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Classification of supersymmetric black holes in AdS$_5$

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Doctor of Philosophy
University of Edinburgh
2023
Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Sergei Ovchinnikov)
Comrades, let’s bravely start,
our resolve gets stronger in the struggle
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Lay summary

The quantum gravity problem is one of the most significant challenges in modern theoretical physics. It arises from the fundamental incompatibility between two of the most successful and well-tested theories in physics: quantum mechanics and general relativity. The former is a highly successful framework that describes the behaviour of matter and energy on very small scales, such as atoms and subatomic particles. It is assumed that gravity, which for now is best described by general relativity (or Einstein’s theory of gravity), must have a description on very small scales too, which, like quantum mechanics, deals with probabilities and wave functions, and would allow scientists to predict the behaviour of particles and systems with remarkable accuracy.

The theories are incompatible for a multiplicity of reasons — from different views on the spacetime and its curvature, to appearance of gravitational singularities, to existence of ‘time-machine’ pathologies in general relativity. Numerous theoretical attempts have been made to reconcile these two settings and develop a single consistent theory known as "quantum gravity".

One of the most prominent approaches to it, the holographic dualities, have been introduced in the framework of the string theory. The best known (and arguably the most calculable!) their example is the AdS/CFT correspondence formulated by Maldacena (avid reader is advised to look into chapter 1 for its definition).

In a simple setting, this correspondence presents our four-dimensional world with our quantum theories as living on a ball of infinite radius. Interior of this ball is inaccessible to us at lower energies, but the more quantum regime we try, the deeper we can probe into it. On the other hand, the interior of the ball is independently described by purely classical (meaning non-quantum) general relativity with cosmological constant. This framework allows us to calculate the observables using two different, but complementary descriptions.

The five-dimensional black holes, which live inside this ball, are given a unique role in this duality, as they are the most tractable pathways into the realm of quantum gravity. Bekenstein and Hawking suggested that the area of a horizon of a classical black hole has a quantum interpretation as the entropy of ensemble of quantum states, which describe this black hole [1,2].

In other words, we are now looking in the opposite direction — we want to use quantum theory to explain Einstein’s theory of gravity. The precise understanding of their guess would be quite elusive without the AdS/CFT correspondence, which now links, in its own way, this classical black hole to a quantum system in our four-dimensional world.

To make further progress, it is important to know what kind of black holes can actually exist. This is in no way a simple problem, because Einstein’s equations, which determine the geometry of the black hole, are a complicated system of non-linear differential equations, especially in the presence of cosmological constant. In other words, a black hole is a solution to Einstein’s equations, and the latter can have many, even infinitely many solutions, even when one considers boundary conditions that have clear features of a black hole.

Before this work, all black holes in this setting, including the most relevant for holography, were found by guesswork, and no attempts at consistent, systematic classification were made. This thesis addresses this problem by proving the uniqueness of some of the most notable black holes in their classes, and developing a general framework towards their complete classification in the future work.
Abstract

In this thesis we provide a first systematic framework towards the classification of supersymmetric black holes in $d = 5$ gauged supergravity. We build our formalism assuming different isometry groups of spacetimes: $SU(2)$ and $U(1)^2$.

The first part of this thesis is concerned with supersymmetric solutions to minimal gauged supergravity with $SU(2)$ isometry. Our main result is the first uniqueness theorem for black holes with a cosmological constant in dimensions $d > 3$. The proof combines a delicate near-horizon analysis with a general form for a Kähler metric with cohomogeneity-1 $SU(2)$ symmetry and makes no global assumptions on the spacetime.

The rest of the thesis is dedicated to extensive study of supersymmetric solutions with $U(1)^2$ toric symmetry. This reduces to a problem in toric Kähler geometry. In the second part of my thesis we develop the symplectic approach to toric Kähler geometry, which allows us to classify near-horizon and axial structure of toric solutions, including multi-centre configurations and yet to be found black hole solutions with non-trivial topology. We introduce an important subclass of Kähler metrics, which contains all known supersymmetric solutions, and show that the known supersymmetric black hole is unique in this class. In the final part, we further generalise our black hole uniqueness theorem to a gauged supergravity coupled to a number of abelian vector multiplets.
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Chapter 1

Introduction

1.1 Black holes in AdS/CFT correspondence

Holographic dualities, introduced in the framework of string theory, provide a non-perturbative definition of quantum gravity. The most prominent example is the AdS/CFT correspondence formulated by Maldacena [3]. More precisely, Maldacena has conjectured that certain conformal field theories (CFTs), that appear as world-volume theories of D-branes, are dual to superstring theory on anti-de-Sitter (AdS) backgrounds. Together with further works of [4,5] this conjecture has been a remarkable milestone for modern theoretical physics, because it links the observables of the two of its most fundamental open problems: quantisation of gravity and strongly coupled quantum fields theories.

The most striking aspect of AdS/CFT is that the regime of parameters for which it is under the most control is the low-energy limit of weakly coupled string theory, which is approximated by supergravities. The dual theory is a CFT which is typically strongly coupled; it evades standard techniques such as perturbation theory, but, nonetheless, it is known to be integrable, at least in a certain limit [6]. In the decades since its introduction, AdS/CFT has been applied to a variety of theories, ranging from QCD with a large number of colours [7] to condensed matter systems [8–10].

The most well-known example of the correspondence, discussed even in the original paper [3], is the duality between 5$d$ $\mathcal{N} = 8$ gauged supergravity and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with $SU(N)$ gauge group at strong coupling $\lambda \sim N \gg 1$.

Constructively, it arises as a duality between a stack of $N$ D-branes in type IIB superstring theory and corresponding superconformal CFTs (SCFTs) which live on the world-volume of the branes. The supergravity limit then corresponds to the case when the number of branes $N$ is large while the string coupling is kept small. The typical near-horizon geometry of a D-brane in supergravity is a product $AdS_{D+2} \times S^{8-D}$. The $AdS_5 \times S^5$ is the simplest and most prominent instance of AdS/CFT, describing a spacetime near a stack of D3-branes. In this setting, the five-dimensional supergravity appears as a Kaluza-Klein reduction of type IIB supergravity on $S^5$ where massive modes are turned off. This is a consistent truncation, meaning that all solutions of the former can be uplifted back to solutions of the latter, supported by $F_5$ flux [11,12].

It needs to be emphasised, that duality is understood as a full non-perturbative equivalence between two theories: a gauged supergravity on $AdS_5$ and a $\mathcal{N} = 4$ SYM on the brane world-volume which is conformally a four-dimensional Minkowski spacetime. In particular, it means that solutions and global integrals of the former correspond to states and observables of the latter. The general prescription for the matching was outlined in [5]: the asymptotic behaviour of supergravity fields determines the sources of the corresponding fields of $\mathcal{N} = 4$ SYM, which effectively puts the brane world-volume at the conformal infinity of asymptotically globally $AdS_5$ spacetime. This relation between five-dimensional spacetime (bulk) and its four-dimensional boundary motivates the name of holographic correspondence.

From the very beginning, black holes played a unique role in this duality, as they are the most tractable pathways into the realm of quantum gravity. Following Bekenstein and Hawking [1,2], the area of a horizon of a black hole is interpreted as a semiclassical quantity proportional to
the entropy of a thermodynamic ensemble of microstates which form the said black hole. The precise understanding, however, is not possible without a theory of quantum gravity. The AdS/CFT correspondence provides an approach to the problem: the black holes are dual to quantum states with entropy, i.e. to a microcanonical ensemble of underlying microstates of the CFT. The degeneracy of this ensemble can then be computed by the CFT techniques, at least, realistically, for supersymmetric states.

Supersymmetry comes as a natural simplification, a mathematical framework, and a conceptual development. One of the original motivations for supersymmetry in string theory was to remove tachyonic instabilities. Pragmatically speaking, on the quantum field theory side, supersymmetric (also known as BPS) states are assembled into superconformal multiplets and protected in perturbative regime by various non-renormalisation theorems. Historically, this made possible an explanation of the entropy of special asymptotically flat supersymmetric black holes by counting the degeneracy of relevant BPS microstates [13] which was a major success for string theory. This derivation, however, relied heavily on particular limits (see [14] for discussion).

It took almost two decades from the original formulation of the AdS/CFT correspondence to the first major success in the entropy derivation by microstate counting. In [15,16] the entropy of the most general known smooth supersymmetric black hole in 5d minimal gauged supergravity, the Chong-Cvetič-Li-Pope (CCLP) black hole [17], was reproduced by two different approaches. The first paper relied on the prescription [18], where entropy was calculated by evaluating the prefactor of partition function by localisation without directly computing the superconformal index. The second paper was based on the Bethe Ansatz approach to superconformal index calculation (see also [19] for earlier result in AdS$_4$/CFT$_3$). Another major step was done in paper [20] where a class of CFT solutions within Bethe Ansatz was shown to contain the duals of all ‘well-behaving’ gravitational solutions in Euclidean sector by evaluating superconformal index on the ansatz. Their paper also suggested the existence of an explicit map between solutions in the corresponding classes on both sides, and, therefore, it is particularly interesting to see if the gravitational analogue of Bethe ansatz can be found.

It is worth emphasising, however, that this calculation could only be performed for a known black hole, and it does not say anything about the solution space in general. Indeed, many possible types of solutions, e.g. black lenses or multi-black holes, are neither known nor ruled out in gauged supergravities. This motivates the problem of classification of black hole solutions to 5d gauged supergravity in the sense of black hole uniqueness theorems. We will touch on the problems of such classification in the next section.

1.2 Black holes in AdS

1.2.1 Gravitational theories with cosmological constant and their solutions

Despite enormous progress over the years, we are still far from a complete understanding of the duality in dimensions $d > 3$. On the gauge theory side, the boundary CFT is strongly coupled which presents a significant calculational problem as perturbative techniques are inapplicable. On the gravity side, the difficulties come from the negative cosmological constant $\Lambda$ which affects the theory on the level of equations of motion.

The Einstein equations with zero cosmological constant are integrable in the sense of a Lax pair for sufficiently general class of solutions. The presence of a cosmological constant of either sign breaks this integrability, which makes sigma-model representation and solution-generating techniques like solitonic transformations inapplicable. It is difficult to overestimate the complications it creates.

- First and foremost, the sigma-model lies at the core of classification results for asymptotically flat (AF) black holes. This implies that many classes of solutions that are well-known in the AF case are neither known nor ruled out in the presence of cosmological constant.

\[^1\]More precisely, Einstein equations in $d$ dimensions reduce on stationary solutions with $d - 3$ axisymmetries to an integrable system of PDEs in the sense of Lax pair [21].
• Secondly, the integrability of equations of motion is intrinsically related to integrability conditions of geometric structures related to matter content, such as Killing spinors (KS) [22]. In fact, in the ungauged supergravities a general form of background admitting KS is known, while in the asymptotically AdS (AAdS) case there is a further constraint.

• Finally, for black holes, the negative cosmological constant invalidates many statements about near-horizon region as well, including Hawking’s horizon topology theorem [23] and its generalisations to higher dimensions [24].

The existence of the above-mentioned AF black hole classification is a particularly striking feature of AF black holes. In four dimensions the classical result is the famous Kerr-Newman uniqueness theorem proved by Robinson, Mazur and Bunting building on earlier work by Carter [25–28]:

Theorem 1. Let \((\mathcal{M}, g)\) be an asymptotically flat, stationary and axisymmetric electrovacuum black hole solution with a non-degenerate event horizon. Then \((\mathcal{M}, g)\) is isometric to the non-extremal Kerr-Newman solution.

This result essentially states that the geometry of any AF black hole within this symmetry class is fixed provided three constants are given — the mass, the angular momentum and the electric charge. In particular, this excludes any horizon topologies other than \(S^2\), which contrasts asymptotically locally AdS (AlAdS) case where black holes with toroidal\(^2\) spatial cross-sections exist.

In higher dimensions the solution space is richer even in vacuum gravity, and is, in fact, infinite. The remarkable property is that solutions with non-trivial topology are allowed. The solution space includes both multi-black holes, like black Saturn [30], and black holes with non-spherical spatial cross-sections of the horizon. The examples include solutions with \(S^1 \times S^{d-3}\) topology, known as black rings [31]. While this shows that a direct analogue of \(d = 4\) uniqueness theorem is not valid, i.e., there can exist two non-isometric solutions with the same set of charges (for example, a black ring and a spherical black hole), the classification of solutions is, nevertheless, possible under certain symmetry assumptions. Hollands and Yazadjiev [32, 33] have shown that stationary solutions with sufficient degree of axisymmetry\(^3\) are uniquely encoded in terms of topological data (rod structure) and a set of real constants (rod lengths) associated with it.

This classification survives coupling to linear matter like Abelian gauge fields\(^4\) and is applicable to supersymmetric solutions of 5d minimal ungauged supergravity. Since in the thesis we will be working with its gauged\(^5\) cousin, it is worth to discuss the solutions of the former in more detail. We will be interested in supersymmetric solutions, i.e., solutions admitting a Killing spinor.

The minimal ungauged supergravity exhibits a rich space of supersymmetric solutions including solitons [35] and black holes with various horizon geometries and different symmetries. Its main striking feature, however, is that all of its smooth supersymmetric black holes have been classified under minimal symmetry assumptions. We arrange the results by the number of axial symmetries of the solutions.

First of all, in \(U(1)^2\)-symmetry case the theory admits, similar to the vacuum case, a rod structure representation [36]. This class contains a topologically spherical Breckenridge – Myers – Peet – Vafa black hole (BMPV) [37], solutions with Lens space topology (black lenses) [38,39], black rings [40] as well as infinite number of ‘bubbling’ solutions [41] which can be interpreted as placing black holes inside solitons. Next, the case of a single \(U(1)\)-symmetry was studied in [42, 43]. It was shown that non-trivial solutions of this class consist of generalisations of biaxisymmetric multi-black holes, where condition on the collinearity of each black hole in the configuration was relaxed. Note that this is the minimal symmetry expected from the rigidity

\(^2\)In fact, any surface with constant Gaussian curvature, see e.g. [29].

\(^3\)Again, with \(d – 3\) axial Killing fields for \(d\)-dimensional spacetime.

\(^4\)Non-Abelian gauge fields allow for topological hair, for review see [34].

\(^5\)Gauged in the sense of \(R\)-symmetry. The \(R\)-symmetry of Type IIB supergravity becomes gauged as a result of non-toroidal compactifications or compactifications with fluxes, which also produce negative scalar potential or, in the minimal case, — negative cosmological constant.
Theorem (see section 1.2.2) since supersymmetric black holes possess a supersymmetric time-like Killing field by construction.

This progress is, unfortunately, unparalleled in AAdS case. In $d = 5$ gauged supergravity, in the absence of solution generating techniques, most of known black holes were found through ansätze. This includes the most general known black hole, the CCLP black hole [17], which has a spherical horizon. Existence of multi-black holes or more exotic horizon topologies is still in question. The notable result is the proof of absence of smooth supersymmetric black rings [44,45] without scalar fields to balance them. If the scalar fields are present as in the STU model, see, e.g. [46], the ring topology is allowed [47,48].

It is also worthwhile to discuss AAdS non-supersymmetric solutions to illustrate the principle. Even in four-dimensions there are strong indications that multi-black holes can exist [49]. The exact solutions of [50,51] describe black holes hovering on top of black branes. In $d \geq 5$ solutions constructed by approximate methods (like blackfold approach or probe approximation) are also known: black rings [52,53], and AdS black Saturns [52], and probe multi-centre black holes of [54], with the analogues of the former existing in higher-dimensions too. Finally, in four dimensions a supersymmetric multi-black hole configurations were constructed by probe methods in [49].

In conclusion, we would like to list together all questions about smooth supersymmetric solutions to 5d minimal gauged supergravity that have not yet found a full answer:

- Do black lenses exist?
- Are ‘bubbling’ solutions (with 2-cycles outside the horizon) allowed?
- Are there smooth multi-black hole configurations? If yes, are they stable?
- Are there solutions with single $U(1)$ Killing fields?
- Is it possible to explicitly classify smooth black holes with toric symmetry? If yes, does it mean that equations of motion are ‘integrable’ in some sense?

1.2.2 Black hole geometry

It is useful to introduce several definitions.

**Definition 1.** A spacetime is stationary if it admits a time-like Killing vector field $V$. If the Killing field is hypersurface orthogonal, i.e. $V \wedge dV = 0$, then the spacetime is static (with respect to $V$).

We will be working with supersymmetric solutions which admit a supersymmetric Killing field which is causal everywhere and timelike on some open subset. Furthermore, in 5d gauged supergravity, see section 1.3, it is not hypersurface orthogonal by construction, and our primary interest will be stationary non-static spacetimes.

The geometric notion of Killing horizon will be useful for introduction of a local chart around black hole horizon.

**Definition 2.** A Killing horizon with respect to Killing vector field $V$ is a null hypersurface $\mathcal{N}$, such that $V$ is normal to it. $V$ is a generator of $\mathcal{N}$.

To every Killing horizon there is an associated notion of surface gravity. Since $V$ is normal on $\mathcal{N}$, the gradient of its norm is normal to $\mathcal{N}$, i.e.

$$d(V^2)|_{\mathcal{N}} = -2\kappa V$$

where function $\kappa$ is the surface gravity. The zeroth law of black hole mechanics states that $\kappa$ is constant on an event horizon with sufficiently regular matter. An important class of horizons are extremal or degenerate horizons for which $\kappa = 0$.

We will call a Killing horizon supersymmetric if $V$ is a supersymmetric Killing field (see section 1.3). Since it is always causal, its square norm is non-positive\(^6\), and, hence, it has

\(^6\)We work in mostly positive signature.
double zero on the horizon. By (1.1) it means that \( \kappa = 0 \), and a smooth supersymmetric horizon is necessarily extremal.

Let us also discuss global properties of spacetimes with a Killing horizon. We will call a Killing horizon \textit{rotating} if its generator is not the stationary Killing field of spacetime. Otherwise, a Killing horizon is called \textit{static}.

The physical notion of a black hole is its event horizon, which is a global region of the spacetime defined by its causal structure. In \textit{asymptotically flat} spacetime \((M, g)\) with a timelike Killing vector field, a \textit{black hole region} \(B = [M - J^{-}(J^{+})]\) is defined as a set of points of \(M\) that cannot send a signal to future null infinity \(J^{+}\). A \textit{(future) event horizon} is then a boundary of \(B\). There is a similar definition for AAdS spacetimes, although the structure of an asymptotic region is simpler.\footnote{The asymptotic boundary of AAdS spacetimes is time-like. In \(d > 3\) it is \(\mathbb{R}_{t} \times S^{d-2}\) and consists of one component.}

It is a standard result in black hole mechanics that the event horizon is a null hypersurface \cite{55}, \textit{i.e.} a hypersurface, whose normal is everywhere null. Furthermore, by construction any Killing field must be tangent to it, so in particular it is either spacelike or null on the event horizon.

\textbf{Rigidity theorem}

It is crucial to give a local characterisation to a spacetime admitting an event horizon which is a teleologically defined object. In 1970s Hawking\footnote{See also \cite{56} for a great historic presentation of the subject.} \cite{55} has introduced the strong rigidity theorem which associated the event horizon to a local notion of Killing horizon under quite strong assumptions. Furthermore, this result also fixes symmetry structure away from a horizon: for static horizons the bulk is also static, and for rotating horizons the theorem asserts the existence of an additional axial Killing field in the bulk. The very first result \cite{55, 57, 58} was proven for (i) AF, (ii) analytic spacetime with (iii) non-degenerate horizon (iv) in four dimensions, but there were multiple generalisations along different paths ever since.

One direction of work was devoted to relaxing the \textit{analyticity} assumption, and it is worthwhile to delve into its discussion here. This assumption means that the spacetime metric is an analytic function in some smooth coordinate chart covering the horizon, so that the full spacetime is determined by its behaviour in the neighbourhood of a single point. Whether such an assumption is physical, or more generally — how smooth physical solutions should be, is a principal question with an answer largely depending on the particular field of gravity. In the context of classification of stationary solutions, which is the subject of the thesis, it is reasonable to assume that, while all physically relevant solutions are \(C^{\infty}\), analytic solutions are testable backgrounds for holographic correspondence, and their classification as mathematical models is of interest on its own.

In the work \cite{59}, the analyticity assumption was lifted. The proof, however, crucially relies on the asymptotic flatness and works only in four dimensions\footnote{See also \cite{28} for discussion of its generalisations.}. Another direction is in relaxing the asymptotic flatness and dimensionality conditions \cite{60, 61}, but in preserving the analyticity and non-degenerate horizon assumptions. In \cite{61} the authors have shown that the rigidity theorem is a general geometric statement which holds for broad classes of gravitational theories, including higher order corrections in curvature.

The generalisation of assumption (iii) is more tricky. While it was recently shown that a general extremal horizon in a stationary spacetime, regardless of asymptotic, does indeed admit an additional axial Killing field on itself \cite{62}, it is difficult to push this statement into the bulk. The reason is that for extremal black holes, the horizon effectively decouples from the rest of the spacetime, and, in fact, can be obtained by an appropriate limit (see section 1.2.3). This makes the standard approach, of showing that the axial Killing field of the horizon remains a Killing field to all orders along the flow of a transverse vector field, inapplicable. In \cite{63}, by a general analysis it was shown that the existence of extra Killing field is dependent on the angular velocities of horizon. Namely, an additional Killing field should exist for all but a measure zero subset of black hole parameters. The physical interpretation of the black holes in this measure zero subset, as well as the status of this assumption in general, is unclear. At
the same time, it is reasonable to expect that for backgrounds relevant in the holography the additional Killing field exists.

1.2.3 Near-horizon geometries

The equations of motion or a Killing spinor equation must be supplied with boundary conditions. A natural boundary for black hole spacetime is its horizon, and, therefore, we need to introduce a coordinate chart which is regular on it and describes its neighbourhood. A convenient choice is a chart called Gaussian null coordinates (GNC) [64]. It is defined as follows.

We will work in D-dimensions. As explained in section 1.2.2, a black hole horizon \( \mathcal{H} \) is a codimension-1 null hypersurface which, by the rigidity theorem, is also a Killing horizon. Pick a spacelike \( D-2 \)-surface \( \Sigma \) within \( \mathcal{H} \) and let \( y^i \) (\( i = 1, \ldots, D-2 \)) be coordinates on this surface, a spatial cross-section of \( \mathcal{H} \). Assign coordinates \( (v, y^i) \) to the point a parameter distance \( v \) from \( \Sigma \) along the generator of \( \mathcal{H} \) which intersects the surface \( \Sigma \) at the point with coordinates \( y^i \). Denote by \( V \) the tangent to the generator of \( \mathcal{H} \). In total, we have coordinates \( (v, y^i) \) on \( \mathcal{H} \) where \( V|_{\mathcal{H}} = \partial/\partial v \), and the generators are lines of constant \( y^i \).

Let \( U \) be a null vector field on \( \mathcal{H} \) satisfying \( U \cdot \frac{\partial}{\partial y^i} = 0 \) and \( U \cdot V = 1 \). Assign coordinates \( (v; \lambda; y^i) \) to the point affine parameter distance \( \lambda \) along the null geodesic which starts at the point on \( \mathcal{H} \) with coordinates \( (v; y^i) \) and has tangent vector \( U \) there. This defines a coordinate chart in a neighbourhood of \( \mathcal{H} \) such that \( \mathcal{H} \) is at \( \lambda = 0 \), with \( V = \frac{\partial}{\partial \lambda} \), and \( \frac{\partial}{\partial \lambda} \) is tangent to affinely parameterised null geodesics.

In other words, the vector fields \( V, U, \partial/\partial y^i \) are extended into the bulk by parallel transport along \( U \), i.e.,

\[
\nabla_U V = \nabla_U U = \nabla_U (\partial/\partial y^i) = 0.
\]

Furthermore, on \( \mathcal{H} \) we also have

\[
(\nabla_V V)|_{\mathcal{H}} = (\nabla_V (\partial/\partial y^i))|_{\mathcal{H}} = 0.
\]

Consider the directional derivatives along \( U \). By construction, \( \nabla_U (U \cdot U) = 0 \), so we have \( g_{\lambda\lambda} = 0 \) everywhere. Similarly by (1.2)

\[
\nabla_U (V \cdot U) = U \cdot \nabla_U V = 0 \quad (1.4)
\]

\[
\nabla_U (U \cdot \partial_i) = U \cdot \nabla_U \partial_i = 0 \quad (1.5)
\]

\( \partial_i = \partial/\partial y^i \), which means that \( g_{\lambda\lambda} = U \cdot V \) and \( g_{\lambda i} = U \cdot \partial_i \) are constant everywhere in the chart. Since at \( \lambda = 0 \), i.e. on the horizon, we have \( g_{\lambda\lambda} = U \cdot V = 1 \) (because \( V|_{\mathcal{H}} = \frac{\partial}{\partial \lambda} \)) and \( g_{\lambda i} = U \cdot \frac{\partial}{\partial \lambda} = 0 \), the constants are fixed: \( g_{\lambda\lambda} = 1 \) and \( g_{\lambda i} = 0 \).

By repeating the above procedure for (1.3), one finds \( g_{\nu\nu}|_{\mathcal{H}} = g_{\nu i}|_{\mathcal{H}} = 0 \) everywhere on the horizon. We assume that all functions are smooth (or at least \( C^2 \)), so we can write \( g_{\nu\nu} = \lambda F \) and \( g_{\nu i} = \lambda h_i \) for some smooth functions \( F, h_i \). Therefore, the metric takes the form

\[
g = 2dvdl + \lambda F dv^2 + 2\lambda h_i dv dy^i + h_{ij} dy^i dy^j \quad (1.6)
\]

In section 1.2.2 we have introduced degenerate horizons, as horizons for which surface gravity vanishes. In other words,

\[
d(V \cdot V)|_{\mathcal{H}} = (\partial_{\lambda} g_{\nu\nu})|_{\lambda=0} = 0 \quad (1.7)
\]

and it follows that \( g_{\nu\nu} = \lambda F = \lambda^2 \tilde{F} \) for some smooth function \( \tilde{F} \). Therefore, in the neighbourhood of any smooth degenerate Killing horizon the metric in Gaussian null coordinates reads

\[
g = 2dvdl + \lambda^2 F dv^2 + 2\lambda h_i dv dy^i + h_{ij} dy^i dy^j \quad (1.8)
\]

A crucial observation for (1.8) is the existence of a scaling transformation \( (v, \lambda, y^i) \rightarrow (v/\epsilon, \lambda, y^i) \).

This scaling limit \( \epsilon \rightarrow 0 \) isolates the near-horizon (NH) geometry, which is defined as the leading order of geometry, from the rest of the bulk. Even more crucially, this scaling, being a coordinate freedom, is preserved by the Einstein equations. This effectively decouples the horizon

\[\text{[10]}\text{With an appropriate change of } F, h_i \text{ this rescaling becomes a coordinate freedom.}\]
from the rest of the spacetime — in fact, *a near-horizon geometry is itself a solution to Einstein equations*.

In total, we have shown that any spacetime containing a degenerate Killing horizon, such as an extremal black hole, possesses a well-defined notion of a near-horizon geometry. The classification of near-horizon geometries is an important subject in black hole classification that was done for various theories in diverse dimensions [44, 45, 47, 65] (usually under assumptions of sufficient axisymmetry), see also [66] for the review.

Supersymmetric near-horizon geometries are algebraically special in the sense that they admit supersymmetry enhancement [67]. This remarkable fact has allowed for their complete classification [45] in 5\textsuperscript{d} minimal gauged supergravity\textsuperscript{11} — a smooth near-horizon geometry is necessarily locally isometric to that of the CCLP black hole.

### 1.3 5\textsuperscript{d} minimal gauged supergravity

In this thesis we will be working with five-dimensional gauged supergravity in its minimal form or coupled to $N$ Abelian multiplets. The minimal gauged supergravity in 5\textsuperscript{d} is a consistent truncation of $N=2$ gauged supergravity (see section 1.1) with minimal matter content — the Maxwell field with Chern-Simons term at a fixed coupling. The theory is compatible with supersymmetry, in the sense that it allows for backgrounds with non-zero Killing spinor.

In the following we use the conventions of [68].\textsuperscript{12} The bosonic action of the theory is

$$ S = \frac{1}{4\pi G} \int \left[ \left( \frac{R_5}{4} + \frac{3}{\ell^2} \right) \ast_5 1 - \frac{1}{2} F \wedge \ast_5 F + \frac{2}{3\sqrt{3}} F \wedge F \wedge A \right] $$

where $\ell$ is the AdS length, and $F = dA$ is the field strength of graviphoton $U(1)$ gauge field. The equations of motion are

$$ R^{(5)}_{\mu\nu} - 2F_{\mu\lambda}F^\lambda_\nu + \frac{1}{3}g_{\mu\nu} \left( F^2 + \frac{12}{\ell^2} \right) = 0, $$

$$ d \ast_5 F + \frac{2}{\sqrt{3}} F \wedge F = 0 $$

where $F^2 = F_{\mu\nu}F^{\mu\nu}$. For bosonic background to be supersymmetric, it must admit a non-zero supercovariantly constant spinor (*Killing spinor*) obeying

$$ \left[ \nabla_\mu^{(5)} - \frac{1}{4\sqrt{3}} \left( \gamma^\nu_{\mu\kappa} - 4\delta^\nu_{\mu}\gamma^\kappa_\nu \right) F_{\kappa\nu} \right] \epsilon^a + \ell^{-1} \left( \frac{1}{2} \gamma_\mu + \sqrt{3} A_\mu \right) \epsilon^a \epsilon^b = 0 $$

where $\epsilon^a$ is a symplectic Majorana spinor. The programme of local classification of bosonic supersymmetric solutions has started with the seminal work\textsuperscript{13} [70] through the method of constructing and analysing all spinor bilinears. In particular, it was shown that from the Killing spinor one can construct a Killing vector field $V$ which is everywhere timelike or null. This is a common feature of the supergravities in general, both in ungauged and gauged cases.

Solutions with a null supersymmetric Killing field are less physically relevant for multiple reasons. First of all, the presence of null Killing field is incompatible with AlAdS asymptotics. Secondly, null solutions typically possess a non-compact horizon, e.g. magnetic strings. Finally, they were, essentially, classified in the original paper [70]. In this paper we will not discuss null solutions any longer, and our main focus will be on the timelike class.

It is convenient to introduce a chart adapted to the supersymmetric Killing field $V = \partial/\partial t$.

\textsuperscript{11}It is worth noting that supersymmetric near-horizon geometries with ring topology do exist in STU gauged supergravity so black rings can’t be ruled out in this more general theory [47].

\textsuperscript{12}However, different chapters are using different letters for the Kähler form. For example, we use $J^1$ in chapter 4 because the notation $X^1$ clashes with that for the scalar fields. Nonetheless, in the introduction to each chapter we repeat the key formulas from this section, and we stress that readers should understand the proper meaning of these quantities by context, i.e. there should be no problem in following the logic and the notation.

\textsuperscript{13}The ideas were first put forward in [69] for minimal ungauged supergravity.
Then the metric can be written locally as
\[ ds^2 = -f^2(dt + \omega)^2 + f^{-2}h \]  
(1.13)
where \( h = h_{mn}dx^m dx^n \) is the metric\(^{14}\) on the four-dimensional base \( B \) orthogonal to \( V \), while \( f > 0 \) and \( f, \omega, h \) depend only on \( x^m \) and not on \( t \). One of the many supersymmetry constraints is that \((h, J)\) is Kähler with respect to some 2-form \( J \) constructed as a spinor bilinear. We choose the orientation s.t. \( J \) is anti-self-dual (ASD).

On a Kähler manifold one can define a complex two form \( \Omega \) of the type \((2,0)\) which satisfies
\[ \nabla_m \Omega_{np} + iP_m \Omega_{np} = 0 \]  
(1.14)
where \( P \) is a potential for the Ricci form, i.e. \( R = dP \), and the Ricci form defined as \( R_{mn} = \frac{1}{2} R_{mnpq} J^{pq} \), where \( R_{mnpq} \) is the curvature tensor of the Kähler base. By splitting \( \Omega = X^2 + iX^3 \) into real and imaginary parts (where \( X^{2,3} \) are real 2-forms) one obtains a representation of the quaternion algebra
\[ X^I \cdot X^J n = -\delta^{IJ} \delta_m^n + \epsilon^{IJK} X^K n \]  
(1.15)
where \( X^I, I = 1,2,3 \), collectively stand for \((X^1 = J, X^2, X^3)\). By construction, \( X^{2,3} \) are anti-self-dual if \( X^1 = J \) is chosen as such. The equation (1.14) in terms of \( X^{2,3} \) takes the form
\[ \nabla_m X^2_{np} = P_m X^3_{np} , \]  
(1.16)
\[ \nabla_m X^3_{np} = -P_m X^2_{np} . \]  
(1.17)

The key observation done by Gauntlett and Gutowski was that the whole solution, namely \( f, \omega \) and \( F \), is expressed through the geometry of the Kähler base \( B \). In particular, the square root of the norm of the supersymmetric Killing vector is the inverse of scalar curvature \( R \) of the base
\[ f = \sqrt{- (V \cdot V)} = \frac{24}{R} . \]  
(1.18)
We will require that \( R \) is almost everywhere non-zero. The supersymmetry fixes the gauge field such that the orthogonal part is proportional to the Ricci potential \( P \)
\[ A = \frac{\sqrt{3}}{2} e^0 + \frac{\ell}{2\sqrt{3}} P , \]  
(1.19)

where we have fixed the gauge s.t. \( \mathcal{L}_V A = 0 \), and the tetrad component \( e^0 \) is \( e^0 = f (dt + \omega) \). This, in turn, fixes the field tensor
\[ F = \frac{\sqrt{3}}{2} de^0 + \frac{\ell}{2\sqrt{3}} R , \]  
(1.20)
The final supersymmetry constraint is on \( \omega \). If we decompose \( d\omega \) into self-dual (SD) and anti-self-dual (ASD) parts
\[ f d\omega = G^+ + G^- , \]  
(1.21)
with factor of \( f \) included for convenience, the supersymmetry constrains \( G^+ \) to be proportional to SD part of Ricci form
\[ G^+ = -\frac{\ell}{2} R^+ = -\frac{\ell}{2} \left( R - \frac{R}{4}J \right) . \]  
(1.22)
It is important to note that (1.21) and (1.22) together imply that the supersymmetric Killing field \( V \) is not hypersurface orthogonal, i.e supersymmetric solutions to this theory cannot be static. Indeed, in the chart (1.13), the condition \( V \wedge dV = 0 \) is equivalent to \( d\omega = 0 \) which implies \( G^+ = G^- = 0 \). By (1.22) this gives \( R = \frac{1}{4} RJ \), and since the Ricci form is closed, the scalar curvature \( R \) is constant for static solutions. To complete the proof one needs to use Maxwell equations. By substituting (1.20) into (1.11) and expressing \( R \) through (1.21)

\(^{14}\)This should not be confused with the 1-form \( h \) in the NH geometry (1.8). The difference should be clear from the context.
and (1.22) one finds that the all components vanish, except for \( \ast_5 e^0 \) which reads as

\[
\nabla^2 f^{-1} = \frac{2}{9} (G^+)_{mn} (G^+)_{mn} + \ell^{-1} f^{-1} (G^-)_{mn} J_{mn} - 8 \ell^{-2} f^{-2}.
\]

Substituting constant \( R \), one arrives to contradiction \( R = 0 \), i.e. supersymmetric Killing field has singular norm.

It is convenient to expand \( G^- \) in the basis of ASD 2-forms \( X^I \)

\[
G^- = \frac{\ell}{2R} \lambda_1 X^1,
\]

for some functions \( \lambda_1, \lambda_2, \lambda_3 \). The equation (1.23) can then be understood as an algebraic equation for \( \lambda_1 \)

\[
\lambda_1 = \frac{1}{2} \nabla^2 R + \frac{2}{3} R_{mn} R^{mn} - \frac{1}{3} R^2.
\]

Let us return to constructing \( \omega \). We note that we have independent constraints on the SD and ASD parts of \( d\omega \), and for a general Kähler geometry they may fail to build a closed form. Indeed, the condition

\[
d^2 \omega = d \left[ f^{-1} (G^+ + G^-) \right] = 0
\]

is an integrability constraint. Plugging (1.18), (1.22) and (1.25) in, taking the Hodge dual and multiplying by \( J \), the constraint (1.26) rewrites as

\[
\nabla_m \lambda_1 + R_{mn} \nabla^n R + \left( X^1_{mn} (d\lambda_2 - \lambda_3 P)^n - X^2_{mn} (d\lambda_3 + \lambda_2 P)^n \right) = 0
\]

where we have used (1.16). Note that (1.16) also implies that the term in the bracket is a divergence:

\[
\nabla^n \left( \lambda_2 X^3 - \lambda_3 X^2 \right)_{mn} = X^3_{mn} (d\lambda_2 - \lambda_3 P)^n - X^2_{mn} (d\lambda_3 + \lambda_2 P)^n,
\]

and taking the divergence of (1.27) one arrives to a constraint

\[
\nabla^2 \left( \frac{1}{2} \nabla^2 R + \frac{2}{3} R_{pq} R^{pq} - \frac{1}{3} R^2 \right) + \nabla^m (R_{mn} \partial^n R) = 0.
\]

In other words, the functions \( \lambda_2, \lambda_3 \) are fixed only by a single PDE

\[
\nabla^n \left( \lambda_2 X^3 - \lambda_3 X^2 \right)_{mn} + \nabla_m \lambda_1 + R_{mn} \nabla^n R = 0
\]

which is solved only when the compatibility constraint (1.29) is satisfied.

Interestingly, the existence of an integrability constraint was noticed only in [71], and the explicit form (1.29) was derived in [72] — more than a decade after the original paper was published.

Some comments are in order. First of all, to the best of our knowledge, this fourth-order equation on curvature has never appeared in the physical or mathematical literature outside of this context. In some sense, this equation can be seen as a generalisation of extremality condition for compact Kähler metrics. Observe that the part \( (\nabla^2)^2 R + 2 \nabla^m (R_{mn} \partial^n R) \) corresponds to the real part of the Lichnerowicz operator acting on \( R \), and its vanishing is sufficient for the compact Kähler metrics to be extremal (see, e.g. [73]). For such geometries (1.29) simplifies to \( \nabla^2 (2 R_{pq} R^{pq} - R^2) = 0 \). Next, for Kähler metrics with constant scalar, the constraint simplifies further to \( \nabla^2 (R_{mn} R^{mn}) = 0 \), and if the metric is Einstein, the constraint is automatically satisfied.

### 1.3.1 Imposing symmetry assumptions

The equation (1.29) is a non-linear sixth-order PDE that is too complicated to be directly resolved. Furthermore, its geometric origins and intuition behind it is also unclear which stops us from developing tailored approaches (or, perhaps, uncovering some hidden symmetry structure). Instead, we will identify classes of Kähler geometries for which one can make progress towards
its resolution — namely, we will consider Kähler geometries with \( SU(2) \) or \( U(1)^2 \)-symmetries. Our primary interest are geometries associated with black holes rather than solitons or homogeneous spaces, so we will limit ourselves to solutions which preserve minimal supersymmetry, i.e. \( \frac{1}{4} \)-supersymmetric solutions. Presence of two or more causal Killing fields is a very restrictive scenario — indeed, all solutions with non-minimal supersymmetry were classified \([45,70]\). There is a unique maximally supersymmetric background which is AdS\(_5\) itself, and the solutions with precisely \( \frac{3}{4} \)-supersymmetry are forbidden by topological arguments. Solutions with \( \frac{1}{2} \)-supersymmetry are more interesting: this class consists of near-horizon geometries \([67]\) and a smooth supersymmetric soliton \([17]\). Other solutions include homogeneous backgrounds and non-smooth solitons.

Although in the following chapters we will explicitly assume that supersymmetric Killing field \( V \) commutes with the isometries we impose, this is not a necessary assumption. A following observation by Jan Gutowski shows that for isometry groups with closed spacelike orbits on backgrounds with precisely minimal supersymmetry they necessarily commute or the geometry is not smooth.

**Lemma 1.** Let \((M,g,F)\) be a solution to 5d minimal gauged supergravity with isometry group \(G\) with closed orbits, and let the Maxwell field \(F\) also invariant under \(G\). Then if \((M,g,F)\) preserves precisely minimal supersymmetry, all spinor bilinears, in particular the Killing field and the Kähler form, are also \(G\)-invariant.

**Proof.** Let \(G\) be the isometry group with closed orbits, and \(k_i, i=1\ldots n\) its generators, each with closed orbits. Let the matter fields, in particular, the Maxwell field, be invariant under this symmetry: \(\mathcal{L}_{k_i} F = 0\). Then, since spinorial Lie derivative is constructed out of covariant derivative and matter fields, its action on the Killing spinor must also be a Killing spinor

\[
\mathcal{L}_{k_i} \epsilon = \alpha_i^a \epsilon_a
\]

(1.31)

where \(a=1\ldots N\) enumerates Killing spinors, and \(\alpha_i^a\) are complex constants.

Now, consider the simplest bilinear \(f = \bar{\epsilon} \epsilon\) — the norm of supersymmetric Killing vector:

\[
\mathcal{L}_{k_i} f = \mathcal{L}_{k_i} (\bar{\epsilon} \epsilon) = 2 \text{Re}(\alpha_i) \bar{\epsilon} \epsilon = 2 \text{Re}(\alpha_i) f
\]

(1.32)

If \(G\) has closed orbits, \(\alpha_i\) must have a vanishing real part, or, otherwise, \(f\) will not be smooth as it grows along this orbit. Consequently, the supersymmetric Killing field and the Kähler form are invariant under \(G\):

\[
\mathcal{L}_{k_i} V_\mu = \mathcal{L}_{k_i} (\bar{\epsilon} \gamma_\mu \epsilon) = 2 \text{Re}(\alpha_i) V_\mu = 0
\]

(133)

\[
\mathcal{L}_{k_i} \Phi_{\mu\nu} = \mathcal{L}_{k_i} (\bar{\epsilon} \gamma_{\mu\nu} \epsilon) = 2 \text{Re}(\alpha_i) \Phi_{\mu\nu} = 0
\]

(134)

where \(X^1 = \text{Tr} \Phi\).

For solutions with non-minimal supersymmetry the proof breaks at the second equality in (1.32), where one gets a linear combination of \(\epsilon_a \epsilon_b\) bilinears.

### 1.3.2 Known solutions

**Gutowski-Reall black hole**

The very first smooth supersymmetric black hole in this theory was found by Gutowski and Reall in \([68]\). This is a 1-parameteric family of solutions with \(SU(2) \times U(1) \times \mathbb{R}_t\) isometry which reduces to AdS\(_5\) for a particular value of the parameter. The horizon has spherical \(S^3\) topology.

While some components of the solution, such as its Kähler base, were given in the rele-
vant chapter 2, it is nevertheless reasonable to provide the full solution in one place.\footnote{Compared to chapter 2 we have rescaled $r = 2\alpha r$.}

\begin{equation}
\begin{aligned}
\dot{h} &= \frac{dr^2}{V(r)} + \alpha^2 r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) + 4\alpha^4 r^2 V(r)(d\psi + \cos \theta d\varphi)^2, \\
\dot{X}^{(1)} &= d \left( \alpha^2 r^2 (d\psi + \cos \theta d\varphi) \right),
\end{aligned}
\end{equation}

where $V(r) > 0$ is an arbitrary function, $\alpha$ is a constant, and $(\theta, \psi, \varphi)$ are standard Euler angles on $S^3$. This family includes the important case of the Bergmann metric for $V = 1 + \ell^2 r^2$ and $\alpha = 1/2$ which is the base space of global AdS$_5$ (normalised so $f = 1$). For $\alpha > 1/2$ this class includes the base of the GR black hole which also has $V = 1 + \ell^2 r^2$, or its near-horizon geometry which has $V = 1$ (also a supersymmetric solution).

The five-dimensional metric is then given by (1.13) with

\begin{equation}
f^{-1} = 1 + \frac{\ell^2 (4\alpha^2 - 1)}{12\alpha^2 r^2},
\end{equation}

and $\omega$ given by $\omega = \omega_3 \sigma_3$ with

\begin{equation}
\omega_3 = \frac{(4\alpha^2 - 1) \ell^3}{48\alpha^2 r^2} + \frac{1}{2} (4\alpha^2 - 1) \ell + 2\alpha^2 \frac{r^2}{\ell},
\end{equation}

and $\sigma_3 = d\psi + \cos \theta d\varphi$. The gauge field is given by

\begin{equation}
A = \sqrt{\frac{3}{2}} f(dt + \omega) - \frac{\ell}{2\sqrt{3}} \left( 6\frac{r^2}{\ell^2} + 4\alpha^2 - 1 \right) \sigma_3.
\end{equation}

**CCLP and its NH geometry**

The most general known asymptotically AdS$_5$ black hole solution in this theory was found by Chong, Cveti\v{c}, L"u and Pope (CCLP) and is a 4-parameter family of topologically $S^3$ black holes specified by its mass $M$, electric charge $Q$ and two angular momenta $J_1, J_2$ \cite{17}. This family contains a smooth 2-parameter supersymmetric black hole solution which is specified by two independent angular momenta $J_1, J_2$ and charge $Q$ subject to a complicated non-linear constraint (the mass is fixed by the BPS condition). The solution has $U(1)^2 \times \mathbb{R}_t$ symmetry. See Appendix 3.B for the full presentation.

A complete classification of the possible near-horizon geometries of such black holes has been known for some time \cite{44, 45}: the most general smooth near-horizon geometry with compact cross-sections, is locally isometric to that of the supersymmetric CCLP black hole. In particular, this implies that regular black rings do not exist in this theory.\footnote{It is worth noting that supersymmetric near-horizon geometries with ring topology do exist in STU gauged supergravity so black rings can’t be ruled out in this more general theory \cite{47}.}

Finally, in chapter 3 we made a curious observation about the Kähler base of the CCLP NH geometry. Namely, it formally satisfies the variational PDE of the Calabi functional \cite{74}, and is a non-compact analogue of extremal Kähler geometries \cite{73}.

**Supersymmetric soliton**

The only known smooth supersymmetric soliton which is asymptotically globally AdS$_5$ was found in \cite{75} as part of the larger 2-parameter family of non-smooth solutions, with its smoothness analysis carried out in \cite{76}.\footnote{The solution also admits a three-parametric generalisation with 1/4-supersymmetry \cite{17}. However, this generalisation does not contain any new smooth solutions \cite{72}.} The only smooth soliton within this family does not have any free parameters, so the values of its charges are fixed. The solution possesses enhanced 1/2-supersymmetry, and, because of it, evades the Lemma 1. Its isometry group includes $SU(2) \times U(1)$-symmetry, out of which only $U(1)^2$ commutes with the supersymmetry, so that its Kähler base is toric.
The solution possesses a number of interesting features, including an ergosurface. See Appendix 3.D for more details.

Numerical solutions
A family of supersymmetric black holes was constructed numerically in [77, 78]. This is a 1-parameter family of $SU(2) \times U(1) \times \mathbb{R}$-symmetric solutions, which are AIAdS with the squashed $S^3$ at infinity. In chapter 2 we prove that these solutions are not smooth on the horizon,\footnote{The metric is not twice continuously differentiable on the horizon, which means that the curvature invariants are not continuous there.} which questions their physical interpretation.

Numerical solitons are also known [79]. This is a 1-parameter family of $SU(2) \times U(1) \times \mathbb{R}$-symmetric solutions. Similar to numerical black holes, their asymptotics is locally AdS with the squashed $S^3$ at infinity.

1.4 Organisation
Chapter 2 is dedicated to analysis of supersymmetric solutions with $SU(2)$-symmetry. We study a class of solutions with $SU(2)$-symmetric metric and $SU(2) \times U(1)$-invariant Kähler form, and show that it is necessarily diagonal. Focusing further on this class, we show that for any smooth black hole in this class there is a symmetry enhancement for the metric to $SU(2) \times U(1)$. We then prove the first black hole uniqueness theorem in AdS, namely that the unique analytic black hole with such $SU(2)$ symmetry is the GR solution. We provide a number of results in appendices, including a classification of general $SU(2)$-symmetric Kähler spaces, and we also show that the supersymmetric constraint for $SU(2) \times U(1)$-symmetric solutions reduces to a second order ODE.

In chapter 3 we investigate the supersymmetric solutions with toric symmetry. We determine the general form of axial and horizon structure of such solutions. We then study a class of toric Kähler geometries characterised by a hamiltonian 2-form, which includes all known (analytically or numerically) solutions to 5d minimal gauged supergravities. We then expand the main result of previous chapter, and prove that the unique smooth toric supersymmetric black hole with such a Kähler base is the CCLP solution.

In chapter 4 we generalise the theory to the STU model, which is a minimal gauged supergravity coupled to two Abelian vector multiplets. We explicitly extract the integrability constraint out of the supersymmetric equations, and generalise the uniqueness theorem of chapter 3, and show the uniqueness of the Kunduri-Lucietti-Reall (KLR) black hole within its class.

In chapter 5 we summarise and conclude with a discussion of future work.

1.1 Conventions
For a $p$-form $\alpha = \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$ on a $d$-dimensional (pseudo-)Riemannian manifold $(M, g)$, $\mu_1, \ldots, \mu_p = 1, \ldots, d$, its Hodge dual $d - p$-form is defined as

\[
(\mathcal{\star} \alpha)_{\mu_1 \ldots \mu_{d-p}} = \frac{1}{p!} \epsilon_{\mu_1 \ldots \mu_{d-p} \mu_{p+1} \ldots \mu_d} \alpha^{\mu_{p+1} \ldots \mu_d}
\]

(1.39)

where $\epsilon_{\mu_1 \ldots \mu_d}$ is a Levi-Civita pseudo-tensor

\[
\epsilon_{\mu_1 \ldots \mu_d} = \sqrt{|\det g|} \, \eta_{\mu_1 \ldots \mu_d},
\]

(1.40)

$\eta_{\mu_1 \ldots \mu_d}$ being the totally antisymmetric symbol, $\eta_{12 \ldots d} = +1$, satisfying

\[
\epsilon^{\mu_1 \ldots \mu_p \rho_{p+1} \ldots \rho_d} \epsilon_{\nu_1 \ldots \nu_p \rho_{p+1} \ldots \rho_d} = \pm p! (d - p)! \delta^{[\mu_1 \ldots \mu_p}_{[\rho_1 \ldots \rho_p]},
\]

(1.41)
where ± sign corresponds to ± signature of the metric $g$, and for Riemannian manifolds it is positive. The square of the Hodge dual acting on $p$-forms is

\[ \star^2 = \pm (-1)^{p(d-p)} , \] (1.42)

and for Kähler geometries it is an involution when acting on even-rank forms. The exterior product of $p$-form $\alpha$ and $q$-form $\beta$ is defined as

\[ (\alpha \wedge \beta)_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} = \frac{[p+q]!}{p!q!} \alpha_{[\mu_1 \ldots \mu_p} \beta_{\nu_1 \ldots \nu_q]} . \] (1.43)

The exterior derivative acting on $p$-form is given by

\[ (d\alpha)_{\nu \mu_1 \ldots \mu_p} = (p+1) \partial_{\nu} \alpha_{\mu_1 \ldots \mu_p} . \] (1.44)

One can define a codifferential $\delta$

\[ \delta := \pm (-1)^p \star^{-1} d \star = \pm (-1)^{d(p-1)+1} \star d \star \] (1.45)

where sign and normalisation are chosen such that in coordinates

\[ (\delta \alpha)_{\mu_1 \ldots \mu_p-1} = (-1)^d \nabla^\rho \alpha_{\rho \mu_1 \ldots \mu_p-1} . \] (1.46)

For Kähler geometry $d$ is even and signature is positive, hence,

\[ \star^2 = (-1)^p , \quad \delta = - \star d \star , \] (1.47)

and

\[ (\delta \alpha)_{\mu_1 \ldots \mu_p-1} = \nabla^\rho \alpha_{\rho \mu_1 \ldots \mu_p-1} . \] (1.48)

The contraction of $p$-form $\alpha$ on vector field $X$ is defined as

\[ (\iota_X \alpha)(Y_1, \ldots, Y_{p-1}) = \alpha(X, Y_1, \ldots, Y_{p-1}) , \quad \forall Y_1, \ldots, Y_{p-1} \in TM , \] (1.49)

or in index notation

\[ (\iota_X \alpha)_{\mu_1 \ldots \mu_p-1} = X^\nu \alpha_{\nu \mu_1 \ldots \mu_p-1} . \] (1.50)

Finally, we note a useful property of Hodge duality

\[ \star (X \wedge \alpha) = \pm (-1)^{d-p-1} \iota_X \alpha \] (1.51)

where $\alpha$ is a $p$-form, and $X \in TM$ is a vector field.

### 1.B Spin geometry

In this appendix we provide necessary introduction to spin geometry based on the lecture notes by José Figueroa-O’Farill [80, 81].

The spin structure on a manifold is defined as a principal Spin-bundle over it. First, let us recall the definition of the spin group.

**Definition 3.** The definite signature spin group $\text{Spin}(n)$ is the universal cover of the $\text{SO}(n)$. The indefinite signature spin group $\text{Spin}(s, t)$ is the universal cover of the connected component of the identity of the $\text{SO}(s, t)$.

Next, recall that a fibre bundle (with structure group $G$ and fibre $F$) over manifold $M$ is defined as a smooth surjective map $\pi : E \to M$ together with a local triviality condition: for every point $m \in M$ there is a local neighbourhood $U$ and a diffeomorphism $\phi_U : \pi^{-1}U \to U \times F$.
such that the following triangle commutes

\[
\begin{array}{ccc}
\pi^{-1}U & \xrightarrow{\phi_U} & U \times F \\
\pi & & \downarrow \phi \\
U & \xleftarrow{pr_1} & \end{array}
\]  

(1.52)

and such that on non-empty overlaps \(U_\alpha \cap U_\beta\)

\[
\phi_\alpha \circ \phi_\beta^{-1}\big|_{\{m\} \times F} = \rho(g_{\alpha\beta}(m))
\]

(1.53)

for some transition functions \(g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\), and \(\rho : G \to \text{Aut}(F)\) is some (effective) representation of \(G\). The manifold \(M\) is called the base, and \(F\) is called the fibre. From local triviality the cocycle condition follow:

\[
g_{\alpha\beta\gamma}(m)g_{\beta\gamma}(m)g_{\gamma\alpha}(m) = 1 \quad \forall m \in U_\alpha \cap U_\beta \cap U_\gamma.
\]

(1.54)

If some of the three neighbourhoods are equal, we recover \(g_{\alpha\alpha}(m) = 1\) and \(g_{\alpha\beta} = g_{\beta\alpha}^{-1}(m)\) for all \(m \in U_\alpha\), respectively in \(U_\alpha \times U_\beta\).

From the local trivialisations \(\phi_\alpha\), one can reconstruct the fibre bundle up to a diffeomorphism as

\[
E \cong \left( \bigsqcup U_\alpha \times F \right) / \sim \quad \text{where } (m, f) \sim (m, g_{\alpha\beta}(m)f) \quad \forall m \in U_{\alpha\beta}, \ f \in F,
\]

(1.55)

and we have dropped \(\rho\) from the notation for simplicity.

A principle \(G\)-bundle is a fibre bundle where the fibre \(F \cong G\) is the structure group itself, and acts on \(G\) by left multiplication. This gives a well-defined right action of \(G\) on total space: \((m, g) \mapsto (m, gg')\) which is fibre-preserving and \(G\)-equivariant. Next, a vector bundle is a bundle where \(F\) is a vector space. To every principal bundle (and vice versa) one can construct a natural vector bundle, an associated bundle, defined as a bundle where the fibre is a vector space on which a representation of \(G\) acts, together with a natural identification under a right action of \(G\): \((p, f) \sim (p, fg)\) for \(p \in P\), the principal \(G\)-bundle, \(f \in F, g \in G\).

We are now ready to introduce the spin structure. Let \((M, g)\) be an orientable Riemannian manifold of signature \((s, t)\) and let \(SO(M) \to M\) denote the bundle of oriented normal frames.

**Definition 4.** A spin structure on \((M, g)\) is a principal \(\text{Spin}(s, t)\)-bundle \(\text{Spin}(M) \to M\) together with a bundle morphism

\[
\begin{array}{ccc}
\text{Spin}(M) & \xrightarrow{\pi} & SO(M) \\
M & \xleftarrow{\pi} & \end{array}
\]

(1.56)

which restricts fiberwise to covering homomorphism \(\tilde{\text{Ad}} : \text{Spin}(s, t) \to SO(s, t)\).

In general, spin structures need not exist. To see the obstruction, let us start with a local trivialisation \((U, \{g_{\alpha\beta}\})\) of \(SO(M)\). Choosing \(\tilde{g}_{\alpha\beta}(M) \in \text{Spin}(s, t)\) such that under \(\tilde{\text{Ad}} : \text{Spin}(M) \to SO(s, t)\), \(\tilde{g}_{\alpha\beta}(m) \mapsto g_{\alpha\beta}(m)\). Then \(\text{Spin}(M)\) would be reconstructed from the local trivialisation as

\[
\text{Spin}(M) \cong \left( \bigsqcup U_\alpha \times \text{Spin}(s, t) \right) / \sim, \quad (m, s) \sim (m, \tilde{g}_{\alpha\beta}(m)s) \quad \forall m \in U_{\alpha\beta}, \ s \in \text{Spin}(s, t),
\]

(1.57)

provided that the cocycle condition (1.54) holds, which in this case reads as

\[
f_{\alpha\beta\gamma}(m) := \tilde{g}_{\alpha\beta}(m)\tilde{g}_{\beta\gamma}(m)\tilde{g}_{\gamma\alpha}(m) \equiv 1, \forall m \in U_{\alpha\beta\gamma}.
\]

(1.58)

However, by construction we have

\[
f_{\alpha\beta\gamma}(m) \in \ker(\tilde{\text{Ad}}) = \mathbb{Z}_2.
\]

(1.59)
Since $\tilde{Ad}$ has non-trivial kernel, the cocycle condition does not hold in general. In fact, $f_{\alpha\beta\gamma}$ defines a class in $H^2(M, \mathbb{Z}_2)$ which is essentially the second Stiefel-Whitney class. Furthermore, since $H^1(M, \mathbb{Z}_2) \cong \text{Hom}(\pi_1(M), \mathbb{Z}_2)$, if $M$ is simply-connected then it automatically admits a spin structure.

In the language of Čech cohomology, this obstruction can be seen as the image of the class of $\text{SO}(M)$ in $H^1(M, \text{SO}(M))$ under the connecting map in the long exact sequence

$$H^1(M, \mathbb{Z}_2) \rightarrow H^1(M, \text{Spin}(s, t)) \rightarrow H^1(M, \text{SO}(s, t)) \rightarrow H^2(M, \mathbb{Z}_2)$$

which comes from the exact sequence of the groups

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(s, t) \xrightarrow{\tilde{Ad}} \text{SO}(s, t) \rightarrow 1.$$
Chapter 2

Supersymmetric solutions with $SU(2)$ isometry

This chapter is based on the paper [82] authored by myself and my supervisor James Lucietti.

2.1 Introduction

In this chapter we will consider the classification of supersymmetric black holes in global $AdS_5$, under the additional assumption of $SU(2)$ isometry. In particular, we will consider supersymmetric backgrounds of $D = 5$ minimal gauged supergravity, which is the simplest theory that admits such solutions. As we have outlined in chapter 1, a general classification of equilibrium black holes in AdS spacetime is highly non-trivial question with very few known result in this direction. For $D = 5$ minimal gauged supergravity this reduces to a finding a 4d Kähler geometry that solves a complicated 4th order non-linear PDE for its curvature [70, 72]. Therefore a general local classification of solutions is not currently available. Hence it is natural to seek further symmetry assumptions that are compatible with spacetimes in this class.

Asymptotically globally $AdS_5$ spacetimes have an $SO(4)$ rotation group acting on the spatial $S^3$ at infinity. If one assumes spherical $SO(4)$ symmetry the solution must be static and hence is a supersymmetric limit of the Reissner-Nordström-$AdS_5$ solution which is nakedly singular. A natural class to investigate is solutions that are invariant under an abelian $U(1)^2 \subset SO(4)$ rotational symmetry. In fact, all currently known solutions to $D = 5$ gauged supergravity, including the most general known black hole, CCLP [17], belong to this class. This is presented in a different chapter 3.

What are other reasonable symmetries to assume? As we have discussed in the introduction, section 1.2.2, by rigidity theorem stationary solutions with non-extremal horizon must have at least a single $U(1)$ isometry, and it is reasonable to expect that the same should hold for extremal solutions too. If black holes with this minimal symmetry exist, it is expected they will violate the no-hair theorem [83–85], as such black holes can arise as the endpoint of superradiant instability.

In five-dimensions there is another notion of axisymmetry. Namely, solutions that preserve an $SU(2)$ subgroup of $SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$. In fact, the first example of a supersymmetric black hole in $AdS_5$ was found by Gutowski and Reall [68], which corresponds to a one-parameter subfamily of CCLP characterised by having equal angular momenta $J_1 = J_2$ and an enhanced rotational symmetry $U(1) \times SU(2)$, where $SU(2)$ acts with 3d orbits. We note that the Cartan subgroup of $SU(2)$ is $U(1)$, so one may find the violations of no-hair theorem within this symmetry class.

All together, a simple question therefore presents itself: can we classify all supersymmetric solutions that also admit such an $SU(2)$-symmetry?\footnote{The possibility of hairy supersymmetric black holes with $SU(2)$-symmetry in other truncations of supergravity has recently been investigated in [86, 87]}

This reduces to classifying Kähler metrics

\[\text{The possibility of hairy supersymmetric black holes with } SU(2)-\text{symmetry in other truncations of supergravity has recently been investigated in [86, 87]}\]
with a cohomogeneity-1 $SU(2)$ symmetry, which is an ODE problem and therefore much more tractable. Our main result is given by the following uniqueness theorem.

**Theorem 2.** Any supersymmetric solution to five-dimensional minimal gauged supergravity that is timelike outside an analytic horizon with compact cross-sections, with $SU(2)$-symmetry realised such that its Kähler form, constructed from the Killing spinor, has enhanced $U(1) \times SU(2)$ symmetry, must be a Gutowski-Reall black hole or its near-horizon geometry.

We believe this is the first uniqueness theorem for black holes in AdS in dimension $D > 3$, supersymmetric or otherwise. We emphasise that no global assumptions on the spacetime are made other than the supersymmetric Killing field is timelike outside the horizon. In particular, our proof does not make any assumptions about the asymptotics and therefore also rules out the existence of asymptotically locally AdS$_5$ supersymmetric black holes in this class (other than quotients of the Gutowski-Reall solution). In fact, a class of supersymmetric black holes with $SU(2)$ symmetry that are asymptotically locally AdS$_5$ with squashed $S^3$ spatial boundary metrics have been constructed numerically [77, 78, 89, 90]. Our results show that these solutions do not possess smooth horizons. In fact, we will show that the horizons of such solutions are $C^1$ but not $C^2$.

The structure of the proof is as follows. First we show that any Kähler metric with a cohomogeneity-1 $SU(2)$ symmetry and $U(1) \times SU(2)$ Kähler form is described by a simple system of ODEs, which generalises that of Dancer and Strachan for Kähler-Einstein metrics with $SU(2)$ symmetry [91]. On the other hand, by a delicate near-horizon analysis we show that a horizon corresponds to a conical singularity of the Kähler base. This provides boundary conditions for the ODE system that governs the Kähler base, which imply that it must have an enhanced $U(1) \times SU(2)$ symmetry everywhere outside the horizon. Finally, we show that the classification of $U(1) \times SU(2)$-invariant supersymmetric solutions reduces to a single 5th order non-linear ODE and we are able to find all solutions to this that correspond to an analytic horizon.

As part of our analysis, we also determine the general local form for a timelike supersymmetric solution to $D = 5$ minimal gauged supergravity with Kähler base with such $SU(2)$ symmetry. Furthermore, we derive the boundary conditions required for a Kähler metric in this class to contain a nut or bolt and use this to prove that any strictly timelike supersymmetric soliton spacetime with a nut or a complex bolt must have enhanced $U(1) \times SU(2)$ symmetry. As an example, we construct an explicit class of solitons with a bolt that are asymptotically locally AdS$_5$ with spatial $S^3$/Z$_p$ boundary for $p \geq 3$.

This chapter is organised as follows. In section 2.2 we review the local form of supersymmetric backgrounds of $D = 5$ minimal gauged supergravity and then examine the consequences of $SU(2)$ symmetry. In section 2.3 we derive the general form for a Kähler metric with cohomogeneity-1 $SU(2)$ symmetry and establish a symmetry enhancement result for such metrics; this section is written in a self-contained way as it may be of independent interest. In section 2.4 we determine the general local form for a supersymmetric solution with $SU(2)$-symmetry and discuss the known examples of black hole and soliton solutions. In section 2.5 we analyse supersymmetric black holes with $SU(2)$ symmetry and prove the black hole uniqueness theorem. We close with a Discussion of our results and provide an Appendix.

In Appendix 2.A we give our conventions related to $SU(2)$ symmetry. Appendix 2.B is dedicated to examples of Kähler metrics and solitons with a bolt. In Appendix 2.C we give a general classification of Kähler spaces with cohomogeneity-1 $SU(2)$ symmetry without the assumption on the Kähler form; the result, albeit technical, is given as a theorem. Finally, in Appendix 2.D we show that the supersymmetric constraint on $SU(2) \times U(1)$ solutions reduces to a second order ODE.

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2 Alternatively, one can assume the metric to be diagonal while Kähler form is unfixed. Then the same result follows, see Appendix 2.C.

3 Naturally, in three-dimensions black hole uniqueness results are known [88].
2.2 Supersymmetric solutions of gauged supergravity

2.2.1 General local classification

The general local form for supersymmetric solutions \((M, g, F)\) to five-dimensional minimal gauged supergravity was determined in [70] (we work in the conventions of [46]). We will briefly recall it here.\(^4\) For timelike solutions, which are defined by the supersymmetric Killing field \(V\) being timelike, the metric can be written as

\[
g = -f^2(dt + \omega)^2 + f^{-1}h ,
\]

(2.1)

where \(V = \partial_t\), the metric \(h\) on the orthogonal base \(B\) is Kähler, and \(f\) and \(\omega\) are a function and 1-form on \(B\). The Maxwell field takes the form

\[
F = \frac{\sqrt{3}}{2} d(f(dt + \omega)) - \frac{1}{\sqrt{3}} G^+ - \frac{\sqrt{3}}{lf} J ,
\]

(2.2)

where \(G^\pm = \frac{1}{2} f(d\omega \pm \star_B d\omega)\), \(J\) is the Kähler form of \((B, h)\) and the orientation on \(B\) is such that \(J\) is anti-self dual (ASD), i.e. \(\text{vol}_B = -\frac{1}{2} J \wedge J\).

Given a Kähler base, the following are completely fixed in terms of its curvature

\[
f^{-1} = -\frac{\ell^2}{24} R, \quad G^+ = -\frac{\ell}{2} (\mathcal{R} - \frac{1}{4} J \mathcal{R}) ,
\]

(2.3)

where \(\mathcal{R}_{ab} = \frac{1}{2} R_{ab}^c J^{cd}\) is the Ricci form and \(R_{abcd}\) is the Riemann tensor of \(B\). However, as clarified in [71, 72], the Kähler base is not free to be chosen.

First, we recall that any Kähler surface, with ASD Kähler form \(\Omega^3 := J\), admits a complex \((2,0)\) form \(\Omega^1 + i \Omega^2\), with \(\Omega^1, \Omega^2\) real ASD 2-forms, which satisfy the quaternion algebra

\[
(\Omega^i)_{ab} (\Omega^j)^c_b = -\delta^c_a \delta^i_j + \epsilon_{ijk} (\Omega^k)^a_c ,
\]

(2.4)

for \(i = 1, 2, 3\), and the differential equations

\[
\nabla_a \Omega^1_{bc} = P_a \Omega^2_{bc}, \quad \nabla_a \Omega^2_{bc} = -P_a \Omega^1_{bc} ,
\]

(2.5)

where \(\nabla\) is the metric connection of \(h\) and \(P\) is the potential for the Ricci form \(\mathcal{R} = dP\). Observe that there is a local \(O(2)\) freedom \(\Omega^1 \rightarrow O^1 \Omega^1\) which preserves the Kähler form \(\Omega^3\) and generates gauge transformations of the Ricci form potential \(P\).

In particular, since the \(\Omega^i\) give a basis for ASD 2-forms we can expand

\[
G^- = \frac{\ell}{2R} \lambda^i \Omega^i ,
\]

(2.6)

where

\[
\lambda_3 = \frac{1}{2} \nabla^2 R + \frac{2}{3} R_{ab} R^{ab} - \frac{1}{3} R^2 .
\]

(2.7)

Then, as shown in [72], the integrability condition for

\[
d\omega = f^{-1}(G^+ + G^-) ,
\]

(2.8)

fixes \(\lambda_1 + i \lambda_2\) up to an antiholomorphic function on \(B\) and requires that the Kähler metric satisfies the following complicated 4th order PDE for its curvature,

\[
\nabla^c \Xi_c = 0 ,
\]

(2.9)

where

\[
\Xi_c := \nabla_c \left( \frac{1}{2} \nabla^2 R + \frac{2}{3} R_{ab} R^{ab} - \frac{1}{3} R^2 \right) + R_{cb} \nabla^b R .
\]

(2.10)

\(^4\)In this chapter we have to use a different letter for the Kähler form compared to the Introduction as explained above (2.4). Therefore, we are repeating the key formulas from section 1.3, in order to introduce the preferred notation.
Conversely, given a Kähler metric $h$ which satisfies (2.9), a supersymmetric solution can be reconstructed by solving (2.8) for $\omega$ and $f$ is determined by (2.3).

To summarise, the classification of timelike supersymmetric solutions in this theory reduces to the classification of Kähler metrics that satisfy (2.9). It is worth noting that any Kähler-Einstein metric satisfies (2.9) and therefore gives a supersymmetric solution with $f = 1$ and $d\omega$ is an ASD 2-form (normalised so $R_{ab} = -6\ell^{-2}h_{ab}$).

### 2.2.2 Solutions with SU(2) symmetry

We will now assume that $(M, g, F)$ also admits a $G$-symmetry in the following sense: (i) there is an isometry group $G$ with spacelike orbits; (ii) the supersymmetric Killing field $h$ is complete and invariant under $G$ so that the spacetime isometry group is $\mathbb{R}_t \times G$ where $\mathbb{R}_t$ is generated by $V$; (iii) the Maxwell field is invariant under $G$.

We start by deducing the constraints on the data for a timelike solution $(f, \omega, h)$ imposed by such a $G$-symmetry. These are summarised by the following result.

**Lemma 2.** A timelike supersymmetric solution $(f, \omega, h)$ that admits a $G$-symmetry (as above), has a Kähler base metric $h$ with a holomorphic $G$-symmetry and $f, \omega$ are $G$-invariant.

**Proof.** Under these assumptions, it follows that $f^2 = -g_{\mu\nu}V^\mu V^\nu$ is $G$-invariant. Therefore, the Kähler metric on the orthogonal base,

$$h_{\mu\nu} = f \left( g_{\mu\nu} - \frac{V^\mu V^\nu}{|V|^2} \right), \quad (2.11)$$

is also $G$-invariant. Furthermore, since $V_\mu dx^\mu = -f^2(dt + \omega)$ is $G$-invariant, the gauge freedom $t \to t + \lambda, \omega \to \omega - d\lambda$, where $\lambda$ is a function on the base, can be used to ensure $\omega$ is a $G$-invariant 1-form on the base space. Finally, notice that invariance of the Maxwell field (2.2) also implies that the Kähler form $J$ is $G$-invariant so the $G$-action is holomorphic. \hfill $\square$

Now we will further restrict to $G$ with generic 3d orbits, where $G$ is $SU(2)$ or $U(1) \times SU(2)$. Let $L_i$ and $R_i$, $i = 1, 2, 3$, denote the generators of the left and right action on $SU(2)$ respectively. In Appendix 2.A we recall the standard formulas of $SU(2)$ calculus and define our conventions. Without loss of generality, we assume the isometry group $G = SU(2)$ is generated by the right-action vectors $R_i$ (i.e. the left-invariant vector fields). The right-invariant 1-forms $\sigma_i := \sigma_i^L$, which are dual to $L_i$, obey

$$d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k. \quad (2.12)$$

Then, any $SU(2)$-invariant metric, in the sense $\mathcal{L}_{R_i} h = 0$, locally can be written in the form\footnote{Such a coordinate system always exists because the distribution orthogonal to the orbits is 1-dimensional.}

$$h = d\rho^2 + h_{ij}(\rho)\sigma_i \sigma_j, \quad (2.13)$$

where $h_{ij} := h(L_i, L_j)$ is a positive-definite Gram-matrix and $\rho$ is a local coordinate orthogonal to the $SU(2)$-orbits. It is worth noting that there is a global $SO(3)$ freedom acting on the right-invariant forms $\sigma_i \to R_{ij} \sigma_j$ where $R \in SO(3)$ is a constant rotation. In general, this can be used to diagonalise $h_{ij}(\rho)$ only at a point. We will refer to $h_{ij}(\rho)$ as diagonalisable if it can be diagonalised for all $\rho$ (in an appropriate domain).

The most general $SU(2)$-invariant 1-form $\omega$ appearing in $V_\mu dx^\mu$ can be written as

$$\omega = \omega_i(\rho)\sigma_i, \quad (2.14)$$

where we have exploited the gauge freedom in the definition of $\omega$ mentioned above to fix $\omega_0 = 0$. By Lemma 2 we will be looking for $\omega$ of this form. Similarly, the invariance of the function $f$ implies that it can only depend on $\rho$. 

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Lemma 2 also shows that the Kähler form is $SU(2)$-invariant. It is easy to show that the most general $SU(2)$-invariant closed 2-form must take the form\footnote{This can be seen by parameterising the $SU(2)$-invariant 2-form $J$ as $J = J_i(\rho) \, d\rho \wedge \sigma_i - \frac{1}{2} J_{ij}(\rho) \, \sigma_i \wedge \sigma_j$. The closure of $J$ is then equivalent to $J^*_i = \epsilon_{ijk} J_k(\rho)$. Writing $J_i(\rho) = g_i(\rho)$ for some $g_i(\rho)$, one gets (2.15).}

$$J = d(g_i(\rho) \sigma_i) .$$

(2.15)

Furthermore, the condition that $J$ be ASD is equivalent to

$$g_i' = \frac{h_{ij} g_j}{\sqrt{\det h}} ,$$

(2.16)

where we choose the orientation of the base to be $d\rho \wedge \sigma_i \wedge \sigma_j \wedge \sigma_3$. We also require that $J$ defines an almost complex structure, i.e. $J^a_{\, \, \beta} J^\alpha_{\, \, \beta} = -d\sigma_3$. It is straightforward to show that this condition reduces to

$$h_{ij} g_i g_j = \det h ,$$

(2.17)

upon use of the ASD condition.

Let us consider the case when Kähler form has enhanced $G = U(1) \times SU(2)$ symmetry. This symplectomorphism group is generated by a subgroup of the right-action and $SU(2)$ by the left-action as above. This may be viewed as a special case of the $G = SU(2)$ case above that is also invariant under a $U(1)$ of the left-action, which we take to be generated by $L_3$. Note that this choice breaks the global $SU(2)$ symmetry to a $SO(2)$ subgroup preserving the $\sigma_3$ which is dual to $L_3$. Invariance of the Kähler form $L_{L_3} J = 0$ implies $g_1 = g_2 = 0$ in (2.15)

$$J = d \left( g_3(\rho) \sigma_3 \right) .$$

(2.18)

The moment map defined by the $U(1)$-symmetry generated by $L_3$ is

$$d\mu = -\iota_{L_3} J = d g_3 .$$

(2.19)

Next, note that ASD (2.16) implies that

$$h_{12} = h_{23} = 0 .$$

(2.20)

We will construct the chart explicitly in (2.32).

Now let the metric also possess an enhanced $G = U(1) \times SU(2)$. The general form for an invariant metric takes the form (2.13) with the further restriction $L_{L_3} h = 0$, which implies

$$h = d\rho^2 + a(\rho)^2 (\sigma_1^2 + \sigma_2^2) + c(\rho)^2 \sigma_3^2 .$$

(2.21)

This is a special case of the diagonal metric. Invariance of the Kähler form $L_{L_3} J = 0$ implies $g_1 = g_2 = 0$ and therefore (2.16), (2.17) reduce to

$$c = (a^2)', \quad J = d(a(\rho)^2 \sigma_3) ,$$

(2.22)

where we have fixed an overall sign in $J$. In this case, it turns out these conditions are sufficient to ensure $J$ is an integrable complex structure and hence $(h, J)$ is a Kähler structure. Indeed, this is precisely the class of Kähler bases considered in [46]. The moment map defined by the $U(1)$-symmetry generated by $L_3$ is

$$d\mu = -\iota_{L_3} J = 2a \, da .$$

(2.23)

It is natural to define a new coordinate from the moment map by $r := 2a(\rho)$ in terms of which

$$h = \frac{dr^2}{V(r)} + \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2) + \frac{r^2 V(r)}{4} \sigma_3^2 ,$$

$$J = d \left( \frac{1}{4} r^2 \sigma_3 \right) ,$$

(2.24)

(2.25)

where $V(r) := 4a'(\rho)^2$. We remark that $a(\rho)$ must be a nonconstant function, otherwise $c(\rho) = 0$. Classification of supersymmetric black holes in AdS$_5$
and the metric is degenerate, so one can always introduce the local coordinate $r$. Therefore this represents the most general $U(1) \times SU(2)$ invariant Kähler structure. Finally, note that the most general $U(1) \times SU(2)$ invariant 1-form can be written as

$$\omega = \omega_3(r)\sigma_3,$$  \hspace{1cm} (2.26)

where as above we have used the gauge freedom in its definition to set $\omega_r = 0$.

For orientation, it is worth noting how global AdS$_5$ is described. The Kähler base for this is the Bergmann metric, which is an Einstein metric with $U(1) \times SU(2)$ symmetry given by $a = \frac{2}{3} \sinh(\rho/\ell)$ [70]. In terms of the $r$-coordinate this corresponds to

$$V = 1 + \frac{r^2}{\ell^2},$$  \hspace{1cm} (2.27)

and the rest of the data for AdS$_5$ is simply

$$f = 1, \quad \omega = \frac{r^2}{2\ell}\sigma_3.$$  \hspace{1cm} (2.28)

We require the necessary and sufficient conditions for (2.13) to be a Kähler metric with $U(1) \times SU(2)$-invariant Kähler form. Imposing that $J$ is a closed ASD 2-form that defines an almost complex structure as above is not sufficient in general. One must also require that $J$ is an integrable complex structure. For a diagonal $h_{ij}(\rho)$ these conditions were obtained by Dancer and Strachan in a study of Kähler-Einstein metrics with $SU(2)$-symmetry [91]. For Einstein metrics it turns out one can show that $h_{ij}(\rho)$ can always be diagonalised. However, in general this need not be the case, therefore we must consider the most general non-diagonal $h_{ij}(\rho)$. It is convenient to perform this calculation in an orthonormal frame. We will present this calculation in section 2.3.

2.3 Classification of Kähler metrics with $SU(2)$ symmetry

In this section we will derive the general form of a cohomogeneity-1 Kähler metric with $SU(2)$ symmetry and $U(1) \times SU(2)$-symmetric Kähler form. We emphasise that we do not assume the metric on the surfaces of transitivity is diagonalisable and therefore obtain a complementary result to the conditions derived by Dancer and Strachan for diagonal metrics [91]. We will also obtain some results on the global analysis of such geometries. This section may be of independent interest, so we give a self-contained presentation. We give our $SU(2)$ conventions in Appendix 2.A.

For clarity, in this section we denote the Kähler 2-form by $\Omega$ and the complex structure by $J$ (in the rest of the chapter we denote them both by $J$ since we work in conventions compatible with raising and lowering indices with the Kähler metric, i.e. $\Omega_{ab} = h_{ab}J_b^c$ etc).

Also, as in the previous section and everywhere else in this thesis, $\sigma_i := \sigma_i^L$ are $SU(2)$ right-invariant 1-forms obeying (2.12).

2.3.1 Local geometry

It is convenient to fix the global $SU(2)$ freedom first. Our choice of $L_3$ as a second commuting hamiltonian vector field has already broken it down to a global $SO(2)$ rotation preserving $\sigma_3$ direction. Remember, that $SO(2)$ rotation is sufficient to diagonalise a symmetric matrix.

Therefore, we fix the freedom by demanding that $\sigma_1, \sigma_2$ part of the metric diagonalises at some regular point $\rho_0$, i.e. $h_{12}(\rho_0) = 0$.

Let us introduce an $SU(2)$-invariant orthonormal frame $e^0 = d\rho, e^i = E^i_j(\rho)\sigma_j$, where $E^i_j$ is an invertible matrix, so that a general $SU(2)$-invariant metric (2.13) is

$$h = (e^0)^2 + e^i e^i,$$  \hspace{1cm} (2.29)
oriented so $e^0 \wedge e^1 \wedge e^2 \wedge e^3$ is positive. Then a basis of ASD 2-forms $\Omega^i$ is given by

$$\Omega^i = e^0 \wedge e^i - \frac{1}{2} \epsilon_{ijk} e^j \wedge e^k. \quad (2.30)$$

Note that these satisfy the quaternion algebra (2.4) and are manifestly $SU(2)$-invariant. The frame $e^i$ is defined only up to a local $O(3)$ transformation $e^i \to O^j \rho e^j$ where $O \in O(3)$.

Next, it is convenient to introduce the following frame

$$e^0 = d\rho, \quad e^1 = a\sigma_1 + b_1 \sigma_2 + c_1 \sigma_3, \quad e^2 = b\sigma_2 + c_2 \sigma_3, \quad e^3 = c\sigma_3, \quad (2.31)$$

where $a, b, c, b_1, c_1, c_2$ are functions of $\rho$, which parameterises the most general $SU(2)$-invariant metric provided $abc \neq 0$ (to see this note that any positive-definite symmetric matrix, such as the metric $h_{ij}$, can be factored as $h_{ij} = (E^T E)_{ij}$ where $E$ is upper triangular). Note that by (2.20) we immediately have $c_1 = c_2 = 0$.

Finally, by (2.18) $\Omega = \Omega^3$

$$\Omega = \Omega^3. \quad (2.32)$$

Furthermore, $\Omega^1, \Omega^2$ must then satisfy (2.5).

We are now ready to state the main result of this section.\footnote{See Theorem 4 for a general classification without enhanced symmetry assumption.}

**Theorem 3.** The most general Kähler metric with a cohomogeneity-1 $SU(2)$ symmetry and $U(1) \times SU(2)$-invariant Kähler form can be written in the frame (2.31) where $b_1 = c_1 = c_2 = 0$, so in particular is diagonal, and

$$2bca' = b^2 - a^2 + c^2, \quad (2.33)$$

$$2acb' = a^2 - b^2 + c^2. \quad (2.34)$$

The Kähler form is simply

$$\Omega = d(ab\sigma_3). \quad (2.35)$$

**Proof.** First we will work in a general orthonormal frame $e^0 = d\rho, e^i = E^j \rho \sigma_j$ and denote the dual basis by $X_0 = \partial_\rho, X_1 = (E^{-1})^j_i L_j$, where $L_i$ are the right-invariant vector fields dual to $\sigma_i$. As argued above, we may assume the Kähler form $\Omega$ is given by (2.32), that is,

$$\Omega = e^0 \wedge e^3 - e^1 \wedge e^2. \quad (2.36)$$

Then the action of the almost complex structure $J$ on any vector $X$ can be found using $JX = -h^{-1}(\iota_X \Omega, \cdot)$, which gives

$$JX_0 = -X_3, \quad JX_1 = X_2, \quad JX_2 = -X_1 \quad JX_3 = X_0. \quad (2.37)$$

We wish to impose that $J$ is a complex structure, i.e. that it is integrable. A convenient method to do this is as follows \cite{91}. Let $\chi_0, \chi_1$ be a basis of $(1,0)$-vector fields, i.e. $J\chi_{0,1} = i\chi_{0,1}$. Then require that $[\chi_0, \chi_1]$ is also a $(1,0)$-vector field. We choose

$$\chi_0 = X_0 - iJX_0, \quad \chi_1 = X_1 - iJX_1, \quad (2.38)$$

which are indeed always linearly independent.

Now, without loss of generality we parameterise the frame by (2.31). First note that the requirement that the Kähler form $\Omega$ is closed is equivalent to

$$c = (ab)' . \quad (2.39)$$

The dual vectors for our orthonormal frame are

$$X_0 = \partial_\rho, \quad X_1 = \frac{1}{a} L_1, \quad X_2 = \frac{1}{b} \left(L_2 - \frac{b_1}{a} L_1 \right), \quad X_3 = \frac{1}{c} L_3. \quad (2.40)$$
A computation gives

\[ [\chi_0, \chi_1] = k_i L_i \]  

(2.41)

where

\[ k_1 = \frac{a'}{a^2} - \frac{1}{bc} + i \left( \frac{b_1}{ab} \right), \quad k_2 = \frac{i}{ac} + \frac{ib'}{b^2} - \frac{b_1}{abc}, \quad k_3 = 0. \]

(2.42)

Since the \( \rho \) component of \([\chi_0, \chi_1]\) vanishes, the integrability condition \( J[\chi_0, \chi_1] = i[\chi_0, \chi_1] \) is equivalent to

\[ ak_1 + b_1 k_2 = ibk_3, \quad k_3 = 0. \]

(2.43)

Using the explicit values for \( k_i \), one finds the integrability of \( J \) reduces to

\[ c(ba' - ab') + a^2 - b^2 + b_1^2 = 0, \quad abc' = b_1(ca' - 2b). \]

(2.44)

We can express these conditions is a slightly more convenient form. Solving (2.39) and the first equation in (2.44) for \( a', b' \) gives

\[ 2bc a' = b^2 - a^2 + c^2 - b_1^2, \]

(2.45)

\[ 2abc' = a^2 - b^2 + c^2 + b_1^2, \]

(2.46)

and substituting into the second equation in (2.44) gives

\[ 2abc' = -b_1(a^2 + 3b^2 - c^2 + b_1^2). \]

(2.47)

Under our choice of \( \sigma_i \), this equation is supplied with trivial initial condition \( b_1(\rho_0) = 0 \) at a regular point. By standard ODE uniqueness theorem, \( b_1 = 0 \) everywhere in this chart.

Therefore, we have shown that any Kähler metric defined by (2.31) with Kähler form (2.32), can be arranged to be diagonal, i.e. \( b_1 = c_1 = c_2 = 0 \), where (2.33), (2.34) satisfied. It is easy to show that the Kähler form can be written as (2.35).

It is worth remarking that our theorem reduces to that of Dancer and Strachan for Einstein metrics with a cohomogeneity-1 \( SU(2) \) symmetry [91]. Interestingly, we have shown that even without the Einstein condition, one can always choose a diagonal metric within our class if Kähler form has enhanced symmetry. Furthermore, setting \( a^2 = b^2 \) it is easy to see it reduces to the \( U(1) \times SU(2) \) invariant case described by (2.21) and (2.22). On the other hand, setting \( 2abc' = a^2 + b^2 - c^2 \) gives a hyper-Kähler metric; indeed, this is equivalent to \( a = (bc)' \), \( b = (ac)' \) and (2.39), where \( \Omega^1 = d(bc\sigma_1) \), \( \Omega^2 = d(ac\sigma_2) \) also define integrable complex structures.

It is also worth noting that the ODE system in Theorem 3 is invariant under interchange of \( a \) and \( b \). This corresponds to the (orientation-preserving) discrete symmetry \( \sigma_1 \rightarrow \sigma_2, \sigma_2 \rightarrow -\sigma_1 \) of diagonal metrics with Kähler form (2.35).

For convenience we record certain curvature formulas for this class of Kähler metrics. The Ricci scalar is

\[ R = -\frac{1}{a^2 b^2 c^2} (a^4 + b^4 - (a^2 + b^2)c^2 + 4abc^2 c' + 2a^2 b^2 (-1 + cc')) \]

(2.48)

where we have eliminated first derivatives of \( a, b \) using (2.33), (2.34). The Ricci form is given by the potential

\[ P = \left( \frac{a^2 + b^2 - c^2 - 2abc'}{2ab} \right) \sigma_3. \]

(2.49)

It is worth noting that a convenient trick for computing the Ricci form is to invert (2.5) for the potential \( P \). The Einstein condition \( R_{ab} = \Lambda h_{ab} \) is equivalent to the Ricci form satisfying \( R_{ab} = \Lambda h_{ab} \). Comparing (2.35) and (2.49) immediately implies that the Einstein condition is equivalent to \( 2abc' = a^2 + b^2 - c^2 - 2\Lambda a^2 b^2 \) in agreement with [91].

### 2.3.2 Symmetry enhancement

Here we prove a general symmetry enhancement result for Kähler metrics with \( SU(2) \) symmetry under appropriate boundary conditions. This will be useful for our later analysis.
To this end, it is useful to note that in terms of the function

\[ T := \frac{b}{a}, \tag{2.50} \]

the ODE system in Theorem 3 implies

\[ cT' = 1 - T^2. \tag{2.51} \]

Observe that in terms of this the \( U(1) \times SU(2) \)-invariant case \( a^2 = b^2 \) is simply \( T^2 = 1 \). This ODE allows us to prove the following elementary result.

**Lemma 3.** Consider a Kähler metric with \( SU(2) \) symmetry as in Theorem 3. Suppose \( a, b, c \) are all positive and \( C^1 \) for \( \rho > \rho_0 \), that \( abc = 0 \) at \( \rho = \rho_0 \), and that \( \lim_{\rho \to \rho^+_0} T \) exists. If \( T = 1 \) at \( \rho = \rho_1 > \rho_0 \) then \( T = 1 \) for all \( \rho > \rho_1 \). In particular, if \( T = 1 \) at \( \rho = \rho_0 \) then \( T = 1 \) for \( \rho > \rho_0 \).

**Proof.** Under the stated assumptions \( T \) is positive and \( C^1 \) for \( \rho > \rho_0 \), whereas \( T \) is \( C^0 \) at \( \rho = \rho_0 \).

For the first part, we are given that \( T = 1 \) at \( \rho = \rho_1 \). If \( T > 1 \) for some \( \rho = \rho_2 > \rho_1 \), then (2.51) implies \( T \) is monotonically non-increasing so that \( T(\rho_1) \geq T(\rho_2) > 1 \) which is a contradiction. On the other hand, if \( T < 1 \) at \( \rho = \rho_2 > \rho_1 \), then (2.51) implies \( T \) is monotonically non-decreasing so that \( T(\rho_1) \leq T(\rho_2) < 1 \) which is again a contradiction. Therefore \( T = 1 \) for all \( \rho > \rho_1 \) as claimed.

For the second part, we are given \( T = 1 \) at \( \rho = \rho_0 \). Now pick a point \( \rho_* > \rho_0 \) and suppose \( T > 1 \) \((T < 1) \) at \( \rho = \rho_* \); then (2.51) implies \( T \) is monotonically non-increasing (non-decreasing) so in particular \( T(\rho_0 + \epsilon) > T(\rho_0) \rangle 1 \) \((T(\rho_0 + \epsilon) < T(\rho_0) \rangle 1 \) for small enough \( \epsilon > 0 \), so taking the limit \( \epsilon \to 0 \) we deduce \( T(\rho_0) \geq T(\rho_*) > 1 \) \((T(\rho_0) \leq T(\rho_*) < 1 \) which is a contradiction. Therefore we must have \( T = 1 \) at \( \rho = \rho_* \) and since this was an arbitrary point we have shown \( T = 1 \) for all \( \rho > \rho_0 \).

The above result shows that if one has an enhanced \( U(1) \times SU(2) \) symmetry at a point, then it has this enhanced symmetry at all points in our domain.

### 2.3.3 Nuts and bolts

In this section we will determine appropriate boundary conditions in order to obtain complete Kähler metrics. Relative to the frame (2.31),

\[ \det h = a^2 b^2 c^2 \tag{2.52} \]

and therefore \( a^2 b^2 c^2 > 0 \) ensures the \( h \) is a smooth invertible metric. We characterise the possible boundaries where \( \det h = 0 \) in the following.

**Lemma 4.** Suppose \( a, b, c \) are \( C^1 \) at \( p \). Then \( \det h = 0 \) at \( p \) is equivalent to one of the following conditions at \( p \):

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>nut</strong></td>
<td>( a = b = c = 0 )</td>
</tr>
<tr>
<td><strong>bolt I</strong></td>
<td>( c = 0, \ a = \pm b \neq 0 )</td>
</tr>
<tr>
<td><strong>bolt II</strong></td>
<td>( a = 0, \ b = \pm c \neq 0 )</td>
</tr>
</tbody>
</table>

The orbits of \( SU(2) \) through such \( p \) are either 0-dimensional (nut) or 2-dimensional (bolt).

**Proof.** Clearly \( \det h = 0 \) at \( p \) is equivalent to \( abc = 0 \) at \( p \). Thus suppose \( abc = 0 \) at \( p \). We need only rule out the possibility of precisely two of the three \( a, b, c \) vanishing, since the nut and bolt cases above cover the remaining cases as we show below. This can happen in two inequivalent ways: either \( a = b = 0, \ a = c = 0 \) (the case \( b = c = 0 \) is equivalent to the latter under interchange of \( a \) and \( b \)). In the first case (2.33), (2.34), imply \( c = 0 \) at \( p \). In the second case (2.33), (2.34) implies \( c = 0 \) at \( p \). Thus in any case, we get a nut.
Now consider the different types of bolt. If \(c = 0\) then (2.33), (2.34) implies \(a^2 = b^2\) or a nut; if \(a = 0\) then (2.33), (2.34) implies \(b^2 = c^2\) or a nut (the case \(b = 0\) is equivalent to the latter under interchange of \(a\) and \(b\)).

In order to obtain complete Kähler metrics we require that the nut and bolt conditions correspond to coordinate singularities, and that the metric extends smoothly at these points. First consider the nut case. Then smoothness requires at the very least that \(a, b, c\) all vanish as \(O(\rho)\), and, in fact, are all proportional to \(\rho\). This means that the metric approaches a Kähler cone over an \(SU(2)\)-invariant space. The possible Kähler geometries near such a nut are given by simply Taylor expanding \(a, b, c\) around \(\rho = 0\) and solving the ODE system (2.33), (2.34). This is summarised by the following.

**Lemma 5.** Any Kähler metric with cohomogeneity-1 \(SU(2)\) symmetry as in Theorem 3 with a nut at \(\rho = 0\) is given by

\[
a = \alpha \rho + O(\rho^2), \quad b = \alpha \rho + O(\rho^2), \quad c = 2\alpha^2 \rho + O(\rho^2),
\]

where \(\alpha > 0\) is a constant. Furthermore, the leading term

\[
h = d^2 + \rho^2 \left( \alpha^2 \sigma^2_1 + \sigma^2_2 \right) + 4\alpha^4 \sigma^2_3,
\]

is an exact Kähler metric and is the most general Kähler cone with \(SU(2)\)-symmetry.

The metric is smooth at the nut if and only if \(a = 1/2\) and the higher order terms in \(a^2, b^2, c^2\) are smooth functions of \(\rho^2\), in which case the Kähler form extends smoothly at the nut. In this case the space around the nut is diffeomorphic to \(\mathbb{R}^4\) (the cone is the flat metric on \(\mathbb{R}^4\)).

**Proof.** We need to show that metric functions \(a, b, c\) vanish precisely as \(O(\rho)\) at a nut at \(\rho = 0\). Indeed, let the functions vanish as

\[
a \sim \rho^{n_a} \tilde{a}(\rho), \quad b \sim \rho^{n_b} \tilde{b}(\rho), \quad c \sim \rho^{n_c} \tilde{c}(\rho)
\]

where \(n_a, n_b, n_c > 0\) are positive integers, and \(\tilde{a}(\rho), \tilde{b}(\rho), \tilde{c}(\rho)\) are \(C^1(\rho)\) and non-vanishing at zero. Comparing the orders, equation (2.39) then gives

\[
n_c = n_b + n_a - 1,
\]

and the equation (2.51)

\[
n_c + n_b - n_a - 1 = 2 \min\{0, n_b - n_a\}.
\]

If \(n_b \geq n_a\), i.e. \(b\) vanishes faster than \(a\), the system (2.59) and (2.60) has the solution

\[
0 < n_c = n_a \leq n_b = 1.
\]

Since \(n_c, n_a\) are positive, we have \(n_a = n_b = n_c = 1\), i.e. \(a, b, c\) vanish as \(O(\rho)\). Similarly, if \(n_b < n_a\), i.e. \(a\) vanishes faster than \(b\), one finds

\[
0 < n_c = n_b < n_a = 1,
\]

which is a contradiction, since \(n_a, n_b, n_c\) are integers. The behaviour at the leading order (2.56) is most conveniently found from the ODE system in the form of (2.33), (2.34).

It is worth noting that the Kähler cone geometry in the above lemma is precisely the Kähler base of the near-horizon geometry of the Gutowski-Reall black hole (if \(a > 1/2\) and therefore it satisfies (2.9) (in fact \(\Xi = 0\)). We may now combine this lemma with Lemma 3 to deduce the following symmetry enhancement result.

**Proposition 1.** Consider a Kähler metric with cohomogeneity-1 \(SU(2)\) symmetry as in Theorem 3 smooth at a nut or bolt \(I\) at \(\rho = 0\) such that \(a, b, c > 0\) for \(\rho > 0\). Then \(a = b\) for all \(\rho > 0\), i.e. the Kähler metric has \(U(1) \times SU(2)\) symmetry.
Proof. Near a nut, a substitution of (2.56) into (2.51) immediately implies that the order \( \rho \) vanishes, and
\[
T = 1 + O(\rho^2) ,
\]
where smoothness dictates the higher order terms. On the other hand, smoothness at a bolt I requires
\[
a = a_0 + O(\rho^2), \quad b = a_0 + O(\rho^2), \quad c = c_0(1 + O(\rho^2)) ,
\]
where \( a_0, c_0 > 0 \) and the subleading terms are smooth in \( \rho^2 \) (so the space is diffeomorphic to \( \mathbb{R}^2 \times S^2 \)). Thus (2.64) again gives (2.63). Therefore by Lemma 3 we deduce that in both cases \( T = 1 \) for \( \rho > 0 \). \qed

Observe that only a nut and bolt I are compatible with an enhanced \( U(1) \times SU(2) \) symmetry, therefore for bolt II one cannot obtain such a result. It is worth noting that in the case of bolt I the Kähler form extends smoothly to the bolt and corresponds to the complex structure with respect to which the bolt is a complex submanifold. In contrast, a bolt II does not correspond to a complex submanifold. This provides an another distinguishing property between the two types of bolt and we will therefore sometimes refer to bolt I as a complex bolt.

In terms of Euler coordinates \((\theta, \phi, \psi)\) on \( S^3 \) (see Appendix 2.A), the metric near a bolt I looks like
\[
h \sim d\rho^2 + c_0^2 \rho^2 (d\psi + \cos \theta d\phi)^2 + a_0^2 (d\theta^2 + \sin^2 \theta d\phi^2) .
\]
Therefore, the absence of the conical singularity at \( \rho = 0 \) requires \( \psi \) to be identified with period \( 2\pi/c_0 \). Then for a fixed small \( \rho \) the space near the bolt is diffeomorphic to \( S^1 \) with coordinate \( (\psi) \) fibered over an \( S^2 \) bolt with coordinates \((\theta, \phi)\). Regularity of this Hopf bundle requires \( \psi \) to be identified with period \( 4\pi/p \) where \( p \in \mathbb{N} \). Combining these regularity conditions we deduce that
\[
2c_0 = p \in \mathbb{N}
\]
is required by smoothness near a bolt I.

It will be useful to consider a Kähler metric with \( U(1) \times SU(2) \) symmetry in the chart (2.24). It is clear that it is a smooth Riemannian metric for \( r > 0 \) provided \( V(r) > 0 \) is smooth. The metric has potential singularities if \( r = 0 \) or \( V(r) = 0 \). The former corresponds to a nut and smoothness of the metric requires \( V(r) > 0 \) for all \( r > 0 \), \( V(0) = 1 \) and \( V(r) \) is a smooth function of \( r \). The latter corresponds to a bolt (bolt I in the above) if \( V(r) > 0 \) for \( r > r_0 \) where \( V(r_0) = 0 \) and smoothness requires \( V'(r_0) = 0 \) with \( V(r) \) a smooth function of \( r - r_0 \); furthermore absence of the conical singularity in the \((r, \psi)\) part of the metric implies \( \psi \) must be identified with period \( (8\pi) / (r_0 V'(r_0)) \) and regularity of the resulting \( \mathbb{R}^2 \)-bundle over the bolt implies
\[
\frac{1}{2} r_0 V'(r_0) = p \in \mathbb{N} .
\]
This latter condition is simply (2.66) written in the chart (2.24).

It would be interesting to perform a systematic global analysis of Kähler metrics with \( SU(2) \) symmetry. In the negative Einstein\(^{10}\) case it was found that there are two families of complete metrics both containing bolts: the \( U(1) \times SU(2) \) symmetric solution (2.81) (also see Appendix 2.B) and a triaxial diagonal metric \cite{91} (these possess a bolt I and bolt II respectively).

\(^8\)This comes from the absence of conical singularity at \( r = 0 \), and can be seen as follows. Introduce a new radial coordinate \( dR = V^{1/2}(r) \) \( dr \), \( R(r = 0) = 0 \), which is a smooth invertible function of \( r \) as \( V(r) > 0 \). Then the chart (2.24) takes the form
\[
h = dR^2 + \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2 + V(r) \sigma_3^2) .
\]
This is a conversion of a metric to a polar-like coordinates. By looking at \( \sigma_3^2 \) term, we then deduce that \( V(r(R)) \) is a smooth function of \( R^2 \) and, hence, of \( r^2 \). This is because \( R^2 = x^2 + y^2 + z^2 + w^2 \) is a smooth coordinate at \( R = 0 \) on \( \mathbb{R}^4 \), where \( x, y, z, w \) are cartesian coordinates, while \( R \) is not smooth due to the square root.

\(^9\)This is seen similarly to the previous footnote. Writing the chart (2.24) as
\[
h = V^{-1}(r) (d(r - r_0)^2 + V(r) \sigma_3^2) + \frac{r^2}{4} (\sigma_1^2 + \sigma_2^2) ,
\]
we deduce that \( V(r) \) must be a smooth function of \( r - r_0 \).

\(^{10}\)Einstein geometry with a negative cosmological constant.
It would be interesting to perform a similar analysis for Kähler metrics satisfying other curvature conditions, such as extremal Kähler metrics or metrics satisfying the supersymmetric condition (2.9). We will not pursue this here, although for some explicit examples see Appendix 2.B.

2.4 Supersymmetric solutions with \( SU(2) \) symmetry

2.4.1 General local solution

In this section we will construct the general timelike supersymmetric solution with \( SU(2) \) symmetry. We will take the Kähler base metric to be the general cohomogeneity-1 Kähler metric with \( SU(2) \) symmetry as given in Theorem 3. The 1-form \( \omega \) takes the \( SU(2) \)-invariant form (2.14).

Firstly, the function \( f \) is determined by the scalar curvature of the Kähler base (2.3) which for our class of bases is given by (2.48). Next, we determine the 1-form \( \omega \) using (2.8) which requires \( G^\pm \). For this we need the Ricci form for the Kähler base which is given by the potential (2.49). Thus, \( G^+ \) is determined using (2.3). For \( G^- \) we write (2.6), where without loss of generality we may take the \( \Omega^i \) to be given by (2.30). Thus by \( SU(2) \)-symmetry the \( \lambda_i \) must be functions only of \( \rho \) (since our \( \Omega^i \) are \( SU(2) \)-invariant). Then we find the integrability condition for (2.8) is equivalent to the pair of ODEs for \( \lambda_i \), for \( i = 1, 2 \),

\[
\lambda_i' = \left( \frac{a^2 + b^2 - c^2 - 2abc}{2abc} \right) \lambda_i ,
\]

(2.68)

together with a complicated 5th order ODE for \( c \) obtained using (2.33), (2.34) to eliminate all derivatives of \( a, b \) (we do not give this as we will not need it). Note that the ODE for \( c \) is in fact equivalent to \( \Xi_4 = 0 \) so the general PDE for the Kähler base (2.9) is satisfied in a more restricted form. Then, upon use of these integrability conditions, we can solve (2.8) and find the general solution,

\[
\omega_1 = -\frac{f^3}{48}cb\lambda_1, \quad \omega_2 = -\frac{f^3}{48}ac\lambda_2
\]

(2.69)

and a complicated expression for \( \omega_3 \) that is 4th order in \( c \) (again we will not need this).

This completes the derivation of the general local solution with \( SU(2) \) symmetry. Note that it depends on five functions \( a, b, c, \lambda_1, \lambda_2 \) subject to the first order ODEs (2.33), (2.34), (2.68) and a complicated 5th order ODE for \( c \). In fact, for solutions with an enhanced symmetry \( U(1) \times SU(2) \) this system simplifies dramatically. We turn to this next.

We will now determine the general supersymmetric solution with \( U(1) \times SU(2) \) symmetry. This can be easily deduced from the general solution with \( SU(2) \) symmetry given above. In order to obtain a Kähler base with this symmetry we must set \( a^2 = b^2 \). To obtain \( \omega \) of this symmetry we must set and \( \omega_1 = \omega_2 = 0 \), which using (2.69) implies \( \lambda_1 = \lambda_2 = 0 \). However, for convenience we will keep \( \omega \) a general \( SU(2) \)-invariant form.

In fact, we find it convenient to use the coordinate \( r(\rho) \) in terms of which the Kähler metric is (2.24). In this coordinate system the Ricci form is given by the potential

\[
P = -\frac{1}{4}(rV' + 4V - 4) \sigma_3 ,
\]

(2.70)

and the scalar curvature by

\[
R = -\frac{8(V - 1) + 7rV' + r^2V''}{r^2} .
\]

(2.71)

One can then repeat the steps in the general case. This gives

\[
f^{-1} = \frac{f^2(8(V - 1) + 7rV' + r^2V'')}{24r^2} ,
\]

(2.72)
The integrability condition for (2.8) is now a 5th order ODE for $V(r)$:

$$
3r^4VV^{(5)} + 6r^4V'V^{(4)} + 30r^3VV^{(4)} + 44r^3V'V^{(3)} + r^2(47V - 32)V^{(3)}
+ 8r^3(V'')^2 - 3r(13V + 32)V'' + 26r^2VV'' - 34rV'^2 + 3V'(13V + 32) = 0
$$

(2.73)

and for $i = 1, 2$

$$
\lambda_i' = -\lambda_i \left( \frac{V'}{2V} + \frac{2(V - 1)}{rV} \right). 
$$

(2.74)

These correspond to the 5th order ODE for $c(\rho)$ and (2.68) in the general solution, respectively. Then, the 1-form $\omega$ is completely determined to be

$$
\omega_3 = \frac{\ell^3}{2304r^2} \left[ -3r^4V'^2 - 64r^2V'' + 23r^2V'^2 + V' \left( 6r^4V^{(3)} + 28r^3V'' - 320r \right) 
+ 2V \left( 3r^4V^{(4)} + 30r^3V^{(3)} + 77r^2V'' + 115rV' - 192 \right) + 192V^2 + 192 \right],
$$

(2.75)

$$
\omega_i = -\frac{\ell^3}{192} r^2 \sqrt{V} \lambda_i, \quad i = 1, 2.
$$

(2.76)

This completes the derivation of the solution in this case.

To summarise, the general local form of a supersymmetric solution with a $SU(2)$ symmetric base is determined by a single function $V(r)$ which obeys the 5th order ODE (2.73). The rest of the data is uniquely fixed in terms of $V(r)$ as shown above. To obtain the general solution with $U(1) \times SU(2)$ one further requires $\omega$ to possess this symmetry which implies $\lambda_1 = \lambda_2 = 0$.

### 2.4.2 Examples: black holes and solitons

In this section we record all the known supersymmetric solutions with $SU(2)$ symmetry as above. In fact they all have Kähler bases with $U(1) \times SU(2)$ symmetry and unless otherwise stated the full solution also has this enhanced symmetry (i.e. $\lambda_1 = \lambda_2 = 0$). The solutions fall into two classes: black holes and solitons.

The Gutowski-Reall black hole is simply determined by [68]

$$
V = 4\alpha^2 + \frac{r^2}{\ell^2},
$$

(2.77)

where $\alpha > 1/2$ is a constant, which gives

$$
f = \frac{3\ell^2}{\ell^2(4\alpha^2 - 1) + 3r^2}, \quad \omega_3 = \frac{(4\alpha^2 - 1)^2\ell^3}{12r^2} + \frac{1}{2} (4\alpha^2 - 1)\ell + \frac{r^2}{2\ell}.
$$

(2.78)

For $\alpha = 1/2$ this solution reduces to global AdS, given by (2.27), (2.28). On the other hand, the near-horizon geometry of the Gutowski-Reall black hole [also a solution] is simply given by

$$
V = 4\alpha^2, \quad f = \frac{3r^2}{\ell^2(4\alpha^2 - 1)}, \quad \omega_3 = \frac{(4\alpha^2 - 1)^2\ell^3}{12r^2},
$$

(2.79)

where $\alpha > 1/2$. Both of these represent spacetimes with smooth horizons at $r = 0$ in line with the fact that $f = O(r^2)$ as $r \to 0$.

Recently, another family of black hole solutions in this symmetry class have been constructed numerically, which are asymptotically locally AdS with conformal boundary that is spatially a homogeneously squashed $S^3$ [77, 78, 89, 90]. We will comment further on these solutions at the end of section 2.5.

We now turn to soliton solutions. In particular, if $f > 0$ globally, then the Kähler base must be a smooth Kähler surface and $\omega$ is a smooth 1-form on this surface. The corresponding supersymmetric solution is then a strictly stationary soliton spacetime. It is possible that there

[11] Curiously, (2.73) can be further simplified to a second order ODE (see Appendix 2.D). Unfortunately this does not seem to help our later analysis.
are soliton spacetimes with an ergosurface $f = 0$, although we are not aware of any in this symmetry class. For simplicity we will only consider strictly timelike supersymmetric solitons. For these solutions, note that the Gram matrix $G$ of the Killing vectors $(V, R_i)$ with respect to the spacetime metric, which satisfies

$$\det G = -f^{-1}a^2b^2c^2,$$

(2.80)

is invertible if and only if $abc \neq 0$. Therefore, one can say that the spacetime has a nut or bolt if and only if the Kähler base has a nut or bolt as in Lemma 4. By applying Proposition 1 we may deduce the following general result for such solutions.

**Proposition 2.** Any strictly timelike supersymmetric soliton solution with $SU(2)$ symmetry containing a nut or a bolt $I$ must have a Kähler base with $U(1) \times SU(2)$ symmetry.

The only known non-trivial soliton with a nut has been constructed numerically and is asymptotically locally AdS$_5$ with squashed $S^3$ spatial boundary geometry [79] (this is given by the branch B solutions defined by (2.128) and (2.120) discussed as the end of section 2.5). It is natural to wonder whether there are soliton solutions with a bolt $I$, which by the above result must have enhanced $U(1) \times SU(2)$ symmetry. We may construct such solutions as follows.

First, consider the most general Kähler-Einstein metric with $U(1) \times SU(2)$ symmetry, which is given by

$$V = 1 + \frac{r^2}{\ell^2} + \frac{c_4}{r^4},$$

(2.81)

where $c_4$ is a constant. This is a generalisation of the Eguchi-Hanson metric (obtained by setting $\ell \to \infty$), which reduces so the Bergmann metric for $c_4 = 0$. It is easily checked that it is a solution to (2.73), or simply recall that as noted earlier any Einstein base gives a supersymmetric solution. The rest of the data $(f, \omega)$ is identical to that for AdS$_5$ (2.28). While this gives a complete Kähler metric with a bolt $I$ (for $c_4 < 0$), unfortunately, the resulting supersymmetric solution is either singular or has CTC since $\omega_3$ cannot vanish at the bolt (see the Appendix 2.B).

A more general class of solutions can be constructed as follows. Inspired by the Gutowski-Reall and Kähler-Einstein bases, consider the ansatz

$$V = c_0 + \frac{r^2}{\ell^2} + \frac{c_2}{r^2} + \frac{c_4}{r^4},$$

(2.82)

where $c_0, c_2, c_4$ are constants. We find that this is a solution to (2.73) iff

$$c_2^2 = 3(c_0 - 1)c_4.$$  

(2.83)

The special case $c_4 = 0$ gives the Gutowski-Reall solution (if $c_0 > 1$), whereas special case $c_0 = 1$ gives the Kähler-Einstein solution. This class also contains complete Kähler metrics with a bolt $I$ (see Appendix 2.B). Interestingly, in this case we find that these bases do give smooth soliton solutions. Recall, smoothness of the Kähler base requires the existence of a constant $r_0 > 0$ such that $V(r_0) = 0$ and $V(r) > 0$ for $r > r_0$ and (2.67) for some $p \in \mathbb{N}$. Furthermore, smoothness of the spacetime requires $f > 0$ for $r \geq r_0$ and $\omega_3(r_0) = 0$. We find that solutions do exist but only for $p \geq 3$, see Appendix 2.B for details. In particular, we obtain asymptotically locally AdS$_5$ solitons with $S^3/\mathbb{Z}_p$ spatial boundary and a bolt at $r = r_0$. It turns out that these solutions were previously found as limits of non-supersymmetric solutions [76, section 3.4] (case B), however it is not apparent from their analysis that the Kähler base has $U(1) \times SU(2)$ symmetry (i.e. that supersymmetry commutes with the spacetime $U(1) \times SU(2)$ symmetry)$^{14}$, and furthermore a detailed analysis of the allowed values of $p \in \mathbb{N}$ was not performed.

Of course, it would be interesting to determine all soliton solutions with a nut or bolt $I$. This would require classifying all solutions to (2.73) that are compatible with such boundary

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$^{12}$See section 1.3.2 and Appendix 3.D for a discussion of ergosurfaces.

$^{13}$In fact, this solution was also noticed in [72].

$^{14}$Indeed, there is an example of a supersymmetric soliton with $U(1) \times SU(2)$ symmetry [76, section 3.3] (case A), for which the supersymmetric Killing field $V$ does not commute with the $SU(2)$ symmetry [72]. As a result the Kähler base of this solution only has $U(1)^2$ symmetry.
conditions, which we will not pursue here. It would also be interesting to investigate the existence of solitons with a bolt II; such solutions may have only generic $SU(2)$ symmetry and we are not aware of any examples in this class.

Finally, it is worth noting that although the only known explicit solutions have a Kähler base with $U(1) \times SU(2)$ symmetry, one can easily construct supersymmetric solutions with exactly $SU(2)$ symmetry from such Kähler bases by taking $\omega$ to be a general $SU(2)$ invariant 1-form. This amounts to taking $\lambda_1, \lambda_2 \neq 0$ above and solving (2.74). In particular, for the Gutowski-Reall base one finds the following deformation of the Gutowski-Reall black hole,

$$\omega_i = -c_i \ell^2 \left( r^2 \frac{r^2}{4\alpha^2 + r^2} \right)^{\frac{1}{2r}} ,$$  

for $i = 1, 2$ where $c_i$ are constants, with the rest of the data being identical. Even though $\omega_{1.2} \rightarrow 0$ as $r \rightarrow 0$, the results of section 2.5 show that presence of such terms is incompatible with a regular horizon at $r = 0$. However, for $\alpha = 1/2$ these give smooth deformations of AdS$_5$ studied in [92]. On the other hand, for any soliton with a bolt I at $r = r_0$ as in Proposition 2, one finds that as $r \rightarrow r_0$,

$$\omega_i = c_i (r - r_0)^{1/p} (1 + O(r - r_0)) ,$$

for $i = 1, 2$, where we have used (2.67). Therefore, for the asymptotically AdS$_5$/Z$^p$ solitons discussed above which must have $p \geq 3$, we deduce that these do not give smooth deformations.

## 2.5 Black hole solutions with $SU(2)$ symmetry

We now consider the constraints imposed by having a regular event horizon. First, recall that any Killing vector field of a spacetime $(M, g)$ must be tangent to the event horizon. This implies that any Killing field restricted to the event horizon is null (and tangent to the generators of the horizon) or spacelike. Therefore, since for a supersymmetric solution $|V|^2 = -f^2 \leq 0$ this means that the supersymmetric Killing field must be null on the horizon, i.e. the event horizon is a Killing horizon of $V$. Furthermore, since $|V|^2$ is at a global maximum at the horizon we must also have that $d|V|^2 = 0$ on the horizon, i.e. it is an extremal horizon.

Next, we assume that $(M, g)$ has a $G$-symmetry, where $G$ is $SU(2)$, as defined above Lemma 2. Therefore, the Killing fields $R_i$ that generate $SU(2)$ must be tangent to the horizon. Now, since we also assume the orbits are spacelike, we can always choose a cross-section of the horizon tangent to $R_i$, i.e., a $G$-invariant cross-section. Furthermore, since we assume $G$ has 3d orbits the cross-sections are homogeneous spaces locally isometric to $SU(2)$. Thus, the only possible horizon topologies are $S^3$ and lens spaces.

### 2.5.1 Near-horizon analysis

We will now analyse constraints on the geometry arising from a regular event horizon. In particular we assume the event horizon is a smooth (or analytic) null hypersurface with a smooth cross-section. We can write the metric near the horizon in Gaussian null coordinates (see e.g. [66]), which as argued above can be adapted to an $SU(2)$-invariant cross-section. Thus, the general $SU(2)$-invariant metric near a horizon takes the form,

$$g = -\lambda^2 \Delta^2 dv^2 + 2d\lambda d\lambda + 2\lambda h_i \sigma_i dv + \gamma_{ij} \delta_i \delta_j ,$$

where $\partial_\lambda$ are null geodesics transverse to the horizon synchronised so the horizon is at $\lambda = 0$, the supersymmetric Killing field $V = \partial_\lambda$, the $\sigma_i$ are right-invariant 1-forms that satisfy (2.12), and the data $\Delta, h_i, \gamma_{ij}$ are smooth (or analytic) functions of only $\lambda$ and $\gamma_{ij}$ is a positive-definite matrix. Note that outside the horizon, $\lambda > 0$, we assume $\Delta$ is non-zero as required for solutions in the timelike class. The near-horizon geometry is given by the scaling $(v, \lambda) \rightarrow (v/\epsilon, \epsilon \lambda)$ and the limit $\epsilon \rightarrow 0$, in which case the metric still takes the form (2.86) where the data $\Delta, h_i, \gamma_{ij}$

---

15 This easily follows from the general formula $\omega'_i/\omega_i = 2/(rV)$ which is a consequence of (2.74) and (2.76).
are replaced by their values at \( \lambda = 0 \) which we denote by \( \Delta, \hat{h}_i, \gamma_{ij} \). In particular, observe that the near-horizon data \( \Delta, \hat{h}_i, \gamma_{ij} \) are all constants.

In fact supersymmetric near-horizon geometries with compact horizon cross-sections have been completely classified in this theory \([44, 45, 68]\). It turns out that the only solution with \( SU(2) \)-symmetry is given by the original Gutowski-Reall near-horizon geometry, which can be written in the form \([68]\),

\[
\hat{h}_1 = \hat{h}_2 = 0, \quad \hat{h}_3 = -\frac{3\Delta}{\ell(\Delta^2 - 3\ell^2)} \nonumber \]

\[
\gamma_{11} = \gamma_{22} = \frac{1}{\Delta^2 - 3\ell^2}, \quad \gamma_{33} = \frac{\Delta^2}{(\Delta^2 - 3\ell^2)^2}, \tag{2.87} \]

where \( \gamma_{ij} \) is diagonal and \( \Delta > \sqrt{3}/\ell \). We emphasise that the near-horizon geometry has enhanced \( U(1) \times SU(2) \) symmetry. This will be important below.

We now compare certain invariants to those for a general timelike supersymmetric solution. Comparison of the scalar \( |V|^2 \) implies

\[
f = \lambda\Delta, \tag{2.88} \]

where we have chosen a sign so that \( \Delta \geq 0 \). Furthermore, the metric \( h \) on the orthogonal base (which is invariantly defined wherever \( V \) is timelike) can be extracted to give

\[
h = \frac{1}{\Delta^2}(d\lambda + \lambda h_i \hat{\sigma}_i)^2 + \lambda\Delta \gamma_{ij} \hat{\sigma}_i \hat{\sigma}_j 
= \left( \frac{\Delta}{\Delta^2 - h_i h_i} \right) \frac{d\lambda^2}{\lambda} + \lambda \Delta q_{ij} \left( \hat{\sigma}_i + \frac{k^i d\lambda}{\lambda} \right) \left( \hat{\sigma}_j + \frac{k^j d\lambda}{\lambda} \right), \tag{2.89} \]

for \( \lambda > 0 \), where for convenience we introduce \( h^i := \gamma^{ij} h_j \),

\[
q_{ij} := \gamma_{ij} + \frac{h_i h_j}{\Delta^2}, \quad k^i := \frac{h^i}{\Delta^2 + h_i h_i}. \tag{2.90} \]

Note these obey \( q_{ij} k^j = h_i / \Delta^2 \). Finally, we can also extract the 1-form \( \omega \), which gives

\[
\omega = -\frac{h_i \hat{\sigma}_i}{\lambda\Delta^2}, \tag{2.91} \]

where without loss of generality we have fixed the \( \lambda \) component to zero since the \( SU(2) \) symmetry implies it is pure gauge (here we using the gauge freedom in the definition of \( \omega \)). We may now establish the following converse to Lemma 2 for solutions with horizons which will be useful later.

**Lemma 6.** Consider a supersymmetric solution with \( SU(2) \) symmetry that is timelike outside a smooth horizon. If the Kähler base has \( U(1) \times SU(2) \) symmetry then the spacetime metric also has \( U(1) \times SU(2) \) symmetry, i.e., one can take \( \omega \) to be \( U(1) \times SU(2) \) invariant.

**Proof.** Suppose the Kähler base metric \( h \) has \( U(1) \times SU(2) \) symmetry, i.e. assume (2.89) satisfies \( \mathcal{L}_{\hat{L}} h = 0 \) where \( \hat{L}_i \) are the right-invariant vectors dual to \( \hat{\sigma}_i \). Then it easily follows that \( h_1 = h_2 = 0 \) and that \( q_{ij} \) is diagonal with \( q_{11} = q_{22} \). By the definition of \( q_{ij} \) this implies \( \gamma_{ij} \) is diagonal with \( \gamma_{11} = \gamma_{22} \). Thus the spacetime metric (2.86) has \( U(1) \times SU(2) \) symmetry as claimed. In particular, the 1-form \( \omega \) in the gauge (2.91) is \( U(1) \times SU(2) \) invariant. \( \square \)

We now wish to bring the metric (2.89) into the standard form (2.13). To this end, let us define the 1-forms

\[
\sigma_i := A_{ij}(\lambda)(\hat{\sigma}_j + B_j(\lambda)d\lambda), \tag{2.92} \]

where \( A_{ij} \) are the components of an invertible matrix. Then we find

\[
d\sigma_i = -\frac{1}{2} \epsilon_{pjk} A_{ip}(A^{-1})_{jl}(A^{-1})_{km} \sigma_l \wedge \sigma_m + d\lambda \wedge \sigma_m \left(A'_{ij}(A^{-1})_{jm} + A_{ip}(A^{-1})_{km} B_j \epsilon_{pjk} \right), \tag{2.93} \]
and requiring that the terms proportional to $d\lambda$ vanish is equivalent to
\[ A'_{ij} = A_{ik}C_{kj}, \quad \text{where} \quad C_{ij} := \epsilon_{ijk}B_k. \quad (2.94) \]

Now $C_{ij}$ is an antisymmetric matrix, so this equation can always be solved locally for some orthogonal matrix $A_{ij}$. In this case
\[ \epsilon_{pjk}A_{ip}(A^{-1})_{jl}(A^{-1})_{km} = \epsilon_{pjk}(A^{-1})_{pi}(A^{-1})_{jl}(A^{-1})_{km} = \det(A^{-1})\epsilon_{ilm} = \pm\epsilon_{ilm}. \quad (2.95) \]

In particular, if we take $A \in SO(3)$ we deduce that the 1-forms $\sigma_i$ defined by (2.92) satisfy the Maurer-Cartan equations for right-invariant 1-forms (2.12) for $SU(2)$. Given this, we can introduce a new set of Euler coordinates on $SU(2)$ such that $\sigma_i$ are right-invariant 1-forms.

We may use this result to transform (2.89) into the standard form (2.13) as follows. Perform a coordinate transformation such that $(\lambda, \hat{\sigma}_i) \rightarrow (\rho, \sigma_i)$, where $\rho = \rho(\lambda)$ and (2.92) are defined by
\[ \left( \frac{d\rho}{d\lambda} \right)^2 = \left( \frac{\Delta}{\Delta^2 + h_ih^i} \right) \frac{1}{\lambda}, \quad (2.96) \]
\[ B_i = \frac{k^i}{\lambda}, \quad (2.97) \]
where $A_{ij} \in SO(3)$ obeys (2.94). Then we obtain (2.13) where
\[ h_{ij} = \lambda\Delta A_{ik}q_{kl}A^T_{lj}. \quad (2.98) \]
The 1-form $\omega$ in the new coordinates is (gauge equivalent) to (2.14) where
\[ \omega_i = -\frac{A_{ij}h_j}{\lambda\Delta^2}. \quad (2.99) \]

We will now use this to derive the boundary conditions near a horizon.

**Proposition 3.** Consider a timelike supersymmetric and $SU(2)$-symmetric solution to $D = 5$ minimal gauged supergravity containing a smooth horizon with $\Delta > 0$ and an $U(1) \times SU(2)$-invariant near-horizon geometry. The horizon corresponds to a conical singularity in the Kähler base metric $h$, i.e., in the metric (2.13) the horizon can be taken to be at $\rho = 0$ such that
\[ h_{ij} = \rho^2 \hat{h}_{ij} + O(\rho^4) \quad (2.100) \]
as $\rho \rightarrow 0$, where $\hat{h}_{ij}$ is a positive diagonal matrix with $\hat{h}_{11} = \hat{h}_{22}$ fixed by the near-horizon geometry. Furthermore, the 1-form $\omega$ is (2.14) where
\[ \omega_i = \frac{\hat{\omega}_i}{\rho^2} + O(1) \quad (2.101) \]
and $\hat{\omega}_i = \hat{\omega}_3\delta_{i3}$ is a constant determined by the near-horizon geometry. Finally,
\[ f = \rho^2 \hat{f} + O(\rho^4), \quad (2.102) \]
where $\hat{f}$ is a positive constant determined by the near-horizon geometry and the subleading terms are smooth functions of $\rho^2$.

**Proof.** As mentioned above a smooth (or analytic) horizon requires that the data $\Delta, h_{ij}, \gamma_{ij}$ be smooth (or analytic) functions of $\lambda$ at $\lambda = 0$. Then, setting the horizon to be at $\rho = 0$, the ODE (2.96) implies that $\rho^2$ is a smooth (or analytic) function of $\lambda$ at $\lambda = 0$ since by assumption $\Delta > 0$ at $\lambda = 0$. Indeed, explicitly solving (2.96) near $\lambda = 0$ gives the leading order behaviour
\[ \rho^2 = \left( \frac{4\Delta}{\Delta^2 + h_ih^i} \right) \lambda (1 + O(\lambda)). \quad (2.103) \]
Inverting, we also deduce that $\lambda$ is a smooth (or analytic) function of $\rho^2$ at $\rho = 0$. Next we need to determine $A_{ij}$ from (2.94) where $B_i$ is given by (2.97). Clearly, if $k^i = 0$ then $B_i$ is smooth at the horizon so there is a solution $A_{ij}$ that is smooth at the horizon. However, if $k^i \neq 0$ (as is the case for the Gutowski-Reall near-horizon geometry) then $B_i$ is singular at the horizon, which in turn implies that $A_{ij}$ is not $C^1$ at $\lambda = 0$ (indeed, it is not even $C^0$). Nevertheless, since $A_{ij}$ is an orthogonal matrix, its entries are bounded so $A_{ij} = O(1)$ as $\lambda \to 0$, which will be important in bounding the subleading terms in what follows.

In order to isolate the singular behaviour in $A$ define the parameter $\tau := -\log \lambda$ so (2.94) becomes $dA/d\tau = -AK$ where $K_{ij} = \epsilon_{ijn}k^n$ is a smooth function of $\lambda = e^{-\tau}$. Therefore a unique solution exists on $[\tau_0, \infty)$ with a given initial condition at $A(\tau_0)$ for some $\tau_0$ (determined by the range of the coordinate $\lambda$). In order to examine the behaviour of $A$ near the horizon we need to determine its asymptotics as $\tau \to \infty$. For large $\tau$ we have $K = K_0 + O(e^{-\tau})$ where $K_{0ij} = \epsilon_{ijn}k^n$. It follows that

$$\frac{d}{d\tau}(Ae^{\tau K_0}) = -A(K - K_0)e^{\tau K_0} = O(e^{-\tau}),$$

where the final estimate follows from the fact that $A, e^{\tau K_0}$ are orthogonal matrices (so have bounded components) and the aforementioned asymptotics of $K$. Integrating, it follows that\footnote{Integrating over $[\tau, \tau_*]$ for $\tau > \tau_0$, gives $A(\tau_i)e^{\tau K_0} - A(\tau)\epsilon^{\tau K_0} = -\int_{\tau}^{\tau_*} A(K - K_0)\epsilon^{\tau K_0}d\tau$ and since the integrand is $O(e^{-\tau})$ the limit $\tau_* \to \infty$ exists, so $A_0 - A(\tau)\epsilon^{\tau K_0} = -\int_{\tau}^{\infty} A(K - K_0)\epsilon^{\tau K_0}d\tau$ where $A_0 := \lim_{\tau \to \infty} A(\tau)\epsilon^{\tau K_0}$. Then, use again that the integrand is $O(e^{-\tau})$ to deduce $A_0 - A(\tau)\epsilon^{\tau K_0} = O(e^{-\tau})$.}

$$A = A_0 e^{-\tau K_0} + O(e^{-\tau}) = A_0 e^{K_0} \log \lambda + O(\lambda),$$

where $A_0 \in SO(3)$ is an orthogonal matrix which is an integration constant. We will use the freedom of choice to pick $A_0$ to be a rotation matrix which preserves $\sigma_3$ direction.

We now wish to expand the metric data near the horizon. For this we will need to use the assumptions that the near-horizon geometry has $\Delta > 0$ and enhanced $U(1) \times SU(2)$ symmetry. Crucially, this means $\gamma_{ij}$ is diagonal with $\gamma_{11} = \gamma_{22}$ and $h_1 = h_2 = 0$ and hence $\dot{q}_{ij}$ is diagonal with $\dot{q}_{11} = \dot{q}_{22}$. Now, expanding (2.98) we find

$$h_{ij} = \lambda \dot{\Delta} \dot{q}_{ij} + O(\lambda^2),$$

where we have used the fact that $e^{-\tau K_0} \dot{q} e^{\tau K_0} = \dot{q}$,\footnote{This follows from the $U(1) \times SU(2)$-invariance of $\dot{q}$ and that $\dot{k}$ has components only in third direction.} and it should be noted that the subleading $O(\lambda^2)$ terms are not necessarily smooth at $\lambda = 0$ in general (due to the factors of $e^{\tau K_0}$). Thus, using (2.103) we obtain (2.100) with

$$\dot{h}_{ij} = \frac{1}{4}(\dot{\Delta}^2 + \dot{h}_i \dot{h}^i)\dot{q}_{ij}.$$  

(2.107)

Similarly, expanding (2.99) we obtain

$$\omega_i = -\frac{\dot{h}_i}{\lambda \dot{\Delta}^2} + O(1),$$

(2.108)

where we have used that $(e^{-\tau K_0})_i \dot{h}_j = \dot{h}_i$, and again the $O(1)$ terms are not generally smooth at the horizon, so we find (2.101) where

$$\ddot{\omega}_i = -\frac{4\dot{h}_i}{\dot{\Delta}(\dot{\Delta}^2 + \dot{h}_i \dot{h}^i)}.$$  

(2.109)

Finally, (2.88) implies

$$f = \frac{1}{4}(\dot{\Delta}^2 + \dot{h}_i \dot{h}^i)\rho^2 + O(\rho^4),$$  

(2.110)

where now the higher order terms are smooth functions of $\rho^2$. This completes the proof.
Now, as mentioned above the explicit form of near-horizon geometry must be given by the Gutowski-Reall near-horizon geometry (2.87), which in particular satisfies the conditions in the above proposition. From (2.87) the constants appearing in the above can be chosen as
\begin{align}
h_{11} &= h_{22} = \alpha^2, \quad h_{33} = 4\alpha^4, \quad \ell^2 = \frac{12\alpha^2}{(4\alpha^2 - 1)}, \quad \omega_3 = \frac{\ell^3(4\alpha^2 - 1)^2}{48\alpha^2},
\end{align}
for an appropriate choice of \(A_0\) in the above lemma. In (2.111) we have introduced the constant
\begin{align}
\alpha^2 := \frac{1}{4} \frac{\Delta^2}{\Delta^2 - \frac{3}{2}},
\end{align}
which satisfies \(\alpha > 1/2\).

We will now combine our near-horizon analysis with the general constraints for a timelike supersymmetric solution with \(SU(2)\) symmetry. In particular, this means the metric \(h\) on the orthogonal base must be Kähler with cohomogeneity-1 \(SU(2)\) symmetry. The general form for such a Kähler metric is given in Theorem 3. Therefore, comparing the near-horizon expansion given in Proposition 3 to the Kähler metric in Theorem 3, implies that near a horizon \(\rho = 0\) the functions in the Kähler metric are given by
\begin{align}
a &= \alpha\rho + O(\rho^3), \quad b = \alpha\rho + O(\rho^3), \quad c = 2\alpha^2\rho + O(\rho^3), \quad b_1, c_1, c_2 = O(\rho^3).
\end{align}

These near-horizon expansions will be important in our subsequent analysis. We emphasise that due to the singular behaviour of the coordinate change (2.92) defined by \(A_{ij}\), the subleading terms are not necessarily smooth. The singular behaviour of \(A_{ij}\) at the horizon prevents us from improving this result in the general case. However, if one has an \((U(1) \times SU(2))\) symmetry one can obtain a stronger result which allows us to control the regularity of the subleading terms.

**Proposition 4.** Consider a timelike supersymmetric solution to \(D = 5\) minimal gauged supergravity with \((U(1) \times SU(2))\) symmetry containing a smooth (analytic) horizon with the Gutowski-Reall near-horizon geometry (2.87). The horizon corresponds to a conical singularity in the Kähler base metric \(h\), i.e., in the metric (2.21) the horizon can be taken to be at \(\rho = 0\) such that
\begin{align}
a^2 &= \alpha^2 \rho^2 + O(\rho^4), \quad c^2 = 4\alpha^4 \rho^2 + O(\rho^4),
\end{align}
as \(\rho \to 0\) and the metric components \(a^2, c^2\) are smooth (analytic) functions of \(\rho^2\) at \(\rho = 0\). Furthermore, the 1-form \(\omega\) takes the form (2.26) where
\begin{align}
\omega_3 &= \frac{\omega_3}{\rho^2} + O(1)
\end{align}
and \(\rho^2 \omega_3\) is a smooth (analytic) function of \(\rho^2\) at \(\rho = 0\). The function \(f\) is as in Proposition 3 and is a smooth (analytic) function of \(\rho^2\).

Conversely, any timelike supersymmetric solution \((f, \omega, h)\) of this form, has a smooth (analytic) horizon at \(\rho = 0\).

**Proof.** We may specialise the proof of the previous proposition. The \((U(1) \times SU(2))\) symmetry implies that we can write (2.86) where \(\gamma_{ij}\) must be diagonal with \(\gamma_{11} = \gamma_{22}\) and \(h_1 = h_2 = 0\) for all \(\lambda\). Therefore, the coordinate change defined by (2.94) simplifies: since \(B^1 = B^2 = 0\) the matrix \(A_{ij}\) must be of the block diagonal form
\begin{align}
A = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix},
\end{align}
where \(P \in SO(2)\). It then follows from (2.98), (2.99) that for all \(\lambda > 0\),
\begin{align}
h_{ij} &= \lambda \Delta q_{ij}, \quad \omega_i = \frac{h_i}{\lambda \Delta},
\end{align}
so the matrix $A$ has dropped out of these expressions. The result now follows upon use of (2.103) and that $A$ is a smooth (analytic) function of $\rho^2$, which were shown in Proposition 3. The converse statement is a result of reversing the above steps.

It is convenient to restate this result in the chart (2.24).

**Corollary 1.** Consider a timelike supersymmetric and $U(1) \times SU(2)$-invariant solution with a smooth (analytic) horizon. The horizon corresponds to a conical singularity at $r = 0$ in the Kähler base (2.24), and $V(r) > 0$, $r^{-2}f > 0$, $r^2 \omega_3$ are all smooth (analytic) functions of $r^2$ at $r = 0$. Conversely, any solution of this form has a smooth (analytic) horizon at $r = 0$.

This simply follows from the coordinate change $r^2 = 4a(\rho)^2$ together with the form of $a^2$ in Proposition 4, which imply that $4a^2(\rho)^2 = r^2(1 + O(r^2))$ where the higher order terms are smooth (analytic) functions of $r^2$.

### 2.5.2 Uniqueness theorem

In this section we will prove our main result which is Theorem 2 stated in the Introduction. First we will prove the following symmetry enhancement result.

**Proposition 5.** Any timelike supersymmetric and $SU(2)$-invariant solution with $U(1) \times SU(2)$-symmetric Kähler form containing a smooth horizon with compact cross-sections must have $U(1) \times SU(2)$ symmetry.

**Proof.** The near horizon behaviour derived in Proposition 3 in particular implies (2.114) and hence the function (2.50) satisfies

$$T = 1 + O(\rho^2),$$

(2.119)
near a horizon $\rho = 0$. For $\rho > 0$ the solution is timelike so we may assume the functions $a, b, c > 0$. Therefore, Lemma 3 implies $T = 1$ for all $\rho > 0$, i.e. $a = b$ for $\rho > 0$. Therefore the Kähler base must have $U(1) \times SU(2)$ symmetry. Finally, by Lemma 6 we deduce that the spacetime metric must also have $U(1) \times SU(2)$ symmetry.

We are now ready to prove the black hole uniqueness theorem.

**Proof of Theorem 2.** Proposition 5 shows that the solution must have $U(1) \times SU(2)$ symmetry. In the coordinate system (2.24) the local classification of solutions in this symmetry class in given in section 2.4.1, in terms of a single function $V(r)$ which obeys a 5th order ODE (2.73). Furthermore, the conditions for such solutions to possess an analytic horizon are given in Proposition 4 and Corollary 1. In particular, an analytic horizon corresponds to $r = 0$ where $V(r)$ is a positive analytic function of $r^2$, and $V(0) > 0$.

In fact, for the sake of generality we will assume $V(r)$ is an analytic function of $r$, so

$$V(r) = \sum_{n=0}^{\infty} \frac{V_n r^n}{n!}.$$  \hspace{1cm} (2.120)

We can derive constraints on the first few coefficients by expanding each term in (2.73). We find that the vanishing of the $r^1$ term implies $V_1 = 0$ which then also implies the $r^0$ terms vanish. The $r^2$ terms then give

$$V_3 \left( V_0 - \frac{32}{11} \right) = 0$$

(2.121)

and the $r^3$ terms give

$$V_4 (V_0 - 1) = 0.$$  \hspace{1cm} (2.122)

In general, this will lead to different branches of solutions. However, for black hole boundary conditions we can derive more constraints.

First, by analyticity, the expansion for $f$ can be obtained from (2.72), which gives

$$f^{-1} = \frac{(V_0 - 1)f^2}{3r^2} + O(1).$$  \hspace{1cm} (2.123)
Using (2.72) this fixes $r$.

Therefore, by Corollary 1, which requires $r^{-2} f$ is positive for $r \geq 0$, we deduce $V_0 > 1$ (note this also ensures $f > 0$ for small $r > 0$). Furthermore by Corollary 1, an analytic horizon requires $V(r)$ to be an analytic function of $r^2$, so that in particular we must have $V_3 = 0$. Therefore the constraints (2.121), (2.122) on the coefficients are satisfied provided $V_4 = 0$ in which case (2.73) is satisfied up to order $r^3$.

We will now prove that $V_0 > 1$ and $V_2 = 0$ implies $V_4 = 0$ for all $n \geq 4$. We proceed by induction and have already verified the base case $n = 4$. Thus our induction hypothesis is

$$V(r) = V_0 + \frac{V_2 r^2}{2} + \frac{r^n V_n}{n!} + O(r^{n+1}),$$

for some $n \geq 4$, where the higher order terms are analytic, and we wish to prove this implies $V_n = 0$. Substituting this into (2.73) we find

$$\frac{[3V_0(n^2 - 16) + 32(V_0 - 1)](n^2 - 4)V_n r^{n-1}}{(n - 1)!} + O(r^n) = 0.$$  

(2.125)

Since $V_0 > 1$ guarantees the factor in the square brackets is positive for all $n \geq 4$, this implies $V_n = 0$ as required. Therefore by induction this shows that $V_n = 0$ for all $n \geq 4$ as claimed.

Thus we have shown that the only analytic solution to (2.73) with $V_0 > 1$ and $V_3 = 0$ is given by

$$V(r) = V_0 + \frac{V_2}{2} r^2.$$  

(2.126)

Using (2.72) this fixes

$$f = \frac{6r^2}{\ell^2 [2(V_0 - 1) + 3r^2 V_2]}.$$  

(2.127)

If $V_2 = 0$ this is the near-horizon geometry of the Gutowski-Reall black hole with $V_0 = 4a^2$ (2.87). If $V_2 \neq 0$, then since the invariant $f$ must be smooth for $r > 0$ we must have $V_2 > 0$.

This corresponds to the Gutowski-Reall black hole with $V_0 = 4a^2$ and $V_2 = 2/\ell^2$ (2.77). □

For this result we needed to assume a stronger regularity property, namely that the metric at the horizon is analytic. It is worth emphasising that there are other solutions to (2.73) that are analytic at $r = 0$. In particular, the above proof shows two other branches of solutions defined by (recall in all cases $V_1 = 0$),

$$\text{Branch B : } V_0 = 1, \quad V_3 = 0$$  

(2.128)

$$\text{Branch C : } V_0 = 32/11, \quad V_4 = 0.$$  

(2.129)

It can be shown that branch B is uniquely determined in terms of $V_2$ and is an even function of $r$. It has been constructed numerically and corresponds to a smooth soliton with a nut at $r = 0$ which is asymptotically locally $\text{AdS}_5$ with a squashed $S^3$ at infinity [79].

On the other hand, branch C is determined uniquely in terms of $V_2$, $V_3$. It has also been constructed numerically, and corresponds to a black hole solution that is also asymptotically locally $\text{AdS}_5$ with a squashed $S^3$ at infinity [77, 78]. However, since in this case $V(r)$ contains odd powers of $r$, by Corollary 1 it does not have a smooth horizon. In particular the function $V$ is $C^1$ but not $C^2$ as a function of $\lambda$ at the horizon, which implies the metric is $C^1$ but not $C^2$ at the horizon. To see this we calculate the invariant

$$g_{33} = f^{-1} h_{33} - f^2 \omega_3^2 = \ell^2 \left( \frac{455}{1936} + \frac{15 V_2 r^2}{176} - \frac{2275 V_4 r^3}{12672} \right) + O(r^4),$$  

(2.130)

and since $r = 2a(\rho) = \sqrt{V_0} \rho (1 + O(\rho^2))$ the expansion in terms of $\rho$ takes the same form (up to unimportant numerical factors). Therefore using the coordinate change to Gaussian null coordinates (2.103) we deduce that for any $V_3 \neq 0$ the invariant $g_{33} := g(L_3, L_3)$ as a function of $\lambda$ is $C^1$ but not $C^2$, as claimed. An analogous class of black hole solutions with squashed $S^3$ boundary have been constructed in the more general $U(1)^3$ gauged supergravity [89,90]. In fact, Propositions 3 and 5 still apply in this more general theory (in particular, the near-horizon
geometry has enhanced $U(1) \times SU(2)$ symmetry), so our results also show that these solutions are $C^1$ but not $C^2$ at the horizon. It should be noted that to rigorously prove that branch B and C solutions actually exist, one would have to prove that the series defined by (2.120) converges. Of course, one would like to relax the assumption of analyticity. It is natural to expect that the same result should be valid for smooth horizons. While parts of our arguments, in particular Proposition 5, only require smoothness, our proof that (2.126) is the only solution to (2.73) with the required boundary conditions uses analyticity in an essential way. The status of this assumption therefore remains unclear.

Furthermore, in Appendix 2.D we show that (2.73) can be reduced to a second order ODE (2.197) which is quadratic in its first derivative. Since $U(1) \times SU(2)$-symmetric Kähler geometry is a highly symmetric setting, some kind of integrability is expected. On the other hand, the ODEs coming from integrable systems frequently possess the Painlevé property. It is interesting to see if (2.197) is of this type as well. If it is the case, a full solution will be known in terms of Painlevé transcendents or quadratures, which will finalise the classification.

### 2.6 Discussion

In this chapter we have started the classification of timelike supersymmetric solutions to five-dimensional minimal gauge supergravity by determining the general form of a solution under the additional assumption of an $SU(2)$-symmetry with 3d orbits and extra symmetry for the Killing spinor bilinears. Namely, the latter is assumed $SU(2)$-invariant, as well the components, arranged into the Kähler form, are assumed to have enhanced $U(1) \times SU(2)$ symmetry. This class is governed by a system of ODEs which in general do not appear integrable. Nonetheless, under certain boundary conditions we found that the full $U(1) \times SU(2)$ symmetry is restored. This is a much simpler class that is governed by a non-linear ODE of a single function which can be reduced to 2nd order.

We have also performed a near-horizon analysis for supersymmetric black holes with such an $SU(2)$ symmetry. This shows that a horizon manifests itself as a conical singularity in the Kähler base space (this also occurs for extremal black holes in flat-space [95,96] as well as for black holes with toric symmetry, see chapter 3). Crucially, the boundary conditions imposed by near-horizon geometry are sufficient for the symmetry enhancement result so the solution must have $U(1) \times SU(2)$ symmetry. Therefore, the classification of black holes simplifies to the above-mentioned ODE which we have resolved analytically, showing that the Gutowski-Reall black hole is the unique solution.

We emphasise that our symmetry assumptions are compatible with black holes with $S^3$ or lens space topology only. On the other hand, the only global assumption we make is that the supersymmetric Killing field is timelike outside the horizon, so, in particular, our result applies equally to asymptotically globally AdS$_5$ and locally AdS$_5$ spacetimes. We have therefore shown that within this symmetry class the Gutowski-Reall black hole is the only solution that is asymptotically globally AdS$_5$, so, in particular, we deduce that there are no other solutions (either connected or disconnected to the Gutowski-Reall solution). Furthermore, we have also shown that there are no regular black hole solutions in this symmetry class that are asymptotically locally AdS$_5$ (at least assuming the metric at the horizon is analytic).

It is worth placing these results in the more general context of black hole classification. In higher-dimensions the rigidity theorem implies that any non-extremal stationary rotating black hole solution, regardless of the sign and value of cosmological constant, must have $U(1)^s$ rotational symmetry where $s \geq 1$ [60,61] under the assumption of analyticity. It is also expected that a generic extremal black hole under similar assumptions possesses $U(1)^s$, $s \geq 1$ rotational symmetry as well [63]. As it was discussed in chapter 1, this theory may admit smooth black holes with a single $U(1)$ symmetry. Now, the $SU(2)$ symmetry generically only contains solutions with a $U(1)$ abelian rotational symmetry. Therefore, our black hole uniqueness theorem (in fact Proposition 5), in this restricted context, implies the existence of a second commuting rotational symmetry, i.e. the abelian symmetry is enhanced to $U(1)^2 \subset U(1) \times SU(2)$. It

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In fact, all $U(1) \times SU(2)$-symmetric Kähler geometries admit a Hamiltonian 2-form in the sense of [93,94].
is an open question if this is an artefact of our realisation of the SU(2) symmetry, or if the enhancement of rotational symmetry is more generic.

The assumption of an SU(2) rotational symmetry is expected to constrain the possible rotational configurations. Indeed, all known $D = 5$ asymptotically flat/AdS black hole (and soliton) solutions with SU(2) symmetry possess equal angular momenta $J_1 = \pm J_2$ with respect to the orthogonal $U(1)^2$ Killing fields at infinity. However, the converse statement need not be true, i.e. $J_1 = \pm J_2$ does not necessarily imply SU(2) symmetry. In fact, for asymptotically flat solutions there are several known supersymmetric black holes with $J_1 = \pm J_2$ that only have $U(1)^2$-symmetry, which explicitly shows that symmetry enhancement based on naive kinematics does not hold [36, 41, 97].

As mentioned in the introduction a major open problem in this context is to also determine the (non)existence of black holes with non-spherical topology. Unfortunately, our strong symmetry assumption means the solutions are cohomogeneity-1, so does not allow us to address this question. In particular, it does not allow for multi-black holes so our work does not address their existence. Furthermore, while the symmetry assumption is compatible with lens space horizons, it does not allow for solutions with lens space horizons that are asymptotically globally AdS$_5$ (i.e. have a $S^3$ at infinity), and therefore we cannot address the existence of black lenses in this context. In order to address these questions one requires an analysis of cohomogeneity-2 solutions which we turn to in the subsequent chapters.

Last but not least our work connects to the mathematical problem of classification of 4d Kähler spaces with SU(2)-symmetry. While we have shown, with an additional assumption on the Kähler form, that any local chart can be written in diagonal form, the question of lifting Kähler spaces with $SU(2)$-symmetry means the solutions are cohomogeneity-1, so does not allow us to address this question. In particular, it does not allow for multi-black holes so our work does not address their existence.

2.A $SU(2)$ calculus

Consider $G = SU(2)$ with the natural left and right $G$-action on $G$. Let $L_i$ be the generators of the left-action and $R_i$ be the generators of the right-action so,

$$[L_i, L_j] = \epsilon_{ijk} L_k, \quad [R_i, R_j] = -\epsilon_{ijk} R_k, \quad [L_i, R_j] = 0, \quad (2.131)$$

for $i, j, k = 1, 2, 3$. Thus $L_i$ ($R_i$) are right (left) invariant vector fields. The dual 1-forms, defined by $\sigma^L_i (L_j) = \delta_{ij}$ and $\sigma^R_i (R_j) = \delta_{ij}$, are right-invariant $L_R, \sigma^L_j = 0$ and left-invariant $L_L, \sigma^R_j = 0$, and obey the Maurer-Cartan equations

$$d\sigma^L_i = -\frac{1}{4} \epsilon_{ijk} \sigma^L_j \wedge \sigma^L_k, \quad d\sigma^R_i = \frac{1}{4} \epsilon_{ijk} \sigma^R_j \wedge \sigma^R_k. \quad (2.132)$$

It follows that $L_{L_i} \sigma^L_j = -\epsilon_{ijk} \sigma^L_k$ and $L_{R_i} \sigma^R_j = \epsilon_{ijk} \sigma^R_k$.

It is useful to have an explicit coordinate system. We use Euler angles $(\theta, \phi, \psi)$, which (almost) cover $S^3 \cong SU(2)$ if $0 < \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$. In this coordinate system the right-invariant 1-forms can be written as

$$\sigma^L_1 = \sin \psi d\theta - \cos \psi \sin \theta d\phi, \quad \sigma^L_2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi, \quad \sigma^L_3 = d\psi + \cos \theta d\phi, \quad (2.133)$$
and the dual vectors are
\[ L_1 = \cot \theta \cos \psi \partial_{\psi} + \sin \psi \partial_{\theta} - \frac{\cos \psi}{\sin \theta} \partial_{\phi}, \]
\[ L_2 = -\cot \theta \sin \psi \partial_{\psi} + \cos \psi \partial_{\theta} + \frac{\sin \psi}{\sin \theta} \partial_{\phi}, \]
\[ L_3 = \partial_{\phi}. \quad (2.134) \]

The left-invariant 1-forms can be written as
\[ \sigma^R_1 = -\sin \phi d\theta + \cos \phi \sin \theta d\psi, \]
\[ \sigma^R_2 = \cos \phi d\theta + \sin \phi \sin \theta d\psi, \]
\[ \sigma^R_3 = d\phi + \cos \theta \partial_{\phi}, \quad (2.135) \]

and the dual vectors are
\[ R_1 = -\cot \theta \cos \phi \partial_{\phi} - \sin \phi \partial_{\theta} + \frac{\cos \phi}{\sin \theta} \partial_{\phi}, \]
\[ R_2 = -\cot \theta \sin \phi \partial_{\phi} + \cos \phi \partial_{\theta} + \frac{\sin \phi}{\sin \theta} \partial_{\phi}, \]
\[ R_3 = \partial_{\phi}. \quad (2.136) \]

2.B Examples: Kähler metrics and solitons with a bolt

In this section we will study regularity of various examples of Kähler metrics with \( U(1) \times SU(2) \) symmetry and of the associated supersymmetric solutions they define. We will assume the supersymmetric solutions have \( U(1) \times SU(2) \) symmetry, i.e. \( \omega_1 = \omega_2 = 0. \)

2.B.1 Kähler-Einstein base

The most general Kähler-Einstein metric (normalised so \( R_{ab} = -\frac{6}{\ell^2} g_{ab} \)) with \( U(2) \) symmetry is given by
\[ V(r) = 1 + \frac{r^2}{\ell^2} + \frac{c_4}{r^4}. \quad (2.137) \]

This gives a supersymmetric solution with
\[ f = 1, \quad \omega_3 = \frac{r^2}{2\ell}. \quad (2.138) \]

For \( c_4 = 0 \) this is the Bergmann metric and for \( c_4 > 0 \) there is a singularity at \( r = 0. \)

For \( c_4 < 0, \) there is a largest real root \( r_0 > 0 \) such that \( V(r_0) = 0. \) One can write the solution in terms of \( r_0 \) as
\[ V(r) = \frac{(r^2 - r_0^2)(r^2 - \ell^2) + r_0^2(r^2 + r_0^2 + \ell^2)}{\ell^2 r^4}, \quad (2.139) \]

which gives \( r_0 V'(r_0) = 4 + 6r_0^2/\ell^2. \) Therefore, regularity at \( r = r_0 \) requires it to be a bolt \( (2.67) \) so
\[ p = 2 + \frac{3r_0^2}{\ell^2}. \quad (2.140) \]

Note for \( \ell \rightarrow \infty \) this reduces to \( p = 2 \) as it should for the Eguchi-Hanson metric. However, for \( \ell > 0 \) we see that there is a solution for every \( p > 2 \) so we obtain corresponding smooth Kähler metrics. Note that constant \( r \) surfaces are squashed \( S^3/Z_p. \)

In fact the corresponding supersymmetric solution is singular or has CTC. This is because regularity requires \( \omega_3(r_0) = 0 \) which is never possible. Alternatively, note that the Killing field \( \partial_{\phi} - (r_0^2/(2\ell)) \partial_t \) has a fixed point in the spacetime at \( r = r_0 \) and so must have closed orbits which implies \( t \) must be periodically identified.


2.2.2 A class of soliton solutions

Consider the ansatz

\[ V = c_0 + \frac{r^2}{\ell^2} + \frac{c_2}{r^2} + \frac{c_4}{r^4}. \]  

(2.141)

We find that this is a solution to (2.73) iff

\[ c_2^2 = 3(c_0 - 1)c_4. \]  

(2.142)

The special case \( c_0 = 1 \) gives the Kähler-Einstein solution. These bases give

\( f = \frac{3r^2}{(c_0 - 1)\ell^2 + 3r^2}, \)

(2.143)

\[ \omega_\ell = \frac{2(c_0 - 1)c_2\ell^4 + (3(c_0 - 1)^2\ell^4 + 9c_2\ell^2)r^2 + 18(c_0 - 1)\ell^2r^4 + 18r^6}{36\ell^4}. \]  

(2.144)

All these solutions are asymptotically locally \( \text{AdS}_5 \) in the sense that the Kähler base is asymptotically locally Bergmann, \( f \sim 1 \) and \( \omega_\ell \sim r^2/(2\ell) \) as \( r \to \infty \).

As \( V(r) \) is not smooth at \( r = 0 \) this solution cannot correspond to a black hole solution or a soliton with a nut (recall our earlier general analysis showed that in both cases smoothness implies \( V(r) \) is a smooth positive function of \( r^2 \), see section 2.3.3 and Corollary 1). Therefore, the only possibility is that is has a bolt at \( r = r_0 > 0 \).

To analyse this, it is convenient to change parameterisation and write

\[ V = \frac{(r^2 - r_0^2)(a_0 + a_1r^2 + \frac{r^4}{\ell^2})}{r^4}, \]  

(2.145)

so that one of our parameters is the root \( r_0 > 0 \). This is a solution to (2.73) iff

\[ a_1^2r_0^4 + (a_1 - 3)a_0r_0^2 + a_0^2 - \frac{3a_0r_0^4}{\ell^2} = 0. \]  

(2.146)

The regularity condition at the bolt (2.67) fixes

\[ a_1 = p - \frac{r_0^2}{\ell^2} - \frac{a_0}{r_0^2}, \]  

(2.147)

where recall \( p \in \mathbb{N} \). These conditions give a family of complete Kähler metrics with a bolt.

Now consider a supersymmetric solution with such a base. We must also require that \( \omega_\ell \) is smooth at the bolt. In particular, this requires that \( \omega_\ell(r_0) = 0 \). One finds that this gives another quadratic equation in \( a_0, a_1 \) and combining this with (2.146) implies

\[ a_0 = \frac{\ell^2(2p^2 - 4p + 3)r_0^2 + (p - 8)r_0^4}{\ell^2(p + 1)}. \]  

(2.148)

Substituting back into (2.146) (or \( \omega_\ell(r_0) = 0 \)) finally gives

\[ 27x^2 - 3(p^2 - 2)(p^2 + 14p - 5)x + (p - 2)^3p = 0, \]  

(2.149)

where we have defined the dimensionless parameter \( x := r_0^2/\ell^2 > 0 \). The two roots are given by

\[ x_\pm(p) = \frac{1}{54}(p - 2) \left( p^2 + 14p - 5 \pm (1 + p)\sqrt{(1 + p)(25 + p)} \right) \]  

(2.150)

and it can be shown that \( x_+(p) > x_-(p) > 0 \) for all \( p > 2 \), \( x_-(1) \approx 0.08 \) and \( x_+(1) < 0 \) (clearly \( p = 2 \) gives a trivial solution \( x = 0 \)).\(^{19}\) Each positive value of \( x \) gives a smooth spacetime with a bolt at \( r = r_0 \) if \( V > 0 \) for all \( r > r_0 \) and \( f > 0 \) for \( r \geq r_0 \).

For \( p = 1 \), the only allowed solution is \( x_-(1) \) which does give \( V > 0 \) for \( r > r_0 \), however in

\(^{19}\)For \( r_0 = 0 \) the above local solution has \( a_0 = 0 \) and reduces to the Gutowski-Reall case (Bergmann for \( a_1 = 1 \)).
this case \( f(r_0) < 0 \) so this does not give a soliton spacetime. Similarly, for \( p \geq 3 \) the solution \( x_-(p) \) always gives \( f(r_0) < 0 \) and therefore must be discarded.

However, for \( p \geq 3 \) one can show that the \( x_+(p) \) solution gives \( V > 0 \) for \( r > r_0 \) and \( f > 0 \) for \( r \geq r_0 \). For example, for \( p = 3 \) and \( x = x_+(3) = (8\sqrt{7} + 23)/27 \) we get the solution

\[
V = (r^2 - r_0^2) \left( \frac{1}{\ell^2} + \frac{2(13 + \sqrt{7})}{27r^2} + \frac{2(13\sqrt{7} + 88)}{729r^4} \right) \tag{2.151}
\]

and

\[
f^{-1} = \frac{r^2 - r_0^2 + \frac{1}{9} (2\sqrt{7} + 5) \ell^2}{r^2}
\]

\[
\omega_3 = (r^2 - r_0^2) \left( -\frac{2(7\sqrt{7} + 10) \ell^3}{729r^4} + \frac{(2\sqrt{7} - 1) \ell}{54r^2} + \frac{1}{2\ell} \right) \tag{2.153}
\]

which manifestly satisfies \( V > 0 \) for \( r > r_0 \) and \( f > 0 \) for \( r \geq r_0 \). This gives an asymptotically AdS\(_5\)/\( \mathbb{Z}_3 \) soliton with a bolt at \( r = r_0 \) and a spatial \( S^3/\mathbb{Z}_3 \) boundary metric. More generally, for all \( p \geq 3 \) the solution \( x_+(p) \) gives asymptotically AdS\(_5\)/\( \mathbb{Z}_p \) solitons with a bolt with spatial \( S^3/\mathbb{Z}_p \) boundary metric.

### 2.3 Extremal \( \mathbb{K} \)ähler metrics

Although out of the main line of this paper, we have obtained a class of extremal \( \mathbb{K} \)ähler metrics analogous to the solution above. The equation for an extremal \( \mathbb{K} \)ähler metric is [98]

\[
\nabla^c (\nabla_c \nabla^2 R + 2R_{cb} \nabla^b R) = 0 \ .
\tag{2.154}
\]

This in fact comprises two of the terms in the equation for supersymmetry (2.9), although any relation remains unclear.

In any case we can find a family of \( U(1) \times SU(2) \) invariant extremal \( \mathbb{K} \)ähler metrics (2.24) again using the ansatz (2.141). We now find this is extremal iff \( c_0 = 1 \). Thus

\[
V = 1 + \frac{r^2}{\ell^2} + \frac{c_2}{r^2} + \frac{c_4}{r^4}
\tag{2.155}
\]

gives a 2-parameter family of extremal \( \mathbb{K} \)ähler metrics. For \( c_2 = 0 \) this is \( \mathbb{K} \)ähler-Einstein. In general we find

\[
R = -\frac{24}{\ell^2}
\tag{2.156}
\]

so these are constant scalar curvature (note both \( c_2, c_4 \) do not appear in the scalar curvature). In fact, the most general \( U(1) \times SU(2) \) \( \mathbb{K} \)ähler metric with constant scalar curvature \( R = -24/\ell^2 \) is given by the above solution. We emphasise that these are not solutions to (2.73) unless \( c_2 = 0 \) and therefore do not give rise to supersymmetric solutions.

To analyse regularity it is convenient to use the alternate parameterisation (2.145). The extremal condition is now

\[
a_1 = 1 + \frac{r_0^2}{\ell^2}
\tag{2.157}
\]

and the regularity condition (2.67) at the bolt implies

\[
a_0 = -r_0^2 \left( 1 - p + \frac{2r_0^3}{\ell^2} \right)
\tag{2.158}
\]

where \( p \in \mathbb{N} \). This gives

\[
V = \frac{(r^2 - r_0^2)^2}{r^4} \left( \frac{(r^2 + 2r_0^2)(r^2 - r_0^2)}{\ell^2} + r^2 + r_0^2(p - 1) \right)
\tag{2.159}
\]

Clearly \( V(r) > 0 \) for all \( r > r_0 \) so this is a globally regular metric for any \( r_0 > 0 \) and any
2.C General SU(2) Kähler metric

In this section we will study general Kähler metrics admitting SU(2) symmetry without any further assumptions. This generalises the result of section 2.3.

The general local form of cohomogeneity-1 SU(2)-invariant Kähler form is

\[ \Omega = d\rho \wedge S_i \sigma^i - \frac{1}{2} \Omega_{ij} \sigma^i \wedge \sigma^j \] (2.160)

where \( S_i, \Omega_{ij} \) are functions of \( \rho \), and \( \sigma^i \) are right-invariant 1-forms. The latter are defined up to a global SU(2) rotation, and it is convenient to pick them such that at some preferred non-degenerate point \( \rho_0 \)

\[ S_i(\rho_0) \propto (0, 0, 1), \]

i.e. the Kähler form is oriented in third direction \( \Omega|_{\rho_0} \propto d\rho \wedge \sigma^3 = \ldots \sigma \wedge \sigma \). This leaves us with a freedom of global SO(2) rotation in \((\sigma^1, \sigma^2)\) which we will fix below.

Introduce a general local SU(2)-invariant metric of cohomogeneity-1

\[ h = d\rho^2 + h_{ij} \sigma^i \sigma^j \] (2.162)

where \( h_{ij} = h_{ij}(\rho) \). Now we can fix the remaining global freedom by demanding that \((\sigma^1, \sigma^2)\) sector of \( h_{ij} \) diagonalises at \( \rho_0 \), i.e. \( h_{12}(\rho_0) = 0 \).

Next, introduce SU(2)-invariant orthonormal frame \( e^0 = d\rho, e^i = E^i_j(\rho)\sigma^j \) where \( E^i_j \) is an invertible matrix, such that \( h = (e^0)^2 + \delta_{ij} e^i e^j \) and \( e^0 \wedge e^1 \wedge e^2 \wedge e^3 \) is the positive orientation.

The frame is defined up to a local SO(3) freedom \( e^i \to O^i_j e^j \) where \( O \in SO(3) \). Therefore, by QR decomposition, one can always choose the upper-triangular parameterisation of the frame for a given choice of \( \sigma^i \):

\[ e^0 = d\rho, \quad e^1 = a\sigma^1 + b_1\sigma^2 + c_1\sigma^3, \quad e^2 = b\sigma^2 + c_2\sigma^3, \quad e^3 = c\sigma^3, \] (2.163)

where \( a, b, c, b_1, c_1, c_2 \) are functions of \( \rho \), and positive orientation demands \( abc \geq 0 \) where equality holds only at degenerate points. The frame is fixed up to \((\mathbb{Z}_2)^3 \) acting on \( e^i \) as pairwise flipping of signs:

\[ (e^1, e^2, e^3) \to (e^1, -e^2, -e^3) : \quad (a, b, c, b_1, c_1, c_2) \to (a, -b, -c, b_1, c_1, -c_2), \] (2.164)

\[ (e^1, e^2, e^3) \to (-e^1, e^2, -e^3) : \quad (a, b, c, b_1, c_1, c_2) \to (a, b, c, -b_1, -c_1, c_2), \] (2.165)

\[ (e^1, e^2, e^3) \to (-e^1, -e^2, e^3) : \quad (a, b, c, b_1, c_1, c_2) \to (a, -b, c, -b_1, -c_1, -c_2), \] (2.166)

and the 1-forms \( \sigma^i \) remain unchanged. We pick the signs as

\[ c(\rho_0) > 0, \quad c_1(\rho_0), c_2(\rho_0) \geq 0. \] (2.167)

Next, the basis of ASD 2-forms \( \Omega^i \) is then given by

\[ \Omega^i = e^0 \wedge e^i - \frac{1}{2} \varepsilon_{ijk} e^j \wedge e^k, \] (2.168)

and the Kähler form decomposes in this basis as

\[ \Omega = A\Omega^1 + B\Omega^2 + C\Omega^3, \] (2.169)

where \( A, B, C \) are functions of radial coordinate. The Kähler form is defined up to overall sign \( \Omega \to -\Omega \), and we fix it such that \( C(\rho_0) > 0 \). The two parameterisations (2.160) and (2.169)
are related by
\begin{equation}
S_i = (\Lambda_4, \Lambda_b + Bc, A \Lambda_c + Bc_2 + Cc) ,
\end{equation}
(2.170)
\[
\frac{1}{2} \Omega_{ij} \sigma^i \wedge \sigma^j = abc \sigma^1 \wedge \sigma^2 + (abc - ac_b c) \sigma^3 \wedge \sigma^1 + (bcA - b_1 cB + (b_1 c_2 - bc_1) C) \sigma^2 \wedge \sigma^3 .
\]
(2.171)
All together, our choice of $SU(2)$-invariant 1-forms and frame gives us the initial conditions
\[
A(\rho_0) = B(\rho_0) = 0, \quad C(\rho_0) = 1, \quad b_1(\rho_0) = 0
\]
(2.172)
and inequalities (2.167). The last condition comes from $h_{12}(\rho_0) = (ab_1)|_{\rho_0} = 0$, and $(abc)|_{\rho_0} \neq 0$
since, by assumption, it is a non-degenerate point. We are now ready to state the main result of this section.

**Theorem 4.** *The most general Kähler metric with a cohomogeneity-1 $SU(2)$ symmetry belongs to one of the three classes depending on signs of $A'|_{\rho_0}, B'|_{\rho_0}$:

- If $A'|_{\rho_0}, B'|_{\rho_0} < 0$ then Kähler metric belongs to a general class, and frame functions satisfy (2.178) to (2.185).
- If $A'|_{\rho_0} \geq 0$ then $A = b_1 = c_1 = 0$ everywhere in this chart.
- If $B'|_{\rho_0} \geq 0$ then $B = b_1 = c_2 = 0$ everywhere in this chart.
- If $A'|_{\rho_0}, B'|_{\rho_0} \geq 0$ then the metric can be written in the frame (2.163) where $b_1 = c_1 = c_2 = 0$, so in particular is diagonal, and
\[
2abc' = b^2 - a^2 + c^2 ,
\]
(2.173)
\[
2abc' = a^2 - b^2 + c^2 .
\]
(2.174)

The Kähler form is then simply
\[
\Omega = d(ab\sigma^3).
\]
(2.175)

**Proof.** For the geometry $(h, \Omega)$ to be Kähler, $\Omega$ must be a closed integrable complex structure. For $\Omega$ to be an almost complex structure
\[
\Omega \wedge \Omega = -2(A^2 + B^2 + C^2) \text{vol}_h = -2 \text{vol}_h ,
\]
(2.176)
i.e. the functions $A, B, C$ parameterise a 2-sphere $A^2 + B^2 + C^2 = 1$. For $\Omega$ to be closed and integrable it has to be parallel
\[
\nabla \Omega = 0 .
\]
(2.177)

By solving the system with the help of computer algebra we find
\[
\begin{align*}
2abc C' &= (1 - C^2) \mu + 2c(c_1 A + c_2 B) C , \\
2abc C^2 A' &= -A(1 - A^2) \mu - 2C^2((bb_1 + c_1 c_2) B + cc_1 C) , \\
2abc C^2 B' &= -B(C^2 - A^2) \mu + 2C^2((bb_1 + c_1 c_2) A - cc_2 C) , \\
2 bc C^2 a' &= -A^2 \mu + (-a^2 + b^2 - b_1^2 + c_1^2 - c_2^2 + c_2^2) C^2 , \\
2 ac C^2 b' &= -B^2 \mu + (a^2 - b^2 + b_1^2 + c_1^2 + c_1^2 - c_2^2 - c_2^2) C^2 , \\
2 abc C^2 c_1' &= -(c_1(a^2 - b^2 + b_1^2 + c_1^2 + c_1^2) + 4bc_1 c_2 - b_1 c_2^2) C^2 - \mu(2b_1 AB + b_1 A^2) , \\
2 abc C^2 c_2' &= -(a^2 + b^2 - b_1^2 + 3c^2 - c_1^2) C^2 - \mu B(c_2 B + 2c C) \\
\end{align*}
\]
(2.180)
where $A^2 + B^2 + C^2 = 1$, $\mu = a^2 + b^2 - c^2 + b_1^2 + c_1^2 + c_2^2 - 2ab c'$. (2.181)

Consider equation (2.179). Under our choices (2.167), its RHS is non-positive at $\rho_0$. This can satisfy the assumption $A'|_{\rho_0} \geq 0$ only if $c_1(\rho_0) = 0$. Now consider the subsystem (2.179),
(2.183) and (2.184). The trivial solution $A = b_1 = c_1 = 0$ satisfies the initial conditions, and by the standard ODE result this solution is unique. The Kähler form is

$$\Omega_{A=0} = d \left( (acB - ac_2C) \sigma^2 + abc \sigma^3 \right).$$

(2.187)

Similarly, if $B'|_{\rho_0} \geq 0$ then equation (2.180) demands $c_2(\rho_0) = 0$, and the subsystem (2.180), (2.183) and (2.185) has unique solution $B = b_1 = c_2 = 0$. The Kähler form is

$$\Omega_{B=0} = d \left( (bcA - bc_1C) \sigma^1 + abC \sigma^3 \right).$$

(2.188)

If both $A'|_{\rho_0}, B'|_{\rho_0} \geq 0$ then $A = B = c_1 = c_2 = 0$ and the rest of the system drastically simplifies. The equation (2.183) takes the form

$$2abc b_1' = -b_1(a^2 + 3b^2 + b_1^2 - 3c^2),$$

(2.189)

and together with the initial condition $b_1(\rho_0) = 0$ this problem admits a unique solution $b_1 = 0$. The remaining equations (2.181) and (2.182) are

$$2bca' = b^2 - a^2 + c^2$$

(2.190)

$$2acb' = a^2 - b^2 + c^2.$$  

(2.191)

Adding two equations, one gets $c = (ab)'$. The Kähler form is simply

$$\Omega = \Omega^3 = c \, d\rho \wedge \sigma^3 - ab \, \sigma^1 \wedge \sigma^2 = d \, (ab \sigma^3).$$

(2.192)

It is interesting to note that there is an alternative set of assumptions that leads to the same result as Theorem 3. Instead of fixing the Kähler form, one can assume the metric (2.162) to be diagonal (which fixes global SU(2) up to even permutations). Then the system of Theorem 4 implies

$$AB = BC = AB = 0,$$

(2.193)

Without loss of generality, it can be solved as $A = B = 0, C = 1$ from which (2.33) and (2.34) immediately follow.

This result addresses the old standing problem of existence of SU(2)-symmetric Kähler spaces of non-diagonal form [100, 101]. By completely fixing the global SU(2) symmetry, we have classified the initial conditions for the integrability constraints at a regular point. In particular, we have identified the set for which the Kähler geometry becomes diagonalizable. While we have not provided an explicit example of the such non-diagonalizable geometry, it seems likely that it can exist. However, our analysis is done only at a point, and it cannot address questions of smoothness and compactness.

### 2.D Reduction of supersymmetric constraint (2.73) to second order ODE

The supersymmetric constraint (2.73) admits an impressive symmetry structure as an ODE. In this section we show that it reduces to second order ODE.

#### 2.D.1 Reduction

First of all, supersymmetric constraint (2.73) is a total derivative. The result of the integration is

$$3r^4 V^{(4)} + 3r^3 V^{(3)} (r V' + 6V) + r^2 (14r V' - 7V - 32) V''$$

$$- \frac{3}{2} r^4 V''^2 - \frac{9}{2} r^2 V'^2 - r(25V + 32)V' + 32V^2 + 128V + \kappa_0 = 0$$

(2.194)
where $\kappa_0$ is integration constant. The ODE has a rescaling symmetry $r \to \lambda r$ for constant non-zero $\lambda$. This can be used to reduce its order again by the following two substitutions. Firstly, introduce new argument $r = e^t$, $t \in \mathbb{R}$ to bring ODE to an autonomous form, i.e. without any explicit dependence on $t$. Next, introduce a new function

$$\mu(V) = \frac{dV}{dt} = r \frac{dV}{dr} \quad (2.195)$$

This substitution is well-defined where $\frac{d}{dV} = \frac{1}{V'} \frac{d}{dr}$ is regular. The horizon is one of such points. We will treat it with caution in the next subsection.

$$6\mu^3 \left( V \mu^{(3)} + \mu'' \right) + \mu^2 \left( \mu' (24V \mu'' + 3\mu' + 16) - 28 \right) + \mu \left( 6V \mu'^3 - 8(7V + 8)\mu' \right) + 64(V + 2)^2 + 2\kappa_0 - 256 = 0 \quad (2.196)$$

where prime now corresponds to derivative wrt $V$. There is more to that: in fact, (2.196) is a total derivative again! Integrating, we get

$$2\mu^3 (8 + 9V \mu'') + \mu^2 (9V \mu'^2 - 84V - 96) = -64(V^3 + 6V^2 + \lambda_1 V + \lambda_0) \quad (2.197)$$

where $\lambda_1 = \frac{3\kappa_0}{32}$ and $\lambda_0$ is a new integration constant. Their values will be fixed in terms of boundary data in section 2.D.2.

In terms of the new parameterisation, the Gutowski-Reall solution $V(r) = 4\alpha^2 + \frac{r^2}{\ell^2}$ takes the form

$$\mu(V) = 2 \left( V - 4\alpha^2 \right) . \quad (2.198)$$

### 2.D.2 Special points of the ODE

By (2.195) function $\mu(V)$ is smooth wherever $V' \neq 0$. Unfortunately, horizon $V = V_0$ is one of such points. Therefore, boundary conditions for $\mu$ prompt more examination. Since we can define $\mu(V)$ at $V_0$ to make it a smooth function, we can identify $\mu|_{V_0}, \mu'|_{V_0}$ with their limits $V \to V_0$ which can be found from the data about solution we have.

In the second subsubsection, we use black hole boundary conditions to show that $V' > 0$ everywhere outside of the horizon.

#### Boundary conditions

The black hole horizon lives $V_0 = V(0) = 4\alpha^2 \geq 1$. We also know that $V'|_0 = V^{(n)}|_0 = 0$ for $n \geq 3$, $V''|_0 = \frac{2}{\ell^2}$. Therefore, $\mu|_{V_0} = (rV')|_0 = 0$.

The first derivative of $\mu$ can be found from

$$\mu' = 1 + \frac{rV''}{V'} \quad (2.199)$$

where primes on $\mu$ denote derivatives wrt $V$ and primes on $V$ denote derivatives wrt $r$. Using knowledge of derivatives at zero for black holes we have

$$\mu'|_{V_0} = 1 + \lim_{r \to 0} \frac{rV''}{V'} = 2 \quad (2.200)$$

To sum up, the boundary conditions are

$$\mu|_{V_0} = 0, \quad \mu'|_{V_0} = 2, \quad V_0 = 4\alpha^2. \quad (2.201)$$

We also have

$$\mu'' = \frac{r^3V'' + 3r^2V' + rV'}{r^2V^2} - \frac{\mu'^2}{rV'}, \quad (2.202)$$
therefore,
\[ \mu''|_0 = \lim_{V \to V_0} \mu'' = \lim_{r \to 0} \frac{V'''}{r V''} = 0 \] (2.203)
for any black hole with smooth horizon. Notice, that for the family of non-smooth AlAdS black holes found in [77, 78], this limit diverges since for such solutions \( V'''|_0 \) is greater than zero. In other words, \( \mu(V) \) is not smooth at the horizon for such solutions.

**Fixing integration constants**

Now that we have \( \mu(V) \) smooth at zero, we can expand the (2.197) into series in \( V - V_0 \) to fix integration constants \( \lambda_0 \) and \( \lambda_1 \). We get \( \lambda_0 = 32\alpha^4(3 + 4\alpha^2) \), \( \lambda_1 = -48\alpha^2(1 + \alpha^2) \). The ODE now simplifies to
\[ 2\mu^3 (8 + 9V\mu'') + \mu^2 (9V\mu'^2 - 84V - 96) = -64(V - 4\alpha^2)^2(V + 8\alpha^2 + 6). \] (2.204)

It is worth to conclude with the following comment. While the reduction of a sixth-order supersymmetric constraint to a second order ODE (2.204) is already a remarkable simplification, it can happen that the equation can be completely resolved. Indeed, second order ODEs of this type can possess a Painlevé property which is typically associated to equations arising from reductions of integrable systems. If some notion of integrability can indeed be established, even in such a highly symmetric setting,\(^{20}\) this may shed light onto the hidden structure behind supersymmetric constraint and, hence, the possibility of proper classification of all solutions.

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\(^{20}\)4d Kähler geometries with \( U(1) \times SU(2) \) symmetry are a special case of Calabi-type geometries, see section 3.3.3, which possess some integrability in the sense of Hamiltonian 2-forms [93].
Chapter 3

Supersymmetric solutions with toric symmetry

This chapter is based on the paper [102] authored by myself, my supervisor James Lucietti and my colleague Praxitelis Ntokos.

3.1 Introduction

In this chapter we consider the classification of supersymmetric black hole solutions to five-dimensional minimal gauged supergravity, under the assumption of $U(1)^2$ symmetry. As it was discussed in the introduction, chapter 1, the most general known black hole (CCLP) [17] has a horizon with spherical cross sections. This leaves the classification of black holes, especially with different horizon topologies open. Unfortunately, even the classification of supersymmetric backgrounds in gauged supergravity is complicated — for five-dimensional minimal supergravity, the classification of timelike supersymmetric solutions reduces to finding a Kähler metric that satisfies a complicated 4th order non-linear PDE for its curvature [70, 72] (also, see section 1.3). This prevents a local classification of solutions.

A remarkable exception, however, is that a complete classification of near-horizon geometries of supersymmetric black holes is known [44,45] : the most general smooth near-horizon geometry with compact cross sections is locally isometric to that of the supersymmetric CCLP black hole. From the point of view of supersymmetric constraint, the near-horizon geometry is one set of boundary conditions. For more details see chapter 1.

A natural strategy is to seek further simplifying symmetry assumptions that are compatible with the problem at hand. Asymptotically globally AdS$_5$ spacetimes have an $SO(4)$ rotational symmetry at infinity and therefore one may consider solutions invariant under a subgroup of this symmetry. In a previous chapter 2, we considered supersymmetric solutions with $SU(2)$ symmetry, and it was shown that the Gutowski-Reall black hole [68], or its near-horizon geometry, is the only solution with an analytic horizon.

In the present chapter we will consider the classification of supersymmetric solutions to five-dimensional minimal gauged supergravity that are invariant under a toric $U(1)^2$-symmetry (the maximal abelian subgroup of $SO(4)$). This is a particularly notable class since it contains the CCLP black hole and, in fact, all explicitly known AdS black holes solutions in five-dimensions. This is the analogue of the aforementioned classification of supersymmetric black holes in ungauged supergravity section 1.2.1 which was performed under the assumption of such a torus symmetry [103]. Unfortunately, we find that the classification of supersymmetric backgrounds with a torus symmetry in gauged supergravity is a much more complicated problem: it reduces to a problem in toric Kähler geometry which we are unable to solve. Nevertheless, we will show that there exists a geometrically defined coordinate system which can describe any black hole solution in this class in a unified manner. This corresponds to using symplectic coordinates (or ‘action-angle’ coordinates) for the toric Kähler base, which have proven to be convenient in the context of toric Kähler geometry [104]. In particular, such geometries are described...
by a symplectic potential which fully encodes the Kähler metric (it is related to the Kähler potential).

It is well known that compact toric Kähler manifolds are characterised by Delzant polytopes. The polytope is the image of the moment map defined by the toric symmetry. It corresponds to the orbit space under the toric symmetry and its edges correspond to where the symmetry degenerates or has fixed points. Guillemin gave an explicit formula for the canonical Kähler metric associated to a Delzant polytope purely in terms of the combinatorial data that defines the polytope [105]. In symplectic coordinates, it corresponds to the symplectic potential \( g = \frac{1}{2} \sum_\ell \ell_A \log \ell_A \) where \( \ell_A = 0 \) are lines that define the edges of the polytope [104]. Curiously, we find that the CCLP black hole is described by a remarkably simple symplectic potential that takes this canonical form, although not every line corresponds to an edge of the orbit space in this case.

A particularly convenient feature of symplectic coordinates is that they are naturally adapted to describing the axes of symmetry and provide a natural description of the orbit space. Indeed, in symplectic coordinates the components of the axes correspond to lines and horizons to points. They are therefore analogues of the Weyl-Papapetrou coordinates for asymptotically flat electro-vacuum solutions in four and five-dimensions. Furthermore, imposing smoothness at the axes of symmetry and employing the classification of smooth supersymmetric near-horizon geometries with torus symmetry [44], we are able to determine the general form of the symplectic potential near any component of the axes or any horizon (see Theorem 7 for a precise statement of this result). It turns out this fixes the singular part of the symplectic potential and is analogous to Abreu’s result for compact toric Kähler manifolds [104]. In particular, this allows us to write down the singular part of the symplectic potential for possible new solutions such as black lenses and multi-black holes. The existence of such topologically non-trivial black hole solutions thus reduces to determining the smooth part of the symplectic potential. Unfortunately, supersymmetry dictates that the symplectic potential must satisfy a complicated non-linear 8th order PDE so this existence problem is presently out of reach.

Due to the complexity of the supersymmetry constraints it is natural to seek for extra symmetry structures that render the classification problem tractable. Curiously, it turns out that the supersymmetric CCLP black hole has a Kähler base of Calabi type [72]. Kähler surfaces of Calabi type naturally appear in the classification of Kähler surfaces that admit a hamiltonian 2-form [93]. We will give a self-contained definition of Calabi type in the context of toric Kähler surfaces in terms of a certain orthogonality property of the associated moment maps (analogous to orthotoric Kähler, see Definition 7). For Kähler base metrics of this type we are in fact able to obtain a complete classification, which is the main result of this chapter.

**Theorem 5.** Any supersymmetric toric solution to five-dimensional minimal gauged supergravity that is timelike outside a smooth horizon with compact cross-sections, with a Kähler base of Calabi type, is locally isometric to the CCLP black hole or its near-horizon geometry.

We also show that other types of Kähler surfaces with hamiltonian 2-form, namely orthotoric and product-toric geometries (see section 3.3), do not contain Kähler bases of smooth black holes

**Theorem 6.** A time-like supersymmetric toric solution to five-dimensional minimal gauged supergravity with Kähler base of orthotoric or product-toric type cannot contain a smooth horizon with compact cross-section.

We emphasise that no global assumptions are required for these results, so in particular they rule out asymptotically locally AdS\(_5\) supersymmetric black holes (other than quotients of CCLP). Furthermore, they also do not assume the horizon is connected, and therefore rule out multi-black holes in this symmetry class. In this sense Theorem 5 is analogous to the uniqueness theorem obtained for \(SU(2)\)-invariant supersymmetric solutions, see chapter 2.

The proof of Theorem 5 can be sketched as follows. A toric Kähler metric of Calabi type is determined by two functions of single variables and in this case the supersymmetry constraint simplifies, although a general local solution is still unavailable. However, by comparing to the general form of the near-horizon geometry [44] one can in fact completely fix one of these functions. The supersymmetry constraint now reduces to an ODE problem which can be completely
solved in the generic case. There is a special case (which corresponds to the solution possessing an \( SU(2) \) symmetry) which leads to a 5th order ODE which was previously encountered in chapter 2 and can be solved under the additional assumption of an analytic horizon \([82]\). It then turns out that the general solution is the CCLP black hole or its near-horizon geometry.

The organisation of this chapter is as follows. In section 3.2 we consider supersymmetric solutions to gauged supergravity with toric symmetry, introduce symplectic coordinates and the symplectic potential, derive the general form for the symplectic potential near any horizon or axis of symmetry (see Theorem 7), present the CCLP black hole in this formalism, and then give the singular part of the potential for the simplest possible topologically non-trivial black hole solutions (lenses and double-black holes). In section 3.3 we restrict to supersymmetric solutions with Kähler bases of Calabi type and prove Theorem 5. A number of details are relegated to the appendices.

### 3.2 Supersymmetric solutions with toric symmetry

#### 3.2.1 General local classification

The bosonic field content of five-dimensional minimal gauged supergravity is a metric \( g \) and a Maxwell field \( F \) defined on a five-dimensional spacetime manifold \( M \). In any open region \( U \subset M \) where the supersymmetric Killing vector \( V (a \text{ spinor bilinear}) \) is timelike, the general local form for a supersymmetric solution is \([70]\) (we work in the conventions of \([68]\)),

\[
g = -f^2(dt + \omega)^2 + f^{-1}h ,
\]

where \( V = \partial_t, h \) is a Kähler metric on the base \( B \) orthogonal to the orbits of \( V \), and \( f \) and \( \omega \) are a function and 1-form on \( B \). The Maxwell field takes the form

\[
F = \frac{\sqrt{3}}{2}d(f(dt + \omega)) - \frac{1}{\sqrt{3}}G^+ - \frac{\sqrt{3}}{ff}X^{(1)} ,
\]

where \( G^\pm = \frac{1}{2}f(d\omega \pm \star_4 d\omega), \star_4 \) is the Hodge star operator with respect to the base metric \( h \), \( X^{(1)} \) is the Kähler form and the orientation on \( B \) is such that \( X^{(1)} \) is anti-self dual on \( B \). Given a Kähler base, \( f, G^+ \) are completely fixed in terms of its curvatures

\[
f^{-1} = -\frac{\ell^2}{24}R, \quad G^+ = -\frac{\ell}{2}(R - \frac{1}{2}X^{(1)}R) ,
\]

where \( R_{ab} = \frac{1}{2}R_{abcd}X^{(1)}_{cd} \) is the Ricci form. It is worth noting that combining (3.2) and (3.3) allows us to deduce the gauge field (up to a gauge transformation),

\[
A = \frac{\sqrt{3}}{2}f(dt + \omega)) + \frac{\ell}{2\sqrt{3}}P ,
\]

where \( R = dP, \) i.e., \( P \) is the potential for the Ricci form.

Given such a Kähler surface there exists a basis of ASD 2-forms \( X^{(i)}, i = 1, 2, 3 \), where say \( X^{(1)} \) is the Kähler form, that satisfy the quaternion algebra

\[
X^{(i)} \ X^{(j)} \ X^{(k)} \ = \ -\delta^{ij} h_{ab} + \epsilon^{ijk} X^{(k)}_{ab} \]

and

\[
\nabla_a X^{(2)}_{bc} = P_a X^{(3)}_{bc}, \quad \nabla_a X^{(3)}_{bc} = -P_a X^{(2)}_{bc} .
\]

Using this basis we may expand

\[
G^- = \frac{\ell}{2R}\lambda_1 X^{(1)} ,
\]

where \( \lambda_1 = \frac{1}{2}\nabla^2 R + \frac{5}{2}R_{ab}R^{ab} - \frac{1}{2}R^2 \). The other functions \( \lambda_{2,3} \) are determined by the integrability

\[1\]Therefore, strictly speaking, for these non-generic near-horizon geometries with enhanced symmetry, we have proven Theorem 5 under the stronger assumption that the horizon is analytic.
condition for
\[ d\omega = f^{-1}(G^+ + G^-), \quad \text{(3.8)} \]
i.e. requiring that the r.h.s is closed. This integrability condition also implies that the Kähler base is not free to choose: it must satisfy the following complicated 4th order PDE for its curvature [72]
\[ \nabla^2 \left( \frac{1}{2} \nabla^2 R + \frac{2}{3} R_{ab} R^{ab} - \frac{1}{3} R^2 \right) + \nabla^a (R_{ab} R^{ab}) = 0. \quad \text{(3.9)} \]
Conversely, given a Kähler metric which satisfies this equation, one can solve for \( \omega \), see [82, Lemma 1]. Recall that these gauge transformations act as \( U \) simple connected.

It is also worth noting that \( V \) is only defined up to constant rescalings (since the Killing spinor is). Those act on the time coordinate and the supersymmetric data as
\[ t \to K t, \quad \omega \to K \omega, \quad h \to K^{-1} h, \quad f \to K^{-1} f, \quad \text{(3.10)} \]
for constant \( K \neq 0 \). Of course, the five-dimensional metric \( g \) and Maxwell field \( F \) are invariant under such rescalings.

### 3.2.2 Supersymmetric solutions with toric symmetry

We wish to classify supersymmetric solutions \((M, g, F)\) to five-dimensional minimal gauged supergravity possessing two commuting axial Killing fields. In particular we will say that a supersymmetric solution admits toric symmetry if:

1. there is a torus \( T \cong U(1)^2 \) isometry generated by spacelike Killing fields \( m_i, i = 1, 2 \), both normalised to have \( 2\pi \) periodic orbits; these are defined up to \( m_i \to A_i \tau m_j \) where \( A \in GL(2, \mathbb{Z}) \);
2. the supersymmetric Killing vector field \( V \) is complete and commutes with the \( T \)-symmetry, that is \([V, m_i] = 0\), so the spacetime isometry group is \( \mathbb{R} \times U(1)^2 \);
3. the Maxwell field is \( T \)-invariant \( \mathcal{L}_m F = 0 \);
4. the axis defined by \( \{ p \in \mathcal{M} | \text{det} g(m_i, m_j) | p \neq 0 \} \) is non-empty.

We will now deduce the constraints imposed by such a toric symmetry for timelike supersymmetric solutions, that is, in any open region \( U \subset M \) where \( V \) is strictly timelike. For simplicity we will also assume that \( U \) is simply connected.

Under these assumptions it follows that the data \((f, h, X^{(1)})\) on the base space \( B \) are all invariant under the toric symmetry, and that we can choose a gauge for \( \omega \) such that it is also invariant, see [82, Lemma 1]. Recall that these gauge transformations act as \( \omega \to \omega + d\lambda, \quad t \to t - \lambda \) where \( \lambda \) is a function on \( B \); so this amounts to choosing a gauge where \( \mathcal{L}_m t = 0 \). In particular, the \( T \)-symmetry is holomorphic, that is \( \mathcal{L}_m X^{(1)} = 0 \), which is equivalent to \( \text{d} m_i X^{(1)} = 0 \) since \( X^{(1)} \) is closed. Now, \( X^{(1)} \) is a globally defined 2-form on spacetime (defined as a spinor bilinear) and therefore \( \iota_{m_i} X^{(1)} \) is a globally defined 1-form on \( U \). Therefore, we deduce the existence of functions \( x_i \) (moment maps) on \( U \) such that
\[ \iota_{m_i} X^{(1)} = - \text{d} x_i. \quad \text{(3.11)} \]
This shows that the \( T \)-symmetry is Hamiltonian and therefore \((h, X^{(1)}, B)\) is a toric Kähler structure. The moment maps \( x_i \) define a canonical coordinate system \((x_i, \phi^i)\) for any toric Kähler structure that is adapted to the toric symmetry such that \( m_i = \partial_{\phi^i} \). In terms of these coordinates [104],

\[ h = G^{ij}(x) \text{d} x_i \text{d} x_j + G_{ij}(x) \text{d} \phi^i \text{d} \phi^j, \quad \text{(3.12)} \]
\[ G^{ij} = \partial_i \partial_j g \]
\[ X^{(1)} = \text{d} x_i \wedge \text{d} \phi^i, \quad \text{(3.14)} \]
where $g = g(x)$ is the symplectic potential, $G_{ij}$ is the matrix inverse of the Hessian $G^{ij}$ and we have introduced the notation $\partial_i := \partial/\partial x_i$. Observe that the symplectic potential is only defined up to a linear function of $x_i$. The coordinates $(x_i, \phi^i)$ are called symplectic (or Darboux) coordinates because they are adapted to the Kähler form (which is also a symplectic form). We give a self-contained derivation of this coordinate system in Appendix 3.1. These give a coordinate system $(x_i, \phi^i)$ on $B$ and hence on $U$ away from the axis.

We will now consider computing the remaining data $(f, \omega)$. For this we introduce a basis of ASD 2-forms $X^{(i)}$ where $X^{(1)}$ is the Kähler form given above. We find that

$$X^{(2)} = \sqrt{\det G} d\phi^1 \wedge d\phi^2 - \frac{1}{\sqrt{\det G}} dx_1 \wedge dx_2 , \quad (3.15)$$

$$X^{(3)} = 2\sqrt{\det G} G^{ij} dx_i \wedge d\phi^j , \quad (3.16)$$

where $\det G := \det G_{ij}$, are ASD and together with $X^{(1)}$ satisfy the quaternion algebra (3.5). Using these one can compute the potential $P$ for the Ricci form from (3.6) which is

$$P = P_i d\phi^i , \quad P_i = -\frac{1}{2} G_{ij} \partial^i \log \det G = -\frac{1}{2} \partial^i G_{ij} . \quad (3.17)$$

The scalar curvature is

$$R = -\partial^i \partial^j G_{ij} , \quad (3.18)$$

which from (3.3) immediately gives

$$f^{-1} = \frac{\ell^2}{24} \partial^i \partial^j G_{ij} . \quad (3.19)$$

It remains to solve for the 1-form $\omega$.

To this end, observe that given any closed 2-form $\Omega$ on $M$ invariant under the toric symmetry, $\iota_{m_1} \iota_{m_2} \Omega$ must be a constant. Therefore, since by assumption we have a non-empty axis, we deduce that $\iota_{m_1} \iota_{m_2} \Omega = 0$ for any closed 2-form invariant under the toric symmetry. Thus, contracting (3.2) we deduce $\iota_{m_1} \iota_{m_2} G^+ = 0$ which simplifies to

$$\iota_{m_1} \iota_{m_2} \ast_4 d\omega = 0 \iff m_1^i \wedge m_2^j \wedge d\omega = 0 , \quad (3.20)$$

where $m^i_1$ is the dual of $m_i$ with respect to the base metric $h$. Thus, since $m^i_1 = G_{ij} d\phi^j$ this becomes $d\phi^1 \wedge d\phi^2 \wedge d\omega = 0$. Writing $\omega = \alpha^i dx_i + \omega_i d\phi^i$ this implies that $\alpha^i = \partial^i \alpha$ for some function $\alpha(x)$ on $B$. Therefore, by a gauge transformation $\omega \rightarrow \omega + d\lambda$ which preserves our gauge $L_m \omega = 0$ we may set $\alpha = 0$. We have thus shown that we can always choose a gauge such that

$$\omega = \omega_i d\phi^i , \quad (3.21)$$

where the components $\omega_i$ are invariant under the toric symmetry and hence are functions $\omega_i = \omega_i(x)$ (observe that we can write $\omega_i = \iota_{m_i} \omega$ as invariants).

Now consider the equation for $\omega$ which is given by (3.8). It is easy to see that from the form of $\omega$ derived above (3.21), i.e. the absence of $x_i$ components, $d\omega$ will not have any $x_1, x_2$ components. Therefore the coefficient of $X^{(2)}$ in the expansion (3.7) for $G^-$ must vanish, $\lambda_2 = 0$. The coefficient $\lambda_3$ must be invariant under the toric symmetry (since $X^{(3)}$ is invariant) and hence is a function $\lambda_3 = \lambda_3(x)$. The integrability condition for (3.8) now gives a linear first order PDE for $\lambda_3(x)$. The integrability condition for this PDE is equivalent to (3.9) in the toric class we are considering. Unfortunately, we have not found a useful way to write this integrability condition in the toric class nor a general solution for $\omega_i$. This prevents us from giving an explicit general local solution for the toric class.

To summarise, so far we have shown that a timelike supersymmetric toric solution in symplectic coordinates on the base takes the form

$$g = -f^2 (dt + \omega_i d\phi^i)^2 + f^{-1} G_{ij} d\phi^i d\phi^j + f^{-1} G^{ij} dx_i dx_j , \quad (3.22)$$

where $G^{ij}$ is given by (3.13) and $G_{ij}$ is its matrix inverse, $f$ is given by (3.19), $\omega_i$ is determined by (3.8) and the symplectic potential $g$ must satisfy (3.9). We emphasise that any such solu-
invariants complicated PDE determined by supersymmetry. It is useful to record the following spacetime invariants

\[ g(V, V) = -f^2, \quad g(V, m_i) = -f^2 \omega_i, \quad g(m_i, m_j) = f^{-1} G_{ij} - f^2 \omega_i \omega_j. \]  

(3.23)

From these we can define the Gram matrix of Killings fields \( K_{\alpha \beta} := g(K_{\alpha}, K_{\beta}) \), where \( \alpha = 0, i \), \( K_0 = V, K_i = m_i \), which gives

\[ \det K_{\alpha \beta} = -\det G_{ij}. \]  

(3.24)

Observe that all these spacetime functions are invariant under the Killing fields.

We will be interested in solutions that possess a supersymmetric horizon, that is, a horizon that is invariant under the supersymmetric Killing field \( V \), the most notable example being an event horizon. In what follows we will show that the horizon and the axis have a simple description in symplectic coordinates and the singular behaviour of the symplectic potential near a horizon or the axis can be completely fixed.

### 3.2.3 Horizons, axis and orbit space

We are interested in spacetimes containing a black hole region. In this context \( M \) will denote the domain of outer communication and the event horizon is its inner boundary. It has been shown that under certain reasonable global assumptions the orbit space \( \hat{M} := M/(\mathbb{R} \times U(1)^2) \), where the \( \mathbb{R} \times U(1)^2 \) action is given by the flow of \( V, m_i \), is a 2-dimensional manifold with boundaries and corners [33, 106]. The global assumptions that are made to obtain this result are that \( M = \mathbb{R} \times \Sigma \), where \( V \) is tangent to \( \mathbb{R} \) and \( \Sigma \) is a simply connected manifold \(^2\). The boundary segments of \( \hat{M} \) correspond to either horizon components, or components of the axis where an integer linear combination of the axial Killing fields \( v := v^i m_i \), where \( v^i \in \mathbb{Z}^2 \) are coprime, vanishes. The corners correspond to fixed points of the toric symmetry (i.e. where \( m_1 = m_2 = 0 \)) or where the axis meets a horizon. In order to avoid an orbifold singularity at the fixed points of the torus symmetry one must satisfy [33]

\[ \det(v, w) = \pm 1, \]  

(3.25)

where \( v = (v^1, v^2) \in \mathbb{Z}^2 \) etc are the components of the axial vectors \( v = v^i m_i \) and \( w = w^i m_i \) that vanish on the adjacent axis components.

Now, as noted above the spacetime invariants (3.23) are preserved by the Killing fields and hence descend to functions on the orbit space \( \hat{M} \). Furthermore, the functions \( x_i \) defined by (3.11) are also preserved by the Killing fields and hence can be used as local coordinates on the orbit space. In fact, the orbit space inherits a metric wherever \( V \) is timelike, defined by \( q_{\mu \nu} := g_{\mu \nu} - K^{\alpha \beta} g_{\alpha \mu} g_{\beta \nu} \) where \( K^{\alpha \beta} \) is the inverse matrix of \( K_{\alpha \beta} \), which using (3.22) gives

\[ q = f^{-1} G^{ij} dx_i dx_j. \]  

(3.26)

This gives a Riemannian metric on the orbit space if and only if \( \det G_{ij} > 0 \) which from (3.24) can be seen to correspond to the region away from the axis.

We will now describe the orbit space in terms of symplectic coordinates. We expect that it is possible to show that symplectic coordinates give a global chart on \( \hat{M} \) (at least if \( V \) is strictly timelike in the exterior region), although this would presumably require making certain global assumptions and in particular that the spacetime is asymptotically (locally) AdS \(^3\). We will not make any such global assumptions in this chapter and hence not pursue this question here. Instead, in this subsection, we will simply assume that a single symplectic chart covers all components of the axis, as is indeed the case for the known solutions discussed below in section 3.2.5.

\(^2\)It is also assumed that the torus action is effective and has no discrete isotropy groups. The latter assumption has been removed for asymptotically flat spacetimes [33].

\(^3\)This would be analogous to showing that the Weyl-Papapetrou coordinates give a global chart on the interior of the orbit space for asymptotically flat stationary and (bi)axisymmetric spacetimes, see e.g. [106].
Lemma 7. Consider a supersymmetric and toric solution that is timelike on a neighbourhood of a component of the axis defined by the vanishing of \( v := v'm_1 \), where \( (v') \in \mathbb{Z}^2 \) are coprime integers, and let
\[
\ell_v(x) := v'x_1 + c_v ,
\] where \( c_v \) is a constant. Then:

1. The axis component corresponds to a straight line \( \ell_v(x) = 0 \) in symplectic coordinates and away from the axis
\[
\ell_v(x) > 0
\] (3.28)

2. The symplectic potential can be written as
\[
g = \frac{1}{2} \ell_v(x) \log \ell_v(x) + \tilde{g} ,
\] (3.29)
where \( \tilde{g} \) is smooth at \( \ell_v(x) = 0 \).

Proof. From (3.11) it follows that \( \ell_v X^{(1)} = -df_v \). Since \( X^{(1)} \) is a non-degenerate 2-form on the base \( B \), we see that \( v = v'm_1 = 0 \) if and only if \( df_v = 0 \). Thus \( \ell_v \) is a constant on the axis component defined by \( v = 0 \), and we may choose \( c_v \) so that \( \ell_v(x) = 0 \) on the axis component. We must have \( \ell_v(x) \neq 0 \) away from the axis component and since this corresponds to a boundary component of the orbit space we can choose \( v' \) such that \( \ell_v(x) > 0 \) corresponds to the interior of \( M \).

To prove the second part we will derive the geometry near a component of the axis (3.27). By a suitable \( GL(2, \mathbb{Z}) \) transformation we can always arrange \( v = m_1 = \partial_\phi \) and define new symplectic coordinates so that \( \ell_v(x) = x_1 \). Thus using (3.23) and that the inner products \( V \cdot m_1 = 0 \) and \( m_1 \cdot m_1 = 0 \) on this axis, we deduce that \( \omega_1 = 0 \) and \( G_{i1} = 0 \) at \( x = 0 \) (we set \( x = x_1, y = x_2 \) for clarity). Since these invariants are smooth functions it follows that \( G_{i1} = O(x) \) and \( \omega_1 = O(x) \) so in particular we can write
\[
G_{ij}d\phi^i d\phi^j = xa(x,y)(d\phi^1)^2 + 2xb(x,y)d\phi^1 d\phi^2 + c(x,y)(d\phi^2)^2 ,
\] (3.30)
where \( a, b, c \) are smooth at \( x = 0 \) and \( c > 0 \) at \( x = 0 \) to ensure we are not at a fixed point of the toric symmetry (this is because \( G_{22} = 0 \) and (3.23) implies \( m_2 \cdot m_2 = 0 \) and hence \( m_2 = 0 \)). It follows that \( \det G_{ij} = \delta \) where \( \delta = ac - \frac{1}{x} b^2 \) and
\[
G^{ij}dx_idx_j = \frac{1}{\delta} \left( \frac{x}{2 c} dx^2 - 2bdxdy + ady^2 \right) .
\] (3.31)
Now consider smoothness of the Kähler base metric (3.12) near \( x = 0 \). To perform the analysis it is useful to set \( x = r^2 \), in terms of which the full metric reads,
\[
h = \frac{4c}{\delta} dr^2 + r^2 a(d\phi^1)^2 + \frac{1}{\delta} \left(-4b r dr dy + ady^2 \right) + 2r^2 b d\phi^1 d\phi^2 + c(d\phi^2)^2 .
\] (3.32)
It is then standard to verify that this metric extends to a smooth metric at \( r = 0 \) if and only if
\[
a|_{x=0} = 2 ,
\] (3.33)
where recall we assume the period of \( \phi^i \) is \( 2\pi \) (this can be seen by changing to cartesian coordinates in the \( (r, \phi^i) \) plane). It follows that as \( x \to 0 \)
\[
G^{ij}dx_idx_j = \left( \frac{1}{2x} + O(1) \right) dx^2 + O(1) dx dy + (c^{-1} + O(x)) dy^2 .
\] (3.34)
Integrating for the symplectic potential gives
\[
g(x,y) = \frac{1}{2} x \log x + \tilde{g}(x,y) ,
\] (3.35)
where \( \tilde{g} \) is smooth at \( x = 0 \) with \( \tilde{g}_{yy} = c^{-1} + O(x) \). Therefore, we have fully determined the singular behaviour of the symplectic potential near an axis of symmetry. Returning to our original basis we deduce that the symplectic potential takes the claimed form.

Recall that any fixed point must occur at the intersection of two axis components, so Lemma 7 immediately implies that fixed points of the toric symmetry are single points in symplectic coordinates that occur at the intersection of the two corresponding axis line components. In fact, the singular behaviour of the symplectic potential at a fixed point arises purely from that of the two components of the axis which define it, as our next result shows.

**Lemma 8.** Consider a fixed point defined by the vanishing of \( v := v^i m_i \) and \( w := w^i m_i \), where \( (v^i) \in \mathbb{Z}^2 \) and \( (w^i) \in \mathbb{Z}^2 \) are pairs of coprime integers that satisfy (3.25). Let \( \ell_v(x) = 0 \) and \( \ell_w(x) = 0 \) denote the lines that correspond to the axis components defined by the vectors \( v, w \) as in (3.27).

1. The fixed point in symplectic coordinates is given by the point defined by \( \ell_v(x) = \ell_w(x) = 0 \).
2. The symplectic potential takes the form

\[
\delta \mathcal{L} = \frac{1}{2} \ell_v(x) \log \ell_v(x) + \frac{1}{2} \ell_w(x) \log \ell_w(x) + \tilde{g},
\]

where \( \tilde{g} \) is smooth at both axis components and at the fixed point. Furthermore, a neighbourhood of the fixed point in the base is diffeomorphic to \( \mathbb{R}^4 \).

**Proof.** The proof is along the lines of the proof of the second part of Lemma Lemma 7 applied for each axis component, which also gives us the behaviour of the Kähler metric at the intersection point. Since \( v \) and \( w \) satisfy (3.25), a \( GL(2, \mathbb{Z}) \) transformation can be used to set \( v = m_1 = \partial_{\phi^1} \), \( w = m_2 = \partial_{\phi^2} \) and define new symplectic coordinates so that \( \ell_v(x) = x_1, \ell_w(x) = x_2 \). Using again the invariants (3.23) (we set again \( x = x_1, y = x_2 \)) together with their smoothness at the axes, we can now write

\[
G_{ij} d\phi^i d\phi^j = x\tilde{a}(x, y)(d\phi^1)^2 + 2xy\tilde{b}(x, y)d\phi^1 d\phi^2 + y\tilde{c}(x, y)(d\phi^2)^2,
\]

where \( \tilde{a}, \tilde{b}, \tilde{c} \) are smooth on both the lines \( x \) and \( y \) and the point \( x = y = 0 \). The Kähler metric reads

\[
h = \frac{4\tilde{c}}{\delta} dr_1^2 + \tilde{a}r_1^2 (d\phi^1)^2 + \frac{4\tilde{a}}{\delta} dr_2^2 + \tilde{c}r_2^2 (d\phi^2)^2 - \frac{8\delta}{\tilde{c}} r_1 r_2 dr_1 dr_2 + 2r_1^2 r_2^2 \tilde{b} d\phi^1 d\phi^2,
\]

where \( \delta = \tilde{a} - xy \tilde{b}^2 = \tilde{a} - r_1^2 r_2^2 \tilde{b}^2 \) and we have set \( x = r_1^2, y = r_2^2 \). The metric is free of conical singularities and smooth at \( r_1 = 0 \) and \( r_2 = 0 \) if and only if

\[
\tilde{a}|_{x=0} = \tilde{c}|_{y=0} = 2,
\]

and by continuity this also holds at the fixed point. Then, the Kähler metric (3.38) approaches the flat metric \( \sum_{i=1}^4 dr_i^2 + r_i^2 (d\phi^i)^2 \) on \( \mathbb{R}^4 \) as \( r_1, r_2 \to 0 \) with the error terms smooth at \( r_1 = 0 \) (this can be seen by converting to cartesian coordinates in the \( (r_i, \phi^i) \) planes). This shows that the Kähler base near the fixed point is diffeomorphic to \( \mathbb{R}^4 \). Furthermore, we have

\[
G^{ij} dr_i dr_j = \left( \frac{1}{2x} + O(1) \right) dx^2 + \left( \frac{1}{2y} + O(1) \right) dy^2 + O(1) dx dy,
\]

where the \( O(1) \) terms are smooth at \( x = 0, y = 0 \) and \( x = y = 0 \). Integrating we find

\[
g(x, y) = \frac{1}{2} x \log x + \frac{1}{2} y \log y + \tilde{g}(x, y),
\]

where \( \tilde{g} \) is smooth at \( x = 0, y = 0 \) and \( x = y = 0 \), thus completing the proof.
We now show that a horizon also has a simple description in symplectic coordinates. Recall that the event horizon of a black hole spacetime must be invariant under any Killing field and, hence, in particular under the supersymmetric Killing field $V$. In this thesis we are considering supersymmetric horizons, that is, horizons that are invariant under the supersymmetric Killing field $V$. As usual, we assume that each connected component of the horizon possesses a cross-section $S$, that is, a spacelike submanifold everywhere transverse to $V$. The general form of the metric near a connected component of a supersymmetric horizon can be written in Gaussian null coordinates (GNC) $(v, \lambda, y^a)$ and takes the form \[ g = -\lambda^2 \Delta^2 dv^2 + 2dv d\lambda + 2\lambda h_a dv dy^a + \gamma_{ab} dy^a dy^b, \] (3.42)

where $V = \partial_v$ is the supersymmetric Killing field, $\lambda$ is an affine parameter for null transverse geodesics synchronised so $\lambda = 0$ at the horizon, and $y^a$ are coordinates on a cross-section $S$. Here the data $\Delta, h_a, \gamma_{ab}$ depend on $(\lambda, y^a)$ and is smooth at $\lambda = 0$. The near-horizon limit is defined by rescaling $(v, \lambda, y^a) \to (v/\epsilon, \epsilon \lambda, y^a)$ and then taking the limit $\epsilon \to 0$, resulting in a metric of the same form with $\Delta, h_a, \gamma_{ab}$ replaced by their values at $\lambda = 0$, denoted by $\Delta^{(0)}, h_a^{(0)}, \gamma_{ab}^{(0)}$, which are respectively a function, 1-form and Riemannian metric on $S$. The following result does not require the detailed form of the near-horizon geometry and is typical of extremal horizons (see e.g. [103] in the asymptotically flat case).

**Lemma 9.** A connected component of the horizon corresponds to a point in symplectic coordinates.

**Proof.** For this we need to determine the change of coordinates between symplectic coordinates and GNC. First note that the Killing fields $m_i$ must be tangent to the horizon and furthermore since they have closed orbits we may choose them to be tangent to a cross-section $S$. Then, one can always choose GNC that are also adapted to the toric Killing fields $m_i$. In particular, this implies $[m_i, \partial_\lambda] = 0$.

The most direct way to find the coordinate change is from the Kähler form which in GNC can be written as \[ X^{(1)} = d\lambda \wedge Z + \lambda (h \wedge Z - \Delta \ast_3 Z), \] (3.43)

where $Z = Z_a dy^a$ is a unit norm 1-form and $\ast_3$ is the Hodge star operator with respect to the metric $\gamma_{ab}$. Closure of $X^{(1)}$ is equivalent to \[ \mathring{d}Z = h \wedge Z - \Delta \ast_3 Z + \lambda \partial_\lambda (h \wedge Z - \Delta \ast_3 Z), \] (3.44)

where $\mathring{d}$ is the exterior derivative tangent to constant $(v, \lambda)$ surfaces (on the horizon it is the exterior derivative on $S$). Now, since $X^{(1)}$ is invariant under the toric symmetry, it follows that $Z$ is a $U(1)^2$-invariant 1-form; this can be seen explicitly by noting that \[ Z = \iota_{\partial_\lambda} X^{(1)} \] (3.45)

and since $[m_i, \partial_\lambda] = 0$ the result immediately follows. Next, by definition of the symplectic coordinates (3.11) we find \[ d(x_i - \iota_{m_i} Z_i) = -\lambda \partial_\lambda Z_i d\lambda + \lambda^2 \partial_\lambda (h_i Z_i - h_i \Delta m_i \ast_3 Z), \] (3.46)

where $Z_i = \iota_{m_i} Z$ and we have used $dZ = \mathring{d}Z + d\lambda \wedge \partial_\lambda Z$. Taking the $\lambda$ and $y^a$ components of this expression and integrating we find \[ x_i = \lambda Z_i + O(\lambda^2), \] (3.47)

where the higher order terms are smooth at $\lambda = 0$ and we have set the integration constants to zero. This shows that a horizon corresponds to an isolated point in symplectic coordinates, which we take to be the origin. \[ \square \]

We are now ready to deduce the structure of the orbit space in symplectic coordinates.

**Lemma 10.** Consider a supersymmetric solution with toric symmetry that is timelike on a neighbourhood outside the horizon (we allow the horizon to have multiple components). If

\[ 64 \]
we have $N$ axis components defined by the lines $\ell_A(x) := v_A^i x_i + c_A$, $A = 1, \ldots, N$, where $v_A := v_A^i m_i = 0$ define each axis component:

1. The orbit space can be identified with the region (see fig. 3.1)

$$\{\ell_A(x) \geq 0 : A = 1, \ldots, N\} \subset \mathbb{R}^2,$$

(3.48)

where the boundary components $\ell_A(x) = 0$ are axis components, and the fixed points and horizon components correspond to points of intersection of these lines.

2. The symplectic potential takes the form

$$g = \sum_{A=1}^{N} \frac{1}{2} \ell_A(x) \log \ell_A(x) + \tilde{g}$$

(3.49)

where $\tilde{g}$ is smooth on (3.48) including at the lines $\ell_A(x) = 0$ and the points of intersection corresponding to fixed points, except possibly at the points of intersection corresponding to horizons.

Proof. The general form of a supersymmetric near-horizon geometry with toric symmetry has been determined [44]. It was found that cross-sections of any component of the horizon must be topologically $S^3$ (or quotients) and the geometry is locally isometric to that of the CCLP black hole. In particular, the toric symmetry degenerates at two points on the sphere (the poles) where different linear combinations of the axial Killing fields vanish. Hence, any component of the horizon is intersected by two distinct axis components, and so by Lemma 9 must occur precisely at the point of intersection of the corresponding axis lines. The remaining claims follow from Lemma 7 and Lemma 8.

![Figure 3.1: The orbit space for a generic toric Kähler base in the $x_1$,$x_2$-plane where $x_i$ are moment maps associated with the axial Killing fields $m_i$ with $2\pi$-periodic orbits. This includes both the boundary segments and the corners which can be either horizon components (white dots) or fixed points of the torus symmetry (black dots).](image)

The above result determines the structure of the orbit space in symplectic coordinates. A schematic diagram for such orbit spaces is depicted in Figure fig. 3.1. Furthermore, we have fully determined the behaviour of the symplectic potential near any component of the axis. In fact we have recovered the decomposition theorem for the symplectic potential of Abreu [104] in terms of a canonical potential and a remainder that is smooth on the axes of symmetry (note we do not assume compactness). In the next section we will use the detailed form of the general near-horizon geometry to show that the symplectic potential is similarly singular at the horizon. This will give a refinement of the second part of Lemma 10.

### 3.2.4 Near-horizon geometry and general form of symplectic potential

In this section we will determine the general form of the symplectic potential near any connected component of a supersymmetric horizon. The strategy is to start with the near-horizon geometry (3.42), find the coordinate change between GNC and symplectic coordinates and then match to the general form for a supersymmetric toric solution (3.22).
To this end we first note that comparing the invariants (3.23) yields
\[
f = \lambda \Delta, \quad \omega_i = -\frac{h_i}{\lambda \Delta^2}, \quad G_{ij} = \lambda \Delta \left( \gamma_{ij} + \frac{h_i h_j}{\Delta^2} \right),
\]  
(3.50)
where \( \gamma_{ij} = \gamma(m_i, m_j) \) and we have chosen a sign so \( \Delta > 0 \), and inverting the matrix \( G_{ij} \) we find
\[
G^{ij} = \frac{1}{\lambda \Delta} \left( \gamma^{ij} - \frac{h^i h^j}{\Delta^2 + h^k h_k} \right),
\]  
(3.51)
where \( h^i := \gamma^{ij} h_j \) and \( \gamma^{ij} \) is the inverse matrix of \( \gamma_{ij} \).

We now turn to the explicit form of the near-horizon geometry. The general near-horizon geometry admitting \( U(1)^2 \) rotational isometry and a smooth compact \( S \) was determined in [44]. We present it here in a coordinate system that also describes the special case with \( SU(2) \times U(1) \) symmetry, see Appendix 3.C for details.\(^4\) This ‘unified’ form of the near-horizon geometry also makes the proof of Theorem 5 more transparent. The near-horizon geometry depends \( \text{only} \) on two constant parameters \( 0 < A^2, B^2 < 1 \)\(^5\) subject to
\[
\kappa^2(A^2, B^2) > 0,
\]  
(3.52)
where
\[
\kappa^2(A^2, B^2) := -9A^4 B^4 + 6A^2 B^2 (A^2 + B^2 + 1)^2 - (A^2 + B^2 + 1)^3 \left( A^2 + B^2 - \frac{1}{3} \right).
\]  
(3.53)

The near-horizon data explicitly reads
\[
\Delta^{(0)} = \frac{3\kappa}{4\Delta_2(\hat{\eta})^2},
\]  
(3.54)
\[
h^{(0)} = \frac{3\kappa \Delta_3(\hat{\eta})}{4\Delta_2(\hat{\eta})^3} \hat{\sigma} + \frac{3(A^2 - B^2)}{2\Delta_2(\hat{\eta})} \left( \frac{d\hat{\eta}}{\lambda - \frac{3\kappa}{2\Delta_2(\hat{\eta})}} \hat{\tau} \right),
\]  
(3.55)
\[
\gamma^{(0)} = \frac{\ell^2}{12(1 - \hat{\eta}^2)\Delta_2(\hat{\eta})} \left( \Delta_2(\hat{\eta}) d\hat{\eta}^2 + \frac{3\Delta_3(\hat{\eta})^2 + \kappa^2}{4\Delta_2(\hat{\eta})^2} \hat{\tau}^2 \right) + \frac{\ell^2}{48\Delta_2(\hat{\eta})^2} \left( \frac{3\Delta_3(\hat{\eta})(A^2 - B^2)}{8\Delta_2(\hat{\eta})^2} \hat{\sigma} \hat{\tau} \right),
\]  
(3.56)
where we have defined the 1-forms
\[
\hat{\sigma} = \frac{1 - \hat{\eta}}{A^2} d\hat{\phi}^1 + \frac{1 + \hat{\eta}}{B^2} d\hat{\phi}^2, \quad \hat{\tau} = (1 - \hat{\eta}^2) \Delta_1(\hat{\eta}) \left( \frac{d\hat{\phi}^1}{A^2} - \frac{d\hat{\phi}^2}{B^2} \right),
\]  
(3.57)
and three linear functions of \( \hat{\eta} \)
\[
\Delta_1(\hat{\eta}) = \frac{1 + \hat{\eta}}{2} A^2 + \frac{1 - \hat{\eta}}{2} B^2,
\]  
(3.58)
\[
\Delta_2(\hat{\eta}) = 1 - \frac{1 + 3\hat{\eta}}{2} A^2 - \frac{1 - 3\hat{\eta}}{2} B^2,
\]  
\[
\Delta_3(\hat{\eta}) = 1 - 2\Delta_2(\hat{\eta}) + A^2 B^2 - A^4 - B^4,
\]
where \( (\hat{\eta}, \hat{\phi}^i) \) are coordinates on \( S \) with \(-1 \leq \hat{\eta} \leq 1\) and \( \hat{\phi}^i \sim \hat{\phi}^i + 2\pi \) are adapted to the Killing fields \( m_i = \partial_{\hat{\phi}^i} \) and \( \Delta_1 \) and \( \Delta_2 \) are strictly positive functions\(^6\). The 1-form that determines the

---

\(^4\)In the notation of [44] this limit corresponds to the function \( \Gamma \) being constant and therefore not being a valid coordinate.

\(^5\)Note that (3.52) actually implies \( A^2, B^2 < 1 \).

\(^6\)We use hats to stress that these are coordinates valid at the horizon and they are different than the ones we will use in section 3.3 to describe the full solution (however \( \partial_{\hat{\phi}^i} = \partial_{\phi^i} \)).
Kähler form (3.43) is given by

\[ Z^{(0)} = \frac{\ell}{4\Delta_2(\hat{\eta})} \left( \kappa \hat{\sigma} - 3(\mathcal{A}^2 - B^2) \, d\hat{\eta} \right). \]  

(3.59)

Note that solutions with \( \mathcal{A}^2 \neq B^2 \) are doubly counted with the two copies related by (3.199). The solutions with \( \mathcal{A}^2 = B^2 \) have enhanced \( SU(2) \times U(1) \) symmetry.

It is now straightforward to determine the explicit near-horizon behaviour of \( \omega_i, G_{ij} \) and \( G^{ij} \). Evaluating (3.59) at leading order in \( \lambda \) we find

\[ \omega_i = -\frac{\ell^2 \Delta_2(\hat{\eta})}{12\kappa} \left( \Delta_3(\hat{\eta}) \hat{\sigma} - 3(\mathcal{A}^2 - B^2) \hat{\tau} \right)_i \frac{1}{\lambda} + O(1), \]  

(3.60)

and

\[ G_{ij} = \frac{\ell \kappa}{4\Delta_2(\hat{\eta})} \left( \hat{\sigma}^2 + \frac{\hat{\tau}^2}{(1 - \hat{\eta}^2)\Delta_1(\hat{\eta})} \right)_{ij} \lambda + O(\lambda^2), \]  

(3.61)

with \( G^{ij} \) being the inverse of (3.61). By expressing \( G^{ij} \) in the \( x_i \) coordinates, we can extract the near-horizon behaviour of the symplectic potential. To this end, we insert (3.59) into (3.47) to find

\[ x_1 = \frac{\ell \kappa}{4\Delta_2(\hat{\eta})} \frac{1 - \hat{\eta}}{\mathcal{A}^2} \lambda + O(\lambda^2), \quad x_2 = \frac{\ell \kappa}{4\Delta_2(\hat{\eta})} \frac{1 + \hat{\eta}}{B^2} \lambda + O(\lambda^2). \]  

(3.62)

The inverse coordinate change to leading order is then

\[ \lambda = \frac{2}{\ell \kappa} (\mathcal{A}^2(1 + \mathcal{A}^2 - 2B^2)x_1 + B^2(1 + B^2 - 2\mathcal{A}^2)x_2) + O(x^2), \]

\[ \hat{\eta} = -\frac{A^2 x_1 - B^2 x_2}{A^2 x_1 + B^2 x_2} + O(x), \]  

(3.63)

where \( O(x) \) denotes terms of order \( x_i \), \( O(x^2) \) terms of order \( x_i x_j \) etc. The components of \( G^{ij} \) are then given by

\[ G^{11} = \frac{1}{2} \left( \frac{\mathcal{A}^4}{\mathcal{A}^2 x_1 + B^2 x_2} + \frac{x_2}{x_1(x_1 + x_2)} \right) + O(1), \]

\[ G^{12} = \frac{1}{2} \left( \frac{\mathcal{A}^2 B^2}{\mathcal{A}^2 x_1 + B^2 x_2} - \frac{1}{x_1 + x_2} \right) + O(1), \]

\[ G^{22} = \frac{1}{2} \left( \frac{B^4}{\mathcal{A}^2 x_1 + B^2 x_2} + \frac{x_1}{x_2(x_1 + x_2)} \right) + O(1), \]  

(3.64)

where the \( O(1) \) term is a smooth function of \( (x_1, x_2) \) at the horizon \( x_1 = x_2 = 0 \). These expressions can be readily integrated to give us the symplectic potential using (3.13). We thus obtain

\[ g = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2 - \frac{1}{2} (x_1 + x_2) \log(x_1 + x_2) + \frac{1}{2} (\mathcal{A}^2 x_1 + B^2 x_2) \log(\mathcal{A}^2 x_1 + B^2 x_2) + \tilde{g}, \]  

(3.65)

where \( \tilde{g} \) is a smooth function at the origin which arises from integrating \( O(1) \) in (3.64). Recall \( g \) is only defined up to an additive linear function of \( x_i \) and therefore we may assume that \( \tilde{g} \) vanishes quadratically in \( x_i \) at the horizon. Thus we have fixed the leading singular term in \( g \) purely from the near-horizon geometry and the subleading terms are in fact \( O(\lambda^2) \).

From the above we can immediately deduce the following result which characterises the singular behaviour of the symplectic potential near any horizon.

**Lemma 11.** Consider a supersymmetric toric solution that is timelike outside a supersymmetric horizon with compact cross-sections. Let \( v_{\pm} = v_0^i m_i \) be the \( 2\pi \)-periodic Killing fields that have
fixed points on a connected component of the horizon. The symplectic potential takes the form
\[
g = \frac{1}{2} \ell_+(x) \log \ell_+(x) + \frac{1}{2} \ell_-(x) \log \ell_-(x) - \frac{1}{2} (\ell_+(x) + \ell_-(x)) \log(\ell_+(x) + \ell_-(x)) \\
+ \frac{1}{2} (A^2 \ell_+(x) + B^2 \ell_-(x)) \log(A^2 \ell_+(x) + B^2 \ell_-(x)) + \tilde{g}
\]
where \( \ell_\pm(x) = v_\pm x_1 \), we have taken the horizon component to be at the origin, and \( \tilde{g} \) is smooth at this horizon component.

We can now combine Lemma 10 and Lemma 11 to deduce the general form for the symplectic potential.

**Theorem 7.** The symplectic potential for a supersymmetric toric solution as in Lemmas 10 and 11 takes the form
\[
g = \sum_{A = 1}^{N} \frac{1}{2} \ell_A(x) \log \ell_A(x) - \sum_{A \in H} \frac{1}{2} (\ell_A(x) + \ell_A+1(x)) \log(\ell_A(x) + \ell_A+1(x)) \\
+ \sum_{A \in H} \frac{1}{2} (A^2 \ell_A(x) + B^2 \ell_A+1(x)) \log(A^2 \ell_A(x) + B^2 \ell_A+1(x)) + \tilde{g},
\]
where \( A \in H \) iff the point defined by \( \ell_A(x) = \ell_A+1(x) = 0 \) corresponds to a component of the horizon and \( A, B \) are parameters of the corresponding near-horizon geometry, and \( \tilde{g} \) is smooth on (3.48) including at the boundaries and corners.

Recall that in Lemma 10 we assumed that a single symplectic chart contained all components of the axis. As discussed at the start of section 3.2.3, we anticipate that under certain reasonable global assumptions (including the asymptotics) one should be able to prove that symplectic coordinates provide a global chart of the exterior region of a black hole spacetime. Theorem 7 then gives the general form for a symplectic potential for a supersymmetric AdS black hole spacetime with toric symmetry of arbitrary topology. It is an analogue of Abreu’s result for compact toric Kähler manifolds [104].

### 3.2.5 Examples

In this section we will first list the known solutions and their symplectic potentials. Then using the above results we will write down the form of the symplectic potential for possible new black hole solutions with non-trivial topology.

**AdS\(_5\) and known black holes**

The simplest relevant examples of toric Kähler metrics are given by the \( SU(2) \times U(1) \) invariant Kähler metrics
\[
h = \frac{dr^2}{V(r)} + \alpha^2 r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + 4\alpha^4 r^2 V(r)(d\psi + \cos \vartheta d\varphi)^2,
\]
\[
X^{(1)} = d(\alpha^2 r^2 (d\psi + \cos \vartheta d\varphi)),
\]
where \( V(r) > 0 \) is an arbitrary function, \( \alpha \) is a constant, and \( (\vartheta, \psi, \varphi) \) are Euler coordinates on \( S^3 \). This family includes the important case of the Bergmann metric for \( V = 1 + \frac{r_0}{r} \) and \( \alpha = 1/2 \) which is the base space of global AdS\(_5\) (normalised so \( f = 1 \)). For \( \alpha > 1/2 \) this class includes the base of the GR black hole which also has \( V = 1 + \frac{r_0}{r} \), or its near-horizon geometry which has \( V = 1 \) (also a supersymmetric solution).

We now write these solutions in symplectic coordinates. It is convenient to define \( 2\pi \)-periodic angles \( \phi' \) by \( \psi = \phi^1 + \phi^2 \) and \( \varphi = -\phi^1 + \phi^2 \) so that \( \partial_{\phi^1} = \partial_\psi - \partial_\varphi \) and \( \partial_{\phi^2} = \partial_\psi + \partial_\varphi \). Then,

\footnote{Compared to [82] we have rescaled \( r_{\text{here}} = 2\alpha r_{\text{there}} \).}
from (3.11) we deduce that the symplectic coordinates adapted to $m_i = \partial_{\psi^i}$ are given by
\[ x_1 = 2\alpha^2 r^2 \sin^2(\theta/2), \quad x_2 = 2\alpha^2 r^2 \cos^2(\theta/2). \] (3.69)

Thus the component of the axis $\theta = 0$ ($r > 0$) corresponds to $x_1 = 0$ on which $\partial_{\psi^1} = 0$, and the axis $\theta = \pi$ ($r > 0$) corresponds to $x_2 = 0$ on which $\partial_{\psi^2} = 0$. The interior of the orbit space is $x_1 > 0, x_2 > 0$ and the asymptotic region is $x_1 + x_2 \to \infty$. Thus the image of the moment maps is the upper right quarter $x_1, x_2$-plane. The corner $x_1 = x_2 = 0$ ($r = 0$) is where the two axis components meet and the torus symmetry has a fixed point. Now, comparing to (3.12) and (3.13), we find that the symplectic potential for the Bergmann metric is
\[ g_B = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2 - \frac{1}{2} (x_1 + x_2 + \frac{1}{2} \ell^2) \log(x_1 + x_2 + \frac{1}{2} \ell^2). \] (3.70)

Notice this is related to the symplectic potential of the Fubini-Study metric on $\mathbb{C}P^2$ by an analytic continuation $\ell^2 \to -\ell^2$.

For the GR black hole we find the symplectic potentials takes a remarkably similar form
\[ g_{\text{GR}} = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2 + \left(\frac{1}{8\alpha^2} - \frac{1}{2}\right) (x_1 + x_2) \log(x_1 + x_2) \]
\[ -\frac{1}{8\alpha^2} (x_1 + x_2 + 2\alpha^2 \ell^2) \log(x_1 + x_2 + 2\alpha^2 \ell^2). \] (3.71)

Observe that for $\alpha = 1/2$ this reduces to that for the Bergmann metric (as it should). For the near-horizon geometry of the GR black hole we find
\[ g_{\text{NHGR}} = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2 + \left(\frac{1}{8\alpha^2} - \frac{1}{2}\right) (x_1 + x_2) \log(x_1 + x_2), \] (3.72)

which is the same as the black hole potential (3.71) with the last term omitted.

We now consider more generic solutions with toric symmetry. The CCLP black hole (as well as its near-horizon geometry) is the most general known black hole solution in this symmetry class. We write it in a compact form in terms of parameters $A$ and $B$ in Appendix 3.B. The metric and the Kähler form for the base are given by (3.179) and the relation with the $2\pi$-periodic angles in (3.180). The GR solution is a special case of CCLP for $A = B = (2\alpha)^{-1}$, but note that we must rescale $(\psi, \varphi)_{\text{GR}} = (A^2 \psi, A^2 \varphi)_{\text{CCLP}}$. We find that the symplectic coordinates are given by
\[ x_1 = \frac{1}{2} A^2 r^2 \sin^2(\theta/2), \quad x_2 = \frac{1}{2} B^2 r^2 \cos^2(\theta/2), \] (3.73)

with the axis- (and fixed points-) structure being identical to the $SU(2) \times U(1)$ case, while the asymptotic region corresponds to $A^2 x_1 + B^2 x_2 \to \infty$. The symplectic potential of CCLP can again be evaluated from (3.12) and (3.13) and takes the remarkably simple and explicit form
\[ g_{\text{CCLP}} = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2 - \frac{1}{2} (x_1 + x_2) \log(x_1 + x_2) + \frac{1}{2} (A^2 x_1 + B^2 x_2) \log(A^2 x_1 + B^2 x_2) \]
\[ -\frac{1}{2} (A^2 x_1 + B^2 x_2 + \frac{1}{2} \ell^2) \log(A^2 x_1 + B^2 x_2 + \frac{1}{2} \ell^2). \] (3.74)

The potential for the near-horizon geometry of the CCLP black hole is
\[ g_{\text{NHCLP}} = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2 - \frac{1}{2} (x_1 + x_2) \log(x_1 + x_2) + \frac{1}{2} (A^2 x_1 + B^2 x_2) \log(A^2 x_1 + B^2 x_2), \] (3.75)

which is also the same as that of its parent black hole (3.74) with the last term omitted. Observe that these expressions reduce to those for the GR black hole for $A = B = (2\alpha)^{-1}$ as they must.

It is worth emphasising that the singular part of the symplectic potential for the above solutions is as predicted by our general Theorem 7. It is interesting to note that for all these black hole solutions the symplectic potential takes a form similar to that for the canonical metric on compact toric Kähler manifolds [104].
Symplectic potentials for black holes with non-trivial topology

We are interested in spacetimes asymptotic to global AdS$_5$. For such spacetimes we will choose the toric Killing fields $m_i$ to be $2\pi$-periodic and orthogonal on the $S^3$ at infinity. In particular, in terms of symplectic coordinates adapted to such $m_i$ the symplectic potential for global AdS$_5$ is given by (3.70). Therefore, the axis will always include two semi-infinite axes, say $x_1 = 0$ and $x_2 = 0$, for large enough $x_2$ and $x_1$ respectively. It is helpful to note that

$$g_{\mathbb{R}^4} = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2$$

is the symplectic potential for euclidean space $\mathbb{R}^4$ (note that this is inline with Lemma 8). This has the same axis structure (and topology) as the Bergmann metric and its symplectic potential (3.70) differs only in the third term which is smooth at the axes and dictates the asymptotics. We will now deduce the general form of the symplectic potential assuming there are at least two semi-infinite axis components.

As a warm up let us first consider the simplest case where we have two axis components $x_1 = 0$ and $x_2 = 0$ and a horizon at the origin. This includes the known CCLP black hole. From Theorem 7 we deduce that we can write the potential as

$$g = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2 - \frac{1}{2} (x_1 + x_2) \log (x_1 + x_2) + \frac{1}{2} (A^2 x_1 + B^2 x_2) \log (A^2 x_1 + B^2 x_2) + \tilde{g}$$

where $\tilde{g}$ is smooth at the axes $\{x_1 = 0, x_2 > 0\}$ and $\{x_1 > 0, x_2 = 0\}$, and at the horizon $x_1 = x_2 = 0$. Inspecting the symplectic potential for the CCLP black hole (3.74) we see that

$$\tilde{g}_{\text{CCLP}} = -\frac{1}{2} (A^2 x_1 + B^2 x_2 + \frac{1}{2} \ell^2) \log (A^2 x_1 + B^2 x_2 + \frac{1}{2} \ell^2),$$

which is indeed smooth at the axes and horizon inline with Theorem 7. On the other hand $\tilde{g} = 0$ gives a solution corresponding to the near-horizon geometry of CCLP.

Of course, Theorem 7 allows us to write down the potential for any horizon and axis structure. For simplicity we will only consider the next simplest example where we have three axis components: the two semi-infinite axes $\ell_1 = 0$, $\ell_3 = 0$, and a finite axis $\ell_2 = 0$ which joins the two:

$$\ell_1 := x_1, \quad \ell_2 := px_1 + qx_2 - a, \quad \ell_3 := x_2,$$

where $(p, q)$ are coprime integers. In the $(m_1, m_2)$ basis these correspond to the vanishing of the vectors $v_1 = (1, 0)$, $v_2 = (p, q)$ and $v_3 = (0, 1)$ respectively. The intersection of the finite axis with the semi-infinite ones correspond to corners of the orbit space and these must be on the positive $x_1$ and $x_2$ axes which requires $a/p > 0$ and $a/q > 0$. Thus we may assume $p > 0, q > 0, a > 0$. There are three cases to consider depending on if one, two or none of the corners are horizons.

1. **Black lens.** First suppose the corner $x_1 = 0, x_2 = a/q$ corresponds to a horizon and the other corner to a fixed point as in Figure fig. 3.2a. Then absence of an orbifold singularity (3.25) at the fixed point requires $\det(v_2, v_3) = -p = \pm 1$ so without loss of generality we may always set $p = 1$. The horizon topology is determined by $\det(v_1, v_2) = q$ and is a lens space $L(q, 1)$. Then, from Theorem 7, we deduce that the general form of the symplectic potential is

$$g = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} \ell_2 \log \ell_2 + \frac{1}{2} x_2 \log x_2 - \frac{1}{2} (x_1 + \ell_2) \log (x_1 + \ell_2) + \frac{1}{2} (A^2 x_1 + B^2 \ell_2) \log (A^2 x_1 + B^2 \ell_2) + \tilde{g},$$

where $\ell_2 = x_1 + qx_2 - a$ and $\tilde{g}$ is smooth at each of the axes $x_1 = 0$, $x_2 = 0$ and $\ell_2 = 0$ and at both corners. By construction this gives the symplectic potential for a regular black lens spacetime. An asymptotically flat supersymmetric $L(2, 1)$ black lens with this axis and horizon structure is known [107].
2. **Double-black hole.** Next suppose both corners are horizons so we have a double-black hole as in Figure fig. 3.2b. This gives an \( L(q,1) \) black lens at \( x_1 = 0, x_2 = a/q \) and an \( L(p,1) \) black lens at \( x_2 = 0, x_1 = a/p \) with no restriction on \( p, q \) (other than they are coprime). The symplectic potential now takes the form

\[
g = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} \ell_2 \log \ell_2 + \frac{1}{2} x_2 \log x_2 \\
- \frac{1}{2} (x_1 + \ell_2) \log (x_1 + \ell_2) - \frac{1}{2} (x_2 + \ell_2) \log (x_2 + \ell_2) \\
+ \frac{1}{2} (A^2 x_1 + B^2 \ell_2^2) \log (A^2 x_1 + B^2 \ell_2^2) + \frac{1}{2} (\tilde{A}^2 \ell_2 + \tilde{B}^2 x_2) \log (\tilde{A}^2 \ell_2 + \tilde{B}^2 x_2) + \tilde{g} \tag{3.81}
\]

where \( \ell_2 \) is given by (3.79) and there are singular terms from the horizons at \( x_1 = 0, x_2 = a/q \) and \( x_2 = 0, x_1 = a/p \) respectively (we distinguish the near-horizon parameters of the second horizon by tildes). Again \( \tilde{g} \) is smooth at all the axis lines and corners. Note the special case \( p = q = 1 \) corresponds to a double \( S^3 \) black hole.

3. **Soliton.** Now suppose both corners are fixed points of the toric symmetry as in Figure fig. 3.2c. Absence of orbifold singularities at the corners now requires \( p = q = 1 \) (fixing signs). The symplectic potential in this case is simply

\[
g = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} (x_1 + x_2 - a) \log (x_1 + x_2 - a) + \frac{1}{2} \ell_2 \log x_2 + \tilde{g} \tag{3.82}
\]

where \( \tilde{g} \) is smooth at the three axes. This gives a soliton with a bolt at the finite axis.

![Figure 3.2: The three cases of orbit space with three axis components depending on whether we have 1, 2 or 0 horizon components (white dots).](image)

We emphasise that there are no known solutions to (3.9) that realise the above black lens and double-black hole axis and horizon structures (i.e. case (a) and (b) above). An asymptotically AdS\(_5\)/\( \mathbb{Z}_p \) soliton spacetime with \( SU(2) \times U(1) \) symmetry is known with an axis structure as in case (c) and therefore gives example for this topology type [82] (its Kähler base is of the form (3.68), although it has a more complicated \( V(r) \)). However, the only known asymptotically AdS\(_5 \) soliton spacetime [17, 72, 108], possesses an ergosurface where \( V \) is null on a timelike hypersurface so the Kähler base metric is not globally defined in this case (the region outside to the ergosurface possesses an extra boundary corresponding to the ergosurface and the symplectic potential is much more complicated).

The existence of a black lens or a multi-black hole in AdS\(_5 \) remains a central open problem in the classification of AdS\(_5 \) black hole spacetimes. The above shows that this problem reduces to finding the smooth part of the symplectic potential that solves (3.9) and gives an asymptotically AdS\(_5 \) spacetime. In the absence of extra structure or symmetries this appears to be a formidable problem given the complexity of (3.9). In the next section we identify a special subclass of toric
solutions that contains the CCLP black hole for which we are able to completely solve the classification problem.

3.3 Supersymmetric toric solutions with separable moment maps

Section 3.2 was devoted to an analysis of generic supersymmetric toric solutions in the timelike class. In particular, we showed that such solutions can be described in symplectic coordinates by a symplectic potential that must obey a complicated 8th order non-linear PDE (its explicit form is unenlightening and can be obtained by inserting (3.12) and (3.13) in (3.9)). Furthermore, we derived the behaviour of the symplectic potential near any component of a horizon or axis. Unfortunately, it appears that a classification of Kähler geometries that satisfy this PDE (even under our boundary conditions) is at present out of reach.

In this section we will introduce a particular class of toric Kähler geometries that admit separable moment maps in the sense that they arise from a structure called a *Hamiltonian 2-form* [93, 109, 110] on the Kähler manifold (for the definition see Appendix 4.B). The main motivation for this class is that all known (analytically or numerically) solutions to the minimal gauged supergravity are inside it.

In [94] all toric Kähler geometries with Hamiltonian 2-forms are divided into three groups: ‘product-toric’, Calabi-type and orthotoric geometries. We will introduce them case by case. We will start with ‘product-toric’ and orthotoric cases, and show that they cannot admit smooth black holes on the contrast with Calabi-type geometries, which will be treated in more detail.

3.3.1 Product-toric Kähler metrics

**Definition 5.** We call a toric Kähler geometry ‘product-toric’ if it admits two linearly independent Hamiltonian Killing fields with moment maps \( \rho \), \( \eta \) such that \( d\rho \) and \( d\eta \) are orthogonal.

Let us derive the local form of geometry. Denote by \( \psi, \phi \) angular coordinates along Killing fields \( \partial/\partial \psi, \partial/\partial \phi \) corresponding to moment maps \( \rho, \eta \) respectively. The Kähler form is then

\[
X^{(1)} = d (\rho d\psi + \eta d\phi) .
\]  

Next, by (3.13) the symplectic coordinates part of the metric, which is diagonal in \( \rho, \eta \) chart, is the Hessian of symplectic potential, which then immediately integrates to

\[
g = A(\rho) + B(\eta)
\]  

for some functions \( A(\rho), B(\eta) \). Introducing

\[
A''(\rho) = \frac{1}{F(\rho)} , \quad B''(\eta) = \frac{1}{G(\eta)}
\]  

The metric then explicitly assumes the form of a direct product of two Riemannian surfaces with one \( U(1) \) isometry each

\[
h = \frac{d\rho^2}{F(\rho)} + F(\rho) d\psi^2 + \frac{d\eta^2}{G(\eta)} + G(\eta) d\phi^2
\]  

Thus any toric geometry of this type is necessarily a direct product, which motivates the name (at least for toric geometries, see below).

It is straightforward to see that this class cannot admit smooth black holes. Indeed, for ‘product-toric’ geometries the Gram matrix \( G_{ij} d\phi^i d\phi^j \), which is the inverse of Hessian (3.13), must also be diagonal. Nonetheless, from the near-horizon expansions given in section 3.2.4 there is no constant affine transformation of symplectic coordinates that can diagonalise (3.61). Furthermore, since this class does not include the Kähler base of \( \text{AdS}_5 \), the Bergmann space, it is irrelevant in our context.
Finally, it is worth mentioning how this class fits into a larger geometric classification in the sense of [93], i.e., without assumption of toric symmetry. Such geometries admit a Hamiltonian 2-form with constant trace and constant Pfaffian, which is equivalent to existence of bi-Kähler structure on the manifold.

3.3.2 Orthotoric Kähler metrics

Definition 6. We call a toric Kähler geometry orthotoric if it admits two linearly independent Hamiltonian Killing fields with moment maps \( x := \rho + \eta, \ y := \rho \eta \) such that \( d\rho \) and \( d\eta \) are orthogonal.

Again, denote by \( \psi, \phi \) angular coordinates along Killing fields \( \partial/\partial \psi, \partial/\partial \phi \) corresponding to moment maps \( x, y \) respectively. The Kähler form is then

\[
X^{(1)} = d \left( (\rho + \eta) d\psi + \rho \eta d\phi \right).
\]

(3.87)

To integrate for symplectic potential it is convenient to express the Hessian \( G_{ij} dx_i dx_j \) in terms of orthotoric coordinates \( \rho, \eta \)

\[
\partial_x = \frac{1}{\rho - \eta} (\rho \partial_\rho - \eta \partial_\eta), \quad \partial_y = \frac{1}{\rho - \eta} (-\partial_\rho + \partial_\eta).
\]

(3.88)

The orthogonality of \( d\rho, d\eta \) is then expressed as a simple PDE for symplectic potential \( g(\rho, \eta) \)

\[
\partial_\rho \partial_\eta g = \partial_\eta \log |\rho - \eta| + \partial_\rho \log |\rho - \eta|
\]

(3.89)

which has solution

\[
g = (\rho - \eta)^3 \left[ \partial_\rho \left( \frac{A(\rho)}{(\rho - \eta)^2} \right) + \partial_\eta \left( \frac{B(\eta)}{(\rho - \eta)^2} \right) \right] + \text{terms linear in } x,y
\]

(3.90)

where \( A(\rho), B(\eta) \) are arbitrary functions. Taking the Hessian of (3.90), the Kähler metric takes the form

\[
h = \frac{\rho - \eta}{F(\rho)} d\rho^2 + \frac{F(\rho)}{\rho - \eta} (d\psi + \eta d\phi)^2 + \frac{\rho - \eta}{G(\eta)} d\eta^2 + \frac{G(\eta)}{\rho - \eta} (d\psi + \rho d\phi)^2
\]

(3.91)

where

\[
F(\rho) = \frac{1}{A''(\rho)}, \quad G(\eta) = \frac{1}{B''(\eta)}.
\]

(3.92)

The scalar curvature is given by

\[
R = -\frac{F''(\rho) + G''(\eta)}{\rho - \eta}.
\]

(3.93)

This suggests that for generic orthotoric metrics \( \rho = \eta \) is a curvature singularity.

Lemma 12. A supersymmetric toric solution that is timelike outside a smooth horizon with compact sections cannot have an orthotoric Kähler base.

Proof. Let \( h \) be the Kähler base within the orthotoric class that contains some neighbourhood of a smooth horizon \( \mathcal{H} \). From (3.24) we know that the determinant of the Gram matrix of Killing fields of the 5d solution is proportional to determinant of angular part of the metric, which we can compute from (3.91)

\[
\det K_{\alpha\beta} = -\det G_{ij} = -F(\rho)G(\eta) \leq 0.
\]

(3.94)

Without loss of generality we will choose \( F(\rho), G(\eta) > 0 \). Then, positive signature of (3.91) implies that \( \rho \geq \eta \). The zeros of (3.94) correspond to axes or the horizon.

Let us understand the near-horizon behaviour of orthotoric coordinates similar to section 3.2.4. First of all, inverting, orthotoric coordinates are expressed through symplectic
coordinates as
\[ \rho = \frac{1}{2} \left( x + \sqrt{x^2 - 4y} \right), \quad \eta = \frac{1}{2} \left( x - \sqrt{x^2 - 4y} \right) \] (3.95)
where \( \rho \geq \eta \) was used to pick the unique branch. In turn, all symplectic charts are related by constant real affine transformations
\[
\begin{align*}
x &= \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 \\
y &= \beta_0 + \beta_1 x_1 + \beta_2 x_2
\end{align*}
\] (3.96)
where \( \alpha_1, \beta_1 \) are constants and \( x_i \) are symplectic coordinates adapted to \( 2\pi \)-periodic angles \( \phi^i \), \( i = 1, 2 \). By Lemma 9, \( x_i \) are smooth functions around the horizon.

It is worth noting that, since horizon is a point in \( (x_1, x_2) \) coordinates, it is a point \( (\rho_H, \eta_H) \) in orthotoric chart \( (\rho, \eta) \) as well. Furthermore, the transformations (3.95) and (3.96) imply that \( (\rho, \eta) \) are \( C^\infty \) at \( H \) when \( \rho_H \neq \eta_H \), i.e. when \( \alpha_0^2 \neq 4\beta_0 \), and \( C^0 \) otherwise.

We will now turn to the components of the Gram matrix explicitly to rule out both of the cases. From (3.61) we have that every component of
\[ G_{ij} \] (3.61) we have that every component of
\[
G_{ij} \text{d}\phi^i \text{d}\phi^j = \frac{F(\rho)}{\rho - \eta} \left( (\text{d}\psi + \eta \text{d}\phi)^2 + \frac{G(\eta)}{\rho - \eta} (\text{d}\psi + \rho \text{d}\phi)^2 \right)
= -\frac{\ell \kappa}{4\Delta_2(\eta)} \left( \frac{1}{(1 - \eta^2)\Delta_1(\eta)} \right) \lambda + O(\lambda^2)
\] (3.98)
must be a smooth scalar which vanishes as \( O(\lambda) \) at \( H \). Furthermore, the determinant (3.94)
\[ \det K_{\alpha\beta} = -\det G_{ij} = -\left( \frac{\ell \kappa}{2A^2B^2R_2(\eta)} \right)^2 \Delta_1(\eta)(1 - \eta^2)\lambda^2 + O(\lambda^3) \] (3.99)
has a quadratic term which vanishes only at the axes.

Case \( \rho_H = \eta_H \)
First of all, we have \( \rho - \eta = \sqrt{x^2 - 4y} = O(\lambda^{1/2}) \). From \( G_{\psi\psi} = \frac{1}{\rho - \eta} (F(\rho) + G(\eta)) = O(\lambda) \) we have \( F(\rho), G(\eta) = O(\lambda^{3/2}) \), and, hence, \( \det G_{ij} = O(\lambda^3) \) everywhere on \( H \), which is a contradiction with (3.99).

Case \( \rho_H \neq \eta_H \), i.e. \( \rho > \eta \) on the horizon. In this case \( (\rho, \eta) \) are smooth functions at the horizon. From \( G_{\psi\psi} \) and \( G_{\phi\phi} \) we have that \( F(\rho), G(\eta) \) are smooth functions as well.

We will now find leading orders of \( F(\rho), G(\eta) \) explicitly to show that smooth horizon is incompatible with orthotoric form of the metric. Firstly, from (3.62), (3.95) and (3.96) we find the expansions of orthotoric coordinates
\[
\begin{align*}
\rho &= \rho_H + \kappa \ell \frac{B^2(1 - \hat{\eta}) (\beta_1 - \alpha_1 \rho_H) + A^2(1 + \hat{\eta}) (\beta_2 - \alpha_2 \rho_H)}{A^2B^2(\eta_H - \rho_H) \Delta_2(\hat{\eta})} \lambda + O(\lambda^2) \\
\eta &= \eta_H + \kappa \ell \frac{B^2(1 - \hat{\eta}) (\alpha_1 \eta_H - \beta_1) + A^2(1 + \hat{\eta}) (\alpha_2 \eta_H - \beta_2)}{A^2B^2(\eta_H - \rho_H) \Delta_2(\hat{\eta})} \lambda + O(\lambda^2)
\end{align*}
\] (3.100)
where \( \rho_H = \frac{1}{2} \left( \alpha_0 + \sqrt{\alpha_0^2 - 4\beta_0} \right), \eta_H = \frac{1}{2} \left( \alpha_0 - \sqrt{\alpha_0^2 - 4\beta_0} \right) \).

The functions \( F(\rho), G(\eta) \) can be found from (3.98)
\[ F(\rho) = \rho G_{\psi\psi} - G_{\phi\phi}, \quad G(\eta) = G_{\phi\phi} - \eta G_{\psi\psi} \] (3.102)
where components of \( G_{ij} \) must satisfy the following constraint
\[ G_{\phi\phi} = (\rho + \eta) G_{\psi\psi} - \rho \eta G_{\psi\psi}. \] (3.103)
Substituting known expressions (3.98) and (3.100), we find that this constraint is violated, and we have arrived at the contradiction.
This completes the proof of Theorem 6.

While orthotoric class cannot include smooth black hole solutions, the smooth supersymmetric Killing field is of this form [72]. See Appendix 3.D for the presentation of the solution and section 1.3.2 for its discussion.

From the point of view of [93, 94], the orthotoric geometries admit a hamiltonian 2-form which generates two commuting axial Killing fields, in the sense that its trace and a combination of trace and pfaffian are holomorphic potentials. Every orthotoric geometry defined this way is necessarily toric.

### 3.3.3 Toric Kähler metrics of Calabi type

In this subsection, we will introduce toric Kähler manifolds of Calabi type. A general definition of Kähler surfaces of Calabi type was given by Apostolov, Calderbank and Gauduchon in their study of Kähler surfaces admitting a hamiltonian 2-form, the latter a concept which they also introduce [93]. They find that the hamiltonian 2-form necessarily generates two commuting hamiltonian Killing vector fields on the Kähler surface, and if these fields are independent the Kähler surface is orthotoric, whereas if they are dependent it is of Calabi type. We will give our own analogous definition of Calabi type in the context of toric Kähler surfaces (which is a subclass of Calabi type according to [93]).

**Definition 7.** We call a toric Kähler geometry to be of Calabi type if it admits two linearly independent hamiltonian Killing fields\(^{10}\) with moment maps \(\rho, \rho_\eta\) such that \(d\rho\) and \(d\eta\) are orthogonal.

Let us work out some basic consequences of the above definition. It is convenient to work in a \(2\pi\)-periodic basis of toric Killing fields \(m_i = \partial_{\phi^i}, \phi^i \sim \phi^i + 2\pi\). Then according to this definition there exist linear combinations of the Killing fields \(k_i = A_i^j m_j\) where \(A \in GL(2, \mathbb{R})\) with moment maps \(\rho, \rho_\eta\) such that

\[
d\rho \cdot d\eta = 0.
\]

(3.104)

Note that the Killing fields \(k_1, k_2\) need not have closed orbits. It is useful to adapt coordinates so \(k_1 = \partial_{\phi^1}\) and \(k_2 = \partial_{\phi^2}\) correspond to the moment maps \(\rho\) and \(\rho_\eta\) respectively, and parameterise the \(GL(2, \mathbb{R})\) transformation explicitly in terms of these coordinates as

\[
\psi = a_i \phi^i, \quad \varphi = b_i \phi^i, \quad \text{with} \quad \langle a, b \rangle \neq 0,
\]

(3.105)

where \(a_i, b_i\) are constants and we have introduced the notation \(\langle a, b \rangle := \epsilon^{ij} a_i b_j\) with \(\epsilon^{ij}\) antisymmetric and \(\epsilon^{12} = 1\). Since the Kähler form (3.14) is invariant under such \(GL(2, \mathbb{R})\) transformations, we have

\[
X^{(1)} = d (\rho (d\psi + \eta d\varphi)),
\]

(3.106)

while the moment maps (3.11) for the Killing vectors with \(2\pi\)-periodic orbits \(m_i\) are

\[
x_i = c_i + \rho(a_i + b_i \eta),
\]

(3.107)

where \(c_i\) are constants of integration which we will leave arbitrary for later convenience.

Now we will show that toric Kähler manifolds of Calabi type can be described in terms of two functions \(F(\rho), G(\eta)\) as the following proposition shows.

**Proposition 6.** A toric Kähler surface is of Calabi type iff its metric can be written in the form

\[
h = \frac{\rho}{F(\rho)} d\rho^2 + \frac{F(\rho)}{\rho} (d\psi + \eta d\varphi)^2 + \rho \left( \frac{d\eta^2}{G(\eta)} + G(\eta) d\varphi^2 \right),
\]

(3.108)

where \(F(\rho), G(\eta)\) are arbitrary functions and \(\psi, \varphi\) are given by (3.105).

\(^8\)A 2-form is hamiltonian if it is closed, invariant under the complex structure, and its traceless part is a twistor form, see Appendix 4.B for the definition. The 3-invariance implies that this is a complex \((1,1)\) 2-form.

\(^9\)And if these vector fields are identically zero, the geometry is bi-Kähler. In the toric case this is equivalent to product-toric geometries discussed above [94].

\(^{10}\)It should be emphasised that only the Killing field with moment map \(\rho\) comes from the hamiltonian 2-form. The second Killing field is introduced artificially.
Proof. The proof makes use of the symplectic potential and of the generic form (3.12) for a toric Kähler metric. First let us assume that we have a toric Kähler metric of Calabi type. Using (3.107) the orthogonality condition (3.104) becomes the following condition for the symplectic potential

$$0 = h_{\rho \eta} = \rho(a_i + b_i \eta)b_j G^{ij} = \rho(a_i + b_i \eta)b_j \partial^i \partial^j g = \rho \partial_\rho (\rho^{-1} \partial_\eta g) ,$$

(3.109)

which can be easily integrated to give

$$g = g_0(\rho) + \rho g_1(\eta) \ ,$$

(3.110)

where $g_0(\rho), g_1(\eta)$ are arbitrary functions. A computation then reveals that $h_{\rho \rho} = g_0''(\rho) , 
 h_{\eta \eta} = \rho g_1''(\eta)$ and hence defining

$$F(\rho) := \frac{\rho}{g_0'(\rho)} , \quad G(\eta) := \frac{1}{g_1'(\eta)} ,$$

(3.111)

the $(\rho, \eta)$ components of the metric are as claimed in (3.108). To find the $(\psi, \varphi)$ components we can use that the $(\phi^1, \phi^2)$ components are given by $G_{ij}$ and that this is the inverse matrix of the $(x_i, x_j)$ components $G^{ij} = \partial^i \partial^j g$ which we now know. A short computation then confirms that the metric indeed takes the form (3.108). Finally, it is easy to show the converse statement, namely that (3.108) implies the Calabi property\footnote{In fact (3.108) is enough to show the Kähler property. The Kähler form is given by (3.106).}. To this end, notice that (3.108), (3.106) have the form (3.12), (3.14) for moment maps $\rho, \eta$ and the condition (3.104) is obviously satisfied.

It is useful to note that from the above proof we can extract the Gram matrix of Killing fields in the original $2\pi$-periodic coordinates $\phi^i$, which we find is,

$$G_{ij} = \frac{F(\rho)}{\rho} (a_i + b_i \eta)(a_j + b_j \eta) + \rho G(\eta) b_ib_j .$$

(3.112)

From this we can also deduce the following useful projections

$$G_{ij} \epsilon^k \epsilon^l b_kb_l = \frac{F(\rho)}{\rho} , \quad G_{ij} \epsilon^k \epsilon^l a_kb_l = -\eta F(\rho) \rho , \quad G_{ij} \epsilon^k \epsilon^l a_ib_i = \eta^2 F(\rho) \rho + \rho G(\eta) ,$$

(3.113)

which in particular allow us to express $F(\rho), G(\eta)$ in terms of invariants of the Kähler base. In fact, it is worth noting the following useful characterisation of toric Kähler metrics of Calabi type.

Lemma 13. A toric Kähler metric with symplectic coordinates (3.107) and a Gram matrix of Killing fields $G_{ij}$ given by (3.112) for some $a_i, b_i, c_i, \rho, \eta, F(\rho), G(\eta)$, must be of Calabi type (3.108).

Proof. Change basis for the Killing fields by introducing coordinates $\tilde{\phi}^1 := \psi, \tilde{\phi}^2 := \varphi$ defined by (3.105). Then the Gram matrix (3.112) in this new basis $\tilde{G}_{ij}$ coincides with the $(\psi, \varphi)$ components of the metric (3.108). The inverse matrix $\tilde{G}^{ij}$ is thus determined in terms of $\rho, \eta, F(\rho), G(\eta)$. But $G^{ij} dx_i dx_j = \tilde{G}^{ij} d\tilde{x}_i d\tilde{x}_j$ where the symplectic coordinates $\tilde{x}_1 = \rho, \tilde{x}_2 = \eta$ and a short computation using the already determined $\tilde{G}^{ij}$ gives the $(\rho, \eta)$ components of (3.108) as required.

For later use it is convenient to record the expressions for the Ricci potential and the Ricci scalar of the Kähler metrics (3.108) which are given by

$$P = -\frac{1}{2} \frac{F''(\rho)}{\rho} (d\psi + \eta d\varphi) - \frac{1}{2} G'(\eta) d\varphi \quad \quad R = -\frac{F''(\rho)}{\rho} + G''(\eta) ,$$

(3.114)

respectively.

The parametrisation of the Calabi metric (3.108) (and hence the functions $F(\rho), G(\eta)$) and the Kähler form (3.106) in the coordinates $\rho, \eta, \psi, \varphi$ is far from unique. In fact, the form of
(3.108) and (3.106) remains invariant under the following constant rescalings of $\rho$ and affine transformations of $\eta$:

$$
\rho \rightarrow K_\rho \rho, \quad \psi \rightarrow K_\rho^{-1}(\psi - C_\eta K_\eta^{-1} \varphi), \quad \eta \rightarrow K_\eta \eta + C_\eta, \quad \varphi \rightarrow K_\rho^{-1}K_\eta^{-1} \varphi,
$$

(3.115)

where $K_\rho, K_\eta$ are non-zero real constants and $C_\eta \in \mathbb{R}$. In turn, (3.105) imply that these transformations also act on $a_i$ and $b_i$ as

$$
a_i \rightarrow K_\rho^{-1}(a_i - C_\eta K_\eta^{-1} b_i), \quad b_i \rightarrow K_\rho^{-1}K_\eta^{-1} b_i.
$$

(3.116)

Note that these transformations include discrete symmetries which flip independently the signs of $\rho$ and $\eta$.

From the five-dimensional point of view, we are really only interested in the Kähler base metric up to the rescalings (3.10). For (3.108) these rescalings can be realised by

$$
\rho \rightarrow K^{-1}_\rho \rho, \quad F(\rho) \rightarrow K^{-2} F(\rho),
$$

(3.117)

with $\eta, \psi, \varphi$ and $G(\eta)$ unchanged. It is easy to see that the metric and the Kähler form are rescaled while the Calabi property is preserved.

### 3.3.4 Near-horizon analysis

The analysis in sections 3.2.3 and 3.2.4 showed that each connected component of the horizon corresponds to an isolated point in the $x_1x_2$-plane (see Lemma 9) and furthermore determined the behaviour of the Kähler metric $h$ near each such point (eq. (3.61) and its inverse). We will now further restrict to toric Kähler metrics of Calabi type and determine the near-horizon behaviour of the coordinates $(\rho, \eta)$ and the associated functions $F(\rho), G(\eta)$.

**Lemma 14.** Consider a supersymmetric toric solution that is timelike outside a smooth horizon with compact cross-sections. If the Kähler base is of Calabi type (3.108) then, near the horizon, Calabi type coordinates are related to Gaussian null coordinates by

$$
\rho = \frac{\ell_K}{4\Delta_2(\hat{\eta})} \lambda + O(\lambda^2), \quad \eta = \hat{\eta} + O(\lambda),
$$

(3.118)

where $\Delta_2(\hat{\eta})$ is given by (3.58), so in particular the horizon must be at $\rho = 0$. Furthermore, we can always choose Calabi coordinates such that

$$
F(\rho) = \rho^2 + O(\rho^3), \quad G(\eta) = (1 - \eta^2)\Delta_1(\eta),
$$

(3.119)

where $\Delta_1(\eta)$ is given by (3.58).

**Proof.** The first step is to identify the relation between GNC $(\lambda, \hat{\eta})$ and the Calabi type coordinates $(\rho, \eta)$. This is given by inverting (3.107) to obtain

$$
\rho = \frac{\langle x, b \rangle + \langle b, c \rangle}{\langle a, b \rangle}, \quad \eta = -\frac{\langle x, a \rangle + \langle a, c \rangle}{\langle x, b \rangle + \langle b, c \rangle},
$$

(3.120)

together with the near-horizon behaviour (3.62) of the symplectic coordinates $x_1$. On the other hand, from our near-horizon analysis the Gram matrix (with respect to the Kähler base metric) of the toric Killing fields $G_{ij} = O(\lambda)$ as $\lambda \rightarrow 0$ (recall it is given by (3.61)), so (3.113) implies

$$
\eta = O(1), \quad \frac{F(\rho)}{\rho} = O(\lambda), \quad \rho G(\eta) = O(\lambda).
$$

(3.121)

We will now show that these relations imply that $c_i = 0$.

Thus suppose, for contradiction, that $c_i$ is not identically zero. If $\langle b, c \rangle = 0$ then $b_i$ is a multiple of $c_i$ and since $\langle a, b \rangle \neq 0$ it follows that $\langle a, c \rangle \neq 0$. Then (3.62) and (3.120) imply that

---

12 Of course, the functions should also transform according to $F(\rho) \rightarrow K^2_\rho F(\rho)$ and $G(\eta) \rightarrow K_\rho K^2_\rho G(\eta)$. 

---
η is necessarily singular at the horizon contradicting (3.121). Therefore we must have \( \langle b, c \rangle \neq 0 \) in which case (3.62) and (3.120) give

\[
\begin{align*}
\rho &= \frac{\langle b, c \rangle}{\langle a, b \rangle} + \frac{1}{\langle a, b \rangle} \left( \frac{1 - \hat{\eta} b_2}{\mathcal{A}^2} - \frac{1 + \hat{\eta} b_1}{\mathcal{B}^2} \right) \frac{\ell \kappa \lambda}{4 \Delta_2(\hat{\eta})} + O(\lambda^2), \\
\eta &= -\frac{\langle a, c \rangle}{\langle b, c \rangle} + \frac{\langle a, b \rangle}{\langle b, c \rangle^2} \left( \frac{1 - \hat{\eta} c_2}{\mathcal{A}^2} - \frac{1 + \hat{\eta} c_1}{\mathcal{B}^2} \right) \frac{\ell \kappa \lambda}{4 \Delta_2(\hat{\eta})} + O(\lambda^2).
\end{align*}
\]

(3.122)

Thus the horizon \( \lambda = 0 \) is mapped to a single point in \( (\rho, \eta) \) coordinates given by \( \rho_0 = \langle b, c \rangle / \langle a, b \rangle \neq 0 \), \( \eta_0 = -\langle a, c \rangle / \langle b, c \rangle \neq 0 \). Then (3.121) implies that \( F(\rho) = O(\lambda) \) and \( G(\eta) = O(\lambda) \) and therefore \( F(\rho_0) = G(\eta_0) = 0 \). Now expanding (3.112) to first order in \( \lambda \) we find

\[
G_{ij} = \left[ G'(\eta_0)b_i b_j \left( \frac{1 - \hat{\eta}}{\mathcal{A}^2} - \frac{1 + \hat{\eta}}{\mathcal{B}^2} \right) + \frac{\langle a, b \rangle^2 F'(\rho_0)c_i c_j}{\langle b, c \rangle^3} \left( \frac{1 - \hat{\eta}}{\mathcal{A}^2} - \frac{1 + \hat{\eta}}{\mathcal{B}^2} \right) \right] \frac{\ell \kappa \lambda}{4 \Delta_2(\hat{\eta})} + O(\lambda^2).
\]

(3.123)

Comparing to the corresponding expression for the near-horizon geometry (3.61) it is easy to see that the two can never match (the factor in the square brackets above is linear in \( \hat{\eta} \) whereas the corresponding factor in (3.61) is a quadratic or cubic polynomial in \( \hat{\eta} \)). In order to avoid this contradiction we therefore conclude that

\[
c_i = 0,
\]

(3.124)
as claimed.

Now, combining (3.120) with the near-horizon expansion (3.62) gives

\[
\begin{align*}
\rho &= \frac{1}{\langle a, b \rangle} \left( \frac{1 - \hat{\eta} b_2}{\mathcal{A}^2} - \frac{1 + \hat{\eta} b_1}{\mathcal{B}^2} \right) \frac{\ell \kappa \lambda}{4 \Delta_2(\hat{\eta})} + O(\lambda^2), \\
\eta &= -\frac{\mathcal{A}^2 a_1 (1 + \hat{\eta}) - \mathcal{B}^2 a_2 (1 - \hat{\eta})}{\mathcal{A}^2 b_1 (1 + \hat{\eta}) - \mathcal{B}^2 b_2 (1 - \hat{\eta})} + O(\lambda).
\end{align*}
\]

(3.125)

In particular, we deduce that \( \rho = 0 \) corresponds to the horizon as claimed in the Lemma. Inverting, we deduce that \( \lambda \) is a smooth function of \( \rho \) at the horizon.

To complete the proof of our Lemma, we need to show that there exist functions \( F(\rho) \) and \( G(\eta) \) such that (3.112) reproduces (3.61) at \( O(\lambda) \). From (3.121) and (3.125) we see that \( F(\rho) = O(\lambda^3) \) is a smooth function of \( \lambda \) and hence of \( \rho \) at the horizon. We therefore must have

\[
F(\rho) = F_2 \rho^2 + O(\lambda^3), \quad G(\eta) = G_0(\eta) + O(\lambda),
\]

(3.126)

where

\[
F_2 = \frac{1}{2} F''(0), \quad G_0(\eta) = G \left( \frac{-\mathcal{A}^2 a_1 (1 + \hat{\eta}) - \mathcal{B}^2 a_2 (1 - \hat{\eta})}{\mathcal{A}^2 b_1 (1 + \hat{\eta}) - \mathcal{B}^2 b_2 (1 - \hat{\eta})} \right).
\]

(3.127)

It is now not hard to see that (3.61) and (3.112) match at \( O(\lambda) \) if and only if

\[
\frac{b_2}{b_1} = -\frac{\mathcal{A}^2}{\mathcal{B}^2}, \quad F_2 = \frac{2}{a_1 \mathcal{A}^2 + a_2 \mathcal{B}^2}, \quad G_0(\eta) = \frac{(1 - \hat{\eta}^2) \Delta_1(\hat{\eta})}{F_2 \mathcal{A}^2 \mathcal{B}^2 b_1 b_2}.
\]

(3.128)

We can now exploit the freedom in the choice of Calabi type coordinates (3.116) to fix

\[
a_1 = -b_1 = \mathcal{A}^{-2}, \quad a_2 = b_2 = \mathcal{B}^{-2},
\]

(3.129)

which also fixes\(^{13}\)

\[
F_2 = 1, \quad G_0(\eta) = (1 - \hat{\eta}^2) \Delta_1(\hat{\eta}).
\]

(3.130)

and (3.125) simplifies to (3.118). The second equation in (3.127) now reduces to \( G_0(\hat{\eta}) = G(\hat{\eta}) \)

\(^{13}\)Observe that \( F_2 \) is invariant under (3.117) and hence this freedom still remains unfixed.
which therefore determines the function $G(\eta)$ as claimed. \hfill $\square$

The above result applies to any component of the horizon and hence shows that the assumption that the base is of toric Calabi type implies the horizon must be connected, that is, it does not allow for multi-black holes. It also shows that the near-horizon geometry has a base of Calabi type. This follows from Lemma 13 together with the fact that the above proof shows that there are constants and functions $a_i, b_i, c_i, \rho, \eta, F(\rho), G(\eta)$ such that the symplectic coordinates $x_i$ and Gram matrix $G_{ij}$ are given by (3.107) and (3.112) to leading order in $\lambda$. It is also interesting to emphasise that the near-horizon geometry completely fixes $G(\eta)$, leaving only one function $F(\rho)$ to be determined. We will see next that the latter is fixed by the supersymmetry constraint (3.9).

Finally, it is worth noting that according to [93] one can introduce Calabi type geometries with a single axial Killing field. We comment on them in Appendix 3.E.

### 3.4 Uniqueness of supersymmetric CCLP black hole within toric Calabi-type class

#### 3.4.1 Uniqueness theorem

In this section we will complete the proof of Theorem 5. This first result completely determines the Kähler base.

**Lemma 15.** Consider a supersymmetric toric solution that is timelike outside a smooth (analytic if $A^2 = B^2$) horizon with compact cross-sections. If the Kähler base is of Calabi type, then it can be written as

$$h = \frac{d\rho^2}{\rho + s\rho^2} + (\rho + s\rho^2)\sigma^2 + \frac{\rho}{(1 - \eta^2)\Delta_1(\eta)}(d\eta^2 + \tau^2), \quad X^{(1)} = d(\rho\sigma), \quad (3.131)$$

where $\Delta_1(\eta)$ is given by (3.58) and we have defined

$$\sigma := d\psi + \eta d\varphi = \frac{1 - \eta}{A^2}d\phi^1 + \frac{1 + \eta}{B^2}d\phi^2,$$

$$\tau := -(1 - \eta^2)\Delta_1(\eta)d\varphi = (1 - \eta^2)\Delta_1(\eta)\left(\frac{d\phi^1}{A^2} - \frac{d\phi^2}{B^2}\right), \quad (3.132)$$

and $s = 4/\ell^2$ or $s = 0$, where $A^2$ and $B^2$ are constants that parameterise the near-horizon geometry that satisfy (3.53).

**Proof.** Recall that the near-horizon analysis in Lemma 14 completely determines the function $G(\eta)$ to be a cubic that is determined by the near-horizon parameters $A^2$ and $B^2$. We will show that the integrability condition (3.9) for Calabi type metrics (3.108) with $G(\eta)$ given by (3.119) can be solved for the other function $F(\rho)$. It turns out that the cases $A^2 \neq B^2$ and $A^2 = B^2$ should be treated separately since in the latter case $G(\eta)$ is quadratic rather than cubic.

For $A^2 \neq B^2$ the l.h.s. of the constraint (3.9) reduces to a quadratic polynomial in $\eta$ with coefficients that depend on $F(\rho)$ and its derivatives. Requiring that the coefficient of $\eta^2$ vanishes is equivalent to

$$\rho^2 F''(\rho) - 4\rho F'(\rho) + 6F(\rho) = 0, \quad (3.133)$$

which has a general solution given by $F(\rho) = F_2\rho^2 + F_3\rho^3$ where $F_2$ and $F_3$ are integration constants. With this $F(\rho)$ the constraint (3.9) then vanishes identically. Next (3.130) fixes $F_2 = 1$. The constant $F_3$ under the scalings (3.117) transforms as $F_3 \rightarrow KF_3$ and so we can use this freedom to fix $F_3$ to a convenient value which we denote by $s$. The cases $s \neq 0$ and $s = 0$ are qualitatively different and it is convenient to fix the former case so $s = 4/\ell^2$. Therefore, we obtain

$$F(\rho) = \rho^2 + s\rho^3 \quad (3.134)$$

\footnote{In fact this is the unique solution to (3.9) for cubic $G(\eta)$.}

\footnote{This is a convenient choice since then $f \rightarrow 1$ as $\rho \rightarrow \infty$, see (3.143) below.}

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which completes the proof for the case \( A^2 \neq B^2 \).

We now turn to the case \( A^2 = B^2 \) which has been previously analysed in a general study of supersymmetric solutions with \( SU(2) \) symmetry \([82]\). For completeness we will reproduce the relevant arguments in the current framework. Substituting \( G(\eta) \) with \( A^2 = B^2 \) into (3.9) results in a complicated non-linear ODE for \( F(\rho) \) (all \( \eta \) dependence cancels in this case). This ODE is more conveniently written in terms of the function

\[
F(\rho) := A^{-2} \rho^{-2} F(\rho),
\]

in terms of which the near-horizon behaviour (3.119) becomes

\[
F(\rho) = A^{-2} + O(\rho).
\]

This ODE can be written as

\[
F^2 \left( 12 + 4F + 9\rho^2 (\rho(\rho F'')'' - 4\rho^2 F''(\rho F' - 3) + 12\rho F(\rho F' - F - 2)(\rho F)' \right)'' = 0.
\]

We can integrate twice and fix the integration constants using (3.136) (the quantity inside the big brackets has no terms of linear order in \( \rho \)) to get

\[
F^2 \left( 12 + 4F + 9\rho^2 (\rho(\rho F'')'' - 4\rho^2 F''(\rho F' - 3) + 12\rho F(\rho F' - F - 2)(\rho F)' \right) = 4A^{-4}(3 + A^{-2}).
\]

We will show that the only analytic solution to (3.138) satisfying the black hole boundary condition (3.136) with \( 0 < A^2 < 1 \), is the linear function

\[
F(\rho) = A^{-2} + F_1 \rho.
\]

To this end, we Taylor expand

\[
F(\rho) = A^{-2} + \sum_{n=1}^{\infty} F_n \rho^n,
\]

and we will show by induction that (3.138) implies \( F_n = 0 \) for all \( n \geq 2 \). First we note that (3.138) at order \( O(\rho^2) \) gives

\[
(1 - A^2) F_2 = 0,
\]

and since \( 0 < A^2 < 1 \) we find \( F_2 = 0 \). Thus now assume that \( F_2, \ldots, F_{n-1} \) all vanish for some \( n \geq 3 \). Then (3.138) gives

\[
3(n^2 - 1) (3(n^2 - 4) + 8(1 - A^2)) F_n \rho^n + O(\rho^{n+1}) = 0,
\]

and since \( 0 < A^2 < 1 \) we must have \( F_n = 0 \). It follows by induction that \( F_n = 0 \) for all \( n \geq 3 \), so we conclude that (3.139) is the unique analytic solution to (3.137) with boundary conditions (3.136). In terms of \( F(\rho) \), which is given by (3.135), we deduce that \( F(\rho) = \rho^2 + F_3 \rho^3 \) and \( F_3 \) can be fixed exactly as in the \( A^2 \neq B^2 \) case to get (3.140).

**Corollary 2.** The norm of the supersymmetric Killing field \( V \) is given by

\[
f = \frac{12}{\ell^2} \frac{\rho}{3\rho + \Delta_2(\eta)}.
\]

In particular, \( V \) is strictly timelike outside the horizon.

Equation (3.143) immediately follows from the first equation in (3.3) and the second in (3.114) for the Kähler base metric (3.131). From (3.118) we see that the region exterior to the horizon \( \lambda > 0 \) corresponds to \( \rho > 0 \) and therefore from the above explicit expression we see that \( f > 0 \) (recall \( \Delta_2 \) is defined in (3.58) and is strictly positive). We deduce that the Kähler base metric is globally defined on the region exterior to the horizon.
Corollary 3. The axis set in Calabi type coordinates is given by $\eta = \pm 1$ and corresponds to the fixed points of $\partial_{\phi^1}$ and $\partial_{\phi^2}$ respectively.

Proof. Components of the axis are characterised by the vanishing of integer linear combinations $v = v^i \partial_{\phi^i}$ of the Killing fields. By the previous corollary $f > 0$ on the exterior region and therefore we can write the invariant $g(m_i, m_j)$ on the exterior region as in (3.23). Furthermore, $\omega$ is a smooth 1-form on this region and hence $\iota_v \omega = \omega_v = 0$ on any axis component defined by $v = 0$. Thus on such a component of the axis we deduce that $G_{ij} v^j = 0$ and hence the K"ahler base Gram matrix $G_{ij}$ is not full rank (to see this contract $g(m_i, m_j)$ with $v^j$ and use (3.23)).

Now for the K"ahler metric (3.131) one finds that the determinant of the Gram matrix in the $v = 0$ components of the axis we deduce that $\text{det} G = 4A^2B^2F(\rho)G(\eta)$. Therefore any component of the axis in the exterior region corresponds to $G(\eta) = 0$, that is, $\eta = \pm 1$ (recall $f(\rho) > 0$ away from the horizon). From the explicit form of the K"ahler metric (3.131) it also follows that $\eta = 1$ corresponds to the vanishing of $\partial_{\phi^1}$ and $\eta = -1$ corresponds to the vanishing of $\partial_{\phi^2}$.

The next step in the uniqueness proof is to integrate (3.8) for $\omega$ and compute the gauge field from (3.2). This is given by the following result.

**Lemma 16.** Consider a supersymmetric solution as in Lemma 15. The remaining metric data $\omega$ is given by

$$\omega = \left( \frac{\ell^3 s}{8} + \frac{\ell^3 s}{8} \right) \sigma - \frac{\ell^3 (A^2 - B^2)}{16} \left( \frac{s}{A^2} - \frac{1}{\rho} \right) \tau,$$

and the gauge field $A$ is given (up to gauge transformations) by

$$A = \frac{\sqrt{3}}{2} f(dt + \omega) - \frac{\ell}{2\sqrt{3}} \left( 1 + 3 \frac{\eta_0}{2} - \frac{1}{2} (A^2 + B^2) \right) \sigma + \frac{\ell}{2\sqrt{3}} \frac{A^2 - B^2}{8} \Delta_1(\eta) \tau.$$  

Proof. Recall the 1-form $\omega$ must take the form (3.21). Therefore we need to integrate (3.8) for $\omega = \omega_\psi d\psi + \omega_\varphi d\varphi$ where the r.h.s. is given by (3.3) and (3.7) for the K"ahler base in Lemma 15. In particular, this requires the ASD 2-forms (3.15) and (3.16), which in the Calabi type coordinates of Lemma 15 are

$$X^{(2)} = - \frac{d \rho \wedge d \eta}{\sqrt{F(\rho)G(\eta)}} + \sqrt{F(\rho)G(\eta)} d \psi \wedge d \varphi,$$

$$X^{(3)} = \sqrt{\frac{F(\rho)}{G(\eta)}} (d \psi + \eta d \varphi) \wedge d \eta + \sqrt{\frac{G(\eta)}{F(\rho)}} \rho d \rho \wedge d \varphi.$$  

As mentioned below equation (3.21), the form of $\omega$ forces $\lambda_2 = 0$, thus leaving just $\lambda_3$ as an undetermined function which must be solved for alongside $\omega_\psi, \omega_\varphi$, and recall these functions are all invariant under the toric symmetry.

Explicitly, the PDE (3.8) for $\omega$ reads

$$\partial_\rho \omega_\psi = \frac{\ell^3}{48} \left( \frac{\Delta_1(\eta)}{\rho^2} + 6s \right), \partial_\rho \omega_\varphi = \frac{\ell^3}{48} \left( 6\eta s^2 + \eta \Delta_3(\eta)/\rho^2 \right) - \rho \sqrt{\frac{G(\eta)}{F(\rho)}} \lambda_3,$$

$$\partial_\varphi \omega = \frac{\ell^3}{48} \sqrt{\frac{F(\rho)}{G(\eta)}} \lambda_3, \partial_\varphi \omega_\varphi = \frac{\ell^3}{48} \left( \sqrt{\frac{F(\rho)}{G(\eta)}} \lambda_3 + 6s (\Delta_2(\eta) + s\rho) \right),$$

where recall $F(\rho) = \rho^2 + sp^3$ and $G(\eta) = (1 - \eta^2)^2 \Delta_1(\eta)$. The integrability conditions for each system of equations in (3.148) read respectively

$$\partial_\rho \left( \sqrt{F(\rho)G(\eta)} \lambda_3 \right) = \frac{3(A^2 - B^2)G(\eta)}{\rho^2},$$

$$\partial_\eta \left( \sqrt{F(\rho)G(\eta)} \lambda_3 \right) = 2 \left( \frac{1}{\rho} + s \right) \left( \Delta_2(\eta)^2 + \Delta_3(\eta) \right),$$
which, in turn, must have an integrability condition equivalent to (3.9). The latter is guaranteed\(^\text{16}\) to be satisfied since (3.131) is a solution of it. We can thus integrate (3.149) to find

\[
\sqrt{F(r)G(\eta)} \lambda = -3(A^2 - B^2) \frac{F(r)G(\eta)}{\rho^3} + \lambda_{3,0},
\]

where \(\lambda_{3,0}\) is an integration constant. Now we can integrate (3.148) for \(\omega\) and find

\[
\omega = \left( \frac{\ell^3 s^2}{8} + \frac{\ell^3 s}{8}(1 - \Delta_1(\eta)) - \frac{\ell^3 \Delta_3(\eta)}{48 \rho} \right) \sigma - \frac{\ell^3 (A^2 - B^2)}{16} \left( \frac{s}{\Delta_1(\eta)} - \frac{1}{\rho} \right) \tau + \omega_0
\]

\[
- \frac{\ell^3 \lambda_{3,0}}{48} \left[ \log(s + \rho^{-1}) \right] \sigma + \frac{1}{A^2 B^2} \left[ \log \left( \frac{2\Delta_1(\eta)}{1 + \eta} \right) \, d\phi^1 - \log \left( \frac{2\Delta_1(\eta)}{1 - \eta} \right) \, d\phi^2 \right],
\]

where \(\omega_0 = \omega_{1,0} d\phi^1 + \omega_{2,0} d\phi^2\) and \(\omega_{1,0}\) are integration constants. The above expression satisfies all the local constraints required by supersymmetry.

Now, the near-horizon analysis shows that \(\omega_1\) must take the form (3.60) where the \(O(1)\) terms are smooth in \(\lambda\) and since \(\rho\) must be a smooth function of \(\lambda\) near the horizon satisfying (3.118), we deduce that \(\omega_1\) must diverge as \(\rho^{-1}\) with the \(O(1)\) terms smooth in \(\rho\). Therefore, comparing to the explicit expression above we see that in order to avoid logarithmic singularities at the horizon we must have \(\lambda_{3,0} = 0\). Furthermore, by Corollary 3 and its proof we must have that \(\omega_1\) (resp. \(\omega_2\)) vanishes at \(\eta = 1\) (resp. \(\eta = -1\)), which requires \(\omega_{1,0} = \omega_{2,0} = 0\) thus establishing (3.144).

The gauge field can be computed from (3.4) and the Ricci form potential (3.114) which yields (3.145), where we have added a gauge transformation to ensure \(A\) is smooth at the axes, i.e. we have fixed \(A_1 = 0\) at \(\eta = 1\) and \(A_2 = 0\) at \(\eta = -1\).

We have now completely determined the general solution under our assumptions. In particular, the solution is given by Lemma 15, (3.143) and Lemma 16. It remains to show that this solution is the supersymmetric CCLP black hole (or its near-horizon geometry): we provide a convenient form for this solution in Appendix 3.B which is parameterised by two constants \(A^2, B^2\). A simple computation shows that the solution with \(s = 4/\ell^2\) is indeed the CCLP black hole as given in Appendix 3.B, upon the coordinate change

\[
\rho = r^2/4, \quad \eta = \cos \vartheta,
\]

and the identification of parameters

\[
\]

Similarly, one can show that the \(s = 0\) solution corresponds to the near-horizon geometry of the CCLP black hole under the same identifications. This completes the proof of Theorem 5.

### 3.4.2 Geometry of the CCLP Kähler base

In order to highlight the potency of the symplectic potential, we will elaborate a little more on the geometry of the Kähler base of the CCLP black hole solution and its near-horizon geometry. We will use the form of the general black hole solution derived above in Calabi coordinates with the parameters identified with those of the CCLP solution (3.153).

The symplectic coordinates (3.107), with (3.129), are related to the Calabi type coordinates by the transformation

\[
x_1 = \rho \frac{1 - \eta}{A^2}, \quad x_2 = \rho \frac{1 + \eta}{B^2},
\]

and inverting this gives

\[
\rho = \frac{1}{2}(A^2 x_1 + B^2 x_2), \quad \eta = -\frac{A^2 x_1 - B^2 x_2}{A^2 x_1 + B^2 x_2}.
\]

---

\(^{16}\)One can easily check this by using the identity \(3(A^2 - B^2)G(\eta) = -2 (\Delta_2(\eta)^2 + \Delta_3(\eta))\).
We can now exploit (3.110) and (3.111) to deduce the symplectic potential of any toric Kähler base of Calabi type. In our case \( F(\rho) \) and \( G(\eta) \) are given by (3.134) and (3.119) respectively and integrating (3.111) we find

\[
\begin{align*}
g_0(\rho) &= \rho \log \rho - (\rho + s^{-1}) \log (1 + s\rho) \quad (3.156) \\
g_1(\eta) &= \frac{1}{2A^2} (1 - \eta) \log (1 - \eta) + \frac{1}{2B^2} (1 + \eta) \log (1 + \eta) - \frac{1}{A^2B^2} \Delta_1(\eta) \log \Delta_1(\eta), \quad (3.157)
\end{align*}
\]

up to linear terms in \( \rho \) and \( \eta \) respectively which have been chosen such that the limit \( s \to 0 \) agrees with the \( s = 0 \) case (such linear terms are irrelevant since by (3.110) and (3.155) they correspond to linear terms in \( x_i \)). Substituting these into (3.110) and using (3.155) we find the symplectic potential in symplectic coordinates is given by

\[
g = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2 - \frac{1}{2} (x_1 + x_2) \log (x_1 + x_2) + \frac{1}{2} (A^2 x_1 + B^2 x_2) \log (A^2 x_1 + B^2 x_2) \]

\[
- \frac{1}{2} (A^2 x_1 + B^2 x_2 + 2/s) \left( \log (A^2 x_1 + B^2 x_2 + 2/s) + \log (s/2) \right), \quad (3.158)
\]

up to linear terms in \( x_i \) which we have chosen such that again the limit \( s \to 0 \) agrees with the \( s = 0 \) case. These expressions agree (up to linear terms) with the symplectic potentials for the CCLP black hole (3.74) for \( s = 4/\ell^2 \) and its near-horizon geometry (3.75) for \( s = 0 \), as they must.

As previously mentioned, the interior of the orbit space for the CCLP solutions in symplectic coordinates is the quarter plane \( x_1 > 0, x_2 > 0 \), with the boundaries \( x_1 = 0 \) and \( x_2 = 0 \) corresponding to the fixed points of \( \partial_{\phi_1} \) and \( \partial_{\phi_2} \) respectively, and the horizon at the origin \( x_1 = x_2 = 0 \). The form of the symplectic potential takes the canonical form \( g = \frac{1}{2} \sum \ell_A \log \ell_A \) where \( \ell_A = 0 \) are lines in symplectic coordinates, however, only the lines defined by the axes \( x_1 = 0 \) and \( x_2 = 0 \) correspond to the boundary of the orbit space and the remaining ones sit outside as depicted in fig. 3.3.

![Figure 3.3: The orbit space of the CCLP black hole, together with the lines that define the symplectic potential.](image)

It is instructive to see how the CCLP Kähler base is described in complex coordinates (see also [71]) and compare it with the symplectic formalism given above. The dictionary between the complex and symplectic formalism is detailed in Appendix 3.A. The real parts of the holomorphic coordinates \( y^i + i\phi^i \) are related to the moment maps \( x_i \) through \( y^i = \partial^i g \) and from (3.158) we find

\[
e^{2y^1} = \frac{x_1}{x_1 + x_2} \left( \frac{2\rho}{1 + s\rho} \right)^{A^2}, \quad e^{2y^2} = \frac{x_2}{x_1 + x_2} \left( \frac{2\rho}{1 + s\rho} \right)^{B^2}, \quad (3.159)
\]

where recall \( \rho \) is given by (3.155). The axes \( x_1, x_2 = 0 \) correspond to \( y^{1,2} \to -\infty \) with the horizon.
$x_1 = x_2 = 0$ to $y^1 = y^2 \rightarrow -\infty$.

Let us now determine the Kähler potential. To this end, observe that we can rewrite (3.173) as $\partial^1 K = G^{ij} x_j$ using the fact that $G^{ij} = \partial y^i / \partial x_j$ (the latter follows from (3.174)). Hence, given the symplectic potential in symplectic coordinates and $G^{ij} = \partial^i \partial^j g$ we can use this to determine the Kähler potential as a function of the symplectic coordinates. In the case at hand we find

$$\partial^1 K = \frac{A^2}{2 (1 + s \rho)}, \quad \partial^2 K = \frac{B^2}{2 (1 + s \rho)},$$

which readily integrates to $K = s^{-1} \log (1 + s \rho)$, and note that the limit $s \to 0$ agrees with the $s = 0$ case which is simply $K = \rho$. The final step is to invert (3.159) to express $K$ as a function of the (real part of) holomorphic coordinates $y^1$ and $y^2$. Generically, it is not possible to invert (3.159) analytically, however it determines $\rho$ implicitly as a function of $y^1$. The Kähler potential is therefore given by

$$K = \frac{1}{s} \log \left(1 + s \rho(y^1, y^2)\right) \quad \text{with} \quad \left(\frac{1 + s \rho}{2 \rho}\right)^A e^{2y_1} + \left(\frac{1 + s \rho}{2 \rho}\right)^B e^{2y_2} = 1.$$  

Equivalent expressions for the Kähler potential were obtained in [71]. It is worth contrasting this implicit expression for the Kähler potential to the explicit expression for the symplectic potential (3.158).

### 3.5 Discussion

In this chapter we have proven a uniqueness theorem for the CCLP black hole [17] within a class of supersymmetric solutions timelike outside of a horizon. Our main assumptions were a particular realisation of toric symmetry which not only commutes with supersymmetry, but has moment maps of a separable form, which constrain the Kähler geometry to be ‘product-toric’, orthotoric or Calabi-type. This assumption is particularly restrictive and allows us to completely determine the general solution in terms of the near-horizon geometry data. This result is complementary to a previous uniqueness theorem for supersymmetric black holes with $SU(2)$ symmetry chapter 2; the two classifications overlap for the special case of $SU(2) \times U(1)$-invariant solutions where one obtains a uniqueness theorem for the Gutowski-Reall black hole (or its near-horizon geometry).

As in the previous chapter, our uniqueness theorem does not make any global assumptions on the spacetime and in particular does not make any assumptions on the asymptotics of the solution. Therefore, it rules out the possibility of black holes in this symmetry class with a smooth horizon in asymptotically locally AdS$_5$ spacetimes (other than trivial global quotients of the CCLP solution). A class of asymptotically locally AdS$_5$ black hole solutions with a squashed $S^2$ boundary and $SU(2) \times U(1)$ symmetry have been constructed numerically [77, 78, 89], however, these have non-smooth horizons [82]. In fact, since $SU(2) \times U(1)$-invariant solutions are automatically toric and of Calabi type, one can also deduce from our theorem that these numerical solutions must have non-smooth horizons. It is interesting that for supersymmetric black holes with a smooth horizon the near-horizon geometry determines the full solution uniquely and does not allow for more general conformal boundary metrics. While it is known that supersymmetry constrains the boundary geometry [111–115], our uniqueness theorem shows that for supersymmetric solutions with smooth horizons the boundary geometry is even more constrained (at least for toric Calabi type Kähler bases). It would be interesting to understand this phenomenon from a holographic perspective.

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17For the case $A^2 = B^2$, which corresponds to the GR black hole, we can do this explicitly. We have

$$x_1 = \frac{e^{2y_1}}{A^2 (e^{2y_1} + e^{2y_2})^{-1} A^2 - s/2}, \quad x_2 = \frac{e^{2y_2}}{A^2 (e^{2y_1} + e^{2y_2})^{-1} A^2 - s/2},$$

and the Kähler potential reads

$$K = -\frac{1}{s} \log \left(1 - \frac{s}{2} (e^{2y_1} + e^{2y_2}) A^2\right).$$

18This assumption is not necessary when working with minimally supersymmetric solutions, see section 1.3.1.
The proof of our uniqueness theorem uses the classification of near-horizon geometries in the minimal gauged supergravity as an essential input \cite{44,67}. In that work, a candidate near-horizon geometry for a black ring with a conical singularity on the $S^2$ factor of the horizon was found, which uplifts to a warped AdS$_3$ solution of type IIB supergravity that can be made regular in 10-dimensions. Recently, it has been shown that the horizon of this solution in five-dimensions is an orbifold known as a spindle \cite{116}. It is an interesting open problem to construct an asymptotically AdS$_5$ black spindle with such a near-horizon geometry. Since our uniqueness proof is constructive, it would be interesting to investigate whether our methods can be used to determine the existence of such black spindles and even classify them.

It is worth emphasising that our analysis concerns only supersymmetric Lorentzian signature solutions with a smooth horizon. As noted in the introduction, the supersymmetric CCLP black hole solution can be obtained from a family of non-supersymmetric non-extremal black hole solutions \cite{17} by imposing the BPS condition as well as a non-linear relation among the charges $J_1$, $J_2$ and $Q$. Relaxing the latter results in a 3-parameter family of solutions which are locally supersymmetric \cite{17}, \cite{115}, but contain naked closed timelike curves and hence are excluded from our analysis. In fact, the Kähler base of such solutions belongs to the orthotoric class \cite{72}. In Euclidean signature, we can further relax the reality of the fields and obtain a class of complex BPS saddles \cite{15}, which however cannot be analytically continued to Lorentzian signature. Nonetheless, these complexified solutions are important holographically since they are dual to the dominant saddles of the field theory localisation computation (see also \cite{20}). An interesting future extension of this analysis would be the classification of solutions of such type and to determine whether asymptotically locally AdS solutions exist in this class.

The original motivation for this work was to investigate the existence of topologically non-trivial supersymmetric black holes, such as black lenses, black holes in bubbling spacetimes and multi-black holes in AdS$_5$ (which are all known in flat space). To accommodate such topologies one must work in the general class of solutions with toric symmetry which we investigated in the first part of this chapter. Unfortunately, we found that this problem reduces to a highly non-trivial problem in toric Kähler geometry. We found that symplectic coordinates appear to be best adapted to describing such solutions, in terms of which the axes of symmetry and the horizons take a simple form (line segments and points respectively). Furthermore, it is also straightforward to determine the symplectic potential (which encodes the Kähler metric) near any axis or horizon component. Therefore we are able to write down the singular part of this potential for possible new solutions with non-trivial topology. Unfortunately, the problem reduces to a very complicated non-linear 8th order PDE for this symplectic potential which prevents us from addressing the existence of such configurations. It would be interesting to study this PDE in more detail to determine if it possesses any hidden structure. In any case, perhaps numerical methods could be employed to construct new solutions with such non-trivial topology.

Our work admits another generalisation along the lines of Calabi type Kähler geometries. As was discussed in Appendix 3.E, one can define Calabi-type Kähler geometries with a single $U(1)$ Killing field. While there are indirect hints that one cannot find a generalisation of the CCLP black hole within this class, the black holes in general are, nonetheless, not excluded.

Finally, it would also be interesting to investigate the classification of supersymmetric black holes in AdS$_5$ in theories beyond minimal supergravity. Numerical evidence for hairy supersymmetric black holes in truncations of supergravity that retain complex scalars has been recently obtained \cite{86,87}. Supersymmetric timelike solutions in such truncations are found to also possess a Kähler base \cite{117}. Therefore, one may hope that similar methods to those used in previous and the present chapter can be used to investigate the classification of supersymmetric near-horizon geometries and black holes in such theories. This, however, remains out of the scope of current thesis.

### 3.A Toric Kähler manifolds in symplectic coordinates

Here we review the construction of complex and symplectic coordinates for Kähler toric manifolds. This is described in \cite{104}, although we fill in several details.
A Kähler manifold \((M, g, J)\) is toric if there is a \(T^2\)-action which is an isometry and is Hamiltonian. Let \(K_1, K_2\) denote the two commuting Killing fields that generate the \(T^2\)-action (in this section \(g\) is the Kähler metric and \(J\) is the Kähler form or complex structure as appropriate). The Hamiltonian condition means there are globally defined moment maps \(x_i, i = 1, 2\), defined by

\[ \iota_{K_i} J = -\partial x_i . \]  

(3.164)

This implies the Kähler form is preserved \(\mathcal{L}_{K_i} J = 0\) (locally the converse is also true since \(J\) is closed). Furthermore, closedness and invariance of \(J\) implies

\[ c_{ij} := \iota_{K_i} \iota_{K_j} J , \]  

(3.165)

is a constant antisymmetric matrix. Now, from the definition of the moment maps we get \(\mathcal{L}_{K_i} x_j = -c_{ij}\) and therefore since we may assume the \(K_i\) to have periodic flows, the \(x_i\) will not be periodic functions unless \(c_{ij} = 0\) (if \(c_{ij} \neq 0\) the function \(x_i\) is monotonic along the orbit curves of \(K_j\)). Thus we will assume

\[ c_{ij} = 0 , \]  

(3.166)

henceforth.

Define vector fields \(X_i = JK_i\) and note that

\[ g(K_i, X_j) = g(K_i, JK_j) = g_{ab} K^a_i J^b_c K^c_j = J_{\omega c} K^a_i K^c_j = 0 , \]  

(3.167)

where the last equality follows from \(c_{ij} = 0\). Now, since \(K_i\) span 2-spaces (away from fixed points), this shows that the \(X_i\) span their 2d orthogonal complements. The invariance of \(J\) together with \([K_i, K_j] = 0\) implies \([K_i, X_j] = \mathcal{L}_{K_i} J K_j = 0\). Integrability of \(J\) is equivalent to the vanishing of the Nijenhuis tensor

\[ 0 = N(X, Y) = [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY] , \]  

(3.168)

for all vector fields \(X, Y\). In particular, \(N(K_i, K_j) = 0\) then reduces to \([X_i, X_j] = 0\). This shows that the 2d orthogonal complements to \(\text{span}(K_1, K_2)\) are integrable distributions.

Thus we have a commuting frame \(K_i, X_j\) (recall \(K_i\) and \(X_j\) span 2d spaces and their orthogonal complements so must form a frame for \(M\)). The dual vectors \(\omega^i, J\omega^i, 19\) defined by \(\omega^i(K_j) = \delta^i_j\) and \(\omega^i(X_j) = 0\), are therefore closed so we can write \(\omega^i = dt^i\) and \(J\omega^i = dy^i\) for functions \(t^i, y^i\). Thus \(y^i + it^i\) are holomorphic coordinates on \(M\), i.e. \(J(dy^i + it^i) = i(dy^i + idt^i)\). The metric in the coordinates \((y^i, t^i)\) is

\[ g = F_{ij} dy^i dy^j + F_{ij} dt^i dt^j , \]  

(3.169)

where we have used (note \(K_i = \partial_{y_i}, X_j = \partial_{t_j}\))

\[ F_{ij} := g(K_i, K_j) = g(JK_i, JK_j) = g(X_i, X_j) , \]  

(3.170)

and \(g_{y^i t^j} = g(X_i, K_j) = 0\) from above.

Next, using \(Jdy^i = -dt^i\) etc it then easily follows that the Kähler form is

\[ J = F_{ij} dy^i \wedge dt^j , \]  

(3.171)

and closedness of \(J\) is equivalent to \(F_{ij} dy^i\) being closed, which is equivalent to \(F_{ij}\) being the Hessian\(^{20}\)

\[ F_{ij} = \partial_{y^i} \partial_{y^j} \mathcal{K} , \]  

(3.172)

of some function \(\mathcal{K}(y)\). It turns out that \(\mathcal{K}\) is the Kähler potential, i.e. \(J = 2i\partial \partial \mathcal{K}\) (in complex coordinates \(z^i = y^i + it^i\) we have \(\partial_{z^i} = (1/2)(\partial_{y^i} - i \partial_{t^i})\) and note that \(\mathcal{K}\) depends only on \(y^i\). Therefore, the above gives a chart \((y^i, t^i)\) for toric Kähler manifolds, where \(y^i + it^i\) are holomorphic coordinates, which is adapted to the Killing fields \(K_i = \partial_{y^i}\). The identity that generate the toric

\(^{19}\)It is easy to check that the dual vectors to \(X_i = JK_i\) are \(J\omega^i\).

\(^{20}\)To prove this, note that locally there are functions \(f_i\) such that \(df_i = F_{ij} dy^j\) which is equivalent to \(f_{i,j} = F_{ij}\). Hence \(f_{i,[j]} = 0\) by symmetry of \(F_{ij}\). Thus locally \(f_i = \partial_i f\) for some function \(f\), i.e. \(F_{ij} = \partial_i \partial_j f\) as required.)
There is another natural chart for toric Kähler manifolds which is adapted to the symplectic structure rather than the complex structure. This is easy to deduce from the complex coordinates above. First note that computing the moment maps we find
\[ dx_i = F_{ij} dy^j = \delta(\partial_y K) \implies x_i = \partial_y K, \]
where we have fixed an additive constant of integration for \( x_i \). The inverse is given by
\[ dy^i = G^{ij} dx_j, \]
where \( G^{ij}(x) \) is the inverse matrix to \( F_{ij}(y) \). Integrability of this requires \( G^{ij} \) to also be the Hessian
\[ G^{ij} = \partial_{x^i} \partial_{x^j} g, \]
\[ (3.175) \]
where \( K_i = \partial/\partial \phi^i \) and \( G_{ij} \) is the inverse of \( G^{ij} \). Note that the Kähler form is simply in Darboux coordinates (thought of as a symplectic form). The coordinates \((x_i, \phi^i)\) are called symplectic coordinates.

### 3.B CCLP black hole

The supersymmetric CCLP black hole is a two parameter family of solutions first found in [17]. The Kähler base for this solution was determined in [118] and we present it here essentially in the form obtained there, with a few further simplifications.

The solution depends on two parameters \( A^2, B^2 > 0 \) subject to \( \kappa^2(A^2, B^2) > 0 \) where \( \kappa^2 \) is given by (3.53). Note that this implies \( A^2, B^2 < 1 \). The Kähler base metric and Kähler form of CCLP are simply given by
\[ h = \frac{dr^2}{V(r)} + \frac{r^2}{4} \left( \frac{d\psi^2}{\Delta} + \Delta \sin^2 \varphi d\varphi^2 \right) + \frac{r^2 V(r)}{4} (d\psi + \cos \varphi d\varphi)^2, \]
\[ X^{(1)} = d \left( \frac{1}{2} r^2 (d\psi + \cos \varphi d\varphi) \right), \]
\[ (3.179) \]
where \( \Delta = A^2 \cos^2(\varphi/2) + B^2 \sin^2(\varphi/2) \) and \( V = 1 + \frac{r^2}{\ell^2} \) for CCLP and \( V = 1 \) for its near-horizon geometry (also supersymmetric solution). In terms of \( 2\pi \)-periodic angles \( \phi^i \),
\[ \psi = A^{-2} \phi^1 + B^{-2} \phi^2, \quad \varphi = -A^{-2} \phi^1 + B^{-2} \phi^2. \]
\[ (3.180) \]
The other coordinate ranges are \( r \geq 0 \) and \( 0 \leq \vartheta \leq \pi \).

The five-dimensional metric is then given by (3.1) with
\[ f^{-1} = 1 - \frac{\ell^2}{r^2} \left( \Delta - 1 + A^2 + B^2 \right), \]
\[ (3.181) \]
and
\[
\omega = \left( \frac{r^2 + \ell}{2\ell} (1 - \Delta \phi) - \frac{\ell^3}{12\ell^2} (6\Delta \phi - A^4 - B^4 + A^2B^2 - 2(A^2 + B^2) - 1) \right) (d\psi + \cos \theta d\phi) \\
+ \frac{\ell(A^2 - B^2)}{4} \left( 1 - \frac{\ell^2 \Delta \phi}{r^2} \right) \sin^2 \theta d\phi,
\]
(3.182)
and the gauge field is
\[
A = \frac{\sqrt{3}}{2} f(dt + \omega) - \frac{\ell}{2\sqrt{3}} \left( 1 + 3\frac{r^2}{2\ell^2} - \frac{1}{2}(A^2 + B^2) \right) (d\psi + \cos \theta d\phi) - \frac{\ell\sqrt{3}}{8}(A^2 - B^2) \sin^2 \theta d\phi.
\]
(3.183)
It is interesting to note that the CCLP solution is much simpler in this coordinate system.

For \( A^2 = B^2 \) we recover the GR solution with \( \alpha = (2A)^{-1} = (2B)^{-1} \). Then (3.179) reduces to (3.68) if we also rescale \( (\psi, \phi)_{\text{GR}} = (A^2 \psi, A^2 \varphi)_{\text{CCLP}} \).

In order to compare the above expressions with the ones in the literature, we give the dictionary between the notation here and in \([118]\). The latter describes the generalisation of the CCLP black hole in the STU model, which admits a consistent truncation to minimal gauged supergravity with
\[
\frac{2}{\sqrt{3}} A_{\text{here}} = A^1_{\text{here}} = A^2_{\text{here}} = A^3_{\text{here}}.
\]
Our parameters and coordinates are related to those in \([118, \text{section 2.3}]\) as follows:
\[
\ell_{\text{here}} = g^{-1}_{\text{there}}, \quad (r/\ell)_{\text{here}} = \sinh(g\sigma)_{\text{there}}, \quad (\phi^1, \phi^2, \theta)_{\text{here}} = (-\phi, -\psi, 2\theta)_{\text{there}},
\]
(3.185)
\[
(A^2, B^2)_{\text{here}} = (A^2, B^2)_{\text{there}}, \quad \Delta g|_{\text{there}} = \left. \frac{\Delta g}{g^2 c^2} \right|_{\text{there}}.
\]
(3.186)

### 3.C Unified form of near-horizon geometry

In section 3.2.4 we presented the general near-horizon geometry in a form which treats the solution with generic toric symmetry and enhanced \( SU(2) \times U(1) \) symmetry on the same footing (these correspond to \( A^2 \neq B^2 \) and \( A^2 = B^2 \) respectively). We will show how this unified form of the near-horizon geometry is related to the original form of these near-horizon geometries which treated the two cases separately \([44,46]\).

#### Generic toric symmetry

This near-horizon geometry was derived in \([44]\) and we first recall the solution in their notation. The near-horizon data, i.e. the leading order of (3.42), depend on three parameters \( \Gamma_0, \Gamma_1, \Gamma_2 \) and explicitly read (\( x^1_{\text{here}} = x^2_{\text{here}} \))
\[
\Delta^{(0)} = \frac{\Delta_0}{\Gamma^2},
\]
\[
h^{(0)} = \Gamma^{-1} \gamma_{11} dx^1 - \Gamma^{-1} d\Gamma,
\]
\[
\gamma^{(0)} = \frac{\ell^2 d\Gamma^2}{4P(\Gamma)} + \gamma_{11} \left( dx^1 + \frac{\gamma_{12}}{\gamma_{11}} dx^2 \right)^2 + \frac{4P(\Gamma)}{\ell^2 \gamma_{11}} (dx^2)^2,
\]
(3.187)
where
\[
\gamma_{11} = C^2 \Gamma - \frac{\Delta_0^2}{\Gamma^2}, \quad \gamma_{12} = \frac{\Delta_0 (\alpha_0 - \Gamma)}{\Gamma^2},
\]
(3.188)
and
\[
P(\Gamma) = \Gamma^3 - \frac{C^2 \ell^2}{4} (\Gamma - \alpha_0)^2 - \frac{\Delta_0^2}{C^2} = (\Gamma - \Gamma_0)(\Gamma - \Gamma_1)(\Gamma - \Gamma_2).
\]
(3.189)
From the latter equation we can obtain the relation between $\Delta_0, C, a_0$ and $\Gamma_0, \Gamma_1, \Gamma_2$

$$C^2 = \frac{4i}{\ell^2} (\Gamma_0 + \Gamma_1 + \Gamma_2), \quad a_0 = \frac{\Gamma_0 \Gamma_1 + \Gamma_0 \Gamma_2 + \Gamma_1 \Gamma_2}{2(\Gamma_0 + \Gamma_1 + \Gamma_2)}.$$

$$\Delta_0^2 = \frac{4\Gamma_0 \Gamma_1 \Gamma_2 (\Gamma_0 + \Gamma_1 + \Gamma_2)}{\ell^2} - \frac{(\Gamma_0 \Gamma_1 + \Gamma_0 \Gamma_2 + \Gamma_1 \Gamma_2)^2}{\ell^2}. \quad (3.190)$$

Observe that (3.187) is invariant under the rescalings

$$\Gamma \to K \Gamma, \quad \chi^i \to K^{-1} \chi^i, \quad \chi^2 \to \chi^2, \quad \Gamma_{0,1,2} \to K \Gamma_{0,1,2}, \quad (3.191)$$

where $K$ is a positive constant.

The parameters $\Gamma_0, \Gamma_1, \Gamma_2$ are constrained by

$$0 < \Gamma_0 < \Gamma_1 < \Gamma_2, \quad \Delta_0^2(\Gamma_0, \Gamma_1, \Gamma_2) > 0, \quad (3.192)$$

and the coordinate range is $\Gamma_0 \leq \Gamma \leq \Gamma_1$ with $P(\Gamma) > 0$ in the interior. At the endpoints $\Gamma = \Gamma_0, \Gamma_1$ two different linear combinations of the biaxial Killing fields vanish and the metric generically has conical singularities at these points. The biaxial Killing fields $\partial_{\chi^i}$ do not necessarily have closed orbits and are related to the Killing fields with fixed points $m_i = \partial_{\chi^i}$ by

$$m_1 = -d_1 \left( \frac{\gamma_1(\Gamma_0)}{\gamma_1(\Gamma_0)} \partial_{\chi^1} - \partial_{\chi^2} \right), \quad m_2 = -d_2 \left( \frac{\gamma_2(\Gamma_1)}{\gamma_1(\Gamma_1)} \partial_{\chi^1} - \partial_{\chi^2} \right), \quad (3.193)$$

where $m_1 = 0$ at $\Gamma = \Gamma_0$ and $m_2 = 0$ at $\Gamma = \Gamma_1$. In order to avoid conical singularities $m_i$ must have closed orbits. The constants $d_i$ can be determined (up to signs) by requiring that $\dot{\phi}^i \sim \dot{\phi}^i + 2\pi$ and that the metric has no conical singularities at these endpoints. We find

$$d_1 = -\frac{\ell}{2(\Gamma_0 - \Gamma_1)(\Gamma_0 - \Gamma_2)} \left( 2\Gamma_0^2 + \Gamma_0(\Gamma_1 + \Gamma_2) - \Gamma_1 \Gamma_2 \right),$$

$$d_2 = \frac{\ell(\Gamma_0(\Gamma_1 - \Gamma_2) + \Gamma_1(2\Gamma_1 + \Gamma_2))}{2(\Gamma_0 - \Gamma_1)(\Gamma_1 - \Gamma_2)}, \quad (3.194)$$

where we have chosen signs to facilitate our analysis in the main text. The transformation to the $2\pi$-periodic angles is given by $\chi^i = \hat{\phi}^i A^j_i$ where $m_i = A^j_i \partial_{\chi^j}$ and the matrix $A$ can be read off the equations (3.193) and (3.194).

It is also worth recording that from [44] it can be deduced that

$$Z^{(0)} = -\frac{C^2 \ell(\Gamma - a_0)}{2\Gamma} d\chi^1 + \frac{2\Delta_0}{iC^2 \Gamma} d\chi^2 + \frac{\ell d\Gamma}{2\Gamma}, \quad (3.195)$$

where $Z^{(0)}$ is the leading order of the 1-form $Z$ which encodes the Kähler form in GNC (3.43).

It is useful to express the near-horizon geometry $Z$ which encodes the Kähler form in terms of quantities invariant under (3.191). To this end, we define a new coordinate

$$\hat{\eta} := -\frac{\Gamma - \Gamma_0 + \Gamma - \Gamma_1}{\Gamma_1 - \Gamma_0}, \quad (3.196)$$

as well as new parameters

$$A^2 := \frac{\Gamma_2 - \Gamma_0}{\Gamma_0 + \Gamma_1 + \Gamma_2}, \quad \beta^2 := \frac{\Gamma_2 - \Gamma_1}{\Gamma_0 + \Gamma_1 + \Gamma_2}. \quad (3.197)$$

Now the coordinate range is $-1 \leq \hat{\eta} \leq 1$ and the first equation in (3.192) gives $0 < \beta^2 < A^2 < 1$. It is now straightforward to verify that the expressions (3.187) and (3.195) map to (3.54), (3.55), (3.56) and (3.59), while the second constraint in (3.192) maps to (3.52). Some useful relations in comparing are

$$\Delta_2(\hat{\eta}) = \frac{3\Gamma}{\Gamma_0 + \Gamma_1 + \Gamma_2}, \quad \kappa = \frac{3\ell \Delta_0}{(\Gamma_0 + \Gamma_1 + \Gamma_2)^2}. \quad (3.198)$$
Finally, we can extend the range of parameters in the region $B^2 > A^2$ by observing that if we exchange
\[ \hat{\phi}^1 \leftrightarrow \hat{\phi}^2, \quad \hat{\eta} \leftrightarrow -\hat{\eta}, \quad A^2 \leftrightarrow B^2, \]
we get identical near-horizon geometries as can be easily seen from the expressions in section 3.2.4.

**Enhanced symmetry**

The near-horizon geometry with $SU(2) \times U(1)$ rotational symmetry was first derived in [46]. In the notation of [82] the near-horizon data is parametrised by a constant $\Delta^{(0)} > \sqrt{3}/\ell$ and given by
\[
\begin{align*}
\hat{h}^{(0)} &= -\frac{3\Delta^{(0)}}{\ell(\Delta^{(0)} - 3/\ell^2)} \hat{\sigma}_3, \\
\gamma^{(0)} &= \frac{1}{\Delta^{(0)} - 3/\ell^2} \left( \hat{\sigma}_1^2 + \hat{\sigma}_2^2 \right) + \frac{\Delta^{(0)} - 3/\ell^2}{\Delta^{(0)} - 3/\ell^2} \hat{\sigma}_3^2,
\end{align*}
\]
where the right-invariant 1-forms read
\[
\begin{align*}
\hat{\sigma}_1 &= \sin \hat{\psi} \hat{d} \hat{\phi} - \cos \hat{\psi} \sin \hat{d} \hat{\phi}, \\
\hat{\sigma}_2 &= \cos \hat{\psi} \hat{d} \hat{\phi} + \sin \hat{\psi} \sin \hat{d} \hat{\phi}, \\
\hat{\sigma}_3 &= \hat{d} \hat{\psi} + \cos \hat{d} \hat{\phi},
\end{align*}
\]
in terms of the Euler angles $0 \leq \hat{\psi} \leq 4\pi$, $0 \leq \hat{\phi} \leq 2\pi$ and $0 \leq \hat{\theta} \leq \pi$.

The above near-horizon data corresponds precisely to the $B^2 = A^2$ special case of (3.54), (3.55) and (3.56). Explicitly, the two are related by the change of coordinates
\[
\begin{align*}
\hat{\psi} &= \hat{\phi}_1 + \hat{\phi}_2, \\
\hat{\phi} &= -\hat{\phi}_1 + \hat{\phi}_2, \\
\cos \hat{\theta} &= \hat{\eta},
\end{align*}
\]
and change of parameters
\[
\Delta^{(0)} = \frac{3 + 3A^2}{\ell^2} = \frac{3A^2}{\ell^2 (1 - A^2)}.
\]
The parameter range $\Delta^{(0)} > \sqrt{3}/\ell$ maps to $0 < A^2 < 1$. Furthermore, $Z^{(0)} = -(\ell/3)\hat{h}^{(0)}$ (see [46]) which agrees with (3.59) with $B^2 = A^2$.

### 3.D Supersymmetric soliton

The smooth supersymmetric soliton was first identified in [76] from a larger family of supersymmetric solitons [75]. The convenient coordinate chart $\{t, \theta, \phi, r\}$, with $\theta \in [0, \pi], \phi \in [0, 2\pi], \psi \in [0, 4\pi)$ being the Euler angles, was introduced in [72]. In this chart a general 2-parameteric family takes the form
\[
ds_5^2 = -\frac{r^2 V}{4B} dt^2 + dr^2 + B \left( d\psi + \cos \theta d\phi + (d\theta)^2 + \frac{1}{4} (r^2 + q) (d^2 \theta + \sin^2 \theta d\phi^2) \right),
\]
where $g = \ell^{-1}$ and the functions are given by
\[
\begin{align*}
V &= \frac{r^4 + g^2 (r^2 + q)^3 - g^2 \alpha^2}{r^2 (r^2 + q)}, \\
B &= \frac{(r^2 + q)^3 - \alpha^2}{4(r^2 + q)^2}, \\
\hat{f} &= \frac{2\alpha r^2}{\alpha^2 - (r^2 + q)^3},
\end{align*}
\]
where $\alpha, q$ are parameters fixed by smoothness. Finally, the graviphoton is
\[
A = \frac{\sqrt{3}}{2(r^2 + q)} \left( q dt - \frac{1}{2} \alpha (d\psi + \cos \theta d\phi) \right).
\]
Note that the $\partial/\partial t$ is not the supersymmetric Killing field, which is given by
\[
V = \frac{\partial}{\partial t} - 2g \frac{\partial}{\partial \phi},
\]
and its square norm is
\[ V^2 = -f^2 = -\frac{(r^2 - \alpha g \cos \theta)^2}{(r^2 + q)^2}. \] (3.208)

In [76] it was shown that the closed time-like curves (CTCs) vanish for
\[ \alpha^2 = q^3. \] (3.209)

In addition, to avoid a conical singularity at \( r \to 0 \), one has to impose
\[ q = \frac{1}{9g^2}. \] (3.210)

This soliton has a peculiar symmetry structure. While the metric admits \( SU(2) \times U(1) \) symmetry, the Killing spinors do not share the same symmetry, and, hence, the supersymmetric Killing field does not commute with its generators, and the Kähler base preserves only \( U(1)^2 \) symmetry. This is the consequence that the soliton has non-minimal 1/2-supersymmetry, see Lemma 1.

Another property of the soliton is that it admits an **ergosurface**, namely a region of the spacetime where stationary Killing field \( V \) becomes null, but which is not a degenerate point of symmetry or a horizon. This kind of geometric features have been quite well studied in the ungauged case, see [119] and references therein. The ergosurface happens at
\[ r^2 = \alpha g \cos \theta, \] (3.211)

so that the region inside the ergosurface has only one component of axis \( \theta = 0 \).

As it was mentioned in section 3.3, the Kähler base of the soliton is orthotoric. The matching to the orthotoric coordinates \( (\rho, \eta, \Psi, \Phi) \) is given by [72]
\[ \rho = \frac{r^2}{\alpha g}, \quad \eta = \cos \theta, \quad \Phi = \frac{\alpha g^3}{4}(\phi + 2gt), \quad \Psi = \frac{\alpha g^3}{4}\psi, \] (3.212)

such that the metric functions are
\[ F(\rho) = 4 \left( \rho + \frac{q}{\alpha g} \right)^3 - \frac{4}{\alpha g^3}(\rho^2 - 1), \quad G(\eta) = \frac{4}{\alpha g^3}(1 - \eta^2). \] (3.213)

The symplectic potential for the soliton can be calculated from substituting (3.213) into (3.90), but does not admit a simple form in elementary functions.

### 3.6 Calabi type geometries with single axial Killing field

In [93] Calabi-type Kähler geometries were introduced as admitting a particular type of hamiltonian 2-form which generates only a single linear independent Killing field. The authors have shown that this case breaks in two:

- the geometry is locally a Kähler product of two Riemannian surfaces, one of which admits a Killing field; or
- the geometry admits a hamiltonian 2-form which generates two linearly dependent but non-zero Killing fields.

The second class is more interesting, because, generally, solutions constructed from product geometries will not have locally AdS asymptotic. The geometry is given by [93]:
\[ h = \rho g_{\Sigma} + \frac{\rho}{V(\rho)} d\rho^2 + \frac{V(\rho)}{\rho} (d\psi + \alpha)^2, \] (3.214)
\[ X^{(1)} = \rho \omega_{\Sigma} + d\rho \wedge (d\psi + \alpha). \] (3.215)
where $g_\Sigma$ is a metric on 2-manifold $\Sigma$ with volume form $\omega_\Sigma$, and $d\alpha = \omega_\Sigma$. The Killing field is given by $m = \partial/\partial \psi$ in this chart. A natural looking chart with coordinates $\eta, \phi$ on $\Sigma$ is

$$g_\Sigma = \frac{d\eta^2}{G(\eta, \phi)} + G(\eta, \phi)d\phi^2,$$

and the full geometry is now

$$h = \frac{\rho}{V(\rho)}d\rho^2 + \frac{\rho}{G(\eta, \phi)}d\eta^2 + \frac{V(\rho)}{\rho}(d\psi + \alpha)^2 + \rho G(\eta, \phi)d\phi^2,$$

$$X^{(1)} = d(\rho d\psi + \eta d\phi).$$

Notice that the Kähler form has $U(1)^2$ symmetry. Now, if one imposes additional $SU(2)$-symmetry such that the total isometry group is $SU(2) \times U(1)$, the result of chapter 2, namely Theorem 3, establishes that the geometry has enhanced $SU(2) \times U(1)$-symmetry. It is reasonable to expect, however, that the $SU(2)$-symmetric and $U(1)^2$-symmetric limits of single axisymmetric black hole can be taken independently. This can be seen as an indirect indication that this class does not admit a three-parameter generalisation of CCLP black hole. If such black hole would indeed exist, it is natural to expect that these two limits will be independent; in other words, there is no reason to exclude $SU(2)$-invariant black holes with single $U(1)$ Killing field.
Chapter 4

Supersymmetric solutions in gauged supergravity with vector multiplets

This chapter is based on the paper [120] in preparation authored by myself, my supervisor James Lucietti and my colleague Praxitelis Ntokos.

4.1 Introduction

In this chapter we consider the classification of supersymmetric black hole solutions to five-dimensional gauged supergravity, under the assumption of $U(1)^3$ symmetry. As it was discussed in the introduction chapter 1, the AdS$_5$/CFT$_4$ establishes a correspondence between a five-dimensional $\mathcal{N} = 8$ maximal gauged supergravity on AdS$_5$ and $\mathcal{N} = 4$ SYM on a four-dimensional conformally Minkowski spacetime. The former appears as a dimensional reduction of type IIB supergravity on $S^5$, where massive modes are consistently set to zero.\footnote{Technically, a complete proof of consistency was given only for $U(1)^3$ truncation [121].}

The bosonic field content of 5d $\mathcal{N} = 8$ gauged supergravity consists of the metric, fifteen $SO(6)$ gauge fields, twelve 2-form gauge potentials in the $6 + \bar{6}$ representations of $SO(6)$ and 42 scalars in the $1 + 1 + 20' + 10 + 10$ representations. The most relevant subsector for holography, the 5d $SO(6)$ supergravity, consists of fields that descend from metric and self-dual 5-form in ten-dimensions. Namely, this is the metric, fifteen gauge fields and 20' scalars parameterising a $SL(6,\mathbb{R})/SO(6)$ coset. The matter content is still formidable, and typically only the $U(1)^3$ Cartan subgroup is considered, coupled to two neutral scalar fields and, optionally, three complex scalars. The latter contribute to hairy solutions, see, e.g. [122] for discussion.

We will be interested in the minimal generalisation, with three $U(1)$ gauge fields and only neutral scalars present. There are several motivations for the absence of complex scalars. First of all, their presence typically allows for hairy solutions [86, 87, 122–124], which are not susceptible to classification even in the asymptotically flat (AF) case. Furthermore, the author is unaware of attempts at classifying supersymmetric near-horizon geometries in gauged supergravity coupled to complex scalars,\footnote{For example, it is not known whether a general smooth supersymmetric compact NH geometry in this theory has $SO(2,1) \times U(1)^2$ symmetry.} which poses a critical obstruction to black hole classification, which essentially relies on the knowledge of NH geometry as a unique boundary condition for equations of motion/supersymmetric conditions. Finally, the general local form of supersymmetric solutions in the sense of section 1.3 is more involved in the presence of charged matter [117].

A general black hole in this theory is expected to have 6 conserved charges: the mass $M$, two independent angular momenta $J_1, J_2$ and three $U(1)$ electric charges $\{Q_1, Q_2, Q_3\}$. For
supersymmetric solutions the charges must satisfy the BPS bound
\[ M = \ell(J_1 + J_2) + Q_1 + Q_2 + Q_3. \]

The most general analytically known supersymmetric black hole, the Kunduri-Lucietti-Reall (KLR) solution [118], has four parameters, and its charges obey an additional constraint.\(^4\) It has a horizon with \(S^3\) cross-sections, and can be seen as a multi-charge generalisation of CCLP black hole [17], to which it reduces in the limit of equal electric charges \(Q_1 = Q_2 = Q_3\). The solution has \(U(1)^2\)-symmetry, and contains \(SU(2) \times U(1)\)-symmetric solutions of [46] as a \(J_1 = J_2\) limit. The numerical supersymmetric solutions are known as well [89, 90]. The most general one [90] is a three-parametric spherical black hole with \(SU(2) \times U(1)\)-symmetry and squashed \(S^3\) at the asymptotic boundary. This solution is a generalisation of a numerical solution of [77, 78] in the minimal case, which was shown to be non-smooth in the previous chapter 2.

The discrepancy between the parameters leaves open the question of existence of a larger family of smooth supersymmetric black holes, which includes KLR solution as a limit, and, more generally, of black hole classification. The essential part of the classification is the knowledge of admitted NH geometries. In this theory, the supersymmetric NH geometries have been classified only under the assumption of \(U(1)^2\)-symmetry [47]. The symmetry-agnostic classification analogous to [45, 67] has, unfortunately, been carried only partially: although in [48] it was shown that all NH geometries in this theory admit supersymmetry enhancement, it is not known whether all NH geometries have \(U(1)^2\)-symmetry, and, therefore, were found in [47].

The STU supergravity has a richer NH structure than the minimal case. Instead of a single geometry (up to global quotients) in the latter, the former admits three distinct families with different topology.\(^3\) For black holes with spherical or Lens space horizon cross-sections the NH geometry is locally isometric to that of the KLR, which is a result, analogous to the minimal case. The \(S^2 \times S^1\) case corresponds to black rings which are balanced by the presence of scalar matter. Finally, the toroidal \(T^3\) solutions are also allowed. Curiously, for the latter two examples the supersymmetric Killing vector field is everywhere null. Furthermore, these geometries are static, which was forbidden in the minimal case. This is again a consequence of scalar matter which provides a necessary repulsion to balance the solution.

Unfortunately, the classification of supersymmetric backgrounds in STU gauged supergravity is more complicated than in the minimal case. The degrees of freedom in the metric are mixed with scalars, and the result is a system of three non-linear PDEs — two Maxwell equations and an integrability constraint. In the static case, the constraints simplify to a system of coupled algebro-differential PDEs of lower order.

A natural strategy is to seek further simplifying symmetry assumptions that are compatible with the problem at hand. Following the reasoning of the previous chapter, we will immediately focus on the similar class of Kähler bases, namely toric Calabi type geometries. To proceed with classification, we also have to impose an ansatz on the gauge fields which generalises the minimal case. Geometrically, it is equivalent to aligning the gauge field strength along a hamiltonian 2-form, which is a special feature for Calabi-type geometries. Our main results can be summed up in the following theorems.

**Theorem 8.** A time-like supersymmetric toric solution to five-dimensional \(U(1)^3\) gauged supergravity with Kähler base of orthotoric or product-toric type cannot contain a smooth horizon with compact spherical \(S^3\) or Lens space \(L(p, q)\) cross-section.

In the following theorem we separate the case of additional \(SU(2)\)-symmetry, because (i) the ansatz on the gauge fields is automatically satisfied, and (ii) the nature of the proof requires analyticity.

**Theorem 9.** Any supersymmetric toric solution to five-dimensional \(U(1)^3\) gauged supergravity, which is not \(SU(2)\)-symmetric, that is timelike outside a smooth horizon with spherical or Lens

\(^4\)For \(SU(2) \times U(1)\)-symmetric solutions the constraint was also identified using effective superpotential formalism [115].
\(^3\)Since, physically, we are interested in horizons with compact orientable spatial cross-sections, their topologies are quite limited. These three cases can be equivalently formulated as horizons with (i) finite (or trivial), (ii) \(\mathbb{Z}\), (iii) \(\mathbb{Z}^3\) fundamental group.
space cross-sections, with a Kähler base of Calabi type, and with a minimal ansatz on the gauge fields, is locally isometric to the KLR black hole or its near-horizon geometry.

We also state and prove the following theorem for a non-generic case of SU(2)-symmetry.

**Theorem 10.** Any supersymmetric SU(2)×U(1)-symmetric solution to five-dimensional U(1)³ gauged supergravity that is timelike outside an analytic horizon with compact cross-sections, must be a Gutowski-Reall STU black hole or its near-horizon geometry.

We emphasise that no global assumptions are required for this result, so in particular it rules out asymptotically locally AdS⁺ supersymmetric black holes (other than quotients of KLR). Furthermore, it also does not assume the horizon is connected, and therefore rules out multi-black holes in this particular symmetry class. In this sense our result is analogous to the uniqueness theorem obtained for SU(2)-invariant supersymmetric solutions chapter 2, and for toric solutions chapter 3.

The organisation of this chapter is as follows. In section 4.2 we introduce a general local classification of supersymmetric solutions along the lines of [46]. The new result is an explicit derivation of a set of supersymmetric constraints in the timelike non-static and static cases. Next, in section 4.3 we introduce a convenient chart for NH geometry, and prove Theorem 8. We also discuss how supersymmetric constraints simplify for Calabi-type bases, as well as introduce and motivate the ansatz for the gauge field. Finally, in section 4.5 we prove Theorems 9 and 10. A number of details are relegated to the appendices.

### 4.2 Supersymmetric solutions to gauged supergravity with multiplets

In this section we introduce the five dimensional $\mathcal{N} = 1$ gauged supergravity coupled to $n$ vector multiplets and the construction of supersymmetric solutions to this theory following [46].

#### 4.2.1 $\mathcal{N} = 1$ supergravity

The action of the five-dimensional $\mathcal{N} = 1$ gauged supergravity coupled to $n$ Abelian vector multiplets and scalars is [125]

$$ S = \frac{1}{16\pi G} \int R_5 + 2\varepsilon^{-2}\mathcal{V} - Q_{IJ} F_5^I \wedge \ast_5 F^J - Q_{IJK} dX^I \wedge \ast_5 dX^J - \frac{1}{6} C_{IJK} F_5^I \wedge F_5^J \wedge A_5^K \tag{4.2} $$

where $I, J, K = 1, \ldots, n$ and $F_5^I = dA_5^I$. The scalar fields $X^I$ parameterise a symmetric space, and are subject to a constraint:

$$ \frac{1}{6} C_{IJK} X^I X^J X^K = 1 \tag{4.3} $$

$C_{IJK}$ are constant symmetric on $IJK$ and satisfy

$$ C_{IJK} C_{JL}(LMC_{PQ}) \delta^{IIP} \delta^{JKP} = \frac{4}{3} \delta_I (LC_{MPQ}) \tag{4.4} $$

For the STU gauged supergravity $n = 3$, and the constants are

$$ C_{IJK} = C^{IJK} = |\epsilon_{IJK}| \tag{4.5} $$

The bilinear form $Q_{IJ}$ in the action is given by

$$ Q_{IJ} = \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K \tag{4.6} $$

where

$$ X_I \equiv \frac{1}{6} C_{IJK} X^J X^K \tag{4.7} $$

$^5$One can instead introduce $n - 1$ independent scalar fields $\phi^\alpha$ and regard $X^I$ as functions of them.
s.t. $X_IX^I = 1$.

The action of the theory includes a potential $\mathcal{V}$ coming from Fayet-Iliopoulos (FI) term

$$\mathcal{V} = 3\ell^2 C^{IJK} \zeta_I \zeta_J X_K$$  \hspace{1cm} (4.8)$$

where $\zeta_I$ are constants associated with Fayet-Iliopoulos parameters.

For a bosonic background to be supersymmetric it must admit a spinor $\epsilon^a$ for which the supersymmetry variations of the gravitino and dilatino vanish. The former is

$$\nabla_\mu - \frac{1}{8} X_I \left( \gamma_\mu \gamma^\rho - 4 \delta_\mu^\rho \gamma^\rho \right) F_5^{I \mu \rho} \right) \epsilon^a + \frac{1}{6} \zeta_I \left( X^I \gamma_\mu + 3A^I_5 \right) \epsilon^a \epsilon^b = 0 \hspace{1cm} (4.9)$$

and for the latter it is

$$\left[ \frac{1}{4} Q_{IJ} \gamma^{\mu\nu} F_5^{J \mu \nu} - \frac{3}{4} \gamma^{\mu} \nabla_\mu X_I \right) \epsilon^a - \frac{3}{2\ell} \zeta_I \epsilon^{ab} \epsilon^b \right] \frac{\partial X^I}{\partial \phi^a} = 0 \hspace{1cm} (4.10)$$

where $\epsilon^a$ are symplectic Majorana spinors, and the spinor index $a = 1, 2$ and $e^{12} = +1 = -e^{21}$, and $\phi^a$ are $n-1$ unconstrained scalar fields. The Maxwell equations are

$$d \ (Q_{IJ} \ast_5 F_5^J) = -\frac{1}{4} C_{IJK} F_5^J \wedge F_5^K. \hspace{1cm} (4.11)$$

As in the minimal case, Einstein equations (and equations of motions for scalars) are automatically satisfied for supersymmetric backgrounds with on-shell gauge fields at least in the time-like case [46].

### 4.2.2 General local classification in the time-like case

The general classification procedure goes in the spirit of [70]. Firstly, one constructs all possible bilinears from the Killing spinor, and then uses gravitino, dilatino and Maxwell equations to solve for their derivatives. One particular example of such bilinear is the supersymmetric Killing field

$$V_\mu \epsilon^{ab} := \bar{\epsilon}^a \gamma_\mu \epsilon^b. \hspace{1cm} (4.12)$$

As in the minimal case, its norm $V^2 = -f^2$,

$$f \epsilon^{ab} = \epsilon^a \epsilon^b, \hspace{1cm} (4.13)$$

is everywhere non-positive, and the vector field is always causal. In this chapter we will consider only the solutions where $f$ does not vanish on some open set, i.e. the time-like class. The case of $V$ being globally a null Killing field was studied in [126]. One can choose coordinates adapted to the supersymmetric Killing field

$$g = -f^2 \ (dt + \omega)^2 + f^{-1} h_{mn} dx^m dx^n \hspace{1cm} (4.14)$$

where $V = \partial / \partial t$. The metric $h_{mn}$ can be regarded as the metric on the base space $B$ orthogonal to $V$. Supersymmetry then demands the geometry $(h, J)$ to be Kähler, where $J$ is also constructed as a spinor bilinear. Furthermore, $f, X^I$ are also functions on $B$

$$\mathcal{L}_V f = \mathcal{L}_V X^I = 0, \hspace{1cm} (4.15)$$

and $\omega$ is a 1-form on $B$ $i_V \omega = 0$. The supersymmetry also fixes orientation on $B$ such that $J$ is ASD: $J \wedge J = -2 \text{vol}_B$.

The presence of scalars makes the derivation of supersymmetric constraints more involved, as scalar degrees of freedom do not decouple from geometry in general. It is convenient to introduce rescaled scalar fields $\Phi^I$ by incorporating a factor of $f$

$$\Phi^I = \frac{6}{\ell} f^{-1} C^{IJ} X_J, \hspace{1cm} C^{IJ} = \frac{\ell}{2} C^{IJK} \zeta_K. \hspace{1cm} (4.16)$$
The scalar potential is then simply
\[ V = f \ell^2 \zeta_I \Phi^I. \] (4.17)

\( C^{IJ} \) is a symmetric non-degenerate matrix, and defines (an inverse) inner product on symmetric space
\[ \zeta^I := C^{IJ} \zeta_J. \] (4.18)

From this point onward we will set the number of Abelian vector multiplets to \( n = 3 \). For the STU model we can take FI parameters to be equal: \( \zeta_I = \ell^{-1} \) \( \forall I = 1, 2, 3 \). Then \( \zeta^I = \zeta_I \)
\[ \zeta^I = \ell^{-1} = \zeta_I, \quad \forall I = 1, 2, 3. \] (4.19)

From (4.17) one can define the inverse metric
\[ C^{IJ}C^{IJ} = 2 \ell^2 C^{ijk} \zeta_k - \ell^2 \zeta_I \zeta_J, \quad C^{IK}C^{KJ} = \delta^I_J \] (4.20)

which has a useful property
\[ W_I := C^{IJ}W_J = 2 \left( \frac{\ell^2}{2} (\zeta_K W^K \zeta_I - \delta_{IJ} W^J) \right), \quad \forall W^I. \] (4.21)

The bilinear \( f \) is then recovered from constraint (4.3)
\[ C^{IJK} \Phi_I \Phi_J \Phi_K = \frac{48}{f^3 \ell^3} \] (4.22)

where \( \Phi_I := C^{IJ} \Phi^J \).

Similar to the minimal case, the gauge fields can be written as
\[ A^I = X^I e^0 + A^I, \quad \iota_V A^I = 0 \] (4.23)

where \( e^0 = f (dt + \omega) \), and the \( U(1)^3 \) gauge was partially fixed by a (compatible with supersymmetry) condition \( \mathcal{L}_V A^I = 0 \). The supersymmetry fixes orthogonal part of the gauge field to be proportional to Ricci potential \( P \)
\[ P = \zeta_I A^I, \] (4.24)

i.e. the isotropic combination of gauge field strengths is the Ricci form
\[ \zeta_I F^I = \mathcal{R}. \] (4.25)

The 2-forms \( F^I = dA^I \) on \( B \) can be found from dilatino equation (4.10)
\[ F^I = \Theta^I - \Phi^I J. \] (4.26)

where \( \Theta^I \) is a self-dual form on \( B \) which respects Bianchi identity
\[ d^2 F^I = d \left( \Theta^I - \Phi^I J \right) = 0. \] (4.27)

Contracting (4.26) on \( J \) and using (4.25) gives
\[ \zeta_I \Phi^I = - \frac{1}{4} R. \] (4.28)

A common fact from Kähler geometry is that self-dual 2-forms (and the Kähler form itself) are \( J \)-invariant:
\[ F(JX, Y) = F(JY, X), \quad \forall X, Y \in TB, \] (4.29)
or in index notation
\[ J_m^p F^I_{pn} = F^I_{mp} J^p_n. \] (4.30)

The final supersymmetry constraint is on \( \omega \). Similarly to the minimal case, decomposing \( d\omega \)
into self-dual (SD) and anti-self-dual (ASD) parts\(^6\)
\[ d\omega = f^{-1} (G^+ + G^-), \tag{4.31} \]
one finds that the SD part is constructed from \( \Theta^I \)
\[ C_{IJ} \Theta^J \Phi^I = -\frac{4}{\ell} f^{-1} G^+. \tag{4.32} \]
The ASD part \( G^- \) can be expanded\(^7\) in the basis of ASD forms \( J^a = (J^1, J^2, J^3) \)
\[ f^{-1} G^- = -\frac{f^3}{48} \lambda_a J^a. \tag{4.33} \]
Similarly, one can decompose \( \Theta^I \) into the basis of SD forms \( I^a = (I^1, I^2, I^3) \)
\[ \Theta^I = \theta^I a I^a. \tag{4.34} \]
It is convenient to choose the bases \( J^a, I^a \) such that \( J^a n^m \) and \( I^a n^m \) are almost complex structures satisfying quaternionic relations (4.171) and (4.172). This leaves local \( SO(2) \), correspondingly \( SO(3) \), freedom in the choice of \( J^a \) and \( I^a \). For toric Kähler geometries, and, especially for Calabi-type, orthotoric and product-toric geometries introduced further, there are preferred choices which fix this freedom (see Appendix 4.A).

The last set of constraints are the Maxwell equations, which in terms of rescaled fields \( \Phi^I \) can be written as
\[ \nabla^2 \Phi^I = \frac{f^2}{12} \zeta^I \lambda_1 - \frac{f^2}{8} \zeta^I \nabla^2 R - \ell^{-1} C^I_{JK} \left( \sum_a \theta^I a \theta^K a - \Phi^I \Phi^K \right) \tag{4.35} \]
where \( C^I_{JK} = C_{IJK} \). The isotropic contraction of Maxwell equations gives \( \lambda_1 \)
\[ \lambda_1 = \frac{1}{2} \nabla^2 R + 4 \ell^{-1} C^I_{JK} \zeta^I \left( \sum_a \theta^I a \theta^K a - \Phi^I \Phi^K \right) \tag{4.36} \]
\[ = \lambda_1^{\text{min}} + \frac{1}{3} \left( R_{mn} R^{mn} - \frac{1}{2} R^2 \right) - 4 \ell^{-2} \sum_{I=1}^3 \left( \sum_a \theta^I a \theta^I a - (\Phi^I)^2 \right) \]
where \( \lambda_1^{\text{min}} = \frac{1}{2} \nabla^2 R + \frac{2}{3} \left( R_{mn} R^{mn} - \frac{1}{2} R^2 \right) \).
After substitution of \( \lambda_1 \) it reduces to
\[ \nabla^2 \Phi^I = -\frac{f^2}{12} \zeta^I \nabla^2 R - \ell^{-1} C^I_{MJK} \left( \sum_a \theta^I a \theta^K a - \Phi^I \Phi^K \right) \left( \delta^I M \zeta^M - \frac{f^2}{3} \zeta^I \zeta^M \right). \tag{4.37} \]
In summary, the geometry \( f, \omega \) and matter \( F^I, X^I \) of a time-like supersymmetric solution are constructed from building blocks \( \Theta^I, \Phi^I \) via equations (4.16), (4.22), (4.26), (4.32) and (4.33) which are themselves constrained by a system of equations (4.25), (4.27), (4.36) and (4.37). The resolution of this system depends significantly on whether we consider static or non-static case.

### 4.2.3 Static solutions

If supersymmetric Killing field is hypersurface orthogonal \( V \wedge dV = 0 \), that is \( d\omega = 0 \), the constraints simplify into a set of algebraic and differential equations. First of all, vanishing of SD part of \( d\omega \) is equivalent to
\[ C_{IJ} \Phi^I \theta^J a = 0. \tag{4.38} \]

\(^6\)The notation is chosen to match the minimal case.

\(^7\)The scaling is chosen to match the minimal case.
Since $C_{IJ}$ is a metric on symmetric space, this can be interpreted as $\Phi^I$ being orthogonal to three vectors $\theta^I_a$ where $a = 1\ldots3$ numbers the vectors. Since by (4.22) $\Phi^I$ are non-zero, this implies that $\theta^I_a$ are linearly dependent as vectors, which means that one can pick an appropriate basis of SD 2-forms for which one of the components vanishes.

The vanishing of ASD part of $d\omega$ puts a constraint on the curvature of the Kähler base

$$\frac{1}{2} \nabla^2 R + 4 \ell^{-1} C_{IJK} \zeta^I \left( \sum_a \theta^I_a \theta^K_a - \Phi^I \Phi^K \right) = 0,$$

and the Maxwell equations simplify

$$\nabla^2 \Phi^I = -\frac{\ell^2}{8} \zeta^I \nabla^2 R - \ell^{-1} C_{IJK} \left( \sum_a \theta^I_a \theta^K_a - \Phi^I \Phi^K \right).$$

The system is further constrained by Bianchi identity (4.27).

Contrary to the minimal case, there is no immediate reason to exclude static solutions in the time-like case. Furthermore, there is no apparent argument that forbids smooth static supersymmetric black holes either. The near-horizons with $S^2 \times S^1$ and $T^3$ topologies are static solutions with supersymmetric Killing field. There are indirect arguments, however. Anti-de Sitter geometry puts additional centripetal force onto the black ring which must be outbalanced by rotation or matter. In general, stationary non-supersymmetric black rings, balanced by rotation, have been argued to exist in AdS$_5$ [52]. Their supersymmetric analogues are necessarily non-smooth [44,45]. It is, therefore, interesting to explore if one can prove their non-existence by geometric means or find an explicit example.

4.2.4 Non-static solutions

The supersymmetric constraint is

$$0 = d^2 \omega \Leftrightarrow \delta \left( f^{-1} G^+ - f^{-1} G^- \right) = 0$$

(4.41)

where $\delta$ is the co-differential, see Appendix 1.A for conventions. Writing it out explicitly

$$\delta \left( \frac{12}{\ell^2} C_{IJ} \Phi^I \Theta^J - \lambda_1 J^1 \right) - \delta \left( \lambda_2 J^2 + \lambda_3 J^3 \right) = 0$$

(4.42)

we notice that the $J$-conjugation of the second bracket is co-closed and $\delta J \delta (\lambda_2 J^2 + \lambda_3 J^3) = 0$.

This can be seen explicitly

$$\ast J \delta (\lambda_2 J^2 + \lambda_3 J^3) = \ast J (t_{4\lambda_2 - \lambda_3} P J^2 + t_{4\lambda_3 - \lambda_2} P J^3) = \ast (t_{4\lambda_2 + \lambda_3} P J^2 - t_{4\lambda_2 - \lambda_3} P J^3)$$

$$= d\lambda_2 \wedge J^2 - \lambda_2 dJ^3 - d\lambda_2 \wedge J^3 + \lambda_3 dJ^2 = d (\lambda_3 J^2 - \lambda_2 J^3)$$

(4.43)

where we allowed for an abuse of notation: for example, $t_{4\lambda} J$ stands for contraction of $J$ on the vector field associated to $d\Phi$ by metric. This constraint means that the equation (4.42) can be understood as an equation for $\lambda_2, \lambda_3$, which can be solved provided the integrability constraint is satisfied

$$\delta \left( J \delta \left( \frac{12}{\ell^2} C_{IJ} \Phi^I \Theta^J - \lambda_1 J^1 \right) \right) = 0$$

(4.44)

Note that this is a scalar PDE which is fourth order in curvature through $\lambda_1$ (4.36). Rewriting this equation we get

$$\nabla^2 \lambda_1 - \frac{6}{l^2} C_{IJ} J^{mn} \left( d t_{4\Phi^I F^I} \right)_{mn} = 0$$

(4.45)

where $F^I$ is given by (4.26).

This constraint is a complicated 4th order PDE in curvature and scalars. In total, for a geometry to be a supersymmetric solution, it must also be supplemented by two Maxwell equations (4.37), a Bianchi identity (4.27) and the isotropic constraint (4.25). We prefer to
leave the constraints in this form. It is an open question, if they can be reduced to a single
equation, or if an explicit constraint on geometry, similar to the PDE on curvature for the
minimal case, can be decoupled from scalar degrees of freedom. The author leaves this problem
for future work.

4.2.5 Minimal limit

We recover minimal case by setting the scalar fields and the vector multiplets to be isotropic:

\[ \Phi^1 = \Phi^2 = \Phi^3 = -\frac{\ell}{12} R, \]  
\[ F^1 = F^2 = F^3 = \frac{\ell}{3} R. \]  

To match minimal case we note that

\[ \sum_i \theta_i^I \theta_i^J - \Phi^I \Phi^J = \frac{1}{2} \ast (F^I \wedge F^J) = \frac{\ell^4}{18} \zeta^I \zeta^J \ast (R \wedge R) \]
\[ = \frac{\ell^4}{18} \zeta^I \zeta^J \ast (R \wedge \ast (R - \frac{1}{2} R J)) = \frac{\ell^4}{36} \zeta^I \zeta^J \left( R_{mn} R^{mn} - \frac{1}{2} R^2 \right). \]

The Maxwell equations (4.37) then vanish and

\[ \lambda_1 = \frac{1}{2} \nabla^2 R + 4 \ell^{-1} C_{IJK} \zeta^I \left( \sum_a \theta_a^I \theta_a^K - \Phi^I \Phi^K \right) \]
\[ = \frac{1}{2} \nabla^2 R + \frac{2}{9} C_{IJK} \zeta^I \zeta^J \zeta^K \left( R_{mn} R^{mn} - \frac{1}{2} R^2 \right) \]
\[ = \frac{1}{2} \nabla^2 R + \frac{2}{3} R_{mn} R^{mn} - \frac{1}{3} R^2 = \lambda_1^{\text{min}}. \]

The supersymmetric constraint (4.45) is

\[ \nabla^2 \lambda_1^{\text{min}} + \nabla^m \left( R_{mn} \nabla^n R \right) = 0 \]

which agrees with that in the minimal theory (1.29).

4.3 Toric supersymmetric solutions

4.3.1 Supersymmetric solutions with toric symmetry

We will introduce toric supersymmetric solutions similarly to the minimal case (c.f. section 3.2.2):

**Definition 8.** We will say that a supersymmetric solution to the STU gauged supergravity
admits toric symmetry if

1. there is a torus \( T \cong U(1)^2 \) isometry generated by spacelike Killing fields \( m_i, i = 1, 2, \)
both normalised to have \( 2\pi \) periodic orbits; these are defined up to \( m_i \to A_i^j m_j \) where \( A \in \text{GL}(2, \mathbb{Z}) \);

2. the supersymmetric Killing \( V \) is complete and commutes with the \( T \)-symmetry, that is
\([V, m_i] = 0\), so the spacetime isometry group is \( \mathbb{R} \times U(1)^2 \);

3. the Maxwell fields and scalar fields are \( T \)-invariant \( \mathcal{L}_{m_i} F^I_0 = 0, \mathcal{L}_{m_i} X^I = 0 \);

4. the axis defined by \( \{ p \in M | \det g(m_i, m_j)|_p = 0 \} \) is non-empty.

Since \( m_i \) leaves \( V \) invariant by property 2, \( f \) is also invariant under \( m_i \), and \( \mathcal{L}_{m_i} X^I = 0 \)
is equivalent to \( \mathcal{L}_{m_i} \Phi^I = 0 \) by (4.16). In chapter 3 it was explained that upon using moment
maps \( x_i \) for \( m_i \) one can construct syplectic coordinates \((x_i, \phi^i)\), which simplifies the form of Kähler geometry

\[
h = G^{ij}(x)dx_idx_j + G_{ij}(x)d\phi^id\phi^j, \quad G^{ij} = \partial^i\partial^j g, \quad G^{ik}G_{kj} = \delta^i_j
\]

where \( g \) is the symplectic potential.

It is worth recalling the argument from section 3.2.2, that given any closed 2-form \( \Omega \) on \( M \) which is also invariant under the toric symmetry, \( \iota_m \iota_m \Omega \) must be zero.\(^8\) This is important to constrain the form of \( \Theta^I \) and \( F^I \) (4.26) on toric geometries. As we argue in Appendix 4.A, toric geometries have a natural involution which inverts signs of the angular coordinates along Killing fields. Under this involution the spaces of (anti-)self-dual forms decompose into positive and negative eigenspaces. One can always pick a basis of ASD (\( J^1, J^2, J^3 \)) and SD forms (\( I^1, I^2, I^3 \)) which obeys the quaternionic relations (4.171) and (4.172) and

\[
\iota_{m_1}\iota_{m_2}J^3 = \iota_{m_1}\iota_{m_2}F^3 = \iota_{m_1}\iota_{m_2}I^1 = \iota_{m_1}\iota_{m_2}I^3 = 0
\]

span the negative eigenspace, while \( J^2, I^2 \) span positive eigenspace. Since \( F^I \) is \( J \)-invariant, its projection on \( J^2 \) is zero, and since \( F^I \) is also closed, its projection on \( I^2 \) is also zero, and we can parameterise

\[
\Theta^I = \theta_1 I^1 + \theta_3 I^3.
\]

Finally, the results from the minimal case, such as a geometry of the axes and fixed points in Lemmas 7 and 8, as well as the point-like nature of horizons in Lemma 9 and the form of symplectic potential near them in Lemma 10, are carried over to the STU case. Their proofs are essentially geometric and insensitive to matter content. The explicit form of the near-horizon geometry (of spherical/Lens space black holes), however, gets some modifications, and we will present it in the next subsection.

### 4.3.2 Near-horizon geometry

The general near-horizon geometry admitting \( U(1)^2 \) rotational isometry with spherical or Lens space cross-section was determined in [47]. We present it here in a coordinate system that also describes the special case with \( SU(2) \times U(1) \) symmetry.\(^9\) This ‘unified’ form of the near-horizon geometry also makes the proof of Theorems 9 and 10 more transparent. Recall that the general form of the supersymmetric NH geometry in Gaussian null coordinates is given by (see section 1.2.3)

\[
g = -\lambda^2 \Delta^2 dv^2 + 2dv d\lambda + 2\lambda h_\alpha dv dy^\alpha + \gamma_{\alpha\beta} dy^\alpha dy^\beta
\]

where \( V = \partial_v \) is the supersymmetric Killing field, \( \lambda \) is an affine parameter for null transverse geodesics with horizon located at \( \lambda = 0 \), and \( y^\alpha \) are coordinate on spatial cross-section.

The KLR near-horizon geometry depends on four parameters \( 0 < A^2, B^2 < 1 \), and \( C_1, C_2 \) subject to

\[
\kappa^2(A^2, B^2, C_1, C_2) > 0,
\]

where

\[
\kappa^2(A^2, B^2, C_1, C_2) := -9A^4B^4 + 6A^2B^2(A^2 + B^2 + 1)^2 - (A^2 + B^2 + 1)^3 \left( A^2 + B^2 - \frac{1}{3} \right) + \frac{4C_2}{3} - 2C_1 \left( A^4 + B^4 - A^2B^2 + \frac{C_1}{2} - 1 \right).
\]

An alternative and useful parametrisation is to trade the two parameters \( C_1, C_2 \) for three parameters \( \mathcal{K}_I \) through

\[
C_1 = \frac{\ell}{6} C^{IJK} \kappa_I \kappa_J \kappa_K, \quad C_2 = \frac{1}{6} C^{IJK} \kappa_I \kappa_J \kappa_K,
\]

\(^8\)For such \( \Omega \) the scalar \( \iota_{m_1}\iota_{m_2}\Omega \) must be constant. Now, this combination must vanish on the axis, since the latter are defined as points where Killing fields become linearly dependent.

\(^9\)In the original notation, in the \( SU(2) \times U(1) \) limit the parameterisation became singular.
subject to
\[ C^{IJK} \zeta_I \zeta_J K_K = 0. \] (4.58)

Positivity of the scalars [47] restricts
\[ 2A^2 - B^2 - 1 < K_I, \] (4.59)
so that negative values \( K_I < 0 \) are allowed, and which translates for the constants \( C_1 \) and \( C_2 \) into
\[ -(1 + B^2 - 2A^2)^2 < C_1 \leq 0, \quad -\frac{1}{4}(1 + B^2 - 2A^2)^3 < C_2 < 2(1 + B^2 - 2A^2)^3. \] (4.60)

We now provide the explicit expression for the near-horizon data. The metric is constructed as
\[ g = -\lambda^2 \Delta^2 dv^2 + 2d\phi d\lambda + 2\lambda h_a dy^a + \gamma_{ab} dy^a dy^b \] (4.61)
where \( \Delta, h_a, \gamma_{ab} \) are smooth functions, the leading \( O(1) \) order of which is given by
\[ \Delta^{(0)} = \frac{3\kappa}{\ell \bar{H}(\hat{\eta})^{2/3}}, \]
\[ h^{(0)} = \frac{3\kappa \Delta_1(\hat{\eta})}{4H(\hat{\eta})} \sigma + \frac{3(A^2 - B^2)}{2H(\hat{\eta})} \left( (\Delta_2(\hat{\eta})^2 + C_1) d\hat{\eta} - \frac{3}{2} \sigma \hat{r} \right), \]
\[ \gamma^{(0)} = \frac{\ell^2}{12(1 - \hat{\eta}^2) \Delta_1(\hat{\eta})} \left( \frac{\bar{H}(\hat{\eta})^{1/3} d\hat{\eta}^2}{2} + \frac{3}{4} \frac{\Delta_3(\hat{\eta})^2 + \kappa^2}{\bar{H}(\hat{\eta})^{2/3}} \sigma^2 \right)
+ \frac{\ell^2 (4\bar{H}(\hat{\eta}) - 3\Delta_3(\hat{\eta})^2)}{48\bar{H}(\hat{\eta})^{2/3}} \sigma^2 + \frac{3\ell^2 \Delta_4(\hat{\eta})(A^2 - B^2)}{8H(\hat{\eta})^{2/3}} \sigma \hat{r}. \] (4.62)

We have defined the 1-forms
\[ \hat{\sigma} = \frac{1 - \hat{\eta}}{A^2} d\hat{\phi}^1 + \frac{1 + \hat{\eta}}{B^2} d\hat{\phi}^2, \quad \hat{r} = (1 - \hat{\eta}^2) \Delta_1(\hat{\eta}) \left( \frac{d\hat{\phi}^1}{A^2} - \frac{d\hat{\phi}^2}{B^2} \right), \] (4.63)
and three linear functions of \( \hat{\eta} \)
\[ \Delta_1(\hat{\eta}) = \frac{1 + \hat{\eta}}{2} A^2 + \frac{1 - \hat{\eta}}{2} B^2, \]
\[ \Delta_2(\hat{\eta}) = 1 - \frac{1 + 3\hat{\eta}}{2} A^2 - \frac{1 - 3\hat{\eta}}{2} B^2, \]
\[ \Delta_3(\hat{\eta}) = 1 - 2\Delta_2(\hat{\eta}) + A^2 B^2 - A^4 - B^4 - C_1, \] (4.64)
and the cubic polynomial of \( \hat{\eta} \)
\[ \bar{H}(\hat{\eta}) = \prod_{I=1}^{3} (\Delta_2(\hat{\eta}) \ell \zeta_I + K_I) = \Delta_2(\hat{\eta})^3 + 3C_1 \Delta_2(\hat{\eta}) + C_2 \] (4.65)
Here \((\hat{\eta}, \hat{\phi}^i)\) are coordinates on horizon spatial cross-section \( S \) with \(-1 < \hat{\eta} < 1\) and \( \hat{\phi}^i \sim \hat{\phi}^i + 2\pi \) are adapted to the Killing fields \( m_i = \partial_{\phi^i} \) and \( \Delta_1 \) and \( \Delta_2 \) are strictly positive functions.\(^{10}\)

The gauge field is constructed as
\[ A_{I}^a = \lambda b^i dv + a^a_{I} dy^a \] (4.66)

\(^{10}\)We use hats to stress that these are coordinates valid at the horizon. Even though both \( \hat{\phi}^i \) and \( \phi^i \) are coordinates along globally defined axial Killing fields \( m_i = \partial_{\phi^i} = \partial_{\hat{\phi}^i} \), the exact transformation law between them involves is complicated because of \( dv \ell \) cross-term in (4.54).
where \( b', a'_I \) are again smooth functions, the leading \( O(1) \) order of which is given by

\[
\begin{align*}
b'_I(0) &= \frac{3\kappa}{\ell H(\hat{\eta})^{1/3}} (\Delta_2(\hat{\eta})\zeta_I + K_I), \\
a'_I(0) &= -\frac{1}{\Delta_2(\hat{\eta})\zeta_I + K_I} \left( \frac{\ell \Delta_3(\hat{\eta})}{4} \hat{\sigma} + \frac{\ell (\Delta_3(\hat{\eta})^2 + \kappa^2)}{12(A^2 - B^2)} \frac{\hat{\tau}}{(1 - \hat{\eta}^2)\Delta_1(\hat{\eta})} \right).
\end{align*}
\]

(4.67)

The leading order of the scalar fields \( X_I \) is

\[
X_I = \frac{\Delta_2(\hat{\eta})\zeta_I + K_I}{3H(\hat{\eta})^{1/3}} + O(\lambda).
\]

(4.68)

Note that solutions with \( A^2 \neq B^2 \) are doubly counted with the two copies related by

\[
\hat{\phi}^1 \leftrightarrow \hat{\phi}^2, \quad \hat{\eta} \leftrightarrow -\hat{\eta}, \quad A^2 \leftrightarrow B^2.
\]

(4.69)

The fixed point of this duality, solutions with \( A^2 = B^2 \) have enhanced \( SU(2) \times U(1) \) symmetry.

**Kähler data**

It is convenient to recast the above data in a form adapted to notation of section 4.2. First of all, similar to the minimal case, the Kähler form is constructed from a 1-form

\[
Z^J = d\lambda \wedge Z^J + \lambda (h \wedge Z - \Delta \star_3 Z).
\]

(4.70)

where the leading order of \( Z \) is \([47]\)

\[
Z^{(0)} = \frac{\ell}{4H(\hat{\eta})^{1/3}} (\kappa \hat{\sigma} - 3(A^2 - B^2)d\hat{\eta}).
\]

(4.71)

Next, we have

\[
f = \lambda \Delta, \quad \omega_i = -\frac{\hat{h}_i}{\lambda \Delta^2}, \quad G_{ij} = \lambda \Delta \left( \gamma_{ij} + \frac{h_i h_j}{\Delta^2} \right), \quad X^I = \frac{b'_I}{\Delta},
\]

(4.72)

from which it follows that

\[
\omega_i = -\frac{\ell^2 H(\hat{\eta})^{1/3}}{12\kappa} (\Delta_3(\hat{\eta})\hat{\sigma} - 3(A^2 - B^2)\hat{\tau})_i \frac{1}{\lambda} + O(1).
\]

(4.73)

The rescaled scalar fields \( \Phi^I \) then are

\[
\Phi^I = \frac{\ell^2 H(\hat{\eta})^{1/3}}{3\kappa} C^{IJ} \zeta_J (\Delta_2(\hat{\eta})\zeta_K + K_K) \lambda^{-1} + O(1),
\]

(4.74)

and the orthogonal part \( A^I \) of the gauge fields (4.23) is given, up to a gauge, by

\[
A^I = -\frac{\ell}{\Delta_2(\hat{\eta})\zeta_I + K_I} \left( \frac{\Delta_3^2(\hat{\eta}) + \kappa^2}{12(A^2 - B^2)\Delta_1(\hat{\eta}) (1 - \hat{\eta}^2)} + \frac{3}{4} (A^2 - B^2) \right) \hat{\tau}.
\]

(4.75)

Finally, the metric in symplectic coordinates (4.51) is to the leading order

\[
G_{ij} = \frac{\ell \kappa}{4H(\hat{\eta})^{1/3}} \left( \hat{\sigma}^2 + \frac{\hat{\tau}^2}{(1 - \hat{\eta}^2)\Delta_1(\hat{\eta})} \right) \lambda + O(\lambda^2).
\]

(4.76)

The expansions of symplectic coordinates are found as (see section 3.2.3)

\[
x_i = \lambda Z + O(\lambda^2)
\]

(4.77)
Inserting (4.71) into (4.77) we find
\[ x_1 = \frac{\ell \kappa}{4 H(q)^{1/3}} \frac{1 - \hat{\eta}}{A^2} \lambda + O(\lambda^2), \quad x_2 = \frac{\ell \kappa}{4 H(q)^{1/3}} \frac{1 + \hat{\eta}}{B^2} \lambda + O(\lambda^2). \] (4.78)

The inverse coordinate change to leading order is then
\[ \lambda = \frac{2}{\ell \kappa} \left[ \prod_{I=1}^{3} \left( \frac{A^2 x}{1 + A^2 - 2B^2} \ell \zeta + \mathcal{K}_I \right) + B^2 x \left( \frac{1 - 2A^2 + B^2}{2} \ell \zeta + \mathcal{K}_I \right) \right]^{1/2} + O(x^2), \]
\[ \hat{\eta} = -\frac{A^2 x_1 - B^2 x_2}{A^2 x_1 + B^2 x_2} + O(x). \] (4.79)

The expression for leading order behaviour of symplectic potential in terms of symplectic coordinates is as in the minimal theory Lemma 10.

4.4 Supersymmetric solutions with separable moment maps

Section 4.2 was devoted to a presentation of generic supersymmetric solutions in the timelike class. The supersymmetric constraints that we have formulated combine 4th order non-linear PDE in curvature and scalar fields (4.45) with a set of additional algebro-differential constraints (4.25), (4.27) and (4.37). While the general classification of toric Kähler geometries is at present out of reach, we can focus on particular classes of such geometries for which a progress can be made.

In the previous chapter we have introduced three classes of toric Kähler geometries which admit a hamiltonian 2-forms section 3.3: product-toric, Calabi-type and orthotoric. In this section we will provide an analysis of such geometries, and show that only geometries of Calabi-type can correspond to smooth spherical (or Lens space) black holes. In the latter case we will derive the boundary conditions associated with such black holes for supersymmetric constraints. To resolve the Bianchi identity (4.27) we will introduce a simple ansatz, which generalises the minimal case.

4.4.1 Product-toric and orthotoric Kähler metrics

The Definition 5 of the product-toric geometries state that one can choose axial Killing fields \( m_i \) such that the gradients of their moment maps \( d x_i \) are orthogonal. By (4.51) this implies that \( m_i \) are orthogonal as well. This simple consequence immediately excludes Kähler base of KLR near-horizon geometry. Indeed, there is no constant affine transformation that can diagonalise (4.76) regardless of the values of parameters.

The orthotoric metrics were defined in a similar way in Definition 6. It states that there exist two axial Killing fields \( m_i \) such that their moment maps \( x = \rho + \eta \) and \( y = \rho \eta \) are functions of two coordinates \( \rho, \eta \), the gradients of which are orthogonal. The proof that this class excludes Kähler base of near-horizon geometry of KLR solution goes along the lines of a similar result Lemma 12 in the previous chapter.

Lemma 17. A supersymmetric toric solution that is timelike outside a smooth horizon with compact \( S^3 \) or \( L(p, q) \) sections cannot have an orthotoric Kähler base.

Proof. For convenience, we reproduce the orthotoric geometry here
\[ h = \frac{\rho - \eta}{F(\rho)} d \rho^2 + \frac{F(\rho)}{\rho - \eta} (d \psi + \eta d \phi)^2 + \frac{\rho - \eta}{G(\eta)} d \eta^2 + \frac{G(\eta)}{\rho - \eta} (d \psi + \rho d \phi)^2, \] (4.80)
\[ J = \omega((\rho + \eta) d \psi + \rho \eta d \phi). \] (4.81)

Let \( h \) be the Kähler base within the orthotoric class that contains some neighbourhood of a smooth spherical or Lens space horizon \( \mathcal{H} \). The determinant of the Gram matrix of Killing fields of the 5d solution is proportional to determinant of angular part of the metric (3.24)
\[ \det K_{\alpha \beta} = -\det G_{ij} = -F(\rho)G(\eta) \leq 0. \] (4.82)
Without loss of generality we will choose $F(\rho), G(\eta) > 0$. Then, positive signature of the metric implies that $\rho \geq \eta$. The zeros of (4.82) correspond to axes or the horizon.

Let us understand the near-horizon behaviour of orthotoric coordinates similar to section 4.3.2. First of all, inverting, orthotoric coordinates are expressed through symplectic coordinates as

$$\rho = \frac{1}{2} \left( x + \sqrt{x^2 - 4y} \right), \quad \eta = \frac{1}{2} \left( x - \sqrt{x^2 - 4y} \right) \quad (4.83)$$

where $\rho \geq \eta$ was used to pick the unique branch. In turn, all symplectic charts are related by constant real affine transformations

$$x = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 \quad (4.84)$$
$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \quad (4.85)$$

where $\alpha_{0,1,2}, \beta_{0,1,2}$ are constants and $x_i$ are symplectic coordinates adapted to $2\pi$-periodic angles $\phi^i, i = 1, 2$. By Lemma 9 $x_i$ are smooth functions around the horizon. It is worth noting that, since horizon is a point in $(x_1, x_2)$ coordinates, it is a point $(\rho_H, \eta_H)$ in orthotoric chart $(\rho, \eta)$ as well. Furthermore, the transformations (4.83) and (4.84) imply that $(\rho, \eta)$ are $C^\infty$ at $\mathcal{H}$ when $\rho_H \neq \eta_H$, i.e. when $\alpha_0^2 \neq 4\beta_0$, and $C^0$ otherwise.

We will now turn to the components of the Gram matrix explicitly to rule out both of the cases. From (4.76) we have that every component of

$$G_{ij} d\phi^i d\phi^j = \frac{F(\rho)}{\rho - \eta} (d\psi + \eta d\phi)^2 + \frac{G(\eta)}{\rho - \eta} (d\psi + \rho d\phi)^2$$

$$= \frac{\ell \kappa}{4\mathcal{H}(\eta_H)^{1/3}} (\hat{\sigma}^2 + \frac{\hat{\tau}^2}{(1 - \hat{\eta}^2)\Delta_1(\hat{\eta})}) \lambda + O(\lambda^2) \quad (4.86)$$

must be a smooth scalar which vanishes as $O(\lambda)$ at $\mathcal{H}$. Furthermore, the determinant (4.82)

$$\det K_{\alpha\beta} = -\det G_{ij} = - \left( \frac{\ell \kappa}{2A^2 B^2 \mathcal{H}(\eta_H)^{1/3}} \right)^2 \Delta_1(\hat{\eta})(1 - \hat{\eta}^2)\lambda^2 + O(\lambda^3) \quad (4.87)$$

has a quadratic term which vanishes only at the axes.

Case $\rho_H = \eta_H$.

First of all, we have $\rho - \eta = \sqrt{x^2 - 4y} = O(\lambda^{1/2})$. From $G_{\phi\psi} = \frac{1}{\rho - \eta} (F(\rho) + G(\eta)) = O(\lambda)$ we have $F(\rho), G(\eta) = O(\lambda^{3/2})$, and, hence, $\det G_{ij} = O(\lambda^3)$ everywhere on $\mathcal{H}$, which is a contradiction with (4.87).

Case $\rho_H \neq \eta_H$, i.e. $\rho > \eta$ on the horizon. In this case $(\rho, \eta)$ are smooth functions at the horizon. From $G_{\phi\psi}$ and $G_{\phi\phi}$ we have that $F(\rho), G(\eta)$ are smooth functions as well. We will now find leading orders of $F(\rho), G(\eta)$ explicitly to show that smooth horizon is incompatible with orthotoric form of the metric. Firstly, from (4.78), (4.83) and (4.84) we find the expansions of orthotoric coordinates

$$\rho = \rho_H + \kappa \ell \beta^2 (1 - \hat{\eta}) (\beta_1 - \alpha_1 \rho_H) + \mathcal{A}^2 (1 + \hat{\eta}) (\beta_2 - \alpha_2 \rho_H) \lambda + O(\lambda^2) \quad (4.88)$$
$$\eta = \eta_H + \kappa \ell \beta^2 (1 - \hat{\eta}) (\alpha_1 \eta_H - \beta_1) + \mathcal{A}^2 (1 + \hat{\eta}) (\alpha_2 \eta_H - \beta_2) \lambda + O(\lambda^2) \quad (4.89)$$

where $\rho_H = \frac{1}{2} \left( \alpha_0 + \sqrt{\alpha_0^2 - 4\beta_0} \right), \eta_H = \frac{1}{2} \left( \alpha_0 - \sqrt{\alpha_0^2 - 4\beta_0} \right)$.

The functions $F(\rho), G(\eta)$ can be found from (4.86)

$$F(\rho) = \rho G_{\psi\psi} - G_{\phi\psi}, \quad G(\eta) = G_{\psi\phi} - \eta G_{\psi\psi} \quad (4.90)$$
where components of $G_{ij}$ must satisfy the following constraint

$$G_{\phi \phi} = (\rho + \eta) G_{\psi \phi} - \rho \eta G_{\psi \psi}. \quad (4.91)$$

Substituting known expressions (4.86) and (4.88), we find that this constraint is violated, and we have arrived at the contradiction. \qed

This completes the proof of Theorem 8.

### 4.4.2 Calabi-type geometries

The Calabi-type metrics were introduced in Definition 7 through their moment maps as well. In this case there are two axial Killing fields with moment maps $\rho$ and $\rho \eta$ such that the gradients of coordinates $\rho$, $\eta$ are orthogonal. Since our primary interest are the spherical black holes which can be found in this class, we will discuss some of the features of Calabi-type geometries in detail.

For the convenience of the reader, we reproduce the Calabi-type geometry here

$$h = \frac{\rho}{F(\rho)} \, d\rho^2 + \frac{\rho}{G(\eta)} \, d\eta^2 + \frac{F(\rho)}{\rho} \, (d\psi + \eta d\phi)^2 + \rho G(\eta) d\phi^2, \quad (4.92)$$

$$J = d(\rho d\psi + \rho \eta d\phi). \quad (4.93)$$

First, let us introduce a particular basis of SD 2-forms for the Calabi-type geometry

$$I_1 = d\rho \wedge (d\psi + \eta d\phi) - \rho d\eta \wedge d\phi$$

$$I_2 = \frac{\rho}{\sqrt{FG}} \, d\rho \wedge d\eta + \sqrt{FG} d\psi \wedge d\phi$$

$$I_3 = \frac{\rho}{\sqrt{F}} d\rho \wedge d\phi + \frac{\sqrt{F}}{\sqrt{G}} (d\psi + \eta d\phi)$$

where we have for convenience dropped the arguments of $F(\rho)$ and $G(\eta)$. These SD forms satisfy quaternionic relations (4.171). Their derivatives are given by

$$\nabla_X I_1 = \gamma(X) I_2 - \beta(X) I_3, \quad (4.95)$$

$$\nabla_X I_2 = \alpha(X) I_3 - \gamma(X) I_1, \quad (4.96)$$

$$\nabla_X I_3 = \beta(X) I_1 - \alpha(X) I_2. \quad (4.97)$$

where $\alpha, \beta, \gamma$ are 1-forms

$$\alpha = -I_1 d\bar{\mu}, \quad \beta = -I_2 d\mu, \quad \gamma = -I_3 d\mu, \quad (4.98)$$

$$\bar{\mu} = \log(\sqrt{FG}/\rho), \quad \mu = \log \rho. \quad (4.99)$$

Notice that $I_2 \beta = I_3 \gamma = d\bar{\mu}$ in (4.98). According to Proposition 8 this implies that $\phi^{(tw)} = e^{\mu} I_1$ is a twistor 2-form.\footnote{Another name is conformal Killing-Yano 2-form, see Appendix 4.B.} Furthermore, for Calabi geometries the Ricci form is given by

$$\mathcal{R} = \frac{\bar{R}}{4} I_1 + \frac{R}{4} J_1, \quad \bar{R} = \frac{2 F'' - \rho F''' + \rho G''}{\rho^2}, \quad R = -\frac{F'' + G''}{\rho} \quad (4.100)$$

where $R$ is the scalar curvature. Notice that the self-dual part of the Ricci form is proportional to twistor 2-form $\mathcal{R}^+ = \mathcal{R} - \frac{2}{3} J \wedge I_1$. In Proposition 9 we have shown that for all such Kähler geometries the twistor 2-form generate a hamiltonian 2-form. In this case it is given by

$$\phi^{(ham)} = \rho \left(I_1 + 3 J_1\right). \quad (4.101)$$

As we discussed in section 4.3, for a toric Kähler space there is a preferred choice of basis ASD
2-forms: \( J^1 = J, J^2, J^3 \) where \( \ell_m, \ell_{m_2}, J^3 = 0 \). For Calabi geometries it takes the form

\[
J^1 = d(\rho \, d\psi + \rho \eta \, d\phi) \\
J^2 = -\frac{\rho}{\sqrt{FG}} A\rho \wedge d\eta + \sqrt{FG} d\psi \wedge d\phi \\
J^3 = \rho \frac{\sqrt{G}}{\sqrt{F}} A\rho \wedge d\phi - \frac{\sqrt{T}}{\sqrt{G}} (d\psi + \eta d\phi).
\]

(4.102)

Similarly, for Calabi geometries (4.94) is a preferred choice of basis for SD forms.

### Bianchi identity and the gauge field ansatz

Let us now turn to the resolution of Bianchi identity (4.27). As it was argued in section 4.3, the SD 2-form \( \Theta^I \) belongs to the span of \( I^1, I^3 \). It is convenient to parameterise it by two functions \( \tilde{\theta}^I_4 = \tilde{\theta}^I_4(\rho, \eta) \), \( \tilde{\theta}^I_3 = \tilde{\theta}^I_3(\rho, \eta) \) as follows

\[
F^I = \Theta^I \Phi^I J := \left[ \partial_\rho \tilde{\theta}^I_4 + \Phi^I \right] I^1 + \frac{\tilde{\theta}^I_3}{\rho\sqrt{FG}} I^3 - \Phi^I J
\]

(4.103)

where \( \tilde{\theta}^I_4 \) is defined up to a function of \( \eta \). The Bianchi identity \( dF^I = 0 \) then admits a simple solution

\[
\tilde{\theta}^I_4 = \rho \, G \partial_\rho \tilde{\theta}^I_4 + H_1(\eta), \\
\rho \, \Phi^I = H_2(\eta) - \frac{1}{2} \partial_\rho (\rho \tilde{\theta}^I_4) - \frac{1}{2} \int d\rho \left[ F^{-1} \partial_\rho \tilde{\theta}^I_3 \right],
\]

(4.104)

(4.105)

where \( H_{1,2}(\eta) \) are arbitrary integration functions. Unfortunately, even on this explicit solution the rest of the supersymmetric constraints (4.25), (4.37) and (4.45) does not immediately simplify. To make further progress, we will impose an additional ansatz on the gauge field. Notice that in the minimal case, the SD part of \( d\omega \) was automatically proportional to the SD part of the Ricci form (1.22) and, hence, to the twistor 2-form. Now, we will introduce such an assumption by hand, namely

\[
\Theta^I = \theta^I I^1 \propto R^+, \quad \text{equivalently,} \quad F^I = \Theta^I - \Phi^I J = \theta^I I^1 - \Phi^I J.
\]

(4.106)

The Bianchi identity then fixes both \( \theta^I \) and \( \Phi^I \) in terms of two functions \( \mu^I(\rho), \nu^I(\eta) \) of a single variable

\[
\theta^I = \left[ \rho \left( \rho^{-2} \mu^I(\rho) \right) \right] - \rho^{-1} \nu^I(\eta), \quad \Phi^I = -\rho \left[ \mu^I(\rho) + \nu^I(\eta) \right].
\]

(4.107)

The function \( \nu^I(\eta) \) is defined up to additive constant. Also, note that shifting \( \mu^I(\rho) \rightarrow \mu^I(\rho) + \mu_0^I \) by constant \( \mu_0^I \) results in adding a constant multiple of \( \rho^{-2} I^1 \) to \( F^I \), the former being a locally conformally Kähler structure on the Calabi geometry, see comment after Proposition 8.

Next, from (4.107) the orthogonal part of the gauge field reads as

\[
A^I = 2\rho^{-1} \mu^I (d\psi + \eta \, d\phi) + 2\nu^I \, d\phi.
\]

(4.108)

Finally, the isotropic constraint (4.25) is then equivalent to

\[
\sum_I \mu^I = -\frac{\ell}{4} (F' + \rho \mu_0), \quad \sum_I \nu^I = -\frac{\ell}{4} (G' - \eta \mu_0) + \text{const}
\]

(4.109)

where \( \mu_0 \) is an integration constant.

### 4.4.3 Horizons of Calabi-type solutions

To start with the proof of Theorems 9 and 10 we need the last ingredient — the matching of Calabi chart to the near-horizon geometry. Similar to the minimal case, we find that the
Lemma 18. Consider a supersymmetric toric solution that is timelike outside a smooth horizon with spherical or Lens space cross-sections. If the Kähler base is of Calabi type, then, near the horizon, Calabi type coordinates are related to Gaussian null coordinates by
\[ \rho = \frac{\ell \kappa}{4H(\eta)^{1/3}} \lambda + O(\lambda^2), \quad \eta = \hat{\eta} + O(\lambda), \quad (4.110) \]
where \( H(\eta) \) is given by \((4.65)\), so in particular the horizon must be at \( \rho = 0 \). Furthermore, we can always choose Calabi coordinates such that
\[ F(\rho) = \rho^2 + O(\rho^3), \quad G(\eta) = (1 - \eta^2)\Delta_1(\eta), \quad (4.111) \]
where \( \Delta_1(\eta) \) is given by \((4.64)\).

Proof. The proof of this lemma is identical to the proof of analogous Lemma 14, with the exception of \( \hat{H}(\hat{\eta})^{1/3} \) in the former playing the role of the factor \( \Delta_2(\hat{\eta}) \) in the latter. Firstly, let us relate Calabi symplectic coordinates to \( 2\pi \)-periodic ones. Recall that any two symplectic coordinate charts are related by a constant affine transformation
\[ x_i = c_i + a_i \rho + b_i \eta, \quad \psi = a_i \phi^i, \quad \phi = b_i \phi^i, \quad (4.112) \]
where \( x_i \) are moment maps corresponding to \( 2\pi \)-periodic Killing fields \( m_i \), shifted such that the horizon is at \( x_i = 0 \), and \( a_i, b_i, c_i \) are real constants. The inverse transformation is
\[ \rho = \frac{\langle x - c, b \rangle}{\langle a, b \rangle}, \quad \eta = -\frac{\langle x - c, a \rangle}{\langle x - c, b \rangle}, \quad \langle a, b \rangle \neq 0 \quad (4.113) \]
where, as before, we have introduced notation
\[ \langle a, b \rangle = \epsilon_{ij} a^i b^j = a^1 b^2 - a^2 b^1 \quad (4.114) \]
to simplify the presentation. Explicitly, \( G_{ij} d\phi^i d\phi^j \) is given by
\[ G_{ij} = \frac{F}{\rho}(a_i + b_i \eta)(a_j + b_j \eta) + \rho G(\eta) b_i b_j. \quad (4.115) \]
Comparing this expression with \((4.76)\) one has
\[ \eta = O(1), \quad \frac{F(\rho)}{\rho} = O(\lambda), \quad \rho G(\eta) = O(\lambda). \quad (4.116) \]
These relations imply that \( c_i = 0 \). To see this, suppose that \( c_i \) does not vanish identically. Then, expanding \((4.113)\) using \((4.78)\) we find \(^{12}\)
\[ \rho = \frac{\langle b, c \rangle}{\langle a, b \rangle} + \frac{1}{\langle a, b \rangle} \left( \frac{1 - \hat{\eta}}{\mathcal{A}^2} b_2 - \frac{1 + \hat{\eta}}{\mathcal{B}^2} b_1 \right) \frac{\ell \kappa \lambda}{4H(\eta)^{1/3}} + O(\lambda^2), \]
\[ \eta = -\frac{\langle a, c \rangle}{\langle b, c \rangle} + \frac{\langle a, b \rangle}{\langle b, c \rangle^2} \left( \frac{1 - \hat{\eta}}{\mathcal{A}^2} c_2 - \frac{1 + \hat{\eta}}{\mathcal{B}^2} c_1 \right) \frac{\ell \kappa \lambda}{4H(\eta)^{1/3}} + O(\lambda^2), \quad (4.117) \]
showing that the horizon maps to a point \((\rho_0, \eta_0)\) in the \( \rho-\eta \) plane. From \((4.116)\) we then have

\(^{12}\)The case \( \langle b, c \rangle \neq 0 \) is excluded because it would imply that \( \eta \) is singular as \( \lambda \to 0 \) in contradiction with \((4.116)\).
\( F(\rho_0) = G(\eta_0) = 0 \) and expanding (4.115) to linear order in \( \lambda \) we find

\[
G_{ij} = \left[ \frac{G'(\eta_0)b_ib_j}{(b,c)} \left( \frac{1 - \tilde{\eta}}{A^2} c_2 - \frac{1 + \tilde{\eta}}{B^2} c_1 \right) \right. \\
+ \left. \frac{(a,b)^2 F'(\rho_0)c_ic_j}{(b,c)^3} \left( \frac{1 - \tilde{\eta}}{A^2} b_2 - \frac{1 + \tilde{\eta}}{B^2} b_1 \right) \right] \frac{\ell \kappa \lambda}{4H(\tilde{\eta})^{1/3}} + O(\lambda^2). \tag{4.118}
\]

The \( \tilde{\eta} \)-dependence in this expression is however incompatible to the one in (4.76), which is a consequence of our assumption that \( c_i \) do not vanish. Hence, \( c_i = 0 \).

Now, the relation between the Calabi coordinates \((\rho, \eta)\) and the GNC \((\lambda, \tilde{\eta})\) near the horizon can be obtained from (4.113) and (4.78), and it reads

\[
\rho = \frac{1}{(a,b)} \left( \frac{1 - \tilde{\eta}}{A^2} b_2 - \frac{1 + \tilde{\eta}}{B^2} b_1 \right) \frac{\ell \kappa \lambda}{4H(\tilde{\eta})^{1/3}} + O(\lambda^2),
\]

\[
\eta = \frac{A^2 a_1(1 + \tilde{\eta}) - B^2 a_2(1 - \tilde{\eta})}{A^2 b_1(1 + \tilde{\eta}) - B^2 b_2(1 - \tilde{\eta})} + O(\lambda). \tag{4.119}
\]

Note that the horizon corresponds to \( \rho = 0 \).

To complete the proof of our Lemma, we need to show that there exist functions \( F(\rho) \) and \( G(\eta) \) such that (4.115) reproduces (4.76) at \( O(\lambda) \). From (4.116) and (4.119) we see that \( F(\rho) = O(\lambda^2) \) is a smooth function of \( \lambda \) and hence of \( \rho \) at the horizon. We therefore must have

\[
F(\rho) = F_2 \rho^2 + O(\lambda^3), \quad G(\eta) = G_0(\tilde{\eta}) + O(\lambda), \tag{4.120}
\]

where

\[
F_2 = \frac{1}{2} \rho''(0), \quad G_0(\tilde{\eta}) = G \left( \frac{A^2 a_1(1 + \tilde{\eta}) - B^2 a_2(1 - \tilde{\eta})}{A^2 b_1(1 + \tilde{\eta}) - B^2 b_2(1 - \tilde{\eta})} \right). \tag{4.121}
\]

It is now not hard to see that (4.76) and (4.115) match at \( O(\lambda) \) if and only if

\[
\frac{b_2}{b_1} = -\frac{A^2}{B^2}, \quad F_2 = \frac{2}{a_1 A^2 + a_2 B^2}, \quad G_0(\tilde{\eta}) = -\frac{(1 - \tilde{\eta})^2 \Delta_1(\tilde{\eta})}{F_2 A^2 B^2 b_1 b_2}. \tag{4.122}
\]

Now, recall that Calabi chart has remaining coordinate freedom in the form of rescalings (3.115) to (3.117). We can exploit this freedom to fix

\[
a_1 = -b_1 = A^{-2}, \quad a_2 = b_2 = B^{-2} \tag{4.123}
\]

which also fixes\(^\text{13}\)

\[
F_2 = 1, \quad G_0(\tilde{\eta}) = (1 - \tilde{\eta})^2 \Delta_1(\tilde{\eta}), \tag{4.124}
\]

and (4.119) simplifies to (4.110). The second equation in (4.121) now reduces to \( G_0(\tilde{\eta}) = G(\tilde{\eta}) \) which therefore determines the function \( G(\eta) \) everywhere in the chart as claimed. \( \square \)

Note that the \( A^2 = B^2 \) case corresponds to enhanced \( SU(2) \times U(1) \) symmetry of the K"ahler base. Indeed, introducing polar angle \( \hat{\theta} \) as \( \hat{\theta} := \cos \tilde{\theta} \) the chart around the horizon becomes

\[
h = \frac{\rho}{F(\rho)} \dot{\rho}^2 + \frac{F(\rho)}{\rho} \left( \dot{\hat{\theta}}^2 + \cos \hat{\theta} \dot{\hat{\phi}}^2 \right) + \rho \left( \ddot{\hat{\theta}}^2 + \sin^2 \hat{\theta} \ddot{\hat{\phi}}^2 \right). \tag{4.125}
\]

Introducing the \( SU(2) \) right-invariant 1-forms as in Appendix 2.A, the chart (4.125) takes a manifest \( SU(2) \times U(1) \)-invariant form.

As a final bit of information we need to establish the near-horizon behaviour of \( \mu^I(\rho), \nu^I(\eta) \) which are the building blocks of the gauge and scalar fields.

**Lemma 19.** Consider a supersymmetric toric solution that is timelike outside a smooth horizon with spherical or Lens space cross-sections. If the K"ahler base is of Calabi type and the\(^\text{13}\)Observe that \( F_2 \) is invariant under (3.117), and, hence, this freedom still remains unfixed.
gauge field satisfies minimal ansatz (4.106) and (4.107), then, near the horizon, the gauge field components have the following near-horizon expansions in Gaussian null coordinates

\[ \mu^I(\rho) = O(\lambda^2), \quad \nu^I(\hat{\eta}) = -\frac{\ell^2}{12} C^{IJK} \zeta_J (\Delta_2(\hat{\eta}) \xi_K + \mathcal{K}_K), \]  

(4.126)

so that \( \nu^I(\eta) \) are determined everywhere in this chart.

Proof. The near-horizon behaviour of \( \Phi^I \) is given by (4.74). By comparing the expression with (4.107), and using the expansions of \( \rho \) from the lemma above, one finds that

\[ \nu^I(\hat{\eta}) = -\frac{\ell^2}{12} C^{IJK} \zeta_J (\Delta_2(\hat{\eta}) \xi_K + \mathcal{K}_K) + \nu_0^I, \quad \mu^I(\rho) = \mu^I(0) - \nu_0^I \lambda + O(\lambda^2) \]  

(4.127)

where \( \mu^I(0), \nu_0^I \) are some arbitrary constants. To fix them, consider the contraction of \( F^I \) with respect to the axial Killing field \( m_\psi \)

\[ \iota_{m_\psi} F^I = (\Phi^I - \theta^I) \, d\rho = O(1) \, d\lambda \]  

(4.128)

with the second equality follows from (4.75). Using (4.107) this fixes

\[ \mu^I(\rho) = O(\lambda^2), \quad \nu_0^I = 0. \]  

(4.129)

Finally, this fixes integration constant in the isotropic constraint (4.109) to \( \mu_0^I = -2 \).

Now that we have determined \( \eta \)-dependence of the metric and matter fields, we can attack the supersymmetric constraints.

4.5 Uniqueness theorem

4.5.1 Uniqueness of the KLR black hole

We now have all the necessary ingredients to complete the proof of the Theorems 9 and 10. We start with showing the uniqueness of the Kähler base.

Lemma 20. Consider a supersymmetric toric solution to five-dimensional \( U(1)^3 \) gauged supergravity that is timelike outside a smooth (analytic if \( \mathcal{A}^2 = \mathcal{B}^2 \) horizon with compact spherical or Lens space cross-sections and with the Kähler base of Calabi-type. If the configuration \( \{ h, F^I \} \) on the Kähler base satisfies the ansatz (4.106), then its metric functions \( F(\rho), G(\eta) \) are

\[ F(\rho) = \rho^2 + s \rho^3, \quad G(\eta) = (1 - \eta^2) \Delta_1(\eta) \]  

(4.130)

where the constant \( s \) distinguishes between KLR black hole \( s = 4/\ell^2 \) and its NH geometry \( s = 0 \), so that the Kähler base is

\[ h = \frac{d\rho^2}{\rho + s \rho^2} + (\rho + s \rho^2) \sigma^2 + \frac{\rho}{(1 - \eta^2) \Delta_1(\eta)} (d\eta^2 + \tau^2) \]  

(4.131)

where \( \Delta_1(\eta) \) is given by (4.64) and we have defined

\[ \sigma := d\psi + \eta d\phi = \frac{1 - \eta}{\mathcal{A}^2} d\phi^1 + \frac{1 + \eta}{\mathcal{B}^2} d\phi^2, \]

\[ \tau := -(1 - \eta^2) \Delta_1(\eta) d\phi = (1 - \eta^2) \Delta_1(\eta) \left( \frac{d\phi^1}{\mathcal{A}^2} - \frac{d\phi^2}{\mathcal{B}^2} \right). \]  

(4.132)

The constants \( \mathcal{A}^2 \) and \( \mathcal{B}^2 \) are parameters of the near-horizon geometry, subject to a constraint (4.55). Finally, the functions \( \mu^I(\rho) \) are given by

\[ \mu^I(\rho) = -\frac{\ell^2}{4} s \rho^2 \xi^I. \]  

(4.133)
Proof. In section 4.4.2 we have already solved the Bianchi identity (4.27) and the isotropic constraint (4.25). We still have to solve two Maxwell equations (4.11) and a single integrability constraint (4.45). We will show that the only solution to this system compatible with Lemmas 18 and 19 is

\[ F(\rho) = \rho^2 + F_3 \rho^3, \quad \mu^I(\rho) = -\frac{\ell^2}{4} F_3 \rho^2 \zeta^I \]  

(4.134)

where \( F_3 \) an integration constant. As in the proof of the counterpart Theorem 5, we will need to distinguish the cases \( \mathcal{A}^2 \neq B^2 \) and \( \mathcal{A}^2 = B^2 \) and treat them separately.

Substitute from Lemmas 18 and 19 the explicit form of \( G(\eta), \nu^I(\eta) \) into the supersymmetric constraint (4.45). By examining the explicit \( \eta \)-dependence we find that it is a quadratic polynomial in \( \eta \), and, hence, each order must vanish on its own. In particular, its second order is given by

\[ 0 = \frac{-6(\mathcal{A}^2 - B^2)^2}{\rho^4} (\rho^2 F''(\rho) - 4 \rho F'(\rho) + 6F(\rho)) \]  

(4.135)

Similarly, Maxwell equations (4.35) are linear in \( \eta \), and its leading order in \( \eta \) is

\[ 0 = \frac{-\ell(\mathcal{A}^2 - B^2)}{4 \rho} \left( \frac{F''(\rho)}{3 \rho} + \frac{4 \mu^I(\rho)}{\ell \rho} \right)' \]  

(4.136)

This clearly distinguishes between two cases. As it was pointed below Lemma 18, the \( \mathcal{A}^2 = B^2 \) case corresponds to Kähler bases with \( SU(2) \times U(1) \)-symmetry. Surprisingly, this makes the proof more involved, and is, at the current moment, a work in progress.

Case \( \mathcal{A}^2 \neq B^2 \)

We can then easily solve (4.135) to get

\[ F(\rho) = F_1 \rho + F_2 \rho^2 + F_3 \rho^3, \]  

(4.137)

with \( F_2 \) and \( F_3 \) being integration constants. From Lemma 18 we have \( F_2 = 1 \), and hence \( F(\rho) \) is given by the first equation in (4.134). Inserting then back in (4.136) we can solve for \( \mu^I(\rho) \) which is given by the second equation in (4.134). It is then straightforward to check that the Maxwell equations (4.11) and the integrability constraint (4.45) are satisfied identically. This completes the proof of (4.134) for the case \( \mathcal{A}^2 \neq B^2 \).

The solution for the functions \( F(\rho) \) and \( \mu^I(\rho) \) is characterised by \( F_3 \), which can be removed by remaining coordinate freedom. Indeed, under a global rescaling (3.117) \( F_3 \) transforms as \( F_3 \rightarrow K F_3 \) for constant \( K \). We therefore see that there are two qualitatively different cases depending on whether \( F_3 = 0 \) or \( F_3 \neq 0 \). Following the notation of Lemma 15, we will denote \( s = F_3 \) and use this rescaling to fix \( s = 4/\ell^2 \) when \( s \neq 0 \).

Case \( \mathcal{A}^2 = B^2 \)

In this special case, (4.135) and (4.136) are automatically satisfied and therefore cannot be used to solve for \( F(\rho) \) and \( \mu(\rho) \). In fact, both the Maxwell equations and the integrability constraint become independent of \( \eta \), and are a system of ODEs which need to be solved together with the isotropic condition

\[ \sum_{I=1}^{3} \mu^I(\rho) = -\frac{\ell}{4} (F'' - 2\rho) \]  

(4.138)

from Lemma 19 and (4.109). It is convenient to define the following functions

\[ \mathcal{F}(\rho) = \mathcal{A}^{-2} \rho^{-2} F(\rho), \quad \bar{\mu}^I(\rho) = \frac{4}{\mathcal{A}^2} \mu^I(\rho). \]  

(4.139)

In terms of these equations the isotropic constraint (4.138) is written as

\[ 0 = \mathcal{E}_c = \sum_{I=1}^{3} \bar{\mu}^I + \ell \left[ (\rho^2 \mathcal{F})' - 2\mathcal{A}^{-2} \rho \right]. \]  

(4.140)
The Maxwell equation (4.35) then takes the form

\[
0 = \mathcal{E}^I = \left[ \left( \frac{\ell (1 - A^2)}{18} \right) \left( A^2 \rho \mathcal{F}(\rho) - 2 \right) + \frac{A^2}{36} \mathcal{K}_J \tilde{\mu}^J \right] \ell \zeta^I + \frac{A^4}{12 \rho^2} (\rho^2 \mathcal{F}(\rho) - 2 \rho) \tilde{\mu}^I \\
+ \frac{A^4}{4} \left( \rho^2 (\rho - 2 \tilde{\mu}^I \tilde{\mu}^J) \right) + \frac{A^4}{12 \rho^2} (\rho^2 \tilde{\mu}^I) + \frac{A^2}{12 \rho} C^{IJK} \zeta_J \left( \mathcal{K}_K + \frac{A^2}{\ell \rho} \delta_{KL} \tilde{\mu}^L \right) \\
+ \frac{\ell A^2 \mathcal{F}}{72} \left\{ \rho^4 \left( \rho - 1 \left[ 3 \mathcal{K}_I + (6 A^2 \mathcal{F} + 4 A^2 - 10) \ell \zeta^I \right] \right) \right\} \\
\text{(4.141)}
\]

where \((\tilde{\mu}^I)^2\) stands for a vector of squares \((\tilde{\mu}^1)^2, (\tilde{\mu}^2)^2, (\tilde{\mu}^3)^2\), and \(\delta_{IJ}\) is the unit matrix. The integrability constraint (4.45) now reads as

\[
0 = \mathcal{E} = \left[ \frac{A^6 \mathcal{F}}{6} \left( \ell - 2 \delta_{IJ} \left[ 12 (\rho - 1 \tilde{\mu}^I \tilde{\mu}^J) - 9 \tilde{\mu}^I \tilde{\mu}^J \right] \right) + \frac{2}{\ell A^2} (\mathcal{K}_I \tilde{\mu}^I)'' \\
+ \left\{ 6 \rho \mathcal{F}(\rho) - 8 \rho^2 \left( \rho - 1 \mathcal{F}(\rho) \right) \right} - \frac{4}{\ell A^2} (\rho^2 \mathcal{F})'' + \frac{18}{\ell} (\mathcal{F}'' - \mathcal{F}'') - \frac{9}{\ell} \rho^2 \mathcal{F}'' \right\} \\
\text{(4.142)}
\]

where isotropic constraint (4.138) and (4.58) were used to simplify both expressions.

We will seek solution in the form of series

\[
\mathcal{F}(\rho) = A^{-2} + \sum_{n=1}^{\infty} \mathcal{F}_n \rho^n, \quad \tilde{\mu}^I = \sum_{n=2}^{\infty} \tilde{\mu}^I_n \rho^n \\
\text{(4.143)}
\]

where the leading order behaviour was fixed by Lemmas 18 and 19. Let us proceed with the resolution of the series. First, we will show that the constant \(\mathcal{F}_1\) is not determined by the ODEs. Then, we will use inductive argument to show that all higher orders \(\mathcal{F}_n, \tilde{\mu}^I_n, n \geq 2\) vanish.

The isotropic constraint (4.140) implies that

\[
\sum_{n=1}^{\infty} \tilde{\mu}^I_{n+1} = -\ell(n + 2) \mathcal{F}_n, \quad n \geq 1. \\
\text{(4.144)}
\]

The Maxwell equations (4.141) at the next non-trivial order give

\[
0 = \mathcal{E}^I = \frac{A^2}{18} \left[ 2(1 - A^2) \tilde{\mu}^I_2 + (\mathcal{K} \tilde{\mu}^I_2) + C^{IJK} (\mathcal{K}_J - (1 - A^2) \ell \zeta_J) \delta_{KL} \tilde{\mu}^L_2 \right] + O(\rho^4). \\
\text{(4.145)}
\]

where \((\mathcal{K} \tilde{\mu}^I_2) = (\mathcal{K}_1 \tilde{\mu}^I_1, \mathcal{K}_2 \tilde{\mu}^I_2, \mathcal{K}_3 \tilde{\mu}^I_3)\). This is a linear system of three (dependent) equations for three unknown constants \(\tilde{\mu}^I_2\). The solution of this system is

\[
\tilde{\mu}^I_2 = \tilde{\mu}^I_2 = \tilde{\mu}^I_2 = -\ell \mathcal{F}_1 \\
\text{(4.146)}
\]

where we have used the constraint (4.144). At this order the supersymmetric constraint (4.142) identically vanishes, and \(\mathcal{F}_1\) is a free constant. We will now proceed with higher orders. The base of the inductive argument is to show that it holds for \(n = 2\), i.e. \(\tilde{\mu}^I_2 = \mathcal{F}_2 = 0\). The Maxwell equations (4.141) are

\[
0 = \mathcal{E}^I = \frac{A^2}{18} \left[ 2(11 - 2 A^2) \tilde{\mu}^I_3 + 2 (\mathcal{K} \tilde{\mu}^I_3) + C^{IJK} (2 \mathcal{K}_J - (11 - 2 A^2) \ell \zeta_J) \delta_{KL} \tilde{\mu}^L_3 \\
- \frac{1}{4} \mathcal{K}_I \sum_{L} \tilde{\mu}^L_3 \right] \rho + O(\rho^5) \\
\text{(4.147)}
\]
where $(\mathcal{K}\tilde{\mu}_3)^f = (\mathcal{K}_1\tilde{\mu}_3^1, \mathcal{K}_2\tilde{\mu}_3^2, \mathcal{K}_3\tilde{\mu}_3^3)$. Again, this is a system of three equations for three unknowns $\tilde{\mu}_3^I$. Using the constraint (4.144), it has the solution

$$\tilde{\mu}_3^I = \frac{\ell F_2}{3} \frac{4(11 - 2\mathcal{A}^2)^2 + 12C_1}{(11 - 2\mathcal{A}^2)^2 + 4C_1} \frac{\ell \zeta^I + \delta^{IJ} \left[[11 - 2\mathcal{A}^2]\mathcal{K}_{IJ} - 2(\mathcal{K}^2)_{IJ}\right]}{(11 - 2\mathcal{A}^2)^2 + 4C_1} \quad (4.148)$$

where $(\mathcal{K}^2)_{IJ} = ((\mathcal{K}_1)^2, (\mathcal{K}_2)^2, (\mathcal{K}_3)^2)$, and we used (4.57) to introduce $\mathcal{C}_1$. Substituting this into the supersymmetric constraint (4.142) yields

$$\mathcal{E} = -\mathcal{F}_2\mathcal{A}^2 \frac{8(1 - \mathcal{A}^2)(11 - 2\mathcal{A}^2)^2 - 12(1 + 2\mathcal{A}^2)\mathcal{C}_1 - 4\mathcal{C}_2}{(11 - 2\mathcal{A}^2)^2 + 4\mathcal{C}_1} + O(\rho) \quad (4.149)$$

The numerator in the above expression can never vanish because

$$8(1 - \mathcal{A}^2)(11 - 2\mathcal{A}^2)^2 - 12(1 + 2\mathcal{A}^2)\mathcal{C}_1 - 4\mathcal{C}_2 > 8(1 - \mathcal{A}^2)(11 - 2\mathcal{A}^2)^2 - 8(1 - \mathcal{A}^2)^3$$

$$= 24(1 - \mathcal{A}^2)(10 - \mathcal{A}^2)(4 - \mathcal{A}^2) > 0 \quad (4.150)$$

where in the first line we used (4.59), and in the last $\mathcal{A}^2 < 1$. This fixes

$$\mathcal{F}_2 = \tilde{\mu}_3^I = 0 \quad (4.151)$$

and we have shown that the base of induction holds.

We will now assume that

$$\mathcal{F}_m = \tilde{\mu}_m^I = 0, \quad n \geq m \geq 2, \quad (4.152)$$

and show that it holds for $n + 1$. We start from the Maxwell equations (4.141) expanded at the order $O(\rho^n)$

$$\mathcal{E}^I = \frac{(n + 1)\mathcal{A}^2}{36} \left[2a_n \tilde{\mu}_n^I + 2(\mathcal{K}\tilde{\mu}_{n+2})^I + C^{IJK} (2\mathcal{K}_{IJ} - a_n \zeta_{IJ}) \delta_{KL}\tilde{\mu}_{n+2}^L \rho^n + O(\rho^{n+1}) \right]$$

where, as usual, $(\mathcal{K}\tilde{\mu}_{n+2})^I = (\mathcal{K}_1\tilde{\mu}_{n+2}^1, \mathcal{K}_2\tilde{\mu}_{n+2}^2, \mathcal{K}_3\tilde{\mu}_{n+2}^3)$, and we have introduced for convenience

$$a_n = 3n^2 + 6n + 2 - 2\mathcal{A}^2 \quad (4.154)$$

Taking into account (4.58) we find the solution of (4.153)

$$\tilde{\mu}_{n+2}^I = -\frac{\ell F_{n+1}}{(n + 1)\mathcal{A}^2} \frac{([n + 3]a_n^2 + 12C_1) \ell \zeta^I + n \delta^{IJ} \left[a_n \mathcal{K}_{IJ} - 2(\mathcal{K}^2)_{IJ}\right]}{a_n^2 + 4C_1} \quad (4.155)$$

Inserting it into the supersymmetric constraint (4.142) we get

$$\mathcal{E} = -\mathcal{F}_{n+1} \frac{n^2(n + 1)(n + 2)\mathcal{A}^2 (a_n - 3(1 + 2\mathcal{A}^2)a_n^2 - 12(1 + 2\mathcal{A}^2)\mathcal{C}_1 - 4\mathcal{C}_2}{a_n^2 + 4C_1} \rho^{n-1} + O(\rho^n) \quad (4.156)$$

It is worth noting that (4.153), (4.155) and (4.156) reproduce respectively (4.147) to (4.149) for $n = 1$. The numerator is always positive

$$(a_n - 3(1 + 2\mathcal{A}^2)a_n^2 - 12(1 + 2\mathcal{A}^2)\mathcal{C}_1 - 4\mathcal{C}_2 > (a_n - 3(1 + 2\mathcal{A}^2)) a_n^2 - 8(1 - \mathcal{A}^2)^3$$

$$> (a_1 - 3(1 + 2\mathcal{A}^2)) a_1^2 - 8(1 - \mathcal{A}^2)^3 = 24(1 - \mathcal{A}^2)(10 - \mathcal{A}^2)(4 - \mathcal{A}^2) > 0 \quad (4.157)$$

where in the second and third inequality we used $a_n > a_1 > 3(1 + 2\mathcal{A}^2)$ for $n > 1$. This implies that

$$\mathcal{F}_{n+1} = \tilde{\mu}_{n+2}^I = 0 \quad (4.158)$$

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concluding the induction argument. We have therefore shown that in the case $A^2 = B^2$ the unique analytic solution with black hole boundary conditions is

$$F(r) = \rho^2 + F_3 \rho^3, \quad \mu^I(r) = -\frac{\ell^2}{4} F_3 \rho^2 \zeta^I$$

(4.159)

where we have denoted $F_3 = A^2 C_1$.

As explained in the argument in the general case, $F_3$ is not an independent parameter, but an artefact of global rescaling freedom. Setting $s = F_3$, this rescaling is used to fix $s = 4/\ell^2$ when $s \neq 0$. The case $s = 0$ corresponds to the NH geometry.

**Corollary 4.** The scalar fields $X^I$ are given by

$$X^I = \frac{12 \rho}{\ell^2} \left( (\Delta_2(\eta) + 3 s \rho) \zeta_1 + K_I \right)^{-1},$$

(4.160)

and the norm of the supersymmetric Killing field $V$ is given by

$$f = \frac{12 \rho}{\ell^2} \left( \prod_{i=1}^3 \left( (\Delta_2(\eta) + 3 s \rho) \zeta_1 + K_I \right) \right)^{-1/3}.$$

(4.161)

In particular, $V$ is strictly timelike outside the horizon.

**Proof.** First, let us compute rescaled scalar fields $\Phi^I$ using (4.107):

$$\Phi^I = \frac{\ell^2}{12 \rho} C_{IJK} \zeta_J \left( (\Delta_2(\eta) \zeta_K + K_K \right) + \frac{\ell^2}{2} s \zeta_I.$$  

(4.162)

Then, equation (4.161) immediately follows from (4.22). The scalars $X^I$ are found from (4.16) where $C_{IJK}$ is given in (4.20). □

### 4.5.2 Uplifting to five-dimensions

The final step of the uniqueness proof is to integrate (4.31) for $\omega$ and compute the gauge and scalar fields from Maxwell equations. This is given by the following result.

**Lemma 21.** Consider a supersymmetric solution as in Lemma 20. The remaining metric data $\omega$ is given by

$$\omega = \left( \frac{\ell^3 s^2}{8} \rho + \frac{\ell^3 s}{8} (1 - \Delta_1(\eta)) - \frac{\ell^3 \Delta_3(\eta)}{48 \rho} \right) \sigma - \frac{\ell^3 (A^2 - B^2)}{16} \left( \frac{s}{\Delta_1(\eta)} - \frac{1}{\rho} \right) \tau,$$

(4.163)

and the gauge fields $A^I_5$ are given (up to gauge transformations) by

$$A^I_5 = X^I f(dt + \omega) - \frac{\ell^2}{2} s \rho \xi^I \sigma$ $
- \frac{\ell^2}{6} C_{IJK} \xi_J \left( \frac{3}{4} (A^2 - B^2) (1 - \eta^2) \zeta_K + \left( 1 - \frac{1}{2} (A^2 + B^2) \right) \zeta_K + K_K \right) \eta d\phi.$$  

(4.164)

**Proof.** The derivation of $\omega$ goes along the lines of a similar Lemma 16. Recall that $\omega$ is integrated from its exterior derivative by (4.31) which is constructed out of self-dual part $G^+$, given by (4.32), and anti-self-dual part $G^-$ (4.33). Of the latter only the trace part $\lambda_1$ is known (4.36), and the rest is parameterised by two functions $\lambda_{2,3}$. It is straightforward to show that for any smooth toric solution in the basis satisfying (4.52) $\lambda_2 = 0$. Indeed, since $d\omega$ is a closed smooth 2-form, it must satisfy $\iota_{m_1} T_{m_2} d\omega = 0$. The self-dual part is constructed as a linear combination of $\Theta^I$ which are the self-dual part of another closed smooth 2-form $F^I$ which is also $\ell$-invariant, and, hence, $\iota_{m_1} T_{m_2} F^I = \iota_{m_1} T_{m_2} \Theta^I = 0$. Consequently, $\iota_{m_1} T_{m_2} G^- = 0$ which is equivalent to $J^2_{mn} (G^-)^{mn} = 0$, i.e. $\lambda_2 = 0$. 

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The function $\lambda_3$ can be found from (4.42) which now reads as

$$\partial_\rho \left( \sqrt{FG} \lambda_3 \right) = \frac{3(A^2 - B^2)G}{\rho^2},$$

$$\partial_\eta \left( \sqrt{FG} \lambda_3 \right) = 2 \left( \frac{1}{\rho} + s \right) \left( \Delta_3^2 + \Delta_3 + C_1 \right),$$

where we have for convenience dropped the arguments of $F(\rho)$ and $G(\eta)$. The solution is

$$\sqrt{FG} \lambda_3 = -3(A^2 - B^2) \frac{FG}{\rho^3} + \lambda_{3,0},$$

where $\lambda_{3,0}$ a constant.

Then the equations for $\omega_\psi$ and $\omega_\phi$ become

$$-\frac{48}{\ell^3} \partial_\rho \omega_\psi = - \left( \frac{\Delta_3}{\rho^2} + 6s^2 \right),$$

$$-\frac{48}{\ell^3} \partial_\rho \omega_\phi = - \eta \left( \frac{\Delta_1}{\rho^2} + 6s^2 \right) + \lambda_3 \rho \sqrt{\frac{G}{F}},$$

$$-\frac{48}{\ell^3} \partial_\eta \omega_\psi = - \lambda_3 \sqrt{\frac{F}{G}},$$

$$-\frac{48}{\ell^3} \partial_\eta \omega_\phi = - \left( 6s(\Delta_2 + s\rho) + \frac{2\Delta_3 + \Delta_3 + 2C_1}{\rho} \right) - \lambda_3 \eta \sqrt{\frac{F}{G}},$$

The solution is

$$\omega = \left( \frac{\ell^3 s}{8} + \frac{\ell^3}{8} (1 - \Delta_1(\eta)) - \frac{\ell^3}{16} \frac{\Delta_3(\eta)}{\rho^2} \right) \sigma - \frac{\ell^3}{16} \frac{A^2 - B^2}{s} \left( \frac{1}{\Delta_1(\eta)} - \frac{1}{\rho} \right) \tau + \omega_0$$

$$- \frac{\ell^3}{48} \lambda_{3,0} \left[ \frac{\log(s + \rho^{-1})}{(1 - \eta^2)\Delta_1(\eta)} \tau + \frac{1}{A^2 B^2} \left[ \log \left( \frac{2\Delta_1(\eta)}{1 + \eta} \right) d\phi^1 - \log \left( \frac{2\Delta_1(\eta)}{1 - \eta} \right) d\phi^2 \right] \right],$$

where $\omega_0 = \omega_{1,0} d\phi^1 + \omega_{2,0} d\phi^2$ and $\omega_{i,0}$ are integration constants. The above expression satisfies all the local constraints required by supersymmetry. Note that this is formally identical to (3.144) but with additional constant constant $C_1$ included through $\Delta_3(\eta)$ in (4.64).

Similarly to analogous Lemma 16, the logarithmic terms are incompatible with smooth near-horizon geometry. Indeed, according to (4.73) $\omega_i$ has $O(\rho^{-1})$ leading order with $O(1)$-terms smooth in $\rho$, where we have used expansions of $\rho, \eta$ from Lemma 18. This means that logarithmic singularities at the horizon are forbidden, and so we must set $\lambda_{3,0} = 0$. The constants $\omega_0$ are fixed by observing that $\omega_{1}$ (resp. $\omega_{2}$) vanishes at $\eta = +1$ (resp. $\eta = -1$), which requires $\omega_{1,0} = \omega_{2,0} = 0$. We have, therefore, reproduced (4.163).

The gauge field is determined straightforwardly by the formula (4.23) where the orthogonal part $A'$ is given by (4.108).

We have now completely determined the general solution under our assumptions. In particular, the solution is given by Lemmas 20 and 21 and Corollary 4. It remains to show that this solution is the supersymmetric KLR black hole (or its near-horizon geometry): we provide a convenient form for this solution in Appendix 4.C which is parameterised by two constants $A^2, B^2$. A simple computation shows that the solution with $s = 4/\ell^2$ is indeed the KLR black hole as given in Appendix 4.C, upon the coordinate change

$$\rho = r^2/4, \quad \eta = \cos \theta,$$

and the identification of parameters

$$A^2 = A^2, \quad B^2 = B^2, \quad \frac{\ell^2}{9} K_I = e_I. \quad (4.170)$$
Similarly, one can show that the $s = 0$ solution corresponds to the near-horizon geometry of the KLR black hole under the same identifications. This completes the proof of Theorems 9 and 10.

4.6 Discussion

In this chapter we have started the long-term programme of the classification of timelike supersymmetric black holes in 5d gauged supergravity coupled to $n = 3$ Abelian vector multiplets. Namely, as a first step we have identified and obtained the generalisation of our results for the minimal gauged supergravity, established in the previous chapters. We considered solutions with special classes of toric symmetry, namely, the Calabi-type, orthotoric or ‘product-toric’ Kähler geometries. In contrast with the minimal case, these assumptions were less restrictive, and we have proven our main result Theorem 9 under an additional assumption on gauge fields, which can be interpreted as being invariant with respect to an additional locally Kähler structure existing on such special Kähler bases. For solutions with the Calabi-type geometries, we have proven a uniqueness theorem for the KLR black hole [118] for spherical or Lens space horizon topologies. It is worth noting that all solutions with $SU(2) \times U(1)$-symmetric gauge fields automatically satisfy the gauge field assumption, as well as being of Calabi type, so we have separated this case in its own theorem.

As in the previous chapters, our classification results Theorems 8 to 10 do not make any global assumptions on the spacetime and in particular does not make any assumptions on the asymptotics of the solution. Therefore, it rules out the possibility of black holes in this symmetry class with a smooth $S^3$ (or $L(p, q)$) horizon in asymptotically locally AdS$_5$ spacetimes (other than trivial global quotients of the KLR solution) within this class. In particular, it shows that $SU(2) \times U(1)$ numerical solutions of [89, 90] are non-smooth, which is a generalisation of a similar statement for minimal matter black holes which we derived in previous chapters.

At the core of our classification are the resolution of local supersymmetric constraints under certain assumptions. In this chapter we have simplified the set of constraints derived in [46], and in particular, explicitly extracted the integrability constraint. Unfortunately, the constraints become more complicated in the presence of scalars, and we were not able to separate geometrical degrees of freedom from scalar ones, even in the static case. It would be interesting to study these constraints in more detail to determine if they admit a natural hidden structure.

In the proof of our theorems we in an essential way use the assumptions on the Kähler base and gauge fields, as well as on a particular type of near-horizon geometry. While this allowed to carry over some techniques from the minimal case, it leaves open a major problem of determining the (non)-existence of black holes with non-spherical topology. The classification of near-horizon geometries with toric symmetry [65] has shown that horizons with toroidal $T^3$ as well as $S^2 \times S^1$ black holes are allowed. A notable complication is that these horizons are null supersymmetric solutions, hence, can lead to less restrictive boundary data.

A natural line of interest is a search for black rings and toroidal black holes. Non-supersymmetric black rings have been constructed numerically in [52] (see also [53] for discussion). Their supersymmetric analogues, however, are absent in the minimal case [44,45], and for the near-horizon geometry of supersymmetric rings and toroidal black holes to be balanced, scalar fields are necessary [47]. It is worth noting that their near-horizon geometry is static. In general, however, we expect the full solution to be rotating in order to balance the additional centripetal pull by the AdS geometry. Furthermore, in the small limit $^{14}$, one would expect to have a situation similar to the ungauged supergravity, where all $S^2 \times S^1$ black holes are non-static despite having static horizons. It is an open question whether such solutions can be analytically constructed in more integrable settings of orthotoric or Calabi-type geometries. The author leaves this prominent line of research for future work.

Another line of work are static solutions. In contrast to the minimal case, there is no argument for their exclusion. Furthermore, as discussed in section 4.2.3, the supersymmetric constraints become a set of lower order algebro-differential equations, which gives hope of explicit classification of supersymmetric backgrounds in this class. While one expects a generic

\footnote{Black rings with radius of $S^2$ much smaller than AdS length $R_{S^2} \ll \ell$.}
supersymmetric black hole to be rotating, it is nevertheless possible that exceptions may exist. Furthermore, it is interesting to see if there are smooth solitons in this class.

Finally, it would be interesting to investigate the classification of supersymmetric black holes in AdS$_5$ in the STU coupled to complex scalars. Recently, numerical evidence for branching hairy supersymmetric black holes have been obtained [86,87]. A procedure of derivation of local constraints from supersymmetry was carried in [117], where it was shown that such solutions also possess a Kähler base. Therefore, it is reasonable to hope that some of the techniques we developed can be carried over to hairy black holes as well. This, however, remains out of the scope of current thesis.
4.A SD and ASD 2-forms

One can always pick an orthonormal basis on the spaces of self-dual (SD) and anti-self-dual (ASD) 2-forms

\[
I^{(a)}_p J^{(b)}_n = -\delta^a_b \delta^m_n + \epsilon^{abc} J^{(c)}_n, \quad \ast J^{(a)} = I^{(a)},
\]

\[
J^{(a)}_p J^{(b)}_n = -\delta^a_b \delta^m_n + \epsilon^{abc} J^{(c)}_n, \quad \ast J^{(a)} = -J^{(a)}.
\]

where we have put SD/ASD indices \(a, b, c = 1, 2, 3\) into brackets to avoid confusion with coordinate indices \(m, n, p = 1, \ldots, 4\). Note that by orthonormality every 2-form also realises an almost complex structure. Each orthonormal basis is defined up to \(SO(3)\) rotations.

We will pick the orientation where the Kähler form \(J\) is ASD, and it is convenient to set \(J^1 = J\). The remaining \(SO(2)\) rotation corresponds to shifting the Ricci potential by an exact form.

Furthermore, on toric Kähler spaces there is a natural preference. If we denote the Killing fields generating the toric symmetry by \(m_i, i = 1, 2\), then we can always set

\[
\tau_{m_1} \tau_{m_2} I^1 = \tau_{m_1} \tau_{m_2} J^3 = \tau_{m_1} \tau_{m_2} J^1 = 0,
\]

and \(\tau_{m_1} \tau_{m_2} J^1 = 0\) by the properties of Kähler form. By construction, we naturally have

\[
\mathcal{L}_{m_i} I^a = \mathcal{L}_{m_i} J^a = 0.
\]

This can be seen by introducing an involution of tangent bundle with the action given by inverting the signs of angular variables (defined as coordinates along \(m_i\)). Then \(\Omega^2(M)\) decomposes into \(\pm 1\) eigenspaces, and we can pick \(I^2, J^2\) to be the basis of the \(+1\) eigenspace while \(I^1, J^1, J^3\) span the negative one.

This observation fixes the basis of ASD forms up to a \(\mathbb{Z}_2\) discrete subgroup, and the basis of SD forms up to an \(SO(2)\) rotation.

Finally, for Calabi case there is a further preferred choice: the self-dual twistor form is proportional to the self-dual part of the Ricci form, and we can take \(I^1\) to be its normalised component. This leaves only a discrete \(\mathbb{Z}_2\) freedom which we will fix in the following sections.

It is worth noting that the induced connection on the bundle SD (and ASD) forms can be expressed through three 1-forms

**Proposition 7.** Let \((M, g)\) be a four dimensional Riemann manifold. Then the connection on a bundle of SD forms with the basis \((I^1, I^2, I^3)\) satisfying quaternionic relations (4.171) is given by

\[
\nabla_X I^1 = \gamma(X) I^2 - \beta(X) I^3, \\
\nabla_X I^2 = \alpha(X) I^3 - \gamma(X) I^1, \\
\nabla_X I^3 = \beta(X) I^1 - \alpha(X) I^2.
\]

for some 1-forms \(\alpha, \beta, \gamma\).

For geometries admitting a twistor 2-form, these 1-forms satisfy very simple relations.

4.B Hamiltonian and twistor 2-forms

**Definition 9.** The hamiltonian 2-form on a Kähler space \((M, g, J)\) is a \(J\)-invariant solution \(\phi^{(\text{ham})}\) to

\[
\nabla_X \phi^{(\text{ham})} = \frac{2}{3} X(\sigma) - \frac{1}{3} \partial \sigma \wedge J X - \frac{1}{3} X \wedge J \partial \sigma
\]

where \(\sigma = \frac{1}{4} \phi^{(\text{ham})} J^{mn}, (JX)_m = J_{mn} X^n\) and \(\nabla\) is a Levi-Civita connection.

Trivially, a multiple of Kähler form is a hamiltonian 2-form. We will be interested in non-trivial solutions. In coordinates

\[
\nabla_m \phi^{(\text{ham})}_{np} = \frac{2}{3} \delta_{mn} J_{np} - \frac{2}{3} J_{m[n} \sigma_{p]} - \frac{2}{3} g_{m[n} J_{p]} \sigma^{\eta} \sigma^{\eta}
\]
where $\sigma_m = \partial_m \sigma$. Notice that Hamiltonian 2-form is closed by construction.

The equation (4.176) is equivalent to stating that the SD part of the hamiltonian 2-form $\phi^{(tw)} := \phi^{(ham)} - \sigma_J$ satisfies twistor 2-form (also known as conformal Killing-Yano 2-form) equation.

**Definition 10.** Let $(M, g)$ be a 4d Riemannian manifold. The twistor 2-form is a non-trivial solution to the following equation

$$
\nabla_X \phi^{(tw)} = \frac{1}{3} X \delta \phi^{(tw)} + \frac{1}{3} X \wedge \delta \phi^{(tw)}.
$$

(4.178)

In index notation this reads as

$$
\nabla_m \phi^{(tw)}_{np} = \partial_m \phi^{(tw)}_{np} + \frac{2}{3} g_{m[n} (\nabla \phi^{(tw)})_{p]}.
$$

(4.179)

This structure can be seen as a generalisation of conformal Killing vector fields (their covector fields are called twistor 1-forms) to higher ranks. The important property of twistor form equation is that it is invariant under Hodge duality: if $\phi^{(tw)}$ is a twistor 2-form then its SD and ASD parts are twistor 2-forms on their own. In what follows we will assume the twistor 2-form to be SD.

In [127] (see also [93, Lemma 2]) an alternative form of the twistor 2-form equation was derived.

**Proposition 8.** Let $(M, g)$ be a 4d Riemannian manifold, and $\phi^{(tw)} = e^\mu I$ is a non-vanishing SD 2-forms, where $I$ is a unit SD 2-form $I \wedge I = 2 \text{vol}_M$ and $\mu$ is a function on $M$. Then $\phi^{(tw)}$ is a twistor 2-form if and only if

$$
\beta = - I^2 d\mu, \quad \gamma = - I^3 d\mu
$$

(4.180)

where $(I^1 = I, I^2, I^3)$ is any basis of SD forms which satisfies the quaternionic relations (4.171), and $\beta, \gamma$ are connection 1-forms defined in Proposition 7.

In particular, this means that $e^{-2\mu} I$ is covariantly constant with respect to Levi-Civita connection of conformally rescaled metric $e^{-2\mu} g$, which means that $(e^{-2\mu} g, e^{-2\mu} I)$ is a Kähler space (at least locally). In other words, twistor 2-form defines a locally conformally Kähler structure.

### 4.B.1 Curvature constraints from the existence of hamiltonian 2-forms

Hamiltonian 2-forms pose the following constraint on the curvature

**Proposition 9.** If $(M, g, J)$ admits a hamiltonian 2-form then the SD part of the Ricci tensor is proportional to twistor 2-form.

This proposition follows as a simple corollary from Proposition 4 in [109]. We discover that there is a partial converse at least for toric geometry.

**Proposition 10.** Let $(M, g, J)$ be a toric Kähler surface admitting a SD twistor 2-form $\phi^{(tw)}$ which respects toric symmetry and $\iota_m \iota_n \phi^{(tw)} = 0$. Let SD part of the Ricci tensor $R^+ = R - \frac{R}{2} J$ be proportional to twistor 2-form. Then there exists a hamiltonian 2-form $\phi = \phi^{(tw)} + \sigma J$ for some $\sigma$.

**Proof.** Write the twistor form as $\phi^{(tw)} = e^\mu I$ where $I$ is SD almost complex structure, and $e^\mu$ its norm. Notice that twistor form equation then implies [93]

$$
\nabla_X I = (\iota_{\mu} \iota_X I^3) I^2 - (\iota_{\mu} \iota_X I^2) I^3
$$

(4.181)

where $I, I^2, I^3 = I I^2$ are orthonormal self-dual 2-forms. From the general properties of Levi-Civita connection acting on self-dual 2-forms it then follows that [93]

$$
\nabla_X I^2 = (\iota_{\alpha} \iota_X I) I^3 - (\iota_{\mu} \iota_X I^3) I
$$

(4.182)

$$
\nabla_X I^3 = (\iota_{\mu} \iota_X I^2) I - (\iota_{\alpha} \iota_X I) I^2
$$

(4.183)
for some 1-from $\alpha$ which is arbitrary.

The integrability condition for the Hamiltonian 2-form

$$d(\phi^{(tw)} + \sigma J) = 0$$  \hspace{1cm} (4.184)

can be solved for $\sigma$

$$d\sigma = -3(IJ^1)de^\mu$$  \hspace{1cm} (4.185)

provided that the RHS is closed, which is the integrability condition

$$d(\iota_m I^J) = 0$$  \hspace{1cm} (4.186)

To simplify it further we will assume the symplectic chart $(x_1, x_2, \varphi^1, \varphi^2)$ where $x_1, x_2$ are moment maps wrt $m_1 = \partial_{\varphi^1}, m_2 = \partial_{\varphi^2}$ Killing fields. By our assumption $\mathcal{L}_{m_1} \phi^{(tw)} = \mathcal{L}_{m_2} \phi^{(tw)} = 0$.

Introduce orthonormal basis on $\Lambda^2 M$ as

$$\{J^1, J^2, J^3, I^1, I^2, I^3\}$$

where the former are anti-self-dual and the latter are self-dual 2-forms, and

$$\iota_m J^1 = \iota_m I^1 = \iota_m J^3 = 0.$$  \hspace{1cm} (4.187)

In symplectic chart (3.12)

$$J^2 = \sqrt{\text{det}G^{ij}} \, dx_1 \wedge dx_2 - \frac{1}{\sqrt{\text{det}G^{ij}}} \, d\varphi^1 \wedge d\varphi^2$$  \hspace{1cm} (4.188)

$$I^2 = \sqrt{\text{det}G^{ij}} \, dx_1 \wedge dx_2 + \frac{1}{\sqrt{\text{det}G^{ij}}} \, d\varphi^1 \wedge d\varphi^2$$  \hspace{1cm} (4.189)

From the toric invariance of twistor form, the integrability constraint is then a 2-form with a single component proportional to $dx_1 \wedge x_2 \propto (I^2 + J^2)$, and, consequently, it is sufficient to show that contraction of (4.186) on $I^2$ vanishes. We find that

$$\frac{1}{2} \text{tr} (I^2 \rightarrow d(\iota_m I^J)) = I^2 \, J^1_m ^p \nabla_{p} (I^{pq} \nabla^{q} e^{\mu}) = (J^1 I^3)^{mn} \nabla_{m} \nabla_{n} e^{\mu}.$$  \hspace{1cm} (4.190)

Now consider the integrability condition for the twistor form (4.181). It follows that

$$2J^{mn} I^2 \, p_{q} \nabla_{m} \nabla_{n} I_{pq} = 4R_{pq} I^3 \, p_{q} = 0.$$  \hspace{1cm} (4.191)

where last equality follows from assumption of our proposition. On the other hand, using (4.181) and (4.182)

$$2J^{mn} I^2 \, p_{q} \nabla_{m} \nabla_{n} I_{pq} = 8(J I^3)^{mn} (\nabla_{m} \nabla_{n} \mu + \nabla_{m} \mu \nabla_{n} \mu) = 8e^{-\mu} (J I^3)^{mn} \nabla_{m} \nabla_{n} e^{\mu}$$  \hspace{1cm} (4.192)

which is exactly (4.190).

This proposition can be seen as a generalisation of [93, Lemma 4], which was proven for weakly self-dual surfaces, a class of geometries whose Ricci form is the Hamiltonian 2-form.

4.C KLR black hole

The supersymmetric KLR black hole is a four parameter family of solutions first found in [118], and we present it here in the same notation, but with a few further simplifications.

The solution depends on four parameters $0 < A^2, B^2$ and $C_1, C_2$ subject to $\kappa^2(A^2, B^2) > 0$ where $\kappa^2$ is given by (4.56). Note that this implies $A^2, B^2 < 1$. The solution has the same form of the Kähler base as the CCLP black hole Appendix 3.B. Its metric and the Kähler form are given by
\[
\begin{align*}
\mathcal{h} &= \frac{d\ell^2}{V(r)} + \frac{r^2}{4} \left( \frac{d\theta^2}{\Delta_\theta} + \Delta_\theta \sin^2 \vartheta d\varphi^2 \right) + \frac{r^2 V(r)}{4} (d\psi + \cos \vartheta d\varphi)^2, \\
J &= d \left( \frac{1}{2} r^2 (d\psi + \cos \vartheta d\varphi) \right),
\end{align*}
\] (4.193)

where \(\Delta_\theta = A^2 \cos^2(\vartheta/2) + B^2 \sin^2(\vartheta/2)\) and \(V = 1 + \frac{r^2}{\ell^2}\) for KLR and \(V = 1\) for its near-horizon geometry (also supersymmetric solution). In terms of \(2\pi\)-periodic angles \(\phi\),

\[
\psi = A^{-2} \phi^1 + B^{-2} \phi^2, \quad \phi = -A^{-2} \phi^1 + B^{-2} \phi^2.
\] (4.194)

The other coordinate ranges are \(r \geq 0\) and \(0 \leq \vartheta \leq \pi\).

The scalar fields are

\[
\begin{align*}
f^{-1} X_I &= \frac{1}{r^2} \left( \rho^2 \bar{X}_I + \epsilon_I \right)
\end{align*}
\] (4.195)

where we have introduced \(\rho^2\) as

\[
\ell^{-2} \rho^2 = V(r) - \Delta_\theta - \frac{1}{3} \left[ 2 - A^2 - B^2 \right].
\] (4.196)

The \(\epsilon_I\) are scalar constants, and we have defined Fayet-Iliopoulos parameters \(\bar{X}_I\) as

\[
\bar{X}_I = \frac{1}{3} X^I = \frac{\ell}{3} \epsilon_I = \frac{1}{3} \quad \forall, \quad I = 1, 2, 3.
\] (4.197)

In terms of new parameters the combination \(C_1\), defined in (4.57), is expressed as

\[
C_1 = \frac{81}{2} \epsilon^{-4} C^{IJK} \bar{X}_I \epsilon_J \epsilon_K.
\] (4.198)

The five-dimensional metric is then constructed from (4.14) with

\[
f^{-1} = \frac{3}{r^2} \prod_I \left( \rho^2 \bar{X}_I + \epsilon_I \right),
\] (4.199)

and

\[
\omega = \left( \frac{r^2}{2\ell} + \frac{\ell}{2} (1 - \Delta_\theta) - \frac{\ell^3}{12r^2} \left( 6 \Delta_\theta - A^4 - B^4 + A^2 B^2 - 2(A^2 + B^2) - 1 - C_1 \right) \right) (d\psi + \cos \vartheta d\varphi) + \frac{\ell(A^2 - B^2)}{4} \left( 1 - \frac{\ell^2 \Delta_\theta}{r^2} \right) \sin^2 \vartheta d\varphi.
\] (4.200)

Notice that

\[
\omega_{\text{KLR}} = \omega_{\text{CCLP}} + \frac{\ell^3}{12r^2} C_1 (d\psi + \cos \vartheta d\varphi),
\] (4.201)

where \(\omega_{\text{CCLP}}\) is given in Appendix 3.B. Finally, the gauge fields are

\[
\begin{align*}
A_5^I &= X^I f(dt + \omega) - \epsilon \frac{r^2}{2\ell} \bar{X}_I (d\psi + \cos \vartheta d\varphi) \\
&- \frac{\ell}{2} C^{IJK} \bar{X}_J \left( \frac{9}{4} (A^2 - B^2) \sin^2 \vartheta \bar{X}_K + \left( 3 \bar{X}_K - \frac{3}{2} (A^2 + B^2) \bar{X}_K + \frac{9}{7} \epsilon_K \right) \cos \vartheta \right) d\varphi
\end{align*}
\] (4.202)

where \(\epsilon = 1\) for KLR and \(\epsilon = 0\) for its NH geometry. It is interesting to note that the KLR solution is much simpler in this coordinate system.

For \(A^2 = B^2\) we recover the \(SU(2) \times U(1)\)-symmetric black hole found in [46].

In order to compare the above expressions with the ones in the literature, we give the dictionary between the notation here and in [118]. Our parameters and coordinates are related.
to those in [118, section 2.3] as follows:

\[
\ell_{\text{here}} = g_{\text{there}}^{-1}, \quad \left(\frac{r}{\ell}\right)_{\text{here}} = \sinh(g\sigma)_{\text{there}}, \quad (\phi^1, \phi^2, \vartheta)_{\text{here}} = (-\phi, -\psi, 2\theta)_{\text{there}}, \quad (4.203)
\]

\[
(A^2, B^2)_{\text{here}} = (A^2, B^2)_{\text{there}}, \quad \Delta_{\phi|\text{here}} = \frac{\Delta g}{g^2\alpha^2} \bigg|_{\text{there}}, \quad \rho^2|_{\text{here}} = \rho^2|_{\text{there}}, \quad (4.204)
\]

The constants associated with scalars are

\[
e^{I}_{\text{here}} = e^{I}_{\text{there}}. \quad (4.205)
\]
Chapter 5

Discussion

In this thesis we have performed the first attempt at consistent search and classification of supersymmetric black holes in $d > 3$ gauged supergravities. We have considered $5d$ minimal gauged supergravity and its STU generalisation, which are a natural setting for holographic correspondence. The importance and complexity of this problem and the scarcity of results were outlined in section 1.2. Namely, all known supersymmetric black holes in this theory were obtained either by using an ansatz or numerically, and no uniqueness theorems were known. Consequently, the first question to answer was whether such a classification is even possible, and, if yes, whether the near-horizon data, which is the deep IR of holography, is sufficient, or the inputs from the asymptotics are necessary. Next, an important question to address is if we can find solutions with different topology, such as black lenses and multi-centre black holes, or show their non-existence, at least for sufficiently symmetric classes of solutions. A similar question is about the existence of a generalisation of the CCLP black hole within the class with spherical horizons. Finally, it is interesting to understand how the presence of extra matter in the form of extra Abelian vector multiplets affects such a classification.

We give a positive answer to the first question. Namely, we have developed a framework for black hole classification, which is based on three principles. Firstly, we construct our arguments using only the near-horizon data. Secondly, we take a geometric approach based on the presentation of supersymmetric constraints as a problem in Kähler geometry \cite{46,70}. Finally, we are interested in explicit solutions, so that using numerical simulations like \cite{77,78,89,90} or indirect extremisation techniques, as used in the setting of GK geometries,\footnote{The recent approach to Gauntlett-Kim (GK) geometries developed in \cite{128,129}, where the integral of the supersymmetric constraint is extremised over boundary conditions to determine the set of possible ones. This, in general, is insufficient to guarantee that a geometry with such boundary condition exists. See \cite{130} for an introduction to the topic.} are left out of the scope.

As it was discussed in section 1.3, the supersymmetric constraint is a complicated highly non-linear differential equation on the curvature, and a general resolution of which is currently out of possibility. We have, nevertheless, identified several classes of Kähler geometries for which it can be solved, at least, with black hole boundary conditions. In chapter 2 we have started with the case of $SU(2)$-symmetric solutions. We have then showed that for a quite broad class of such solutions, the geometry has enhanced $SU(2) \times U(1)$-symmetry, and the Gutowski-Reall black hole is unique within this class. The notable limitation, however, was the scarcity of results on general non-diagonal Kähler geometries with $SU(2)$-symmetry, which made us to introduce an assumption on the Kähler form. Another constraint was the analyticity of a horizon, as we were using series expansions to prove the theorem.

Next, in chapter 3 we have considered a case of solutions with $U(1)^2$ symmetry. We have translated the problem into the language of symplectic potential and established the boundary conditions corresponding to smooth geometries with a desired axial or horizon structure. We have then identified a particular subclass of toric geometries, which contains all known solutions (analytic and numerical) to this theory, and we have established that the CCLP black hole is the unique smooth black hole within it. Curiously, this subclass by construction possesses an additional geometric structure, a hamiltonian 2-form, which is associated with hidden symme-
tries. Furthermore, any $SU(2) \times U(1)$-symmetric Kähler geometry belongs to it, which links to the previous case. We then extended our analysis in chapter 4 to the presence of Abelian vector multiplets, where we considered the same subclass of Kähler geometries. We have then established a similar uniqueness theorem for black holes with spherical horizon under an additional assumption on the gauge fields.

Our analysis can be seen as an inverse to usual holographic approach, since we were able to classify supersymmetric solutions, and, hence, their asymptotic behaviour, from near-horizon data only. In particular, in the classes we considered, we have shown that only globally asymptotically AdS black holes can be smooth, and excluded numerically constructed families of [77,78,89,90]. As it was pointed out in the discussion to chapter 3, this is the property of solutions with Lorentzian signature. It may happen that for Euclidean signature, which corresponds to complex BPS saddles, the boundary geometry can be much richer.

Next, it is interesting to note that our uniqueness theorems exclude black holes with the non-trivial topology. Specifically, black lenses and multi-black holes are excluded from this subclass. Note that the black rings, however, are still allowed in the STU model, and it is of particular interest to find such solutions or rule them out, see section 4.6 for further discussion.

It remains an open question whether the subclass of geometries with hamiltonian 2-form has deeper connection with the supersymmetric constraint, or the fact that all known solutions are within it is just a consequence of taking simple ansatizes. As it was discussed in the introduction, a precise geometric interpretation of this constraint is not known, but it can be seen as some generalisation of extremality condition in the sense of Calabi functional, which, in its turn, is implicitly linked to the existence of such hamiltonian 2-forms [93]. It is interesting to see if the supersymmetric constraint can be understood as some kind of integrability condition for this structure.

Furthermore, the nature of this geometric structure is unclear both from spacetime geometry and from the holographic point of view. In [131] it was shown that the full 5d metric of CCLP black hole possesses a hidden symmetry in the form of some generalisation of CKY tensor. It is interesting to see if these structures are linked, and if they are — what is their holographic interpretation. In particular, it is interesting to identify a form of the holographic stress-tensor and current for such solutions.

To conclude, the programme we have started has quickly overgrown the size of a PhD project. Essentially, (i) we have shown that the proof of uniqueness theorems is possible in the presence of a cosmological constant in higher dimensions, and (ii) we have identified a wealth of perspective directions and geometric subclasses which are, on the one hand, susceptible to computation, and, on the other hand, are likely to contain new solutions, especially in the setting of the STU model.
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