14c), the divergence of the entropy current in Landau frame (4.4.1) on flat space along with the

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10One way to see this is because $\langle \partial_\mu \delta u_\nu (t) \delta T^{\mu\nu}_{(1)} (0) \rangle = \langle \partial_\mu \delta u_\nu (0) \delta T^{\mu\nu}_{(1)} (t) \rangle$, which is a direct result of time reversal symmetry.
requirement that it be positive definite for the dissipative sector reads

$$-T \partial_\mu S^\mu = \frac{1}{4} \eta^{ijkl} (\partial_i v^j + \partial_j v^i) (\partial_k v^l + \partial_l v^k) + \kappa_{ij} \partial_i v^i \partial_t v^j$$

$$+ \kappa^{ijk} (\partial_i v^j + \partial_j v^i) \partial_t v^k \leq 0 ,$$

(4.4.15)

with equality if and only if all the dissipative coefficients are zero.

As in Eq. (3.1.40), it is useful to decompose the expression (4.4.15) into scalar, vector and tensor sectors,

$$\vec{b}^{(S)} T A^{(S)}_{(3 \times 3)} \vec{b}^{(S)} + \vec{b}^{(V)} T A^{(V)}_{(2 \times 2)} \vec{b}^{(V)} + t A^{(T)} \leq 0 ,$$

(4.4.16)

where the basis vector of scalars is

$$\vec{b}^{(S)} = \begin{pmatrix} \frac{1}{2} \sqrt{\nu} \partial_t v^2 \\ n^i \eta^{ij} \partial_1 v^j \\ p^{ij} \partial_t v^i \end{pmatrix} ,$$

(4.4.17)

with the associated quadratic form

$$A^{(S)}_{(3 \times 3)} = \begin{pmatrix} s_1 & s_4 & s_5 \\ s_4 & s_2 & s_6 \\ s_5 & s_6 & s_3 \end{pmatrix} .$$

(4.4.18)

The basis vector of vectors is

$$\vec{b}^{(V)}_i = \begin{pmatrix} \frac{1}{\sqrt{\nu}} \partial_t v^i \\ p^{ij} \eta^{jk} (\partial_j v^k + \partial_k v^j) \end{pmatrix} ,$$

(4.4.19)

which has the associated quadratic form

$$A^{(V)}_{(2 \times 2)} = \begin{pmatrix} f_1 & f_3 \\ f_3 & f_2 \end{pmatrix} ,$$

(4.4.20)

The single tensor structure is described by

$$A^{(T)} = \left( p^{ik} p^{jl} + p^{il} p^{jk} - \frac{2}{d-1} p^{ij} p^{kl} \right) (\partial_i v^j + \partial_j v^i) (\partial_k v^l + \partial_l v^k) .$$

(4.4.21)

Positivity of entropy production must hold for all fluid configurations and thus for each of the three sectors separately.

Just like in Section 3.1.4, we can now use Sylvester’s criterion (Thm. 1) to determine the inequality-type constraints on the dissipative transport coefficients imposed by the second law of thermodynamics. The tensor contribution requires that

$$t \leq 0 .$$

(4.4.22)
The quadratic form of the vector sector must be negative definite, i.e. all its eigenvalues must be negative, which is the case if and only if

\[ f_1 \leq 0, \quad f_2 \leq 0, \quad \det A^{(V)}_{(2 \times 2)} = f_1 f_2 - f_2^2 \geq 0. \] (4.4.23)

Finally, the quadratic form of the scalar sector must be negative definite as well, which gives the conditions

\[ s_1 \leq 0, \quad s_2 \leq 0, \quad s_3 \leq 0, \] (4.4.24a)

\[ s_1 s_2 - s_3^2 \geq 0, \quad s_1 s_3 - s_2^2 \geq 0, \quad s_3 s_2 - s_1^2 \geq 0, \] (4.4.24b)

\[ \det A^{(S)}_{(3 \times 3)} = s_1 s_2 s_3 - s_3^2 s_4^2 - s_2^2 s_6^2 + 2 s_4 s_5 s_6 - s_1 s_2^2 \leq 0. \] (4.4.24c)

At this stage, one may wonder whether these inequalities can be simultaneously satisfied. It can be checked that this is indeed the case; one explicit example is

\[ t = -1, \quad f_1 = -1, \quad f_2 = -1, \quad f_3 = 0, \quad s_1 = -\frac{1}{2}, \] (4.4.25)

\[ s_2 = -4, \quad s_3 = -3, \quad s_4 = -1, \quad s_5 = -1, \quad s_6 = -1. \]

### 4.4.4 Scale invariance: Lifshitz fluid dynamics

If our fluid enjoys scale symmetry with dynamical exponent \( z \), the following Ward identity must be satisfied (see also [120] for more details)

\[ zT^{(0)}_{00} + T^i_i = 0. \] (4.4.26)

Using the constitutive relations for the dissipative sector (4.4.14a)–(4.4.14c), this gives rise to the following relations at first order

\[ zv^i \kappa^i_k = \delta_{ij} \kappa^{i;k}, \quad zv^i \eta^i_{jk} = \delta_{ij} \eta^{i;k}, \] (4.4.27)

which amounts to

\[ zs_5 = (d - 1)s_3 + s_6, \]

\[ zs_4 = (d - 1)s_6 + s_2, \] (4.4.28)

\[ zs_1 = (d - 1)s_5 + s_4. \]

Hence, if we impose scale symmetry, the 6 dissipative scalar transport coefficients get reduced by 3, while the 3 vector and 1 tensor transport coefficients are unaffected. In other words, uncharged Lifshitz hydrodynamics has 7 dissipative transport coefficients at first order. We note that the inequality type constraints for Lifshitz fluids are obtained by substituting the relations (4.4.28) into (4.4.24a)–(4.4.24c).

We furthermore remark that since all transport coefficients that appear in the \( \eta \)-tensor
are functions of $T$ and $v^2$ and have scaling dimension $d$, they must be of the form
\[ T^{d/2} f(\alpha), \quad \alpha = v^2 T^{\frac{2}{z} - 2}, \tag{4.4.29} \]
for some unknown function $f$, where $\alpha$ has no scaling dimension.

Turning to the NHS sector, which is described by (4.4.12), we find that scale symmetry gets rid of two coefficients. It sets
\[ s_{2\text{NHS}} = -(d-1)s_{1\text{NHS}} \text{ and } s_{3\text{NHS}} = -z s_{1\text{NHS}}, \]
and leaves $f_{\text{NHS}}$ free. Furthermore, scale symmetry reduces the number of hydrostatic transport coefficients from two to one. This can be seen as follows. The $z$-trace Ward identity on curved space reads
\[-z v^\gamma \tau_\mu T^{\mu \gamma} + h_\mu ^{\rho \nu} T^{\mu \nu} = -z \tau_\mu T^{\mu} + h_\mu ^{\nu \gamma} T^{\mu \nu} = 0. \tag{4.4.30}\]
If we substitute (4.3.35a) and (4.3.35b) as well as (4.3.37) into this Ward identity we find that (by looking at the terms proportional to the trace of the extrinsic curvature)
\[ F_1 = -2F_2 \frac{u^2(z-1)}{zT}. \tag{4.4.31} \]
The rest of the terms in (4.4.30) then tell us that
\[ p(T, u^2) = T^{1 + \frac{d}{2}} p(\alpha), \quad F_2(T, u^2)T^{2 - \frac{2}{z}} = T^{\frac{d}{2}} q(\alpha), \tag{4.4.32} \]
where $p(\alpha)$ and $q(\alpha)$ are arbitrary functions of $\alpha$ which is the scale invariant combination $\alpha = u^2 T^{\frac{2}{z} - 2}$. We can then write the hydrostatic partition function (4.4.12) as
\[ S_{\text{HS(Lif)}} = \int d^{d+1}x e \left( T^{1 + \frac{d}{2}} p(\alpha) + T^{\frac{d}{2}} q(\alpha) u^\mu \partial_\mu \alpha \right) + O(\partial^2). \tag{4.4.33} \]
The same can be done for the action (4.3.75) describing NHS transport, which for Lifshitz scaling takes the form
\[ S_{\text{NHS(Lif)}} = \int d^{d+1}x e \left( T^{\frac{d}{2}} r_1(\alpha) u^\mu \partial_\mu \alpha + T^{\frac{d}{2} - \frac{2z+2}{z}} r_2(\alpha) u^\mu \nu \xi_\rho h_{\mu \nu} \right) + O(\partial^2). \tag{4.4.34} \]
exhibiting two NHS transport coefficients in agreement with the statement above.

All together for an uncharged Lifshitz fluid we find 7 dissipative, 1 HS and 2 NHS transport coefficients.

Interestingly, it was shown in [120] that Lorentzian boost symmetry can only be restored when $z = 1$, while Galilei boosts can only be imposed when $z = 2$, which produces a Schrödinger fluid. This provides a no-go theorem for scale invariant fluids, which can only be boost invariant when $z = 1$ or $z = 2$.

### 4.4.5 Lorentz boost invariance

What we have obtained is the most general set of first order transport coefficients without assuming boost invariance. In this subsection, we show how to recover relativistic first
order hydrodynamics from our results. In Landau frame and on Minkowski spacetime the relativistic energy-momentum tensor is given by \((3.1.28)\), which we here reproduce for convenience

\[
T_{\mu\nu} = \tilde{\epsilon} U^\mu U^\nu + P \Pi^\mu \gamma - \tilde{\zeta} \Pi^\mu \gamma \partial_\rho U^\rho - \eta \Pi^\mu \Pi^\nu \sigma_{\rho\sigma},
\]

(4.4.35)

where \(\tilde{\zeta}\) and \(\eta\) are the bulk and shear viscosity terms, respectively, which are independent of \(v^2\) in the Lorentzian case, and where the velocity \(U\) is given by

\[
U^\mu = \gamma (1, v^i), \quad U^\mu = \gamma (-1, v^i),
\]

(4.4.36)

where \(\gamma = (1 - v^2)^{-1/2}\). In writing the relativistic energy-momentum tensor \((4.4.35)\), we defined the projector \(\Pi^\mu \gamma = \delta^\mu_\nu + U^\mu U^\nu\), while the shear tensor \(\sigma_{\rho\sigma}\) is given in \((3.1.26)\).

A necessary condition for this to be recovered from our general framework is the boost Ward identity, \(T^0_i(1) = -T^i_0(1)\), where \(T^0_i(1) = -v^j T^i_j(1)\) in Landau frame. Using \((4.4.8a)\) and \((4.4.8b)\) we see that this translates into the requirements

\[
v^j \eta_{ijkl} = \kappa_{kli}, \quad v^j \kappa_{ijk} = \kappa_{ik}.
\]

(4.4.37)

This implies the relations

\[
f_3 = v^2 f_2, \quad f_1 = v^2 f_3, \quad s_5 = v^2 s_6, \quad s_4 = v^2 s_2, \quad s_1 = v^2 s_4.
\]

(4.4.38)

Hence in Landau frame it is sufficient to compare our expression for \(T^i_j(1)\) with \((4.4.35)\) at first order. A tedious calculation shows that the two expressions agree if and only if

\[
s_3 = -\tilde{\zeta} \gamma - \frac{2}{d(d - 1)} \eta \gamma, \\
s_6 = -\tilde{\zeta} \gamma^3 + \frac{2}{d} \eta \gamma^3, \\
s_2 = -\tilde{\zeta} \gamma^5 - 2 \eta \gamma^5 + \frac{2}{d} \eta \gamma^5, \\
f_2 = -\eta \gamma^3, \\
t = -\eta \gamma,
\]

(4.4.39)

so that we recover the standard transport coefficients \(\tilde{\zeta}\) and \(\eta\) that feature in \((4.4.35)\). The inequalities \((4.4.22)\) and \((4.4.24a)\) for the boost-agnostic transport coefficients imply via the above relations the inequalities

\[
\eta \geq 0, \quad \tilde{\zeta} \geq 0,
\]

(4.4.40)

recovering our previous results for relativistic fluids \((3.1.43)\). In this way, it is also possible to recover (massless) Galilean invariant fluids, although we refrain from giving the details.
Chapter 5

Nonrelativistic closed string theory

In this chapter, which is based on the publications [3, 7], develops the theory of nonrelativistic closed string theory using the technology of $1/c^2$ expansions, which we discussed in Section 2.12.

The chapter is organised as follows. In Section 5.1, we describe the string $1/c^2$ expansion that lies at the heart of the expansion of string theory that this paper investigates, and we discuss the interpretation of the string $1/c^2$ expansion as an expansion around a decompactification limit. We expand the spectrum of a closed relativistic bosonic string on a background with a compact circle that is wound by the string. Then, in Section 5.2, we consider in detail the gauge structure of type II SNC geometry. In Section 5.3, we expand both the Nambu–Goto and Polyakov actions up to NNLO. Furthermore, we employ the string $1/c^2$ expansion of gravity and demonstrate that the LO part of Einstein’s equations imposes a two-dimensional foliation structure in the sense of Frobenius on the longitudinal target space. We also discuss the LO equation of motion for the embedding fields and show that when $\alpha^{A, B}$ in (1.3.3) is traceless, this equation of motion is automatically satisfied. In Section 5.4, we explicitly demonstrate the equivalence between the Gomis–Ooguri string generalised to a curved background, and the NLO theory when the background satisfies (1.3.3) with $\alpha^{A, \lambda} = 0$. Following this, we discuss the role of the WZ term in Section 5.5 and show that we can cancel the LO theory (and thereby remove the need to consider various foliation constraints) by fine-tuning the Kalb–Ramond field. We also discuss Stückelberg symmetries between the subleading longitudinal geometric fields and the Kalb–Ramond field. We then go on to consider the spectrum on flat space in Section 5.6, which involves fixing the residual gauge redundancies. At NLO, this reproduces the spectrum of the Gomis–Ooguri string, while the spectrum of the NNLO theory matches the result obtained by $1/c^2$ expanding the relativistic spectrum in Section 5.1.1. In Section 5.7, we consider the target space symmetries of the $1/c^2$ expanded string theories and show that the symmetry algebra corresponds to the string $1/c^2$ expansion of the Poincaré algebra. In Section 5.8, we develop the phase space formulation of the LO, NLO and NNLO string theories. Concretely, this is achieved by $1/c^2$ expanding the relativistic phase space action, and we go through the Dirac procedure and find the Dirac brackets at each order in Section 5.8.2. We then quantise the theories in Section 5.8.3 by deriving the commutators and writing down the normal ordering constant.
5.1 The string $1/c^2$ expansion

In this section we further develop the string $1/c^2$ expansion that we already described in Section 2.12. This formalism forms the basis of the remainder of this chapter.

5.1.1 Nonrelativistic expansion of the string spectrum

The nonrelativistic limit of a string is much more subtle than the nonrelativistic limit of a particle. If one were to consider the limit of a string in a TNC geometry, cf., Section 2.11, one finds that the string cannot oscillate and simply moves along like an extended solid object. Thus, rather than considering a “particle limit” (or expansion), we must consider a “string limit” (or expansion), where two directions are singled out, as discussed in Section 2.12. Additionally, it can be shown that, unless the spatial direction that is singled out is compact, all closed strings acquire infinity energy and become unphysical [63]. When this direction is compact, the string can wind around it, and this winding will play the rôle of the string mass.

Another way to look at this is that, in order to be able to define a nonrelativistic sector of string theory, which has an intrinsic length scale given by the string length $\ell_s \sim \sqrt{\hbar/(cT)}$, we need an additional length scale to form a dimensionless expansion parameter. We precisely achieve this by considering flat target space with a compact direction, i.e. $\mathbb{R}^{1,24} \times S^1_R$ where the radius of the circle is $R$. We introduce coordinates $x^M = \{t, x^i, v\}$ on this space, where $\{t, x^i\}$ for $i = 1, \ldots, 24$ are coordinates on $\mathbb{R}^{1,24}$ and $v$ is the coordinate on $S^1_R$ which we take to have dimensions of time (although $v$ remains a spatial direction). This means that $v$ is periodically identified according to

$$v \sim v + 2\pi R_{\text{eff}}, \quad (5.1.1)$$

where we defined the effective radius with dimensions of time as

$$R_{\text{eff}} = R/c, \quad (5.1.2)$$

which we assume is independent of $c$. The directions $t$ and $v$, which both have dimensions of time, will be referred to as the longitudinal directions, while the directions $x^i$, which have dimensions of length, will be called transverse.

The line element on $\mathbb{R}^{1,24}$ in the coordinates $\{t, x^i, v\}$ is given by

$$ds^2 = \eta_{MN} dx^M dx^N = c^2 (-dt^2 + dv^2) + dx^i dx^i, \quad (5.1.3)$$

where the components of the Minkowski metric explicitly are given by $\eta_{MN} = c^2 (-\delta^t_t \delta^v_v + \delta^v_v + \delta^i_i) + \delta^i_i \delta^i_i$.

The nonrelativistic expansion, as we will see, ultimately corresponds to an expansion where $R \gg \ell_s$, which, as discussed in [3], thus admits an interpretation as a decompactification limit. Closed relativistic bosonic strings on $\mathbb{R}^{1,24} \times S^1_R$ with metric $\eta_{MN}$ are described by
the Polyakov Lagrangian
\[ L_P = -\frac{cT}{2} \oint d\sigma^1 \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \eta_{MN}, \] (5.1.4)

where the string embedding fields \( X^M(\sigma^0, \sigma^1) = \{X^t(\sigma^0, \sigma^1), X^v(\sigma^0, \sigma^1), X^i(\sigma^0, \sigma^1)\} \) split into longitudinal components \( \{X^t, X^v\} \) corresponding to the embedding fields in the time direction and in the compact direction \( v \), while the \( X^i \) for \( i = 1, \ldots, 24 \) are the transverse embedding fields. Like \( t \) and \( v \), both \( X^t \) and \( X^v \) have dimensions of time. The transverse embedding fields have dimensions of length, while the worldsheet coordinates \( \sigma^\alpha = (\sigma^0, \sigma^1) \) are dimensionless. The worldsheet metric \( \gamma^{\alpha\beta} \) is also dimensionless, and \( \gamma = \det(\gamma^{\alpha\beta}) \) is its dimensionless determinant. Finally, \( T \) is the string tension with dimensions of mass/length. The combination \( \partial_\alpha X^M \partial_\beta X^N \eta_{MN} \) has dimensions of length squared.

The invariant mass squared of a closed bosonic string in such a target space is given by
\[ M^2 = \frac{\hbar^2 n^2}{c^2 R^2} + \frac{w^2 R^2}{\alpha'^2 c^2} + \frac{2}{\alpha' c^2} (N + \tilde{N} - 2\hbar), \] (5.1.5)

where \( \alpha' = \frac{1}{2\pi T} \), and \( w \) and \( n \) are, respectively, the winding number and momentum mode in the compact direction. The winding number \( w \) counts the number of times the closed string winds around the circle \( S^1_R \), while the momentum mode \( n \) comes from the quantised centre of mass momentum of the string in the \( v \)-direction. The number operators \( N \) and \( \tilde{N} \), which satisfy \( N - \tilde{N} = \hbar nW \), have dimensions of energy \( \times \) time. The relativistic dispersion relation that relates the invariant mass squared to the Noether charges corresponding to energy \( E \) and spatial momentum \( p_i \) is
\[ E^2 = M^2 c^4 + p^2 c^2, \] (5.1.6)

where \( p^2 = \vec{p} \cdot \vec{p} \) is the norm squared of the spatial momentum. In addition to (5.1.2), we define the (by assumption) \( c \)-independent combination
\[ \alpha'_{\text{eff}} = \frac{\alpha'}{c}, \] (5.1.7)

which we may equivalently express in terms of the effective tension \( T_{\text{eff}} = cT \), which is related to \( \alpha'_{\text{eff}} \) as
\[ \alpha'_{\text{eff}} = \frac{1}{2\pi T_{\text{eff}}}. \] (5.1.8)

The effective string tension \( T_{\text{eff}} \) has dimensions of mass/time, while \( \alpha'_{\text{eff}} \) has dimensions of time/mass. In terms of the quantities introduced above, we may write down the following dimensionless parameter
\[ \epsilon = \alpha'^2 \hbar = \frac{\alpha'_{\text{eff}}^2 \hbar}{c^2 R_{\text{eff}}^2}, \] (5.1.9)
in terms of which the energy as defined in (5.1.6) can be written as

\[ E = \frac{c^2 w R_{\text{eff}}}{\alpha'_\text{eff}} \sqrt{1 + \frac{2e}{w^2} \left( \frac{N + \tilde{N}}{\hbar} - 2 \right) + \frac{\alpha'_\text{eff}}{\hbar w^2} p^2 + \frac{c^2 n^2}{w^2}}. \quad (5.1.10) \]

The expansion in \( c^{-2} \) is the same as the expansion in \( \epsilon \). If we define

\[ E = c^2 E_{\text{LO}} + c^{-2} E_{\text{NLO}} + O(c^{-4}), \quad (5.1.11) \]

then we find

\[ E_{\text{LO}} = \frac{w R_{\text{eff}}}{\alpha'_\text{eff}}, \quad (5.1.12a) \]

\[ E_{\text{NLO}} = \frac{1}{w R_{\text{eff}}} (N_{(0)} + \tilde{N}_{(0)} - 2h) + \frac{\alpha'_\text{eff}}{2w R_{\text{eff}}}(p_{(0)})^2, \quad (5.1.12b) \]

\[ E_{\text{NNLO}} = \frac{1}{w R_{\text{eff}}} (N_{(2)} + \tilde{N}_{(2)} + \alpha'_\text{eff}p_{(0)}p_{(2)}) + \frac{\alpha'_\text{eff}h^2 n^2}{2w R_{\text{eff}}} \]

\[ - \frac{\alpha'_\text{eff}}{2w^3 R_{\text{eff}}^3} (N_{(0)} + \tilde{N}_{(0)} - 2h + \frac{\alpha'_\text{eff}}{2}(p_{(0)})^2)^2, \quad (5.1.12c) \]

where we expanded

\[ N = N_{(0)} + c^{-2} N_{(2)} + O(c^{-4}), \quad \tilde{N} = \tilde{N}_{(0)} + c^{-2} \tilde{N}_{(2)} + O(c^{-4}), \]

\[ p_i = p_{(0)i} + c^{-2} p_{(2)i} + O(c^{-4}). \quad (5.1.13) \]

Note that we did not expand the momentum mode \( n \) and the winding number \( w \) since they are integer-valued, although we could have done so abstractly in which case they would also lead to subleading contributions in the same way as \( N, \tilde{N}, \) and \( p \) above. As we pointed out in [3], the nonrelativistic limit that we are considering corresponds to an expansion in the dimensionless quantity \( \epsilon \), which we see can equivalently be thought of as a \( 1/c^2 \) expansion or a \( 1/R^2_{\text{eff}} \) expansion—that is to say, an expansion around a decompactification limit.

The expansion in \( \epsilon \ll 1 \) can be viewed as \( c R_{\text{eff}} \gg \sqrt{\alpha'_\text{eff} \hbar} \) which means that the radius of the circle is much larger than the string length. Alternatively we can view it as saying that \( \frac{\alpha'_\text{eff} \hbar}{c R_{\text{eff}}} \ll c \) which means that the velocity of the centre of mass momentum mode \( p_{25} \) along \( X^{25} = c X^v \) is much smaller than the speed of light.

An important ingredient in string theory is the Kalb–Ramond 2-form field \( B_{\text{MN}} \), which together with the metric and the dilaton forms the universal massless sector of closed string theory. The coupling between the string embedding fields and the Kalb–Ramond field is described by the Wess–Zumino Lagrangian

\[ L_{\text{WZ}} = -\frac{cT}{2} \int \! d\sigma^1 \epsilon^{\alpha \beta} \partial_\alpha X^M \partial_\beta X^N B_{\text{MN}}(X). \quad (5.1.14) \]

As we show in Section 5.5, if we add a constant Kalb–Ramond B-field with legs only in the
timelike and compact directions of the form $B_{MN} = 2c^2 B_\delta \delta_{|M}^\nu \delta_{N}^\tau$, the energy, which is defined by $E = - \oint d\sigma^1 \frac{\partial L}{\partial \dot{X}^0} \dot{X}^0$, takes the same form except that the LO energy is now

$$E_{\text{LO}} = \frac{wR_{\text{eff}}}{\alpha_{\text{eff}}^2} (1 - B),$$

(5.1.15)

where we point out that such a Kalb–Ramond 2-form with constant components along $t$ and $v$ is not globally pure gauge because $v$ is periodic.

By tuning the $B$-field, we may for example remove the leading order term entirely, while choosing $B = 1/2$ leads to the spectrum of the Gomis–Ooguri string [63] when truncating at $O(c^0)$.

### 5.1.2 Longitudinal T-duality and the $1/c^2$ expansion

The spectrum of the relativistic closed string (5.1.5) is invariant under T-duality in the $v$-direction, which amounts to the exchanges

$$R \leftrightarrow \frac{\hbar \alpha'}{c R} =: \tilde{R} \quad \text{and} \quad w \leftrightarrow n.$$

(5.1.16)

Adopting the terminology of [72], this is a longitudinal spatial T-duality. To explore the rôle of T-duality in the context of the string $1/c^2$ expansion, it is useful to recast the spectrum (5.1.5) in the form

$$M^2 = \frac{n^2 \tilde{R}}{\alpha'^2} + \frac{w^2 \alpha'^2}{\alpha'^2} + \frac{2}{\alpha' c} (N + \tilde{N} - 2\hbar),$$

(5.1.17)

which is manifestly invariant under (5.1.16). This suggests, in addition to the expansion set up in (5.1.10) in terms of the dimensionless expansion parameter $\epsilon$ defined in (5.1.9), another dual expansion in terms of the dual dimensionless expansion parameter

$$\tilde{\epsilon} = \frac{\alpha' \hbar}{c R^2}.$$

(5.1.18)

The parameter $\tilde{\epsilon}$ is the T-dual of (5.1.9); i.e., longitudinal T-duality (5.1.16) sends $\epsilon \leftrightarrow \tilde{\epsilon}$. The starting point for the expansion in terms of $\tilde{\epsilon}$ is thus

$$E = \frac{\hbar}{R_{\text{eff}}} \sqrt{1 + \frac{2\tilde{\epsilon}}{n^2} \left( \frac{N + \tilde{N}}{\hbar} - 2 \right) + \frac{\alpha' \tilde{\epsilon}}{\hbar n^2} \tilde{\epsilon}^2 + \frac{w^2 \tilde{\epsilon}^2}{n^2}},$$

(5.1.19)

which has an expansion around $\tilde{\epsilon} = 0$ of the form

$$E = c^2 E_{\text{LO}} + \tilde{E}_{\text{NLO}} + c^{-2} \tilde{E}_{\text{NNLO}} + O(c^{-4}),$$

(5.1.20)
where now
\[ \tilde{E}_{\text{LO}} = \frac{n \tilde{R}_{\text{eff}}}{\alpha'_{\text{eff}}}, \]  
\[ \tilde{E}_{\text{NLO}} = \frac{1}{n \tilde{R}_{\text{eff}}} (N(0) + \tilde{N}(0) - 2h) + \frac{\alpha'_{\text{eff}}}{2n \tilde{R}_{\text{eff}}} (p_{(0)})^2, \]  
\[ \tilde{E}_{\text{NNLO}} = \frac{1}{n \tilde{R}_{\text{eff}}} (N(0) + \tilde{N}(0) - 2h + \frac{\alpha'_{\text{eff}}}{2} (p_{(0)})^2), \]  
(5.1.21a)
(5.1.21b)
(5.1.21c)
where we defined the $c$-independent combination $\tilde{R}_{\text{eff}} = \tilde{R}/c$ in the T-dual picture. Thus, longitudinal T-duality switches between the two expansions (5.1.11) and (5.1.20), and although the relativistic energy (5.1.10) remains T-duality invariant, the $1/c^2$ expansion is no longer longitudinal T-duality invariant order by order. Instead, $\tilde{E}_{\text{NLO}}$ transforms into $E_{\text{NLO}}$ under the replacements (5.1.16). We stress that both expansions correspond to decompactification limits in terms of the radii $R$ and $\tilde{R}$.

### 5.2 Gauge structure of $1/c^2$ expanded Lorentzian geometry

In this section, we derive the gauge transformations that appear at each order in the string $1/c^2$ expansion of a $(d + 2)$-dimensional Lorentzian geometry that we considered in Section 5.1. We adopt the same strategy as in [24], where the analogous particle $1/c^2$ expansion of Lorentzian geometry was developed.

It is useful to define Lorentzian vielbeins $\hat{e}_M^{\tilde{A}}$ with $\tilde{A} = 0, 1, \ldots, d + 1$ via
\[
G_{MN} = \eta_{\tilde{A} \tilde{B}} \hat{e}_M^{\tilde{A}} \hat{e}_N^{\tilde{B}},
\]
\[
\hat{e}_M^{\tilde{A}} = cT_M^A \delta^{\tilde{A}}_A + \varepsilon_M^{A' \tilde{A}'} \delta^{\tilde{A}'}_{A'},
\]  
(5.2.1)
where the second line is the string $1/c^2$ expansion of the Lorentzian vielbein. The Lorentzian vielbein $\hat{e}_M^{\tilde{A}} = (cT_M^A, \varepsilon_M^{A'})$ transforms under diffeomorphisms and local Lorentz transformations $\hat{L}_\tilde{B}^\tilde{A} = (L^A_B, L^{A'}_{B'})$. We can write the local Lorentz transformations as
\[
\hat{L}_\tilde{B}^\tilde{A} = \delta^{\tilde{A} \tilde{B}}_C e^{CA}_{\text{LO}} + c^{-1} \delta^{\tilde{A} \tilde{B}}_C e^{CA}_{\text{NNLO}} + c^{-1} \delta^{B' \tilde{A}'}_B e^{BA'}_{\text{LO}} + \delta^{\tilde{A} B'_C} e^{CA'}_{\text{NNLO}},
\]  
(5.2.2)
where factors of $c$ above have been chosen to respect their appearance in (5.2.1), and where we wrote $L^A_B = L^{A}_{\text{LO}}$. Using the expression (5.2.2) for the local Lorentz transformations, the transformation property $\delta \hat{e}_M^{\tilde{A}} = \varepsilon \hat{e}_M^{\tilde{A}} + \hat{L}_\tilde{B}^\tilde{A} \hat{e}_M^{\tilde{B}}$ for the Lorentzian vielbein gives
where we used (5.2.1). Note that

$$L_{\Lambda'}^B = -\delta_{\Lambda'B}^\Lambda \eta_{AB} L_A^B.$$  

(5.2.4)

In terms of these vielbeins, $\Pi_{MN}^\perp$ becomes

$$\Pi_{MN}^\perp = \delta_{\Lambda'B'}^\Lambda \epsilon_M^\Lambda' \epsilon_N^B',$$  

(5.2.5)

where the primed indices range over $\Lambda' = 2, \ldots, d + 1$, and where

$$\epsilon_M^\Lambda' = E_M^{\Lambda'} + c^{-2} \pi_M^{\Lambda'} + O(c^{-4}),$$  

(5.2.6)

which means that

$$H_{MN} = \delta_{\Lambda'B'}^\Lambda E_M^{\Lambda'} E_N^{B'}, \quad \Phi_{MN} = 2 \delta_{\Lambda'B'}^{\Lambda} E_M^{\Lambda'} \pi_N^{B'}.$$  

(5.2.7)

Expanding the gauge parameters above according to

$$\Xi^M = \xi^M + c^{-2} \xi^M + c^{-4} \phi^M + O(c^{-6}),$$

$$L = \Lambda + c^{-2} \bar{\sigma} + c^{-4} \Lambda_{(4)} + O(c^{-6}),$$

$$L_{\Lambda'B'} = \lambda^{\Lambda'B'} + c^{-2} \lambda_{(2)B'} + O(c^{-4}),$$

$$L_{\Lambda'B'} = \lambda^{\Lambda'B'} + c^{-2} \lambda_{(2)B'} + O(c^{-4}),$$

(5.2.8)

the expansions (2.12.4) and (5.2.6) for $T_M^A$ and $\epsilon_M^\Lambda$ combined with the transformation properties in (5.2.3a) and (5.2.3b) yields

$$\delta \tau_M^A = \xi \tau_M^A + \lambda \tau_M^A + \Lambda \tau_M^A,$$  

(5.2.9a)

$$\delta \epsilon_M^{A'} = \xi \epsilon_M^{A'} + \lambda^{A'B'} \epsilon_M^{B'} + \Lambda^{A'B'} \epsilon_M^{B'},$$  

(5.2.9b)

$$\delta m_M^A = \xi \epsilon_M^A + \lambda \epsilon_M^A,$$  

(5.2.9c)

$$\delta \pi_M^{A'} = \xi \epsilon_M^{A'} + \lambda^{A'B'} \pi_M^{B'} + \Lambda^{A'B'} \epsilon_M^{B'},$$  

(5.2.9d)

$$\delta B_M^A = \xi \epsilon_M^A + \lambda \pi_M^A + \Lambda \pi_M^A + \Lambda \epsilon_M^A + \Lambda \pi_M^A + \Lambda \epsilon_M^A + \Lambda \pi_M^A + \Lambda \epsilon_M^A + \Lambda \pi_M^A,$$  

(5.2.9e)

These match the standard SNC transformations (for the fields that exist in SNC geometry), except for the transformation of $m_M^A$, which is analogous to the situation for type II torsional
Newton–Cartan geometry [24]. To make the transformations match, we decompose the subleading diffeomorphisms as

$$\zeta^M = \sigma^A \tau^M A + E^M A \cdot \zeta^A \cdot,$$

(5.2.10)

and impose the strong foliation constraint (2.12.8) written in the form

$$D_{(M} \tau_{N)} A = 0,$$

(5.2.11)

where D includes the SO(1, 1) “spin connection” $\omega^A B$, i.e.,

$$D_M \tau^A N = \partial_M \tau^A N - \omega^A B \cdot \cdot.$$

(5.2.12)

We can rewrite the Lie derivative that appears in the transformation of $m^M A$ as

$$\mathcal{L} \zeta \tau^M A = \partial_M (\zeta^N \tau^A N) + 2 \zeta^N D_{(N} \tau_{M)} A - 2 \zeta^N \epsilon^A B \omega^{(M} \tau_{N)} B$$

$$= D_M \sigma^A + 2 \zeta^N D_{(N} \tau_{M)} A + \zeta^N \epsilon^A B \omega^N \tau_M ^B .$$

(5.2.13)

Defining

$$\sigma = \tilde{\sigma} + \zeta^N \omega_N$$

(5.2.14)

the transformation of $m^M A$ precisely reduces to that of SNC if (2.12.8) holds, namely

$$\delta m^M A = \mathcal{L} \xi m^M A + D_M \sigma^A + \Lambda \epsilon^A B m^B M + \sigma \epsilon^A B \tau_M ^B + \lambda^A B \cdot E_M ^B \cdot,$$

(5.2.15)

and using $\lambda^A B \cdot = - \lambda_B ^A \cdot$, this becomes Eq. (A.6) of [73].

5.3 Expanding the Nambu–Goto and Polyakov actions

5.3.1 General properties of nonrelativistic string expansions

Let $\mathcal{L}[X; c]$ be the string Lagrangian—either Nambu–Goto or Polyakov. As indicated, this Lagrangian depends on the embedding scalars $X^M$, which are expanded as in (5.3.7), as well as explicitly on $c$. Dimensional analysis reveals that the string Lagrangian starts at $O(c^2)$, so we can expand it as

$$\mathcal{L}[X; c] = c^2 \mathcal{L}^{(-2)} (X) + c^0 \mathcal{L}^{(0)} (X) + c^2 \mathcal{L}^{(2)} (X) + O(c^4) .$$

(5.3.1)
We now (functionally) Taylor expand these Lagrangians to get

\[
\mathcal{L}[X; c] = c^2 \mathcal{L}^{(-2)}(x) + \left[ \mathcal{L}^{(0)}(x) + y^M \frac{\delta \mathcal{L}^{(-2)}(x)}{\delta x^M} \right] \\
+ c^{-2} \left[ \mathcal{L}^{(2)}(x) + z^M \frac{\delta \mathcal{L}^{(-2)}(x)}{\delta z^M} + \frac{1}{2} y^M y^N \frac{\delta^2 \mathcal{L}^{(-2)}(x)}{\delta x^M \delta x^N} + y^M \frac{\delta \mathcal{L}^{(0)}(x)}{\delta x^M} \right] + \mathcal{O}(c^{-4})
\]

\[
= c^2 \mathcal{L}_{LO} + \mathcal{L}_{NLO} + c^{-2} \mathcal{L}_{NNLO} + \mathcal{O}(c^{-4}),
\]

(5.3.2)

where, e.g., \( y^M \frac{\delta \mathcal{L}^{(-2)}(x)}{\delta x^M} \) is the \( x^M \) variation of \( \mathcal{L}^{(-2)}(x) \) with \( \delta x^M \) replaced by \( y^M \). We define

\[
\mathcal{L}_{LO} = \mathcal{L}^{(-2)}(x),
\]

(5.3.3a)

\[
\mathcal{L}_{NLO} = \mathcal{L}^{(0)} + y^M \frac{\delta \mathcal{L}_{LO}}{\delta x^M},
\]

(5.3.3b)

\[
\mathcal{L}_{NNLO} = \mathcal{L}^{(2)} + z^M \frac{\delta \mathcal{L}_{LO}}{\delta z^M} + y^M y^N \frac{\delta^2 \mathcal{L}_{LO}}{\delta x^M \delta x^N} - \frac{1}{2} y^M y^N \frac{\delta^2 \mathcal{L}_{LO}}{\delta x^M \delta x^N}.
\]

(5.3.3c)

Note that the minus sign in the last term in (5.3.3c) comes from the second term in (5.3.3b) via the term \( y^M \frac{\delta \mathcal{L}_{NLO}}{\delta x^M} \). In this way, the rôle of the subleading embedding fields is to impose the equations of motion of the Lagrangians that appear at previous orders. Explicitly, the first and second variational derivatives that appear above are given by

\[
y^M \frac{\delta \mathcal{L}_{(N)LO}}{\delta x^M} = y^M \frac{\partial \mathcal{L}_{LO}}{\partial x^M} + \frac{\partial \mathcal{L}_{LO}}{\partial \partial_{\alpha} x^M} \partial_{\alpha} y^M,
\]

(5.3.4a)

\[
y^M y^N \frac{\delta^2 \mathcal{L}_{LO}}{\delta x^M \delta x^N} = y^M y^N \frac{\partial^2 \mathcal{L}_{LO}}{\partial x^M \partial x^N} + 2 y^M \partial_{\alpha} y^N \frac{\partial^2 \mathcal{L}_{LO}}{\partial x^M \partial \partial_{\alpha} x^N} + \partial_{\alpha} y^M \partial_{\beta} y^N \frac{\partial^2 \mathcal{L}_{LO}}{\partial \partial_{\alpha} x^M \partial \partial_{\beta} x^N}.
\]

(5.3.4b)

These expressions are only defined up to total derivatives and will play an important rôle in our considerations below.

### 5.3.2 Nambu–Goto action

We now apply the framework developed above to the Nambu–Goto (NG) action. This was also considered in [3] (see also [75]), but only up to NLO. The relativistic NG action is

\[
S_{NG}[X; c] = \int_{\Sigma} d^2 \sigma \mathcal{L}_{NG}[X; c] = -cT \int_{\Sigma} d^2 \sigma \sqrt{-\det G_{\alpha \beta}(X)},
\]

(5.3.5)

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where $G_{\alpha\beta}(X)$ is the pullback of the relativistic target space metric in terms of the relativistic embedding fields as defined in (5.3.6), i.e.,

$$G_{\alpha\beta}(X) = \partial_\alpha X^M \partial_\beta X^N G_{MN}(X), \tag{5.3.6}$$

where the argument “$X$” on the left-hand side indicates that the pullback is with respect to the embedding field $X^M$. Expanding the embedding fields according to

$$X^M = x^M + c^{-2} y^M + c^{-4} z^M + \mathcal{O}(c^{-6}), \tag{5.3.7}$$

the $1/c^2$ expansion of $G_{\alpha\beta}(X)$ takes the form

$$G_{\alpha\beta}(X) = c^2 \tau_{\alpha\beta}(x) + H_{\alpha\beta}(x, y) + c^{-2} \phi_{\alpha\beta}(x, y, z) + \mathcal{O}(c^{-4}), \tag{5.3.8}$$

where we need to take into account both terms coming from the $1/c^2$ expansion of the pullback maps $\partial_\alpha X$ as well as those that arise from a Taylor expansion of $G_{MN}(X)$.

\begin{align}
\tau_{\alpha\beta}(x) &= \partial_\alpha x^M \partial_\beta x^N \tau_{MN}, \tag{5.3.9a} \\
H_{\alpha\beta}(x, y) &= H_{\alpha\beta}(x) + 2 \tau_{MN}(x) \partial_\alpha x^M \partial_\beta y^N + \partial_\alpha x^M \partial_\beta x^N y^L \partial_L \tau_{MN}(x), \tag{5.3.9b} \\
\phi_{\alpha\beta}(x, y, z) &= \Phi_{\alpha\beta}(x) + \partial_\alpha y^M \partial_\beta y^N \tau_{MN}(x) + 2 \partial_\alpha x^M \partial_\beta z^N \tau_{MN}(x) + 2 \partial_\alpha x^M \partial_\beta y^N y^L \partial_L \tau_{MN}(x) + \frac{1}{2} \partial_\alpha y^M \partial_\beta y^N \partial_L \tau_{MN}(x) + y^L \partial_L H_{MN}(x). \tag{5.3.9c}
\end{align}

These expressions are unwieldy, but as we will show, the subleading embedding fields encode information about dynamics at previous orders, so their appearance is entirely dictated by this data, which makes them easy to handle.

By assumption, the pull-back $\tau_{\alpha\beta}(x) = \partial_\alpha x^M \partial_\beta x^N \tau_{MN}(x)$ is a two-dimensional Lorentzian metric, and so admits an inverse that we denote $\tau^{\alpha\beta}(x)$—this is a condition on the embedding of the worldsheet in target space. This inverse satisfies

$$\tau_{\alpha\delta}(x) \tau^{\delta\beta}(x) = \delta_\beta^\alpha, \tag{5.3.10}$$

and we can explicitly write it as

$$\tau^{\alpha\beta}(x) = \frac{\epsilon^{\alpha\alpha'} \epsilon^{\beta\beta'} \tau_{\alpha'\beta'}(x)}{\det(\tau_{\alpha\beta}(x))}. \tag{5.3.11}$$

This implies that we can write

$$G_{\alpha\beta}(X) = c^2 \tau_{\alpha\gamma}(x) \left( \delta^{\gamma}_{\beta} + \frac{1}{c^2} \tau^{\gamma\delta}(x) H_{\delta\beta}(x, y) + \frac{1}{c^4} \tau^{\gamma\delta}(x) \phi_{\delta\beta}(x, y, z) \right) + \mathcal{O}(c^{-4})$$

$$= c^2 \tau_{\alpha\gamma}(x) \left( \delta^{\gamma}_{\beta} + \frac{1}{c^2} M^\gamma_{\delta} \right),$$

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where the last equality defines the \( c \)-dependent matrix \( M \). The determinant of this expands as follows

\[
\det(1 + c^{-2}M) = 1 + c^{-2}\text{Tr}[M] + \frac{c^{-4}}{2}\left((\text{Tr}[M])^2 - \text{Tr}[M^2]\right),
\]

where \( \text{Tr}[M] = M^{\alpha\alpha} \). This means that

\[
\sqrt{-\det G_{\alpha\beta}(X)} = c^2\sqrt{-\tau} \left( 1 + \frac{1}{2c^2}\text{Tr}[M] + \frac{1}{8c^4}\left((\text{Tr}[M])^2 - 2\text{Tr}[M^2]\right) + O(c^{-6}) \right),
\]

(5.3.12)

where we defined \( \tau = \det \tau_{\alpha\beta} \), and where all higher-order terms involve powers of traces of \( M \) or traces of powers of \( M \). Writing out \( M \) explicitly, we get

\[
\mathcal{L}_{NG}[X; c] = -c^2\text{Teff} \sqrt{-\tau(x)} - \frac{\text{Teff}}{2} \sqrt{-\tau(x)}\tau^{\alpha\beta}(x)H_{\alpha\beta}(x, y)
\]

\[
-\frac{c^4}{8} \text{Teff} \sqrt{-\tau(x)} \left[ 4\tau^{\alpha\beta}(x)\phi_{\alpha\beta}(x, y, z) + (\tau^{\alpha\beta}(x)H_{\alpha\beta}(x, y))^2 - 2\tau^{\alpha\beta}(x)\tau^{\gamma\delta}(x)H_{\alpha\gamma}(x, y)H_{\delta\beta}(x, y) \right] + O(c^{-3}).
\]

(5.3.13)

Based on our previous considerations, we find that

\[
\mathcal{L}_{NG-LO} = -\text{Teff} \sqrt{-\tau(x)},
\]

(5.3.15a)

\[
\mathcal{L}_{NG-NLO} = -\frac{\text{Teff}}{2} \sqrt{-\tau(x)}\tau^{\alpha\beta}(x)H_{\alpha\beta}(x) + y^M \frac{\delta\mathcal{L}_{NG-LO}}{\delta x^M},
\]

(5.3.15b)

\[
\mathcal{L}_{NG-NNLO} = -\frac{\text{Teff}}{8} \sqrt{-\tau(x)} \left[ 4\tau^{\alpha\beta}(x)\phi_{\alpha\beta}(x, y, z) + (\tau^{\alpha\beta}(x)H_{\alpha\beta}(x, y))^2 - 2\tau^{\alpha\beta}(x)\tau^{\gamma\delta}(x)H_{\alpha\gamma}(x, y)H_{\delta\beta}(x) \right]
\]

\[
+ \frac{z^M}{\delta x^M} \frac{\delta\mathcal{L}_{NG-LO}}{\delta x^M} + \frac{y^M}{\delta x^M} \frac{\delta\mathcal{L}_{NG-NLO}}{\delta x^M} + \frac{1}{2}y^M y^N \frac{\delta^2\mathcal{L}_{NG-LO}}{\delta x^M \delta x^N}.
\]

(5.3.15c)

Rather than computing these variations, we can also explicitly compute the terms involving the subleading embedding fields using the expressions (5.3.9b) and (5.3.9c).

### 5.3.3 The leading order equation of motion

We now turn our attention to the equation of motion of the NG-LO Lagrangian (5.3.15a), which is imposed by \( y^M \) in the NG-NLO Lagrangian (5.3.15b). Working up to total derivatives, we find that

\[
y^M \frac{\delta\mathcal{L}_{NG-LO}}{\delta x^M} = -\text{Teff} \partial_{\alpha} y^M \sqrt{-\tau^{\alpha\beta} \partial_{\beta} x^N \tau_{MN}} - \frac{\text{Teff}}{2} \sqrt{-\tau^{\alpha\beta} \partial_{\alpha} x^N \partial_{\beta} x^L y^M \partial_{\delta} \tau_{NL}}
\]

\[
= -\text{Teff} \epsilon^\alpha_{\beta} \epsilon_{AB} y^M \partial_{\alpha} x^K \partial_{\beta} x^L \left[ \tau_{\alpha}^A \delta_{\alpha} \tau_{\beta}^B - \tau_{\alpha}^B \delta_{\beta} \tau_{\alpha}^A \right]
\]

\[
= -\frac{1}{2} \text{Teff} \epsilon^\alpha_{\beta} \epsilon_{AB} y^M \partial_{\alpha} x^K \partial_{\beta} x^L
\]

\[
\times \left[ \tau_{\alpha}^A (\delta_{\alpha} \tau_{\beta}^B - \delta_{\beta} \tau_{\alpha}^A) + \tau_{\beta}^B (\delta_{\alpha} \tau_{\alpha}^A - \delta_{\beta} \tau_{\alpha}^A) - \tau_{\alpha}^A (\delta_{\alpha} \tau_{\beta}^A - \delta_{\beta} \tau_{\alpha}^A) \right],
\]

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where we used the relation for the inverse longitudinal vielbeins
\begin{equation}
\tau^A = \frac{1}{\sqrt{-\tau}} \epsilon^{\alpha \beta} \tau_B^\alpha \epsilon BA.
\end{equation}

Defining the quantity
\begin{equation}
F_{MN} := \epsilon_{AB} \tau_M^A \tau_N^B,
\end{equation}
we can write the LO equation of motion as
\begin{equation}
y^M \frac{\delta L_{\text{NG-LO}}}{\delta x^M} = -\frac{1}{2} T_{\text{eff}} \epsilon^{\alpha \beta} y^M \partial_\alpha x^K \partial_\beta x^L \left[ \partial_K F_{ML} + \partial_L F_{KM} + \partial_M F_{LK} \right].
\end{equation}
The equation of motion for $x$ obtained from the NG-LO Lagrangian can thus be written as
\begin{equation}
\epsilon^{\alpha \beta} \partial_\alpha x^K \partial_\beta x^L (dF)_{MKL} = 0.
\end{equation}

If we contract this with $\partial_\gamma x^M$, the fact that the worldsheet indices $\alpha, \beta, \gamma$ only take two values implies that the pullback vanishes identically. The fact that the pullback of the LO equation of motion is identically satisfied is a consequence of two-dimensional reparameterisation invariance of the Lagrangian.

### 5.3.4 Codimension-2 foliations from the string beta function

Conformal invariance of the relativistic quantum string theory requires that the $\beta$-functions vanish. In particular, the $\beta$-function for $G_{MN}$ vanishes, which to leading order in $\alpha'$ is equivalent to the vacuum Einstein equations [257]
\begin{equation}
0 = \beta_{MN}(G) = \alpha' R_{MN} + O(\alpha'^2),
\end{equation}
where $R_{MN}$ is the Ricci tensor of $G_{MN}$, and where we ignored the Kalb–Ramond B-field and the dilaton. In this section, we demonstrate that the LO part of the vacuum Einstein equation implies that the $1/c^2$ expanded geometry admits a codimension-2 foliation defined by $\tau^A$. Assuming such a foliation, the LO equation for the string reduces to a constraint on the 1-forms $\alpha^A_B$ in Frobenius’ integrability condition in (2.12.7). To see this, write

\begin{align}
G_{MN} &= c^2 T_{MN} + \Pi^2_{MN}, \\
G^{MN} &= c^{-2} T^{MN} + \Pi^1{}^{MN}, \\
\Gamma^P_{MN} &= \frac{c^2}{2} \Pi^P_{MN} \left( \partial_M T_{NQ} + \partial_N T_{MQ} - \partial_Q T_{MN} \right) + O(1), \\
R_{MN} &= -\Gamma^Q_{MR} \Gamma^R_{QP} + O(c^2),
\end{align}

where $\Gamma^P_{MN}$ is the Levi–Civita connection of $G_{MN}$. The LO vacuum Einstein equations $R_{MN} = 0$ give
\begin{equation}
H^+ O_S H^+ O_T \eta_{AB} \eta_{CD} \tau_M^A \tau_P^C \left( d\tau^B \right)_{RS} \left( d\tau^D \right)_{QT} = 0.
\end{equation}
Contracting with $\tau^M \tau^P F$ and dropping the invertible $\eta$ metrics we obtain
\[ H^{-Q} H^{-R} (d\tau^B)_{RS} (d\tau^D)_{QT} = 0. \] (5.3.23)

This is a sum of squares for $B = D = 0, 1$ and so it is equivalent to
\[ H^{-Q} H^{-R} (d\tau^A)_{RS} = 0. \] (5.3.24)

This in turn is equivalent to
\[ d\tau^A = \alpha^A_B \wedge \tau^B, \] (5.3.25)

for arbitrary 1-forms $\alpha^A_B$. We recognise this as the Frobenius integrability condition (2.12.7) for a codimension-2 foliation of $d$-dimensional Riemannian leaves with normal 1-forms $\tau^M A$.

If we assume that (5.3.25) holds, the LO equation of motion for $x^M$ becomes
\[ \epsilon^{\alpha\beta} \partial^{\alpha} x^K \partial^{\beta} x^L \left[ X_{KAB} \tau^M A \tau_L B - X_{LAB} \tau^M A \tau_K B + X_{MAB} \tau_L A \tau_K B \right] = 0, \] (5.3.26)

where we defined
\[ X_{MAB} = \epsilon_{AC} \alpha^M C - \epsilon_{BC} \alpha^M A. \] (5.3.27)

A trivial solution of the LO equation of motion for $x^M$ is to set $X_{MAB} = 0$ which is equivalent to setting the trace of $\alpha^M A B$ equal to zero.\(^2\) In fact we can write
\[ X_{MAB} = \epsilon_{AB} \alpha^M C. \] (5.3.28)

In other words a sufficient condition for the LO action to have an identically satisfied equation of motion for $x^M$ is to demand that the target space obeys (5.3.25) with $\alpha^M A A = 0$, which is more general than the strong foliation constraint (2.12.8) for which $\alpha^M A B$ is proportional to $\epsilon^A B$.

More generally, equation (5.3.26) implies that $X_{MAB} = \tau^M C X_{CAB}$ where $X_{CAB} = -X_{CBA}$. This can be seen as follows: write (5.3.26) as
\[ X_{MAB} \epsilon^{\alpha\beta} \tau^A \tau^B = \epsilon^{\alpha\beta} \partial^{\alpha} x^K \partial^{\beta} x^L \left[ X_{KAB} \tau^M B - X_{LAB} \tau^M K \right] \tau^M A, \] (5.3.29)

and use $\epsilon^{\alpha\beta} \tau^A \tau^B = \sqrt{-\tau} \epsilon^{AB}$ to get
\[ X_{MAB} = -\frac{1}{\sqrt{-\tau}} \epsilon^{\alpha\beta} \partial^{\alpha} x^K \partial^{\beta} x^L X_{KCE} \tau^E \tau^M C \epsilon_{AB}, \] (5.3.30)

where we used that $X_{MAB} = -X_{MBA}$ and thus proportional to $\epsilon_{AB}$ (since the indices $A, B$\(^1\)

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\(^1\)We thank José Figueroa-O’Farrill for useful discussions on this point.

\(^2\)To see this, note that $X_{MAB} = 0$ implies that $0 = \epsilon^{AB} X_{MAB} = -2 \alpha^M A A$, where we used the identity $\epsilon^{AB} \epsilon_{BC} = \delta^A_C$. 

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only take two values). This is of the form \( X_{MAB} = \tau_{MC} X_{CAB} \) with \( X_{CAB} \) given by

\[
X_{CAB} = -\frac{1}{\sqrt{-\tau}} \varepsilon^{\alpha\beta} \partial_\alpha x^K \partial_\beta x^L X_{KCE} \tau_{LE} \varepsilon^{AB},
\]

(5.3.31)

and it then follows that (5.3.26) can be written as

\[
\varepsilon^{\alpha\beta} \partial_\alpha x^K \partial_\beta x^L \tau_{KA} \tau_{LB} \tau_{MC} \left[ X_{ABC} + X_{BCA} + X_{CAB} \right] = 0.
\]

(5.3.32)

This is zero for any \( X_{ABC} \) because the term in square brackets is antisymmetric in \((A, B, C)\) and these can each only take two values.\(^3\) Thus, the LO equation of motion for \( x^M \) is trivially satisfied for any target space that obeys the Frobenius integrability condition (5.3.25) with the additional requirement that \( X_{MAB} = \tau_{MC} X_{CAB} \). In other words, starting from a target space that admits a codimension-2 foliation the LO equation of motion for \( x^M \) will restrict this foliation to one for which \( \alpha_M C \) is proportional to \( \tau_M A X_A \) for some \( X_A \).

When we add a B-field the \( \beta \)-functions of the relativistic string sigma model change and then the results depend on how we expand the B-field (see Section 5.5 for more details).

### 5.3.5 Polyakov action

The relativistic Polyakov action is

\[
S_P[X; c] = \int_\Sigma d^2\sigma \mathcal{L}_P[X; c] = -\frac{c^T}{2} \int_\Sigma d^2\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN}(X).
\]

(5.3.33)

In accordance with previous results, this leads to

\[
\mathcal{L}_P = -\frac{T_{\text{eff}}}{2} \sqrt{-\gamma} \left[ c^2 \gamma^{\alpha\beta} \tau_{\alpha\beta}(x) + \gamma^{\alpha\beta} H_{\alpha\beta}(x, y) + c^{-2} \gamma^{\alpha\beta} \phi_{\alpha\beta}(x, y, z) \right].
\]

(5.3.34)

In the relativistic string theory, the fiducial worldsheet metric \( \gamma_{\alpha\beta} \) is on-shell equivalent (up to a local rescaling) to the pullback of the target space metric \( G_{\alpha\beta} \), which expands according to (5.3.8). Therefore, we assume a similar expansion for \( \gamma_{\alpha\beta} \), namely

\[
\gamma_{\alpha\beta} = \gamma_{(0)\alpha\beta} + c^{-2} \gamma_{(2)\alpha\beta} + c^{-4} \gamma_{(4)\alpha\beta} + \cdots,
\]

(5.3.35)

where the LO component \( \gamma_{(0)\alpha\beta} \) is a Lorentzian metric, while the subleading components \( \gamma_{(2)\alpha\beta} \) and \( \gamma_{(4)\alpha\beta} \) are symmetric tensors. The inverse worldsheet metric takes the form

\[
\gamma^{\alpha\beta} = \gamma^{(0)\alpha\beta} - c^{-2} \gamma^{(2)\alpha\beta} + c^{-4} \left[ \gamma^{(2)\alpha\gamma} \gamma^{(2)\gamma\beta} - \gamma^{(4)\alpha\beta} \right] + O(c^{-6}).
\]

(5.3.36)

---

\(^3\)We remind the reader that capital Latin indices from the start of the alphabet, \( A, B, \ldots \) are longitudinal tangent space indices, while capital Latin indices from the middle of the alphabet \( M, N, \ldots \) are spacetime indices.
where we raised indices on $\gamma_{(2)}$ and $\gamma_{(4)}$ using $\gamma_{(0)}^{\alpha\beta}$. This means that the expansion of the worldsheet metric determinant becomes

$$\sqrt{-\gamma} = \sqrt{-\gamma_{(0)}} \left[ 1 + c^{-2} \frac{1}{2} \gamma_{(2)}^{\alpha\alpha} + c^{-4} \frac{1}{8} \left[ 4\gamma_{(4)}^{\alpha\alpha} + (\gamma_{(2)}^{\alpha\alpha})^2 - 2\gamma_{(2)}^{\alpha\beta} \gamma_{(2)}^{\alpha\alpha} \right] \right] + O(c^{-6}).$$

Expanding all field quantities appropriately in $1/c^2$, the Polyakov Lagrangian acquires the following expansion

$$\mathcal{L}_P = c^2 \mathcal{L}_{P-LO} + c^{-2} \mathcal{L}_{P-NLO} + c^{-4} \mathcal{L}_{P-NNLO} + O(c^{-6}),$$

where

$$\mathcal{L}_{P-LO} = -\frac{T_{\text{eff}}}{2} \sqrt{-\gamma_{(0)}^{\alpha\beta}} \tau_{\alpha\beta}(x),$$

The equation of motion for $x^M$ is

$$0 = \frac{1}{\sqrt{-\gamma_{(0)}}} \partial_{\alpha} \left( \sqrt{-\gamma_{(0)}^{\alpha\beta}} \partial_{\beta} \chi^N \tau_{MN}(x) \right) - \frac{1}{2} \gamma_{(0)}^{\alpha\beta} \partial_{\alpha} \chi^1 \partial_{\beta} \chi^N \delta_{MN} \tau_{LN}(x),$$

while the (Virasoro) constraint from integrating out $\gamma_{(0)}^{\alpha\beta}$ is

$$T_{\alpha\beta}^{(0)} = -\frac{2}{T_{\text{eff}} \sqrt{-\gamma_{(0)}}} \frac{\delta \mathcal{L}_{P-LO}}{\delta \gamma_{(0)}^{\alpha\beta}} = \tau_{\alpha\beta} - \frac{1}{2} \gamma_{(0)}^{\gamma\delta} \gamma_{(0)}^{\alpha\beta} \gamma_{(0)}^{\gamma\delta} = 0.$$

Plugging this into the P-LO Lagrangian density (5.3.38) gives the NG-LO Lagrangian density (5.3.15a). At NLO, the Polyakov Lagrangian becomes

$$\mathcal{L}_{P-NLO} = -\frac{T_{\text{eff}}}{2} \sqrt{-\gamma_{(0)}^{\alpha\beta}} H_{\alpha\beta}(x,y) + \frac{T_{\text{eff}}}{2} \sqrt{-\gamma_{(0)}^{\alpha\beta}} \gamma_{(2)}^{\alpha\beta} \gamma_{(0)}^{\gamma\delta} \left[ \gamma_{(0)}^{\gamma\delta} - \frac{1}{2} \gamma_{(0)}^{\alpha\beta} \gamma_{(0)}^{\gamma\delta} \right]$$

$$= -\frac{T_{\text{eff}}}{2} \sqrt{-\gamma_{(0)}} \left( \gamma_{(0)}^{\alpha\beta} \gamma_{(0)}^{\gamma\delta} - \gamma_{(0)}^{\gamma\delta} \gamma_{(0)}^{\gamma\delta} \right) + y^M \frac{\delta \mathcal{L}_{P-LO}}{\delta x^M},$$

where the second equality is up to total derivatives and where we introduced the Wheeler–DeWitt (WDW) metric

$$G_{(0)}^{\alpha\beta} \gamma_{(0)}^{\gamma\delta} = \gamma_{(0)}^{\gamma\delta} \gamma_{(0)}^{\alpha\beta} + \gamma_{(0)}^{\delta\beta} \gamma_{(0)}^{\gamma\alpha} - \gamma_{(0)}^{\gamma\alpha} \gamma_{(0)}^{\delta\beta},$$

which has the following symmetries

$$G_{(0)}^{\alpha\beta} \gamma_{(0)}^{\gamma\delta} = G_{(0)}^{\beta\alpha} \gamma_{(0)}^{\gamma\delta}, \quad G_{(0)}^{\alpha\beta} \gamma_{(0)}^{\gamma\delta} = G_{(0)}^{\alpha\delta} \gamma_{(0)}^{\gamma\beta}, \quad G_{(0)}^{\alpha\beta} \gamma_{(0)}^{\gamma\delta} = G_{(0)}^{\gamma\delta} \gamma_{(0)}^{\alpha\beta}. $$
Varying $\gamma_{(2)\alpha\beta}$ in the NLO Lagrangian produces the LO Virasoro constraint (5.3.40), while varying $\gamma_{(0)\alpha\beta}$ leads to the NLO Virasoro constraint

$$
T_{(2)\alpha\beta}^{(2)} = -\frac{2}{T_{\text{eff}}\sqrt{-\gamma_{(0)}}} \frac{\delta L_{\text{P-NLO}}}{\delta \gamma_{(0)\alpha\beta}}
$$

$$
= H_{\alpha\beta}(x, y) - \frac{1}{2}\gamma_{(0)\gamma\delta} H_{\gamma\delta}(x, y) \gamma_{(0)\alpha\beta} + \frac{1}{4} \tau_{\alpha'\beta'} \gamma_{(2)\gamma\delta} G_{(0)}^{\alpha'\beta'} \gamma_{(0)\alpha\beta}
$$

$$
- 2\tau_{\gamma(x)\gamma_{(2)\beta}} \gamma_{(0)} + \frac{1}{2} \tau_{\alpha\beta} \gamma_{(2)\gamma} + \frac{1}{2} \gamma_{(2)\alpha\beta} \gamma_{(0)} \gamma_{\gamma\delta} = 0.
$$

If we contract this with $\gamma_{(0)\alpha\beta}$, we get

$$
\gamma_{(0)} T_{(2)\alpha\beta}^{(2)} = -\gamma_{(2)\alpha\beta} T_{(0)}^{(2)}.
$$

(5.3.44)

As we will see later in (5.3.54), this is the Ward identity corresponding to Weyl symmetry of the P-NLO action.

At this stage, it is worth pointing out a subtlety regarding the $y$-terms: their equation of motion of the P-NLO Lagrangian gives the LO-NG $x$ equation of motion up to LO Virasoro constraints, which are now imposed by $\gamma_{(2)\alpha\beta}$. Therefore, we could combine these terms (which turn out to involve derivatives of $y$) that are proportional to the LO Virasoro constraints with $\gamma_{(2)\alpha\beta}$. We pursue this explicitly in Section 5.4, where this observation is important for the identification of the NLO theory (with the strong foliation constraint imposed) as the Gomis–Ooguri string.

Repeating this exercise at NNLO gets us

$$
\mathcal{L}_{\text{P-NNLO}} = \frac{T_{\text{eff}}}{2} \sqrt{-\gamma_{(0)}} \left( \gamma_{(0)}^{\alpha\beta} \Phi_{\alpha\beta}(x, y, z) - \frac{1}{2} G_{(0)}^{\alpha\beta\gamma\delta} \left[ \tau_{\alpha\beta}(x) \gamma_{(4)\gamma\delta} + H_{\alpha\beta}(x, y) \gamma_{(2)\gamma\delta} \right] 
$$

$$
+ \frac{1}{2} \tau_{\alpha\beta}(x) \gamma_{(2)\gamma\delta} \gamma_{(2)\gamma\delta} + \frac{1}{2} \tau_{\gamma(x)\gamma_{(2)\beta}} \gamma_{(0)} + \frac{1}{2} \gamma_{(2)\alpha\beta} \gamma_{(0)} \gamma_{\gamma\delta} \right) 
$$

$$
- \frac{T_{\text{eff}}}{2} \sqrt{-\gamma_{(0)}} \left( \gamma_{(2)\alpha\beta} \Phi_{\alpha\beta}(x) - \frac{1}{2} G_{(0)}^{\alpha\beta\gamma\delta} \left[ \tau_{\alpha\beta}(x) \gamma_{(4)\gamma\delta} + H_{\alpha\beta}(x) \gamma_{(2)\gamma\delta} \right] 
$$

$$
+ \frac{1}{2} \tau_{\alpha\beta}(x) \gamma_{(2)\gamma\delta} \gamma_{(2)\gamma\delta} + \frac{1}{2} \tau_{\gamma(x)\gamma_{(2)\beta}} \gamma_{(0)} + \frac{1}{2} \gamma_{(2)\alpha\beta} \gamma_{(0)} \gamma_{\gamma\delta} \right) 
$$

$$
+ z^{M} \frac{\delta L_{\text{P-LO}}}{\delta x^{M}} + y^{M} \frac{\delta L_{\text{P-NLO}}}{\delta x^{M}} - \frac{1}{2} y^{M} y^{N} \frac{\delta^{2} L_{\text{P-LO}}}{\delta x^{M} \delta x^{N}}.
$$

(5.3.45)

Integrating out $\gamma_{(4)}$ now gives the LO Virasoro constraint (5.3.40), while integrating out $\gamma_{(2)}$ gives the NLO Virasoro constraint (5.3.44), and a lengthy calculation shows that using these returns the NG-NNLO Lagrangian (5.3.15c). Integrating out $\gamma_{(0)}$ in the P-NNLO Lagrangian gives the NNLO Virasoro constraint

$$
T_{(4)\alpha\beta}^{(4)} = \Phi_{\alpha\beta}(x, y, z) - \frac{1}{2} \gamma_{(0)\alpha\beta} \gamma_{(0)}^{\gamma\delta} \Phi_{\gamma\delta}(x, y, z) + \text{terms involving } \gamma_{(2)} \text{ and } \gamma_{(4)}.
$$

(5.3.46)
where the terms involving $\gamma^{(2)}$ and $\gamma^{(4)}$ are not required and we refrain from writing them.

Some general comments about the structure of the constraints are in order. First, we note that we could equivalently have expanded the relativistic Virasoro constraints
\[ G_{\alpha\beta} - \frac{1}{2} \gamma_{\alpha\beta}^{\gamma_{(2)}} G_{\alpha\beta} = 0, \]  
(5.3.47)
to obtain the LO, NLO, and NNLO constraints above. Furthermore, as we saw, the most subleading Virasoro constraint, i.e., the one obtained by integrating out $\gamma^{(0)}$ in the P-NLO and P-NNLO actions, is not required to obtain the corresponding NG actions. This is because imposing all previous constraints (e.g., the LO constraint at NLO) reduces the most subleading constraint, i.e., the one obtained by integrating out $\gamma^{(0)}$, to an equation for the most subleading $\gamma^{(n)}$, which is a Lagrange multiplier for the LO Virasoro constraint and which disappears from the action when imposing all previous constraints.

The worldsheet gauge symmetries of the relativistic Polyakov action are diffeomorphisms $\xi^\alpha$ and Weyl transformations $\omega$, which act infinitesimally on the worldsheet metric $\gamma_{\alpha\beta}$ as
\[ \delta \gamma_{\alpha\beta} = 2 \omega_{(0)} \gamma_{\alpha\beta} + 2 \omega_{(2)} \gamma_{\alpha\beta} + 2 \omega_{(4)} \gamma_{\alpha\beta}. \]  
(5.3.48)

We expand these gauge parameters according to
\[ \xi^\alpha = \xi^\alpha_{(0)} + c^{-2} \xi^\alpha_{(2)} + c^{-4} \xi^\alpha_{(4)} + \cdots, \]
\[ \omega = \omega_{(0)} + c^{-2} \omega_{(2)} + c^{-4} \omega_{(4)} + \cdots, \]  
(5.3.49)
and, since we consider closed strings, the spatial worldsheet coordinate $\sigma^1$ is periodically identified, $\sigma^1 \sim \sigma^1 + 2\pi$. The gauge parameters are periodic in $\sigma^1$. Expanding the expression for the relativistic general worldsheet gauge transformation leads to
\[ \delta \gamma_{(0)\alpha\beta} = \xi_{(0)} \gamma_{(0)\alpha\beta} + 2 \omega_{(0)} \gamma_{(0)\alpha\beta}, \]  
(5.3.50a)
\[ \delta \gamma_{(2)\alpha\beta} = \xi_{(0)} \gamma_{(2)\alpha\beta} + \xi_{(2)} \gamma_{(0)\alpha\beta} + 2 \omega_{(0)} \gamma_{(2)\alpha\beta} + 2 \omega_{(2)} \gamma_{(0)\alpha\beta}, \]  
(5.3.50b)
\[ \delta \gamma_{(4)\alpha\beta} = \xi_{(0)} \gamma_{(4)\alpha\beta} + \xi_{(2)} \gamma_{(2)\alpha\beta} + \xi_{(4)} \gamma_{(0)\alpha\beta} + 2 \omega_{(0)} \gamma_{(4)\alpha\beta} + 2 \omega_{(2)} \gamma_{(2)\alpha\beta} + 2 \omega_{(4)} \gamma_{(0)\alpha\beta}. \]  
(5.3.50c)

The diffeomorphisms and Weyl symmetries of the worldsheet data have Ward identities associated to them. At LO, we have
\[ \delta S_{\text{P-LO}} = \frac{T_{\text{eff}}}{2} \int d^2 \sigma \sqrt{-\gamma^{(0)}} T_{(0)\alpha\beta} \delta \gamma_{(0)\alpha\beta}, \]  
(5.3.51)
where $T_{(0)\alpha\beta} = \gamma^{(\alpha\alpha') \gamma^{(\beta\beta') \gamma^{(0)\alpha\beta}}}$, and so the Ward identity for LO Weyl transformations tells us that the energy-momentum tensor is traceless
\[ \gamma^{(\alpha\alpha') \gamma_{(0)\alpha\beta} = 0}. \]  
(5.3.52)
For the NLO action, we have
\[
\delta S_{\text{P-NLO}} = \frac{T_{\text{eff}}}{2} \int \sqrt{-\gamma(0)} \left( T^{\alpha\beta}_{(2)} \delta \gamma_{(0)\alpha\beta} + T^{\alpha\beta}_{(0)} \delta \gamma_{(2)\alpha\beta} \right),
\]
where \( T^{\alpha\beta}_{(2)} = \gamma^{\alpha\alpha'} \gamma^{\beta\beta'} T^{(2)}_{\alpha\beta} \). The LO Ward identities are reproduced by the subleading Weyl transformations and subleading diffeomorphisms, while the Ward identity for LO Weyl transformations \( \omega_{(0)} \) now takes the form
\[
T^{(2)}_{\alpha\beta} \gamma_{(0)\alpha\beta} + T^{(0)}_{\alpha\beta} \gamma_{(2)\alpha\beta} = 0.
\]
(5.3.54)

There are similar Ward identities at the NNLO. Furthermore, there are also Ward identities for the gauge symmetries associated with the \( c^{-2} \) expansion of the generator of worldsheet diffeomorphism invariance. We will refrain from writing them down as we will not need them explicitly.

### 5.3.6 Partial gauge fixing

The symmetries of the Lorentzian LO worldsheet metric \( \gamma_{(0)\alpha\beta} \) are exactly identical to those of the relativistic Polyakov action, so we can gauge fix the LO gauge redundancy by locally going to flat gauge
\[
\gamma_{(0)\alpha\beta} = \eta_{\alpha\beta}.
\]
(5.3.55)

The residual gauge transformations at LO are those diffeomorphisms that can be undone by a Weyl transformation \( \xi(\sigma) \eta_{\alpha\beta} + 2\omega_{(0)} \eta_{\alpha\beta} = 0 \), and in lightcone coordinates these take the familiar form
\[
\xi_{(0)}(\sigma) = \xi_{(0)}^- (\sigma^-) \partial^- + \xi_{(0)}^+ (\sigma^+) \partial^+,
\]
(5.3.56)
where \( \xi_{(0)}^\pm (\sigma^\pm) \) are periodic in their argument. For the LO Weyl transformation, this corresponds to \( \omega_{(0)} = -\frac{1}{2} \partial_\alpha \xi_\alpha^{(0)} \).

Turning our attention to the NLO gauge redundancies, now have
\[
\delta \gamma_{(2)\alpha\beta} = 2\partial_{(\alpha} \xi_{(2)\beta)} + 2\omega_{(2)} \eta_{\alpha\beta},
\]
(5.3.57)
where \( \xi_{(2)} = \eta_{\beta} \xi_{(2)}^\beta \), and where we left out the transformations under LO residual gauge transformations \( \xi_{(0)}^\pm \). The NLO gauge transformations are independent of \( \gamma_{(2)} \) and act as three local shifts, allowing us to locally gauge fix
\[
\gamma_{(2)\alpha\beta} = 0.
\]
(5.3.58)

The residual gauge transformations satisfy
\[
2\partial_{(\alpha} \xi_{(2)\beta)} + 2\omega_{(2)} \eta_{\alpha\beta} = 0,
\]
(5.3.59)
and so take exactly the same form as those at LO, namely
\[
\xi_{(2)}(\sigma) = \xi_{(2)}^-(\sigma^-) \partial_- + \xi_{(2)}^+(\sigma^+) \partial_+ , \tag{5.3.60}
\]
with \( \omega_{(2)} = -\frac{1}{2} \partial_{\alpha} \xi_{(0)}^0 \). This pattern repeats itself at all orders: in particular, at NNLO, we get
\[
\delta \gamma_{(4)\alpha\beta} = 2 \partial_{\alpha} \xi_{(4)} + 2 \omega_{(4)} \eta_{\alpha\beta} , \tag{5.3.61}
\]
where, again, the NNLO gauge transformations act as local shifts, so that we may locally set
\[
\gamma_{(4)\alpha\beta} = 0 , \tag{5.3.62}
\]
leaving once more the residual gauge transformations
\[
\xi_{(4)}(\sigma) = \xi_{(4)}^-(\sigma^-) \partial_- + \xi_{(4)}^+(\sigma^+) \partial_+ . \tag{5.3.63}
\]

The relativistic embedding field \( X^M \) transforms as a scalar under worldsheet diffeomorphisms \( \xi_{\alpha} \), \( \delta X^M = \xi_{\alpha} \partial_{\alpha} X^M \). We can expand these diffeomorphisms to get
\[
\delta x_M = \xi_{\alpha} \partial_{\alpha} x_M , \tag{5.3.64a}
\]
\[
\delta y_M = \xi_{\alpha} \partial_{\alpha} y_M + \xi_{(0)} \partial_{(0)} y_M , \tag{5.3.64b}
\]
\[
\delta z_M = \xi_{\alpha} \partial_{\alpha} z_M + \xi_{(2)} \partial_{(2)} z_M + \xi_{(0)} \partial_{(0)} z_M . \tag{5.3.64c}
\]
These expressions will come in handy when we fix the residual gauge invariances discussed above.

### 5.4 Relating the NLO theory to the Gomis–Ooguri string

We will show that for appropriate target spacetimes, the P-NLO Lagrangian (5.3.41) can be recast as the SNC Polyakov Lagrangian of \([72, 244] \). To do so, we first write \( \gamma_{(0)\alpha\beta} \) in terms of vielbeins
\[
\gamma_{(0)\alpha\beta} = \eta_{ab} e_{\alpha}^a e_{\beta}^b = -\frac{1}{2} (e_{\alpha}^+ e_{\beta}^- + e_{\alpha}^- e_{\beta}^+) , \tag{5.4.1}
\]
where \( a, b = 0, 1 \) are worldsheet tangent space indices. We also defined the null combinations
\[
e_{\alpha}^\pm = e_{\alpha}^0 \pm e_{\beta}^1 , \tag{5.4.2}
\]
which have inverses given by
\[
e_{\alpha}^\pm = (e_{\alpha}^0 \pm e_{\alpha}^1)/2 . \tag{5.4.3}
\]

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Note that the Minkowski metric $\eta_{ab}$ and its inverse $\eta^{ab}$ in lightcone coordinates $\sigma^\pm = \sigma^0 \pm \sigma^1$ are

$$
\eta^{++} = \eta^{--} = \frac{1}{2}, \quad \eta^{+-} = \eta^{-+} = 0,
$$

$$
\eta^{--} = \eta^{++} = -2, \quad \eta^{\pm \pm} = \eta^{\pm \mp} = 0.
$$

(5.4.4)

Similarly, the Levi-Civita symbol $\epsilon^{ab}$ in lightcone coordinates is

$$
\epsilon^{--} = \epsilon^{++} = 0, \quad \epsilon^{\pm \pm} = \epsilon^{\pm \mp} = \frac{1}{2},
$$

$$
\epsilon^{\pm +} = \epsilon^{+ \pm} = 2.
$$

(5.4.5)

where we use the convention $\epsilon^{01} = 1 = -\epsilon^{01}$. Note furthermore the useful identity

$$
\epsilon^\alpha \gamma^\beta = \delta_\alpha^\beta,
$$

(5.4.6)

which we will use extensively in the present section.

Consider the NLO Polyakov Lagrangian in (5.3.41). We Define the Lagrange multiplier Lagrangian

$$
\mathcal{L}_{LM} = \frac{T_{\text{eff}}}{4} \sqrt{-\gamma^{(0)}} G_{abcd}^{(0)} \tau^\alpha_{\alpha} \gamma^\gamma_{\gamma} \tau^\beta_{\beta} \gamma^\delta_{\delta}.
$$

(5.4.7)

We can write $\gamma^{(2)}_{\alpha \beta}$ in terms of worldsheet vielbeine as follows

$$
\gamma^{(2)}_{\alpha \beta} = 2 \eta_{\alpha \beta} e_\alpha^a e_\beta^b A^a_b = 2 e_\alpha^a e_\beta^b A_{ab}.
$$

(5.4.8)

This leads to

$$
\mathcal{L}_{LM} = \frac{T_{\text{eff}}}{2} \sqrt{-\gamma^{(0)}} G_{abcd}^{(0)} e_\alpha^a e_\beta^b e_\gamma^c e_\delta^d G_{\alpha \beta \gamma \delta}.
$$

(5.4.9)

where $G_{abcd}^{(0)} = e_\alpha^a e_\beta^b e_\gamma^c e_\delta^d G^{\alpha \beta \gamma \delta}$. We factorised $\tau_{\alpha \beta}$ using the symmetry properties of the WDW metric (the cross term $\tau^0_{\alpha} \tau^1_{\beta} - \tau^0_{\beta} \tau^1_{\alpha}$ drops out due to its antisymmetry in $\alpha$ and $\beta$). The flat WDW metric in lightcone coordinates takes the form

$$
G_{abcd}^{(0)} = \eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc} - \eta^{ab} \eta^{cd},
$$

(5.4.10)

and has only two non-zero components

$$
G_{++-+}^{(0)} = G_{-++-}^{(0)} = 8.
$$

(5.4.11)

This means that the Lagrange multiplier Lagrangian involves only two constraints imposed by $A_{++}$ and $A_{--}$, i.e., $A_{+-}$ does not contribute. In other words, we can write the Lagrange
multiplier Lagrangian as
\[
\mathcal{L}_{LM} = 4T_{\text{eff}} \sqrt{-\gamma(0)} \epsilon^\alpha - (-\tau_\alpha^0 + \tau_\alpha^1) \epsilon^\beta - (\tau_\beta^0 + \tau_\beta^1) A_++
+ 4T_{\text{eff}} \sqrt{-\gamma(0)} \epsilon^\alpha_+ (-\tau_\alpha^0 + \tau_\alpha^1) \epsilon^\beta_+ (\tau_\beta^0 + \tau_\beta^1) A_-.
\]  
(5.4.12)

The constraints imposed by \(A_{--}\) and \(A_{++}\) are
\[
A_{++} : 0 = \epsilon_+ (-\tau_\alpha^0 + \tau_\alpha^1) \epsilon^\beta - (\tau_\beta^0 + \tau_\beta^1),
\]
\[
A_{--} : 0 = \epsilon_+ (\tau_\alpha^0 + \tau_\alpha^1) \epsilon^\beta + (\tau_\beta^0 + \tau_\beta^1).
\]  
(5.4.13)

The \(e^\alpha_{\pm}\) projections of \((-\tau_\alpha^0 + \tau_\alpha^1)\) cannot be both zero and likewise for \((\tau_\alpha^0 + \tau_\alpha^1)\) because this would imply that \(\tau_\alpha^A\) is not invertible. Without loss of generality we choose the constraints to be
\[
\epsilon^\alpha_+ (-\tau_\alpha^0 + \tau_\alpha^1) = 0 = \epsilon^- (\tau_\beta^0 + \tau_\beta^1),
\]  
(5.4.14)

with the other projections non-zero, i.e.,
\[
\epsilon^\alpha_+ (-\tau_\alpha^0 + \tau_\alpha^1) \neq 0, \quad \epsilon^\beta_+ (\tau_\beta^0 + \tau_\beta^1) \neq 0.
\]  
(5.4.15)

Now, using
\[
\epsilon^\alpha_a = \frac{1}{\sqrt{-\gamma(0)}} \epsilon^\alpha \epsilon^\beta \epsilon_{b\alpha},
\]  
(5.4.16)
as well as our previous results for the Levi-Civita symbol in lightcone coordinates, we find that
\[
\epsilon^\beta_- = \frac{1}{2\sqrt{-\gamma(0)}} \epsilon^\beta \gamma \epsilon^\gamma_+, \quad \epsilon^\beta_+ = -\frac{1}{2\sqrt{-\gamma(0)}} \epsilon^\beta \gamma \epsilon^\gamma_-. 
\]  
(5.4.17)

If we substitute this into the Lagrange multiplier Lagrangian we obtain
\[
\mathcal{L}_{LM} = 2T_{\text{eff}} \epsilon^\alpha_- (-\tau_\alpha^0 + \tau_\alpha^1) \epsilon^\beta \gamma \epsilon^\gamma_+ (\tau_\beta^0 + \tau_\beta^1) A_++
+ 2T_{\text{eff}} \epsilon^\alpha \epsilon^\gamma \epsilon^- \epsilon^\gamma_- (\tau_\beta^0 - \tau_\beta^1) A_-.
\]  
(5.4.18)

Using (5.4.15) we can make the following field redefinitions
\[
\hat{\lambda}_{++} = 4\epsilon_+ (-\tau_\alpha^0 + \tau_\alpha^1) A_++, \quad \hat{\lambda}_{--} = 4\epsilon_+ (\tau_\beta^0 + \tau_\beta^1) A_--,
\]  
(5.4.19)
which leads to
\[
\mathcal{L}_{LM} = -\frac{T_{\text{eff}}}{2} \left[ \hat{\lambda}_{++} \epsilon^\alpha \epsilon^\beta \epsilon_+ + \hat{\lambda}_{--} \epsilon^\alpha \epsilon^\beta \epsilon^- \right],
\]  
(5.4.20)
where we have introduced the lightcone combinations
\[ \tau^\pm_\beta = \tau^0_\beta \pm \tau^1_\beta. \tag{5.4.21} \]

This produces the Lagrangian
\[
\mathcal{L}_{P-NLO} = \frac{T_{\text{eff}}}{2} \sqrt{-\gamma(0)} \gamma^{\alpha\beta} H_{\alpha\beta} - \frac{T_{\text{eff}}}{2} \left[ \tilde{\lambda}^{++} e^{\alpha\beta}_\alpha e^{+\beta} + \tilde{\lambda}^{--} e^{\alpha\beta}_\alpha e^{-\beta} - \tilde{\lambda}^{+-} e^{\alpha\beta}_\alpha e^{\pm\beta} \right] + y^M \frac{\delta \mathcal{L}_{P-LO}}{\delta x^M}. \tag{5.4.22} \]

A somewhat tedious calculation shows that the last term in the Lagrangian above can be recast as
\[
y^M \frac{\delta \mathcal{L}_{P-LO}}{\delta x^M} = y^M \frac{\delta \mathcal{L}_{NG-LO}}{\delta x^M} - T_{\text{eff}} e^{\alpha\gamma} (\tau^{\alpha\gamma}_M + \partial^\gamma y^M + y^M \partial^\gamma x^N \partial_M \tau^N_\gamma) + y^M \frac{\delta \mathcal{L}_{P-LO}}{\delta x^M}. \tag{5.4.23} \]

With this, the curved space P-NLO Lagrangian becomes
\[
\mathcal{L}_{P-NLO} = \frac{T_{\text{eff}}}{2} \left( \sqrt{-\gamma(0)} \gamma^{\alpha\beta} H_{\alpha\beta}(x) + \lambda^{++} e^{\alpha\beta}_\alpha e^{+\beta} + \lambda^{--} e^{\alpha\beta}_\alpha e^{-\beta} - \lambda^{+-} e^{\alpha\beta}_\alpha e^{\pm\beta} \right) + y^M \frac{\delta \mathcal{L}_{NG-LO}}{\delta x^M}, \tag{5.4.24} \]

where we defined the combinations
\[
\lambda^{++} = \tilde{\lambda}^{++} - 2 e^{\alpha\gamma}_\alpha (\tau^{\alpha\gamma}_N + \partial^\gamma y^N + \partial^\gamma x^M y^N \partial_M \tau^N_\gamma), \quad \lambda^{--} = \tilde{\lambda}^{--} + 2 e^{\alpha\gamma}_\alpha (\tau^{\alpha\gamma}_N + \partial^\gamma y^N + \partial^\gamma x^M y^N \partial_M \tau^N_\gamma). \tag{5.4.25} \]

This Lagrangian reduces to that of the SNC string of [72, 244] when imposing the strong foliation constraint (2.12.8).

### 5.5 The Wess–Zumino term

The Wess–Zumino (WZ) Lagrangian density reads
\[
\mathcal{L}_{WZ} = -\frac{c T}{2} \epsilon^{\alpha\beta} x^M \partial_\alpha x^N \partial_\beta B_{MN}(x). \tag{5.5.1} \]

We expand the Kalb–Ramond field according to
\[
B_{MN} = c^2 B_{(-2)MN} + B_{(0)MN} + c^{-2} B_{(2)MN} + O(c^{-4}). \tag{5.5.2} \]

We do not include terms of order \( c^4 \) as these would be more divergent than the LO terms in the NG action. The WZ Lagrangian acquires the following expansions
\[
\mathcal{L}_{WZ} = c^2 \mathcal{L}_{WZ-LO} + \mathcal{L}_{WZ-NLO} + c^{-2} \mathcal{L}_{WZ-NLO} + O(c^{-4}), \tag{5.5.3} \]
where the Lagrangians that appear at each order in $c$ are given by

\[
\mathcal{L}_{WZ-LO} = -\frac{T_{\text{eff}}}{2} \varepsilon^{\alpha\beta} B_{(-2)\alpha\beta},
\]

(5.5.4a)

\[
\mathcal{L}_{WZ-NLO} = -\frac{T_{\text{eff}}}{2} \varepsilon^{\alpha\beta} B_{(0)\alpha\beta} + \frac{y_{M}}{\delta x^{M}} \frac{\delta \mathcal{L}_{WZ-LO}}{\delta x^{M}},
\]

(5.5.4b)

\[
\mathcal{L}_{WZ-NNLO} = -\frac{T_{\text{eff}}}{2} \varepsilon^{\alpha\beta} B_{(2)\alpha\beta} + z_{M} \frac{\delta \mathcal{L}_{WZ-LO}}{\delta x^{M}} + \frac{y_{M}}{\delta x^{M}} \frac{\delta^{2} \mathcal{L}_{WZ-LO}}{\delta x^{M} \delta x^{N}}.
\]

(5.5.4c)

### 5.5.1 Lifting the foliation constraint

The addition of a WZ term modifies the LO equation of motion according to

\[
\frac{\delta \mathcal{L}_{WZ-LO}}{\delta x^{M}} \frac{\delta x^{M}}{\delta x^{M}} = \frac{\delta \mathcal{L}_{WZ-LO}}{\delta x^{M}} + \frac{T_{\text{eff}}}{2} \varepsilon^{\alpha\beta} \partial_{\alpha} x^{N} \partial_{\beta} B_{(-2)MN} + \frac{T_{\text{eff}}}{2} \varepsilon^{\alpha\beta} \partial_{\alpha} x^{N} \partial_{\beta} x^{L} \partial_{M} B_{(-2)LN} + \frac{T_{\text{eff}}}{2} \varepsilon^{\alpha\beta} \partial_{\alpha} x^{N} \partial_{\beta} x^{L} H_{(-2)MLN},
\]

(5.5.5)

where

\[
H_{(-2)MLN} = 3 \partial_{[M} B_{(-2)NL]} = \partial_{M} B_{(-2)NL} + \partial_{N} B_{(-2)LM} + \partial_{L} B_{(-2)MN},
\]

(5.5.6)

is the 3-form field strength of $B_{(-2)MN}$. Combining this with the LO equation of motion in the absence of a Kalb–Ramond field (5.3.19), we now find that the LO equation of motion becomes

\[
\frac{\delta \mathcal{L}_{NG-LO}}{\delta x^{M}} + \frac{\delta \mathcal{L}_{WZ-LO}}{\delta x^{M}} = 0,
\]

(5.5.7)

which we can write as

\[
\varepsilon^{\alpha\beta} \partial_{\alpha} x^{K} \partial_{\beta} x^{L} \partial_{[M} F_{KL] + B_{(-2)KL}] = 0,
\]

(5.5.8)

where we recall that $F_{MN} = \varepsilon_{AB} T_{A}^{M} \tau_{N}^{B}$. Hence, if $B_{(-2)MN} = -F_{MN}$, we lift the foliation constraint. This corresponds to a $B$-field of the form (see also [75])

\[
B_{MN} = -c^{2}(T_{M}^{0} T_{N}^{1} - T_{M}^{1} T_{N}^{0}) + \bar{B}_{MN},
\]

(5.5.9)

where $\bar{B}_{MN} = \bar{B}_{(0)MN} + c^{-2} \bar{B}_{(2)MN}$, where the components at each order in $1/c^{2}$ are given by

\[
B_{(-2)MN} = 2 \tau_{[M}^{1} \tau_{N]}^{0},
\]

\[
B_{(0)MN} = 2 \tau_{[M}^{1} m_{N]}^{0} - 2 \tau_{[M}^{0} m_{N]}^{1} + \bar{B}_{(0)MN},
\]

\[
B_{(2)MN} = 2 \tau_{[M}^{1} B_{N]}^{0} - 2 \tau_{[M}^{0} B_{N]}^{1} + 2 m_{[M}^{1} m_{N]}^{0} + \bar{B}_{(2)MN}.
\]

(5.5.10)
so that the WZ Lagrangians of (5.5.3) become

\[ L_{WZ-LO} = T_{\text{eff}} \sqrt{-\tau}, \]

\[ L_{WZ-NLO} = -\frac{T_{\text{eff}}}{2} \epsilon^{\alpha\beta} \bar{B}(0)_{\alpha\beta} + T_{\text{eff}} \epsilon^{\alpha \beta} (\tau^0_\alpha m^1_\beta - \tau^1_\alpha m^0_\beta) + y^M \frac{\delta L_{WZ-LO}}{\delta x^M}, \]

\[ L_{WZ-NNLO} = -\frac{T_{\text{eff}}}{2} \epsilon^{\alpha\beta} \bar{B}(2)_{\alpha\beta} + T_{\text{eff}} \epsilon^{\alpha \beta} (\tau^0_\alpha B^1_\beta - \tau^1_\alpha B^0_\beta) + T_{\text{eff}} \epsilon^{\alpha \beta} m^0_\alpha m^1_\beta + y^M \frac{\delta L_{WZ-LO}}{\delta x^M} + y^M \frac{\delta L_{WZ-NLO}}{\delta x^M} + \frac{1}{2} y^M y^N \frac{\delta^2 L_{WZ-LO}}{\delta x^M \delta x^N}, \]

where we used

\[ \tau^0_\beta = -\tau^{\alpha \beta} \tau^0_\alpha , \quad \tau^1_\beta = \tau^{\alpha \beta} \tau^1_\alpha. \]

This choice identically cancels the LO Lagrangian and thus lifts the foliation constraint coming from the LO equation of motion for \( x^M \). As was also pointed out in [75], this particular choice also removes the field \( m^A_M \) at NLO, which is related to a Stückelberg symmetry as we now discuss.

### 5.5.2 A Stückelberg symmetry

The NLO Lagrangian in the presence of the Kalb–Ramond field (5.5.2) is

\[ L_{NG+WZ-NLO} = -\frac{T_{\text{eff}}}{2} \left( \sqrt{-\tau^0(x) \tau^{\alpha \beta}(x)} H_{\alpha \beta}(x) + \epsilon^{\alpha \beta} B(0)_{\alpha \beta} \right) + y^M \frac{\delta L_{NG+WZ-LO}}{\delta x^M}. \]

As pointed out in [75], this Lagrangian possesses the following Stückelberg symmetry

\[ H_{MN} \rightarrow H_{MN} + 2C_{(0)(M}^A \tau^N_{)B} \eta_{AB}, \quad B_{(0)MN} \rightarrow B_{(0)MN} + 2C_{(0)(M}^A \tau^N_{)B} \epsilon_{AB}, \]

which follows from

\[ \sqrt{-\tau^0} \sqrt{-\tau^{\alpha \beta}} H_{\alpha \beta} + \epsilon^{\alpha \beta} B(0)_{\alpha \beta} \rightarrow \sqrt{-\tau^0} \sqrt{-\tau^{\alpha \beta}} H_{\alpha \beta} + \epsilon^{\alpha \beta} B(0)_{\alpha \beta} + 2 \sqrt{-\tau^0} \sqrt{-\tau^{\alpha \beta}} \partial_\alpha x^M \partial_\beta x^N C_{(0)(M}^A \tau^N_{)B} \eta_{AB} + 2 \epsilon^{\alpha \beta} \partial_\alpha x^M \partial_\beta x^N C_{(0)(M}^A \tau^N_{)B} \epsilon_{AB}, \]

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where
\[
\sqrt{-\tau} \partial^\alpha x^M \partial^\beta x^N C_{(0)(M} A \tau_{N)} B \eta_{AB} + \epsilon^{\alpha\beta} \partial^\alpha x^M \partial^\beta x^N C_{(0)(M} A \tau_{N)} B \epsilon_{AB} = \sqrt{-\tau} \partial^A C_{(0)A} - \sqrt{-\tau} \partial^A C_{(0)A} = 0,
\]
(5.5.15)
where we used (5.3.16). In this way, we can remove the field $m_M A$ from the description of the NLO string with the choice
\[
C_{(0)M A} = m_M A.
\]
(5.5.16)
Note that this can also be achieved with the Kalb–Ramond field of (5.5.9). With this choice, we also cancel the LO Lagrangian.

At NNLO, the NG action including the WZ term is
\[
\mathcal{L}_{NG+WZ-NNLO} = -\frac{T_{\text{eff}}}{8} \sqrt{-\tau} \left[ 4\tau^\alpha\beta (x) \Phi_{\alpha\beta}(x) + (\tau^\alpha\beta (x) H_{\alpha\beta}(x))^2 - 2\tau^\alpha\beta (x) \tau^\gamma\delta (x) H_{\alpha\gamma}(x) H_{\delta\beta}(x) \right] \\
- \frac{T_{\text{eff}}}{2} \epsilon^{\alpha\beta} B_{(2)\alpha\beta} \\
+ 2 M \frac{\delta L_{NG+WZ-LO}}{\delta x^M} + y^M \frac{\delta L_{NG+WZ-NLO}}{\delta x^M} - \frac{1}{2} y^M y^N \frac{\delta^2 L_{NG+WZ-LO}}{\delta x^M \delta x^N}.
\]
(5.5.17)
This Lagrangian again has a Stückelberg symmetry acting as
\[
\Phi_{MN} \rightarrow \Phi_{MN} + 2 C_{(2)(M} A \tau_{N)} B \eta_{AB}, \quad B_{(2)MN} \rightarrow B_{(0)MN} - 2 C_{(2)(M} \tau_{N)} B \epsilon_{AB},
\]
(5.5.18)
and choosing
\[
C_{(2)M A} = B_{M A},
\]
(5.5.19)
removes all terms involving $B_{M A}$ from NNLO Lagrangian, which again exactly corresponds to the choice (5.5.9). This means that the NNLO takes the form
\[
\mathcal{L}_{NG+WZ-NNLO} = -\frac{T_{\text{eff}}}{8} \sqrt{-\tau} \left[ 4\tau^\alpha\beta (x) \Phi_{\alpha\beta}(x) + (\tau^\alpha\beta H_{\alpha\beta})^2 - 2\tau^\alpha\beta \tau^\gamma\delta H_{\alpha\gamma} H_{\delta\beta} \right] \\
- \frac{T_{\text{eff}}}{2} \epsilon^{\alpha\beta} B_{(2)\alpha\beta} + T_{\text{eff}} \epsilon^{\alpha\beta} m^\alpha \epsilon^\beta \nu^M \frac{\delta L_{NG+WZ-NNLO}}{\delta x^M}.
\]
(5.5.20)
Since we cannot now remove $m_M A$, there is no longer an exact matching between SNC and TNC when the $v$-direction is an isometry, but we can still rewrite the above to bring it in a form as close to TNC as possible. In the Polyakov formulation, the Stückelberg symmetry also acts on the Lagrange multipliers $\gamma_{(2)}$ and $\gamma_{(4)}$. The $C_{(0)}$ symmetry acts on $\gamma_{(2)\alpha\beta}$ and $C_{(2)}$ acts on $\gamma_{(4)\alpha\beta}$ in exactly the same way, which can in principle be derived from Eq. (3.25) of [75]. Thus, with the special choice (5.5.9), the P-NNLO Lagrangian in the presence of a
Kalb–Ramond field takes the form
\[
\mathcal{L}_{P\text{-}WZ-\text{NNLO}} = -\frac{T_{\text{eff}}}{2} \sqrt{-\gamma(0)} \gamma_{(\alpha \beta)}(\Phi_{\alpha \beta} + \eta_{AB} m^A_{\alpha} m^B_{\beta}) + \frac{T_{\text{eff}}}{4} \sqrt{-\gamma(0)} G^{\alpha \beta \gamma \delta}_{(0)} \left[ \tau_{\alpha \beta} Y_{(4)\gamma \delta} + H_{\alpha \beta} Y_{(2)\gamma \delta} \right]
+ \frac{T_{\text{eff}}}{4} \sqrt{-\gamma(0)} \tau_{\alpha \beta} Y_{(2)\gamma \delta} \gamma_{(\epsilon \eta)} + \frac{1}{2} \gamma_{(0)} G^{\alpha \beta}_{(0)}
- \frac{T_{\text{eff}}}{2} \epsilon^{\alpha \beta} \bar{B}_{(2)\alpha \beta} + T_{\text{eff}} \epsilon^{\alpha \beta} m^0_{\alpha} m^1_{\beta} + y^M \frac{\delta L_{P\text{-}WZ-\text{NLO}}}{\delta \chi^M}.
\]

### 5.6 The spectrum

In this section, we compute the spectrum of the $1/c^2$ expanded string theories on flat target space. Flat target space corresponds to
\[
\tau^M = \delta^M, \quad \tau^M = \delta^M, \quad H_{MN} = \delta^M \delta^N, \quad m^A_M = B^A_M = \phi^0_{MN} = 0, \quad (5.6.1)
\]
where the spatial index ranges over $i = 1, \ldots, 24$. The spectrum of the string theories matches order by order the expansion of the relativistic string spectrum (5.1.11); in other words, the $1/c^2$ expansion and the computation of the spectrum commute.

### 5.6.1 Mode expansions and spectrum

Since we consider closed strings, the string embedding scalars—with the exception of the leading order field $\chi^M$—are periodic in $\sigma^1$,
\[
\begin{align*}
\chi^i(\sigma^0, \sigma^1 + 2\pi) &= \chi^i(\sigma^0, \sigma^1), \\
y^i(\sigma^0, \sigma^1 + 2\pi) &= y^i(\sigma^0, \sigma^1), \\
z^i(\sigma^0, \sigma^1 + 2\pi) &= z^i(\sigma^0, \sigma^1),
\end{align*}
\]
while the leading order embedding scalar in the compact direction satisfies
\[
\chi^\nu(\sigma^0, \sigma^1 + 2\pi) = \chi^\nu(\sigma^0, \sigma^1) + 2\pi w R_{\text{eff}}, \quad (5.6.3)
\]
where $w \in \mathbb{Z}$ is the winding number. The P-LO Lagrangian (5.3.38) in flat space with the gauge choice (5.3.55) is
\[
\mathcal{L}_{\text{P-LO}} = \frac{T_{\text{eff}}}{2} \eta^{\alpha \beta} \partial_\alpha \chi^\nu \partial_\beta \chi^\nu - \frac{T_{\text{eff}}}{2} \eta^{\alpha \beta} \partial_\alpha \chi^\nu \partial_\beta \chi^\nu. \quad (5.6.4)
\]
The equations of motion are
\[
\partial_- \partial_+ \chi^\nu = 0, \quad \partial_- \partial_+ \chi^\nu = 0, \quad (5.6.5)
\]
\footnote{Since we do not expand the winding number, only the leading order field carries winding.}
while the LO Virasoro constraint with
\[ \tau_{\alpha\beta} = -\partial_\alpha x^t \partial_\beta x^t + \partial_\alpha x^v \partial_\beta x^v, \]  
(5.6.6)
from integrating out \(\gamma^{(0)}_{\alpha\beta}\) reduces to \(\tau_{++} = 0 = \tau_{--}\), i.e.,
\[ \partial_+ x^+ \partial_+ x^- = 0, \quad \partial_- x^+ \partial_- x^- = 0, \]  
(5.6.7)
where
\[ x^\pm = x^t \pm x^v. \]  
(5.6.8)
Without loss of generality we can choose the following conditions\(^5\)
\[ \partial_- x^+ = 0, \quad \partial_+ x^- = 0. \]  
(5.6.9)
Note that this combination makes sense because \(x^v\) was defined as having dimensions of time even though it is a spatial direction. Since \(x^t = \frac{1}{2}(x^+ + x^-)\) and \(x^v = \frac{1}{2}(x^+ - x^-)\), the constraints (5.6.9) imply the LO equations of motion (5.6.5), which thus are not required. The LO-NG Lagrangian, which is obtained by integrating out \(\gamma^{(0)}_{\alpha\beta}\) in (5.3.38) can be written as
\[ \mathcal{L}_{\text{NG-LO}} = T_{\text{eff}} \partial_+ (x^+ \partial_- x^-) - T_{\text{eff}} \partial_- (x^+ \partial_+ x^-), \]  
(5.6.10)
and is a total derivative. In deriving this result we assumed that the map from the worldsheet to the 2-dimensional Lorentzian submanifold of the target space (described by \(x^t\) and \(x^v\)) is orientation preserving, i.e.,
\[ \dot{x}^t x^{v'} - x^t \dot{x}^{v'} > 0. \]  
(5.6.11)
The Lagrangian is linear in the velocities and so the Hamiltonian is minus the Lagrangian, which means that the LO contribution to the energy is a constant.

With the \(v\)-direction compact, the following mode expansions for \(x^\pm\) are compatible with the LO Virasoro constraints (5.6.9) and the boundary condition (5.6.3),
\[ x^- = x^-_0 + wR_{\text{eff}} \sigma^- + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} \alpha_k^e e^{-ik \sigma^-}, \]  
\[ x^+ = x^+_0 + wR_{\text{eff}} \sigma^+ + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} \tilde{\alpha}_k^e e^{-ik \sigma^+}, \]  
(5.6.12)
where \(w\) is the winding number and where \(R_{\text{eff}}\) is the radius of the compact \(v\)-direction that we introduced in Section 5.1.1. Note that this in particular implies that
\[ x^v = x^v_0 + wR_{\text{eff}} \sigma^1 + \text{oscillations}, \]  
(5.6.13)
\(^5\)There are 4 solutions to (5.6.7) but two imply either \(x^+\) is constant or \(x^-\) is constant which is not allowed by our choice of boundary conditions while the other two solutions are related by interchanging \(\sigma^+\) with \(\sigma^-\).
where, in agreement with the results of [73, 75], the LO embedding field of our theory carries no momentum but only winding. We have yet to fix the residual LO gauge transformations (5.3.56) with parameters \( \xi^\pm_{(0)}(\sigma^\pm) \), which act on \( x^\pm \) as

\[
\delta_{\xi_{(0)}} x^\pm = \xi^\pm_{(0)}(\sigma^\pm) \partial_\pm x^\pm ,
\]

where we used (5.6.9), and where \( \xi^\pm_{(0)}(\sigma^\pm) \) are periodic. Therefore, we can fix \( \xi^-_{(0)}(\sigma^-) \) by removing all oscillations from \( x^- \) and \( \xi^+_{(0)}(\sigma^+) \) by removing all oscillations from \( x^+ \), which means that the fully gauge fixed mode expansions read

\[
x^\pm = x^\pm_0 + w_{\text{R eff}} \sigma^\pm . \tag{5.6.15}
\]

It is important to point out that this is an on-shell statement (since it relies on the use of the Virasoro constraints (5.6.9)) and that it is not possible to remove all oscillations from \( x^- \) off-shell. The LO energy is given by

\[
E_{\text{LO}} = - \oint d\sigma^1 \frac{\partial \mathcal{L}_{P-\text{LO}}}{\partial (\partial_\sigma x^t)} = \frac{w_{\text{R eff}}}{\alpha'_{\text{eff}}}, \tag{5.6.16}
\]

and is the stringy analogue of the rest mass of a point particle: the NG-LO Lagrangian is a total derivative, as is the LO point point particle Lagrangian which is responsible for producing the rest mass term \( m c^2 \) (see, e.g., [258]).

Now, at NLO, the Polyakov Lagrangian (5.3.41) in flat target space takes the form

\[
\mathcal{L}_{P-NLO} = - \frac{T_{\text{eff}}}{2} \eta^{\alpha \beta} \partial_\alpha x^i \partial_\beta x^i + T_{\text{eff}} \eta^{\alpha \beta} \partial_\alpha y^t \partial_\beta x^t - T_{\text{eff}} \eta^{\alpha \beta} \partial_\alpha y^v \partial_\beta x^v . \tag{5.6.17}
\]

where we used the worldsheet gauge choices (5.3.55) and (5.3.58). The equations of motion for \( y^t \) and \( y^v \) are (5.6.5), while the equations of motion for \( x^t \), \( x^v \), and \( x^i \) give

\[
\partial_- \partial_+ y^t = 0 , \quad \partial_- \partial_+ y^v = 0 , \quad \partial_- \partial_+ x^i = 0 . \tag{5.6.18}
\]

These equations of motion imply the following mode expansions

\[
\begin{align*}
\chi^i &= x^i_0 + \frac{1}{2 \pi T_{\text{eff}}} p_{(0)i}^0 \sigma^0 + \frac{1}{\sqrt{4 \pi T_{\text{eff}}}} \sum_{k \neq 0} i \frac{\hat{\alpha}_k}{k} e^{-ik\sigma^-} + \frac{\hat{\alpha}_k}{k} e^{-ik\sigma^+} , \tag{5.6.19a} \\
y^t &= y^t_0 - \frac{1}{2 \pi T_{\text{eff}}} p_{(0)t}^0 \sigma^0 + \frac{1}{\sqrt{4 \pi T_{\text{eff}}}} \sum_{k \neq 0} i \frac{\hat{\beta}_k}{k} e^{-ik\sigma^-} + \frac{\hat{\beta}_k}{k} e^{-ik\sigma^+} , \tag{5.6.19b} \\
y^v &= y^v_0 + \frac{1}{2 \pi T_{\text{eff}}} p_{(0)v}^0 \sigma^0 + \frac{1}{\sqrt{4 \pi T_{\text{eff}}}} \sum_{k \neq 0} i \frac{\hat{\beta}_k}{k} e^{-ik\sigma^-} + \frac{\hat{\beta}_k}{k} e^{-ik\sigma^+} , \tag{5.6.19c}
\end{align*}
\]

where we have not included a term linear in \( \sigma^1 \) in the mode expansion for \( y^v \). Had we kept such a term, it would have corresponded to expanding the winding number of the relativistic parent theory in powers \( 1/c^2 \), but since the winding number is an integer, we refrain from doing so. The momenta \( p_{(0)t} \), \( p_{(0)v} \), and \( p_{(0)v} \) featuring in the mode expansions
above correspond to the canonical momenta \( p_{(0)i} = \oint d\sigma \frac{\partial L_{\text{P-NLO}}}{\partial \dot{x}_i}, \quad p_{(0)t} = \oint d\sigma \frac{\partial L_{\text{P-NLO}}}{\partial \dot{x}_t}, \) and
\[
p_{(0)v} = \oint d\sigma \frac{\partial L_{\text{P-NLO}}}{\partial \dot{x}_v},
\]
respectively.

The equation of motion for \( \gamma_{(2)\alpha\beta} \) in the P-NLO Lagrangian (5.3.41) in flat space gives the LO Virasoro constraints (5.6.9), while the equation of motion for \( \gamma_{(0)\alpha\beta} \) leads to the NLO Virasoro constraints (5.3.44), which in flat space can be written as
\[
\partial_+ y^- = \frac{1}{w_{\text{eff}}} \partial_+ x^i \partial_+ x^i, \quad \partial_- y^+ = \frac{1}{w_{\text{eff}}} \partial_- x^i \partial_- x^i,
\]
(5.6.20)
where we, as above, defined
\[
y^\pm = y^t \pm y^v,
\]
(5.6.21)
with the following mode expansions
\[
y^- = y_0^- - \frac{1}{2\pi \eta_{\text{eff}}} p_{(0)+}(\sigma^+ + \sigma^-) + \frac{1}{\sqrt{4\pi \eta_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} \left[ \beta_k^- e^{-ik\sigma^-} + \tilde{\beta}_k^- e^{-ik\sigma^+} \right],
\]
y^+ = y_0^+ - \frac{1}{2\pi \eta_{\text{eff}}} p_{(0)-}(\sigma^+ + \sigma^-) + \frac{1}{\sqrt{4\pi \eta_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} \left[ \beta_k^+ e^{-ik\sigma^-} + \tilde{\beta}_k^+ e^{-ik\sigma^+} \right],
\]
(5.6.22)
where
\[
p_{(0)\pm} = \frac{1}{2} (p_{(0)t} \pm p_{(0)v}),
\]
(5.6.23)
are the canonical momenta in lightcone coordinates, \( p_{(0)i} = \oint d\sigma \frac{\partial L_{\text{P-NLO}}}{\partial \dot{x}_i}, \) while the oscillator modes are
\[
\beta_k^+ = \beta_k + \beta_v, \quad \beta_k^- = \beta_k^t - \beta_v^t,
\]
\[
\tilde{\beta}_k^+ = \tilde{\beta}_k + \tilde{\beta}_v, \quad \tilde{\beta}_k^- = \tilde{\beta}_k^t - \tilde{\beta}_v^t.
\]
(5.6.24)

Using (5.3.64b) and the leading order mode expansion (5.6.15), the NLO residual gauge transformations (5.3.60) with \( \xi^\pm (\sigma^\pm) \) act on \( y^\pm \) as
\[
\delta_{\xi^+ (2)} y^- = w R_{\text{eff}} \xi^- (2) (\sigma^-), \quad \delta_{\xi^- (2)} y^+ = w R_{\text{eff}} \xi^+ (2) (\sigma^+).
\]
(5.6.25)
Hence, the quantities \( \partial_\pm y^\pm \), which feature prominently in the NLO Virasoro constraints (5.6.20), remain invariant under the subleading residual gauge symmetries (5.3.60). Thus we can use \( \xi^- (2) \) to set \( \beta_k^- = 0 \) for \( k \neq 0 \) and \( \xi^+ (2) \) to set \( \tilde{\beta}_k^+ = 0 \) for \( k \neq 0 \), which fixes all residual gauge transformations. Now, the constraints (5.6.20) imply that the zero modes satisfy
\[
p_{(0)-} = -\frac{N_{(0)}}{w R_{\text{eff}}} - \frac{\alpha_{\text{eff}}'}{4w R_{\text{eff}}} (p_{(0)i})^2, \quad p_{(0)+} = -\frac{\tilde{N}_{(0)}}{w R_{\text{eff}}} - \frac{\alpha_{\text{eff}}'}{4w R_{\text{eff}}} (p_{(0)i})^2,
\]
(5.6.26)
where \((p_0) = p_0 i (p_0)\), and where we defined

\[
N(0) = \sum_{n=1}^{\infty} \alpha_n^{i} \alpha_n^{*}, \quad \tilde{N}(0) = \sum_{n=1}^{\infty} \tilde{\alpha}_n^{i} \tilde{\alpha}_n^{*}.
\]

(5.6.27)

The higher oscillator modes satisfy

\[
\tilde{\beta}_- = \frac{1}{\sqrt{4\pi T_{\text{eff}}} w_{\text{R}_{\text{eff}}}} \sum_{n \in \mathbb{Z}} \alpha_k^{i} \alpha_n^{*}, \quad \beta_+ = \frac{1}{\sqrt{4\pi T_{\text{eff}}} w_{\text{R}_{\text{eff}}}} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_k^{i} \tilde{\alpha}_n^{*},
\]

(5.6.28)

where the zero mode is given by

\[
\alpha_0^{i} = \tilde{\alpha}_0^{i} = \frac{1}{\sqrt{4\pi T_{\text{eff}}} p_0^{i}}.
\]

(5.6.29)

The relativistic energy \(E\) can be expanded as

\[
E = c^2 E_{\text{LO}} + E_{\text{NLO}} + c^{-2} E_{\text{NNLO}} + O(c^{-4})
\]

(5.6.30)

This means that the contribution to the spectrum from the NLO Lagrangian becomes

\[
E_{\text{NLO}} = -\frac{N(0)}{w_{\text{R}_{\text{eff}}}} - \frac{\tilde{N}(0)}{w_{\text{R}_{\text{eff}}}} + \frac{\alpha_{\text{eff}}'}{2 w_{\text{R}_{\text{eff}}}} (p_0^{i})^2,
\]

(5.6.31)

Note also that

\[
E_{\text{LO}} = -\frac{\alpha_{\text{eff}}'}{2 w_{\text{R}_{\text{eff}}}} (p_0^{i})^2,
\]

(5.6.32)

which follows from \(\frac{\delta L_{\text{P-NLO}}}{\delta (\partial Y_0)} = \frac{\delta L_{\text{P-LO}}}{\delta (\partial Y_0)}\). Thus the energy up to NLO is

\[
E = c^2 E_{\text{LO}} + E_{\text{NLO}} + O(c^{-2}) = c^2 w_{\text{R}_{\text{eff}}} + \frac{N + \tilde{N}}{w_{\text{R}_{\text{eff}}}} + \frac{\alpha_{\text{eff}}'}{2 w_{\text{R}_{\text{eff}}}} p_0^2 + O(c^{-2}),
\]

(5.6.33)

where

\[
N = N(0) + O(c^{-2}), \quad \tilde{N} = \tilde{N}(0) + O(c^{-2}), \quad p_1 = p_0^{i} + O(c^{-2}).
\]

(5.6.34)

The momentum \(p_0^{i}\) is quantized. To see this, note that the momentum along the string in the \(v\)-direction is \(p_0^{i} = \int d\sigma^{j} \frac{\delta L_{\text{P}}}{\delta (\partial Y^j)} = p_0^{i} + O(c^{-2})\), where we used the mode expansions (5.6.13) and (5.6.19c). Standard quantum mechanics arguments tell us that the wave function \(\Psi = \frac{e^{i h \hat{p} \cdot X}}{\sqrt{\alpha_{\text{eff}}}}\) in particular includes a term of the form \(e^{i \hat{p} \cdot x}\), and since the
string wave function is single-valued, \( p_{(0)v} \) must be quantized according to

\[
p_{(0)v} = \frac{\hbar n}{\mathcal{R}_{\text{eff}}},
\]  

(5.6.35)

and since

\[
p_{(0)v} = p_{(0)+} - p_{(0)-} = \frac{N_{(0)} - \tilde{N}_{(0)}}{w\mathcal{R}_{\text{eff}}},
\]  

(5.6.36)

we obtain the level matching condition

\[
N_{(0)} - \tilde{N}_{(0)} = \hbar n.
\]  

(5.6.37)

The NNLO Lagrangian (5.3.45) with worldsheet gauge choices (5.3.55), (5.3.58) and (5.3.62) for a flat target space is

\[
\mathcal{L}_{\text{P-NNLO}} = \frac{T_{\text{eff}}}{2} \eta^{\alpha \beta} \left( \partial^\alpha y^t \partial^\beta y^t - \partial^\alpha y^v \partial^\beta y^v \right) + \frac{T_{\text{eff}}}{2} \eta^{\alpha \beta} \left( \partial^\alpha x^t \partial^\beta z^t - \partial^\alpha x^v \partial^\beta z^v \right) - \frac{T_{\text{eff}}}{2} \eta^{\alpha \beta} \partial^\alpha x^i \partial^\beta y^i.
\]  

(5.6.38)

In addition to the LO and NLO equations of motion, (5.6.5) and (5.6.18), which arise as the equations of motion for \( z^t, z^v, y^t, y^v, y^i \), the equations of motion for \( x^t, x^v, x^i \) are

\[
\partial_- \partial_+ z^t = 0, \quad \partial_- \partial_+ z^v = 0, \quad \partial_- \partial_+ y^i = 0.
\]  

(5.6.39)

As above, this leads to the following mode expansions

\[
y^i = y^i_0 + \frac{1}{2\pi T_{\text{eff}}} p_{(2)i} \sigma^0 + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} \left[ \hat{\rho}^i_k e^{-ik\sigma^0} + \hat{\beta}^i_k e^{ik\sigma^0} \right],
\]  

(5.6.40a)

\[
z^t = z^t_0 - \frac{1}{2\pi T_{\text{eff}}} p_{(2)t} \sigma^0 + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} \left[ \hat{\chi}^t_k e^{-ik\sigma^0} + \hat{\chi}^t_k e^{ik\sigma^0} \right],
\]  

(5.6.40b)

\[
z^v = z^v_0 + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} \left[ \hat{\chi}^v_k e^{-ik\sigma^0} + \hat{\chi}^v_k e^{ik\sigma^0} \right],
\]  

(5.6.40c)

where, as before, we do not include a term linear in \( \sigma^1 \) in the mode expansion of \( z^v \), while we now also choose to exclude a term linear in \( \sigma^0 \), since this would correspond to expanding the integer-valued momentum mode \( n \). As above, we have \( p_{(2)t} = \oint d\sigma^1 \frac{\delta \mathcal{L}_{\text{P-NNLO}}}{\delta x^t} \) as well as \( p_{(2)i} = \oint d\sigma^1 \frac{\delta \mathcal{L}_{\text{P-NNLO}}}{\delta x^i} \). The constraints from both LO (5.6.9) and NLO (5.6.20) arise as the Virasoro constraints from integrating out \( \gamma^{(4)\alpha\beta} \) and \( \gamma^{(2)\alpha\beta} \), respectively, while the novel constraint at NNLO is given by equation (5.3.46) which becomes

\[
0 = 2\partial_- x^t \partial_- y^i - \partial_- y^+ \partial_- y^- - w\mathcal{R}_{\text{eff}} \partial_- z^+,
\]

\[
0 = 2\partial_+ x^t \partial_+ y^i - \partial_+ y^+ \partial_+ y^- - w\mathcal{R}_{\text{eff}} \partial_+ z^-.
\]  

(5.6.41)
where we used the LO Virasoro constraints, and where we defined $z^\pm = z^1 \pm z^\nu$. We can write the mode expansions for the $z^1$ and $z^\nu$ fields as

$$z^- = z_0^- - \frac{1}{4\pi T_{\text{eff}}} \mathcal{P}_{(2)\downarrow} (\sigma^+ + \sigma^-) + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} \left[ \chi_k^- e^{-ik\sigma^-} + \tilde{\chi}_k^- e^{-ik\sigma^+} \right],$$

$$z^+ = z_0^+ - \frac{1}{4\pi T_{\text{eff}}} \mathcal{P}_{(2)\downarrow} (\sigma^+ + \sigma^-) + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} \left[ \tilde{\chi}_k^+ e^{-ik\sigma^+} + \chi_k^+ e^{-ik\sigma^-} \right],$$

(5.6.42)

where

$$\chi_k^+ = \chi_k^1 + \chi_k^\nu, \quad \chi_k^- = \chi_k^1 - \chi_k^\nu, \quad \tilde{\chi}_k^+ = \tilde{\chi}_k^1 + \tilde{\chi}_k^\nu, \quad \tilde{\chi}_k^- = \tilde{\chi}_k^1 - \tilde{\chi}_k^\nu.$$

(5.6.43)

We can now fix residual gauge transformations (5.3.63) at NNLO $\xi_{\pm}^{\pm} (\sigma^\pm)$ by setting $\chi_k^1 = 0$ for $k \neq 0$ and $\tilde{\chi}_k^1 = 0$ for $k \neq 0$, which fixes all residual gauge transformations. This is entirely analogous to what happened at NLO. The gauge fixing of the subleading residual gauge transformations, discussed below (5.6.25), imply that $\partial^- y^-$ and $\partial^+ y^+$ contain no oscillations, and so the NLO Virasoro constraints give us

$$\partial^- y^- = \frac{\alpha' \eta N_0}{\omega R_{\text{eff}}} + \frac{\alpha' \eta^2}{4\omega R_{\text{eff}}} (p_0)^2,$$

$$\partial^+ y^+ = \frac{\alpha' \eta N_0}{\omega R_{\text{eff}}} + \frac{\alpha' \eta^2}{4\omega R_{\text{eff}}} (p_0)^2,$$

(5.6.44)

where we used the NLO Virasoro constraints in the guise of (5.6.26). Using the relation

$$4\bar{N}_0 N_0 = (N_0 + \bar{N}_0)^2 - \hbar^2 w^2 n^2,$$

(5.6.45)

where we imposed the level matching condition of (5.6.37), the NNLO Virasoro constraints (5.6.41) then imply the following two expressions for the zero mode $p_{(2)\downarrow}$

$$-p_{(2)\downarrow} = \frac{\alpha' \eta}{\omega R_{\text{eff}}} p_0 (p_0) + \frac{2}{\omega R_{\text{eff}}} \sum_{k \neq 0} \alpha_k \beta_k^- - \frac{\alpha' \eta^3}{8\omega R_{\text{eff}}} (p_0)^4 - \alpha' \eta^2 (N_0 + \bar{N}_0)^2 - \frac{2}{2\omega R_{\text{eff}}} \frac{\hbar^2 n^2}{2},$$

$$- \frac{\alpha' \eta^2}{\omega R_{\text{eff}}} (p_0) + \frac{2}{\omega R_{\text{eff}}} \sum_{k \neq 0} \alpha_k \beta_k^- - \frac{\alpha' \eta^3}{8\omega R_{\text{eff}}} (p_0)^4 - \alpha' \eta^2 (N_0 + \bar{N}_0)^2 - \frac{2}{2\omega R_{\text{eff}}} \frac{\hbar^2 n^2}{2},$$

(5.6.46a)

where $p_{(2)\downarrow} = \sqrt{4\pi T_{\text{eff}}} \beta_0^1 = \sqrt{4\pi T_{\text{eff}}} \beta_0^1$. The contribution to the spectrum from the NNLO
Lagrangian is obtained by adding the two expressions above (and dividing by two)

\[
E_{\text{NNLO}} = -\int d\sigma^1 \frac{\partial L_{\text{NNLO-P}}}{\partial (\partial_0 x^1)} = -p_{(2)|t}
\]

\[
= \frac{\alpha'}{w R_{\text{eff}}} p_{(0)}(p_{(2)|t} + \frac{N_{(2)}}{w R_{\text{eff}}} - \frac{(\alpha'')^3}{8 w^3 R_{\text{eff}}^3} (p_{(0)})^4 - \alpha' \frac{(N_{(0)} + \tilde{N}_{(0)})^2}{2 w^3 R_{\text{eff}}^3}
\]

\[-(\alpha'')^2 (p_{(0)})^2 \frac{N_{(0)} + \tilde{N}_{(0)}}{2 w^3 R_{\text{eff}}^3} + \alpha' \frac{h^2 n^2}{2 w R_{\text{eff}}^3},
\]

where we defined the subleading number operators

\[
N_{(2)} = \sum_{k \neq 0} \alpha_i^k \beta_k^i = \sum_{k=1}^{\infty} \alpha_i^k \beta_k^i + \sum_{k=1}^{\infty} \alpha_i^k \beta_{-k}^i,
\]

\[
\tilde{N}_{(2)} = \sum_{k \neq 0} \tilde{\alpha}_i^k \tilde{\beta}_k^i = \sum_{k=1}^{\infty} \tilde{\alpha}_i^k \tilde{\beta}_k^i + \sum_{k=1}^{\infty} \tilde{\alpha}_i^k \tilde{\beta}_{-k}^i.
\]

In terms of the NNLO Lagrangian, the LO and NLO contributions to the spectrum are the canonical momenta conjugate to \(z^t\) and \(y^t\), respectively,

\[
E_{\text{LO}} = -c^2 \int d\sigma^1 \frac{\partial L_{\text{NNLO-P}}}{\partial (\partial_0 z^1)} , \quad E_{\text{NLO}} = -\int d\sigma^1 \frac{\partial L_{\text{NNLO-P}}}{\partial (\partial_0 y^1)}.
\]

The means that the energy up to NNLO becomes

\[
E = c^2 E_{\text{LO}} + E_{\text{NLO}} + c^{-2} E_{\text{NNLO}} + O(c^{-4})
\]

\[
= \frac{c^2 w R_{\text{eff}}}{\alpha'_{\text{eff}}} + \frac{N + \tilde{N}}{w R_{\text{eff}}} + \frac{\alpha''}{2 w^2 R_{\text{eff}}^2} p^2 - \frac{(\alpha'')^3}{8 w^3 c^2 R_{\text{eff}}^3} p^4 - \alpha' \frac{(N + \tilde{N})^2}{2 w^3 c^2 R_{\text{eff}}^3}
\]

\[-(\alpha'')^2 p^2 \frac{N + \tilde{N}}{2 w^3 c^2 R_{\text{eff}}^3} + \alpha' \frac{h^2 n^2}{2 w c^2 R_{\text{eff}}^3} + O(c^{-4}),
\]

where at NNLO, we have

\[
N = N_{(0)} + c^{-2} N_{(2)} + O(c^{-4}), \quad \tilde{N} = \tilde{N}_{(0)} + c^{-2} \tilde{N}_{(2)} + O(c^{-4}),
\]

\[
p_{(1)} = p_{(0)}(1) + c^{-2} p_{(2)}(1) + O(c^{-4}).
\]

Subtracting the two expressions for \(p_{(2)|t}\) in (5.6.46a) and (5.6.46b) gives the subleading level matching condition

\[
N_{(2)} = \tilde{N}_{(2)}.
\]

Note that this is a consequence of our choice to not expand the momentum mode \(n\) and the winding number \(w\). The level matching conditions (5.6.37) and (5.6.52) are compatible with
the 1/c² expansion of the relativistic level matching condition

\[ N - \tilde{N} = h \omega n. \] (5.6.53)

### 5.6.2 The Gomis–Oguri spectrum in the presence of a B-field

In flat target space (5.6.1), we take the Kalb–Ramond field to have the following expansion

\[ B_{MN} = 2c^2 \delta^t_M \delta^v_N B_{(-2)tv} + c^{-2} B_{(0)MN} + O(c^{-4}), \] (5.6.54)

where the LO term only involves \( x^t \) and \( x^v \). If we now choose \( B_{(-2)tv} = B = \text{const.} \) and \( B_{(0)MN} = B_{(2)MN} = 0 \), we get the following expression for the spectrum

\[
E = c^2 E_{LO} + E_{NLO} + c^{-2} E_{NNLO} + O(c^{-4})
= \frac{c^2 W_{\text{eff}}}{\alpha'_{\text{eff}}} (1 - B) + \frac{N + \tilde{N}}{w_{\text{eff}}} + \frac{\alpha'_{\text{eff}}}{2w_{\text{eff}}} p^2 - \frac{(\alpha'_{\text{eff}})^3}{8w^3 c^2 R^3_{\text{eff}}} p^4 - \frac{\alpha'_{\text{eff}} (N + \tilde{N})^2}{2w^2 c^2 R^3_{\text{eff}}}
- (\alpha'_{\text{eff}})^2 p^2 \frac{N + \tilde{N}}{w^3 c^2 R^3_{\text{eff}}} + \alpha'_{\text{eff}} \frac{h^2 n^2}{2w c^2 R^3_{\text{eff}}} + O(c^{-4}),
\] (5.6.55)

which gives rise to the modification of \( E_{LO} \) discussed in (5.1.15). Up to \( O(1) \), this is the result of [63] when taking \( B = 1/2 \) (see also [69]).

### 5.7 Target space symmetries and algebra of charges

The target space symmetries manifest themselves as global symmetries for the embedding fields, i.e., in the case of Poincaré symmetry for the relativistic string we have \( X^M \rightarrow L^M \delta N^N + \alpha^M \) for constant \( L^M \) and \( \alpha^M \), and the Noether charges corresponding to these global symmetries generate the Poincaré algebra under the Poisson bracket. The string theories we consider in this work arise as 1/c² expansions of relativistic string theory, and hence it is natural to expect that the charge algebra of the string theory at a given order in 1/c² corresponds to an expansion of the Poincaré algebra. As we demonstrate in this section, this is indeed the case. In particular, the algebra of charges at NLO gives rise to the string Bargmann algebra.

#### 5.7.1 Expansion of target space symmetries

We now write \( X^M = (cX^A, X^i) \), having dimensions \([X^A] = \text{time} \) and \([X^i] = \text{length} \), and where, with a slight abuse of notation, we have introduced longitudinal indices \( A, B = (t, v) \). The (infinitesimal) Poincaré transformations act in the following way on \( X^A \) and \( X^i \)

\[
\delta X^A = \Lambda^A_B X^B + c^{-2} \Lambda^A_i X^i + \alpha^A, \\
\delta X^i = \Lambda^i_A X^A + \alpha^i,
\] (5.7.1)
where $\Lambda^A B$ are longitudinal Lorentz transformations (which are dimensionless), $\Lambda^i_j$ are transverse rotations (again dimensionless), while

$$\Lambda^A_i = -\delta_{ij} A^j_B \eta^{AB}, \quad (5.7.2)$$

are string Galilei boosts with dimensions of velocity. Finally, the $a^A$ and $a^i$ are longitudinal and transverse translations, respectively.

We now expand the embedding fields as in (5.3.7) and the transformation parameters as

$$\begin{align*}
a^A &= a^A_{(0)} + c^{-2} a^A_{(2)} + c^{-4} a^A_{(4)} + \mathcal{O}(c^{-6}), \\
\Lambda^A B &= \Lambda^A_{(0)B} + c^{-2} \Lambda^A_{(2)B} + c^{-4} \Lambda^A_{(4)B} + \mathcal{O}(c^{-6}), \\
\Lambda^A_i &= \Lambda^A_{(0)i} + c^{-2} \Lambda^A_{(2)i} + \mathcal{O}(c^{-4}), \\
a^i &= a^i_{(0)} + c^{-2} a^i_{(2)} + \mathcal{O}(c^{-4}), \\
\Lambda^A &= \Lambda^A_{(0)} + c^{-2} \Lambda^A_{(2)} + c^{-4} \Lambda^A_{(4)} + \mathcal{O}(c^{-6}), \\
\Lambda^i_j &= \Lambda^i_{(0)j} + c^{-2} \Lambda^i_{(2)j} + \mathcal{O}(c^{-4}).
\end{align*} \quad (5.7.3)$$

This leads to

$$\begin{align*}
\delta x^A &= \Lambda^A_{(0)B} x^B + a^A_{(0)}, \quad (5.7.4a) \\
\delta x^i &= \Lambda^i_{(0)j} x^j + \lambda^i_{(0)A} x^A + a^i_{(0)}, \quad (5.7.4b) \\
\delta y^A &= \Lambda^A_{(0)B} y^B + \lambda^A_{(2)B} x^B + \lambda^A_{(0)B} x^B + a^A_{(2)}, \quad (5.7.4c) \\
\delta y^i &= \lambda^i_{(0)j} y^j + \lambda^i_{(2)j} x^j + \lambda^i_{(0)A} y^A + \lambda^i_{(2)A} x^A + a^i_{(2)}, \quad (5.7.4d) \\
\delta z^A &= \Lambda^A_{(0)B} z^B + \lambda^A_{(2)B} z^B + \lambda^A_{(4)B} z^B + \lambda^A_{(0)B} z^B + a^A_{(4)} + \lambda^A_{(2)B} z^B + a^A_{(4)} + \lambda^A_{(4)B} z^B + a^A_{(4)} + \lambda^A_{(4)B} z^B + a^A_{(4)}.
\end{align*} \quad (5.7.4e)$$

In the expanded string theory, each of the transformations above will have an associated Noether charge. These in turn correspond to expanded components of the relativistic Noether charges, which arise from the symmetries (5.7.1) of the relativistic Polyakov Lagrangian (5.3.33), which on flat target space and in conformal gauge is

$$\mathcal{L}_p = -\frac{T_{\text{eff}}}{2} \eta^{\alpha \beta} \partial_\alpha X^\lambda \partial_\beta X^\mu \eta_{MN}. \quad (5.7.5)$$

The Noether currents corresponding to the translations $a^A$ and $a^i$ are

$$\begin{align*}
\Pi^A_\alpha &= \frac{\partial \mathcal{L}_p}{\partial \partial_\alpha X^A} = -c^2 T_{\text{eff}} \partial^A \partial_\beta X^\lambda \eta_{\beta A}, \\
\Pi^i_\alpha &= \frac{\partial \mathcal{L}_p}{\partial \partial_\alpha X^i} = -T_{\text{eff}} \partial^\alpha X^i \delta_{ii},
\end{align*} \quad (5.7.6)$$

while the Noether currents for longitudinal Lorentz transformations $\Lambda^A B$ and transverse
rotations $A^i_j$ take the form
\[
J_{AB}^\alpha = X_A \Pi_B^\alpha - X_B \Pi_A^\alpha,
\]
\[
J_{ij}^\alpha = X_i \Pi_j^\alpha - X_j \Pi_i^\alpha.
\] (5.7.7)

Finally, the Noether current for the transformations $A^A_1$ is
\[
J_{A_1}^\alpha = X_A \Pi_i^\alpha - c^{-2} X_i \Pi_A^\alpha.
\] (5.7.8)

Expanding the Noether currents in powers of $1/c^2$, we get
\[
\Pi_1^\alpha = c^2 \pi_{(-2)}^\alpha + \pi_{(0)}^\alpha + c^{-2} \pi_{(-2)}^\alpha + \mathcal{O}(c^{-4}),
\]
\[
\Pi_i^\alpha = \pi_{(0)i}^\alpha + c^{-2} \pi_{(-2)i}^\alpha + \mathcal{O}(c^{-4}),
\]
\[
J_{(AB)}^\alpha = c^2 j_{(-2)}^\alpha + j_{(0)AB}^\alpha + c^{-2} j_{(-2)AB}^\alpha + \mathcal{O}(c^{-4}),
\]
\[
J_{(ij)}^\alpha = j_{(0)ij}^\alpha + c^{-2} j_{(-2)ij}^\alpha + \mathcal{O}(c^{-4}),
\]
\[
J_{A_1}^\alpha = j_{(0)A_1}^\alpha + c^{-2} j_{(-2)A_1}^\alpha + \mathcal{O}(c^{-4}),
\] (5.7.9)

where
\[
\pi_{(-2)}^\alpha = -\mathcal{E}_{\text{eff}} \partial^\alpha x_A,
\]
\[
\pi_{(0)i}^\alpha = -\mathcal{E}_{\text{eff}} \partial^\alpha y_i,
\]
\[
j_{(-2)AB}^\alpha = 2x_{[A} \pi_{(-2)]B}^\alpha,
\]
\[
j_{(0)ij}^\alpha = 2x_{[i} \pi_{j]}^\alpha,
\]
\[
j_{(0)A_1}^\alpha = x_A \pi_{(0)i}^\alpha - x_i \pi_{(-2)}^\alpha,
\]
\[
j_{(2)A_1}^\alpha = x_A \pi_{(2)i}^\alpha - x_i \pi_{(-2)}^\alpha + y_A \pi_{(0)i}^\alpha - y_i \pi_{(-2)}^\alpha,
\]
\[
j_{(2)AB}^\alpha = 2x_{[A} \pi_{(2)]B}^\alpha + 2y_{[A} \pi_{(0)]B}^\alpha + 2z_{[A} \pi_{(-2)]B}^\alpha,
\]
\[
\pi_{(2)A}^\alpha = -\mathcal{E}_{\text{eff}} \partial^\alpha z_A.
\] (5.7.10)

As we will see below, the modes of the $1/c^2$ expanded Noether currents are precisely the Noether currents associated to the expanded transformations; in this sense, the $1/c^2$ expansion commutes with the Noether procedure.

### 5.7.2 Expansion of the Poincaré algebra

The $(d+2)$-dimensional Poincaré algebra is $\text{so}(d+1, 1)$ in the basis $(J_{MN}, P_M)$ has the brackets
\[
[J_{MN}, J_{KL}] = \eta_{MK} J_{NL} - \eta_{NK} J_{ML} - \eta_{ML} J_{NK} + \eta_{NL} J_{MK},
\]
\[
[J_{MN}, P_K] = 2\eta_{KM} P_N.
\] (5.7.11)

As explained in [75], we set up the string $1/c^2$ expansion of the Poincaré algebra by splitting the index $M = (A, i)$ and reinstating factors of $c$, which amounts to
\[
J_{A_1} = c \mathcal{B}_{A_1} \quad \text{and} \quad P_A = H_A/c,
\] (5.7.12)

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for the stringy boosts and the longitudinal translations, respectively. We then expand the generators according to either

\[ X = \sum_{k \in \mathbb{N}_0} c^{-2k} x^{(2k)}, \]

\[ Y = \sum_{k \in \mathbb{N}_0} c^{-2k} y^{(2k-2)}, \]

where \( X \) consists of the generators \( \{ J_{ij}, P_{li}, B_{A[i]} \} \), while \( Y \) consists of the generators \( \{ J_{AB}, H_A \} \).

We define the level of a generator to be the (even) integer in parentheses in the superscript; for example, \( P_{li}^{(10)} \) is a “level-10 generator”. Thus, \( Y \) begins at level \(-2\), while \( X \) begins at level 0. The reasoning behind this off-set for \( Y \) will become clear below. This means that we obtain the following brackets (where \( m, n \in \mathbb{N}_0 \))

\[ [J_{AB}^{(2m-2)}, B_{C}^{(2n-2)}] = 2\eta_{C[H_{B}]}, \]

\[ [J_{ij}^{(2m)}, P_{k}^{(2n)}] = 2\delta_{k|i}p_{ij}^{(2m+2n)}, \]

\[ [H_A^{(2m-2)}, B_{Bl}^{(2n)}] = -\eta_{AB}P_{li}^{(2m+2n)}, \]

\[ [B_{A}^{(2m)}, J_{ij}^{(2n-2)}] = -2\eta_{AB}B_{C[l]}^{(2m+2n)}, \]

\[ [J_{AB}^{(2m-2)}, I_{CD}^{(2n-2)}] = 0, \]

\[ [B_{A}^{(2m)}, J_{kl}^{(2n-2)}] = 4\delta_{[j]k}\eta_{AB}J_{ij}^{(2m+2n)} + \delta_{ij}J_{AB}^{(2m+2n)}. \]

We will denote the \( 1/c^2 \) expanded Poincaré algebra, which contains infinitely many generators, by \( \text{iso}_{1/c^2}(d + 1, 1) \). Now, notice that for a given, fixed integer \( n \geq 0 \), the set of all generators of level \( 2k \geq 2n \) forms an (infinite dimensional) ideal in \( \text{iso}_{1/c^2}(d + 1, 1) \) that we call \( i_n \). The biggest ideal is \( i_0 \), and in general we have the filtration

\[ i_0 \supset i_1 \supset i_2 \supset \ldots \]

(5.7.15)

This means that for each integer \( n \geq 0 \), we can form the quotient algebra

\[ q_n = \text{iso}_{1/c^2}(d + 1, 1)i_n. \]

(5.7.16)

For \( n = 0 \), only the level \(-2\) generators \( J_{AB}^{(-2)} \) and \( H_A^{(-2)} \) remain and generate the two-dimensional Poincaré algebra, i.e., \( q_0 \cong \text{iso}(1, 1) \). For \( n = 1 \), the algebra \( q_1 \) has the following non-zero brackets

\[ [J_{AB}^{(-2)}, H_{C}^{(-2)}] = 2\eta_{C[H_{B}]}, \]

\[ [J_{ij}^{(-2)}, P_{k}^{(0)}] = 2\delta_{k|i}p_{ij}^{(0)}, \]

\[ [H_A^{(-2)}, B_{C[l]}^{(0)}] = -\eta_{AB}P_{li}^{(0)}, \]

\[ [B_{A}^{(0)}, J_{ij}^{(0)}] = -2\delta_{[j]k}\eta_{AB}J_{ij}^{(0)} + \delta_{ij}J_{AB}^{(0)}. \]

(5.7.17a)

(5.7.17b)

(5.7.17c)

(5.7.17d)

(5.7.17e)
This algebra is known as the string Bargmann algebra \[164\]. Below we will show that the algebra of Noether charges at order $N^n_{\text{LO}}$ of the string $1/e^2$ expansion is isomorphic to $q_n$ for $n = 0, 1, 2$, and we expect this pattern to persist at all orders.\(^6\)

### 5.7.3 Noether charge algebra

The LO Polyakov Lagrangian on flat space in flat conformal gauge reads

$$L_{P-\text{LO}} = -\frac{T_{\text{eff}}}{2}\eta_{AB}\eta^{\alpha\beta}\partial_\alpha x^A \partial_\beta x^B,$$  \hspace{1cm} (5.7.18)

As we have already remarked, the LO Polyakov Lagrangian is identical to the relativistic Polyakov Lagrangian with two target space dimensions. The conserved current corresponding to translations $a^A(0)$ is

$$\pi_0^A = -\frac{T_{\text{eff}}}{2}\partial_\alpha x^A,$$  \hspace{1cm} (5.7.19)

while the current for longitudinal Lorentz transformations is

$$j_{AB} = x_A\pi_0^B - x_B\pi_0^A.$$  \hspace{1cm} (5.7.20)

The equal-time Poisson brackets\(^7\) between the canonically conjugate variables $x^A$ and $\pi_0^B$ at LO are

$$\{x^A(\sigma^1), \pi_0^B(\tilde{\sigma}^1)\} = \delta_A^B \delta(\sigma^1 - \tilde{\sigma}^1).$$  \hspace{1cm} (5.7.21)

The charges at LO are

$$P_{(-2)A} = \oint d\sigma^1 \pi_0^A, \quad J_{(-2)AB} = \oint d\sigma^1 j_{AB},$$  \hspace{1cm} (5.7.22)

which, using the LO brackets (5.7.21) generate the 2-dimensional Poincaré algebra, which we called $q_0$ above. The specific map between charges and generators is

$$P_{(-2)A} \leftrightarrow H_A^{(-2)}, \quad J_{(-2)AB} \leftrightarrow J_{AB}^{(-2)}.$$  \hspace{1cm} (5.7.23)

This is as it should be: since the LO Lagrangian is the standard Polyakov action with a two-dimensional target space, the charge algebra better be the two-dimensional Poincaré algebra.

We now turn our attention to the NLO Polyakov Lagrangian, which we write as

$$L_{P-\text{NLO}} = -\frac{T_{\text{eff}}}{2}\eta^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^i - T_{\text{eff}}\eta_{AB}\eta^{\alpha\beta}\partial_\alpha x^A \partial_\beta y^B.$$  \hspace{1cm} (5.7.24)

\(^6\)An identical picture exists for the particle: there, the first non-trivial quotient algebra is the Bargmann algebra (see also \[24\]).

\(^7\)We will have more to say about Poisson brackets in Section 5.8.
Invariance under stringy boosts $\lambda^A_{(0)}$ follows from
\[ \delta_{\lambda,(0)} L_{\text{P-NLO}} = -T_{\text{eff}} \eta^\alpha_\beta \partial_\alpha x^A (\lambda^i_{(0)} A + \eta_{AB} \lambda^B_{(0)} i) = 0, \] (5.7.25)
and equation (5.7.2), while invariance under subleading longitudinal Lorentz transformations $\lambda^A_{(2)B}$ is a consequence of
\[ \delta_{\lambda,(2)} L_{\text{P-NLO}} = -T_{\text{eff}} \eta^\alpha_\beta \partial_\alpha x^A \lambda^B_{(2)AB} = 0. \] (5.7.26)
Invariance under the remaining symmetries follows from identical arguments. The transverse translations $a^i_{(0)}$ and rotations $\lambda^i_{(0)j}$ act only on $x^i$, and the Noether currents are
\[ \pi^\alpha_{(0)i} = \frac{\delta L_{\text{P-NLO}}}{\delta \partial^\alpha x^i} = -T_{\text{eff}} \partial^\alpha x^i, \] (5.7.27)
\[ j^\alpha_{(0)ij} = x^i \pi^\alpha_{(0)j} - x^j \pi^\alpha_{(0)i}. \]
The LO longitudinal translations $a^A_{(0)}$ now give rise to the current
\[ \pi^\alpha_{(0)A} = \frac{\delta L_{\text{P-NLO}}}{\delta \partial^\alpha x^A} = -T_{\text{eff}} \partial^\alpha y^A, \] (5.7.28)
while the LO longitudinal Lorentz transformations $\lambda^A_{(0)B}$ produce the current
\[ j^\alpha_{(0)AB} = 2x^A \pi^\alpha_{(0)B} + 2y^A \partial^\alpha y^B. \] (5.7.29)
These are different to the currents corresponding to those transformation in the leading order theory; these instead arise as the currents for subleading longitudinal transformation. Specifically, NLO translations $a^A_{(2)}$, acting on $y^A$, produce the current $\pi^\alpha_{(-2)A}$ given in (5.7.19), and the current for NLO longitudinal Lorentz transformations $\lambda^A_{(2)B}$ is $j^\alpha_{(-2)AB}$ given in (5.7.20). Finally, the current for stringy boosts $\Lambda^A_{(0)}$ is
\[ j^\alpha_{(0)A} = x_A \pi^\alpha_{(0)i} - x_i \pi^\alpha_{(-2)A}. \] (5.7.30)
The charges at NLO are
\[ \mathcal{P}_{(-2)A} = \int d\sigma^1 \pi^0_{(-2)A}, \quad j_{(-2)AB} = \int d\sigma^1 j^0_{(-2)AB}, \]
\[ \mathcal{P}_{(0)i} = \int d\sigma^1 \pi^0_{(0)i}, \quad \mathcal{P}_{(0)} = \int d\sigma^1 \pi^0_{(0)} A, \]
\[ \mathcal{J}_{(0)AB} = \int d\sigma^1 j^0_{(0)AB}, \quad \mathcal{J}_{(0)i} = \int d\sigma^1 j^0_{(0)i}, \]
\[ \mathcal{J}_{(0)A} = \int d\sigma^1 j^0_{(0)A}. \] (5.7.31)
The Poisson brackets between canonically conjugate variables at NLO change compared to
The Poisson brackets at NNLO are
\[
\{y^A(\sigma^1), \pi_{(-2)B}^0(\bar{\sigma}^1)\} = \delta^A_B \delta(\sigma^1 - \bar{\sigma}^1),
\]
\[
\{x^A(\sigma^1), \pi_{(0)1B}^0(\bar{\sigma}^1)\} = \delta^A_B \delta(\sigma^1 - \bar{\sigma}^1),
\]
\[
\{x^i(\sigma^1), \pi_{(0)ij}^0(\bar{\sigma}^1)\} = \delta_i^j \delta(\sigma^1 - \bar{\sigma}^1),
\]
and using these brackets, the charges at NLO generate the string Bargmann algebra \( q_1 \). The dictionary between charges and generators at NLO is
\[
\begin{align*}
\mathcal{P}_{(-2)A} & \leftrightarrow H_A^{(0)}, & \mathcal{J}_{(-2)AB} & \leftrightarrow J_{AB}^{(0)}, \\
\mathcal{P}_{(0)A} & \leftrightarrow H_A^{(-2)}, & \mathcal{J}_{(0)AB} & \leftrightarrow J_{AB}^{(-2)}, \\
\mathcal{P}_{(0)i} & \leftrightarrow P_i^{(0)}, & \mathcal{J}_{(0)ij} & \leftrightarrow J_{ij}^{(0)}, \\
\mathcal{J}_{(0)AI} & \leftrightarrow B_{AI}^{(0)}.
\end{align*}
\]
Note that the charges \( \mathcal{P}_{(-2)A} \) and \( \mathcal{J}_{(-2)AB} \) now correspond to different generators compared to what they did at LO. However, the charges corresponding to the transformations \( \alpha^A_{(0)} \) and \( \lambda^A_{(0)B} \) still correspond to the same generators; a pattern that persists to all orders, as we shall see.

Moving on to NNLO, we may write the Lagrangian as
\[
\mathcal{L}_{\mathcal{P}-\text{NNLO}} = -\frac{T_{\text{eff}}}{2} \eta^{\alpha\beta} \eta_{AB} \partial_\alpha y^A \partial_\beta y^B - T_{\text{eff}} \eta^{\alpha\beta} \eta_{AB} \partial_\alpha x^A \partial_\beta x^B - T_{\text{eff}} \eta^{\alpha\beta} \partial_\alpha x^i \partial_\beta y^i.
\]
The symmetries are given in (5.7.4a)–(5.7.4e), and we find that, again, the most subleading transformations give rise to the most leading charges: i.e., \( \alpha^A_{(1)} \) and \( \lambda^A_{(2)} \) give rise to \( \mathcal{P}_{(-2)A} \) and \( \mathcal{J}_{(-2)AB} \), \( \alpha^A_{(2)} \) and \( \lambda^A_{(2)B} \) give rise to \( \mathcal{P}_{(0)A} \) and \( \mathcal{J}_{(0)AB} \), \( \alpha^l_{(2)} \) and \( \lambda^l_{(2)j} \) give rise to \( \mathcal{P}_{(0)i} \) and \( \mathcal{J}_{(0)ij} \), and \( \lambda^A_{(0)i} \) gives rise to \( \mathcal{J}_{(0)AI} \). The new charges at NNLO are those associated to the most leading transformations and are given by
\[
\begin{align*}
\mathcal{P}_{(2)A} & = \oint d\sigma^1 \pi_{(2)A}^0, & \mathcal{J}_{(2)AB} & = \oint d\sigma^1 j_{(0)AB}^0, \\
\mathcal{P}_{(2)i} & = \oint d\sigma^1 \pi_{(0)i}^0, & \mathcal{J}_{(2)ij} & = \oint d\sigma^1 j_{(0)ij}^0, \\
\mathcal{J}_{(2)AI} & = \oint d\sigma^1 j_{(0)AI}^0.
\end{align*}
\]
The Poisson brackets at NNLO are
\[
\begin{align*}
\{x^A(\sigma^1), \pi_{(-2)B}^0(\bar{\sigma}^1)\} & = \delta^A_B \delta(\sigma^1 - \bar{\sigma}^1), \\
\{y^A(\sigma^1), \pi_{(0)1B}^0(\bar{\sigma}^1)\} & = \delta^A_B \delta(\sigma^1 - \bar{\sigma}^1), \\
\{x^i(\sigma^1), \pi_{(0)ij}^0(\bar{\sigma}^1)\} & = \delta_i^j \delta(\sigma^1 - \bar{\sigma}^1), \\
\{y^i(\sigma^1), \pi_{(0)ij}^0(\bar{\sigma}^1)\} & = \delta_i^j \delta(\sigma^1 - \bar{\sigma}^1), \\
\{x^i(\sigma^1), \pi_{(0)ij}^0(\bar{\sigma}^1)\} & = \delta_i^j \delta(\sigma^1 - \bar{\sigma}^1),
\end{align*}
\]
and with these we find that the charges at NNLO generate the algebra $q_2$ with the following identification

\[
\mathcal{P}_{(2)} A \leftrightarrow H_A^{(-2)} , \quad \mathcal{P}_{(0)} A \leftrightarrow H_A^{(0)} , \quad \mathcal{P}_{(-2)} A \leftrightarrow H_A^{(2)} , \\
\mathcal{J}_{(2)} AB \leftrightarrow J_{AB}^{(-2)} , \quad \mathcal{J}_{(0)} AB \leftrightarrow J_{AB}^{(0)} , \quad \mathcal{J}_{(-2)} AB \leftrightarrow J_{AB}^{(2)} , \\
\mathcal{P}_{(2)} i \leftrightarrow p_i^{(0)} , \quad \mathcal{P}_{(0)} i \leftrightarrow p_i^{(2)} , \quad \mathcal{J}_{(2)} ij \leftrightarrow J_{ij}^{(-2)} , \quad \mathcal{J}_{(0)} ij \leftrightarrow J_{ij}^{(0)} , \quad \mathcal{J}_{(-2)} AB \leftrightarrow J_{AB}^{(2)} .
\]

(5.7.36)

In this way, the Noether charge corresponding to, e.g., LO longitudinal translations $a^A_0$ still plays the rôle as the generator $H_0^{(-2)}$, but this is a new charge compared to those that appeared at LO and NLO. We expect this pattern to continue at all orders, which leads to the following general identification between charges and generators at $N^n$LO

\[
\mathcal{P}_{(-2+2k)} A \leftrightarrow H_A^{(-2+2(n-k))} , \quad \mathcal{J}_{(-2+2k)} AB \leftrightarrow J_{AB}^{(-2+2(n-k))} , \\
\mathcal{P}_{2k} i \leftrightarrow p_i^{(-2+2(n-k))} , \quad \mathcal{J}_{2k} ij \leftrightarrow J_{ij}^{(-2+2(n-k))} ,
\]

(5.7.37)

where $n$ is fixed and $k \in \{0, 1, \ldots, n\}$ in the first line and $k \in \{0, 1, \ldots, n-1\}$ (for $n \geq 1$) in the second line. For example, $n = 2$ corresponds to NNLO and reproduces (5.7.36). The fact that the most subleading Noether charge is associated with the most leading generator—implying that the counting starts at opposite ends, as it were—is a consequence of the fact that the Poisson brackets change at each order.

### 5.8 Phase space formulation

As we saw in the previous section, the Poisson brackets change at each order in $1/c^2$. In this section, we develop a phase space formulation for the $1/c^2$ expansion of the closed bosonic string, and from the expansion of the symplectic form we will explicitly see how the Poisson brackets are modified at each order. In particular, this affects the brackets between the constraints.

As a consequence, the $1/c^2$ expansion of the constraint (soft) algebra does not agree with that obtained from a Dirac-type analysis of the constraints obtained from a $1/c^2$ expansion: the $1/c^2$ expansion does not commute with the Dirac procedure.

The gauge symmetries generated by the expanded constraints can be gauged fixed using standard methods, leading to gauge fixed fixed Poisson brackets that can be used to quantise the string theory by passing to commutators. As is the case for the relativistic string, this leads to a normal ordering constant.
5.8.1 Expanding the relativistic phase space action

The relativistic phase space Lagrangian describing a closed bosonic string is

\[ L = \oint d\sigma^{\perp} [\dot{X}^{M}P_{M} - \theta^{-}F_{\perp} - \theta^{+}F_{\perp}] , \]  

(5.8.1)

where the constraints \( F_{\perp} \) are imposed by the Lagrange multipliers \( \theta^{\pm} \). They are given by

\[ F_{\perp}^{\pm} = \frac{1}{4cT} (P \pm cTX')^{M}(P \pm cTX')^{N}\eta_{MN} . \]  

(5.8.2)

As above, we split the spacetime index \( M = (A, i) \). From the expansion of the Noether currents (5.7.9) and the expansion of \( X^{M} \) in (5.3.7), we find that the quantities involved in (5.8.1) expand as

\[ X^{A} = x^{A} + c^{-2}y^{A} + c^{-4}z^{A} + O(c^{-6}) , \]  

(5.8.3a)

\[ X^{i} = x^{i} + c^{-2}y^{i} + c^{-4}z^{i} + O(c^{-6}) , \]  

(5.8.3b)

\[ P_{A} = c^{2}P_{(0)}^{A} + c^{-2}P_{(2)}^{A} + O(c^{-4}) , \]  

(5.8.3c)

\[ P_{i} = P_{(0)}^{i} + c^{-2}P_{(2)}^{i} + c^{-4}P_{(4)}^{i} + O(c^{-6}) . \]  

(5.8.3d)

Furthermore, the Minkowski metric \( \eta_{MN} \) and its inverse expand according to

\[ \eta_{MN} = c^{2} (\delta_{M}^{A} \delta_{N}^{t} + \delta_{M}^{i} \delta_{N}^{i}) + \delta_{M}^{a} \delta_{N}^{a} , \]  

\[ \eta^{MN} = \frac{1}{c^{2}} (\delta_{t}^{M} \delta_{t}^{N} + \delta_{v}^{M} \delta_{v}^{N}) + \delta_{l}^{M} \delta_{l}^{N} . \]  

(5.8.4)

which implies that

\[ P^{t} = \eta^{tM}P_{M} = -\frac{1}{c^{2}}P_{t} \quad \text{and} \quad P^{v} = \eta^{vM}P_{M} = \frac{1}{c^{2}}P_{v} , \]  

(5.8.5)

leading to the following expansions for the contravariant longitudinal momenta

\[ P^{t} = -P_{(-2)t} - c^{-2}P_{(0)t} - c^{-4}P_{(2)t} + O(c^{-6}) , \]  

\[ P^{v} = P_{(-2)v} + c^{-2}P_{(0)v} + c^{-4}P_{(2)v} + O(c^{-6}) . \]  

(5.8.6)

Using the Minkowskian metric in the form (5.8.4), we can write the constraints as

\[ \mathcal{H}_{\pm} = \frac{c^{2}}{4T_{\text{eff}}} \left( -(P^{t} \pm T_{\text{eff}}X'^{t})^{2} + (P^{v} \pm T_{\text{eff}}X'^{v})^{2} \right) + \frac{1}{4T_{\text{eff}}} \left( P^{t} \pm T_{\text{eff}}X'^{t} \right)^{2} . \]  

(5.8.7)

Expanding the constraints in powers of \( 1/c^{2} \), we get

\[ \mathcal{H}_{\pm} = c^{2}\mathcal{H}_{(-2)\pm} + \mathcal{H}_{(0)\pm} + c^{-2}\mathcal{H}_{(2)\pm} + c^{-4}\mathcal{H}_{(4)\pm} + \cdots , \]  

(5.8.8)
where

\[ \mathcal{H}_{(-2)\pm} = \frac{1}{4T_{\text{eff}}} \eta_{AB}(P^A_{(-2)} \pm T_{\text{eff}}\nu^A)(P^B_{(-2)} \pm T_{\text{eff}}\nu^B), \]

\[ \mathcal{H}_{(0)\pm} = \frac{1}{4T_{\text{eff}}} (P_{(0)i} \pm T_{\text{eff}}\nu^A)^2 + \frac{1}{2T_{\text{eff}}} \eta_{AB}(P^A_{(-2)} \pm T_{\text{eff}}\nu^A)(P^B_{(0)} \pm T_{\text{eff}}\nu^B), \]

\[ \mathcal{H}_{(2)\pm} = \frac{1}{4T_{\text{eff}}} (P_{(0)i} \pm T_{\text{eff}}\nu^A)(P_{(2)i} \pm T_{\text{eff}}\nu^A) + \frac{1}{4T_{\text{eff}}} \eta_{AB} \left[ (P^A_{(2)} \pm T_{\text{eff}}\nu^A)(P^B_{(2)} \pm T_{\text{eff}}\nu^B) + 2(P^A_{(0)} \pm T_{\text{eff}}\nu^A)(P^B_{(4)} \pm T_{\text{eff}}\nu^B) \right] \]

(5.8.9)

where \( \eta_{AB} = \text{diag}(-1, 1) \) is the two-dimensional Minkowski metric. The Lagrange multipliers are expanded according to

\[ \vartheta_{\pm} = \vartheta_{(0)\pm} + c^{-2}\vartheta_{(2)\pm} + c^{-4}\vartheta_{(4)\pm} + \cdots, \]

(5.8.10)

and, as we shall see, at each order the \( \vartheta_{(n)} \) will impose a different constraint. Combining our findings above, the phase space Lagrangian expands as

\[ L = c^2L_{\text{LO}} + L_{\text{NLO}} + c^{-2}L_{\text{NNLO}} + O(c^{-4}), \]

(5.8.11)

where

\[
L_{\text{LO}} = \int d\sigma^1 \left[ \dot{x}^i P_{(-2)A} - \vartheta_{(0)i} \mathcal{H}_{(-2)-} - \vartheta_{(0)}^+ \mathcal{H}_{(-2)+} \right],
\]

(5.8.12a)

\[
L_{\text{NLO}} = \int d\sigma^1 \left[ \dot{x}^i P_{(0)i} + \dot{\vartheta}^i P_{(0)A} + \dot{y}^i P_{(-2)A} - \vartheta_{(0)i} \mathcal{H}_{(0)-} - \vartheta_{(0)}^+ \mathcal{H}_{(0)+} - \vartheta_{(2)i} \mathcal{H}_{(-2)-} - \vartheta_{(2)}^+ \mathcal{H}_{(-2)+} \right],
\]

(5.8.12b)

\[
L_{\text{NNLO}} = \int d\sigma^1 \left[ \dot{x}^i P_{(2)i} + \dot{y}^i P_{(0)A} + \dot{z}^i P_{(-2)A} + \dot{\vartheta}^i P_{(2)i} + \dot{y}^i P_{(2)i} - \vartheta_{(0)i} \mathcal{H}_{(2)-} - \vartheta_{(2)i} \mathcal{H}_{(0)-} - \vartheta_{(4)i} \mathcal{H}_{(-2)-} - \vartheta_{(4)}^+ \mathcal{H}_{(-2)+} \right].
\]

(5.8.12c)

We see that the symplectic structure changes at each order in \( 1/c^2 \), which means that the Poisson brackets change at each order; something we already made use of in Section 5.7.

### 5.8.2 Dirac brackets

The relativistic first class constraints \( \mathcal{H}_{\pm} \) give, as we shall see, rise to novel first-class constraints when expanded in powers of \( 1/c^2 \). These first class constraints generate gauge redundancies, which can be fixed by an admissible gauge fixing condition which typically restricts the canonical variables which now form the so called reduced phase space. On the reduced phase space, we may then read off the Dirac brackets [259], and from those we can pass to the quantum theory by promoting the Dirac brackets to quantum commutators.
The gauge choice must satisfy the following conditions (in which case the choice is “canonical”)

1. The gauge choice must be “accessible”: this means that there exists a gauge transformation that transforms the canonical variables into a set that satisfies the gauge condition.

2. The gauge choice must fix the gauge completely. This amounts to the requirement that the determinant of the brackets between the (first class) constraints and the gauge choice is non-zero; equivalently, the brackets considered as a matrix indexed by the (equal) number of constraints and gauge fixing conditions is invertible.

Having fixed the gauge, we then solve the constraints inside the action and read off the Dirac brackets from the kinetic term of the unconstrained action.

**LO**

From the kinetic term of the LO phase-space Lagrangian (5.8.12a), we read off the (equal-σ0)

\[
\{x^A(\sigma^1), P_{(-2)}B(\tilde{\sigma}^1)\} = \delta^A_B \delta(\sigma^1 - \tilde{\sigma}^1).
\]

Using these brackets, the algebra of the LO constraints \( \{H_{(-2)}^+, H_{(-2)}^-(\sigma^1)\} \) becomes

\[
\{H_{(-2)}^+, H_{(-2)}^-(\sigma^1)\} = (H_{(-2)}^+(\sigma^1) + H_{(-2)}^-(\tilde{\sigma}^1)) \delta'(\sigma^1 - \tilde{\sigma}^1),
\]

\[
\{H_{(-2)}^-, H_{(-2)}^+(\sigma^1)\} = - (H_{(-2)}^-(\sigma^1) + H_{(-2)}^+(\tilde{\sigma}^1)) \delta'(\sigma^1 - \tilde{\sigma}^1),
\]

\[
\{H_{(-2)}^-, H_{(-2)}^-(\sigma^1)\} = 0, \tag{5.8.14}
\]

where we used that \( \{x^t(\sigma^1), P_{(-2)}t(\tilde{\sigma}^1)\} = \delta'(\sigma^1 - \tilde{\sigma}^1) \), where the derivative is with respect to \( \sigma^1 \), as well as the identity

\[
\frac{\partial}{\partial \sigma^1} \delta(\sigma^1 - \tilde{\sigma}^1) = - \frac{\partial}{\partial \tilde{\sigma}^1} \delta(\sigma^1 - \tilde{\sigma}^1). \tag{5.8.15}
\]

The action of the gauge transformations generated by \( H_{(-2)}^\pm \) on a function \( F \) on phase space is

\[
\delta_{\epsilon_{(-2)}}F = \left\{ F, \mathcal{F}, \delta(\sigma^1 - \tilde{\sigma}^1) \right\}, \tag{5.8.16}
\]

where \( \epsilon_{(-2)}^\pm \) are the parameters of the gauge transformation. One can check that the gauge transformations act separately on the combinations \( P_{(-2)}^A \pm T_{\text{eff}} x^A \), and since we are dealing with closed strings, these are periodic functions in \( \sigma^1 \) and thus we may without loss of
generality express them as Fourier series

\begin{align}
p_{(-2)}^A - T_{\text{eff}} \chi^A &= \sqrt{\frac{T_{\text{eff}}}{\pi}} \sum_{k \in \mathbb{Z}} e^{ik\sigma^0} \alpha_k^A(\sigma^0), \\
p_{(-2)}^A + T_{\text{eff}} \chi^A &= \sqrt{\frac{T_{\text{eff}}}{\pi}} \sum_{k \in \mathbb{Z}} e^{-ik\sigma^0} \tilde{\alpha}_k^A(\sigma^0),
\end{align}

(5.8.17)

where reality constrains the modes to satisfy \((\alpha_k^A)^* = \alpha_{-k}^A\) and \((\tilde{\alpha}_k^A)^* = \tilde{\alpha}_{-k}^A\). Now, since \(\chi^\nu\) carries winding, we can integrate the relations above to find that the total momentum \(p_{(-2)}\) (although these are on-shell equivalent to the Noether charges \(p_{(-2)}^A\)) we considered in the previous section, we will use the symbol \(\wp\) to denote the total momentum along the string in the phase space formulation) is given by

\begin{equation}
\wp_{(-2)}^A(\sigma^0) = \oint d\sigma^1 p_{(-2)}^A = \sqrt{4\pi T_{\text{eff}}} \alpha_0^A(\sigma^0) = \sqrt{4\pi T_{\text{eff}}} \tilde{\alpha}_0^A(\sigma^0),
\end{equation}

(5.8.18)

and

\begin{equation}
\wp_{(-2)}^\nu(\sigma^0) = \oint d\sigma^1 p_{(-2)}^\nu = \sqrt{4\pi T_{\text{eff}}} \alpha_0^\nu(\sigma^0) + 2\pi T_{\text{eff}} w_{\text{eff}} = \sqrt{4\pi T_{\text{eff}}} \tilde{\alpha}_0^\nu(\sigma^0) - 2\pi T_{\text{eff}} w_{\text{eff}},
\end{equation}

(5.8.19)

where we assumed that \(\chi^\nu\) contains a winding term linear in \(\sigma^1\). Adding the expressions in (5.8.17) produces the Fourier series for \(p_{(-2)}^t\) and \(p_{(-2)}^\nu\)

\begin{align}
p_{(-2)}^t(\sigma^0, \sigma^1) &= \frac{\wp_{(-2)}^t(\sigma^0)}{2\pi} + \sqrt{\frac{T_{\text{eff}}}{4\pi}} \sum_{k \neq 0} e^{ik\sigma^1} (\alpha_k^t(\sigma^0) + \tilde{\alpha}_{-k}^t(\sigma^0)), \\
p_{(-2)}^\nu(\sigma^0, \sigma^1) &= \frac{\wp_{(-2)}^\nu(\sigma^0)}{2\pi} + \sqrt{\frac{T_{\text{eff}}}{4\pi}} \sum_{k \neq 0} e^{ik\sigma^1} (\alpha_k^\nu(\sigma^0) + \tilde{\alpha}_{-k}^\nu(\sigma^0)),
\end{align}

(5.8.20)

while subtracting gives us

\begin{align}
\chi^t &= -\frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} e^{ik\sigma^1} (\alpha_k^t(\sigma^0) - \tilde{\alpha}_{-k}^t(\sigma^0)), \\
\chi^\nu &= w_{\text{eff}} -\frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} e^{ik\sigma^1} (\alpha_k^\nu(\sigma^0) - \tilde{\alpha}_{-k}^\nu(\sigma^0)),
\end{align}

(5.8.21)

integrating to

\begin{align}
\chi^t &= \chi_0^t(\sigma^0) + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} e^{ik\sigma^1} (\alpha_k^t(\sigma^0) - \tilde{\alpha}_{-k}(\sigma^0)), \\
\chi^\nu &= \chi_0^\nu(\sigma^0) + w_{\text{eff}} \sigma^1 + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} e^{ik\sigma^1} (\alpha_k^\nu(\sigma^0) - \tilde{\alpha}_{-k}(\sigma^0)).
\end{align}

(5.8.22)
If we then Fourier expand the constraints \( \mathcal{J}_{(-2)} \), we get
\[
\mathcal{J}_{(-2)_-} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\sigma^3} L_n^{(-2)} \quad \text{and} \quad \mathcal{J}_{(-2)_+} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-in\sigma^3} \tilde{L}_n^{(-2)}, \tag{5.8.23}
\]
where
\[
L_n^{(-2)} = \frac{1}{2} \sum_{k \in \mathbb{Z}} \eta_{AB} \alpha_k^A \alpha_{n-k}^B, \quad \tilde{L}_n^{(-2)} = \frac{1}{2} \sum_{k \in \mathbb{Z}} \eta_{AB} \tilde{\alpha}_k^A \tilde{\alpha}_{n-k}^B. \tag{5.8.24}
\]
If we also expand the Lagrange multipliers (where \( \tilde{\vartheta}^{(0)}_- \) has modes \( \tilde{\vartheta}_n^{(0)}_- \) and \( \tilde{\vartheta}^{(0)}_+ \) has modes \( \tilde{\vartheta}_n^{(0)}_+ \)), the Lagrangian (up to total derivatives) takes the form
\[
L_{\text{LO}} = \dot{x}_0^A \varphi_{(-2)A} + \sum_{k=1}^{\infty} \frac{i}{k} \eta_{AB} (\dot{\alpha}_k^A \alpha_{n-k}^B + \dot{\tilde{\alpha}}_k^A \tilde{\alpha}_{n-k}^B) - \sum_{n \in \mathbb{Z}} (\vartheta_{-n}^{(0)} L_n^{(-2)} + \tilde{\vartheta}_{-n}^{(0)} \tilde{L}_n^{(-2)}). \tag{5.8.25}
\]
From this we may read off the Poisson brackets
\[
\{ x_0^A, \varphi_{(-2)B} \} = \delta_B^A, \quad \{ \alpha_k^A, \alpha_{-k}^B \} = \{ \tilde{\alpha}_k^A, \tilde{\alpha}_{-k}^B \} = -i k \eta_{AB}, \tag{5.8.26}
\]
which lead to
\[
\{ L_k^{(-2)}, L_n^{(-2)} \} = -i (k-n) L_{k+n}^{(-2)} \quad \text{and} \quad \{ \tilde{L}_k^{(-2)}, L_n^{(-2)} \} = -i (k-n) \tilde{L}_{k+n}^{(-2)}. \tag{5.8.27}
\]
To fix the gauge redundancy, we set
\[
\alpha_k^+ = \tilde{\alpha}_k^+ = 0 \quad \forall k \neq 0. \tag{5.8.28}
\]
To check that this indeed fixes the gauge, we compute
\[
\{ L_n^{(-2)}, \alpha_{-k}^+ \} = -i k \alpha_{-n-k}^+ = -i k \alpha_0^+ \delta_{nk}, \quad \{ \tilde{L}_n^{(-2)}, \tilde{\alpha}_{-k}^+ \} = -i k \tilde{\alpha}_0^+ \delta_{nk}, \tag{5.8.29}
\]
where we used the gauge fixing conditions. Since these are invertible matrices for \( k \neq 0 \), we conclude that this fixes all the gauge invariances except those corresponding to \( L_0^{(-2)} \) and \( \tilde{L}_0^{(-2)} \), which in gauge fixed form are given by
\[
L_0^{(-2)} = \frac{1}{\sqrt{2} \pi T_{\text{eff}}} \eta_{AB} \varphi_{(-2)A} \varphi_{(-2)B} + \frac{\pi T_{\text{eff}} \omega^2 R_{\text{eff}}}{2} - \frac{1}{2} \varphi^\nu_{(-2)} \varphi^\nu_{(-2)} R_{\text{eff}},
\]
\[
\tilde{L}_0^{(-2)} = \frac{1}{\sqrt{2} \pi T_{\text{eff}}} \eta_{AB} \varphi_{(-2)A} \varphi_{(-2)B} + \frac{\pi T_{\text{eff}} \omega^2 R_{\text{eff}}}{2} + \frac{1}{2} \varphi^\nu_{(-2)} \varphi^\nu_{(-2)} R_{\text{eff}}. \tag{5.8.30}
\]
Setting these constraints equal to zero implies that \( \varphi^\nu_{(-2)} = 0 \) and \( \varphi^\nu_{(-2)} = 2 \pi T_{\text{eff}} \omega R_{\text{eff}} \), in agreement with what we previously found.

For \( n \neq 0 \), we can solve the conditions \( 0 = L_n^{(-2)} = -\frac{1}{2} \alpha_0^+ \alpha_n^- \) and \( 0 = \tilde{L}_n^{(-2)} = -\frac{1}{2} \tilde{\alpha}_0^+ \tilde{\alpha}_n^- \) for \( \alpha_n^- \) and \( \tilde{\alpha}_n^- \), which gives us \( \alpha_n^- = \tilde{\alpha}_n^- = 0 \). Hence, the gauge fixed LO Lagrangian is
\[
L_{\text{LO}} = \dot{x}_0^A \varphi_{(-2)A} - \vartheta_0^{(0)} \varphi_{(-2)} - \tilde{\vartheta}_0^{(0)} \tilde{L}_0^{(-2)}, \tag{5.8.31}
\]
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leaving us with the gauge-fixed Poisson brackets

$$\{ x^A_i, \wp_{(-2)B} \} = \delta^A_B .$$  \hfill (5.8.32)

In agreement with our previous findings, there are no oscillations in the LO theory.

**NLO**

We now repeat the above analysis for the NLO theory. We read off the following NLO Poisson brackets from the NLO Lagrangian (5.8.12b)

$$\{ x^i(\sigma^1), P_{(0)j}^{(-1)}(\tilde{\sigma}^1) \} = \delta^i_j \delta(\sigma^1 - \tilde{\sigma}^1) ,$$

$$\{ x^A(\sigma^1), P_{(0)B}^{(-1)}(\tilde{\sigma}^1) \} = \{ y^A(\sigma^1), P_{(-2)B}^{(-1)}(\tilde{\sigma}^1) \} = \delta^A_B \delta(\sigma^1 - \tilde{\sigma}^1) .$$  \hfill (5.8.33)

Relative to these brackets, we may compute the (soft) algebra of the four constraints $\mathcal{H}_{(-2)}^{\pm}$ and $\mathcal{H}_{(0)}^{\pm}$

$$\{ \mathcal{H}^{\pm}_{(-2)}(\sigma^1), \mathcal{H}^{\pm}_{(-2)}(\tilde{\sigma}^1) \} = \{ \mathcal{H}^{\pm}_{(0)}(\sigma^1), \mathcal{H}^{\pm}_{(-2)}(\tilde{\sigma}^1) \} = 0 ,$$

$$\{ \mathcal{H}^{\pm}_{(0)}(\sigma^1), \mathcal{H}^{\pm}_{(0)}(\tilde{\sigma}^1) \} = \pm (\mathcal{H}^{\pm}_{(-2)}(\sigma^1) + \mathcal{H}^{\pm}_{(-2)}(\tilde{\sigma}^1)) \delta'(\sigma^1 - \tilde{\sigma}^1) ,$$

$$\{ \mathcal{H}^{\pm}_{(0)}(\sigma^1), \mathcal{H}^{\pm}_{(0)}(\tilde{\sigma}^1) \} = 0 .$$  \hfill (5.8.34)

At NLO, the gauge transformations act separately on the combinations $P_{(0)i} \pm T_{\text{eff}} x^i$ and $P^A_{(0)} \pm T_{\text{eff}} y^A$, which we take to have the mode expansions

$$P_{(0)i} \pm T_{\text{eff}} x^i = \sqrt{\frac{T_{\text{eff}}}{\pi}} \sum_{k \in \mathbb{Z}} e^{\pm ik\sigma^1} \chi^i_k(\sigma^0) ,$$

$$P^A_{(0)} \pm T_{\text{eff}} y^A = \sqrt{\frac{T_{\text{eff}}}{\pi}} \sum_{k \in \mathbb{Z}} e^{\mp ik\sigma^1} \chi^A_k(\sigma^0) .$$  \hfill (5.8.35)
For the momenta and embedding fields, this leads to

\[
p^A_{(0)}(\sigma^0, \sigma^1) = \frac{p^A_{(0)}(\sigma^0)}{2\pi} + \frac{1}{\sqrt{4\pi}} \sum_{k \neq 0} e^{ik\sigma^1} \left( \beta^A_k(\sigma^0) + \tilde{\beta}^A_{-k}(\sigma^0) \right),
\]

\[
p^i_{(0)}(\sigma^0, \sigma^1) = \frac{g^i_{(0)}(\sigma^0)}{2\pi} + \frac{1}{\sqrt{4\pi}} \sum_{k \neq 0} e^{ik\sigma^1} \left( \alpha^i_k(\sigma^0) + \tilde{\alpha}^i_{-k}(\sigma^0) \right),
\]

\[
\chi^i(\sigma^0, \sigma^1) = \chi^i_0(\sigma^0) + \frac{1}{\sqrt{4\pi} \eta^{A \beta}} \sum_{k \neq 0} e^{ik\sigma^1} \left( \alpha^i_k(\sigma^0) - \tilde{\alpha}^i_{-k}(\sigma^0) \right),
\]

\[
y^A(\sigma^0, \sigma^1) = y^A_0(\sigma^0) + \frac{1}{\sqrt{4\pi} \eta^{A \beta}} \sum_{k \neq 0} e^{ik\sigma^1} \left( \beta^A_k(\sigma^0) - \tilde{\beta}^A_{-k}(\sigma^0) \right).
\]

If we then Fourier expand the constraints \( \mathcal{H}_0 \), we now get

\[
\mathcal{H}_{(0)\pm} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{i n \sigma^1} L_n^{(0)} \quad \text{and} \quad \mathcal{H}_{(0)\mp} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-i n \sigma^1} \tilde{L}_n^{(0)},
\]

where

\[
L_n^{(0)} = \sum_{k \in \mathbb{Z}} \left( \frac{1}{2} \alpha_k^i \alpha^i_{n-k} + \eta_{AB} \alpha_k^A \rho^B_{n-k} \right),
\]

\[
\tilde{L}_n^{(0)} = \sum_{k \in \mathbb{Z}} \left( \frac{1}{2} \tilde{\alpha}_k^i \tilde{\alpha}^i_{n-k} + \eta_{AB} \tilde{\alpha}_k^A \tilde{\rho}^B_{n-k} \right).
\]

Using the same procedure as above, we now find that the NLO Lagrangian can be written as (up to total derivatives)

\[
L_{\text{NLO}} = \chi^i_0 \varphi_{(0)i} + \chi^A_0 \varphi_{(0)A} + y^A_0 \varphi_{(-2)A}
\]

\[
+ \sum_{k=1}^{\infty} \frac{1}{k} \left( \alpha_k^i \alpha^i_k + \tilde{\alpha}_k^i \tilde{\alpha}^i_{-k} + 2\eta_{AB} \alpha_k^A \rho^B_k + \tilde{\alpha}_k^A \tilde{\rho}^B_k \right) - \sum_{n \in \mathbb{Z}} \left( \tilde{\varphi}_{(1)}^{(0)} L_n^{(0)} + \tilde{\varphi}_{(1)}^{(0)} \tilde{L}_n^{(0)} + \varphi_{(2)}^{(0)} L_n^{(-2)} + \varphi_{(2)}^{(0)} \tilde{L}_n^{(-2)} \right),
\]

which allows us to read off the Poisson brackets for the modes

\[
\{ \chi^i_0, \varphi_{(0)i} \} = \delta^i_j, \quad \{ \alpha_k^i, \alpha^i_{-k} \} = \{ \tilde{\alpha}_k^i, \tilde{\alpha}^i_{-k} \} = -ik \delta^i_j,
\]

\[
\{ \chi^A_0, \varphi_{(0)A} \} = \delta^A_B, \quad \{ \alpha_k^A, \beta^B_{-k} \} = \{ \tilde{\alpha}_k^A, \tilde{\beta}^B_{-k} \} = -ik \eta^{A B},
\]

\[
\{ y^A_0, \varphi_{(-2)B} \} = \delta^A_B.
\]

We now come to the gauge fixing. In the Polyakov formalization, we found that we could gauge fix the residual gauge invariance by removing the oscillations \( \beta^A_k \) and \( \tilde{\beta}^A_{-k} \) for all nonzero \( k \). Hence, to find the gauge fixed Poisson brackets for the remaining modes, we also set these to
We may now solve the constraints for the remaining modes. At LO, we find again that all oscillations vanish (i.e., \( \alpha_k^- = \tilde{\alpha}_k^- = 0 \) \( \forall k \neq 0 \), \( \beta_k^- = \tilde{\beta}_k^- = 0 \) \( \forall k \neq 0 \)).

We may check that this really does fix the gauge

\[
\{ \bar{L}^{(-2)}_n, \beta_- \} = -\frac{i\mathbf{k}}{2} \delta_{n\mathbf{k}}, \quad \{ \bar{L}^{(-2)}_n, \tilde{\beta}^- \} = -\frac{i\mathbf{k}}{2} \delta_{n\mathbf{k}},
\]

\[
\{ L^{(0)}_n, \alpha_- \} = -\frac{i\mathbf{k}}{2} \delta_{n\mathbf{k}}, \quad \{ \bar{L}^{(0)}_n, \tilde{\alpha}^- \} = -\frac{i\mathbf{k}}{2} \delta_{n\mathbf{k}},
\]

where we used the gauge-fixing conditions on the right-hand side. Since these are invertible for \( k = 0 \), we have fixed all the gauge redundancy except for that corresponding to \( k = 0 \). Furthermore, from the results above, we see that setting \( \alpha^- = \tilde{\alpha}^+ = 0 \) will now fix the NLO gauge redundancy generated by \( \mathcal{H}_0(\pm) \), while the LO gauge redundancy is now fixed by setting \( \beta^- = \tilde{\beta}^- = 0 \). The gauge-fixed zero-mode constraints are

\[
L^{(-2)}_0 = \frac{1}{8\pi\mathcal{T}_{\text{eff}}} \eta_{AB} \varphi^A \varphi^B - \frac{\mathcal{R}^{2}_{\text{eff}}}{2} - \frac{1}{2} \bar{\varphi} \varphi^{(-2)} - \mathcal{W}^{\mu\nu}_{\text{eff}},
\]

\[
\bar{L}^{(-2)}_0 = \frac{1}{8\pi\mathcal{T}_{\text{eff}}} \eta_{AB} \varphi^A \varphi^B + \frac{\mathcal{R}^{2}_{\text{eff}}}{2} + \frac{1}{2} \bar{\varphi} \varphi^{(-2)} - \mathcal{W}^{\mu\nu}_{\text{eff}},
\]

\[
L^{(0)}_0 = N_0 + \frac{1}{2} \alpha_0 A^0 B^0 + \eta_{AB} A_0 B_0,
\]

\[
\bar{L}^{(0)}_0 = \bar{N}_0 + \frac{1}{2} \tilde{\alpha}_0 \tilde{A}^0 \tilde{B}^0 + \eta_{AB} \tilde{A}_0 \tilde{B}_0.
\]

We may now solve the constraints for the remaining modes. At LO, we find again that all oscillations vanish (i.e., \( \alpha_k^+ = \tilde{\alpha}_k^- = 0 \) for all \( k \neq 0 \)), while at NLO we get

\[
\beta_n^+ = \frac{1}{\mathcal{T}_0} \sum_{k \in \mathbb{Z}} \alpha^+_k \tilde{\alpha}^i_{n-k}, \quad \tilde{\beta}_n^- = \frac{1}{\mathcal{T}_0} \sum_{k \in \mathbb{Z}} \tilde{\alpha}^i_k \tilde{\alpha}^i_{n-k}.
\]

This means that the gauge-fixed NLO Lagrangian takes the form

\[
L_{\text{NLO}} = \bar{x}_0^i \varphi(0) A^i + \tilde{x}_0^i \varphi(0) B^i + \tilde{y}_0^i \varphi(2) A^i + \sum_{k=1}^{\infty} \frac{i}{k} (\tilde{\alpha}^i_k \tilde{\alpha}_k + \tilde{\alpha}^i_k \tilde{\alpha}_k) - \left( \tilde{\beta}_0^i L_0^{(0)} + \tilde{\beta}_0^i \bar{L}_0^{(0)} + \tilde{\beta}_0^i \bar{L}_0^{(0)} + \tilde{\beta}_0^i \bar{L}_0^{(0)} \right).
\]

---

\(^8\)We could also have fixed the gauge at LO in this way.
From this, we read off the gauge-fixed Poisson brackets
\[
\{ x^i_0, \varphi(0) \} = \delta^i_0, \quad \{ x^A_0, \varphi(0)_B \} = \delta^A_B, \quad \{ y^A_0, \varphi(-2)_B \} = \delta^A_B, \quad \{ \alpha^i_k, \alpha^j_{-k} \} = (\tilde{\alpha}^i_k, \tilde{\alpha}^j_{-k} = -ik\delta^i_j. \tag{5.8.46}
\]

We can work out the remaining brackets by using the relations (5.8.44).

**NNLO**

Finally, we consider the Dirac procedure at NNLO, where the Lagrangian is given by (5.8.12c). From this, we get the following Poisson brackets
\[
\{ x^i(\sigma^1), P^j(2)_{\tilde{\sigma}^1} \} = \{ y^i(\sigma^1), P^j(0)_{\tilde{\sigma}^1} \} = \delta^i_j \delta(\sigma^1 - \tilde{\sigma}^1), \quad \{ x^A(\sigma^1), P^B(2)_{\tilde{\sigma}^1} \} = \{ y^A(\sigma^1), P^B(0)_{\tilde{\sigma}^1} \} = \{ z^A(\sigma^1), P^B(-2)_{\tilde{\sigma}^1} \} = \delta^A_B \delta(\sigma^1 - \tilde{\sigma}^1). \tag{5.8.47}
\]

The non-zero constraint brackets are
\[
\{ F^{(0)}_{\pm}(\sigma^1), F^{(2)}_{\pm}(\tilde{\sigma}^1) \} = \pm(\mathcal{F}^{(0)}_{\pm}(\sigma^1) + \mathcal{F}^{(0)}_{\pm}(\tilde{\sigma}^1)) \delta(\sigma^1 - \tilde{\sigma}^1), \quad \{ F^{(2)}_{\pm}(\sigma^1), F^{(2)}_{\pm}(\tilde{\sigma}^1) \} = \pm(\mathcal{F}^{(2)}_{\pm}(\sigma^1) + \mathcal{F}^{(2)}_{\pm}(\tilde{\sigma}^1)) \delta(\sigma^1 - \tilde{\sigma}^1), \tag{5.8.48}
\]

and so the constraints remain first-class. At NNLO, we write down mode expansions for the combinations
\[
P^{(2)}_{\pm} = \pm T_{\text{eff}} y^{\sigma^1}, \quad P^{A}_{\pm} = \pm T_{\text{eff}} z^{\sigma^1}.
\]
This leads to the following mode expansions for the momenta and the embedding fields

\[ p^A_{(2)}(\sigma^0, \sigma^1) = \frac{\varphi^A_{(2)}(\sigma^0)}{2\pi} + \sqrt{\frac{T_{\text{eff}}}{4\pi}} \sum_{k \neq 0} e^{ik\sigma^1} (\chi^A_k(\sigma^0) + \bar{\chi}^A_{-k}(\sigma^0)), \]

\[ p^i_{(2)}(\sigma^0, \sigma^1) = \frac{\varphi^i_{(2)}(\sigma^0)}{2\pi} + \sqrt{\frac{T_{\text{eff}}}{4\pi}} \sum_{k \neq 0} e^{ik\sigma^1} (\beta^i_k(\sigma^0) + \bar{\beta}^i_{-k}(\sigma^0)), \]

\[ y^i(\sigma^0, \sigma^1) = y_0^i(\sigma^0) + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} e^{ik\sigma^1} (\beta^i_k(\sigma^0) - \bar{\beta}^i_{-k}(\sigma^0)), \]

\[ z^A(\sigma^0, \sigma^1) = z_0^A(\sigma^0) + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k \neq 0} \frac{i}{k} e^{ik\sigma^1} (\chi^A_k(\sigma^0) - \bar{\chi}^A_{-k}(\sigma^0)). \]

If we then Fourier expand the constraints \( \mathcal{H}_{(0)\pm} \), we now get

\[ \mathcal{H}_{(2)-} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\sigma^1} L_n^{(2)} \quad \text{and} \quad \mathcal{H}_{(2)+} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-in\sigma^1} \bar{L}_n^{(2)}, \]

where

\[ L_n^{(2)} = \sum_{k \in \mathbb{Z}} \left( \alpha^A_k \beta^i_{n-k} + \frac{1}{2} \eta_{AB} (2\alpha^A_k \chi^B_{n-k} + \beta^A_k \beta^B_{n-k}) \right), \]

\[ \bar{L}_n^{(2)} = \sum_{k \in \mathbb{Z}} \left( \bar{\alpha}^A_k \bar{\beta}^i_{n-k} + \frac{1}{2} \eta_{AB} (2\bar{\alpha}^A_k \bar{\chi}^B_{n-k} + \bar{\beta}^A_k \bar{\beta}^B_{n-k}) \right). \]

Using the same procedure as above, we now find that the NNLO Lagrangian can be written as (up to total derivatives)

\[ L_{\text{NNLO}} = \dot{x}_0 \varphi^{(2)A} + \dot{y}_0 \varphi^{(0)A} + \dot{x}^A \varphi^{(2)A} + \dot{y}^A \varphi^{(0)A} + \dot{z}^A \varphi^{(-2)A} \]

\[ + \sum_{k=1}^{\infty} \frac{i}{k} \left( 2\alpha^A_k \beta^B_{-k} + 2\bar{\alpha}^A_k \bar{\beta}^B_{-k} + \eta_{AB} (2\alpha^A_k \chi^B_{-k} + 2\bar{\alpha}^A_k \bar{\chi}^B_{-k} + \beta^A_k \beta^B_{-k} + \bar{\beta}^A_k \bar{\beta}^B_{-k}) \right) \]

\[ - \sum_{n \in \mathbb{Z}} \left( \delta_{-n}^{(0)} L_n^{(2)} + \bar{\delta}_{-n}^{(0)} \bar{L}_n^{(2)} + \delta_{-n}^{(2)} \tilde{L}_n^{(0)} + \bar{\delta}_{-n}^{(2)} \tilde{\bar{L}}_n^{(0)} + \delta_{-n}^{(4)} \tilde{L}_n^{(-2)} + \bar{\delta}_{-n}^{(4)} \tilde{\bar{L}}_n^{(-2)} \right). \]
This gives the following Poisson brackets for the modes

\[
\{ x^i_0, \varphi_{(2)} \} = \{ y^i_0, \varphi_{(0)} \} = \delta^i_j,
\]

\[
\{ \alpha^i_k, \beta^j_{-k} \} = \{ \tilde{\alpha}^i_k, \tilde{\beta}^j_{-k} \} = -\frac{ik}{2} \delta^{ij},
\]

\[
\{ x^A_0, \varphi_{(2)} B \} = \{ y^A_0, \varphi_{(0)} B \} = \{ x^A_0, \varphi_{(-2)} B \} = \delta^A_B,
\]

\[
\{ \alpha^A_k, \chi^B_k \} = \{ \tilde{\alpha}^A_k, \tilde{\chi}^B_k \} = -\frac{ik}{2} \eta^{AB},
\]

\[
\{ \beta^A_k, \beta^B_k \} = \{ \tilde{\beta}^A_k, \tilde{\beta}^B_k \} = -i k \eta^{AB}.
\]

The gauge fixing follows the same pattern as what we worked out above

\[
\alpha^-_k = \tilde{\alpha}^+_k = 0 \quad \forall k \neq 0,
\]

\[
\beta^-_k = \tilde{\beta}^+_k = 0 \quad \forall k \neq 0,
\]

\[
\chi^-_k = \tilde{\chi}^+_k = 0 \quad \forall k \neq 0.
\]

Once more, we may check that this really does fix the gauge

\[
\{ L_n^{(-2)}, x^-_k \} = -\frac{ik}{2} \alpha^-_k \delta_{nk}, \quad \{ \tilde{L}_n^{(-2)}, \tilde{x}^+_k \} = -\frac{ik}{2} \tilde{\alpha}^+_k \delta_{nk},
\]

\[
\{ L_n^{(0)}, \beta^-_k \} = -i k \alpha^-_k \delta_{nk}, \quad \{ \tilde{L}_n^{(0)}, \tilde{\beta}^+_k \} = -i k \tilde{\alpha}^+_k \delta_{nk},
\]

\[
\{ L_n^{(2)}, \alpha^-_k \} = -\frac{ik}{2} \alpha^-_k \delta_{nk}, \quad \{ \tilde{L}_n^{(0)}, \tilde{\alpha}^+_k \} = -\frac{ik}{2} \tilde{\alpha}^+_k \delta_{nk},
\]

which are indeed invertible, which means that we have fixed all the gauge invariance except for that corresponding to \( k = 0 \). Furthermore, from the results above, we see that setting \( \alpha^- = \tilde{\alpha}^+ = 0 \) will now fix the NLO gauge redundancy generated by \( \mathcal{T}_{(0)+} \), while the LO gauge redundancy is now fixed by setting \( \beta^- = \tilde{\beta}^+ = 0 \). The gauge-fixed zero-mode constraints are

\[
L_0^{(-2)} = \frac{1}{8 \pi \text{eff}} \eta_{AB} \varphi_{(-2)}^A \delta_{(-2)}^B + \frac{\pi \text{eff}_{w} R_{w}^2}{2} \varphi_{(-2)}^A w_{\text{eff}}^2 + \frac{1}{2} \varphi_{(-2)}^A w_{\text{eff}}^2 R_{w}^2,
\]

\[
\tilde{L}_0^{(-2)} = \frac{1}{8 \pi \text{eff}} \eta_{AB} \varphi_{(-2)}^B \delta_{(-2)}^A + \frac{\pi \text{eff}_{w} R_{w}^2}{2} \varphi_{(-2)}^B w_{\text{eff}}^2 + \frac{1}{2} \varphi_{(-2)}^B w_{\text{eff}}^2 R_{w}^2,
\]

\[
L_0^{(0)} = N_0 + \frac{1}{2} \alpha_0^A \alpha_0^A + \eta_{AB} \alpha_0^A \beta_0^B,
\]

\[
\tilde{L}_0^{(0)} = \tilde{N}_0 + \frac{1}{2} \tilde{\alpha}_0^A \tilde{\alpha}_0^A + \eta_{AB} \tilde{\alpha}_0^A \tilde{\beta}_0^B,
\]

\[
L_0^{(2)} = N_2 + \alpha_0^A \beta_0^B + \frac{1}{2} \eta_{AB} (2 \alpha_0^A \chi_0^B + \beta_0^A \tilde{\beta}_0^B),
\]

\[
\tilde{L}_0^{(2)} = \tilde{N}_2 + \tilde{\alpha}_0^A \tilde{\beta}_0^B + \frac{1}{2} \eta_{AB} (2 \tilde{\alpha}_0^A \chi_0^B + \tilde{\beta}_0^A \tilde{\beta}_0^B).
\]

As before, we may now solve the constraints for the remaining modes. At LO, we find again that all oscillations vanish (i.e., \( \alpha_k^+ = \tilde{\alpha}_k^- = 0 \) for all \( k \neq 0 \)), while at NLO and NNLO, we get
We can work out the remaining brackets by using the relations (5.8.58).

\[ k \]

where

\[ (for \, n \neq 0) \]

\[ \beta_n^+ = \frac{1}{\alpha_0} \sum_{k \in \mathbb{Z}} \alpha_k^i \alpha_{n-k}^i, \quad \hat{\beta}_n^- = \frac{1}{\alpha_0} \sum_{k \in \mathbb{Z}} \hat{\alpha}_k^i \hat{\alpha}_{n-k}^i, \]

\[ \chi_n^+ = \frac{1}{\alpha_0} \sum_{k \in \mathbb{Z}} \left[ 2\alpha_k^i \beta_{n-k}^i - \frac{\beta_0^i}{\alpha_0} \alpha_k^i \alpha_{n-k}^i \right], \]

\[ \hat{\chi}_n^- = \frac{1}{\alpha_0} \sum_{k \in \mathbb{Z}} \left[ 2\hat{\alpha}_k^i \hat{\beta}_{n-k}^i - \frac{\beta_0^i}{\alpha_0} \hat{\alpha}_k^i \hat{\alpha}_{n-k}^i \right]. \]  

(5.8.58)

This means that the gauge-fixed NNLO Lagrangian takes the form

\[ L_{\text{NNLO}} = \dot{x}^A_{(2)A} + \dot{y}^A_{(0)A} + \dot{x}^A_{(2)A} + \dot{y}^A_{(0)A} + \dot{z}^A_{(2)A} \]

\[ + \sum_{k=1}^{\infty} \frac{2i}{k} \left( \hat{\alpha}_k^i \hat{\beta}_{-k}^i + \hat{\alpha}_k^i \beta_{-k}^i \right) \]

\[ - \sum_{n \in \mathbb{Z}} \left( \dot{\varphi}_{-n}^{(0)2} L_{(2)} + \ddot{\varphi}_{-n}^{(0)2} \bar{L}_{(n)} + \dot{\varphi}_{-n}^{(0)2} L_{(0)} + \ddot{\varphi}_{-n}^{(0)2} \bar{L}_{(n)} + \dot{\varphi}_{-n}^{(0)2} \bar{L}_{(2)} + \ddot{\varphi}_{-n}^{(0)2} \bar{L}_{(2)} \right). \]  

(5.8.59)

From this, we read off the gauge-fixed Poisson brackets

\[ \{ \dot{x}^A_{(2)}, \varphi_{(2)A} \} = \{ \dot{y}^A_{(0)}, \varphi_{(0)A} \} = \delta^A_j, \]

\[ \{ \dot{x}^A_{(2)}, \varphi_{(0)B} \} = \{ \dot{y}^A_{(0)}, \varphi_{(0)B} \} = \{ \varphi_{(2)A}, \varphi_{(2)B} \} = \delta^A_B, \]

\[ \{ \alpha_k^i, \beta_{-k}^i \} = \{ \hat{\alpha}_k^i, \hat{\beta}_{-k}^i \} = -\frac{ik}{2} \delta^{ij}. \]  

(5.8.60)

We can work out the remaining brackets by using the relations (5.8.58).

### 5.8.3 Commutators and the normal ordering constant

In passing from the classical theory to the quantum theory, the standard folklore tells us to replace Poisson brackets with commutators according to the rule

\[ [\cdot, \cdot] = i\hbar \{\cdot, \cdot\}. \]  

(5.8.61)

This is, however, not true in general; rather it is based on the canonically conjugate variables having dimensions length and momentum, respectively. Explicitly, the commutator \([q, p]\) has dimensions of mass \(\times\) \(\text{length}^2 / \text{time} = [\hbar]\), while \([q, p] = 1\) is dimensionless (due to the definition of the Poisson bracket), and hence we must compensate with a factor of \(\hbar\), leading to (5.8.61). In the general case, (5.8.61) becomes instead

\[ [a, b] = ik_{[a][b]} [a, b], \]  

(5.8.62)

where \(k_{[a][b]}\) is some combination of fundamental constants with dimensions of \([a][b]\).
NLO

Since there are no oscillations at LO, we jump straight to NLO, where the oscillations are $\alpha_i^k$ and $\tilde{\alpha}_i^k$ for $k \neq 0$. As usual, modes with $k > 0$ are interpreted as annihilation operators (and vice versa), which annihilate the NLO vacuum state $|0\rangle_{NLO}$

$$\alpha_i^k |0\rangle_{NLO} = \tilde{\alpha}_i^k |0\rangle_{NLO} = 0 \quad \forall k > 0. \quad (5.8.63)$$

Since

$$[\alpha_i^k] = [\tilde{\alpha}_i^k] = \text{length} \times \sqrt{\frac{\text{mass}}{\text{time}}}, \quad (5.8.64)$$

the relation (5.8.62) tells us that the modes satisfy the commutation relations

$$[\alpha_i^k, \alpha_j^{-k}] = \hbar k \delta_{ij}, \quad [\tilde{\alpha}_i^k, \tilde{\alpha}_j^{-k}] = \hbar k \delta_{ij}. \quad (5.8.65)$$

In terms of these modes, the number operators are given by

$$N(0) = \frac{1}{2} \sum_{k \neq 0} \alpha_i^{-k} \alpha_i^k, \quad \tilde{N}(0) = \frac{1}{2} \sum_{k \neq 0} \tilde{\alpha}_i^{-k} \tilde{\alpha}_i^k. \quad (5.8.66)$$

Adopting normal ordering, where creation operators (i.e., $\alpha_i^{-k}$ and $\tilde{\alpha}_i^{-k}$ for $k > 0$) are moved to the left, we find that

$$\frac{1}{2} \sum_{k \neq 0} \alpha_i^{-k} \alpha_i^k = \sum_{k=1}^{\infty} \alpha_i^{-k} \alpha_i^k + \frac{1}{2} \sum_{k=1}^{\infty} [\alpha_i^k, \alpha_i^{-k}] = \sum_{k=1}^{\infty} \frac{\hbar d}{2} \sum_{k=1}^{\infty} k = \sum_{k=1}^{\infty} \alpha_i^{-k} \alpha_i^k - \frac{\hbar d}{24}, \quad (5.8.67)$$

where we used that $\sum_{k=1}^{\infty} k = -\frac{1}{12}$, and where $d = D - 2$ is the dimension of the transverse space.

NNLO

The NNLO vacuum $|0\rangle_{NNLO}$ is annihilated by both $\alpha$ and $\beta$ annihilation operators,

$$\alpha_i^k |0\rangle_{NNLO} = \tilde{\alpha}_i^k |0\rangle_{NNLO} = \beta_i^k |0\rangle_{NNLO} = \tilde{\beta}_i^k |0\rangle_{NNLO} = 0 \quad \forall k > 0. \quad (5.8.68)$$

The subleading oscillator modes have dimensions of

$$[\beta] = \sqrt{\text{mass}} \times \text{length} \times (\text{time})^{-5/2}, \quad (5.8.69)$$

and hence (5.8.62) gives us

$$[\alpha_i^k, \beta_j^{-k}] = c^2 \frac{\hbar k}{2} \delta_{ij}, \quad [\tilde{\alpha}_i^k, \tilde{\beta}_j^{-k}] = c^2 \frac{\hbar k}{2} \delta_{ij}. \quad (5.8.70)$$

The LO number operators no longer suffer from an ordering ambiguity since the $\alpha$’s commute. Now, however, the operators making up the subleading number operators do not commute.
The subleading number operators are given by

\[ N^{(2)} = \sum_{k=1}^{\infty} \alpha_k^\dagger \beta_k^\dagger + \sum_{k=1}^{\infty} \alpha_k \beta_{-k}^\dagger, \]

\[ \tilde{N}^{(2)} = \sum_{k=1}^{\infty} \tilde{\alpha}_k^\dagger \tilde{\beta}_k^\dagger + \sum_{k=1}^{\infty} \tilde{\alpha}_k \tilde{\beta}_{-k}^\dagger, \]

(5.8.71)

where the second term on each line is not normal ordered. Performing this normal ordering is entirely analogous to what we did above:

\[ \sum_{k=1}^{\infty} \alpha_k^\dagger \beta_{-k}^\dagger = \sum_{k=1}^{\infty} \beta_{-k}^\dagger \alpha_k^\dagger + \frac{c^2 \hbar d}{2} \sum_{k=1}^{\infty} k, \]

\[ \sum_{k=1}^{\infty} \tilde{\alpha}_k^\dagger \tilde{\beta}_{-k}^\dagger = \sum_{k=1}^{\infty} \tilde{\beta}_{-k}^\dagger \tilde{\alpha}_k^\dagger - \frac{c^2 \hbar d}{24}, \]

(5.8.72)

and so we recover the same normal ordering constant as above. This pattern should persist to all orders in \( 1/c^2 \).

Performing a full quantisation and classification of states remains an interesting open problem.
Chapter 6

Conclusion & outlook

In this chapter, we take stock of what we have achieved in this thesis and provide an overview of several open questions which deserve further investigation.

Starting in Chapter 2, we provided an introduction to non-Lorentzian geometry, emphasising the power of Cartan geometry and G-structures. We also made the intriguing observation that the homogeneous spaces of the Poincaré group capture the asymptotic structure of Minkowski space, and that these spaces are of Carrollian type.

In Chapter 3, we provided a review of relativistic hydrodynamics using both the traditional approach of constitutive relations and the entropy current, as well as the more modern Lagrangian approach, where the non-dissipative sector of transport is captured by an “action”. This chapter served as warm-up for Chapter 4, where we presented the complete first-order energy-momentum tensor for a boost-agnostic fluid in curved spacetime, going beyond the linearised results obtained in [122]. Implementing the constraint of non-negativity of the divergence of the entropy current, we find 10 dissipative, 2 hydrostatic non-dissipative and 4 non-hydrostatic non-dissipative transport coefficients. In the linearised regime the latter four coefficients vanish as a result of implementing the Onsager relations. Using the curved spacetime formulation we explicitly obtained all non-dissipative transport coefficients, notably both in Landau frame and Lagrangian frame, by using a Lagrangian whose form was derived by starting with the hydrostatic partition function. Furthermore, we checked that our final results reproduce the well-known relativistic first-order transport coefficients when Lorentz boost symmetries are present. We also treated the special case when the hydrodynamic theory exhibits an additional Lifshitz scale invariance, in which case there are 7 dissipative, 1 hydrostatic non-dissipative and 2 non-hydrostatic non-dissipative transport coefficients.

In Chapter 5, we developed the string $1/c^2$ expansion of closed relativistic bosonic strings up to NNLO. The target space geometry of these strings is obtained from a string $1/c^2$ expansion of a Lorentzian geometry. At NLO this leads to string Newton–Cartan geometry or generalisations thereof. The string theories we obtain arise as $1/c^2$ expansions of relativistic strings in either the Nambu–Goto, Polyakov or phase space formulation. At NLO, when the target space foliation is such that $\alpha^{\Lambda A} = 0$ in (5.3.25), the theory takes the form of the Gomis–Ooguri string. We have computed the spectrum of the string by computing the energy at every order and shown that it agrees with the $1/c^2$ expansion of the relativistic energy of a
string in a flat target spacetime. In order to perform this expansion we had to assume that
the target space has a circle and that string has a nonzero (positive) winding along that circle.
In the phase space formulation, we were able to perform the Dirac procedure and, with the
Dirac brackets in hand, quantise the theory order by order. This, in particular, led to the
same normal ordering constant as in the closed bosonic relativistic string, but the way this
constant appears is different at each order.

There are a plethora of very interesting future directions, many of which are currently
work in progress. We list some of these below.

**Flat and celestial holography** As we pointed out in the introduction, Carrollian structures
appear to be particularly relevant for flat holography. Understanding this better is
therefore both interesting and important. A particularly interesting approach could
involve devising a theory of Carrollian strings, where conformal Carrollian field theories
could be engineered in a suitable brane construction.

**Gravitational vacua and (pseudo-)Carrollian spaces** As we mentioned in the introduc-
tion, the enhancement of Poincaré to BMS symmetry at null infinity in asymptotically
flat spacetimes implies that the (radiative) vacuum of gravity in asymptotically flat
spacetimes is infinitely degenerate [260]. The different gravitational vacua are related
by supertranslations and superrotations. At null infinity, these vacua can be un-
derstood in terms of Cartan geometry based on a certain homogeneous space of the
Poincaré group [261, 262]. In [263], it was shown that the superrotation sector can be
derived from a three-dimensional action.

The enhancement of Poincaré to BMS can also be demonstrated at spatial [264–266]
and time-like infinity [267]. In the recent work [5] (see also Section 2.4), we showed
that both space-like and time-like infinities can be understood as (pseudo-)carrollian,
homogeneous spaces of the Poincaré group. In particular, the blow-up of time-like
infinity can be described by AdS/C. It is thus suggestive that (a constrained version
of) the Lagrangian for AdS describes the BMS vacuum sector of General Relativity
near time-like infinity. Similar comments apply to the case of space-like infinity with
associated pseudo-Carrollian Klein pair Spi, that is related to the Ashtekar–Hansen
structure at spatial infinity [213,214].

**General Relativity: The view from timelike infinity** More speculative, but also more re-
warding, is the idea that the relation between General Relativity on asymptotically
flat spacetimes and the gaugings of AdS and Spi extends also away from the vacuum
sector. In [5] it was shown how points in Minkowski spacetime correspond to certain
higher-dimensional geometries in AdS/C/Spi. Perhaps the Lagrangian constructed
in [6] based on the Klein pair of AdS and its solutions encode certain aspects of
General Relativity on asymptotically flat spacetimes. Since the map between the flat
models of the two Cartan geometries, Minkowski and AdS/C/Spi, is non-local, the putat-
ive construction would likely have a twistorial flair. We leave this interesting possibility
for further study.
Fracton fluids The results of Section 4 and [4] in principle provides all the tools we need to build a theory of fracton fluids. As discussed in the introduction, fractons defy a conventional quantum field theory description, and hydrodynamics is one of the few tools we have that will allow us to probe such theories. However, the mobility restrictions imposed by the symmetry algebra (1.0.1) imply that fractons do not admit a conventional fluid description. As we already discussed in the introduction, we can spontaneously break the dipole symmetry, we can use fluid dynamics to describe what then becomes a fracton superfluid. An analysis of perfect fracton superfluids does not, at the time of writing, exist; neither using either the traditional constitutive relations approach that we met in Section 3.1 nor the Lagrangian methods discussed in Sections 3.2 and 4.3. As we also remarked in the introduction, it is particularly intriguing that existing Lagrangian methods seem to be inadequate to describe fracton superfluids. In collaboration with Jay Armas, I am working towards such a formalism, and we hope to report on this in the near future.

Kubo formulae and linear response theory for boost-agnostic fluids With the full geometrical information at our disposal, we can now for example compute Kubo formulae and relate individual transport coefficients to a particular linear response. Consider for example the response of the system in flat space to a purely time-dependent perturbation $\delta h_{\mu\nu}$. The perfect fluid equations of motion (4.1.30) remain valid to first order if we also impose

$$\delta P = \delta (e\rho) = 0,$$

which evaluates to

$$\delta T_{\mu\nu} = -\rho \delta h_{\mu\nu} - \frac{1}{2} \delta h_{kk} \delta \rho,$$

(6.1.1)

$$\delta T_{ij} = -\rho \delta h_{ij} - \frac{1}{2} \delta h_{kk} \rho v^i v^j - \rho \delta h_{0i} - \rho v^j \delta h_{ij} - \rho v^i \delta h_{0i} - \rho v^i v^j \delta h_{ki} - \rho v^j \delta h_{ki}.$$

(6.1.3)

From these variations one can read off that there are leading order contributions to the two-point function $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ which are $\omega$-independent. If we subtract these leading order contributions, we find from (4.4.4) that we expect a contribution proportional to $\omega \eta_{\mu\nu\rho\sigma}$ to the two-point function, and hence a Kubo formula of the type

$$\lim_{\omega \to 0} \frac{1}{\omega} \langle (TT) - \langle TT \rangle_{\text{leading}} \rangle \eta_{\mu\nu\rho\sigma}.$$

This is not yet the complete answer though. First of all, the leading order change in $v^i$ is also relevant and using the explicit expression one can show that while at first order $\langle T_{\mu\nu} T_{00} \rangle$ is indeed proportional to $\eta_{\mu\nu00}$, $\langle T_{\mu\nu} T_{0i} \rangle$ does not contain any contribution of $\eta$, and

$$\langle T_{\mu\nu} T_{ij} \rangle = \eta_{\mu\nu00} - \eta_{\mu\nu00} (i v^j).$$

(6.1.6)
In addition, we have not yet included the hydrostatic contribution to the stress tensor. These contributions can in principle be extracted from the analysis in Section 4.3. Alternatively, one can also try to derive Kubo formulae for the HS coefficients directly from the partition function, which in particular guarantees that the stress tensor will be covariantly conserved [268]. We leave a more detailed analysis of all these issues to future work.

Besides the application to Kubo formulae, the geometrical formulation based on Aristotelian (or absolute) spacetime also paves the way for computing hydrodynamic modes—our framework provides the ideal starting point for such an analysis. One is now provided with the tools to answer questions regarding the stability of the hydrodynamical spectrum at first order in curved spacetime, as was studied for boost invariant systems in [129, 269].

**Boost-agnostic fluid/gravity correspondence** Another worthwhile open direction is to consider the relation to holography for the case of Lifshitz fluids, for which we have identified the reduced set of first-order transport coefficients. Such fluids were discussed in a holographic context in Refs. [132, 151–154]. Following these works, it would be interesting to study hydrodynamic modes of Lifshitz fluids using quasinormal modes in order to find the extra dissipative and non-dissipative transport coefficients, as was done in [155]. It would furthermore be interesting to see if a damping/overdamping transition as reported in [270, 271] could be reproduced. More generally, addressing the question of universal properties obeyed by transport coefficients in holographic setups and the development of a full-fledged fluid/gravity correspondence would be relevant to pursue in this case.

**Dynamics of Aristotelian surfaces** In another direction, it would be worthwhile to examine submanifolds and fluids living on them in the spirit of [1]. In particular, it was shown in the context of Newton–Cartan geometry that the normal projection of \( v^\mu \) can be interpreted as the transverse velocity of the submanifold. In the absence of boost symmetry, the rôle of such a velocity turns out to be more prominent and alters the notion of pullback map. The details will appear in future work.

**Beta functions in nonrelativistic string theory** The beta functions of NR string theory were considered in [74, 169, 171], and an action has been proposed that recovers all but one of the beta functions in [166], namely the stringy counterpart of the Poisson equation. We expect that the string \( 1/c^2 \) expansion of NS-NS gravity would lead to an action principle for all the beta functions of NR string theory including the Poisson equation. This expectation is based on an analogous situation observed in the context of gravity: the action for NR gravity obtained using the particle \( 1/c^2 \) expansion of GR in [22, 24] is precisely able to reproduce the Poisson equation, while previous approaches based on strict limits were unable to do so.

**Odd powers** Another natural generalisation would be to consider the inclusion of odd powers of \( 1/c \) in the expansion, thereby turning the string \( 1/c^2 \) expansion into a string
1/c expansion. For gravity, the analogous generalisation for the particle expansions was considered in [272].

**Open strings and D-branes** In this thesis, we have only considered closed strings, but open strings and branes also have a rôle to play in NR string theory [182, 183, 273]. The 1/c² expansion of open strings and D-branes would be a natural next step (see [274]). In a related direction, it would be interesting to explore the p-brane 1/c² expansion of Lorentzian geometry. This would presumably lead to a “type II p-brane Newton–Cartan geometry”, which would generalise the geometry developed in [275] in much the same way that both type II string and particle Newton–Cartan geometries generalise their “type I” counterparts.

**Landscape of non-Lorentzian string theories** More generally, it has become increasingly clear that NR strings are part of a landscape of non-Lorentzian string theories. The existence of such a landscape of non-Lorentzian string theories has for example been demonstrated within double field theory, where many non-Lorentzian geometries were found within double geometry [77, 276]. Based on these observations, it would be particularly interesting to apply the 1/c² expansion in the context of double field theory, which might give another perspective on the emergence of non-Lorentzian geometries.
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