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Achievable Information Rates for Nonlinear Frequency Division Multiplexed Fibre Optic Systems

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Doctor of Philosophy

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To All the Stars Laughing in the Sky
and the Sea Reflecting their Light
Abstract

Fibre optic infrastructure is critical to meet the high data rate and long-distance communication requirements of modern networks. Recent developments in wireless communication technologies, such as 5G and 6G, offer the potential for ultra-high data rates and low-latency communication within a single cell. However, to extend this high performance to the backbone network, the data rate of the fibre optics connection between wireless base stations may become a bottleneck due to the capacity crunch phenomena induced by the signal dependent Kerr nonlinear effect. To address this, the nonlinear Fourier transform (NFT) is proposed as a solution to resolve the Kerr nonlinearity and linearise the nonlinear evolution of time domain pulses in the nonlinear frequency domain (NFD) for a lossless and noiseless fibre. Nonlinear frequency division multiplexing (NFDM), which encodes information on NFD using the discrete and continuous spectra revealed by NFT, is also proposed. However, implementing such signalling in an optical amplifier noise-perturbed fibre results in complicated, signal-dependent noise in NFD, the signal-dependent statistics and unknown model of which make estimating the capacity of such a system an open problem.

In this thesis, the solitonic part of the NFD, the discrete spectrum is first studied. Modulating the information in the amplitude of soliton pulse, the maximum time-scaled mutual information is estimated. Such a definition allows us to directly incorporate the dependence of soliton pulse width to its amplitude into capacity formulation. The commonly used memoryless channel model based on noncentral chi-squared distribution is initially considered. Applying a variance normalising transform, this channel is approximated by a unit-variance additive white Gaussian noise (AWGN) model. Based on a numerical capacity analysis of the approximated AWGN channel, a general form of capacity-approaching input distributions is determined. These optimal distributions are discrete comprising a mass point at zero (off symbol) and a finite number of mass points almost uniformly distributed away from zero. Using this general form of input distributions, a novel closed-form approximation of the capacity is determined showing a good match to numerical results. A mismatch capacity bounds are developed based on split-step simulations of the nonlinear Schrödinger equation considering both single soliton and soliton sequence transmissions. This relaxes the initial assumption of memoryless channel to show the impact of both inter-soliton interaction and Gordon-Haus effects. Our results show that the inter-soliton interaction effect becomes increasingly significant at higher soliton amplitudes and would be the dominant impairment compared to the timing jitter induced by the Gordon-Haus effect.
Next, the intrinsic soliton interaction, Gordon Haus effect and their coupled perturbation on the soliton system are visualised. The feasibility of employing an artificial neural network to resolve the inter-soliton interaction, which is the dominant impairment in higher power regimes, is investigated. A method is suggested to improve the achievable information rate of an amplitude modulated soliton communication system using a classification neural network against the inter-soliton interaction. Significant gain is demonstrated not only over the eigenvalue estimation of nonlinear Fourier transform, but also the continuous spectrum and eigenvalue correlation assisted detection scheme in the literature.

Lastly, for the nonisolonic radiation of the NFT, the continuous spectrum is exploited. An approximate channel model is proposed for direct signalling on the continuous spectrum of a NFDM communication system, describing the effect of noise and nonlinearity at the receiver. The optimal input distribution that maximises the mutual information of the proposed approximated channel under peak amplitude constraint is then studied. We present that, considering the input-dependency of the noise, the conventional amplitude-constrained constellation designs can be shaped geometrically to provide significant mutual information gains. However, it is observed that further probabilistic shaping and constellation size optimisation can only provide limited additional gains beyond the best geometrically shaped counterparts, the 64 amplitude phase shift keying. Then, an approximated channel model that neglects the correlation between subcarriers is proposed for the matched filtered signalling system, based on which the input constellation is shaped geometrically. We demonstrate that, although the inter-subcarrier interference in the filtered system is not included in the channel model, shaping of the matched filtered system can provide promising gains in mismatch capacity over the unfiltered scenario.
It is commonly known that the Internet is built on the submarine optical fibre connection between data centres located at different continents. Relying on well developed conventional linear fibre optics techniques, fibre communication system have supported data demand since the invention of the Internet. The recent rapid development of the wireless communication technology and the increasing data traffic from high data rate demand activities such as video streaming and cloud computing, put more challenges on the throughput of fibre communication system. Due to the signal-dependent nonlinearity effect that would be enhanced when the signal power is increased, the capacity of the optical fibre system can no longer be improved with conventional techniques. Recently, a novel technique based on the nonlinear Fourier transform (NFT) is suggested to take advantage on this nonlinearity to reveal new possibilities instead of trying to compensate for it. In this thesis, the two degrees of freedom, continuous spectrum and discrete spectrum, are investigated. The discrete spectrum corresponds to the soliton pulse, whose shape is invariant against both linear and nonlinear distortion of a noiseless and lossless fibre. Being an important candidate in resolving the nonlinearity, the amplitude modulated soliton communication system will experience non-trivial signal dependent noise and complicated nonlinear interaction between soliton pulses. The channel model of the amplitude modulated soliton communication is first transformed to an equivalent trivial Gaussian channel. Then the maximum data rate and the corresponding input distribution is estimated numerically using the simplified channel model. Furthermore, an analytical approximation of the maximum data rate and optimal input distribution is inspired by the insight from the numerical optimisation. Previously, the inter-soliton interaction is shown to be the dominant distortion over the signal dependent noise in higher power regime. With the assistance of high performance computing platform, a neural network trained with large number of simulated transmissions of interacted soliton pulses is implemented to improve the robustness against the nonlinear interaction. Lastly, the continuous spectrum, on the other hand, possesses similar properties to the linear frequency spectrum, suggesting that the corresponding time domain pulse will be distorted by both dispersion and nonlinearity. However, in a lossless and noiseless fibre, the evolution of the continuous spectrum is linear despite the nonlinear evolution of the time domain pulse. Similar to the amplitude modulated soliton system, the noise in the continuous spectrum modulated system is also signal dependent with memory. Extending the aforementioned transformation, an analytical approximated channel is proposed when the correlation between the subcarriers could be neglected. The input constellation is then shaped geometrically to adapt to the approximated channel model. It is demonstrated that despite the misalignment between the approximated channel and true channel, significant gain could be achieved over an unshaped constellation.
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Upon this voyage through celestial skies,
I wandered far and wide, a curious flight,
From planet to planet, seeking wisdom’s prize,
Encountering guiding stars, their lessons shining bright.

In the constellation of minds that lit my path,
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Whose tender care and love, like petals’ soft caress,
Nurtured my growth, my aspirations to compose.

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Dr. Mohammadamin Baniasadi, the lamplighter, true,
Illuminated darkness, knowledge burning bright,
To past and future members of the group, I value,
Your cosmic dance, like stars that filled my night.

To friends in Edinburgh, my precious rose garden,
Each unique and cherished, like planets in the sky,
Our laughter echoed, little bells to sweeten,
The heavy air, as days and nights passed by.

As the sea reflects the starry skies above,
I recognise the gifts I’ve been bestowed with love,
The China Scholarship Council, Leverhulme Trust,
Your faith in me, a treasure, never rusts.

When I return to the sky, shedding weighty skin,
Like a snake’s transformation, a new life shall begin,
The nights shall ring with laughter, like little bells’ sweet sound,
A testament to the love that in learning I have found.
So, let us celebrate this cosmic tale,
With hearts ablaze, as we set sail,
We’ll join the stars, their laughter echoing through,
A boundless universe, where dreams come true.

– YuGPT Chenat
I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Yu Chen
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Nomenclature

\[ \begin{align*}
\alpha & \quad \text{Fibre loss factor} \\
\beta_2 & \quad \text{Group velocity dispersion} \\
\delta & \quad \text{Energy truncation factor in time domain} \\
\varepsilon & \quad \text{Energy truncation factor in frequency domain} \\
\gamma & \quad \text{Kerr nonlinearity} \\
\lambda & \quad \text{Nonlinear frequency} \\
\lambda_{m,m} = 1, 2, \ldots, M & \quad \text{Eigenvalues in discrete spectrum} \\
R & \quad \text{Time-scaled mutual information} \\
E[\cdot] & \quad \text{Expectation operation} \\
\mu_A & \quad \text{Mean of random variable } A \\
\mu_{NCX} & \quad \text{Mean of a specified distribution} \\
\nu_0 & \quad \text{Carrier frequency} \\
\omega_0 & \quad \text{Carrier angular frequency} \\
\rho(\lambda) & \quad \text{Continuous spectrum} \\
\sigma^2 & \quad \text{Normalised ASE noise PSD per distance} \\
\sigma_A^2 & \quad \text{Variance of random variable } A \\
\sigma_N^2 & \quad \text{Normalised ASE noise PSD at given normalised distance} \\
a(\lambda) & \quad \text{Jost scattering coefficient} \\
A, a & \quad \text{Transmitted symbol in original domain and its realisation} \\
A_{\text{lb}} & \quad \text{Minimum amplitude constraint} \\
A_{\text{max}} & \quad \text{Maximum nonzero amplitude in given input constellation in VNT domain} \\
A_{\text{min}} & \quad \text{Minimum nonzero amplitude in given input constellation in VNT domain} \\
A_{\text{ub}} & \quad \text{Peak amplitude constraint} \\
b(\lambda) & \quad \text{Jost scattering coefficient} \\
C_{m,m} = 1, 2, \ldots, M & \quad \text{Norming constant in discrete spectrum} \\
C_{\text{bpcu}} & \quad \text{Capacity in bits per channel use} \\
C_{\text{bps}} & \quad \text{Capacity in bits per second} \\
C_M & \quad \text{Mismatch capacity in bits per symbol} \\
C_{\text{TSM}} & \quad \text{Mismatch time-scaled capacity} \\
C_{\text{TS}} & \quad \text{Time-scaled capacity} \\
f_w & \quad \text{Unitless bandwidth given energy truncation} \\
H(X) & \quad \text{Entropy of random variable } X \\
h(Y) & \quad \text{Differential entropy of random variable } Y \\
h_{\text{Planck}} & \quad \text{Planck constant} \\
h_S & \quad \text{Split-step Fourier step size}
\end{align*} \]
NOMENCLATURE

\[ I(X;Y) \] Mutual information between random variables \( X \) and \( Y \)

\[ K_T \] Phonon occupancy factor

\[ L \] fibre length

\[ l \] Normalised fibre length

\[ L_D \] Dispersion length

\[ N(T,Z) \] Amplifier spontaneous emission noise with unit

\[ n(T,Z) \] ASE noise without unit

\[ N_{\text{ASE}} \] ASE noise PSD per distance

\[ Q(T,Z) \] Complex envelope of the optical field with unit

\[ q(T,Z) \] Complex envelope of the optical field without unit

\[ R, r \] Received symbol in original domain and its realisation

\[ T \] Time with unit

\[ t \] Time without unit

\[ T(\cdot) \] Variance normalising transformation

\[ T^{-1}(\cdot) \] Inverse variance normalising transformation

\[ T_0 \] Normalisation time

\[ t_w \] Unitless time width given energy truncation

\[ X, x \] Transmitted symbol in transformed domain and its realisation

\[ X_{\text{lb}} \] Minimum amplitude constraint in VNT domain

\[ X_{\text{max}} \] Maximum nonzero amplitude in given input constellation in VNT domain

\[ X_{\text{min}} \] Minimum nonzero amplitude in given input constellation in VNT domain

\[ X_{\text{ub}} \] Peak amplitude constraint in VNT domain

\[ Y, y \] Received symbol in transformed domain and its realisation

\[ Z \] Propagation distance with unit

\[ z \] Propagation distance without unit
ADAM  Adaptive moment estimation.
AIR  Achievable information rate.
AMMSE  Affine minimum mean squared error.
ANN  Artificial neural network.
APSK  Amplitude phase shift keying.
ASE  Amplifier spontaneous emission.
ASK  Amplitude shift keying.
AWGN  Additive white Gaussian noise.

BLSTM  Bidirectional long short-term memory.
BPSK  Binary phase shift keying.

CCA  Canonical correlation analysis.
CNN  Convolutional neural network.
CS  Continuous spectrum.

DBP  Digital backpropagation.
DS  Discrete spectrum.
DSP  Digital signal processing.

EDFA  Erbium-doped fibre amplifiers.

FDM  Frequency division multiplexing.
FO  Full Optimisation, i.e. probabilistic geometric hybrid shaping.
FT  Fourier transform.
FWHM  Full-width at half maximum.

GLM  Gelfan-Levitan-Marchenko.
GS  Geometric shaping.
GVD  Group velocity dispersion.

IFT  Inverse Fourier transform.
INFT  Inverse nonlinear Fourier transform.
ISI  Inter-subcarrier interference.
IVNT  Inverse variance normalising transform.
IVSTF  Inverse Volterra series transfer function.
KdV  Korteweg-deVries.
KL Kullback-Leibler.

LMMSE Linear minimum mean squared error.
LSTM Long short-term memory.

MCF Multi-core fibre.
MI Mutual information.
MIMO Multiple-input multiple-output.
MMF Multi-mode fibre.
MSE Mean squared error.

NCG Noncircular Gaussian.
NCX Noncentral chi.
NCX2 Noncentral chi-squared.
NFD Nonlinear frequency domain.
NFDM Nonlinear frequency division multiplexing.
NFT Nonlinear Fourier transform.
NIS Nonlinear inverse synthesis.
NLSE Nonlinear Schrödinger equation.

OADM Optical add-drop multiplexer.
OAM Orbital angular momentum.
ODE Ordinary differential equation.
OFDM Orthogonal frequency division multiplexing.
OOK On-off Keying.
OSNR Optical signal to noise ratio.

PAM Pulse amplitude modulation.
PDE Partial differential equation.
PDF Probability density function.
PS Probabilistic Shaping.
PSD Power spectrum density.

QAM Quadrature amplitude modulation.
QPSK Quadrature phase shift keying.

ReLU Rectified linear unit.
RHS Right hand side.
RNN Recurrent neural network.

SBS Stimulated Brillouin scattering.
SDM Spatial division multiplexing.
SE Spectral efficiency.
SMF  Single-mode fibre.
SPM  Self-phase modulation.
SRS  Stimulated Rayleigh scattering.
SVM  Support vector machine.
VNT  Variance normalising transform.
WDM  Wavelength division multiplexing.
XPM  Cross-phase modulation.
In this chapter, a brief overview of the history of fibre optics development is presented, followed by a literature review of relevant studies related to this thesis. The contributions of this thesis are then summarised and outlined in their respective chapters. Lastly, a list of publications produced during the course of this research is presented.

1.1 Overview

Since 1841, when Jean-Daniel Colladon performed the famous experiment of launching a beam of light into a stream of water and observing its direction being bent at the water-air interface, the fundamental physical phenomenon behind fibre optics has been known as internal reflection. This phenomenon occurs because water and air have different refractive indices. A similar experiment was reproduced by John Tyndall in his lecture at the Royal Institution. In the 1920s, the first glass optical fibre was initially developed for medical purposes, aiming to illuminate body cavities, similar to the dentistry illumination developed later in the twentieth century. Following the path of fibre optics in medical use, the first practical semi-flexible gastroscope was developed in 1956, followed by the production of the first glass clad fibres. In 1966, Charles K. Kao and George A. Hockham, who would later become the 2009 Nobel Prize winners, pointed out that the performance of fibre optics at the time was limited by heavy attenuation [3]. They proposed the theoretical conclusion that the fibre loss had to be reduced to below 20 dB/km to provide a capacity over 1 Gbps [3]. Due to the heavy attenuation, optical fibre was primarily employed in short-range connections, such as the television cameras used in NASA’s moon exploration programme in 1968. In 1970, Corning Glass Works demonstrated a titanium-doped silica glass fibre that achieved 17 dB/km attenuation, reaching the crucial fibre loss threshold described by Kao and Hockham. Since achieving this milestone in attenuation, fibre loss has been reduced so rapidly that an outside vapour deposition fibre with 0.16 dB/km was reported in the 1980s [4].
Advances in both laser source and optical fibre technologies encouraged the commercialisa-
tion of fibre optics communication systems. The first-generation system was implemented with
optics-electronics-optics regenerators, featuring a bit rate of 45 Mbps and an 850 nm carrier
wavelength. The second-generation system was subsequently developed and operated at a
wavelength of 1300 nm. Multi-mode fibres (MMF) employed in the first and early second-
generation systems introduced multi-mode dispersion effects between different modes of the
fibre. This complicated crosstalk limited the capacity of the early systems. Single-mode fibres
(SMF) were shown to have lower attenuation and essentially zero multi-mode dispersion
effects [4]. It was experimentally demonstrated that the data rate of the SMF system could
achieve up to 2 Gbps, with the repeater spacing exceeding 100 km [5–10]. By 1987, a
commercial second-generation system could support a 1.7 Gbps data rate and a repeater
spacing of 50 km [2]. The development of single longitudinal mode laser sources, as op-
posed to the multiple longitudinal mode laser employed in the first and second-generation
systems, revealed the 1550 nm wavelength that could achieve the lowest loss in silica fibres.
The dispersion-shifted fibre was also implemented to reduce the dispersion effect with this
wavelength. Combining these two techniques, the third-generation fibre optics systems could
operate at a 10 Gbps bit rate in the 1990s [2].

Multiple beams of light can propagate within the fibre simultaneously, revealing a potential
multiplexing gain over employing a single beam of light. A system that utilises multiple lasers
at different wavelengths to carry more information at the same time is known as wavelength
division multiplexing (WDM). WDM is commonly used in radio frequency communication,
where it is more widely known as frequency division multiplexing (FDM). Implementing WDM
in optical fibre communication requires additional hardware, such as optical (de)multiplexers.
In a WDM system, the optical add-drop multiplexer (OADM) is commonly used to route or
de-route different beams of light without physically changing the wiring of fibres. In addition to
(de)multiplexing, the OADM can also be employed in an optical fibre network to route beams
to target users.

Although the loss of fibre could be addressed with regenerators, the reach or coverage area
of fibre optic systems was still limited by the high cost of these regenerators. The first effective
erbium-doped fibre amplifier (EDFA), which was pumped by a laser diode, was demonstrated
by Nakazawa and Snitzer in 1989 [11], two years after it was proposed in 1987 [12, 13].
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Compared to regenerators, optical amplifiers are more cost-effective because they do not
require additional hardware for optics-electronics conversion and do not need multiple sets
of devices for each wavelength. However, due to variations in the amplifier frequency re-
sponse, each wavelength in a WDM system could potentially result in different optical signal-
to-noise ratios (OSNR) at the receiver, limiting the selection of wavelengths. Gain-flattening
filters were developed in 1992 to address this issue by launching each beam with an optimal
power level, allowing an equal level of OSNR to be achieved [14, 15]. The implementation of
gain-flattening filtered optical amplifiers and WDM ignited an era of rapid capacity boosting in fourth-generation fibre optic communication, as shown in Figure 1.1. In 1991, an early demonstration of a WDM system showed a data rate of 1.7 Gbps per channel over a system span of 840 km [16,17]. In early 1996, three research groups from Bell Labs, Fujitsu, and NTT demonstrated WDM systems that could support a total capacity of over 1 Tbps [18–20].

Coherent detection, also known as heterodyne detection and previously used in radio frequency and microwave communication, can detect both amplitude and phase, allowing for more degrees of freedom and complex modulation formats. Coherent detection for fibre optics was first studied in the 1980s but was found to be unnecessary because of the popularity and effectiveness of optical amplifiers at the time [2]. Employing complex modulation can improve spectral efficiency (SE) by transmitting more bits with a single symbol. Compared to WDM systems that employ amplitude shift keying (ASK) in previous generation technology with a spectral efficiency around 0.8 bits/s/Hz, fifth-generation fibre optic systems can achieve at least 3 bits/s/Hz with polarisation multiplexing and quadrature phase shift keying (QPSK) [2,21]. More importantly, coherent detection also allows for a digital copy of the signal to be obtained. When high-speed digital signal processing (DSP) chips become available, dispersion compensation can be performed digitally. Additionally, encoder and decoder pairs can be implemented to reduce the bit-error rate of the system. A notable milestone for the fifth generation is a 320 km link with 640 WDM channels that achieved a total data rate of 64 Tbps in 2011 [22]. It is also worth highlighting the achievement of 1 Tbps per WDM channel demonstrated in [23], showing the first result close to the Shannon limit via probabilistic shaping (PS) of the constellations. As a brief summary of the development of the modern fiber optics communication system, the records of data rates and data reaches achieved from experimental links are summarised in Figure 1.1.

![Figure 1.1](image_url):

Figure 1.1: The highest (a) capacity-distance product (adapted from [1]; ©2020 Wiley) and (b) capacity (adapted from [2], ©2016 G. P. Agrawal) records from research demonstrations.
1.1. Overview

In the aforementioned fibre optics technologies, the launch power of the laser source was kept relatively low, assuming that the signal-dependent Kerr nonlinearity of the fibre was weak and negligible. However, with the emergence of new data demands, such as video streaming and cloud computing, data demand is predicted to increase exponentially [24]. According to this prediction, data demand will increase fourfold every five years, and the network will require a traffic demand of 1 Pbps by 2024 [24]. To meet such increasing data demands, it is essential to improve the capacity of fibre optic networks. Initially, the research community believed that raising the OSNR could improve channel capacity, as predicted by Shannon’s theory for an additive white Gaussian noise (AWGN) channel [25]. However, it has been reported that increasing the launch power can also increase nonlinearity, resulting in decreased spectral efficiency at high OSNR regimes [21,26].

The sixth-generation fibre optics system, which takes advantage of space division multiplexing (SDM) techniques, could potentially be the solution to meet the increasing demand for data transmission. The SDM gain can be provided regardless of the operational OSNR regime. SDM fibre can be classified into two categories based on the number of cores within one fibre cladding: multi-core fibre (MCF) and multi-mode fibre (MMF). In MCF, multiple fibre cores are physically far apart within one cladding, allowing for very low coupling between the cores. A research group at NICT Japan demonstrated an optical link over 2000 km that achieved a data rate of 172 Tbps with a 3-core fibre in 2020 [27]. The same group reported a longer and faster transmission with a data rate of 319 Tbps and a reach distance of over 3000 km with a 4-core fibre in 2021 [28]. On the other hand, MMF, similar to the first-generation fibre system in earlier years, unavoidably introduces inter-mode interference. However, the capability of mitigating inter-mode interference has been greatly enhanced, thanks to coherent receivers and fast DSP chips. In [29], an SDM system with simultaneous transmission of six spatial and polarization modes was reported, where each mode was operated with 40 Gbps QPSK modulation. Using modern offline DSP and multiple-input multiple-output (MIMO) processing algorithms, the penalty was shown to be lower than 1.2 dB. One of the latest records for MMF is the transmission of 1.53 Pbps over a 55-mode fibre [30]. It is worth mentioning that the use of orbital angular momentum (OAM) could potentially provide better multi-mode multiplexing gain due to the orthogonality between different OAM modes [31–33].

The above-mentioned sixth-generation fibre optics techniques require a new set of fibre infrastructure to be implemented, as the current fibre infrastructure is built on SMF. To enable a smoother transition from standard SMF to MMF, a novel technique called nonlinear Fourier Transform (NFT) has gained attention. The NFT, also known as inverse scattering transform, can transform the signal into the nonlinear frequency domain (NFD), where NFD spectra evolve linearly while pulse evolution in the time domain is nonlinear [34–37]. This property of the NFD suggests great potential for achieving the capacity of nonlinear fibre. However, this linear evolution is only possible within noiseless propagation, which is not feasible for
1.1. Overview

long-haul fibre with optical amplifiers. The amplifier spontaneous emission (ASE) noise in the time domain will introduce signal-dependent noise in NFD with an unknown channel model, making the capacity of such a system an open problem that will be discussed in this thesis. Despite the nontrivial noise modelling, the NFT approach could still be a potential candidate for improving the capacity of nonlinear fibre.

1.2 Objectives and Key Contributions

1.2.1 Objectives

In nonlinear fibre optics communication system, it is already shown that the signal dependent nonlinearity will soon limit the further improvement of the system SE [21]. The proposition of the nonlinear Fourier transform (NFT) introduces the potential of taking advantage of the nonlinearity rather than treating the nonlinearity as a type of distortion. Due to the nontrivial noise of the NFD modulated system, the capacity problem and the channel model for the nonlinear frequency division multiplexed (NFDM) system are not well defined in the literature. Thus, the main objectives of this thesis include two aspects, formulating and resolving the capacity problem, and modelling the communication channel. More specifically, this thesis has the objectives as listed below

- Formulating and resolving the capacity problem for the amplitude modulated single soliton system, the simplest discrete spectrum (DS) modulated communication system, for the existing signal-dependent perturbative theory channel model.
- Investigating the nonlinear interaction between solitons when transmitting in sequence, and how such nonlinear effect could be resolved.
- Modelling the continuous spectrum (CS) modulated NFDM system, formulating and resolve its corresponding capacity problem.

1.2.2 Key Contributions

The main contributions of this thesis are listed as

- Achievable information rate (AIR) maximisation for a DS-modulated NFDM systems perturbed by ASE noise by reducing the signal dependence of the original noncentral chi-squared (NCX2) channel to an AWGN model via variance normalising transform (VNT) and integrating soliton time-width in capacity problem formulation. The equivalence between two models is proved using Kullback-Leibler (KL) divergence. Additionally, an analytical approximation for capacity and is also proposed along with mismatch capacity lower bounds for the system when interaction is not considered. When inter-soliton interaction is taken into account, the mismatch capacity could be reduced to zero, revealing the limit of the perturbative theory channel model.
1.2. Objectives and Key Contributions

- **Visualisation** of the under-studied issue of inter-soliton interaction and its coupling with noise-induced Gordon-Haus timing jitter in soliton sequence transmission via numerical experiments, a factor that undermines the validity of conventional perturbative theory channel models. **Bidirectional long short-term memory (BLSTM) neural network** trained to address the detection of interacted solitons, showing higher AIR in communication when benchmarked against two model-based soliton detection schemes. Additionally, the hyperparameter analysis on the training soliton sequence length reveals insights into the size of the channel memory required for more efficient system operation.

- **Channel modelling** of the CS-modulated NFDM system via a set of signal dependent signal decomposition and VNT. Both geometric shaping and probabilistic-geometric hybrid shaping are performed to maximise the proposed channel mutual information. The **AIR** of the channel is then estimated with the shaped constellation. Both shaping schemes show better performance over the benchmark unshaped constellations, while the hybrid shaping provides limited gain over its counterpart. Furthermore, we also show that a matched filter in NFD could provide AIR gain despite the inter-subcarrier interference induced in such signal dependent channel.

1.3 Thesis Outline

The remainder of the thesis is arranged as follow

**Chapter 2**

This chapter provides the necessary background knowledge for this thesis. The background of nonlinear fibre optics, nonlinear Fourier transform and neural network are described, followed by some discussion on their numerical implementation in MATLAB and open-source communities.

**Chapter 3**

This chapter is based on [38] which provides an estimation of the channel capacity of an amplitude modulated single soliton communication system. The signal-dependent channel model of this system is first normalised into an equivalent AWGN channel using VNT. Then, factorising the signal-dependent time width for signalling the soliton, the time-scaled mutual information is formulated as the objective function. Subjecting to a peak amplitude constraint, the input constellation is shaped accordingly. Relying on the features of the shaped constellations, an analytical approximation of the optimised time-scaled mutual information is also derived. Then, the proven capacity lower bound, the mismatch capacity, is estimated with
1.3. Thesis Outline

the shaped constellation and Monte-Carlo simulation. At last, assuming a channel memory
of single soliton, three-soliton sequence simulations are generated to show the impact of
inter-soliton interaction on the communication, highlighting the limit of the perturbative theory
channel model.

Chapter 4

This chapter is based on [39] which investigates the feasibility of neural network in the detec-
tion of the interacted soliton. The interaction between soliton is first visualised in more detail to
reveal the coupling effect between the intrinsic interaction that occurs in the absences of noise
and the noise-induced Gordon-Haus timing jitter. Then, the AIR of the trained network and
compared with two other model-based detection schemes, direct detection and correlation
based detection. Followed by the hyperparameter analysis, the potential optimisation of the
network parameter and the size of the channel memory are exploited.

Chapter 5

This chapter is based on [40, 41] which focus on estimation of the channel capacity of a CS
modulated NFDM system. Based on the perturbative theory statistics in the literature, we
propose a channel model via a set of signal-dependent decomposition and the VNT. The
performance of the proposed channel model is then compared with the noncentral Gaussian
(NCG) channel with the same statistics. The mutual information for the proposed model is then
optimised by shaping the input constellation geometrically and geometric-probabilistically.
The AIR is estimated with Monte-Carlo mismatch capacity and compared with the unshaped
standard constellations. Lastly, the impact of implementing a matched filter in the NFD is
investigated.

Chapter 6

In this chapter, the thesis will be concluded along with further insight into potential future
research directions.
Chapter 2

Background

In this chapter, a comprehensive description of the physical phenomenon in nonlinear optical fibre is presented. Mathematical and numerical tools are introduced for simulating the physical channel, utilising the nonlinear frequency domain (NFD) spectra, and estimating system performance.

2.1 Optical Fibre Channel

The optics fibre contains two types of material with different refractive indices constructed in a concentric manner as shown in the Figure 2.1. The refractive index of the core material is higher than that of the cladding material, which allow the light to be fully internally reflected at the material interface. Allowing the low loss long distance propagation of light. The figures show an example of multi-mode fibre (MMF) in earlier fibre communication system reviewed in Chapter 1. The number of mode supported by the fibre will be determined by the design of the fibre including the diameter of the core and the difference between the core and cladding refractive indices difference. The fibre which supports only one mode is known as single-mode fibre (SMF). The evolution of the light pulse in fibre is primarily determined by three psychical phenomena. Note that the detailed derivation from the Maxwell equation is out of the scoope of this thesis, the interested readers are prompted to [42] for more detailed information.
2.1. Optical Fibre Channel

Summarising the three physical phenomena that contribute to the distortion of the light pulse, the pulse evolution in nonlinear optical fibre is typically described with the nonlinear Schrödinger equation (NLSE) as [42]

$$\frac{\partial Q(T,Z)}{\partial Z} = -\frac{\alpha}{2} Q(T,Z) - j\beta_2 \frac{\partial^2 Q(T,Z)}{\partial T^2} + j\gamma |Q(T,Z)|^2 Q(T,Z),$$

(2.1)

where $Q(T,Z)$ denotes the complex envelope of the optical field, $T$ and $Z$ are time and propagation distance, and $\alpha$, $\beta_2$ and $\gamma$ indicate fibre loss, group velocity dispersion (GVD) and Kerr nonlinearity, respectively. The physical channel model considered throughout this thesis relies on the NLSE. Hence, it is necessary to discuss the main physical impairments considered in this thesis, as well as the numerical simulation method used for the NLSE channel.

2.1.1 Fibre loss and Optical Amplification

As light pulses propagate within an optical fibre in communication systems, they suffer from power loss, which has been decreased by several orders of magnitude since the first practical fibre demonstration that had a loss lower than 20 dB/km [43]. Fibre loss is introduced due to non-ideal manufacturing processes, resulting in impurities of the silica and uneven distribution of the material’s refractive index. The impurities of silica, such as the OH ion, generate material absorption at specific wavelengths, causing attenuation of the light pulse. If we isolate fibre loss from the NLSE (2.1) with $\beta_2 = 0$ and $\gamma = 0$, the pulse evolution can be described by the differential equation as

$$\frac{\partial Q(T,Z)}{\partial Z} = -\frac{\alpha}{2} Q(T,Z),$$

(2.2)

whose solution suggests the power loss is characterised by the loss factor $\alpha$ as

$$Q(T,Z) = Q(T,0) \exp\left(-\frac{\alpha}{2} Z\right),$$

where $Q(T,Z)$ and $Q(T,0)$ denotes the received and transmitted light pulse in a lossy fibre where only the fibre loss is considered and propagation distance of $Z$. Note that typically, the unit of the loss factor is given in dB/m, the following relationship should be employed to convert to units of $m^{-1}$,

$$\alpha_{\text{dB}} = -\frac{10}{Z} \log\left(\frac{P_{\text{Rx}}}{P_{\text{Tx}}}\right) = 4.343 \alpha,$$

(2.3)
where $\alpha_{\text{dB}}$ denotes the loss factor in dB/m, while $\alpha$ indicates the factor in m$^{-1}$. The ratio between the received power and transmitted power is denoted with $P_{\text{Rx}}/P_{\text{Tx}}$. To reduce the intrinsic loss, attempts have been made to reduce the impurity and Rayleigh scattering. It was reported that the typical impurity ion, the OH ion, could be reduced to less than one part in one hundred million [44]. Additionally, glass with the scattering-induced loss of the level of 0.05 dB/km was also developed [45]. As previously reviewed, 0.2 dB/km loss at 1.55 µm could be achieved with the dominant single mode fibre (SMF) implemented in practice.

Although low intrinsic loss fibre can be achieved with modern technology, fibre loss can still accumulate over transoceanic connections with propagation distances of the order of 1000 km. To achieve such communication reach, optical amplifiers are required to be installed between every fibre span. Optical amplifiers can be categorised into different types based on their operation mechanism. One of the most important optical amplifiers in fibre optics communication is the Erbium-doped fibre amplifier (EDFA) mentioned previously. The EDFA relies on the stimulated emission of a piece of Erbium-doped fibre, the so-called gain medium of the optical amplifier, contained between two Faraday isolators. The Erbium-doped fibre is pumped with an external laser source to excite the Erbium ions so that the input light beam can be amplified with photons emitted when the Erbium ions settle down to the ground state. Another player in optical amplification is the Raman amplifier, which relies on stimulated Rayleigh scattering (SRS). In contrast to EDFA, a long piece of undoped fibre is employed as the amplifier gain medium. Similar to stimulated emission, electrons in undoped fibre will be stimulated with photons from an external laser source. When the electrons de-excite from the active state, photons will be emitted, hence amplifying the input signal.

Other than the intentionally generated stimulated emission photons for amplification purposes, the gain medium also emits photons spontaneously. However, these randomly emitted photons possess random direction and phase, which leads to amplifier spontaneous emission (ASE) noise that can degrade the signal. To account for optical amplification and induced ASE noise, the pulse evolution in an amplified lossy fibre from (2.2) can be rewritten as follows:

$$\frac{\partial Q(T,Z)}{\partial Z} = \left( G(Z) - \frac{\alpha}{2} \right) Q(T,Z) + N(T,Z), \tag{2.4}$$

where $G(Z)$ denotes the amplifier gain coefficient that depends on the type of amplifier used, and $N(T,Z)$ denotes the amplifier spontaneous emission (ASE) noise term, which is well modelled with a circular white Gaussian process [46, 47]. There are two different methods for implementing optical amplification: lumped amplification and distributed amplification. Lumped amplification systems typically use EDFA, while distributed amplification systems typically use Raman amplifiers.
In a periodically lumped EDFA-amplified fibre, dedicated EDFA amplifiers are implemented periodically between each span of the fibre. The typical length of the doped fibre gain medium within the EDFA is of the order of 10 m, with each span of fibre length from 40 km to 120 km [26]. The relatively short gain medium reveals one of the key advantages of EDFA, the low cost. Denoting the length of each fibre span with $L_{sp}$ and the number of spans within a specified fibre link with $N_{sp}$, one could write the autocorrelation of the ASE noise at the receiver end as

$$E[N(T,Z = N_{sp}L_{sp})N^*(T',Z' = N_{sp}L_{sp})] = N_{ASE}^{EDFA} \delta(T - T'),$$

where the power spectrum density (PSD) of the noise is given as

$$N_{ASE}^{EDFA} = N_{sp} h_{Planck} v_{0} n_{sp} (e^\alpha L_{sp} - 1),$$

where the constant $n_{sp} < 1$ denotes the spontaneous factor, $h_{Planck}$ indicates the Planck constant and $v_{0}$ indicates the carrier frequency of the amplified light. The noise will be pumped at each span of the fibre, hence, only the noise statistics at the receiver is produced.

Distributed Raman amplification, as opposed to lumped EDFA amplification, amplifies the signal continuously along the fibre. External laser pumps are installed periodically to support active amplification. The laser pump energy is transferred to amplify the light pulse via several fibre couplers and the SRS process. Generally speaking, distributed Raman amplification could provide superior noise performance to that of lumped EDFA. However, because of the high implementation cost and other practical issues, including laser safety during implementation, it is not as commercialised as its EDFA counterparts [48]. In this thesis, distributed Raman amplification is chosen over lumped EDFA because of its superior noise performance. If the distributed Raman amplifiers are ideally implemented such that the fibre loss is completely compensated, i.e., $G(Z) = \alpha/2$, one could write the autocorrelation of the ASE noise as

$$E[N(T,Z)N^*(T',Z')] = N_{ASE} \delta(T - T') \delta(Z - Z'),$$

where the power spectrum density (PSD) per distance of the noise is given as

$$N_{ASE} = \alpha h v_{0} K_{T},$$

and the $K_{T}$ denotes the phonon occupancy factor.
2.1. Optical Fibre Channel

2.1.2 Chromatic dispersion

Dispersion is a common physical phenomenon that occurs in materials such as glass, and it refers to the dependence of the speed of light on its frequency. An example of this phenomenon is the use of a dispersive prism to separate white light into a spectrum of colours. In this thesis, we will only consider chromatic dispersion that occurs in standard single-polarisation SMF, as intermodal dispersion can occur in MMF and polarisation mode dispersion can occur with dual-polarisation transmission. To characterise chromatic dispersion, the refractive index that depends on the frequency is expanded using a Taylor series around the centre carrier frequency of the light pulse. If the carrier frequency is denoted by $\omega_0$, and the mode-propagation constant $\beta(\omega)$ is formulated with refractive index $n_r(\omega)$, $\beta(\omega)$ can be expanded as

$$\beta(\omega) = n_\nu(\omega) \frac{\omega}{c} = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2} \beta_2(\omega - \omega_0)^2 + \frac{1}{6} \beta_3(\omega - \omega_0)^3 + ..., \tag{2.7}$$

where $c$ denotes the speed of light in a vacuum medium and $\beta_i = \frac{\partial \beta}{\partial \omega}|_{\omega = \omega_0}$ denotes the $i$-th derivative of $\beta(\omega)$ with respect to the angular frequency $\omega$ at carrier frequency $\omega_0$. The $\beta_0$ could be interpreted as a constant phase shift, and the $\beta_1$ indicates the inverse of group velocity of the light pulse, which describes the travelling speed of the envelope of the wave. The $\beta_2$ parameter refers to the GVD, which will introduce pulse broadening effect. Besides the parameter $\beta_2$, the dispersion parameter $D$ is also commonly used in the literature [42,49]. It is related to $\beta_2$ as

$$D = -\frac{2\pi c}{\lambda_0^2} \beta_2,$$

where $\lambda_0$ denotes the carrier wavelength corresponding to the carrier frequency $\omega_0$. In addition to the intrinsic refractive index of the material, the dielectric waveguiding also contributes to GVD [42]. This contribution is known as the waveguide dispersion and is determined by the core radius and the core-cladding index difference of the fiber. Interestingly, dispersion-shifted fiber can be designed to reduce dispersion by driving the zero-dispersion wavelength towards the carrier wavelength with waveguide dispersion [42,50].

The $\beta_3$ and higher order derivative terms correspond to third and higher-order dispersion whose effect will become more significant for the ultrashort light pulse which is less than 5 ps or when the fibre is operated close to zero GVD wavelength [42]. The third and higher order dispersion will be ignored in this thesis.

If a lossless fibre ($\alpha = 0$) with no nonlinearity ($\gamma = 0$) is considered, the pulse evolution described with (2.1) could be written as

$$\frac{\partial Q(T, Z)}{\partial Z} = -j \beta_2 \frac{\partial^2 Q(T, Z)}{\partial T^2}.$$
which could be resolved with the Fourier transform of $Q(T,Z)$. If the Fourier transform of the $Q(T,Z)$ is denoted with $\tilde{Q}(\omega,Z)$, it should be defined as

$$\tilde{Q}(\omega,Z) = \int Q(T,Z) \exp(-j\omega T) dT.$$ 

Performing the Fourier transform on both sides of equation (2.7), the partial differential equation (PDE) is reduced to an ordinary differential equation (ODE) as

$$\frac{d\tilde{Q}(\omega,Z)}{dZ} = j\frac{\beta_2}{2}\omega^2 \tilde{Q}(\omega,Z),$$

whose solution is given as

$$\tilde{Q}(\omega,Z) = \tilde{Q}(\omega,0) \exp \left( j\frac{\beta_2}{2}\omega^2 Z \right).$$

The dispersed envelope at any $Z$ could be calculated via the inverse Fourier transform on the $\tilde{Q}(\omega,Z)$ as

$$Q(T,Z) = \int \tilde{Q}(\omega,0) \exp \left( j\frac{\beta_2}{2}\omega^2 Z + j\omega T \right) d\omega.$$ 

Since the Fourier transform of a Gaussian pulse is also a Gaussian pulse, the Gaussian pulse with a normalising time $T_0$ that controls the pulse width could be formulated as

$$Q(T,0) = \exp \left( -\frac{T^2}{2T_0^2} \right); \quad (2.8)$$

$$\tilde{Q}(\omega,0) = T_0 \sqrt{2\pi} \exp \left( -\frac{T_0^2 \omega^2}{2} \right). \quad (2.9)$$

The spectrum of the evolved pulse is then written as

$$\tilde{Q}(\omega,Z) = T_0 \sqrt{2\pi} \exp \left( -\frac{T_0^2 \omega^2}{2} + j\frac{\beta_2}{2}\omega Z \right).$$

Performing the inverse Fourier transform, the dispersive differential equation is solved for the Gaussian input pulse, and the evolved pulse shape is derived as

$$Q(T,Z) = \frac{T_0}{\sqrt{T_0^2 - j\beta_2 Z}} \exp \left[ -\frac{T^2}{2(T_0^2 - j\beta_2 Z)} \right]. \quad (2.10)$$

It could be observed that the dispersed pulse will still preserve the Gaussian shape described by the functionality of the pulse, while the pulse width is expanded. In the literature, the pulse broadening effect is modelled by the pulse broadening factor defined as the ratio between the full-width at half maximum (FWHM) of the input pulse and output pulse. For the Gaussian input
and output pulse discussed in this example, the FWHM widths could be calculated simply as

\[ BF = \frac{T_{\text{out \ FWHM}}}{T_{\text{in \ FWHM}}} = \frac{2\sqrt{2(1+(Z/L_D)^2)\ln 2}T_0}{2\sqrt{2\ln 2}T_0} = \sqrt{1 + \left(\frac{Z}{L_D}\right)^2}, \]

where the FWHM width is taken at the point that the amplitude of the Gaussian pulse becomes 1/2 of the maximum, \( BF \) denotes the broadening factor and \( L_D = T_0^2/|\beta_2| \) is defined as the dispersion length as commonly seen in the literature. The dispersion length provides a distance measurement of the strength of GVD-induced pulse broadening. It is interpreted as the length when a unchirped Gaussian pulse is input into the dispersive medium, the pulse width is broadened with a factor of \( \sqrt{2} \). The frequency chirping could also be observed from the spectrum of the dispersed Gaussian pulse, this will not be discussed in detail within this thesis.

As revised in the previous chapter, dispersion can be mitigated with dispersion-shifted fibre that shifts the zero-dispersion wavelength of the fibre to the operating wavelength. The zero-dispersion wavelength of conventional fibre is typically around 1.3 \( \mu \text{m} \), while the optical amplifier performance at 1.5 \( \mu \text{m} \) is much more promising. Using dispersion-shifted fibre, the zero-dispersion wavelength is shifted to 1.5 \( \mu \text{m} \), enabling low dispersion and better optical amplifier performance. In a dispersion-shifted fibre, GVD dispersion can be neglected, but other impairments, including higher-order dispersion and nonlinearity, will become more significant [48]. Dispersion can also be compensated by employing consecutive segments of fibres with opposite dispersion, known as dispersion compensation fibre. However, this technique requires higher infrastructure cost as it involves implementing two types of fibre with opposite dispersion in one link. Overall, a more cost-effective dispersion compensating method became available since the invention of coherent receiver. The coherent receiver allows both the amplitude and phase of the light pulse to be detected, enabling the dispersion to be compensated with modern digital signal processing (DSP) techniques. Note that when nonlinearity, which occurs simultaneously with dispersion, becomes non-negligible, signal-dependent nonlinearity will make the equalisation of the channel effect intractable with conventional DSP schemes.

### 2.1.3 Kerr nonlinearity

In the previous section, the causes and effects of the dominant linear impairment, GVD, are reviewed. It has been accepted that chromatic dispersion is induced by the frequency-dependent refractive index. As a matter of fact, the refractive index is not only dependent on the carrier wavelength but also dependent on the signal intensity [42]. A more complete approximation of the refractive index is written as

\[ \tilde{n}_r(\omega, E) = n_r(\omega) + \tilde{n}_2|E|^2, \]
where \( n_r(\omega) \) describes the frequency-dependent part of the index, and \( \tilde{n}_2|E|^2 \) implies the signal dependence with \( E \) as a slow-varying signal function of time. Assuming the electromagnetic wave \( E \) is linearly polarised, the nonlinear-index coefficient \( \tilde{n}_2 \) is determined by only one component of the third order susceptibility, which is a forth-rank tensor. The contribution of the signal dependent part of the refractive index is included in the pulse evolution PDE via perturbation theory analysis. The kerr nonlinearity parameter \( \gamma \) is hence defined to incorporate the nonlinear-index coefficient as

\[
\gamma(\omega_0) = \frac{\omega_0 \tilde{n}_2}{c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^4 \, dx \, dy
\]

where \( F(x,y) \) denotes the modal distribution of the fibre which is determined by fibre parameters such as geometric dimensions and the core-cladding index difference. The last term containing \( \gamma \) in (2.1) only refers to self-phase modulation (SPM). Note that higher order nonlinearity described by the \( \gamma(\omega_0) \) at a given carrier frequency \( \omega_0 \) includes cross-phase modulation (XPM), SRS and stimulated Brillouin scattering (SBS). When a pulse width greater than 1 ps is employed, higher-order nonlinearity could be ignored similar to the higher order dispersion [42].

Similar to the analysis conducted in previous section, one could isolate the SPM effect by assuming a lossless and zero dispersion fibre. The NLSE (2.1) will be reduced to

\[
\frac{\partial Q(T,Z)}{\partial Z} = j\gamma|Q(T,Z)|^2 Q(T,Z).
\]

The general solution to the (2.11) could be found by rewriting \( Q(T,Z) \) in terms of its amplitude \( V(T,Z) \) and phase \( \phi(T,Z) \) similar to [1]. Using the chain rule, the PDE (2.11) is rewritten as

\[
\frac{\partial V}{\partial Z} + jV \frac{\partial \phi}{\partial Z} = j\gamma V^3,
\]

which is not difficult to resolve within time domain by considering the real and imaginary parts of both sides of the equation being equal as

\[
\frac{\partial V}{\partial Z} = 0; \quad \frac{\partial \phi}{\partial Z} = \gamma V^2.
\]

Under SPM, the amplitude of the pulse will remain invariant, while the phase will evolve in a signal-dependent manner with the signal amplitude \( V(T,Z) \) showing up within its derivative, resulting in the so-called "self phase modulation". If the initial pulse shape is denoted with \( Q(T,0) = V(T,0) \exp[j\phi(T,0)] \), the solution of amplitude \( V(T,Z) \) and phase \( \phi(T,Z) \) could be determined as

\[
V(T,Z) = V(T,0); \quad \phi(T,Z) = \gamma|Q(T,0)|^2 Z + \phi(T,0).
\]
The complex SPM-distorted pulse could be obtained by combining the amplitude and phase functions as

\[ Q(T, Z) = Q(T, 0) \exp \left( j \gamma |Q(T, 0)|^2 Z \right). \] (2.12)

The solution (2.12) suggests that during propagation, the signal spectrum varies in a signal-dependent manner, as the initial signal level is incorporated into the frequency term when a Fourier transform is performed, resulting in a frequency shift that depends on the initial signal itself. It is counter-intuitive that the pulse amplitude remains unchanged, as suggested by the partial differentiation, as opposed to the pulse expansion demonstrated in the GVD dispersed pulse.

In conventional fibre optic systems, nonlinearity has not been the main impairment of the link due to the relatively low signal power. However, when considering nonlinearity as an impairment, various mitigation techniques have been developed to address both nonlinearity and the GVD that occurs simultaneously are reviewed in [51]. For example, digital back-propagation (DBP) is considered to be a straightforward technique to invert the channel effect by numerically reversing the deterministic part of the pulse evolution to estimate the noisy version of the transmitted pulse. Another method for compensating for nonlinearity is the Volterra series. The Volterra series approximates the memory effect of a nonlinear channel with a high-order polynomial expansion. It is reported that by expanding the frequency domain of NLSE with a third order nonlinear transfer function, the inverse Volterra series transfer function (IVSTF) based nonlinearity compensation could outperform the split-step Fourier method based DBP in a single-polarisation optical transmission in some specific scenarios [52,53].

2.1.4 Stochastic NLSE and Normalisation

In the previous sections, three categories of signal impairments within a single polarisation standard SMF were reviewed and connected to the terms in the lossy NLSE (2.1). The differential equation corresponding to each individual impairment was solved analytically. Note that for the validity of the NLSE (2.1), certain slow-varying assumptions are made, such as the signal bandwidth being smaller than the carrier frequency, allowing the propagation of the entire signal to be modelled with a similar level of dispersion and nonlinearity. Additionally, the signal pulse width is assumed to be larger than 5 ps, and the signal power is constrained so that higher-order dispersion and higher-order nonlinearity could be ignored. Assuming that ideal distributed Raman amplification perfectly compensates for fibre loss, the stochastic NLSE with the ASE noise term is given as

\[ \frac{\partial Q(T, Z)}{\partial Z} = -j \beta_2 \frac{\partial^2 Q(T, Z)}{\partial T^2} + j \gamma |Q(T, Z)|^2 Q(T, Z) + N(T, Z), \] (2.13)
where \( Q(T,Z) \) denotes the complex envelope of the optical field, \( N(T,Z) \) represents the ASE noise term, \( T \) and \( Z \) are time and propagation distance with units of s and m, and \( \beta_2 \) and \( \gamma \) indicate GVD and Kerr nonlinearity, respectively. The fibre loss term \( \alpha \) is omitted since ideal distributed Raman amplification is assumed. The ASE noise term, as highlighted previously, could be modelled as a zero mean white Gaussian noise with autocorrelation (2.5) and spectral density of the noise in \( W/(m \cdot Hz) \) as (2.6) for the ideal distributed Raman amplification assumed in this thesis.

For ease of analysis and to align with the established literature \([54, 55]\), the NLSE could be transformed into a normalised form as

\[
j \frac{\partial q(t,z)}{\partial z} = \frac{\partial^2 q(t,z)}{\partial t^2} + 2|q(t,z)|^2 q(t,z) + n(t,z),
\]

with the corresponding normalised parameters as

\[
q = \sqrt{\gamma L_D} Q, \quad z = Z/2L_D, \quad t = T/T_0,
\]

where the dispersion length is defined as \( L_D = T_0^2/|\beta_2| \) as mentioned previously, and the normalising time \( T_0 \) can be selected independent of other parameters. However, it should be noted that \( T_0 \) is selected with respect to the target physical pulse width as it is typically taken into account in the signalling. Consequently, the autocorrelation of the normalised noise is given by

\[
E[n(t,z)n^*(t',z')] = \sigma^2 \delta(t-t') \delta(z-z'),
\]

where \( \sigma^2 = N_{ASE} \frac{2\gamma L_D}{T_0} \) according to the normalisation scheme (2.15). It is worth emphasising that the NLSE describes the pulse evolution for a single polarisation system. When dual polarisation is employed, the Manakov equation should be used instead \([56, 57]\).

### 2.1.5 Numerical Simulation of Fibre

The NLSE (2.13) is a nonlinear PDE that cannot be solved analytically when both linear GVD and nonlinear SPM occur simultaneously during propagation. There are two main categories of numerical methods proposed to solve the NLSE: finite-difference and pseudospectral methods. However, the most commonly used method to date is the split-step Fourier method. This numerical method is based on the approximation that dispersion and nonlinearity can be treated sequentially rather than simultaneously when the fibre is segmented into sufficiently small segments. The split-step Fourier method works by rewriting the NLSE (2.13) using the linear operator \( \hat{D} \) and the nonlinear operator \( \hat{N} \), such that

\[
\frac{\partial \hat{Q}}{\partial Z} = (\hat{D} + \hat{N})\hat{Q} + \hat{N};
\]
where the linear operator and nonlinear operator are defined as

\[
\hat{D} = -j\beta_2 \frac{\partial^2}{\partial T^2}, \\
\hat{N} = j\gamma |Q(T,Z)|^2.
\]

The dispersion, described by the linear operator \( \hat{D} \), and the nonlinearity, described by the signal dependent operator \( \hat{N} \), act on the pulse simultaneously. The split-step Fourier method, as suggested by its name, splits the fibre into a number of sufficiently short segments with a step size \( h_S \). Within each sufficiently small step of size \( h_S \), it is assumed that the pulse evolves through a \( h_S \)-long purely nonlinear fibre with SPM, followed by a \( h_S \)-long purely dispersive fibre, which acts on the SPM-perturbed pulse from the previous step.

The philosophy of the split-step Fourier method is illustrated in Figure 2.2, where \( Q_D(T) \), \( Q_{D1}(T) \), \( Q_{D2}(T) \), and \( Q_N(T) \) are the intermediate time domain pulses obtained during the split-step Fourier method. The branch (b) in Figure 2.2 corresponds to the original split-step Fourier method discussed earlier. As demonstrated in previous sections, differential equations with only linear operators are easier to solve in the frequency domain, while those with only nonlinear operators are easier to solve in the time domain. This is why the method is called the “split-step Fourier method” as it involves transitioning between the time domain and Fourier domain at each step.
More specifically, the exact solution to the rewritten NLSE within a single step $h_S$ could be obtained as

$$Q(T, Z + h_S) = \exp\left(h_S(\hat{D} + \hat{N})\right)Q(T, Z) + N,$$

where the ASE noise term $N$ should correspond to a white Gaussian random process with a PSD of $N_{ASE}h_S$. If the original split-step approximation of is employed, the solution could be approximated as

$$Q(T, Z + h_S) \approx \exp\left(h_S \hat{D}\right)Q_N + N,$$

where the intermediate time domain pulse $Q_N$ corresponds to the output of the nonlinear operator $\hat{N}$ alone, without the presence of the linear operator

$$Q_N(T) = \exp\left(h_S \hat{N}\right)Q(T, Z).$$

Substitute in the form of the nonlinear operator, the input-output relationship is derived as

$$Q_N(T) = \exp(h_S j\gamma|Q(T, Z)|^2)Q(T, Z).$$

The linear operator contains a second derivative with respect to time, which makes it more convenient to resolve in the frequency domain. By taking the Fourier transform of the second derivative, the $\frac{\partial^2}{\partial T^2}$ is replaced with $(j\omega)^2$. This leads to the derivation of the input-output relationship in the frequency domain as

$$\tilde{Q}_D(\omega) = \tilde{Q}_N(\omega) \exp\left(j\frac{\beta_2}{2} \omega^2 h_S\right),$$

where $\tilde{Q}_D(\omega)$ and $\tilde{Q}_N(\omega)$ denote the Fourier transforms of the time domain pulses $Q_D$ and $Q_N$. To obtain the approximated output, the ASE noise term $\hat{N}$ is added after taking the inverse Fourier transform of $\tilde{Q}_D(\omega)$.

The variant of the split-step Fourier method described in branch (c) of Figure 2.2 above has been shown to provide more accurate results than the original method [58]. The computation of pulse evolution in each step is split into three parts, with a symmetrically placed linear step with half the step size before and after the nonlinear step. This ensures that the approximation of the pulse is more accurate than in the original method. The pulse evolution in each step could be computed as

$$\tilde{Q}_{D1}(\omega) = \tilde{Q}(\omega, Z) \exp\left(j\frac{\beta_2}{2} \omega^2 h_S\right);$$

$$Q_N(T) = \exp(h_S j\gamma|Q_{D1}(T)|^2)Q_{D1}(T);$$

$$\tilde{Q}_{D2}(\omega) = \tilde{Q}_N(\omega) \exp\left(j\frac{\beta_2}{2} \omega^2 h_S\right);$$

$$Q(T, Z + h_S) = Q_{D2}(T) + N,$$
Algorithm 1 Split-step Fourier Simulation

1: Compute the number of steps, \( N_{\text{step}} = \lceil L/h_S \rceil \)
2: Compute the remainder of \( L/h_S \)
3: if remainder of \( L/h_S \neq 0 \) then
   4: \( h_R = \) remainder
5: else if remainder of \( L/h_S = 0 \) then
6: \( h_R = h_S \)
7: end if
8: \( i \leftarrow 1; l \leftarrow 0 \) ▷ Initialise the step counter
9: while \( i \leq N_{\text{step}} \) do
10: if \( i < N_{\text{step}} \) then
11: \( h \leftarrow h_S \)
12: else if \( i = N_{\text{step}} \) then
13: \( h \leftarrow h_R \)
14: end if
15: \( \tilde{Q}(\omega, l) = \text{FFT} \left[ Q(T, l) \right] \)
16: \( \tilde{Q}_{D1}(\omega) = \tilde{Q}(\omega, l) \exp \left( j \frac{\beta_2}{2} \omega^2 l \right) \)
17: \( Q_{D1}(T) = \text{IFFT} \left[ \tilde{Q}_{D1}(\omega) \right] \)
18: \( Q_N(T) = \exp (h_S j \gamma |Q_{D1}(T)|^2) Q_{D1}(T) \)
19: \( Q_N(\omega) = \text{FFT} \left[ Q_N(T) \right] \)
20: \( \tilde{Q}_{D2}(\omega) = \tilde{Q}_N(\omega, l) \exp \left( j \frac{\beta_2}{2} \omega^2 l \right) \)
21: \( Q(T, l + h) = \text{IFFT} \left[ \tilde{Q}_{D2}(\omega) \right] + N \)
22: \( l = l + h \)
23: end while

where \( \tilde{U}(\omega) \) is used to denote the Fourier transform of the time domain pulse \( U(T) \). The split-step Fourier method is computationally efficient compared to other methods such as the finite difference method, mainly due to the use of the fast Fourier transform algorithm [59]. The split-step Fourier method could be implemented as summarised in Algorithm 1. It should be noted that the current loop is dependent on the previous loop, so parallel loops cannot be employed to improve efficiency. To acquire a reliable estimation of the channel, a large number of channel realisations should be simulated. To utilise all available memory resources, multiple channel realisations are packed within one large matrix, and an automatic batch divider is implemented to divide the task into smaller sizes. The detailed MATLAB code for implementing the split-step Fourier method is given in Appendix A.

To demonstrate the result of the algorithm, an unchirped Gaussian pulse with \( T_0 = 0.1 \) ns as in (2.8) is transmitted in a 2000 km noise-free and lossless optical fibre link. The pulse evolution is first simulated in a pure dispersive fibre with the split-step Fourier method described in Algorithm 1. The amplitude of the simulated received pulse is presented along with the analytical solution obtained from (2.10) in Figure 2.3a. Then, the pulse evolution in a pure
2.1. Optical Fibre Channel

Figure 2.3: Example of transmitting an unchirped Gaussian pulse (2.8) with $T_0 = 0.1$ ns (Tx) in a (a) GVD perturbed and (b) SPM perturbed, noiseless, and lossless fibre. The split-step Fourier method simulations (Num. Rx) and analytical solutions (Ana. Rx) for (a) (2.10) and (b) (2.12) are shown in the (a) amplitude and (b) phase of the time domain pulse shape, respectively.

Figure 2.4: Example of transmitting an unchirped Gaussian pulse (2.8) with $T_0 = 0.1$ ns (Tx) in a 2000 km, NLSE described, noise-free and lossless nonlinear fibre with both GVD and SPM. (a) The amplitude of transmitted (Tx) and numerically simulated received (Num. Rx) time domain pulse; (b) the simulated pulse evolution throughout the propagation.

Another example is shown in Figure 2.4, where an unchirped Gaussian pulse with $T_0 = 0.1$ ns, as defined in Equation (2.8), is launched into a 2000 km, noise-free, and lossless nonlinear fibre described by the NLSE, which includes both linear GVD and nonlinear SPM effects. The resulting complicated distortion can be observed in both the amplitude of the time domain pulse in Figure 2.4 and the pulse evolution depicted in Figure 2.4b. It should be noted that
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these simulations assume a carrier wavelength of 1550 nm. The dispersive fibre is assumed to have a $\beta_2$ value of $-2.1 \times 10^{-23}$ s$^2$/km and a $\gamma$ value of 0 /W/km, while the nonlinear fibre has a $\beta_2$ value of 0 s$^2$/km and a $\gamma$ value of 1.27 /W/km. The NLSE-described nonlinear fibre is assumed to have a $\beta_2$ value of $-2.1 \times 10^{-23}$ s$^2$/km and a $\gamma$ value of 1.27 /W/km.

2.2 Nonlinear Fourier Transform

In this section, first, a brief background is given about the predecessor of the NFT, inverse scattering method, to establish the philosophy of solving the nonlinear PDEs with scattering data. Then, some basic information and properties of the NFT are discussed. Lastly, the data transmission method using the NFT is introduced, followed by a short overview of potential obstacles.

2.2.1 Overview and Motivation

In the previous sections, it could be observed that the deterministic pulse evolution becomes intractably complicated when linear dispersion effect is coupled with nonlinear SPM effect. However, there exists a special family of optical pulses known as solitons which could preserve its shape or evolve periodically with the interplay between GVD and SPM.

Solitons are commonly observed in media with nonlinear dynamics, such as the Fermi-Pasta-Ulam lattice dynamics, which could be modelled by a nonlinear PDE known as the Korteweg-deVries (KdV) equation [60, 61]. It was observed that when two of such pulses are launched towards each other, after a complicated nonlinear interaction when they collide, they could recover their original shape [61]. The mathematical explanation for the soliton solution to the KdV equation was not fully understood until it was discovered that there exists an invariant eigenvalue of the Schrödinger operator [62]. It was demonstrated that the eigenvalues could be obtained by substituting the soliton solution of the KdV equation into a Linear Schrödinger equation, suggesting a possible connection between the invariant soliton solution and the invariant eigenvalue [34, 62]. At the time, it was not clear whether the existence of the soliton could be extended to other nonlinear PDEs until the contribution by the mathematician Lax. He proposed the mathematical concept Lax pair, which consists of an auxiliary operator (for example, the Schrödinger operator for KdV equation) eigenvalue pair, corresponding to integrable nonlinear PDEs [63]. If the Lax pair of a nonlinear PDE could be identified, the PDE could be solved in a similar manner as solving the KdV equation [64].
Similar to the KdV equation, solitons are also a family of solutions to the NLSE as it also belongs to the family of integrable nonlinear PDE. The fundamental soliton containing a sech pulse is believed to be the simplest form of the solitonic solution to the NLSE [42]. The fundamental soliton solution to the unitless NLSE (2.14) is given as [42]

\[ q(t, z) = 2\eta e^{-2j\zeta t + 4j(\zeta^2 - \eta^2)z - j(\psi + \pi/2)} sech(2\eta t - 8\eta \zeta z - 2\Upsilon), \]  

(2.17)

where the only eigenvalue is \( \lambda_1 = \zeta + j\eta \) (\( \eta > 0 \)) and \( C_1 = 2\eta e^{2\Upsilon} = |C_1| e^{j\psi} \) is known as norming constant or spectral amplitude, which corresponds to the eigenvalue \( \lambda_1 \). Assuming the soliton pulse is centred at \( t = 0 \) at the transmitter, i.e. \( \psi = -\pi/2 \) and \( \Upsilon = 0 \), the soliton at the beginning of the fibre, \( z = 0 \), could be written as

\[ q(t, 0) = 2A sech(2At) \]  

(2.18)

where \( A = \eta \) indicates the amplitude of the fundamental soliton pulse.

De-normalising the soliton pulse with (2.15) and setting the soliton amplitude \( A = 1 \), an example fundamental soliton pulse is launched into a noiseless nonlinear optical fibre used in previous example described in Figure 2.4a. The pulse evolution over the fibre is shown in Figure 2.5. As opposed to the intractable nonlinear evolution of the unchirped Gaussian pulse, the pulse shape of the soliton remains invariant as predicted by the invariant eigenvalue.
2.2. Nonlinear Fourier Transform

2.2.2 Nonlinear Fourier Transform and NLSE

The inverse scattering transform was originally proposed to solve nonlinear PDE with specific integrable boundary conditions, such as the KdV equations [62, 64, 65]. Similarly, the NLSE, which describes pulse evolution, has similar integrable properties. Therefore, the NLSE could be solved in a similar way, revealing the existence of solitons. In [66], solitons were proposed as information carriers in communication systems. It was highlighted that the imaginary parts (the amplitude of the soliton) of the eigenvalues could be used to encode and decode information, which could mitigate the effects of amplifier noise and fibre Raman effect by discarding the real parts. However, the amplifier noise introduces random timing jitter during the propagation of the soliton, known as the Gordon-Haus effect, which limits the data rate of soliton communication [67]. Additionally, solitons exert interaction forces between each other when they are temporally close, especially when they are transmitted in sequence within a communication link [68]. When combined with the Gordon-Haus effect, the data rate limit is exacerbated because the random timing jitter increases the chance that neighbouring solitons coincide. Due to the data rate limit and the development of the aforementioned techniques, soliton communication remained quiet until the proposition of NFT.

In 2014, a series of works [34–36] reviewed the mathematical tools and numerical implementation of NFT. By applying NFT, the time-domain pulse can be decomposed into two types of spectra in NFD, where the spectra evolution is linearised. These two spectra are the discrete spectrum (DS) and the continuous spectrum (CS). The DS consists of eigenvalues and their corresponding norming constants, while the complex CS can be thought of as analogous to the frequency spectrum obtained from linear Fourier transform (FT). The DS part of NFD corresponds to the solitonic part of the signal, where a single soliton would remain invariant as a result of perfect balance between linear and nonlinear impairments, and multiple solitons would evolve periodically. On the other hand, the CS portion of NFD corresponds to the nonsolitonic radiation of the signal, suggesting that the corresponding time-domain pulse would suffer from both linear and nonlinear impairments during its evolution. The two spectra construct the two categories of degrees of freedom where information can be modulated in a nonlinear frequency division multiplexed (NFDM) system [37].

In order to reveal the new degrees of freedom, NFD, it is necessary to identify the Lax pair for the NLSE (2.14). Zakharov and Shabat identified the Lax pair for the NLSE [69], which was later further developed by Ablowitz and others [70]. The resulting scheme is referred to as the "Nonlinear Fourier Transform" (NFT). This transformation is analogous to the Fourier Transform, as both involve solving differential equations by diagonalising them. In Fourier Transform, an ODE is transformed into the Fourier domain, where the differential equation
2.2. Nonlinear Fourier Transform

can be diagonalised. The solution is then obtained by taking the inverse Fourier transform of the evolved Fourier spectrum. Similarly, when solving integrable nonlinear PDEs, the NFT is used to linearise the equation in the transformed domain, and the solution is obtained by taking the inverse NFT (INFT).

To simplify the identification of the Lax pair for the nonlinear PDE, the NLSE (2.14) can be rewritten in the form of an operator $\hat{K}$ by setting the noise term $n$ to zero. The resulting equation is given by

$$ \frac{\partial q}{\partial z} = \hat{K}q, \quad (2.19) $$

where the signal $q$ and propagation distance $z$ dependent operator $\hat{K}$ is expressed as

$$ \hat{K} = -j \frac{\partial^2}{\partial t^2} - 2j|q(t,z)|^2. \quad (2.20) $$

The Lax pair for the NLSE was discovered by Zakharov and Shabat and it is given as [71]

$$ L = j \begin{bmatrix} \frac{\partial}{\partial t} & q \\ -q^* & -\frac{\partial}{\partial t} \end{bmatrix}; \quad (2.21) $$

$$ M = \begin{bmatrix} 2j\lambda^2 - j|q|^2 & -2\lambda q - \frac{\partial q}{\partial t} \\ 2\lambda^*q - \frac{\partial q}{\partial t} & -2j\lambda^2 + j|q|^2 \end{bmatrix}, \quad (2.22) $$

where $\lambda$ is introduced to denote the spectral parameter, which is analogous to the frequency in Fourier transform and is typically interpreted as the nonlinear frequency in the literature. In addition, $\lambda$ is considered invariant during the evolution over $z$ when it takes the eigenvalues.

The Lax pair consists of two operators $L$ and $M$ which correspond to two problems. The operator $L$ appears in the formulation of the eigenvalue problem

$$ Lv = \lambda v, \quad (2.23) $$

which is essentially equivalent to an eigenvalue decomposition problem of the operator. The eigenvalue of the operator is denoted with $\lambda$ while the eigenfunction (the scattering data) is denoted with $v$ defined as

$$ v(t,\lambda) = \begin{bmatrix} v_1(t,\lambda) \\ v_2(t,\lambda) \end{bmatrix}. $$

The evolution of the scattering data is then characterised by the second operator in Lax pair, the $M$ operator, as

$$ \frac{dv}{dz} = Mv. \quad (2.24) $$
2.2. Nonlinear Fourier Transform

Employing the features described above, the relationship between the diagonalisable operator \( L \) and operator \( M \) could be derived as

\[
\frac{dL}{dz} = ML - LM, \tag{2.25}
\]

by differentiating equation (2.23) with respect to \( z \) and using the spatial invariant eigenvalue \( \frac{d\lambda}{dz} = 0 \) and equation (2.24). Substituting the resulting equation into (2.23) completes the derivation.

The Zakharov Shabat problem could be expressed by rewriting equation (2.23) with a more commonly used variant of the \( L \) operator, the \( P \) operator, as

\[
\frac{\partial v(t, \lambda)}{\partial t} = P v(t, \lambda) = \begin{bmatrix} -j\lambda & q(t, z) \\ -q^*(t, z) & j\lambda \end{bmatrix} v(t, \lambda), \tag{2.26}
\]

where the operator \( P \) is obtained from \( L \) using the following relationship,

\[
P = \text{diag}(-j, j)(L - \lambda I) + \frac{\partial}{\partial t} I.
\]

In addition, a boundary condition must be assumed to solve the problem. In this work, the vanishing boundary condition is assumed, where the time-domain pulse \( q(t) \) approaches zero as \( |t| \to \infty \). Another commonly used boundary condition is the periodic boundary condition, where \( q(t) \) is assumed to be periodically repeating itself with a predefined, sufficiently large period. The periodic boundary condition results in the periodic NFT, which is beyond the scope of this thesis. Unless specified otherwise, the NFT with the vanishing boundary condition is assumed in this thesis. Readers interested in the periodic NFT are referred to relevant literature such as [72, 73].

If the Zakharov Shabat eigen problem is resolved under the vanishing boundary condition, the two Jost coefficients \( a(\lambda) \) and \( b(\lambda) \) can be obtained from the solution of the eigenfunction \( v(t, \lambda) \). However, when the \( L^1 \) norm of the time domain pulse \( ||q||_{L^1} \) is greater than \( \pi/2 \), \( a(\lambda) \) can have zero(s) on the upper half complex plane \( \mathbb{C}^+ \), where \( \text{Im}(\lambda) > 0 \) [34]. If \( a(\lambda) \) has \( M \) zeros, then the zeros \( \lambda_m \in \mathbb{C}^+, m = 1, 2, ..., M \) correspond to the eigenvalues of the time domain pulses, and the ratios

\[
C_m = \frac{b(\lambda_m)}{a'(\lambda_m)}, \quad m = 1, 2, ..., M,
\]
indicate the norming constant, which is sometimes referred to as spectral amplitudes, reflection coefficient or residuals. Here, $a'(\lambda)$ denotes the first derivative of the Jost coefficient $a(\lambda)$ with respect to $\lambda$ at the eigenvalue $\lambda = \lambda_m$. The complete set of the eigenvalues and their respective norming constants $\lambda_m, C_m, m = 1, 2, ..., M$ make up the discrete spectrum (DS) of the time domain pulse $q$. Mapping to the eigenvalue, the DS corresponds to the solitonic components in the pulse, while the norming constant could indicate the temporal position of the solitons [34].

The non-solitonic radiation is decomposed onto the CS. The spectrum amplitude of the CS is defined as

$$\rho(\lambda) = \frac{b(\lambda)}{a(\lambda)},$$

where the denominator is $a(\lambda)$ instead of $a'(\lambda_m)$ in the DS norming constant. The relationship between the DS and CS becomes more clear that the undefined ratio $b(\lambda_m)/a(\lambda_m)$ when $a(\lambda_m) = 0$ will result in the eigenvalues in DS. Furthermore, it is also worth noting that there is a unity constraint between $a(\lambda)$ and $b(\lambda)$ that

$$|a(\lambda)|^2 + |b(\lambda)|^2 = 1. \quad (2.27)$$

This unity constraint is the source of the energy barrier issue in $b$-modulated systems, which are a variant of CS-modulated systems [74].

In this thesis, the efficient and reliable fast NFT/INFT algorithm developed by Sander and et al. will be employed to perform the transformation between time domain and NFD [75]. However, the details of implementing NFT/INFT are beyond the scope of this thesis. Interested readers are encouraged to refer to the award-winning paper series on the NFT [34–36] for more information.

Following the philosophy of solving nonlinear PDE using NFT, the evolution described by the PDE should be converted to the NFD domain as well. It is shown that the pulse evolution described by the noiseless NLSE is linearised into a propagation length determined phasor in the Jost coefficients as [34]

$$a(\lambda, z) = a(\lambda, 0); \quad (2.28)$$
$$b(\lambda, z) = b(\lambda, 0) \exp(4j\lambda^2 z). \quad (2.29)$$

Consequently, the evolution of CS and DS are derived as

$$\rho(\lambda, z) = \rho(\lambda, 0) \exp(4j\lambda^2 z); \quad (2.30)$$
$$C_m(\lambda_m, z) = C_m(\lambda_m, 0) \exp(4j\lambda_m^2 z). \quad (2.31)$$
2.2. Nonlinear Fourier Transform

Figure 2.6: (a) Time domain pulse amplitude and (b) NFD spectral amplitude of a propagation example. The transmitted time domain pulse (Tx) corresponding to 128-subcarrier sinc pulse shaping CS modulated signal with a single eigenvalue $\lambda_1 = 1j$ and norming constant $C_1 = -2$. The equalised received pulse (Eq. Rx) is obtained from performing INFT of the NFD equalised received pulse (Rx).

where the eigenvalues $\lambda_m$ remain invariant \(^1\). The effectiveness of this is demonstrated with an example of transmitting time domain pulse corresponding to a 128-subcarrier sinc pulse shaping CS modulated signal with a single eigenvalue of $\lambda_1 = 1j$ and a norming constant of $C_1 = -2$ in Figure 2.6a. The time domain pulse is launched into the same noiseless fibre employed in the last section, and the received pulse is equalised in CS and DS with $\exp(-4j\lambda_1^2 z)$ and $\exp(-4j\lambda_2^2 z)$ respectively. It could be observed that the time domain pulse could be equalised perfectly despite the intractable nonlinear evolution, suggesting the channel evolution could be well modelled by the phasor in NFD. In addition, the spectral amplitude of CS and DS remain unaltered with the invariant eigenvalue after propagating through the nonlinear NLSE channel as shown in Figure 2.6b.

In addition to the linearisation of the nonlinear PDE, NFT also satisfies the Parseval’s identity, similar to the Fourier transform, as \([34]\)

$$
\int_{-\infty}^{\infty} |q(t)|^2 dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \log(1 + |\rho(\lambda)|^2) d\lambda + 4 \sum_{m=1}^{M} \text{Im}(\lambda_m), \tag{2.32}
$$

where the first term on the right hand side (RHS) denotes the amount of nonsolitonic energy radiation, and the second term denotes the amount of solitonic energy radiation. Note the constant $1/\pi$ in the computation of the CS energy. This is also implied in the approximation relationship between CS spectrum and Fourier spectrum. It is suggested that when the $L_1$ norm of the signal $q$ is sufficiently small, i.e. $||q||_{L_1} \ll 1$, the CS is approximately equal to the

\(^1\) The sign in the phasor might be different based on the selection of the evolution direction and other formulation such as the ordering of the eigenvalues’ and eigenfunctions’ elements. The phasor presented in this works corresponds to the NFT/INFT formulation in \([75]\).
2.2. Nonlinear Fourier Transform

Fourier transform of the negative conjugate of the signal \( q \), as [34]

\[
\rho(\lambda) \approx \hat{Q}(\lambda) = \int_{-\infty}^{\infty} q^*(t) \exp(-2j\lambda t) dt,
\]

(2.33)

where the relationship between the nonlinear frequency and angular frequency is defined as \( \lambda = \pi \omega \). Combining this property and the approximation that \( \log(1 + x) \approx x \) for small \( x \), one could identify the equivalence between Parseval’s identity of NFT and the Fourier transform.

Although some similar properties are shared between the NFT and the Fourier transform, it should be emphasised that NFT possesses different properties. Unlike the Fourier transform, NFT only has weak linearity. If \( |a| \ll 1 \) and the NFD spectrum of \( q(t) \) is given by \( \{ \rho(\lambda) \} \), the NFT of \( aq(t) \) is approximately given by \( \{ a\rho(\lambda) \} \) [34]. However, if the NFD spectrum of \( q(t) \) is given by \( \{ \rho(\lambda), C_m(\lambda_m), m = 1, 2, ..., M \} \), the NFT of \( aq(t) \) is generally not given by \( \{ a\rho(\lambda), aC_m(\lambda_m), m = 1, 2, ..., M \} \). Furthermore, unlike the complete linearity of the Fourier transform, the weak linearity of NFT does not include additivity. Due to the absence of additivity and incomplete homogeneity, this property is named weak linearity.

Other properties worth noting include the time shift property and frequency shift property [34]. If one denotes the NFD spectrum of a given time domain pulse \( q(t) \) with \( \{ \rho(\lambda), C_m(\lambda_m), m = 1, 2, ..., M \} \), then the NFD spectrum of the time-shifted pulse \( q(t - t_0) \) and frequency-shifted pulse \( q(t)e^{-2j\omega t} \) is given by \( e^{-2j\lambda t_0}\{ \rho(\lambda), C_m(\lambda_m), m = 1, 2, ..., M \} \) and \( \{ \rho(\lambda - \omega), C_m(\lambda_m - \omega), m = 1, 2, ..., M \} \), respectively.

2.2.3 Nonlinear Frequency Division Multiplexed System

The NFD spectrum of the signal will be well preserved, as the channel effect could be compensated linearly with the phasor. This idea could be inherited in establishing communication over nonlinear but integrable channel like NLSE described optical fibre. Using orthogonal frequency division multiplexing (OFDM) as an analogy, nonlinear frequency division multiplexing (NFDM) is proposed.

Modulating the signal on the NFD spectrum allows the input-output relationship of this linearised noisy channel to be expressed in an additive manner as

\[
\rho(\lambda, z) = \rho(\lambda, 0) \exp(4j\lambda^2z) + N_\rho;
\]

\[
C_m(\lambda_m(z), z) = C_m(\lambda_m(0), 0) \exp(4j\lambda_m(0)^2z) + NC_m;
\]

\[
\lambda_m(z) = \lambda_m(0) + N\lambda_m.
\]
2.2 Nonlinear Fourier Transform

where $N_{\rho}$, $N_{Cm}$ and $N_{\lambda m}$ denote the ASE noise induced noise in the CS, norming constants and eigenvalues. Generally speaking, the signalling of the NFDM system involves two steps: modulating the transmitted symbols onto the selected degree of freedom of the NFD domain with appropriate mapping, and then performing the INFT to obtain the corresponding time domain pulse. After propagating through the channel, the NFT is utilised to extract the evolved NFD spectrum, and symbol detection is performed after removing the channel effect with phasor equalisation.

In this thesis, the amplitude-modulated single soliton system is first discussed. The channel model of such a system is derived using perturbation theory to be a noncentral chi-squared (NCX2) distribution, revealing the signal dependence nature of the $N_{\lambda m}$ term [76,77]. However, the nontrivial noise model of the system makes the estimation of the channel capacity an open problem. When the amplitude-modulated solitons are transmitted in sequence, as expected in a communication system, the intrinsic attraction force between neighbouring solitons could introduce non-tractable inter-soliton interaction when the solitons are pulled sufficiently close to each other. This effect is further enhanced by the Gordon-Haus timing jitter, which increases the probability of them being temporally close.

Next, the CS modulated system is investigated. In the contrast to the amplitude-modulated soliton system where only the imaginary part of the eigenvalue is used for modulation, complex symbol is modulated on the CS spectrum. It is demonstrated that the noise term $N_{\rho}$ not only possesses signal dependence similar to the eigenvalue modulated system, but also shows correlation between real parts and imaginary parts. In addition, unlike the closed-form channel model available in the amplitude-modulated soliton system, only the channel statistics up to second order are available for the CS modulated system.

In this thesis, attempts were made trying to resolve the above-mentioned issues. Overall, the NFDM will suffer from similar signal-dependent non-trivial noise, and nonlinear interference issues. To fully utilise the potential of the NFDM system, these problems need to be resolved.

2.3 Information Theoretical Tools

Information theory provides a good quantification mechanism for estimating the amount of information contained in a random variable (in communication system, the random variable could refers to an information source). It could be interpreted as the theoretical limit on the performance of data compression schemes. When applying information theory on a pair of information sources, the mutual information could be estimated to provide a rigorous limit on how much information is shared between the sources. Considering the information sources as input and output of a communication link, the limit on the information rate could be estimated without designing specific error correction schemes.
2.3. Information Theoretical Tools

2.3.1 Shannon Entropy

The term entropy was first proposed by physicist Ludwig Boltzmann in research for thermodynamics as a measure of the number of possible microscopic arrangements of individual molecules of a system. The Boltzmann entropy essentially measures the amount of randomness required to statistically connect the randomly fluctuated microscopic interaction and the macroscopically observed behaviour. This concept is further cultivated into a new branch in thermodynamics known as statistical mechanics. The Shannon entropy, on the other hand, quantifies the amount of information based on the amount of randomness in the information source as

$$H(X) = E_X \left[ \log_2 \frac{1}{P_X(x)} \right]$$

where $X$ denotes the random variable with realisation $x$ occurring with probability $P_X(x)$ and $E_X[\cdot]$ denotes taking the expectation over the distribution of random variable $X$. The entropy quantifies the average amount of information generated by the realisations of the random variable, where one realisation $x$ is quantified with $\log_2 \frac{1}{P_X(x)}$. The base of the logarithmic operation determines the unit of the measured information. If not specified otherwise, the base 2 is typically used, resulting in the unit of bit/symbol. This lower bounds the minimum number of digital 0 and 1 bits required to represent the realisation $x$.

Returning to the definition of entropy, $P_X(x)$ corresponds to the probability of realisation $x$. For a discrete random variable $X$, the definition can be rewritten as

$$H(X) = \sum_{x \in X} P_X(x) \log_2 \frac{1}{P_X(x)}$$

where the expectation operation is replaced with a weighted sum of the information of individual possible realisations. On the other hand, the entropy of a continuous random variable is undefined, as the probability of an individual realisation is zero. For a continuous random variable $Y$, differential entropy is defined as

$$h(Y) = \int_{y \in Y} p_Y(y) \log_2 \frac{1}{p_Y(y)} \, dy,$$

where $p_Y(y)$ is the probability density function (PDF) that describes the probability density at a particular realisation $y$. There is no physical implication for differential entropy as it could be negative as opposed to the non-negative entropy.

If the input random variable of a communication link is denoted as $X$ and the output random variable is denoted with $Y$, then the conditional and joint differential entropy are given as

$$h(Y|X) = \int_{x \in X} \int_{y \in Y} p_{X,Y}(x,y) \log_2 \frac{1}{p_Y(y|X(x))} \, dy \, dx;$$
2.3. Information Theoretical Tools

\[ h(X, Y) = \int_{x \in X} \int_{y \in Y} p_{X,Y}(x, y) \log_2 \frac{1}{p_{X,Y}(x,y)} \, dy \, dx, \]

where \( p_{Y|X}(y|x) \) denotes the conditional PDF of \( y \) given \( x \) and \( p_{X,Y}(x,y) \) represents the joint probability of the two random variables. The conditional differential entropy denotes the amount of information added over \( X \) in \( Y \), while the joint differential entropy denotes the amount of information contained in the joint space of \( X \) and \( Y \). Note that the conditional PDF \( p_{Y|X}(y|x) \) is typically known as channel law due to its physical meaning. In this work, discrete constellation is typically assumed for the input distribution for its practicality in implementation. Hence, the conditional and joint differential entropy should be rewritten as

\[ h(Y|X) = \sum_{x \in X} P_X(x) \int_{y \in Y} p_{Y|X}(y|x) \log_2 \frac{1}{p_{Y|X}(y|x)} \, dy, \]

\[ h(X,Y) = \sum_{x \in X} \int_{y \in Y} p_{X,Y}(x,y) \log_2 \frac{1}{p_{X,Y}(x,y)} \, dy, \]

where the integration over \( X \) space is replace with summation. Correspondingly, the output differential entropy could be expressed with channel law and input distribution as

\[ h(Y) = \int_{y \in Y} \sum_{x \in X} P_X(x) p_{Y|X}(y|x) \log_2 \frac{1}{\sum_{x' \in X} P_X(x') p_{Y|X}(y|x')} \, dy. \]

2.3.2 KL Divergence and Mutual Information

Kullback-Leibler (KL) divergence, also known as relative entropy or crossentropy\(^2\), is a non-negative distance-like measure of the difference between two probability distributions. It is defined as

\[ D_{KL}(P_{X1}, P_{X2}) = \int_{x \in X} p_{X1}(x) \ln \frac{p_{X1}(x)}{p_{X2}(x)} \, dx \geq 0, \]

where \( P_{X1}(x) \) and \( P_{X2}(x) \) denote two distributions that describe the random variable \( X \). It could be shown that the KL divergence is a non-negative term via the convexity of the logarithmic function and Jensen’s inequality. The KL divergence is equal to zero if and only if the two distributions are identical. If one measures the KL divergence between the \( P_{X,Y}(x,y) \) and \( P_X(x)P_Y(y) \), the mutual information with bits as units between random variable \( X \) and \( Y \) is defined as

\[ I(X; Y) = D_{KL}(P_{X,Y}(x,y), P_X(x)P_Y(y)); \]

\[ = \int_{x \in X} \int_{y \in Y} P_{X,Y}(x,y) \log_2 \frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} \, dy \, dx. \]

\(^2\) The crossentropy here specifically refers to the crossentropy loss function in machine learning application, in other application, it might be defined differently.
2.3. Information Theoretical Tools

Mutual information could be interpreted as a measure of the correlation between the random variables $X$ and $Y$. When $X$ and $Y$ are independent of each other, i.e. $P_{X,Y}(x,y) = P_X(x)P_Y(y)$, the mutual information $I(X;Y)$ is zero because there is no correlation between the variables.

If one rewrite the mutual information with discrete input $X$ and continuous output $Y$ of a communication link as discussed in previous section as

$$I(X;Y) = h(Y) - h(Y|X);$$

$$= H(X) - h(X|Y).$$

$h(Y|X)$ could be interpreted as the extra differential entropy introduced by the noise at the receiver. Removing the excessive terms will result in the mutual information shared between the input $X$ and output $Y$. Although $h(X|Y)$ could not be interpreted similarly, the expression could still provide a trivial bound that the mutual information between input and output should not exceed the source entropy $H(X)$.

### 2.3.3 Capacity and Mismatch Capacity

The channel capacity is defined as the mathematical maximum information rate that could be reliably transmitted between the input $X$ and output $Y$ given a particular channel law as

$$C_{\text{bpsu}} = \max_{\{P_X(x)\}} I(X;Y),$$

(2.34)

where the mutual information is maximised over different input distribution. The block diagram for a general communication is shown in Figure 2.7. Assuming a bandlimited additive white Gaussian noise (AWGN) channel and an average signal power constraint, the famous Shannon capacity equation is derived as

$$C_{\text{bps}} = W \log_2 \left( 1 + \frac{P}{N_0 W} \right),$$

where $W$ denotes the bandwidth of the channel and $P$ denotes the average signal power constraint. The capacity could be achieved with a Gaussian input at the signal power $P$, which corresponds to an average power constraint on the input signal. However, when a peak power constraint is considered, the optimal input distribution becomes a discrete uniform-like distribution, and the capacity is estimated with the specified peak power constraint [78].

The channel capacity problem (2.34) is not typically tractable due to two reasons. On the one hand, the channel models are generally not as mathematically convenient as the AWGN channel which results in an input $X$ independent conditional differential entropy $h(Y|X)$ term.

On the other hand, a sufficiently accurate channel model with closed-form analytical PDF is not always available. Hence, the capacity bounds are usually estimated to provide an indirect estimation of the capacity. When the exact channel model is not available, an auxiliary channel
is selected to assist the design of the decoding scheme. If the channel law generated with the auxiliary channel is denoted with $Q_{Y|X}(y|x)$, the corresponding a posterior and joint probability are then connected via Bayesian rule as

$$Q_{X|Y}(x|y) = \frac{Q_{X,Y}(x,y)}{Q_Y(y)} = \frac{Q_{Y|X}(y|x)P_X(x)}{\int_{x \in X} Q_{Y|X}(y|x)P_X(x)dx},$$

where the auxiliary output distribution $Q_Y(y)$ is derived via marginalising the auxiliary joint distribution $Q_{X,Y}(x,y)$. By substituting the auxiliary decoding rules into the mutual information expression, one could write down the mismatch mutual information as

$$I_M(X,Y) = E_{X,Y} \left[ \frac{Q_{X,Y}(x,y)}{P_X(x)Q_Y(y)} \right];$$

where the expectation $E_{X,Y}[\cdot]$ over the joint distribution $P_{X,Y}(x,y)$ corresponds to the true channel. Maximising the $I_M$ over the input distribution allow the mismatch capacity $C_M$ to be obtained. However, due to the difficulty in modelling the channel, the optimality of the input distribution cannot be proved easily. In this work, the terminology of mismatch capacity $C_M$ and mismatch mutual information $I_M$ is used inter-changeably. The expectation is usually estimated numerically with Monte Carlo method and a large number of channel realisations when the true channel model is not available. The lower bounding effect of the mismatch capacity contain two steps as

$$C(X,Y|P_X^*(x)) \geq I(X,Y|P_X^*(x)) \geq C_M(X,Y|P_X^*(x)).$$

The first lower bounding could be proven trivially by having a sub-optimal input distribution $P_X^*(x)$ rather then the optimal $P_X^*(x)$. The equality will hold when the input distribution is optimal for the channel model $P_{Y|X}(y|x)$, i.e. $P_X^*(x) = P_X^*(x)$. The second lower bounding is proved via the non-negativity of KL divergence with the equality occur at auxiliary channel $Q_{Y|X}(y|x)$ being equal to the true channel $P_{Y|X}(y|x)$ [79–82].
2.4 Neural Networks in Communication Systems

The artificial neural network (ANN) is believed to be a universal function approximator capable of estimating any intricate nonlinear function which lacks analytical form [83]. Neural networks have demonstrated their advance in vast range of tasks such as computer vision tasks in autopilot technology, natural language processing tasks in the latest conversational robot, ChatGPT. Inspired by the structure of biological neural networks, ANNs are composed of artificial neurons, which are designed to replicate the reaction of biological neurons to external stimulation. When a biological neuron cell is stimulated by the neurotransmitters or external stimulation, the ion channels on the cell membrane will open to relocate the ions inside and outside the cell to create a potential difference, i.e. an electric signal. By integrating electric signals from different branches of the cell, the neuron is activated and releases neurotransmitters to the next cell if the integrated signal exceeds a certain threshold. To mimic this process of the biological neuron cell, an artificial neuron cell is designed as Figure 2.8.

\[
\begin{align*}
\sum \quad f(\cdot) & \rightarrow y \\
\end{align*}
\]

**Figure 2.8:** Schematic of an artificial neuron which combines its inputs \( x \) into an output \( y \).

The inputs and weights are denoted with vectors \( x \) and \( w \), then the output \( y \) is written as

\[
y = f(wx + b),
\]

where the activation function \( f(\cdot) \) reproduces how the electric signal is processed in a biological neuron cell. It is essential for the activation function to be nonlinear, if one intends to approximate a nonlinear function. Assuming an ANN with only linear activation function is implemented, it is clear that such network could be written with matrix multiplication, resulting in a linear function that cannot characterise any nonlinearity.

There are a large variety of nonlinear activation functions, including sigmoid, tanh, rectified linear unit (ReLU) and leaky ReLU. One of the most famous choices for the activation function is the sigmoid function. If define \( u = wx + b \), the sigmoid activation function is given as

\[
f_{\text{sigmoid}}(u) = \frac{1}{1 + \exp(-u)}.
\]
As shown in Figure 2.9a, the sigmoid function is a smooth function whose output is limited within $[0, 1]$. The hyperbolic tanh function is also a popular option, due to its smoothness and the capability of generating negative output.

Non-smooth functions are also commonly used in ANN as activation functions. One of the most famous options is the ReLU function, denoted as $f_{ReLU}$. This function sets the output to zero when the input $u$ is negative, and to the input $u$ otherwise:

$$ f_{ReLU} = \begin{cases} 0 & u < 0 \\ u & u \geq 0 \end{cases} $$

However, a problem known as the "Dying ReLU" can occur if the input to the neuron is always negative, causing the output to be zero and the neuron to become "dead". This problem can be mitigated by using a variant of the ReLU called the leaky ReLU. The leaky ReLU introduces a small proportional factor for the negative part of the input as

$$ f_{LeakyReLU} = \begin{cases} 0.01u & u < 0 \\ u & u \geq 0 \end{cases} $$

where the factor 0.01 can be changed to other values to form the parametric ReLU activation function. Despite the potential problem of "Dying ReLU", the ReLU activation function has a constant gradient that enables faster convergence during training and helps to mitigate the problem of gradient vanishing. The gradient vanishing problem is more common with activations that have a small gradient asymptotically, such as the sigmoid and tanh activations at high $|u|$, which can cause the network to stop learning effectively. Figure 2.9b illustrates a comparison between the ReLU and leaky ReLU activation functions.
Other than the aforementioned activation functions, the softmax activation function is commonly used in ANN classifier as the output could be interpreted as probability. The softmax function is an element-wise operation that operates on a set of inputs, if the input vector and output vector are denoted with \( u \) and \( y \), the softmax activation is defined as

\[
y_i = f_{\text{softmax}}(u) = \frac{\exp(u_i)}{\sum_j \exp(u_j)},
\]

where \( y_i \) and \( u_i \) denotes the \( i \)-th element of the corresponding vector. It is not difficult to prove that \( \sum_i y_i = 1 \), allowing the output of the softmax activation to be interpreted as probability when probability output is required. Hence, the softmax activation is commonly used as the output layer of ANN classifier.

In communication systems, the most straightforward application of the ANN is the ANN classifier that maps the time domain samples directly to the symbol. The softmax output layer is typically employed at the end of the classifier network followed by a hard decision with maximum a posterior probability criteria. In the classifier network, the crossentropy loss function is typically used for evaluation of the network performance. The crossentropy gives an estimation of the difference between the ideal a posterior probability vector (a vector whose elements corresponds to the transmitted symbol is 1) and the network output a posterior probability vector. With a sufficient number of training data, the network is trained to minimise this loss.

The classifier ANN could be realised using different network architectures. One such architecture is the multilayer perceptron, which is a simple feedforward ANN constructed using multiple layers of fully connected neurons and a softmax output layer. Although it is demonstrated that a feedforward network is able to perform symbol detection tasks in a memoryless channel, its structure inherently prevents the information from previous input vectors from persisting to explore potential correlation in time. This issue is particularly significant in symbol detection for communication over channels with memory. To overcome this limitation, the recurrent neural network (RNN) can be implemented. RNNs are constructed with a recurrent connection that effectively introduce memory into the network. The recurrent connection feeds the previous hidden state back into the recurrent unit to update the present hidden state as

\[
h_t = f_h(w_h x_t + u_h h_{t-1} + b_h),
\]

3. Recall the equivalence between terms KL divergence, crossentropy and relative entropy, crossentropy term will be used in the machine learning related content to align with the terminology in the relevant literature.
2.4. Neural Networks in Communication Systems

where \( h_t \) and \( h_{t-1} \) denote the current and last output of the cell, \( x_t \) denotes the current input, \( w_h, u_h \) and \( b_h \) denotes the input weights, hidden state weights and biases vector respectively.

The hidden state is computed using the activation function \( f_h(\cdot) \). The current output \( y_t \) is produced via another activation as

\[
y_t = f_y (w_y h_t + b_y)
\]

where the \( w_y \) and \( b_y \) indicate the output weights and biases. The network implemented with the described cell is known as Elman network, which is considered an example of one of the simplest RNN architectures.

![Figure 2.10: Schematic figure of the LSTM cell.](image)

The Elman RNN cell contains a simple concatenation of the current input with the last output of the network, suggesting limited performance in taking advantage of the potential long-term dependencies in the data. To adapt to long-term correlations, the long short-term memory (LSTM) architecture has been proposed. The LSTM cell introduces more gates to control the flow of information within the cell. A basic LSTM cell contains four gates, an input gate \( i \), a forget gate \( f \), a cell candidate \( \tilde{c} \), and an output gate \( o \), which are activated with two activation functions. As shown in the Figure 2.10, their input-output relationships are given as

\[
i_t = f_g (w_i x_t + u_i h_{t-1} + b_i);
\]

\[
f_t = f_g (w_f x_t + u_f h_{t-1} + b_f);
\]

\[
\tilde{c}_t = f_c (w_c x_t + u_c h_{t-1} + b_c);
\]

\[
o_t = f_g (w_o x_t + u_o h_{t-1} + b_o);
\]
where the subscript $t$ denotes the time while the others denote the gate type. The forgetting gate, input gate and the output gate use the same gate activation function $f_g$, while a different cell activation function $f_c$ is used for the cell activation $\tilde{c}$. The cell state vector $c$ and hidden state vector $h$ are acquired via

$$c_t = f_t \odot c_{t-1} + i_t \odot \tilde{c}_t;$$

$$h_t = o_t \odot f_c(c_t),$$

where $\odot$ denotes the element-wise product and the cell input activation function is reused to obtain the current hidden state denoted by $f_h(\cdot)$. The activation functions $f_g$ and $f_c$ are typically selected to be $\tanh$ and sigmoid correspondingly. The LSTM network built on the discussed LSTM unit could only takes advantage of the proceeding data in the forward order.

![Diagram of LSTM and BLSTM cells](image)

**Figure 2.11:** Schematic of a) the connections of LSTM cells in BLSTM cell; b) a two-layer stacked RNN.

A bidirectional RNN is proposed to exploit data sequences in both forward and backward direction for potential correlation [89–91]. In Figure 2.11a, the structure of a bidirectional LSTM (BLSTM) unit is presented, where one LSTM cell is trained with data in the forward order and the other cell is trained with data in the backward order. However, this approach has an intrinsic limitation on its application: the network requires the full data sequence to
complete the given task, making it less appealing in high-timely critical applications like real-time translation. The merger combines the hidden states from two LSTM cells into one for further processing. The combination scheme varies from network to network, typical methods include concatenation, $h_t = [\tilde{h}_t, \hat{h}_t]$, and element-wise averaging, $h_t = 1/2(\tilde{h}_t + \hat{h}_t)^4$.

In some scenarios, a single layer of LSTM (or BLSTM) may not be sufficient to capture all the information about a nonlinear mapping relationship. Besides increasing the number of LSTM hidden units, stacking additional layers could also enhance the network’s ability to learn complex functions, provided that there is enough training data available. An example of a general two-layer stacked RNN network is depicted in Figure 2.11b, where the RNN blocks could be replaced with any RNN cells, including Elman RNN cells, LSTM cells, and BLSTM cells [92].

Once the network architecture and loss function are defined, and the training data is prepared, the network is ready to be trained. The network training problem is essentially a high-dimensional, non-convex optimisation for minimising the objective function, which is the loss function. Conveniently, the MATLAB Deep Learning toolbox provides a comprehensive implementation of the network, allowing the network to be easily trained with the adaptive moment estimation (ADAM) algorithm. In addition to the ready-to-use training algorithm, MATLAB also provides a convenient experiment designer that enables Bayesian optimisation of the hyperparameters, such as learning rate and number of hidden layers.

## 2.5 Summary

In this chapter, we first introduce the mathematical model that describes pulse evolution in nonlinear fibre optics. Due to the analytical complexity of the NLSE, the split-step Fourier method is presented as a frequently employed tool for simulating pulse evolution in the nonlinear fibre. This mathematical model underlines the signal dependence of fibre nonlinearity, which remains the key factor limiting further improvements in spectral efficiency. The NFT is then introduced as a potential method that leverages, rather than being hindered by the nonlinearity. The NFT unveils the discrete and continuous spectra, onto which the system could modulate information.

Within this thesis, we first exploit the simplest DS modulated system, taking advantage of the invariant property of the eigenvalue in the DS, in Chapters 3 and 4. In Chapter 3, we employ tools from information theory such as mutual information, KL divergence, and mismatch capacity. We utilise the KL divergence to highlight the discrepancy between the approximated and

4. In MATLAB, the concatenation merger is used by default.
2.5. Summary

perturbative theory channel models, with the latter primarily modelling the impact of the ASE noise. The mismatch capacity is used to estimate the achievable information rate (AIR) for the perturbative theory channel model. Additionally, we employ split-step Fourier simulations to elucidate the nonlinear interaction between solitons when transmitted sequentially.

In Chapter 4, nonlinear impairments beyond ASE noise are further explored. This examination leverages the split-step Fourier simulations and the timing shift properties of the NFT introduced in this chapter. These nonlinear impairments comprise intrinsic inter-soliton interaction in the absence of noise and noise-induced Gordon-Haus timing jitter. Together, these impairments create a non-tractable problem, prompting an exploration of the potential of ANN detectors. This exploration relies on the effectiveness of ANN in general applications and the related background reviewed in this chapter, laying the foundation for discussions in Chapter 4.

The evolution of the CS, akin to the norming constants in the DS, is linearised in the NFD. Given the similarity between the CS and the ordinary Fourier spectrum, Chapter 5 delves into a CS modulated NFDM system. This system is posited as an analogue to the linear OFDM system. Firstly, an approximated complex channel for the system is proposed, based on perturbative theory statistics. Then, by maximising the mutual information of this proposed channel model, the mismatch capacity is estimated, offering a lower bound for the channel capacity.
This chapter explores the capacity of the amplitude modulated soliton communication system, which takes advantage of the invariant property of solitons by modulating the information symbol using amplitude. The chapter begins by discussing the noise perturbed channel model of the system, followed by formulating the capacity problem using time-scaled mutual information (MI). By considering the signal-dependent time width, the data rate performance of the system can be estimated. The capacity is then numerically evaluated using an approximated analytical solution. Finally, the chapter concludes with a discussion of the limitations of the channel model based on simulated transmissions of soliton sequence.

3.1 Literature Review

The spectral efficiency (SE) of conventional standard single-mode fibres (SMF) is often constrained by the Kerr nonlinearity, a signal-dependent effect that intensifies as the launch power increases [26]. Predicted by Zahkarov and Shabat, the existence of the soliton pulse is first demonstrated via numerical simulation by Hasegawa [93]. The stability of the soliton evolution in nonlinear fibre shows its ability of balancing linear dispersion and Kerr nonlinearity, hence implying its potential application as information carrier [66]. However, it was pointed out that the amplifier noise will introduce Gordon-Haus timing jitter, which will limit the data rate of the proposed on-off keying soliton communication [67].

It was not until the new emergence of nonlinear Fourier transform (NFT) that the soliton communication started to regain some research interests. In [76, 77], the channel statistics of an ASE noise-perturbed amplitude-modulated single soliton communication system was discussed. Using perturbation theory and Fokker–Planck equation, it was derived that statistics of the noise on soliton amplitude are signal dependent, and the noisy output of the channel is well described with a noncentral chi-squared (NCX2) distribution whose variance is dependent on the mean. Due to the nontrivial channel model and signal dependent noise,
the capacity of such a system remains an open problem. In [94, 95], the channel model for both single- and dual- polarisation amplitude-modulated soliton systems were revisited, and the capacity lower bounds for both systems were estimated using a series of input distributions including half-Gaussian distribution and Rayleigh distribution.

Other than soliton amplitude, i.e. the imaginary part of the single eigenvalue in discrete spectrum (DS), the norming constant could carry information due to its linear evolution in nonlinear fibre. In [96–98], the statistics of the channel noise in the norming constants when \( N \) eigenvalues are employed were derived using perturbation theory, followed by an approximated Gaussian channel model for the special case that \( N = 1 \). The approximated channel was used in the estimation of the achievable information rate (AIR)\(^1\) of a system that performs modulation on both eigenvalue and the corresponding norming constant, showing the AIR gain over the system that only takes advantage of eigenvalue [99]. In [100, 101], the spectral efficiency of a single eigenvalue system was further investigated by taking both eigenvalue and norming constant degrees of freedom for modulation, combined with certain optimisation on constellation. It was highlighted that modulating on the real part of the eigenvalue could be more beneficial than conventional modulation on the imaginary part. In [102], a multistage received symbol estimator for the amplitude and phase modulated single soliton system was designed and AIR was improved by 1.03 bits/symbol with lower complexity. In [103], the probabilistic shaping (PS) technique was implemented to adapt to the signal-dependent channel, showing promising gain of 6.4 dB over unshaped constellation.

Although the complexity of signalling increases with the number of eigenvalues used for modulation, multi-eigenvalue systems are believed to make better use of nonlinear frequency domain (NFD) spectra. For example, in [104], a dual-polarisation multi-soliton system was proposed, where information was encoded in norming constant pairs corresponding to their eigenvalues. To reduce the correlation between norming constant pairs and improve the data rate, a precoding scheme was proposed. In [105], two system designs of information encoding schemes on the eigenvalues were proposed for a nonlinearity-dominant fibre, reporting a maximum spectral efficiency of over 3 bits/s/Hz. Additionally, in [106], an experimental demonstration of a multi-soliton system was carried out where information was encoded on the on/off of three eigenvalues over an 1800 km link.

\(^1\) AIR can be measured in bits/symbol or bits/s. In this series of literature, AIR in bits/symbol is estimated.
3.2 System Modelling

Recall the normalised stochastic nonlinear Schrödinger equation (NLSE) that describes the pulse evolution in a distributed Raman amplified nonlinear optical fibre is given as (2.14), with the noise statistics of (2.16) in Chapter 2. Using the inverse scattering method, the NFT transforms the time domain optical signal into scattering data, consisting of a continuous spectrum (CS) \( \rho(\lambda, z) \), eigenvalues \( \lambda_m(z) \) and corresponding norming constants \( C_m(z) \) which evolve linearly along the fibre in nonlinear spectral domain. It can be shown that, in a noise-free and interaction-free scenario, the eigenvalues \( \lambda_m \) are preserved during the evolution along the fibre [107]. If only one eigenvalue exists at \( z = 0 \) and \( \rho(\lambda, 0) = 0 \), the solution of NLSE is a first-order soliton, which can be described analytically as

\[
q(t, z) = 2\eta e^{-2i\zeta t + 4i(\zeta^2 - \eta^2)z - i(\psi + \pi/2)} \text{sech}(2\eta t - 8\eta\zeta z - 2\Upsilon),
\]

(3.1)

where the only eigenvalue is \( \lambda_1 = \zeta + i\eta \) (\( \eta > 0 \)). Also, \( e^{2\Upsilon} = \frac{C_1}{\eta^2} \) and \( \psi = \arg C_1(z) \) where \( C_1 \) denotes the norming constant corresponding to eigenvalue \( \lambda_1 \).

The energy of the soliton in (3.1) is equal to \( 4\eta \), where the temporal width and bandwidth are proportional to \( 1/\eta \) and \( \eta \) respectively. Note that within this work, only the imaginary part of the eigenvalue is modulated and the real part is set to zero to create the simplest form of the fundamental soliton, i.e., \( \eta = A, \zeta = 0 \). Thus, at \( z = 0 \), the input pulse can be expressed as [76,108]

\[
q(t, z = 0) = 2A\text{sech}(2At).
\]

(3.2)

The propagation of the soliton pulse over the fibre is described by the NLSE, and at the receiver side, the eigenvalue can be detected by NFT or pulse energy estimation. If the detected eigenvalue is denoted as \( R \), the channel model for this amplitude modulated first-order soliton transmission system can be described by a conditional probability density function (PDF) \( p_{R|A}(r|a) \), which is non-Gaussian with a variance dependent on its mean [76,108]. Ignoring inter-soliton interactions, a memoryless channel model can be defined for the amplitude modulated soliton system with perturbative theory analysis based on a NCX2 distribution with 4 degrees of freedom as [76,94]

\[
p_{R|A}(r|a) = \frac{2}{\sigma_N^2} \sqrt{\frac{\pi}{a}} \exp \left( -\frac{2a + 2r}{\sigma_N^2} \right) I_1 \left( \frac{4\sqrt{ar}}{\sigma_N^2} \right),
\]

(3.3)

where \( I_1(\cdot) \) denotes the modified first order Bessel function of the first kind. The mean and variance of this distribution for large \( a \) are \( \mu_{\text{NCX2}}(a) = \sigma_N^2 + a \) and \( \sigma_{\text{NCX2}}^2(a) = \frac{1}{2}(\sigma_N^2 + a) \) respectively, where \( \sigma_N^2 = \frac{1}{2} \sigma^2 \frac{A}{2L} \) at distance \( Z = L \) and \( \sigma^2 \) is the power spectral density of the normalised ASE noise as defined in the noise autocorrelation. It can be seen that the channel model (3.3) for the imaginary part of the eigenvalue (soliton amplitude, or soliton
energy) is non-Gaussian with signal dependent variance. In the next section, we develop different approaches to estimate the capacity of the channel described by (3.3). It is worth emphasizing that the perturbative theory NCX2 channel is also an approximated channel model with certain effective operational regime based on the small perturbation assumption.

3.3 Capacity Formulation for Memoryless Soliton Communication Channel

Here, the capacity problem for the channel defined by the conditional PDF (3.3) is formulated considering a peak amplitude constraint since the bandwidth occupied by soliton pulses is directly related to their amplitudes. That is, the modulating data on higher amplitudes requires larger bandwidth while the maximum signal bandwidth is restricted by physical limitations. Moreover, in practical scenarios, peak power is also constrained due to device limitations. Another important issue that needs to be considered for soliton communications systems is that soliton pulses defined as in (3.2) are not time-limited, and thus, they should be truncated for practical implementation.

We define the practical width of a soliton pulse (denoted by $t_w$) as the temporal width that contains $(1 - \delta)$ of the soliton energy. Recalling the energy of the normalised soliton (3.2) is equal to $4A$, this practical width can be obtained by solving the equation below for $t_w$

$$\int_{-t_w/2}^{+t_w/2} \left[ 2A\text{sech}(2At) \right]^2 \, dt = (1 - \delta)4A,$$

which is given by

$$t_w(A, \delta) = \frac{1}{2A} \ln \left( \frac{2}{\delta} - 1 \right),$$

where the fixed value $\delta$ should be sufficiently small to make the truncation error negligible compared to noise. For example, assuming that the soliton pulse width is defined based on containing 99.9% of its energy ($\delta = 0.001$), we have $t_w = 3.8/A$. Noting that the temporal width of soliton pulses is inversely related to their amplitudes, we can also introduce a minimum amplitude constraint to limit the utilisation of the temporal resources. Based on the constraints mentioned above, the capacity problem can be formulated similar to the (2.34) as

$$C_{bpcu} = \sup_{P_a(a): A \in \{0\} \cup [A_{lb}, A_{ub}]} I(A; R),$$

(3.6)
where $C_{bpcu}$ denotes the capacity in bits per symbol per channel use, $I(A;R)$ represents the MI. Denoting the transmitted and received eigenvalues with random variables $A$ and $R$ respectively, $A_{ub}$ is the maximum amplitude constraint determined by maximum bandwidth or peak power and $A_{lb}$ is the minimum amplitude constraint determined by the maximum allowed symbol duration. Note that we also consider the possibility of transmitting no soliton over a symbol duration (i.e., off symbol) with probability $P_0$, which is denoted by $A = 0$ here.

Noting that the signal space and the temporal resources are inter-related in the underlying soliton communication system, we will use an alternative capacity formulation that maximises time-scaled MI [109] to get better insights into AIRs of the system in bits per second. Unlike [109], we assume a fixed symbol duration for all transmitted solitons to facilitate practical implementation. Since the pulse width is inversely related to the amplitude of the soliton, the minimum nonzero soliton amplitude $A_{min} \geq A_{lb}$ (i.e., maximum pulse width) in a given input distribution determines the symbol duration. Note that $A_{min}$ is not necessarily equal to the minimum amplitude constraint $A_{lb}$ and $P(A < A_{min}) = P_0$. The time-scaled MI (MI) is thus defined as

$$\mathbb{R}(A;R) = \frac{I(A;R)}{I_w(A_{min}, \delta)},$$

(3.7)

where MI is divided by the normalised symbol duration, resulting in a unit of bits/normalised time. The data rate in bits/second can be estimated by dividing the time-scale MI (3.7) with the normalising time $T_0$. The corresponding time-scaled capacity formulation is then given by

$$C_{TS} = \sup_{P_0(a): A \in [0,A_{lb}], A_{ub}} \mathbb{R}(A;R).$$

(3.8)

Note that the minimum amplitude constraint $A_{lb}$ can be also relaxed, since it is already inherently imposed by the modified objective function, i.e., the time-scaled MI. This is because the optimal solution would not include the small soliton amplitudes that consume the available temporal resources inefficiently due to their very large pulse width. Hence the capacity problem can be also written as

$$C_{TS} = \sup_{P_0(a): A \in [0,A_{ub}]} \mathbb{R}(A;R).$$

(3.9)

In Section 3.3.2, it is shown that a minimum nonzero soliton amplitude $A_{min}$ naturally appears in the optimal distribution of the capacity problem in (3.9). Numerically, the optimisation results shown in later sections also show that the minimum nonzero-amplitude $A_{min} > A_{lb}$ for the interested regime, where the $A_{lb}$ is intended to exclude soliton amplitudes which are too small to support the existence of the discrete spectrum [34]. Hence, the relaxation of the minimum amplitude constraint $A_{lb}$ is valid in this work.
3.3. Capacity Formulation for Memoryless Soliton Communication Channel

3.3.1 Equivalent Channel Model Based on VNT

To simplify the capacity analysis, similar to the method used in [54, 108, 110, 111], a variance normalising transform (VNT) is applied here to transform the original signal-dependent noise channel to a channel with a fixed noise power at sufficiently large signal-to-noise ratios. In general, the VNT can be applied to any random variable $R$ where its variance $\sigma_R^2$ is related to the mean $\mu_R$ as $\sigma_R^2 = f^2(\mu_R)$. Then the variance of the transformed random variable, $Y = T(R)$, is normalised to one (i.e., mean independent) at sufficiently large values of $\mu_R$. The general form of VNT can be written based on [112] as

$$T(u) = \int \frac{1}{f(u)} \, du.$$  \hspace{1cm} (3.10)

Therefore the normalised random variable $Y = T(R)$ has moments $\mu_Y = E[y] \approx T(\mu_R)$ and $\sigma_Y^2 \approx 1$ for sufficiently large value of $\mu_R$. Substituting the statistics of the NCX2 channel $\mu_{NCX2}(a) = \sigma_N^2 + a$ and $\sigma_{NCX2}^2(a) = \frac{1}{2} \sigma_N^4 + a \sigma_N^2 = \sigma_N^2 (\frac{1}{2} \sigma_N^2 + a) = \sigma_N^2 (\mu_{NCX2}(a) - \frac{1}{2} \sigma_N^2)$ considered in this work, the VNT will be given as

$$T(u) = \int \frac{1}{\sigma_N^2 (u - \frac{1}{2} \sigma_N^2)} \, du = 2 \sqrt{\frac{u}{\sigma_N^2} - \frac{1}{2}} \approx 2 \sqrt{\frac{u}{\sigma_N^2}},$$  \hspace{1cm} (3.11)

where the approximation is made for mathematical simplicity and due to the fact that the variance normalisation itself defined by VNT is only precise at large values of $u/\sigma_N^2$ where the adopted approximation is also precise [54, 110, 111, 113].

As shown in Figure 3.1, an equivalent soliton communication system can be defined based on the VNT approach where the noise power is signal-independent at large signal levels. Note that, in order to perform the coding and decoding at the same signal space, it is convenient to include both VNT and inverse VNT (IVNT) meaning that the soliton amplitude, $A$, is determined from the original input data $X = T(A)$ as

$$A = T^{-1}(X) = \sigma_N^2 \frac{X^2}{4}. $$  \hspace{1cm} (3.12)

Noting the square root form of the VNT defined in (3.11) and considering that the NCX2 model in (3.3) defines the channel between the soliton eigenvalues $A$ and $R$ in Figure 3.1, the equivalent channel model between the transformed random variables $X$ and $Y$ is described by a noncentral chi (NCX) conditional PDF as

$$p_{Y|X}(y|x) = \frac{y^2}{x} \exp \left( -\frac{y^2 + x^2}{2} \right) I_1(xy),$$  \hspace{1cm} (3.13)

where $X = T(A) = 2 \sqrt{A}/\sigma_N$ and $Y = T(R) = 2 \sqrt{R}/\sigma_N$. 

3.3. Capacity Formulation for Memoryless Soliton Communication Channel

!!Figure 3.1!!: Block diagram of an amplitude modulated soliton communication system with IVNT and VNT, $A$ and $R$ denote the transmitted and received soliton amplitude, $X$ and $Y$ denote the transformed input and output signals, and $q$ denotes the time domain signal.

The capacity in bits per symbol of the system in (3.6) can then be rewritten based on the random variables $X$ and $Y$ as

$$C_{\text{bpcu}} = \sup_{P_X: X \in \{0\} \cup [X_{lb}, X_{ub}]} I(X; Y),$$  \hspace{1cm} (3.14)

where $X_{lb} = T(A_{lb})$ and $X_{ub} = T(A_{ub})$. Moreover, the corresponding time-scaled capacity formulation is given by

$$C_{\text{TS}} = \sup_{P_X: X \in \{0\} \cup [X_{lb}, X_{ub}]} \mathbb{R}(X; Y),$$  \hspace{1cm} (3.15)

or based on the relaxed constraint as

$$C_{\text{TS}} = \sup_{P_X: X \in [0, X_{ub}]} \mathbb{R}(X; Y),$$  \hspace{1cm} (3.16)

where the time-scaled MI can be written as

$$\mathbb{R}(X; Y) = \frac{I(X; Y)}{t_w(A_{min}, \delta)} = \frac{\sigma_N^2 X_{min}^2}{2\ln(2/\delta - 1)} I(X; Y),$$  \hspace{1cm} (3.17)

and $X_{min}$ denotes the minimum nonzero symbol amplitude, i.e., $A_{min} = T^{-1}(X_{min}) = \sigma_N^2 X_{min}^2 / 4$. It is important to notice that the VNT transformation does not affect the MI between input and output, i.e., $I(A; R) = I(X; Y)$, since the VNT function (3.11) is a monotonic and invertible function within the domain of interest (See Lemma in [54]). Hence, the capacity formulations in (3.9) and (3.15) are equivalent.
3.3. Capacity Formulation for Memoryless Soliton Communication Channel

3.3.2 Approximate AWGN Channel Model

It has been shown that the probability distribution of the normalised random variable after VNT tends to Gaussian distribution for a family of originally non-Gaussian probability distributions [54,110]. In this section, we first show that this is also true for the NCX distribution (3.13) in a Kullback–Leibler (KL) divergence sense. This inspires us to propose an approximate additive white Gaussian noise (AWGN) channel model to describe the amplitude modulated soliton communication system after VNT transformation as

\[ Y = X + \Gamma, \tag{3.18} \]

where the additive noise \( \Gamma \) is Gaussian with zero mean and unit variance.

**Proposition 1.** The KL divergence between the NCX distribution, \( p_{Y|X}(y|x) \), given in (3.13) and a Gaussian distribution \( q_{Y|X}(y|x) \) with mean \( x \) and unit variance tends to zero for a sufficient large \( x \), that is

\[ \lim_{x \to +\infty} D_{KL}(p, q|x) = 0, \tag{3.19} \]

where the KL divergence, \( D_{KL}(p, q|x) \), is defined as

\[ D_{KL}(p, q|x) = \int_{-\infty}^{+\infty} p_{Y|X}(y|x) \ln \frac{p_{Y|X}(y|x)}{q_{Y|X}(y|x)} \, dy, \tag{3.20} \]

**Proof of Proposition 1.** The detailed proof is shown in Section 3.3.3

Proposition 1 indicates that the NCX channel model (3.13) behaves similarly to the approximate AWGN channel for a sufficiently large \( x \). For example, The KL divergence \( D_{KL} \) is estimated as small as \( 1.77 \times 10^{-12} \) for \( x = 86.67 \). This is by assuming that the pulse width contain 99.9% of the energy (\( \delta = 0.001 \)) and some typical fibre parameters as in Table 3.1. Next, we will show that the proposed approximate AWGN channel converges to the original NCX channel at sufficiently large large \( X_{lb} \).

**Theorem 1.** Given the input \( X \in \{ 0 \cup [X_{lb},X_{ub}] \} \) at a sufficiently large \( X_{lb} \), the mismatch capacity of the NCX channel with the approximate AWGN channel defined by (3.18) as auxiliary channel converges to the actual capacity of the NCX channel.

**Proof of Theorem 1.** The detailed proof is shown in Section 3.3.4
In [78, 114], it is shown for the AWGN channel with amplitude constraints that the capacity-achieving distribution is discrete with a finite number of mass points for such channels. An upper bound is proposed in [115] for the number of mass points. However, these works focus on the MI-based capacity formulation. In the next Proposition, we extend the result in [78] to show the discreteness of the optimal solution to the time-scaled MI maximisation problem for the proposed approximate AWGN channel.

**Proposition 2.** Given an AWGN channel with the input amplitude constraint of \( X \in \{0 \cup [X_{lb}, X_{ub}] \} \) and \( X_{lb} \to \infty \), the optimal input distribution for the capacity formulation in (3.15) is discrete with a finite number of mass points.

**Proof of Proposition 2.** The detailed proof is shown in Section 3.3.5

It is commonly accepted that the capacity achieving distribution for the AWGN channel subject to an average power constraint should belong to a continuous Gaussian distribution. In [78], Smith proved that the capacity achieving distribution for the peak amplitude constrained AWGN channel is discrete with a finite number of mass points due to the additional peak amplitude constraint on the input distribution. Additionally, it is also proved that for the peak amplitude and average power constrained AWGN channel, i.e. Shannon’s capacity problem with an additional peak amplitude constraint, the optimal input distribution is still discrete, implying the peak amplitude constraint introduces the discreteness into the optimal input distribution [78]. It is also worth highlighting that the other constrained channels whose capacity achieving distributions are discrete with finite number of mass points were revised in [115, 116].

Now, approximating the channel in (3.15) with an AWGN model based on Theorem 1 and considering the conclusion of Proposition 2 on the discreteness of the optimal input distribution asymptotically, the MI between \( X \) and \( Y \) can be expressed as

\[
I(X; Y) = h(Y) - h(Y|X) = h(Y) - h(\Gamma)
\]

\[
= \sum_{k=0}^{M} \int P_X(x_k)q_Y|X(y|x_k) \log_2 \left( \frac{1}{\sum_{j=0}^{M} P_X(x_j)q_Y|X(y|x_j)} \right) dy - \log_2 \sqrt{2\pi e},
\]

where \( h(Y) \) denotes the output differential entropy, \( h(\Gamma) \) denotes the differential entropy of the unit variance AWGN noise, \( x_k \) and \( P_X(x_k) \) denote the input symbols and their corresponding probabilities within the input source alphabet, \( M \) denote the size of the nonzero alphabet, \( x_0 = 0 \) and \( P_X(x_0) = P_0 \) denotes the corresponding probability. Hence, the problem in (3.15) can be rewritten as

\[
C_{TS} = \max_M \left[ \left\{ \max_{x_k \in \{0\} \cup [X_{lb}, X_{ub}]} \mathbb{E}(X; Y) \right\} \right].
\]
where the time-scaled MI function $\mathbb{R}(X; Y)$ is a function of two $(M+1)$-length vectors $\mathbf{x}$ and $\mathbf{P}_X$ which denote the mass points and their probabilities. As mentioned in the previous sections, the minimum amplitude constraint can be also relaxed yielding

$$C_{TS} = \max_M \left[ \max_{\mathbf{x}, \mathbf{P}_X: \mathbf{x} \in [0, X_{ub}]} \mathbb{R}(X; Y) \right]. \quad (3.23)$$

Since the input distribution is discrete, the vector $[\mathbf{x}, \mathbf{P}_X]$ is sufficient to describe the input random variable $X$. The discreteness of the capacity-achieving input distribution allows for numerical evaluation of the capacity expression using similar algorithms as in [78, 108]. In this work, the optimisation over $[\mathbf{x}, \mathbf{P}_X]$ is performed using an interior-point optimiser in MATLAB given the number of nonzero mass point is fixed at $M$. The optimisation on $M$ is then performed based on an exhaustive search approach which will keep increasing $M$ until additional mass points can no longer improve the optimised time-scaled MI.

<table>
<thead>
<tr>
<th>Table 3.1: Fibre Parameter.</th>
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<tr>
<td><strong>length $L$</strong></td>
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<tr>
<td><strong>Loss $\alpha$</strong></td>
</tr>
<tr>
<td><strong>Group velocity dispersion factor $\beta_2$</strong></td>
</tr>
<tr>
<td><strong>Kerr nonlinearity factor $\gamma$</strong></td>
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<tr>
<td><strong>Phonon occupancy $K_T$</strong></td>
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<tr>
<td><strong>Signal wavelength $\nu_0$</strong></td>
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<tr>
<td><strong>Normalising time $T_0$</strong></td>
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Figure 3.2 shows the capacity-achieving distributions obtained by solving (3.22) and the corresponding capacity estimation using the optimised input distribution. For these results, we assume an ideal distributed Raman amplified 2000 km fibre with the parameters detailed in Table 3.1. Using the constraint from $X_{ub} = 200$ to $X_{ub} = 500$. This range of peak amplitude constraint corresponds to the range of maximum eigenvalue from $A_{ub} = 0.4$ to $A_{ub} = 2.5$, which represent the peak optical power $-5$ dBm and $+10$ dBm, respectively.

In Figure 3.2a–c, the optimal distributions are shown for various peak amplitude constraints $X_{ub}$. The figures show that the optimal distributions consist of an isolated mass point at zero (off symbol), and a uniform-like distribution starting from a minimum nonzero symbol (denoted by $X_{\text{min}}$) to the maximum symbol amplitude (denoted by $X_{\text{max}} = X_{ub}$). It is also important to point out that the probabilities at $X_{\text{min}}$ and $X_{\text{max}}$ getting closer to the probabilities of the mass points in between as $X_{ub}$ increases, showing a convergence towards a uniform distribution. Note that the results in [109] shows a nonuniform distribution of optimal mass points since the pulse width is assumed to be variable.
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Figure 3.2: The optimal input distribution and the corresponding optimised time-scaled MI obtained as the numerical solution of (3.22) subject to the peak amplitude constraint $X_{ub}$ assuming $\delta = 0.001$. (a) The location of the optimal mass points (the peak amplitude is shown as the purple solid line with star) (b) The optimal probability of the mass point at zero (i.e., off symbol) (c) The optimal probabilities of the nonzero mass points, (d) The maximum Time-scaled MI given based on the solution of (3.22) and the lower bounds on the time-scaled capacity of the original noncentral chi-squared distribution (NCX) channel achieved by using different input distributions, including, on-off keying (OOK), 4 pulse amplitude modulation (4-PAM) and the input distribution given in (a) to (c). Note that the additional power axis denotes the power level of the solitons corresponds to the peak amplitude $X_{ub}$ assuming $\delta = 0.001$.

Figure 3.2d presents the capacity of the approximate AWGN channel based on the solution of (3.22) as well as some lower bounds on the capacity of the original NCX channel (3.13). The best lower bound is obtained by applying the optimal distribution of the approximate AWGN channel as in Figure 3.2a–c to the time-scaled MI of the NCX channel. This lower bound precisely overlaps with the capacity of the approximate AWGN channel, further confirming the result of Theorem 1, in a MI sense, i.e., that the AWGN channel is a very good approximation of the NCX channel within the range of consideration. Figure 3.2d also includes
the time-scaled MI estimated for the transmission of conventional on-off keying (OOK) and 4 pulse amplitude modulation (4-PAM) signals over the original NCX channel. As expected, both conventional modulations show lower time-scaled MI comparing to the optimised input distribution. However, the conventional 4-PAM signal achieves even lower time-scaled MI than OOK. This is due to the fact that the fixed symbol duration is inversely related to the amplitude of the minimum nonzero amplitude $X_{\text{min}}$, which is $X_{\text{min}} = X_{\text{ub}}/3$ for 4-PAM but $X_{\text{min}} = X_{\text{ub}}$ for OOK. In general, for a $K$-PAM modulation scheme, the time-scaled MI can be upper bounded by the time-scaled source entropy, $H(X) = \frac{\sigma^2 X^2_{\text{min}}}{2 \ln(2/\delta - 1)} \log_2(K)$, where the $X_{\text{min}} = X_{\text{ub}}/(K-1)$. It can be then shown that the time-scaled source entropy for $K$-PAM will always decrease with respect to $K$ for $K \geq 2$. This suggests that $K$-PAM with higher $K$ cannot achieve better time-scaled MI than OOK. It is also worth noting that some of the sub-optimal distributions proposed in the literature (e.g., the half-Gaussian bound proposed in [94]) is not included here as the half-Gaussian input source would give a zero time-scaled MI when a fixed symbol duration is considered as in this paper.

3.3.3 Proof of Proposition 1

Proof of Proposition 1. Following a similar method as in [111], a non-negative term KL divergence is employed to evaluate difference between two distributions, as defined in (3.20). Within this proof, $p_{Y|X}(y|x)$ is considered to be a noncentral chi (NCX) distribution as

$$p_{Y|X}(y|x) = \frac{y^2}{x} \exp\left\{ -\frac{y^2 + x^2}{2} \right\} I_1(xy),$$

(3.24)

where $I_1(\cdot)$ denotes the modified Bessel function of the first kind, and the mean and variance of $p_{Y|X}$ are denoted with $\mu_{\text{NCX}}(x)$ and $\sigma^2_{\text{NCX}}(x)$ respectively. $q_{Y|X}(y|x)$ is considered as a Gaussian distribution with identical mean $\mu_{\text{NCX}}(x)$ and variance $\sigma^2_{\text{NCX}}(x)$, i.e.,

$$q_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma^2_{\text{NCX}}(x)} \exp\left\{ -\frac{(y - \mu_{\text{NCX}}(x))^2}{2\sigma^2_{\text{NCX}}(x)} \right\}.$$

(3.25)

To prove the convergence of the NCX distribution to a Gaussian distribution with mean $x$ and unit variance at a sufficiently large $x$, we first verify the convergence of the moments of the NCX distribution and then show it tends to a Gaussian distribution for large $x$. Taking the limits of the first and second moments at large values of $x$, we obtain

$$\lim_{x \to \infty} \mu_{\text{NCX}}(x) = \lim_{x \to \infty} \sqrt{4 + x^2 - \sigma^2_{\text{NCX}}(x)} = \lim_{x \to \infty} \sqrt{3 + x^2} = x,$$

(3.26)

$$\lim_{x \to \infty} \sigma^2_{\text{NCX}}(x) = \lim_{x \to \infty} 4 + x^2 - \mu^2_{\text{NCX}}(x) = \lim_{x \to \infty} 4 + x^2 - \frac{\pi}{2} \left[ L_{1/2}^1\left(-\frac{x^2}{2}\right) \right]^2 = 1,$$

(3.27)
which verifies the convergence of moments to the corresponding values in the theorem statement. Now substituting the NCX distribution (3.24) and its corresponding Gaussian distribution (3.25) into (3.20), the KL divergence can be expressed as

\[
D_{KL}(p, q|x) = \int_{-\infty}^{+\infty} p_{Y|X}(y|x) \ln(p_{Y|X}(y|x)) \, dy - \int_{-\infty}^{+\infty} p_{Y|X}(y|x) \ln(q_{Y|X}(y|x)) \, dy
= -h_{NCX}(x) - E_{NCX}[\ln(q_{Y|X}(y|x))],
\]

(3.28)

where \(h_{NCX}(x)\) denotes the differential entropy of the NCX distribution (3.24) given parameter \(x\), and \(E_{NCX}(\cdot)\) denotes the expectation over the NCX distribution (3.24). The first term can be expressed as

\[
h_{NCX}(x) = -E_{NCX}[\ln(p_{Y|X}(y|x))]
= -E_{NCX} \left[ \ln(\frac{y^2}{x} \exp \left( \frac{-(y^2 + x^2)}{2} \right) I_1(xy) \right]
= -2E_{NCX}[\ln(y)] - E_{NCX}[\ln(I_1(xy))] + x^2 + \ln(x) + 2,
\]

(3.29)

while the second term can be written as

\[
E_{NCX}[\ln(q_{Y|X}(y|x))] = E_{NCX} \left[ \ln \left( \frac{1}{\sqrt{2\pi\sigma_{NCX}^2(x)}} \exp \left( -\frac{(y - \mu_{NCX}(x))^2}{2\sigma_{NCX}^2(x)} \right) \right) \right]
= \ln \frac{1}{\sqrt{2\pi\sigma_{NCX}^2(x)}} - \frac{1}{2}.
\]

(3.30)

Since the function \(f(y) = \ln(I_1(xy))\) — where \(x\) is a given non-negative constant — and function \(g(y) = \ln(y)\) are concave functions [117], Jensen’s inequality is applied to obtain an upper bound on the KL divergence as

\[
D_{KL}(p, q|x) = 2E_{NCX}[\ln(y)] + E_{NCX}[\ln(I_1(xy))] - x^2 - \ln \frac{x}{\sqrt{2\pi\sigma_{NCX}^2(x)}} - \frac{3}{2}
\leq \ln \frac{(\mu_{NCX}(x))^2I_1(x\mu_{NCX}(x))}{\sqrt{2\pi\sigma_{NCX}^2(x)}} \leq D_{ub}(x).
\]

(3.31)

Next, we find the limit of the upper bound of KL divergence \(D_{ub}(x)\) using the limits of the mean \(\mu_{NCX}\) and variance \(\sigma_{NCX}^2(x)\) already calculated in (3.26) and (3.27), that is

\[
\lim_{x \to +\infty} D_{ub}(x) = \lim_{x \to +\infty} \frac{(3 + x^2)I_1(x\sqrt{3 + x^2})\sqrt{2\pi}}{xe^{x^2/2} + 3/2} = 0.
\]

(3.32)
At last, using the non-negativity of KL divergence, we have \( 0 \leq \lim_{x \to +\infty} D_{KL} \leq \lim_{x \to +\infty} D_{ub} = 0 \), i.e., \( \lim_{x \to +\infty} D_{KL} = 0 \). Therefore, we can conclude that the KL divergence between (3.24) and (3.25) goes to zero when \( x \) is sufficiently large and this concludes the proof.

### 3.3.4 Proof of Theorem 1

**Proof of Theorem 1.** In this proof, we show that the gap between the NCX channel capacity and the mismatch capacity of the NCX channel given the approximate AWGN channel as auxiliary channel tends to zero as \( X_{lb} \to \infty \). Consider that the input random variable \( X \in \{0 \cup [X_{lb}, X_{ub}]\} \) is separated into zero and nonzero sets. Then the PDF of \( X \) can be written as

\[
p_X(x) = P_0 \delta(x) + (1 - P_0) p_{\hat{X}}(x),
\]

where \( \delta(x) \) denotes the Dirac delta function, and \( p_{\hat{X}}(x) \) denotes the PDF of the nonzero input \( \hat{X} \). Similarly, the output random variable \( Y \) can also be separated in a similar manner as

\[
p_Y(y) = P_0 p_{Y|X}(y|0) + (1 - P_0) p_{Y}(y),
\]

where the \( p_{Y}(y) = \int_{\hat{X}} p_{Y|X} p_{\hat{X}}(x) \, dx \) denotes the PDF of the output corresponding to the nonzero input. The MI between input \( X \) and output \( Y \) is then given as

\[
I(X;Y) = h(Y) - h(Y|X) = \int_X \int_Y p_X(x) p_{Y|X}(y|x) \log_2 \frac{1}{p_Y(y)} \, dy \, dx - \int_X \int_Y p_X(x) p_{Y|X}(y|x) \log_2 \frac{1}{p_Y(y)} \, dy \, dx.
\]

Substituting equation (3.34) in the output differential entropy \( h(Y) \), it is then rewritten as

\[
\begin{align*}
h(Y) &= \int_{X_{lb}}^{X_{ub}} \int_{-\infty}^{+\infty} p_X(x) p_{Y|X}(y|x) \log_2 \frac{1}{p_Y(y)} \, dy \, dx + \int_{-\infty}^{X_{lb}/2} P_0 p_{Y|X}(y|0) \log_2 \frac{1}{p_Y(y)} \, dy \\
&\quad + \int_{X_{lb}/2}^{X_{ub}} P_0 p_{Y|X}(y|0) \log_2 \frac{1}{P_0 p_{Y|X}(y|0) + (1 - P_0) P_{Y}(y)} \, dy' \, dx' \\
&\quad + \int_{-\infty}^{X_{lb}/2} P_0 p_{Y|X}(y|0) \log_2 \frac{1}{P_0 p_{Y|X}(y|0) + (1 - P_0) P_{Y}(y)} \, dy,
\end{align*}
\]

(3.36)
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where we changed the variable of integral in the first term of the last equality as \( y' = y - X_{lb} \).

Taking the Taylor expansion of the logarithmic functions inside the two integrals of the right hand side of (3.36) at \( y' = 0 \) and \( y = 0 \), respectively, we obtain

\[
h(y) = \int_{X_{lb}}^{X_{ub}} \int_{-X_{ub}/2}^{X_{ub}/2} p_X(x)p_{Y|X}(y|x) \left[ \log_2 \frac{1}{1 - P_0} \log_2 \left( \frac{1 - P_0}{1 - P_0} \right) \right] \, dx \, dy' + \int_{X_{lb}}^{X_{ub}} p_{Y|X}(y|0) \left[ \log_2 \frac{1}{1 - P_0} \right] \, dy' + \hat{\Delta}
\]

where \( \hat{\Delta} \) and \( \Delta_0 \) are higher order terms for nonzero and zero input respectively. At \( X_{lb} \to \infty \), the two terms correspond to the integration on higher order terms

\[
\hat{\Delta} = E_{Y,X} \left[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \log_2 \frac{P_0 p_{Y|X}(y|0)}{1 - P_0} \right]
\]

and

\[
\Delta_0 = E_{Y,X=0} \left[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \log_2 \frac{(1 - P_0) P_0}{P_0 p_{Y|X}(y|0)} \right]
\]

can be written in the form of expectations. They will therefore vanish since, for the NCX distribution, \( \lim_{X_{lb} \to \infty} p_{Y|X}(y' \leq -X_{lb}/2|x \geq X_{lb}) = 0 \) and \( \lim_{X_{lb} \to \infty} p_{Y|X}(y \geq X_{lb}/2|0) = 0 \). Inserting (3.37) in (3.35), the MI is given as

\[
I(X;Y) = \int_{X_{lb}}^{X_{ub}} \int_{-X_{ub}/2}^{X_{ub}/2} p_X(x)p_{Y|X}(y'|x) \log_2 \frac{p_{Y|X}(y'|x)}{(1 - P_0) p_{Y'}(y')} \, dx \, dy' + \hat{\Delta}
\]

Performing mismatch decoding with an auxiliary channel, which is not necessarily equivalent to the exact channel, provides a lower bound for the capacity of the exact channel as discussed in Section 2.3.3 [79–82]. Assuming the mismatch decoder design is based on the Gaussian distribution, \( q_{Y|X}(y|x) \), the mismatch capacity \( I_{LB} \) is defined as

\[
I_{LB} = \int_X \int_Y p_X(x)p_{Y|X}(y|x) \log_2 \frac{q_{Y|X}(y|x)}{q_Y(y)} \, dy \, dx,
\]

where the mismatch output distribution \( q_Y(y) \) can be written in similar manner as (3.34) as

\[
q_Y(y) = P_0 q_{Y|X}(y|0) + (1 - P_0) q_Y(y),
\]
where the \( q_0(y) = \int_X q_{Y|X}(y|x)p_X(x) \, dx \) denotes the PDF of the output corresponding to the nonzero input. The mismatch capacity at \( X_0 \) can be obtained via similar approach as before as

\[
I_{LB}(X;Y) = \int_{-X_0/2}^{X_0} \int_{-X_0/2}^{X_0} p_X(x)p_{Y|X}(y|x) \left[ \log_2 \frac{q_{Y|X}(y'|x)}{(1-P_0)q_{Y|X}(y'|0)} + \sum_{k=1}^\infty \frac{(-1)^k}{k} \left( \frac{P_0 q_{Y|X}(y'|0)}{(1-P_0)q_{Y|X}(y')} \right)^k \right] \, dy \, dx
\]

\[
+ \int_{-X_0/2}^{X_0} p_X(x) \left[ \log_2 \frac{q_{Y|X}(y|0)}{P_0 q_{Y|X}(y|0)} + \sum_{k=1}^\infty \frac{(-1)^k}{k} \left( \frac{P_0 q_{Y|X}(y|0)}{P_0 q_{Y|X}(y|0)} \right)^k \right] \, dy
\]

\[
= \int_{-X_0/2}^{X_0} \int_{-X_0/2}^{X_0} p_X(x)p_{Y|X}(y|x) \log_2 \frac{p_{Y|X}(y'|x)}{(1-P_0)q_{Y|X}(y')} \, dy \, dx + \hat{\Delta} - \hat{\nabla} + \Delta_0 - \nabla_0
\]

where the \( \hat{\nabla} \) and \( \nabla_0 \) are higher order terms of the Taylor expansion for nonzero and zero inputs. Similarly, at \( X_0 \to \infty \), the two integration on the higher order terms

\[
\hat{\nabla} = E_{Y,X} \left[ \sum_{k=1}^\infty \frac{(-1)^k}{k} \left( \frac{P_0 q_{Y|X}(y|0)}{P_0 q_{Y|X}(y|0)} \right)^k \right]
\]

and

\[
\nabla_0 = E_{Y,X=0} \left[ \sum_{k=1}^\infty \frac{(-1)^k}{k} \left( \frac{(1-P_0)q_{Y}(y)}{P_0 q_{Y|X}(y|0)} \right)^k \right]
\]

can be written in the form of expectations. They will therefore vanish, for the AWGN channel, since \( \lim_{X_0 \to \infty} q_{Y|X}(y' \leq -X_0/2|x \geq X_0) = 0 \) and \( \lim_{X_0 \to \infty} q_{Y|X}(y \geq X_0/2|0) = 0 \) for the Gaussian distribution. The gap between the MI \( I(X;Y) \) and its lower bound \( I_{LB}(X;Y) \) is then defined as

\[
I_{gap} = I(X;Y) - I_{LB}(X;Y)
\]

\[
= \int_{-X_0/2}^{X_0} \int_{-X_0/2}^{X_0} p_X(x)p_{Y|X}(y'|x) \log_2 \frac{p_{Y|X}(y'|x)}{(1-P_0)Q_0(y')} \, dy \, dx + \hat{\Delta} - \hat{\nabla} + \Delta_0 - \nabla_0
\]

\[
- \int_{-X_0/2}^{X_0} \int_{-X_0/2}^{X_0} p_X(x)p_{Y|X}(y'|x) \log_2 \frac{p_{Y|X}(y'|x)}{q_{Y|X}(y')} \, dy \, dx.
\]

At \( X_0 \to \infty \), the vanishing terms \( \hat{\Delta}, \hat{\nabla}, \Delta_0 \) and \( \nabla_0 \) tends to 0, Hence, the limit of \( I_{gap} \) is given by

\[
\lim_{X_0 \to \infty} I_{gap} = \int_{-X_0}^{X_0} p_X(x) \left( \int_{-\infty}^{+\infty} p_{Y|X}(y'|x) \log_2 \frac{p_{Y|X}(y'|x)}{q_{Y|X}(y')} \, dy \right) \, dx
\]

\[
- \int_{-\infty}^{+\infty} \left( \int_{-X_0}^{X_0} p_X(x)p_{Y|X}(y'|x) \, dx \right) \log_2 \frac{p_{Y|X}^*(y')}{q_{Y|X}^*(y')} \, dy'
\]

\[
= \int_{-X_0}^{X_0} p_X(x) D_{KL}(p^*(y'), q^*(y')) \, dx - (1-P_0) D_{KL}(p^*(y'), q^*(y'))
\]
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where the second term in (3.42) is a nonnegative KL divergence term, hence, the $I_{\text{gap}}$ at $X_{lb} \to \infty$ is bounded by

$$0 \leq \lim_{X_{lb} \to \infty} I_{\text{gap}} \leq \int_{X_{lb}}^{X_{ub}} p_X(x) \log_2 \frac{p_X(x)D_{\text{KL}}(p', q'|x)}{p_{\hat{X}}(x)D_{\text{KL}}(p, q|x)} \, dx,$$

(3.44)

which is the expectation of the KL divergence over the nonzero range of $X$. According to Proposition 1, $\lim_{x \to \infty} D_{\text{KL}}(p, q|x [X_{lb}, X_{ub}])$ for the NCX PDF $p_{Y|X}(y|x)$ and Gaussian PDF $q_{Y|X}(y|x)$ tends to 0, therefore, the upper bound of the $I_{\text{gap}}$ also tends to 0. This completes the proof.

3.3.5 Proof of Proposition 2

Proof of Proposition 2. Consider the input random variable $X \in \{0 \cup [X_{lb}, X_{ub}]\}$ is separated into zero and nonzero sets. Then the probability density function of $X$ can be

$$p_X(x) = P_0 \delta(x) + (1 - P_0)p_{\hat{X}}(x),$$

(3.45)

where $\delta(x)$ denotes the Dirac delta function, and $p_{\hat{X}}(x)$ denotes the PDF of the nonzero input $\hat{X}$. Similarly, the output random variable $Y$ can also be separated in a similar manner as

$$p_Y(y) = P_0 p_{Y|X}(y|0) + (1 - P_0)p_Y(y),$$

(3.46)

where the term $p_Y(y) = \int_{X_{lb}}^{X_{ub}} p_X(x)p_{Y|X}(y|x)p_{\hat{X}}(x) \, dx$ denotes the PDF of the output corresponding to the nonzero input. Using the Taylor expansion as in Equation (3.37), and considering the AWGN channel model, defined by $p_{Y|X}(y|x)$, the MI is given as

$$I(X; Y) = h(Y) - h(Y|X) = h(Y) - h(\Gamma)$$

$$= \int_{-\infty}^{+\infty} p_Y(y) \log_2 \frac{1}{p_Y(y)} \, dy - \log_2 \sqrt{2\pi e}$$

$$= \int_{X_{lb}}^{X_{ub}} p_X(x) p_{Y|X}(y|x) \log_2 \frac{1}{(1 - P_0)p_Y(y)} \, dy' \, dx + \Delta$$

$$+ \int_{-\infty}^{X_{lb}/2} P_0 p_{Y|X}(y|0) \log_2 \frac{1}{P_0 p_Y(y|0)} \, dy + \Delta_0 - \log_2 \sqrt{2\pi e},$$

(3.47)

where we changed the variable of integral in the first term of the last equality as $y' = y - X_{lb}$. At $X_{lb} \to \infty$, the integration over the higher order terms,

$$\Delta = E_{\hat{X} \times Y'} \left[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[ \frac{P_0 p_{Y|X}(y'|0)}{(1 - P_0)p_Y(y')} \right]^k \right].$$

$$\Delta = 0,$$
and
\[
\Delta_0 = E_{X_0 \times Y} \left[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left( \frac{(1-P_0)P_X(y)}{P_0P_{Y|X}(y|0)} \right)^k \right]
\]
can be written in the form of expectations. They will therefore vanish since, for AWGN channel, 
\[
\lim_{X_{\text{fin}} \to \infty} P_{Y|X}(y' \leq X_{\text{fin}}/2|x) = 0 \quad \text{and} \quad \lim_{X_{\text{fin}} \to \infty} P_{Y|X}(y \geq X_{\text{fin}}/2|0) = 0.
\]
Hence, we will have
\[
I_\infty(X;Y) = \lim_{X_{\text{fin}} \to \infty} I(X;Y)
\]
\[
= \int_{X_{\text{fin}}} \int_{-\infty}^{+\infty} p_X(x)P_{Y|X}(y|x) \log_2 \frac{1}{(1-P_0)P_Y(y)} \, dy \, dx
\]
\[
+ \int_{-\infty}^{+\infty} P_0P_{Y|X}(y|0) \log_2 \frac{1}{P_0P_{Y|X}(y|0)} \, dy - \log_2 \sqrt{2\pi e},
\]
\[
=R_0 \log_2 \frac{1}{P_0} + (1-P_0) \log_2 \frac{1}{1-P_0} + (1-P_0) \left( h(\hat{Y}) - \log_2 \sqrt{2\pi e} \right)
\]
\[
=h_0 + (1-P_0)I(\hat{X};\hat{Y}), \tag{3.48}
\]
where \( h_0 = R_0 \log_2 \frac{1}{P_0} + (1-P_0) \log_2 \frac{1}{1-P_0} \), and \( I(\hat{X};\hat{Y}) \) denotes the MI between the nonzero input \( \hat{X} \) and its corresponding output \( \hat{Y} \). The time- scaled capacity formulation in (3.15) is then given as
\[
C_{TS} = \sup_{p_X(x):X \in \{0\} \cup [X_{\text{fin}},X_{\text{ab}}]} R_\infty(X;Y) = \sup_{p_X(x):X \in \{0\} \cup [X_{\text{fin}},X_{\text{ab}}]} \frac{I_\infty}{I_w(A_{\text{min}},\delta)}
\]
\[
= \sup_{R_0,p_X(x):x \in [X_{\text{fin}},X_{\text{ab}}]} \frac{\sigma_{\text{N}}^2 X_{\text{min}}^2}{2 \ln(2/\delta - 1)} \left( h_0 + (1-P_0)I(\hat{X};\hat{Y}) \right). \tag{3.49}
\]
Let \( p_X^*(x) \) denote the capacity achieving distribution for the problem in (3.49), which is fully defined by \( P_0^*, X_{\text{min}}^* \) and the PDF of nonzero input, \( p_X^* \). Next, we will show that this distribution is discrete with a finite number of mass points, which in turn implies the statement of this proposition. We first show that the capacity achieving distribution \( p_X^* \) is also the solution of the following optimisation problem
\[
\sup_{p_X(x):x \in [X_{\text{min}}^*,X_{\text{ab}}]} I(\hat{X};\hat{Y}). \tag{3.50}
\]
Let \( p_X^0 \) be any arbitrary distribution within the feasible set of the problem in (3.50), implying that \( X_{\text{min}}^* \geq X_{\text{min}}^* \). Since \( p_X^* \) defines the capacity achieving distribution for the problem in (3.49), it yields a time- scaled MI larger than that of any arbitrary distribution such as \( p_X^0 \). Therefore, we can write
\[
\frac{\sigma_{\text{N}}^2 X_{\text{min}}^2}{2 \ln(2/\delta - 1)} \left( h_0|p_0^* + (1-P_0^*)I^*(\hat{X};\hat{Y}) \right) \geq \frac{\sigma_{\text{N}}^2 X_{\text{min}}^*}{2 \ln(2/\delta - 1)} \left( h_0|p_0^* + (1-P_0^*)I^*(\hat{X};\hat{Y}) \right), \tag{3.51}
\]
where $I^* (\hat{X}; \hat{Y})$ and $I^\circ (\hat{X}; \hat{Y})$ are the MI given by $p^*_X$ and $p^\circ_X$, respectively. Simplifying (3.51) and using the fact that $X^\circ_{\min} \geq X^*_\min$, we have

$$h_0 |p_0^* + (1 - P_0^*) I^* (\hat{X}; \hat{Y}) \geq h_0 |p_0^* + (1 - P_0^*) I^\circ (\hat{X}; \hat{Y}) \geq h_0 |p_0^* + (1 - P_0^*) I^\circ (\hat{X}; \hat{Y}),$$

which implies that $I^* (\hat{X}; \hat{Y}) \geq I^\circ (\hat{X}; \hat{Y})$. Since this is true for any arbitrary $p^\circ_X$ within the feasible set of the problem in (3.50), we can conclude that $p^*_X$ is also the optimal distribution for the problem in (3.50). Note that the problem in (3.50) is equivalent to the amplitude constrained AWGN channel capacity problem presented in [78]. Thus, the optimal $p^*_X$ and thereby $p^*_X$ should be discrete with a finite number of mass points as well, which concludes this proof.

### 3.3.6 Analytical Capacity Approximation

Inspired by the optimal input distributions obtained in the last section as presented in Figure 3.2, in this section, we focus on developing an analytical approach for time-scaled capacity estimation of the soliton communication system. Assuming that the peak amplitude constraint $X_{\text{ub}}$ is sufficiently large, Figure 3.2 shows that the capacity-achieving input distribution obtained by solving (3.22) is discrete with a finite number of mass points including an almost uniform distribution within $[X_{\min}, X_{\max} = X_{\text{ub}}]$, and an additional mass points at zero, where the optimal $X_{\min}$ needs to be found by solving the optimisation problem. We therefore consider a general form of discrete input distribution with a mass point at zero with probability $P_0$ and a discrete uniform distribution within $[X_{\min}, X_{\max}]$ to find an analytical estimation of the solution to the capacity problem given in (3.22). Note that the upper boundaries of the distribution is denoted by $X_{\max} \leq X_{\text{ub}}$ rather than $X_{\max} = X_{\text{ub}}$ to keep it inline with the peak amplitude constraint introduced earlier.

To write the corresponding MI based on (3.21), we first need to define the statistics of the channel output given the input signal parameters, $p_Y(y|P_0, X_{\min}, X_{\max})$. In order to make the capacity analysis tractable, we make an approximation that the distribution of the noisy output signal $Y$ given the transmission of nonzero mass points, i.e., $p_Y(y|X \in [X_{\min}, X_{\max}])$ is approximated by a continuous uniform distribution within the range $[X_{\min}, X_{\max}]$. This approximation is reasonable when the number of mass points $M$ are large and the noise variance is small compared to the signal level. Based on this approximation and also considering the Gaussian noise added to the zero mass point, we can write

$$p_Y(y|P_0, X_{\min}, X_{\max}) \approx P_0 f_G(y) + \frac{1 - P_0}{X_{\max} - X_{\min}} u(y|X_{\min}, X_{\max}),$$

(3.53)
3.3. Capacity Formulation for Memoryless Soliton Communication Channel

where the $f_G(\cdot)$ denotes the PDF of a zero mean, unit variance Gaussian distribution and $u(y|X_{\text{min}}, X_{\text{max}})$ denotes the step function that is equal to 1 when $y$ is within $[X_{\text{min}}, X_{\text{max}}]$ and 0 otherwise.

Considering the approximate PDF in (3.53), we now calculate the differential entropy of the received signal as

$$ h(Y) = \int_{-\infty}^{+\infty} p_Y(y) \log_2 \frac{1}{p_Y} dy $$

$$ \approx \int_{-\infty}^{X_{\text{min}}} P_0 f_G(y) \log_2 \frac{1}{P_0} dy + \int_{X_{\text{min}}}^{+\infty} p_Y(y) \log_2 \frac{1}{p_Y} dy $$

$$ \approx \int_{-\infty}^{X_{\text{max}}} P_0 f_G(y) \log_2 \frac{1}{P_0 f_G(y)} dy + \int_{X_{\text{min}}}^{X_{\text{max}}} \frac{1 - P_0}{X_{\max} - X_{\min}} \log_2 \frac{X_{\max} - X_{\min}}{1 - P_0} dy $$

$$ = P_0 \log_2 \frac{1}{P_0} + P_0 \log_2 \sqrt{2\pi e} + (1 - P_0) \log_2 \frac{X_{\max} - X_{\min}}{1 - P_0}, $$  

(3.54)

where the approximation $a$ leads from applying the approximate output distribution in (3.53), and the approximation $b$ is valid under the assumption that $X_{\min} \gg 0$, i.e., $f_G(y \geq X_{\min}) \approx 0$.

Substituting (3.54) into the equation (3.21), the approximated MI is then given as a function of $P_0$, $X_{\text{min}}$ and $X_{\text{max}}$ as

$$ I_{\text{app}}(X; Y) = P_0 \log_2 \frac{1}{P_0} + (1 - P_0) \log_2 \frac{X_{\max} - X_{\min}}{1 - P_0} - (1 - P_0) \log_2 \sqrt{2\pi e}. $$  

(3.55)

Noting that the scaling time (3.5) is a function of the minimum mass point $X_{\min}$, the approximate time-scaled MI function $R_{\text{app}}(X; Y)$ is then given as

$$ R_{\text{app}}(X; Y) = \frac{\sigma_N^2 X_{\min}^2}{2 \ln(2/\delta) - 1} \left[ P_0 \log_2 \frac{1}{P_0} + (1 - P_0) \log_2 \frac{X_{\max} - X_{\min}}{1 - P_0} - (1 - P_0) \log_2 (\sqrt{2\pi e}) \right]. $$  

(3.56)

**Theorem 2.** Given the approximated time-scaled MI function in (3.56), the solution to the capacity problem given in (3.22), is obtained as

$$ C_{\text{app}} = R_{\text{app}}(X; Y)|_{P_0^*, X_{\text{min}}^*, X_{\text{max}}^*}, $$  

(3.57)

where the optimal parameters of the input distribution are given as

$$ X_{\text{max}}^* = X_{\text{ub}}, $$  

(3.58)

$$ X_{\text{min}}^* = (X_{\text{ub}} + \sqrt{2\pi e}) \left[ 1 - \frac{1}{2W\left( \frac{X_{\text{ub}}}{2\sqrt{2\pi}} + \frac{\sqrt{e}}{2} \right)} \right], $$  

(3.59)

$$ P_0^* = \frac{2\sqrt{2\pi e}W \left( \frac{X_{\text{ub}}}{2\sqrt{2\pi}} + \frac{\sqrt{e}}{2} \right)}{X_{\text{ub}} + \sqrt{2\pi e}}, $$  

(3.60)
where \( W(\cdot) \) denotes the Lambert W function.

Proof of Theorem 2. Recalling the approximate time-scaled MI function is given as

\[
R_{\text{app}}(X;Y) = \frac{\sigma_N^2 X_{\text{min}}^2}{2 \ln(2/\delta + 1)} \left[ P_0 \log_2 \frac{1}{P_0} + (1 - P_0) \log_2 \frac{X_{\text{max}} - X_{\text{min}}}{1 - P_0} - (1 - P_0) \log_2(\sqrt{2\pi e}) \right].
\] (3.61)

In order to find the maximum of the function (3.61) analytically, its first order partial derivatives with respect to \( P_0, X_{\text{min}}, \) and \( X_{\text{max}}, \) are first derived. These first order partial derivatives are given as

\[
\frac{\partial R_{\text{app}}}{\partial X_{\text{max}}} = \frac{\sigma_N^2 X_{\text{min}}^2}{2 \ln(2/\delta + 1)} \left[ 1 - P_0 \right],
\] (3.62)

\[
\frac{\partial R_{\text{app}}}{\partial X_{\text{min}}} = \frac{2 \sigma_N^2 X_{\text{min}}^2}{2 \ln(2/\delta + 1)} \left( P_0 \log_2 \frac{1}{P_0} + (1 - P_0) \log_2 \frac{X_{\text{max}} - X_{\text{min}}}{1 - P_0} - (1 - P_0) \log_2(\sqrt{2\pi e}) \right) - \frac{\sigma_N^2 X_{\text{min}}^2}{2 \ln(2/\delta + 1)} \left( \frac{1 - P_0}{X_{\text{max}} - X_{\text{min}}} \right),
\] (3.63)

\[
\frac{\partial R_{\text{app}}}{\partial P_0} = \frac{\sigma_N^2 X_{\text{min}}^2}{2 \ln(2/\delta + 1)} \left( \log_2 \frac{1}{P_0} - \log_2 \frac{X_{\text{max}} - X_{\text{min}}}{1 - P_0} + \log_2(\sqrt{2\pi e}) \right),
\] (3.64)

Notice that (3.62) is positive because \( X_{\text{max}} > X_{\text{min}} \) and \( P_0 < 1 \). This implies that the approximate time-scaled MI, \( R_{\text{app}}(\cdot) \), monotonically increases with respect to \( X_{\text{max}} \), thus, \( R_{\text{app}}(\cdot) \) maximises at the boundary as

\[
X_{\text{max}}^* = X_{\text{ub}}.
\] (3.65)

Now setting the partial derivative in (3.63) to zero and using the boundary condition above, we obtain the following nonlinear equation that needs to be solved to obtain the possible optimal value of \( X_{\text{min}} \) denoted by \( X_{\text{min}}^* \).

\[
2 \ln \frac{X_{\text{ub}} - X_{\text{min}}^* + \sqrt{2\pi e}}{\sqrt{2\pi e}} = \frac{X_{\text{min}}^*}{X_{\text{ub}} - X_{\text{min}}^* + \sqrt{2\pi e}}.
\] (3.66)

Note that the solution to this nonlinear equation can be written based on the Lambert W function \( W(\cdot) \) as

\[
X_{\text{min}}^* = (X_{\text{ub}} + \sqrt{2\pi e}) \left[ 1 - \frac{1}{2W \left( \frac{X_{\text{ub}}}{2\sqrt{2\pi} + \sqrt{2}} \right)} \right].
\] (3.67)

Then, the corresponding probability of the zero mass point can be derived by setting (3.64) to zero and using the results above as

\[
P_0^* = \frac{\sqrt{2\pi e}}{X_{\text{ub}} - X_{\text{min}}^* + \sqrt{2\pi e}} = \frac{2\sqrt{2\pi e}W \left( \frac{X_{\text{ub}}}{2\sqrt{2\pi} + \sqrt{2}} \right)}{X_{\text{ub}} + \sqrt{2\pi e}}.
\] (3.68)
Now, in order to show the optimality of $P_0^*$ and $X_{\text{min}}^*$ derived above, the second order partial derivative test should be performed. The second order partial derivative with respect to $P_0$ and $X_{\text{min}}$ is taken first and are given as

$$\frac{\partial^2 R_{\text{app}}}{\partial P_0^2} = -\frac{\sigma_N^2 X_{\text{min}}^2}{2\ln(2/\delta + 1)} \left( \frac{1}{P_0^*} + \frac{1}{1 - P_0^*} \right),$$  

(3.69)

$$\frac{\partial^2 R_{\text{app}}}{\partial X_{\text{min}}^2} = \frac{\sigma_N^2}{2\ln(2/\delta + 1)} \left[ 2(P_0 \ln \frac{1}{P_0} + (1 - P_0) \ln \frac{X_{\text{ub}} - X_{\text{min}}}{1 - P_0} - (1 - P_0) \ln \sqrt{2\pi e} - 4X_{\text{min}} \frac{1 - P_0}{X_{\text{ub}} - X_{\text{min}}} X_{\text{min}}^2 \frac{1}{X_{\text{ub}} - X_{\text{min}}} \right].$$  

(3.70)

The mixed second order partial derivatives are also required, which are given as

$$\frac{\partial^2 R_{\text{app}}}{\partial P_0 \partial X_{\text{min}}} = \frac{\partial^2 R_{\text{app}}}{\partial X_{\text{min}} \partial P_0} = \frac{\sigma_N^2}{2\ln(2/\delta + 1)} \left[ 2X_{\text{min}} \ln \frac{\sqrt{2\pi e} (1 - P_0)}{P_0 (X_{\text{ub}} - X_{\text{min}})} + \frac{X_{\text{min}}^2}{X_{\text{ub}} - X_{\text{min}}} \right].$$  

(3.71)

By inspecting equations (3.69) and (3.70), one may find that the second order partial derivatives are less than zero at $P_0 = P_0^*$ and $X_{\text{min}} = X_{\text{min}}^*$, while the determinant of the Hessian matrix, $\left| \frac{\partial^2 R_{\text{app}}}{\partial P_0^2} \frac{\partial^2 R_{\text{app}}}{\partial X_{\text{min}}^2} - \frac{\partial^2 R_{\text{app}}}{\partial P_0 \partial X_{\text{min}}} \frac{\partial^2 R_{\text{app}}}{\partial X_{\text{min}} \partial P_0} \right|$, is larger than zero. Hence, the maximum of the time-scaled MI in (3.61) is obtained at the optimal points $X_{\text{max}}^* = X_{\text{ub}}$, $P_0^*$ and $X_{\text{min}}^*$ defined above as

$$C_{\text{app}} = R_{\text{app}}(X; Y) \big|_{P_0^*, X_{\text{min}}^*, X_{\text{max}}^*}. \quad (3.72)$$

Using Theorem 2, the approximate solution to the capacity problem in (3.22) can be calculated analytically. As it can be observed in Figure 3.3, this approximate capacity result demonstrates a close match to the exact capacity results obtained numerically. It is also observed that the analytical results are lower than the other curves due to the difference between the output distribution. When deriving the analytical approximation, the output distribution of the non-zero symbol is assumed to be a uniform distribution, while the output distribution corresponds to the input distribution sketched in Figure 5.4 clearly result in two peak at the edges of the non-zero symbols. Do note that the difference between the output distributions does not imply it will be a lower bound in all cases.
3.4 Mismatch Capacity for Soliton Communication over the NLSE Channel

So far, we have focused on the capacity estimation of the first-order soliton transmission based on the commonly used memoryless channel model defined by the NCX2 distribution in (3.3). In this section, we study the capacity limits of the soliton transmission over a more realistic description of the fibre-optic channel defined by the NLSE. Hence, both the Gordon-Haus effect and the nonlinear interactions between adjacent soliton pulses can be incorporated into the capacity analysis. For this purpose, we use the numerical evaluation of mismatch capacity bounds based on split-step simulation of the NLSE. The mismatch capacity approach is commonly used to provide a lower bound on the capacity of a communication system, by assuming a mismatch distribution for decoding the received signal [79, 111]. If the mismatch distribution is denoted by \( q_{Y|X}(y|x) \) and the real channel statistics is denoted by \( p_{Y|X}(y|x) \), the time-scaled mismatch capacity bound for a discrete input signal is expressed as

\[
C_{TSM} = \frac{1}{t_w(A_{\text{min}}, \delta)} \sum_{k=0}^{M} \int_{-\infty}^{+\infty} P_X(x_k) p_{Y|X}(y|x_k) \log \frac{q_{Y|X}(y|x_k)}{\sum_{j=0}^{M} P_X(x_j) q_{Y|X}(y|x_j)} dy
\]

\[
= \frac{1}{t_w(A_{\text{min}}, \delta)} \sum_{k=0}^{M} P_X(x_k) E_{p_{Y|X}(y|x_k)} \left[ \log \frac{q_{Y|X}(y|x_k)}{\sum_{j=0}^{M} P_X(x_j) q_{Y|X}(y|x_j)} \right],
\]  

where \( P_X(x_j) \) denotes the input probability of symbol \( x_j \) taken from optimisation (3.22), and \( E_{p_{Y|X}(y|x_k)}[\cdot] \) denotes an expectation operation over the channel model \( p_{Y|X}(y|x_k) \). Recall from Section 3.3.1 that the unit-variance Gaussian distribution and the NCX distribution are well matched for the range of interest. Thus, a unit-variance Gaussian distribution \( q_{Y|X}(y|x) \) is a reasonable mismatch distribution to be employed in the calculation of the mismatch capacity.

To take into account the impairments introduced by ASE noise, such as Gordon-Haus timing jitter, as well as intersoliton interaction effects, we use the split-step method to simulate the propagation of single soliton or soliton sequence transmission over the fibre. Hence many realisations of the fibre-optic channel can be generated based on the simulation of NLSE to establish the statistics of the realistic channel given the capacity-approaching input distribution obtained in Section 3.3.2. The generated channel statistics can then be used to numerically estimate the mismatch capacity in (3.73) through a Monte Carlo approach. Noting that the input distribution applied here is not necessarily the optimal distribution for the realistic channel. Our results, \( C_{TSM} \) (3.73), provide a lower bound on the mismatch capacity, which in turn gives a lower bound on the capacity of perturbative theory based isolated soliton communication channel with Gordon Haus effect. The simulation of the channel realisation required for the Monte Carlo estimation of mismatch capacity is generated following each function block of the proposed system as in Figure 3.1. The pulses correspond to the input alphabet will be transmitted into a simulated fibre perturbed by ASE noise via the split step.
3.4. Mismatch Capacity for Soliton Communication over the NLSE Channel

Figure 3.3: Time-scaled MI estimated from the additive white Gaussian noise (AWGN) model optimisation in (3.22), the analytical approximation in (3.57), and the corresponding mismatch capacity bound in (3.73) for a 2000 km long fibre, assuming $\delta = 0.001$. The subplot shows the zoomed figure of $X_{ub} \in [330, 380]$. Fourier method based on the NLSE (2.13). The output pulse from the simulated fibre will then be put through an NFT detector, which extracts the imaginary part $R$ of the eigenvalue from the detected pulse. The received eigenvalue's imaginary part $R$ will then be VNT transformed into the transformed domain for decoding the information. Unless otherwise mentioned, $\delta = 0.001$ is assumed to calculate the soliton duration, i.e., 99.9% soliton energy pulse width.

3.4.1 Mismatch Capacity for Single Soliton Transmission

We first focus on single soliton transmission over the NLSE which takes into account the Gordon-Haus effect while ignoring the inter-soliton interaction effects. Using identical fibre parameters as in Table 3.1, Figure 3.3 compares the time-scaled mismatch capacity calculated based on 1000 realisations per possible symbol for $X_{ub} \in [200, 500]$ with the time-scaled capacity of AWGN model obtained in Section 3.3.2 and the analytical approximation derived in Section 3.3.6. From Figure 3.3, it can be observed that the time-scaled MI increases as the peak amplitude constraint increases. It is also observed that all the curves provide a well-matched estimations of the capacity, confirming that the Gordon-Haus effect is not so significant within the range of interest here. Nevertheless, we can see that, for larger $X_{ub}$, the gap between mismatch and AWGN curves increases, which can be due to the stronger Gordon-Haus effect, that will be experienced by larger amplitude soliton pulses. Note that the timing jitter introduced by the Gordon-Haus effect can shift the soliton beyond the limited timing window over which the NFT is applied, which leads to energy loss and possible errors in eigenvalue detection.
3.4.2 Mismatch Capacity for Soliton Sequence Transmission

The memoryless channel model of soliton communication considered in Section 3.3 and in most of the literature is only valid when there is no intersoliton interactions, limiting the accuracy of the model to the cases where the sequence of soliton pulses are well separated. In this section, we use the mismatch capacity approach introduced above to provide some insights on the impact of inter-soliton interaction effects on the capacity of soliton communication systems. In the previous section, the performance of the system is discussed based on simulating the transmission of a single soliton pulse through a long haul fibre-optic channel, which neglects the inter-solitonic interactions.

The modelling of the interacted solitons remains an open problem due to the nonlinear nature of the interaction, making it difficult to consider the interaction in the constellation shaping. In the literature both analytical [118] and numerical modelling [119] of the soliton parameters for two-soliton sequence are presented. In [120], the Fokker–Planck equation of a three-soliton sequence under distributed noise is derived analytically. To obtain the analytical perturbative theory model for the evolution of the three soliton sequence similar to [76, 77], the Fokker-Planck equation needs to be resolved under specific conditions. For a longer soliton sequence, the asymptotic behaviour of a $N$-soliton sequence is derived under a variety of perturbation for the solitons, including linear and nonlinear dispersive perturbation and etc. [121]. However, the derived asymptotic behaviour is not applicable for the soliton communication system discussed in this work, as the solitons should corresponds different amplitudes, as opposed to [121] where similar amplitudes are considered. Due to the stochastic and nonlinear nature of the interaction under ASE noise, and the deterministic nature of the intrinsic interaction given the amplitudes of the soliton sequence and signalling time-width, there might be a promising potential in using the neural network in detecting the interacted solitons. In this section, we assume that the size of the channel memory is 1 soliton on both side, i.e., the solitons that are one soliton away from the information soliton does not introduce interaction. (This assumption is later shown to be invalid in Chapter 4, as the length of the interaction induced memory is estimated to be 16-soliton long.) Hence, the transmission of a sequence of three soliton pulses is considered, where the middle soliton is considered to be the target soliton for detection. Meanwhile, the neighbouring solitons (i.e., the first and the third solitons) are assumed to be independently and randomly selected based on the statistics of the input signal distribution taken from the solution of the AWGN capacity formulation in (3.22). Note that the pulse width of a soliton is a function of $\delta$ and $X_{\text{min}}$ in the input signal distribution. The simulation is performed based on the same split step Fourier method employed in Section 3.4.1, while the NFT-based detection is only performed over the pulse width of the middle
3.4. Mismatch Capacity for Soliton Communication over the NLSE Channel

Soliton. Note that the channel memory of size of 1-soliton is an inaccurate approximation of the much longer sequence of solitons one may encounter in practice, but for the purpose of demonstrating the detrimental effect on the system AIR, this approximation should be sufficient.

![Figure 3.4: Inter-soliton interaction mean squared error (MSE) for different soliton pulse width determined by different values of $\delta$ and based on the link parameters stated in Table 3.1.](image)

It has been shown in [68] that, even in the absence of any noise, solitons can exert attracting or repelling forces on each other when they are not placed far enough, and this leads to inter-soliton interaction effects. Thus, before implementing the soliton sequence transmission in the presence of the ASE noise, we intend to estimate the mean squared error (MSE) induced by the noiseless inter-soliton interaction to evaluate the significance of this effect for different soliton separations. Recall that the ASE noise power after VNT is normalised to 1. Hence, the inter-soliton interaction effect would be negligible relative to noise, if the inter-soliton interaction MSE is much less than 1, i.e.,

$$\text{MSE} = \mathbb{E}[(Y_{nl} - X)^2] \ll 1,$$  

(3.74)

where $\mathbb{E}[\cdot]$ denotes expectation over all possible combination of the three-soliton sequences, $Y_{nl}$ denotes the received VNT transformed eigenvalue in a noiseless scenario. The noiseless simulation is based on the identical simulation parameters as in Table 3.1 but in the absence of ASE noise (i.e., assuming noiseless ideal distributed Raman amplification) and using the input soliton amplitudes taken from the capacity-approaching distribution given in Section 3.3.2. In this section, the signalling of the solitons are based on four different $\delta$ parameters and their corresponding pulse widths. Note that a smaller $\delta$ leads to a longer symbol duration as defined by (3.5), which results in more separation between solitons and thus less inter-soliton interaction.
3.4. Mismatch Capacity for Soliton Communication over the NLSE Channel

Figure 3.4 shows the inter-soliton interaction MSE estimated by simulating the transmission of all possible three-soliton sequences following the input distribution given in Section 3.3.2 assuming different values of \( \delta \). The overall trend of the MSE is increasing as the peak amplitude constraint \( X_{ub} \) is increasing. Moreover, as expected, decreasing the \( \delta \) parameter reduces the MSE. In fact, reducing \( \delta \) corresponds to the decreasing the fraction of energy truncation that essentially extends the soliton temporal separation. The additional temporal separation will reduce the force between the solitons [68], thus, the inter-soliton interaction is mitigated. Note that, for \( \delta = 10^{-3} \), the MSE goes beyond unity for \( X_{ub} > 300 \) as shown in Figure 3.4, meaning that the inter-soliton interaction effect becomes comparable to noise beyond that point, hence, the \( \delta \) parameter needs to be reduced to maintain a low interaction effect. Similarly, it is observed that the MSE becomes comparable to noise for \( \delta = 3 \times 10^{-4} \) beyond \( X_{ub} = 400 \). The decreasing trend observed in the Figure 3.4 might not be deterministic as the detection is performed on the segmented interacted soliton, whose level of distortion might not be well modelled by the amount of MSE introduced in the detected symbol.

![Figure 3.4](image1.png)

![Figure 3.5](image2.png)

**Figure 3.5:** The capacity estimation of the soliton communication based on the AWGN model optimisation in (3.22), and the mismatch capacity bounds in the presence (mis. inter) or absence (mis. no inter) of inter-soliton interaction effects in terms of (a) time-scaled MI and (b) MI, for different values of \( \delta \) and the link parameters stated in Table 3.1.

In order to evaluate the impact of inter-soliton interaction effect on the capacity of the system, Figure 3.5 shows the time-scaled capacity results and the corresponding MI calculated based on different proposed methods including the AWGN model and mismatch decoding with or without inter-soliton interaction effects for different values of \( \delta \). A second data rate axis is included to allow a direct transfer between the time-scaled MI and the data rate in bits/second, where the data rate is obtained by denormalising the time-width in time-scaled MI with normalisation time \( T_0 \). Figure 3.5a shows the significant impact of inter-soliton interaction effects on the time-scaled capacity at higher peak amplitudes. For example, for \( \delta = 10^{-3} \), the time-scaled MI gradually drops beyond \( X_{ub} = 300 \) and tends to zero before \( X_{ub} = 400 \).
3.4. Mismatch Capacity for Soliton Communication over the NLSE Channel

It is also observed that when $\delta$ decreases, the longer symbol duration scales down the time-scaled MI in the whole range of $X_{ub}$ but the efficiency of the communication system in combating intersoliton interaction effects improves (i.e., capacity drop shifts to higher soliton amplitudes). This indicates that there is a trade-off in selecting the parameter $\delta$. On the one hand, a smaller $\delta$ mitigates more effectively both inter-soliton interaction and Gordon-Haus effects, and on the other hand, it reduces how efficiently the temporal resources are being used. Hence, in future work, $\delta$ also needs to be included in the capacity problem formulation. Nevertheless, Figure 3.5a gives an estimation of sensitivity of the time-scaled capacity with respect to $\delta$ by providing the mismatch results at different values of this parameter. Therefore, by taking the supremum of the curves with different $\delta$ values in different parts of the dynamic range, we can obtain a good estimation of the capacity lower bound in the presence of soliton interaction. For example, based on the available results, the capacity result at $\delta = 10^{-3}$ is best up to $X_{ub} = 300$ while the capacity results for $\delta = 3 \times 10^{-4}$ and $\delta = 10^{-4}$ are best in ranges $X_{ub} \in [300, 400]$ and $X_{ub} > 400$, respectively. Due to the variable signalling time-width scheme employed in [109], the comparison between the data rates in [109] and in this Chapter under fixed signalling interval could not be made.

The MI results presented in Figure 3.5b is produced from scaling back the optimised time-scaled MI results in Figure 3.5a. It therefore focuses on how efficiently each soliton is decoded rather than how efficiently the temporal resources are being used. The figure shows that, for $\delta = 10^{-3}$, the inter-soliton interaction effect strongly degrades the mismatch capacity beyond $X_{ub} = 300$ as expected from Figures 3.4 and 3.5a. By reducing $\delta$, it is observed that the inter-soliton interaction effect decreases and it almost matches the mismatch capacity results with no interaction at $\delta = 10^{-4}$. This is also expected from Figure 3.4, as $\delta = 10^{-4}$ shows MSE $\ll 1$ for most of the range of interest. In addition, the mismatch capacity at $\delta = 10^{-5}$ even outperforms the mismatch capacity with no interaction and almost matches the AWGN result. This is because the mismatch with interaction at $\delta = 10^{-5}$ corresponds to the transmission of a soliton sequence with longer symbol duration. The longer duration essentially eliminates both the Gordon-Haus effect as well as the interaction effects. However, this is not the case in the mismatch results with no interaction where we still assume a shorter pulse width with $\delta = 10^{-3}$. This also verifies the accuracy of the proposed AWGN approximation model compared to the realistic simulated channel when both the Gordon-Haus and inter-soliton interaction effects are negligible.
3.5 Summary

In this chapter, the NCX2 channel model is first simplified into a mathematical equivalent unit variance AWGN channel by employing a VNT. The time-scaled capacity is then numerically calculated relying on the simplified channel model. Additionally, an analytical expression of the capacity is derived under certain approximations based on the numerical results. The effectiveness of the channel simplification and the approximated capacity result is demonstrated by the good agreement between the numerically estimated AWGN capacity, the analytically approximated capacity for the AWGN channel, and the Monte Carlo method estimated mismatch capacity of the split-step simulated NLSE channel. Lastly, the impact of inter-soliton interaction on the soliton communication system is briefly demonstrated via simulations of three-soliton sequences. The dominance of the intrinsic inter-soliton interaction over Gordon-Haus timing jitter is shown through the detrimental mismatch capacity drop in the interacted soliton system. In the next chapter, the intrinsic interaction and its coupling effect with the Gordon Haus effect is demonstrated further with numerical simulation. The feasibility of addressing such nonlinear effect with artificial neural network will also be exploited.
In practical implementations of amplitude-modulated soliton systems, sequences of modulated solitons are typically transmitted. However, the inter-soliton force exerts between successive solitons in addition to the Gordon-Haus effect induced by amplifier spontaneous emission (ASE) noise. As discussed in Chapter 3, this force can cause soliton interaction even in the absence of noise, making the perturbative theory noise model effectively invalid. Interacted solitons cannot be detected by simply extracting eigenvalues with the nonlinear Fourier transform (NFT). The rapid development of artificial neural networks (ANNs) provides a new potential solution for detecting soliton sequences in interaction-perturbed systems.

In this chapter, we present a comprehensive visualisation of the coupling between the two effects and estimate the achievable information rate (AIR) of an ANN-assisted detector based on simulations of soliton sequence transmissions. This provides a feasibility analysis of using ANNs to address the detection problem in an interaction-impaired system.

### 4.1 Literature Review

In the majority of the literature, it is assumed that each soliton pulse is well-separated temporally, so that interaction between neighbouring solitons can be ignored. As a result, the information rate is typically estimated in bits/symbol due to difficulties in determining the appropriate pulse width for signalling. In [109], the time-scaled mutual information (MI) was used as the objective function for the constellation shaping problem. The signal-dependent pulse width of a single soliton pulse was taken into account, and it was assumed that the receiver was capable of variable-length pulse detection. An estimation of the data rate in bits/s was provided. In [122], an achievable rate in bits/s was estimated by considering the estimated statistics of the Gordon-Haus timing jitter effect at the receiver. It was reported...
that the energy radiation leaked into the continuous spectrum due to solitonic interaction and pulse truncation is correlated with the discrete spectrum. Such correlation could be exploited to improve soliton detection when interaction occurs [123]. Furthermore, it is worth mentioning [124] that a novel technique was proposed to allow the occurrence of solitonic interaction by carefully designing the phase relationships between neighbouring solitons. It was reported that with the proposed phase design, the positions of solitons after interaction will exchange in a deterministic manner.

Recent developments in digital signal processing (DSP) chips have made it possible to implement artificial neural networks (ANNs) in applications with high computational complexity, such as pattern recognition. ANNs have shown significant advances in identifying highly complex patterns, as they can approximate any nonlinear function using different activation functions and neuron connection schemes, according to the universal approximation theorem [83]. As reviewed previously, implementing an nonlinear Frequency division multiplexed (NFDM) system involves a series of highly nonlinear and complicated operations, such as NFT and inverse NFT (INFT), detecting solitons after interaction, and addressing crosstalk between continuous spectrum (CS) and discrete spectrum (DS). The success of data-driven techniques, such as ANNs, in various applications, including object recognition in autopilot vehicles and natural language processing in ChatGPT\(^1\), implies their great potential in improving the performance of NFDM systems. In [49], ANN classifiers and ANNs regression equalisers were implemented and compared for symbol detection in conventional wavelength division multiplexing (WDM) systems. In [125], the application of ANN in both single carrier WDM system and NFDM system are revised. ANNs can also be employed in learned digital backpropagation (DBP) to unroll channel effects in conventional WDM systems [125]. In [126], the bidirectional long short-term memory (BLSTM) network is employed to performed learned DBP to compensate the nonlinear impairment on a polarisation multiplexed WDM system. Various schemes of employing ANNs in NFDM systems including end to end learning optimisation have also been reviewed [125].

Due to the computational complexity involved in computing eigenvalues in multi-eigenvalue DS NFDM systems, the use of NFT for symbol detection can be less efficient and suffer from higher processing noise. Moreover, NFT may not be able to reliably extract DS from interacted soliton pulses. As a result, data-driven ANN receivers have gained more attention. In [84, 127], a time-domain receiver was implemented with a single hidden-layer feedforward ANN to detect a two-eigenvalue pulse. Even with a single hidden layer, the ANN outperformed direct detection with NFT and the minimum Euclidean distance benchmark scheme. In [128], an autoencoder relying on unsupervised ANN was used to adaptively adjust the DS modulation scheme according to the true nonlinear optical fibre channel. Typically, in the numerical

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1. Please see the Acknowledgement of this thesis, a poem with certain rhythm is generated by ChatGPT with materials provided by the author.
4.1 Literature Review

NFT algorithms, filtering is implemented to remove excessive noise in eigenvalue estimation. However, in [129], noise that would otherwise be discarded was taken into a trained ANN to extract any remaining potential information to assist with eigenvalue detection. In [130], the ANN was employed as an equaliser to equalise channel perturbation, rather than operating in classifier mode to make hard decisions for multi-eigenvalue DS NFDM systems, whose information is carried by the $b$ coefficient of the norming constant.

On the other hand, a CS NFDM system may suffer from high inter-subcarrier interference, which can be interpreted as a nonlinear signal-dependent channel with memory, making symbol detection with conventional NFT difficult. Data-driven ANN can potentially exploit the memory effect of the channel to assist symbol detection and provide a performance gain. In [131], a convolutional neural network (CNN) was operated in serial mode for symbol detection of CS NFDM systems and in parallel mode to detect symbols for parallel channels simultaneously. It was shown that high-accuracy detection could be achieved with reasonable computational complexity compared to conventional NFT. In [132], $k$-means clustering and support vector machine (SVM) classifiers were implemented in a CS NFDM system and compared, showing that SVM could adapt to non-circular received symbol clouds better than its $k$-means counterpart. Furthermore, in [133], the performance of a feedforward ANN as a hard decision detector and noise equaliser were demonstrated individually. It was shown that error performance could be improved with the assistance of ANN, and ANN operated as an equaliser could provide superior gain over its hard decision classifier counterpart. In [134], a two-stage detection scheme was proposed. The received time-domain pulse should be first equalised with an ANN before extracting CS. After additional ANN equalisation on the CS, it was demonstrated that error performance could be further improved.

4.2 System Model

Recalled the pulse evolution in a lossless, nonlinear standard single mode fibre (SMF) is governed by the stochastic nonlinear Schrödinger equation (NLSE) defined as (2.14) with ASE noise statistics given as (2.16) in Chapter 2. The NFT transforms the time domain signal into scattering data which evolves linearly in the nonlinear NLSE channel. The scattering data includes the continuous spectrum (CS), $\rho(\lambda, z)$, and the discrete spectrum (DS), $\{\lambda_m(z)_{m=1}^{M}, C_m(z)_{m=1}^{M}\}$. The DS contains eigenvalues $\lambda_m$ and corresponding norming constants $C_m$. If only the imaginary part of a single eigenvalue is employed and the soliton is centred at time zero, the solution of (2.14) is known as first order soliton, defined as

$$q(t, 0) = 2 \text{Asech}(2At),$$

(4.1)
where $A = \text{Im}(\lambda_1) > 0$ and norming constant $C_1 = -2A$ at transmitter $z = 0$. Similar to a Gaussian pulse, the soliton pulse (4.1) contains a sech hyperbolic shape in both time and frequency domains, which depends on its amplitude $A$. For the practical implementation of the soliton, the pulse should be truncated within $[\frac{-1}{2T}, \frac{1}{2T}]$. If $(1 - \delta)$ of the soliton energy is preserved within $t_w$, the time width could be defined as [38]

$$t_w(A, \delta) = \frac{1}{2A} \ln \left( \frac{2}{\delta} - 1 \right). \quad (4.2)$$

Similarly, the bandwidth of soliton that preserved $(1 - \epsilon)$ of energy could be defined as

$$f_w(A, \delta) = \frac{2A}{\pi^2} \ln \left( \frac{2}{\epsilon} - 1 \right). \quad (4.3)$$

Using time width (4.2) and bandwidth (4.3), the $(1 - \delta)$ time width and $(1 - \epsilon)$ bandwidth describe the resources occupied by soliton. These time and spectral resources can be extended to a constellation of soliton amplitudes, where the time width for the signalling is determined by the minimum amplitude, $A_{\text{min}}$, while the occupied bandwidth is determined by the maximum amplitude, $A_{\text{max}}$. Similar to [123, 135, 136], a fixed time-bandwidth product is considered to estimate the system performance with limited resources. Hence, the time width for different constellations will be determined differently corresponding to their $A_{\text{min}}$ values.

To utilise available resources to achieve maximum spectral efficiency (SE), constellation shaping should be employed to obtain the best constellation.

$$C_{\text{bpcu}} = \sup_{P_A(A):A \in [0];[A_{\text{min}}, A_{\text{max}}]} I(A;R), \quad (4.4)$$

where $I(A;R)$ indicates the MI between transmitted and received eigenvalues $A$ and $R$, $A_{\text{min}}$ and $A_{\text{max}}$ denotes the minimum and maximum soliton amplitude of the input distribution respectively. The objective function, the mutual information,

$$I(A;R) = I(T(A); T(R)) = I(X;Y),$$

is estimated using an approximated Gaussian channel for the isolated amplitude-modulated soliton system with a variance normalising transform (VNT) $T(u) = 2\sqrt{2u/\sigma^2T}$, where $l$ denotes the unitless propagation distance\(^2\). The range of soliton amplitude is defined by the effective time-bandwidth product,

$$TB(A_{\text{min}}, A_{\text{max}}, \delta, \epsilon) = \frac{1}{\pi^2} \frac{A_{\text{max}}}{A_{\text{min}}} \ln \left( \frac{2}{\delta} - 1 \right) \ln \left( \frac{2}{\epsilon} - 1 \right), \quad (4.5)$$

\(^2\) More details are discussed in Chapter 3, where the equivalence of the channel is proven.
where the energy truncation factors $\delta$ and $\epsilon$ can be selected independently. In this work, unless specified otherwise, $\delta$ and $\epsilon$ are selected to be $10^{-2}$, i.e. 99% of the soliton energy should be preserved in both time and frequency domains. When the time-bandwidth product $TB$ and available bandwidth $f_w$ is given, the value of $A_{\min}$ could be calculated accordingly.

### 4.3 Channel Perturbation Beyond Additive Noise

In Chapter 3, the mismatch capacity results estimated with simulated transmission of the time-scaled MI maximised constellation show that the degradation of system performance is not mainly constrained by the ASE noise directly. When sufficient temporal separation is assumed for the system, the mismatch capacity estimated shows that the Gordon-Haus effect is less significant in the operational regime of interest. When random neighbours are considered, the mismatch capacity will degrade to zero in a higher power regime if insufficient temporal spacing is implemented. The AIR estimation shows that the intractable non-noise channel perturbations, such as the intrinsic inter-soliton interaction and Gordon-Haus timing jitter, deviate the actual channel model away from that predicted by the perturbative theory, which describes the noise effect. Therefore, this section mainly discusses the numerical simulation of these intractable non-noise channel perturbations.

#### 4.3.1 Intrinsic Inter-Soliton Interaction

In Chapter 3.4.2, we simulated transmissions of a series of three-soliton pulse sequences in an ideal lossless nonlinear fibre with no ASE noise to isolate the intrinsic interaction from the coupling with the noise-induced Gordon-Haus effect. The received three-soliton sequence was segmented into three equally long segments, where the second pulse segment was regarded as the information soliton. We evaluated the effect of intrinsic interaction by calculating the mean squared error (MSE) between the transmitted symbol and the NFT-estimated received soliton amplitude. This segmentation-NFT scheme is referred to as the direct NFT scheme in this chapter. Our results show that the MSE of the direct NFT scheme in the noise-free three-soliton sequence simulation is significantly higher than the ASE noise-induced MSE in the isolated soliton simulation.

The experiments in Chapter 3 reveal the dominance of intrinsic interaction in an amplitude-modulated soliton system that is implemented in a close-to-practice scenario. However, visualisation of the inter-soliton interaction effect on the solitonic pulse is lacking. To illustrate this effect, a simple 4-pulse amplitude modulation (PAM) constellation of $[0, 0.75, 1, 1.25]$ is selected as an example constellation taken from the mid-amplitude regime discussed in Chapter 3. To improve the signalling efficiency $t_w(0.75, 10^{-2})$ is selected as the soliton time-width, resulting in the signal power of 1.48, 2.73 and 3.70 dBm for the nonzero solitons, respectively. When comparing to the $\delta = 10^{-3}$ in Chapter 3, the reduced pulse width results...
Figure 4.1: Examples of transmitted and received three-soliton sequences through a noiseless and lossless nonlinear fibre with a 4-PAM constellation of \([0, 0.75, 1, 1.25]\). The edges of each soliton segments are highlighted with grid lines, while the transmitted soliton amplitudes are noted at the top of each figure.

in higher signal power. The selected constellation and signalling time-width are then used to simulate the propagation of a sequence of three randomly selected solitons in a noiseless NLSE channel. Figure 4.1 provides examples of four soliton amplitude combinations for each amplitude of the centre soliton that generate the highest MSE between the transmitted symbol and the symbol detected via direct NFT.

Since the pulse width of each soliton is determined by the minimum soliton amplitude in the constellation, solitons with \(A \geq 0.75\) will naturally impose guarding intervals on both sides of the pulse. This is due to the characteristics of the sech pulse, where the signal decreases towards zero at the tails of the pulse. From the example simulation results shown in Figure 4.1, it is not difficult to see that lower interaction occurs with larger solitons, while stronger interaction can be observed for smaller solitons, which correspond to a less effective guarding interval. For example, the combination \([0.75, 1.25, 0.75]\) results in almost overlapping transmitted and received pulses. For the combination \([1, 1, 1]\), it could be observed that neighbour solitons lean toward the centre soliton, while the interaction is not strong enough to completely diminish the bell shape of the soliton pulse. In this particular constellation set, the soliton amplitude 0.75 preserves the least of its soliton energy, in other words, no effective guarding interval is present. As a result, when more than one soliton with amplitude 0.75 is transmitted
successively, the completeness of the pulse cannot be guaranteed. For example, only two main lobes could be observed at receiver for the \( \left[ 0.75, 0.75, 0.75 \right] \) combination, while three main lobes are transmitted, as shown in the pulse evolution Figure 4.2a. When two solitons with amplitude 0.75 are transmitted, the two solitons will collide and form a single lobe as seen in examples like the combinations \( \left[ 1, 0.75, 0.75 \right] \) and \( \left[ 1.25, 0.75, 0.75 \right] \). The evolution of the soliton sequence \( \left[ 1, 25, 0.75, 0.75 \right] \) is shown in Figure 4.2b for visualisation of this phenomenon.

It should be noted that a well-spaced constellation was employed in this demonstration, allowing for sufficient differences between soliton amplitudes. When a fixed soliton temporal interval is considered for pulse generation, a sufficiently large guarding interval could be introduced for soliton amplitude 1 and 1.25, enabling them to be less or negligibly perturbed by the interaction. However, when a more closely spaced constellation is employed in such a fixed interval soliton system, higher intrinsic interaction could be observed due to the reduced effective guarding interval. Additionally, three-soliton sequences are mainly demonstrated in this section for simplicity in visualisation. In practical applications, much longer soliton sequences
with more solitons are expected to be transmitted. For longer soliton sequences, inter-soliton interaction could become more complicated and intractable. Example simulated transmissions of five-soliton sequences with more closely spaced a 4-PAM constellation \([0, 1.05, 1.15, 1.25]\) are presented in Figure 4.3, where more interacted combination could be identified.

\[ \begin{array}{c}
\text{Figure 4.3:} \text{ Examples of transmitted and received five-soliton sequences through a noiseless and lossless nonlinear fibre with a 4-PAM constellation of } [0, 1.05, 1.15, 1.25]\text{. The edges of each soliton segments are highlighted with grid lines, while the transmitted soliton amplitudes are noted at the top of each figure.} \\
\end{array} \]

### 4.3.2 Gordon Haus Effect

In the presence of ASE noise, random fluctuations in the group velocity of solitons can cause them to arrive at the receiver with random timing jitter, introducing a constraint on the maximum allowable soliton transmission rate for a given propagation distance [67]. The Gordon-Haus effect could increase error probability via two mechanisms. On the one hand, the nature of the timing jitter suggests that the soliton could be shifted out of the sampling window of the NFT when the soliton is well isolated. On the other hand, when solitons are transmitted in sequence, the jitter could potentially shift the soliton towards its neighbours to enhance the inter-soliton interaction. In [67], the variance of the timing jitter is estimated, which is later
extended by taking into account simple solitonic interaction in an on-off keying (OOK) soliton system in [137]. In [122], the guarding interval is proposed based on the jitter variance, with the belief that it can ensure that the soliton is free from the effects of inter-soliton interaction induced by jitter.

To demonstrate the timing jitter at the receiver, solitons with amplitudes from the constellation $[0, 0.75, 1, 1.25]$ are generated with time width of $t_w (0.75, 10^{-2})$. For each nonzero amplitude, 1000 realisations are generated in an ASE noise perturbed nonlinear fibre. In each realisation, a three-soliton sequence is launched into the fibre, where two neighbouring solitons are set to be 0 to serve as guarding intervals, isolating the centre soliton and allowing the Gordon-Haus timing jitter to be isolated from intrinsic solitonic interaction. Next, the NFT is performed on the received three-soliton sequence with two different sampling windows, $t_w$ and $3t_w$. The NFT with sampling windows $t_w$ is performed on the segment of the sequence corresponding to the soliton capturing the potential extra error induced by timing jitter moving the soliton out of the window. The NFT with sampling windows $3t_w$ results in a reference detection scheme where a more than sufficient guarding interval is provided to capture all the shifted soliton.

![Figure 4.4](image)

Figure 4.4: Histograms of the NFT detected soliton amplitude of transmitting 0.75, 1 and 1.25 with two 0 solitons as guarding intervals on both sides. The NFT detection is performed on both the centre soliton, with a $t_w$ sampling window, and the whole sequence, with a $3t_w$ sampling window.

In the Figure 4.4, histograms of the NFT detected soliton amplitudes are presented. The variances of the received symbol summarise the total power of the Gordon-Haus effect induced error and ASE noise. When transmitting 0.75, there is effectively no guarding interval between the main soliton lobe and its neighbour zero solitons, making the soliton more sensitive to jitter. Comparing the corresponding variances for both NFT detection schemes, it could be quantitatively estimated that the Gordon Haus effect introduces around 34% more error power by simply shifting the soliton away from the sampling window for soliton amplitude of 0.75. As
expected, higher soliton amplitudes are more resistant to jitter due to the effective guarding intervals, with increments of 3.4% and 0.4% observed for soliton amplitudes of 1 and 1.25, respectively. In addition to the variance difference, the mean differences between the two NFT detection schemes are due to the different level of energy truncation. Similar to the Gordon-Haus induced variance difference, it is more obvious in lower soliton amplitudes because of the signal dependence of time width.

To visualise the Gordon-Haus timing jitter, the centre of the soliton pulse is required. Relying on the equation for the soliton centre [70] and assuming that the centre of soliton is initially set at 0, the Gordon Haus effect induced jitter could be estimated with

\[
t_{\text{GH}} = \frac{1}{2R} \log_2 \left| \frac{C_1}{2R} \right|
\]

(4.6)

where the \( C_1 \) denotes the norming constant detected for the single eigenvalue. The amount of Gordon Haus timing jitter is estimated and sketched in Figure 4.5.

![Figure 4.5: Histograms of the Gordon Haus timing jitter \( t_{\text{GH}} \) of transmitting 0.75, 1 and 1.25 with zero solitons as guarding intervals on both sides. The NFT detection is performed on both the centre soliton with a \( t_w \) sampling window, and the whole sequence with a 3\( t_w \) sampling window.](image)

Estimating the statistics with the same setup as before, larger jitter could be observed for the larger transmitted soliton. For example, the variance of the jitter for soliton amplitude 0.75 is estimated to be \( 2.54 \times 10^{-4} \) for NFT detection with a sampling window \( t_w \), while the variance increases to \( 3.82 \times 10^{-4} \) for soliton amplitude 1.25. The dependence of the jitter on the signal properties has been reported previously through perturbative theory studies [67,99]. Comparing between the NFT results for two sampling windows, a larger difference between
the variances could be observed for smaller signal due to the smaller effective guarding interval. Furthermore, the larger sampling window provides more temporal space for the NFT to capture more severe jitter accurately, leading to a larger variance when using a larger sampling window.

Following the second mechanism of generating errors by Gordon Haus effect, the timing jitter could potentially move the soliton towards its neighbours, increasing the interaction between the solitons when combined with the intrinsic inter-soliton force and noise-induced timing jitter. However, due to coupling of the intrinsic interaction and Gordon Haus induced interaction, it is difficult to visualise such an effect. Moreover, in the previous chapter, it is also demonstrated that the Gordon Haus timing jitter induced out-of-window error is not as significant as the intrinsic interaction. Figure 4.6 shows 1000 transmissions of a three-soliton sequence over a noisy NLSE channel using a higher power constellation of \([0, 1.75, 1.95, 2.25]\). The received pulse over a noiseless fibre is also included for reference, and the edges of each soliton segment are highlighted with grid lines.

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4.3. Channel Perturbation Beyond Additive Noise

Conventionally, the Gordon Haus timing jitter limits the performance of soliton systems by introducing a limit on the soliton rate propagation distance product [67]. For a given link distance, this constant limits the soliton rate, severely constraining the data rate of OOK soliton systems. To improve the data rate, one can either increase the symbol rate or the amount of information carried by one symbol. Using higher-order constellations or shorter pulse widths for higher-order solitons can be used to achieve this goal. However, as seen in Figure 4.5, larger solitons are more susceptible to timing jitter, which can be further increased by the Gordon Haus effect. Additionally, as demonstrated in Chapter 3, coupling between the Gordon Haus effect and intrinsic interaction poses a significant challenge for implementing soliton communication systems. The extreme nonlinear distortion observed in this section highlights the difficulties involved in practical implementations of soliton communication systems.

4.3.3 Digital Back-propagation

As discussed in the previous section, the primary challenge in a soliton communication system is the nonlinear effects caused by both intrinsic and Gordon Haus-induced soliton interactions. Digital back-propagation (DBP) has been proposed as a general technique to reverse nonlinear effects in a sampled digital domain, where the signal is represented as a sequence of discrete values. DBP involves applying the inverse of the NLSE, which is the mathematical equation that models the nonlinear effects in the optical fibre, to the received signal. This process effectively unwinds any deterministic nonlinear effects introduced by the NLSE, including soliton interaction, and restores the original pulses.

Referring to the split step Fourier simulation of the NLSE discussed in Chapter 2, inverting the NLSE could be achieved in a similar manner. The split step Fourier implementation of the DBP relies on the same approximation that when a segment is sufficiently small, the occurrences of linear dispersion effect and the nonlinearity could be considered sequential. The split step Fourier DBP is implemented by recursively solving the following equations initialised with a received pulse \( Q(T, Z) \) with its physical unit,

\[
\begin{align*}
\widetilde{Q}_{D1}(\omega) &= \hat{Q}(\omega, Z + h_S) \exp\left(-j\frac{\beta_2}{2} \omega^2 h_S^2 \right); \\
Q_N(T) &= \exp(-h_S j \gamma |\hat{Q}_{D1}(T)|^2) \hat{Q}_{D1}(T); \\
\hat{Q}(\omega, Z) &= \tilde{Q}_N(\omega) \exp\left(-j\frac{\beta_2}{2} \omega^2 h_S^2 \right),
\end{align*}
\]

where \( h_S \) denotes the step size of the split step Fourier method, the \( \hat{Q}(\omega) \) denotes the Fourier transform of the \( Q(T) \), and the \( \hat{Q}_{D1}, \hat{Q}_{D2} \) and \( \hat{Q}_N \) are the intermediate auxiliary pulse introduced for the convenience of computation.
4.3. Channel Perturbation Beyond Additive Noise

Figure 4.7: Examples of DBP interaction compensation for received three-soliton sequences through a noiseless and lossless nonlinear fibre with 4-PAM constellation of $[0, 0.75, 1, 1.25]$. The edges of each soliton segment are highlighted with grid lines.

In Figure 4.7, the intrinsic interaction in the received pulses corresponding to soliton combinations $[0.75, 0.75, 0.75]$ and $[0.75, 0.75, 1]$ under a noiseless and lossless nonlinear fibre is effectively removed with the split step Fourier DBP. In both figures, it could be concluded that the DBP can remove the intrinsic interaction whose pulse evolution is completely described by the deterministic NLSE. Small errors could be observed in the DBP compensated pulse where abrupt changes occur, for example, at connections between solitons. As the sampling frequency is determined by the bandwidth of the soliton, high frequency components corresponding to abrupt changes induced by truncation cannot be captured with the specified sampling frequency. In other words, the NLSE inverted pulse via DBP results in a low-pass filtered transmitted pulse.

In principle, the deterministic DBP is incapable of removing stochastic noise. Nevertheless, it can still significantly reduce the effect of nonlinear interaction, even if it is enhanced by random Gordon Haus timing jitter. Recall that Gordon Haus induced interaction is introduced when solitons are shifted temporally close to each other. By reversing the interaction with DBP, the temporally close solitons can be restored. The performance of DBP can be demonstrated with combinations in Figure 4.6, where the Gordon Haus effect enhances inter-soliton interaction. In Figure 4.8, the received three-soliton sequences are equalised with DBP with a step size of $h_s = 200$ m. Despite the enhanced interaction, it can be observed that DBP can still reduce the interaction to estimate the transmitted pulse. Due to stochastic ASE noise and a limited sampling frequency, the DBP equalised pulse could be regarded as an estimation of low-pass filtered transmitted pulse with noise, which can then be processed to estimate the transmitted symbol. In general, the DBP method requires a complete soliton sequence to achieve a complete inversion of the NLSE effects. As discussed earlier, when more complex constellations are used, and more solitons are transmitted in sequence, requiring the complete interacted
4.3. Channel Perturbation Beyond Additive Noise

sequence will increase communication overhead and memory demand. Another drawback of DBP is the computational time. Similar to the split-step Fourier simulation of NLSE, DBP is a recursive algorithm, where each step is dependent on the previous step, making the algorithm non-parallelizable. Other than the DBP, the other conventional equalisation techniques include Volterra equalisation could also restore the transmitted pulse to some extent. Although the state-of-art Volterra equalisation could achieve similar performance as the DBP, it is still limited by the computational complexity required by the long channel memory [51]. Similarly, another equalisation technique, the decision feedback equaliser, might also suffer from the same limitation. In [138], the decision feedback equaliser implemented with the Volterra filter is essentially a nonlinear filter with specified number of taps. The decision feedback equaliser might also suffer from a similar limitation due to the long channel memory, which will be demonstrated later in Section 4.7.

![Figure 4.8: Examples of transmitted and DBP compensated received three-soliton sequences through a noisy, lossless nonlinear fibre with 4-PAM of constellation of \([0, 0.75, 1.25, 1.75]\). The received pulse over a noiseless fibre is also included for reference, and the edges of each soliton segment are highlighted with grid lines.](image)

**Figure 4.8**: Examples of transmitted and DBP compensated received three-soliton sequences through a noisy, lossless nonlinear fibre with 4-PAM of constellation of \([0, 1.75, 1.95, 2.25]\). The received pulse over a noiseless fibre is also included for reference, and the edges of each soliton segment are highlighted with grid lines.

### 4.3.4 Sequence NFT Detection

As a competitive candidate, NFT also has potential in detecting interacted solitons. As reviewed in [34], the physical meaning of the DS is the solitonic components of the time domain pulse. When interaction occurs, the nature of these solitonic components should remain. To investigate the potential of NFT in detecting interacted solitons, two three-soliton sequences are formed by taking combinations \([0.75, 0.75, 0.75]\) and \([0.75, 0.75, 1.25]\) from the 4-PAM constellation \([0, 0.75, 1, 1.25]\). The NFT is then applied over the entire three-soliton sequence and the resulting sequences are then transmitted through a noiseless and lossless nonlinear fibre to explore the feasibility of using NFT against intrinsic inter-soliton interaction.
The results of this experiment are shown in Figure 4.9. Firstly, it is worth noting that the eigenvalues and norming constants do not correspond to linear combinations of the DS of each individual time-shifted solitons. The transmitted three-soliton sequence with soliton amplitudes of 0.75 gives rise to eigenvalues of \([0.54j, 0.77j, 0.88j]\), as shown in Figure 4.7a. These eigenvalues do not correspond to a linear combination of \(0.75j\) for each soliton, which is due to the fact that the NFD spectrum does not obey the superposition property of linear systems. Similarly, the norming constants cannot assist the detection of the transmitted combination of the solitons due to the same reason. For instance, the combination \([0.75, 0.75, 0.75]\) gives rise to norming constants of \([-31, -147.9, -120.6]\), although the order of transmission could be estimated with Equation (4.6) for this particular example, the deviated eigenvalues results increases the complexity of detection.
At the receiver side, the eigenvalues show an invariant property despite small errors in the real parts, which could be due to computation errors. For example, the received eigenvalues for the combination $[0.75j, 0.75j, 0.75j]$ are $[0.00026 + 0.52j, -0.00042 + 0.77j, 7.2 \times 10^{-5} + 0.88j]$, where the imaginary parts of the three eigenvalues are well-preserved, and the real parts show errors of the order of $10^{-3}$. However, the norming constants are retrieved with larger amounts of error. For the same combination, the equalised norming constants at the receiver are estimated to be $[-20.88 + 6.88j, -122.55 - 1.63j, -105.977 - 5.91j]$, indicating a maximum error in imaginary parts of the order of 6.88. In another combination $[0.75, 0.75, 1]$, the maximum error in the imaginary parts is of the order of $159.26$, while the corresponding transmitted norming constant is 68.48.

A similar observation could be made for the transmission in a noisy NLSE channel, where the Gordon Haus effect is taken into consideration. The same combinations from the constellation as in Figure 4.6 are chosen to create the transmitted three-soliton sequence via both noiseless and noisy NLSE channels. The noisy transmission is simulated 1000 times to generate a scatter plot for received eigenvalues and norming constants, as shown in Figure 4.10.

Similar results are also observed for the combination in higher power regime. For example, the combination $[1.75, 1.75, 1.75]$ results in transmitted eigenvalues and norming constants of $[1.27j, 1.78j, 2.04j]$ and $[-73, -345, -281]$, respectively. After noiseless propagation, the received eigenvalues and norming constants pair is estimated to be $[0.0012 + 1.24j, -0.0004 + 1.77j, -0.0002 + 2.05j]$ and $[-40.77 + 44.61j, -205.72 + 183.19j, -228.48 + 6.23j]$. The perturbation of the ASE noise and intrinsic soliton interaction is illustrated in Figure 4.10. The eigenvalues remain separated despite ASE noise, and the noisy received norming constants surround the noiseless received norming constants. Although well separated, the eigenvalues detected cannot be interpreted into the transmitted combination easily. The experiments reveal the potential of employing higher-order solitons, which correspond to more than one eigenvalue for communication. However, for the purpose of detecting interacted solitons, the NFT extracted DS does not provide enough information to retrieve the transmitted solitons in their transmitted order.

Similar to the DBP scheme, the eigenvalues in the interacted solitons could be extracted if the complete interacted pulse shape is provided to the NFT detector. However, the order of the eigenvalues cannot be determined based on the norming constants corresponding to eigenvalues because of the nonlinearity of the NFT operation itself. Although the Fast NFT algorithm [75] is computationally more efficient than the split-step Fourier DBP algorithm, the NFT has limited performance in terms of detecting interacted solitons.

As a benchmark to other detection schemes and to highlight of the detrimental impact of the soliton interaction if not carefully handled, the first detection scheme employed is direct NFT detection with VNT to resolve signal dependence of noise [38]. The direct NFT scheme involves segmenting the soliton sequence into equally long pulses and then performing the
4.3. Channel Perturbation Beyond Additive Noise

Figure 4.10: Examples of detecting the eigenvalues (left) and norming constants (right) with NFT for transmitted and received three-soliton sequences through a noiseless and lossless nonlinear fibre with 4-PAM of constellation of $[0, 1.75, 1.95, 2.25]$.

NFT on each segment to extract eigenvalues. Using the auxiliary mismatch decoding scheme as described in Section 2.3.3, the AIR for this scheme is estimated with [81]

$$I_{\text{isolated}} = E_{X,Y} \left[ \log \frac{Q_1(y|x)}{Q_1(y)} \right], \tag{4.7}$$

where an unit-variance AWGN channel $Q_1(y|x)$ is employed as an auxiliary mismatch channel, the auxiliary output channel model is calculated via $Q_1(y) = \sum_x P_X(x)Q_1(y|x)$ and numerical expectation is taken over a larger number of channel realisations. Recall that it is shown in Section 3.4.1 that the unit-variance AWGN channel provides a tight lower bound when solitonic interaction is not taken into account.
4.4 Continuous Spectrum Assisted Detection

Ideally, no CS should be extracted from both transmitted and received pulses. In practice, however, imperfections such as the discretization of the continuous signal, ASE noise introduced by optical amplifiers in long haul fibre, and the truncation of soliton pulses at the transmitter can introduce noise on both the imaginary and real parts of the eigenvalue, generating non-zero CS. Conventional detection schemes for the soliton amplitude typically discard the received CS as it is regarded as uncorrelated noise. However, it has been reported that non-zero CS generated from practical implementation, such as ASE noise and soliton truncation, is potentially correlated to the error between transmitted and received soliton amplitude even under soliton interaction [123]. Promising gains have been demonstrated in [123] under no assumption of ignoring the soliton interaction. The second detection scheme aims at exploiting this potential correlation to reduce the error in detecting the received soliton amplitude. This detection scheme is considered in this work to provide a benchmark schemes that takes into account soliton interaction to some extent using traditional statistics techniques.

The estimated correlation is used to compute an affine minimum mean squared error (AMMSE) estimation of the noise [123]. To formulate the estimator, the estimated term should be first defined. In this particular detection scheme, the correlation is used in the estimation of the error between the transmitted and received soliton amplitude, i.e. the error between the imaginary parts of the transmitted and received eigenvalues. Hence, the estimated term is defined as

\[ n_\eta = R - A, \]  

(4.8)

where \( n_\eta \) denotes the error in the imaginary parts of the eigenvalues, while \( R = \text{Im}(\lambda_1(l)) \) and \( A = \text{Im}(\lambda_1(0)) \) denote the imaginary parts of the received and transmitted eigenvalues.

The noise \( n_\eta \) is estimated with sampled CS spectrum vectors \( \rho' \)

\[
\rho' = \begin{bmatrix} \text{Re}(\rho(\lambda, l)) \\ \text{Im}(\rho(\lambda, l)) \end{bmatrix},
\]  

(4.9)

where the received CS \( \rho(\lambda, l) \) is sampled at \( N_{CS} \)-element vector of nonlinear frequencies \( \lambda \) and rewritten in its real and imaginary parts. An affine transformation is then introduced to convert the real CS vector \( \rho' \) with the dimension of \( 2N_{CS} \times 1 \) to the scalar estimator \( \hat{n}_\eta \) as

\[
\hat{n}_\eta = W \rho' + B
\]  

(4.10)

where \( W \) and \( B \) denote the weights vector and bias correspondingly which are estimated from a set of known training data. Finally, selecting the mean squared error (MSE) as the objective function, the AMMSE estimator could be defined with

\[
[W^*, B^*] = \arg \min_{[W,B]} \mathbb{E}_{\lambda,R}\rho' |n_\eta - \hat{n}_\eta|^2,
\]  

(4.11)
and substituting the optimal weights $W^*$ and bias $B^*$ into the AMMSE estimator $\hat{n}_\eta$, the output of the AMMSE detector is defined as

$$R_{\text{AMMSE}} = R - \hat{n}_\eta^*,$$

(4.12)

where the detection error $\hat{n}_\eta^* = W^* \rho' + B^*$ is estimated with the CS and is then subtracted from the received symbol $R$.

The solution for the linear minimum mean squared error ($\text{LMMSE}$) problem (4.11) is trivial. It is widely known that the minimum MSE could be achieved with

$$\hat{n}_\eta = \Gamma(n_\eta, \nu) \frac{\sigma_{n_\eta}}{\sigma_\nu} (v - \mu_\nu) + \mu_n,$$

(4.13)

where an auxiliary scalar $\nu$ is introduced and it is defined as an alternative linear formulation of $\hat{n}_\eta$. $\Gamma(n_\eta, \nu)$ denotes the Pearson correlation coefficient, while $\mu$ and $\sigma$ denotes the mean and standard deviation of the variables in their subscripts estimated from the known training data. The minimum MSE achieved is written as

$$E[|n_\eta - \hat{n}_\eta^*|^2] = (1 - \Gamma^2(n_\eta, \nu)) \sigma_{n_\eta}^2,$$

(4.14)

It is apparent that the minimum MSE (4.14) could be minimised by maximising Pearson correlation coefficient between the noise $n_\eta$ and the auxiliary scalar $\nu$.

Canonical correlation analysis (CCA) is a statistical technique that maximises the correlation between the canonical scores of multivariate vectors. In our case, we select the noise $n_\eta$ as one of the scalar canonical scores, while the other is determined by the sampled CS and represented as:

$$v = \Delta \rho';$$

(4.15)

where $\Delta$ is a vector that maps the vector $\rho'$ into a scalar. The CCA could resolve the following Pearson correlation coefficient maximisation problem

$$\Gamma^* = \max_{\Delta} \Gamma(n_\eta, \nu),$$

(4.16)

where the optimal transformation vector is denoted with $\Delta^*$. Using built-in statistics toolbox in MATLAB, CCA could be implemented directly over known training data to obtain $\Gamma^*$ and its corresponding $\Delta^*$. Substituting $\Gamma^*$ and $\Delta^*$ into the minimum MSE and the LMMSE estimator, the optimal weights $W^*$ and bias $B^*$ could be calculated by rewriting the estimator in the affine
4.4. Continuous Spectrum Assisted Detection

The form of (4.10)

\[ \hat{n}_\eta^* = \Gamma^* \frac{\sigma_{n_\eta}}{\sigma_{\nu^*}} (\nu^* - \mu_{\nu^*}) + \mu_{n_\eta}; \]

\[ = \Gamma^* \frac{\sigma_{n_\eta}}{\sigma_{\nu^*}} \nu^* - \Gamma^* \frac{\sigma_{n_\eta}}{\sigma_{\nu^*}} \mu_{\nu^*} + \mu_{n_\eta}, \]  

(4.17)

where the \( \nu^* = \Delta^* \rho' \). Hence, the optimal weights and bias are given by

\[ W^* = \Gamma^* \frac{\sigma_{n_\eta}}{\sigma_{\nu^*}} \Delta^*; \]  

(4.18)

\[ B^* = -\Gamma^* \frac{\sigma_{n_\eta}}{\sigma_{\nu^*}} \mu_{\nu^*} + \mu_{n_\eta}. \]  

(4.19)

It is worth emphasising that the AMMSE estimator is computed using a large number of channel realisations for more reliable correlation estimation. CCA should be performed first to obtain \( \Gamma^* \) and \( \Delta^* \). Next, \( \Delta^* \) is used to transform the CS of each realisation to the auxiliary scalar \( \nu^* \), where the standard deviation \( \sigma_{\nu^*} \) and mean \( \mu_{\nu^*} \) are estimated numerically. Finally, if the sample standard deviation \( \sigma_{n_\eta} \) and mean \( \mu_{n_\eta} \) of the detection error are estimated and substituted into equations (4.18), (4.19) and (4.10), the AMMSE estimator can be defined.

If the AMMSE estimated noise is removed from the received soliton amplitude \( R \), the resultant received symbol \( R_{AMMSE} \) will have lower noise power despite the effect of solitonic interaction [123]. To formulate a comparable performance metric for this scheme, the AIR for this scheme could be estimated with

\[ I_{AMMSE} = E_{A,R_{AMMSE}} \left[ \log \frac{Q_2(r_{AMMSE}|a)}{Q_2(r_{AMMSE})} \right], \]  

(4.20)

where a Gaussian channel with statistics estimated from the training data is selected to be the auxiliary channel model \( Q_2(r_{AMMSE}|a) \), the auxiliary output channel model is calculated via \( Q_2(r_{AMMSE}) = \sum_a p_a(a)Q_2(r_{AMMSE}|a) \) and the expectation should be taken on a different set of channel realisations to show transferability. Note that the interaction of the soliton is not taken into account in \( I_{isolated} \) since the auxiliary channel only characterises the channel model for isolated soliton communication system. However, it is demonstrated that the AMMSE method could take advantage of CS generated by the interaction to some extent [123].
4.5 BLSTM Network Classifier Detection

As revised in the literature, the BLSTM network could take advantage of the correlation between samples of the sequence in both forward and backward directions, suggesting benefit in exploiting the potential correlation inside the sequence. Additionally, the network architectures including feedforward network, basic LSTM network and BLSTM network are also compared under a toy example. The initial experiment results suggest that the BLSTM network could outperform other architectures. Hence, we selected the BLSTM network classifier as the third detection scheme. The sampled time domain pulse is first fed into a sequence input layer that matches the number of extracted features. Recalled that as reviewed in the Chapter 1, the coherent signalling is a commonly employed technique in modern optics fibre communication system, the light pulse propagated in the fibre is complex signal. In the case of having only a time-domain complex pulse, the number of features is two, corresponding to the real and imaginary parts. Then, the input layer is connected to two layers of BLSTM layer to further explore the relevant correlation between samples. Finally, the outputs of the BLSTM layers are then fully connected to a softmax layer. The output of the softmax layer could be interpreted as the a posterior probability \( Q_3(A|q(t,l)) \) that implies the probability of symbol \( A \) is transmitted given \( q(t,l) \) is received \([49]\). The system block diagram, including the other two detection schemes mentioned earlier, is illustrated in Figure 4.11. Note that the correlation between the CS and DS could be used to improve the reliability of the detection. Hence, an optional connection between the received CS and the input of the BLSTM network is highlighted in dash line. If the received CS is also fed into the network, the complex samples of the CS are converted into their equivalent real sequence by separating them into their real and imaginary parts, providing two extra features for the sequence input layer.

The detected symbol \( R_{BLSTM} \) could be obtained after a hard decision of taking the maximum a posterior among all. Moreover, the a posterior outputs of the softmax layer could also be interpreted as the auxiliary mismatch decoding rules, which could be used to estimate AIR \([49]\). The AIR of BLSTM detector \( I_{BLSTM} \) could be calculated as

\[
I_{BLSTM} = H(A) - E_{A,R_{BLSTM}} \left[ \log \frac{1}{Q_3(a|R_{BLSTM})} \right],
\]

where \( H(A) \) denotes the source entropy and the expectation is also taken on a separated test set that is not used in either training or validation of the network.
4.6 Numerical Discussion

The three detection schemes discussed above are then simulated using a series of 8- and 16-PAM constellations obtained by solving the optimisation problem (4.4) using the interior-point algorithm in MATLAB. The transmitted symbol is first randomly drawn from the probabilistic and geometric shaped constellation and encoded on a purely imaginary eigenvalue. Applying the inverse NFT (INFT) on the eigenvalue yields the corresponding time domain pulse, i.e. a soliton. The pulse evolution over a long haul fibre is simulated with the split step Fourier method, after which the three detection schemes are employed. In the direct NFT detection and also the AMMSE detection schemes, a fast NFT algorithm developed for MATLAB is used to compute the nonlinear frequency spectrum efficiently [75]. According to insights from observing numerical simulation results and from the literature [123], only a fraction of the sampled CS is necessary for correlation estimation. In this chapter, 1/4 of the CS samples will be taken for the AMMSE estimator if not specified otherwise. As for the BLSTM network detector, the network employed here consists of a sequence input layer (flatten layer) with input size of two (real parts and imaginary parts of the pulse), two BLSTM layers with 100 hidden units each layer, a fully connected layer, and a softmax output layer with 8 or 16 outputs depending on the modulation format.

Figure 4.11: The system block diagram for the three categories of detection schemes discussed in this chapter. Examples of transmitted and received soliton sequence pulses are presented along with the corresponding pulse evolution.
4.6. Numerical Discussion

Figure 4.12: Achievable information rate of AM soliton communication system with probabilistically and geometrically shaped 8 and 16 PAM.

The numerical results are obtained through simulations of three detection schemes on an ideal distributed Raman amplified 2000 km fibre with fibre attenuation coefficient $\alpha = 0.2$ dB/km, dispersion parameter $\beta_2 = -2.1 \times 10^{-26}$ s$^2$/m, and nonlinearity coefficient $\gamma = 1.27 \times 10^{-3}$ /W/m, assuming a phonon occupancy $K_T$ of 1.13 and a wavelength of 1.55 $\mu$m. Each of the training and validation channel realisations includes transmission of a 16-soliton sequence through the split-step Fourier simulated fibre with the step size of 200 m. In contrast, each testing channel realisation includes transmission of a 256-soliton sequence, which is much longer and closer to practical conditions. Note that the number of samples per soliton is computed according to the 99% bandwidth and more than 10 times oversampling to the next power of 2. For instance, it is estimated that 99% and 99.9% time widths would both require 64 samples per soliton. Hence, the size of the network input data is $64 \times 2$, where the 64 specifies the number of samples per soliton, and the 2 indicates the number of features, which are real and imaginary parts of the signal samples in coherent signalling. The training of the BLSTM network is conducted with 80,000 channel realisations. To avoid and monitor overfitting, a validation set of 80,000 channel realisations is employed, and the training will be stopped when the validation loss ceases to improve. Then, $I_{\text{BLSTM}}$ is estimated with 1,000 test realisations. The neural network is trained with crossentropy loss and the adaptive moment estimation (ADAM) optimiser provided in MATLAB deep learning toolbox. The same training data could also be employed to compute the AMMSE estimator, and $I_{\text{AMMSE}}$ is also estimated with the same test data.

Considering a fixed maximum and minimum soliton amplitude ratio $A_{\text{max}}/A_{\text{min}} = 1.24$ (i.e. a fixed effective time bandwidth product $TB = 3.52$ for $\delta = \varepsilon = 10^{-2}$), the AIRs are estimated for different maximum soliton amplitudes $A_{\text{max}}$ (i.e. bandwidth), as shown in Figure 4.12. Note that the SE could be calculated by dividing the AIR by the effective time-bandwidth product $TB$. It is observed that the inter-soliton interaction is very strong in the power regime considered in
### Table 4.1: Achievable Data Rates in Gbps for BLSTM and AMMSE Detectors

<table>
<thead>
<tr>
<th>PAM, $\delta$</th>
<th>$A_{\text{max}}$</th>
<th>$R_{\text{BLSTM}}$</th>
<th>$R_{\text{AMMSE}}$</th>
<th>$A_{\text{max}}$</th>
<th>$R_{\text{BLSTM}}$</th>
<th>$R_{\text{AMMSE}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8, $10^{-2}$</td>
<td>8.95 3.34</td>
<td>10.87 3.12</td>
<td>8.52 7.76</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8, $10^{-3}$</td>
<td>6.24 6.17</td>
<td>11.72 2.80</td>
<td>7.94 6.73</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16, $10^{-2}$</td>
<td>11.56 3.21</td>
<td>11.72 2.80</td>
<td>11.56 3.21</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16, $10^{-3}$</td>
<td>7.94 6.73</td>
<td>10.89 7.82</td>
<td>7.94 6.73</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This work, making the $I_{\text{isolated}}$ degrade to 0, implying the auxiliary mismatched isolated soliton channel model becomes effectively invalid. The AMMSE estimator could compensate for the interaction to some extent, producing 2.93 bits/symbol at $A_{\text{max}} = 0.51$, which is very close to the limit of 8-PAM. However, $I_{\text{AMMSE}}$ will decrease when the soliton amplitude increases because the AMMSE estimator cannot cope with the increasing interaction in the higher soliton amplitude regime. On the contrary, the BLSTM network detector could extend the power regime for which AIR increases. It is worth noticing that $I_{\text{BLSTM}}$ saturates at 3 bits/symbol for $A_{\text{max}}$ lower than 0.98 for the 8-PAM constellation, due to the limit of the constellation size. This suggests that the interaction could be fully compensated by the neural network within this operation regime. For 16-PAM where the constellation points are distributed closer, the task becomes more challenging for the network. A clear peak at around $A_{\text{max}} = 0.98$ could be identified, and 3.87 bits/symbol could be achieved. When $A_{\text{max}}$ is less than 0.98, $I_{\text{BLSTM}}$ could be improved by increasing the power, suggesting that the system is more perturbed by noise in this regime. If the power is further increased, $I_{\text{BLSTM}}$ could no longer be improved because the system becomes more interaction-dominant in this regime. It is also seen that the 16-PAM performance degrades to 1.08 bits/symbol, which is lower than the 1.33 bits/symbol provided by 8-PAM. This is because of the dominant soliton interaction having a higher impact on the more closely spaced constellation.

Previously, the AIR in bits/symbol under certain time-bandwidth constraint was discussed. However, for practical insight, the data rate in bits/second should also be investigated. The data rate is determined by SE and the time width of the pulse. In the amplitude modulated soliton system considered in this chapter, the time width of the pulse could be altered by selecting different time domain energy truncation factors $\delta$. The sech pulse has leading and trailing edges that degrade to effectively zero, hence this is equivalent to introducing guarding intervals between soliton pulses. Consequently, gain in SE could be expected because of lower interaction. This reveals a trade-off between SE and time width. Since the optimal SE in the regime of interest occurs at $A_{\text{max}} = 0.98$, and the AIR for 8-PAM and 16-PAM approximately coincide at $A_{\text{max}} = 1.34$ as shown in Figure 4.12, the data rates are investigated by selecting a smaller $\delta$ and are shown in Table 4.1. They correspond to bandwidths around 10GHz and 14GHz, respectively. As expected, reducing $\delta$ results in lower interaction, allowing a higher AIR to be achieved. For example, an AIR of 2.96 bits/symbol and a data rate of 6.17 Gbps...
could be achieved with 8-PAM and the AMMSE estimator at $A_{\text{max}} = 0.98$ and $\delta = 10^{-3}$. However, for the BLSTM detector, the harm in data rate from extending the time width cannot be compensated by the SE gain in the operational setup considered in this work. For instance, although a higher AIR of 3.83 bits/symbol can be achieved with 16-PAM and $\delta = 10^{-3}$ at $A_{\text{max}} = 0.98$, the corresponding data rate of 10.89 Gbps is lower than the data rate of 11.72 Gbps with $\delta = 10^{-2}$. It should be noted that choosing a smaller $\delta$ leads to a higher time-bandwidth product $T_B$, making the use of a BLSTM network detector with a more compact truncation factor of $\delta = 10^{-2}$ a reasonable choice in the considered system.

Figure 4.13: Example of transmitting a raised cosine pulse (4.22) with $\alpha_r = 1$, $A = 0.98$ and $T_0 = 0.1$ ns (Tx) in a 2000 km, NLSE described, noise-free and lossless nonlinear fibre with both GVD and SPM. (a) The amplitude of transmitted (Tx) and numerically simulated received (Num. Rx) time domain pulse; (b) the simulated pulse evolution throughout the propagation.

To demonstrate the benefit of employing the soliton pulse over the conventional pulse, such as raised cosine pulse, we estimate the data rate of a OOK system with constellation $[0, 0.98]$. For the conventional pulse shaping, raised cosine pulse is chosen and it is given as

$$q_{\text{RC}}(t) = \begin{cases} \frac{\pi A}{T_0} \sin \left( \frac{1}{\alpha_r} \right), & t = \pm \frac{T_0}{2} \\ \frac{A}{T_0} \sin(t) \frac{\cos(\pi \alpha_r t)}{1 - (2\alpha_r t)^2}, & \text{otherwise} \end{cases} \quad (4.22)$$

where $A$ denotes the transmitted symbol, $\alpha_r$ indicates the roll-off factor, and $T_0$ denotes the normalisation time. Selecting the $\alpha_r = 1$, $A = 0.98$ and $T_0 = 0.1$ns, the evolution of the raised cosine pulse in a noiseless and lossless nonlinear fibre could be simulated as shown in the Figure 4.13. Since the evolved raised cosine pulse cannot be determined analytically, the symbol interval required to include the evolved pulse is estimated numerically to be $10T_0$. If we then assume the symbol $A$ modulated on the pulse could be detected reliably under the nonlinear channel, the data rate of the OOK amplitude modulated raised cosine system is estimated to be $1/10T_0 = 1$Gbps occupying the bandwidth of 10GHz. As for the soliton communication system, it is demonstrated that the nonlinear channel effect including
4.6 Numerical Discussion

Figure 4.14: The training validation and test sets AIRs of 8 PAM at $A_{\text{max}} = 2.40$ with BSLTM network with a) two BLSTM layers of different number of hidden units; b) different number of BLSTM layer with 100 hidden units.

inter-soliton interaction could be efficiently reduced at $A_{\text{max}} = 0.98$ for 8-PAM. Hence, the data rate of the OOK soliton system occupying the same 10GHz bandwidth is estimated as $1/t_w(0.98, 10^{-2})T_0 = 1/2.77T_0 = 3.7$Gbps. This example implies the benefit of the soliton over conventional pulse shape in terms of improving the temporal efficiency in signalling.

4.7 Hyperparameter Analysis

The performance of the BLSTM network detector depends on various of hyperparameters. Generally speaking, it is believe that implementing more ANN units will result in a more complicated network which could model a more complex nonlinearity of the target function. Likewise, increasing the number of hidden layers could also potentially increase the capability of the ANN. Thus, the first set of hyperparameters investigated includes the number of hidden units in the both BLSTM layers and the number of BLSTM layers embedded in the classifier. To observe any potential improvement without introducing too much interaction, the most challenging case of 8-PAM at $A_{\text{max}} = 2.4$ is considered in this set of analysis. The AIRs estimated for training, validation and test sets are shown in Figure 4.14. Given a fixed size of the training dataset, increasing the number of hidden units in the BLSTM layers could increase the network capability until around 63 units. The AIR estimated with the test set shows no significant improvement for more than 63 BLSTM hidden units. When the number of hidden units are increased over 63, the test set AIR starts to decrease. This reveals a trade-off between the network complexity and the size of the training set. As the network becomes more complex, more hidden variables need to be trained, requiring more training data. A similar conclusion could be drawn for the analysis with the depth of the BLSTM.
network, i.e. the number of BLSTM layers with 100 hidden units. When a single BLSTM layer is implemented, the network is not as capable as the network with two BLSTM layers, while three BLSTM layers can limit the network performance due to the number of training data provided.

Next, we investigate the hyperparameters that could affect the convergence of the network. Due to computational limitations, only a limited number of realisations are generated for training. Therefore, we investigate the performance of the network under different sizes of training data to provide insight into whether providing more training data could further improve the network’s performance. Figure 4.15a shows the BLSTM network trained with different sizes of training data. We observe that the network’s performance on the validation data can be improved by providing more training data. However, we cannot determine such a trend for the testing data since it contains the propagation of much longer soliton sequences, which may contain interacted pulses that are not learned by the network, resulting in more fluctuations in the AIR curves.

Although training is terminated if the validation loss does not improve with further training to reduce the risk of overfitting, it is worth investigating whether additional measures should be taken. Dropout layers are typically introduced between hidden layers to allow the network to ignore some neurons during the training process with a certain dropout probability. The dropout probability sets a limit on the learning process such that the network cannot over-learn the training data. By implementing an additional dropout layer after each BLSTM layer in the classifier, AIRs for 8-PAM at $A_{\text{max}} = 2.40$ are plotted over different dropout probabilities in Figure 4.15b. It is observed that for this particular scenario, overfitting is not the dominant factor that limits performance on the unseen validation and test data. Adding dropout layers with dropout probability up to 0.2 generates similar performance as the network without dropout layers, while further increasing the dropout probability over 0.3 degrades the network performance.
Lastly, the choice of input sequence features and soliton sequence length is discussed. The aforementioned network requires two features to be extracted: the real and imaginary parts of the time domain pulses. It is highlighted that using the CS of the time domain pulse to implement an AMMSE estimator could potentially improve the performance [123]. Thus, the real and imaginary parts of the CS are also included as additional features for the network to identify a better nonlinear estimator that could potentially outperform the AMMSE estimator. In Figure 4.16a, the AIR of the BLSTM network with the real and imaginary parts of both the time domain pulse and CS spectrum is plotted and compared with the AIR achieved by the network that only uses the time domain pulse. It is observed that the additional CS feature does not provide any significant gain. This could suggest that the network structure is sufficiently complicated to learn the CS of the corresponding pulse.

The soliton sequence length of the training data indicates the memory of the channel. However, due to limited computational power, the length of the soliton sequence in the training data cannot match the length of the soliton sequence in the testing data. Therefore, it is important to identify the minimum length of the soliton sequence that can accurately characterise the nonlinear channel for the much longer testing soliton sequence, in order to train the BLSTM network detector. To estimate the longest memory numerically, the largest $A_{\text{max}} = 2.40$ is selected in this set of experiments. Considering similar total numbers of soliton transmissions, $2^6 \times 10^4$, $2^5 \times 10^4$, $2^4 \times 10^4$, and $2^2 \times 10^4$ realisations of 3, 8, 16, and 64-soliton sequences are generated as training and validation data, respectively. The trained network is then tested with the same 256-soliton sequence testing data employed previously. The performance of the network on both the validation set and test set is shown in Figure 4.16b. The memory of the interacting channel can be estimated to be around 16, as the AIRs on the validation set and test set start to converge.
4.8 Summary

This chapter presents an investigation into the estimation of the AIR of amplitude modulated soliton communication systems, with a focus on accounting for the inter-soliton interaction that is often neglected in existing literature. Recall that it was also demonstrated under naive assumption of channel memory of one soliton in Chapter 3 that such inter-soliton interaction can occur even in the absence of noise, detrimentally reducing the AIR of the system. The chapter begins with a brief review of the fundamental soliton pulse and its properties, including its signal-dependent time width and bandwidth. The time and frequency resources occupied by soliton pulses are then translated into peak and minimum amplitude constraints in the capacity formulation. The intrinsic inter-soliton interaction and the Gordon Haus effect are then examined with example propagation, revealing the highly nonlinear nature of these perturbations beyond the ASE noise.

Motivated by the nature of the problem, the chapter then presents estimations of the AIR for three different detection schemes that account for the inter-soliton interaction to varying extents: direct NFT detection, CS-assisted detection, and BLSTM classifier detection. These detection schemes are evaluated using optimised 8- and 16-PAM constellations. The chapter concludes with an analysis of the hyperparameters of the ANN used in the BLSTM classifier detection, demonstrating the advantages of the data-driven approach over its counterparts. Overall, this chapter provides a comprehensive examination of the challenges and potential solutions involved in estimating the AIR of amplitude modulated soliton communication systems in the presence of inter-soliton interaction. In both Chapter 3 and 4, the AIRs of the DS modulated system are mainly discussed, leaving another degree of freedom for modulation in NFD, the CS, untouched. In the next chapter, the focus of the research is turned towards the potential of the CS spectrum.
Chapter 5

On Optimally Shaped Signals for Nonlinear Frequency Division Multiplexed Fibre Systems

In Chapters 3 and 4, the signal is modulated on the invariant eigenvalue of the discrete spectrum (DS) of the nonlinear frequency domain (NFD). Recall that the evolution of the continuous spectrum (CS) in an ideal nonlinear fibre is linear, suggesting great potential for signal modulation. In this chapter, the signal dependent channel model for a CS modulated system is first investigated. An approximated channel model for the system is defined after imposing a signal-dependent decomposition and variance normalising transform (VNT). Next, the mutual information (MI) of this approximated channel is then maximised with geometric shaping (GS) technique, followed by full optimisation (FO) of both constellation positions and probabilities. The two categories of optimised constellations are then employed to estimate the mismatch capacity lower bounds with numerical channel simulations. Finally, a matched filter is implemented in the CS domain to reduce overall noise power. By ignoring inter-subcarrier interference (ISI) induced by matched filtering, the approximated channel model is used for GS of the constellation. Similar to the system without the CS filter, mismatch capacity lower bound is estimated to discuss the trade-off between denoising and interference.

5.1 Literature Review

Modulating the signal in the CS spectrum is also promising, despite the fact that its corresponding time-domain pulse will still suffer from the pulse-broadening impairment due to dispersion. Owing to the similarity between CS and the linear frequency spectrum, modulation techniques from the linear orthogonal frequency division multiplexed (OFDM) system could potentially be migrated into CS NFDM system smoothly. In [55], different signalling methods were investigated to adapt to the signal-dependent channel of CS NFDM system, including a non-uniform constellation that is adapted to the signal dependency, matched filtering in the CS spectrum, and Gelfand-Levitan-Marchenko (GLM) equation-based signalling. It is worth
mentioning that the GLM method was referred to as nonlinear inverse synthesis (NIS) in another series of papers [139–143]. The main advantage of the NIS implementation of the CS NFDM system is higher robustness against the noise compared to direct modulation on CS. In [144], the channel statistics of the NIS system were derived from the noise statistics of the CS spectrum, and the statistics were used to estimate the capacity lower bound with Pinsker formula. As opposed to the NIS implementation, the capacity bounds of directly CS-modulated NFDM systems were exploited with the signal-dependent channel statistics in [54]. In [145], the real and imaginary channels of the complex CS spectrum were assumed to be two independent real channels, allowing a variance normalising transform (VNT) to be performed on each channel [113]. After the VNT, each individual channel was approximated with an AWGN model, simplifying capacity estimation under a peak power constraint [78]. Despite the progress made in the reviewed works, it was only pointed out that the channel noise of the CS domain shows strong dependence on the input signal. The statistics of CS noise were described in both [144] and [145] up to the second moment, but the closed-form expression of the conditional probability density function (PDF) of the noisy signal that defines the channel is not available. The inaccessible channel model with amplifier spontaneous emission (ASE) noise makes the capacity of CS remains an open problem. To exacerbate the situation, a recent work reported that processing noise of numerical NFT that is not well modelled will contribute dominantly to the degradation of the performance in high power regime [146].

The CS spectrum $\rho(\lambda)$ is determined by the ratio of two Jost coefficients $b(\lambda)/a(\lambda)$, where $\lambda$ denotes the nonlinear frequency. Instead of modulating information on CS spectrum like the direct modulation or NIS implementation mentioned before, modulating pulse on the Jost coefficient $b(\lambda)$ is also a popular design choice. $b$-modulation is preferable over the conventional $\rho$ modulation because of its high spectrum efficiency and compact support of its corresponding time domain pulses [147,148]. However, the $b$-modulation scheme possesses an intrinsic limitation of unity constraint from $|a|^2 + |b|^2 = 1$, which is also known as energy barrier [74]. To avoid or bypass the energy barrier, either a bijective nonlinear function should be used to compound the signal into an open interval of $(-1,1)$ [149], or a tailored pulse shaping scheme that does not allow signal to exceed the range should be employed [74]. It was concluded with a dual-polarisation system that the nonlinear mapping reduces the benefit on spectral efficiency [149]. The dual-polarisation system was also shown to provide 1 dB Q-factor gain over conventional $\rho$ modulation [149] and was demonstrated experimentally to achieve data rate of 220 Gbps with the spectral efficiency of 4 bits/s/Hz [150]. In [151], the Q-factor gain of up to 1.5 dB was reported with experimental link of a single-polarisation $b$-modulated system. In [147], it was shown that channel model for the $b$-modulated system is well described by a Gaussian channel with signal dependent variance. Employing a similar recipe as [144], analytical lower bound of capacity was also estimated.
5.2 Channel Model

The pulse evolution in an ideal distributed Raman amplified standard single mode fibre is governed by the stochastic nonlinear Schrödinger equation (NLSE) (2.14) with noise statistics (2.16) in Chapter 2. Performing the nonlinear Fourier transform (NFT) on the pulse \( q(t, z) \) will decompose the signal into

\[
\rho(\lambda, z) = e^{4j\lambda^2l} \rho(\lambda, z = 0) \quad [34].
\]

Taking advantage of this property, the transmitted information symbols are first encoded directly on the CS shape \( \rho(\lambda, 0) \) via certain pulse shaping and inverse NFT (INFT) is taken to convert it into the time domain pulse \( q(t, l) \). The pulse is then transmitted through a standard single mode fibre, where the NFT is performed at the receiver to acquire the CS \( \rho(\lambda, l) \) of the received pulse \( q(t, l) \). After equalisation with \( e^{-4j\lambda^2l} \), the received information symbol can be extracted by down-sampling the equalised CS.

5.2.1 Channel Statistics and Approximated Model

In the proposed system, the information symbols are directly modulated at different nonlinear frequencies, allowing them to be detected by down-sampling the equalised CS. In [144], the continuous channel statistics is discussed using perturbative theory. If one denotes the equalised output as \( Y_\lambda \) and the input as \( X_\lambda \), the input-output relationship of the system is given as [144]

\[
Y_\lambda = e^{-4j\lambda^2l} \rho(\lambda, l) = X_\lambda + N_\lambda, \quad (5.1)
\]

where the conditional statistics of the zero mean noise \( N_\lambda \) given \( X_\lambda \) are as [144]

\[
\begin{align*}
E[(Y_\lambda - X_\lambda)(Y_{\lambda'} - X_{\lambda'}^*)] &= E[N_\lambda N_{\lambda'}^*] = \sigma^2 \delta(\lambda - \lambda')(1 + |X_\lambda|^2 + |X_{\lambda'}|^4); \\
E[(Y_\lambda - X_\lambda)(Y_{\lambda'} - X_{\lambda'})] &= E[N_\lambda N_{\lambda'}] = \sigma^2 \delta(\lambda - \lambda')X_\lambda^2, \quad (5.2)
\end{align*}
\]

where \( \sigma^2 \) denotes the unitless spectral density per propagation length as in (2.14), \((\cdot)^*\) denotes complex conjugation, and \( \delta(\cdot) \) denotes the Dirac-Delta function. Note that the condition is omitted for simplicity, the \( X_\lambda \) should be considered given unless specified otherwise. The \( \delta \) function in the correlation indicates that the noise at one \( \lambda \) is uncorrelated to that at another \( \lambda' \).

**Lemma 1.** Given a continuous channel

\[
Y_t = X_t + N_t, \quad (5.4)
\]

where the autocorrelation and pseudo-autocorrelation of the zero-mean noise \( N_t \) is defined as

\[
\begin{align*}
E[N_t N_{t'}^*] &= \sigma^2 \delta(t - t')(1 + |X_t|^2 + |X_{t'}|^4); \\
E[N_t N_{t'}] &= \sigma^2 \delta(t - t')X_t^2, \quad (5.6)
\end{align*}
\]
when the system is sampled with sufficiently high sampling frequency, the correlation of noise between different $t$ is zero as

$$E[N_t N_{t'}^*] = 0, \quad t \neq t'.$$  \hfill (5.7)

**Proof.** The proof of this Lemma is shown in the proof of Lemma 4.

Relying on Lemma 1, it is assumed that when sufficiently sampled, the discretized CS preserves its continuous nature as well, hence approximately, the inter-subcarrier interference (ISI) could be neglected. Employing such an approximation, the statistics of the noise conditional on the input $X_\lambda$ are written as

$$E[h | Y_\lambda - X_\lambda] = E[N_\lambda] = \sigma_N^2;$$ \hfill (5.8)

$$E[(Y_\lambda - X_\lambda)^2] = E[N_\lambda^2] = \sigma_N^2 X_\lambda^2;$$ \hfill (5.9)

where the $\sigma_N^2 = \sigma^2 l_\mu$ corresponds to the received noise power when the transmitted symbol $X_\lambda = 0$, $l$ denotes the unitless propagation distance and $\mu$ indicates the unitless time domain pulse width. The $\sigma_N^2$ can be calculated using Parseval’s theorem and the Fourier transform (FT) approximation of NFT when the $L_1$ norm of the signal is small without solitonic signal [34].

Within the reviewed literature, although the statistics of the channel noise is available up to the second moment [144, 145], an accurate channel model with closed-form expression for the CS nonlinear Frequency division multiplexing (NFDM) system remains unknown. To allow further analysis, approximations should be made to derive the closed-form channel law. Inspired by the Pinsker formula [144], a non-circular Gaussian (NCG) model with the available first and second order statistics can provide a closed-form expression. This will allow a more accurate channel law can be possibly found. In this section, a more accurate channel model compared to the benchmark NCG model is proposed by employing VNT after a change of basis. The advantages of our proposed model over NCG are demonstrated through numerical simulations in Section 5.2.2. Additionally, assuming a Gaussian distributed input, the analytical Pinsker capacity lower bound will also be derived using the NCG channel as the auxiliary channel for the mismatch decoding in Section 5.3.2.

In [55, 145], the correlation between the real and imaginary parts of the channel is reported with numerical simulation results. Rewriting the noise statistics (5.8) and (5.9) with the real and imaginary parts as

$$E[N_r^2] = \frac{\sigma_N^2}{2} \left( 1 + 2X_r^2 + 2X_r^2X_i^2 + X_r^4 + X_i^4 \right);$$ \hfill (5.10)

$$E[N_i^2] = \frac{\sigma_N^2}{2} \left( 1 + 2X_r^2 + 2X_r^2X_i^2 + X_r^4 + X_i^4 \right);$$ \hfill (5.11)
5.2. Channel Model

the correlation between them is then given as

$$E[N_rN_i] = \sigma^2_{N} X_r X_i,$$  \hspace{1cm} (5.12)

where $$Y_\lambda = Y_r + j Y_i = X_\lambda + N_\lambda = (X_r + N_r) + j (X_i + N_i)$$. Despite the non-zero signal dependent correlation (5.12), it is pointed out in [55] that the statistics of the phase noise is dependent on the transmitted symbol amplitude rather than its phase. The insight reveals a set of signal dependent decomposition where the decomposed components are uncorrelated to each other. One could decompose the received symbol $$Y_\lambda$$ into the parallel and orthogonal directions of $$X_\lambda$$ as

$$Y_\lambda = Y_p \cdot \frac{X_r + j X_i}{|X_\lambda|} + Y_o \cdot \frac{-X_i + j X_r}{|X_\lambda|},$$  \hspace{1cm} (5.13)

where $$Y_p = \frac{Y_r X_r + Y_i X_i}{|X_\lambda|}$$ denotes the component on the parallel direction and $$Y_o = -\frac{Y_r X_i + Y_i X_r}{|X_\lambda|}$$ denotes the component on the orthogonal direction as shown in the Figure 5.1.

![Figure 5.1: Decomposition operation of the received symbol $$Y_\lambda$$ into the parallel and orthogonal direction of the transmitted symbol $$X_\lambda$.](image)

Taking the conditional expectation given $$X_\lambda$$ over the parallel component $$Y_p$$ allows the statistics to be derived as

$$E[Y_p] = |X_\lambda|; \hspace{0.5cm} \text{var}[Y_p] = \frac{\sigma^2_N}{2} (1 + |X_\lambda|^2)^2.$$  \hspace{1cm} (5.14)

Similarly, the conditional statistics of the orthogonal component $$Y_o$$ are obtained as

$$E[Y_o] = 0; \hspace{0.5cm} \text{var}[Y_o] = \frac{\sigma^2_N}{2} (1 + |X_\lambda|^4).$$  \hspace{1cm} (5.15)

Note that the condition on the input $$X_\lambda$$ is omitted for simplicity. In terms of the correlation between the decomposed components, it is shown that the parallel component is uncorrelated to the orthogonal component as

$$E[Y_p Y_o] = \frac{X_r X_i}{|X_\lambda|^2} E[N_r^2 - N_i^2] - \frac{X_i^2 - X_r^2}{|X_\lambda|^2} E[N_r N_i] = 0.$$  \hspace{1cm} (5.16)
If the zero correlation property is extended to assume that the two components are independent, the complex channel condition on a given input can be decomposed into two independent channels. The joint distribution of the decomposed channels is hence given as the product of their corresponding marginal distributions as

$$ P(Y_p, Y_o|X_\lambda) = P(Y_p|X_\lambda)P(Y_o|X_\lambda). $$ (5.17)

As emphasised previously, even though the conditional statistics of the decomposed components are obtained using the perturbative theory results [144], accurate models with closed-form expressions for the decomposed components are still unavailable. In order to further the capacity analysis by performing MI maximisation, additional approximations for the channel model should be made. For the parallel component $Y_p$, the dependence between the mean $E[Y_p]$ and the variance $\text{var}[Y_p]$ reveals the potential of employing the VNT [145].

**Lemma 2.** Given a real random variable $Y$ with mean $\mu_Y$ and mean dependent variance $\sigma_Y^2 = f(\mu_Y)$, the VNT transformed random variable $R = T(Y)$ should have mean $\mu_R \approx T(\mu_Y)$, and variance $\sigma_R^2 \approx 1$, where VNT $T(\cdot)$ is defined as

$$ T(u) = \int \frac{1}{\sqrt{T(u)}} du. $$ (5.18)

**Proof.** Consider a transformed random variable $R = T(Y)$ where the Taylor series expansion around $Y = \mu_Y$ has been performed, then $R$ can be approximated with

$$ R = T(Y) \approx T(\mu_Y) + T'(\mu_Y)(Y - \mu_Y), $$ (5.19)

where the higher order terms are omitted. The mean and variance of $R$ are then derived as

$$ E[R] \approx E[T(\mu_Y) + T'(\mu_Y)(Y - \mu_Y)] = T(\mu_Y); $$ (5.20)

$$ \text{var}[R] \approx T'^2(\mu_Y)\sigma_Y^2 = T'^2(\mu_Y)f(\mu_Y). $$ (5.21)

It is clear that the transformation $T(\cdot)$ normalises the mean dependent variance to 1. Note that the variance could be normalised to any other convenient constant according to the use case by scaling $T(\cdot)$ with the square root of the constant.

Using the statistics of $Y_p$, the VNT transform $T(\cdot)$ is derived according to Lemma 2 as

$$ T(u) = \int \frac{2}{\sigma_N^2 (1+u^2)^2} du = \sqrt{\frac{2}{\sigma_N^2}} \text{atan}(u). $$ (5.22)
Moreover, the distribution of the VNT transformed random variable can be approximately described with a Gaussian distribution. The Gaussian approximation is accurate for a variety of mean-variance dependent non-Gaussian distributions, such as Poisson [111], Gamma [110] and noncentral-chi squared distributions [38]. In this work, the Gaussian approximation for the VNT transformed random variable is employed for the parallel component to allow further investigation of the optimal constellation design. Thus, it is assumed that the conditional probability density function PDF of $R_p = T(Y_p)$ is an unit variance Gaussian with the mean at $T(|X_\lambda|)$. The probability transformation rule is then employed to derive the marginal conditional distribution

$$P(Y_p|X_\lambda) = P(T^{-1}(R_p)|X_\lambda)$$

(5.23)

Note that the functionality of the VNT transformation could introduce a peak amplitude constraint on the $R_p$ as reported in [145]. Due to the $\tan$ function in $T(\cdot)$, the value of $R_p$ is constrained within $[-\sqrt{\frac{\pi}{\sigma_N^2}}, \sqrt{\frac{\pi}{\sigma_N^2}}]$ to ensure the bijectivity of the transformation. Consequently, for the Gaussian approximation made for $R_p$ to be effective, the constraint on $R_p$ further limits the regime of $T(|X_\lambda|)$ to $[-\sqrt{\frac{\pi}{\sigma_N^2}} + 4, \sqrt{\frac{\pi}{\sigma_N^2}} - 4]$ to ensure almost 100% of the Gaussian PDF $P(R_p|X_\lambda)$ is contained within the valid range.

In [55], it is pointed out that the distribution of the phase angle of the received symbol is mainly dependent on the input symbol amplitude rather than on the phase. Combining this insight, the component statistics (5.15) and also some numerical simulation results, the $Y_o$ is approximated with a zero mean symmetric distribution. In this work, a signal dependent Gaussian distribution as

$$P(Y_o|X_\lambda) = \frac{1}{\sqrt{\pi \sigma_N^2 (1 + |X_\lambda|^4)}} \exp\left(-\frac{Y_o^2}{\sigma_N^2 (1 + |X_\lambda|^4)}\right)$$

(5.24)

is selected to approximate the distribution of the orthogonal component $Y_o$ for its simplicity. Note that $Y_o$ does not possess mean-dependent variance, thus, VNT cannot be derived using Lemma 2 for this component.

So far, the approximated marginal conditional PDFs of the parallel channel $P(Y_p|X_\lambda)$ and the orthogonal channel $P(Y_o|X_\lambda)$ have been proposed. Recall the assumed independence between the two channels when $|X_\lambda|$ is given, the conditional channel is then approximated with

$$P(Y_p, Y_o|X_\lambda) = \frac{\cos^2(\tan(Y_p))}{\pi \sigma_N^2 \sqrt{1 + |X_\lambda|^4}} \exp\left\{ -\frac{(T(Y_p) - T(|X_\lambda|))^2}{2} - \frac{Y_o^2}{\sigma_N^2 (1 + |X_\lambda|^4)}\right\}.$$  

(5.25)
Note that the MI cannot be derived within the decomposed domain as the parallel and orthogonal directions will vary depending on different input symbol phases $\angle X_\lambda$. To calculate the MI, one will have to convert the distribution (5.25) back to the original real and imaginary domain. Since the transformation $[Y_p, Y_o] \rightarrow [Y_r, Y_i]$ has the Jacobian of unity, the conversion is given as

$$P(Y_r, Y_i|X_\lambda) = P(Y_p = \frac{Y_r X_r + Y_i X_i}{|X_\lambda|}, Y_o = \frac{-Y_r X_i + Y_i X_r}{|X_\lambda|}),$$

(5.26)

where the relationships between $[Y_p, Y_o]$ and $[Y_r, Y_i]$ are substituted in equation (5.25).

### 5.2.2 Numerical Analysis of the Approximated Channel Model

With the approximated channel model being established, the accuracy of the approximated channel model should be investigated with numerical simulations of transmitting CS modulated signals. In order to achieve reasonable accuracy in terms of describing the CS of a time domain pulse, the sampling frequency of the signal should be sufficiently high. Furthermore, as pointed out in Section 5.2.1, sufficient samples should also be taken in the CS domain such that the interference between nonlinear frequency subcarriers can be neglected. In this set of numerical experiments, the transmissions of an NFDM data frame are considered. The NFDM data frame is constructed as an analogy of an orthogonal frequency division multiplexing (OFDM) signal, $M = 128$ subcarriers are considered within the nonlinear frequency window $[-4, 4]$ with sinc pulse shaping as

$$\rho(t, 0) = \sum_{m=1}^{M} x_m \text{sinc} \left( \frac{M}{\Lambda} + \frac{M - (2m - 1)}{2} \right),$$

(5.27)

where $x_m$ denotes the information symbol encoded on the $m$-th subcarrier and $\Lambda = 8$ indicates the nonlinear frequency width of the NFDM symbol. The corresponding time domain pulses $q(t, 0)$ at the transmitter should be obtained using the INFT, then transmitted through a split step Fourier simulated fibre. In this work, we considered a long haul fibre of 2000 km with loss of $\alpha = 0.02$ dB/km, group velocity dispersion factor $\beta_2 = -2.2 \times 10^{-26}$ s$^2$/m, and Kerr nonlinear factor $\gamma = 1.27 \times 10^{-3}$ W/m. The time domain pulse width is selected to be 20 ns, including total guarding interval of 10 ns to avoid interference between neighbour pulses caused by dispersion induced pulse broadening. As for the normalising factors, the normalisation time is selected to be $T_0 = 0.1$ ns, the dispersion length and normalisation power are $L_D = 455$ km and $1/\gamma L_D = 2.39$ dBm accordingly. Additionally, the wavelength employed for the signalling is 1.55 $\mu$m with the phonon occupancy factor $K_T = 1.13$.

In this work, the NFT/INFT algorithm provided in [75] is employed to transform efficiently between the CS and time domains. As highlighted previously, high resolution in both time and CS domains is essential for two reasons. Firstly, to reduce the computation error of NFT, the time domain signal should be sampled sufficiently higher than Nyquist sampling frequency.
Secondly, the CS domain should have high resolution such that the discretization error can be neglected. In this work, 7.8 times Nyquist sampling is employed in the time domain sampling and 2 times of time domain samples are used to ensure sufficient sampling in both the time and CS domain.

Figure 5.2: The histograms (first row) of the decomposed parallel and orthogonal components $Y_p$ and $Y_o$ of the received symbol $Y_\omega$ given the transmitted symbol $X_\lambda$ with different amplitudes being 0.9 and 1.5 and phases being 0, $2\pi/3$ and $4\pi/3$. The distributions (second row) of the received symbol given the transmitted symbol with different amplitudes being 0.9 and 1.5 and phases being $2\pi/3$. The corresponding non-circular Gaussian (NCG) model and the proposed model are also included as solid curves and contour lines.

To obtain the signal dependent channel statistics, 200 transmissions of the NFDM data frame are performed, which include 25600 random realisations for each symbol considered. Recall the peak amplitude constraint introduced by the functionality of the VNT (5.22), $|X_\lambda|$ for the given fibre parameters should be limited within $[0, 2.7]$. Hence, the symbol amplitudes considered here are 0.9 and 1.5 while phases selected are 0, $2\pi/3$ and $4\pi/3$. In the first row of Figure 5.2, the histograms with the same amplitude $|X_\lambda|$ are presented in the same plot, the overlap between histograms given different phases $\angle X_\lambda$ are shown in both $Y_p$ and $Y_o$ components. This verifies the insight that the channel model is mainly dependent on the amplitude of the transmitted symbol rather than the phase. The marginalised PDFs of the NCG model with given statistics (5.2) and (5.3) and the proposed model are also plotted. Furthermore, using $\angle X_\lambda = 2\pi/3$ as an example, the NCG model and the proposed model (5.26) are sketched in two-dimensional contours over colour-scaled histogram of the received
symbol. A clear preference for the proposed model could be observed for describing the peak of the histogram. In addition to the advantage in capturing the peak, the proposed model contours also show good agreement with the simulated histogram, while the NCG model contours fail to capture its elliptical direction.

5.3 Capacity Analysis

In the previous section, the approximated channel model has been proposed based on the perturbation theory channel statistics and the VNT transformation, which will be used here to inspire the optimal constellation designs for signalling on CS. The model provides a full analytical description of the channel statistics which is matched up to the second moment to the statistics derived from perturbation theory. Therefore, the channel model can be used for shaping the input constellation to adapt to the signal dependence of the noise. However, due to the assumptions made in the approximation, the shaped input constellation is not necessarily capacity-achieving, but the optimised constellation would lead to a lower bound on the channel capacity. To further study the capacity of the CS channel, other than the Pinsker lower bound [144, 152], mismatch capacity [38, 111] with optimised input constellation will also be included in the next sections.

5.3.1 Input Constellation Shaping

Using the proposed model, the MI between the complex input $X_\lambda$ and output $Y_\lambda$ is written as

$$I(Y_\lambda; X_\lambda) = I(Y_r, Y_i; X_\lambda) = h(Y_r, Y_i) - h(Y_r, Y_i|X_\lambda),$$

(5.28)

where $h(Y_r, Y_i)$ denotes the output differential entropy calculated by

$$h(Y_r, Y_i) = \int_{Y_r} \int_{Y_i} P(Y_r, Y_i) \log_2 \frac{1}{P(Y_r, Y_i)} \, dY_r dY_i,$$

(5.29)

and $h(Y_r, Y_i|X_\lambda)$ denotes the conditional differential entropy given as

$$h(Y_r, Y_i|X_\lambda) = \sum_{x \in X_\lambda} P(X_\lambda = x) h(Y_r, Y_i|X_\lambda = x).$$

(5.30)

Note that discrete constellation is assumed to be employed in this chapter, considering a closer to practice constellation. Because of the discreteness assumed, besides the conditional differential entropy (5.30), the output distribution $P(Y_r, Y_i)$ is also written with summation as

$$P(Y_r, Y_i) = \sum_{x \in X_\lambda} P(X_\lambda = x) P(Y_r, Y_i|X_\lambda = x).$$

(5.31)
Moreover, recall the unity Jacobian of the transformation and the assumed independence between the two components, when the input symbol is given the conditional differential entropy \( h(Y_r, Y_i | X_\lambda = x) \) is then rewritten into
\[
h(Y_r, Y_i | X_\lambda = x) = h(Y_p | X_\lambda = x) + \log_2 \sqrt{\pi e \sigma_N^2 (1 + |x|^4)}, \tag{5.32}
\]
where the differential entropy \( h(Y_p | X_\lambda = x) \) is expressed as
\[
  h(Y_p | X_\lambda = x) = \int_{Y_p} P(Y_p | X_\lambda = x) \log_2 \frac{1}{P(Y_p | X_\lambda = x)} dY_p. \tag{5.33}
\]

Using the MI as the objective function, the constellation shaping problem is formulated as
\[
  \max_K \max_{[X,P(X)]} I(Y_\lambda; X_\lambda), \tag{5.34}
\]
subject to the constraints
\[
  \sum P(X) = 1, \quad |X_1|, \ldots, |X_K| \leq X_{\text{max}}, \tag{5.35}
\]
where \( X = [X_1, X_2, \ldots, X_K] \) denotes the vector whose elements indicate the input alphabet, while the \( P(X) = [P(X_1), P(X_2), \ldots, P(X_K)] \) indicates the probabilities correspond to the symbols in the alphabet. The latter constraint in (5.35) is interpreted as a peak amplitude constraint, which is inherently imposed by VNT [145].

Solving the optimisation problem defined in the equation (5.34) subject to the constraints (5.35) is equivalent to the optimal constellation design denoted as FO in this chapter. In some scenarios, FO is not favoured due to its complexity in implementation. Since the symbols of the constellation are originally equiprobable, a probability matcher is required to be implemented such that an information source with optimised probabilities can be established.

To avoid the additional probability matcher, GS is preferred in some scenarios by constraining probability further to \( P(X) = \frac{1}{K} \mathbf{1} \), where the bold \( \mathbf{1} \) denotes a \( K \)-length vector of ones. Furthermore, only \( K \) that is positive power of 2 is considered in GS to simplify the source coding further. GS optimises the positions of the input symbols \( X_\lambda \) with the prior that they are equiprobable. Besides the simplified system structure, the GS optimisation also consumes less time to converge as the dimension of the feasible region is smaller than that of the probabilistic geometric hybrid shaping in the FO scheme for the same \( K \).
5.3. Capacity Analysis

5.3.2 Pinsker Lower Bound

In the previous section, the MI of the approximated channel model is maximised by performing two constellation shaping optimisations, while the channel model is proposed under assumptions of sufficient sampling, independence between parallel and orthogonal components, and Gaussian PDF of the appropriate stage. If only the sufficient sampling approximation is made, a capacity lower bound can also be derived using the Pinsker formula and the channel statistics (5.8) and (5.9). The Pinsker formula provides a capacity lower bound for a given channel with known first and second order statistics while the exact model is unknown [82, 144, 152]. The lower bounding is provided in two aspects. On the one hand, the input distribution is assumed to be a zero-mean circular Gaussian with given constellation variance [82]. On the other hand, the channel is assumed to be a real vector Gaussian channel with the given statistics (hence only first and second order statistics are required) [82].

Lemma 3. Given an arbitrary complex channel $Y = X + N$ with signal-dependent zero-mean noise $N$. If the signal dependence of the noise is defined as equation (5.8) and (5.9), the channel capacity is lower bounded by

$$C_{\text{pin}} = \log_2 \left( 1 + \frac{\sigma_S^2}{\sigma_N^2(1 + \sigma_S^2 + 2\sigma_N^2)} \right),$$

(5.36)

where the transmitted signal $X$ is of a zero-mean circular Gaussian with variance of $\sigma_S^2$.

Proof. In order to derive a capacity lower bound using the Pinsker formula, one will first have to rewrite the complex system with a vector real systems as [82, 144]

$$Y = X + N,$$

(5.37)

where the conditional statistics of the noise vector are described by (5.10), (5.11) and (5.12). Without knowing the exact channel model (i.e. the conditional PDF), the capacity of such a system could be lower bounded by the MI assuming input is a complex circular Gaussian distribution and the channel is a Gaussian distribution with specified correlations. In this work, we refer to this lower bound as the Pinsker lower bound $C_{\text{pin}}$, and it is defined by substituting the Gaussian distributions into the random vector MI equation as

$$C_{\text{pin}} = \frac{1}{2} \log_2 \frac{\det \Sigma_X \det \Sigma_Y}{\det \Sigma_U}$$

(5.38)

where the vector $U = [X_r, X_i, Y_r, Y_i]^T$ is an auxiliary random vector introduced for the convenience in the notation, $\Sigma_X$, $\Sigma_Y$ and $\Sigma_U$ are the input, output, and input-output joint covariance matrices respectively. The complex input is considered to be a complex circular Gaussian with zero mean and $\sigma_S^2$ variance, the covariance matrix of the rewritten input vector $X$ is given as $\Sigma_X = \sigma_S^2 I_2$. The zero mean signal dependent noise is characterised by the conditional
statistics (5.10), (5.11) and (5.12), hence the covariance matrix of the noise vector $N$ is rearranged as

$$
\Sigma_{N|X} = \frac{\sigma_N^2}{2} \begin{bmatrix}
1 + 2X_r^2 + 2X_i^2X_r^2 + X_i^4 + X_r^4 & 2X_rX_i \\
2X_iX_r & 1 + 2X_i^2 + 2X_r^2X_i^2 + X_i^4 + X_r^4
\end{bmatrix}.
$$

Using the covariance matrix of the input $\Sigma_X$ and the noise $\Sigma_{N|X}$, the covariance matrix of the output $Y$ can be acquired by marginalising the conditional output correlation matrix as

$$
\Sigma_Y = \left[ \frac{\sigma_N^2}{2} (1 + \sigma_S^2 + 2\sigma_S^4) + \frac{\sigma_S^2}{2} \right] I_2.
$$

Recall the circular complex Gaussian assumed for the input, the real and imaginary parts of the input are independent to each other. The independence leads to zero marginal correlation of the real and imaginary channels, i.e. $E[Y_rY_i] = 0$. Furthermore, this property also simplifies the computation of the marginalisation of the second moment of each output channel, resulting in equal second moment, $E[Y_rY_i] = E[Y_rY_i]$.

The final component required for the Pinsker lower bound (5.38) is the input-output joint covariance matrix $\Sigma_U$. The joint covariance matrix consists of three sub-matrices as

$$
\Sigma_U = \begin{bmatrix}
\Sigma_X & \Sigma_{XY} \\
\Sigma_{XY}^T & \Sigma_Y
\end{bmatrix}.
$$

where $\Sigma_{XY}$ denotes the input-output covariance matrix. Due to the assumed circular symmetric input, it is easy to show that the input-output correlation matrix is a diagonal matrix as

$$
\Sigma_{XY} = \frac{\sigma_S^2}{2} I_2
$$

Finally, substitute the covariance matrices derived above into (5.38) using the same techniques as in [144], the Pinsker lower bound $C_{\text{pin}}$ is obtained as

$$
C_{\text{pin}} = \log_2 \left( 1 + \frac{\sigma_S^2}{\sigma_N^2(1 + \sigma_S^2 + 2\sigma_S^4)} \right).
$$

Using Lemma 3 the Pinsker lower bound can be written considering the input being zero-mean circular Gaussian with variance of $\sigma_S^2$ as equation (5.36). Note that the Gaussian input assumed in this lower bound does not take into account the peak amplitude constraint discussed in this chapter. If identical constellation variance as the optimised constellation is employed, the Pinsker formula will provide a lower bound without the peak amplitude constraint used in the constellation shaping schemes.
5.3.3 Mismatch Capacity Lower Bound

So far, the capacity analysis of the CS channel has been performed under some approximations which allow the analytical result to be derived. In this section, the mismatch capacity will be discussed based on a set of realistic channel realisations, including transmissions of CS pulses in a split step Fourier simulated fibre with ASE noise added at each step. Furthermore, a practical and efficient NFT/INFT [75] algorithm is also employed to compute the CS domain pulse shaped, which includes the effect of computational error in practice. The mismatch capacity is given as

\[
C_M = \sum_{k=1}^{K} \int_{Y_r, Y_i} P(X_k) P(Y_r, Y_i | X_k) \log_2 \frac{P(Y_r, Y_i | X_k)}{\sum_{m=1}^{K} P(X_m) P(Y_r, Y_i | X_m)} \, dY_r dY_i;
\]

\[
= \sum_{k=1}^{K} P(X_k) E_T \left[ \log_2 \frac{P(Y_r, Y_i | X_k)}{\sum_{m=1}^{K} P(X_m) P(Y_r, Y_i | X_m)} \right], \tag{5.44}
\]

where the \(P(Y_r, Y_i | X_k)\) denotes the true channel model given symbol \(X_k\) is transmitted, and the \(E_T(\cdot)\) denotes the expectation over the true channel, which is approximated with the Monte Carlo method by numerically averaging over large number of realisations. Mismatch capacity quantifies the amount of information that could be reliably transmitted in the true physical channel using the decoding rules optimally designed for the auxiliary channel without specifying the exact decoder [38, 79, 111]. This is also known as the achievable information rate (AIR), where the achievability relies on the assumed auxiliary channel based on which the optimal decoding rule is already known. The input distributions employed here are obtained from the constellation shaping discussed in the previous section, while the approximated channel \(P(Y_r, Y_i | X_k)\) in (5.26) and (5.25) is selected to be the auxiliary mismatch channel. Such an estimation not only provides a proven capacity lower bound, but also gives an estimate of the impact of the approximations made in deriving the channel model (5.26) and (5.25), considering the equivalence between the \(C_M\) and the MI achieved by the same input \(P(X)\) when \(P(Y_r, Y_i | X_k) = P_T(Y_r, Y_i | X_k)\).

5.3.4 Numerical Results and Discussion

In this work, both FO and GS schemes are implemented to estimate their corresponding optimal MI under the peak amplitude constraint in the CS domain. By performing optimisations for both schemes, the shaping gain will be estimated to evaluate the necessity to perform the additional probabilistic shaping and constellation size optimisation. Note that the optimal shaping in the FO scheme and the GS shaping corresponding to the inner optimisation in (5.34) are solved with an interior-point algorithm with random initialisation to enhance
5.3. Capacity Analysis

![Figure 5.3: (a) MI achieved by the FO optimized (optimal sizes specified), GS optimized constellations and unshaped (US) 16, 32, 64 APSK. (b) Mismatch capacity lower bound employing FO optimized, GS optimized, US 64 APSK, and also Pinsker Capacity lower bound when the input variance is equal to the input variance of FO optimized constellation.](image)

The algorithm convergence. The optimisation on the constellation size $K$ in FO scheme is carried out by identifying the non-increasing trend of the optimised MI for different $K$’s. The channel parameter $\sigma_N^2$ is selected to be 0.0154, which corresponds to the same long-haul fibre employed in Section 5.2.2.

Implementing the optimisation with the parameters described above, the optimised MI under the peak amplitude constraint $X_{\text{max}} \in [0.3, 2.4]$ is shown in Figure 5.3a. The MI for unshaped 16, 32, 64 amplitude phase shift keying (APSK) are calculated to represent the performance of unshaped constellation as benchmarks\(^1\). The APSKs are chosen over quadrature amplitude modulations (QAM) which are more common in optical fibre communication as the optimally shaped constellations also converge to an APSK multi-ring constellation. The range of interest considered here roughly corresponds to the time domain power regime of $-30$ to $-10$ dBm.

Recall that the conversion between the symbol in CS domain and the time domain power is not straightforward, the power regime mentioned above is estimated under the assumptions of sinc pulse shaping and 128 subcarriers as in (5.27).

Overall, the MIs for both GS and FO shaping techniques increase at higher $X_{\text{max}}$, which corresponds to a higher signal power. In addition, both schemes outperform the best unshaped APSK constellation. The gain over unshaped APSK schemes at lower values of $X_{\text{max}}$ is not as significant. For example, at $X_{\text{max}} = 0.6$, an MI of 3.59 bits per channel use can be achieved with unshaped 32-APSK, while FO can only provide an improvement of 0.07 bits per channel use. Conversely, more significant gains are observed at higher values of $X_{\text{max}}$. For instance, at $X_{\text{max}} = 2.4$, the unshaped 64-APSK achieves an MI of 4.51 bits per channel use, while FO

\(^1\) The APSKs employed here correspond to 4+12APSK, 4+12+16APSK, and 8+16+20+20APSK respectively. The radius and phase angle of the points are selected according to the highest coding spectral efficiency in [153].
can provide additional gain of 0.41 bits per channel use. Moreover, it is also worth pointing out that the unshaped APSK will result in a decreasing MI for the high $X_{\text{max}}$ because of the unshaped APSK not being able to adaptively change the spacing between the constellation rings to account for the increased nonlinear effects.

When comparing between the two shaping techniques, the additional probabilistic shaping and constellation size optimisation in FO result in less than 0.1% MI gain over their GS counterparts for all the $X_{\text{max}}$. Moreover, it is worth noticing that the rate of improvement will start to reduce, showing a trend towards saturation. This could be the consequence of the noise power increasing faster than the signal power with the increasing signal amplitude.

Comparing the optimised distributions of both schemes shown in Figure 5.4, it is observed that under the peak amplitude constraint, both schemes converge to multi-ring constellations. The FO scheme will allocate the probabilities of the mass points on the ring adaptively, for example, in the Figure 5.4b, the mass points on the outer ring are assigned with higher probabilities than those on the central rings. The GS shaping, on the other hand, will also arrange the spacing between the constellation rings with respect to the signal dependence similar to FO scheme.

In both Figure 5.4c and Figure 5.4f, it should be noticed that the spacing between the rings is increased along with the signal amplitude to adapt to the signal dependency of the noise.
5.3. Capacity Analysis

Employing the shaped input constellations discussed above, the mismatch capacity is estimated as a proven lower bound for the true capacity with $1000 \times M = 128,000$ realisations of the channel. Additionally, the Pinsker lower bound [144] is also calculated as a lower bound subjected to the identical variance of the input as the FO shaped constellation. Recall this variance constraint is a weaker constraint compared to the peak amplitude constraint when identical variance is considered. The mismatch capacity and Pinsker lower bound are sketched in Figure 5.3b. As expected from the previous discussion of the additional shaping gain from performing the extra constellation size optimisation and probabilistic shaping, almost identical mismatch lower bounds are achieved by the FO shaped constellation and the GS shaped 64 APSK respectively. It is worth pointing out that the MI gap of 0.02 is observed at $X_{\text{max}} = 0.5$ for the FO while the gap of 0.15 is produced at $X_{\text{max}} = 2.1$. This implies the gap between the optimised MI, which is calculated from the approximated channel model (5.26), and the mismatch capacity will become larger at higher $X_{\text{max}}$.

The trend of the gap matches with the discussion in Section 5.2.1. The perturbative channel statistics (5.2) and (5.3) [144] correspond to the continuous signal. However, the signal will be discretized before further digital signal processing such as numerical NFT [75] in the practical system and also in the simulations considered within this chapter. The discretized channel (5.8) and (5.9) are obtained based on the infinitely high sampling rate approximation as discussed in Lemma 1. To capture the deviation between the continuous perturbative model and the discretized model with 400 GHz sampling rate, i.e., 7.8 times Nyquist rate, 25,600 random realisations of transmitting randomly selected symbols from a constellation with continuous uniform distributed phase angles and the same symbol amplitudes from 0 to 2.4. The variance of the received symbol and its decomposed components are shown in Figure 5.5. When the transmitted symbol amplitude is lower than 1.2, the good match between the numerical and analytical estimated statistics implies the discretized channel

Figure 5.5: The variance of the received symbol $Y_{\lambda}$, and its parallel components $Y_p$, and orthogonal components $Y_o$ given the amplitude of the transmitted symbols $|X_{\lambda}|$ with uniformly distributed phase $\angle X_{\lambda}$. 

![Figure 5.5: The variance of the received symbol Y₁, and its parallel components Y_p, and orthogonal components Y_o given the amplitude of the transmitted symbols |X₁| with uniformly distributed phase ∠X₁.](image)
5.3. Capacity Analysis

model provides an accurate estimation for the statistics of the noise in this regime. Hence, the gap between the optimised MI and mismatch capacity should also be small. However, in the higher signal regime, it is observed that the gap between the numerical and analytical statistics is increasing. As explained in the proof of Lemma 4, the error is signal dependent and it is introduced by discretising a signal dependent continuous random process. Referred as processing noise, such signal dependent deviation was also reported in the literature due to its increasing dominance in higher power regime [146].

Despite the deviation from the analytical MI, the shaping constellation can provide gain over the unshaped constellation as expected, considering the unshaped 64 APSK only achieves a maximal of 4.59 bits per channel use at \( X_{\text{max}} = 1.5 \) while the maximal of 4.72 can be achieved by the GS 64 APSK at \( X_{\text{max}} = 2.1 \). When comparing the Pinsker lower bound, a better lower bound is generated by the Pinsker lower bound when the \( X_{\text{max}} \) is small due to the additional peak amplitude constraint on the shaping scheme. However, the shaped constellations achieve higher lower bounds even with the extra constraint at higher \( X_{\text{max}} \), this reveals the potential of performing constellation shaping even using an approximated signal dependent channel model.

5.4 Signalling With Matched Filter

If sinc pulse shaping is used in frequency domain, it is commonly known that the corresponding inverse FT (IFT) is a rectangle function which is band-limited in time domain. Exploiting this property of the sinc pulse shaping (5.27), the signal dependent noise in the CS modulated system could be potentially reduced with a matched filter implemented in the CS domain [55]. However, the signal dependence of the CS noise will induce ISI at the matched filter output. Convolving the received symbol \( Y_{\lambda} \) with the matched filter, the input-output relationship is then given as

\[
Y_{f\lambda} = h_f(\lambda) * Y_{\lambda} = X_{\lambda} + h_f(\lambda) * N_{\lambda} = X_{\lambda} + N_{f\lambda},
\]  

(5.45)

where \( h_f(\lambda) \) denotes the impulse response of the matched filter in CS domain given as

\[
h_f(\lambda) = \frac{M}{\Lambda} \text{sinc} \left( \frac{M}{\Lambda} \lambda \right).
\]  

(5.46)

The convolution is equivalent to multiplying a rectangular brick wall filter with the IFT domain of the received CS pulses. As the sinc pulse is band-limited in the IFT domain, the signal \( X_{\lambda} \) will remain unchanged after matched filtering. Using the input-output relationship (5.45), autocorrelation (5.2) and (5.3), it is clear that the filtered noise will remain zero mean, while
the second order statistics are given as

$$E[N_{\lambda\lambda}N_{\lambda\lambda}^*] = \int \sigma^2 h(t,\lambda-\xi)h(t,\lambda'-\xi) \left(1 + |X_t|^2 + |X_t|^4 \right) d\xi; \quad (5.47)$$

$$E[N_{\lambda\lambda}N_{\lambda\lambda}^*] = \int \sigma^2 h(t,\lambda-\xi)h(t,\lambda'-\xi)X_t^2 d\xi; \quad (5.48)$$

where $\xi$ is the auxiliary variable introduced due to the convolution and $\frac{M}{\Lambda}$ corresponds to the separation between the subcarriers. It can be observed from (5.47) and (5.48) that although the subcarriers are allocated on the zero crossing points of the $\text{sinc}(\cdot)$ functions of their neighbors, the signal dependence will still introduce ISI. In the previous sections, when filtering is not employed, the discretized channel statistics (5.8) and (5.9) are obtained assuming sampling in the CS domain is sufficiently high, the ISI can be neglected. However, for the filtering to be effective, the passband of the filter should be comparable to the bandwidth of the signal and this will make the no ISI approximation inaccurate.

Lemma 4. Given an continuous channel filtered with an arbitrary filter $h(t)$ that does not alter the input signal $X_t$, as

$$Y_{ht} = Y_t * h(t) = X_t + N_t * h(t) \quad (5.49)$$

where the zero-mean noise $N_t$ is defined by autocorrelation (5.5) and pseudo-autocorrelation (5.6) and the input signal $X_t$ is defined with $\text{sinc}$ pulse shaping as

$$X_t = \sum_{m=1}^{M} x_m \text{sinc} \left( \frac{M}{\Lambda} t + \frac{M - (2m-1)}{2} \right), \quad (5.50)$$

where $\frac{M}{\Lambda}$ denotes the separation between the time domain subcarriers, and $x_m$ denotes the information symbol on the $m$-th subcarrier at $t = \frac{(2m-1)-M}{2}$.

When a matched filter $h_f(t)$ that matches to the pulse shaping function given as

$$h_f(t) = \frac{M}{\Lambda} \text{sinc} \left( \frac{M}{\Lambda} t \right) \quad (5.51)$$

is imposed into the system. The correlation of the filtered noise $N_{ht}$ between different subcarriers $t$ and $t'$ ($t \neq t'$) becomes nonzero and dependent on the signal over the whole signalling width $\Lambda$.

Proof. Consider the continuous channel after the implementation of an arbitrary filter $h(t)$ as defined in (5.49). If $\tau$ is introduced as an auxiliary variable in convolution, the random process $N_{ht}$ after filtering with an arbitrary filter $h(t)$ would possess an autocorrelation as

$$E[N_{ht}N_{ht}^*|h(t)] = \int \sigma^2 h(t-\tau)h^*(t'-\tau) \left(1 + |X_t|^2 + |X_t|^4 \right) \ d\tau; \quad (5.52)$$

$$E[N_{ht}N_{ht}^*|h(t)] = \int \sigma^2 h(t-\tau)h(t'-\tau)X_t^2 \ d\tau. \quad (5.53)$$
5.4. Signalling With Matched Filter

In practice, the random process will be sampled. Sampling of an unlimited bandwidth signal would correspond to multiplying a brick wall filter \( \tilde{h}_i(f) = \text{rect}(f/f_s) \) whose bandwidth is equal to the sampling frequency \( f_s \) in the frequency domain, which will then correspond to convolving the corresponding time domain impulse response \( h_i(t) = f_s \text{sinc}(f_s t) \). When \( f_s \) is sufficiently large, the approximation \( \lim_{f_s \to \infty} h_i(t) = \delta(t) \) can be employed to show that

\[
E \left[ N_{lw} N_{lw}' | h_i(t) \right] = E \left[ N_i N_i' \right], \quad \text{and} \quad E \left[ N_{lw} N_{lw}' | h_i(t) \right] = E \left[ N_i N_i' \right].
\]

The equivalence implies that the ISI can be neglected if sampled at sufficiently high sampling rate. Note that the intention of the Lemma 1 is to highlight that the noise statistics before the digital matched filtering will remain approximately the same if the sampling rate is kept sufficiently high.

The approximation made in Lemma 1 given sufficient sampling is shown, however, such an approximation cannot be made for the matched filtered signal-dependent noise. To show the Lemma 4, consider sinc pulse shaping in the input signal \( X_i \) as defined in (5.50). The matched filter \( h_i(t) \) is hence defined as (5.51). For a conventional system where the noise is modelled with i.i.d AWGN noise \( G \) with zero mean and \( \sigma^2 \) power spectrum density, it is trivial to find out that the power of the \( h_i(t) \) filtered noise \( G_i \) is given as \( \frac{M}{\Lambda} \sigma^2 \). On the contrary, if the noise is signal dependent with the correlation (5.5) and (5.6), the noise is no longer i.i.d. The correlation of the filtered noise is given as \( E \left[ N_{lw} N_{lw}' | h_i(t) \right] \) and \( E \left[ N_{lw} N_{lw}' | h_i(t) \right] \). Additionally, the passband of the filter \( h_i(t) \) is equal to the bandwidth of the signal \( X_i \), making the \( \delta(t) \) approximation employed previously not applicable. Hence, the noise statistics of each subcarrier will become dependent on not only the signal on the subcarrier, but also on the signals on the neighbour subcarriers. As an example, the autocorrelation of the filtered noise for the signal (5.50) is written as

\[
E \left[ N_{lw} N_{lw}' | h_i(t) \right] = \int \sigma^2 \frac{M}{\Lambda^2} \text{sinc} \left( \frac{M}{\Lambda} (t - \tau) \right) \frac{M}{\Lambda^2} \text{sinc} \left( \frac{M}{\Lambda} (t' - \tau) \right) \left( 1 + |X_i|^2 + |X_i'|^4 \right) d\tau,
\]

\[
= \sigma^2 \text{sinc} \left( \frac{M}{\Lambda} (t - t') \right)
\]

\[
+ \int \sigma^2 \frac{M}{\Lambda^2} \text{sinc} \left( \frac{M}{\Lambda} (t - \tau) \right) \frac{M}{\Lambda^2} \text{sinc} \left( \frac{M}{\Lambda} (t' - \tau) \right) \left( |X_i|^2 + |X_i'|^4 \right) d\tau. \quad (5.54)
\]

Now considering \( t = \frac{(2m-1)M}{2} \), \( m = 1, 2, ..., M \) and \( t' = \frac{(2m'-1)M}{2} \), \( m' = 1, 2, ..., M \) at subcarriers, in the absence of signal dependency (i.e., the terms \( |X_i|^2 \) and \( |X_i'|^4 \) are dropped), the integral term in the right hand side (RHS) of (5.54) vanishes. In addition, the sinc term in the RHS of (5.54) equals \( \frac{M}{\Lambda} \sigma^2 \) at \( m = m' \) and equals zero at \( m \neq m' \), suggesting no ISI in the absence of signal dependency. On the contrary, the integral term in the RHS of the (5.54) shows nonzero autocorrelation in both \( m = m' \) and \( m \neq m' \) cases. Hence, it is clear that ISI will be introduced in the filtered system by the signal dependence of the noise.
In order to implement constellation shaping, the closed-form expression of the channel law is required for the formulation of the MI, which would be the objective function of the shaping optimisation. When ISI is considered, such a closed-form expression would be intractable, hence, making it infeasible to optimise the constellation. To simplify the implementation of the constellation shaping, only the signal dependent part of the channel noise will be considered despite the existence of the ISI. The noise at each subcarrier is considered as a white signal dependent noise that only depends on the signal on the same subcarrier. Using similar approaches as Section 5.2.1, the CS noise statistics after filtering are then derived as

\[ E \left[ (Y_f - X)^2 \right] = E \left[ N_f^2 \right] \approx \sigma_{fi}^2, \]  
\[ E \left[ (Y_f - X)^2 \right] = E \left[ |N_f|^2 \right] \approx \sigma_{fil}^2 |X|^2, \]  

where \( \sigma_{fil}^2 = \sigma^2 / \pi M / \Lambda \) corresponds to the received noise power when transmitted symbol \( X = 0 \), the \( \pi M / \Lambda \) denotes the unitless passband width of the filter (5.46) when the small signal asymptotic IFT is performed [34]. Employing the same methodology as the unfiltered channel, an identical approximate channel model with attenuated noise power can be obtained by substituting the approximated filtered received noise power \( \sigma_{fil}^2 \), the filtered received symbol \( Y_f \), and its corresponding decomposed components into equations (5.26) and (5.25) as

\[ P(Y_f, Y_i | X) = P(Y_f = Y_f, Y_i = Y_i | X, \sigma_{i}^2 = \sigma_{fil}^2) \text{ where } P(Y_p, Y_o | X) = P(Y_p = Y_f, Y_o = Y_i | X, \sigma_{N}^2 = \sigma_{fil}^2). \]

Similar to the previous section, numerical simulations are performed to estimate the channel statistics to evaluate the influence of the neglected ISI. Since the conditional channel statistics are not dependent on the phase of the input, they are estimated with the same approach as in the previous section. This approach allows signal dependence of the noise and the ISI to

Figure 5.6: (a) The variance of the matched filtered received symbol \( Y_f \), and its parallel components \( Y_{fp} \), and orthogonal components \( Y_{fo} \) given the amplitude of the transmitted symbols \( |X| \) with uniformly distributed phase \( \angle X \). (b) Mismatch capacity employing GS 16, 32, 64 and 128 APSKs optimised constellations and the optimised MI corresponding to the no ISI approximated filtered channel.
be decoupled to some extent, and the numerical estimation of the filtered noise statistics is shown in Figure 5.6a. When the signal is small, the error of the NFT can be neglected, it is observed that the variance of the filtered noise is well described by the model (5.55), while the decomposed components show otherwise. The decomposed components show a slightly larger deviation from the model when compared to the unfiltered noise shown in Figure 5.5, which implies the effect of ISI when the signal is small. As the signal becomes larger, the error introduced by the discretized NFT becomes more significant, the model becomes more inaccurate due to the combined effect of the NFT error and the ISI.

Recall in Section 5.3.4, it is pointed out that the additional constellation size optimisation and probability shaping do not provide significant gain for the signal dependent noise considered in this chapter. Furthermore, the filtered channel (5.45) with no ISI approximation has the same functionality as the unfiltered channel (5.1) considered previously. Thus, for more practical implementation of the optimiser, only the GS-APSK will be considered for the filtered system. The optimisation problem is hence formulated as

$$\max_X I(Y_{fr}; X_\lambda)$$

subject to the same constraint (5.35), where probabilities $P_X = \frac{1}{K} \mathbf{1}$ and $K = 16, 32, 64, 128$ are considered. Note that $K = 128$ is taken into account based on the fact that the noise power is reduced by the matched filter, suggesting that more constellation points could be supported.

Using the identical optimiser and system parameters as in the previous section, the optimal discrete constellation for the channel $P(Y_{fr}, Y_{fi}|X_\lambda)$ can be obtained. It is worth emphasising that ISI is not considered in the channel model, and the numerical simulation shows that the ISI is non-negligible in reality. For the optimised constellation to produce an effective capacity lower bound of the system, the mismatch capacity is necessary. Considering the channel model $P(Y_{fr}, Y_{fi}|X_\lambda)$ as an auxiliary channel, the mismatch capacity for the filtered system is given as

$$C_{Mfil} = \sum_{k=1}^{K} P(X_k) E_{Tfil} \left[ \log_2 \frac{P(Y_{fr}, Y_{fi}|X_k)}{\sum_{n=1}^{K} P(X_n) P(Y_{fr}, Y_{fi}|X_n)} \right],$$

where the $E_{Tfil}(\cdot)$ denotes the expectation over the true filtered channel, which is approximated with the Monte Carlo method by numerically averaging over $1000M$ realisations of the filtered channel, where the ISI will be included. In Figure 5.6b, the mismatch capacity of the filtered system and the MI for the no ISI approximated filtered channel model for GS 16, 32, 64 and 128 - APSKs are shown. The optimised MIs will keep increasing along with the relaxation of the peak amplitude constraint till saturation. The saturation is due to the capacity of the approximated channel is higher than the source entropy, as seen in the figure that 4 and 5 bits per channel use are provided by 16 and 32-APSK at $X_{max} = 2.4$. 
5.4. Signalling With Matched Filter

The capacity lower bound of the true system that takes into account the ISI, however, shows a different trend. At low $X_{\text{max}}$, relaxing the constraint will improve the mismatch capacity as the ISI is small when the signal is of low amplitude. At higher $X_{\text{max}}$, the mismatch capacity will decrease because of the increasing ISI along with the increased signal. For instance, the maximum capacity lower bound of 6.19 bits per channel use is achieved by GS 128APSK at $X_{\text{max}} = 1.5$, while only 5.82 can be achieved at $X_{\text{max}} = 2.4$. Considering the pulse width being 20ns, the maximum data rate of 2.41 Mbits/s/subcarrier is achieved, which is significantly higher than the maximum data rate of 1.85 Mbits/s/subcarrier achieved in the direct signalling system.

When comparing the optimised MI and the mismatch capacity, the alignment at low $X_{\text{max}}$ are all reasonably good, which highlights the effectiveness of the approximation when ISI is weak. When ISI becomes more dominant with the increasing $X_{\text{max}}$, the gap between the optimised MI and the mismatch capacity becomes larger because the no ISI approximation is no longer accurate as well as the signal dependent discretization error becomes larger. The additional error power introduced by the ISI will limit the mismatch capacity achieved. For example, 6.4 bits per channel use MI are predicted by no ISI optimisation for GS 128 APSK at $X_{\text{max}} = 1.5$, while only 6.19 can be achieved by mismatch capacity.

Furthermore, it is also worth noticing that the size of the constellation affects the alignment of the optimised MI and mismatch capacity. For a smaller constellation size like 16-APSK, the distances between the constellation points are sufficiently large to sustain the MI under the combined effect of ISI and signal dependent noise. Hence the corresponding mismatch capacity and optimised MI will both saturate at 4 bits per channel use. On the contrary, increasing the constellation size will reduce the distance between the constellation points, the influence of the ISI will be more significant. As an example, the gaps between optimised MI and mismatch capacity are 0.14, 0.48 and 0.79 for GS 32, 64 and 128 APSKs, respectively.

5.5 Summary

In this chapter, the optimal achievable MI of the CS-modulated system is explored with and without a matched filter in the CS domain. Firstly, an approximated channel model is proposed by combining channel statistics based on perturbation theory [144], VNT, and some insights from numerical simulations for both systems. The optimal discrete input distribution is obtained by maximising the MI of the approximated channel model under the peak amplitude constraint with a finite number of mass points. For the system without a CS domain filter, two types of input distribution shaping are considered, namely GS and FO. The GS scheme attempts to maximise the MI by optimising the positions of the mass points of the input distribution, assuming an equiprobable distribution with a given distribution size. The FO scheme maximises the MI by performing probabilistic geometric hybrid shaping for a number of distribution
sizes. Moreover, the mismatch capacities are estimated for both shaped constellation with the Monte Carlo method to provide capacity lower bounds and the Pinsker formula of the channel model is also derived as benchmark from the existing literature. The results suggest that the additional probabilistic shaping gain of FO is insignificant when compared with the best geometrically shaped scheme, i.e., 64 APSK. Relying on this result, only GS is performed to maximise the MI for the approximated filtered channel model. Note that the developed signal shaping methods are based on the proposed approximated channel model rather than the actual NLSE model. Although introducing matched filtering in the nonlinear frequency domain could result in extra interference, the gain in capacity lower bounds implies that filtering is still beneficial in terms of removing excessive noise.
Chapter 6

Conclusion and Future Works

This thesis investigates various aspects of nonlinear frequency division multiplexed (NFDM) communication systems, demonstrating their potential to improve data rates in long-haul fibre communication. Most of the contributions focus on a specific category of discrete spectrum (DS) modulated NFDM system, which utilises amplitude modulation of a single soliton for communication in the nonlinear optical fibre channel. In addition, the continuous spectrum (CS) modulated system is discussed as another aspect of the available degrees of freedom in the nonlinear frequency domain (NFD). The findings of this thesis confirm the potential of the nonlinear Fourier transform (NFT) in estimating the capacity of nonlinear optical fibre links, which could facilitate a smooth transition to next-generation technology for long-haul optical fibre communication. However, this research also highlights some unresolved issues related to the nonlinear nature of the problem, which require further exploration in future research projects. In this chapter, we provide a detailed summary of the main works conducted in this thesis, followed by some insights on potential future works.

6.1 Capacity of Amplitude Modulated Soliton System

The channel models for NFDM systems, even for a simple amplitude-modulated soliton system, can be quite complex, making it challenging to determine the system’s capacity. To address this problem, new approaches have been proposed for estimating the achievable information rates (AIRs) of such systems, accounting for effects such as Gordon-Haus jitter and inter-soliton interaction. One commonly used channel model for isolated solitons in the literature is the noncentral chi-squared (NCX2) model, which has been approximated by an auxiliary unit-variance additive white Gaussian noise AWGN channel using variance normalising transform (VNT). By proving the equivalence in terms of Kullback-Leibler (KL) divergence between the two channels, it is revealed that the gap between the auxiliary mismatch capacity and the NCX channel capacity approaches zero asymptotically in the high-power regime.
6.1. Capacity of Amplitude Modulated Soliton System

It is known that the capacity-achieving distribution for an AWGN channel with a peak amplitude constraint is discrete. This AWGN channel is used as an approximation for the channel model of an amplitude-modulated soliton system. Using the approximated channel model and subject to a peak amplitude constraint, optimal input distribution and the corresponding capacity are obtained numerically. The optimised distributions are found to be discrete, with a mass point at zero corresponding to no soliton transmission, as well as an almost uniform distribution of mass points spread in a range from zero up to the peak amplitude constraint.

An analytical approximation of the optimal distribution for an amplitude-constrained AWGN channel is proposed using the general form of the capacity-achieving distribution. The optimal output distribution is approximated as a mixture of a unit-variance Gaussian centred at zero and a continuous uniform distribution over the nonzero region. This allowed for the derivation of an analytical expression for the probability of the zero mass point and the minimum and maximum of the nonzero region. Despite additional approximations, this analytical approach closely matched the numerical results based on the AWGN model. Furthermore, the optimal input distribution based on the AWGN model is used to calculate the mismatch capacity of the soliton communication system using a split-step Fourier simulation of the realistic channel defined by the nonlinear Schrödinger equation (NLSE). The results showed a close match between the NLSE simulation-estimated mismatch capacity and the optimised time-scaled mutual information (MI) based on the approximated AWGN channel model. This verified the equivalence between the two channels and highlighted the effectiveness of the VNT technique.

To estimate the mismatch capacity of the system taking into account inter-soliton interaction, channel realisations are generated by propagating three-soliton sequences, where neighbouring solitons are randomly selected. The results indicate that, for long-haul fibres operating in a range of launch powers up to 10 dBm, the effect of inter-soliton interaction caused by limiting the soliton pulse width is stronger than the Gordon-Haus effect. Moreover, there is a trade-off between extending the pulse width to avoid inter-soliton interaction and compressing the pulse width to improve the temporal efficiency.

6.2 AIR of ANN-Aided Interaction-Perturbed Soliton System

Previous studies have shown that inter-soliton interaction is the primary factor limiting the performance of such systems, but it is not well understood or visualised in the literature. The aim of the Chapter 4 is to provide a better understanding of inter-soliton interaction and its impact on the performance of amplitude-modulated soliton communication systems, along with the effect of ASE noise-induced Gordon-Haus jitter.
The inter-soliton interaction is first classified into two categories: intrinsic interaction, which occurs naturally under an ideal, noise-free scenario, and interaction that is enhanced under the perturbation of ASE noise. The intrinsic inter-soliton interaction induced by the inter-soliton force exerted from the temporally closed solitons is first visualised with examples of the transmission of three-soliton sequences. Similar to signal-dependent channel noise, it is observed that the interaction force is also signal-dependent, especially in fixed-interval signalling considered in this thesis. The transmission of three-soliton sequences demonstrates that lower signal amplitude could experience more interaction due to lower effective guarding intervals induced by fixed-interval and truncation of the soliton. The limitation of visualisation is then highlighted with examples of transmitting five-soliton sequences, showing that a closer-to-practice system where longer soliton sequence should be considered, more complicated interaction patterns should be expected.

When considering ASE noise, the Gordon-Haus timing jitter is introduced, which unavoidably couples with intrinsic inter-soliton interaction. Thus, the error introduced by the timing jitter of the Gordon-Haus effect is also classified into two mechanisms. When interaction does not occur, the error beyond ASE noise is mainly due to shifting the soliton out of the NFT sampling window. This is visualised through the propagation of three-soliton sequences with neighbouring solitons being zero. Performing the NFT over the centre soliton intervals enables the capturing of errors caused by ASE noise and out-of-interval shifting. The timing jitter can be visualised by extracting the norming constant of the whole sequence interval with NFT. The signal dependence of the Gordon-Haus effect timing jitter is verified with numerical simulations. When interaction occurs, it can be enhanced by the Gordon-Haus effect if the timing jitter shifts the soliton towards its neighbours. This mechanism of introducing error is demonstrated with simulations of transmitting three-soliton sequences in a higher power regime. The combining effect of ASE noise, timing jitter, and intrinsic inter-soliton force generates new interaction patterns.

Motivated by the nonlinear nature of the problem and the capabilities of artificial neural networks (ANN), a bidirectional Long short-term memory (BLSTM) network classifier detector has been implemented in an amplitude soliton communication system. The detector provides a significant gain under soliton interaction compared to conventional digital signal processing (DSP) techniques, such as the direct NFT and affine minimum mean squared error (AMMSE) estimator. However, it has been demonstrated that the signal-dependent interaction could exceed the capability of the BLSTM network, leading to performance degradation in AIR. Taking into account the signal-dependent signalling time width, a trade-off between interaction and time efficiency has been identified. The potential of the proposed network has been further investigated with different hyperparameter settings, revealing that no significant gain could be achieved over the baseline system setup selected for the hyperparameter analysis.
6.3 On Optimally Shaped Signals for CS NFDM Systems

Nontrivial signal-dependent noise is also reported for the CS modulated NFDM system. If the signal is directly modulated on the CS, the noise statistics up to second order are available in the literature. However, unlike the NCX2 perturbative model for the amplitude-modulated soliton system, an accurate channel model with a closed-form expression is not known despite contributions on channel statistics in the literature. Inspired by the insights observed in numerical simulations, that the distribution of the received symbol depends mainly on the transmitted symbol amplitude rather than its phase. Therefore, a transmitted symbol-dependent signal decomposition is employed to change the basis of the complex received symbol. Under the assumptions of sufficient sampling and available signal-dependent channel statistics, the component that is in line with the transmitted symbol is normalised with VNT to reduce its signal dependence, while the component orthogonal to the transmitted symbol remains unaltered. By approximating the VNT transform inline components and orthogonal components with two independent Gaussian distributions, an approximated channel model for the direct signalling CS modulated NFDM system is proposed. By comparing it with the noncircular Gaussian (NCG) model with the same channel statistics, it is observed that the proposed model could describe the features of the CS channel more accurately.

Based on the proposed approximated channel model, the MI gain from shaping the distribution of the input is discussed. Promising MI gains are observed for both full optimisation (FO) and geometric shaping (GS) shaped constellations over unshaped ring-based amplitude phase shift keying (APSK) constellations. However, the gain from additional probabilistic shaping is not as significant when the number of constellation points in each ring is optimally determined. Conventionally, the probabilistic shaped quadrature amplitude modulation (QAM) constellation is preferred to avoid redesigning the analogue digital converter and optical signal processing algorithm, in addition to the effective shaping gain provided by Maxwell-Boltzmann shaping [154]. The GS technique has attracted certain attention from the development of end-to-end learning and autoencoders, which allow the constellation to be shaped geometrically according to learned features of the channel [128]. However, the autoencoder is considered to be complex due to the large amount of training data required and the complicated structure in the implementation. Comparing to the learned GS, the GS constellation demonstrated in this chapter is more practical to implement, as the shaped constellation can be practically obtained using practical numerical optimisers based on the proposed channel model. Moreover, the capacity lower bound is also estimated with the mismatch capacity and the Pinsker Formula. The gap between the mismatch capacity lower bound and the estimated MI for the approximated channel model implies that the proposed approximated channel is more accurate in the lower to mid power regime.
Lastly, a matched filter in the CS domain is employed to attempt to denoise the received CS signal using the bandlimited property of the pulse shaping function. However, it is shown that the employing a matched filter with comparable bandwidth as the CS signal in a signal dependent channel will introduce inter-subcarrier interference (ISI) into the system. If ISI is neglected, an approximated channel model with reduced noise power is employed to geometrically shape the constellation, adapting only to the signal dependent noise. The estimated mismatch capacity lower bound outperforms that of the unfiltered system although the ISI is neglected in the constellation shaping.

6.4 Future Works

This thesis has focused on the capacity analysis of NFDM optical fibre communication systems, specifically, the amplitude modulated soliton system and CS direct modulated NFDM system. The results obtained provide valuable insights into the behaviour of the nonlinear frequency domain (NFD) signal as it propagates through an ASE noise-perturbed nonlinear optical fibre described by the stochastic NLSE. It is observed that there is still a large gap between the highest AIR achieved in this thesis and the record of the conventional fibre optics techniques without NFT. Within the discussion of this thesis, the degrees of freedom of the NFD are not occupied completely, we mainly focused on modulating on the imaginary part of the eigenvalue, or the CS spectrum, while leaving the real part of the eigenvalue, the corresponding norming constant outside of the scope. Due to the lack of channel modelling of the NFD channel, and the inter-spectrum interference between the CS and DS, the communication techniques employed in the NFDM are not as optimal as the techniques employed in the better modelled, well studied conventional fibre optics system. The insight suggest many unanswered questions and potential topics for future research in this field. In the following section, potential directions for future research will be discussed to possibly provide inspiration for readers.

For the DS modulated system, a potential extension of this thesis is to include the soliton pulse truncation factor $\delta$ in the capacity problem formulation as an additional variable. This would allow for a more comprehensive analysis of the soliton interaction effects if the research scope is limited to the amplitude modulated single soliton system. Additionally, the capacity problem based on the assumption of variable pulse width can be considered in the presence of soliton interaction effects. As demonstrated in this thesis, the deterministic intrinsic inter-solitonic interaction is the dominant factor that limits the system performance. Resolving this interaction described by the NLSE for multi-soliton sequences could potentially invert the interacted pulse with high computational efficiency, closing the gap between the AIR and the channel capacity. Another interesting problem related to DS modulated NFDM systems is to employ other degrees of freedom of the DS domain. The amplitude modulated soliton system
discussed in this thesis only uses the imaginary part of the only eigenvalue in the DS, while other degrees of freedom remain untouched. One could investigate modulating the signal in both positions of the eigenvalues and/or the norming constants. If a channel model of such a higher order DS modulated system is developed, the constellation shaping technique could potentially improve the AIR of the system.

For the CS modulated system, a possible extension of this work would be to derive a closed-form expression for the ISI. Specifically, for a CS modulated system with a matched filter, if an analytical closed-form expression for the ISI on each subcarrier can be obtained, the Pinsker Formula can be employed to derive a capacity lower bound for this vector channel. Moreover, if a channel model can be developed and the input constellation can be shaped accordingly, a tighter capacity lower bound could be achieved. Similarly, another interesting problem to investigate is the \( b \)-modulated NFDM system, which is a variant of the CS modulated system discussed in this thesis. By employing a matched filter in the \( b \) domain, a similar channel with ISI and power-reduced signal-dependent noise could be obtained. In [123], it is reported that the non-ideal propagation of the solitons results in a nonzero CS domain, which is correlated to the noise in the imaginary part of the eigenvalue. If such leakage can be analytically modelled, the potential of the full NFD modulated NFDM system that modulates signals in both CS and DS domains can be investigated to estimate the capacity of a complete NFDM system that takes advantage of all the revealed degrees of freedom.

If all degrees of freedom in NFD are occupied, the full potential of NFDM system could be taken. However, due to complicated interference between DS and CS introduced by ASE noise, performance of full NFD modulated NFDM system cannot be estimated with simple scaling. The modelling of the DS and CS hybrid modulated system remains as an open problem for future research. The feasibility and potential of full nonlinear spectrum modulated NFDM system was prompted in [155], where the potential for error performance was studied to reveal signal-dependent noise in the NFD. In the same year, [156] demonstrated detection of nonlinear spectrum over a 3-span 240 km experimental link for binary phase shift keying (BPSK) NFDM system where the symbols are encoded on both CS and DS. In [157], another experimental full NFD system was demonstrated with 16 QAM CS and QPSK modulated multi-soliton DS. The experiment showed that utilising the additional DS degree of freedom could provide up to 3 dB gain in Q-factor decreasing threshold comparing to the case where only CS is modulated. In [158], crosstalk between CS and DS channels was furthered studied with more complicated modulation format, and information loss was also estimated. In [159], shaped constellation was modulated in both CS and DS spectrum resulting in a data rate improvement of around 12% compared to a NFDM system that only modulates on CS.
Regarding future work related to ANN, the use of data-driven networks has the potential to solve nonlinear problems in NFDM systems. As discussed in Chapter 4, ANN can potentially solve the digital backpropagation (DBP) problem, which could be more computationally efficient than split-step Fourier DBP. Another potential area of investigation is the use of the BLSTM regression network, which has been shown to achieve better AIR than a classifier network with a similar architecture in some scenarios [49]. Additionally, ANN can be used to resolve the nonlinear ISI in matched filtered CS NFDM systems, resulting in higher AIR. Recent developments in ANN research have also revealed its potential for solving partial differential equations (PDE), indicating the potential use of ANN in the implementation of the NFT/INFT.

Generally, due to the nonlinear nature of the NFT related problems, the NFDM is not yet a well studied system, many works in other aspects of the NFDM system could still be done apart from what are mentioned above.


[28] B. J. Puttnam, R. S. Luis, G. Rademacher, Y. Awaji, and H. Furukawa, “319 tb/s transmission over 3001 km with s, c and l band signals over >120nm bandwidth in 125 μm wide 4-core fiber,” in *2021 Optical Fiber Communications Conference and Exhibition (OFC)*, 2021, pp. 1–3.


Appendix A

MATLAB Code for Split-Step Fourier Simulation

An example of implementation of split-step Fourier simulation of the Nonlinear Schrödinger equation is shown as following:

```matlab
function Q_Rx = ssfm_matrix_v4(Q_Tx, dT, alphadB, b2, gamma, L_seq, h, ..., noi_control, exe_mode)

n_realization = size(Q_Tx, 1);

% System and GPU availability check, if MacOS or GPU is not available in windows or Linux, execution mode will be set to 1, i.e. CPU mode,
% otherwise, execution mode will be left unaltered
if ismac
    if exe_mode ~= 1
        disp('cannot use GPU, switch to CPU mode')
        exe_mode = 1;
    end
elseif isunix || ispc
    try
```

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canUseGPU = parallel.gpu.GPUDevice.isAvailable;
catch ME
    canUseGPU = false;
end
if canUseGPU == false
    disp('cannot use GPU, switch to CPU mode')
    exe_mode = 1;
end
if exe_mode == 2
    Q_Tx = gpuArray(Q_Tx);
    batch_num = 1;
    batch_size = n_realization;
    stop_flag = 1; % flag created to monitor the process of batch finishing
    ME = [];
    while (stop_flag <= batch_num)
        try
            for batch_ind = 1:batch_num
                tic
                disp(['batch ', num2str(batch_ind), '/',
                      num2str(batch_num), ' attempt'])
                Q_ind = [sum(batch_size(1:batch_ind-1))+1,
                          sum(batch_size(1:batch_ind))];
                Q_Rx(Q_ind(1):Q_ind(2), :, :) = ...
                ssfm_matrix_main(...
                                Q_Tx(Q_ind(1):Q_ind(2), :, :), ...
                                dT, ...
                                alphadB, b2, gamma, L_seq, h,...
                                noi_control);
                batch_time = toc;
                disp(['batch ', num2str(batch_ind), '/',
                      num2str(batch_num), ...
                      ' finished in ', num2str(batch_time /3600), ' hours'])
                stop_flag = stop_flag+1;
        end
    end
catch ME
    disp('attempt fail')
    if batch_num >= n_realization
        error('not batch size related issue encountered')
    else
        batch_num = batch_num*2;
        batch_size = [floor(n_realization/batch_num) + mod(n_realization, batch_num), ...
                       floor(n_realization/batch_num)*ones(1, batch_num-1)];
        if abs(sum(batch_size, 'all') - n_realization) >= 1e-8
            error('batch splitting error')
        end
        disp(['split task into ', num2str(batch_num) ' batches'])
    end
end
if isempty(ME)
    break %if no error message encounter, break the loop
end
Q_Rx = ssfm_matrix_main(Q_Tx, dT, alphadB, b2, gamma, L_seq, h, ...
elseif exe_mode == 1
    Q_Rx = ssfm_matrix_main(Q_Tx, dT, alphadB, b2, gamma, L_seq, h, ...
end
function Q_Rx = ssfm_matrix_main(Q_Tx, dT, alphadB, b2, gamma, L_seq, h, ...
A. MATLAB Code for Split-Step Fourier Simulation

noi_control)

%Q_Tx--------non-normalized time domain signal
% could be a matrix, where each row represent one signal

%T0 --------normalizing time
%Tmax--------unnormalized half time window, unit second
%alphaB ----fiber loss in dB
%b2--------GVD parameter beta2, unit s^-2/m, assume
          focusing NLSE
%gamma------nonlinearity parameter, unit /W/m
%D----------number of samples
%L_seq------fiber length where the pulse is recorded, unit meter
%h----------split step fourier step length, unit meter
%exe_mode----execution mode control, 1 - CPU mode, 2 - GPU mode

h_planck = 6.62607004e-34; % Planck constant
alpha = alphadB/(4.343*10^3); % Ref page #55 eqn 2.5.3
% Fiber loss value in dB/km
alpha=alphadB/(4.343*10^-3); % Ref page #55 eqn 2.5.3
Fiber optic Comm
%by GP Agrawal

% gamma=1.27e-3; % fiber non linearity in /W/m
% b2=-21e-27; % 2nd order disp. (s^2/m)

KT = 1.13; % photon occupancy factor
mu_0 = 3e8/1550e-9; % light frequency in the fibre, 1550 nm=1.55 um

L = max(L_seq); % Propagation Distance

[num,D] = size(Q_Tx); % num is number of pulses,
                  % D is number of sample per pulse

Tmax = (D-1)*dT/2;
fs = 1/dT;
A. MATLAB Code for Split-Step Fourier Simulation

%%% Noise Parameter %%%
noise_miu = 0;
N_ASE = alpha*L*h_planck*mu_0*KT;
noise_power = N_ASE*h/L/fs;

%%% Implement the SSFM for Q(z,t) %%%
D_fft = 2^(ceil(log2(D)))+1;  % number of sample for fft
w=(-D_fft/2:1:D_fft/2-1)/D_fft*fs*2*pi; % Frequency grid of spectrum, rad/s

% zero padding of the signal
Q_Tx = [Q_Tx, zeros(num,D_fft-D)];
pulse_evo = zeros(num, D, length(L_seq));
spectrum = fftshift(fft(Q_Tx,D_fft,2),2); % input Pulse spectrum, the system is a band
   %pass system, bandwidth is limited within %-B to B

% Main loop of the non-normalized SSFM
i = 1;
for l = h:h:L % Note that the length with
   % Symmetrized Split step fourier
   % Generate Noise at beginning of each step
   N = wgn(num,D_fft,noise_power,'linear','complex');
   % pad N with zero as well, so that the size matches
   N = [N, zeros(num,D_fft-D)];
   % D step
   spectrum=spectrum.*exp(1j*b2/2*w.^2*h/2);
   Q_Tx=ifft(ifftshift(spectrum,2),D_fft,2);
   Q_Tx = Q_Tx(:,1:D_fft);
   % N step
Q_Tx = Q_Tx .* exp(1j*gamma*(abs(Q_Tx)).^2*h);
spectrum = fftshift(fft(Q_Tx, D_fft, 2), 2);

% D step
spectrum = spectrum .* exp(1j*b2/2*w.^2*h/2);

Q_Tx = ifft(ifftshift(spectrum, 2), D_fft, 2);
% update both the time domain signal and spectrum with noise
Q_Tx = Q_Tx(:,1:D_fft) + noi_control*N;
spectrum = fftshift(fft(Q_Tx, D_fft, 2), 2);

% Record the Pulse Shape at the interested FIbre Length
if l == L_seq(i)
    pulse_evo(:, :, i) = Q_Tx(:, 1:1:D);
i = i + 1;
end

% pulse_Leading_edge(i, :) = f(:, 1);
% pulse_Trailing_edge(i, :) = f(:, D);
%
% power_Leading_edge(i, :) = mean(abs(f(:, 1:32)).^2, 2);
% power_Trailing_edge(i, :) = mean(abs(f(:, D-31:D))
% .^2, 2);

end

% Additional steps to ensure when final l is not a integer multiple of h,
% the rest of the distance can be included into the calculation as well.
dis_left = L-l;
if dis_left > 0 && dis_left < h

% Generate Noise at beginning of each step
N = wgn(num, D_fft, noise_power, 'linear', 'complex');

% D step
spectrum = spectrum .* exp((1j*b2/2*w.^2)*(dis_left/2));
A. MATLAB Code for Split-Step Fourier Simulation

```matlab
Q_Tx = ifft(ifftshift(spectrum,2),D_fft,2);
Q_Tx = Q_Tx(:,1:D_fft);

% N step
Q_Tx = Q_Tx .* exp(1j*gamma*((abs(Q_Tx)).^2)*(dis_left));
spectrum = fftshift(fft(Q_Tx, D_fft,2),2);

% D step
spectrum = spectrum .* exp((1j*b2/2*w.^2)*(dis_left/2));
Q_Tx = ifft(ifftshift(spectrum,2),D_fft,2);
Q_Tx = Q_Tx(:,1:D_fft) + noi_control*N;
spectrum = fftshift(fft(Q_Tx, D_fft,2),2);

% pulse_shape = [pulse_shape; f(:,1:D)];
% dis = [dis,L];
end

% f = single(f);
% f(abs(f)<1e-11) = abs(f(abs(f)<1e-11));
Q_Rx = pulse_evo;
end
```
Appendix B

MATLAB Code for Constellation Optimising

An example of constellation optimising algorithm used in amplitude modulated soliton system is given as following:

```matlab
%% This is a Smith algorithm program that search for the optimum source
%% distribution that could maximise the MI(Y|X).

%% In this version of the optimization code, it contain 2 different
%% optimization, PGS and GS with constellation that includes 0. Note that
%% all of the constellation shaping is done given a particular constellation
%% size.

%% Function Body
function Capacity = MI_Max_fun_with0(X_Amp, Z_ref, sigma_N_2, del, depth)

VNT = @(x) 2* sqrt(x/ sigma_N_2);
IVNT = @(x) sigma_N_2/4* x.^2;

Amp_ub = X_Amp;
Amp_lb = VNT(IVNT(X_Amp)/1.2417);

% Tw = @(A) max(1/2./A(A>0).*log(2/del - 1));
```
% _____________Initialization________________
% Define the initial point

% flag1 = 0;
% flag2 = 0;
C = [];
C_GS = [];
um = [];
Z_opt_rec = {};
X_opt_GS_rec = {};
n_in = max([length(Z_ref)/2, 2]);
n = (n_in~=0)*n_in + (n_in==0)*2;
i = 1;
% depth = 2;

if isempty(Z_ref) ~= 1
    MI_ref = MI_fun(Z_ref);
else
    MI_ref = 0;
end

if isfile('Smith_intermediate.mat')
    %If this intermediate data file exists, suggest the
    %optimizer is
    %terminated unfinished, load in data and continue
    load Smith_intermediate.mat ;
n = n+1;
i = i+1;
else
    %If the intermediate data file doesn' existsm
    initialize the optimizer
    %normally
end
tic
while (1)
    \% GS Step
    \%GS Constraint
    \%The vector we want to optimize consist of value of the soliton
    \%amplitude, with amplitude being 0 representing no soliton signal

    \%Define the constraints feasible space
    \%    ub_GS = Amp_ub*ones(n,1); %upperbound of the mass point
    \%    lb_GS = Amp_lb*ones(n,1); %lowerbound of the mass point

    ub_GS = [Amp_ub*ones(n-1,1)]; % upperbound of the mass point
    lb_GS = [Amp_lb*ones(n-1,1)]; % lowerbound of the mass point

    Aeq_GS = [];
    beq_GS = [];

    \% A_GS = (eye(n-1,n)-[zeros(n-1,1),eye(n-1,n-1)]);
    \% b_GS = zeros(n-1,1);
    A_GS = [];
    b_GS = [];

    \%Determine the initial mass points and mass value according to the number of mass point and constraints.
    \%When n not equal to n of Z_ref
    \%GS
    \% x0 = sort(Amp_ub*rand(n-2, 1));
% X0_GS = [x0; X_Amp];

x0 = Amp_lb : (Amp_ub - Amp_lb) / (n-2) : Amp_ub;
X0_GS = x0 .';
else
    %GS
    X0_GS = Z_ref(n+2:end);
end

% _____________Objective function and Optimization__________________________

%GS Step
neg_MI_GS = @(X) - MI_fun_GS_with0(X);
disp(['GS Step initialization: ' num2str([n, -neg_MI_GS(X0_GS)])])

opt_fmincon = optimoptions('fmincon',...
    'Algorithm', 'interior-point',...
    'MaxFunctionEvaluations', (n+1)*1e6, ...
    'MaxIterations', 1e6, ...
    'StepTolerance', 1e-13, ...
    'OptimalityTolerance', 1e-10, ...
    'FunValCheck', 'on', ...
    'UseParallel', true);

[X_opt_GS, neg_C_GS, exit_flag_GS(i)] = ... 
    fmincon(neg_MI_GS, X0_GS,... 
    A_GS, b_GS, Aeq_GS, beq_GS, lb_GS, ub_GS, [],... 
    opt_fmincon);
disp(['GS Results for ' , num2str([n, -neg_C_GS])]);

C_GS(i) = -neg_C_GS;
X_opt_GS_rec(i) = {X_opt_GS};

% PGS Optimization
% PGS Constraint
%The vector we want to optimize consist of two part, with the size of
```matlab
% 2*n, 1 n elements are probability, 2 n elements are the soliton
% amplitude with amplitude 0 representing no soliton is transmitted

% Define the constraints feasible space

% ub_PGS = [ones(n,1); Amp_ub*ones(n,1)]; % upperbound of the mass value, and point
% lb_PGS = [zeros(n,1); Amp_lb*ones(n,1)]; % lowerbound of the mass value, and point

ub_PGS = [ones(n,1); Amp_ub*ones(n-1,1)]; % upperbound of the mass value, and point
lb_PGS = [zeros(n,1); Amp_lb*ones(n-1,1)]; % lowerbound of the mass value, and point

Aeq_PGS = [ones(1,n), zeros(1,n-1)]; % The matrix expression of mass values
beq_PGS = 1; % sum to one

% A_PGS = [zeros(n-1,n), (eye(n-1,n)-[zeros(n-1,1), eye(n-1,n-1))];
% b_PGS = zeros(n-1,1);
A_PGS = [];
b_PGS = [];

% Determine the initial mass points and mass value according to the number of mass point and constraints.

if n ~= (length(Z_ref)+1)/2
    % When n not equal to n of Z_ref
    %PGS
    p0 = rand(n, 1);
p0 = p0./sum(p0);
    x0 = sort(Amp_ub*rand(n-1, 1));
    Z0 = [p0; x0];
```
else
    \% When n equal to Z_ref
    \% PGS
    Z0 = Z_ref;
end

\% PGS Step
Z0 = [1/(length(X_opt_GS)+1)*ones(length(X_opt_GS)+1,1); X_opt_GS];
neg_MI = @(Z) -MI_fun_with0(Z);
disp(['PGS Step initialization: ' num2str([n, neg_MI(Z0)])])

[Z_opt, neg_C, exit_flag(i)] = ...
    fmincon(neg_MI,Z0,...
    A_PGS,b_PGS,Aeq_PGS,beq_PGS,lb_PGS,ub_PGS,[],...
    opt_fmincon);
C(i) = - neg_C;
num(i) = n;
Z_opt_rec(i) = {Z_opt};
disp(['Results for ', num2str([n, C(i)])]);

% if C(i) < C_GS(i)
%     disp('PGS smaller than GS, attempt SQP algorithm to solve')
%     opt_fmincon.Algorithm = 'sqp';
%     [Z_opt, neg_C, exit_flag(i)] = ...
%     fmincon(neg_MI,Z0,...
%     A_PGS,b_PGS,Aeq_PGS,beq_PGS,lb_PGS,ub_PGS
%     ,[],...
%     opt_fmincon);
%
% C(i) = - neg_C;
% Z_opt_rec(i) = {Z_opt};
% disp(['SQP Results for ', num2str([n, C(i)])]);
% end
% Determine when the two flag, flag1 corresponds to the
capacity
% esimtaed being larger than the unshaped MI, while the
flag2
%corresponds to the decreasing trend of the optimized
MI

% Flag 1:
if i <= depth + 1
    flag1 = 0;
elseif C(i - depth) >= max(MI_ref)
    flag1 = 1;
else
    flag1 = 0;
end

% Flag 2: by changing the parameter depth, we can change
%to check
if i <= depth + 1
    flag2 = 0;
elseif sum(C(i-(depth - 1):i) - C(i-(depth):i-1)<=0) == depth
    flag2 = 1;
elseif sum((C(i-(depth - 1):i) - C(i - depth))/C(i - depth) <= 1e-6) == depth
    flag2 = 1;
else
    flag2 = 0;
end

if flag1 == 1 && flag2 == 1
    %Optimization finished, now output results
    [~, i_max] = max(C);
    i_max = i - depth;
    n = num(i_max);
B. MATLAB Code for Constellation Optimising

```
Cap = C(i_max);
Z_opt = cell2mat(Z_opt_rec(i_max));
break;
end

% Save the intermediate data in case of the disruption
save('Smith_intermediate.mat')

n = n+1;
i = i+1;
end

optim_time = toc;
disp(['Optimisation time = ', num2str(optim_time/3600), ' hours'])

% % Mesh Value for generating a output distribution figure
% Yr_grid = -X_Amp-3*sqrt(sigma_N_2*(1+X_Amp^2)^2):...
% 6*sqrt(sigma_N_2*(1+X_Amp^2)^2)/500:...
% X_Amp+3*sqrt(sigma_N_2*(1+X_Amp^2)^2);
% Yi_grid = Yr_grid;
% [Yr_mesh, Yi_mesh] = meshgrid(Yr_grid, Yi_grid);
% P_ri_mesh = Pri_fun(Yr_mesh, Yi_mesh,...
% Z_opt(2*n+1:3*n).*exp(1j.*Z_opt(n+1:2*n)), Z_opt(1:n) , sigma_N_2);

% Capacity.Amp_Con = X_Amp;
% Capacity.Alp_real = (Z_opt(2*n+1:3*n).*cos(Z_opt(n+1:2*n) ));'
% Capacity.Alp_imag = (Z_opt(2*n+1:3*n).*sin(Z_opt(n+1:2*n) ));'
% Capacity.Prob = Z_opt(1:n);
% Capacity.num_mass_point = n;
Capacity.Z_opt = Z_opt;
Capacity.Cap = Cap;

% Optimizer Record
Capacity.num_rec = num;
Capacity.Optimization_time = optim_time;
```
B. MATLAB Code for Constellation Optimising

```matlab
% PGS record
Capacity.Z_opt_rec = Z_opt_rec;
Capacity.Cap_rec = C;
Capacity.Exit_Flag = exit_flag;

% GS record
Capacity.Z_opt_GS_rec = X_opt_GS_rec;
Capacity.Cap_GS_rec = C_GS;
Capacity.Exit_Flag_GS = exit_flag_GS;

% Capacity.Yr_mesh = Yr_mesh;
% Capacity.Yi_mesh = Yi_mesh;
% Capacity.P_ri_mesh = P_ri_mesh;

% The optimization is finished, delete the intermediate data
delete Smith_intermediate.mat

figure
subplot(1, 2, 1)
hold on
plot(num, C_GS, '^-', 'DisplayName', 'GS')
plot(num, C, '-o', 'DisplayName', 'PGS')
hold off
xlabel('num of mass point')
ylabel('MI')
legend
box on
grid on

subplot(1, 2, 2)
hold on
for n = 1:length(num)
    Z_plot = Z_opt_rec{n};
    mass_value = Z_plot(1:(length(Z_plot)+1)/2);
    mass_point = [0; Z_plot((length(Z_plot)+1)/2+1:end)];
    stem3(mass_point, num(n)*ones(size(mass_point)),
          mass_value)
```
end
hold off
xlabel('mass point position')
ylabel('number of mass point')
zlabel('mass value')
box on
grid on

%% Customed Function Definition
%1. Self-Defined differential Entropy function that deal with 0 ponits for
%this complex case
function h = h_fun(P)
    h = -P.*log2(P);
    h(abs(P)<=1e-20) = 0;
    h(isnan(h)) = 0;
end

% 2. P(Y|X), the conditional distribution for the VNT transformed soliton
% amplitude, this distribution is a unit variance Gaussian when X is
% nonzero, when X is zero, the probability is 1 at Y is zero, and 0
% everywhere else
function P = PY_X_Gau_fun(Y, X)
    if X == 0
        P = dirac(Y);
    else
        P = 1/sqrt(2*pi)*...
            exp(-1/2*(Y - X).^2);
    end
end
function P = PY_Gau_fun(Y, Z)
    n = length(Z)/2;
    PX = Z(1:n);
    X = Z(n+1:2*n);
    if isempty(find(abs(X) <= 1e-20, 1)) ~= 1
        disp('0 in constellation')
        P0 = PX(1);
        PX = PX(2:end);
        X = X(2:end);
    end
    P = zeros(size(Y));
    for i = 1:length(X)
        P_comp = PY_X_Gau_fun(Y, X(i));
        P = P + PX(i)*P_comp;
        if isempty(find(isnan(P_comp), 1)) == 0
            error('NaN encounter')
        end
    end
end

function H = HY_Gau_fun(Z)
    n = length(Z)/2;
    PX = Z(1:n);
    X = Z(n+1:2*n);
    H = integral(@(Y) h_fun(PY_Gau_fun(Y, Z)), ...
        -inf, inf,..., ...
        'RelTol',0,'AbsTol',1e-12, 'ArrayValued',true,...
        'Waypoints', unique([X.',(X.'+10), (X.'-10)], 'sorted'));
B. MATLAB Code for Constellation Optimising

```matlab
% H = integral(@(Y) h_fun(PY_Gau_fun(Y, Z)),...
% X(1)-20, X(end)+20,...
% 'RelTol',0,'AbsTol',1e-12,'ArrayValued',true,...
% 'Waypoints', unique([X.',(X.'+10), (X.'-10)], 'sorted'));
% H = integral(@(Y) h_fun(PY_Gau_fun(Y, Z)),...
% -inf, inf,...
% 'Waypoints', unique([X.',(X.'+10), (X.'-10)], 'sorted'));

[int_lb, int_ub] = int_bound(X-30, X+30);

H = 0;
for n = 1:length(int_lb)
    H_temp = integral(@(Y) h_fun(PY_Gau_fun(Y, Z)),...
                      int_lb(n), int_ub(n));
    H = H+H_temp;
end

% if isempty(find(abs(X) <= 1e-20, 1)) ~= 1
% H = H - h_fun(PX(1));

if isnan(H)
    % Show distribution
    disp(X_xi)
    disp(P_X)
    % Show pdf validity test result
    P_ri_val = integral(@(Y) PY_Gau_fun(Y, Z), ...  
                         -inf, inf);
    if abs(P_ri_val - 1) >= 1e-3
        disp(P_ri_val)
        disp('range error');
    end
end

save('Singularity_Smith.mat')
```
B. MATLAB Code for Constellation Optimising

```matlab
% Singularity error handling
if abs(sum(P_X) - 1) >= 1e-4
    error('Singularity encountered in H(Y)
end

% 5. Mutual Information
function MI = MI_fun(Z)
    m = length(Z)/2;
    P_X = Z(1:m);
    X = Z(m+1:2*m);
    if abs(sum(P_X) - 1) >= 1e-4
        error('Error in MI calculation: Input distribution invalid')
    end

    if isempty(find(abs(X) <= 1e-20, 1)) ~= 1
        % X contains 0
        P0 = P_X(abs(X) <= 1e-20);
        X_mod = X;
        X_mod(abs(X) <= 1e-20) = [];
        P_X_mod = P_X;
        P_X_mod(abs(X) <= 1e-20) = [];
        MI = P0*log2(1./P0) + HY_Gau_fun([P_X_mod; X_mod])
            - (1 - P0)*log2(sqrt(2*pi*exp(1)));
    else
        % X does not contain 0
        MI = HY_Gau_fun(Z) - log2(sqrt(2*pi*exp(1)));
    end

% 6. Mutual Information for uniform distribution
function MI = MI_fun_GS(X)
```

m = length(X);

% Set the input distribution to uniform
Z_GS = [1/m*ones(size(X)); X];

MI = MI_fun(Z_GS);
end

function MI = MI_fun_GS_with0(X)
m = length(X)+1;

% Set the input distribution to uniform
Z_GS = [1/m*ones(m, 1); 0; X];

MI = MI_fun(Z_GS);
end

function MI = MI_fun_with0(Z)
m = (length(Z)+1)/2;

% Set the input distribution to uniform
Z_mod = [Z(1:m); 0; Z(m+1:end)];

MI = MI_fun(Z_mod);
end

function T = Tw_with0(Z, sigma_N_2, del)

% Tha A must be a vector which size is nx1, n > 1, if n = 1,
% then A should
% not be 0
n = (length(Z)+1)/2;

% Variance Normalizing Transfigure; plot(Res.nn.P_noi);
% hold on; plot(Res.P_noi); ormation
% VNT = @(x) 2*sqrt(x/ sigma_N_2);
% IVNT = @(x) sigma_N_2/4*x.^2;
B. MATLAB Code for Constellation Optimising

A = IVNT(Z(n+1:end));
T = 1./(2.*A).*log(2/del - 1);
T = max(T(~isinf(T))&(~isnan(T)));
end

function T = Tw_GS_with0(X, sigma_N_2, del)
% Tha A must be a vector which size is nx1, n > 1, if n = 1, then A should
% not be 0
m = length(X)+1;
T = Tw_with0([1/m*ones(m, 1); X], sigma_N_2, del);
end

function [int_lb, int_ub] = int_bound(l, u)
% This function is built to check whether there is crossing
% in the
% integration bound defined
if length(l) ~= length(u)
    error('input bounds sizes donnot match')
end
l = sort(l, 'ascend');
u = sort(u, 'ascend');
n = length(l);
ind = find(u(1:end-1)>l(2:end));
int_lb=l;
int_ub=u;
int_lb(ind+1) = [];
int_ub(ind) = [];
end
An example of constellation optimising algorithm used in continuous spectrum modulated nonlinear frequency division multiplexed system is given as following:

```matlab
% This is a Smith algorithm program that search for the optimum source
% distribution that could maximise the MI(Yr, Yi; Xr, Xi) considering the peak amplitude constraint assuming the optimal input distribution is discrete; the optimized MI will increase with respect to the number of increase; the channel model P(Yr, Yi|X) can be obtained from two independent Gaussian at the projected, and VNT transformed domain

%% Function Body
function Capacity = Smith_PA_fun4(X_Amp, Z_ref, N_para, depth)
% clc
% clear
% close all
% X_Amp = 0.3;
% Z0 = [];
% N_para = 0.0154;

Amp_lb = 0;
Amp_ub = X_Amp;

% _______________Initialization_________________
% Define the initial point
```
flag1 = 0;
flag2 = 0;
C = [];
C_GS = [];
um = [];
Z_opt_rec = {};
Z_opt_GS_rec = {};
n_in = max([length(Z_ref)/3, 9]);

n = (n_in~=0)*n_in + (n_in==0)*2;
i = 1;

% depth = 2;

% Calculate the MI for the unshaped constellation for reference

% 16 APSK
Con_ri = (apskmod((0:15), [4 12], X_Amp*[1, 3.2]/3.2)).'; % DVB S2X
Z_16APSK = [1/16*ones(16, 1); angle(Con_ri); abs(Con_ri)];
MI_16APSK = MI_fun(Z_16APSK, N_para);

% 32 APSK
Con_ri = (apskmod((0:31), [4 12 16], X_Amp*[1, 2.53, 4.3]/4.3)).'; % DVB S2X
Z_32APSK = [1/32*ones(32, 1); angle(Con_ri); abs(Con_ri)];
MI_32APSK = MI_fun(Z_32APSK, N_para);

% 64 APSK
Con_ri = (apskmod((0:63), [8 16 20 20], X_Amp*[1, 2.2, 3.6, 5.2]/5.2)).'; % DVB S2X
Z_64APSK = [1/64*ones(64, 1); angle(Con_ri); abs(Con_ri)];
MI_64APSK = MI_fun(Z_64APSK, N_para);

if isempty(Z_ref) ~= 1
    MI_ref = MI_fun(Z_ref, N_para);
else
    MI_ref = 0;
end
if isfile('Smith_intermediate.mat')
    \%If this intermediate data file exists, suggest the
    \%terminated unfinished, load in data and continue
    load Smith_intermediate.mat;
    n = n+1;
    i = i+1;
else
    \%If the intermediate data file doesn't existsm
    \%normaly
    \%initialize the optimizer
end

disp([MI_16APSK, MI_32APSK, MI_64APSK, MI_ref])
tic
while (1)
    \%GS Constraint
    \%The vector we want to optimize consist of three part,
    \%with the size of
    \%2*n, 2 n elements are angle, 3 n elements are
    \%amplitude
    \%Define the constraints feasible space
    ub_GS = [pi*ones(n,1);Amp_ub*ones(n,1)]; \% upperbound of the mass value, and point
    lb_GS = [-pi*ones(n,1); Amp_lb*ones(n,1)]; \% lowerbound of the mass value, and point
    Aeq_GS = []; \%The
    beq_GS = []; \%sum to one
    % A_GS = [zeros(n-1,n),(eye(n-1,n)-[zeros(n-1,1),eye(n -1,n-1)]);
% b_GS = zeros (n-1,1);
A_GS = [];
b_GS = [];

% PGS Constraint
% The vector we want to optimize consist of three part,
% with the size of
% 3*n, 1 n elements are probability, 2 n elements are
% angle, 3 n elements
% are amplitude

% Define the constraints feasible space
ub_PGS = [ones(n,1); pi*ones(n,1); Amp_ub*ones(n,1)];
% upperbound of the mass value, and point
lb_PGS = [zeros(n,1); -pi*ones(n,1); Amp_lb*ones(n,1)];
% lowerbound of the mass value, and point
Aeq_PGS = [ones(1,n),zeros(1,n), zeros(1,n)]; % The
beq_PGS = 1; % sum to
one

% A_PGS = [zeros(n-1,n), zeros(n-1,n), (eye(n-1,n) -[
zeros(n-1,1), eye(n-1,n-1)])];
% b_PGS = zeros(n-1,1);
A_PGS = [];
b_PGS = [];

% Determine the initial mass points and mass value
% according to the
% number of mass point and constraints.
if n~=length(Z_ref)/3
  % When n not equal to n of Z_ref
  % PGS
  p0 = rand(n, 1);
p0 = p0./sum(p0);
  pha0 = 2*pi*rand(n, 1) - pi;
\[\text{rho0} = \text{sort}(\text{Amp_ub} \cdot \text{rand}(n, 1));\]

\[\text{Z0} = [\text{p0}; \text{pha0}; \text{rho0}];\]

%GS
\[\text{pha0} = 2\pi \cdot \text{rand}(n, 1) - \pi;\]

\[\text{rho0} = \text{sort}(\text{Amp_ub} \cdot \text{rand}(n, 1));\]

\[\text{Z0_GS} = [\text{pha0}; \text{rho0}];\]

\text{else}

% When n equal to Z_ref
% PGS
\[\text{Z0} = \text{Z_ref};\]

%GS
\[\text{Z0_GS} = \text{Z_ref}(n+1:end);\]
\text{end}

% _____________Objective function and Optimization__________________________
%GS Step
\[\text{neg_MI_GS} = @(Z) -\text{MI_fun_GS}(Z, N_{para});\]

\text{disp(['GS Step initialization: ' num2str([n, -neg_MI_GS(Z0_GS)])])}

\text{opt_fmincon = optimoptions('fmincon',...}
\text{'Algorithm'},'interior-point',...\n\text{'MaxFunctionEvaluations'},(n+1)*1e6,...\n\text{'MaxIterations'},1e6,...\n\text{'StepTolerance'}, 1e-13,...\n\text{'OptimalityTolerance'}, 1e-6,...\n\text{'FunValCheck'},'on',...\n\text{'UseParallel', true};\n\text{[Z_opt_GS, neg_C_GS, exit_flag_GS(i)] = ...}
\text{fmincon(neg_MI_GS, Z0_GS,...}
\text{A_GS,b_GS,Aeq_GS,beq_GS,lb_GS,ub_GS,[],...}
\text{opt_fmincon});}

%PGS Step
B. MATLAB Code for Constellation Optimising

```matlab
Z0 = [2/length(Z_opt_GS)*ones(length(Z_opt_GS)/2, 1); Z_opt_GS];

neg_MI = @(Z) -MI_fun(Z, N_para);
disp(['PGS Step initialization:' num2str([n, -neg_MI(Z0)]))

[Z_opt, neg_C, exit_flag(i)] = fmincon(neg_MI, Z0, A_PGS, b_PGS, Aeq_PGS, beq_PGS, lb_PGS, ub_PGS, [], opt_fmincon);

C(i) = -neg_C;
C_GS(i) = -neg_C_GS;
num(i) = n;
Z_opt_rec(i) = {Z_opt};
Z_opt_GS_rec(i) = {Z_opt_GS};
disp(['Results for ', num2str([n, C(i)])]);

if C(i) < C_GS(i)
    disp('PGS smaller than GS, attempt SQP algorithm to solve')
opt_fmincon.Algorithm = 'sqp';
[Z_opt, neg_C, exit_flag(i)] = fmincon(neg_MI, Z0, A_PGS, b_PGS, Aeq_PGS, beq_PGS, lb_PGS, ub_PGS, [], opt_fmincon);

C(i) = -neg_C;
Z_opt_rec(i) = {Z_opt};
disp(['SQP Results for ', num2str([n, C(i)])]);
end

% Determine when the two flag, flag1 corresponds to the capacity estimated being larger than the unshaped MI, while the flag2 corresponds to the decreasing trend of the optimized MI
```
% Flag 1:
if i <= depth + 1
    flag1 = 0;
elseif C(i - depth) >= max([MI_16APSK, MI_32APSK, MI_64APSK, MI_ref])
    flag1 = 1;
else
    flag1 = 0;
end

% Flag 2: by changing the parameter depth, we can change how far we want to check
if i <= depth + 1
    flag2 = 0;
elseif sum(C(i-(depth-1):i) - C(i-(depth):i-1)<=0) == depth
    elseif sum((C(i-(depth-1):i) - C(i - depth))/C(i - depth) <= 1e-6) == depth
    elseif sum((C(i-(depth-1):i) - C(i - depth))/C(i - depth) <= 1e-6) == depth && C(i - depth) == max(C)
        flag2 = 1;
    else
        flag2 = 0;
    end

if flag1 == 1 && flag2 == 1
    % Optimization finished, now output results
    [~, i_max] = max(C);
    i_max = i - depth;
    n = num(i_max);
    Cap = C(i_max);
    Z_opt = cell2mat(Z_opt_rec(i_max));
    break;
end

% Save the intermediate data in case of the disruption
save ('Smith_intermediate.mat')

n = n + 1;
i = i + 1;
end

optim_time = toc;
disp(['Optimisation time = ', num2str(optim_time/3600), ' hours'])

% Mesh Value for generating a output distribution figure
Yr_grid = -X_Amp - 3*sqrt(N_para*(1+X_Amp^2)^2):...
6*sqrt(N_para*(1+X_Amp^2)^2)/500:...
X_Amp + 3*sqrt(N_para*(1+X_Amp^2)^2);
Yi_grid = Yr_grid;
[Yr_mesh, Yi_mesh] = meshgrid(Yr_grid, Yi_grid);
P_ri_mesh = Pri_fun(Yr_mesh, Yi_mesh,...
Z_opt(2*n+1:3*n).*exp(1j.*Z_opt(n+1:2*n)), Z_opt(1:n),
N_para);

Capacity.Amp_Con = X_Amp;
Capacity.Alp_real = (Z_opt(2*n+1:3*n).*cos(Z_opt(n+1:2*n)))
'.
Capacity.Alp_imag = (Z_opt(2*n+1:3*n).*sin(Z_opt(n+1:2*n)))
'.
Capacity.Prob = Z_opt(1:n);
Capacity.num_mass_point = n;
Capacity.Z_opt = Z_opt;
Capacity.Cap = Cap;

% Optimizer Record
Capacity.Z_opt_rec = Z_opt_rec;
Capacity.num_rec = num;
Capacity.Cap_rec = C;
Capacity.Z_opt_GS_rec = Z_opt_GS_rec;
Capacity.Cap_GS_rec = C_GS;
Capacity.Optimization_time = optim_time;
Capacity.Exit_Flag = exit_flag;
Capacity.Yr_mesh = Yr_mesh;
Capacity.Yi_mesh = Yi_mesh;
Capacity.P_ri_mesh = P_ri_mesh;

% The optimization is finished, delete the intermediate data
delete Smith_intermediate.mat

% figure
% hold on
% plot(num, C_GS, '-^', 'DisplayName', 'GS')
% plot(num, C, '-o', 'DisplayName', 'PGS')
% hold off
% xlabel('num of mass point')
% ylabel('MI')
% legend
% box on
% grid on

end

%%% Customed Function Definition

% %1. P(Rp|X), the conditional distribution for the VNT
% transformed inline components
% function P = PRp_X_fun(Rp, X_amp, N_para)
% VNT = @(u) sqrt(2/N_para)*atan(u);
% P = 1/sqrt(2*pi)...
% exp(-1/2*(Rp - VNT(X_amp)).^2);
% end

%2. P(Yp|X) the marginal conditional distribution for the
% Projection component
function P = PYp_X_fun(Yp, X_amp, N_para)
B. MATLAB Code for Constellation Optimising

VNT = @(u) sqrt(2/N_para)*atan(u);

IVNT = @(u) tan(sqrt(N_para/2)*u);

P = (cos(atan(Yp)).^2)./sqrt(pi*N_para).*...
  exp(-1/2*(VNT(Yp) - VNT(X_amp)).^2);

end

%3. P(Yo|X), the marginal conditional distribution for the orthogonal

function P = PYo_X_fun(Yo, X_amp, N_para)

    sig_Yo = sqrt(N_para/2*(1 + X_amp.^4));

    P = 1./sqrt(2*pi*sig_Yo.^2) *...
        exp(-1/2*(Yo).^2/sig_Yo.^2);

end

%4. P(Yp, Yo|X), the joint distribution in the projection and orthogonal

function P = Ppo_X_fun(Yp, Yo, X_amp, N_para)

    P = PYp_X_fun(Yp, X_amp, N_para).*PYo_X_fun(Yo, X_amp, N_para);

end

%5. P(Yr, Yi|X), the joint distribution in the real and imaginary space

function P = Pri_X_fun(Yr, Yi, X, N_para)

    sig_Yo = sqrt(N_para/2*(1 + abs(X).^4));

    Pp = ((cos(atan(Yr.*cos(angle(X)) + Yi.*sin(angle(X)))).^2)
         ./sqrt(pi*N_para).*...
B. MATLAB Code for Constellation Optimising

```matlab
exp(-1/N_para*(atan(Yr.*cos(angle(X)) + Yi.*sin(angle(X)) - atan(abs(X))).^2));
Po = 1/sqrt(2*pi*sig_Yo.^2)*...
exp(-1/2*(Yr.*cos(angle(X)+pi/2) + Yi.*sin(angle(X)+pi/2)).^2./sig_Yo.^2);
P = Pp.*Po;
end

%6. Self-Defined differential Entropy function that deal with 0 points for this complex case
function h = h_fun(P)
h = -P.*log2(P);
h(abs(P) <= 1e-14) = 0;
h(isnan(h)) = 0;
end

%7. Yr, Yi joint output PDF
function P = Pri_fun(Yr, Yi, X, PX, N_para)
P = zeros(size(Yr));
for i = 1:length(X)
P_comp = Pri_X_fun(Yr, Yi, X(i), N_para);
P = P + PX(i)*P_comp;
if isempty(find(isnan(P_comp), 1)) == 0
    error('NaN encounter')
end
end

%8. Conditional Entropy
function H_ri_X = H_Y_X_fun(X_xi, P_X, N_para)
H_ri_X = 0;
H_ri_X_comp = zeros(length(X_xi));
```

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P_ri_X_val = zeros(length(X_xi));

for i = 1:length(X_xi)
    disp(X_xi(i))

    H_ri_X_comp(i) = integral(@(Yp) h_fun(PYp_X_fun(Yp, abs(X_xi(i))), N_para)), ...
    -inf, inf) + log2(sqrt(2*pi*exp(1)*N_para/2*(1 + abs(X_xi(i)).^4)));

    if isempty(find(isnan(H_ri_X_comp), 1)) == 0
      % Disp the X_xi
      disp(X_xi(i))
      % Disp the PDF validity test result
      P_ri_X_val(i) = ...
      integral2(@(Yr, Yi) Pri_X_fun(Yr, Yi, X_xi(i), N_para), ...
      -inf, inf, ... ...
      -inf, inf);

      if abs(P_ri_X_val(i) - 1) >= 1e-3
        disp(P_ri_X_val(i))
        disp('range error');
      end

      save('Singularity_Smith.mat')
      error('singularity encounter in H(Y|X)')
    end

    H_ri_X = P_X(i)*H_ri_X_comp(i) + H_ri_X;
end

% 9. Output Entropy
function H_ri = H_Y_fun(X_xi, P_X, N_para)
    H_ri = integral2(@(Yr, Yi) h_fun(Pri_fun(Yr, Yi, X_xi, P_X, N_para)), ...
    -inf, inf, ... ...
    -inf, inf);
if isnan(H_ri)
    % Show distribution
    disp(X_xi)
    disp(P_X)

    % Show pdf validity test result
    P_ri_val = integral2(@(Yr, Yi) Pri_fun(Yr, Yi, X_xi, P_X, N_para), ...
       -inf, inf, ...
       -inf, inf);

    if abs(P_ri_val - 1) >= 1e-3
        disp(P_ri_val)
        disp('range error');
    end

    save('Singularity_Smith.mat')
    error('singularity encounter in H(Y)')
end

% 10. Mutualn Information
function MI = MI_fun(Z, N_para)
    m = length(Z)/3;

    P_X = Z(1:m);
    X_amp = Z(2*m+1:3*m);
    X_ang = Z(m+1:2*m);
    X_xi = X_amp.*exp(1j.*X_ang);

    if abs(sum(P_X) - 1) >= 1e-4
        error('Error in MI calculation: Input distribution invalid')
    end

    H_ri_X = H_Y_X_fun(X_xi, P_X, N_para);
    H_ri = H_Y_fun(X_xi, P_X, N_para);
B. MATLAB Code for Constellation Optimising

```
MI = H_ri - H_ri_X;

end

% 10. Mutual Information for uniform distribution
function MI = MI_fun_GS(Z, N_para)
    m = length(Z)/2;
    P_X = 1/m*ones(m,1);
    X_amp = Z(m+1:2*m);
    X_ang = Z(1:m);
    X_xi = X_amp.*exp(1j.*X_ang);
    H_ri_X = H_Y_X_fun(X_xi, P_X, N_para);
    H_ri = H_Y_fun(X_xi, P_X, N_para);
    MI = H_ri - H_ri_X;
end
```
Appendix C

MATLAB Code for NFDM Channel Simulator

An example of channel simulator for amplitude modulated soliton communication is given as following:

```matlab
% This function is created to perform soliton sequence transmission and NFT
% detection.

% In this set of code, the number of samples of the time domain is
% automatically selected based on the maximum soliton amplitude since the
% bandwidth of the first order solitonic pulses is defined by the soliton
% amplitude.

% Description of Simulator Input and Output
% Input Parameter
% Name Size Meaning
% Sol_Con 1xN Soliton Amplitude Constellation
% Prob Nx1 Probabilities of the Constellation
% seq_leng 1x1 Constant that described how many soliton in this soliton sequence
% n_realization 1x1 Number of realization per constellation
```

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% L_seq 1xK Interested propagation
distance, (unit m)
% del 1x1 The percentage of energy
transection that
% defined the soliton time
domain width
%Output Parameter
%Name Size

Meaning
% Q_Tx number of realization x time samples
Transmitted pulses, with unit
% Q_Rx number of realization x time samples x
L_seq index Received pulses, with unit
% eig_Tx_exact 1xConstellation size
Soliton Amplitude constellation
% eig_Tx_NFT 1xConstellation size
Eigenvalue computed using the del parameter
% eig_Tx number of realization x seq_leng
Eigenvalues of the transmitted
segenated
% eig_Rx number of realization x seq_leng x L_seq
index Eigenvalues of the Received pulses
detected
% by segmentation and NFT detection, if number
% of eigenvalue is larger than 1, take the
C. MATLAB Code for NFDM Channel Simulator

```matlab
function Res = SolSeqSimulator_v2(Sol_Con, Prob, seq_leng, n_realization,...
    L_seq, del, exe_mode)
%% ----------------------- Parameter Initialization
------------------------

% Fibre parameter setting
gam=1.27e-3; % fiber non linearity in /W/ m
b2= -21e-27; % 2nd order disp. (s2/m)
alphadB=0.2; % Fiber loss value in dB/km
alpha=alphadB/(4.343*10^-3); %Ref page#55 eqn 2.5.3 Fiber optic Comm
h_planck = 6.62607004e-34; % Planck constant
KT = 1.13; % photon occupancy factor
mu_0 = 3e8/1550e-9; % light frequency in the fibre
T0 = 1e-10; % Normalizing time constant, sec
Ld = T0^2/abs(b2); % Dispersion length, metre
L = max(L_seq); % Transmission length, m
h = 200; % SSFM step length, m
sigma_N = sqrt(alpha*h_planck*mu_0*KT*gam*Ld*L/2/T0); % Noise PSD
```
N_ASE = alpha*L*h_planck*mu_0*KT;
A_alp = Sol_Con;
%% ----------------------- System Parameter Setting -----------------------
% Pulse width, contain of two part, pulsewidth and guarding interval
epsilon = 0.001;
f_guard = 0;
T_p = @(A, del) 1/2./A(2:end).*log(2/del - 1);
T_g = @(A) sqrt((4*alpha*h_planck*mu_0*KT*gam)/(9*Ld*T0)*(L^3).*A)...*erfcinv(2*epsilon);
T_w = @(A) T_p(A, del) + 2*f_guard*T_g(A); % timing window, unitless
Tmax = max(T_p(A_alp, del) + 2*f_guard*max(T_g(A_alp)))*T0/2;
% disp(['soliton pulsewidth is ', num2str(2*Tmax/10^-9), ' ns'])
%
determine the number of sample per soliton
A_max = max(A_alp);
fs = 10*12*A_max/pi^2/T0; % Sampling frequency in Hz
%
determine the number of sample per soliton
D = 2^ceil(log2(2*Tmax/(1/fs))); % disp(['2', num2str(log2(D)), ' samples per solitons'])
%
time grid
dtau = 2*Tmax/D;
tau = -Tmax:dtau:Tmax-dtau; % Non-normalized time, second
t = tau./T0; % Normalized time, unitless
% Nonlinear Fourier Transform Parameters
M = 2*D; % Number of samples in the nonlinear frequency domain
[XI, xi] = mex_fnft_nsev_inverse_XI(D, [t(1),t(end)], M);
  % Location of the 1st and last sample in the nonlinear frequency domain, as well as the grid of all locations
% time_grid = -seq_leng*Tmax-neig_num/2*dtau:dtau:seq_leng* Tmax+neig_num/2*dtau; %Non-normalized time, second

% Build the alphabet for the transmitted pulses using INFT
Q_sol_alp = zeros(length(A_alp),D);
for i = 1:length(A_alp)
  if i == 1
    Q_sol_alp(i,:) = zeros(1, D);
  else
    Q_sol_alp(i,:) = mex_fnft_nsev_inverse([], XI,
      complex(A_alp(i)*1j), ...
      complex(-1), D, [t(1),t(end)], 1);
  end
end
Q_sol_alp = Q_sol_alp / sqrt(gam*Ld);

% Estimate the energy of the transmitted pulse
E_theo = 4*A_alp*T0/gam*Ld;
E_Tx = mean(Q_sol_alp.^2,2)*Tmax*2;
del_Tx_est = (E_theo - E_Tx')/E_theo;

% NFT detection on the transmitted signal to built transmitted eigenvalue
% alphabet
A_Tx = zeros(size(A_alp));
for n = 1:length(A_alp)
  [con_spc_Tx, A_Tx_temp, norm_const_Tx] = ....
mex_fnft_nsev(complex(Q_sol_alp(n,:)*sqrt(gam*Ld)), ... 
[t(1),t(end)], XI, 1, 'M', M);
if isempty(A_Tx_temp) ~= 1
    A_Tx(n) = imag(A_Tx_temp);
else
    A_Tx(n) = 0;
end

%% ----------------------- Channel Realization
-----------------------------------------------
Q_Tx_mat = zeros(n_realization, seq_leng*D);
Q_Tx = zeros(n_realization, seq_leng*D);
Q_Rx = zeros(n_realization, seq_leng*D, length(L_seq));
A_Tx_seq = zeros(n_realization, seq_leng);

%1. Build Transmitted Pulse Sequence
Q_Tx_ind = reshape(invCDFrand(Prob, n_realization*seq_leng), [n_realization, seq_leng]);
for m = 1:seq_leng
    Q_Tx_mat(:, (m-1)*D+1:m*D) = Q_sol_alp(Q_Tx_ind(:,m),:);
end
Q_Tx(:, :) = Q_Tx_mat;
A_Tx_seq(:, :) = A_alp(Q_Tx_ind);

%2. Perform transmission
% Determine how the task can be splitted to run in GPU
Q_Rx = ssfm_matrix_v4(Q_Tx(:, :), dtau, ... 
    alphadB, b2, gam, L_seq, h,... 
    1, exe_mode);

%3. Perform Detection
R_Rx = zeros(n_realization, seq_leng, length(L_seq));
n_eig = zeros(n_realization, seq_leng, length(L_seq));
for k = 1:length(L_seq)
    res_NFT = seg_eig_detector(Q_Rx(:, :, k)*sqrt(gam*Ld),
                                seq_leng, Tmax/T0, A_Tx);
    R_Rx(:, :, k) = res_NFT.eig;
    norm_Rx(:, :, k) = res_NFT.norm_const;
    n_eig(:, :, k) = res_NFT.n_eig;
end

%% ------------------------- Result Output
----------------------------------

% Time domain Pulse Shape
Res.Q.Tx = Q.Tx;
Res.Q.Rx = Q.Rx;

% Direct NFT detection Results
Res.eig.Tx.exact = Sol.Con;
Res.eig.Tx.NFT = A.Tx;

% Direct NFT detection Results
Res.eig.Tx = A.Tx_seq;
Res.eig.Rx = R.Rx;
Res.num.eig.Rx = n.eig;
Res.norm.Rx = norm.Rx;
end

%% Function definition
% 1. Inverse CDF method to generate random number
% corresponds to specified discrete probability distribution
function ind_rand = invCDFrand(P_X, len)
% elements follows a discrete input distribution whose CDF is given by
% vector CDF

CDF = zeros(size(P_X));
for i = 1:length(P_X)
    CDF(i) = sum(P_X(1:i));
end

p_rand = rand(1,len);
ind_inter = p_rand < CDF;
ind_rand = sum(~ind_inter) + 1;

ind_rand = ind_rand';

% Test invCDFrand
% figure
% hold on
% histogram(ind_rand, 'Normalization', 'probability');
% stem(1:i, P_Tx)
% hold off

end

function y = allcomb(x,K)
% This function takes in vector x with size Nx1 and number of selecting K,
% it will generate a matrix with the size N^KxK that contains all the
% possible combination with repetition

% Create all possible permutations (with repetition) of letters stored in x
C = cell(K, 1);          % Preallocate a cell array
[C{:}] = ndgrid(x);      % Create K grids of values
y = cellfun(@(x){x(:)}, C);   % Convert grids to column vectors
y = [y{:}];               % Obtain all permutations
function res = seg_eig_detector(q, sol_num, tmax, LUT)

% Check whether LUT is column vector, if not, convert to column vector
if size(LUT, 2) ~= 1
    LUT = LUT.
end

[n_realiz, D] = size(q);
D = D/sol_num;
dt = 2*tmax/D;
eig = zeros(n_realiz, sol_num);
norm = zeros(n_realiz, sol_num);
n_eig = zeros(n_realiz, sol_num);

[XI, ~] = mex_fnft_nsev_inverse_XI(D, [-tmax, tmax-dt], 2*D);

for m = 1:sol_num
    parfor i = 1:n_realiz
        q_seg = squeeze(q(i, (m-1)*D+1:m*D));
        [~, eig_Rx, norm_Rx] = ...
            mex_fnft_nsev(complex(q_seg), ...
            [-tmax, tmax-dt], XI, 1, 'M', 2*D);

        if length(eig_Rx) == 1
            eig(i,m) = eig_Rx;
            norm(i,m) = norm_Rx;
        elseif length(eig_Rx) > 1
            disp(length(eig_val_Rx))
            diff = min(abs(eig_Rx - LUT), [], 2);
            [~, min_diff_ind] = min(diff);
            [~, min_diff_ind] = ind2sub(size(diff), min_diff_ind);
            eig(i,m) = eig_Rx(min_diff_ind);
norm(i,m) = norm_Rx(min_diff_ind);
else
    eig(i,m) = 0;
    norm(i,m) = 0;
end

n_eig(i,m) = length(eig_Rx); % record number of eigen
% value detected
end

res.eig = eig;
res.norm_const = norm;
res.n_eig = n_eig;
end

An example of channel simulator for continuous spectrum modulated nonlinear frequency division multiplexed system is given as following:

% In this program, we create a CS transmission and detection simulator, the
detection scheme contain both detection results with and without filtered.

% In this version the RAM usage is improved

function STAT = CS_Simulator6(QAM_Con, P_Tx, N_realization)
% clc
% clear
% close all
%
% % QAM_Con = [0.8; 1.2];
% % P_Tx = [1/2; 1/2];
% % QAM_Con = 0;
% % P_Tx = 1;
% % N_realization = 50;
%
% % Parameter Initialization
% Fibre parameter setting
% fiber non linearity in W/m
gam = 1.27e-3;

% 2nd order disp. (s2/m)
b2 = -22e-27;

% Fiber loss value in dB/km
alphadB = 0.2;

% Ref page#55 eqn 2.5.3
alpha = alphadB / (4.343e10^3);
Fiber optic Comm

% by GP Agrawal
h_planck = 6.62607004e-34; % Planck constant
KT = 1.13; % photon occupancy factor
mu_0 = 3e8 / 1550e-9; % light frequency in the fibre

T0 = 1e-10; % Normalizing time constant, sec
Ld = T0^2 / abs(b2); % Dispersion length, metre

L = 2e6; % Transmission length, m
h = 200; % SSFM step length, m
l = L / 2 / Ld; % Normalized transmission length

N_ASE = alpha * L * h_planck * mu_0 * KT;

sigma2 = alpha * h_planck * mu_0 * KT * (2 * gam * T0^3 / abs(b2)^2);

Tw = 200 * T0; % Full time domain pulse width, s
Tmax = Tw / 2; % Half-time domain pulse width, time domain window [-Tmax, Tmax], s

tw = Tw / T0; % Full time domain pulse width, unitless

tmax = Tmax / T0; % Half-time domain pulse width, unitless time domain window [-tmax, tmax]

N_para = sigma2 * l * tw;

% Number of symbol and Number of samples
K = 2^7; % number of symbol per vector
D = 2^13; % Number of sample in time domain
N_Tx = 2*D; % Number of sample in nonlinear frequency domain, Tx
N_Rx = 16*D; % Number of sample in nonlinear frequency domain, Rx

%% Nonlinear frequency domain grid for Signaling Parameter
Lam_Sinc_wid = 8;
Lam_Sinc = [-4, 4];
Lam_Rx_exact = Lam_Sinc(1)+(2*(1:K)-1)*Lam_Sinc(2)/K;
    % Nonlinear Frequency sampling points

%% Transmitter Sampling grid setting
% Normalized Time domain time grid, identical for both transmit and received
del_t = tw/(D-1);
t_grid = -tmax : del_t : tmax;

Tw_pad = (N_Rx-1)*del_t;
t_grid_pad = -(N_Rx-1)/2*del_t: del_t: (N_Rx-1)/2*del_t;

% Transmit nonlinear frequency grid, approximately equivalent to frequency
grid times pi
[Lam_Tx, Lam_grid_Tx] = mex_fnft_nsev_inverse_XI(D, [-tmax, tmax], N_Tx);
    % Location of the 1st and last sample in the nonlinear frequency domain, as well as the grid of all locations
Lam_S_Tx = Lam_Tx(2) - Lam_Tx(1);
del_Lam_Tx = mean(Lam_grid_Tx(2:end) - Lam_grid_Tx(1:end - 1));

%% Receiver Sampling grid setting
% Received nonlinear frequency grid, approximately equivalent to frequency
grid times pi
C. MATLAB Code for NFDM Channel Simulator

```matlab
[Lam_Rx, Lam_grid_Rx] = mex_fnft_nsev_inverse_XI(D, [-tmax, tmax], N_Rx);

% Location of the 1st and last sample in the nonlinear
% frequency domain, as well as the grid of all locations

Lam_S_Rx = Lam_Rx(2) - Lam_Rx(1);
del_Lam_Rx = mean(Lam_grid_Rx(2:end) - Lam_grid_Rx(1:end - 1));

rho_window = ones(size(Lam_grid_Rx));
rho_window(abs(Lam_grid_Rx) > Lam_Rx_exact(end)+20* Lam_Sinc_wid/K) = 0;

ind_up = find(Lam_grid_Rx > Lam_Rx_exact(end)+20* Lam_Sinc_wid/K);
ind_up = ind_up(1);
ind_low = find(Lam_grid_Rx < Lam_Rx_exact(1)-20* Lam_Sinc_wid/K);
ind_low = ind_low(end);

% Sampling grid in the Asymptotic time domain (IFFT of the CS)

tau_w_Rx = pi*(D - 1)/Lam_S_Rx;
del_tau_Rx = pi/Lam_S_Rx;
tau_grid_Rx = -(D-1)/2:1:(D-1)/2)*del_t;
fil_mask = ones(size(tau_grid_Rx));
fil_mask(abs(tau_grid_Rx) > K*pi/Lam_Sinc_wid/2) = 0;
tpass = K*pi/Lam_Sinc_wid;
N_para_fil = sigma2*l*tpass;

%% Transmitter Signal modulation
% Encoder with INFT

% 1. Randomly generate the symbols according to the specified distribution

Tx_Sym_exact = zeros(N_realization, K);
for n = 1:N_realization
```

%Take K th random symbol from the Constellation
Sym_Ind = invCDFrand(P_Tx, K); 
Tx_Sym_exact(n,:) = QAM_Con(Sym_Ind); 
%Symbol vector
end

% 2. Pulseshaping to acquired the direct CS modulate pulse
rho_0 = zeros(N_realization, N_Tx);
for n = 1:N_realization
rho_0(n,:) = ... 
    Tx_Sym_exact(n,:)... 
    *sinc(K/Lam_Sinc_wid*(ones(K,1)*Lam_grid_Tx)+ K/2 - 
    (2*(1:K)'-1)/2); 
end

% 3. Generate the Time domain pulse via INFT
% INFT to convert to time domain
q_Tx = zeros(N_realization, D);
for n = 1:N_realization
    q_Tx(n,:) = mex_fnft_nsev_inverse(complex(rho_0(n,:)), 
    Lam_Tx, [], ... 
    [], D, [-tmax,tmax], 1);
end

%% Transmission in time domain
%Denormalization
Q_Tx = q_Tx/sqrt(gam*Ld);
%channel propagation
Res_mat = ssfm_matrix(Q_Tx, T0, Tmax, alphadB, b2, gam, L, 
    h, 1);
% Q_Rx = Res_mat.Q_Rx;
Q_Rx = signalBW_filter(Res_mat.Q_Rx, Tmax*T0, 50e9);

%Normalization
q_Rx = Q_Rx*sqrt(gam*Ld);
q_Rx_zp = [zeros(N_realization, (N_Rx-D)/2), q_Rx, zeros( 
    N_realization, (N_Rx-D)/2)];
%% Receiver demodulation
% rho_Rx = zeros(N_realization, N_Rx);
% rho_Rx_eq = zeros(N_realization, N_Rx);
% rho_Rx_eq_fil = zeros(N_realization, N_Rx);
%
% rrho_IFFT_pad = zeros(N_realization, N_Rx);

Rx_Sym = zeros(N_realization, K);
Rx_Sym_fil = zeros(N_realization, K);

% Conventional with NFT
for n = 1:N_realization
    % Direct detection without any filtering
    [Rx_Sym(n,:), ~, ~] = mex_fnft_nsev(q_Rx(n,:), ...[-tmax, tmax], [Lam_Rx_exact(1), Lam_Rx_exact(end)],
        1, 'M', K);
    Rx_Sym(n, :) = Rx_Sym(n, :) .* exp(-4j*Lam_Rx_exact.^2*l));

    % Detection with filtering
    [rho_Rx, ~, ~] = mex_fnft_nsev(complex(q_Rx(n,:)), ...[-tmax, tmax], Lam_Rx, 1, 'M', N_Rx);
    % rho_Rx_eq(n,:) = rho_Rx(n,:).*exp(-4j*Lam_grid_Rx.^2*1).*rho_window;
    rho_Rx_eq = rho_Rx.*exp(-4j*Lam_grid_Rx.^2*1);

    %IFFT of the CS domain
    % cyclic shift the CS domain signal to align the zero frequency
    rho_Rx_shift = [rho_Rx_eq(:,end), rho_Rx_eq(:,1:end-1)]
    rrho_IFFT_pad = ifftshift(ifft(ifftshift(rho_Rx_shift,
        2),N_Rx,2), 2)*N_Rx/pi*del_Lam_Rx;
    rrho_IFFT = rrho_IFFT_pad(:, (N_Rx - D)/2 + D);

    % Perform Filtering then get back to the CS domain
C. MATLAB Code for NFDM Channel Simulator

```matlab
rrho_IFFT_fil = [zeros(1, (N_Rx-D)/2), rrho_IFFT.*
    fil_mask, zeros(1, (N_Rx-D)/2)];

rho_Rx_fil_shift = fftshift(fft(fftshift(rrho_IFFT_fil,
    2), N_Rx, 2), 2)*del_t;

rho_Rx_eq_fil = [rho_Rx_fil_shift(1,2:end),
    rho_Rx_fil_shift(1,1)];

% Resampled the CS domain by INFT then NFT
q_Tx_est_fil = mex_fnft_nsev_inverse(rho_Rx_eq_fil,
    Lam_Rx, [], ...
    [], D, [-tmax,tmax], 1);

[Rx_Sym_fil(n,:), ~, ~] = mex_fnft_nsev(q_Tx_est_fil,
    ...
    [-tmax,tmax], [Lam_Rx_exact(1), Lam_Rx_exact(end)],
    1, 'M', K);

% Variance of the noise in the detected symbols
MSE_Sym = mean(abs(Rx_Sym - Tx_Sym_exact).^2, 'all');
MSE_Sym_fil = mean(abs(Rx_Sym_fil - Tx_Sym_exact).^2, 'all'
);

% Result Output
Tx_Sym_reshaped = reshape(Tx_Sym_exact, [1, K*N_realization
    ]);
Rx_Sym_reshaped = reshape(Rx_Sym, [1, K*N_realization]);
Rx_Sym_fil_reshaped = reshape(Rx_Sym_fil, [1, K*
    N_realization]);

STAT.N_para = N_para;
STAT.N_para_fil = N_para_fil;
STAT.Tx_Sym = Tx_Sym_reshaped;
STAT.Rx_Sym = Rx_Sym_reshaped;
STAT.Rx_Sym_fil = Rx_Sym_fil_reshaped;
```
function ind_rand = invCDFrand(P_X, len)
% This function generates a vector of random number of
% elements whose distribution follows a discrete input vector whose CDF
% is given by
% vector CDF

CDF = zeros(size(P_X));
for i = 1:length(P_X)
    CDF(i) = sum(P_X(1:i));
end

p_rand = rand(1, len);
ind_inter = p_rand < CDF;
ind_rand = sum(~ind_inter) + 1;

function q_fil = signalBW_filter(q, Tmax, fc)
[-, D] = size(q);
N = 2*D;
fs = (D-1)/2/Tmax;
del_f = fs/N;
f_grid = -fs/2: del_f: fs/2 - del_f;

if fc > fs
    error('cutoff frequency too large')
end

H_fil = ones(size(f_grid));
H_fil(abs(f_grid) > fc) = 0;

Q = fftshift(fft(q, N, 2), 2);
q_fil = ifft(ifftshift(Q.* H_fil), N, 2);
q_fil = q_fil(:, 1:D);
end
An example of the bidirectional neural network employed in this work are shown as following:

```
% This is a script that is used to train a Deep BiLSTM network to detect
% the soliton for the 8PAM amplitude modulated soliton system receiver to
% perform the soliton detection when inter-soliton interaction is taken
% into account

% Version Log:
% v5
% In this version, we try to introduce the CS spectrum into the input
% features to assist the detection of the interacted soliton referring
% to the Zhang2019improved paper (doi: 10.1109/JLT.2019.2910519)

% v4
% In this version, we try to introduce soliton train as the training
% and validation data to see whether performance for 256 soliton sequence
% test set can be improved

% v3
% In this version, we try different optimizer, we found out that adam
% optimizer provide the best validation results
```
% v2
% In this version, the naming system is updated with an
% automatic generated
% string of name that contain the number of BiLSTM layers.
% v1
% In this version, the testing that will be run to test the
% NN is contained
% within a script. In this version, we also take advantage
% of such testing
% script.

clc
close all
clear

load('OptimResult_AsymXmin.mat', 'mass_poi', 'mass_val','
Xmax_seq');
Xmax_seq_Full = Xmax_seq;
mass_poi_Full = mass_poi;
mass_val_Full = mass_val;

% Distance index, k = 1 (500 km), 2 (1000 km), 3 (1500 km),
% 4 (2000 km)
L_seq = [0.5e6, 1e6, 1.5e6, 2e6];
k = 4;

%%%% ----------------------- Parameter Initialization
%------------------------
% Fibre parameter setting
% fiber non linearity in /W/m
gam=1.27e-3;
% 2nd order disp. (s2/m)
b2= -21e-27;
% Fiber loss value in dB/km
alphadB=0.2;
alpha=alphadB/(4.343*10^-3); %Ref page#55 eqn 2.5.3
Fiber optic Comm

% by GP Agrawal
h_planck = 6.62607004e-34; % Planck constant
KT = 1.13; % photon occupancy factor
mu_0 = 3e8/1550e-9; % light frequency in the fibre
T0 = 1e-10; % Normalizing time constant, sec
Ld = T0^2/abs(b2); % Dispersion length, metre
L = L_seq(end); % Transmission length, m
h = 200; % SSFM step length, m

N_ASE = alpha * h_planck * mu_0 * KT;
sigma_2 = N_ASE * 2 * gam * Ld^2 / T0; % Noise PSD
sigma_N_2 = 1/2 * sigma_2 * L / 2 / Ld;
sigma_N = sqrt(sigma_N_2);

VNT = @(x) 2 * sqrt(x / sigma_N_2);
IVNT = @(x) sigma_N_2 / 4 * x.^2;

%% build file name based on the specification
seq_len = 16; % Number of neighbours
Xmax_seq = Xmax_seq_Full;
Amax_seq = IVNT(Xmax_seq);
Amax = Amax_seq(15); % Peak soliton amplitude
del = 1e-2; % delta parameter defined pulsewidth
del_str = num2str(del, '%.0e');
del_str(3:4) = [];
PAM_Size = 16;

% Name of the data
training_data_name = [num2str(PAM_Size), 'PAM_TrainSet_PP', ...
                      num2str(Amax, '%.3f')...
                      'Solx', num2str(seq_len), 'x160000del', del_str, '.mat'];
test_data_name = [num2str(PAM_Size), 'PAM_TestSet_PP', ...
                 num2str(Amax, '%.3f')...
                 'Solx256x1000del', del_str, '.mat'];

% ZP_test_data_name = ['8PAM_TestSet_PP', num2str(Amax, '%.3f'),...
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```matlab
% 'Solx256x1000del', del_str,'0Pad.mat']

%% Read in the data and process the data
disp('load training data start')
load(trainning_data_name)
disp('data training data finish')

[n_realization, D] = size(Channel_Realiz.Q_Tx);
n_train = n_realization*1/2;
D = D/seq_leng;
epsilon = 0.001;
f_guard = 0;
T_p = @(A, del) 1/2./A(2:end).*log(2/del - 1);
T_g = @(A) sqrt((4*alpha*h_planck*mu_0*K*T*gam)/(9*Ld*T0)*(L^3).*A).
    *erfcinv(2*epsilon);
T_w = @(A) T_p(A, del) + 2*f_guard*T_g(A);  \% timing window, unitless
Tmax = max(T_p(A_alp, del) + 2*f_guard*max(T_g(A_alp)))\*T0/2;
[XI, xi] = mex_fnft_nsev_inverse_XI(D, [-Tmax/T0,Tmax/T0], 2*D);

\%normalized pulseshape data
q_Tx = reshape(permute(Channel_Realiz.Q_Tx(1:n_train,:),[2,1]),...)
    [D,n_train*seq_leng]).'*sqrt(gam*Ld);
q_Rx = reshape(permute(Channel_Realiz.Q_Rx(1:n_train,:,end),[2,1,3]),...)
    [D,n_train*seq_leng]).'*sqrt(gam*Ld);

\%eigenvalue data
A_Tx = reshape(permute(Channel_Realiz.eig_Tx(1:n_train,:),[2,1]),...)
```
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```matlab
[1, n_train*seq_leng]).';
R_Rx = reshape(permute(Channel_Realiz.eig_Rx(1:n_train,:),
end,[2,1,3]),...
[1, n_train*seq_leng]).';
A_alp_exact = Channel_Realiz.eig_Tx_exact.';

% Validation data
% normalized pulseshape data
q_Tx_Val = reshape(permute(Channel_Realiz.Q_Tx(n_train+1:
end,:),[2,1]),...
[D, (n_realization - n_train)*seq_leng]).'*sqrt(gam*Ld);
q_Rx_Val = reshape(permute(Channel_Realiz.Q_Rx(n_train+1:
end,:,end),[2,1,3]),...
[D, (n_realization - n_train)*seq_leng]).'*sqrt(gam*Ld);

% eigenvalue data
A_Tx_Val = reshape(permute(Channel_Realiz.eig_Tx(n_train+1:
end,:),[2,1]),...
[1, (n_realization - n_train)*seq_leng]).';
R_Rx_Val = reshape(permute(Channel_Realiz.eig_Rx(n_train+1:
end,:),[2,1,3]),...
[1, (n_realization - n_train)*seq_leng]).';
clear Channel_Realiz

%% Specify and train the Neural Network
input_mode = 1;

% Process the training data input
XTrain = Input_Processing(q_Rx, 1, Tmax/T0, A_alp_exact,
input_mode);

% Inspect data reshaping is performing correctly
XTrain(1:5)
figure
plot_ind = round(size(A_Tx, 1)*rand(1,1));
plot(XTrain{plot_ind}')
```
xlabel("Time Step")
title("Training Observation 1")
numFeatures = size(XTrain{1},1);
legend("Feature " + string(1:numFeatures), 'Location', 'northeastoutside')

% Process training data output
[~, A_Tx_ind] = min(abs(A_Tx - A_alp_exact), [], 2);
YTrain = categorical(A_Tx_ind);

% Process validation data input
XVal = Input_Processing(q_Rx_Val, 1, Tmax/T0, A_alp_exact, input_mode);

% Process validation data output
[~, A_Tx_ind_Val] = min(abs(A_Tx_Val - A_alp_exact), [], 2);
YVal = categorical(A_Tx_ind_Val);

% Specify the network structure
inputSize = size(XTrain{1}, 1);
% numHiddenUnits = 2*size(XTrain{1}, 2);
numHiddenUnits = 100;
LSTMDepth = 2;
drop_prob = 0;
numClasses = length(A_alp_exact);
layers = sequenceInputLayer(inputSize);
if drop_prob == 0
% When dropout probability 0, no dropout layer
    for i = 1:LSTMDepth-1
        layers = [layers;...
            bilstmLayer(numHiddenUnits, OutputMode="last")];
    end

    layers = [layers
            bilstmLayer(numHiddenUnits, 'OutputMode', 'last')
        fullyConnectedLayer(numClasses)
        softmaxLayer]
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```
classificationLayer];

else
  % When dropout probability not 0, include dropout layer
  for i = 1:LSTMDepth-1
    layers = [layers;...
      bilstmLayer(numHiddenUnits, OutputMode="last");
      ...;
      dropoutLayer(drop_prob)];
  end

  layers = [layers
    bilstmLayer(numHiddenUnits, 'OutputMode', 'last')
    dropoutLayer(drop_prob)
    fullyConnectedLayer(numClasses)
    softmaxLayer
    classificationLayer];
end

maxEpochs = 1000;
miniBatchSize = 2500;

% Training Setting
options = trainingOptions('adam', ...
  'ExecutionEnvironment', 'auto', ...
  'GradientThreshold', 1, ...
  'MiniBatchSize', miniBatchSize, ...
  'MaxEpochs', maxEpochs, ...
  'Verbose', 0, ...
  'Plots', 'training-progress', ...
  'ValidationData', {XVal, YVal}, ...
  'ValidationPatience', 50);

Network_name = [num2str(PAM_Size), 'PAM_PP', num2str(Amax, '%.3f'), ...
  'x', num2str(seq_leng), '_', ...
  num2str(LSTMDepth), 'BiLSTM_', ...
  'del', del_str,...
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```
'\text{in}', \text{num2str}(\text{input\_mode}), '.\text{mat}'];

% Train Network
[net, tr] = trainNetwork(XTrain, YTrain, layers, options);
% save(NetWork\_name, 'net', 'tr')
% load(NetWork\_name, 'net', 'tr')

% Test Network with validation data
disp('load validation data for testing start')
load(trainning\_data\_name)
disp('data validation data for testing finish')

% q Tx = reshape(permute(Channel\_Realiz.Q\_Tx(n\_train+1:end,:,)[2,1]),...  
% [D,(n\_realization - n\_train)\*seq\_leng]).'*sqrt(gam*Ld);

q Rx = reshape(permute(Channel\_Realiz.Q\_Rx(n\_train+1:end,:,end),[2,1,3]),...
  [D,(n\_realization - n\_train)\*seq\_leng]).'*sqrt(gam*Ld);

% eigenvalue data
A Tx = reshape(permute(Channel\_Realiz.eig\_Tx(n\_train+1:end,:,)[2,1]),...  
  [1,(n\_realization - n\_train)\*seq\_leng]).';

% R Rx = reshape(permute(Channel\_Realiz.eig\_Rx(n\_train+1:end,:,k),[2,1,3]),...  
% [1,(n\_realization - n\_train)\*seq\_leng]).';
A_alp_exact = Channel\_Realiz.eig\_Tx\_exact.';

clear Channel\_Realiz

% Process the testing data input
XTest1 = Input\_Processing(q Rx, 1, Tmax/T0, A_alp_exact,  
  input\_mode);

% Process training data output
[-, A\_Tx\_ind] = min(abs(A\_Tx - A\_alp\_exact), [], 2);
YTest1 = categorical(A\_Tx\_ind);
```
YPred1 = classify(net, XTest1, 'MiniBatchSize', miniBatchSize);

% Performance Metric Estimation
% Estimate the symbol error rate (SER)
SER1 = sum(YPred1 ~= YTest1)./numel(YTest1);

% Estimate MI with the classification result
[MI1, P_X_Y1] = MI_est(YTest1, YPred1);

% Estimate Mismatched MI with a posterior probability
P_X_Y_mesh = predict(net, XTest1, 'MiniBatchSize', miniBatchSize);
% h_XgivY1 = mean(-log2(max(P_X_Y_mesh,[],2)), 'all');
P_X_Y = zeros(size(P_X_Y_mesh, 1), 1);
for ii = 1:size(P_X_Y_mesh, 1)
P_X_Y(ii) = P_X_Y_mesh(ii, A_Tx_ind(ii));
end
h_XgivY1 = mean(-log2(P_X_Y), 'all');
MisMatch_MI_1 = Prob.'*log2(1./Prob) - h_XgivY1;

% Test Network with 256 Soliton Sequence testing data
disp('load test data start')
Channel_Realiz_Seq = load(test_data_name, 'Channel_Realiz');
Channel_Realiz_Seq = Channel_Realiz_Seq.Channel_Realiz;
disp('load test data finish')

% q_Tx_test = reshape(permute(Channel_Realiz_Seq.Q_Tx(:,:),[2,1]),... 
% [D,1000*256]).'*sqrt(gam*Ld);
% load received time domain pulse
q_Rx_test = reshape(permute(Channel_Realiz_Seq.Q_Rx(:,end,:),[2,1]),... 
% [D,1000*256]).'*sqrt(gam*Ld);
% load transmitted eigenvalue
eig_Tx_test = reshape(permute(Channel_Realiz_Seq.eig_Tx(:,,:),[2,1]),...
% load received eigenvalue
% \texttt{eig\_Rx\_test} = reshape(permute(Channel\_Realiz\_Seq.eig\_Rx(:,,:,:),[2,1]),
% \texttt{[1,1000*256])}.';

A\_alp\_exact = Channel\_Realiz\_Seq.eig\_Tx\_exact.';

\texttt{clear Channel\_Realiz\_Seq}

% Process the testing data input
XTest2 = Input\_Processing(q\_Rx\_test, 1, Tmax/T0,
A\_alp\_exact, input\_mode);

% Process training data output
[\texttt{~}, A\_Tx\_ind] = min(abs(eig\_Tx\_test - A\_alp\_exact), [], 2);
YTest2 = categorical(A\_Tx\_ind);

YPred2 = classify(net, XTest2, \texttt{'MiniBatchSize'},
\texttt{miniBatchSize});

% Performance Metric Estimation
\texttt{SER2} = sum(YPred2 ~= YTest2)./numel(YTest2);

% Estimate MI with the classification result
[MI2, P\_X\_Y2] = MI\_est(YTest2, YPred2);

% Estimate Mismatched MI with a posterior probability
P\_X\_Y\_mesh = predict(net, XTest2, \texttt{'MiniBatchSize'},
\texttt{miniBatchSize});
\texttt{h\_XgivY1} = mean(-log2(max(P\_X\_Y\_mesh,[], 2)), \texttt{'all'});

P\_X\_Y = zeros(size(P\_X\_Y\_mesh, 1), 1);
\texttt{for ii = 1:size(P\_X\_Y\_mesh, 1)}
\texttt{P\_X\_Y(ii) = P\_X\_Y\_mesh(ii, A\_Tx\_ind(ii));}
\texttt{end}
\texttt{h\_XgivY2} = mean(-log2(P\_X\_Y), \texttt{'all'});

MisMatch\_MI\_2 = Prob.'*log2(1./Prob) - h\_XgivY2;
%% Function Definition
function XInput = Input_Processing(q, sol_num, tmax, LUT, input_mode)

[n_realiz, D] = size(q);
D = D/sol_num;
dt = 2*tmax/D;
[XI, ~] = mex_fnft_nsev_inverse_XI(D, [-tmax, tmax-dt], 2*D);

switch input_mode
  case 1 % real and imaginary part
    q_ri = reshape([real(q), imag(q)].', D, []).';
    XInput = mat2cell(q_ri, 2*ones(n_realiz, 1), D);
  case 2 % amplitude and phase angle
    q_ap = reshape([abs(q), angle(q)].', D, []).';
    XInput = mat2cell(q_ap, 2*ones(n_realiz, 1), D);
  case 3 % amplitude only
    q_abs = reshape(abs(q).', D, []).';
    XInput = mat2cell(q_abs, ones(n_realiz, 1), D);
  case 4 % real and imaginary and eigen value domain
    % 1. calculate DS
    DS_eig = zeros(size(q));
    DS_norm = zeros(size(q));
    for m = 1:sol_num
      parfor i = 1:n_realiz
        [~, eig, norm] = ...
        mex_fnft_nsev(complex(q(i,:)), ...
        [-tmax, tmax-dt], XI, 1, 'M', 2*D);
        DS_eig(i, :) = [eig, zeros(1, D - length(eig))];
        DS_norm(i, :) = [norm, zeros(1, D - length(norm))];
    end
  end
end
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```matlab
end
end
q_ri_DS = reshape([real(q),imag(q), real(DS_eig),
imag(DS_eig)],', D, []);'
XInput = mat2cell(q_ri_DS, 4*ones(n_realiz, 1), D);
case 5  \%real and imaginary and full DS domain
  DS_eig = zeros(n_realiz, 2*D);
  DS_norm = zeros(n_realiz, 2*D);
for m = 1:sol_num
    parfor i = 1:n_realiz
        [~, eig, norm] = ...
        mex_fnft_nsev(complex(q(i,:)), ...
        [-tmax,tmax-dt], XI, 1, 'M', 2*D);
        DS_eig(i, :) = [eig, zeros(1,D - length(eig ))];
        DS_norm(i, :) = [norm, zeros(1,D - length( norm))];
    end
end
q_ri_DS = reshape([real(q),imag(q),...
real(DS_eig),imag(DS_eig),...
real(DS_norm),imag(DS_norm)],', D, []);'
XInput = mat2cell(q_ri_DS, 6*ones(n_realiz, 1), D);
case 6  \%real and imaginary time and full CS domain
CS_real = zeros(size(q));
CS_imag = zeros(size(q));
for m = 1:sol_num
    parfor i = 1:n_realiz
        [CS, _, _] = ...
        mex_fnft_nsev(complex(q(i,:)), ...
        [-tmax,tmax-dt], XI, 1, 'M', D);
        CS_real(i, :) = real(CS);
        CS_imag(i, :) = imag(CS);
    end
```
end
q_ri_CS = reshape([real(q), imag(q), ...
    CS_real, CS_imag].', ...
    D, []).';
XInput = mat2cell(q_ri_CS, 4*ones(n_realiz, 1), D);
end
end

function res = seg_eig_detector(q, sol_num, tmax, LUT)
[n_realiz, D] = size(q);
D = D/sol_num;
dt = 2*tmax/D;
eig = zeros(n_realiz, sol_num);
norm = zeros(n_realiz, sol_num);
n_eig = zeros(n_realiz, sol_num);
[XI, ~] = mex_fnft_nsev_inverse_XI(D, [-tmax, tmax-dt], 2*D);
for m = 1:sol_num
    for i = 1:n_realiz
        q_seg = squeeze(q(i, (m-1)*D+1:m*D));
        [~, eig_Rx, norm_Rx] = ...
            mex_fnft_nsev(complex(q_seg), ...
                [-tmax, tmax-dt], XI, 1, 'M', 2*D);
        if length(eig_Rx) == 1
            eig(i, m) = eig_Rx;
            norm(i, m) = norm_Rx;
        elseif length(eig_Rx) > 1
            % disp(length(eig_val_Rx))
            diff = min(abs(eig_Rx.' - LUT), [], 2);
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```matlab
[~, min_diff_ind] = min(diff);
eig(i,m) = eig_Rx(min_diff_ind);
norm(i,m) = norm_Rx(min_diff_ind);
else
eig(i,m) = 0;
norm(i,m) = 0;
end

n_eig(i,m) = length(eig_Rx);  % record number of eigen %value detected
end
end
res.eig = eig;
res.norm_const = norm;
res.n_eig = n_eig;
end

% This is the function created to estimate the mutual information between
% transmitted and received symbol
function [MI, p_x_y] = MI_est(X, Y)
x_alp = unique(X);

p_x = zeros(length(x_alp), 1);
p_y = zeros(length(x_alp), 1);
p_x_y = zeros(length(x_alp), length(x_alp));

Sym_Tx = double(X);
Sym_Rx = double(Y);

for ii = 1:length(x_alp)
    p_x(ii) = numel(find(Sym_Tx == ii))./length(Sym_Tx);
p_y(ii) = numel(find(Sym_Rx == ii))./length(Sym_Rx);
    for jj = 1:length(x_alp)
```
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```matlab
464  p_x_y(ii, jj) = numel(find(Sym_Tx == ii &
465          Sym_Rx == jj))./length(Sym_Rx);
466  p_xgivy(ii, jj) = numel(find(Sym_Tx == ii &
467          Sym_Rx == jj))./numel(find(Sym_Rx == jj));
468  end
469  end
470
471  H_XgivY = p_x_y.*log2(1./p_xgivy);
472  H_XgivY(p_x_y == 0) = 0;
473  H_XgivY = sum((H_XgivY), 'all');
474
475  MI2 = p_x.'*log2(1./p_x) - H_XgivY;
476
477  MI = p_x_y.*log2(p_x_y./(p_x*p_y.'));
478  MI(p_x_y == 0) = 0;
479  MI = sum((MI), 'all');
480  % MI = sum(MI(~isnan(MI)), 'all');
481  end
```