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Spencer cohomology, supersymmetry and the structure of Killing superalgebras

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Declaration

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where stated otherwise by reference or acknowledgement, the work presented is entirely my own.

(Andrew D. K. Beckett)
For Grandad.
Abstract

We review and expand upon work from the last decade on the application of Spencer cohomology to the study of supersymmetric bosonic backgrounds of supergravity. The central observation of this project is that the symmetry superalgebras of such backgrounds, known as Killing superalgebras, are filtered subdeformations of the Poincaré superalgebra. Such deformations of Z-graded Lie superalgebras are governed by Spencer cohomology, thus the structure of Killing superalgebras can be determined from the Spencer cohomology of (graded subalgebras of) the Poincaré superalgebra. Moreover, the cohomology calculation often allows one to write down the Killing spinor equation and determine much of the structure of supersymmetric supergravity backgrounds, and in some cases it allows one to generalise the notion of supersymmetric geometry.

This thesis contributes to the existing literature on Spencer cohomology and supersymmetry both in terms of general theoretical results and in terms of explicit calculations and examples. We consider the algebraic structure of Killing superalgebras, with particular focus on the highly supersymmetric case, generalising to arbitrary dimension and amount of supersymmetry a number of results previously only proven for 11-dimensional supergravity. We characterise possible obstructions to “integrating” infinitesimal filtered deformations ((2,2)-Spencer cohomology classes) of the Poincaré superalgebra to full deformations. We also give a geometric description of backgrounds with gauged R-symmetry, clarifying and improving upon previous treatments in the literature, and generalise the aforementioned results to Killing superalgebras of such backgrounds. Some worked examples in 2 dimensions are presented. We determine the (2,2)-Spencer cohomology of the N-extended Poincaré superalgebra in 5 and 6 spacetime dimensions with gauged R-symmetry. We also show that this cohomology is trivial in Type IIA. We then consider the structure of filtered deformations and maximally supersymmetric backgrounds in the minimal 5-dimensional case with gauged R-symmetry, showing that obstructions to integrating infinitesimal deformations exist and arise in both algebraic and geometric guises. We also classify some sub-classes of the maximally supersymmetric backgrounds and their superalgebras in this case.
The dichotomy in modern physics

At the heart of modern physics, there is a fundamental dichotomy between our understanding of phenomena on very large scales and on very small scales.

On one hand, we have our current best theory of the very large, general relativity, which describes gravity as the curvature of spacetime; the reader might be familiar with the “rubber sheet analogy” (Figure 1). So far, every astronomical observation we have ever is consistent with general relativity. It has successfully predicted not only the motions of the planets and stars (including phenomena such as the precession of the orbit of Mercury which are not accounted for by Newtonian gravity, our previous best theory), but also the bending of light around heavy objects, the existence of black holes and gravitational waves, and even the evolutionary history of the universe itself.

Figure 1: The rubber sheet analogy. The Earth drags on and bends the spacetime around it. This causes other objects to be deflected towards it, which we experience as gravity [1].

On the other hand, our theory of the microscopic world, quantum theory, has been equally successful. With it, we have understood the nature of light, atoms, and the properties of the materials and chemicals which form the world around us. The Standard Model of Particle Physics, which builds upon quantum theory to give a description of the elementary particles which make up atoms, has been repeatedly vindicated by experiments like those at the Large Hadron Collider (LHC) at CERN.

Both general relativity and quantum theory arose in the early 20th century and upended our understanding of our universe like nothing ever before. They have also changed our lives; to name just a few of their most immediate effects, quantum
theory enabled both the semiconductor revolution and the development of nuclear power and weaponry, while our understanding of gravitational time dilation effects through general relativity is necessary to accurately calibrate GPS satellites. Despite their scientific success and societal impact, there are still some major questions about the universe which they do not give us answers to in their current form: the origin of dark matter and dark energy, the nature of the interiors of black holes, and the differences in strength between the fundamental forces, to name just a few.

More fundamentally, although each one of the theories works well wherever the effects of the other are weak enough to be ignored, we know that there are some features of the universe – near the hypothetical singularities inside black holes and at the beginning of the universe, for example – where both gravitational and quantum effects should be very strong, and this is where our twin theories really start to fail. In their current form, they seem to be incompatible with each other, producing mathematical inconsistencies and making nonsensical predictions whenever we try to put them together. Since we can’t probe these extreme parts of the universe directly, theoretical physicists have spent a lot of time coming up with various ways of altering the theories in order to unify them into a single coherent theory of “quantum gravity” (sometimes known as a “Theory of Everything”). While none of these unambiguously solve all of the problems mentioned above or have been experimentally verified, the candidate which has undoubtedly received the most attention is string theory.

String theory tells us that fundamental particles should be thought of not as point-like object, as particle physics usually treats them, but as tiny vibrating strings. In this picture, the different modes of vibration in the strings correspond to different types of particles; particle properties like mass and electrical charge arise like notes on a string instrument. Originally proposed in the 1970s not as a theory of quantum gravity but as a theory of the strong nuclear force which binds together the nuclei of atoms, string theory was eventually abandoned as a nuclear theory in favour of the quark model and quantum chromodynamics, which form part of the Standard Model. However, even then, the physicists working on it recognised it as a potential quantum gravity theory because some of those vibrating modes looked like hypothetical particles called gravitons – more on them below. Despite this exciting development, technical problems with the theory meant that it languished in relative obscurity until it was combined with an idea called supersymmetry, leading to the so-called “superstring revolutions” of the 1980s and 1990s during which the theory became not just a candidate for quantum gravity but a paradigm for understanding and unifying many aspects of theoretical physics and mathematics.

**Bosons, fermions and supersymmetry**

In the quantum world, there is another dichotomy, this time between types of particles. On one side we have the fermions, which we think of as matter particles because their quantum properties allow them to form structures such as atoms and molecules. Electrons, quarks, protons and neutrons are all fermions. On the other side are the bosons which are force particles; they cannot form structure like fermions but are responsible for “mediating” the forces between other particles. For example, light is made up of bosonic particles called photons, and the electromagnetic attraction or repulsion between charged particles such as electrons can be
understood as an exchange of bosons between them. Other, perhaps less well-known bosons are those responsible for the nuclear forces: the gluons and the W and Z bosons. Of course, there is also famous Higgs boson which gives mass to (at least some of) the fundamental particles and is also responsible for some of the more unusual properties of the nuclear forces. In a theory of quantum gravity, we expect the gravitational force to be communicated by a boson called a graviton, as mentioned above. Whether a particle is fermion or boson is determined by its spin, a quantum mechanical property relating to how the behaviour of the particle changes when it’s rotated around.

Supersymmetry is a hypothesised symmetry between fermions and bosons, and therefore between particles with different spins. In a supersymmetric world, we expect that every fermion should have a bosonic counterpart with similar properties called a “superpartner”, and vice-versa. Physicists have had a lot of fun coming up with silly names for these: the superpartner of a quark is a squark, and the Higgs boson’s superpartner is a higgsino. The reason that this idea caught on was because it had the potential to resolve some outstanding questions in particle physics as well as making the theory more well-behaved mathematically. Some of the hypothesised superpartners of Standard Model particles were also proposed as candidates for dark matter, a type of matter whose existence is suggested by astronomical observations but whose exact nature still remains mysterious to us. Unfortunately, what were at one time thought to be promising supersymmetric extensions of the Standard Model have been ruled out over the past decade because the LHC has not produced the superpartners they predict. Nonetheless, there are other supersymmetric theories which have not been ruled out, and some theoretical and mathematical physicists are still very interested in supersymmetry because of its ability to help us solve mathematical problems and its relationship to string theory.

From string theory to supergravity

The original version of string theory which was developed to explain the nuclear strong force could only describe bosons and had various other technical issues, but superstring theory, the combination of string theory with supersymmetry, incorporates fermions and is much better behaved. As already mentioned, gravitons appear in string theory as certain vibrational modes in the strings, and in superstring theory they have a superpartner called the gravitino.

The full version of superstring theory is very mathematically complex, so we often study it by simplifying it in various ways. One way to do this is to “zoom out”, ignoring the quantum, stringy features of the theory and focussing on the combination of supersymmetry with gravity, which gives us a theory of supergravity. It should be noted that string theory isn’t the only way to get to supergravity; it can (and historically was) studied independently by directly combining the principles of general relativity with supersymmetry. It turns out that, as with particle theory and string theory, supersymmetry can make gravity behave better mathematically. For example, naïve approaches to quantum gravity often predict the values of physically measurable quantities to be infinite but many of these infinities disappear in supergravity, and black holes which have some supersymmetry are often mathematically much simpler than generic black holes.
Deformations and supersymmetric geometry

Recalling the rubber sheet analogy, we can think of gravity as a deformation of spacetime which would otherwise be flat. In an analogous way, the supersymmetry of a supergravity theory is a deformed version of the supersymmetry in flat spacetime that particle physicists usually work with. The work in this thesis is part of a project to understand the mathematics of supersymmetry in curved spacetime by thinking of the mathematical constructs which describe it, known as Killing superalgebras, as deformations of the construct which describes supersymmetry in flat spacetime, the Poincaré superalgebra, using a mathematical tool known as Spencer cohomology. We find that Spencer cohomology not only gives us a new perspective on mathematical problems coming from supergravity but allows us to go beyond it, opening the door to a much richer world of supersymmetric geometry that could give us a better understanding of supersymmetric physics in curved spacetime.
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Chapter 1

Introduction

Supergravity

Supergravity theories are supersymmetric theories of gravity. More precisely, they are physical theories in the general relativistic paradigm in which the metric is coupled to various other fields such that the total system exhibits supersymmetry. The representation of the supersymmetry algebra to which the metric, or more properly the vielbein, belongs is called the gravity supermultiplet, which also contains a fermionic vector-spinor field (spin-$\frac{3}{2}$ in 4 dimensions) known as the gravitino and possibly other fields as well. While supergravity is not a single theory but rather a class of theories possessing the properties discussed above, supersymmetry heavily constrains the field content and dynamics of these theories, so there are many common features and close relationships between the theories.

Before discussing some of these general aspects in more detail, let us consider as an example one of the simplest supergravity theories, the minimal 4-dimensional theory.

Minimal supergravity in 4 dimensions

The dynamical fields in this theory are the vierbein and the gravitino. The theory has the following action \[ S_{D=4} = \frac{1}{16\pi G} \int \left( R - \Psi_{\mu} \Gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho + \mathcal{L}_\Psi \right) \text{vol,} \tag{1.1} \]

where \( \text{vol} = \sqrt{\det g} d^4 x \) is the standard volume form, \( R \) is the Ricci scalar, \( \Psi \) is the gravitino, a Majorana (real) vector-spinor fermion field and \( \Gamma^{\mu\nu\rho} = \Gamma^{[\mu\nu}\Gamma^{\rho]} \) is a third-rank Dirac matrix. The indices \( \mu, \nu, \ldots \) are coordinate indices and we denote by \( a, b, \ldots \) the indices in a local orthonormal frame, in which the metric has components of the standard Lorentzian inner product \( \eta_{ab} \). The components of the vierbein which maps the coordinate frame to the orthonormal frame are denoted \( e_a^\mu \) so that \( g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b \), and the components of the Levi-Civita spin connection are \( \omega_{\mu ab} \).

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1Properly speaking, this is the on-shell second-order formulation of the theory; in a second-order formulation, the spin connection is also dynamical but there is no \( \mathcal{L}_\Psi \) term, while off-shell formulations include more fields in the supermultiplet, but they are unphysical auxiliary fields.
The first term in the action is the Einstein–Hilbert action, the second is the kinetic term for a vector-spinor, known as the Rarita–Schwinger action, and $L_{\Psi^4}$ is a sum of interaction terms which are quartic in the gravitino. All supergravity theories include the Einstein–Hilbert and Rarita–Schwinger terms in their actions, and all contain quartic fermion interactions, although the precise form of the latter varies, and the action is typically much more complicated.

The action is invariant under the following local supersymmetry transformation:

$$\delta \epsilon e^a_\mu = \frac{1}{2} \epsilon \Gamma^a \Psi_\mu,$$
$$\delta \epsilon \Psi_\mu = \nabla^K \epsilon = \partial_\mu \epsilon + \frac{1}{4} (\omega_{\mu ab} + K_{\mu ab}) \Gamma^{ab} \epsilon,$$ \hspace{1cm} (1.2)

where $\epsilon$ is an arbitrary local supersymmetry parameter – a spinor-valued function, or more precisely, a local section of a spinor bundle – and $\nabla^K$ is a torsional spin connection with components $\omega^K_{\mu ab} = \omega_{\mu ab} + K_{\mu ab}$ where

$$K_{\mu\nu\rho} = -\frac{1}{4} \left( \overline{\Psi}_\mu \Gamma_\rho \Psi_\nu + \overline{\Psi}_\rho \Gamma_\nu \Psi_\mu - \overline{\Psi}_\nu \Gamma_\mu \Psi_\rho \right).$$ \hspace{1cm} (1.3)

The precise form of $L_{\Psi^4}$ and the supersymmetry transformation are essentially fixed by the requirement that the action is invariant and that the supersymmetry algebra closes on-shell, as we will discuss below.

### On-shell and local supersymmetry

A theory is said to be supersymmetric on-shell when the supersymmetry algebra closes on-shell, meaning that the supercommutator of supersymmetry transformations acting on fields $[\delta \epsilon_1, \delta \epsilon_2]$ is an infinitesimal bosonic symmetry of the action for all parameters $\epsilon_1, \epsilon_2$, but only when the equations of motion are satisfied. In the 4-dimensional example above, this supercommutator, when on-shell, is an infinitesimal diffeomorphism. This may seem like a slightly peculiar feature if one is not familiar with supersymmetric theories, but it is nonetheless common in such theories; one can interpret it as saying that the space of general field configurations is not a representation of the supersymmetry algebra but the subspace of “on-shell” configurations (those satisfying the equations of motion) is.$^2$

A more unique feature of supergravity is that the supersymmetry is local, meaning that the supersymmetry parameter $\epsilon$ is an arbitrary (local) section of a spinor bundle, contrasted with the global or rigid supersymmetry of other well-known theories on fixed backgrounds such as super Yang–Mills (SYM) or Wess–Zumino in which the parameter of the transformation is (covariantly) constant. In the local case, the space of supersymmetries is infinite-dimensional, while in the global case it is finite-dimensional. Since, as in general relativity, there is no fixed background in supergravity theory, local supersymmetry is more akin to the diffeomorphism invariance of general relativity than the gauge invariance of a Yang–Mills theory. Moreover, local supersymmetry essentially implies (local) diffeomorphism invari-

---

$^2$As mentioned in the previous footnote, for some theories there are equivalent off-shell formulations in which the algebra acts on the full configuration space of a larger number of fields (preserving the subspace of on-shell configurations) but the additional fields are unphysical.
ance; for the supersymmetry algebra to close, the theory must also be invariant under
the infinitesimal diffeomorphism along the vector field given by pairing two local
supersymmetry parameters, and since arbitrary vector fields can be generated this way,
the theory must be invariant under any local diffeomorphism. We note, however, that
supergravity theories can also include a Maxwell- or Yang–Mills-type gauge sector,
and in such theories the supercommutator of two supersymmetry transformations
will generally include infinitesimal gauge transformations of all types.

Supergravity theories can be considered as gauge theories of rigid Poincaré su-
persymmetry, subject to additional constraints; from this point of view, the vielbeine,
spin connection and gravitino are the gauge fields associated to the local translations,
Lorentz transformations and supersymmetries respectively. One can also construct
some supergravity theories (in particular matter-coupled theories) as a constrained
gauge theory of the superconformal algebra instead [2].

Extended, matter-coupled and higher-dimensional theories

Supergravity theories are studied as both classical and quantum field theories; much
could be said about both aspects here, but we note just some particular features. In
the classical setting, these theories often give rise to integrable structures and many
other geometric structures, some of which are discussed in detail below and will be
the focus of this thesis. In the quantum setting, the ultraviolet divergences which
usually confound any attempt to quantise gravitational theories are softened in
supergravity in that they occur at higher loop levels, and this was an early motivation
for the study of supergravity [3]. In theories with $N$-extended supersymmetry, in
which the supersymmetry algebra contains $N$ copies of the minimal supercharge
multiplet and there are $N$ gravitini in the gravity supermultiplet, these divergences
can be pushed to even higher loop number. In the $N = 8$ theory, which has the
most supersymmetry possible in 4 dimensions (see below), they may even vanish
completely [4, 5]. While this theory is physically unrealistic for various reasons (lack
of chiral fermions, lack of a mechanism to spontaneously break the supersymmetry,
a gauge group which does not contain that of the standard model, . . .), it is a useful
toy model. Incidentally, $N = 4$ SYM is well-known as a toy model for gauge theories
for essentially the same reason, and indeed the finiteness properties of $N = 8$ have
been elucidated by exploiting the so-called “double copy” relation between the two
theories [4, 5].

Many supergravity theories, including the aforementioned $N = 8$, appear as low
energy compactification limits of (super)string theory and M-theory, which preserve
the good ultraviolet behaviour of $N = 8$ while overcoming its shortfalls, and modern
interest in supergravity is largely driven by this connection. As such, supergravity is
often studied in higher dimensions, especially in 10 and 11 dimensions (the critical di-
mensons for superstring and M-theory respectively), although dimensions between
4 and 10 are also of interest for a variety of reasons. For example, 5-dimensional black
holes are significantly richer than 4-dimensional black holes, and supersymmetry
provides a very useful ansatz for studying them [6, 7].

Apart from its motivation through M-theory, 11-dimensional supergravity is a
useful example to consider because it is richer than the minimal $D = 4$ case but still
relatively simple. Along with the vielbeine and gravitino, the theory includes a gauge
3-form $A$ with 4-form field strength $F = dA$. The action is [2, 8]

$$S_{D=11} = \frac{1}{16\pi G} \int \left[ \left( R - \nabla_{\mu} \Gamma^{\mu\nu\rho} \nabla_{\nu} \Psi_{\rho} \right) \text{vol} - \frac{1}{2} F \wedge * F - \frac{1}{6} F \wedge F \wedge A \right. \\
\left. + \left( -\frac{1}{96} \nabla_{\mu} \left( F_{\alpha\beta\gamma\delta} \Gamma^{\mu\alpha\beta\gamma\delta} + 12 F^{\mu\nu} \Gamma^{\alpha\beta} \Psi_{\nu} + \mathcal{L}_{\Psi} \right) \text{vol} \right] \right), \quad (1.4)$$

where $*$ is the Hodge operator, $\text{vol} = * 1 = \sqrt{|\det g|} d^{11}x$ is the standard volume form, $R$ is the Ricci scalar, and $\Psi$ is a Majorana gravitino. In the first line, we have the Einstein–Hilbert and Rarita–Schwinger terms once again, but we also have a Maxwell-type kinetic term and a Chern–Simons term for the 3-form gauge field. In the second line, we have interactions between the gauge field and gravitino as well as the quartic gravitino interactions. This action is invariant under the supersymmetry transformations

$$\delta_{\epsilon} e^K_{\mu} = \frac{1}{2} \epsilon \Gamma^{a} \Psi_{\mu},$$

$$\delta_{\epsilon} \Psi_{\mu} = \nabla^K_{\mu} \epsilon + \frac{1}{288} \left( F_{\alpha\beta\gamma\delta} \Gamma^{\mu}_{\alpha\beta\gamma\delta} - 8 F_{\mu\alpha\beta\gamma} \Gamma^{\alpha\beta\gamma} \right) \epsilon,$$  \quad (1.5)

where $\epsilon$ is the local supersymmetry parameter and the torsional spin connection $\nabla^K$ is as in the 4-dimensional example.

**The landscape of supergravity theories**

The full landscape of supergravity theories is very rich; depending on the spacetime dimension and amount of supersymmetry $N$ demanded, other than the metric and gravitini there can also be many other fields in the gravity supermultiplet, while “matter” supermultiplets with lower-spin fields can also be coupled in. These $N$-extended and matter-coupled theories can be particularly rich and complicated, even if one ignores the fermions and focusses solely on the bosonic sector. There are typically many scalar fields which participate in a sigma model subsector of the theory, meaning that they can be understood as the components of a map from the base spacetime to a so-called “target space” or “scalar manifold” equipped with a Riemannian metric (or more generally, as components of a section of some bundle equipped with such a metric). In these cases, supersymmetry heavily constrains the target space geometry, and a plethora of geometric structures arise in this way. Higher gauge structures often appear (as we have already seen in 11-dimensional supergravity) and are typically coupled to the scalar manifold in highly non-trivial ways.

The review of Van Proeyen [9] gives an excellent survey of this landscape, and Table 1.1 is extracted from that article along with its original caption. The table is organised by spacetime dimension and “number of supercharges” (that is, the dimension of the spinor representation in which the supersymmetry parameter takes its values) which comes in multiples of the dimension of the irreducible spinor representation, and it essentially arises from the analysis of the representation theory of the Poincaré superalgebra in [10]. The main label of each entry indicates the existence of a supergravity multiplet, while the extra symbols label the matter supermultiplets; empty spaces indicate that no multiplets exist. In all non-empty cases, a
Table 1.1: Summary of the landscape of supersymmetric theories, including supergravity and rigid supersymmetric theories. The table and caption are taken from [9].

Table 2: Supersymmetry and supergravity theories in dimensions 4 to 11. An entry represents the possibility to have supergravity theories in a specific dimension $D$ with the number of supersymmetries indicated in the top row. We first repeat for every dimension the type of spinors that can be used. Every entry allows different possibilities. Theories with more than 16 supersymmetries have different gaugings. Theories with up to 16 (real) supersymmetry generators allow ‘matter’ multiplets. The possibility of vector multiplets is indicated with $\nabla$. Tensor multiplets in $D = 6$ are indicated by $\odot$. Multiplets with only scalars and spin-$\frac{1}{2}$ fields are indicated with $\clubsuit$. At the bottom is indicated whether these theories exist only in supergravity, or also with just rigid supersymmetry.

<table>
<thead>
<tr>
<th>$D$</th>
<th>susy</th>
<th>$\Delta$</th>
<th>$\nabla$</th>
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<td>11</td>
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<tr>
<td>10</td>
<td>MW</td>
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<td>9</td>
<td>M</td>
<td>N = 2</td>
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<td>7</td>
<td>S</td>
<td>N = 4</td>
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<tr>
<td>6</td>
<td>SW</td>
<td>(2, 2)</td>
<td>(3, 1)</td>
<td>(4, 0)</td>
<td>(2, 1)</td>
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<td>S</td>
<td>N = 8</td>
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<tr>
<td>4</td>
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</table>

SUGRA SUGRA/SUSY SUGRA SUGRA/SUSY

“pure” supergravity theory whose field content is just the supergravity multiplet is possible, and matter couplings are possible wherever matter supermultiplets exist; in general one can choose arbitrary combinations of these multiplets and there are choices to be made about how they are coupled. Note that when the number of supercharges is greater than 16, no matter supermultiplets are listed; this is because the lowest-spin supermultiplet is the gravity supermultiplet. Analogously, there do not exist supergravity theories with more than 32 supercharges since the lowest-spin multiplet in this case contains fields with spin greater than 2 which are ruled out for physical reasons. In dimensions greater than 11, minimal spinors have at least 64 components, hence no supergravity theories can exist. At the bottom of the table it is indicated whether there exist only supergravity (“SUGRA”) theories or also rigid supersymmetric (“SUSY”) theories for a given number of supercharges. As previously explained, rigid supersymmetry means that a theory has a fixed (usually Minkowski) background and finite-dimensional space of supersymmetries, usually parametrised by a (covariantly) constant spinor parameter, although we will discuss a generalisation of this in the next section. The labelling of the 10- and 11-dimensional theories is due to their relationship to M-theory and various string theories. Finally, the table also contains information about dimensional reduction. Kaluza-Klein reduction (compactification on a circle) of a theory in the table produces a theory which belongs to the entry underneath it. Those which appear at the top of a column cannot be obtained by such a reduction and can be considered “primitive”.

While Table 1.1 gives us significant grounding for the possible supergravity theories, it is still not quite the full story. To begin with, there is actually no supermultiplet
containing the graviton in the (3,0), (3,1), and (4,0)-extended cases in 6 dimensions; the simplest supermultiplets contain some exotic tensor fields with complicated Lorentz transformations, hence there is strictly speaking no supergravity multiplet and no supergravity theory admitting this supersymmetry. In other cases, a choice of supergravity multiplet and some ensemble of matter multiplets does not fully specify a theory; while the possible supersymmetric actions are heavy constrained by supersymmetry, they are not entirely determined by it. Typical action-building principles such as demanding that only derivatives of up to second order appear constrain the possibilities, but there are still choices to be made. For example, in matter-coupled theories, one typically has a choice of scalar target space manifold, and the group of isometries of the target space acts as global symmetries on the simplest possible theories, but subgroups of this can be gauged by the vectors appearing in the supergravity or matter multiplets. Even the maximally supersymmetric theories, which do not admit matter couplings, are not completely unique; for example, the standard Type IIA theory obtained by dimensional reduction of the standard 11-dimensional theory admits a 1-parameter family of “massive” deformations [11]. Moreover, while the maximum number of dimensions in Lorentzian signature is 11, in signature (10,2) the minimal spinors have 32 components, and a generalisation of supergravity theory in this signature [12, 13] related to an analogue of M-theory known as F-theory [14] has been considered.

Killing spinors and superalgebras

Let us now move on from the supergravity theories themselves and discuss some aspects of their solutions. We will be particularly interested in this work in mathematical structures associated with supersymmetric solutions.

Supersymmetric backgrounds and Killing spinors

Bosonic backgrounds of supergravity theories are solutions of those theories in which all fermionic fields vanish. If a supersymmetry of the theory preserves a given background, it is said to be a supersymmetry of that background. The requirement that an infinitesimal supersymmetry of a background preserves the vanishing-fermion condition gives rise to a set of linear equations in the supersymmetry parameter which can be interpreted as equations on a spinor field on the background. If these equations have non-trivial solutions then the background is said to be supersymmetric. The requirement that the variation of the gravitino vanishes is expressed as a differential equation of the form $D \epsilon = 0$, where $\epsilon$ is the supersymmetry parameter and $D$ is a connection on the bundle of spinors. This is called the Killing spinor equation, and the spinor fields satisfying it are Killing spinors. The connection $D$ can be expressed as $D = \nabla - \beta$, where $\nabla$ is (the spin-lift of) the Levi-Civita connection, and $\beta$ is a 1-form with values in spinor endomorphisms parametrised by the bosonic fields of the background (or their field strengths). In the minimal 4-dimensional theory, after setting $\Psi = 0$ in (1.2) we find that $\delta_\epsilon \Psi = \nabla \epsilon$, so $D = \nabla, \beta = 0$, and Killing spinors are simply parallel spinors. The Killing spinor equation of 11-dimensional
supergravity is more interesting; writing (1.5) in coordinate-free notation, we find

\[ D_X \epsilon = \nabla_X \epsilon - \frac{1}{24} X \cdot F \cdot \epsilon + \frac{1}{8} F \cdot X \cdot \epsilon, \]

(1.6)

where \( X \) is a vector field, \( \epsilon \) is a section of the spinor bundle and \( \cdot \) denotes both Clifford multiplication and the Clifford action on spinors [8, 15, 16]. In this case, \( \beta_X \epsilon = \frac{1}{24} X \cdot F \cdot \epsilon - \frac{1}{8} F \cdot X \cdot \epsilon. \)

In a theory which has fermions other than the gravitino, the equations arising from the vanishing of their variations are algebraic in \( \epsilon \), in the sense that they involve only pointwise linear operations (in particular Clifford multiplication) and not differential operators. We will refer to these as algebraic constraints and say that Killing spinors satisfying them are constrained Killing spinors.

Let us note here that in our 4- and 11-dimensional examples, we only described the supersymmetry variations locally. Indeed, a complete, mathematically rigorous global formulation of supergravity theories elucidating all of the geometric structures involved is not yet known. On the other hand, in many cases a global interpretation of the bosonic sector including the Killing spinor equation, constraint equations and equations of motion for bosonic backgrounds is possible (see e.g. [17] and the references therein), so the problem of finding supersymmetric backgrounds is mathematically well-formulated. As such, the discussion of Killing spinors is written here in terms of a global geometric problem, and this is the approach we will take in much of this work.

**Amount of supersymmetry and classification**

Supersymmetric backgrounds are often organised by the “amount of supersymmetry” they possess, that is, the dimension of their space of Killing spinors. Since a Killing spinor is a parallel section with respect to a connection, on a connected background it is uniquely determined by its value at one point, thus the space of Killing spinors has dimension less than or equal to the rank of the spinor bundle. If this bound is attained, we say that the background is maximally supersymmetric, and if it has dimension strictly more than half the upper bound, it is highly supersymmetric. We note that in \( N \)-extended theories the fibre of the spinor bundle here is not the irreducible spinor representation but the one in which the supersymmetries of the theory take their values, hence in a general theory the maximum amount of supersymmetry is the same as the “number of supercharges” of the theory. This will play a major role in the approach of this work.

Classification of supersymmetric backgrounds is a subject of great interest, not least because they are closely related to the problem of classifying string theory vacua, but also because a broad variety of different mathematical techniques can be applied to their study. Different techniques are required for different supergravity theories, and how tractable the problem is varies greatly across the different cases. For the minimal theories in 4, 5 and 6 spacetime dimensions, a form of limited classification has been achieved [18–20]; rather than an exhaustive list of possible solutions, these results consist of a local description of the solutions in terms of a small number of functions satisfying some minimal set of equations which can often be described as some more classical geometric problem. In contrast, while the 10-
and 11-dimensional theories have received a great amount of attention (these are after all the low-energy limits of string theory and M-theory), they are much less well-understood. Still, a lot is known; for example, the maximally supersymmetric 10- and 11-dimensional solutions are classified [21]. In all the 32-supercharge theories \((d = 11, \text{Type IIA and IIB})\) there is “gap phenomenon”: there do not exist any solutions with exactly 31 supercharges [22–24]. In Type IIB there are also known to be no solutions with 29 or 30 supercharges [25], while up to local isometry there is a unique solution with 28 supercharges [26]; in the other two theories no such sharp result yet exists. The spaces of solutions in the Type I and heterotic cases also display gap phenomena and are generally better-understood [27, 28]. Many other particular solutions and results on the space of solutions in particular theories exist; see the fairly recent review [29] and the references therein for a more comprehensive overview of the current status of the field.

Rigid supersymmetric theories on curved backgrounds

Aside from their mathematical interest and use in probing supergravity theories, supersymmetric backgrounds can serve as a source of rigid supersymmetric theories on non-trivial backgrounds. The rigid theories mentioned earlier were Poincaré-supersymmetric theories on Minkowski spacetime which can be catalogued relatively simply by understanding the representation theory of the Poincaré superalgebra (with caveats about choices of couplings and potentials). On the other hand, if one wishes to choose a curved Lorentzian manifold and write down a supersymmetric theory on it, it is not clear where one should start since there is no obvious choice of supersymmetry algebra acting on fields on this background; indeed, it is not clear whether one should expect such a superalgebra to exist. Even if one could find a suitable superalgebra, one would still have to work to understand its representation theory. An anti-de Sitter superalgebra can be constructed whose even part is the isometry algebra of anti-de Sitter space (\(\text{AdS}\)) [30] which allows one to study supersymmetric theories on \(\text{AdS}\), but this approach does not obviously generalise.\(^3\)

On the other hand, supersymmetric backgrounds come equipped with their own Killing superalgebras (see below) of which both the Poincaré and \(\text{AdS}\) superalgebras are the most well-known examples. Indeed, Festuccia and Seiberg [31] showed that by essentially freezing out the gravitational degrees of freedom (graviton and gravitino) in a supergravity theory, one can obtain a rigid supersymmetric theory whose degrees of freedom are perturbations of the remaining fields around a supersymmetric background, and the supersymmetry transformation parameters are the Killing spinors.

As we will discuss below, one can view the work described in this thesis as providing an alternative way of obtaining superalgebras associated with curved backgrounds and their spinors, which may provide an alternative source of rigid supersymmetric field theories.

\(^3\)For some choices of signature and bilinears on spinors, de Sitter superalgebras are also possible, but for typical cases in physics only anti-de Sitter is allowed [30].
Killing superalgebras and homogeneity

The Killing spinors of a supergravity background, along with the Killing vectors which preserve all of the bosonic fields, form a Lie superalgebra known as the Killing superalgebra which can be thought of as the supersymmetry algebra of the background itself. As with supersymmetries of the theory, the closure of a Killing superalgebra and its Jacobi identities are generically not trivial in the sense that they do not follow directly from the Killing spinor equation alone; one usually needs to use at least some of the bosonic equations of motion and Bianchi identities of the gauge field strengths of the supergravity theory.

As mentioned above, Killing superalgebras find use as the supersymmetry algebras of rigid supersymmetric theories on the backgrounds they are associated with, but they also find application in the classification problem and are especially useful in the highly supersymmetric case due to the Homogeneity Theorem (see Theorem 3.21). Originally conjectured (and proven in a much weaker form) in [32] and fully proven by José Figueroa-O’Farrill and Noel Hustler in [33–35], this theorem tells us that in the highly supersymmetric case the Dirac currents of Killing spinors span the tangent space at every point on the background. These currents are Killing vector fields which lie in the Killing superalgebra, so this implies in particular that the background is locally a homogeneous Lorentzian space for the even part of its Killing superalgebra, whence up to local isometry it can be reconstructed from its Killing superalgebra, at least in principle.

While the Homogeneity Theorem is a special feature of Lorentzian signature\(^4\) it should not be entirely surprising that the existence of Killing spinors, especially large numbers of them, strongly constrains the geometry. In general, the existence of parallel sections with respect to some connection imposes integrability condition which can be understood as constraints on the curvature of the connection, thus on the geometry, because the curvature must annihilate the parallel sections. As an example, this is discussed explicitly for Killing spinors in 11-dimensional supergravity in [8, Appx.B], and we will discuss a similar phenomenon for geometric Killing spinors below.

Geometric Killing spinors

In the mathematical literature, the term “Killing spinor” is often used for a similar but distinct concept for which we will use the term “geometric Killing spinor” in order to avoid ambiguity. We now take a slight detour to discuss these, since parts of our approach will be inspired by techniques in the geometric Killing spinor literature, and we will also encounter some examples of Killing superalgebras which happen to arise from them. See [36] and [37] for an overview of the points discussed here.

Let \((M, g)\) be a pseudo-Riemannian spin manifold and \(\nabla\) the spin-lift of the Levi-Civita connection. If \(\epsilon\) is a non-zero section of the spinor bundle and \(\lambda\) a constant

\(^4\)The proof relies on the Dirac current being symmetric in the spinor arguments and causal (meaning that “square” of a spinor is either null or timelike), as well as an analysis of subspaces of the (Lorentzian) tangent space. One can construct such a Dirac current map in Lorentzian signature in any dimension. See Hustler’s PhD thesis [33] and the references therein for details.
such that

\[ \nabla_X \epsilon = \frac{\lambda}{n} X \cdot \epsilon \quad (1.7) \]

for all \( X \in \mathfrak{X}(M) \) then \( \epsilon \) is called a geometric Killing spinor with Killing number \( \lambda \). In this context, one typically works with complexified spinors, and a priori \( \lambda \) may be any complex number. However, one can show using the Lichnerowicz formula \([38]\) that an integrability condition for this equation to have non-trivial solutions is that

\[ \lambda^2 = \frac{n}{4(n-1)} R \quad (1.8) \]

where \( R \) is the Ricci scalar, hence the Ricci scalar must be constant and \( \lambda \) either real or pure imaginary since \( \lambda^2 \) is real. Standard spheres in all dimensions are examples of geometries admitting geometric Killing spinors with real Killing number.

Geometric Killing spinors arise naturally from the study of Dirac operators on compact Riemannian spin manifolds with scalar curvature; if \( \lambda \) is an eigenvalue of such an operator on then it satisfies Friedrich's inequality \([39]\) \( \lambda^2 \geq \frac{n}{4(n-1)} R_0 \), where \( R_0 = \min_M R \) is the minimum value of the scalar curvature. It is not difficult to see that a geometric Killing spinor with Killing number \( \lambda \) is an eigenfunction of the Dirac operator with eigenvalue \( \lambda \) saturating Friedrich's inequality, and the converse is also true.

We see from (1.8) that the existence of a Killing spinor constrains the geometry, and that all Killing spinors must have the same Killing number (up to a sign). The result is strengthened in the positive-definite case: a Riemannian spin manifold \((M,g)\) admitting a Killing spinor is Einstein – the Ricci curvature \( \text{Ric} = \frac{1}{\dim M} R g \) – and furthermore, if \( \lambda \neq 0 \) is real then \( M \) is compact and if \( \lambda \neq 0 \) is pure imaginary then \( M \) is non-compact. The situation in indefinite signature is more complicated; for example, \((M,g)\) need not be Einstein. Nonetheless, the warped product techniques generalise, although the geometries in higher and lower dimension need only admit Killing (not parallel) spinors. See \([40]\) and the references therein for a comparison of classification results across signatures.

One can consider a generalisation of geometric Killing spinors to allow for non-constant Killing function instead of a Killing number. Equation (1.8) no longer holds in this scenario, so the Killing function is not constrained to take only real or pure imaginary values. However, in the Riemannian case, for \( \dim M \geq 3 \) another integrability condition enforces this constraint and moreover, for \( \dim M \geq 2 \), a real-valued Killing function must be constant, while a pure imaginary-valued Killing function is identically 0 (hence \( \epsilon \) is parallel) if the scalar curvature is non-negative \([37]\). As such, there has been relatively little interest in this generalisation, although further generalisations have appeared in the literature more recently \([41]\).

Geometric Killing spinors and their generalisations do not fit neatly into the formalism presented in this work because, unlike the Killing spinors of supergravity, they do not in general generate a Lie algebra or superalgebra. However, there are special cases in which they do – see \([42]\) for a construction of exceptional Lie algebras using Killing spinors on higher-dimensional spheres and \$3.2.4$ of this work for a simple 2-dimensional example – and in these cases, our framework applies. We also note that the notion of super-isometry of pseudo-Riemannian spin manifolds in the sense of \([43]\) also does not correspond to our notion of a supersymmetry of
a background but includes geometric Killing spinors. On the other hand, there has been recent work on studying Killing spinors of the geometric and supergravity types in a unified framework by translating their study into an algebro-geometric problem on the space of polyforms [44].

**Spencer cohomology and supersymmetry**

This work is part of a project approaching the classification problem for supersymmetric backgrounds from a new angle; namely, through the algebraic structure of their Killing superalgebras. Beyond this original motivation, the project has provided a number of new insights into the information that the cohomology of the Poincaré superalgebra contains about supergravity theories and supersymmetric geometry more generally.

**Structure of Killing superalgebras and classification of backgrounds**

The algebraic structure of Killing superalgebras was first studied systematically for maximally and highly supersymmetric backgrounds of 11-dimensional supergravity (although many of the results generalise), by José Figueroa-O’Farrill (JMF) Andrea Santi (AS) [15, 16, 45]. They key observations of this work are as follows.

1. Killing superalgebras are filtered Lie superalgebras and their associated graded objects are subalgebras of the Poincaré superalgebra; we say that the Killing superalgebra is a filtered subdeformation of the Poincaré superalgebra; and

2. By work of Cheng and Kac [46], the filtered deformations of a graded Lie superalgebra (under certain homological conditions) are determined by its Spencer cohomology, a refinement of the Chevalley-Eilenberg cohomology with values in the adjoint representation, thus Killing superalgebras can be studied via the Spencer cohomology of subalgebras of the Poincaré superalgebra.

This suggests a new approach to the classification of (at least highly supersymmetric) solutions; one can use homological methods to classify the geometrically realisable filtered subdeformations of the Poincaré superalgebra, i.e. those which can be realised as (subalgebras of) Killing superalgebras, and then reconstruct the background from this algebraic data. This is particularly well-suited to the highly supersymmetric case since, as mentioned above, the Homogeneity Theorem tells us that highly supersymmetric backgrounds are (locally) homogeneous spaces for the even part of their Killing superalgebras. One also finds that homogeneity significantly simplifies the algebraic analysis, in particular ensuring that the technical homological conditions mentioned in the second point above are satisfied.

Using Spencer cohomology and (local) homogeneity, the local classification of highly supersymmetric backgrounds in the 11-dimensional case has been reduced to the problem of classifying so-called abstract symbols up to a certain equivalence relations. These “symbols” are pairs \((S’,\phi)\), where \(S’\) is a subspace of the odd part \(s_\tau\) of the Poincaré superalgebra \(s\) with \(\text{dim } S > 16\) and \(\phi\) is essentially a Spencer cohomology class of \(s\), satisfying some algebraic conditions [45, 47]. While this has the
effect of transforming a differential-geometric problem into a possibly more tractable algebraic problem, the transformed problem is still highly non-trivial. Nonetheless, AS has demonstrated that the transformation allows for alternative constructions of known backgrounds and an alternative proof of the gap phenomenon [47]. Moreover, many more insights have been obtained via the Spencer method in different numbers of dimensions, as we shall now discuss.

**More insights from Spencer cohomology**

The work on Killing superalgebras and Spencer cohomology mentioned above along with further work by JMF, AS, Paul de Medeiros (PdM) and the author [47–50] has revealed that Spencer cohomology “knows” more about supergravity than one might expect a priori. In [45] it was shown that there is a homological characterisation the class of highly supersymmetric subdeformations of the Poincaré superalgebra in 11 dimensions (those with odd dimension $> 16$) which are geometrically realisable, meaning that they satisfy some natural homological conditions which ensure that they embed into the Killing superalgebra of a background. Moreover, these backgrounds automatically satisfy the bosonic equations of motion. In [15, 16], it was observed that the Spencer cohomology of the full Poincaré superalgebra can be used in a more naïve way to simply write down the Killing spinor equation for $D = 11$ supergravity, and that the equations of motion plus the Bianchi identity for the 4-form field strength are implied by the Clifford-trace condition $\Gamma^\mu R^D_{\mu \nu} = 0$, which itself can be understood to arise from the integrability conditions discussed in [8, Appx.B]. Thus one can, at least indirectly, recover the entire bosonic sector of $D = 11$ supergravity from Spencer cohomology alone – one does not need to know the full Lagrangian or the supersymmetry variations.

It was shown in [48] that Spencer cohomology recovers the field content and Killing spinor equation of the “old minimal off-shell” formulation of supergravity in $D = 4$ and the Killing spinor equation, and that Killing superalgebra exists even if one does not impose any further equations, hence in a sense the Killing superalgebra is also “off-shell”. However, the Clifford-trace condition does not seem to recover the supergravity equations of motion in this case. In $D = 6$, as well as recovering the supergravity field content and Killing spinor equation, the Spencer cohomology detects extra bosonic data, an $\text{sp}(1)$-valued 1-form [49]. In the same work, it was shown that the Spencer method can be extended to include the outer automorphisms of the Poincaré superalgebra (the $R$-symmetry algebra $\text{sp}(1)$) and thus potentially describe Killing superalgebras for supergravity theories with gauged $R$-symmetry. Sufficient conditions for the existence of a Killing superalgebra (which is non-trivial) were also found. In previous work with JMF [50] (a large part of which is found in the author’s MSc thesis [51]), we showed that there is also an $\text{sp}(1)$-valued 1-form in $D = 5$ which does not come from supergravity. We also showed that the Clifford-trace condition implies (when the 1-form vanishes) the minimal $D = 5$ supergravity bosonic equations of motion plus the Bianchi identity for the supergravity data (a 2-form field strength) and is sufficient but not necessary for the existence of a Killing superalgebra; for example, if the 1-form vanishes, the Bianchi identity alone suffices.

The non-supergravity Spencer data in $D = 5$ and $D = 6$ are particularly interesting when we consider the possibility of supersymmetric backgrounds which support
the Killing superalgebras corresponding to them. The maximally supersymmetric backgrounds were (locally) classified in [49, 50] and dimensional reduction of the 5-dimensional backgrounds was performed in [52], resulting in a number of novel 4-dimensional geometries in both Lorentzian and Euclidean signature. While this possibility has not been investigated so far, these geometries may admit (non-Poincaré) rigid supersymmetric field theories which are apparently not accessible through the Festuccia–Seiberg method mentioned earlier. In the present work, we will demonstrate many more examples of this phenomenon.

Apart from the work discussed above and the new work presented in this thesis which is summarised below, JMF, PdM, AS and the author all have various unreleased Spencer cohomology calculations which we hope to compile and publish in the near future.

Summary of thesis

This thesis picks up on and develops many of the threads discussed above.

Chapter 2 consists of some brief preliminaries and sets notation and terminology for (pseudo-)inner products, Clifford algebras and spinors, Poincaré superalgebras (and various extensions and generalisations), filtered deformations, Spencer cohomology and homogeneous spaces.

In Chapter 3, after introducing some standard technology for working with spinors on manifolds, we develop a general framework for the study of Killing superalgebras in arbitrary signature. In particular, we consider the conditions under which a connection on a spinor bundle gives rise to such a structure and show that these superalgebras are filtered subdeformations of the Poincaré superalgebra. The approach is independent of dimension, signature, and other choices, in particular also covering the case of Killing algebras, in which the bracket of two spinor fields is skew-symmetric (examples of which appear in [42]). We then go on to examine such deformations using Spencer cohomology before showing, following [45], that a certain class of filtered subdeformations can be realised as (subalgebras of) Killing superalgebras on homogeneous spaces. Some simple 2-dimensional examples are discussed in detail.

Chapter 4 generalises the treatment of Chapter 3 to include Killing superalgebras with $R$-symmetry using the notion of generalised spin or spin-$G$ structures [53]. This puts some of the ideas of [49] on a more firm geometric footing and shows how those ideas can be generalised. We make note of the additional difficulties and possibilities opened up here and suggest areas where the formalism could still be improved.

Chapters 5-7 are dedicated to explicit calculations applying the theory developed in the earlier chapters to cases of physical interest. In Chapter 5, we generalise the Spencer cohomology calculations in [48, 51], computing the relevant cohomology group for the $N$-extended and $R$-symmetry extended Poincaré superalgebras in $D = 5$ and $D = 6$. In Chapter 6, we perform the Spencer cohomology calculation for the Type IIA ($D = 10, N = (1,1)$) Poincaré superalgebra, showing that it is trivial, consistent with the non-existence of non-trivial maximally supersymmetric Type IIA backgrounds [21]. In Chapter 7, we take the minimal $R$-symmetry extended $D = 5$ case from Chapter 5 and perform similar analysis of the maximally supersymmetric
geometries and their Killing superalgebras to those found in [48–50], finding some
novel features including obstructions to deformations on the algebraic side and to
the existence of Killing superalgebras on the geometric side as well as a very rich
space of maximally supersymmetric solutions.

Chapter 8 comprises a summary of the results of the thesis including tables, some
further discussion placing it in the context of the literature, and an outlook for future
work.

Most of the work in Chapter 3 onwards is new and at the time of writing has
not been published; except where noted, all propositions and theorems are either
completely novel or non-trivial generalisations of existing results in the literature,
and the explicit calculations in §3.2.4, §3.3.6 and Chapters 5-7 are all new.
Chapter 2

Preliminaries

2.1 Clifford algebras, spin and spinors

We will first give a basic outline of a number of structures associated to inner product spaces, including the (special) orthogonal group and algebra, Clifford algebra, spin group and spinor an pinor representations. We will not give a complete treatment of any of these; the structure and representation theory of the Clifford algebra and spin group in particular is a beautiful but rather technical story which we do not have space to do justice to here but which has received very comprehensive treatments elsewhere. A couple of classic references on general theory are the books of Lawson–Michelson [54] and Harvey [55]. The textbook on supergravity of Freedman and van Proeyen [2] gives a physics application-oriented treatment in Lorentzian signature. JMF also has an excellent set of introductory lecture notes on the general theory [56] and a second set of notes on the formalism of Majorana spinors [57] which we will borrow from heavily in places. We provide only the details necessary here to set notation, conventions and terminology for the present work. Some additional details for particular cases will be provided where required for calculations in later chapters.

2.1.1 Inner product spaces and Clifford algebras

Inner product spaces

Let $V$ be a vector space of dimension $n$ with (pseudo-)inner product $\eta$ of signature $(p, q)$, where $n = p + q$. In some places, we will be interested in particular in the Euclidean case with signature $(n, 0)$ or the Lorentzian\(^1\) case with signature $(n - 1, 1)$ or $(1, n - 1)$. The inner product induces the musical isomorphism $\flat : V \to V^*$ sending $v \mapsto v^\flat$, defined by

$$v^\flat(w) = \eta(v, w)$$

\(^1\)We remain agnostic about whether the signature is “mostly plus” or “mostly minus” for now, making the convenient choice wherever we must specify
for all \( v, w \in V \). The inverse isomorphism \( \varepsilon : V^* \rightarrow V \) sends \( \alpha \mapsto \alpha^\varepsilon \). We will often work in an orthonormal basis \( \{e_\mu\}_{\mu=1}^n \) for \( V \) relative to which, as a matrix,

\[
[\eta_{\mu\nu}] := [\eta(e_\mu, e_\nu)] = \text{diag}(1, \ldots, 1, -1, \ldots, -1) .
\] (2.2)

Such a choice of basis induces a (non-canonical) isomorphism \((V, \eta) \cong \mathbb{R}^{p,q}\), where \(\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, \langle \cdot, \cdot \rangle)\) denotes the standard inner product space with signature \((p, q)\);

\[
\langle x, x' \rangle = x_1 x'_1 + x_2 x'_2 + \ldots + x_p x'_p - x_{p+1} x'_{p+1} - \cdots - x_{p+q} x'_{p+q}
\] (2.3)

for \( x, x' \in \mathbb{R}^{p+q} \). The isomorphism is given by \( e_\mu \mapsto e_\mu^\ast \), where the latter is the standard basis element with \( \mu \)th entry 1 and all other entries 0.

The inner product \( \eta \) induces an inner product on \( V^* \) via the musical isomorphisms which is often denoted (especially in physics) by \( \eta^{-1} \) and is defined by

\[
\eta^{-1}(\alpha, \beta) = \eta(\alpha^\varepsilon, \beta^\ast)
\] (2.4)

for \( \alpha, \beta \in V^* \) whence we also have \( \eta(v, w) = \eta^{-1}(v^\ast, w^\varepsilon) \) for all \( v, w \in V \); thus by construction \( \varepsilon \) and \( \ast \) are isomorphisms of inner product spaces. Note that

\[
\alpha(\beta^\ast) = \eta^{-1}(\alpha, \beta).
\] (2.5)

Fixing a (not necessarily orthonormal) basis \( \{e_\mu\}_{\mu=1}^n \) for \( V \) and denoting its canonical dual basis by \( \{\theta^\mu\}_{\mu=1}^n \) so that \( \theta^\mu(e_\mu) = \delta_\mu^\nu \), the components of \( \eta^{-1} \) are denoted by \( \eta^{\mu\nu} = \eta^{-1}(\theta^\mu, \theta^\nu) \). Then

\[
\theta^\mu((\theta^\nu)^\ast) = \eta^{-1}(\theta^\mu, \theta^\nu) = \eta^{\mu\nu}
\] (2.6)

whence

\[
(\theta^\mu)^\ast = \eta^{\mu\nu} e_\nu,
\] (2.7)

and similarly

\[
(e_\mu)^\ast = \eta_{\mu\nu} \theta^\nu.
\] (2.8)

It follows that \( \eta^{\mu\nu} \eta_{\nu\rho} = \delta^\mu_\rho \), or as a matrix equation, \( [\eta_{\mu\nu}]^{-1} = [\eta^{\mu\nu}] \), justifying the notation \( \eta^{-1} \). Clearly, \( \{e_\mu\}_{\mu=1}^n \) is an orthonormal basis for \((V, \eta)\) if and only if \( \{\theta^\mu\}_{\mu=1}^n \) is an orthonormal basis for \((V^*, \eta^{-1})\), and when the bases are orthonormal we have \( \eta_{\mu\nu} = \eta^{\mu\nu} \).

We will often identify \( V \cong V^* \) using the musical isomorphisms, and we can also identify the spaces of tensors \( \otimes^p V \otimes \otimes^q V^* \cong \otimes^{p+q} V \), etc. in similar fashion. In expressions given in components with respect to the chosen basis, this isomorphism is implemented by raising and lowering indices using by \( \eta_{\mu\nu} \) and \( \eta^{\mu\nu} \) in the usual way.

When we specialise to the Lorentzian case, we follow the slightly different conventions more common in the physics literature: the index \( \mu \) runs from 0 to \((n-1)\), and for an orthonormal basis we have \( [\eta_{\mu\nu}] = \text{diag}(\pm 1, \mp 1, \mp 1 \ldots, \mp 1) \) with the choice of sign depending on the signature convention. In Lorentzian signature, we will also
make use (in Chapter 7) of Witt bases \(\{e_+, e_-, e_1, \ldots, e_{n-2}\}\), for which we have

\[
[\eta_{\mu\nu}] = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -\eta_{00} \eta_{n-2}
\end{pmatrix}.
\] (2.9)

A Witt basis can be obtained from a Lorentzian orthonormal basis by setting

\[
e_+ = \frac{1}{\sqrt{2}}(e_0 + e_{n-1}), \quad e_- = \eta_{00} \frac{1}{\sqrt{2}}(e_0 - e_{n-1}),
\] (2.10)

where we note that \(\eta_{00}\) is a sign which depends on the signature convention, and

an orthonormal basis is obtained from a Witt basis by inverting this transformation. Note that \(e_+, e_-\) are null and \(\eta_{+-} = \eta(e_+, e_-) = 1\). The canonical dual basis \(\{\theta^+, \theta^-, \theta^1, \ldots, \theta^{n-2}\}\) to a Witt basis for \((V, \eta)\) is a Witt basis for \((V^*, \eta^{-1})\).

**Orthonormal groups and algebras**

We associate to \((V, \eta)\) the *orthogonal group*

\[
O(V, \eta) = \{A \in \text{GL}(V) \mid \eta(Av, Aw) = \eta(v, w) \quad \forall v, w\}
\] (2.11)

that is, the group of linear automorphisms of \(V\) which preserve \(\eta\) and the *special orthogonal group*

\[
SO(V, \eta) = O(V, \eta) \cap \text{SL}(V, \eta)
\] (2.12)

that is, subgroup of \(O(V, \eta)\) consisting of elements of determinant one – a general element of \(O(V, \eta)\) has determinant \(\pm 1\). If \(\eta\) has (positive- or negative-) definite signature then \(O(V, \eta)\) has two connected components distinguished by the sign of the determinant, and \(SO(V, \eta)\) is the connected component of the identity; in indefinite signature \(O(V, \eta)\) has four connected components and \(SO(V, \eta)\) has two. In the latter case, connected components are distinguished by whether the actions of their elements preserve or reverse spatial and temporal orientations of \((V, \eta)\). We denote the connected component of \(SO(V)\) (and thus of \(O(V)\)) by \(SO_0(V)\).

The *special orthogonal Lie algebra* is the commutator Lie algebra of skew-symmetric (relative to \(\eta\)) endomorphisms of \(V\)

\[
\mathfrak{so}(V, \eta) = \{A \in \mathfrak{gl}(V) \mid \eta(Av, w) = -\eta(v, Aw) \quad \forall v, w \in V\}.
\] (2.13)

and is the Lie algebra of all of the groups mentioned above. We often suppress \(\eta\) in the notation and write \(O(V), SO(V)\) and \(\mathfrak{so}(V)\) for the spaces above where there is no ambiguity.

In the case of Lorentzian signature, \(\mathfrak{so}(V)\) is known as the *Lorentz algebra*, \(O(V)\) is the *Lorentz group*, \(SO(V, \eta)\) is the *special Lorentz group* and \(SO_0(V, \eta)\) the *special orthochronous Lorentz group* (since it preserves time orientation).

There is an isomorphism of \(\mathfrak{so}(V)\)-modules \(\mathfrak{so}(V) \cong \wedge^2 V\) given by associating a 2-vector \(\omega_A\) to each \(A \in \mathfrak{so}(V)\), defined by

\[
Av = -\iota_{\omega_A} \omega_A
\] (2.14)
for all \( v \in V \). The inverse map associating a skew-symmetric endomorphism \( A_\omega \) to each 2-vector \( \omega \) is similarly given by

\[
A_\omega v = -\iota_v \omega
\]

for all \( v \in V \). Relative to the orthonormal basis for \( V \), we have

\[
\omega_A = \frac{1}{2} A^{\mu \nu} e_\mu \wedge e_\nu \quad \text{where} \quad A e_\mu = e_\nu A^\nu_\mu.
\] (2.16)

**Clifford algebras**

We define the *Clifford algebra* \( \text{Cl}(V, \eta) \) (where again we normally suppress \( \eta \)) as the quotient of the tensor algebra \( \bigotimes^* V = \bigoplus_{k=0}^{\infty} V \otimes^k \) of \( V \) by the relation

\[
v \cdot v = \pm \eta(v, v)1
\] (2.17)

where \( \cdot \) denotes multiplication; that is, \( \text{Cl}(V) \cong \bigotimes^* V / I \) where \( I \) is the ideal generated by elements of the form \( v \otimes v \mp \eta(v, v)1 \). This ideal is preserved by the action of \( \text{O}(V) \), giving \( \text{Cl}(V) \) the structure of an \( \text{O}(V) \)-module. There is a natural embedding of \( \text{O}(V) \)-modules \( V \hookrightarrow \text{Cl}(V) \) given by composing the natural maps \( V \hookrightarrow \bigotimes^* V \to \text{Cl}(V) \). This extends to an isomorphism of \( \text{O}(V) \)-modules \( \text{Cl}(V) \cong \bigwedge^* V \) which is most simply stated in terms of the orthonormal basis \( \{ e_\mu \} \) for \( V \) (although is independent of this choice): to the pure \( p \)-form

\[
e_{\mu_1} \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_p}
\] (2.18)

with pairwise distinct indices \( \mu_i \), we associate the Clifford algebra element

\[
e_{\mu_1} \cdot e_{\mu_2} \cdot \cdots \cdot e_{\mu_p}.
\] (2.19)

This allows us to identify \( \bigwedge^* V \) and \( \text{Cl}(V) \) with each other as \( \text{O}(V) \)-modules and consider them to be two different algebra structures on the same underlying space, with one using the wedge product \( \wedge \) and the other the Clifford product \( \cdot \).

The Clifford algebra possesses a natural \( \mathbb{Z}_2 \)-grading

\[
\text{Cl}(V) = \text{Cl}_0(V) \oplus \text{Cl}_1(V)
\] (2.20)
given by \( \text{Cl}_k = \bigoplus_{l=k \mod 2} \bigwedge^l V \) (that is, we take the \( \mathbb{Z} \)-grading of \( \bigwedge^* V \) modulo 2), or equivalently as the eigen-decomposition with respect to the canonical grading automorphism \( a \mapsto \bar{a} \) of \( \text{Cl}(V) \) induced by the automorphism \( v \mapsto \bar{v} = -v \) on \( V \).

There is a Lie algebra embedding \( \frak{so}(V) \hookrightarrow \text{Cl}(V) \), where the Lie bracket on \( \text{Cl}(V) \) is the commutator with respect to Clifford multiplication. This is given by \( A \mapsto \pm \frac{1}{2} \omega_A \), where \( \pm \) is the same as the sign appearing in (2.17). Thus any representation of the Clifford algebra restricts to a (Lie algebra) representation of \( \frak{so}(V) \), where \( A \in \frak{so}(V) \)

\footnote{Note that there is a conventional choice of sign here. It is most common in the mathematical literature to choose \( - \), while often a \(+\) is used in physics. In order to aid comparison to other works, we use \( - \) in the theoretical discussion of Clifford algebras in this chapter, but we will often use \(+\) sign where convenient. This and other sign conventions are indicated where necessary.}

\footnote{The Clifford product does not respect this \( \mathbb{Z} \)-grading, but it does respect the induced \( \mathbb{Z}_2 \)-grading. It also respects the induced \( \mathbb{Z} \)-filtration – see §2.3.1, especially Example 1.}
Spencer cohomology and Killing superalgebras

acts as $\pm \frac{1}{2} \omega_A$. We will use $\cdot$ to denote the Clifford action of $p$-forms as well as the action of $so(V)$ on Clifford modules.

**Classification of real Clifford algebras**

One aspect of the theory of Clifford algebras that we will not be able to do full justice to is their classification; we will simply quote the main results. Since real inner product spaces are determined up to isomorphism by the choice of signature, so too are their Clifford algebras. We thus work with the standard model inner product spaces $(V, \eta) = \mathbb{R}^{p,q}$, and following the usual convention in the mathematical literature, for the purposes of the present discussion we take a $-$ sign in the relation (2.17) and define $\text{Cl}(p, q) = \text{Cl}(\mathbb{R}^{p,q})$.

We denote by $K(m)$ the algebra of $m \times m$ square matrices over the field $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Note that $\mathbb{C}(m)$ is naturally a $\mathbb{C}$-algebra and by restriction of scalars an $\mathbb{R}$-algebra, while $\mathbb{R}(m)$ and $\mathbb{H}(m)$ are only $\mathbb{R}$-algebras. For any signature, $\text{Cl}(p, q)$ is isomorphic as an $\mathbb{R}$-algebra to either $K(m)$ or to $K(m) \oplus K(m)$ for some integer $m$ and some $K$. The integer $m$ depends only on the dimension $d$ while the field $K$ and whether $\text{Cl}(p, q)$ is isomorphic to $K(m)$ or two copies thereof depends only on $p - q \mod 8$ – this is known as *Bott periodicity*. A completely analogous pattern occurs for the even subalgebra $\text{Cl}_0(p, q)$. We summarise the classification of the real Clifford algebras and their even parts in Table 2.1.

<table>
<thead>
<tr>
<th>$p - q \mod 8$</th>
<th>$\text{Cl}(p, q)$</th>
<th>$\text{Cl}_\mathbb{R}(p, q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mathbb{R}(2^{n/2})$</td>
<td>$\mathbb{R}(2^{(n-2)/2}) \oplus \mathbb{R}(2^{(n-2)/2})$</td>
</tr>
<tr>
<td>1</td>
<td>$\mathbb{C}(2^{(n-1)/2})$</td>
<td>$\mathbb{R}(2^{(n-1)/2})$</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{H}(2^{(n-2)/2})$</td>
<td>$\mathbb{C}(2^{(n-2)/2})$</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{H}(2^{(n-3)/2}) \oplus \mathbb{H}(2^{(n-3)/2})$</td>
<td>$\mathbb{H}(2^{(n-3)/2})$</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{H}(2^{(n-2)/2}) \oplus \mathbb{H}(2^{(n-2)/2})$</td>
<td>$\mathbb{H}(2^{(n-4)/2}) \oplus \mathbb{H}(2^{(n-4)/2})$</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{C}(2^{(n-1)/2})$</td>
<td>$\mathbb{H}(2^{(n-3)/2})$</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{R}(2^{n/2})$</td>
<td>$\mathbb{C}(2^{(n-2)/2})$</td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{R}(2^{(n-1)/2}) \oplus \mathbb{R}(2^{(n-1)/2})$</td>
<td>$\mathbb{R}(2^{(n-1)/2})$</td>
</tr>
</tbody>
</table>

**Complex and complexified Clifford algebras**

It will be useful to also consider the complex Clifford algebras; these are significantly less rich than their real counterparts, but they will be useful when it comes to representation theory.

Let $W$ be a finite-dimensional vector space over $\mathbb{C}$ and let $B$ be a non-degenerate $\mathbb{C}$-bilinear form on $W$. Unlike inner products over the real numbers, $B$ does not
have a signature; there is always a choice of basis of $W$ such that $B$ is the identity matrix\textsuperscript{4}. Thus any pair $(W, B)$ is isomorphic to $(\mathbb{C}^n, (-, -))$ where $n = \dim W$ and $(-, -)$ is the standard "dot product" bilinear form given by $(z_1, z_2) = z_1^T z_2$. We define the complex Clifford algebra $\text{Cl}(W, B)$ as the quotient of the complex tensor algebra of $W$ by the relation (taking the same sign convention as in (2.17), although it is less consequential here)

$$w \cdot w = \pm B(w, w)\mathbb{1}. \quad (2.21)$$

The isomorphism of $(W, B)$ to $(\mathbb{C}^n, (-, -))$ induces an isomorphism of $\text{Cl}(W, B)$ to $\text{Cl}(n, \mathbb{C}) := \text{Cl}(\mathbb{C}^n, (-, -))$. Thus the complex Clifford algebras are classified by their dimension rather than by their signature. Analogously to (but simpler than) the case of real Clifford algebras, as a $\mathbb{C}$-algebra $\text{Cl}(n, \mathbb{C})$ is isomorphic to either a complex matrix algebra or a direct sum of two copies of such an algebra, depending on whether the dimension $n$ is even or odd. The complex Clifford algebras are also $\mathbb{Z}_2$-graded in a similar way to their real counterparts, and $\text{Cl}_0(n, \mathbb{C})$ is also isomorphic to either one or two copies of a complex matrix algebra. The precise classification is given in Table 2.2.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{Cl}(n; \mathbb{C})$</th>
<th>$\text{Cl}_0(n; \mathbb{C})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2m$</td>
<td>$\mathbb{C}(2^m)$</td>
<td>$\mathbb{C}(2^{m-1}) \oplus \mathbb{C}(2^{m-1})$</td>
</tr>
<tr>
<td>$2m+1$</td>
<td>$\mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$</td>
<td>$\mathbb{C}(2^m)$</td>
</tr>
</tbody>
</table>

If $(V, \eta)$ is a real inner product space, note that we can define a non-degenerate $\mathbb{C}$-bilinear form $\eta_{\mathbb{C}}$ on the complexified space $V \otimes \mathbb{C}$ by $\eta_{\mathbb{C}}(v_1 \otimes z_1, v_2 \otimes z_2) = \eta(v_1, v_2)z_1z_2$ and that

$$\text{Cl}(V, \eta) \otimes \mathbb{C} \cong \text{Cl}(V \otimes \mathbb{C}, \eta_{\mathbb{C}}) \quad (2.22)$$

as complex algebras; in particular,

$$\text{Cl}(p, q) \otimes \mathbb{C} \cong \text{Cl}(n, \mathbb{C}). \quad (2.23)$$

Complexifying Clifford algebras in this way is often used, especially in the physics literature, to make working with their representations simpler, especially when one is not working with fixed dimension and signature. Recovering the representations of a real Clifford algebra from those of its complex counterpart requires the use of the so-called Majorana condition, which we will discuss in more detail below.

**Clifford morphisms**

Let us briefly discuss a tool which will be useful when we wish to work with an explicit representation of a (real or complex) Clifford algebra as a matrix algebra.

\textsuperscript{4}Or indeed any other diagonal matrix whose diagonal entries are each either +1 or −1.
Let \((V, \eta)\) be a real (pseudo-)inner product space. Then a linear map \(f : V \to A\), where \((A, \cdot)\) is an associative algebra, is a \textit{Clifford morphism} if
\[
\sigma(v) \cdot \sigma(v) = \pm \eta(v, v)
\] (2.24)
for all \(v \in V\) where the sign is the same one as in the relation (2.17) – note that this is equivalent to \(f(v) \cdot f(w) + f(w) \cdot f(v) = \pm 2\eta(v, w)\) for all \(v, w \in V\). Clifford morphisms satisfy the following universal property: if \(f : V \to A\) if a Clifford morphism, there exists a unique morphism of associative algebras \(\tilde{f} : \text{Cl}(V) \to A\) completing the following commutative diagram
\[
\begin{array}{ccc}
V & \xrightarrow{i} & \text{Cl}(V) \\
\downarrow{f} & & \downarrow{\tilde{f}} \\
A & \xleftarrow{\phi} & \end{array}
\] (2.25)
where \(i : V \to \text{Cl}(V)\) is the natural embedding. The image of an arbitrary element of the Clifford algebra under \(\tilde{f}\) can be deduced explicitly by using the fact that it is an associative algebra morphism, and it is well-defined precisely because \(f\) is Clifford.

We have described real Clifford maps and their properties here, but the complex case is of course entirely analogous.

Thus, when specifying an explicit morphism \(\text{Cl}(V) \to A\) for \(A = \mathbb{K}(n)\) or \(A = \mathbb{K}(n) \oplus \mathbb{K}(n)\), it suffices to define a map \(V \to A\) and show that it is Clifford.

\textbf{The pin and spin groups}

We define \textit{pin} and \textit{spin} groups of \((V, \eta)\) as the following subgroups of the group of units \(\text{Cl}(V, \eta)^\times\) of \(\text{Cl}(V, \eta)\):

\[
\begin{align*}
\text{Pin}(V, \eta) &= \{v_1 \cdot v_2 \cdots v_r \mid v_i \in V, \eta(v_i, v_i) = \pm 1\}, \quad (2.26) \\
\text{Spin}(V, \eta) &= \{v_1 \cdot v_2 \cdots v_{2r} \mid v_i \in V, \eta(v_i, v_i) = \pm 1\}. \quad (2.27)
\end{align*}
\]

Clearly the latter is a subgroup of the former; in fact

\[
\text{Spin}(V, \eta) = \text{Pin}(V, \eta) \cap \text{Cl}_0(V, \eta).
\] (2.28)

Again, we usually suppress \(\eta\) in the notation. We can define an action \(\widehat{\text{Ad}}\) of \(\text{Pin}(V)\) on \(\text{Cl}(V)\) by \(\widehat{\text{Ad}}_A(b) = \hat{A} \cdot b \cdot A^{-1}\) for \(A \in \text{Pin}(V)\) and \(b \in \text{Cl}(V)\); one can show that this action preserves \(V \subseteq \text{Cl}(V)\) and that for \(v \in V\) with \(\eta(v, v) = \pm 1\), the action of \(\widehat{\text{Ad}}_v\) on \(V\) is the reflection in the perpendicular hyperplane to \(v\). It follows that the image of \(\widehat{\text{Ad}}\) in \(\text{GL}(V)\) is contained in \(\text{O}(V)\), and since the latter group is generated by reflections, \(\widehat{\text{Ad}} : \text{Pin}(V) \to \text{O}(V)\) is surjective. Moreover, we see that the action of \(\text{Spin}(V)\) is by \textit{even} products of reflections, whence restricting to this group gives us a surjection \(\text{Spin}(V) \to \text{SO}(V)\). Similarly to the (special) orthogonal groups, in definite signature \(\text{Pin}(V)\) has two connected components and \(\text{Spin}(V)\) has one, while in indefinite signature \(\text{Pin}(V)\) has four components and \(\text{Spin}(V)\) two. Denoting by \(\text{Spin}_0(V)\) the connected component of the identity of the spin group, we find that \(\text{Ad}\) restricts once again to \(\text{Spin}_0(V) \to \text{SO}_0(V)\).
Now note that for any $v \in V$ with $\eta(v, v) = \pm 1$, the element
\[
\pm 1 = v \cdot v \in \text{Spin}(V)
\] (2.29)
(which actually lies in the connected component) acts trivially under $\widetilde{\text{Ad}}$; in fact, we have
\[
Z_2 = \{-1, 1\} = \ker \widetilde{\text{Ad}} \subseteq \text{Spin}_0(V) \subseteq \text{Spin}(V) \subseteq \text{Pin}(V),
\] (2.30)
so there are canonical short exact sequences of Lie groups
\[
\begin{align*}
1 & \longrightarrow Z_2 \longrightarrow \text{Pin}(V) \longrightarrow \text{O}(V) \longrightarrow 1, \quad (2.31) \\
1 & \longrightarrow Z_2 \longrightarrow \text{Spin}(V) \longrightarrow \text{SO}(V) \longrightarrow 1, \\
1 & \longrightarrow Z_2 \longrightarrow \text{Spin}_0(V) \longrightarrow \text{SO}_0(V) \longrightarrow 1.
\end{align*}
\]
In particular, each of the surjective maps above is a double covering of Lie groups. In
{\text{definite or Lorentzian signature}, one can show that, for $n = \dim V > 2$, the connected
spin group $\text{Spin}_0(V)$ is simply connected, so the last covering map is in fact the
universal cover. This is not the case in other signatures, however.

Remark 1. We note that for any $(p, q)$, there are isomorphisms
\[
\text{Cl}_{\mathbb{R}}(p, q) \cong \text{Cl}_{\mathbb{R}}(q, p) \quad \text{and} \quad \text{Spin}(p, q) \cong \text{Spin}(q, p),
\] (2.32)
but the Clifford algebras and pin groups are not isomorphic. Thus, if we fix an
inner product space (or signature) and we are only interested in, for example, the
representation theory of the spin group rather than that of the pin group or the
Clifford algebra, we could choose to work with either $\text{Cl}(p, q)$ or $\text{Cl}(q, p)$. This is useful
since it can, for example, give us pinor representations (see §2.1.3) and (symplectic)
Majorana conditions (see §2.1.4) which are more convenient to work with.

2.1.2 Interlude on representation theory

Let us briefly take a detour from our discussion of Clifford algebras and (s)pin groups
to collect some facts about representation theory which we will need for the next part
of the story. Much of this material is standard and can be found in many textbooks,
but discussions oriented towards the representation theory of the Clifford algebras
can be found in [56, 57].

Matrix algebras and their representations

Every (real and complex) Clifford algebra is of the form $\mathbb{K}(m)$ or $\mathbb{K}(m) \oplus \mathbb{K}(m)$. The
representation theory of these algebras is very simple. The matrix algebra $\mathbb{K}(m)$
has an irreducible representation given by the natural action on $\mathbb{K}^m$ – we call this the $\text{fundamental}$ – and $\mathbb{K}(m) \oplus \mathbb{K}(m)$ has two irreducible representations given by
projecting to either the first or second factor and then acting by the fundamental – we will call these the left- and right-fundamental respectively.

For \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{H} \) then the fundamental is (up to isomorphism) the unique irreducible real representation of \( \mathbb{K}(m) \), while the left- and right- fundamentals of \( \mathbb{K}(m) \neq \mathbb{K}(m) \) are not isomorphic to one another and are the unique irreducible representations. For \( \mathbb{K} = \mathbb{H} \), the same is also true when we consider representations over \( \mathbb{C} \) or \( \mathbb{H} \). Note that for quaternionic vector spaces, we take \( \mathbb{H} \) to act on the right and, when a basis is chosen, matrices act on the left.

The situation for \( \mathbb{K} = \mathbb{C} \) depends on how we view it. Considered as an \( \mathbb{R} \)-algebra, \( \mathbb{C}(m) \) has two inequivalent \( \mathbb{C} \)-representations: the fundamental and the conjugate representation where \( A \in \mathbb{C}(m) \) acts as its conjugate \( \overline{A} \) on \( \mathbb{C}^m \). However, as \( \mathbb{R} \)-representations, these two representations are isomorphic, the isomorphism being conjugation on \( \mathbb{C}^m \). Finally, considered as a \( \mathbb{C} \)-algebra, the fundamental is the unique representation of \( \mathbb{C}(m) \); the conjugate is not a \( \mathbb{C} \)-algebra representation.

We will always consider \( \text{Cl}(V) \) and \( \text{Cl}_q(V) \) as real algebras and \( \text{Cl}(V, \mathbb{C}) \) and \( \text{Cl}_q(V, \mathbb{C}) \) as complex algebras, although when the real algebras appear as \( \mathbb{C}(m) \) it may sometimes be useful to consider their representations over \( \mathbb{C} \) and at other times over \( \mathbb{R} \), but this will be made clear.

**Reality structures**

Let \( W \) be an \( \mathbb{R} \)-module for some real group, Lie algebra or associative algebra. An invariant complex structure is an equivariant linear map \( J : W \to W \) such that \( J^2 = -1 \). Such a structure makes \( W \) into a \( \mathbb{C} \)-vector space where we define \( (a+bi)w = aw + bwJw \) for all \( w \in W \), \( a, b \in \mathbb{R} \); since the module action commutes with \( J \), this makes \( W \) into a \( \mathbb{C} \)-module.

An invariant quaternionic structure on \( W \) is a pair of invariant complex structures \( J, \overline{J} \) such that \( J \overline{J} = -\overline{J} J \). Such a structure induces a third invariant complex structure \( K = J \overline{J} \) on \( W \) such that \( \{J, \overline{J}, K\} \) satisfy the quaternion relations, hence the name. It also makes \( W \) into an \( \mathbb{H} \)-module with right \( \mathbb{H} \)-vector space structure given by \( w(a+bi+cj+dk) = aw - bJw - cJ\overline{w} - dKw \) for all \( w \in W \), \( a, b, c, d \in \mathbb{R} \).

Now let \( W \) be a \( \mathbb{C} \)-module for a real group, Lie algebra or associative algebra. An invariant \( \sigma \)-reality structure on \( W \) is an anti-linear map \( J : W \to W \) which commutes with the module action such that \( J^2 = \sigma \overline{1} \) for some sign \( \sigma = \pm 1 \); it is a real structure if \( \sigma = +1 \) and it is a quaternionic structure if \( \sigma = -1 \).

If \( J \) is an invariant real structure, it has eigenvalues \( \pm 1 \), the real eigenspaces \( W^\pm \) are invariant under the module action, and as \( \mathbb{R} \)-modules we have \( W = W^+ \oplus W^- \), \( W^+ \cong W^- \). As \( \mathbb{C} \)-modules, \( W \cong W^\pm \otimes \mathbb{C} \). The real submodule \( W^+ \) is called the real part of \( W \) (with respect to \( J \)) and is sometimes denoted \([W]\). If \( J \) is quaternionic, it defines a right \( \mathbb{H} \)-vector space structure on \( W \) via \( w(z_1 + jz_2) = z_1^* w + z_2^* Jw \) for \( z, w \in \mathbb{C} \).

\[ \text{We could also make } W \text{ into a left } \mathbb{H} \text{-vector space by setting } (a+bi+cj+dk)w = aw + bJw + cJ\overline{w} + dKw, \text{ but our convention for quaternionic vector spaces is for scalars to act on the right and matrices on the left. Since conjugation reverses the order of products in } \mathbb{H}, \text{ this is the correct definition.} \]

\( J(zw) = z^* J(w) \) for all \( z \in \mathbb{C} \), \( w \in W \).
We largely follow the book by Harvey [55] and use the same notation. (which is a real representation of)

A complex representation

The pin group \( \text{Pin}(\mathbb{C}) \) acts on any Clifford module by restriction; it can be shown that

Let us now consider the representation theory of some of the groups and algebras introduced above. We assume that all modules (representations) are finite-dimensional.

2.1.3 Clifford, pinor and spinor modules

Let us now consider the representation theory of some of the groups and algebras introduced above. We assume that all modules (representations) are finite-dimensional. We largely follow the book by Harvey [55] and use the same notation.

Bilinear and reality

A complex representation \( W \) of a finite group or compact or semisimple Lie group \( G \) admits an invariant \( \sigma \)-reality structure if and only if it admits a non-degenerate invariant \( \mathbb{C} \)-bilinear with symmetry \( \sigma \). The proof is somewhat involved so we will now reproduce it here (see e.g. [57] for details), but we will give a brief description to make it plausible. A finite-dimensional representation of \( G \) admits a unitary structure, that is, a \( G \)-invariant Hermitian inner product \( \langle - , - \rangle \). The relationship between the \( \sigma \)-reality structure \( \mathcal{J} \) and \( \sigma \)-symmetric bilinear \( B \) is essentially given by

\[
B(w_1, w_2) = \langle \mathcal{J}(w_1), w_2 \rangle
\]

for \( w_1, w_2 \in W \). There are subtleties however; given either \( B \) or \( \mathcal{J} \), the equation above defines the other with all of the desired properties except for the symmetry of \( B \) or reality of \( \mathcal{J} \), so more work must be done. See [57] for the details.

The upshot for our purposes is that the complex (s)pinor representations carry a non-degenerate invariant \( \sigma \)-symmetric inner product whenever they carry a \( \sigma \)-reality structure.

Real pinors and spinors

For a fixed inner product space \( (V, \eta) \), we refer to any module of the real associative algebra \( \text{Cl}(V) \) as a Clifford module. By the classification of real Clifford algebras summarised in Table (2.1) and the discussion of matrix algebras above, up to isomorphism there is either a unique irreducible Clifford module or two such modules. The pin group \( \text{Pin}(V) \) acts on any Clifford module by restriction; it can be shown that
an irreducible Clifford module is also irreducible under the action of Pin(V), thus we also call such a module an \textit{(irreducible) pinor module}. If the Cl(V) \cong \mathbb{K}(m), we denote its fundamental (pinor) representation by \( P = \mathbb{K}^m \); furthermore, if \( \mathbb{K} = \mathbb{C} \) then we denote the conjugate by \( \overline{P} \), recalling that this is isomorphic to \( P \) as an \( \mathbb{R} \)-representation but not as a \( \mathbb{C} \)-representation. If Cl(V) \cong \mathbb{K}(m) \oplus \mathbb{K}(m) \) (with \( \mathbb{K} = \mathbb{R}, \mathbb{H} \)) then we denote the left-fundamental by \( P_+ = \mathbb{K}^m \) and the right-fundamental by \( P_- = \mathbb{K}^m \).

The irreducible representations of Cl\( _0 \)(V) are of course entirely analogous. They are also irreducible as representations of the spin group Spin(V) which acts on them by restriction, and we denote them by \( S, \overline{S} \) and \( S_{\pm} \) as appropriate. They will be referred to as \textit{(irreducible) spinor modules}.

Note we can also restrict the pinor representation to a representation of \( \text{Cl}_0(V) \) or of Spin(V); under this restriction, a pinor module is either irreducible, hence it is also a spinor module, or it is the direct sum of two spinor modules, which may or may not be isomorphic to one another. The relationship between the pinor and spinor modules as \( \text{Cl}_0(V) \)- or Spin(V)-modules is given in Table 2.3. Since conjugate representations are isomorphic over \( \mathbb{R} \), we see that there is a unique real spinor module \( S \) for \( p - q = 1, 2, 3, 5, 6, 7 \mod 8 \) and two such representations \( S_{\pm} \) for \( p - q = 0, 4 \mod 8 \). For \( p - q = 2, 3, 5, 6, 7 \mod 8 \), \( S \) is a restricted pinor representation; for \( p - q = 1 \), the pinor representation breaks up into two copies of \( \mathbb{S} \); and for \( p - q = 0, 4 \mod 8 \), we have \( P = S_+ \oplus S_- \) under the spin group.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\( p - q \mod 8 \) & \( P \) as rep of Spin(V) \\
\hline
0, 4 & \( P \cong S_+ \oplus S_- \) \\
1 & \( P \cong \overline{P} \cong S \oplus S \) \\
2 & \( P \cong S \oplus \overline{S} \) \\
3, 7 & \( P_\pm \cong S \) \\
5 & \( P \cong \overline{P} \cong S \) \\
6 & \( P \cong S, P \otimes \mathbb{C} = S \oplus \overline{S} \) \\
\hline
\end{tabular}
\end{table}

\footnote{Note that the distinction between the fundamental and conjugate representation is arbitrary, in that it depends on the \( \mathbb{R} \)-algebra isomorphism between Cl(V) and \( \mathbb{C}(m) \); composing any such isomorphism with conjugation gives another isomorphism with respect to which \( P \) and \( \overline{P} \) are swapped. The distinction between \( P_+ \) and \( P_- \) is likewise arbitrary.}

\footnote{Many different conflicting conventions for this nomenclature exist which can cause some confusion. Many authors use “spinor” for the irreducible modules of Cl(V) (or more often those of its complexification) and if these modules are reducible under Cl\( _0 \)(V) (resp. its complexification) then the irreducible modules of the latter are “half-spinors”. We will not use this language. For the current discussion, “spinor” and “pinor” modules will always be irreducible, but later it will be convenient for us to refer to any finite-dimensional Spin(V)-module which is a direct sum of these modules as a spinor module.}
Complex pinors and spinors

The representation theory of the complex Clifford algebra is much simpler. For \( n = 2m \) even, \( \text{Cl}(n; \mathbb{C}) \cong \mathbb{C}(2^m) \) has a unique irreducible module \( P^C \cong \mathbb{C}(2^m) \) which under the action of \( \text{Cl}_0(n; \mathbb{C}) \) splits into a sum of two inequivalent irreducible submodules \( S^C_\pm \cong \mathbb{C}(2^{m-1}) \). For \( n = 2m + 1 \) odd, there are two irreducibles \( P^C_\pm \cong \mathbb{C}(2^m) \) (the left- and right- fundamentals for \( \text{Cl}(n; \mathbb{C}) \)), each of which also serves as the unique irreducible spinor module \( S^C \) of \( \text{Cl}_0(n; \mathbb{C}) \). We refer to \( P^C \) or \( P^C_\pm \) of \( \text{Cl}(n; \mathbb{C}) \) and the irreducible modules \( S^C \) or \( S^C_\pm \) of \( \text{Cl}_0(n; \mathbb{C}) \) as the complex pinor and complex spinor modules respectively.

Note that since \( \text{Pin}(p, q) \subseteq \text{Cl}(p, q) \subseteq \text{Cl}(p, q) \otimes \mathbb{C} \cong \text{Cl}(q + p, \mathbb{C}) \), the complex pinor modules are \( \mathbb{C} \)-modules of \( \text{Cl}(p, q) \) and \( \text{Pin}(p, q) \) and similarly \( \text{Cl}_0(p, q) \) and \( \text{Spin}(p, q) \) act on both the complex pinors and spinors. Under these actions, the pinor and spinor modules

When \( \text{Cl}(V) \cong \mathbb{R}(k) \) or \( \mathbb{R}(k) \oplus \mathbb{R}(k) \), the complex pinor modules of carry real structures whose real part is the real spinor module, so they are complexifications of the real pinor modules. When \( \text{Cl}(V) \cong \mathbb{H}(k) \) or \( \mathbb{H}(k) \oplus \mathbb{H}(k) \), the complex pinors also with the real pinors and carry complex structures. When \( \text{Cl}(V) \cong \mathbb{C}(k) \), the complex pinor modules coincide with the real pinor modules as modules over \( \mathbb{C} \) and do not carry a reality structure. Analogous statements hold for the spinors. This information is summarised for each signature in Table 2.4. We remind the reader that conjugate representations are isomorphic when considered as representations over \( \mathbb{R} \).

<table>
<thead>
<tr>
<th>( p - q \mod 8 )</th>
<th>( P^C_{\pm} ) as ( \mathbb{C} )-rep of ( \text{Pin}(V) )</th>
<th>( S^C_{\pm} ) as ( \mathbb{C} )-rep of ( \text{Spin}(V) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \mathbb{R} ) ( P^C \cong P \otimes \mathbb{C} )</td>
<td>( \mathbb{R} ) ( S^C_\pm \cong S_\pm \otimes \mathbb{C} )</td>
</tr>
<tr>
<td>1</td>
<td>( - ) ( P^C_+ \cong P, P^- \cong \overline{P} )</td>
<td>( \mathbb{R} ) ( S^C_\pm \cong S_\pm \otimes \mathbb{C} )</td>
</tr>
<tr>
<td>2</td>
<td>( \mathbb{H} ) ( P^C \cong P )</td>
<td>( - ) ( S^C_\pm \cong S, S^C_\mp \cong \overline{S} )</td>
</tr>
<tr>
<td>3</td>
<td>( \mathbb{H} ) ( P^C_\pm \cong P_\pm )</td>
<td>( \mathbb{H} ) ( S^C_\pm \cong S )</td>
</tr>
<tr>
<td>4</td>
<td>( \mathbb{H} ) ( P^C \cong P )</td>
<td>( \mathbb{H} ) ( S^C_\pm \cong S_\pm )</td>
</tr>
<tr>
<td>5</td>
<td>( - ) ( P^C_\pm \cong P, P^- \cong \overline{P} )</td>
<td>( \mathbb{H} ) ( S^C_\pm \cong S )</td>
</tr>
<tr>
<td>6</td>
<td>( \mathbb{R} ) ( P^C \cong P \otimes \mathbb{C} )</td>
<td>( - ) ( S^C_\pm \cong S, S^C_\mp \cong \overline{S} )</td>
</tr>
<tr>
<td>7</td>
<td>( \mathbb{R} ) ( P^C_\pm \cong P_\pm \otimes \mathbb{C} )</td>
<td>( \mathbb{R} ) ( S^C \cong S \otimes \mathbb{C} )</td>
</tr>
</tbody>
</table>
Volume element and centres

The image of the volume element $\text{vol} \in \wedge^n V$ defined by $\text{vol} = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ in the Clifford algebra, that is $e_1 \cdot e_2 \cdots \cdot e_n$, plays a special role. It satisfies

$$\text{vol} \cdot \text{vol} = (-1)^{\frac{n(n-1)}{2} + p} \cdot \frac{1}{\frac{(p-q)(p-q+1)}{2}} = \begin{cases} 1 & \text{for } p - q = 0, 3 \mod 4, \\ -1 & \text{for } p - q = 1, 2 \mod 4 \end{cases} \quad (2.34)$$

For $n$ odd, $\text{vol}$ commutes with the image of $V$, so since that image generates $\text{Cl}(V)$, $\text{vol}$ is central; for $n$ even, it anti-commutes with the image of $V$, thus it anti-commutes elements of $\text{Cl}_n(V)$ and commutes with those of $\text{Cl}_p(V)$. Moreover, $\text{vol} \in \text{Cl}_n \mod \pm 1(V)$, and the centres of the algebras are as follows, where we use the identification of graded vector spaces $\text{Cl}(V) = \wedge^* V$:

$$Z(\text{Cl}(V)) = \begin{cases} \mathbb{R}[\{1, \text{vol}\}] = \wedge^0 V \oplus \wedge^n V & \text{for } n \text{ odd,} \\ \mathbb{R}[\{1\}] = \wedge^0 V & \text{for } n \text{ even,} \end{cases} \quad (2.35)$$

$$Z(\text{Cl}_p(V)) = \begin{cases} \mathbb{R}[\{1\}] = \wedge^0 V & \text{for } n \text{ odd,} \\ \mathbb{R}[\{1, \text{vol}\}] = \wedge^0 V \oplus \wedge^n V & \text{for } n \text{ even.} \end{cases}$$

One important use of the volume element is to distinguish between inequivalent pinor and spinor representations, and indeed we can use it remove the ambiguity in labelling which we noted in a footnote:

- For $p - q = 3 \mod 4$, $\text{Cl}(V) = \mathbb{K}(m) \oplus \mathbb{K}(m)$ for $\mathbb{K} = \mathbb{R}, \mathbb{H}$ and $\text{vol}$ acts as $\pm i$ on $\mathbb{P}_\pm$;
- For $p - q = 1 \mod 4$, $\text{Cl}(V) = \mathbb{C}(m)$ and $\text{vol}$ acts as $\pm \mathbb{1}$ on $\mathbb{P}, \overline{\mathbb{P}}$;
- For $p - q = 0 \mod 4$, $\text{Cl}_p(V) = \mathbb{K}(m) \oplus \mathbb{K}(m)$ for $\mathbb{K} = \mathbb{R}, \mathbb{H}$ and $\text{vol}$ acts as $\pm 1$ on $\mathbb{S}_\pm$, and the latter can be identified as the $\pm 1$-eigenspaces of $\text{vol}$ in $\mathbb{P}$;
- For $p - q = 2 \mod 4$, $\text{Cl}_p(V) = \mathbb{C}(m)$ and $\text{vol}$ acts as $\pm i$ on $\mathbb{S}, \overline{\mathbb{S}}$, and the latter can be identified as the $\pm i$-eigenspaces of $\text{vol}$ in $\mathbb{P}$.

Analogous statements to all of the above hold if we consider $\text{vol}$ as an element of the complex Clifford algebra $\text{Cl}(V) \otimes \mathbb{C}$, but in this case we can define the element $\omega = i^{\frac{n(n-1)}{2} + p} \text{vol}$ which always squares to $\mathbb{1}$. For $n$ even, this is central in $\text{Cl}(V) \otimes \mathbb{C}$ and its distinguishes the two pinor modules, acting as $\pm \mathbb{1}$ on $\mathbb{P}_\pm \mathbb{C}$. For $n$ odd, it is central in the even subalgebra and it distinguishes the spinor modules, acting as $\pm 1$ on $\mathbb{S}_\pm \mathbb{C}$, which can be identified as the $\pm 1$ eigenspaces of $\omega$ in $\mathbb{P}_\pm \mathbb{C}$.

Note that a change of orthonormal frame which changes orientation defines a volume element with opposite sign, which has the effect of switching the labelling of inequivalent spinor or pinor modules.

**Gamma matrix notation**

The following notation is useful in explicit calculations. Under the representation $\text{Cl}(V) \to \text{End}(\mathbb{P}_\pm) \cong \mathbb{K}(m)$, the image of the basis element $e_\mu$ is denoted $\Gamma_\mu$, and
we denote skew-symmetrised products with weight 1 of such endomorphisms as follows:

\[ \Gamma_{\mu_1 \mu_2 \cdots \mu_p} := \Gamma_{[\mu_1, \Gamma_{\mu_2} \cdots \Gamma_{\mu_p}]} \]  

(2.36)

Note that \( \Gamma_{\mu_1 \mu_2 \cdots \mu_p} \) is the image of the \( p \)-form \( e_{\mu_1} \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_p} \). A particularly important such endomorphism is the image of the volume element of \( e_1 \wedge e_2 \wedge \cdots e_n \) which we denote

\[ \Gamma_* := \Gamma_{1 \cdots n} \]  

(2.37)

Note that this differs from some conventions by a factor of \( \pm i n(n-1)/2 + 1 \). This \( \Gamma \)-matrix is also denoted \( \Gamma_{n+1} \), especially in Lorentzian signature.

We could go on to derive relations between the \( \Gamma \)-matrices for use in explicit calculations, but this is extensively covered in much of the literature and we will not need to work with this notation for arguments in general signature, so we will leave this to be covered on a case-by-case basis when we perform explicit calculations.

The action of \( A \in so(V) \) on a Clifford module is given by \( \pm 1/4 A^{\mu \nu} \Gamma_{\mu \nu} \) where the sign is the same one as in the Clifford relation (2.17).

2.1.4 Majorana conditions and bilinears

Majorana spinors are a type of real spinor very commonly used in the physics literature, along with the closely related concepts of pseudo-Majorana and symplectic Majorana spinors, but many definitions and descriptions can be found in the literature, not all of which agree. Our approach is mostly inspired by the formalism of Majorana and symplectic Majorana spinors given by JMF [57]; as in that work, we will avoid the concept of pseudo-Majorana altogether. See also the treatment in [58]. Essentially, Majorana spinors are elements of real spinor representations which are obtained from complex spinor representations using an invariant real structure.

The main reason for employing Majorana conditions is that it is much simpler to work with complex Clifford algebras and their representations, but at the end of the day we are interested in real spinors and spinors, so we need to be able to recover the “real” content from our complex calculations. For our purposes, it will be necessary to describe, for example all real endomorphisms \( \text{End}_\mathbb{R} \mathbb{S}_{(\pm)} \) of spinor representations. When \( \text{Cl}(V) \) or \( \text{Cl}_0(V) \) is real, these can be described using \( \Gamma \)-matrices. In other cases, \( \Gamma \) matrices only give us real endomorphisms preserving a complex or quaternionic structure. On the other hand, \( C \)-endomorphisms of complex spinors and spinors can always be described using \( \Gamma \)-matrices, and we use Majorana conditions to obtain a description of \( \text{End}_\mathbb{R} \mathbb{S}_{(\pm)} \) in terms of \( \Gamma \)-matrices and the action of some additional algebra.

**Majorana and symplectic Majorana conditions**

Let \( \mathbb{S}^C \) be the irreducible complex pinor or spinor representation admitting a \( \text{Pin}(V) \) or \( \text{Spin}(V) \)-invariant real structure \( \mathcal{J} \) respectively. A (s)pinor \( \epsilon \in \mathbb{S}^C \) satisfying \( \mathcal{J}(\epsilon) = \epsilon \) is called a **Majorana (s)pinor**, and this equation is called the **Majorana condition**. The space of (s)pinors satisfying the Majorana condition is the real part

\[ \mathbb{S}^C_{(\pm)} \]
Spencer cohomology and Killing superalgebras

$[S^C]$ of $S^C$ and is called the \textit{Majorana (s)pinor representation}. It is of course a real representation.

If $S^C$ instead carries an invariant quaternionic structure, we cannot impose the condition $J(\epsilon) = \epsilon$, since this has only trivial solutions. Instead, we recall from §2.1.2 that the tensor product of two quaternionic structures is a real structure and let $\Delta$ be $H$ considered as a $C$-module of some real compact subgroup Sp(1) $\cong$ USp(2) with invariant quaternionic structure $J_\Delta$. Then the tensor product $S^C \otimes C \Delta$ carries a Spin$(V) \times G$-invariant real structure $J_\otimes = J \otimes J_\Delta$. We call $\epsilon \in S^C \otimes C \Delta$ a \textit{symplectic Majorana (s)pinor} if it satisfies the \textit{symplectic Majorana condition} $J_\otimes(\epsilon) = \epsilon$, and the \textit{symplectic Majorana representation} is the real subspace of such spinors, $[S^C \otimes C \Delta]$.

In Table 2.4, we expressed complex (s)pinor modules in terms of their real counterparts. Table 2.5 essentially does the inverse, expressing the real (s)pinor modules as (symplectic) Majorana subspaces. Note that since there is no Pin$(V)$-invariant reality structure on $P^C_\pm$ for $p - q = 1, 5 \mod 8$, we cannot make $P$ into a (symplectic) Majorana representation as described above, although if we wished to, we could consider reality structures on $P^C_\pm \oplus P^C_-$. In both of these cases, however, we have $P \cong S$ or $P \cong S$ as real Spin$(V)$-representations, so can have a Majorana \textit{spinor} condition on $P$. For $p - q = 2, 6 \mod 8$, there is no reality structure on $S^C_\pm$, but in both cases we can instead use reality conditions on $P^C$ (considered as a $C$-module of Spin$(V)$) to obtain $S$, as indicated. Another choice in the latter cases is to abandon the Majorana formalism and simply use $S^C_\pm$ as the real spinors, since as a $R$-module it is isomorphic to $S$, but this will not be useful for our purposes since it does not give a useful description of the real endomorphisms of $S$.

Table 2.5: Real pinor and spinor modules as (symplectic) Majorana subspaces of complex pinor and spinor modules. Reality structures on complex modules are also indicated.

<table>
<thead>
<tr>
<th>$p - q \mod 8$</th>
<th>$P_{(\pm)}$ as Majorana Reality of $P^C_{(\pm)}$</th>
<th>Isomorphisms</th>
<th>$S_{(\pm)}$ as Majorana Reality of $S^C_{(\pm)}$</th>
<th>Isomorphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mathbb{R}$</td>
<td>$P \cong [P^C]$</td>
<td>$\mathbb{R}$</td>
<td>$S \cong [S^C]$</td>
</tr>
<tr>
<td>1</td>
<td>$-$</td>
<td>$-$</td>
<td>$\mathbb{R}$</td>
<td>$S \cong [S^C]$</td>
</tr>
<tr>
<td>2</td>
<td>$H$</td>
<td>$P \cong [P^C \otimes C \Delta]$</td>
<td>$-$</td>
<td>$S \cong [P^C \otimes C \Delta]$</td>
</tr>
<tr>
<td>3</td>
<td>$H$</td>
<td>$P_{\pm} \cong [P^C_\pm \otimes C \Delta]$</td>
<td>$H$</td>
<td>$S_{\pm} \cong [S^C_\pm \otimes C \Delta]$</td>
</tr>
<tr>
<td>4</td>
<td>$H$</td>
<td>$P \cong [P^C \otimes C \Delta]$</td>
<td>$H$</td>
<td>$S \cong [S^C \otimes C \Delta]$</td>
</tr>
<tr>
<td>5</td>
<td>$-$</td>
<td>$-$</td>
<td>$H$</td>
<td>$S \cong [S^C \otimes C \Delta]$</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{R}$</td>
<td>$P \cong [P^C]$</td>
<td>$-$</td>
<td>$S \cong [P^C]$</td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{R}$</td>
<td>$P_{\pm} \cong [P^C_\pm]$</td>
<td>$\mathbb{R}$</td>
<td>$S \cong [S^C]$</td>
</tr>
</tbody>
</table>

\textbf{Dirac, Majorana and Weyl Spinors}

In the physics literature, there is some alternative terminology for different types of spinors which we also briefly make note of. Below, we consider all spaces as
representations of Spin(V).

- A Dirac spinor is a complex spinor, i.e. an element of \( \mathbb{P}^C_{(\pm)} \).

- In even dimension, a Weyl spinor is an element of \( \mathbb{S}^C_\pm \). We also call such spinors chiral, with chirality \( \pm 1 \).

- For \( p - q = 0 \mod 8 \), \( \mathbb{P}^C_{(\pm)} \) carries a Spin(V)-invariant real structure and a Majorana spinor is an element of the real subspace \( [\mathbb{P}^C_{(\pm)}] \).

- For \( p - q = 0 \mod 8 \), \( \mathbb{S}^C_\pm \) carries a Spin(V)-invariant real structure, a Majorana-Weyl spinor is an element of the real subspace \( [\mathbb{S}^C_\pm] \).

- For \( p - q = 2, 3, 4, 5 \mod 8 \), \( \mathbb{P}^C_{(\pm)} \) carries a Spin(V)-invariant quaternionic structure, a symplectic Majorana spinor is an element of \( [\mathbb{P}^C_{(\pm)} \otimes \Delta] \).

- For \( p - q = 4 \mod 8 \), \( \mathbb{S}^C_\pm \) carries a Spin(V)-invariant quaternionic structure, a symplectic Majorana-Weyl spinor is an element of \( [\mathbb{S}^C_\pm \otimes \Delta] \).

We note that the term pseudo-Majorana spinor also appears in the literature and is used in various ways, which for most purposes can be absorbed into our choice of flipping signature in Remark 1. There is also disagreement about the precise meaning of “Majorana spinor”; see [59] for a disambiguation.

**Extended spinor modules**

In the sequel, it will be necessary to work with \( N \)-extended spinor modules. They will be used to construct, for example, \( N \)-extended supersymmetry algebras. Our approach here is inspired by the more extensive description in [58].

**Definition 2.1** (Extended spinor representation). Let \( p - q \neq 0, 4 \mod 8 \), and let \( \mathbb{S} \) be the unique real irreducible spinor module. The (real) \( N \)-extended spinor module is the Spin(V)-module

\[
\mathbb{S}_N = N \mathbb{S} = \mathbb{S} \otimes \mathbb{R}^N.
\] (2.38)

Let \( p - q = 0, 4 \mod 8 \), and let \( \mathbb{S}_+, \mathbb{S}_- \) be the two inequivalent irreducible real spinor modules\(^{10}\). The (real) \((N_+, N_-)\)-extended spinor module is the Spin(V)-module

\[
\mathbb{S}_{(N_+, N_-)} = N_+ \mathbb{S}_+ \oplus N_- \mathbb{S}_- = \mathbb{S}_+ \otimes \mathbb{R}^{N_+} \oplus \mathbb{S}_- \otimes \mathbb{R}^{N_-}.
\] (2.39)

We will refer to both of these as extended spinor representations.

We can obtain the spaces defined above through generalised Majorana and symplectic Majorana conditions on extended complex spinors, as we will now describe. In doing so, our description of the Majorana and symplectic Majorana cases will be unified. Once again, our goal is to use this relationship to obtain a better description of real endomorphisms of the real spinors.

In each case, we will tensor complex spinor or pinor modules with an auxiliary module of which we define two types, each carrying an appropriate reality structure.

---

\(^{10}\)These are Majorana-Weyl for \( p - q = 0 \mod 8 \) and symplectic Majorana-Weyl for \( p - q = 4 \mod 8 \)
The first of these we denote by $\Delta^R_N$, which is $\mathbb{C}^N$ together with its canonical real structure (conjugation). The second is $\Delta^H_N = \mathbb{H}^N$ considered a $\mathbb{C}$-vector space (under scalar multiplication on the right) with quaternionic structure $J_A(q) = qJ_A$. Note that $\dim_{\mathbb{C}} \Delta^R_N = N$, while $\dim_{\mathbb{C}} \Delta^N_N = 2N$. The reason for labelling these spaces as we have will soon become clear. In Chapter 5, where we make use of the quaternionic case in calculations, we will describe $\Delta^N_N$ in more detail, including an explicit identification with $\mathbb{C}^{2N}$.

First, let the dimension $n = p + q$ be odd. In each case, $\mathbb{S}^C$ carries either an invariant real structure or quaternionic structure. For the real case, the tensor product of real structures is a real structure on $\mathbb{S}^C \otimes_{\mathbb{C}} \Delta^R_N$, and one can show that $[\mathbb{S}^C \otimes_{\mathbb{C}} \Delta^R_N] \cong \mathbb{S}^N$ as a real Spin($V$)-module. In the quaternionic case, we similarly take a tensor product of quaternionic structures to get a real structure and find $[\mathbb{S}^C \otimes_{\mathbb{C}} \Delta^H_N] \cong \mathbb{S}^N$.

For $n = p + q$ even, it is worth considering different signatures separately. For $p - q = 0 \mod 8$, $\mathbb{S}^C$ has invariant real structure, thus so does $\mathbb{S}^C \otimes_{\mathbb{C}} \Delta^R_N$, and we have $[\mathbb{S}^C \otimes_{\mathbb{C}} \Delta^R_N] \cong \mathbb{S}^N$. Similar reasoning goes for the quaternionic case $p - q = 4 \mod 8$, and we have $[\mathbb{S}^C \otimes_{\mathbb{C}} \Delta^H_N] \cong \mathbb{S}^N$. In both cases, the sum of invariant real structures on the summands gives $[(\mathbb{S}^C \otimes_{\mathbb{C}} \Delta^R_N) \oplus (\mathbb{S}^C \otimes_{\mathbb{C}} \Delta^H_N)] \cong \mathbb{S}^N$. For $p - q = 2, 6 \mod 8$, we recall that $\mathbb{S}^C$ carries no invariant reality structure but $\mathbb{P}^C$ has quaternionic structure for $p - q = 2 \mod 8$ and real structure for $p - q = 6 \mod 8$; we then have, as appropriate $[\mathbb{P}^C \otimes_{\mathbb{C}} \Delta^H_N] \cong \mathbb{P} \otimes \mathbb{R}^N$ or $[\mathbb{P}^C \otimes_{\mathbb{C}} \Delta^R_N] \cong \mathbb{P} \otimes \mathbb{R}^N$ as pinor representations, and these real subspaces are $\mathbb{S}^N$ as pinor representations in both cases since $\mathbb{P} \equiv \mathbb{S}$.

**Spinor bilinears**

Recall from 2.1.2 that a complex module admits an invariant reality structure if and only if it admits a non-degenerate, invariant bilinear with symmetry $\sigma$, so on general principles we expect many of our complex (s)pinor representations admit such bilinears. We will discuss a slightly more general class of forms on these representations which are only $\mathfrak{so}(V)$-invariant in general, and satisfy some different properties with respect to the Clifford action.

A complex pinor module $\mathbb{P}^C_{(\pm)}$ always admits a non-degenerate *sesquilinear* form $\langle -, - \rangle$ satisfying[57, 58]

$$\langle \epsilon, \epsilon' \rangle = \sigma_A \langle \epsilon', \epsilon \rangle^* \quad \langle \nu \cdot \epsilon, \epsilon' \rangle = \sigma_B \langle \nu \cdot \epsilon', \epsilon \rangle^* \quad (2.40)$$

for all $\epsilon, \epsilon' \in \mathbb{P}^C_{(\pm)}$ and $\nu \in V$ where $\sigma_A, \tau_A$ are some signs called the *symmetry* and *type* of the $\langle -, - \rangle$ respectively, and also a non-degenerate $\mathbb{C}$-*bilinear* form satisfying

$$C(\epsilon, \epsilon') = \sigma_C C(\epsilon', \epsilon), \quad C(\nu \cdot \epsilon, \epsilon') = \tau_C C(\epsilon, \nu \cdot \epsilon'), \quad (2.41)$$

for all $\epsilon, \epsilon' \in \mathbb{P}^C_{(\pm)}$ and $\nu \in V$, where again we have two signs $\sigma_C, \tau_C$ called the symmetry and type of $C$. It follows from the second property of each form that both $\langle -, - \rangle$ and $C$ are $\mathfrak{so}(V)$-invariant regardless of the signs $\tau_A, \tau_C$, since $\mathfrak{so}(V)$ acts as $\wedge^2 V \subseteq \text{Cl}(V)$.

We also define the symmetry and type of an $\mathfrak{so}(V)$-invariant $\mathbb{R}$-bilinear of a real pinor module in a similar way. Such a bilinear always exists (some combination of type and symmetry), and we will demonstrate this using the structures on the
complex module.

We have already seen that $\mathbb{P}_C^{(\pm)}$ also admits a reality structure satisfying

$$J^2 = \sigma_B J,$$
$$J(v \cdot \epsilon) = \tau_B v \cdot J,$$  \hspace{1cm} (2.42)

where again $\sigma_B, \tau_B$ are signs, and the latter property ensures that $J$ is $\mathfrak{so}(V)$-equivariant.

In fact, $J$ can be chosen such that for all $\epsilon, \epsilon' \in \mathbb{S}$,

$$\mathbb{C}(\epsilon, \epsilon') = \langle J(\epsilon), \epsilon' \rangle.$$  \hspace{1cm} (2.43)

With this choice, we must have $\tau_B = \tau_A \tau_C$.

If $J$ is a real structure ($\sigma_B = +1$) then for $\epsilon, \epsilon' \in \mathbb{P}_C^{(\pm)}$ we have $\mathbb{C}(\epsilon, \epsilon') = \langle \epsilon, \epsilon' \rangle$, and this expression is either real or pure imaginary depending on the signs $\sigma_A, \sigma_C$, and in the latter case, if we redefine $C \mapsto \pm i$ (so that the above relation also changes by a relative factor of $i$) then we can take this restriction of $C$ to be real without loss of generality, giving us a non-degenerate, $\mathfrak{so}(V)$-invariant real bilinear on the Majorana submodule satisfying (2.41); its type and symmetry are the same as $C$.

If $J$ is quaternionic ($\sigma_B = -1$) then we can do something similar after tensoring with an auxiliary module $\Delta \simeq \mathbb{H} \simeq \mathbb{C}^2$ or more generally $\Delta_N^H \simeq \mathbb{H} \simeq \mathbb{C}^{2N}$. For now treating the auxiliary module as $\mathbb{H}$, we let $H(q, q') = q^* q'$ denote the standard $\mathbb{H}$-Hermitian inner product and define $\mathbb{C}$-bilinears $h, b$ on $\Delta_N^H$ by the equation

$$H(q, q') = h(q, q') + j b(q, q').$$  \hspace{1cm} (2.44)

Then one can show that $h$ is a $\mathbb{C}$-Hermitian inner product and $b$ is a ($\mathbb{C}$-bilinear) symplectic form, and that these satisfy

$$b(q, q') = h(q, q').$$  \hspace{1cm} (2.45)

Noting that right-multiplication by $j$ is the standard Hermitian structure $J_\Delta$ on $\Delta_N^H$, we see that this of the same form as (2.43). Then we can define a non-degenerate sesquilinear form $\langle -, - \rangle_\otimes = \langle -, - \rangle \otimes h$, a bilinear form $C_\otimes = C \otimes b$, and a real structure $J_\otimes = J \otimes J_\Delta$ on $\mathbb{P}_C \otimes \mathbb{C} \Delta_N^H$ such that

$$C_\otimes(\epsilon, \epsilon') = \langle J_\otimes(\epsilon), \epsilon' \rangle_\otimes,$$  \hspace{1cm} (2.46)

where $\epsilon, \epsilon' \in J_\otimes = J \otimes J_\Delta$, now $C_\otimes$ restricts to a (without loss of generality) real bilinear on the symplectic Majorana submodule $\mathbb{P}_C \otimes \mathbb{C} \Delta_N^H$ with symmetry $-\sigma_C$ (since $b$ is skew-symmetric) and type $\tau_C$.

The matrices $A, B, C$

It is common when performing explicit calculations with spinors to choose a set of matrices $A, B, C$ to represent some of the structures we have discussed at a more abstract level. Let us assume that we have an explicit irreducible representation map $\text{Cl}(V) \otimes \mathbb{C} \to \mathbb{C}(m)$ so that $\mathbb{P}_C^{(\pm)} = \mathbb{C}^m$, and the $\Gamma$-matrices are literally $m \times m$ complex matrices. In fact, representations can be chosen in which the rank-1 matrices $\Gamma_\mu$ are
unitary. Then $A, B, C$ are matrices in $\mathbb{C}(m)$ such that
\begin{equation}
\langle \epsilon, \epsilon' \rangle = \epsilon^\dagger A \epsilon', \quad \mathcal{J}(\epsilon) = (Be)^*, \quad C(\epsilon, \epsilon') = \epsilon C^T \epsilon',
\end{equation}
where $\epsilon, \epsilon' \in \mathbb{P}$ and the notation on the LHS is as above. In particular, $C$ is known as the charge conjugation matrix. The invariance properties of the various structures can be translated into matrix equations:
\begin{align}
A^\dagger &= \sigma_A \tilde{A}, \\
B^* B &= \sigma_B \tilde{1}, \\
C^T &= \sigma_C \tilde{C}, \\
(\gamma_\mu)^\dagger &= \tau_A \tilde{A} \gamma_\mu A^{-1}, \\
(\gamma_\mu)^* &= \tau_B \tilde{B} \gamma_\mu B^{-1}, \\
(\gamma_\mu)^T &= \tau_C \tilde{C} \gamma_\mu C^{-1},
\end{align}
and moreover the equation $C(\epsilon, \epsilon') = \langle \mathcal{J}(\epsilon), \epsilon' \rangle \forall \epsilon, \epsilon' \in \mathbb{P}_{(\pm)}$ becomes $C = B^T A$, and since the $\Gamma$ matrices generate $\mathbb{C}(m)$ as a $\mathbb{C}$-algebra, $A, B, C$ can be expressed in terms of these matrices; in fact, they can be chosen such that they are products of the rank-1 matrices. We note that although some of the signs appearing in these expressions can be freely chosen, some of them are determined by the signature, and even when they are choices, the choices are not independent.

In this notation, the Majorana reality condition becomes $\epsilon^* = Be$, or alternatively $\epsilon^T C = \epsilon^\dagger A$ which are more commonly seen in physics, where the second equation is the statement that $C(\epsilon, \epsilon') = \langle \epsilon, \epsilon' \rangle \forall \epsilon, \epsilon' \in \mathbb{P}_{(\pm)}$. For the symplectic Majorana condition, an analogous condition can be stated, but in its most recognisable form this, requires an adapted basis on $\Delta^H_N$. We will describe this in more detail in Chapter 5 where we will apply these ideas.

This point of view very convenient for calculations and is essentially the form in which spinors were first introduced in physics, and most standard physics references, for example the textbook of Freedman and Van Proeyen [2] still treat them in this way. We will not make use of this formalism in full generality, hence we do not provide any more detail, but we will use some of it in example calculations. Our references [56, 58] provide much more detail, including many examples.

### 2.2 Flat model algebras

The Euclidean and Poincaré algebras are the isometry algebras of Euclidean and Lorentzian spaces respectively. They can be considered as “flat models” for isometry algebras in Riemannian and Lorentzian geometry respectively in the informal sense that they are the isometry algebras of the maximally symmetric flat spaces in their respective settings, but there is a more precise sense in which this is true. Pseudo-Riemannian geometry can be subsumed into the more general setting of Cartan geometry, where the objects of study are principal bundles $P \rightarrow M$ equipped with a Cartan connection which encodes the idea that the geometry locally looks a homogeneous manifold for a Lie pair $(G, H)$ known as the local model – see [60] for a pedagogical introduction emphasising links to pseudo-Riemannian geometry and application to general relativity in particular and [61] for a more systematic treatment. The Cartan connection has a curvature 2-form with respect to which the local model is flat, hence the local model can also be called the flat model. The Cartan connection takes values in the Lie algebra $\mathfrak{g} = \text{Lie } G$, and under some regularity conditions
(which obtain in pseudo-Riemannian geometry), one can associate a Lie algebra of symmetries to the Cartan geometry which, as shown by Čap and Neusser [62], is a filtered deformation\(^{11}\) of a subalgebra of the flat model algebra \(\mathfrak{g}\).

In the pseudo-Riemannian case with signature \((p, q)\), the local model is the homogeneous space \(\text{ISO}(p, q)/\text{O}(p, q)\), the principal bundle is the orthonormal frame bundle, and the Cartan connection, which takes values in \(\mathfrak{g} = \text{iso}(p, q)\), encodes both the coframe (also known as frame field, vielbein or soldering form) and a spin connection. If the spin connection is the Levi-Civita connection, the symmetry algebra is the isometry algebra (algebra of Killing vectors), and the result discussed above says that is filtered, with its associated graded being a subalgebra of \(\text{iso}(p, q)\).

Although we do use the formalism of Cartan geometry in this work, we can view the result on the structure of Killing superalgebras already mentioned in the introduction as a “super” generalisation of this phenomenon in pseudo-Riemannian geometry, and it would be subsumed by a generalisation of the Čap–Neusser result [62] to super-Cartan geometry. Our (much more explicit) version of this result is Theorem 3.10 (and its generalisation Theorem 4.18), which tells us that Killing (super)algebras are filtered subdeformations of the \(\mathbb{Z}\)-graded (super)algebras we will introduce in this section. Motivated by the connection to Cartan geometry, we will refer to these \(\mathbb{Z}\)-graded (super)algebras as flat model (super)algebras.

2.2.1 Definitions of the algebras

**Isometry algebra of an inner product space**

We denote by \(\text{iso}(V, \eta)\) the isometry algebra of (the affine space modelled on) \((V, \eta)\). As a vector space \(\text{iso}(V, \eta) = V \oplus \text{so}(V, \eta)\), where again we will usually suppress \(\eta\) in the notation, and it is equipped with the Lie bracket

\[
[A, B] = AB - BA, \quad [A, v] = Av, \quad [v, w] = 0,
\]

where \(A, B \in \text{so}(V)\) and \(v, w \in V\). In the Euclidean case this is known as the Euclidean algebra; in the Lorentzian case it is the Poincaré algebra.

We write \(\text{iso}(p, q) := \text{iso}(\mathbb{R}^{p, q})\), and if \(\eta\) has signature \((p, q)\), the isomorphism \((V, \eta) \cong \mathbb{R}^{p, q}\) induces a Lie algebra isomorphism \(\text{iso}(V) \cong \text{iso}(p, q)\).

**Isometry algebra with spinors**

We now construct a \(\mathbb{Z}\)-graded Lie algebra or Lie superalgebra \(\mathfrak{s}\) whose even part is \(\text{iso}(V)\) as follows. Let

\[
\mathfrak{s}_0 = \text{so}(V), \quad \mathfrak{s}_{-1} = S, \quad \mathfrak{s}_{-2} = V, \quad \mathfrak{s}_i = 0 \text{ otherwise},
\]

where \((V, \eta)\) has signature \((p, q)\) and \(S\) is an extended spinor representation as in Definition 2.1; that is, we take

\[
\bullet \quad S = \mathbb{S}_N = N\mathbb{S} \quad \text{for some integer } N \geq 1 \text{ for } p - q \neq 0, 4 \mod 8,
\]

\(^{11}\)Filtrations are discussed in §2.3.1 and filtered deformations in §2.3.3.
We let the bracket on \( s_0 = \text{iso}(V) \) be the one defined by (2.49) and note that this respects the \( \mathbb{Z} \)-grading. In order to extend this to a Lie (super)bracket \( s \otimes s \rightarrow s \) in a way which respects the \( \mathbb{Z} \)-grading, we must specify maps \( s_{-1} \otimes s_0 \rightarrow s_{-2} \) and \( s_{-1} \otimes s_{-1} \rightarrow s_{-2} \) — all other components must be zero. We take the first of these maps to be the natural action of \( s_0 \) on \( s \), so the brackets are given by

\[
[\epsilon, \zeta] = \kappa(\epsilon, \zeta), \quad [A, \epsilon] = A \cdot \epsilon = \pm \frac{1}{2} \omega_A \cdot \epsilon, \quad [\nu, \epsilon] = 0,
\]

for \( A \in s_0(V) \), \( \nu \in V \) and \( \epsilon, \zeta \in s_0 \), where we leave the map \( \kappa : \bigwedge^2 s \rightarrow V \) (for a Lie algebra) or \( \kappa : \bigodot^2 s \rightarrow V \) (for a Lie superalgebra) to be determined.

It remains to check the Jacobi identities involving elements of \( s_{-1} \). The \( [s_0, s_0, s_{-1}] \) identity is satisfied precisely because the \( s_0 \otimes s_{-1} \rightarrow s_{-2} \) component of the bracket is a module action. The \( [s_0, s_{-1}, s_{-1}] \) and \( [s_{-2}, s_{-1}, s_{-1}] \) identities are trivial since \( [s_{-1}, s_{-2}] = [s_{-2}, s_{-2}] = 0 \). The remaining component is \( [s_0, s_{-1}, s_{-1}] \), which is satisfied if and only if \( \kappa \) is \( s_0(V) \)-equivariant. Thus we make the following definition.

**Definition 2.2** (Flat model (super)algebra, squaring maps). Let \((V, \eta)\) be an inner product space and \( S \) an (extended) real spinor representation of \( s_0(V) \). Then

- A flat model algebra is a \( \mathbb{Z} \)-graded Lie (superalgebra) \( s \) with underlying \( \mathbb{Z} \)-graded vector space \( V \oplus s \oplus s_0(V) \) and bracket defined by (2.49) and (2.51) for some \( s_0(V) \)-equivariant map \( \kappa : \bigwedge^2 s \rightarrow V \).

- A flat model superalgebra is a \( \mathbb{Z} \)-graded Lie (superalgebra) \( s \) with underlying \( \mathbb{Z} \)-graded vector space \( V \oplus s \oplus s_0(V) \) and bracket defined by (2.49) and (2.51) for some \( s_0(V) \)-equivariant map \( \kappa : \bigodot^2 s \rightarrow V \).

In either case, \( \kappa \) is referred to as the squaring map.

Furthermore, we say that \( s \) is minimal if \( S = S_1 = S \) or \( S_{(1,0)} = S_+ \) and that it is \( N \)-extended if \( S = S_N \) for \( N > 1 \) or \( S = S_{(N_+, N_-)} \) for \( N = N_+ + N_- > 1 \); in the latter case we also say it is \((N_+, N_-)\)-extended.

Algebras of this type were classified by Alekseevsky and Cortés [63]. Since inner product spaces of equal signature are isomorphic, the data required to specify a flat model (super)algebra are thus the signature\(^{12}\) \((p, q)\), the integer \( N \) or \((N_+, N_-)\) and the squaring map \( \kappa \), so the classification reduces to classifying these maps. The name “squaring map” is used because in the more commonly studied superalgebra case they can be thought of as allowing one to “square” a vector into a spinor. We will present the classification scheme in §2.2.2. We note that all flat model (super)algebras are referred to as \( N \)-(super)-extended Poincaré algebras or \( \pm N \)-extended Poincaré algebras in [63], but we reserve the name Poincaré for Lorentzian signature.

**Supersymmetry and no-go theorems**

In Riemannian signature, it is always possible to choose a skew-symmetric \( \kappa \), making \( s \) a Lie algebra. In Lorentzian signature, a symmetric \( \kappa \) always exists, giving us a
superalgebra known as a \textit{Poincaré superalgebra}. The latter is important for physics due to a pair of “no-go” theorems which restrict the possible symmetries and supersymmetries of quantum field theories satisfying some generally reasonable physical assumptions.

The Coleman–Mandula Theorem \cite{Coleman1967} states that in a 4-dimensional quantum field theory\textsuperscript{13} admitting an \textit{S}-matrix description in which the \textit{S}-matrix has a connected symmetry group \(G\) containing a subgroup locally isomorphic to the Poincaré group \(P(V)\), we must have \(\mathfrak{g} = \text{Lie}\, \mathfrak{g} = \mathfrak{p}(V) \oplus \mathfrak{h}\) for some Lie algebra \(\mathfrak{h}\) under the following assumptions: all particles are of positive mass and there are a finite number of particle species with mass below any finite bound; amplitudes are analytic in the scattering angle; scattering occurs between any pair of particle species at almost all energies; and the Lie algebra \(\mathfrak{g}\) of \(G\) acts in a momentum space representation of states via integral operators with distributional kernels. Here, a symmetry of the \(S\)-matrix is a unitary operator on Fock space (the space of multiparticle states) which preserves \(S\). We interpret this result as saying that ordinary symmetries in these theories can be divided into “spacetime” and “internal” symmetries which must commute with one another.

Under the same assumptions, the Haag–Łopuszánski–Sohnius Theorem \cite{Haag1978} states that if \(\mathfrak{g}\) is a Lie \textit{superalgebra} of \textit{S}-matrix preserving operators, \(\mathfrak{g}\) must be \(\mathfrak{s} \oplus \mathfrak{h}\), \(\hat{\mathfrak{s}} \oplus \mathfrak{h}\), or a central extension of one of these, where again \(\mathfrak{h}\) is a (ordinary) Lie algebra and \(\hat{\mathfrak{s}}\) is a Poincaré superalgebra extended by \(R\)-symmetry (see below). Both of the no-go theorems also allow the (extended) Poincaré (super)algebra to be replaced by the conformal (super)algebra if all particles are assumed to be massless \cite{Haag1978}. We note that both of these theorems only in Lorentzian signature; indeed, it is not even clear how one would formulate an analogous statement in other signatures.

\textbf{Supersymmetry algebras in this and other work}

Due to the above as well as our original interest in supergravity as discussed in the introduction, our main focus in this work will be on Lorentzian signature and (extended) Poincaré \textit{superalgebras}. There are also important technical simplifications in this case arising from the Homogeneity Theorem (Theorem 3.21). Nonetheless, we will provide a significant amount of formalism in general signature. We will also only discuss (super)algebras of the type described in Definition 2.2, possibly with \(R\)-symmetry extension (to be discussed later in this section), and their filtered deformations. In particular, we will not consider the conformal superalgebras or central extensions mentioned above, nor any other generalisation. There is a significant amount of literature on these generalisations for the interested reader; for the Lorentzian case, see the classic work of Strathdee (which also discusses the representation theory of the algebras) \cite{Strathdee1979} and the review of Van Proeyen \cite{VanProeyen2004}; for general signature, see \cite{Strathdee1979, VanProeyen2004, VanProeyen2005} and also the recent work \cite{Strathdee2007}.

\textsuperscript{13}The CM and HLS theorems in 4 dimensions are sufficient motivation on their own, but the arguments generalise to higher dimensions.
2.2.2 Squaring maps, Dirac currents and their classification

We now discuss the Alekseevsky–Cortés classification [63] of the squaring maps $\kappa$ on extended spinor modules, hence the flat model algebras of Definition 2.2. Complex squaring maps are also classified there, but we will not use that result. We will provide no proofs here and merely present enough of the formalism to understand the result. The description here will be rather abstract, but in §2.2.4, we will explicitly construct all of the possible squaring maps in $n = p + q = 2$.

It will be necessary to make some slight modifications to our conventions in order to state the result, which we use in this section only. If there are two inequivalent irreducible real pinor modules $P_\pm$, let us set $P = P_+$ so that we can refer to the (irreducible) pinor module $P$ in any signature. If $P$ is irreducible under the action of $\text{Spin}(V)$ then there is a unique irreducible real $\text{Spin}(V)$-module $S \cong P$. If $P$ is reducible then we have a real $\text{Spin}(V)$-module decomposition $P = S_+ \oplus S_-$, where in contrast to our previous conventions we now allow $S_+ \cong S_-$ – this only affects the cases $p - q = 1, 2 \mod 8$. In these cases, we also modify Definition 2.1 to define $S_{(n,+,-)}$ as in the $p - q = 0, 4 \mod 8$ cases (in which $S_+ \not\cong S_-$), and then we have $S_{(n,+,-)} = S_{n,+,-}$.

Reducing the problem

With the above established, we now seek to classify the possible squaring maps on $S_{(n,+,-)}$ or $S_N$. We begin by reducing the problem to a classification of squaring maps on the pinor module $P$. First, we note that we can decompose the symmetric and exterior square of extended spinor modules as follows:

\[
\begin{align*}
\bigodot^2 S_N &= \bigodot^2 S \circ \bigodot^2 R^N \oplus \bigwedge^2 S \circ \bigwedge^2 R^N, \\
\bigwedge^2 S_N &= \bigwedge^2 S \circ \bigodot^2 R^N \oplus \bigodot^2 S \circ \bigwedge^2 R^N, \\
\bigodot^2 S_{(n,+,-)} &= \bigodot^2 S_+ \circ \bigodot^2 R^{N_+} \oplus \bigodot^2 S_- \circ \bigodot^2 R^{N_-} \\
&\quad \oplus \bigwedge^2 S_+ \circ \bigwedge^2 R^{N_+} \oplus \bigwedge^2 S_- \circ \bigwedge^2 R^{N_-} \\
&\quad \oplus (S_+ \oplus S_- \circ R^{N_+}) \oplus \bigwedge^2 S_{(n,+,-)}, \\
\bigwedge^2 S_{(n,+,-)} &= \bigwedge^2 S_+ \circ \bigodot^2 R^{N_+} \oplus \bigwedge^2 S_- \circ \bigodot^2 R^{N_-} \\
&\quad \oplus \bigodot^2 S_+ \circ \bigwedge^2 R^{N_+} \oplus \bigodot^2 S_- \circ \bigwedge^2 R^{N_-} \\
&\quad \oplus (S_+ \oplus S_- \circ R^{N_+}) \oplus \bigwedge^2 S_{(n,+,-)}.
\end{align*}
\]

Now, we seek to classify $so(V)$-equivariant maps $W \to V$ where $W$ is any of the squares of spinor modules above. But these are equivalent to equivariant maps $V^* \to W^*$, so since $V^* \cong V$ is irreducible as an $so(V)$-module, such a map must be either zero or an equivariant embedding $V^* \hookrightarrow W^*$. By the above, we see that $W$ is a sum of modules of the form $\bigodot^2 S$ and $\bigwedge^2 S$, or $\bigodot^2 S_+ \circ \bigodot^2 S_- \circ \bigodot^2 S_+ \circ \bigodot^2 S_-$, so $V^* \hookrightarrow W^*$ is a sum of embeddings of $V^*$ into squares of irreducible spinor modules. Finally, noting that $\bigodot^2 P^*$ contains all such squares in any case, we conclude that it is sufficient to study embeddings $V^* \hookrightarrow \bigodot^2 P^*$.

\[^{14}\text{We always have } S^* \cong S \text{ or } S_\pm \cong S_\pm, \text{ so the dual of a square of spinor modules is also such a square.}\]
Description in terms of bilinears

The utility in describing the maps in terms of the pinor module as above is that it allows us to use the following result.

**Lemma 2.3 ([63]).** Let \((V, \eta)\) be an inner product space and \(\mathbb{P}\) an irreducible pinor module considered as an \(\mathfrak{so}(V)\)-module. Then the following vector spaces are isomorphic:

- \(\mathcal{J} = \text{Hom}(V^*, \otimes^2 \mathbb{P}^*)^{\mathfrak{so}(V)} \) (\(\mathfrak{so}(V)\)-equivariant “squaring maps” \(V^* \hookrightarrow \otimes^2 \mathbb{P}^*)\),
- \(\mathcal{B} = (\otimes^2 \mathbb{P}^*)^{\mathfrak{so}(V)}\) (\(\mathfrak{so}(V)\)-invariant bilinears on \(\mathbb{P}\)),
- \(\mathcal{M} = \text{Hom}(V^* \otimes \mathbb{P}, \mathbb{P})^{\mathfrak{so}(V)} \) (\(\mathfrak{so}(V)\)-equivariant (“multiplications”) \(V^* \otimes \mathbb{P} \to \mathbb{P}\)).

Moreover, a choice of a non-degenerate element in any of these three spaces induces an isomorphism between the other two through the relation

\[
\kappa(\alpha)(\epsilon, \epsilon') = B(\epsilon, \mu(\alpha, \epsilon'))
\]  

(2.53) for all \(\alpha \in V^*\) and \(\epsilon, \epsilon' \in \mathbb{P}\), where \(j \in \mathcal{J}\), \(B \in \mathcal{B}\) and \(\mu \in \mathcal{M}\). Here, we call \(\mu \in \mathcal{M}\) non-degenerate if \(\mu(V^*, \mathbb{P}) = \mathbb{P}\), and \(j \in \mathcal{J}\) is non-degenerate if \(j(V^*)\mathbb{P} = \mathbb{P}^*\) (where we consider \(j : V^* \to \text{Hom}(\mathbb{P}, \mathbb{P}^*)\) and \(j\) is injective as a map \(\mathbb{P} \to V \otimes \mathbb{P}^*\)).

Now let us fix \(\mu : V^* \otimes \mathbb{P} \to \mathbb{P}\) to be the restriction of Clifford multiplication to the rank-1 subspace of the Clifford algebra. Then \(\mu\) is \(\mathfrak{so}(V)\)-equivariant, so by the lemma above it induces an isomorphism \(\mathcal{J} \cong \mathcal{B}\). To make connection with notation used elsewhere, let us denote the squaring map associated to a bilinear \(B \in \mathcal{B}\) by \(\kappa \in \mathcal{J}\). Considered as a map \(\kappa : \otimes^2 \mathbb{P} \to V\), the correspondence takes the more familiar form

\[
\eta(\kappa(\epsilon, \epsilon'), v) = B(\epsilon, v \cdot \epsilon')
\]  

(2.54) for all \(\epsilon, \epsilon' \in \mathbb{P}\), \(v \in V\), where recall that \(\eta\) is the inner product on \(V\). The reader may be more familiar with the form \(\kappa(\epsilon, \epsilon') = \bar{\epsilon} \bar{\epsilon}'\), where we have chosen an orthonormal basis for \(V\) and write \(\bar{\epsilon}\) for the \(B\)-conjugate \(\bar{\epsilon} \in \mathbb{P}^*\) which is the map \(\epsilon' \mapsto \bar{\epsilon} \epsilon' = B(\epsilon, \epsilon)\).

**Definition 2.4** (Dirac current). The \(\mathfrak{so}(V)\)-invariant squaring map \(\kappa : \otimes^2 \mathbb{P} \to V\) formed from an \(\mathfrak{so}(V)\)-invariant bilinear \(B\) on \(\mathbb{P}\) is called the Dirac current associated to \(B\).

We now recall the definition of the symmetry and type of a bilinear on a module of \(\text{Cl}(V)\) from §2.1.4.

**Definition 2.5** (Admissible bilinear). A real bilinear form \(B\) on \(\mathbb{P}\) is admissible if

- \(B\) is either symmetric or skew-symmetric (it has symmetry \(\sigma_B = \pm 1\));
- Clifford multiplication by an element \(v \in V\) is either \(B\)-symmetric or \(B\)-skew-symmetric (it has type \(\tau_B = \pm 1\));
- If \(\mathbb{P} = \mathbb{S}_+ \oplus \mathbb{S}_-\) as an \(\mathfrak{so}(V)\)-module, the submodules \(\mathbb{S}_\pm\) are either mutually \(B\)-orthogonal \((B(\mathbb{S}_+, \mathbb{S}_-) = 0)\) or \(B\)-isotropic \((B(\mathbb{S}_+, \mathbb{S}_-) = 0)\). We define the isotropy \(\iota_B \) of \(B\) to be \(+1\) in the first case and \(-1\) in the second.

Note that an admissible bilinear is automatically \(\mathfrak{so}(V)\)-invariant, hence it lies in \(\mathcal{B}\). Moreover, it can be shown that \(\mathcal{B}\) has a basis consisting of admissible elements.

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Admissible currents and the classification

We say that $\kappa \in \mathcal{J}$ has symmetry $\sigma = 1$ if $\kappa(V^*) \subseteq \bigwedge^2 S^*$ and $\sigma = -1$ if $\kappa(V^*) \subseteq \bigwedge^2 S^*$; this sign is undefined otherwise. Note, however, that any element of $\mathcal{J}$ can be split into a symmetric and skew-symmetric part, giving us

$$\mathcal{J} = \mathcal{J}^+ \oplus \mathcal{J}^-$$  \hspace{1cm} (2.55)

where the subspaces are of course

$$\mathcal{J}^\sigma = \{ \kappa \in \mathcal{J} \mid \sigma = \sigma \}.$$  \hspace{1cm} (2.56)

If $\mathcal{P}$ is reducible with $\mathcal{P} = S_+ \oplus S_-$, a map $\kappa \in \mathcal{J}$ with definite symmetry $\sigma$ has isotropy $\iota = +1$ if $\kappa(V^*) \subseteq (S_+^* \oplus S_+^*) \oplus (S_-^* \oplus S_-^*)$, and $\iota = -1$ if $\kappa(V^*) \subseteq (S_+^* \oplus S_-^*)$. We then have

$$\mathcal{J}^\sigma = \mathcal{J}^{\sigma +} \oplus \mathcal{J}^{\sigma -}$$  \hspace{1cm} (2.57)

and

$$\mathcal{J}^{\sigma \iota} = \{ \kappa \in \mathcal{J} \mid \sigma = \sigma, \iota = \iota \}.$$  \hspace{1cm} (2.58)

For each admissible bilinear in $B \in \mathcal{B}$, the corresponding Dirac current $\kappa \in \mathcal{J}$ defined by (2.54) has symmetry $\sigma = \sigma_B \tau_B$ and if $\mathcal{P}$ is reducible it has isotropy $\iota = -\iota_B$. We will call $\kappa \in \mathcal{J}$ admissible if the corresponding bilinear $B \in \mathcal{B}$ is admissible. An admissible basis for $\mathcal{B}$ induces an admissible basis of $\mathcal{J}$ which is adapted to the above splittings.

We can now state the classification result of [63]. Define $L^\sigma = \dim \mathcal{J}^\sigma$ and $L^{\sigma \iota} = \dim \mathcal{J}^{\sigma \iota}$. The classification is contained in Table 2.6, which plots $(L^+, L^-)$ or $(L^{++}, L^{+-}, L^{-+}, L^{--})$ depending on the reducibility of $\mathcal{P}$ for each signature $(p, q)$. Note that these tuples depend only on $p$ and $q$ modulo 8, or equivalently $n = p + q$ and $s = p - q$ modulo 8.

**Table 2.6: Alekseevsky–Cortés classification of squaring maps (Dirac currents) on pinor modules [63].** The tuples $(L^+, L^-)$ or $(L^{++}, L^{+-}, L^{-+}, L^{--})$ are plotted against dimension $n = p + q$ and signature $s = p - q$ modulo 8.

<table>
<thead>
<tr>
<th>s/n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0,1,0,1)</td>
<td>(2,0,0,0)</td>
<td>(0,1,0,1)</td>
<td>(0,0,2,0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(2,1,0,1)</td>
<td>(2,1,0,1)</td>
<td>(0,1,2,1)</td>
<td>(0,1,2,1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(2,2,2,2)</td>
<td>(4,2,0,2)</td>
<td>(2,2,2,2)</td>
<td>(0,2,4,2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(3,1)</td>
<td>(3,1)</td>
<td>(1,3)</td>
<td>(1,3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(0,4,0,4)</td>
<td>(6,0,2,0)</td>
<td>(0,4,0,4)</td>
<td>(2,0,6,0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(3,1)</td>
<td>(3,1)</td>
<td>(1,3)</td>
<td>(1,3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(1,1)</td>
<td>(2,0)</td>
<td>(1,1)</td>
<td>(0,2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(1,0)</td>
<td>(1,0)</td>
<td>(0,1)</td>
<td>(0,1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Reducible pinor modules

In the cases where $P$ is reducible, non-zero $\mathfrak{so}(V)$-invariant maps $\bigwedge^2 S_\pm \to V$ or $\bigwedge^2 S_\pm \to V$ are obtained by restricting elements of $\mathcal{J}^{++}$ or $\mathcal{J}^{--}$ respectively, though it should be noted that two distinct elements may give the same map when restricted. The $S_+ \oplus S_- \to V$ maps are given by restricting those in $\mathcal{J}^{++}$ or $\mathcal{J}^{--}$, again possibly non-uniquely.

For $p - q = 1,2 \mod 8$ (where $S = S_+ \cong S_-$), this complication can be avoided by simply reading off the classification in $p - q = 6,7 \mod 8$. Indeed, recall Remark 1 we noted that $\text{Spin}(p,q) \cong \text{Spin}(q,p)$, so we can always “flip” the signature to the more convenient one for questions about spin representations. For example, identifying spinor modules $S$ in signatures $p - q = 1 \mod 8$ and $p - q = 7 \mod 8$ with each other, we see that for $p - q = 7 \mod 8$, $P = S$ is irreducible under the spin group and has only one independent Dirac current, say $\kappa_0$, in any dimension, while for $p - q = 1 \mod 8$, $P = S_+ \oplus S_- \cong S \otimes \mathbb{R}^2$ has a basis of four admissible elements. The latter can be understood as tensor products of $\kappa_0$ with four independent non-degenerate bilinear forms on $\mathbb{R}^2$: for example, in the $p + q = p - q = 1 \mod 8$ case, $\kappa_0$ is symmetric and we have

$$\mathcal{J}^{++} = \mathbb{R}\left\{\kappa_0 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \kappa_0 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}, \quad \mathcal{J}^{--} = \mathbb{R}\left\{\kappa_0 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\},$$

(2.59)

$$\mathcal{J}^{+-} = 0, \quad \mathcal{J}^{-+} = \mathbb{R}\left\{\kappa_0 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\},$$

(2.60)

where we write the matrices representing the bilinears on $\mathbb{R}^2$. Going the other way, only elements of $\mathcal{J}^{++}$ restrict non-trivially to $S$, and any such restriction is proportional to $\kappa_0$. Likewise, for $p - q = 6 \mod 8$ we have $P = S$ and $\dim \mathcal{J} = 2$, while for $p - q = 2 \mod 8$, $P \cong S \otimes \mathbb{R}^2$ and $\dim \mathcal{J} = 8$, with admissible currents in the latter case being tensor products of those in the former with bilinears on $\mathbb{R}^2$.

For $p - q = 0,4 \mod 8$ (the cases with (symplectic) Majorana-Weyl spinors, $S_+ \not\cong S_-$), no such trick is available. For $p + q = 0,4 \mod 8$, there are no non-zero equivariant maps $\bigwedge^2 S_+ \to V$ or $\bigwedge^2 S_- \to V$ since $L^{++} = L^{--} = 0$. For $p + q = 2,6 \mod 8$, there are no equivariant maps $S_+ \oplus S_- \to V$ since $L^{+-} = L^{-+} = 0$.

Building back up

Let us now work backwards from the classification of Dirac currents on $P$ (hence on irreducible spinor modules) to squaring maps on extended spinor representations. An $\mathfrak{so}(V)$-equivariant map $\kappa : \bigwedge^2 S_N \to V$ can be decomposed as

$$\kappa = \sum_i \kappa_i^\circ \otimes a_i + \sum_j \kappa_j^\wedge \otimes b_j$$

(2.61)

where $\{\kappa_i^\circ : \bigwedge^2 S \to V\} \subset \mathcal{J}^+$ and $\{\kappa_j^\wedge : \bigwedge^2 S \to V\} \subset \mathcal{J}^-$ are admissible bases, $a_i$ are arbitrary symmetric bilinears on $\mathbb{R}^N$ and $b_i$ are arbitrary skew-symmetric bilinears.
on $\mathbb{R}^n$. An $\mathfrak{so}(V)$-equivariant map $\kappa : \bigodot^2 \mathbb{S}_{(N_+, N_-)} \to V$ decomposes as

$$
\kappa = \sum_{i, \pm} \kappa^{\pm}_i \otimes a^\pm_i + \sum_{j, \pm} \kappa^{\pm}_j \otimes b^\pm_j + \sum_k \kappa^{+-}_k \otimes c_k \quad (2.62)
$$

where $\{\kappa^{\pm}_i : \bigodot^2 \mathbb{S}_+ \to V\}$ and $\{\kappa^{\pm}_j : \bigwedge^2 \mathbb{S}_+ \to V\}$, $\{\kappa^{+-}_k : \mathbb{S}_+ \otimes \mathbb{S}_- \to V\}$ are bases of $\mathfrak{so}(V)$-equivariant maps which can be taken to be restrictions of admissible maps in $\mathscr{S}^+$, $\mathscr{S}^-$ and $\mathscr{S}^+ \oplus \mathscr{S}^-$ respectively, while $a^\pm_i \in \bigodot^2 \mathbb{R}^N_+$, $b^\pm_j \in \bigwedge^2 \mathbb{R}^N_-$ and $c_k \in (\mathbb{R}^{N_+} \otimes \mathbb{R}^{N_-})^*$. There are of analogous decompositions of skew-symmetric $\mathfrak{so}(V)$-equivariant maps $\kappa : \bigwedge^2 \mathbb{S}_N \to V$, $\kappa : \bigwedge^2 \mathbb{S}_{(N_+, N_-)} \to V$.

We emphasise once again that every basis element $\kappa_i$ appearing above is (the restriction of) the Dirac current associated to an admissible bilinear $B_i$ on the pinor module $\mathbb{P}$. In applications, one usually fixes a single admissible $\kappa_i$ and tensors it with a non-degenerate bilinear of the appropriate symmetry rather than taking more complicated linear combinations as above, but we note that this is not the most general choice. The more general type of squaring map can be obtained by defining a complex Dirac current (also classified in [63]) on an extended complex spinor module and imposing non-standard Majorana reality conditions so that the choice of squaring map appears as a choice of reality structure. This is the approach taken in [58].

2.2.3 $R$-symmetry

In the physics literature, the $R$-symmetry group is a group of automorphisms of a Poincaré superalgebra. As already alluded to in our discussion of the HLS Theorem, the $R$-symmetry acts effectively odd part of the superalgebra – it “rotates the supercharges” in the physics parlance – and trivially on the even part. The $R$-symmetry group in 4-dimensional Lorentzian signature and the corresponding extended Poincaré superalgebra $\mathfrak{p}$ appears in [65], and is generalised to arbitrary dimension by Strathdee [10]; a brief summary of this appears in [30]. A complete treatment in arbitrary signature and with various choices of Dirac current recently appeared in the literature [58]. Previous work in this direction includes [68], but the conventions adopted there make it harder to compare results. We note that some sources give slightly different definitions for these groups; our definition is the same as in [58], allowing for the $R$-symmetry group to be disconnected, in contrast with the other references.

Schur algebra and $R$-symmetry group

Let $S$ be a spinor representation of $\mathrm{Spin}(V)$ and define the Schur algebra $\mathcal{C}(S)$ as the algebra of $\mathfrak{so}(V)$-equivariant endomorphisms of $S$ and the Schur group $\mathcal{C}^s(S)$ as its group of units [58, 63]:

$$
\mathcal{C}(S) = \mathrm{End}_{\mathfrak{so}(V)}(S) = \{a \in \mathrm{End}(S) | \forall A \in \mathfrak{so}(V), \forall \epsilon \in S, a(A \cdot \epsilon) = A \cdot (ae)\}, \quad (2.63)
$$

$$
\mathcal{C}^s(S) = \mathrm{GL}_{\mathfrak{so}(V)}(S) = \{a \in \mathrm{GL}(S) | \forall A \in \mathfrak{so}(V), \forall \epsilon \in S, a(A \cdot \epsilon) = A \cdot (ae)\}.
$$

(2.64)
Now suppose we have an \(\mathfrak{so}(V)\)-equivariant map \(\kappa : \bigwedge^2 S \to V\) or \(\kappa : \mathcal{O}^2 S \to V\) (i.e. a squaring map). We define the \(R\)-symmetry group of \((S, \kappa)\) as subgroup of the Schur group preserving \(\kappa\):

\[
R_\kappa = \{ a \in \mathcal{O}(S) \mid \forall \varepsilon, \varepsilon' \in S, \, \kappa(a\varepsilon, a\varepsilon') = \kappa(\varepsilon, \varepsilon') \};
\]

(2.65)

its Lie algebra, the \(\tau\)-symmetry algebra of \((S, \kappa)\), is of course

\[
\tau_\kappa = \{ a \in \mathcal{O}(S) \mid \forall \varepsilon, \varepsilon' \in S, \, \kappa(a\varepsilon, \varepsilon') + \kappa(\varepsilon, a\varepsilon') = 0 \}.
\]

(2.66)

We will omit the subscript \(\kappa\) when the choice of squaring map is unambiguous.

Since the action of \(R_\kappa\) preserves \(\kappa\), it acts by automorphisms on the Lie (super)algebra \(\mathfrak{s} = \mathfrak{V} \oplus \mathfrak{S} \oplus \mathfrak{so}(V)\) defined by \(\kappa\) where the action on the \(f\mathfrak{s}_0 = \mathfrak{V} \oplus \mathfrak{so}(V)\) is trivial. Note that this action preserves the \(\mathbb{Z}\)-grading on \(\mathfrak{s}\). Likewise, \(\tau_\kappa\) acts by graded derivations. In fact, at least for \(n = \dim V > 2\), \(\tau_\kappa\) acts by outer derivations, meaning that if \(a \in \tau_\kappa\), there is no \(Y \in \mathfrak{s}\) such that \(a \cdot X = \text{ad}_Y X\) for all \(X \in \mathfrak{s}\).

The \(R\)-symmetry groups for symmetric squaring maps (hence superalgebras) were determined in [58]. As mentioned at the end of §2.2.2, in that work the choice of squaring map is factored into a choice of reality conditions on a complex spinor modules carrying complex Dirac current. Consulting [58, Table 10], we see that \(R\)-symmetry groups are classical Lie groups,\(^{15}\) as in the standard physics treatment in Lorentzian signature [10, 30], the groups \(\text{O}(N), \text{U}(N), \text{Sp}(N) = \text{USp}(2N)\) appear, but with more unusual choices of reality conditions, \(\text{O}(r, s), \text{U}(r, s), \text{Sp}(r, s)\) (and products) are possible, and in other signatures we one can also have \(\text{GL}(N, \mathbb{R}), \text{SO}(N, \mathbb{C}), \text{Sp}(N, \mathbb{R}), \text{Sp}(N, \mathbb{C}), \text{U}^*(2N) = \text{GL}(N, \mathbb{H}), \text{SO}^*(2N) = \text{SO}(N, \mathbb{H})\) and certain products. To the author’s knowledge, this analysis has not been done for the Lie algebra case, but one expects a similar list. We observe here that in Lorentzian signature, it is always possible to choose a symmetric squaring with compact \(R\)-symmetry group, which we will make use of in Chapter 4 (see Lemmas 4.25 and 4.26).

**Extending the algebra**

As previously alluded to, the flat model (super)algebra \(\mathfrak{s}\) admits an extension by the \(R\)-symmetry algebra \(\tau\) (where we now suppress the \(\kappa\) subscript) which we denote by \(\hat{\mathfrak{s}}\) and call the \(\tau\)-symmetry extended flat model algebra. The underlying vector space is \(\mathfrak{s} \oplus \tau\) with the bracket being given by (2.49), (2.51) and

\[
[a, b] = [a, b]_\tau, \quad [a, A] = 0, \quad [a, v] = 0, \quad [a, \varepsilon] = a\varepsilon,
\]

(2.67)

where \(a, b \in \tau, \, A \in \mathfrak{so}(V), \, v \in V, \, \varepsilon \in S\). The Jacobi identities are satisfied precisely because \(\tau\) acts by automorphisms on \(\mathfrak{s}\). Note that this action also preserves the \(\mathbb{Z}\)-grading (2.50) on \(\mathfrak{s}\), so \(\hat{\mathfrak{s}}\) admits the \(\mathbb{Z}\)-grading

\[
\hat{\mathfrak{s}}_0 = \mathfrak{so}(V) \oplus \tau, \quad \hat{\mathfrak{s}}_{-1} = S, \quad \hat{\mathfrak{s}}_{-2} = V, \quad \hat{\mathfrak{s}}_i = 0 \text{ otherwise},
\]

(2.68)

\(^{15}\text{see [69, App.A2] for the definitions of these groups.}\)
so that \( \text{iso}(V) \subset \mathfrak{s} \subset \widehat{\mathfrak{s}} \) are embeddings of \( \mathbb{Z} \)-graded (super)algebras. Note that the subspace \( \widehat{\mathfrak{s}}_- = \mathfrak{s}_- := V \oplus \mathfrak{s} \) is an ideal subalgebra of both \( \mathfrak{s} \) and \( \widehat{\mathfrak{s}} \) which is often known as the supertranslation ideal when \( \mathfrak{s} \) is the Poincaré superalgebra.

### 2.2.4 Dirac currents in 2 dimensions

We now demonstrate the theory of this section and the last section by explicitly describing the pinor and spinor modules as well as the admissible bilinears and Dirac currents in 2 dimensions and the three possible signatures.

We will make extensive use of the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(2.69)

Note that \( \sigma_1, \sigma_3 \) are real symmetric matrices and \( \Omega = i\sigma_2 \) is the (real and skew-symmetric) standard symplectic matrix. The sets \( \{1, \sigma_1, \sigma_2, \sigma_3\} \) and \( \{1, \sigma_1, \sigma_3, \Omega = i\sigma_2\} \) are both \( \mathbb{C} \)-bases for \( \mathbb{C}(2) \), while the latter is also an \( \mathbb{R} \)-basis for \( \mathbb{R}(2) \). The Pauli matrices satisfy the algebraic identities

\[
\sigma_i^\dagger = \sigma_i, \quad \sigma_i \sigma_j = \delta_{ij} 1 + i \epsilon_{ijk} \sigma_k,
\]

(2.70)

where \( \dagger \) denotes Hermitian adjoint (conjugate-transpose) and we use the Einstein summation convention on repeated indices. The second identity will also be useful to express in the form

\[
[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij} 1,
\]

(2.71)

where \([-, -]\) is the matrix commutator and \([-,-]\) is the anti-commutator.

#### Signature \((2,0)\)

Consulting Table 2.1, we see that \( \text{Cl}(2,0) \cong \mathbb{H} \) and \( \text{Cl}_{\mathbb{R}}(2,0) \cong \mathbb{C} \). We first note that defining \( I = i\sigma_1, J = i\sigma_2 = \Omega, K = -i\sigma_3 \) (with signs chosen for later convenience) gives us a faithful \( \mathbb{R} \)-algebra representation of \( \mathbb{H} \) as \( \mathbb{C} \)-matrices, which we will use as our model for the quaternions, and set

\[
\Gamma_1 = i\sigma_1 = I, \quad \Gamma_2 = i\sigma_2 = J, \quad \Gamma_{ab} = \frac{1}{2} [\Gamma_a, \Gamma_b] = -i \epsilon_{ab} \sigma_3 = \epsilon_{ab} K.
\]

(2.72)

We then define an isomorphism \( \text{Cl}(2,0) \to \mathbb{H} \subseteq \mathbb{C}(2) \) by extending the real Clifford map \( V = \mathbb{R}^2 \to \mathbb{C}(2) \) given by \( v = (v^1, v^2) \mapsto v^a \Gamma_a \). Note that extending the same Clifford map to a complex one defined on \( V \otimes \mathbb{C} = \mathbb{C}^2 \to \mathbb{C}(2) \) gives an isomorphism of complex algebras \( \text{Cl}(2, \mathbb{C}) \to \mathbb{C}(2) \), but we will not use this explicitly here. The even subalgebra \( \text{Cl}_{\mathbb{R}}(2,0) \) is the real span of \( \{1, \Gamma_{12} = K\} \), which we can check is indeed isomorphic to \( \mathbb{C} \) as an \( \mathbb{R} \)-algebra since \( (\Gamma_{12})^2 = K^2 = -1 \).

We can take the (unique) irreducible Clifford (pinor) module to be \( \mathbb{P} = \mathbb{C}^2 \), on which our representation of \( \text{Cl}(V) \) acts by matrix multiplication from the left. We recall that \( \text{so}(2) \) acts on \( \mathbb{P} \) by

\[
A \cdot e = -\omega_A \cdot e = -t \Gamma_{12} e
\]

(2.73)
where \( A \in \mathfrak{so}(2), \epsilon \in \mathbb{P} \) and we have parametrised \( A \) as \( A = t\Omega \) for \( t \in \mathbb{R} \).

The matrix \(-\Gamma_{12}\) has eigenvalues \( \pm i \) with corresponding eigenspaces \( \mathbb{C}e_\pm \), where \( e_+ = (1, 0) \) and \( e_- = (0, 1) \), and we also identify these eigenspaces as the two irreducible real representations \( \mathbb{S}, \overline{\mathbb{S}} \) of \( \text{Cl}_0(2, 0) \cong \mathbb{C} \).

We now describe the admissible bilinears and corresponding Dirac currents on \( \mathbb{P} \). The bilinears and currents on \( \mathbb{S} \) and \( \overline{\mathbb{S}} \) will be restrictions of these. We will determine the bilinears as the real or complex parts of non-degenerate \( \mathbb{C} \)-sesquilinear or \( \mathbb{C} \)-bilinear inner products which are \( \text{so}(2) \)-invariant and with respect to which Clifford multiplication by vectors is either symmetric or skew.

A sesquilinear form can always be written as \( (\epsilon, \epsilon') \mapsto \epsilon^\dagger B \epsilon' \) where \( B \) is a \( \mathbb{C} \)-matrix, and it is \( \text{so}(2) \)-invariant if and only if \( \Gamma_{12}^\dagger B + B \Gamma_{12}^\dagger = 0 \), but \( \Gamma_{12} \) is anti-Hermitian, so this condition is equivalent to

\[
[\Gamma_{12}, B] = 0;
\]  

(2.74)

Clifford multiplication by vectors is symmetric if and only if \( \Gamma_a^\dagger B = B \Gamma_a \) for \( a = 1, 2 \), or, since both matrices \( \Gamma_a \) are also anti-Hermitian,

\[
[\Gamma_a, B] = 0, \quad (2.75)
\]

and it is skew if and only if \( \Gamma_a^\dagger B = -B \Gamma_a \), or equivalently

\[
[\Gamma_a, B] = 0. \quad (2.76)
\]

Conditions (2.74) and (2.75) are satisfied if and only if \( B \in \mathbb{C} \sigma_3 \), while conditions (2.74) and (2.76) are satisfied if and only if \( B \in \mathbb{C} \).

Similarly, \( \mathbb{C} \)-bilinearls can be written as \( (\epsilon, \epsilon') \mapsto \epsilon^T B \epsilon' \) for \( B \in \mathbb{C}(2) \). They are \( \text{so}(2) \)-invariant if and only if \( B \Gamma_{12} + \Gamma_{12}^T B = 0 \), or since \( \Gamma_{12} \) is symmetric,

\[
[\Gamma_{12}, B] = 0; \quad (2.77)
\]

Clifford multiplication by vectors is symmetric if and only if \( \Gamma_a^T B = B \Gamma_a \), but \( \Gamma_1^T = \Gamma_1 \) while \( \Gamma_2^T = -\Gamma_2 \), so this is equivalent to

\[
[\Gamma_1, B] = 0 \text{ and } [\Gamma_2, B] = 0, \quad (2.78)
\]

and it is skew if and only if \( \Gamma_a^T B = -B \Gamma_a \), or

\[
[\Gamma_1, B] = 0 \text{ and } [\Gamma_2, B] = 0. \quad (2.79)
\]

Conditions (2.77) and (2.78) are equivalent to \( B \in \mathbb{C} \sigma_1 \), while (2.77) and (2.79) are equivalent to \( B \in \mathbb{C} \sigma_2 = \mathbb{C} \Omega \).

An admissible basis for invariant bilinear forms on \( \mathbb{P} \), along with the properties for each element, is given in Table 2.7. For each admissible bilinear \( B \) on \( \mathbb{P} \), the Dirac current \( \kappa : \mathcal{O}^2 \mathbb{P} \to \mathbb{R}^2 \) or \( \kappa : \Lambda^2 \mathbb{P} \to \mathbb{R}^2 \) is defined by the equation

\[
\eta(v, \kappa(\epsilon, \epsilon')) = v^T \kappa(\epsilon, \epsilon') = B(\epsilon, v \cdot \epsilon') \quad (2.80)
\]
for all $\nu \in \mathbb{R}^2$, where $\cdot$ is Clifford multiplication, or more explicitly,

$$\kappa(\epsilon, \epsilon') = B(\epsilon, \Gamma_a \epsilon') e_a. \quad (2.81)$$

We briefly recall that the signs appearing in the table are as follows:

$$B(\epsilon, \epsilon') = \sigma_B B(\epsilon', \epsilon), \quad (2.82)$$

$$B(\epsilon, \nu \cdot \epsilon') = \tau_B B(\nu \cdot \epsilon', \epsilon), \quad (2.83)$$

$$\kappa(\epsilon, \epsilon') = \sigma_B \kappa(\epsilon', \epsilon), \quad (2.84)$$

for $\epsilon, \epsilon' \in \mathbb{P}$; the isotropy $\iota_B$ is $+$ if $S_+, S_-$ are mutually $B$-orthogonal and $-$ if they are $B$-isotropic; similarly, the isotropy $\iota_K$ of the Dirac current is $+$ if $\kappa(S_+, S_-) = 0$ and $-$ if $\kappa(S_\pm, S_\pm) = 0$. We always have $\sigma_K = \sigma_B \tau_B$ and $\iota_K = -\iota_B$.

Table 2.7: Properties of admissible bilinears and their Dirac currents in signature $(2, 0)$.

<table>
<thead>
<tr>
<th>$B(\epsilon, \epsilon')$</th>
<th>$\sigma_B$</th>
<th>$\tau_B$</th>
<th>$\iota_B$</th>
<th>$\sigma_K$</th>
<th>$\iota_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re $\left(\epsilon^\dagger \sigma_3 \epsilon'\right)$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Im $\left(\epsilon^\dagger \sigma_3 \epsilon'\right)$</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Re $\left(\epsilon^\dagger \epsilon'\right)$</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Im $\left(\epsilon^\dagger \epsilon'\right)$</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Re $\left(\epsilon^T \sigma_1 \epsilon'\right)$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Im $\left(\epsilon^T \sigma_1 \epsilon'\right)$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Re $\left(\epsilon^T \Omega \epsilon'\right)$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Im $\left(\epsilon^T \Omega \epsilon'\right)$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

The final column of the table verifies the result of the Alekseevsky–Cortés classification given in Table 2.6: $L^{++} = 4, L^{+-} = 2, L^{-+} = 0, L^{--} = 2$.

Note that a bilinear $B$ from the table restricts to a non-trivial bilinear on either $\mathbb{S}$ or $\overline{\mathbb{S}}$ if and only if $\iota_B = +$ and similarly for the Dirac currents; since we always have $\iota_B = -\iota_K$, it is not possible to restrict both non-trivially. Moreover, from the explicit representations we see that when restricted to $\mathbb{S}$, the first and third bilinears on the table agree, as do the second and fourth, and they agree up to a sign on $\overline{\mathbb{S}}$. A similar observation holds for the Dirac currents with $\iota_K = +1$, and so we find that there are 2 independent Dirac currents on $\mathbb{S}$ or $\overline{\mathbb{S}}$, both of which are symmetric. Thus if we wish to form a flat model (super)algebra in this case, we clearly have a lot of choices, but if we demand that the algebra is minimal then we must pick from the Dirac currents which restrict to the irreducible spinor modules (those with $\iota_K = +$) and thus we can only have a superalgebra.
**Signature (1, 1)**

In this case $\text{Cl}(1,1) \cong \mathbb{R}(2)$ with even subalgebra $\mathbb{R}^2$ (embedded as the diagonal matrices). We choose the representation defined by

$$
\Gamma_0 = i\sigma_2 = \Omega, \quad \Gamma_1 = \sigma_1, \quad \Gamma_{\mu\nu} = \epsilon_{\mu\nu}\sigma_3,
$$

(2.85)

where the Levi-Civita symbol is defined so that $\epsilon_{01} = -\epsilon_{10} = 1$. Using the metric to raise and lower indices, we must define the Levi-Civita symbol with raised indices so that $\epsilon^{01} = -\epsilon^{10} = -1$.

The pinor module is $\mathbb{P} = \mathbb{R}^2$, which decomposes as a representation of the even subalgebra as $\mathbb{P} = S_+ \oplus S_-$, where the irreducible spinor modules $S_{\pm}$ are the $\pm 1$ eigenspaces of $\Gamma_{01} = \sigma_3$. From Table 2.6, we expect to find two admissible bilinears, both of which have symmetric Dirac currents with respect to which $S_+, S_-$ are mutually orthogonal. It is simple to verify that the bilinear products defined by the matrices $\sigma_1$ and $\Omega = i\sigma_2$ are admissible, with symmetry and isotropy properties given in Table 2.8.

**Table 2.8: Properties of admissible bilinears and their Dirac currents in signature (1, 1).**

<table>
<thead>
<tr>
<th>$B(\epsilon, \epsilon')$</th>
<th>$\sigma_B$</th>
<th>$\tau_B$</th>
<th>$l_B$</th>
<th>$\sigma_\kappa$</th>
<th>$l_\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon^T \sigma_1 \epsilon'$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\epsilon^T \Omega \epsilon'$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

Either current restricts non-trivially to $S_{\pm}$ but neither bilinear does. Note that since both Dirac currents are symmetric, if we demand a minimal flat model (super)algebra, only superalgebras are possible. On the other hand, in the $N$-extended case there are many possibilities. For example, let $S = S_{(2,0)} = S_+ \otimes \mathbb{R}^2$; then we can define a skew Dirac current $\kappa : \wedge^2 S_{(2,0)} \to \mathbb{R}^{1,1}$ by

$$
\kappa_{\mu}(\epsilon \otimes x, \epsilon' \otimes y) = B(\epsilon, \Gamma_{\mu}\epsilon')x^T \Omega y
$$

(2.86)

where $B$ is either of the bilinears from the table (or indeed any non-degenerate linear combination of them).

**Signature (0, 2)**

Here $\text{Cl}(0,2) \cong \mathbb{R}(2)$, and we define the representation

$$
\Gamma_1 = \sigma_3, \quad \Gamma_2 = \sigma_1, \quad \Gamma_{ab} = \epsilon_{ab}\Omega.
$$

(2.87)

The even subalgebra is the span of $\mathbb{1}$ and $\Omega$, which is isomorphic to $\mathbb{C}$ as an $\mathbb{R}$-algebra. The pinor module is $\mathbb{P} = \mathbb{R}^2$, and this is irreducible under the action of $\text{Cl}_0(0,2)$, so $\mathbb{P} = S$ as a spinor module. The standard inner product and symplectic form are both admissible bilinears; since we expect two such independent bilinears (both with symmetric Dirac current) from Table 2.6, this exhausts the possibilities – we can also check that, up to rescaling, these are the only possibilities in a similar manner to how
we treated the (2, 0) case. Table 2.9 lists their properties. There are no isotropy signs because \( P \) is irreducible under the action of \( \mathfrak{so}(2) \). Once again, only superalgebras are possible in the minimal case. We also note that as expected, the result here agrees with our observation in the (2, 0) case that the irreducible spinor module \( S \) carries two independent admissible Dirac currents, both of which are symmetric.

Table 2.9: Properties of admissible bilinears and their Dirac currents in signature (0, 2).

<table>
<thead>
<tr>
<th>( B(\epsilon, \epsilon') )</th>
<th>( \sigma_B )</th>
<th>( \tau_B )</th>
<th>( \sigma_K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon^T \epsilon' )</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( \epsilon^T \Omega \epsilon' )</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

2.3 Filtered deformations and Spencer cohomology

The central theme of this work is the study of the structure of Killing superalgebras through Spencer cohomology. This section provides all of the algebraic background theory necessary to understand this theory. We first introduce gradings and filtrations of Lie superalgebras and Spencer cohomology, then we discuss filtered deformations and finally how the latter can be understood using Spencer cohomology. Our main source for this section is the work of Cheng and Kac [46].

The use of Spencer cohomology in this and related work is motivated by the result of Cheng and Kac that filtered deformations of \( \mathbb{Z} \)-graded Lie (super)algebras are governed by that cohomology [46]. For the sake of brevity, we will not state the full result here; we will instead simply follow the treatment of Cheng and Kac in describing the cohomology theory here and discuss later how it is used.

In a sentence, the Spencer cohomology of a \( \mathbb{Z} \)-graded Lie superalgebra is the Chevalley–Eilenberg cohomology of its negative-graded part with values in the full algebra under the adjoint action, equipped with a grading inherited from the grading on the algebra. The remainder of this section is dedicated to unpacking this statement, so readers familiar with these concepts can simply skim it for clarification or bypass it entirely. The base field is assumed to be either \( \mathbb{R} \) or \( \mathbb{C} \).

2.3.1 Filtered and graded superalgebras

We assume that the reader is familiar with ordinary \( \mathbb{Z} \)-graded vector spaces and with vector superspaces but recall and set some terminology for \( \mathbb{Z} \)-graded superspaces and algebras.

**\( \mathbb{Z} \)-graded superspaces and superalgebras**

A vector superspace \( V \) is said to be \( \mathbb{Z} \)-graded with grading \( V = \bigoplus_{k \in \mathbb{Z}} V_k \) if the latter is a \( \mathbb{Z} \)-grading of \( V \) as a vector space and the \( \mathbb{Z} \)-grading is compatible with the \( \mathbb{Z}_2 \) grading on \( V \) in the sense that an element of degree \( d \) in the \( \mathbb{Z} \)-grading has degree
We say that a vector space \( V \) is \( Z \)-graded if there exists a decreasing sequence of vector subspaces
\[
V \supseteq \cdots \supseteq V^{k-1} \supseteq V^k \supseteq V^{k+1} \supseteq \cdots \supseteq 0.
\]
To emphasise that we are taking \( V \) with its filtration, we will denote it by \( V^* \). We say that \( V^* \) is finite if there exist some \( k_1, k_2 \) such that \( V = V^{k_1} \) and \( V = V^{k_2} \). Of course, if \( V \) is finite-dimensional then \( V^* \) must be finite. We say that \( V^* \) (not necessarily finite) has finite depth \( h \in \mathbb{Z}_{\geq 0} \) if \( V_{-h} \neq 0 \) but \( V_k = 0 \) for all \( k < -h \).

**Exterior algebra**

If \( V \) is a vector superspace, we define the exterior superalgebra \( \wedge^V := \mathcal{F}(V)/I \) where \( \mathcal{F}(V) \) is the tensor algebra and \( I \) is the ideal generated by elements of the form \( v \otimes w + (-1)^{|v||w|} w \otimes v \), where \( v, w \) are homogeneous elements with parities \( |v|, |w| \) respectively. This is a \( Z \)-graded supercommutative associative algebra of depth 0; the grading is \( \wedge^k V = \bigoplus_{k=0}^{\infty} \wedge^k V \) where \( \wedge^k V \) is the image of \( \mathcal{F}^k(V) = V \otimes k \) under the quotient map. One can show that, for \( V = V_{\bar{0}} \oplus V_{\bar{1}} \),
\[
\wedge^k V = \bigoplus_{i+j=k} \wedge^i V_{\bar{0}} \otimes \wedge^j V_{\bar{1}}.
\]  

It will occasionally be useful to remind the reader of this definition of \( \wedge^k \) of a vector superspace; when we do so, we will say that we mean it in the “super-sense”. There is of course an analogous “super” version of the symmetric algebra \( \mathcal{S}^* V \) but we will not require it in this work.

**Z-filtration**

We say that a vector space \( V \) is \( Z \)-filtered if there exists a decreasing sequence of vector subspaces
\[
V \supsetneq \cdots \supsetneq V^{k-1} \supsetneq V^k \supsetneq V^{k+1} \supsetneq \cdots \supsetneq 0.
\]
To emphasise that we are taking \( V \) with its filtration, we will denote it by \( V^* \). We say that \( V^* \) is finite if there exist some \( k_1, k_2 \) such that \( V = V^{k_1} \) and \( V = V^{k_2} \). Of course, if \( V \) is finite-dimensional then \( V^* \) must be finite. We say that \( V^* \) (not necessarily finite) has finite depth \( h \in \mathbb{Z}_{\geq 0} \) if \( V_{-h} = V \) but \( V_k \neq V \) for all \( k > -h \).

Given a \( Z \)-filtered vector space, there is an associated graded vector space
\[
\text{Gr} V^* = \bigoplus_{k \in \mathbb{Z}} \text{Gr}_k V \text{ where } \text{Gr}_k = \frac{V^k}{V^{k+1}}.
\]
If the filtration \( V^* \) is finite then so is the grading on \( \text{Gr} V^* \), and if \( V^* \) has finite depth \( h \), so does \( \text{Gr} V^* \). The associated graded vector space \( \text{Gr} V^* \) is (non-canonically) isomorphic to to \( V^* \), which can be seen as follows. For each \( V_k \), we choose a complement \( W_k \) to \( V^{k+1} \) so that \( V^k = W_k \oplus V^{k+1} \). Then we have \( V = \bigoplus_{k \in \mathbb{Z}} W_k \) and the quotient map \( V^k \to \text{Gr}_k V^* \) restricts to a linear isomorphism \( W_k \to \text{Gr}_k V^* \). Taking
the sum of these maps gives us a vector space morphism (in fact a graded morphism) \( V \to \text{Gr} V^\ast \).

If \( V^\ast \) is equipped with a Lie (resp. associative) algebra structure, then it is \( \mathbb{Z} \)-filtered as an algebra if \( [V^k, V^l] \subseteq V^{k+l} \) (resp. \( V^k \cdot V^l \subseteq V^{k+l} \)). Its associated graded vector space is naturally a \( \mathbb{Z} \)-graded algebra; for \( X \in \text{Gr}_k V^\ast, Y \in \text{Gr}_l V^\ast \) we choose representatives \( \tilde{X} \in V^k, \tilde{Y} \in V^l \) and then define a bracket \([\cdot, \cdot]_{\text{Gr}}\) (resp. multiplication ‘\( \text{Gr} \)) as described above. In fact, it is easy to see that this is an isomorphism of graded algebras.

Filtered morphisms

A filtered morphism \( f : V^\ast \to W^\ast \) of filtered vector spaces is a vector space morphism \( f : V \to W \) such that \( f(V^l) \subseteq W^l \). We analogously define a filtered morphism of filtered Lie superalgebras. A filtered morphism induces an associated graded morphism \( \text{Gr} f : \text{Gr} V^\ast \to \text{Gr} W^\ast \) by \( \text{Gr} f([v]) = [f(v)] \). One can easily check that this is well-defined and respects gradings. We say that a filtered morphism is strict if \( f(V^k) = f(V) \cap W^k \).

There is a technical issue to be aware of which makes the notion of strict filtered morphism necessary. An isomorphism of vector spaces or algebras which is filtered need not have filtered inverse, and the associated graded morphism need not be an isomorphism, as in the following example.

Example 1. [70, Example 0108] Consider two different filtrations \( V^\ast \) and \( W^\ast \) on the underlying vector space \( \mathbb{R} \) given by

\[
V^k = \begin{cases} 
\mathbb{R} & \text{for } k < 0, \\
0 & \text{for } k \geq 0,
\end{cases} \quad W^k = \begin{cases} 
\mathbb{R} & \text{for } k \leq 0, \\
0 & \text{for } k > 0.
\end{cases}
\]

The identity morphism \( f = \text{Id}_\mathbb{R} \) is clearly an isomorphism of vector spaces which respects the filtration, but \( f(V^0) = \{0\} \subset W^0 = \mathbb{R} \), so it is not strict. The inverse \( f^{-1} = \text{Id}_\mathbb{R} \) is not filtered since \( f(W^0) = \mathbb{R} \) while \( V^0 = 0 \). The graded components of the associated graded vector spaces are

\[
\text{Gr}_k V^\ast = \begin{cases} 
\mathbb{R} & \text{for } k = -1, \\
0 & \text{for } k \neq -1,
\end{cases} \quad \text{Gr}_k W^\ast = \begin{cases} 
\mathbb{R} & \text{for } k = 0, \\
0 & \text{for } k \neq 0.
\end{cases}
\]

and \( \text{Gr} f = 0 \) since if \( x \in \text{Gr} V^\ast \) is non-zero then \( x \in \text{Gr}_{-1} V^\ast \) but then \( (\text{Gr} f)([x]) = [f(x)] = 0 \) since \( \text{Gr}_{-1} W^\ast = 0 \).
On the other hand, for a strict filtered morphism $f$, if $f$ is injective (respectively surjective) then $\text{Gr} f$ is injective (respectively surjective). Moreover, a filtered isomorphism is strict if and only if its inverse is filtered, and the inverse of a strict filtered isomorphism is also strict. We say that two filtered vector spaces (resp. Lie superalgebras) are isomorphic if there exists a strict filtered isomorphism between them.

Filtered superspaces and filtered superalgebras

If $V = V_0 \oplus V_\mathbb{T}$ is a vector superspace, we say that a vector space filtration $V^*$ is a filtration of the superspace if it is compatible with the $\mathbb{Z}_2$-grading in the following sense. We demand that the filtered components are graded subalgebras:

$$V^k = V_0^k \oplus V_\mathbb{T}^k,$$

(note that $V_0^*$ and $V_\mathbb{T}^*$ are filtrations of the odd and even subspace) such that

$$V_0^{2k-1} = V_0^{2k}, \quad \text{and} \quad V_\mathbb{T}^{2k} = V_\mathbb{T}^{2k+1}.$$  

(2.95)

Note that this means $\text{Gr} V_0^* V_\mathbb{T}^* = 0$, while $\text{Gr} V_0^* V_\mathbb{T}^* = \text{Gr} V_0^* V_\mathbb{T}^*$. This ensures that the $\mathbb{Z}_2$-grading induced on $\text{Gr} V^*$ is compatible with the $\mathbb{Z}$-grading.

**Remark 2.** These compatibility conditions ensure that if $g^*$ is a filtered Lie superalgebra, $\text{Gr} g^*$ is a $\mathbb{Z}$-graded Lie superalgebra; in particular, the $\mathbb{Z}$-grading is compatible with the $\mathbb{Z}_2$ grading. This technicality seems to be somewhat under-emphasised in the literature; it is mentioned in the classic reference [71] that the $\mathbb{Z}$-grading on $\text{Gr} g^*$ is not compatible with the $\mathbb{Z}_2$-grading, but sufficient conditions for compatibility are only given for a particular class of filtrations there, not for a general filtration. Our main reference [46] for §2.3 also mentions that a compatibility condition is required but does not spell it out. The conditions above appear in [72].

**Notation**

To avoid overloading notation, outside of the preceding discussion we drop the $\bullet$ superscript from filtered vector spaces and the $\text{Gr}$ subscript from the operation on graded algebras; the filtration structure should be implicitly understood whenever we discuss such spaces. In the following subsection, a bullet superscript is used for cohomological grading.

2.3.2 Spencer cohomology

Now let $g = \bigoplus_{k=h}^{\infty} g_k$ be a $\mathbb{Z}$-graded Lie superalgebra of finite depth\(^{16}\) $h$ such that $\dim g_k < \infty$ for all $k$. Ultimately, we will be interested in the case where $\dim g < \infty$, but our source [46] mainly treats examples where $g$ itself is infinite-dimensional, so

---

\(^{16}\)The Spencer cohomology theory in its original form was only applicable to algebras of depth 1; for those of greater depth Cheng and Kac use the term *generalised* Spencer cohomology. We do not make this distinction.
following loc. cit., we keep the discussion more general for now. We let $g_- = \bigoplus_{k<0} g_k$ and note that it is finite-dimensional.

The Spencer complex

We define the Spencer cochain complex $(C^*(g_-; g), \partial^*)$ as follows. For $p < 0$, we set $C^p(g_-; g) := 0$, while for $p \geq 0$, we have

$$C^p(g_-; g) := (\wedge^p g_-^\ast) \otimes g \cong \text{Hom}(\wedge^p g_-; g), \quad (2.96)$$

where $\wedge^p$ is meant in the super-sense. The differential $\partial^* : C^*(g_-; g) \to C^{*+1}(g_-; g)$ is given by the following formula: for $\phi \in C^p(g_-; g)$, $X_1, \ldots, X_r \in g_0$ and $Y_1, \ldots, Y_s \in g_T$, where $r + s = p + 1$,

$$(\partial^* \phi)(X_1, \ldots, X_r, Y_1, \ldots, Y_s) = \sum_{i=1}^{r} (-1)^{i+1} [X_i, \phi(X_1, \ldots, \hat{X}_i, \ldots, X_r, Y_1, \ldots, Y_s)] + (-1)^r \sum_{s=1}^{r} [Y_j, \phi(X_1, \ldots, X_r, Y_1, \ldots, \hat{Y}_j, \ldots, Y_s)] + \sum_{1 \leq i < j \leq r} (-1)^{i+j} \phi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r, Y_1, \ldots, Y_s) + \sum_{i=1}^{r} \sum_{j=1}^{s} (-1)^i \phi(X_1, \ldots, \hat{X}_i, \ldots, X_r, [X_i, Y_j], Y_1, \ldots, \hat{Y}_j, \ldots, Y_s) + \sum_{1 \leq i < j \leq s} \phi([Y_i, Y_j], X_1, \ldots, X_r, Y_1, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_s), \quad (2.97)$$

where the hat decorations denote omission of an entry. It can be checked that $\partial^2 = 0$, and so $(C^*(g_-; g), \partial^*)$ is indeed a cochain complex. This is nothing but the standard Chevalley–Eilenberg complex for $g_-$ with values in $g$, where the former acts on the latter via the restriction of the adjoint representation. However, since $g$ is graded, there is additional structure.

The grading on $g$ induces a grading on $g^\ast$ defined by $(g^\ast)_k := (g_-^\ast)_k$. This in turn induces a grading on the space of $p$-cochains, in the usual manner for tensor products of graded spaces. Equivalently, we can consider a $p$-cochain to be homogeneous of degree $d$ if it has degree $d$ as a map $\wedge^p g_- \rightarrow g$. We then denote the space of $p$-cochains of degree $d$ as $C^{d,p}(g_-; g)$,

$$C^{d,p}(g_-; g) = (\wedge^p g_-^\ast \otimes g)_d = \text{Hom}(\wedge^p g_-; g)_d, \quad (2.98)$$

so that

$$C^p(g_-; g) = \bigoplus_{d \in \mathbb{Z}} C^{d,p}(g_-; g) \quad \text{and} \quad C^{d,\ast}(g_-; g) := \bigoplus_{p \in \mathbb{Z}} C^{d,p}(g_-; g). \quad (2.99)$$

This grading is preserved by the Spencer differential, so by restricting the differential to the subspaces of cochains with degree $d$, for each $d \in \mathbb{Z}$ we have a subcomplex $(C^{d,\ast}(g_-; g), \partial^*)$. 

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Finally, we note that the adjoint action of $g_0$ on $g$ preserves the grading, thus so does the natural action of $g_0$ on cochains induced by the adjoint representation (which also trivially preserves the homological grading); each $C^p(g_-, g)$ is a graded $g_0$-module. This action also preserves the differential and thus all of the structure on the subcomplex $(C^d,\partial, g_-, g)$.

The cohomology

We define the cocycles, coboundaries and cohomology of the $d$-subcomplex in the usual way:

\[
Z^{d,p}(g_-, g) = \ker (\partial^p : C^p(g_-, g) \to C^{p+1}(g_-, g)),
\]

\[
B^{d,p}(g_-, g) = \text{Im} (\partial^p : C^{p-1}(g_-, g) \to C^p(g_-, g)),
\]

\[
H^{d,p}(g_-, g) = Z^{d,p}(g_-, g) / B^{d,p}(g_-, g),
\]

and these are all naturally $g_0$-modules, since the action of $g_0$ preserves $(C^d,\partial, g_-, g)$. Of course, the space of $p$-cocycles of the whole complex is simply the direct sum of the graded $(d,p)$-cocycles, and similarly for the coboundaries, whence we have a natural isomorphism of $g_0$ modules

\[
H^p(g_-, g) \cong \bigoplus_{d \in \mathbb{Z}} H^{d,p}(g_-, g)
\]

where the former is the cohomology of the whole complex.

Examples and interpretation in low homological degree

The definition of the differential (2.97) on its own is not very enlightening, so let us demonstrate it explicitly on cochains of low homological degree and give and interpretation of the spaces of cocycles and the cohomologies. Fix $d \in \mathbb{Z}$ and homogeneous elements $X, Y, Z \in g_-$; we denote their degree by $|X|$ etc. Then for $\phi \in C^{d,0}(g_-, g) = g_d$, we have

\[
(\partial \phi)(X) = [X, \phi],
\]

which shows that $(d,0)$-cocycles are $g_-$-invariants in $g_d$; equivalently, they are degree-$d$ elements of $g$ which centralise $g_-$, and $(d,1)$-coboundaries are (super)derivations $g_\rightarrow g$ given by restricting inner derivations of $g$ induced by elements of degree $d$. For $\phi \in C^{d,1}(g_-, g) = \text{Hom}(g_-, g)_d$,

\[
(\partial \phi)(X, Y) = [X, \phi(Y)] - [Y, \phi(X)] - \phi([X, Y]),
\]

so $(d,1)$-cocycles are (super)derivations $g_\rightarrow g$ of degree $d$. Thus we find that $H^{d,0}(g_-, g) \cong (g_d)^g$ and $H^{d,1}(g_-, g) = \text{der}(g_-, g)_d / g_-$. For cocycles in higher degree, we do not have such simple descriptions, but for the calculations in this work we will need expressions only for homological degree $p \leq 2$. Thus we make the $p = 2$ case
our final example; for \( \phi \in C^{d,2}(g_-; g) = \text{Hom}(\wedge^2 g_-; g)_d \) we have

\[
(\partial \phi)(X, Y, Z) = [X, \phi(Y, Z)] - \phi([X, Y], Z) + (-1)^{|X||Y|+|Z|} \big( [Y, \phi(Z, X)] - \phi([Y, Z], X) \big) + (-1)^{|X|+|Y|} \big( [Z, \phi(X, Y)] - \phi([Z, X], Y) \big)
\]

(2.106)

2.3.3 Filtered deformations

Let us now consider a \( \mathbb{Z} \)-graded algebra (Lie, super-Lie or associative) \( G \). A filtered deformation of \( g \) is a filtered algebra \( F \) (of the same type) for which \( \text{Gr} F \cong G \) as graded algebras.

**Example 2.** Let \( (V, \eta) \) be an inner-product space of dimension \( n \). Then the Clifford algebra \( \text{Cl}(V, \eta) \) is a filtered deformation of the exterior algebra \( \Lambda^* V \). Indeed, identifying the two as \( \text{so}(V) \)-modules as in \( \S 2.1.1 \), we note that the exterior algebra is naturally a supercommutative \( \mathbb{Z} \)-graded associative algebra. However, due to our conventions (in particular because we use decreasing filtrations), we choose to give \( p \)-vectors degree \( -p \); this makes \( \Lambda^* V \) into a graded algebra of finite depth \( n \) and all positive-graded components zero. The natural filtration has \( k \)-th filtered component (for \( k \geq 0 \)) \( \oplus_{p=-k}^0 \Lambda^{-p} V \). From the defining Clifford relation (2.17), we have

\[
v \cdot w = v \wedge w + \eta(v, w)1
\]

(2.107)

for \( v, w \in V \). Since \( v, w \) have degree \(-1\) but \( \eta(v, w)1 \) has degree \( 0 \), the Clifford multiplication does not preserve the grading. However the problematic term is of lower degree, and since the Clifford algebra is generated by \( V \), this holds true for all products: the Clifford product of an element of degree \(-p\) and an element of degree \(-q\) is a sum of an element of degree \(-(p + q)\) and elements of higher degree, whence the Clifford multiplication preserves the filtration. Now, it is not hard to check that the natural identification of vector spaces \( \text{Gr}(\Lambda^* V) \cong \text{Gr Cl}(V, \eta) \) is in fact an isomorphism of graded algebras, whence our claims since \( \Lambda^* V \cong \text{Gr}(\Lambda^* V) \cong \text{Gr Cl}(V, \eta) \). Essentially, passing to the associated-graded kills off the terms of higher degree, leaving \( v \cdot w = v \wedge w \) (for example).

The example above can be used to motivate the term filtered deformation and model how we will treat them; we view the filtered algebra as an alternative algebra structure on the same underlying vector space as its associated graded, and this explicitly takes the form of a deformation of the graded operation in which the terms which violate the grading are of strictly higher degree than the graded product. Let us now spell this out more explicitly for Lie superalgebras, which will be the objects of interest in this work.

**Explicit deformations and defining sequences**

Let us fix a filtered Lie superalgebra \( \widetilde{g} = \widetilde{g}^{-h} \supseteq \widetilde{g}^{-h+1} \supseteq \ldots \) of finite depth \( h \) and choose, non-canonically, some complementary subspaces \( W_k \) such that \( \widetilde{g}^k = W_k \oplus \widetilde{g}^{k+1} \) and \( W_k \subseteq \widetilde{g}_k^\mathbb{Z} \) (which we can demand due to the compatibility conditions (2.94) and (2.95)).
As we have already seen, this induces a graded vector superspace isomorphism
\( \Phi: \tilde{g} = \bigoplus_{k=-h}^{\infty} W_k \to \text{Gr} \tilde{g} \) where for \( X \in W_k \), \( \Phi(X) = \tilde{X} \in \text{Gr} \tilde{g} \).

Now let \( X \in W_k \) and \( Y \in W_l \). Then since \( \tilde{g}^{k+l} = W_{k+l} \oplus \tilde{g}^{k+l+1} = W_{k+l} \oplus W_{k+l+1} \oplus \ldots \) and the bracket is filtered, we have

\[
[X, Y] = \mathfrak{pr}_{k+l} [X, Y] + \mathfrak{pr}_{k+l+1} [X, Y] + \ldots,
\]

(2.108)

where \( \mathfrak{pr}_j : \tilde{g}^{k+l} \to W_j \) are the projection maps. We thus define a \textit{graded} bracket \( \{-, -\}_0 \) on \( \tilde{g} = \bigoplus_{k=-h}^{\infty} W_k \) as well as a sequence of maps \( \{\mu_d : \wedge^2 \tilde{g} \to \tilde{g}\}_{d>0} \), where \( \mu_d \) has degree \( d \), by declaring that, on homogeneous elements \( X \in W_k \) and \( Y \in W_l \),

\[
[X, Y]_0 := \mathfrak{pr}_{k+l} [X, Y], \quad \mu_d(X, Y) := \mathfrak{pr}_{k+l+d} [X, Y]
\]

(2.109)

and extending linearly. It can be easily verified that \( \{-, -\}_0 \) is a Lie superalgebra bracket; alternatively, it follows from our next observation. Then on on homogeneous elements \( X \in W_k \) and \( Y \in W_l \) again, we have

\[
[\Phi(X), \Phi(Y)] = \left[ \tilde{X}, \tilde{Y} \right]_{\text{Gr}} = [X, Y] = [X, Y]_0 = \Phi([X, Y]_0)
\]

(2.110)

where in each equality we have applied a definition, and in the last equality we have used the fact that \( [X, Y]_0 \in W_{k+l} \) is homogenous. Thus, since homogeneous elements span \( \tilde{g} \), \( \Phi \) is a Lie superalgebra isomorphism\(^{17}\) \( (\tilde{g}, \{-, -\}_0) \cong (\text{Gr} \tilde{g}, \{-, -\}_{\text{Gr}}) \).

Let us now change perspective slightly. What we have essentially shown here is that if \( g = \bigoplus_{k=-h}^{\infty} g_k \) is a fixed \( \mathbb{Z} \)-graded superalgebra, a filtered deformation \( \tilde{g} \) of \( g \) can be thought of as an alternative Lie algebra structure on the same underlying vector space as \( g \); now denoting by \( \{-, -\} \) the graded bracket on \( g \) and by \( \{-, -\}' \) the bracket of \( \tilde{g} \), for \( X, Y \in g \) we have

\[
[X, Y]' = [X, Y] + \mu_d(X, Y);
\]

(2.111)

for some positive degree maps \( \{\mu_d : \wedge^2 g \to g\}_{d>0} \); that is, the primed bracket can be thought of as a deformation of the unprimed bracket by these maps. We will call this sequence a \textit{defining sequence} of the filtered deformation. This sequence is not unique; as we have already seen, the description of the filtered deformation in this way depends on the choices of complementary subspaces. We also cannot simply define a deformed bracket by choosing some arbitrary sequence \( \{\mu_d\}_{d>0} \); the Jacobi identity for \( \{-, -\}' \) imposes relations on the defining sequence. Thus, we need a particular algebraic tool to understand the possible filtered deformations of \( g \), and that is Spencer cohomology.

### Relation to Spencer cohomology

**Proposition 2.6.** Let \( g = \bigoplus_{k=-h}^{\infty} g_k \) be a graded Lie superalgebra and \( \tilde{g} \) a filtered deformation with defining sequence \( \{\mu_d : \wedge^2 g \to g\}_{d>0} \). Then

\(^{17}\)We note that this shows in particular that although the graded bracket \( \{-, -\}' \) defined by a different choice of complementary subspaces is not equal to \( \{-, -\}_0 \), the graded Lie superalgebra structures defined by the two brackets are isomorphic.
(1) [46, Prop.2.1] The first non-zero term $\mu_k$ in $\{\mu_d\}_{d>0}$ is an even Chevalley–Eilenberg 2-cocycle of $g$ with values in the adjoint representation:
$$\mu_k \in Z^2(g; g)_{\mathfrak{g}}.$$

(2) [46, Prop.2.2] Restricting $\mu_k$ to $g_-$ gives us a Spencer cocycle:
$$\mu_k|_{\wedge^2 g_-} \in Z^{k,2}(g_-; g).$$

(3) [46, Prop.2.2] The cohomology class of this cocycle is $g_0$-invariant:
$$[\mu_k|_{\wedge^2 g_-}] \in H^{k,2}(g_-; g)_{g_0}.$$

We will not prove this or any of the following statements here; we simply note that the above essentially follows by expanding the Jacobi identity for the deformed bracket and treating it degree-by-degree. We will do this for some explicit examples in Sections 3.3 and 4.3. The results below are more involved but ultimately follow by the same kind of reasoning. We extract from [46] only the statements we will need in the rest of this work; see loc. cit. for full proofs and a more general treatment.

**Definition 2.7.** Let $g$ be a graded Lie superalgebra of finite depth $h$. Then $g$ is said to be:

- fundamental if $g_-$ is generated by $g_{-1}$;
- transitive if for all $X \in g_k$ with $k \geq 0$, $[X, g_-] = 0 \Rightarrow X = 0$;
- a full prolongation (of $\bigoplus_{k=-h}^0 g_k$) of degree $k$ if $H_{d,1}^d(g_-; g) = 0$ for all $d \geq k$;

We will not need fundamentality for the following result, but it is useful to record it here since we will often consider this property alongside the other two defined above.

**Proposition 2.8.** Let $\tilde{g}, \tilde{g}'$ be two filtered deformations of $g$ with defining sequences $(\mu_d : \wedge^2 g \to g)_{d>0}$, $(\mu_d' : \wedge^2 g \to g)_{d>0}$ respectively. Then

(1) [46, Prop.2.3] If for some $k \geq 1$, $\mu_k - \mu'_k|_{\wedge^2 g_-} \in B^{k,2}(g_-; g)$, then $\tilde{g}'$ has a defining sequence $(\mu''_d)_{d>0}$ such that $\mu''_i = \mu'_i$ for $i < k$ and $\mu''_k|_{\wedge^2 g_-} = \mu_k|_{\wedge^2 g_-}$.

(2) [46, Prop.2.5] If $g$ is transitive then the defining sequence $(\mu_d)_{d>0}$ is completely determined by its restriction to $g_- \otimes g$.

(3) [46, Prop.2.6] If $g$ is transitive and for some $k \geq 1$, $g$ is an almost full prolongation of degree $k$, $\mu_i|_{g_- \otimes g} = \mu'_i|_{g_- \otimes g}$ for $i < k$ and $\mu_k|_{\wedge^2 g_-} = \mu'_k|_{\wedge^2 g_-}$, then $\tilde{g}'$ has a defining sequence $(\mu''_d)_{d>0}$ such that $\mu''_i = \mu'_i$ for $i \leq k$ and $\mu''_i|_{\wedge^2 g_-} = \mu'_i|_{\wedge^2 g_-}$ for all $i$.

This is a rather technical set of results, but it essentially allows us to redefine the defining sequence of a filtered deformation to partially align it with that of another; under certain conditions, this will allow to completely align the two sequences and show that two deformations are isomorphic. We will make use of this in Sections 3.3 and 4.3.
2.3.4 A note on “polarised” and “depolarised” expressions

We finish this section by making note of a useful technique we will often employ in calculations involving Lie superalgebras and elsewhere maps on symmetric products of vector spaces appear.

Let \( U, W \) be some (finite-dimensional) vector spaces and let \( \Psi \in \bigotimes^n U \to W \) be some completely symmetric multilinear map. Then \( \Psi \) is completely determined by “polarised” values of the form \( \Psi(u, u, \ldots, u) \) for elements \( u \in U \). The most well-known application of this is the polarisation identity which is used to recover an inner product from its induced norm in the \( n = 2 \) case, but it holds more generally. This greatly simplifies calculations involving such maps. When we seek to show that a symmetric tensor vanishes or that two such tensors agree, we will generally do this by showing that the desired property holds in its polarised form, where all inputs are taken to be the same. When we need to “depolarise” such an expression, we will indicate that this has been done but will not offer further explanation than what has been indicated here. Let us prove the polarisation identity for \( n = 2, 3 \), which are the cases we will use; the general case can be proved inductively.

Let \( u_1, u_2 \in U \) and let \( n = 2 \). Then, using bilinearity and symmetry,

\[
\Psi(u_1 + u_2, u_1 + u_2) = \Psi(u_1, u_1) + \Psi(u_2, u_2) + 2\Psi(u_1, u_2) \tag{2.112}
\]

which we can rearrange to

\[
\Psi(u_1, u_2) = \frac{1}{2}(\Psi(u_1 + u_2, u_1 + u_2) - \Psi(u_1, u_1) - \Psi(u_2, u_2)) \tag{2.113}
\]

which is a sum of polarised expressions. For \( n = 3 \), we must prove the identity in multiple steps. We have

\[
\Psi(u_1 + u_2, u_1 + u_2, u_3) = \Psi(u_1, u_1, u_3) + \Psi(u_2, u_2, u_3) + 2\Psi(u_1, u_2, u_3) \tag{2.114}
\]

for \( u_1, u_2, u_3 \in U \), so

\[
\Psi(u_1, u_2, u_3) = \frac{1}{2}(\Psi(u_1 + u_2, u_1 + u_2, u_3) - \Psi(u_1, u_1, u_3) - \Psi(u_2, u_2, u_3)) \tag{2.115}
\]

where the right-hand side is now partially polarised, in that two of the arguments in each term are equal. Note that we could have obtained this by using the \( n = 2 \) identity on the first two arguments, which suggests and inductive approach for the proof for general \( n \). It now suffices to show that we can write partially-polarised expressions such as \( \Psi(u_1, u_1, u_2) \) as a sum of fully polarised expressions, whence we consider

\[
\Psi(u_1 + u_2, u_1 + u_2, u_1 + u_2) = \Psi(u_1, u_1, u_1) + \Psi(u_2, u_2, u_2) + 3\Psi(u_1, u_1, u_2) + 3\Psi(u_1, u_2, u_2); \tag{2.116}
\]

subtracting a similar expansion for \( \Psi(u_1 - u_2, u_1 - u_2, u_1 - u_2) \) gives

\[
\Psi(u_1 + u_2, u_1 + u_2, u_1 + u_2) - \Psi(u_1 - u_2, u_1 - u_2, u_1 - u_2)
= 2\Psi(u_2, u_2, u_2) + 6\Psi(u_1, u_1, u_2) \tag{2.117}
\]
which we can rearrange to

\[
\Psi(u_1, u_1, u_2) = \frac{1}{6}(\Psi(u_+, u_+, u_+) - \Psi(u_, u_, u_) - 2\Psi(u_2, u_2, u_2)) \tag{2.118}
\]

where \(u_\pm = u_1 \pm u_2\), and the right-hand side is a sum of fully polarised expressions, as required.

## 2.4 Homogeneous spaces

Homogeneous spaces are manifolds equipped with a transitive group action; informally, this means that they “look the same” at every point. We will make this precise by showing that structures on the tangent bundle and other vector bundles over a homogeneous space are in one-to-one correspondence with linear structures on the fibre over a point. They are relevant to this work due to the Homogeneity Theorem (Theorem 3.21); a highly supersymmetric supergravity background is (at least locally) homogeneous.

### 2.4.1 Homogeneous spaces and Klein pairs

**Definition 2.9.** Let \(G\) be a Lie group. A **homogeneous** \(G\)-space is a manifold \(M\) on which \(G\) acts smoothly and transitively. A morphism of such spaces is a \(G\)-equivariant smooth map, and an isomorphism is a \(G\)-equivariant diffeomorphism.

The isotropy group of a point \(p\) in such a space is the stabiliser subgroup \(G_p = \{ g \in G \mid g \cdot p = p \}\).

Fixing a point \(p \in M\), we note that the orbit map \(\alpha_p : G \to M\) given by \(\alpha_p(g) = g \cdot p\) is surjective (by transitivity), and \(G_p = \alpha_p^{-1}(p)\), so \(G_p\) is a closed subgroup of \(G\). If \(G\) is connected, \(M\) must also be connected since it is the image of \(G\) under \(\alpha_p\). We will often take \(G\) to be connected when discussing homogeneous \(G\)-spaces. Note that all of the isotropy groups of the action of \(G\) on \(M\) are conjugate; for all \(g \in G\) we have \(G_{g.p} = g G_p g^{-1}\).

**Isotropy representation and Frobenius reciprocity**

The derivative at \(p \in M\) by the diffeomorphism \(\phi_g\) of \(M\) associated to an element \(g \in G\) is an invertible linear map \(d_p\phi_g : T_p M \to T_{g \cdot p} M\); in particular, if \(h \in G_p\) then \(d_p\phi_{h} \in \text{GL}(T_p M)\). This defines a representation of \(G_p\) on \(T_p M\) known as the **isotropy representation**.

The derivative of \(\alpha_p\) at the identity of \(G\) is a linear map \(d_e \alpha_p : g = T_e G \to T_p M\) with kernel \(g_p = \text{Lie } G_p\). One can show that \(d_e \alpha_p \circ \text{Ad}_h = d_p\phi_h \circ d_e \alpha_p\) for all \(h \in G_p\); in other words, \(d_e \alpha_p\) is \(G_p\)-equivariant where \(G_p\) acts on \(g\) via the restriction of the adjoint representation of \(G\). Thus there exists a short exact sequence of \(G_p\)-modules

\[
0 \longrightarrow g_p \longrightarrow g \xrightarrow{d_e \alpha_p} T_p M \longrightarrow 0. \tag{2.119}
\]
invariant tensor fields on $M$ are in one-to-one correspondence with $G_p$-invariant tensors on $T_pM$. This correspondence is easy to describe: if $T$ is a $G$-invariant tensor field then its value $T_p$ at $p$ is clearly an $G_p$-invariant tensor on $T_pM$; conversely, if $t$ is a $G_p$-invariant tensor on $T_pM$ then we can define a tensor field $T$ on $M$ by defining $T_{g^{-1}p} := (d_p\phi_g)(t)$. This tensor field $T$ is well-defined and $G$-invariant precisely because $t$ is $G_p$-invariant and $d\phi_g g^\prime = d\phi_g \circ d\phi_g$. We will see a more general version of Frobenius reciprocity in due course.

**Klein pairs and metric Klein pairs**

**Definition 2.10.** A Lie pair is a pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$.

A Klein Pair is a pair $(G, H)$ where $G$ is a connected Lie group and $H$ is a closed subgroup of $G$.

The homogeneous $G$-space associated to $(G, H)$ is the left coset space $G/H$.

The coset space $G/H$ is indeed a homogeneous $G$-space; clearly $G$ acts on it transitively, and it is a smooth manifold because $H$ is closed. It is connected because $G$ is connected, and $H$ is the isotropy group for $\alpha := H$ (the coset of the identity). Conversely, if $M$ is a homogeneous space for a connected group $G$ and $p \in M$ then $(G, G_p)$ is a Klein pair, and $M$ is of course the associated homogeneous space, up to isomorphism of $G$-spaces. Fixing a connected Lie group $G$, there is thus a one-to-one correspondence between conjugacy classes of closed subgroups of $G$ and isomorphism classes of homogeneous $G$-spaces.

Note that we assume that $G$ is connected, but $H$ may not be connected. We will discuss the effect of this on the topology of $M$ below.

The discussion of the isotropy representation above shows that $T_o(G/H) \cong \mathfrak{g}/\mathfrak{h}$ as $H$-modules, and that $H$-invariant tensors on $\mathfrak{g}/\mathfrak{h}$ induce $G$-invariant tensor fields on $G/H$. We will be particularly interested in $H$-invariant (pseudo-)inner products $\eta$ on $\mathfrak{g}/\mathfrak{h}$, which induce $G$-invariant pseudo-Riemannian metrics $g$ on $G/H$.

**Example 3.** The Klein pair $(\text{SO}(3), \text{SO}(2))$, where the latter is embedded as the subgroup of block-diagonal matrices $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, has associated homogeneous $\text{SO}(3)$-space the unit 2-sphere in $\mathbb{R}^3$; $\text{SO}(2)$ is the stabiliser of the point $(0,0,1)$. The Killing form on $\mathfrak{so}(3)$ induces a positive-definite inner product on $\mathfrak{so}(3)/\mathfrak{so}(2)$ which in turn induces, up to a scalar multiple, the standard Riemannian metric on the 2-sphere.

**Definition 2.11.** A metric Lie pair is a triple $(\mathfrak{g}, \mathfrak{h}, \eta)$ where $(\mathfrak{g}, \mathfrak{h})$ is a Lie pair and $\eta$ is an $\mathfrak{h}$-invariant inner product on $\mathfrak{g}/\mathfrak{h}$.

A metric Klein pair is a triple $(G, H, \eta)$ where $(G, H)$ is a Klein pair and $\eta$ is an $H$-invariant inner product on $\mathfrak{g}/\mathfrak{h}$.

The homogeneous pseudo-Riemannian $G$-space associated to $(G, H, \eta)$ is the pair $(G/H, g)$, where $g$ is the pseudo-Riemannian metric induced by $\eta$.

Given a metric Klein pair $(G, H, \eta)$, set $M = G/H$, $o = H$ and note that since $\eta$ is $H$-invariant, the image of the isotropy representation, which we now denote by $\varphi : H \to \text{GL}(T_oM)$, is contained in $O(T_oM) \subseteq \text{GL}(T_oM)$. If $M$ is oriented, then since
$G$ is connected, its action is orientation-preserving, hence the action of $H$ on $T_o M$ must therefore also be orientation-preserving, so $\text{Im} \varphi \subseteq \text{SO}(T_o M)$. If the signature of $\eta$ is indefinite and $M$ is both oriented and time-oriented, we similarly find that $\text{Im} \varphi \subseteq \text{SO}_0(T_o M)$.

### 2.4.2 Homogeneous spaces as principal bundles

#### Principal bundles and connectivity

Given a Klein pair $(G, H)$, right multiplication by elements of $H$ on $G$ gives the coset map $G \rightarrow M = G / H$ the structure of a (right) $H$-principal bundle over $M$. Note that we only consider the left action of $G$ on itself, and $G$ does not act on the right of $M$ in a natural way, while $H$ acts on both $G$ and $M$ from the left and right. The coset map $G \rightarrow M = G / H$ is equivariant with respect to both the left action of $G$ and the right action of $H$.

We have a fibration $H \rightarrow G \rightarrow M$, hence a long exact sequence of groups in homotopy ending in

\[ \cdots \rightarrow \pi_1(G) \rightarrow \pi_1(M) \rightarrow \pi_0(H) \rightarrow \pi_0(G) = 1, \tag{2.120} \]

where we take the basepoints of $G$ and $H$ to be the identity element and that of $M$ to be $o = H$. The group structure on $\pi_0(H)$ is induced by that of $H$, and $\pi_0(G)$ is trivial since we assume $G$ to be connected. Thus if $M$ is simply connected, $H$ must be connected. The converse is not true in general, but if we assume that $G$ is simply connected, $\pi_1(M) \cong \pi_0(H)$, so $M$ is simply connected if and only if $H$ is connected.

When working with a homogeneous $G$-space $M = G / H$, it is often useful to pass to the universal cover $\tilde{G}$ of $G$ by pulling back the action of $G$ on $M$ to $\tilde{G}$. We can then consider $M$ as a homogeneous space corresponding to the Klein pair $(\tilde{G}, H')$ where $H'$ is the preimage of $H$ under the covering map $\tilde{G} \rightarrow G$. Note that the universal cover $\tilde{M}$ of $M$ is then a homogeneous space corresponding to $(\tilde{G}, H'_0)$, where $H'_0$ is the connected component of $H'$.

#### Associated bundles and Frobenius reciprocity

Let $V = T_o M$ for our convenience. Many natural bundles over $M$ can be described as associated bundles with respect to representations of $H$. For example, there is a $G$-equivariant isomorphism of vector bundles

\[ T M \cong G \times_H V \tag{2.121} \]

where $H$ acts on $V$ via the isotropy representation and $G$ acts on $T M$ via the push-forward of the action on $M$ and on the associated bundle via $g \cdot [g', v] = [gg', v]$.

The isomorphism in the direction $G \times_H V \rightarrow T M$ is given by $[g, v] \mapsto g_* v$. The isotropy representation induces representations of $H$ on $V^* = T_o^* M$ and its exterior powers, giving us further $G$-equivariant isomorphisms

\[ T^* M \cong G \times_H V^*, \tag{2.122} \]
\[ V^* \cong \mathcal{A}_H V^*. \] (2.123)

The action of \( H \) on \( \text{End}(V) \) by conjugation by elements of \( \text{GL}(V) \) similarly gives us
\[ \text{End}(TM) \cong \mathcal{A}_H \text{End}(V) \] (2.124)

Considering the isotropy representation as a Lie group morphism \( H \to \text{GL}(V) \) gives various actions of \( H \) on \( \text{GL}(V) \) – by conjugation or by left or right multiplication by the images of elements in \( H \), for example – but we will concentrate on the left action by left multiplication. This allows us to form a \( \text{GL}(V) \)-principal bundle \( G \times_H \text{GL}(V) \) equipped with a compatible (left) action of \( G \), where the two groups act by right multiplication on the right factor and left multiplication on the left factor respectively. Now, fix a frame \( f = (f_1, \ldots, f_n) \) at \( o \) \( (n = \dim M) \), that is, \( f \) is an ordered basis for \( V \). This choice induces a group isomorphism \( \text{GL}(V) \cong \text{GL}(\mathbb{R}^n) \) by representing elements of \( \text{GL}(V) \) in the frame \( f \); in Einstein notation, for \( A \in \text{GL}(V) \) we define the components of the matrix \( A \in \text{GL}(\mathbb{R}^n) \) by \( A^i_j f_j \). This allows us to treat the frame bundle \( FM \to M \) as a \( \text{GL}(V) \)-principal bundle; if \( f' \in FM \) is another frame (at any point), we have
\[ (f' \cdot A)_i = f'_j A^j_i. \] (2.125)

This choice of frame \( f \) also allows us to define a \( G \)-equivariant isomorphism of \( \text{GL}(V) \)-principal bundles (that is, a \((G, \text{GL}(V))\)-equivariant bundle morphism)
\[ FM \cong \mathcal{A}_H \text{GL}(V) \] (2.126)

where the left action of \( G \) on frames is induced by its action on tangent vectors,
\[ g \cdot (f'_1, \ldots, f'_n) = (g \cdot f'_1, \ldots, g \cdot f'_n) = (g \ast f_1', \ldots, g \ast f_n'); \] (2.127)

the isomorphism is given by mapping \( G \times_H \text{GL}(V) \ni [g, A] \mapsto g \cdot f \cdot A \).

Given a metric Klein pair \((G, H, \eta)\) with orientation, the isotropy representation \( H \to \text{SO}(V) \) allows us to construct a \( G \)-equivariant isomorphism of \( \text{SO}(V) \)-bundles
\[ F_{\text{SO}} \cong \mathcal{A}_H \text{SO}(V), \] (2.128)

where \( F_{\text{SO}} \to M \) is the special orthonormal frame bundle of \((M, g)\), in a completely analogous way – the construction requires a choice of oriented orthonormal frame \( f \). Similarly, without an orientation we have \( F_0 \cong \mathcal{A}_H \text{O}(V) \), and with time orientation we have \( F_{\text{SO}_0} \cong \mathcal{A}_H \text{SO}_0(V) \), where the notation should be self-explanatory.

We can make particularly fruitful use of these isomorphisms with Frobenius reciprocity, which we have already seen an application of. This a powerful result which appears in a number of different guises in the literature but in general relates representations of a group \( G \) restricted to a subgroup \( H \) to representations of \( G \) induced by representations of \( H \). The most useful version to us here is due to Bott.

**Proposition 2.12** (Frobenius reciprocity theorem [73]). *Let \( (G, H) \) be a Klein pair, let \( W \) be a \( G \)-module and let \( W' \) be an \( H \)-module. Then there is a an isomorphism of*
vector spaces
\[ \text{Hom}_G(W, \Gamma(G \times H W' \to G/H)) \cong \text{Hom}_H(W, W'). \] (2.129)

In the statement above, we treat \( W \) as an \( H \)-module by restricting the action of \( G \) on \( W \) and we treat the space of sections \( \Gamma(G \times H W' \to G/H) \) as a \( G \)-module by acting on the left factor of each fibre. We recover our previous statement about invariant tensor fields by noting, for instance, that, setting \( W = \mathbb{R} \) the trivial module and \( W' = V = T_o M \), the \( G \)-equivariant bundle isomorphism \( TM \cong G \times H V \) gives
\[ X(M)^G \cong V^H, \] (2.130)
so \( G \)-invariant vector fields on \( M \) are in correspondence with \( H \)-invariant vectors at \( o \). Similarly, using the \( G \)-equivariant bundle morphism \( G \times H V \to V = T^*_o M \), we have
\[ \Omega^*(M)^G \cong (\wedge^* V^*)^H, \] (2.131)
and similarly \( \text{End}(TM)^G \cong \text{End}(V)^H \) etc.

### 2.4.3 Reductive and symmetric spaces

Let us briefly define some particular classes of homogeneous spaces which are much more well-understood.

**Definition 2.13.** A Klein pair \((G, H)\) is called **reductive** if \( \mathfrak{g} \) splits under the adjoint action of \( H \); that is, if there exists some subspace \( \mathfrak{m} \subseteq \mathfrak{g} \) such that \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) and \( \text{Ad}_H(\mathfrak{m}) \subseteq \mathfrak{m} \). Such a Lie pair is symmetric if we also have \([\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}\).

We will not work directly with these definitions or much of the theory of reductive and symmetric homogeneous spaces; we will simply quote some results which will be used in later chapters. See [74] for a comprehensive treatment. Note, however, that if \( H \) is compact then any Klein pair \((G, H)\) is reductive (since any finite-dimensional representation of a compact group is fully reducible). More generally, if there exists an \( \text{Ad}_H \)-invariant (positive-definite) inner product on \( \mathfrak{h} \), \((G, H)\) is reductive since we can set \( \mathfrak{m} = \mathfrak{h}^\perp \).

Riemannian and Lorentzian symmetric spaces have been classified, as we will discuss in §2.4.5. There is no such classification of reductive homogeneous spaces, but reductivity leads to significant simplifications – we will see an example in the next subsection – so is often assumed.

### 2.4.4 Connections, Wang's Theorem and Nomizu maps

Just like tensor fields, invariant connections on homogeneous spaces have a simpler description in terms of equivariant linear maps. This formalism was introduced for affine connections on the homogeneous spaces of reductive Lie pairs by Nomizu [75]. We will discuss a generalisation due to Wang [76]. A good reference for all statements is [74].

For a Lie pair \((G, H)\), we once again denote \( M = G/H \), \( V = T_o M \cong \mathfrak{g}/\mathfrak{h} \) where \( o = H \). Note that since \( H \) acts on \( V \) via the isotropy representation which we now denote \( \varphi : H \to \text{GL}(V) \); it acts on \( \mathfrak{gl}(V) \) by conjugation.
Wang’s Theorem

Suppose that \( \pi : P \to M \) is a right principal \( K \)-bundle for some Lie group \( K \), and suppose that the (left) action of \( G \) on \( M \) lifts to a (left) action on \( P \) which is compatible with the action of \( K \); that is, we have an action of \( G \) on \( P \) such that \( \pi \) is \( G \)-equivariant and \((g \cdot p) \cdot k = g \cdot (p \cdot k)\) for all \( g \in G \) and \( k \in K \). Then note that \( H \) preserves the fibre \( P_0 \), and by fixing a basepoint \( p \in P_0 \) we can define a group homomorphism \( \psi : H \to K \) by declaring \( h \cdot p = p \cdot \psi(h) \). A different choice of basepoint changes \( \psi \) by conjugation by an element of \( K \). We can now state the following.

**Theorem 2.14** ([76]). Let \((G, H)\) be a Lie pair and let \( P \to M = G/H \) be a right principal \( K \)-bundle for some Lie group \( K \). Suppose that the left action of \( G \) lifts to \( P \) compatibly with the action of \( K \). Fix a basepoint \( p \in P_0 \) and let \( \psi : H \to K \) be the induced group morphism. Then there is a one-to-one correspondence between \( G \)-invariant principal connections \( \omega \in \Omega^1(P;\mathfrak{k}) \) on \( P \) and linear maps \( \Psi : \mathfrak{g} \to \mathfrak{k} \) such that

\[
\begin{aligned}
(1) \quad & \Psi \circ \text{Ad}^G_h = \text{Ad}^K_{\psi(h)} \circ \Psi \text{ for all } h \in H, \\
(2) \quad & \Psi(X) = (d_\xi\psi)(X) \text{ for all } X \in \mathfrak{h}.
\end{aligned}
\]

The correspondence is given by

\[
\Psi(X) = \omega_p(\xi_X)
\]

where \( X \in \mathfrak{g} \) and \( \xi_X \) is the fundamental vector field on \( P \) associated to \( X \). Moreover, the curvature \( F_\omega \in \Omega^2(P;\mathfrak{k})^G \) is given by

\[
(F_\omega)_p(\xi_X, \xi_Y) = [\Psi(X), \Psi(Y)]_\mathfrak{k} - \Psi([X, Y]_\mathfrak{g}).
\]

The final equation uniquely determines \( \omega_p \) since \( G \)-transitivity on the base implies that any fibre is carried to \( P_0 \) by the action of some element of \( G \), \( K \) acts freely and transitively on each fibre, and \( \omega \) is a \( G \)-invariant principal connection.

**Affine connections and Nomizu maps**

Now let us consider some special cases. If \( P = FM \) is the frame bundle, recall from §2.4.2 that a choice of frame \( f \) in \( F_0 M \) gives \( FM \) the structure of a principal \( K = \text{GL}(V) \)-bundle and that we have a compatible left action of \( G \). Then we immediately have \( \Psi = \varphi : H \to \text{GL}(V) \), the isotropy representation. Moreover, if \((G, H, \eta)\) is an orientable metric Lie pair, we can similarly take \( P = FSO \), \( K = \text{SO}(V) \) and \( \Psi = \varphi : H \to \text{SO}(V) \). Now recall that an affine connection is equivalent to a principal connection on \( FM \), and a metric affine connection is equivalent to a principal connection on \( FSO \).

Finally, we note that if the Lie pair is reductive with the reductive split \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) then we have \( \mathfrak{m} \cong V \) as \( H \)-modules, and taking \( \Psi \) as above there is a unique map \( \Psi_\mathfrak{m} : \mathfrak{m} \to \text{gl}(\mathfrak{m}) \) (or \( \mathfrak{m} \to \text{so}(\mathfrak{m}) \) in the metric case) such that \( \Psi = \varphi + \Psi_\mathfrak{m} \).

These observations give us the following.

**Corollary 2.15.** Let \((G, H)\) be a Lie pair and \( M = G/H \). Then
(1) Affine connections on $M$ are in one-to-one correspondence with maps $\Psi : \mathfrak{g} \to \mathfrak{gl}(V)$ such that

(a) $\Psi \circ \text{Ad}_h^G = \text{Ad}_{\phi(h)}^{\mathfrak{gl}(V)} \circ \Psi$ for all $h \in H$,
(b) $\Psi(X) = (d\phi)(X)$ for all $X \in \mathfrak{h}$,

where $\phi : H \to \text{GL}(V)$ is the isotropy representation;

(2) If $(G, H)$ is a metric Lie pair, metric affine connections on $M$ are in one-to-one correspondence with such maps $\Psi$ taking values in $\mathfrak{so}(V)$;

(3) If $(G, H)$ is a reductive (metric) Lie pair, (metric) affine connections on $M$ are in one-to-one correspondence with $H$-invariant maps $\Psi_m : m \to \mathfrak{gl}(m)$ ($H$-invariant maps $\Psi_m : m \to \mathfrak{so}(m)$).

The final part of this corollary is Nomizu’s original result. As such, it is standard to call $H$-invariant maps $\Psi_m : m \to \mathfrak{gl}(m)$ Nomizu maps. This terminology is sometimes extended to the map $\Psi$ for non-reductive pairs. We note that in the reductive case, the curvature and torsion of an affine connection connection can be expressed as $H$-invariant linear maps using the Nomizu maps, but we will not directly need this.

**Nomizu map for the Levi-Civita connection**

We finish this section by describing the Nomizu map associated to the Levi-Civita connection for a metric Lie pair $(G, H, \eta)$. This is a generalisation of the description in the reductive case given in [74] to the non-reductive case that appears in e.g. [45].

We first define a (degenerate) symmetric bilinear form on $\mathfrak{g}$ by

$$
\langle X, Y \rangle = \eta([X, Y])
$$

where $X, Y$ denote the images in $V = \mathfrak{g}/\mathfrak{h}$ of $X, Y \in \mathfrak{g}$; note that $\langle -, - \rangle$ is $H$-invariant since $\eta$ is. We then define $U : \mathfrak{g} \to V$ by the equation

$$
2\eta(U(X, Y), v) = \langle X, [Z, Y] \rangle + \langle [Z, X], Y \rangle
$$

(2.134)

for all $X, Y \in \mathfrak{g}$, $v \in V$ and $Z \in \mathfrak{g}$ an arbitrary element such that $\overline{Z} = u$. This is well-defined by $\mathfrak{h}$-invariance of $\langle -, - \rangle$. We can then define $\tilde{L} : \mathfrak{g} \to \text{Hom}(\mathfrak{g}, V)$ by

$$
\tilde{L}(X)Y = \frac{1}{2} \langle X, Y \rangle + U(X, Y).
$$

(2.135)

One can check that $\tilde{L}(\mathfrak{h}) = 0$, so $\tilde{L}$ factors through a map $L : \mathfrak{g} \to \mathfrak{gl}(V)$ which is a Nomizu map and also satisfies

(1) $\text{Im} L \subseteq \mathfrak{so}(V)$,

(2) $\tilde{L}(X)(Y) - \tilde{L}(X)(Y) - [X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.

These two properties precisely say that the affine connection corresponding to $L$ is metric-compatible and torsion-free, hence it is the Levi-Civita connection.

**2.4.5 Classification results**

We now make note of some classification results on pseudo-Riemannian homogeneous spaces which we make use of in Chapter 7.
Cahen–Wallach classification of Lorentzian symmetric spaces

Riemannian and Lorentzian symmetric spaces have been classified, the first by Cartan and the second by Cahen and Wallach [77], building on Cartan’s classification. The Riemannian classification reduces to the problem of classifying simply connected irreducible symmetric spaces by the observations that (1) the universal cover of a Riemannian symmetric space is Riemannian symmetric and (2) if a simply connected Riemannian symmetric space is reducible, say \((M, g) = (M_1, g_1) \times (M_2, g_2)\) then its factors \((M_i, g_i)\) are simply connected and symmetric. Thus any Riemannian symmetric space is the product of simply connected irreducibles quotiented by the action of a finite group. An entirely analogous argument holds in the Lorentzian case, except that in part (2) one of the factors \((M_i)\) is Lorentzian and the other Riemannian, so a Lorentzian symmetric space is a quotient of a product of one simply connected irreducible Lorentzian factor with a product of simply connected Riemannian irreducibles. The classification of the irreducibles then follows via an application of the Cartan classification of finite-dimensional simple Lie algebras.

If one is only concerned with identifying a Lorentzian symmetric space up to local isometry (as we will be), one can assume without loss of generality that the space is simply connected and then decompose it into a product (for example, via the de Rham–Wu Decomposition Theorem [78–80]). The Cahen–Wallach result is used to identify the Lorentzian irreducible factor and the Cartan result to identify the Riemannian factors. We will not actually need to use the Cartan classification so will not discuss it any further here. We quote the statement of the Cahen-Wallach classification given by JMF in his review on Lorentzian symmetric spaces in supergravity [81].

**Theorem 2.16 ([77, 81]).** Let \((M, g)\) be a simply connected Lorentzian symmetric space. Then \(M\) is isometric to the product of a simply connected Riemannian symmetric space and one of the following:

- \(\mathbb{R}\) with the standard timelike metric;
- The universal cover of \(dS_n\) or \(AdS_n\), where \(n \geq 2\);
- A Cahen–Wallach space \(CW_n(A)\) where \(n \geq 3\).

The first two spaces should be familiar; we will briefly give a local description of the latter which we will make use of in Chapter 7. See [81] for a more modern review of Lorentzian symmetric spaces including an intrinsic, coordinate-free definition of \(CW_n(A)\) as a symmetric homogeneous space.

We describe \(CW_n(A)\) in an adapted coordinate system with mostly-minus metric. This is an \(n\)-dimensional Lorentzian manifold (for \(n \geq 3\)) associated to an \((n-2) \times (n-2)\) real symmetric matrix \(A\) with components \(A_{ij}\) with metric \(g\) in a coordinate system \(\{x^+, x^-, x^1, ..., x^{n-2}\}\)

\[
g = 2 dx^+ dx^- + A_{ij} x^ix^j (dx^-)^2 - dx^i dx^i, \tag{2.136}
\]

where \(i, j = 1, ..., n-2\). By a coordinate transformation, one can see that \(CW_n(A) \cong CW_n(B)\) if and only if \(B = cMAM^T\) for some orthogonal \((n-2) \times (n-2)\) matrix \(M\) and
some \( c > 0 \). Using this observation, it is not hard to see that \( CW_n(A) \) is irreducible if and only if \( A \) is non-degenerate; if \( A \) is degenerate with nullity \( k \) then we (locally) have \( CW_n(A) = CW_{n-k}(A') \times \mathbb{R}^k \) as Riemannian product for some non-degenerate \((n-k-2) \times (n-k-2)\) matrix \( A' \).

The CW metric has a Witt coframe given by
\[
\theta^+ = dx^+ + \frac{1}{2} A_{ij} x^i x^j \theta^-, \quad \theta^- = dx^-, \quad \theta^i = dx^i.
\] (2.137)

Working in this coframe, the connection 1-forms are given by
\[
\omega^+ = \omega_{ij} = \omega_{i+} = 0, \quad \omega^- = 2 A_{ij} x^j \theta^-,
\]
and the only non-zero curvature 2-forms are
\[
\Omega_{-i} = -A_{ij} \theta^- \wedge \theta^i,
\] from which we read off that non-zero components of the Riemann tensor are \( R_{-i-j} = A_{ij} \). One can easily check that \( CW_n(A) \) is conformally flat if and only if \( A \) is scalar.

**Komrakov’s classification in low dimensions**

Pseudo-Riemannian homogeneous spaces (or more precisely, isotropically faithful metric Klein pairs) in low dimensions have been classified globally in 2 and 3 dimensions and locally (i.e. in terms of metric Lie pairs) in 4 dimensions. We will not give a detailed discussion of this highly non-trivial problem here; we simply make a note of the sources and give a very brief sketch of the classification scheme.

We say that a Klein pair is *isotropically faithful* if the isotropy representation \( \varphi : H \to GL(V = \mathfrak{h}/\mathfrak{g}) \) is faithful; \( \ker \varphi = 1 \). This implies that the action of \( G \) on \( M = G/H \) is effective, since it is not hard to see that if \( g \in G \) acts trivially, \( g \in \ker \varphi \).

This is an important class of homogeneous spaces; it contains all those homogeneous spaces possessing an invariant affine connection. If we fix \( n = \dim M \), the local classification of isotropically faithful homogeneous spaces reduces to classifying subalgebras \( \mathfrak{h} \) of \( gl(V) \cong gl(n, \mathbb{R}) \) and then pairs \((\mathfrak{g}, \mathfrak{h})\) of codimension \( n \) such that \( \mathfrak{g}/\mathfrak{h} \cong V \) as \( \mathfrak{h} \)-modules.

Most of the progress on the classification problem is due to an extensive project undertaken by a group working at the International Sophus Lie Centre under Boris Komrakov in the 1990s. Much of the output of this collaboration is not published in journals and can only be found in preprints, beginning with the 3-part work [82–84]. As recounted there, the work on this classification began in the 19th century with Sophus Lie himself, who locally classified homogeneous spaces of dimension 2 over \( \mathbb{C} \) and the subalgebras of \( gl(3, \mathbb{C}) \) [85, 86]. Lie’s local classification in 2 dimensions was globally completed by Mostow [87]. The work of the Komrakov group began with a modern exposition of the (real) 2-dimensional case [88] followed by a classification of the subalgebras of \( gl(3, \mathbb{R}) \) which was used to classify the isotropically faithful Lie pairs over \( \mathbb{R} \) of codimension 3 [82–84]. They then classified the metric Lie pairs and their global counterparts in dimensions 2 and 3 [89]. B. Komrakov Jnr. then partially extended this to 4 dimensions, classifying the metric Lie pairs [90–93].

In Chapter 7, we will use these classification results to (at least locally) identify
some 5-dimensional supersymmetric geometries from the even subalgebra of their Killing superalgebras after a de Rham–Wu decomposition.
Chapter 3

Structure of Killing superalgebras

In this chapter, we introduce a general framework for understanding Killing algebras and superalgebras (depending on the symmetry of the squaring map) on spin manifolds in general signature and the connections which induce them, developing a notion of “admissible connection” and “Killing spinor” which does not make reference to any additional structure, such as that which might come along with a supergravity background. We define the Killing (super)algebra associated to an admissible connection and describe its algebraic structure, and then further study this structure using Spencer cohomology, focussing on the highly supersymmetric Lorentzian case, over which we have the most control. We also consider some two-dimensional examples to demonstrate the general theory. The final section is dedicated to reconstructing a background geometry on which a Killing superalgebra can be realised.

The framework developed here is intended to include the Killing spinors of supergravity theories (with some caveats which will be discussed), but not necessarily geometric Killing spinors, since the latter cannot always be organised into an algebra. However, in particular cases where these do form a Killing (super)algebra (see §3.2.4 and also [42]), they are indeed subsumed by the present discussion.

The approach of the geometric part of this chapter is inspired by previous work, especially [45], and many of the arguments are analogous, but it is significantly more general as that work treats only the case of 11-dimensional supergravity, whereas much of the present discussion is agnostic on dimension and signature and does not assume any structure arising from supergravity or any other context, although in later sections we do specialise to Lorentzian signature and a particular class of squaring maps motivated by technical simplifications offered by the Homogeneity Theorem. In particular, of the major geometric results in this chapter, Theorem 3.6 is new but has some overlap with discussion in the thesis of Noel Hustler [33], while Theorems 3.10 and 3.47 are generalisations of results from [45] to the new context using similar arguments. The major algebraic results, Theorem 3.36 and Proposition 3.42, are generalisations of results in [45, 47] which required some additional novel analysis.
3.1 Manifolds with spinors

We first establish the geometric setting in which we will define admissible connections, Killing spinors and Killing superalgebras, as well as some notation and formulae.

3.1.1 Spin structures and associated bundles

Let \((M, g)\) be a connected pseudo-Riemannian spin manifold of signature \((p, q)\). We denote the special orthonormal frame bundle by \(F_{SO} \to M\) and the spin structure by \(\varpi : P \to F_{SO}\). We denote the Lie algebra of Killing vector fields\(^1\) by \(\mathfrak{iso}(M, g)\). The Levi-Civita connection will be denoted \(\nabla\).

Let \(S\) be a (finite-dimensional) spinor representation of \(\text{Spin}(p, q)\) and let \(S := P \times_{\text{Spin}(p, q)} S\) be the associated vector bundle over \(M\), which we will call the spinor bundle. Sections of \(S\) will be called spinor fields, and we denote the space of such fields by \(\mathfrak{S} = \Gamma(S)\). We also let \(\text{Cl}(M, g) = P \times_{\text{Spin}(p, q)} \text{Cl}(p, q) = F_{SO} \times_{\text{SO}(p, q)} \text{Cl}(p, q)\) be the associated Clifford bundle and recall that we can identify it with the exterior bundle \(\wedge^\ast T^* M\). Sections of this bundle can be Clifford-multiplied and Clifford-act on spinor fields via pointwise Clifford multiplication and action. There is a Leibniz rule for \(\nabla\) with respect to Clifford multiplication and action. The bundle of skew-symmetric endomorphisms of \(T M\), which can be identified with the adjoint bundles \(\text{ad} F_{SO} \cong \text{ad} P\), embeds in \(\text{Cl}(M, g)\) as \(\wedge^2 T M\). We denote the space of sections of this bundle by \(\mathfrak{so}(M, g)\) and note that it is an (infinite-dimensional) Lie algebra under the commutator Lie bracket and that it acts on spinor fields via the Clifford action.

3.1.2 Lie derivatives

A standard concept in differential geometry, Lie derivatives will play a central role in the definition of the Killing (super)algebra. We will first introduce some useful formulae for Lie derivatives of tensors fields and then generalise these to allow us to take Lie derivatives of spinor fields along Killing vectors.

**Lie derivative of vectors and tensors**

Given a vector field \(X \in \mathfrak{X}(M)\), we define an endomorphism of \(T M\) by

\[
A_X(Y) := -\nabla_Y X
\]

for all \(Y \in \mathfrak{X}(M)\). This gives us a useful formula for the Lie derivative along \(X\): for \(Y \in \mathfrak{X}(M)\),

\[
\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X = \nabla_X Y + A_X Y,
\]

\(^1\)Despite the notation, this is not the Lie algebra of the isometry group of \((M, g)\) which is in general only a subalgebra of \(\mathfrak{iso}(M, g)\); Killing vector fields are infinitesimal isometries which may not integrate to a global action by a one-parameter group of isometries. We will not need to discuss the isometry group itself so this should not cause confusion.
where we have used the torsion-freeness of $\nabla$. We will also use

$$[X, Y] = A_X Y - A_Y X. \quad (3.3)$$

A similar formula holds a tensor field $T$ of any type:

$$\mathcal{L}_X T = \nabla_X T + A_X \cdot T \quad (3.4)$$

where the action of $A_X$ on $T$ is defined via a Leibniz formula. Note that we have a different Leibniz rule of the form

$$\nabla_X (A_Y \cdot T) = \nabla_X (A_Y) \cdot T + A_Y \cdot \nabla_X T, \quad (3.5)$$

for all $X, Y \in \mathfrak{X}(M)$, or, abstracting $T$, $\mathcal{L}_X [A_Y, A_X] = \nabla_X A_Y A_X - \nabla_X A_Y - \nabla_Y A_X$.

We can use the formula (3.4) to show that $X$ is a Killing vector if and only if $A_X \in \mathfrak{so}(M, g)$: using the metric-compatibility of $\nabla$, we have $\mathcal{L}_X g = A_X \cdot g$, so

$$\mathcal{L}_X g = 0 \iff g(A_X Z, Y) + g(Y, A_X Z) = 0 \quad \forall Y, Z \in \mathfrak{X}(M). \quad (3.6)$$

One can also show that, for $X \in \text{iso}(M, g)$ and $Y \in \mathfrak{X}(M)$,

$$\nabla_Y A_X = - R(Y, X) \quad (3.7)$$

where $R$ is the Riemann curvature considered as a 2-form with values in $\text{ad} F_{SO}$, so $R(Y, X) \in \mathfrak{so}(M, g)$. Then for two Killing vectors $X, Y \in \mathfrak{iso}(M, g)$ and a vector field $Z \in \mathfrak{X}(M)$, using equations (3.3) and (3.7) along with the algebraic Bianchi identity,

$$A_{[X,Y]} Z = -\nabla_Z (A_X Y - A_Y X)$$

$$= - (\nabla_Z A_X) Y - A_X \nabla_Z Y + (\nabla_Z A_Y) X + A_Y \nabla_Z Y$$

$$= R(Z, X) Y + A_X A_Y Z - R(Z, Y) X - A_Y A_X Z \quad (3.8)$$

$$= [A_X, A_Y] Z - R(X, Y) Z.$$

We thus have

$$A_{[X,Y]} = [A_X, A_Y] - R(X, Y) \quad (3.9)$$

for all Killing vectors $X$ and $Y$.

### Lie derivative of spinors

The formulae (3.2) and (3.4) suggest the following definition of the Lie derivative to spinor fields.

**Definition 3.1** (Spinorial Lie derivative [94]). The Lie derivative of a spinor field $\epsilon \in \mathfrak{S} = \Gamma(S)$ along a Killing vector $X \in \mathfrak{iso}(M, g)$ is given by

$$\mathcal{L}_X \epsilon := \nabla_X \epsilon + A_X \cdot \epsilon. \quad (3.10)$$

Note, however, that while the formulae for the Lie derivatives of vector and tensor fields hold for the Lie derivative along any vector field $X$, the right-hand side of
the spinor formula is only defined for Killing vectors\(^2\) \(X\), since sections of \(\text{End}(TM)\) do not act on \(\mathcal{G}\), while \(\mathfrak{so}(M, g)\) does acts on \(\mathcal{S}\). Note that \(\mathfrak{so}(M, g)\) is a (infinite-dimensional) Lie algebra and the action on \(\mathcal{G}\) defines a representation. Similarly to the action of sections of \(\text{End}(M)\) on tensors, this action obeys a Leibniz rule of the form \([\nabla_X A] = \nabla_X A\) as endomorphisms of \(\mathcal{G}\) for all \(A \in \mathfrak{so}(M, g)\) and \(X \in \mathfrak{X}(M)\); more explicitly,

\[
\nabla_X (A \cdot \epsilon) = (\nabla_X A) \cdot \epsilon + A \cdot (\nabla \epsilon)
\]

(3.11)

for all \(\epsilon \in \mathcal{G}\). It follows immediately from the above that we have yet another Leibniz rule

\[
\mathcal{L}_X (A \cdot \epsilon) = (\mathcal{L}_X A) \cdot \epsilon + A \cdot (\mathcal{L}_X \epsilon)
\]

(3.12)

for all \(X \in \mathfrak{so}(M, g)\), \(A \in \mathfrak{so}(M, g)\), or equivalently \([\mathcal{L}_X, A] = \mathcal{L}_X A\) as endomorphisms of \(\mathcal{G}\).

**Proposition 3.2.** The spinorial Lie derivative satisfies the following compatibility condition with the Levi-Civita connection:

\[
[\mathcal{L}_X, \nabla_Y] = \nabla_{[X,Y]}
\]

(3.13)

for all \(X \in \mathfrak{so}(M, g)\), \(Y \in \mathfrak{X}(M)\) as well as the following identity:

\[
\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X
\]

(3.14)

for all Killing vectors \(X, Y\). In particular, the spinorial Lie derivative gives \(\mathcal{G}\) the structure of an \(\mathfrak{so}(M, g)\) module.

**Proof.** Using the Leibniz rule in the form \([\nabla_Y A] = \nabla_Y A\) for sections \(A\) of \(\mathfrak{so}(M, g)\) along with equation (3.7) and the definition of the Riemann tensor, implicitly acting on \(\mathcal{G}\) we have

\[
[\mathcal{L}_X, \nabla_Y] = [\nabla_X, \nabla_Y] + [A_X, \nabla_Y]
= R(X, Y) + \nabla_{[X,Y]} - \nabla_Y A_X
= \nabla_{[X,Y]}
\]

(3.15)

for all \(X \in \mathfrak{so}(M, g)\), \(Y \in \mathfrak{X}(M)\). On the other hand, for all \(X, Y \in \mathfrak{so}(M, g)\), using the definitions of the Riemann tensor as well as equations (3.7) and (3.9), we have

\[
[\mathcal{L}_X, \mathcal{L}_Y] = [\nabla_X, \nabla_Y] + [\nabla_X, A_Y] + [A_X, \nabla_Y] + [A_X, A_Y]
= R(X, Y) + \nabla_{[X,Y]} + \nabla_X A_Y - \nabla_Y A_X + [A_X, A_Y]
= \nabla_{[X,Y]} + [A_X, A_Y] - R(X, Y)
= \nabla_{[X,Y]} + A_{[X,Y]}
= \mathcal{L}_{[X,Y]},
\]

(3.16)

hence the claim.

\[\square\]

We can also use the Leibniz rule to define an action of \(\mathfrak{so}(M, g)\), and thus a Lie derivative along Killing vectors, on sections of \(\text{End}\mathcal{G}\) and on spaces of forms with

---

\(^2\)The spinorial Lie derivative is actually defined for conformal Killing vector fields in [94], but we will only use it for Killing vectors here.
values in this bundle $\Omega^p(M; \text{End} S)$. More generally, the space of sections of any bundle constructed by taking duals and tensor products of $TM$ and $S$ is equipped with such a structure, and all the Leibniz rules and compatibility conditions discussed above also hold when acting on these spaces.

### 3.2 Killing spinors and Killing (super)algebras

We now consider the conditions under which a connection on spinors gives rise to a Killing (super)algebra, using this to motivate a definition of Killing spinor.

#### 3.2.1 Killing spinors

**Connections on spinors**

Let $D$ be a connection on the spinor bundle $S$. We denote by $\mathcal{S}_D$ the real vector space of parallel spinors with respect to $D$:

$$\mathcal{S}_D := \{ \epsilon \in S | D\epsilon = 0 \}. \quad (3.17)$$

This is finite-dimensional; indeed, by the usual arguments for spaces of parallel spinors, $\dim \mathcal{S}_D \leq \text{rank} S = \dim S$. We say that $\mathcal{S}_D$ is maximal if it has the maximal dimension $\dim \mathcal{S}_D$.

Since $\nabla$ and $D$ are both connections on $S$, their difference is an $\text{End} S$-valued 1-form. We thus define $\beta \in \Omega^1(M; \text{End} S)$ by

$$\beta := \nabla - D.$$ 

For all $X \in \text{iso}(M, g)$, $Y \in X(M)$, and $\epsilon \in \mathcal{S}_D$, we have

$$D_Y L_X \epsilon = \nabla_Y L_X \epsilon - \beta(Y) L_X \epsilon$$

$$= L_X \nabla_Y \epsilon - \nabla L_X Y \epsilon - \mathcal{L}_X (\beta(Y) \epsilon) + (\mathcal{L}_X \beta)(Y) \epsilon + \beta(L_X Y) \epsilon \quad (3.18)$$

so the spinorial Lie derivative along $X$ preserves $\mathcal{S}_D$ if and only if $\mathcal{L}_X \beta$ annihilates $\mathcal{S}_D$. For the sake of simplicity, and because it will capture the case of supergravity which is our ultimate interest, we will choose to work with Killing vectors which preserve $\beta$; we denote by $\mathcal{U}_D$ the real subspace of $\text{iso}(M, g)$ consisting of such Killing vectors:

$$\mathcal{U}_D := \{ X \in \text{iso}(M, g) | \mathcal{L}_X \beta = 0 \}. \quad (3.19)$$

This is in fact an ideal of $\text{iso}(M, g)$ since it is the annihilator of $\beta$ in the representation of $\text{iso}(M, g)$ on $\Omega^1(M; \text{End} S)$ defined by the Lie derivative. Clearly the representation of $\text{iso}(M, g)$ on $\mathcal{S}$ (also via Lie derivative) restricts to a representation of $\mathcal{U}_D$ on $\mathcal{S}_D$.

The curvature of the connection $D$ is the section $R^D \in \Omega^2(M; \text{End} S)$ defined by

$$R^D(X, Y) \epsilon = D_X D_Y \epsilon - D_Y D_X \epsilon - D_{[X,Y]} \epsilon \quad (3.20)$$

where $X, Y \in X(M)$ and $\epsilon \in \mathcal{S}$. One can see immediately that $R^D \epsilon = 0$ for all $\epsilon \in \mathcal{S}_D$, and the former equation is very useful as an integrability condition for the existence of $D$-parallel spinors as we will see when we tackle some examples. We will use the following result to compute $R^D$ and perform manipulations involving it.
Proposition 3.3. The curvature $R^D$ is given by the following formula, for $X, Y \in \mathfrak{X}(M)$:

$$R^D(X, Y)\epsilon = R(X, Y) \cdot \epsilon + [\beta(\epsilon), \beta(\epsilon)] \cdot \epsilon - (\nabla_X \beta)(Y)\epsilon - (\nabla_Y \beta)(X)\epsilon,$$  \hfill (3.21)

where $R$ is the Riemann curvature (understood as a 2-form with values in $\text{ad} F_{SO}$ and hence acting on $\mathfrak{S}$) and $[\beta_X, \beta_Y]$ is a commutator of endomorphisms.

Proof. For $\epsilon \in \mathfrak{S}$, we expand $D = \nabla - \beta$ in (3.20) to find

$$R^D(X, Y)\epsilon = \nabla_X(\nabla_Y \epsilon - \beta(Y)\epsilon) - \beta(X)(\nabla_Y \epsilon - \beta(Y)\epsilon)$$
$$- \nabla_Y(\nabla_X \epsilon - \beta(X)\epsilon) + \beta(Y)(\nabla_X \epsilon - \beta(X)\epsilon) - \nabla_{[X, Y]} \epsilon + \beta([X, Y])\epsilon$$
$$= (\nabla_X \nabla_Y \epsilon - \nabla_Y \nabla_X \epsilon - \nabla_{[X, Y]} \epsilon) + (\beta(X)\beta(Y) - \beta(Y)\beta(X)) \epsilon$$
$$= R(X, Y) \cdot \epsilon + [\beta(\epsilon), \beta(\epsilon)] \cdot \epsilon - (\nabla_X \beta)(Y)\epsilon - (\nabla_Y \beta)(X)\epsilon$$  \hfill (3.22)

where we have used the definition of the Riemann tensor and the Leibniz rule in the final line. A further application of the Leibniz rule as well as a the torsion-free property of $\nabla$ gives (3.21).

Remark 3. It is worth further clarifying the notation $R(X, Y) \cdot \epsilon$ above. As explained below (3.7), the Riemann curvature can be considered as a 2-form with values in $\text{ad} F_{SO}$ so that for all $X, Y \in \mathfrak{X}(M)$, $R(X, Y) \in \mathfrak{so}(M, g)$, thus it acts on $\mathfrak{S}$ via Clifford multiplication. Above, we have implicitly used the non-trivial fact that $R(X, Y) \cdot \epsilon = R^V(X, Y)\epsilon$ where $R^V \in \Omega^2(M; \text{End} \mathfrak{S})$ is the curvature 2-form of the spin-lift of the Levi-Civita connection, $R^V(X, Y)\epsilon := \nabla_X \nabla_Y \epsilon - \nabla_Y \nabla_X \epsilon - \nabla_{[X, Y]} \epsilon$. Thanks to this relation to the Riemann tensor, we will never have to use this curvature explicitly. In a local expression, we have $\pm \frac{1}{4} R_{\mu \nu \rho \sigma} \Gamma^r_{\rho \sigma} \epsilon = 2 \nabla_{[\mu} \nabla_{\nu]} \epsilon$ where the sign is the one in the Clifford relation (2.17).

Existence of the Killing (super)algebra

In order to define a Killing (super)algebra, we will need to be able to “square” (or pair) spinors to tangent vectors, just as we required a squaring map to define the flat model (super)algebras $s$ in Section 2.2. For this purpose, we will define a bundle map $\bigotimes^2 S \to TM$ using a squaring map on $S$ by exploiting the associated bundle structure of $\mathcal{S}$ and $TM$.

Lemma 3.4. Let $\kappa : \bigotimes^2 S \to V$ (resp. $\kappa : \bigwedge^2 S \to V$) be an $\mathfrak{so}(V)$-equivariant map. Then there exists a bundle map $\kappa : \bigotimes^2 \mathcal{S} \to TM$ (resp. $\kappa : \bigwedge^2 \mathcal{S} \to TM$) which is covariantly constant and which satisfies $\mathcal{L}_X \kappa = 0$ for all $X \in \mathfrak{iso}(M, g)$.

Proof. We prove the case with $\kappa : \bigotimes^2 S \to V$; the other case is completely analogous. Recalling that $\phi : P \to F_{SO}$ is the spin structure over $M$, by standard theory of associated bundles, we have a one-to-one correspondence

$$\Gamma(\bigotimes^2 \mathcal{S}^* \otimes TM) = \Gamma(P \times \text{Spin}(V) \bigotimes^2 \mathcal{S}^* \otimes V) \cong C^\infty_{\text{Spin}(V)}(P; \bigotimes^2 S^* \otimes V)$$  \hfill (3.23)
where the space on the far right-hand side consists of Spin(V)-equivariant maps \( P \to \mathcal{O}^2S^* \otimes V \), where the target is viewed as a right Spin(V)-module under the inverse of the natural left action. Note that \( \kappa \) is an invariant element of \( \mathcal{O}^2S^* \otimes V \), so the constant map \( \Psi_\kappa : P \to \mathcal{O}^2S^* \otimes V \) given by \( \Psi(p) = \kappa \) for all \( p \in P \) is trivially equivariant. The required bundle map is then the corresponding element \( \kappa \in \text{Hom}(\mathcal{O}^2S, TM) \cong \Gamma(\mathcal{O}^2S^* \otimes TM) \). One can then show by a local computation that \( \nabla \kappa = 0 \) since \( \Psi_\kappa \) is constant and \( \kappa \) is \( \mathfrak{so}(V) \)-equivariant, and the last claim follows similarly.

To lighten the notation, we will omit the underline from the bundle map \( \kappa \) and trust that this will not cause confusion. With the map \( \kappa \) fixed, it will be useful for the proofs of our next few results (and to make a connection to the notation used in Section 3.3) to define a section \( \gamma \) of the bundle \( \text{Hom}(\mathcal{O}^2S, \text{End}(TM)) \) as follows:

\[
\gamma(\epsilon, \zeta)X := -\kappa(\beta(X)\epsilon, \zeta) - \kappa(\epsilon, \beta(X)\zeta)
\]  

(3.24)

for \( \epsilon, \zeta \in S \) and \( X \in \mathfrak{X}(M) \). Note that \( \gamma \) has the same symmetry as \( \kappa \). We then have the following.

**Lemma 3.5.** For all \( \epsilon, \zeta \in S_D \), \( \gamma(\epsilon, \zeta) = A_{\kappa(\epsilon, \zeta)} \).

**Proof.** Let \( Z \in \mathfrak{X}(M) \). Then since \( \nabla_Z \epsilon = \beta(Z)\epsilon \),

\[
A_{\kappa(\epsilon, \zeta)}Z = -\nabla_Z \kappa(\epsilon, \zeta) = -\kappa(\nabla_Z \epsilon, \zeta) - \kappa(\epsilon, \nabla_Z \zeta)
\]

\[
= -\kappa(\beta(Z)\epsilon, \zeta) - \kappa(\epsilon, \beta(Z)\zeta) = \gamma(\epsilon, \zeta)
\]

(3.25)

hence the claim.

We now state our first main result followed by a major definition inspired by it.

**Theorem 3.6 (Existence of (super)algebra associated to \( D \)).** Let \((M, g)\) be a connected pseudo-Riemannian spin manifold, \( S \to M \) a bundle of spinors with the space of sections \( S = \Gamma(S) \) and let \( \kappa : \mathcal{O}^2S \to TM \) (resp. \( \kappa : \Lambda^2S \to TM \)) be a bundle map induced by a spinor squaring map as in Lemma 3.4. Let \( D \) be a connection on \( S \) and define \( \beta = \nabla - D \), where \( \nabla \) is the Levi-Civita spin connection. Define the vector spaces

\[
S_D = \{ \epsilon \in S | D\epsilon = 0 \}, \quad \mathfrak{V}_D = \{ X \in \mathfrak{so}(M, g) | \mathcal{L}X\beta = 0 \}.
\]

and the brackets

\[
[X, Y] = \mathcal{L}X Y, \quad [X, \epsilon] = \mathcal{L}X \epsilon, \quad [\epsilon, \zeta] = \kappa(\epsilon, \zeta),
\]

for \( X, Y \in \mathfrak{V}_D \), \( \epsilon, \zeta \in S_D \). Then \((\mathfrak{K}_D = \mathfrak{V}_D \oplus S_D, [-, -])\) is a Lie superalgebra (resp. algebra) if and only if the following conditions are satisfied for all \( \epsilon, \zeta, \eta \in S_D \):

\[
\gamma(\epsilon, \zeta) = A_{\kappa(\epsilon, \zeta)} \in \mathfrak{so}(M, g),
\]

(3.26)

\[
\mathcal{L}_{\kappa(\epsilon, \zeta)}\beta = 0,
\]

(3.27)

\[
\beta(\kappa(\epsilon, \zeta))\eta + \beta(\kappa(\zeta, \eta))\epsilon + \beta(\kappa(\eta, \epsilon))\zeta + \gamma(\epsilon, \zeta) \cdot \eta + \gamma(\zeta, \eta) \cdot \epsilon + \gamma(\eta, \epsilon) \cdot \zeta = 0,
\]

(3.28)

where \( \gamma \) is the map defined by equation (3.24).
Proof. We first check closure and then the Jacobi identities. We have \([\mathfrak{V}_D, \mathfrak{V}_D] \subseteq \mathfrak{V}_D\) and \([\mathfrak{S}_D, \mathfrak{S}_D] \subseteq \mathfrak{S}_D\) by construction, while \([\mathfrak{S}_D, \mathfrak{S}_D] \subseteq \mathfrak{V}_D\) if and only if
\[
\mathcal{L}_{\kappa(\epsilon, \zeta)} g = 0 \quad \text{and} \quad \mathcal{L}_{\kappa(\epsilon, \zeta)} \beta = 0 \quad \forall \epsilon, \zeta \in \mathfrak{S}_D.
\] (3.29)

By equation (3.6), \(\mathcal{L}_{\kappa(\epsilon, \zeta)} g = 0\) if and only if \(A_{\kappa(\epsilon, \zeta)} \in \mathfrak{so}(M, g)\). By Lemma 3.5, this is equivalent to condition (3.26). The latter condition in (3.29) is just (3.27).

The Jacobi identity for three vectors is clearly satisfied. For the others, we denote by \([-,-,-]\) the Jacobiator and find:
\[
[X, Y, \epsilon] = 0 \iff \mathcal{L}_{[X,Y]} \epsilon = \mathcal{L}_X \mathcal{L}_Y \epsilon - \mathcal{L}_Y \mathcal{L}_X \epsilon,
\] (3.30)
which simply says that \(\mathfrak{S}_D\) is a representation of \(\mathfrak{V}_D\), hence is true by construction;
\[
[X, \epsilon, \zeta] = 0 \iff \mathcal{L}_{\kappa(\epsilon, \zeta)} \kappa = \kappa (\mathcal{L}_{\kappa(\epsilon, \zeta)} \epsilon) + \kappa (\epsilon, \mathcal{L}_{\kappa(\epsilon, \zeta)} \epsilon),
\] (3.31)
which is satisfied by the last part of Lemma 3.4;
\[
[\epsilon, \zeta, \eta] = 0 \iff \mathcal{L}_{\kappa(\epsilon, \zeta)} \eta + \mathcal{L}_{\kappa(\eta, \epsilon)} \zeta + \mathcal{L}_{\kappa(\epsilon, \eta)} \epsilon = 0,
\] (3.32)
which, using the equation \(\nabla \epsilon = \beta \epsilon\) and Lemma 3.5, is condition (3.28). \(\square\)

We interpret this result as giving us a set of conditions on the spinor connection \(D\) for the existence of an associated (super)algebra. This inspires the following definition. Note that conditions (1) and (2) are stronger than (3.26) and (3.28) above; we demand that they hold for all spinors, not just Killing spinors. This makes the definition simpler to check, and it will also be useful for some later results.

**Definition 3.7** (Admissible connections, Killing spinors and Killing (super)algebras). Using the notation of the theorem above, a connection \(D = \nabla - \beta\) on \(\mathcal{S}\) is admissible if the following hold:

1. The map \(\gamma\) defined by equation (3.24) takes values in \(\text{ad} F_{SO}\) (that is, \(\gamma(\epsilon, \zeta) \in \mathfrak{so}(M, g)\) for all \(\epsilon, \zeta \in \mathfrak{S}\));
2. \(\beta(\kappa(\epsilon, \zeta)) \eta + \beta(\kappa(\zeta, \eta)) \epsilon + \beta(\kappa(\eta, \epsilon)) \zeta + \gamma(\epsilon, \zeta) \cdot \eta + \gamma(\zeta, \eta) \cdot \epsilon + \gamma(\eta, \epsilon) \cdot \zeta = 0\) for all \(\epsilon, \zeta, \eta \in \mathfrak{S}\).
3. \(\mathcal{L}_{\kappa(\epsilon, \zeta)} \beta = 0\) for all \(\epsilon, \zeta \in \mathfrak{S}_D\);

If \(D\) is admissible, the differential equation \(D \epsilon = 0\) (equivalently \(\nabla \epsilon = \beta \epsilon\)) is called the Killing spinor equation and \(\mathfrak{S}_D\) the space of Killing spinors. Furthermore, \(\mathfrak{V}_D\) is the space of restricted Killing vectors, and \(\mathfrak{K}_D\) is the Killing (super)algebra.

We now make some comments to connect the above with treatments of Killing spinors and Killing superalgebras in the supergravity literature. In a supergravity theory, the Killing spinor equation \(\nabla \epsilon = \beta \epsilon\) arises by demanding that the supersymmetry variation (with parameter \(\epsilon\)) of the gravitino vanishes, with the 1-form \(\beta\) being parametrised by some bosonic fields \(\Phi\). One then essentially checks that (1) and (2) hold and that the Killing spinor equation and bosonic equations of motion imply
that $\mathcal{L}_\kappa \Phi = 0$, hence $\mathcal{L}_\kappa \beta = 0$ where $\kappa$ is the square of a Killing spinor $\epsilon$, which is (3).

Often in the supergravity literature, the Killing superalgebra is considered to contain not all of the Killing vectors $X \in \mathfrak{V}_D$, but only those for which $\mathcal{L}_X \Phi = 0$ (hence $\mathcal{L}_X \beta = 0$), giving a subalgebra of what we call $\mathfrak{R}_D$ here in general. Another common and even more restrictive definition is that the Killing superalgebra only contains those Killing vectors which are generated by pairs of spinors. This corresponds to our notion of the Killing ideal (this terminology is also used in [45]).

**Definition 3.8 (Killing ideal).** Let $D$ be an admissible connection and $\mathfrak{R}_D$ its Killing (super)algebra. The Killing ideal is the ideal $\mathfrak{K}_D$ of $\mathfrak{R}_D$ generated by the Killing spinors; that is, $\mathfrak{K}_D = \mathfrak{V}_D \oplus \mathfrak{S}_D$ where $\mathfrak{S}_D = [\mathfrak{S}_D, \mathfrak{S}_D]$.

**Remark 4.** The conditions (3.26)-(3.28) (and the related conditions in the definition of an admissible connection) for existence of a Killing superalgebra actually concern only the Killing ideal; that is, given a (not necessarily admissible) spinor connection $D$, $\mathfrak{R}_D$ is a Lie (super)algebra (equivalently $D$ is admissible) if and only if $\mathfrak{K}_D$ is a Lie (super)algebra. Moreover, we will see later that conditions (1) and (2) are essentially Spencer cocycle conditions.

A similar general formulation of Killing superalgebras to the one given here was given in [33], however our treatment here is more general because we allow for arbitrary metric signature and squaring map $\kappa$ of arbitrary symmetry, and that work also makes the assumption mentioned above that $\beta$ is parametrised by some bosonic fields $\Phi$ and that the Killing vectors in the algebra satisfy $\mathcal{L}_X \Phi = 0$.

In §3.2.4, we will see some examples of admissible connections in 2 dimensions, and the Killing superalgebras of some of these examples will be described §3.3.6 after we have discussed the link between Killing superalgebras and Spencer cohomology in more detail.

### 3.2.2 Algebraic structure of Killing (super)algebras

We now turn to describing the algebraic structure of the (super)algebras we have just defined. Our treatment here is a direct generalisation of that of [45], which considered the case of Killing superalgebras of highly supersymmetric backgrounds of 11-dimensional supergravity.

**Killing transport**

Let $D$ be an admissible connection and let us define the vector bundle

$$\mathcal{E} := TM \oplus S \oplus \text{ad} F_{SO}. \tag{3.33}$$

We define a connection $\mathcal{D}$ on $\mathcal{E}$ as follows:

$$\mathcal{D}_Y(X, \epsilon, A) := (\nabla_Y X + AY, D_Y \epsilon, \nabla_Y A + R(Y, X)) \tag{3.34}$$

for $X, Y \in \mathfrak{X}(M), \epsilon \in \mathfrak{S}, A \in \mathfrak{so}(M, g)$.  

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Proposition 3.9. If \( D \) is an admissible connection, the parallel sections of \( \mathcal{D} \) are triples \((X, \epsilon, A_X)\), where \( X \) is a Killing vector, \( \epsilon \) a Killing spinor and \( A_X = -\nabla X \). Thus the \( \mathbb{R} \)-vector space of \( \mathcal{D} \)-parallel sections is isomorphic to \( \mathfrak{iso}(M, g) \oplus \mathcal{E}_D \).

Proof. If the section \((X, \epsilon, A)\) of \( \mathcal{E} \) is parallel with respect to \( \mathcal{D} \), clearly \( \epsilon \) is a Killing spinor and we have \( AY = -\nabla_Y X \) for all \( Y \in \mathfrak{X}(M) \), so \( A = -\nabla X = A_X \). Since \( A \in \mathfrak{so}(M, g) \), \( X \) must be a Killing vector. Conversely, if \( X \) is a Killing vector and \( \epsilon \) a Killing spinor, equation (3.7) gives \( \nabla_Y A_X = -R(Y, X) \), and so \((X, \epsilon, A_X)\) is parallel with respect to \( \mathcal{D} \). \( \square \)

Thanks to this result and since \( M \) is connected, if one knows the values \((X, \epsilon, A_X)_p\) at any point \( p \in M \) for \( X \in \mathfrak{iso}(M, g) \) and \( \epsilon \in \mathcal{E}_D \), one can recover the section \((X, \epsilon, A_X)\) by parallel transport with respect to \( \mathcal{D} \). For this reason, \( \mathcal{D} \) is referred to as the Killing transport connection, and \((X, \epsilon, A_X)_p\) is the Killing transport data of \((X, \epsilon)\) at \( p \). We define an injective \( \mathbb{R} \)-linear map sending a Killing pair to its Killing transport data at \( p \):

\[
\Phi_p : \mathcal{D}_D \rightarrow \mathcal{E}_p, \quad \Phi_p(X, \epsilon) := (X_p, \epsilon_p, (A_X)_p). \tag{3.35}
\]

Localising the Killing algebra at a point

If we restrict our attention to Killing pairs \((X, \epsilon) \in \mathcal{D}_D \), it is possible to recover the Killing transport data of \([(X, \epsilon), (Y, \zeta)]\) at a point from that of \((X, \epsilon)\) and \((Y, \zeta)\). We have

\[
[(X, \epsilon), (Y, \zeta)] = ([X, Y] + \kappa(\epsilon, \zeta), \mathcal{L}_X \zeta - \mathcal{L}_Y \epsilon), \tag{3.36}
\]

and the corresponding parallel section of \( \mathcal{E} \) is

\[
([X, Y] + \kappa(\epsilon, \zeta), \mathcal{L}_X \zeta - \mathcal{L}_Y \epsilon, A_{[X, Y]} + A_{\kappa(\epsilon, \zeta)}). \tag{3.37}
\]

We now expand some of the terms in this expression. By equations (3.3) and (3.9), we have

\[
[X, Y] = \nabla_X Y - \nabla_Y X = A_X Y - A_Y X, \tag{3.38}
\]
\[
A_{[X, Y]} = [A_X, A_Y] - R(X, Y). \tag{3.39}
\]

The Killing spinor equation in the form \( \nabla \epsilon = \beta \epsilon \), gives

\[
\mathcal{L}_Y \epsilon = \nabla_Y \epsilon + A_Y \cdot \epsilon = \beta(Y) \epsilon + A_Y \cdot \epsilon, \tag{3.40}
\]

and by Lemma 3.5,

\[
A_{\kappa(\epsilon, \zeta)} = -\kappa(\beta \epsilon, \zeta) - \kappa(\epsilon, \beta \zeta). \tag{3.41}
\]

Thus the triple (3.37) is given by

\[
[X, Y] + \kappa(\epsilon, \zeta) = A_X Y - A_Y X + \kappa(\epsilon, \zeta), \tag{3.42}
\]
\[
\mathcal{L}_X \zeta - \mathcal{L}_Y \epsilon = \beta(X) \zeta - \beta(Y) \epsilon + A_X \cdot \zeta - A_Y \cdot \epsilon, \tag{3.43}
\]
\[
A_{[X, Y]} + A_{\kappa(\epsilon, \zeta)} = [A_X, A_Y] - R(X, Y) - \kappa(\beta \epsilon, \zeta) - \kappa(\epsilon, \beta \zeta). \tag{3.44}
\]
We now evaluate the expressions above at \( p \) to obtain the Killing transport data of \( [(X,\epsilon),(Y,\zeta)] \). Let us define the inner product space \((V,\eta)=(T_pM,g_p)\) and identify \( S_p \) with the spinor module \( S \) of \( \text{Spin}(V) \) (where we omit \( \eta \) as usual); we then have \( \mathcal{E}_p = V \oplus S \oplus \mathfrak{so}(V) \) as a \( \text{Spin}(V) \)-module. For the sake of readability, we consider the data for the brackets \([X,Y],[X,\epsilon], \) and \([\epsilon,\zeta]\) separately; we find

\[
\Phi_p([(X,Y)]) = ((A_X)_p Y_p - (A_Y)_p X_p, 0, [(A_X)_p,(A_Y)_p] - R_p(X_p,Y_p)), \tag{3.45}
\]
\[
\Phi_p([(X,\epsilon)]) = (0, \beta_p(X_p) \epsilon_p + (A_X)_p \cdot \epsilon_p, 0), \tag{3.46}
\]
\[
\Phi_p([(\epsilon,\zeta)]) = (\kappa_p(\epsilon_p,\zeta_p), 0, -\kappa_p(\beta_p \epsilon_p, \zeta_p) - \kappa_p(\epsilon_p, \beta_p \zeta_p)), \tag{3.47}
\]

where \( \kappa_p \in \bigotimes^2 S \otimes V, \beta_p \in V \otimes \text{End} S, \) and \( R_p \in \bigwedge^2 V^* \otimes \mathfrak{so}(V) \) are the values of \( \kappa, \beta, \) and \( R \) at \( p \in M \). Thus we can indeed express the Killing transport data of any \([(X,\epsilon),(Y,\zeta)]\) at \( p \in M \) in terms of the Killing transport data of \((X,\epsilon)\) and \((Y,\zeta)\) at \( p \) if we know the “background” data \((\kappa_p,\beta_p,R_p)\).

### The structure theorem

We will now prove our first major result. This is a generalisation of the first part of [45, Thm.12], which is particular to 11-dimensional supergravity.

**Theorem 3.10** (Structure of Killing (super)algebras). Let \( D \) be an admissible connection on a spinor bundle \( S \) over a pseudo-Riemannian spin manifold \((M,g)\). Then the Killing (super)algebra \( \mathcal{R}_D \) is a filtered deformation of a graded subalgebra \( s \) of the flat model \( s = V \oplus S \oplus \mathfrak{so}(V) \).

**Remark 5.** Before proving this result, we note that there is a more naïve approach to it which unfortunately fails. We have already observed that \( \mathcal{E}_p = V \oplus S \oplus \mathfrak{so}(V) \), so one might be tempted to identify \( \mathcal{E}_p \) with the flat model algebra \( s \) and proceed as follows: we have a map \( \Phi_p : \mathcal{R}_D \to \mathcal{E}_p = s \), and we can therefore compute brackets on \( \text{Im} \Phi_p \):

\[
[\Phi_p(X),\Phi_p(Y)] = ((A_X)_p Y_p - (A_Y)_p X_p, 0, [(A_X)_p,(A_Y)_p]), \tag{3.48}
\]
\[
[\Phi_p(X),\Phi_p(\epsilon)] = (0, (A_X)_p \cdot \epsilon, 0), \tag{3.49}
\]
\[
[\Phi_p(\epsilon),\Phi_p(\zeta)] = (\kappa_p(\epsilon_p,\zeta_p), 0, 0). \tag{3.50}
\]

Comparing to equations (3.45)-(3.47), we see that \( \Phi_p \) is not a Lie algebra homomorphism in general, but perhaps it is the natural map carrying a filtered Lie algebra to its associated graded inside \( s \). The problem is that Im \( \Phi_p \) need not be closed under the bracket \([-,-]\) – the expressions above need not be the transport data of any Killing pairs – and it may not even be a graded subspace of \( \mathcal{E}_p \), since for example Killing vectors embed diagonally; \( X \mapsto (X_p,0,(A_X)_p) \). A more careful method is therefore required.

**Proof of Theorem 3.10.** We fix an arbitrary point \( p \in M \) and set \( V = T_pM, S = S_p \) as before. Define the evaluation maps

\[
ev^V_p : \mathcal{W}_D \to V \quad \quad X \mapsto X_p, \tag{3.51}
\]
\[
ev^S_p : \mathcal{S}_D \to S \quad \quad \epsilon \mapsto \epsilon_p. \tag{3.52}
\]
Let $V'$ be the subspace of $V$ consisting of values of restricted Killing vectors:

$$V' = \text{Im} \text{ev}^V_p = \{ X_p \in V \mid X \in \mathfrak{D}_D \},$$

(3.53)

and let $\mathfrak{h}$ be the subspace of $\mathfrak{so}(V)$ consisting of values at $p$ of endomorphisms $A_X$ for Killing vectors which vanish at $X$:

$$\mathfrak{h} = \{ (A_X)_p \in \mathfrak{so}(V) \mid X \in \mathfrak{D}_D : X_p = 0 \}.$$

(3.54)

Since a Killing vector $X$ is determined by its Killing transport data $(X_p, (A_X)_p)$, there is an isomorphism of vector spaces $\mathfrak{h} \cong \text{ker} \text{ev}^V_p$. For $A \in \mathfrak{h}$, we denote by $X_A$ the corresponding Killing vector in $\text{ker} \text{ev}^V_p$. We also define

$$S' = \text{Im} \text{ev}^S_p = \{ \epsilon_p \in S \mid \epsilon \in \mathfrak{S}_D \}$$

(3.55)

and note that there exists a vector space isomorphism $S' \cong \mathfrak{S}_D$ since a Killing spinor is defined by its value at $p$. For ease of notation, when there is no ambiguity we now identify these spaces; the symbol $\epsilon$ will denote both a Killing spinor and its value at $p$.

Now let $a = V' \oplus S' \oplus \mathfrak{h} \subseteq \mathfrak{s}$. We claim that this graded subspace is in fact a graded subalgebra of $\mathfrak{s}$. If $A, B \in \mathfrak{h}$, by equation (3.45) we have

$$[X_A, X_B]_p = 0,$$

(3.56)

$$[A_{(X_A, X_B)}] = [A_{X_A}, A_{X_B}]_p = [A, B],$$

(3.57)

so $[X_A, X_B] \in \text{ker} \text{ev}^V_p$, and the corresponding element in $\mathfrak{h}$ is $[A, B]$, showing that $\mathfrak{h}$ is closed under the bracket. Note that this also shows that $\mathfrak{h} \cong \text{ker} \text{ev}^V_p$ as Lie algebras. Now let $A \in \mathfrak{h}$ and $\nu \in V'$. Then there is some $Y \in \mathfrak{D}_D$ with $Y_p = \nu$. Again by equation (3.45), we have

$$[X_A, Y]_p = (A_{X_A})_p Y_p = A \nu = [A, \nu],$$

(3.58)

so $[A, \nu]$ is the value at $p$ of $[X_A, Y]$. Thus $[\mathfrak{h}, V'] \subseteq V'$. For $A \in \mathfrak{h}$ and $\epsilon \in S' \cong \mathfrak{S}_D$, by equation (3.46) we have

$$[X_A, \epsilon]_p = (A_{X_A})_p \epsilon_p = A \epsilon = [A, \epsilon],$$

(3.59)

so $[A, \epsilon] \in S'$. Thus $[\mathfrak{h}, S'] \subseteq S'$. Finally, for $\epsilon, \zeta \in S$, we have

$$[\epsilon, \zeta]_p = \kappa_p(\epsilon_p, \zeta_p) = [\epsilon_p, \zeta_p],$$

(3.60)

so $[S', S'] \subseteq V'$. Hence $a$ is indeed a graded Lie subalgebra of $\mathfrak{s}$.

We can equip $\mathfrak{R}_D$ with a filtration $\mathfrak{R}_D^*$ of depth 2 ($\mathfrak{R}_D^i = \mathfrak{R}_D$ for $i \leq -2$) as follows: we set $\mathfrak{R}_D^0 = 0$ for $i > 0$ and

$$\mathfrak{R}_D^2 = \mathfrak{R}_D = \mathfrak{D}_D \oplus \mathfrak{S}_D \supset \mathfrak{R}_D^1 = \text{ker} \text{ev}^V_p \oplus \mathfrak{S}_D \supset \mathfrak{R}_D^0 = \text{ker} \text{ev}^V_p \cong \mathfrak{h}.$$  

(3.61)
The calculations above show that, as \( h \cong \ker ev_p \)-modules,

\[
\begin{align*}
\text{Gr}_{-2} \mathcal{R}_D^* &= \mathcal{R}_D^* / \ker ev_p V' = V' \\
\text{Gr}_{-1} \mathcal{R}_D^* &= \ker ev_p \oplus \mathcal{S}_D / \ker ev_p S' \\
\text{Gr}_0 \mathcal{R}_D^* &= \ker ev_p \cong h
\end{align*}
\]

(3.62)

thus \( \text{Gr} \mathcal{R}_D^* \cong a \) as \( h \)-modules. We will show that this isomorphism preserves Lie brackets, hence \( \mathcal{R}_D \) is a filtered deformation of \( a \).

As in §2.3.1, we denote the associated graded bracket on \( \text{Gr} \mathcal{R}_D^* \) by \([-,-]_{\text{Gr}}\). For \( A, B \in h, X, Y \in \mathcal{Q}_D \) and \( \epsilon, \zeta \in \mathcal{S}_D \), we have the following for our map \( \text{Gr} \mathcal{R}_D^* \rightarrow a \):

\[
\begin{align*}
[ X_A, Y ]_{\text{Gr}} &= [ X_A, Y ] = A_{X_A} Y &\quad \rightarrow AY_p = [ A, Y_p ] &\quad (\text{in degree } -2), \\
[ \epsilon, \zeta ]_{\text{Gr}} &= [ \epsilon, \zeta ] = \kappa (\epsilon, \zeta) &\quad \rightarrow \kappa_p(\epsilon, \zeta) = [ \epsilon, \zeta ] &\quad (\text{in degree } -2), \\
[ X_A, \epsilon ]_{\text{Gr}} &= [ X_A, \epsilon ] = A_{X_A} \epsilon &\quad \rightarrow A \cdot \epsilon = [ A, \epsilon ] &\quad (\text{in degree } -1), \\
[ X_A, X_B ]_{\text{Gr}} &= [ X_A, X_B ] = X_{[A,B]} &\quad \rightarrow [ A, B ] &\quad (\text{in degree } 0).
\end{align*}
\]

(3.63-3.66)

The only brackets we have not considered above are \([ X, \epsilon ]_{\text{Gr}}\) and \([ X, Y ]_{\text{Gr}}\), but these lie in \( \text{Gr}_{-3} \mathcal{R}_D^* \) and \( \text{Gr}_{-4} \mathcal{R}_D^* \) respectively, both of which are both zero by definition.

We recall from the discussion in §2.3.3 that filtered deformations can be explicitly described as a deformation of the bracket of the graded model algebra by a sequence of maps of positive degrees. It will be useful to give our Killing superalgebras such a presentation.

First note that the isomorphism \( h \cong \ker ev_p \) and the evaluation map \( ev_p \) itself give us the short exact sequence of vector spaces

\[
0 \longrightarrow h \longrightarrow \mathcal{Q}_D \longrightarrow V' \longrightarrow 0.
\]

(3.67)

Any short exact sequence of vector spaces splits, so there exists a (linear) splitting map \( \Lambda : V' \rightarrow \mathcal{Q}_D \) with \( ev_p(\Lambda(v)) = \Lambda(v)_p = v \). We can also parametrise the splitting in terms of a map \( \lambda : V' \rightarrow \mathfrak{so}(V) \) by setting \( \lambda(v) = (\Lambda(v)_p)_p \); the vector field \( \Lambda(v) \) can be recovered from its Killing transport data \((v, \lambda(v))\). The splitting induces a linear isomorphism \( V' \oplus h \rightarrow \mathcal{Q}_D \) given by \((v, A) \rightarrow \Lambda(v) + X_A \). This extends to a \( \mathbb{Z}_2 \) graded linear isomorphism \( \Psi : a \rightarrow \mathcal{R}_D \) given by

\[
\Psi(v, \epsilon, A) = (\Lambda(v) + X_A, \epsilon),
\]

(3.68)

where \( v \in V', A \in h, \) and \( \epsilon \in S' \cong \mathcal{S}_D \) which we will use to show that \( \mathcal{R}_D \) is a filtered deformation of \( a \). We use \( \Psi \) to define a new bracket \([-,-]'\) on \( a \) by pulling back the bracket on \( \mathcal{R}_D \):

\[
[(v, \epsilon, A), (w, \zeta, B)]' = \Psi^{-1}([\Psi(v, \epsilon, A), \Psi(w, \zeta, B)])
\]

(3.69)

with respect to which \( \Psi \) is a Lie (super)algebra isomorphism \( (a, [-,-]') \cong \mathcal{R}_D \) by
construction. Explicitly, the bracket is given by

\[
[A, B]' = [A, B], \quad [A, v]' = A v + [A, \lambda(v)] - \lambda(A v), \quad [v, w]' = \lambda(v) w - \lambda(w) v + \theta_\lambda(v, w), \quad [A, \epsilon]' = A \cdot \epsilon, \quad [v, \epsilon]' = \beta_p(v) \epsilon + \lambda(v) \cdot \epsilon,
\]

(3.70)

where \( \gamma_p : \mathfrak{so}^2 S \to \mathfrak{so}(V) \) is the evaluation at \( p \) of the map \( \gamma \) defined by equation (3.24) and \( \theta_\lambda : \Lambda^2 V' \to \mathfrak{h} \) is the map

\[
\theta_\lambda(v, w) = [\lambda(v), \lambda(w)] - R_p(v, w) - \lambda(\lambda(v) w - \lambda(w) v).
\]

(3.71)

Using the notation of Lie algebra cohomology, we have a map \( \partial : \Lambda^2 \mathfrak{g}^0 \to \mathfrak{h} \) of degree +2 defined as follows:

\[
\partial \lambda(A, v) = [A, \lambda(v)] - \lambda(A v), \quad \partial \lambda(v, w) = \lambda(v) w - \lambda(w) v,
\]

\[
\partial \lambda(v, \epsilon) = \lambda(v) \cdot \epsilon, \quad \partial \lambda(\epsilon, \zeta) = -\lambda(\kappa_p(\epsilon, \zeta)),
\]

(3.72)

whence we have

\[
[-, -]' = [-, -] + \beta_p + \gamma_p + \partial \lambda + \theta_\lambda
\]

(3.73)

where \( \beta_p, \gamma_p \) and \( \theta_\lambda \) have been trivially extended to maps \( \Lambda^2 \mathfrak{g}^0 \to \mathfrak{h} \). The deforming maps are \( \mu = \beta_p + \gamma_p + \partial \lambda \), of degree +2, and \( \theta_\lambda \), of degree +4. That is, the deformation has defining sequence \( (\mu, \theta_\lambda, 0, \ldots) \).

The splitting \( \Lambda : V' \to \mathfrak{g}_D \) and the corresponding map \( \lambda : V' \to \mathfrak{h} \) can be viewed as a “correction” to the failed naïve approach discussed in the remark before the proof of Theorem 3.10. Note that these maps are not canonical; if \( \Lambda' : V' \to \mathfrak{g}_D \) is another splitting and \( \lambda' : V' \to \mathfrak{so}(V) \) is the corresponding map, we have

\[
\Phi_p(\Lambda'(v) - \Lambda(v)) = (v - v', \lambda'(v) - \lambda(v) = (0, \lambda'(v) - \lambda(v)),
\]

(3.74)

so \( (\Lambda' - \Lambda)(v) \in \text{ker} \nu \), and \( (\Lambda' - \Lambda)(v) \) is the corresponding element of \( \mathfrak{h} \). Thus \( \lambda \), and therefore the splitting, is unique up to a choice of map \( V' \to \mathfrak{h} \). As we will see later that such a map is a Spencer (2,1)-cocycle for \( a \), so it is consistent that changing the defining sequence by the coboundary of such a map does not change the isomorphism class of the deformation, only its presentation. Also note that this choice of splitting is exactly the choice of complementary submodules discussed in §2.3.3.

### 3.2.3 Constrained Killing spinors

So far in this section, we have only considered the differential Killing spinor equation \( D \epsilon = 0 \) which in supergravity theories arises from demanding that the variation of the gravitino \( \partial_{\epsilon} \Psi = D \epsilon \) vanishes. While in some pure minimal supergravity theories (in particular \( D = 4, 5, 11 \)) this is the only local condition imposed by supersymmetry, in most supergravity theories (including any extended, gauged or matter-coupled theories and even some pure minimal theories), there are additional fermions present. A background of such a theory is called supersymmetric if the variations of \( \text{all} \) fermions
vanish for some supersymmetry parameter. Since the variations of the additional fermions are always algebraic in the Killing spinor, we will refer to the condition that they vanish as algebraic constraints and we will call the spinors satisfying them as well as \( D\epsilon = 0 \) constrained Killing spinors.

We will make some comments here about how these might be brought into our present framework; we leave a full treatment for future work.

**Algebraic constraints**

Schematically, the algebraic constraints are of the form \( \mathcal{P}\epsilon = 0 \) where \( \mathcal{P} \) is some fibrewise linear operator acting on \( S \). More concretely, all fermions may take values in the same representation as the gravitini in which case \( \mathcal{P} \in \text{End} S \). In a more general situation, the additional fermions take values in some vector bundle \( E \rightarrow M \) (for example, an associated bundle for some gauge symmetry). Then the constraint equation is \( \mathcal{P}\epsilon = 0 \) where \( \mathcal{P} \in \text{Hom}(S, S \otimes E) \cong \text{End} S \otimes E \). This bundle map is parametrised by the bosonic fields of the background, just as the algebraic part\(^3\) \( \beta \) of the connection \( D \) is.

We now make some general comments about how one might go about determining the geometries that support constrained Killing spinors. Let \( D = \nabla - \beta \) be a spinor connection which is not necessarily admissible in the sense Definition 3.7 and let us denote the space of \( D \)-parallel sections of \( S \) which are annihilated by \( \mathcal{P} \) by

\[
\mathcal{G}_{D, \mathcal{P}} := \{ \epsilon \in \mathcal{S} | D\epsilon = 0 \text{ and } \mathcal{P}\epsilon = 0 \}. \tag{3.75}
\]

We previously discussed the integrability condition \( R^D\epsilon = 0 \) for the existence of \( D \)-parallel spinors. In the case of constrained Killing spinors \( \epsilon \in \mathcal{G}_{D, \mathcal{P}} \), this can be supplemented by further integrability conditions, the first-order such conditions being schematically (see e.g. [29])

\[
[\mathcal{P}, \mathcal{P}]\epsilon = 0, \quad \tilde{D}\mathcal{P}\epsilon := [D, \mathcal{P}]\epsilon = 0, \tag{3.76}
\]

where \([-,-]\) in both expressions is a commutator. These conditions require some interpretation which depend on the structure on the bundle \( E \) in which the additional fermions take their values. If \( \mathcal{P} \in \text{End} S \), so \( E \rightarrow M \) is the trivial line bundle, then the meaning of the expressions is clear. The same holds if \( E \) is simply the sum of such bundles. In the more general situation, we may interpret \( \mathcal{P} \) as acting on \( S \otimes E \) as \( \mathcal{P} \otimes \text{Id}_E \) and extend \( D \) to a connection on \( S \otimes E \) which acts trivially on the second factor. In practice though, there may be some more natural interpretation in which these operations are “twisted” by some structure on \( E \).

**Constrained Killing superalgebra**

As we did for \( \mathcal{G}_D \) in §3.2.1, we now seek to find a subalgebra of the algebra of Killing vectors which preserves this space under the action of the Lie derivative; a natural

\(^3\)Let us note that the fermion variations \( \mathcal{P}\epsilon \) and the expression \( \beta \epsilon \) are algebraic in \( \epsilon \) but not necessarily in the bosonic background fields. For example, gauge field strengths (hence derivatives of gauge connections) and derivatives of scalars typically appear as coefficients in both.
choice is
\[ \mathcal{V}_{D,\mathcal{P}} := \left\{ X \in \text{iso}(M, g) \bigg| \mathcal{L}_X \beta = 0 \text{ and } \mathcal{L}_X \mathcal{P} = 0 \right\}, \]  
(3.77)

where again we must interpret the expression \( \mathcal{L}_X \mathcal{P} \). In particular, we assume that there is some structure on \( E \) such which gives rise to some generalisation \( \mathcal{P} \) of the Lie derivative which acts on \( \text{Hom} E \otimes E \) – if \( E \) happens to be a trivial bundle, the spinorial Lie derivative naturally generalises, while if it is an associated bundle to the principal bundle for some gauge symmetry, a covariant derivative similar to the one we will see in §4.1.2 will suffice. We note that in the latter case, \( \mathcal{V}_{D,\mathcal{P}} \) may not close under the Lie bracket, so we may need to refine the definition of \( \mathcal{V}_{D,\mathcal{P}} \) in a similar manner to the treatment of §4.2.1.

If \( \mathcal{V}_{D,\mathcal{P}} \) (or some more carefully chosen subspace) is indeed a subalgebra of \( \text{iso}(M, g) \), its action via the spinorial Lie derivative preserves \( S \). We now seek to give \( \mathcal{R}_{D,\mathcal{P}} = \mathcal{V}_{D,\mathcal{P}} \oplus S \) the structure of a superalgebra using the Lie derivative and squaring map as in Theorem 3.6. But as in the proof of that theorem, all that needs to be checked is that the \( \{ \mathcal{S}_{D,\mathcal{P}}, S \} \) bracket closes on \( \mathcal{V}_{D,\mathcal{P}} \), and that the \( \{ \mathcal{S}_{D,\mathcal{P}}, S \} \) Jacobi identity is satisfied. But we quickly see that this is the case if equations (3.26)-(3.28) as well as
\[ c_L(\epsilon, \zeta) = 0 \]  
(3.78)

are satisfied for all \( \epsilon, \zeta \in \mathcal{S}_{D,\mathcal{P}} \). We note that in a supergravity theory, \( \beta \) and \( \mathcal{P} \) are defined in terms of background fields which we expect to be preserved by the Lie derivatives along Killing vectors generated by spinors in \( \mathcal{S}_{D,\mathcal{P}} \), hence the condition above is also satisfied.

Assuming that the expressions above can be made precise as suggested, we should then generalise Definition 3.7 to the constrained case, considering not just admissible connections \( D \) but admissible pairs \( (D, \mathcal{P}) \), incorporating the condition (3.78) and possibly demanding that the admissibility conditions of Definition 3.7 should only be imposed on \( \ker \mathcal{P} \), rather than the whole spinor bundle.

**Killing transport and filtered deformation**

We now consider whether an analogue of Theorem 3.10 holds, that is, whether it is the case that the Killing superalgebra is a filtered deformation of the Poincaré superalgebra in the constrained case. Recall that an important part of this proof was the localisation of the Killing vectors and spinors in \( \mathcal{R}_D \) at a point via Killing transport. Since constrained Killing spinors are still \( D \)-parallel with respect to a connection \( D \), the same arguments hold, the only modification being that the Killing transport data of a constrained Killing spinor at \( p \in M \) must lie in \( \ker \mathcal{P}_p \). Note that this does not mean that any spinor at \( p \) annihilated by \( \mathcal{P} \) can be Killing-transported to define a constrained Killing spinor; as in the unconstrained case, a \( D \)-parallel global section with this transport data may not exist, and in the constrained case, even if one does exist, it need not be \( \mathcal{P} \)-parallel.

One should thus expect an analogue to Theorem 3.10 for \( \mathcal{R}_{D,\mathcal{P}} \) with essentially the same proof, the main difference being that the filtered subdeformation of \( S \) one thus obtains must have \( S' = \text{ev}_p \mathcal{S}_{D,\mathcal{P}} \subseteq \ker \mathcal{P}_p \subseteq S_p \).
We will not consider constrained Killing spinors in the rest of this work (aside from some comments in Chapter 8), primarily because, in contrast to the differential Killing spinor equation, the algebraic constraints apparently do not have an interpretation in terms of Lie algebra cohomology. We thus leave the details of the treatment suggested above to be fully worked out in future publications.

### 3.2.4 Killing (super)algebras in $D = 2$

Let us now illustrate our theory on admissible connections and Killing superalgebras by considering some examples in two dimensions. We build on our discussion of Dirac currents in 2 dimensions from §2.2.4, using the same conventions and explicit representations of the Clifford algebras. We will consider only signatures $(0,2)$ and $(1,1)$ here since, as we saw in §2.2.4, the Clifford algebra is $\mathbb{R}(2)$ in both cases, hence the situation is much simpler than the $(2,0)$ case where the Clifford algebra is $\mathbb{H}$. Indeed, the cases are so similar that we will be able to perform many of the calculations in a signature-agnostic way. The quaternionic case, as well as other generalisations, will be treated in future work.

**Conventions and formulae**

In our conventions, we can write the relations among the $\Gamma$-matrices as follows:

\[
\Gamma_\mu \Gamma_\nu = \Gamma_{\mu\nu} + \eta_{\mu\nu} \mathbb{1} = \epsilon_{\mu\nu} \Gamma_\ast + \eta_{\mu\nu} \mathbb{1} = -\Gamma_\ast \Gamma_\mu = \sigma \epsilon_{\mu\nu} \Gamma_\nu \quad (\Gamma_\ast)^2 = -\sigma \mathbb{1} \quad (3.79)
\]

where $\sigma = \det[\eta]$ is a sign which is $+1$ in signature $(0,2)$ and $-1$ in signature $(1,1)$. We recall that work in $(0,2)$ signature by taking $\eta$ to be positive-definite but choosing a $+$ sign in the Clifford relation (2.17) (which is the $+$ sign in the first relation above). In signature $(1,1)$, the sign convention is less consequential, but we note that we take $\eta_{00} = -\eta_{11} = -1$. It is also useful to note the following relation between the top-rank matrices with indices raised and lowered:

\[
\Gamma^* = \sigma \Gamma_\ast. \quad (3.80)
\]

We work with $S = \mathbb{P}$, the irreducible pinor module. As a representation of $\mathfrak{so}(V)$, this is irreducible, $S = \mathbb{S}$ in signature $(0,2)$ and reducible, $S = \mathbb{S}_+ \oplus \mathbb{S}_-$, in signature $(1,1)$, with $\mathbb{S}_\pm$ being the $\pm 1$-eigenspaces of $\Gamma_\ast$.

Recall that in either signature, there are two independent admissible bilinears, thus two possible Dirac currents, on $S$; both Dirac currents are symmetric ($\sigma_\kappa = +1$) but one bilinear is symmetric and the other skew-symmetric (thus $\sigma_B = -\tau_B = \pm 1$).

The *Fierz identity* is a formula which allows us to rearrange products of spinors involving bilinears. It is simplest to express using conjugate spinor notation; for $\epsilon \in S$, we denote by $\bar{\epsilon}$ the element of $S^\ast$ given by $\epsilon' \mapsto \bar{\epsilon} \epsilon' := B(\bar{\epsilon}, \epsilon')$. Then $\epsilon \bar{\epsilon} := \epsilon \otimes \bar{\epsilon} \in S \otimes S^\ast = \text{End} S$.

**Proposition 3.11** (Fierz identity). Let $S = \mathbb{P}$ be the (real) pinor representation in signature $(1,1)$ or $(0,2)$ and $B$ either of the admissible bilinears. Then for $\epsilon, \epsilon' \in S$, we have

\[
\epsilon \bar{\epsilon}' = \frac{1}{2} \left( (\bar{\epsilon}' \epsilon) \mathbb{1} + (\bar{\epsilon}' \Gamma_\mu \epsilon) \Gamma^\mu - \sigma (\bar{\epsilon}' \Gamma_\ast \epsilon) \Gamma_\ast \right) \quad (3.81)
\]
Proof. Since $e\bar{e}'$ is an endomorphism of $S$, it is an $R$-linear combination of $\Gamma$-matrices; we have

$$e\bar{e}' = a1 + b_\mu \Gamma^\mu + c\Gamma_*$$

(3.82)

for some $a, b_\mu, c \in R$. One can show (for example in the explicit matrix representation used in §2.2.4) that $\Gamma_\mu$ and $\Gamma_*$ are traceless, while $\text{tr} 1 = \dim S = 2$, whence we compute

$$\text{tr}(e\bar{e}') = 2a,$$

(3.83)

$$\text{tr}(\Gamma_\nu e\bar{e}') = 2b_\nu,$$

(3.84)

$$\text{tr}(\Gamma_* e\bar{e}') = -\sigma 2c.$$  

(3.85)

On the other hand, for any $\Gamma \in \text{End} S$, $\text{tr}(\Gamma e\bar{e}') = \bar{e}' \Gamma e$, which can be verified in the explicit representation or by noting that in Einstein notation with an abstract basis with index $a, b, \ldots = 1, 2$, both expressions are equal to $B_{ab} \Gamma^b e^{\mu} e^c$. The result follows immediately.

The Fierz identity has consequences for the causal properties of the Dirac current. Recall that for $e \in S$, $\kappa_e := \kappa(e, e) \in V$ has components $(\kappa_e)^\mu = \bar{e}' \Gamma^\mu e = B(e, \Gamma^\mu e)$.

**Corollary 3.12.** Let $e \in S$. Then

$$\|\kappa_e\|^2 = (\bar{e} e)^2 + \sigma (\bar{e} \Gamma_* e)^2.$$  

(3.86)

In particular, in the Riemannian case $\kappa_e = 0$ if and only if $e = 0$, while in the Lorentzian case, if $e$ is chiral then $\kappa$ is null, otherwise $\kappa$ is spacelike for $\sigma_B = +1$ and timelike for $\sigma_B = -1$.

**Proof.** For the first claim, applying the Fierz identity to the expression $(\bar{e} e)^2$ gives us

$$(\bar{e} e)^2 = \bar{e}(e e) = \frac{1}{2} (\bar{e} e (\bar{e} e) + (\bar{e} \Gamma_\mu e)(\bar{e} \Gamma^\mu e) - \sigma (\bar{e} \Gamma_* e) (\bar{e} \Gamma_* e))$$

(3.87)

which can be rearranged to give the desired expression. Now, consider the two quadratic forms $\bar{e} \Gamma_* e$ and $\bar{e} e$ and note that $\bar{e} \Gamma_* e = 0$ if $\sigma_B = +1$ and $\bar{e} e = 0$ if $\sigma_B = -1$. In the Riemannian case, the expression which does not vanish identically vanishes if and only if $e = 0$ (check this in the explicit representation). In the Lorentzian case, since the isotropy $i_B = -1$ for either admissible bilinear (see Table 2.8) and $\Gamma_*$ preserves the chiral subspaces, the expression which does not vanish identically vanishes if and only if $e \in S_\pm$. This proves the second claim.

**Geometric constructions**

Throughout, we let $(M, g)$ be a 2-dimensional pseudo-Riemannian spin manifold of either Riemannian or Lorentzian signature. Fixing a spin structure $P \to M$, we let $S := P \times \text{Spin}(V)$ be the spinor bundle with fibre $S$ and $S = \Gamma(S)$ be its space of sections. We let $B$ be either of the admissible bilinears on $S$ in each signature and denote the bilinear form induced by it on $S$ by $\langle - , - \rangle$. The corresponding Dirac current will be denoted $\kappa : \Omega^2 S \to TM$ and is given by the equation

$$g(\kappa(e, e'), X) = \langle e, X \cdot e' \rangle$$

(3.88)
for all \( X \in \mathfrak{X}(M) \).

Let vol denote the canonical volume form on \((M, g)\). Then in a local orthonormal frame, \( \text{vol} \cdot e = \Gamma \cdot e \) for all \( e \in \mathcal{S} \). We note that for \( \alpha^{(p)} \in \Omega^p(M) \), we have

\[
\text{vol} \cdot \alpha^{(0)} = \alpha^{(0)} \text{vol} = * \alpha^{(0)} \quad \text{vol} \cdot \alpha^{(1)} = - * \alpha^{(1)} \quad \text{vol} \cdot \alpha^{(2)} = - * \alpha^{(2)}
\]

where \( \cdot \) denotes Clifford multiplication.

### Admissible connections

Recall Definition 3.7 of an admissible connection \( D = \nabla - \beta \) on a spinor bundle \( S \) equipped with Dirac current \( \kappa \). There are two algebraic conditions on \( \beta \), one of which implies in particular that \( \kappa(e, \zeta) \) is Killing for all \( e, \zeta \in \mathcal{S}_D \), where \( \mathcal{S}_D \) is the space of \( D \)-parallel spinors (Killing spinors). There is also one differential condition, namely that \( \mathcal{L}_\kappa(e, \zeta) \beta = 0 \) for all \( e, \zeta \in \mathcal{S}_D \). The solution to the algebraic conditions is as follows:

\[
\beta(X)e = \begin{cases} 
  bX \cdot e & \text{for } \sigma_B = -1, \\
  b(X) \cdot e & \text{for } \sigma_B = +1,
\end{cases}
\]

where \( b \in C^\infty(M) \). This purely algebraic calculation will fit more naturally into our discussion of Spencer cohomology, so we postpone the proof of this fact until §3.3.6 and for now discuss its geometric consequences. We note that in the first case, the Killing spinor equation is \( \nabla_X e = bX \cdot e \), so we are working with geometric Killing spinors (discussed in the introduction to this work) where the Killing number is allowed to be a function rather than a constant. The second case is a mild further generalisation.

We will derive some results using the integrability condition \( R^D e = 0 \) for the existence of \( D \)-parallel spinors – this is a common method for studying sections which are parallel with respect to a connection which was already mentioned in §3.2.1. We start with the curvature equation of Proposition 3.3:

\[
R^D(X, Y)e = R(X, Y) \cdot e + \left[ \beta(X), \beta(Y) \right] e - (\nabla_X \beta)(Y)e + (\nabla_Y \beta)(X)e
\]

for all \( X, Y \in \mathfrak{X}(M), e \in \mathcal{S} \).

For \( \sigma_B = -1 \), we compute the following commutator:

\[
\left[ \beta(X), \beta(Y) \right] e = b^2(X \cdot Y - Y \cdot X) \cdot e = 2b^2(X \land Y) \cdot e \quad (\sigma_B = -1),
\]

for \( X, Y \in \mathfrak{X}(M) \) and \( e \in \mathcal{S} \), and \( (\nabla_X \beta)(Y) = (\nabla_X b)Y \cdot e \). Our integrability condition is thus

\[
R(X, Y) \cdot e + 2b^2(X \land Y) \cdot e - ((\nabla_X b)Y - (\nabla_Y b)X) \cdot e = 0 \quad (\sigma_B = -1).
\]

In the \( \sigma_B = +1 \) case, we similarly find that

\[
\left[ \beta(X), \beta(Y) \right] e = 2b^2((X \land Y) \cdot e = \sigma 2b^2(X \land Y) \cdot e \quad (\sigma_B = +1).
\]

The second equality is justified as follows, using the definition of the hodge star and
denoting the volume form by \( \text{vol} \):

\[
(*)X \wedge (*)Y = g(*)X, Y \text{ vol} = g(Y, *)X \text{ vol} = Y \wedge (*)X = -(* *)X \wedge Y = \sigma X \wedge Y \tag{3.95}
\]
since we can show that \( *^2 = -\sigma \text{ Id} \) when acting on 1-vectors. The integrability equation is then

\[
R(X, Y) \cdot \epsilon + \sigma 2b^2 \langle X \wedge Y \rangle \cdot \epsilon - ((\nabla_X b) \ast Y - (\nabla_Y b) \ast X) \cdot \epsilon = 0 \quad (\sigma_B = +1). \tag{3.96}
\]

We use the integrability conditions above to show the following.

**Lemma 3.13.** If \( \epsilon \) is a spinor field such that \( \nabla_X \epsilon = \beta(X)\epsilon \) then \( \nabla_{\kappa} b = 0 \).

**Proof.** First, note that \( R(X, Y) \cdot \epsilon \) and \( (X \wedge Y) \cdot \epsilon \) are proportional to \( \text{vol} \cdot \epsilon \), so in the \( \sigma_B = +1 \) case, pairing the integrability condition with \( \epsilon \) and using the fact that \( \langle \epsilon, \text{vol} \cdot \epsilon \rangle = 0 \) gives us

\[
(\nabla_X b)g(*)Y, \kappa_\epsilon - (\nabla_Y b)g(*)X, \kappa_\epsilon = 0 \tag{3.97}
\]
for arbitrary \( X, Y \in \mathcal{X}(M) \). In the \( \sigma_B = -1 \) case, we get the same equation by pairing with \( \text{vol} \cdot \epsilon \) and using \( \langle \epsilon, \epsilon \rangle = 0 \) along with \( \text{vol} \cdot \text{vol} = -\sigma \mathbb{I} \) and \( \text{vol} \cdot X = -*X \). Then, using the definition of the Hodge star operator and \( \nabla_X b = \iota_{db}X \), the equation above is equivalent to

\[
\iota_{db}(X \wedge Y) \wedge \kappa_\epsilon = 0, \tag{3.98}
\]
which using a Leibniz rule is equivalent to \( \iota_{\kappa_\epsilon} db(X \wedge Y) = 0 \). Finally, since \( X, Y \) are arbitrary, this gives us \( \nabla_{\kappa_\epsilon} b = \iota_{\kappa_\epsilon} db = 0 \).

**Proposition 3.14.** Let \((M, g)\) be a 2-dimensional spin manifold with signature \((0, 2)\) or \((1, 1)\), let \( B \) be an admissible bilinear on the irreducible pinor module \( S = \mathbb{P} \) (of which there are two distinguished by \( \sigma_B \)) and \( \kappa \) the corresponding Dirac current. Let \( S \) denote the spinor bundle associated to \( S \). Then if we define \( \beta \in \Omega^1(M; \text{End } S) \) by equation (3.90), \( D = \nabla - \beta \) is an admissible connection.

**Proof.** As already noted, the conditions for admissibility which are algebraic in \( \beta \) are uniquely satisfied by equation (3.90). We will prove this later in this chapter (see Proposition 3.43). It remains to show that \( \mathcal{L}_{\kappa_\epsilon} \beta = 0 \) for \( \epsilon, \zeta \in \mathbb{S}_D \); since \( \sigma_\kappa = +1 \) it suffices to show this for \( \epsilon = \zeta \). For \( \sigma_B = -1 \) we have

\[
(\mathcal{L}_{\kappa_\epsilon} \beta)(X)\zeta = \mathcal{L}_{\kappa_\epsilon} \beta(X)\zeta - \beta(\mathcal{L}_{\kappa_\epsilon} X)\zeta - \beta(X)(\mathcal{L}_{\kappa_\epsilon} \zeta) \\
= \mathcal{L}_{\kappa_\epsilon} (bX \cdot \zeta) - b(\mathcal{L}_{\kappa_\epsilon} X) \cdot \zeta - bX \cdot (\mathcal{L}_{\kappa_\epsilon} \zeta) \tag{3.99}
\]
for all \( X \in \mathcal{X}(M) \) and \( \epsilon, \zeta \in \mathbb{S} \), where we have used the Leibniz rule twice. Similarly, for \( \sigma_B = +1 \) we have \( (\mathcal{L}_{\kappa_\epsilon} \beta)(X)\zeta = (\mathcal{L}_{\kappa_\epsilon} b)(*X) \cdot \zeta \). But the above lemma tells us that if \( \epsilon \in \mathbb{S}_D \) then \( \mathcal{L}_{\kappa_\epsilon} b = \nabla_{\kappa_\epsilon} b = 0 \), whence \( \mathcal{L}_{\kappa_\epsilon} \beta = 0 \) as required.

Thus we have determined the precise form of the admissible connections with respect to the two different choices of Dirac currents. Since both currents are symmetric, these connections give rise to Killing superalgebras.
Further insights from integrability

We can obtain yet further results on the geometries which support Killing superalgebras using the integrability equations. If we have some non-zero $\epsilon \in \mathfrak{S}_D$ then since $R^D(X, Y)\epsilon = 0$, the determinant of $R^D(X, Y)$ (as a spinor endomorphism) must vanish everywhere. We first the following in a local frame:

$$e^{\mu\nu}R_{\mu\nu}^D = \begin{cases} \frac{1}{2}e^{\mu\nu}e^{\sigma\tau}R_{\mu\nu\sigma\tau} + 2b^2e^{\mu\nu}\epsilon_{\mu\nu} \Gamma_* - 2e^{\mu\nu}\nabla^\mu b \Gamma_\nu & \text{for } \sigma_B = -1, \\
\frac{1}{2}e^{\mu\nu}e^{\sigma\tau}R_{\mu\nu\sigma\tau} + \sigma^2b^2e^{\mu\nu}\epsilon_{\mu\nu} \Gamma_* + \sigma 2\nabla^\mu b \Gamma_\mu & \text{for } \sigma_B = +1, \end{cases}$$

(3.100)

where in the second line of each case we have used the well-known fact that the Riemann tensor in 2 dimensions can be expressed in terms of the scalar curvature $R$:

$$R_{\mu\nu\sigma\tau} = \frac{1}{2}R(\eta_{\mu\sigma}\eta_{\nu\tau} - \eta_{\mu\tau}\eta_{\nu\sigma}) = \sigma \frac{1}{2}R\epsilon_{\mu\nu}\epsilon_{\sigma\tau},$$

(3.101)

and a combinatorial identity for $\epsilon$. Recalling our explicit descriptions of the Clifford algebras $\text{Cl}(0,2)$ and $\text{Cl}(1,1)$ in terms of the Pauli matrices, we can compute this determinant in a local frame using the formula

$$\det (a^1 + b\sigma_1 + c\sigma_2 + d\sigma_3) = a^2 - b^2 - c^2 - d^2$$

(3.102)

for $a, b, c, d \in \mathbb{C}$, giving us (we multiply by a sign for a slight simplification):

$$\sigma \det (e^{\mu\nu}R_{\mu\nu}^D) = \begin{cases} \frac{1}{2}R + 4b^2)^2 - 4\|db\|^2 & \text{for } \sigma_B = -1, \\
\frac{1}{2}R + \sigma 4b^2)^2 - \sigma 4\|db\|^2 & \text{for } \sigma_B = +1. \end{cases}$$

(3.103)

We would now like to set the expression above to zero and examine the resulting equations in the $c$ and the Killing number $b$. Let us first note that in Riemannian signature, the norm on differential forms is positive-definite, thus $\|db\|^2 \geq 0$ with equality if and only if $b$ is constant. A similar result holds in the Lorentzian case, but the statement is non-trivial since the norm is not positive-definite in this case.

**Lemma 3.15.** In Lorentzian signature, if $\mathfrak{S}_D \neq 0$ then $\sigma_B\|db\|^2 \leq 0$, and $db$ is null at a point $p \in M$ if and only if $p$ is a critical point for $b$ or $\mathfrak{S}_D$ is spanned by a Killing spinor which is chiral at $p$.

**Proof.** We will prove the $\sigma_B = -1$ case; the $\sigma_B = +1$ case is entirely analogous. By Lemma 3.13, $(db)^2$ is orthogonal to $\kappa_\epsilon$ for all $\epsilon \in \mathfrak{S}_D$. By Corollary 3.12, for $\sigma_B = -1$, $\kappa_\epsilon$ everywhere either timelike or null, and it is null if and only if $\epsilon$ is chiral. Thus where $\kappa_\epsilon$ is timelike, $db$ must be either spacelike or zero, and where $\kappa_\epsilon$ is null, $db$ must be collinear with $\kappa_\epsilon$ case. This proves that $\|db\|^2 \geq 0$. At a point $p$ where the inequality is saturated, we must have either $(db)_p = 0$, whence $p$ is a critical point, or $(db)_p \neq 0$ is null and $\epsilon$ is chiral at $p$. Thus all Killing spinors must be chiral at $p$. But if there are two independent Killing spinors, there are linear combinations of such spinors which are not chiral at $p$, a contradiction.

In particular, when $\mathfrak{S}_D \neq 0$, we may rearrange the equations obtained by setting
the determinant in (3.103) to zero to find
\[
R = \begin{cases} 
\pm 4|db| - 8b^2 & \text{for } \sigma_B = -1, \\
\pm 4|db| - \sigma 8b^2 & \text{for } \sigma_B = +1.
\end{cases} 
\]
(3.104)

where \(|db| = \sqrt{\|db\|^2}|. Constraints of the type we have just derived are well-known in the literature on geometric Killing spinors and their generalisations, but we have arrived at this expression in a slightly non-standard way. The usual method is to use the Lichnerowicz formula \([36, 38]\) as the integrability condition:
\[
\nabla^2 \epsilon + \Delta \epsilon = \frac{1}{4} R \epsilon 
\]
(3.105)

where \(\nabla\) is the Dirac operator (locally \(\Gamma^\mu \nabla_\mu\), \(\Delta\) the Laplace operator (locally \(g^{\mu\nu} \nabla_\mu \nabla_\nu\)) and \(R\) the scalar curvature, and this identity holds for all \(\epsilon \in \mathcal{S}\). Indeed, in two dimensions it is completely equivalent to the integrability condition \(R^D \epsilon = 0\), but we have used the latter here since it will be more useful to us in the higher-dimensional examples we will consider later.

Finally, we recall that for constant \(b = \frac{1}{2}\) and \(\sigma_B = -1\), we are in the standard geometric Killing spinor setup with Killing number \(b\), but we have \(R = -2\lambda^2\), which appears to have the wrong sign compared to equation (1.8). This is essentially because we have chosen to work in “(0,2) signature” by using a positive-definite metric and the “wrong” sign (+) in the Clifford relation (2.17). If we work with complex Clifford algebras and spinors, we recover the standard treatment by replacing \(\Gamma_\mu \rightarrow i \Gamma_\mu\) which has the same effect on the Killing spinor equation as replacing \(\lambda \rightarrow i \lambda\), whence we recover the usual relation \(R = 2\lambda^2\). If we had worked with a negative-definite metric and the “correct” sign in the Clifford algebra, we would have found \(R = 2\lambda^2\).

However, in negative-definite signature this is the correct sign for the curvature of hyperbolic space when \(b \in \mathbb{R}\), consistent with the interpretation which arises when using our original sign convention. In the terminology of the geometric Killing spinor literature, we are therefore working in the case of imaginary Killing spinors (i.e. Killing spinors with pure imaginary Killing number), which even in the Riemannian case admits non-trival generalisation to non-constant \(\lambda\) as provided \(R < 0\) on some locus \([39]\).

**Maximally supersymmetric case**

We call \((M, g, D)\) maximally supersymmetric if \(\dim \mathcal{S}_D = \dim \mathcal{S} = 2\). In this case, the values of \(D\)-parallel spinors span the fibre of \(\mathcal{S}\) at every point of \(M\), and since \(R^D\) annihilates these values, we must have \(R^D = 0\) identically. Then from our local expression (3.100), we see that \(b\) must be constant and we have \(R = -8b^2\) (for \(\sigma_B = -1\)) or \(R = -\sigma 8b^2\) (for \(\sigma_B = +1\)). Thus the maximally supersymmetric \(\sigma_B = -1\) case is precisely the classic geometric Killing spinor regime with imaginary Killing number as we just discussed above. For real \(b\), the scalar curvature is constant and negative, so the geometry must be hyperbolic in Riemannian signature and anti-de Sitter in Lorentzian signature. For \(\sigma_B = +1\) the maximally supersymmetric Riemannian geometry is hyperbolic again, while the maximally supersymmetric
Lorentzian geometry is de Sitter.

In §3.3.6, we will study the maximally supersymmetric filtered deformations of the 2-dimensional flat model $s$, which should be Killing superalgebras of the geometries described above. That calculation will confirm our conclusions about the maximally supersymmetric geometries.

### 3.3 Filtered subdeformations of the Poincaré superalgebra

We specialise now to the case of Lorentzian inner product spaces $(V, \eta)$ and spinor representations $S$ with symmetric squaring map $\kappa : \bigwedge^2 S \to V$, making our flat models Poincaré superalgebras; this is not essential for the first two subsections, although it will simplify some of the exposition, but it will become much more important later on due to the Homogeneity Theorem (Theorem 3.21), which imposes strong homological conditions on graded subalgebras of $s$ with sufficiently large odd subspace. We will discuss this theorem and its consequences in §3.3.3 and later subsections. We are also ultimately interested in the Killing superalgebras of supersymmetric supergravity backgrounds, for which the signature is of course Lorentzian.

We saw in Section 2.3 that filtered deformations of graded Lie superalgebras are governed by their Spencer cohomology. We therefore need to understand the Spencer cohomology of the Poincaré superalgebra and its graded subalgebras before discussing their deformations.

#### 3.3.1 The Spencer (2,2)-cohomology

The complex

A graded subalgebra $a$ of $s$ takes the form $a = a_{-2} \oplus a_{-1} \oplus a_0$, where $a_{-1} = S'$ is a vector subspace of $S$, $a_{-2} = V'$ is a subspace of $V$ containing $\text{Im} \kappa | \bigwedge^2 S'$ and $a_0 = \mathfrak{h}$ is a subalgebra of $\mathfrak{so}(V)$ preserving $S'$ and $V'$. By Proposition 2.6, to study filtered deformations of $a$ we must first understand the degree-2 Spencer complex $(C^{2,2}, \partial)$. Recalling the definition of the Spencer complex from §2.3.2, the space of $(2, p)$-cochains consists of degree-2 maps $\bigwedge^p a \to a$, whence we deduce that the degree-2 complex can be written as

\[
0 \to \text{Hom}(V', \mathfrak{h}) \to \text{Hom}(\bigwedge^2 V', V') \oplus \text{Hom}(V' \otimes S', S') \oplus \text{Hom}(\bigwedge^2 S', \mathfrak{h}) \to \text{Hom}(V' \otimes \bigwedge^2 S', V') \oplus \text{Hom}(\bigwedge^3 S', S') \to 0.
\] (3.106)

We denote the projections to the components of $C^{2,2}(a; a)$ as follows:

\[
\begin{align*}
\pi_1 : C^{2,2}(a; a) &\to \text{Hom}(\bigwedge^2 V', V'), \\
\pi_2 : C^{2,2}(a; a) &\to \text{Hom}(V \otimes S', S'), \\
\pi_3 : C^{2,2}(a; a) &\to \text{Hom}(\bigwedge^2 S', \mathfrak{h}).
\end{align*}
\] (3.107)
We now record a pair of results which will be useful for studying the cohomology of a class of graded subalgebras which we will be particularly interested in later on.

**Lemma 3.16.** Let \( \mathfrak{h} \) be a subalgebra of \( \mathfrak{so}(V) \) and \( V' \) an \( \mathfrak{h} \)-submodule of \( V \) such that

1. the restriction of \( \eta \) to \( V' \) is non-degenerate,
2. the action of \( \mathfrak{h} \) on \( V' \) is faithful,

(which hold in particular for \( V' = V \)) and suppose that \( \lambda : V' \to \mathfrak{h} \) is a linear map such that

\[
\lambda(v)w - \lambda(w)v = 0 \quad \forall v, w \in V'.
\]

Then \( \lambda = 0 \).

**Proof.** Using that \( \lambda \) takes values in \( \mathfrak{h} \subseteq \mathfrak{so}(V) \), for all \( u, v, w \in V' \) we have

\[
\eta(u, \lambda(v)w) = \eta(u, \lambda(w)v) = -\eta(\lambda(w)u, v) = -\eta(\lambda(u)w, v) = \eta(u, \lambda(u)v) = \eta(w, \lambda(v)u) = -\eta(\lambda(v)w, u).
\]

(3.109)

Thus by non-degeneracy we have \( \lambda(v)w = 0 \) for all \( v, w \in V \), so since \( \mathfrak{h} \) acts faithfully on \( V \), \( \lambda = 0 \).

**Corollary 3.17.** Let \( \alpha = V \oplus S' \oplus \mathfrak{h} \) be a graded subalgebra of \( \mathfrak{s} \) with \( \alpha_{-2} = V \). Then the composition

\[
\pi_1 \circ \partial : \text{Hom}(V, \mathfrak{h}) \to \text{Hom}(\wedge^2 V, V)
\]

is injective, whence \( H^2(\alpha_{-2}; \alpha) = Z^2(\alpha_{-2}; \alpha) = 0 \). If \( \mathfrak{h} = \mathfrak{so}(V) \), \( \pi_1 \circ \partial \) is an isomorphism.

**Proof.** Let \( \lambda \in \ker(\pi_1 \circ \partial) \). Then for all \( v, w \in V \),

\[
(\pi_1 \circ \partial)(\lambda)(v, w) = \partial \lambda(v, w) = \lambda(v)w - \lambda(w)v = 0.
\]

(3.111)

The first claim then follows from Lemma 3.16. The second claim follows by a dimension count.

**Remark 6.** The second claim in the result above can be restated as follows: for any linear map \( \alpha : \wedge^2 V \to V \), there exists a unique linear map \( \lambda : V \to \mathfrak{so}(V) \) satisfying \( \alpha(v, w) = \partial \lambda(v, w) = \lambda(v)w - \lambda(w)v \). We will use this fact repeatedly.

**Remark 7.** More generally, \( \pi_1 \circ \partial \) is injective for \( \alpha = V' \oplus S' \oplus \mathfrak{h} \) with \( (\mathfrak{h}, V') \) satisfying the conditions of Lemma 3.16 and is an isomorphism if and only if we also have \( \mathfrak{h} = \mathfrak{so}(V', \eta \otimes^2 V') \). If we relax assumption (2) in the lemma, we have the weaker conclusion that \( \ker \pi_1 \circ \partial \cong \text{Hom}(V'; \text{ann}_\mathfrak{h}(V')) \).

**The cohomology**

**Lemma 3.18.** Let \( \alpha + \beta + \gamma \) be a cocycle in \( C^{2,2}(\alpha_{-2}; \alpha) \) where

\[
\alpha \in \text{Hom}(\wedge^2 V, V), \quad \beta \in \text{Hom}(V \otimes S, S), \quad \gamma \in \text{Hom}(\otimes^2 S, \mathfrak{so}(V)).
\]

(3.112)
Then the cohomology class \([\alpha + \beta + \gamma] \in H^{2,2}(S; S)\) has a unique representative \(\beta' + \gamma'\) with no \(\text{Hom}(\wedge^2 V, V)\) component. We denote the space of such normalised cocycles by \(\mathcal{H}^{2,2}\):

\[
\mathcal{H}^{2,2} = \{\beta + \gamma \in Z^{2,2}(S; S) \mid \beta \in \text{Hom}(V \otimes S, S), \gamma \in \text{Hom}(\wedge^2 S, \mathfrak{so}(V))\},
\]

and then we have

\[
H^{2,2}(S; S) \cong \mathcal{H}^{2,2}
\]

as \(\mathfrak{so}(V)\)-modules.

**Proof.** Using the isomorphism \(\pi_1 \circ \partial\), there exists a unique map \(\lambda \in \text{Hom}(V, \mathfrak{so}(V))\) such that

\[
\partial \lambda(v, w) = \lambda(v)w - \lambda(w)v = \alpha(v, w).
\]

Thus, if we define \(\beta'\) and \(\gamma'\) by

\[
\beta'(v, s) = \beta(v, s) - \partial \lambda(v, s) = \beta(v, s) - \lambda(v) \cdot s,
\]

\[
\gamma'(s, s') = \gamma(s, s') - \partial \lambda(s, s') = \gamma(s, s) + \lambda(\kappa(s, s')),
\]

we have

\[
[\alpha + \beta + \gamma] = [\alpha + \beta + \gamma - \partial \lambda] = [\beta' + \gamma'],
\]

which proves the first claim. It immediately follows that, as \(\mathfrak{so}(V)\)-modules, we have

\[
Z^{2,2}(S; S) = \mathcal{H}^{2,2} \oplus B^{2,2}(S; S) \quad \text{and} \quad H^{2,2}(S; S) \cong \mathcal{H}^{2,2}.
\]

It is worth pausing to explicitly examine the “normalised” Spencer cocycle conditions for \(\beta + \gamma \in \mathcal{H}^{2,2}\). Written in polarised form (explained in §2.3.4), these are

\[
2\kappa(s, \beta(v, s)) + \gamma(s, s)v = 0,
\]

\[
\beta(\kappa_s, s) + \gamma(s, s) \cdot s = 0,
\]

for \(v \in V\) and \(s \in S\), where we introduce the notation \(\kappa_s := \kappa(s, s)\). We will refer to these as the first and second (normalised) Spencer cocycle conditions. We note that the first equation uniquely defines \(\gamma : \wedge^2 S \to \mathfrak{so}(V)\) in terms of \(\beta : V \otimes S \to S\) and constrains the endomorphism \(v \mapsto \kappa(s, \beta(v, s))\) of \(V\) to lie in \(\mathfrak{so}(V)\) for all \(s \in S\). In later chapters, where we explicitly calculate this cohomology in particular cases, we will first determine the space of maps \(\beta\) satisfying this constraint and then substitute the resulting expressions for \(\beta\) and \(\gamma\) into the second equation and solve it.

We also note here that \(\beta + \gamma \in \mathcal{H}^{2,2}\) (or the corresponding cohomology class) is invariant under the action of a subalgebra \(\mathfrak{h} \in \mathfrak{so}(V)\) if and only if \(\beta\) is \(\mathfrak{h}\)-invariant (that is, \(\beta\) is \(\mathfrak{h}\)-equivariant when considered as a map \(V \otimes S \to S\)); indeed, if \(A \in \mathfrak{h}\), \(v \in V\) and \(s \in S\) then we have

\[
(A \cdot (\beta + \gamma))(v, s) = (A \cdot \beta)(v, s) = A \cdot (\beta(v, s)) - \beta(Av, s) - \beta(A, v \cdot s)
\]

so clearly \(\beta\) is invariant if \(\beta + \gamma\) is, while on the other hand if \(\beta\) is invariant then using the first cocycle condition (once in the form above and once in depolarised form) as
well as $\mathfrak{so}(V)$-invariance of $\kappa$, we have

$$
(A \cdot \gamma)(s, s) v = [A, \gamma(s, s)] v - 2\gamma(A \cdot s, s) v
= 2\Lambda(\kappa(s, \beta(v, s))) - 2\kappa(s, \beta(Av, s)) - 2\kappa(A \cdot s, \beta(v, s)) - 2\kappa(s, \beta(v, A \cdot s)) \quad (3.121)
= 2\kappa(s, A \cdot \beta(v, s)) - 2\kappa(s, \beta(Av, s)) - 2\kappa(s, \beta(v, A \cdot s))
= 0,
$$

this $\gamma$ is $\mathfrak{h}$-invariant, whence $\beta + \gamma$ is too.

**Remark 8.** Let us pause briefly to relate these cocycle conditions back to our theory of Killing superalgebras and to explain the coincidences in notation. First note that there is a canonical homomorphism $\text{Hom}(V \otimes S, S) \cong V^* \otimes \text{End}(S)$, so we can view the cocycle component $\beta$ as a spinor endomorphism-valued 1-form on $V$. Under this interpretation, we write $\beta(v)s := \beta(v, s)$.

The definition of admissible connections, Definition 3.7, involves three conditions on the 1-form $\beta = D - \nabla \in \Omega^1(M; \text{End}(S))$ and the section $\gamma \in \Gamma(\bigotimes^2 S^* \otimes TM)$ which is defined in terms of $\beta$ by equation (3.24). Conditions (1) and (2) are algebraic in $\beta, \gamma$ and condition (3) is differential. For a symmetric squaring map, we can polarise the definition of $\gamma$ and the algebraic conditions. Then, taking them pointwise, the definition of $\gamma$ and condition (1) are equivalent to cocycle condition (3.118) above, while condition (2) is precisely (3.119) (were the fibre of $\mathfrak{g}$ is $S$ and the same squaring map is chosen). Thus, if we wish to find admissible connections on a spinor bundle $S$ equipped with a squaring map $\kappa$ over a spin manifold $(M, g)$, we need only compute $\mathcal{H}^{2,2}$ to determine a local ansatz for the allowed 1-forms $\beta$ and then check whether or under what circumstances the remaining condition (namely $\mathcal{L}_{\kappa_0} \beta = 0$ for all $D$-parallel spinors $s$) holds, which we typically do using integrability conditions. Indeed, this is precisely what we did in the 2-dimensional example of §3.2.4.

### 3.3.2 General filtered deformations and cohomology

A filtered deformation $\bar{a}$ of a graded subalgebra $\mathfrak{a} = V' \oplus S' \oplus \mathfrak{h}$ of $\mathfrak{s}$ has the brackets

$$
\begin{align*}
[A, B] &= AB - BA & [s, s] &= \kappa_s + \gamma(s, s) \\
[A, \nu] &= A\nu + \delta(A, \nu) & [v, s] &= \beta(v, s) \\
[A, s] &= A \cdot s & [v, w] &= \alpha(v, w) + \theta(v, w)
\end{align*}
$$

(3.122)

where $A, B \in \mathfrak{h}$, $\nu, w \in V'$, $s \in S'$, and

$$
\alpha : \bigwedge^2 V' \to V', \quad \beta : V' \otimes S' \to S', \quad \gamma : \bigotimes^2 S' \to \mathfrak{h}, \quad \delta : \mathfrak{h} \otimes V' \to \mathfrak{h}
$$

(3.123)

are the components of the degree-2 deformation map, while

$$
\theta : \bigwedge^2 V' \to \mathfrak{h}
$$

(3.124)

is the degree-4 deformation map. We denote the full degree-2 deformation by

$$
\mu = \alpha + \beta + \gamma + \delta : \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a}
$$

(3.125)
so the defining sequence for the deformation is \((\mu, \theta, 0, \ldots)\). From Proposition 2.6, we have the following homological information:

\[
\mu \in Z^{2,2}(a; a), \quad (3.126)
\]
\[
\mu|_{a_+ \otimes a_+} \in Z^{2,2}(a_+; a), \quad (3.127)
\]
\[
[\mu|_{a_+ \otimes a_+}] \in H^{2,2}(a_+; a), \quad (3.128)
\]

where in the first expression we have graded the full Chevalley-Eilenberg complex in the same way that we grade the Spencer complex.

Unpacking the cocycle conditions

Let us consider these conditions in more detail. Equation (3.126) is equivalent to the following system of linear equations:

\[
\alpha(\kappa, s, v) + 2\kappa(s, \beta(v, s)) + \gamma(s, s) v = 0, \quad (3.129)
\]
\[
\beta(\kappa, s) + \gamma(s, s) s = 0, \quad (3.130)
\]
\[
A\alpha(v, w) - \alpha(Av, w) - \alpha(v, Aw) + \delta(A, w) v - \delta(A, v) w = 0, \quad (3.131)
\]
\[
A \cdot (\beta(v, s)) - \beta(Av, s) - \beta(v, A \cdot s) - \delta(A, v) s = 0, \quad (3.132)
\]
\[
[ A, \gamma(s, s) ] - 2\gamma(A \cdot s, s) + \delta(A, \kappa_s) = 0, \quad (3.133)
\]
\[
\delta([A, B], v) - [A, \delta(B, v)] + [B, \delta(A, v)] - \delta(A, B v) + \delta(B, A v) = 0, \quad (3.134)
\]

for all \(A, B \in \mathfrak{h}, \ v, w \in V'\) and \(s \in S'\), where again \(\kappa_s = \kappa(s, s)\). Equation (3.127) is equivalent to the first pair of equations – we will refer to these as the Spencer cocycle conditions. Equation (3.128) is equivalent to the following: for each \(A \in \mathfrak{h}\), there exists a cochain \(\chi_A \in C^{2,1}(a_+; a) = \text{Hom}(V', \mathfrak{h})\) such that

\[
A \cdot (\alpha + \beta + \gamma) = \partial \chi_A. \quad (3.135)
\]

This is implied by equation (3.126): simply taking \(\chi_A = l_A \delta\), the above equation is equivalent to equations (3.131)-(3.133). In the case that \(V' = V\), we have a converse: it follows from Corollary 3.17 that (3.135) defines \(\chi_A\) uniquely and that the assignment \(A \mapsto \chi_A\) is linear. Thus we can define a unique linear map \(\delta : \mathfrak{h} \otimes V \rightarrow \mathfrak{h}\) by \(\delta(A, v) := \chi_A(v)\). Then

\[
\delta([A, B], v) - [A, \delta(B, v)] + [B, \delta(A, v)] - \delta(A, B v) + \delta(B, A v)
\]
\[
= \chi_{[A,B]}(v) - [A, \chi_B(v)] + [B, \chi_A(v)] - \chi_A(B v) + \chi_B(A v) \quad (3.136)
\]
\[
= (\chi_{[A,B]} - A \cdot \chi_B + B \cdot \chi_A)(v).
\]

On the other hand, by (3.135) we have

\[
\partial(\chi_{[A,B]} - A \cdot \chi_B + B \cdot \chi_A) = 0, \quad (3.137)
\]

so \(\chi_{[A,B]} - A \cdot \chi_B + B \cdot \chi_A \in Z^{2,1}(a_+; a)\). But this space is zero by Corollary 3.17. Hence \(\delta\) satisfies (3.134). Thus we have shown the following.
Lemma 3.19. Let $\mu \in Z^{2,2}(a; a)$. Then $\mu|_{a_{-a}} \in Z^{2,2}(a; a)$ with $\mathfrak{h}$-invariant cohomology class. If $V' = V$, we conversely have that if $\mu \in Z^{2,2}(a; a)$ is a Spencer cocycle such that with $\mathfrak{h}$-invariant cohomology class then there exists a unique cochain $\delta \in Z^1(\mathfrak{h}; \mathfrak{h}^* \otimes \mathfrak{h}) \subseteq \text{Hom}(\mathfrak{h} \otimes V, \mathfrak{h})$ such that $\mu = \mu_- + \delta \in Z^{2,2}(a; a)$.

We can be more explicit about the form of $\delta$ in the $V = V'$ case: by Corollary 3.17, there exists a unique map $\lambda : V \to \mathfrak{so}(V)$ such that

$$\alpha(v, w) = \lambda(v)w - \lambda(w)v. \quad (3.138)$$

One can then easily check that

$$\delta(A, v) = [A, \lambda(v)] - \lambda(Av) \quad (3.139)$$

solves equations (3.131)-(3.133).

Jacobi identities

We now consider the conditions imposed on the deforming maps by the Jacobi identities. We denote by $[\tilde{a}_i, \tilde{a}_j, \tilde{a}_k]$ the Jacobi identity on the $i, j, k$-th homogeneous subspace of $\mathfrak{a}$.

- $[\tilde{a}_0, \tilde{a}_0, \tilde{a}_0]$: Follows from Jacobi identity on $\mathfrak{h} \subseteq \mathfrak{so}(V)$.

- $[\tilde{a}_0, \tilde{a}_0, \tilde{a}_-1]$: Follows from $S'$ being a representation of $\mathfrak{h}$.

- $[\tilde{a}_0, \tilde{a}_0, \tilde{a}_{-2}]$: Using that $V'$ is a representation of $\mathfrak{h}$, this is equivalent to the cocycle condition (3.134): for $A, B \in \mathfrak{h}, v \in V$,


$$= ([A, B]v + \delta([A, B], v)) - (ABv + \delta(A, Bv) + [A, \delta(B, v))]$$

$$+ (BAv + \delta(B, Av) + [B, \delta(A, v)])$$

$$= \delta([A, B], v) - \delta(A, Bv) + \delta(B, Av) - [A, \delta(B, v)] + [B, \delta(A, v)].$$

- $[\tilde{a}_0, \tilde{a}_{-1}, \tilde{a}_{-1}]$: Using the $\mathfrak{so}(V)$-equivariance of $\kappa$, this is equivalent to cocycle condition (3.133): for $A \in \mathfrak{h}$, $s \in S'$, we have

$$[A, [s, s]] = 2[[A, s], s]$$

$$= (Ak_s - \delta(A, k_s) + [A, \gamma(s, s)]) - 2\kappa(As, s) + \gamma(As, s))$$

$$= (Ak_s - 2\kappa(As, s)) + ([A, \gamma(s, s)] - 2\gamma(As, s) + \delta(A, k_s)).$$

- $[\tilde{a}_0, \tilde{a}_{-1}, \tilde{a}_{-2}]$: Equivalent to cocycle condition (3.132): for $A \in \mathfrak{h}$, $s \in S'$, $v \in V$, we have

$$[A, [v, s]] - [[A, v], s] - [v, [A, s]]$$

$$= A \cdot (\beta(v, s)) - (\beta(Av, s) + \delta(A, v) \cdot s) - \beta(v, A \cdot s)$$

$$= A \cdot (\beta(v, s)) - \beta(Av, s) - \beta(v, A \cdot s) - \delta(A, v) \cdot s.$$
• \([\bar{a}_0, \bar{a}_{-2}, \bar{a}_{-2}]:\) For \(A \in \mathfrak{h}\) and \(v, w \in V',\) we have

\[
[A, [v, w]] - [[A, v], w] + [v, [A, w]]
= (AA(v, w) + \delta(A, \alpha(v, w)) + [A, \theta(v, w)])
- (\alpha(Av, w) + \theta(Av, w) + \delta(A, v) w + \delta(\delta(A, v), w))
- (\alpha(v, Aw) + \theta(v, Aw) - \delta(A, w) v - \delta(\delta(A, w), v))
= (AA(v, w) - \alpha(Av, w) - \alpha(v, Aw) - \delta(A, v) w + \delta(A, w) v)
+ ([A, \theta(v, w)] - \theta(Av, w) - \theta(v, Aw)
+ \delta(A, \alpha(v, w)) - \delta(\delta(A, v), w) + \delta(\delta(A, w), v)),
\]

so this Jacobi identity has a \(V'\)-component, which is equivalent to the cocycle condition (3.131), and an \(\mathfrak{h}\)-component

\[
(A \cdot \theta)(v, w) = \delta(\delta(A, v), w) - \delta(\delta(A, w), v) - \delta(A, \alpha(v, w)). \quad (3.140)
\]

• \([\bar{a}_{-1}, \bar{a}_{-1}, \bar{a}_{-1}]:\) Equivalent to the second (2,2)-Spencer cocycle condition (3.130):

\[
[[s, s], s] = \beta(\kappa_s, s) + \gamma(s, s) \cdot s.
\]

• \([\bar{a}_{-1}, \bar{a}_{-1}, \bar{a}_{-2}]:\) For \(s \in S', v \in V',\)

\[
[[s, s], v] + 2[s, [v, s]]
= (\alpha(\kappa_s, v) + \theta(\kappa_s, v) + \gamma(s, s) v + \delta(\gamma(s, s), v)) + 2(\kappa(s, \beta(v, s)) + \gamma(s, \beta(v, s)))
= (\alpha(\kappa_s, v) + 2\kappa(s, \beta(v, s)) + \gamma(s, s) v + (\theta(\kappa_s, v) + \delta(\gamma(s, s), v)) + 2\gamma(s, \beta(v, s))).
\]

Like the \([\bar{a}_0, \bar{a}_{-2}, \bar{a}_{-2}]\) identity, there is a \(V'\)-component and a \(\mathfrak{h}\)-component to this Jacobi identity. The former is the first Spencer cocycle condition (3.129), while the latter is

\[
\theta(\kappa_s, v) + \delta(\gamma(s, s), v) + 2\gamma(s, \beta(v, s)) = 0. \quad (3.141)
\]

• \([\bar{a}_{-1}, \bar{a}_{-2}, \bar{a}_{-2}]:\) For \(s \in S', v, w \in V',\)

\[
[[v, w], s] - [v, [w, s]] + [w, [v, s]]
= (\beta(\alpha(v, w), s) + \theta(v, w) \cdot s) - \beta(v, \beta(w, s)) + \beta(w, \beta(v, s)),
\]

so this identity is

\[
\theta(v, w) \cdot s + \beta(\alpha(v, w), s) - \beta(v, \beta(w, s)) + \beta(w, \beta(v, s)) = 0. \quad (3.142)
\]

• \([\bar{a}_{-2}, \bar{a}_{-2}, \bar{a}_{-2}]:\) For \(u, v, w \in V',\)

\[
[[u, v], w] = \alpha(\alpha(u, v), w) + \theta(\alpha(u, v), w) + \theta(u, v) w + \delta(\theta(u, v), w),
\]

so separating the \(V'\) and \(\mathfrak{h}\) components, this identity is

\[
\alpha(\alpha(u, v), w) + \theta(u, v) w + \text{cyclic perms.} = 0, \quad (3.143)
\]

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\[ \theta(\alpha(u, v), w) + \delta(\theta(u, v), w) + \text{cyclic perms.} = 0. \] (3.144)

Note that the identities split into two types: those which involve only the deforming maps of degree 2, which are linear in those maps and are either identically satisfied or equivalent to the homological conditions (3.129)-(3.134), and those which involve the degree-4 map \( \theta \), all of which are quadratic in the deforming maps. In particular, recalling Lemma 3.19, we have the following.

**Proposition 3.20.** Let \( a = V' \oplus S' \oplus \mathfrak{h} \subseteq s \) be a graded subalgebra and let \( \alpha + \beta + \gamma + \delta \in \mathbb{Z}^{2,2}(\alpha; \alpha) \). Then the brackets (3.122) define a filtered deformation \( \tilde{\alpha} \) of \( a \) if and only if there exists a map \( \alpha : \bigwedge^2 V' \to \mathfrak{h} \) satisfying equations (3.140)-(3.144).

Thus the task of determining the possible filtered deformations of a graded subalgebra \( a \subseteq s \) consists of solving the linear equations (3.129)-(3.134) for \( \mu = \alpha + \beta + \gamma + \delta \) and then the quadratic (3.140)-(3.144) for \( \theta \). We can interpret a \((2,2)\)-cocycle \( \mu \) as an infinitesimal filtered deformation of \( a \), and this deformation integrates to an actual filtered deformation \( \tilde{\alpha} \) of \( a \) if and only if there exists a solution to \( \theta \) to the equations (3.140)-(3.144). These equations are highly non-trivial in general, and even if an infinitesimal deformation integrates, the full deformation need not be unique. It is also difficult to give a homological characterisation of when two pairs \((\mu, \theta)\) and \((\mu', \theta')\) give rise to isomorphic deformations.

We can get much better homological control over the problem of determining filtered deformations by passing to a particular class of graded subalgebras \( a \). We have already seen in Lemmas 3.16 and 3.19 that taking \( V = V' \) already gives us significant simplifications. We will now see that this means that \( a \) is transitive; in what follows we will make the further assumption that \( S' \) is large enough to guarantee that \( a \) is fundamental, which will allow us to fully characterise filtered deformations of \( a \) in terms of Spencer cohomology.

### 3.3.3 Highly supersymmetric graded subalgebras

**The Homogeneity Theorem**

We say that a symmetric squaring map \( \kappa \) is causal if \( \kappa_s = \kappa(s, s) \) is either timelike or null for all \( s \in S \). In Lorentzian signature, it can be shown that a symmetric, Dirac current (squaring map on the real pinor module formed from a real bilinear on the pinor module as in (2.54)) always exists in any dimension. The existence of symmetric currents can be deduced from Table 2.6; that they can be chosen to be causal must be shown case-by-case. These can be used to build symmetric squaring maps on (extended) spinor representations \( S \) by restriction and tensoring with bilinears on an auxiliary module. However such a map is constructed, we have the following.

**Theorem 3.21** (Homogeneity Theorem [33, 34]). Let \((V, \eta)\) be a Lorentzian vector space of dimension \( \dim V > 2 \) and \( S \) a (not necessarily irreducible) spinor representation of \( \mathfrak{so}(V) \) with a symmetric, causal \( \mathfrak{so}(V) \)-invariant squaring map \( \kappa : \bigotimes^2 S \to V \). If \( S' \) is a vector subspace of \( S \) with \( \dim S' > \frac{1}{2} \dim S \), then \( \kappa|_{\bigotimes^2 S'} \) is surjective onto \( V \).

For all \( A \in \mathfrak{so}(V) \) and \( s \in S \), \( A \cdot s = 0 \) implies that \( A \kappa_s = 0 \) by invariance of the squaring map. Since \( \mathfrak{so}(V) \) acts faithfully on \( V \), the Homogeneity Theorem has the following corollary.
Corollary 3.22. If the squaring map $\kappa$ is causal and $\dim S' > \frac{1}{2} \dim S$ then the annihilator of $S'$ in $\mathfrak{so}(V)$ is trivial. In particular, any subalgebra $\mathfrak{h}$ of $\mathfrak{so}(V)$ which preserves $S'$ acts faithfully on $S'$.

Guided by the Homogeneity Theorem (and by physics), let us now fix $s$ to be the Poincaré superalgebra associated to a Lorentzian inner product space $(V, \eta)$ and spinor representation $S$ with symmetric causal squaring map $\kappa$. Note that for such algebras we have $[S', S'] = V$ for $\dim S' > \frac{1}{2} \dim S$, which prompts the following definition.

Definition 3.23. A graded subalgebra $a = V \oplus S' \oplus \mathfrak{h}$ of $s$ with $\dim S' > \frac{1}{2} \dim S$ is said to be highly supersymmetric, and it is maximally supersymmetric if $S' = S$.

Homological properties

Recall Definition 2.7 of some homological properties of graded Lie superalgebras. We now show that a highly supersymmetric graded subalgebras of $s$ satisfy some of these.

Lemma 3.24 ([45]). Let $a$ be a highly supersymmetric graded subalgebra of $s$. Then

- $a$ is fundamental and transitive,
- $a$ is a full prolongation of degree 2,
- $H^{d,2}(a_{-};a) = 0$ for all even $d > 2$.

Proof. The Homogeneity Theorem precisely says that $a$ is fundamental, and transitivity follows from faithfulness of the action of $h = a_0 \subseteq \mathfrak{so}(V)$ on $V = a_{-2}$ or $S' = a_{-1}$. Corollary 3.17 then says that $H^{2,1}(a_{-};a) = 0$. For $d > 2$, $H^{d,1}(a_{-};a) = 0$ trivially since $C^{d,1}(a_{-};a) = 0$. The beginning of the degree-4 Spencer complex is

$$0 \longrightarrow \text{Hom}(\Lambda^2 V, \mathfrak{h})$$
$$\longrightarrow \text{Hom}(\Lambda^3 V, V) \oplus \text{Hom}(\Lambda^2 V \otimes S', S') \oplus \text{Hom}(V \otimes \bigotimes^2 S', \mathfrak{h}) \longrightarrow \cdots,$$

and for $\theta \in C^{4,2}(a_{-};a) = \text{Hom}(\Lambda^2 V, \mathfrak{h})$, we have

$$\partial \theta(v, s, s) = \theta(v, \kappa s),$$

for all $v \in V$, $s \in S'$, so again by homogeneity, $\partial \theta = 0$ if and only if $\theta = 0$. This shows that $H^{4,2}(a_{-};a) = 0$, and we have $H^{d,2}(a_{-};a) = 0$ for all $d > 4$ since then $C^{d,2}(a_{-};a) = 0$.

We now note that although we have specialised to highly supersymmetric graded subalgebras of the Poincaré superalgebra, an appropriately modified version of the proof of Lemma 3.24 will hold for any graded subalgebra of a flat model (super)algebra of the form $a = V \oplus S' \oplus \mathfrak{h}$ where $[S', S'] = V$ (in particular if $S' = S$), so the same homological conditions hold for those algebras. We will comment no further on this, other than to observe that most of the results of this section will apply in such cases as well.
We will be particularly interested in the deformations of highly (maximally) supersymmetric graded subalgebras – which we refer to as highly (maximally) supersymmetric subdeformations of $s$ – since by Lemma 3.24 the Spencer cohomology of such subalgebras is particularly simple. The following pivotal observation is a straightforward generalisation to all spacetime dimensions of Proposition 6 of [45] and also of the analogous results for maximally supersymmetric graded subalgebras (those for which $n_1 = S$) in [16, 48]. For the proof, we recall Propositions 2.6 and 2.8 which characterise the defining sequences of filtered deformations in terms of Spencer cohomology and tell us how much freedom we have to redefine them.

**Proposition 3.25** ([45]). Let $a = V' \oplus S' \oplus h$ be a graded subalgebra of $s$ and $\tilde{a}$ a filtered deformation of $a$ with defining sequence $(\mu, \theta, 0, \ldots)$ as above. Then

1. $\mu|_{a \otimes a_-}$ is a cocycle in $C^{2,2}(a_-; a)$, and

   $$\left[ \mu|_{a \otimes a_-} \right] \in H^{2,2}(a_-; a)^h.$$  

   Furthermore, $\mu$ is a cocycle in $C^2(a; a)$.

2. If $\tilde{a}$ is highly supersymmetric and $\tilde{a}'$ is another filtered deformation of $a$ with degree-2 deformation map $\mu'$ such that $[\mu'|_{a_- \otimes a_-}] = [\mu|_{a_- \otimes a_-}]$ then $\tilde{a} \cong \tilde{a}'$ as filtered Lie superalgebras.

**Proof.** The first statement follows by Proposition 2.6. For the second, if $[\mu'|_{a_- \otimes a_-}] = [\mu|_{a_- \otimes a_-}]$ then $(\mu' - \mu)|_{a_- \otimes a_-}$ is a Spencer coboundary, so by point (1) of Proposition 2.8 there is a new defining sequence $\{\mu'', \theta'', 0, \ldots\}$ of $\tilde{a}'$ such that $\mu''|_{a_- \otimes a_-} = (\mu' - \mu)|_{a_- \otimes a_-}$. Then, since $a$ is a full prolongation of degree 2 by Lemma 3.24, by point (3) of Proposition 2.8 we can again find a new defining sequence $\{\mu''', \theta''', 0, \ldots\}$ of $\tilde{a}'$ such that $\mu''' = \mu$ and $\theta''' = \theta''|_{a_- \otimes a_-} = \theta|_{a_- \otimes a_-}$. $\square$

Note that we already saw the first statement of the proposition above in §3.3.2. The second statement allows us to strengthen Proposition 3.20; we already saw in Lemma 3.19 that a choice of $h$-invariant $\mu_{a \otimes a_-}$ determined $\mu$ for $V = V'$; we now see that the cohomology class $[\mu_{a \otimes a_-}]$ determines the entire deformation up to isomorphism in the highly supersymmetric case.

What this means in practice is that in order to find the infinitesimal deformations of a highly supersymmetric graded subalgebra $a$, we compute $H^{2,2}(a_-; a)$ and its $h$-invariants. The other deforming maps $\delta$ and $\theta$ will then be determined by the Jacobi identities. Note, though, that is not guaranteed that every $h$-invariant Spencer $(2, 2)$-cohomology class is the deforming map for some filtered deformation of $a$ – the Jacobi identities which are quadratic in the deformation maps may still obstruct the “integration” of the infinitesimal deformation.

**Maps in cohomology**

It will be useful for us to relate $H^{2,2}(a_-; a)$ to some other cohomology groups. The inclusion $i : a \hookrightarrow s$ of a graded subalgebra into $s$ induces the following maps of cochains:

$$i_* : C^{\bullet, \bullet}(a_-; a) \rightarrow C^{\bullet, \bullet}(a_-; s), \quad \phi \mapsto i_* \phi = i \circ \phi,$$  

(3.148)
where this isomorphism is the restriction of the isomorphism $i, i^*$ respect both this grading and the homological grading – that is, they commute with the differential $\partial$. Thus these maps in turn induce maps in cohomology also denoted $i, i^*$. We will be particularly interested in the following diagram in cohomology in degree $(2,2)$:

$$
\begin{array}{c}
H^{2,2}(s_-; s) \\
\downarrow^{i_*}
\end{array} \rightarrow \begin{array}{c}
H^{2,2}(a_-; a)
\end{array}
$$

Let us consider the kernels of the maps above. Recall that, by Lemma 3.18, $H^{2,2}(s_-; s) \cong H^{2,2}(a_-; a)$ where the latter is the space of normalised Spencer $(2,2)$-cocycles of $s$.

For $\beta : V \otimes S \to S$, $\gamma : \bigwedge^2 S \to \mathfrak{so}(V)$ such that $\beta + \gamma \in H^{2,2}$, we have $i^* \{\beta + \gamma\} = 0$ if and only if $\beta|_{V' \otimes S'} + \gamma|_{\bigwedge^2 S'} = 0$, where $\lambda : V' \to \mathfrak{so}(V)$. If $V' = V$, this implies that $\partial \lambda|_{\bigwedge^2 V} = 0$, so $\lambda = 0$ by Lemma 3.16, so $\beta|_{V' \otimes S'} = 0$ and $\gamma|_{\bigwedge^2 S'} = 0$. Note that the $\beta|_{V' \otimes S'} = 0$ implies $\gamma|_{\bigwedge^2 S'} = 0$ by the cocycle condition for $\beta + \gamma$, so we find that

$$
\mathcal{H}^{2,2}(a_-) := \{ \beta + \gamma \in H^{2,2} \mid \beta|_{V' \otimes S'} = 0 \} \cong \ker i^* \text{ if } V' = V,
$$

where this isomorphism is the restriction of the isomorphism $\mathcal{H}^{2,2} \cong H^{2,2}(s_-; s)$. For $\alpha : \bigwedge^2 V' \to V'$, $\beta : V' \otimes S' \to S'$, $\gamma : \bigwedge^2 S' \to \mathfrak{h}$ with $\alpha + \beta + \gamma \in Z^{2,2}(a_-; a)$, $i_* [\alpha + \beta + \gamma] = 0$ if and only if $i_* (\alpha + \beta + \gamma) = \partial \lambda$ where $\lambda : V' \to \mathfrak{so}(V)$. Then $\gamma(s, s) = -\lambda(\kappa(s, s))$ for all $s \in S'$; in particular, $\lambda([S', S']) \subseteq \mathfrak{h}$. If $V' = [S', S']$ then we find that $\lambda$ takes values in $\mathfrak{h}$, so we have $\lambda = i_* \lambda'$ for some $\lambda' \in C^{2,1}(a_-; a) = \text{Hom}(V', \mathfrak{h})$, whence $\alpha + \beta + \gamma = \partial \lambda'$, so $[\alpha + \beta + \gamma] = 0$. Thus we find

$$
i_* : H^{2,2}(a_-; a) \to H^{2,2}(a_-; a) \text{ is injective if } V' = [S', S'].
$$

In particular, we have shown the following.

**Lemma 3.26.** Let $\alpha$ be a highly supersymmetric graded subalgebra of $s$. Then we have a diagram with exact rows and columns

$$
\begin{array}{c}
0 \\
\downarrow
\end{array} \rightarrow \begin{array}{c}
\mathcal{H}^{2,2}(a_-) \\
\downarrow^{i_*}
\end{array} \rightarrow \begin{array}{c}
H^{2,2}(s_-; s) \cong \mathcal{H}^{2,2} \\
\downarrow^{i^*}
\end{array} \rightarrow \begin{array}{c}
H^{2,2}(a_-; s)
\end{array}
$$

By the same argument as in the proof of Lemma 3.18, in the highly supersymmetric case any cohomology class in $H^{2,2}(a_-; s)$ has a unique representative in the space
of normalised cocycles

$$\mathcal{H}^{2,2}(a_\ldots) = \left\{ \beta + \gamma \in Z^{2,2}(a_\ldots; s) \left| \begin{array}{c} \beta \in \text{Hom}(V \otimes S', S), \\ \gamma \in \text{Hom}(\bigotimes^2 S', \mathfrak{so}(V)) \end{array} \right. \right\}, \quad (3.152)$$

and we have a splitting of $\mathfrak{h}$-modules $Z^{2,2}(a_\ldots; s) = \mathcal{H}^{2,2}(a_\ldots) \oplus B^{2,2}(a_\ldots; s)$, whence as $\mathfrak{h}$-modules $H^{2,2}(a_\ldots; s) \cong \mathcal{H}^{2,2}(a_\ldots)$. We thus have an exact sequence at the level of cocycles corresponding to the bottom row of the diagram in the lemma above

$$0 \longrightarrow \mathcal{H}^{2,2}(a_\ldots) \longrightarrow \mathcal{H}^{2,2} \overset{i^*}{\longrightarrow} \mathcal{H}^{2,2}(a_\ldots). \quad (3.153)$$

On the other hand, any cohomology class in $H^{2,2}(a_\ldots; a)$ defines a unique normalised cocycle in $\mathcal{H}^{2,2}(a_\ldots)$ by pushing forward to $H^{2,2}(a_\ldots; s)$ and then identifying the normalised cocycle. Since $i_*$ is injective in cohomology, if a cocycle in $\mathcal{H}^{2,2}(a_\ldots)$ is obtained from a class in $H^{2,2}(a_\ldots; s)$, that class is unique. We can be much more explicit about this: if $\alpha + \beta + \gamma \in Z^{2,2}(a_\ldots; a)$ then there exists a unique element $\lambda \in C^{2,2}(a_\ldots; s) = \text{Hom}(V, \mathfrak{so}(V))$ such that

$$\alpha(v, w) = \lambda(v)w - \lambda(w)v, \quad (3.154)$$

and the maps $\tilde{\beta} : V \otimes S' \longrightarrow S$ and $\tilde{\gamma} : \bigotimes^2 S' \longrightarrow \mathfrak{so}(V)$ such that $\tilde{\beta} + \tilde{\gamma} \in \mathcal{H}^{2,2}(a_\ldots)$ is the unique normalised cocycle such that $i_* [\alpha + \beta + \gamma] = [\tilde{\beta} + \tilde{\gamma}]$ are then given by

$$\tilde{\beta}(v, s) = \beta(v, s) - \lambda(v) \cdot s, \quad (3.155)$$

$$\tilde{\gamma}(s, s) = \gamma(s, s) + \lambda(\kappa_s), \quad (3.156)$$

for $v \in V$, $s \in S'$. The maps $\lambda, \tilde{\beta}, \tilde{\gamma}$ are unique for a given cocycle $\alpha + \beta + \gamma$, while a different choice of representative for the same cohomology class corresponds to shifting $\lambda$ by a map $V \longrightarrow \mathfrak{h}$. We note that

$$i_* (\alpha + \beta + \gamma) = \tilde{\beta} + \tilde{\gamma} + \partial \lambda, \quad (3.157)$$

and that we have the constraints

$$\tilde{\beta}(v, s) + \lambda(v) \cdot s \in S', \quad (3.158)$$

$$\tilde{\gamma}(s, s) - \lambda(\kappa_s) \in \mathfrak{h}, \quad (3.159)$$

for all $v \in V$, $s \in S'$. Finally, observe that $[\alpha + \beta + \gamma]$ is $\mathfrak{h}$-invariant if and only if $\tilde{\beta} + \tilde{\gamma} \in \mathcal{H}^{2,2}(a_\ldots)$ is (which is the case if and only if $\tilde{\beta}$ is $\mathfrak{h}$-invariant). If this is the case, the discussion following Lemma 3.19 tells us that the map $\delta : \mathfrak{h} \otimes V \longrightarrow \mathfrak{h}$ defined by

$$\delta(A, v) = [A, \lambda(v)] - \lambda(Av). \quad (3.160)$$

is the unique such map so that $\alpha + \beta + \gamma + \delta \in Z^{2,2}(a; a)$. 

Reparametrising the deformation

The discussion above allows us to reparametrise the deformed brackets (3.122) in the highly supersymmetric case.

**Proposition 3.27.** If $\tilde{a}$ is a highly supersymmetric filtered deformation of a graded subalgebra $a = V \oplus S' \oplus \mathfrak{h}$ of $s$ then $\tilde{a}$ has a presentation of the form

\[
\begin{align*}
[A, B] &= \underbrace{AB - BA}_{\mathfrak{h}}, \\
[s, s] &= \underbrace{\kappa_s + \tilde{\gamma}(s, s) - \lambda(\kappa_s)}_{\mathfrak{h}}, \\
[A, v] &= \underbrace{Av + [A, \lambda(v)] - \lambda(Av)}_{\mathfrak{V}}, \\
[v, s] &= \underbrace{\tilde{\beta}(v, s) + \lambda(v) \cdot s}_{S'}, \\
[A, s] &= \underbrace{A \cdot s}_{S}, \\
[v, w] &= \underbrace{\lambda(v) w - \lambda(w) v}_{\mathfrak{V}} + \tilde{\theta}(v, w) - \lambda(\lambda(v) w - \lambda(w) v) + [\lambda(v), \lambda(w)],
\end{align*}
\]

for $A, B \in \mathfrak{h}$, $v, w \in V$, $s \in S'$ where the deforming maps are

\[
\begin{align*}
\tilde{\beta} : V \otimes S' &\rightarrow S, \\
\lambda : V &\rightarrow \mathfrak{so}(V), \\
\tilde{\theta} : \otimes^2 S' &\rightarrow \mathfrak{so}(V),
\end{align*}
\]

where $\tilde{\beta} + \tilde{\gamma} \in \mathcal{H}^{2,2}_{(a_{\ldots})} \mathfrak{h}$, $\tilde{\theta}$ is $\mathfrak{h}$-invariant and the remaining Jacobi identities are equivalent to

\[
\begin{align*}
\tilde{\theta}(\kappa_s, v) + 2\tilde{\gamma}(s, \tilde{\beta}(v, s) + \lambda(v) \cdot s) - [\lambda(v), \tilde{\gamma}(s, s)] &= 0, \\
\tilde{\theta}(v, w) \cdot s &= \tilde{\beta}(v, \tilde{\beta}(w, s) + \lambda(w) \cdot s) - \lambda(w) \cdot \tilde{\beta}(v, s) + \tilde{\beta}(\lambda(w) v, s) \\
&- \tilde{\beta}(w, \tilde{\beta}(v, s) + \lambda(v) \cdot s) + \lambda(v) \cdot \tilde{\beta}(w, s) - \tilde{\beta}(\lambda(w) w, s) \\
\tilde{\theta}(u, v) w + \tilde{\theta}(v, w) u + \tilde{\theta}(w, u) v &= 0, \\
(\lambda(u) \cdot \tilde{\theta})(v, w) + (\lambda(v) \cdot \tilde{\theta})(w, u) + (\lambda(w) \cdot \tilde{\theta})(u, v) &= 0,
\end{align*}
\]

for all $u, v, w \in V$ and $s \in S'$. Two deformations $\tilde{a}, \tilde{a}'$ described in this way are isomorphic as filtered Lie superalgebras if and only if they determine the same element $\tilde{\beta} + \tilde{\gamma} \in \mathcal{H}^{2,2}_{(a_{\ldots})} \mathfrak{h}$.

**Proof.** We begin with a presentation of the form (3.122), so that $\alpha + \beta + \gamma$ is a $(2,2)$-Spencer cocycle for $a$ with $\mathfrak{h}$-invariant cohomology class and then define $\lambda, \tilde{\beta}, \tilde{\gamma}$ by equations (3.154)-(3.156); $\delta$ must then be given by (3.160). The brackets then take the claimed form if we define

\[
\tilde{\theta}(v, w) = \theta(v, w) + \lambda(\lambda(v) w - \lambda(w) v) - [\lambda(v), \lambda(w)].
\]

It remains to check the Jacobi identities (3.140)-(3.144) in our new variables. Substituting the definition of $\delta$ into the first identity (3.140) gives us $A \cdot \theta$ for all $A \in \mathfrak{h}$. The
LHS of equation (3.142) can be rearranged as follows:

\[
\begin{align*}
\theta(v, w) \cdot s + \beta(\alpha(v, w), s) - \beta(v, \beta(w, s)) + \beta(w, \beta(v, s)) \\
= \theta(v, w) \cdot s + \beta(\lambda(v) w - \lambda(w) v, s) + \lambda(\lambda(v) w - \lambda(w) v) \cdot s \\
- \tilde{\beta}(v, \tilde{\beta}(w, s) - \lambda(v) \cdot \tilde{\beta}(w, s) - \lambda(v) \cdot (\lambda(w) \cdot s)} \\
+ \tilde{\beta}(w, \tilde{\beta}(v, s) + \lambda(v) \cdot s) + \lambda(w) \cdot \tilde{\beta}(v, s) + \lambda(w) \cdot (\lambda(v) \cdot s) \\
= (\theta(v, w) + \lambda(\lambda(v) w - \lambda(w) v) - [\lambda(v), \lambda(w)]) \cdot s \\
- (\tilde{\beta}(v, \tilde{\beta}(w, s) + \lambda(w) \cdot s) - \tilde{\beta}(w, \tilde{\beta}(v, s) + \lambda(w) \cdot s)) \\
- (\lambda(v) \cdot \tilde{\beta}(w, s) - \tilde{\beta}(\lambda(v) w, s)) + (\lambda(w) \cdot \tilde{\beta}(v, s) - \tilde{\beta}(\lambda(w) v, s)).
\end{align*}
\]

(3.169)

Using the definition of \(\tilde{\theta}\), we see that setting this to zero gives us equation (3.165). Next, we will show that (3.141) implies (3.164); this is a little more involved and also requires the use of some further identities. Indeed,

\[
\begin{align*}
\theta(k, v) + \delta(\gamma(s, s), v) + 2\gamma(s, \beta(v, s)) \\
= \theta(k, v) + [\tilde{\gamma}(s, \lambda(v))] - \lambda(\tilde{\gamma}(s, s) v) - [\lambda(k), \lambda(v)] + \lambda(\lambda(k) v) \\
+ 2\gamma(s, \tilde{\beta}(v, s) + \lambda(v) \cdot s) - 2\lambda(k, s, \tilde{\beta}(v, s) + \lambda(v) \cdot s)) \\
= \theta(k, v) + \lambda(\lambda(k) v - 2k(s, \lambda(v) \cdot s)) \\
- [\lambda(k), \lambda(v)] + 2\gamma(s, \tilde{\beta}(v, s) + \lambda(v) \cdot s) \\
- [\lambda(v), \tilde{\gamma}(s, s)] - \lambda(\kappa(s, s) v + 2k(s, \tilde{\beta}(v, s))) \\
= \tilde{\theta}(k, v) + 2\gamma(s, \tilde{\beta}(v, s) + \lambda(v) \cdot s) - [\lambda(v), \tilde{\gamma}(s, s)],
\end{align*}
\]

where in the last line we have used a cocycle condition for \(\tilde{\beta} + \gamma\) and \(\mathfrak{so}(V)\)-invariance of \(\kappa\). Substitution of the change of variables followed by some trivial rearrangement shows that (3.142) and (3.143) are equivalent to (3.165) and (3.166) respectively. Finally, (3.144) gives

\[
(\lambda(u) \cdot \tilde{\theta})(v, w) + \lambda(\tilde{\theta}(u, v) w) + \text{cyclic perm's.} = 0,
\]

(3.171)

but then substitution of (3.166) yields (3.167). The final claim follows by Proposition 3.25 since \(\tilde{\beta} + \gamma \in H^{2,2}(A_\cdot; a)\) is uniquely determined by the cohomology class \([\alpha + \beta + \gamma] \in H^{2,2}(A_\cdot; a)\).

3.3.4 Realisable subdeformations

We have just seen that if \(\alpha + \beta + \gamma \in Z^{2,2}(A_\cdot; a)\) is the Spencer cocycle associated to some presentation of a filtered deformation of a highly supersymmetric graded subalgebra of \(s\) then there is a \(h\)-invariant cocycle \(\tilde{\beta} + \tilde{\gamma} \in Z^{2,2}(A_\cdot; s)\) such that \(i_* [\alpha + \beta + \gamma] = [\tilde{\beta} + \tilde{\gamma}]\). We will now restrict our attention to cocycles obeying a stronger version of this and the corresponding filtered subdeformations of \(s\) (where they exist). Essentially, we demand that \(\tilde{\beta} + \tilde{\gamma}\) is the pull-back of some \(h\)-invariant element \(\hat{\beta} + \hat{\gamma} \in H^{2,2}\).
Admissibility

**Definition 3.28.** A cohomology class \([\alpha + \beta + \gamma] \in H^{2,2}(a_-;a)\) for a highly supersymmetric graded subalgebra \(a\) of \(s\) is admissible if we have

\[
i_*[\alpha + \beta + \gamma] = i^*[\hat{\beta} + \hat{\gamma}]
\]

(3.172)

in \(H^{2,2}(a_-;s)\) for some \(h\)-invariant cocycle \(\hat{\beta} + \hat{\gamma} \in \mathcal{H}^{2,2}\).

A cocycle in \(Z^{2,2}(a_-;a)\) is admissible if its cohomology class is admissible.

Note that an admissible cohomology class is in particular \(h\)-invariant. Unpacking the definition a little, for admissible cocycles we have

\[
i_*[\alpha + \beta + \gamma] = i^*[\hat{\beta} + \hat{\gamma}] + \partial \lambda
\]

(3.173)

for a unique \(\lambda \in Z^{2,2}(a_-;s) = \text{Hom}(V, so(V))\) and \(\hat{\beta} + \hat{\gamma} \in (\mathcal{H}^{2,2})^h\) determined uniquely up to elements in \(\mathcal{H}^{2,2}(a_-)\). If we fix an admissible cohomology class but not a particular representative, \(\lambda\) is only determined up to elements in \(Z^{2,2}(a_-;a) = \text{Hom}(V,h)\).

Recalling that a highly supersymmetric subdeformation of \(s\) uniquely determines an element of \(H^{2,2}(a_-;a)^h\) by Proposition 3.25, we make the following definition.

**Definition 3.29 (Geometric realisability).** A highly supersymmetric filtered subdeformation \(\tilde{a}\) of \(s\) is (geometrically) realisable if the associated cohomology class in \(H^{2,2}(a_-;a)^h\) is admissible.

The definitions above are inspired by, although not equivalent to [45, Def. 9] in the context of 11-dimensional supergravity. There, one finds that \(\mathcal{H}^{2,2} \cong \wedge^4 V\), and it is the 4-form \(\varphi\) associated to \(\hat{\beta} + \hat{\gamma}\) which is called admissible if the conditions above are met and an additional algebraic condition related to the fact that \(\varphi\) should considered to be the value at a point of a closed 4-form field strength on some supergravity background is imposed. Our definition here is weaker because it is not clear a priori how this condition should be generalised.

Let us make some particular remarks about these definitions in the maximally supersymmetric case, where \(a_- = s_-\). Here, the map \(i^* : C^{2,2}(s_-;s) \to C^{2,2}(a_-;s)\) whose induced map in cohomology appears in the definition of admissibility, is the identity map. An admissible cocycle for \(a\) is then simply a cocycle in \(Z^{2,2}(a_-;a)\) whose cohomology class is \(a_0\)-invariant. It follows that all maximally supersymmetric filtered deformations of \(s\) are realisable.

We will justify the term “geometrically realisable” in the Section 3.4.2 (see Theorem 3.47). Our goal in this section will be to determine the conditions under which an admissible cohomology class of a highly supersymmetric graded subalgebra \(a \subseteq s\) is integrable, in the sense that there exists a corresponding geometrically realisable filtered deformation \(\tilde{a}\). Parametrising an admissible cocycle as in (3.173) and taking \(\tilde{\beta} = i^*\beta, \tilde{\gamma} = i^*\gamma\), we will depart from Proposition 3.27, asking whether there exists a filtered deformation given by (3.161).

**Lemma 3.30.** Let \(\alpha + \beta + \gamma\) be an admissible cocycle parametrised as in (3.173). Then

- For all \(v \in V, s \in S', \tilde{\beta}(v,s) + \lambda(v) \cdot s \in S'\);
• For all $s, s' \in S', \hat{\gamma}(s, s') - \lambda(\kappa(s, s')) \in \mathfrak{h}$;

• For all $A \in \mathfrak{h}$, $v \in V$, $[A, \lambda(v)] - \lambda(Av) \in \mathfrak{h}$.

Proof. The first two claims follow directly from the definition. For the third, by the Homogeneity Theorem we can take $v = \kappa_s$ for some $s \in S'$. Recalling that the action of $\mathfrak{h}$ preserves $S'$ and using $\mathfrak{so}(V)$-equivariance of $\kappa$ and $\mathfrak{h}$-invariance of $\hat{\gamma}$, we have

$$[A, \lambda(\kappa_s)] - \lambda(A\kappa_s) = [A, \lambda(\kappa_s)] - 2\lambda(\kappa(As, s)) = [A, \lambda(\kappa_s) - \gamma(s, s)] + 2(\hat{\gamma}(As, s) - \lambda(\kappa(As, s))),$$

and both of the terms in the last expression lie in $\mathfrak{h}$ by the second claim of the lemma.

Integrability

It remains to check whether there exists an $\mathfrak{h}$-invariant map $\tilde{\theta} : \wedge^2 V \to \mathfrak{so}(V)$ such that

$$\tilde{\theta}(v, w) - \lambda(\lambda(v)w - \lambda(w)v) + [\lambda(v), \lambda(w)] \in \mathfrak{h}.$$  \hfill (3.175)

satisfying the equations (3.164) to (3.167). We first observe that equations (3.164) and (3.165) simplify slightly:

$$\tilde{\theta}(v, \kappa_s) = 2\hat{\gamma}(s, \hat{\beta}(v, s)) - (\lambda(v) - \hat{\gamma})(s, s),$$

$$\tilde{\theta}(v, w) \cdot s = \hat{\beta}(v, \hat{\beta}(w, s)) - \hat{\beta}(w, \hat{\beta}(v, s))$$

$$+ (\lambda(v) - \hat{\gamma})(w, s) = (\lambda(w) - \hat{\gamma})(v, s),$$

and by the Homogeneity Theorem, either equation fully determines $\tilde{\theta}$ (if it exists). In fact, $\tilde{\theta}$ is uniquely determined by the admissible cohomology class: it depends only on $i^*\hat{\beta}, i^*\hat{\gamma}$ and $\lambda$, so a shift in $\hat{\beta} + \hat{\gamma}$ by an element of $\mathcal{K}^{2,2}(a_{-})$ does not change it, while by $\mathfrak{h}$-invariance of $\hat{\beta}$ and $\hat{\gamma}$, changing $\lambda$ by addition of a map $V \to \mathfrak{h}$ also does not change $\tilde{\theta}$. There may, however, be obstructions to the existence of $\tilde{\theta} : \wedge^2 V \to \mathfrak{so}(V)$ even for an admissible cocycle. For now, we will postpone considering this issue and show that a map $\tilde{\theta}$ determined by the equations above automatically satisfies all of the other required properties.

Lemma 3.31. Any map $\tilde{\theta} : \wedge^2 V \to \mathfrak{so}(V)$ satisfying equation (3.176) also satisfies equation (3.175).

Proof. By the Homogeneity Theorem, it is sufficient to show the result for $w = \kappa_s$ for some $s \in S'$; we have

$$\tilde{\theta}(v, \kappa_s) - \lambda(\lambda(v)\kappa_s - \lambda(\kappa_s)v) + [\lambda(v), \lambda(\kappa_s)]$$

$$= 2\{[\hat{\gamma} - \lambda \circ \kappa](s, \hat{\beta}(v, s) + \lambda(v) \cdot s)$$

$$+ \{[\hat{\gamma} - \lambda \circ \kappa](s, s), \lambda(v)] - \lambda([\hat{\gamma} - \lambda \circ \kappa](s, s)v)\},$$

and since $(\hat{\gamma} - \lambda \circ \kappa)|_{\wedge^2 S}$ takes values in $\mathfrak{h}$, Lemma 3.30 implies that both of the expressions within the braces lie in $\mathfrak{h}$. \hfill □
Lemma 3.32. Any map $\tilde{\theta} : \wedge^2 V \to \mathfrak{so}(V)$ satisfying equation (3.177) also satisfies

$$
\tilde{\theta}(v, w)\kappa_s = 2\tilde{\gamma}(\tilde{\beta}(v, s), s) w - 2\tilde{\gamma}(\tilde{\beta}(w, s), s) v - (\lambda(v) \cdot \tilde{\gamma})(s, s) w + (\lambda(w) \cdot \tilde{\gamma})(s, s) v,
$$

(3.179)

for all $v, w \in V$, $s \in S'$, and this equation determines $\tilde{\theta}$ uniquely.

Proof. First, it will be useful to depolarise one of the cocycle conditions for $\tilde{\beta} + \tilde{\gamma}$:

$$
2\kappa(\tilde{\beta}(v, s), s') + 2\kappa(\tilde{\beta}(v, s'), s) + 2\tilde{\gamma}(s, s') v = 0.
$$

(3.180)

Using $\mathfrak{so}(V)$-invariance of $\kappa$, equation (3.177) gives

$$
\tilde{\theta}(v, w)\kappa_s = 2\kappa(\tilde{\theta}(v, w) \cdot s, s)
$$

$$
= 2\kappa(\tilde{\beta}(v, \tilde{\beta}(w, s)), s) + 2\kappa(\lambda(v) \cdot \tilde{\beta}(w, s), s)
$$

$$
- 2\kappa(\tilde{\beta}(\lambda(v) w, s), s) - 2\kappa(\tilde{\beta}(w, \lambda(v) \cdot s), s) - (v \leftrightarrow w).
$$

(3.181)

We now use equation (3.180) with $s' = s$, $\lambda(v) \cdot s$ and $\tilde{\beta}(w, s)$ to replace some of the terms above and then use $\mathfrak{so}(V)$-invariance of $\kappa$ again to conclude that

$$
\tilde{\theta}(v, w)\kappa_s = -2\tilde{\gamma}(\tilde{\beta}(w, s), s) v - 2\kappa(\tilde{\beta}(v, s), \tilde{\beta}(w, s))
$$

$$
+ 2\kappa(\lambda(v) \cdot \tilde{\beta}(w, s), s) + \tilde{\gamma}(\lambda(v) \cdot s, s) v
$$

$$
+ \tilde{\gamma}(s, s)(\lambda(v) w) + 2\kappa(\tilde{\beta}(w, s), \lambda(v) \cdot s) - (v \leftrightarrow w)
$$

$$
= 2\tilde{\gamma}(\tilde{\beta}(v, s), s) w + \lambda(v)\kappa(\tilde{\beta}(w, s), s)
$$

$$
+ \gamma(s, s)(\lambda(v) w) + \tilde{\gamma}(\lambda(v) \cdot s, s) w - (v \leftrightarrow w),
$$

(3.182)

which is the desired expression. In the final equality, we have used the skew-symmetry of the expression in $v$ and $w$. By the Homogeneity Theorem, this defines $\tilde{\theta}$ uniquely.

Lemma 3.33. For any map $\tilde{\theta} : \wedge^2 V \to \mathfrak{so}(V)$, we have the following:

- Either equation (3.176) or (3.177) implies that $\tilde{\theta}$ is $\mathfrak{h}$-invariant.

- Assuming equation (3.176), equation (3.179) is equivalent to the algebraic Bianchi identity (3.166). In particular, equations (3.176) and (3.177) together imply the Bianchi identity.

- Equations (3.176) and (3.177) together imply (3.167).

Proof. The first claim follows by some fairly straightforward but long manipulations which use $\mathfrak{so}(V)$-invariance of $\kappa$, $\mathfrak{h}$-invariance of $\tilde{\beta}$ and $\tilde{\gamma}$ and the fact that $\lambda(Av) - [A, \lambda(v)] \in \mathfrak{h}$ for all $A \in \mathfrak{h}$. For the second claim, let us assume equation (3.176) holds. Then equation (3.179) gives

$$
\tilde{\theta}(v, w)\kappa_s = \tilde{\theta}(v, \kappa_s) w - \tilde{\theta}(w, \kappa_s) v
$$

(3.183)

for all $v, w \in V$, $s \in S'$, which by the Homogeneity Theorem is the Bianchi identity. Conversely, if we assume the Bianchi identity, the above equation along with (3.176) gives us (3.179).
For the final claim let us assume that \( \tilde{\theta} \) is given by (3.176) and note that by the Homogeneity Theorem we can assume without loss of generality that \( w = \kappa_s \) for arbitrary \( s \in S' \). Then we can use \( so(V) \)-invariance of \( \kappa \) to compute

\[
\left( \Lambda(u) \cdot \tilde{\theta} \right)(v, \kappa_s) + \left( \Lambda(v) \cdot \tilde{\theta} \right)(\kappa_s, u) = \left( \left( \Lambda(\Lambda(u)v - \Lambda(v)u - [\lambda(u), \lambda(v)]) \cdot \tilde{\gamma} \right)(s, s) + 2\tilde{\gamma}(s, (\Lambda(u) \cdot \tilde{\beta})(v, s) - (\Lambda(v) \cdot \tilde{\beta})(u, s)) + 2(\Lambda(u) \cdot \tilde{\gamma})(s, \tilde{\beta}(v, s)) - 2(\Lambda(v) \cdot \tilde{\gamma})(s, \tilde{\beta}(u, s)).
\]

(3.184)

On the other hand, we recall that \( \tilde{\gamma}(s, s) = -\lambda(\kappa_s) \in \mathfrak{h} \), and that \( \tilde{\theta} \) is \( \mathfrak{h} \)-invariant by the first part of the lemma, whence \( \lambda(\kappa_s) \cdot \tilde{\theta} = \tilde{\gamma}(s, s) \cdot \tilde{\theta} \). Using\(^4\) (3.176) and a cocycle condition, we find

\[
\left( \lambda(\kappa_s) \cdot \tilde{\theta} \right)(u, v) = \left[ \tilde{\gamma}(s, s), \tilde{\theta}(u, v) \right] + 2\tilde{\gamma}(s, \tilde{\beta}(u, v), s) - \tilde{\beta}(v, \tilde{\beta}(u, s)) - 2(\lambda(u) \cdot \tilde{\gamma})(s, \tilde{\beta}(u, s)) + 2(\lambda(v) \cdot \tilde{\gamma})(s, \tilde{\beta}(u, s)).
\]

(3.185)

Putting the two calculations together and then using (3.177), we find

\[
\left( \lambda(u) \cdot \tilde{\theta} \right)(v, \kappa_s) + \left( \lambda(v) \cdot \tilde{\theta} \right)(\kappa_s, u) + \left( \lambda(\kappa_s) \cdot \tilde{\theta} \right)(u, v) = \left( \left( \lambda(\lambda(u)v - \lambda(v)u - [\lambda(u), \lambda(v)]) \cdot \tilde{\gamma} \right)(s, s) - \left[ \tilde{\theta}(u, w), \tilde{\gamma}(s, s) \right] + 2\tilde{\gamma}(s, \tilde{\beta}(u, v), s) - \tilde{\beta}(v, \tilde{\beta}(u, s)) \right)
\]

(3.186)

and that \( \tilde{\theta} \) is skew-symmetric. We define the Dirac kernel of \( S' \) by \( \mathcal{D} = \ker \kappa|_{\otimes^2 S'} \) and note that \( \Theta \) factorises as follows:

\[
V \otimes \otimes^2 S' \xrightarrow{\Theta} so(V) \xrightarrow{\tilde{\theta}} \mathfrak{v}
\]

(3.188)

and that \( \tilde{\theta} \) is skew-symmetric. We define the Dirac kernel of \( S' \) by \( \mathcal{D} = \ker \kappa|_{\otimes^2 S'} \) and note that \( \Theta \) factorises as follows:

\[
\frac{\partial (v, s')}{v \in V, s' \in S'} \quad \text{for}
\]

\( ^4 \)Actually, it’s necessarily to depolarise (3.176) in order to evaluate expressions like \( \tilde{\theta}(v, \kappa(s, s')) \) for \( v \in V, s, s' \in S' \).
Lemma 3.34. Suppose that the Θ annihilates the Dirac kernel and thus factors as described above. Then equation (3.176) defines a map $\tilde{\theta} : \wedge^2 V \to \mathfrak{so}(V)$.

Proof. The factorisation of Θ gives us a map $\tilde{\theta} : V \otimes V \to \mathfrak{so}(V)$ satisfying (3.176).

To show that $\tilde{\theta}$ it is alternating, it is sufficient to show that $\tilde{\theta}(\kappa_s, \kappa_s) = 0$ for all $s \in S'$. Using the fact that $\tilde{\gamma}(s, s) - \lambda(\kappa_s) \in \mathfrak{h}$ for $s \in S'$ and $\mathfrak{h}$-invariance of $\tilde{\gamma}$, we have $\lambda(\kappa_s) \cdot \tilde{\gamma} = \tilde{\gamma}(s, s) \cdot \tilde{\gamma}$, so

$$
\tilde{\theta}(\kappa_s, \kappa_s) = 2\tilde{\gamma}(s, \tilde{\beta}(\kappa_s, s)) - (\tilde{\gamma}(s, s) \cdot \tilde{\gamma})(s, s) = 2\tilde{\gamma}(s, s) - \tilde{\gamma}(s, s) - [\tilde{\gamma}(s, s), \tilde{\gamma}(s, s)] = 0,
$$

(3.189)

where we have used a cocycle condition in the last equality. □

Definition 3.35. Let $[\alpha + \beta + \gamma]$ be an admissible cohomology class and, recalling that it does depend on the choice of cocycle representative, let $\Theta : V \otimes \wedge^2 S' \to \mathfrak{so}(V)$ be the map defined by equation (3.187). We say that $[\alpha + \beta + \gamma]$ is integrable if

1. $\Theta$ annihilates the Dirac kernel $\mathcal{D}$ of $\wedge^2 S'$,

2. The map $\tilde{\theta} : \wedge^2 V \to \mathfrak{so}(V)$ defined by $\tilde{\theta} \circ \kappa = \Theta$ satisfies (3.177).

An admissible cocycle is integrable if its cohomology class is.

By construction, we have the following refinement of Proposition 3.27.

Theorem 3.36 (Integration of admissible cocycles). Let $\alpha + \beta + \gamma$ be an admissible, integrable Spencer (2,2)-cocycle for a highly supersymmetric graded subalgebra $\mathfrak{a}$ of $\mathfrak{s}$ with $i_\ast(\alpha + \beta + \gamma) = i_\ast(\tilde{\beta} + \tilde{\gamma}) + \partial \lambda$ and let $\tilde{\theta} : \wedge^2 V \to \mathfrak{so}(V)$ be the map defined by equation (3.176). Then the brackets

$$
[A, B] = AB - BA,
$$

$$
[s, s] = \kappa_s \cdot \tilde{\gamma}(s, s) - \lambda(\kappa_s),
$$

$$
[A, v] = A\gamma + \left[A, \lambda(v)\right] - \lambda(Av),
$$

$$
[v, v] = \tilde{\beta}(v, s) + \lambda(v) \cdot \gamma,
$$

$$
[A, s] = A \cdot s,
$$

$$
[v, w] = \lambda(v) w - \lambda(w) v + \tilde{\beta}(v, w) - \lambda(\lambda(v) w - \lambda(w) v) + [\lambda(v), \lambda(w)],
$$

(3.190)

define a geometrically realisable filtered deformation of $\mathfrak{a}$. Two such deformations $\tilde{\mathfrak{a}}$, $\tilde{\mathfrak{a}}'$ are isomorphic if and only if $(\tilde{\beta}' + \tilde{\gamma}') - (\tilde{\beta} + \tilde{\gamma}) \in \mathfrak{h}^{2,2}(\mathfrak{a}_\ast)$.

Where $\mathfrak{a}$ is a maximally supersymmetric graded subalgebra of the (minimal) Poincaré superalgebra $\mathfrak{s}$ in 4, 5, 6 or 11 spacetime dimensions, there is no obstruction to an admissible cocycle being integrable [16, 48–50]. We already observed that admissible cohomology classes are simply elements of $H^{2,2}(\mathfrak{s}_\ast; \mathfrak{a})^{\mathfrak{h}_0}$ if $\mathfrak{a}$ is highly supersymmetric, so any such cohomology class defines a realisable filtered deformation in these cases. The question of whether obstructions exist in the highly supersymmetric (but sub-maximal) case is still open even in these cases.
On the other hand, if we allow for Poincaré subalgebras extended by $R$-symmetry (as we do in Chapter 4), analogous obstruction do exist in 5 spacetime dimensions, even for maximally supersymmetric subalgebras, as we will show in Chapter 5.

### 3.3.5 The classification problem for highly supersymmetric odd-generated filtered subdeformations

We will now outline a scheme for the classification of highly supersymmetric odd-generated realisable filtered subdeformations $\tilde{a}$ of $s$. Odd-generated here means that $\tilde{a}$ is generated by its odd subspace; $\tilde{a}_\tau = [\tilde{a}_\tau, \tilde{a}_\tau]$. We note that this is very close to a classification of all highly supersymmetric realisable filtered subdeformations, since if $\tilde{a}$ is a general such subdeformation, the ideal subalgebra $\tilde{a}' = \tilde{a}_1 \oplus \tilde{a}_0$ is odd-generated realisable, and for the associated graded subalgebras of $s$ we have $a'_{-2} = a_{-2}, a'_{-1} = a_{-1}$ and $a'_0 \subseteq a_0$. Thus only the zero-graded part is not determined by the odd-generated subalgebra, and by inspection of (3.190), it is clear that the deformation maps for $\tilde{a}$ are also fully determined by those of $\tilde{a}'$. Thus the only data which distinguishes $\tilde{a}$ from its odd-generated ideal is the choice of some subalgebra $a_0 \subseteq s(V)$ which contains $a'_0$ and preserves both $\delta'$ and $\beta + \gamma \in \mathbb{H}^{2,2}$.

We generalise the results of [45, §4] and [47, §5] which treat the 11-dimensional case.

#### Isomorphisms and embeddings

We first generalise Definitions 5 and 11 from [45] with a minor adaptation to include the action of the $R$-symmetry group (which is trivial in 11 dimensions).

First let us consider $s = V \oplus S \oplus \mathfrak{so}(V)$ as a module of $G = \text{Spin}(V) \times R$, where $\text{Spin}(V)$ acts via its natural representations on $V$ and $S$, the adjoint representation on $\mathfrak{so}(V)$ and $R$ acts via the natural representation on $S$ and trivially on $V$ and $\mathfrak{so}(V)$. Since the actions of $\text{Spin}(V)$ and $R$ on $S$ commute, this does indeed give us an action of the product $G$ on $s$. Moreover, it is not hard to see that this action is by graded Lie algebra automorphisms. In particular, if $a$ is a graded subalgebra of $s$ then $g \cdot a$ is also a graded subalgebra for all $g \in G$, with left multiplication by $g$ defining a graded isomorphism $a \cong g \cdot a$.

**Definition 3.37.** Two subdeformations $\tilde{a}, \tilde{a}'$ of the Poincaré superalgebra $s$ are isomorphic if there exists a (strict) filtered isomorphism of Lie superalgebras $\Phi : \tilde{a} \rightarrow \tilde{a}'$ such that the associated graded morphism $\text{Gr } \Phi : a \rightarrow a'$ is given, for all $x \in a$, by $\text{Gr } \Phi(x) = g \cdot x$ for some $g \in \text{Spin}(V) \times R$.

An embedding of subdeformations $\tilde{a} \hookrightarrow \tilde{a}'$ of $s$ is an injective (strict) filtered morphism of Lie superalgebras which is an isomorphism of subdeformations onto its image.

Note that two filtered subdeformations of $s$ being isomorphic as subdeformations is stronger than them simply being isomorphic as filtered Lie superalgebras.

With this in mind, we can strengthen point (2) of Proposition 3.25 by noting that two highly supersymmetric deformations of the same subalgebra $a \subseteq s$ inducing the same cohomology class in $H^{2,2}(a; a)$ are isomorphic as subdeformations and
not just as filtered Lie superalgebras. Indeed, this follows immediately from the proof of the proposition, which also shows in particular that the associated graded automorphism of \( a \) is the identity map.

**Kernels and sections of the squaring map**

Let \( S' \subseteq S \) with \( \dim S' > \frac{1}{2} \dim S \) and recall that by the Homogeneity Theorem, \( \kappa |_{\bigwedge^2 S'} \) is surjective, so we have a short exact sequence of vector spaces

\[
0 \longrightarrow \mathcal{D} \longrightarrow \bigwedge^2 S' \xrightarrow{\kappa |_{\bigwedge^2 S'}} V \longrightarrow 0
\]

(3.191)

where \( \mathcal{D} = \ker \kappa |_{\bigwedge^2 S'} \) is the Dirac kernel of \( S' \). If \( \mathfrak{h} \subseteq \mathfrak{so}(V) \) is a subalgebra preserving \( S' \) then this is a sequence of \( \mathfrak{h} \)-modules. We define a *section of the squaring map* \( \kappa \) (understood to mean \( \kappa \) restricted to \( S' \)) as a linear map \( \Sigma : V \to \bigwedge^2 S' \) which splits the above as a sequence of vector spaces, i.e. \( \kappa \circ \Sigma = \text{Id}_V \), giving a non-canonical decomposition of vector spaces (not of \( \mathfrak{h} \)-modules) \( \bigwedge^2 S' \cong V \oplus \mathcal{D} \). Two such sections differ by a map \( V \to \mathcal{D} \).

**Lemma 3.38.** Let \( a = V \oplus S' \oplus \mathfrak{h} \) be a highly supersymmetric graded subalgebra of \( s \) and let \( \bar{\beta} + \bar{\gamma} \in \mathcal{C}^{2,2} \). Then there exists a cohomology class \( [a + \beta + \gamma] \in H^{2,2}(a_\bullet; a) \) such that \( i_* [a + \beta + \gamma] = i_* [\bar{\beta} + \bar{\gamma}] \) if and only if

1. \( \bar{\gamma} |_{\mathcal{D}} \subseteq \mathfrak{h} \)
2. for all \( v \in V \), \( (i_v \bar{\beta} + \bar{\gamma}(\Sigma(v)))(S') \subseteq S' \) for some (hence any) section \( \Sigma \) of \( \kappa \).

Moreover, if such a cohomology class exists, for a fixed section \( \Sigma \) there exists a unique representative \( a + \beta + \gamma \) such that \( i_* (a + \beta + \gamma) = i_* (\bar{\beta} + \bar{\gamma}) + \partial (\kappa \circ \Sigma) \).

**Proof.** Suppose \( i_* [a + \beta + \gamma] = i_* [\bar{\beta} + \bar{\gamma}] \), so that \( i_* (a + \beta + \gamma) = i_* (\bar{\beta} + \bar{\gamma}) + \partial \lambda \) for some unique \( \lambda \in \mathcal{C}^{2,1}(a_\bullet; s) = \text{Hom}(V, \mathfrak{so}(V)) \). On one hand, restricting to \( \mathcal{D} \subseteq \bigwedge^2 S' \), we find that \( \bar{\gamma}|_{\mathcal{D}} = \gamma|_{\mathcal{D}} \), which takes values in \( \mathfrak{h} \). Now let \( \Sigma : V \to \bigwedge^2 S' \) be a section of \( \kappa \) and note that we have

\[
\bar{\gamma} \circ \Sigma - \lambda = (\bar{\gamma} - \lambda \circ \kappa) \circ \Sigma = \gamma \circ \Sigma
\]

(3.192)

which also takes values in \( \mathfrak{h} \). On the other hand, restricting to \( V \otimes S' \), for all \( v \in V \) and \( s \in S \) we have \( \bar{\beta}(v, s) + \lambda(v) \cdot s = \beta(v, s) \in S' \), i.e. \( i_v \bar{\beta} + \lambda(v) \) preserves \( S' \), but then

\[
i_v \bar{\beta} + \bar{\gamma}(\Sigma(v)) = (i_v \bar{\beta} + \lambda(v)) + (\gamma \circ \Sigma - \lambda)(v)
\]

(3.193)

also preserves \( S' \) since the action of \( \mathfrak{h} \) preserves \( S' \).

Conversely, suppose that \( \bar{\gamma}(\mathcal{D}) \subseteq \mathfrak{h} \) and \( i_v \bar{\beta} + \bar{\gamma}(\Sigma(v)) \) preserves \( S' \) for all \( v \in V \) for some (hence any\(^5\)) section \( \Sigma \). Then define \( \lambda : V \to \mathfrak{so}(V) \) by \( \lambda = \gamma \circ \Sigma \), giving us \( (\bar{\gamma} - \lambda \circ \kappa)(\mathcal{D}) = \bar{\gamma}(\mathcal{D}) \in \mathfrak{h} \) and \( (\gamma - \lambda \circ \kappa) \circ \Sigma = \gamma \circ \Sigma - \lambda = 0 \) which together imply

\(^5\)Since any two sections differ by a map \( V \to \mathcal{D} \), the conditions together imply that \( i_v \bar{\beta} + \bar{\gamma}(\Sigma'(v)) \) preserves \( S' \) for any other section \( \Sigma' \) as well.
We now introduce some technology for building subdeformations from normalised with hats, trusting that this will not cause confusion.

Fix a subspace $S' \subseteq S$ with $\dim S' > \frac{1}{2} \dim S$ and $\beta + \gamma \in H^{2,2}$. We define the envelope of $(S', \beta + \gamma)$ as the subspace

$$h_{(S', \beta + \gamma)} = \gamma(\mathcal{D}) \subseteq \mathfrak{so}(V).$$

We say that $(S', \beta + \gamma)$ is a Lie pair if for all $A \in h_{(S', \beta + \gamma)}$

1. $A \cdot \beta = 0$,
2. $A \cdot S' \subseteq S'$.

**Proposition 3.40.** If $(S', \beta + \gamma)$ is a Lie pair then $h_{(S', \beta + \gamma)}$ is a Lie subalgebra of $\mathfrak{so}(V)$. If additionally $i_\nu \beta + \gamma(\Sigma(v))$ preserves $S'$ for some section $\Sigma$ of $\kappa$ then $i^* (\beta + \gamma) + \partial(\gamma \circ \Sigma)$ defines an admissible cocycle for the graded subalgebra $a_{(S', \beta + \gamma)} = V \oplus S' \oplus h_{(S', \beta + \gamma)}$ of $\mathfrak{s}$. Furthermore, if this cocycle is integrable, the associated realisable deformation $\tilde{\alpha}_{(S', \beta + \gamma)}$ is odd-generated.

Conversely, for any odd-generated realisable filtered subdeformation $\tilde{\alpha}$ of $\mathfrak{s}$ there exists a Lie pair $(S', \beta + \gamma)$ such that $\text{Gr} \tilde{\alpha} \cong a_{(S', \beta + \gamma)}$.

**Proof.** Let $A, B \in h_{(S', \beta + \gamma)}$. Then there exists $\omega \in \mathcal{D}$ such that $B = \gamma(\omega)$; note that $A \cdot \beta = 0 \implies A \cdot \gamma = 0$, so $[A, B] = [A, \gamma(\omega)] = \gamma(A \cdot \omega)$. On the other hand, the action of $A$ preserves $S'$, whence $A \cdot \omega \in O^2 S'$, and $\kappa(A \omega) = \kappa(\omega) = 0$, so $A \cdot \omega \in \mathcal{D}$. Thus $[A, B] \in h_{(S', \beta + \gamma)}$, so the latter is indeed a subalgebra of $\mathfrak{so}(V)$. Since the action of $h_{(S', \beta + \gamma)}$ preserves $S'$, $a_{(S', \beta + \gamma)}$ is a graded subalgebra of $\mathfrak{s}$.

If $i_\nu \beta + \gamma(\Sigma(v))$ preserves $S'$ for some (hence any) section section $\Sigma$, Lemma 3.38 tells us that $i^* (\beta + \gamma) + \partial(\gamma \circ \Sigma)$ is the image under $i_\ast$ of a 2,2-cocycle of $a_{(S', \beta + \gamma)}$.\]
since $\beta$ is $\mathfrak{h}(S',\beta + \gamma)$-invariant, this cocycle is admissible. If it also integrable, we must show that $[S, S] = V \oplus \mathfrak{h}(S',\beta + \gamma)$ where the bracket is the deformed one defined by $i^* (\beta + \gamma) + \partial (\gamma \circ \Sigma)$. The odd-odd bracket is nothing but the map $\kappa + \gamma - \gamma \circ \Sigma \circ \kappa : O S' \to V \oplus \mathfrak{h}(S',\beta + \gamma)$. Recall that the section gives $O S' = \mathfrak{D} \oplus \Sigma(V)$ as a vector space, so since we have

$$\kappa + \gamma - \gamma \circ \Sigma \circ \kappa = \gamma(\mathfrak{D}) = \mathfrak{h}(S',\beta + \gamma) \quad \text{(3.197)}$$

by definition, and for $v \in V$,

$$\kappa + \gamma - \gamma \circ \Sigma \circ \kappa = v + \gamma(\Sigma(v)) - \gamma(\Sigma(v)) = v, \quad \text{(3.198)}$$

we have $\kappa + \gamma - \gamma \circ \Sigma \circ \kappa = \gamma(\mathfrak{D}) = \mathfrak{h}(S',\beta + \gamma) \oplus V$. 

Now assume that $\tilde{\alpha}$ is an odd-generated realisable filtered subdeformation with associated graded class $\tilde{\alpha} = V \oplus \mathfrak{S} \oplus \mathfrak{h}$; by Lemma 3.38, the associated admissible cohomology class has a representative $\alpha + \beta + \gamma$ satisfying $i_* (\alpha + \beta + \gamma) = i^* (\tilde{\beta} + \tilde{\gamma}) + \partial (\tilde{\gamma} \circ \Sigma)$ for some $\tilde{\beta} + \tilde{\gamma} \in \mathcal{K}_{2,2}$ (which we take to be $\mathfrak{h}$-invariant by admissibility) and some section $\Sigma$. We will show that $\mathfrak{h} = \mathfrak{h}(S',\beta + \gamma)$, and we note that this implies that $(S',\beta + \gamma)$ is a Lie pair and that $\tilde{\alpha} = \alpha(S',\beta + \gamma)$. The odd-odd bracket $\kappa + \gamma - \gamma \circ \Sigma \circ \kappa$ surjects onto $V \oplus \mathfrak{h}$ so if $A \in \mathfrak{h}$ there exists $\omega_A \in O S'$ such that

$$A = \kappa(\omega_A) + \gamma(\omega_A) - \gamma(\Sigma(\kappa(\omega_A))) \in V \oplus \mathfrak{h}, \quad \text{(3.199)}$$

but since $A \in \mathfrak{h}$, we must have $\kappa(\omega_A) = 0$, whence $\omega_A \in \mathfrak{D}$. Conversely, for any $\omega \in \mathfrak{D}$ we must have $\gamma(\omega) \in \mathfrak{h}$, whence $\mathfrak{h} = \gamma(\mathfrak{D}) = \mathfrak{h}(S',\beta + \gamma)$. 

With the result above in mind, we say that a Lie pair $(S',\beta + \gamma)$ is admissible if $i_* \beta + \gamma(\Sigma(v))$ preserves $S'$ and integrable if it is admissible and the corresponding admissible cocycle is integrable.

**Equivalence of Lie pairs and correspondence**

Now fix $S'$ and a section $\Sigma : V \to O S'$, and suppose that $\beta + \gamma \in \mathcal{K}_{2,2}(\alpha_\Sigma) \subseteq \mathcal{K}_{2,2}$; that is, $\beta|_S \cdot \alpha_\Sigma = 0$, which also implies that $\gamma|_O S' = 0$. Then $\mathfrak{h}(S',\beta + \gamma) = 0$, and $(S',\beta + \gamma)$ is trivially an integrable Lie pair; we will call it a trivial Lie pair. For such a pair, the associated deformation obtained from Proposition 3.40 is trivial; $\tilde{\alpha}(S',\beta + \gamma) \equiv \alpha(S',\beta + \gamma)$ which has underlying vector space $V \oplus S'$.

Note that if we have two admissible Lie pairs $(S',\beta + \gamma)$ and $(S',\beta' + \gamma')$ such that $(\beta' - \beta) + (\gamma' - \gamma) \in \mathcal{K}_{2,2}(V \oplus S')$ (that is, $\beta'|_S \cdot \alpha_\Sigma = \beta|_S \cdot \alpha_\Sigma$) then $\mathfrak{h}(S',\beta' + \gamma) = \mathfrak{h}(S',\beta + \gamma)$, and $(\beta' - \beta) + (\gamma' - \gamma)$ is invariant under the action of this algebra. Both Lie pairs define the same admissible cocycle in Proposition 3.40; it follows that if one of these pairs is integrable, so is the other, and $\tilde{\alpha}(S',\beta' + \gamma) = \tilde{\alpha}(S',\beta + \gamma)$ Conversely, two Lie pairs which have the same envelope $\mathfrak{h}$ and define the same admissible cocycle for the resulting graded subalgebra of $\mathfrak{s}$ define an $\mathfrak{h}$-invariant element in $\mathcal{K}_{2,2}(V \oplus S')$.

Now recall the natural action of $G = \text{Spin}(V) \times R$ on $\mathfrak{s}$; this induces an action of $G$ on the Spencer complex of $\mathfrak{s}$. If $(S',\beta + \gamma)$ is a (resp. integrable, admissible) Lie pair then $(g \cdot S', g \cdot (\beta + \gamma))$ is also a (resp. integrable, admissible) Lie pair with

$$\mathfrak{h}(g \cdot S', g \cdot (\beta + \gamma)) = g \cdot \mathfrak{h}(S',\beta + \gamma) \quad \text{and} \quad \alpha(g \cdot S', g \cdot (\beta + \gamma)) = g \cdot \alpha(S',\beta + \gamma). \quad \text{(3.200)}$$
Note that \( g \cdot \Sigma : V \rightarrow \mathbb{O}^2(g \cdot S') \) is a section for \( g \cdot S' \). Using \( \Sigma \) and \( g \cdot \Sigma \) to construct the deformations of Proposition 3.40, in the integrable case we can construct an isomorphism of subdeformations of \( s \)

\[
\Phi : \tilde{a}(S', \beta + \gamma) \rightarrow \tilde{a}(g \cdot S', g \cdot (\beta + \gamma))
\]  

(3.201)

as follows. Since we treat a filtered deformation \( \tilde{a} \) as an alternative bracket structure on the same underlying space as its associated graded \( a \), we have \( \Phi = \text{Gr} \Phi \) as linear maps. Thus we may define \( \Phi : a(S', \beta + \gamma) \rightarrow g \cdot a(S', \beta + \gamma) \) on the underlying graded vector spaces by \( \Phi(x) = g \cdot x \) for \( X \in a(S', \beta + \gamma) \). This preserves the grading, hence also the natural filtration (in the strong sense), and one can check that we have

\[
\Phi([X, Y]_{\beta + \gamma + \partial(\gamma \circ \Sigma)}) = [\Phi(X), \Phi(Y)]_{g \cdot \beta + g \cdot \gamma + \partial((g \cdot \gamma) \circ (g \cdot \Sigma))}
\]

(3.202)

where the brackets are the deformed ones using the the cocycle in the subscript. The inverse of \( \Phi \) is of course given by the action of \( g^{-1} \), which also preserves all structure, hence \( \Phi \) is indeed an isomorphism of subdeformations.

The observations above suggest putting an equivalence relation on Lie pairs; in fact, we will define two different such notions.

**Definition 3.41.** Two Lie pairs \((S', \beta + \gamma)\) and \((S'', \beta' + \gamma')\) are weakly equivalent if \((S'', \beta' + \gamma') = (g \cdot S', g \cdot (\beta + \gamma) + \Phi)\) for some \( g \in G \) and \( \Phi \in \mathcal{H}^{2,2}(V \oplus S')h(S', \beta + \gamma)\), and they are strongly equivalent if they are weakly equivalent and \( \Phi = 0 \).

This notions of equivalence are a generalisation of the equivalence relation on Lie pairs for the 11-dimensional Poincaré superalgebra found in [45, 47]. In those works, \( G = \text{Spin}(V) \) rather than \( G = \text{Spin}(V) \times R \) (the \( R \)-symmetry group being \( \mathbb{Z}_2 \{ \pm 1 \} \) on \( S \) in that case, so its action can be absorbed into that of the spin group) and the \( \phi \) term does not appear in the equivalence relation since \( \mathcal{H}^{2,2}(V \oplus S') = 0 \) for all \( S' \), so there is no distinction between strong and weak equivalence. We will see that weak equivalence is a more natural concept than strong equivalence, but strong equivalence will give us some additional information.

Fixing a Lie pair \((S', \beta + \gamma)\), we define

\[
h_{(S', \beta + \gamma)}^{\text{max}} := \{ A \in \mathfrak{so}(V) \mid A \cdot S' \subseteq S' \text{ and } A \cdot \beta = 0 \} \subseteq \mathfrak{so}(V);
\]

(3.203)

note that by definition of a Lie pair, this algebra contains the envelope \( h_{(S', \beta + \gamma)} \), and

\[
a_{(S', \beta + \gamma)}^{\text{max}} = V \oplus S' \otimes h_{(S', \beta + \gamma)}^{\text{max}}
\]

(3.204)

is a graded subalgebra of \( s \) containing \( a_{(S', \beta + \gamma)} \); in fact, it is the maximal such subalgebra whose odd part is \( S' \) and whose zero-graded part preserves \( \beta \).

If \((S', \beta + \gamma)\) is integrable then the integrable cohomology class of \( a_{(S', \beta + \gamma)} \) corresponding to the Lie pair pushes forward to an integrable cohomology class of \( a_{(S', \beta + \gamma)}^{\text{max}} \) which defines a geometrically realisable filtered deformation \( a_{(S', \beta + \gamma)}^{\text{max}} \) which we call maximal (with respect to \((S', \beta + \gamma)\)), since it is maximal among realisable subdeformations into which \( \tilde{a}_{(S', \beta + \gamma)} \) embeds as the odd-generated ideal.
Proposition 3.42. There is a one-to-one correspondence between weak equivalence classes of integrable Lie pairs \((S', \beta + \gamma)\) and isomorphism classes of highly supersymmetric odd-generated realisable subdeformations of \(s\).

If \((S', \beta + \gamma), (S'', \beta' + \gamma')\) are strongly equivalent Lie pairs then their maximal realisable subdeformations \(\bar{\alpha}^\text{max}_{(S', \beta + \gamma)}, \bar{\alpha}^\text{max}_{(S'', \beta' + \gamma')}\) are isomorphic as subdeformations.

Proof. The discussion above Definition 3.41 shows that weakly equivalent integrable Lie pairs define isomorphic odd-generated realisable subdeformations. By the converse part of Proposition 3.40, every odd-generated realisable subdeformation is induced by a Lie pair. It remains to show that if \((S', \beta + \gamma), (S'', \beta' + \gamma')\) are Lie pairs such that \(\bar{\alpha}_{(S', \beta + \gamma)} \cong \bar{\alpha}_{(S'', \beta' + \gamma')}\) as subdeformations then they are weakly equivalent. If \(\Phi: \bar{\alpha}_{(S', \beta + \gamma)} \rightarrow \bar{\alpha}_{(S'', \beta' + \gamma')}\) is such an isomorphism, \(Gr\Phi : \alpha_{(S', \beta + \gamma)} \rightarrow \alpha_{(S'', \beta' + \gamma')}\) is given by \(x \mapsto g \cdot x\) for \(g \in G\), so

\[
\alpha_{(S'', \beta' + \gamma')} = g \cdot \alpha_{(S', \beta + \gamma)} = \alpha_{(G, S', g \cdot (\beta + \gamma))}. \tag{3.205}
\]

But we have already seen that \(\bar{\alpha}_{(S', \beta + \gamma)} \cong \bar{\alpha}_{(G, S', g \cdot (\beta + \gamma))}\) for all \(g \in G\), with the associated graded isomorphism of graded superalgebras given by the action of \(g\), so we must also have \(\bar{\alpha}_{(S', \beta' + \gamma')} \cong \bar{\alpha}_{(G, S', g \cdot (\beta' + \gamma))}\) with associated graded isomorphism being the identity map on \(a' := g \cdot \alpha_{(S', \beta' + \gamma')}\). It follows from Proposition 3.25 that the admissible cohomology classes in \(H^{2,2}(a'_+, a')\), associated to the two deformations are equal, whence their images under \(i_* : H^{2,2}(a'_+, a') \rightarrow H^{2,2}(a'_+, s')\) are also equal, i.e. we have

\[
i^*[\beta' + \gamma'] = i^*[g \cdot \beta + g \cdot \gamma] \in H^{2,2}(a'_+, s'). \tag{3.206}
\]

Thus we find that \(\phi := (\beta' - g \cdot \beta) + (\gamma' - g \cdot \gamma) \in \mathcal{H}^{2,2}(a'_+ \oplus S''\ell, a')\), and \(\phi\) is invariant under the action of the envelope \(h_{(S'', \beta' + \gamma')}\) since \(\beta' + \gamma'\) and \(g \cdot \beta + g \cdot \gamma\) both are.

For the maximal case, we note that \(g \cdot h^\text{max}_{(S', \beta + \gamma)} = h^\text{max}_{(G, S', g \cdot (\beta + \gamma))}\), whence it follows that strongly equivalent Lie pairs define isomorphic maximal subdeformations by an entirely analogous argument to the weak case.

We note that there does not seem to be a converse statement to the second part of the result above; for this to hold, we would have to show that \(\phi\) as in the proof vanishes if it is \(g \cdot h^\text{max}_{(S', \beta + \gamma)}\)-invariant, which there does not seem to be a reason to expect. Nonetheless, we have reduced the problem of classifying odd-generated realisable subdeformations of \(s\) up to isomorphism to classifying integrable Lie pairs up to weak equivalence, and we can also obtain all maximal realisables with respect to a weak equivalence class from the strong equivalence classes it contains.

Maximally supersymmetric case

The case \(S' = S\) brings a number of simplifications to the above discussion. First, for any \(\beta + \gamma \in \mathcal{H}^{2,2}, (S, \beta + \gamma)\) is a Lie pair if and only if the envelope \(h_{(S, \beta + \gamma)}\) preserves \(\beta\), and any Lie pair is admissible (recall our previous observation that if \(a\) is maximally supersymmetric, all \(a_0\)-invariant \((2,2)\)-cohomology classes are admissible), although one still must check that it is integrable. Since \(\mathcal{H}^{2,2}(a_{(S, \beta + \gamma)}) = 0\), there is no distinction between strong and weak equivalence of Lie pairs; to any equivalence class of integrable Lie pairs there corresponds a unique (up to isomorphism) odd-generated
realisable subdeformation and a unique maximal realisable subdeformation containing it.

3.3.6 Spencer cohomology and filtered subdeformations in $D = 2$

In this section, we determine the Spencer cohomology group $H^{2,2}(s_-,s)$ for the flat model superalgebras $s$ in 2 dimensions determined by some of the Dirac currents described in §2.2.4. We already used some of the results below in our discussion of Killing (super)algebras in §3.2.4. We again treat only signatures (1,1) and (0,2) since the (2,0) case is more complicated.

Setup

To construct the flat model (super)algebra $s = V \oplus S \oplus so(V)$ with brackets given by (2.49) and (2.51), we must choose a (not necessarily irreducible) spinor representation $S$ of $so(V)$ and a Dirac current for $S$. All Dirac currents will be symmetric, hence we work with superalgebras. The Dirac currents on the pinor representation $P$ (2-dimensional over $\mathbb{R}$ in both signatures) and their basic properties are given in Tables 2.8 and 2.9. Note that there are two choices in either signature, both of which are symmetric.

In the (0,2) case, the pinor representation is irreducible under the action of the spin group and we obtain a minimal flat model superalgebra (for either choice of current) by setting $S = P$, whence we have $\text{End}(S) \cong \text{Cl}(V)$ and we can choose either of the Dirac currents from Table 2.9. Note that $\text{Cl}_0(V) \cong \mathbb{C}$, and this (hence also the spin group) has two inequivalent irreducible representations, $S = P$ and $\bar{S}$, whence we could also choose $S = \bar{S}$. We will mostly ignore this possibility but we will briefly return to it at the end of this section.

In the (1,1) case, the pinor representation is reducible under the spin group with $P = S_+ \oplus S_-$. Table 2.8 gives the two possible Dirac currents on $P$; one of these has $\iota_{\kappa} = +1$ so restricts non-trivially to $S_{\pm}$, while the other has $\iota_{\kappa} = -1$, thus its restriction is trivial. For $\iota_{\kappa} = +1$ we can choose $S = P$ (which we will call the “non-chiral” case) or, without loss of generality, $S = S_+$ (which we call “chiral”), while for $\iota_{\kappa} = -1$ we must have the non-chiral case $S = P$ (if we wish the odd-odd bracket to be non-trivial). In the non-chiral case we have $\text{End}(S) \cong \text{Cl}(V)$, while in the chiral case $\text{End}(S) \cong \text{Cl}_0(V)$. In what follows, we will work mainly with the non-chiral case and then bootstrap the results for the chiral case from it at the end. We note that only the chiral cases gives a minimal flat model superalgebra.

For our calculations, we will use the same conventions as in §2.2.4 and §3.2.4. With the choices discussed above, we have $\text{Cl}(V) \cong \text{End}(S) \cong \text{End}(P) \cong \mathbb{R}(2)$; in particular $\text{End}(S)$ has a basis given by $\{1, \Gamma_{\mu}, \Gamma_{*}\}$. We will use the following identities of $\Gamma$ matrices:

$$\begin{align*}
\Gamma_{\mu}\Gamma_{\nu} &= \eta_{\mu\nu}1 + \epsilon_{\mu\nu}\Gamma_{*}, \\
\Gamma_{\mu}\Gamma_{\nu} &= -\Gamma_{*}\Gamma_{\mu} = \sigma\epsilon_{\mu\nu}\Gamma_{*}, \\
\Gamma_2 &= -\sigma 1,
\end{align*}$$

where the sign $\sigma$ is the determinant of the matrix with entries $\eta_{\mu\nu}$; this is +1 in signature (0,2) and -1 in signature (1,1), and we recall that the (0,2) signature is
implemented by our choice of sign in the Clifford algebra; \(\eta\) is positive-definite in this case.

Fixing one of the admissible bilinears \(B\) (and thus the corresponding Dirac current \(\kappa\)), for \(s \in S\) we denote by \(\bar{s}\) the dual element in \(S^*\) given by \(s' \mapsto \bar{s}s' := B(s, s')\), whence the Dirac current \(\kappa : \mathcal{O}^2 S \to V\) is given by \(\kappa(s, s') = (3\Gamma^\mu s')e_\mu\) (where \(e_\mu\) is an orthonormal basis for \(V\)).

### Spencer cohomology

We recall Lemma 3.18 and the following discussion, in particular the (normalised and polarised) cocycle conditions in the form of equations (3.118), (3.119). These results were originally stated only for the case of Lorentzian signature but apply to the \((0,2)\) case here as well; in order to determine \(H^{2,2}_\sim = H^{2,2}(s_-; s)\), we must solve the equations

\[
\begin{align*}
2\kappa(s, \beta(v, s)) + \gamma(s, s)v &= 0, \\
\beta(\kappa, s) + \gamma(s, s) \cdot s &= 0,
\end{align*}
\]

where \(\beta \in \text{Hom}(V \otimes S, S), \gamma \in \mathcal{O}^2 S \to \mathfrak{so}(V)\). We begin by parametrising \(\beta\). Since \(\text{Hom}(V \otimes S, S) \cong \text{Hom}(V, \text{End}(S))\), for \(v \in V\) we can write \(\beta_v \in \text{End}(S)\) for the endomorphism \(s \mapsto \beta_v(s) := \beta(v, s)\), and \(\beta_\mu := \beta e_\mu\). Then \(\beta\) can be parametrised as

\[
\beta_\mu = a_\mu + b_{\mu\nu} \Gamma^\nu + c_\mu \Gamma^*.
\]

where we use the Einstein summation convention and the coefficients take real values; we can consider them to be the components of some \(a, c \in V^*, b \in \otimes^2 V^*\).

Now, the first cocycle condition is equivalent to

\[
\gamma(s, s)_{\mu\nu} = -2\bar{s}\Gamma_\mu \beta_{\nu} s.
\]

This equation completely determines \(\gamma\) in terms of \(\beta\), and since \(\gamma\) must take values in \(\mathfrak{so}(V)\), it places the following constraint on \(\beta\):

\[
\bar{s}\Gamma_\mu (\mu \beta_\nu) s = 0.
\]

Expanding this equation using our parametrisation for \(\beta\) and evaluating products of gamma matrices gives us

\[
\bar{s}\Gamma_\mu \beta_\nu s = a_\nu \bar{s}\Gamma_\mu s + b_{\nu\rho} \bar{s}\Gamma_\mu \Gamma^\rho s + c_\nu \bar{s}\Gamma_\mu \Gamma^* s
\]

\[
= b_{\nu\mu} \bar{s}s + (a_\nu \eta_{\mu\rho} + \sigma c_\nu \eta_{\rho\mu}) \bar{s}\Gamma^\rho s + c_\nu b_{\nu\rho} \bar{s}\Gamma^* s.
\]

Now, if \(s \neq 0\) and \(\sigma_B = +1\), \(\bar{s}s \neq 0\) but \(\bar{s}\Gamma^* s = 0\), so we have

\[
b_{(\nu\mu)} = 0 \quad \text{and} \quad a_{(\nu \eta_{\mu\rho})} + \sigma c_{(\nu \epsilon_{\mu\rho})} = 0 \quad \text{for} \ \sigma_B = +1.
\]

Since we are working in two dimensions, the first equation implies that \(b_{\mu\nu} = b_{\epsilon_{\mu\nu}}\) for some \(b \in \mathbb{R}\). Fully symmetrising the latter equation gives \(a_{(\mu \eta_{\nu\rho})} = 0\), and tracing this gives \(a = 0\). Substituting this back into the full equation then gives us \(c = 0\). On
the other hand, if \( s \neq 0 \) and \( \sigma_B = -1 \), \( \overline{s}s = 0 \) and \( \overline{\Gamma}_*s \neq 0 \), so

\[
\epsilon_{(\mu} b_{\nu)} = 0 \quad \text{and} \quad a_{(\nu} \epsilon_{\mu)} + \sigma c_{(\nu} \epsilon_{\mu)} = 0 \quad \text{for} \quad \sigma_B = -1.
\]  

(3.215)

One can show (for instance by substituting values for \( \mu, \nu \)) that the first equation is satisfied if and only if

\[
b_{\mu \nu} = b_{\nu \mu}\]

and again the latter equation has only the trivial solution. Thus we have shown that \( \beta \) is parametrised by a single parameter \( b \in \mathbb{R} \), with

\[
\beta_\mu = \begin{cases} 
be_{\mu \nu} \Gamma^\nu & \text{for} \quad \sigma_B = +1, \\
b \Gamma_\mu & \text{for} \quad \sigma_B = -1.
\end{cases}
\]  

(3.216)

Substituting this back into our equation for \( \gamma \), we have

\[
\gamma(s, s)_{\mu \nu} = \begin{cases} 
2be_{\mu \nu} \overline{s}s & \text{for} \quad \sigma_B = +1, \\
-2be_{\mu \nu} \overline{\Gamma}_*s & \text{for} \quad \sigma_B = -1.
\end{cases}
\]  

(3.217)

The remaining cocycle condition is then identically satisfied by the Fierz identity, as we will now demonstrate. We have

\[
\beta(\kappa, s) + \gamma(s, s) \cdot s = \overline{\Gamma}^S b_{\mu} s + \frac{1}{4} \gamma(s, s)_{\mu \nu} \Gamma^{\mu \nu} s
\]

\[
= \begin{cases} 
be_{\mu \nu} (\overline{\Gamma}^S s) \Gamma^\nu s + \sigma b(\overline{s}s) \Gamma_* s & \text{for} \quad \sigma_B = +1, \\
b(\overline{\Gamma}_\mu s) \Gamma^{\mu} s - \sigma b(\overline{\Gamma}_* s) \Gamma_* s & \text{for} \quad \sigma_B = -1.
\end{cases}
\]  

(3.218)

Now, the Fierz identity with either bilinear (Proposition 3.11) gives us

\[
s\overline{s} = \frac{1}{2} \left( (\overline{s}s) \mathbb{I} + (\overline{\Gamma}^S s) \Gamma_{\mu} - \sigma (\overline{\Gamma}_* s) \Gamma_* s \right)
\]  

(3.219)

but we note that the first term on the right-hand side vanishes for \( \sigma_B = -1 \), and the third vanishes for \( \sigma_B = +1 \). Thus for \( \sigma_B = +1 \),

\[
\epsilon_{\mu \nu} (\overline{\Gamma}^S s) \Gamma^\nu s = \epsilon_{\mu \nu} \Gamma^\nu (s\overline{s}) \Gamma^{\mu} s
\]

\[
= \frac{1}{2} (\overline{s}s) \epsilon_{\mu \nu} \Gamma^\nu \Gamma_{\mu} s + \frac{1}{2} (\overline{\Gamma}^S \mu s) \epsilon_{\mu \nu} \Gamma^\nu \Gamma^\mu \Gamma_{\mu} s
\]

\[
= \sigma (\overline{s}s) \Gamma_* s,
\]  

(3.220)

and for \( \sigma_B = -1 \),

\[
(\overline{\Gamma}_\mu s) \Gamma^{\mu} s = \Gamma_{\mu} (s\overline{s}) \Gamma_{\mu} s
\]

\[
= \frac{1}{2} (\overline{\Gamma}^S \mu s) \Gamma_{\mu} \Gamma_{\mu} s - \sigma \frac{1}{2} (\overline{\Gamma}_* s) \Gamma^{\mu} \Gamma_{\mu} s
\]

\[
= \sigma (\overline{\Gamma}_* s) \Gamma_* s,
\]  

(3.221)

whence \( \beta(\kappa, s) + \gamma(s, s) \cdot s = 0 \), so we have solved the Spencer cocycle equations in the minimal case for \( (0, 2) \) and the minimal non-chiral case of for \( (1, 1) \). In the minimal chiral \( (1, 1) \) case, we can follow essentially the same argument except that we must set \( b_{\mu \nu} = 0 \) in the parametrisation of \( \beta \), whence we have only the trivial solution. In particular, we have shown the following.
Proposition 3.43. In signature (1, 1) or (0, 2), let $s$ be the flat model superalgebra defined by $S = \mathbb{P}$ and $\kappa$ any of the Dirac currents from Tables 2.8 and 2.9. Then

$$H^{2,2}(s_-, s_+) \cong \mathbb{R}$$

(3.222)

as an $\mathfrak{s}_0 = \mathfrak{so}(V)$-module. The space of normalised cocycles $\mathcal{H}^{2,2}$ consists of elements $\beta + \gamma$ given by (3.216), (3.218) for $b \in \mathbb{R}$.

In signature (1, 1), if $S = S_+$ then there is a unique non-trivial Dirac current and we have

$$H^{2,2}(s_-, s_+) = 0.$$  

(3.223)

We already used this result to construct admissible connections on 2-dimensional spin manifolds (in the proof of Proposition 3.14 in particular.)

Maximally supersymmetric filtered subdeformations

Let us now use the result above to describe the maximally supersymmetric filtered subdeformations of $s$. Note that there is no sub-maximal highly supersymmetric case here since $\dim S = 2$. From our discussion at the end of §3.3.5, we know that odd-generated maximally supersymmetric filtered subdeformations are determined by normalised cocycles $\beta + \gamma \in \mathcal{H}^{2,2}$ which is $h_{(\beta + \gamma)} := \gamma(\mathcal{D}) \in \mathfrak{so}(V)$ invariant, where $\mathcal{D} = \ker \kappa$. If we have such a cocycle, we can define an admissible cocycle of the graded subalgebra $a = V \oplus S \oplus h_{(\beta + \gamma)}$, but we must check that this cocycle is admissible. More general maximally supersymmetric deformations are obtained by replacing $h_{(\beta + \gamma)}$ with any larger subalgebra $h$ preserving $\beta + \gamma$.

In the present case, $\dim \mathfrak{so}(V) = 1$, so we must either have $h_{(\beta + \gamma)} = 0$, so $a = s_-$ or $h_{(\beta + \gamma)} = \mathfrak{so}(V)$, so $a = s$. Moreover, any element of $\mathcal{H}^{2,2}$ is actually $\mathfrak{so}(V)$-invariant since $\mathcal{H}^{2,2} \cong \mathbb{R}$, the trivial $\mathfrak{so}(V)$-module. Thus it suffices to study filtered deformations of the whole algebra $s$, since any deformation of $s_-$ can be extended to a deformation of $s$, and such deformations are parametrised up to isomorphism by the integrable elements of $\mathcal{H}^{2,2}$ (in particular, we can take the map $\lambda : V \rightarrow \mathfrak{so}(V)$ which appeared in much of the theoretical discussion to be zero without loss of generality).

We will now show that every element of $\mathcal{H}^{2,2}$ is integrable. Recall that the definition of integrability, Definition 3.35, concerns a map $\Theta : V \otimes \mathcal{O}^2 S \rightarrow \mathfrak{so}(V)$ defined as

$$\Theta(v, e, e) = 2\gamma(e, \beta(v, e))$$

(3.224)

where $v \in V, e \in S$ (for $\lambda = 0$); a simple computation in an orthonormal basis shows that

$$\Theta_{\mu
u}^{\rho\sigma}(e, e) = 2\gamma_{\mu
u}(e, \beta_{\rho\sigma}e) = \theta_{\mu
u, \rho\sigma\epsilon} \kappa_{\epsilon}$$

(3.225)

where

$$\theta_{\mu
u, \rho\sigma\epsilon} = \begin{cases} 4b^2 \epsilon_{\mu\rho}
\epsilon_{\nu\sigma} & \text{for } \sigma_B = +1, \\
4b^2 \epsilon_{\mu\rho}
\epsilon_{\nu\sigma} & \text{for } \sigma_B = -1. \end{cases}$$

(3.226)

Thus we have shown that $\Theta$ factors through a map $\theta : \Lambda^2 V \rightarrow \mathfrak{so}(V)$ with components given above, hence the first integrability condition is satisfied. In component form,
the second integrability condition which we must check is
\[ \frac{1}{4} \theta_{\mu \nu \rho \sigma} \Gamma^{\rho \sigma} \epsilon = [\beta_\mu, \beta_\nu] \epsilon \]
(3.227)
for all \( \epsilon \in S \). The left-hand side can be rewritten
\[ \frac{1}{4} \theta_{\mu \nu \rho \sigma} \Gamma^{\rho \sigma} \epsilon = \frac{1}{4} \theta_{\mu \nu \rho \sigma} \epsilon^{\rho \sigma} \Gamma_* \epsilon = \begin{cases} \sigma 2 b^2 \epsilon_{\mu \nu} \Gamma_* \text{ for } \sigma_B = +1, \\ 2 b^2 \epsilon_{\mu \nu} \Gamma_* \text{ for } \sigma_B = -1. \end{cases} \]
(3.228)
For the right-hand side, we must therefore compute the commutator \([\beta_\mu, \beta_\nu]\), but we already did this in §3.2.4; the result written in our orthonormal basis is
\[ [\beta_\mu, \beta_\nu] = \begin{cases} \sigma 2 b^2 \epsilon_{\mu \nu} \Gamma_* \text{ for } \sigma_B = +1, \\ 2 b^2 \epsilon_{\mu \nu} \Gamma_* \text{ for } \sigma_B = -1, \end{cases} \]
(3.229)
whence the integrability condition is identically satisfied.

Thus for either signature and any choice of Dirac current, there is a one-parameter family of filtered deformations of \( s \) with parameter \( b \in \mathbb{R} \). Let us describe these in terms of an explicit basis for the even part of the algebra. In our chosen orthonormal basis, let \( P_\mu \) denote the infinitesimal translation generators (of which there are two in either signature) and let \( L_{\mu \nu} \) denote the infinitesimal generators of \( so(\mathbb{V}) \); in our case there is one such generator \( L_{\mu \nu} = \epsilon_{\mu \nu} L_* \) where \( L_* = L_{01} \) in signature (1,1) and \( L_* = L_{12} \) in signature (0,2).

The \([s_0, s_0]\), \([s_0, s_{-1}]\) and \([s_0, s_{-2}]\) are not deformed; the first must be trivial since \( s_0 \) one-dimensional while the others are
\[ [L_*, s] = \frac{1}{2} \Gamma_* s, \quad [L_*, P_\mu] = -\sigma \epsilon_{\mu \nu} P_\nu. \]
(3.230)
The deformed brackets take the following form:
\[ [P_\mu, P_\nu] = \sigma 4 b^2 \epsilon_{\mu \nu} L_*, \quad [P_\mu, \epsilon] = b \epsilon_{\mu \nu} \Gamma_* \epsilon, \]
(3.231)
or
\[ [P_\mu, P_\nu] = 4 b^2 \epsilon_{\mu \nu} L_*, \quad [P_\mu, \epsilon] = b \Gamma_* \epsilon, \]
(3.232)
Note that since \([V, V] = R L_* = \tilde{s}_0\) if \( b \neq 0 \), there are no maximally supersymmetric proper subalgebras; the only non-trivial maximally supersymmetric filtered subdeformations are deformations of the whole graded superalgebra \( s \).

In §3.2.4, we discussed supersymmetric geometries (geometries admitting parallel spinors with respect to an admissible connection) in 2 dimensions. There, we found that the maximally supersymmetric geometries were 2-dimensional hyperbolic space with scalar curvature \( R = -8 b^2 \) for either sign \( \sigma_B \) in the Riemannian case, and in the Lorentzian case they were \( dS_2 \) \((R = 8 b^2)\) for \( \sigma_B = +1 \) and \( AdS_2 \) \((R = -8 b^2)\) for \( \sigma_B = -1 \). We reach the same result by inspecting the even part of the deformed
algebras described; in each case, we find the isometry algebra of the appropriate geometry. Indeed, each of those geometries is a homogeneous space for the metric Lie pair \((\mathfrak{e}_0, \mathfrak{s}_0 = \mathfrak{s}_0(V), \eta)\), where we note that \(V \cong \mathfrak{s}_0/\mathfrak{s}_0(V)\) as an \(\mathfrak{s}_0(V)\)-module. Moreover, for \(\sigma = \sigma_B = -1\), \(\mathfrak{s}_0\) is actually the standard anti-de Sitter superalgebra (see [30]).

### 3.4 Highly supersymmetric Lorentzian spin manifolds

In this section, we justify our definition of geometrically realisability of a highly supersymmetric subdeformation \(\mathfrak{a}\) of \(\mathfrak{s}\) by showing that there exists a Lorentzian spin manifold with admissible connection such that \(\mathfrak{a}\) is (a subalgebra of) its Killing superalgebra. We essentially generalise the results of [45, §3.3].

#### 3.4.1 Homogeneous spin structures

We will use the language of homogeneous spaces introduced in Section 2.4 extensively here. Spin structures on homogeneous manifolds were first treated systematically by Bär [95] and by Cahen et al. [96]; our present treatment follows [45].

Let \((G, H, \eta)\) be a metric Klein pair (of any signature), let \((M = G/H, g)\) be the associated homogeneous space, which we assume to be oriented, and let \(V = T_o M\) where \(o = H\). Suppose that there exists a lift of the isotropy representation \(H \to SO(V)\) to \(Spin(V)\); that is, a Lie group morphism \(H \to Spin(V)\) making the following diagram commute:

\[
\begin{align*}
\text{Spin}(V) & \quad \rightarrow \\ H & \quad \downarrow \\ \text{SO}(V) & \quad 
\end{align*}
\]  

Then the canonical map \(\text{Spin}(V) \to \text{SO}(V)\) induces a \(\text{Spin}(V)\)-equivariant bundle map

\[
G \times_H \text{Spin}(V) \to G \times_H \text{SO}(V) \equiv F_{SO}
\]  

which is a spin structure on \(M\). We call this the \emph{homogeneous spin structure associated to the lift} \(H \to \text{Spin}(V)\). Note that \(G\) (left) acts naturally on both of these bundles compatibly with the right action by the spin group, and that the bundle isomorphism is \(G\)-equivariant.

We already saw in Section 2.4 that, using the isotropy representation and related representations of \(H\), many natural bundles on \(M\), particularly those associated to \(F_{SO}\) via representations of \(\text{SO}(V)\), can be viewed as associated bundles to the \(H\)-principal bundle \(G \to M\). Homogeneous spin structures allow us to do the same with representations of the spin group; for example, if \(S\) is a spinor representation of \(\text{Spin}(V)\), there is a \(G\)-equivariant isomorphism of vector bundles

\[
\mathfrak{S} := (G \times_H \text{Spin}(V)) \times_{\text{Spin}(V)} S \cong G \times_H S.
\]  

If \(G\) is simply connected, the construction above classifies the spin structures on \((M, g)\) in definite signature.
Lemma 3.44 ([95, 96]). If \((M, g)\) is a homogeneous Riemannian \(G\)-space where \(G\) is simply connected, the equivalence classes of spin structures on \(M\) are in one-to-one correspondence with spin lifts of the isotropy representation.

The proof of this result relies on the ability to lift the action of \(G\) on \(F_{SO}\) induced by the action on \(M\) to an action on \(P\) which can then be localised at \(0\) to find a lift of the isotropy representation. In definite signature, this follows from the fact that the spin group is connected and simply connected (for \(\dim V \geq 3\)). In indefinite signature, the above fails due to the non-trivial topology of the spin group; in signature \((p \geq 1, q \geq 1)\), the spin group is disconnected. One might nonetheless hope to rescue the situation by assuming a time-orientation on \(M\) and working with \(F_{SO_0}\) and a connected spin structure (an invariant lift to a principal \(\text{Spin}_0(V)\)-structure), but this still fails because the connected component of the spin group is not simply connected for \((p \geq 2, q \geq 2)\). However, in the case that \(M\) is a reductive \(G\)-space, the above still holds [96, 97].

To avoid these issues, one can simply hypothesise that the action of \(G\) lifts to the spin structure; we then say that we have a homogeneous spin structure. Such structures are in one-to-one correspondence with lifts of the isotropy representation. We will not prove this here since it is standard; we will however give a full proof of a generalisation of this result in the next chapter (see Proposition 4.41).

3.4.2 Reconstruction of highly supersymmetric backgrounds

We work with the same structures described in §3.1.1, except we take \((V, \eta)\) and \((M, g)\) to be Lorentzian and the squaring map \(\kappa : \bigotimes^2 S \to V\) to be symmetric and causal, so that \(\mathfrak{s}\) and \(\mathfrak{K}_D\) are Lie superalgebras. We do this both because of our ultimate interest in supergravity backgrounds and because we wish to make use of the Homogeneity Theorem (Theorem 3.21).

We say that a Lorentzian spin manifold \((M, g)\) with spinor bundle \(S\) with fibre \(S\) and admissible connection \(D\) is highly supersymmetric if \(\dim \mathfrak{K}_D > \frac{1}{2} \dim \mathfrak{s}\). Equivalently, \(\mathfrak{K}_D\) is highly supersymmetric as a subdeformation of \(\mathfrak{s}\). By the Homogeneity Theorem, the values of the Killing vector fields arising as squares of Killing spinors (i.e. those in the even part of the Killing ideal \(\mathfrak{K}_D\)) span the tangent space at every point. Thus \(M\) is locally homogeneous; informally, it locally looks like a homogeneous space for a group with Lie algebra \(\mathfrak{K}_D\). If it happens that \(M\) is geodesically complete, it is a homogeneous Lorentzian spin manifold for the connected and simply connected group \(G\) with \(\mathfrak{K}_D = \text{Lie} G\) by [98, Thm.2.2], but we will be content with the local statement here.

In this section, rather than constructing filtered deformations of \(\mathfrak{s}\) from sections on a spin manifold, we will start with a geometrically realisable highly supersymmetric filtered subdeformation of \(\mathfrak{s}\) (see Definition 3.29) and construct a homogeneous spin manifold on which the elements of the algebra can be realised as sections. Our goal will be to generalise [45, Thm.13], which says that geometrically realisable highly supersymmetric filtered deformations of the 11-dimensional Poincaré superalgebra are subalgebras of Killing superalgebras. We will find some potential obstructions to doing this in full generality, but the obstructions themselves will prove instructive.
Superc H-Chandra pairs

We begin by introducing what can be viewed as a Lie supergroup associated to a Lie superalgebra. In a superspace formalism, our the supersymmetric backgrounds we will construct could be considered as homogeneous spaces for such a supergroup. We will not fully use this formalism, but the following definition will be useful for us. See [99] and the references therein for more on the superspace perspective.

Definition 3.45 (Super Harish-Chandra pair). A super Harish-Chandra pair is a pair $(G_0, g)$ where $g$ is a finite-dimensional Lie superalgebra and $G_0$ is a connected Lie group integrating $g_{0\text{r}}$ equipped with a representation $\text{Ad}: G_0 \rightarrow \text{GL}(g)$ (which we call the adjoint representation of the pair) integrating the restriction $\text{ad}: g_{0\text{r}} \rightarrow \mathfrak{gl}(g)$ of the adjoint representation of $g$ to $g_{0\text{r}}$.

Note that if $(G_0, g)$ is a super Harish-Chandra pair, the adjoint representation $\text{Ad}$ preserves the grading on $g$ since $\text{ad}$ does. The induced subrepresentation on $g_{0\text{r}}$ is simply the adjoint representation of $G_0$.

Example 4. The (connected spin lift of the) Poincaré group $\text{Spin}_0(V) \ltimes V$ of course has the Poincaré algebra $\mathbb{R}_0$ as its Lie algebra, and equipping it with the action $(a, v) \cdot s = a \cdot s$ on $\mathbb{R}_0 = S$ makes $(\text{Spin}_0(V) \ltimes V, s)$ into a super Harish-Chandra pair.

Example 5. Let $(M, g, D)$ be a highly supersymmetric Lorentzian spin manifold with admissible connection and let $\mathcal{R}_D$ be its Killing superalgebra. Then, by homogeneity, the values of the Killing vectors of the even part $\mathcal{V}_D \subseteq \text{iso}(M, g)$ of $\mathcal{R}_D$. Let us assume that $M$ is complete and connected and thus let $G$ be the connected subgroup of $\text{ISO}(M, g)$ generated by $\mathcal{V}_D$. Then the action of $G$ is transitive [98, Thm 2.2] and pulls back to an isometric transitive action of the universal covering group $\tilde{G}$ on $M$. Since $\tilde{G}$ is simply connected, the restriction $\text{ad}: \mathcal{V}_D \rightarrow \mathfrak{gl}(\mathcal{R}_D)$ of the adjoint representation of $\mathcal{R}_D$ to $\mathcal{V}_D$ integrates to a representation $\text{Ad}: \tilde{G} \rightarrow \mathcal{R}_D$. Thus $(\tilde{G}, \mathcal{R}_D)$ is a super Harish-Chandra pair, and $M$ is a homogeneous Lorentzian $\tilde{G}$-space.

Consider now a highly supersymmetric filtered subdeformation $\mathfrak{s}$ of the Poincaré superalgebra $\mathfrak{g}$, say with $\text{Gr} \mathfrak{s} \equiv \mathfrak{a} = V \oplus S' \oplus h \subseteq \mathfrak{s}$. In particular, the filtration on $\mathfrak{g}$ is of the form $\mathfrak{g} = \mathfrak{g}^{-2} \supseteq \mathfrak{g}^{-1} \supseteq \mathfrak{g}^{0} \supseteq \mathfrak{g}^{-1} = 0$ and as Lie algebras, $h = \text{Gr}_0 \mathfrak{g}^* = \mathfrak{g}^0 / \mathfrak{g}^{-1} \equiv \mathfrak{g}^0$, so there is a canonical embedding of $h$ as the zeroth part of the filtration of $\mathfrak{g}$.\(^5\) We will thus treat $h$ as a subalgebra of $\mathfrak{g}$.

Lemma 3.46. Let $(G_0, g)$ be a super Harish-Chandra pair where $g$ is a highly supersymmetric filtered deformation of $\mathfrak{s}$, and suppose that the connected subgroup $H$ of $G_0$ generated by $h$ is closed. Then the inclusion $h \hookrightarrow \text{so}(V)$ integrates to a morphism $H \rightarrow \text{Spin}(V)$. In particular, this lifts the linear isotropy representation and thus defines a homogeneous spin structure for $(G_0, H, \eta)$.

*Proof.* First note that $V \equiv g_{0\text{r}} / h$ and $S' \equiv g_{0\text{r}} / g_{0\text{r}}$ as $h$-modules – this is immediate upon inspecting an explicit presentation of $g$ as a filtered subdeformation of $\mathfrak{s}$. In particular, since $\eta$ is $h$-invariant, $(g_{0\text{r}}, h, \eta)$ is a metric Lie pair, so $(G_0, H, \eta)$ is a metric Klein pair, and the inclusion $h \hookrightarrow \text{so}(V)$ is the linear isotropy representation of $h$.

\(^5\)This can also be seen by considering an explicit presentation of $g$ as a filtered deformation of $a$. 

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Similarly, at the level of groups, the adjoint representation induces a Lie group morphism \( \varphi : H \to \text{SO}(V) \) integrating \( h \hookrightarrow \text{so}(V) \). This lifts to a morphism \( \tilde{\varphi} : \tilde{H} \to \text{Spin}(V) \) where \( \pi_H : \tilde{H} \to H \) is the universal cover of \( H \). We will show that there exists a diagonal arrow making the diagram

\[
\begin{array}{ccc}
\tilde{H} & \xrightarrow{\tilde{\varphi}} & \text{Spin}(V) \\
\downarrow{\pi_H} & & \downarrow{\pi} \\
H & \xrightarrow{\varphi} & \text{SO}(V)
\end{array}
\]

(3.236)

commute, which proves the result. Recalling that there is a canonical short exact sequence of Lie groups

\[
1 \to \pi_1(H) \to \tilde{H} \to H \to 1,
\]

(3.237)

we will show that \( \pi_1(H) \subseteq \ker \varphi \), hence \( \varphi \) factors through \( H \), giving the required arrow in the diagram.

Note that the actions of \( H \) on \( V \) and \( S' \) pull back to actions of \( \tilde{H} \) (integrating the actions of \( h \)) in which \( \pi_1(H) \) acts trivially. On the other hand, \( \tilde{H} \) also acts on \( V \) and \( S \) via the action of \( \text{Spin}(V) \) on those spaces. The two actions of \( \tilde{H} \) on \( V \) agree by the commutation of the square diagram above. Since \( \tilde{\varphi} \) integrates \( h \hookrightarrow \text{so}(V) \) and the action of \( h \) (via \( \text{so}(V) \)) on \( S \) preserves \( S' \), so too does the action of \( \tilde{H} \) via \( \text{Spin}(V) \). The two actions of \( \tilde{H} \) on \( S' \) both integrate the same action of \( h \) on \( S' \), so they agree.

The trivial action of \( \pi_1(H) \) on \( V \) gives us

\[
\pi_1(H) \subseteq \ker(\varphi \circ \pi_H) = \ker(\pi \circ \tilde{\varphi})
\]

(3.238)

thus \( \tilde{\varphi}(\pi_1(H)) \subseteq \ker \pi = \{ \pm 1 \} \). On the other hand, the trivial action of \( \pi_1(H) \) on \( S' \) shows that \( -1 \notin \tilde{\varphi}(\pi_1(H)) \), thus \( \pi_1(H) \subseteq \ker \tilde{\varphi} \). This shows that \( \tilde{\varphi} \) factors through \( H \) as claimed.

Reconstruction theorem

Our final result in this chapter is a partial converse to Theorem 3.10 in the highly supersymmetric Lorentzian case. In this context, that result tells us that a Killing superalgebra is a filtered subdeformation of a Poincaré superalgebra \( s \); indeed, it is a geometrically realisable filtered subdeformation (Definition 3.29). We will show that a geometrically realisable highly supersymmetric subdeformation of \( s \) can be expressed as an algebra of sections on a spin manifold with similar structure to a Killing superalgebra, with a slight caveat.

**Theorem 3.47** (Reconstruction of highly supersymmetric background). Let \( \mathfrak{g} \) be a geometrically realisable filtered deformation of a highly supersymmetric graded subalgebra \( \mathfrak{a} = V \oplus S' \oplus \mathfrak{h} \) of the Poincaré superalgebra \( s \). Let \( G_0 \) be the connected and simply connected Lie group corresponding to the Lie algebra \( \mathfrak{g}_0 \) and suppose that the connected subgroup \( H \) corresponding to \( \mathfrak{h} \) is closed. Then there exists a homogeneous spin structure on the Lorentzian homogeneous space \( M = G_0/H \) with a connection \( D \) on the spinor bundle \( \mathcal{S} = G_0 \times_H S \) and an injective \( \mathbb{Z}_2 \)-graded linear
map $\Psi : g \hookrightarrow \mathfrak{D} \oplus \mathfrak{S}$ which restricts to a Lie algebra embedding $\mathfrak{g}_\mathfrak{S} \hookrightarrow \mathfrak{D}$ and which satisfies

$$\Psi([X, s]) = \mathcal{L}_{\Psi(X)}\Psi(s), \quad \Psi([s, s']) = \kappa(\Psi(s), \Psi(s')),$$

(3.239)

for all $X \in \mathfrak{g}_\mathfrak{S}$ and $s, s' \in \mathfrak{S}$.

Furthermore, if $\Psi(\mathfrak{g}_\mathfrak{S}) = \mathfrak{S}_D$ or if $D$ is admissible otherwise, we find in particular that $\mathfrak{g}$ embeds into the Killing superalgebra $\mathfrak{r}_D$ of $(M, g, D)$.

Let us make some remarks about this result before giving the proof. First, it is necessary to hypothesise that $H$ is closed; it may be possible to choose $\mathfrak{h}$ such that this is the case, but we have not been able to show this. We note that this is also implicitly assumed in [45, Thm.13]; although it is not in the statement of the theorem, it is stated in the preamble to the theorem. Moreover, the present theorem is very close to saying that all geometrically realisable highly supersymmetric subdeformations of $s$ embed into Killing superalgebras, but there is an obstruction in that it does not seem to be guaranteed that the connection $D$ on $\mathfrak{S}_D$ which we will construct in the proof is admissible (see Definition 3.7 for our characterisation of admissibility).

The algebraic conditions (1) and (2) are satisfied because we work with admissible cocycles which are expressed in terms of normalised cocycles of the whole Poincaré superalgebra $s$; the difficulty is in showing condition (3), namely that $\mathcal{L}_{\kappa_\epsilon} \beta = 0$ for all $\epsilon \in \mathfrak{S}_D$; we can only show this for $\epsilon \in \Psi(\mathfrak{g}_{\mathfrak{S}})$. On the other hand, by a dimension count we find that maximally supersymmetric subdeformations do in fact embed in Killing superalgebras, since in this case we must have $\Psi(\mathfrak{g}_{\mathfrak{S}}) = \mathfrak{S}_D$.

To put a positive spin on this issue, it provides us with new motivation to study Killing superalgebras constructed from constrained Killing spinors, which we briefly discussed in §3.2.3; while $D$ might not be guaranteed to satisfy the conditions for admissibility on sections of $\mathfrak{S}$, it does satisfy these conditions on sections of the subbundle $\mathfrak{S}' := G \times_H \mathfrak{S}$ to which elements of $\Psi(\mathfrak{g}_{\mathfrak{S}})$ belong and which we could describe as the kernel of some fibrewise linear operator on $\mathfrak{S}$.

For the proof, recall Wang’s Theorem 2.14 on connections on principal bundles, and the following discussion on Nomizu maps. If we have a spin-lift $\phi$ of the isotropy representation, take $P = G \times_H \text{Spin}(V)$ to be the $G$-invariant principal bundle of the theorem and $\phi$ to be the map from the isotropy group to the structure group of $P$. Then one can easily check that the Nomizu map $L : \mathfrak{g} \to \mathfrak{so}(V)$ of the Levi-Civita connection (considered as a $G$-invariant principal connection on $F_{SO}$) satisfies the conditions of Wang’s theorem so also corresponds with a principal connection on $P$, which is of course the spin-lift of the Levi-Civita connection.

**Proof.** By hypothesis we have a metric Klein pair $(G_{\mathfrak{S}}, H, \eta)$. Then by Lemma 3.46, there is a spin lift of the linear isotropy representation of the metric Lie pair $(G_{\mathfrak{S}}, H, \eta)$ with corresponding homogeneous spin structure $P = G_{\mathfrak{S}} \times_H \text{Spin}(V)$ and spinor bundle $\mathfrak{S} = G_{\mathfrak{S}} \times_H \mathfrak{S}$ on $M = G_{\mathfrak{S}}/H$.

Now let $[\mu_-]$ be the admissible Spencer cohomology class for $a = \text{Gr}_{\mathfrak{g}} \subseteq \mathfrak{s}$ corresponding to $\mathfrak{g}$, so that there exists $\beta + \gamma \in (\mathfrak{h}^\otimes 2)^\mathfrak{h}$ satisfying $i_* [\mu_-] = i^* (\beta + \gamma) \in H^2(\mathfrak{a}^\perp; \mathfrak{S})$. Recall that $\beta \in \text{Hom}(V \otimes \mathfrak{S}, \mathfrak{S}) \cong V^* \otimes \text{End} \mathfrak{S}$ is $\mathfrak{h}$-invariant and thus $H$-invariant, so by Frobenius reciprocity it defines a unique $G_{\mathfrak{S}}$-invariant 1-form with
values in $\text{End} S$, although we will actually adopt a sign convention where we work with $\beta \in \Omega^1(M; \text{End} S)^G$ corresponding to $-\beta$. We define the connection $D := \nabla - \beta$.

Since $G_\beta$ acts by isometries by construction and $\beta$ is $G_\beta$-equivariant, for each $X \in g_\beta$ the fundamental vector field $\xi^M_X \in X(M)$ is Killing and preserves $\beta$, so it lies in $\mathfrak{U}_D$. Recalling that the assignment $X \to \xi^M_X$ is a Lie algebra anti-homomorphism $g_\beta \to X(M)$, we define $\Psi(X) = -\xi^M_X$. This gives us a Lie algebra morphism $\Psi : g_\beta \to \mathfrak{U}_D$.

To construct the spinor fields associated to elements of $g_\beta = S'$ and show that they are $D$-parallel, we will exploit the homogeneous bundle structure of $S$. Let us define a left action of $H$ on $G_\beta$ by $h \cdot g = gh^{-1}$ where we use group multiplication in $G_\beta$ on the right-hand side. Then, using some standard associated bundle theory, we have identifications between spaces of sections and $H$-equivariant maps:

$$X(M) \cong C^\infty(G_\beta; V)^H,$$
$$\mathfrak{S} \cong C^\infty(G_\beta; S)^H.$$

(3.240)

For $X \in g_\beta$, the fundamental vector field $\xi_X$ corresponds to the map $G_\beta \to V \cong g_\beta / h$ given by $X \to \text{Ad}_{g^{-1}} X \mod h$. Similarly, for $s \in g_\beta$ we define $\Psi(s) \in \mathfrak{S}$ as the spinor field associated to the map $G_\beta \to S$ defined by $g \to \text{Ad}_{g^{-1}} s$. We now use Wang’s Theorem and the Levi-Civita Nomizu map to show that these spinor fields are $D$-parallel. First, let us take a presentation of the form (3.190) for $g$; in particular, we have a (non-reductive) split $G_\beta = h \oplus V$ and a map $\lambda : V \to \mathfrak{so}(V)$. Then, writing elements $X \in g_\beta$ in the form $X = A + \nu$ for $A \in h$, $\nu \in V$, we can follow the construction of the Nomizu map $L : g_\beta \to \mathfrak{so}(V)$ from §2.4.4 corresponding to the Levi-Civita connection to compute

$$L(A + \nu) = A + \lambda(\nu).$$

(3.241)

In particular, for $s \in S'$ we have $L(A + \nu) \cdot s = [A + \nu, s] - \beta(\nu, s)$, where we use the definition of the bracket on $g$. Since this is also the map corresponding to the spin-lift of the Levi-Civita connection $\nabla$ under Wang’s Theorem 2.14, we can compute the $H$-equivariant map $G_\beta \to S$ corresponding to the spinor field as $\nabla \xi_X \Psi(s)$ as

$$g \mapsto \left( L(\text{Ad}_{g^{-1}} X) - \text{ad}_{\text{Ad}_{g^{-1}} X} \right) \left( \text{Ad}_{g^{-1}} s \right) = -\beta \left( \text{Ad}_{g^{-1}} X \mod h, \text{Ad}_{g^{-1}} s \right),$$

(3.242)

where the equality is due to the expression for $L$ above. But this is nothing but the map corresponding to $\beta(\xi_X, \Psi(s))$ (recalling our sign convention for $\beta$). By homogeneity, the values of the fundamental vector fields span every tangent space, so we have shown that $\nabla \Psi(s) = \beta \Psi(s)$, whence $\Psi(s) \in \mathfrak{S}_D$ as required. We now have a $\mathbb{Z}_2$-graded map $\Psi : g \to \mathfrak{U}_D \oplus \mathfrak{S}_D$. One can show that it satisfies the equations (3.239), by examining the Killing transport data at the basepoint (see §3.2.2, in particular the explicit presentation of the brackets of the Killing superalgebra as a filtered deformation (3.70)). This will also show that $\Psi$ is injective since no non-zero element of $g$ has trivial transport data.

Finally, we note that if $D$ is admissible, $\Psi : g \to \mathfrak{S}_D$ is a Lie superalgebra embedding. The Spencer cocycle conditions for $\beta + \gamma$ ensure $\beta$ satisfies conditions (1) and (2) of Definition 3.7. Condition (3) ($\mathcal{L}_\xi \beta = 0$ for all $\epsilon \in \mathfrak{S}_D$) is not automatically satisfied, but observe that we have $\mathcal{L}_\xi \beta = 0$ for all $X \in g_\beta$ by $G_\beta$-invariance of $\beta$; in
particular, $\mathcal{L}_\xi \beta = 0$ holds for $c \in \Psi(g_T)$.

We note that we have glossed over some details of the calculations above, in particular when constructing the spinorial part of the map $\Psi$; these details are routine exercises in principal connections but somewhat cumbersome to write out in full. A more elegant way of dealing with them is through the superspace formalism introduced in [99] which is used, somewhat implicitly, in the proof of [45, Thm.13], but which is beyond the scope of this work. We will provide a further generalisation of this theorem (Theorem 4.44) to the case of filtered deformations and Killing superalgebras with $R$-symmetry in Chapter 4.

We have already noted some differences between the result above and the much cleaner statement in the 11-dimensional case. Another difference to be aware of is that a geometry $(M, g, D)$ does not seem to be determined by the odd-generated ideal $[g_T, g_T] \oplus g_T$ (or weak equivalence class of a maximal Lie pair $(S' = S_{p}, \beta)$) in general since we may have $\mathcal{K}^{2,2}(a_{-}) \neq 0$; the discussion of §3.3.5 shows that it is possible for two realisable subdeformations of $s$ to have the same odd-generated ideal but for neither to embed in the other. On the geometric side, this means that two simply connected backgrounds may be isometric to one another but have different admissible connections $D$ (corresponding to different flux configurations, in supergravity terminology). On the other hand, it is not hard to see that a strong equivalence class of a maximal Lie pairs does determine the connection uniquely.
Chapter 4

Structure of Killing superalgebras with $R$-symmetry

This chapter essentially generalises the framework of the previous chapter to the case of Killing superalgebras “twisted” by $R$-symmetry. It is structured in a similar way, and its main results are direct generalisations of those of Chapter 3. This framework is intended to be applicable to Killing superalgebras in supergravity theories with gauged $R$-symmetry, but as in the previous chapter, much of the discussion is far more general, with some results being independent of choices such as signature. Note, however, that once again we will eventually specialise to Lorentzian signature and a restricted choice of squaring maps, motivated this time by both the Homogeneity Theorem and the need for a compact $R$-symmetry group.

Our overall approach is inspired by, but more general and fully-developed than, parts of [49] which considers the minimal $D = 6$ Lorentzian case. Much of the discussion of spin-$R$ structures in §4.1.1 is a recapitulation of background material from [53], with some additional features due to the “twisting” group being the $R$-symmetry group. Parts of §4.1.3, §4.1.2, §4.2.1 develop and generalise ideas introduced in [49], with the main novel feature being that the structures involved are described in a more natural way via the spin-$R$ structure. To the author’s knowledge, the “covariant Cartan calculus” and symmetry algebra of a spin-$R$ structure (in particular Proposition 4.8) developed here do not appear elsewhere in the literature, at least in their present form, but the former is a natural generalisation of standard concepts and the latter is at least related to the symmetry algebra of a regular Cartan geometry discussed by Čap and Neusser [62]. The first major result of the chapter, Theorem 4.16, is essentially new. As in the last chapter, the other main results, Theorems 4.18, 4.44 and 4.37 are generalisations of those in [45] to a much more general context, with many new technicalities having to be overcome.

4.1 Manifolds with $R$-gauged spinors

Killing superalgebras “with $R$-symmetry”, or gauged Killing superalgebras, arise as the symmetry algebras of bosonic backgrounds of gauged supergravity theories, where in addition to the gravity supermultiplet, there is a gauge supermultiplet for the $R$-symmetry group, and the theory is invariant under both the supersymmetry
transformations of both multiplets and the $R$-gauge transformations. Such algebras were brought into the Spencer cohomology framework in [49]. In that work, two different treatments are given; the results of the work stand on either one individually. One approach, given in [49, §6], takes place at a higher level of geometric formalism which allows one to be more concise but somewhat obscures the gauge-theoretic nature of the construction by assuming a flat connection from beginning, and gives a definition of the Killing superalgebra which is clear but not clearly motivated, while the local approach of [49, §7] makes the gauge theory very explicit and builds a definition of the Killing superalgebra gradually, showing that it is the natural choice. In this chapter, we hope to synthesise the two approaches, motivating the definition in a similar way to the latter but proceeding with the formalism of the former.

4.1.1 Spin-$R$ structures

Throughout this chapter, we fix a model inner product space $(V, \eta) = \mathbb{R}^{p,q}$ and let $\text{SO}(V)$ be the associated special orthogonal group and $\text{Spin}(V)$ its spin cover. We also fix $S$ a (not necessarily irreducible) spinor representation of $\text{Spin}(V)$ with a squaring map $\kappa : \otimes^2 S \to V$. As in §2.2.3, we let $R$ denote the $R$-symmetry group, i.e. the group of $\kappa$-preserving $\text{Spin}(V)$-module automorphisms of $S$, and $\mathfrak{r} = \text{Lie} R$. We will ultimately, of course, be interested in the Lorentzian case with symmetric squaring map, but we keep the discussion general for now.

Letting $(M, g)$ be an oriented pseudo-Riemannian manifold of the same signature as $(V, \eta)$, we denote its special orthonormal frame bundle by $F_{\text{SO}}$. We first seek to find an appropriate geometric setting in which to describe the gauging of the $R$-symmetry. This is a somewhat more intricate process than it may first appear, owing to the fact that the $R$-symmetry is a group action intrinsic to a spinor representation itself, rather than an action on some space in which spinors can take values (an “internal” symmetry) as is the case for the “colour” of quarks in a Yang-Mills theory. Our goal will be to construct a spinor bundle upon which we have (at least locally) both a spin connection and a gauge connection for the $R$-symmetry. We will first describe some more naïve approaches to this, both of which assume we have a spin structure, before generalising to the setting we will actually use.

**Gauged $R$-symmetry from a spin structure only**

For our first construction, let us assume that $(M, g)$ is a spin manifold with spin bundle $P \to M$. Let $S := P \times_{\text{Spin}(V)} S$ with fibre $S$, and let us try to enlarge the structure group of $S$ to $\text{Spin}(V) \times R$. First, note that the inclusion $\text{Spin}(V) \hookrightarrow \text{Spin}(V) \times R$ induces both a left action and a right action of $\text{Spin}(V)$ on $\text{Spin}(V) \times R$ by multiplication on the left and right, respectively. We thus have a bundle morphism

$$P \hookrightarrow P_R := P \times_{\text{Spin}(V)} (\text{Spin}(V) \times R),$$

covering the identity over $M$, where $P_R$ is a $\text{Spin}(V) \times R$-principal bundle, and this morphism is invariant under the right action by $\text{Spin}(V)$ and injective on fibres. But
since Spin(V) acts trivially on R, we find that

$$P_R \cong \left( P \times_{\text{Spin}(V)} \text{Spin}(V) \right) \times R \cong P \times R \quad (4.2)$$

as (Spin(V) × R)-principal bundles over M. Taking the quotient by the Spin(V) action on this bundle, we obtain the trivial principal R-bundle over MR = M × R → M. Considering S as a representation of Spin(V) × R and using properties of balanced products, we have canonical isomorphisms of vector bundles

$$P_R \times_{\text{Spin}(V) \times R} S \cong \left( P \times_{\text{Spin}(V)} \left( \text{Spin}(V) \times R \right) \right) \times_{\text{Spin}(V) \times R} S$$

$$\cong P \times_{\text{Spin}(V)} \left( \left( \text{Spin}(V) \times R \right) \times_{\text{Spin}(V) \times R} S \right)$$

$$\cong P \times_{\text{Spin}(V)} S$$

$$\cong P \times_{\text{Spin}(V) \times R} S \cong S. \quad (4.3)$$

Clearly this approach is too restrictive; we would like to allow for a non-trivial R-principal bundle so that we can have some more interesting gauge theory.

**Gauged R-symmetry from a spin structure and R-principal bundle**

Let P → M be a spin structure once again, but now let Q → M be some R-principal bundle and recall that the fibre product P ×MQ → M is a Spin(V) × R-principal bundle. We then define a twisted spinor bundle by

$$S_Q := (P \times_M Q) \times_{\text{Spin}(V) \times R} S. \quad (4.4)$$

This coincides with S only if Q ≃ MR.

Clearly this a much more interesting scenario, but we can still do better; since our spinors will be “charged” under the R-symmetry, we can actually weaken the assumption that the manifold is admits a spin structure.

**The spin-R group**

It is well-known that a generalisation of spin structures, known as spin-c structures, allows one to overcome topological obstructions to the existence of spin structures at the cost of making the spinors “charged” under a U(1) gauge field. Since we wish our spinors to be “charged” under the R-symmetry, we will make use of a further generalisation known as spin-G (or generalised spin) structures, where U(1) is replaced with a some other (compact) Lie group G. To the author’s knowledge, this was first done for the special case G = SU(2) in [100], and the general theory was first properly explored by [53]. We note that since SU(2) ≃ Sp(1), spin-SU(2) is essentially spin-h (also known as spin-q), the quaternionic analogue of spin-c; these were more comprehensively considered in the mathematical literature by [101]; see also the recent work [102]. See [103] for an extensive treatment of spin-G structures in general relativity. We note that a spin-G structure is still not the most general way to put (s)pinors on a manifold (see [104, 105]), although it will suffice for our purposes.

Recall that Spin(V) has a central Z2 subgroup which is the kernel of the canonical
morphism \( \pi : \text{Spin}(V) \to \text{SO}(V) \):

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}(V) \longrightarrow \text{SO}(V) \xrightarrow{\pi} 1.
\]  

(4.5)

We will denote the non-trivial element of the \( \mathbb{Z}_2 \) subgroup of \( \text{Spin}(V) \) by \(-1\) – indeed, viewing the spin group as a subgroup of the units of the Clifford algebra, this element really is \(-1\) times the identity. Let us denote the spinor representation map by \( \sigma : \text{Spin}(V) \to \text{GL}(S) \). Since \( S \) is a spinor representation, we have \( \sigma(\pm 1) = \pm 1 \), where we also denote the identity in \( \text{GL}(S) \) by \( 1 \). Recall that \( R \) consists of automorphisms of this representation, so \( \sigma(a) r = r \sigma(a) \) for all \( r, a \in \text{Spin}(V) \), and \( \{ \pm 1 \} \subseteq R \).

The subgroup \( \mathbb{Z}_2 \cong \{(\pm 1, \pm 1)\} \subseteq \text{Spin}(V) \times R \) is contained in the kernel of the action morphism \( \text{Spin}(V) \times R \to \text{GL}(S) \) given by \( (a, r) \mapsto \sigma(a) r = r \sigma(a) \), so this action factors through the \textit{spin-R group of } V, defined as

\[
\text{Spin}^R(V) := \text{Spin}(V) \times R / \{(\pm 1, \pm 1)\}.
\]

(4.6)

giving us the following commutative diagram:

\[
\begin{array}{cccccc}
\text{Spin}(V) & \xrightarrow{\sigma} & \text{Spin}(V) \times R & \xrightarrow{\sigma} & \text{Spin}^R(V) & \xrightarrow{\text{GL}(S)} \text{GL}(S).
\end{array}
\]

(4.7)

In particular, we have an action \( \bar{\sigma} \) of \( \text{Spin}^R(V) \) on \( S \) given by \( [(a, r)] \cdot s = \sigma(a) r s = r \sigma(a) s \). Considering an action of \( \mathbb{Z}_2 \) by \( \pm 1 \) on the right of \( \text{Spin}(V) \) and on the left of \( R \), we can also view \( \text{Spin}^R(V) \) as the balanced product \( \text{Spin}(V) \times \mathbb{Z}_2 \times R \). With either description, we thus have a short exact sequence

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}(V) \times R \longrightarrow \text{Spin}^R(V) \longrightarrow 1.
\]

(4.8)

In particular, \( \text{Spin}(V) \times R \to \text{Spin}^R(V) \) is a double cover of \( \text{Spin}^R(V) \). We have two natural injective Lie group morphisms \( \text{Spin}(V) \hookrightarrow \text{Spin}^R(V) \) and \( R \hookrightarrow \text{Spin}^R(V) \) given by \( a \mapsto [a, 1] \) and \( r \mapsto [r, 1] \) respectively, and two surjective morphisms \( \pi : \text{Spin}^R(V) \to \text{SO}(V) \) and \( \text{Spin}^R(V) \to R/\mathbb{Z}_2 \) mapping \( [a, r] \mapsto \pi(a) \) and \( [a, r] \mapsto [r] \) respectively. All of these maps together with the sequence (4.5) fit into the following diagram with
exact rows and columns:

\[
\begin{array}{ccccccccc}
1 & & 1 & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \text{Spin}(V) & \longrightarrow & \pi & \longrightarrow & 1 \\
\downarrow & & \downarrow & & & & & & \\
1 & \longrightarrow & R & \longrightarrow & \text{Spin}^R(V) & \longrightarrow & \tilde{\pi} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & & & & & \\
R/\mathbb{Z}_2 & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & 1 \\
\end{array}
\]

We also have the following commutative diagram:

\[
\begin{array}{ccc}
1 & \longrightarrow & \mathbb{Z}_2 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \text{Spin}(V) \times R \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \text{Spin}^R(V) \\
\downarrow & & \downarrow \\
R/\mathbb{Z}_2 & \longrightarrow & 1 \\
\end{array}
\]

Finally, \(\text{Spin}^R(V)\) is a double cover; we have short exact sequence

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}^R(V) \longrightarrow \text{SO}(V) \times R/\mathbb{Z}_2 \longrightarrow 1
\]

where the surjection is given by \([a, r] \mapsto ([a], [r])\) and the kernel \(\mathbb{Z}_2\) is generated by the element \([1, -1] = [-1, 1] \in \text{Spin}^R(V)\). We thus have a chain of covering group maps

\[
\text{Spin}(V) \times R \longrightarrow \text{Spin}^R(V) \longrightarrow \text{SO}(V) \times R/\mathbb{Z}_2
\]

so the Lie algebras of these three groups are naturally isomorphic:

\[
\text{Lie Spin}^R(V) \cong \text{Lie} (\text{Spin}(V) \times R) \cong \text{Lie} (\text{SO}(V) \times R/\mathbb{Z}_2) \cong \mathfrak{so}(V) \oplus \mathfrak{r}.
\]

Denoting the adjoint action of any group \(G\) by \(\text{Ad}^G\), we have

\[
\text{Ad}^{\text{Spin}^R(V)}_\imath (X, x) = \left( \text{Ad}^{\text{Spin}(V)}_a X, \text{Ad}^R_r x \right) = \left( \text{Ad}^{\text{SO}(V)}_\imath X, \text{Ad}^{R/\mathbb{Z}_2}_\imath x \right)
\]

for all \(a \in \text{Spin}(V), r \in R, X \in \mathfrak{so}(V)\) and \(x \in \mathfrak{r}\).
Spin-\(R\) structures

Let \((M, g)\) be an oriented pseudo-Riemannian manifold and \(F_{SO} \to M\) its frame bundle given the structure of an \(SO(V)\)-principal bundle. Recall that a spin structure on \((M, g)\) is a \(Spin(V)\)-principal bundle \(P \to M\) with a \(Spin(V)\)-equivariant bundle map \(P \to F_{SO}\) where \(Spin(V)\) acts on \(F_{SO}\) via the natural covering group morphism \(\pi : Spin(V) \to SO(V)\). Now note that \(Spin^R(V)\) also acts on \(F_{SO}\) via the natural map \(\tilde{\pi} : Spin^R(V) \to SO(V)\).

**Definition 4.1** (Spin-\(R\) structure). A spin-\(R\) structure\(^1\) on an oriented pseudo-Riemannian manifold \((M, g)\) is a \(Spin^R(V)\)-principal bundle \(\tilde{P} \to M\) with a \(Spin^R(V)\)-equivariant bundle map \(\tilde{\rho} : \tilde{P} \to F_{SO}\).

Two such structures \(\tilde{\rho} : \tilde{P} \to M, \tilde{\rho}' : \tilde{P}' \to M\) are equivalent if there exists an isomorphism of \(Spin^R(V)\)-principal bundles \(\Phi : \tilde{P} \to \tilde{P}'\) such that the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{P} & \xymatrix{ & \Phi} & \tilde{P}' \\
\tilde{\rho} & F_{SO} & \tilde{\rho}'
\end{array}
\]

Unlike a spin structure, a spin-\(R\) structure is not a covering of the principal bundle since one can easily see that the fibre of the map \(\tilde{\rho} : \tilde{P} \to F_{SO}\) is diffeomorphic to \(R\); in fact, \(\tilde{\rho}\) is an \(R\)-principal bundle on \(F_{SO}\) where \(R\) acts on \(P\) via the natural homomorphism \(R \leftarrow Spin^R(V)\). On the other hand, the homomorphism \(Spin(V) \to Spin^R(V)\) gives an action of the spin group on \(\tilde{P}\), and the quotient by this action gives us a \(Spin^R(V)\)-equivariant bundle map \(\tilde{P} \to \tilde{Q}\), where the latter is the principal \(R/\mathbb{Z}_2\)-bundle \(\tilde{Q} = \tilde{P}/Spin(V) \to M\) (the notation will become clear shortly). We thus have the following commutative diagram of \(Spin^R(V)\)-equivariant bundle maps:

\[
\begin{array}{ccc}
\tilde{P} & \xymatrix{ & \tilde{Q} \\
\tilde{\rho} & Spin^R(V) & \Phi
\end{array}
\]

where each arrow gives a principal bundle structure over the target with and is labelled with the structure group.

Now let \(\tilde{\rho} : P \to F_{SO}\) be a spin structure and \(Q\) a principal \(R\)-bundle. Then there is a spin-\(R\) structure

\[
\tilde{\rho} : (P \times_M Q)/\mathbb{Z}_2 \to F_{SO}
\]

(4.17)

where \(\mathbb{Z}_2\) is understood to be the diagonal subgroup of \(Spin(V) \times R\). The action of \(Spin^R(V)\) on \((P \times_M Q)/\mathbb{Z}_2\) making it into a principal bundle is \([p, q] \cdot [a, r] = [p \cdot a, q \cdot r]\), and \(\tilde{\rho}\) is given by \(\tilde{\rho}(p, [q]) = \tilde{\rho}(p)\). Note that there are canonical isomorphisms of

\[^1\text{We note that our notion of spin-}\(R\) this is closely related to but distinct from the notion of generalised spin-}\(r\), or spinorially twisted spin structures of e.g. \([106, 107]\), which are spin-\(G\) structures with \(G = \text{Spin}(r)\) for \(r \in \mathbb{N}\), which produces spin-\(c\) and spin-\(h\) for \(r = 1, 2\) respectively. See the latter reference for such structures on Riemannian homogeneous spaces.
SO(V)-bundles $\tilde{P}/R \cong P/Z_2 \cong F_{SO}$ and a similar canonical isomorphism of $R/Z_2$-bundles
\[ \overline{Q} = \tilde{P}/\text{Spin}(V) \cong Q/Z_2 \] (4.18)
given by sending $[p, q] \to [q]$, and we can extend the diamond above to the following diagram:
\[ \begin{array}{ccc}
P \xrightarrow{z_2} & F_{SO} \\
\downarrow & & \downarrow \\
\text{Spin}(V) \xrightarrow{z_2} & \tilde{P} \xrightarrow{\text{Spin}^R(V)} & M \\
\downarrow & & \downarrow \\
Q \xrightarrow{z_2} & Q \xrightarrow{R/Z_2} & \\
\end{array} \] (4.19)
where again the arrows represent bundle structures and are labelled by the structure group.

**Definition 4.2** (Reducible spin-$R$ structure). A spin-$R$ structure $\tilde{\omega}: \tilde{P} \to F_{SO}$ is reducible if it admits a lift of the structure group of $\tilde{P}$ to $\text{Spin}(V) \times R$.

That is, $\tilde{P} \to M$ is reducible if there exists a $\text{Spin}(V) \times R$-principal bundle $\tilde{P}$ and a $\text{Spin}(V) \times R$-equivariant bundle map $\Phi: \tilde{P} \to \tilde{P}$. Clearly, the spin-$R$ structure in the diagram (4.19) is reducible. In fact, the converse is also true.

**Proposition 4.3.** A spin-$R$ structure $\tilde{\omega}: \tilde{P} \to F_{SO}$ is reducible if and only if it is equivalent to $(P \times_M Q)/Z_2 \to F_{SO}$ for a spin structure $\omega: P \to M$ and an $R$-principal bundle $Q \to M$.

**Proof.** Clearly if $\tilde{P} := (P \times_M Q)/Z_2$ then the covering map $P \times_M Q \to \tilde{P}$ is the required lift of the structure group. On the other hand, if $\Psi: \tilde{P} \to \tilde{P}$ is a lift of the structure group, we can define a $\text{Spin}(V)$-principal bundle $P := \tilde{P}/R$ and an $R$-principal bundle $Q := \tilde{P}/\text{Spin}(V)$ over $M$.

For any element $p \in P$, choose a lift $\tilde{p} \in \tilde{P}$; note that any other lift of $p$ is of the form $\tilde{p} \cdot (1, r)$ for some $r \in R$, and by the equivariance of the maps,
\[ \tilde{\omega}(\Psi(\tilde{p} \cdot (a, r))) = \tilde{\omega}(\Psi(\tilde{p}) \cdot [a, r]) = \tilde{\omega}(\Psi(\tilde{p})) \cdot a; \] (4.20)
for all $a \in \text{Spin}^R(V)$ and $r \in R$; in particular, $\omega(p) := \tilde{\omega}(\Psi(\tilde{p}))$ is independent of the choice of lift and thus defines a bundle map $\omega: P \to F_{SO}(V)$. The calculation above also shows that $\tilde{\omega}$ is Spin-$R$-equivariant since $\tilde{p} \cdot (a, 1)$ is a lift of $p \cdot [a, 1]$, so $\tilde{\omega}$ is a spin structure. Thus $\tilde{\omega}': P \times_M Q/Z_2 \to F_{SO}$ given by $[p, q] \mapsto \tilde{\omega}(p)$ is a spin-$R$ structure; we will show that there exists an equivalence of spin-$R$ structures $\Phi: \tilde{P} \to P \times_M Q/Z_2$.

It is not difficult to see that $\tilde{\Phi}: \tilde{P} \to P \times_M Q$ given by $\tilde{p} \mapsto ([\tilde{p}], [\tilde{p}])$ is an isomorphism of $\text{Spin}(V) \times R$-principal bundles. We can construct $\Phi$ in an analogous way to the construction of $\tilde{\omega}$ above. Indeed, fix $\tilde{p} \in \tilde{P}$ and let $\tilde{p} \in \tilde{P}$ be a lift ($\Psi(\tilde{p}) = \tilde{p}$). Any other lift is of the form $\tilde{p} \cdot (\pm 1, 1)$, and in $P \times_M Q/Z_2$ we have
\[ [\tilde{\Phi}(\tilde{p} \cdot (a, r))] = [\tilde{\Phi}(\tilde{p}) \cdot (a, r)] = [\check{\Phi}(\tilde{p})] \cdot [a, r] \] (4.21)
so since $[\pm 1, 1] = 1$ in $\text{Spin}^R(V)$, we see that $\Phi(\tilde{p}) := [\check{\Phi}(\tilde{p})]$ is well-defined. Clearly this defines as bundle map $\Phi$, and it is Spin-$R(V)$-equivariant since $\tilde{p} \cdot (a, r)$ is a lift of
if $\hat{\rho} \cdot [a, r]$, whence it is an isomorphism of principal bundles. It remains only to show that $\hat{\omega}' \circ \Phi = \hat{\omega}$. We once again take $\Psi(\hat{\rho}) = \hat{\rho}$ and, using in order the definitions of $\Phi, \hat{\Phi}, \hat{\omega}'$ and $\hat{\omega}$,

$$\hat{\omega}'(\Phi(\hat{\rho})) = \hat{\omega}(|\Psi(\hat{\rho})|) = \hat{\omega}([\hat{\rho}], [\hat{\rho}]) = \hat{\omega}(\hat{\rho}) = \hat{\omega}(\hat{\rho}),$$  \hspace{1cm} (4.22)

for all $\hat{\rho} \in \hat{P}$, as required. \hfill $\square$

**Representations of Spin$^R(V)$ and their associated bundles**

With our definition of spin-$R$ structures in hand, we can finally define a bundle of spinors associated to the spinor representation $S$;

$$\hat{S} := \hat{P} \times_{\text{Spin}^R(V)} S \hspace{1cm} (4.23)$$

where $\hat{P}$ is the spin-$R$ structure. If the spin-$R$ structure is reducible so that $\hat{P} \cong (P \times_M Q)/\mathbb{Z}_2$, this coincides with $S_Q$, and it coincides with $S$ if $Q = M_R$.

As with spin structures, many other natural bundles over $M$ can be described as associated bundles to $\hat{P}$, essentially lifting their structure group from $\text{SO}(V)$ to Spin$^R(V)$. Before discussing this in detail, we take a short detour to discuss the representation theory of Spin$^R(V)$.

Since Spin$^R(V)$ is a quotient of Spin$(V) \times R$, any representation of it pulls back to a representation of Spin$(V) \times R$ and thus of both Spin$(V)$ and $R$. Conversely, two representations $\rho_1 : \text{Spin}(V) \to \text{GL}(W)$ and $\rho_2 : R \to \text{GL}(W)$ on the same space $W$ assemble into a representation $\rho_1 \times \rho_2$ of Spin$(V) \times R$ given by $(\rho_1 \times \rho_2)(a, r) = \rho_1(a)\rho_2(r)$ if and only if $\rho_1(a) \circ \rho_2(r) = \rho_2(r) \circ \rho_1(a)$ for all $a \in \text{Spin}(V)$ and $r \in R$, and the latter factors through a representation $\rho : \text{Spin}^R(V) \to \text{GL}(W)$ if and only if the kernel of the quotient map to Spin$^R(V)$ acts trivially; that is, if

$$\rho_1(-1) \circ \rho_2(-1) = \rho_2(-1) \circ \rho_1(-1) = \text{Id}_W, \hspace{1cm} (4.24)$$

or equivalently $\rho_2(-1) = \rho_1(-1)^{-1}$. On the other hand,

$$\rho_1(-1)^2 = \rho_1((-1)^2) = \rho_1(1) = \text{Id}_W, \hspace{1cm} (4.25)$$

so by uniqueness of inverses, the condition (4.24) is equivalent to

$$\rho_1(-1) = \rho_2(-1). \hspace{1cm} (4.26)$$

This is satisfied in particular if

$$\rho_1(-1) = \rho_2(-1) = \pm \text{Id}_W. \hspace{1cm} (4.27)$$

Such representations will be particularly important (and we will show below that they essentially give us all representations of Spin$^R(V)$), so we make the following definition.

**Definition 4.4.** A representation $\rho$ of Spin$^R(V)$ is even if $\rho_1(-1) = \rho_2(-1) = \text{Id}_W$ and it is odd if $\rho_1(-1) = \rho_2(-1) = -\text{Id}_W$. 

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For example, $S$ is odd, while $\hat{\pi} : \text{Spin}^R(V) \to \text{SO}(V)$ gives us an even representation on $V$ in which $R$ acts trivially, and the adjoint representation of $\text{Spin}^R(V)$ is also even. Since

$$\rho_1(\mp) = \Id_W \Longleftrightarrow \rho_1 \text{ factors through a representation of } \text{SO}(V),$$

$$\rho_2(\mp) = \Id_W \Longleftrightarrow \rho_2 \text{ factors through a representation of } R/\mathbb{Z}_2,$$

even representations of $\text{Spin}^R(V)$ are those which factor through $\text{SO}(V) \times (R/\mathbb{Z}_2)$. Not all representations of $\text{Spin}^R(V)$ have definite parity; for example, on $V \oplus S$, $\rho_1(\mp) = \rho_2(\mp) = (\Id_V, -\Id_S)$. Nonetheless, by the following result, it is sufficient to study the representations of definite parity.

**Lemma 4.5.** Any representation of $\text{Spin}^R(V)$ is a direct sum of even and odd representations. In particular, irreducible representations of $\text{Spin}^R(V)$ are either even or odd.

**Proof.** Since $\rho_1(\mp) = \rho_2(\mp)$ is an involution of $W$ ($\rho_1(\mp)^2 = \Id_W$), $\rho_1(\mp)$ has eigenvalues $\pm 1$, and $W = W_+ \oplus W_-$, where $W_\pm$ is the $\pm 1$ eigenspace of $\rho_1(\mp)$, upon which the involution acts as $\pm \Id_{W_\mp}$. Since $\mp$ is central in both $\text{Spin}(V)$ and $R$, it follows that $\rho_1$ and $\rho_2$ preserve $W_\pm$, hence so does $\rho$. Thus if $\rho$ is irreducible, we have $W = W_+$ or $W = W_-$. \(\square\)

We thus have a natural $\mathbb{Z}_2$-grading on representations of $\text{Spin}^R(V)$; morphisms and tensor products of representations respect parity in the obvious ways, whence representations of $\text{Spin}^R(V)$ form a monoidal subcategory of the category of super vector spaces.

We will also assign a parity to associated vector bundles to a spin structure by saying that such a bundle is odd (or even) if the representation that induces it is. The parity of associated bundles is important for gauge theory; if $W$ is even then we have a natural isomorphism of vector bundles

$$\hat{P} \times_{\text{Spin}^R(V)} W \cong (F_{\text{SO}} \times_M \overline{Q}) \times_{\text{SO}(V) \times R/\mathbb{Z}_2} W \quad (4.28)$$

where we recall that $F_{\text{SO}}$ is the orthonormal frame bundle and $\overline{Q} = \hat{P}/\text{Spin}(V)$ is a $R/\mathbb{Z}_2$-principal bundle, while if $W$ is odd, no such isomorphism exists. Note that if either $R$ or $\text{Spin}(V)$ act trivially on $W$, we have

$$\hat{P} \times_{\text{Spin}^R(V)} W \cong F_{\text{SO}} \times_{\text{SO}(V)} W \quad (4.29)$$

or

$$\hat{P} \times_{\text{Spin}^R(V)} W \cong Q \times_{R/\mathbb{Z}_2} W \quad (4.30)$$

respectively. In particular,

$$\hat{P} \times_{\text{Spin}^R(V)} V \cong TM \quad (4.31)$$

and similarly for $T^*M$, $\wedge^* T^*M$, etc. Particularly important for connections on a spin structure, to be discussed soon, is the isomorphism

$$\hat{P} \times_{\text{Spin}^R(V)} (\mathfrak{so}(V) \oplus \mathfrak{r}) \cong (F_{\text{SO}} \times_{\text{SO}(V)} \mathfrak{so}(V)) \oplus_M \left( \overline{Q} \times_{R/\mathbb{Z}_2} \mathfrak{r} \right) \quad (4.32)$$
where we use the adjoint representations of Spin$^R(V)$, SO($V$) and $R/\mathbb{Z}_2$; that is,
\begin{equation}
\text{ad} \hat{P} \cong \text{ad} F_{SO} \oplus \text{ad} Q.
\end{equation}

Action of Spin$^R(V)$ by conjugation on (linear) endomorphisms $\text{End}(W)$ of any representation $W$ induces
\begin{equation}
\hat{P} \times \text{Spin}^R(V) \text{End}(W) \cong \text{End} \left( \hat{P} \times \text{Spin}^R(V) W \right)
\end{equation}
where the latter is the bundle of fibrewise endomorphisms; in particular, the Spin$^R(V)$-equivariant embedding $\mathfrak{so}(V) \oplus \mathfrak{r} \hookrightarrow \text{End} S$ induces a vector bundle embedding
\begin{equation}
\text{ad} \hat{P} \cong \text{ad} F_{SO} \oplus \text{ad} Q \hookrightarrow \text{End} \hat{S}.
\end{equation}

\section*{R-symmetry as tensoring with an R-module}

It is often the case that $S = \Sigma \otimes \Delta$ where $\Sigma$ is an irreducible spinor representation of Spin($V$) and $\Delta$ is some representation of $R$, usually the fundamental representation $\mathbb{R}^N$, $C^N$ or $\mathbb{H}^N$ where $R = \text{SO}(N)$, $U(N)$ or $\text{USp}(2N)$; in particular $-1 \in R$ acts on $\Delta$ as $-\text{Id}$, so $S = \Sigma \otimes \Delta$ is an odd representation of Spin$^R(V)$ where the action is given by $[a, r] \cdot (s \otimes w) = (\sigma(a)s) \otimes (r \cdot w)$.

Then if $\varpi : P \to F_{SO}$ is a spin structure and $Q \to M$ a principal $R$-bundle, we can form the associated bundles $\Sigma = P \times_{\text{Spin}(V)} \Sigma$ and $\Delta = Q \times_R \Delta$ and show that there is an isomorphism of vector bundles
\begin{equation}
\hat{S}_Q \cong (P \times M Q) \times_{\text{Spin}^R(V)} (\Sigma \otimes \Delta) \cong \Sigma \otimes_M \Delta
\end{equation}
where the latter is the tensor product of vector bundles.

If we have a spin-$R$ structure $\hat{\varpi} : \hat{P} \to F_{SO}$ which is not reducible, we can still form the associated bundle $\hat{S} = \hat{P} \times_{\text{Spin}^R(V)} S$; however, this does not decompose as a tensor product of bundles with fibres $\Sigma$ and $\Delta$ in general.

### 4.1.2 Connections, sections and derivatives

We now let $\hat{\mathcal{S}} = \Gamma(\hat{S})$ denote the space of spinor fields. Since $\hat{S}$ is an associated bundle, there is a natural $C^\infty(M)$-module isomorphism $\hat{\mathcal{S}} \cong C^\infty_{\text{Spin}^R(V)}(\hat{P}; S)$ where the latter is the space of Spin$^R(V)$-equivariant smooth functions from $\hat{P}$ to $S$. We similarly have $\mathcal{X}(M) \cong C^\infty_{\text{Spin}^R(V)}(\hat{P}; V)$, etc.

#### Connections

Let $\hat{\varpi} : \hat{P} \to F_{SO}$ be a spin-$R$ structure and $\hat{S}$ the associated “charged” spinor bundle. Recall that $\text{LieSpin}^R(V) \cong \mathfrak{so}(V) \oplus \mathfrak{r}$ and that its adjoint representation is given by equation (4.14). A (principal or Ehresmann) connection on $\hat{P}$ is a Lie algebra-valued 1-form $\mathcal{A} \in \Omega^1(\hat{P}; \mathfrak{so}(V) \oplus \mathfrak{r})$ satisfying
\begin{equation}
R^*_{[a, r]} \mathcal{A} = \text{Ad}^{\text{Spin}^R(V)}_{[a, r]^{-1}} \mathcal{A}
\end{equation}
Spencer cohomology and Killing superalgebras

\[ \mathcal{A}(\xi_{(X,x)}) = (X, x) \]

for all \( a \in \text{Spin}(V), \ r \in \mathbb{R}, \ X \in \mathfrak{so}(V) \) and \( x \in \mathfrak{t} \). Here \( \xi_{(X,x)} \in \mathfrak{X}(\hat{P}) \) denotes the fundamental vector field generated by \((X,x)\). Letting \( \pi_Q : \hat{P} \to \hat{Q} = \hat{P} / \text{Spin}(V) \) denote the canonical map, one can easily show that

\[ \mathcal{A} = \hat{\omega}^* \omega + \pi_Q^* \alpha, \quad (4.37) \]

where \( \omega \in \Omega^1(F_{SO}; \mathfrak{so}(V)), \ \alpha \in \Omega^1(\hat{Q}, \mathfrak{r}) \) are principal connections, and conversely, any two such connections on \( F_{SO} \) and \( \hat{Q} \) induce a connection on \( \hat{P} \). In what follows, we will demand that \( \omega \) is the Levi-Civita connection. Equivalently, can assign a torsion to \( \mathcal{A} \) using isomorphism of vector bundles

\[ \hat{P} \times_{\text{Spin}^R(V)} V \cong TM, \quad (4.38) \]

and demand that this torsion vanishes. It is easy to see that this torsion is equal to the torsion of the associated metric connection \( \omega \), so vanishes if and only if \( \omega \) is the Levi-Civita connection.

Finally, we note that considered as 2-forms with values in the appropriate adjoint bundles, the curvatures satisfy

\[ F_{\mathcal{A}} = F_\omega + F_\alpha, \quad (4.39) \]

where we implicitly use the isomorphism of vector bundles \( \text{ad} \hat{P} \cong \text{ad} F_{SO} \oplus \text{ad} \hat{Q} \). If \( \omega \) is the Levi-Civita connection, \( F_\omega \) is equal to the Riemann tensor.

The principal connection \( \mathcal{A} \) on \( \hat{P} \) induces Koszul connections on associated bundles, in particular on the spinor bundle \( \hat{S} \), which we will denote by \( \hat{\nabla} \). On even vector bundles on which the action of \( \mathfrak{r} \) is trivial, in particular on \( TM \), \( \hat{\nabla} = \nabla \) where the latter is the Levi-Civita connection. More generally, we may locally abuse notation and write \( \hat{\nabla} = \nabla + \alpha \), where \( \alpha \) is the local pull-back of the connection on \( \hat{Q} \), even on odd associated bundles where neither \( \nabla \) nor the action of \( \alpha \) are globally defined.

**Local description and some formulae**

Let us make the local description of the connections discussed above more explicit. Making manifest the indices of the \( \mathfrak{r} \)-symmetry in some basis and of the tangent space some local orthonormal frame, on a spinor field \( \epsilon \in \hat{S} \), we have

\[ \hat{\nabla}_\mu \epsilon^A = \nabla_\mu \epsilon^A + \alpha_\mu^A \beta^B \epsilon^B, \quad (4.40) \]

where \( \alpha_\mu \) acts on \( \epsilon \) via the action of \( \mathfrak{r} \) on \( S \). Under a local gauge transformation \( \theta : U \to \mathfrak{r} \), we have

\[ \epsilon \to \theta \epsilon, \quad \alpha_\mu \to \text{Ad}_\theta \alpha_\mu - (\partial_\mu \theta) \theta^{-1}, \quad (4.41) \]

whence \( \hat{\nabla}_\mu \epsilon \to \theta \hat{\nabla}_\mu \epsilon \). We will depart slightly from the notation used for curvature in the discussion above, using \( \hat{R} \) for the curvature 2-form of \( \hat{\nabla} \) (which is simply the
representation of $F_{\alpha\beta}$ on $\mathfrak{g}$, $R$ for the Riemann curvature 2-form\(^2\) and $F$ for the curvature of $\alpha$. Thus
\begin{align*}
\hat{R}(X, Y) &= [\hat{\nabla}_X, \hat{\nabla}_Y] - \hat{\nabla}_{[X,Y]}, \\
R(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \\
F(X, Y) &= d\alpha + \frac{1}{2}[\alpha \wedge \alpha],
\end{align*}
(4.42)
where we note that although the LHSs and the RHS the first equation are defined globally, the RHS of the second equation is well-defined globally only on sections of even associated bundles or in local trivialisations, and the third equation is valid only in local trivialisations. The representation of equation (4.39) on $\mathfrak{g}$ gives
\begin{align*}
\hat{R}(X, Y) e &= R(X, Y) \cdot e + F(X, Y) e
\end{align*}
(4.43)
for all $X, Y \in \mathfrak{iso}(M)$ and $e \in \mathfrak{g}$.

**Covariant Lie derivative**

In Section 3.1, we saw that for Killing vectors $X$, $A_X = -\nabla X \in \mathfrak{so}(M, g)$ where the latter is the Lie algebra of skew-symmetric sections of $\text{End}(TM)$, which can also be identified with the space of sections of $\text{ad} F_{SO}$. Since $\mathfrak{so}(V)$ is an even representation of $\text{Spin}^R(V)$ on which $R$ acts trivially, there are natural isomorphisms
\begin{align*}
\hat{P} \times \text{Spin}^R(V) \mathfrak{so}(V) \cong F_{SO} \times \text{SO}(V) \mathfrak{so}(V) = \text{ad} F_{SO},
\end{align*}
(4.44)
thus we can consider $A_X$ to be a section of $\hat{P} \times \text{Spin}^R(V) \mathfrak{so}(V)$, so it naturally acts pointwise on $\mathfrak{g}$. We can thus define the **covariant Lie derivative** $\hat{\mathcal{L}}_X : \mathfrak{g} \to \mathfrak{g}$ along a Killing vector $X$ by
\begin{align*}
\hat{\mathcal{L}}_X &= \hat{\nabla}_X + A_X.
\end{align*}
(4.45)
In fact, this formula defines the covariant Lie derivative on sections of any associated bundle to $\hat{P}$ and agrees with the ordinary Lie derivative on even bundles for which $R$ acts trivially on the fibre (in particular on $TM$). It satisfies the Leibniz rule with respect to any actions of sections on other sections which we will encounter, as well as the following.

**Proposition 4.6.** The covariant Lie derivative described above satisfies the following properties:
\begin{align*}
\left[ \hat{\mathcal{L}}_X, \hat{\nabla}_Y \right] &= \hat{\nabla}_{[X,Y]} + F(X, Y) \\
\text{for } X \in \mathfrak{iso}(M, g) \text{ and } Y \in \mathfrak{X}(M) \text{ (equivalently, } \left[ \hat{\mathcal{L}}_X, \hat{\nabla} \right] = i_X F \text{ for all } X \in \mathfrak{iso}(M, g) \text{) and}
\end{align*}
(4.46)
\begin{align*}
\left[ \hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y \right] &= \hat{\mathcal{L}}_{[X,Y]} + F(X, Y)
\end{align*}
(4.47)
for $X, Y \in \mathfrak{iso}(M, g)$.

\(^2\)See Remark 3 for comments on the action of the Riemann tensor on spinor bundles which still hold at least locally here.

\(^3\)Note that the $F(X, Y)$ terms in these equations are to be understood as operators acting on sections of an associated bundle via the appropriate representation of $\tau$. 
Proof. Recall that $\nabla_Y A_X = R(X, Y)$ (equation (3.7)), so

$$
\bigl[ \mathcal{L}_X, \nabla_Y \bigr] = \bigl[ \nabla_X, \nabla_Y \bigr] + [A_X, \nabla_Y] = \bigl[ \nabla_X, \nabla_Y \bigr] - \nabla_Y A_X
$$

$$
= \nabla_{[X,Y]} + R(X, Y) - R(X, Y)
$$

$$
= \nabla_{[X,Y]} + F(X, Y).
$$

(4.48)

Noting that $\mathcal{L}_X A_Y = \nabla_X A_Y + [A_X, A_Y] = -R(X, Y) + [A_X, A_Y] = A_{[X,Y]}$, where we have used equation (3.9), and using the identity derived above, we have

$$
\bigl[ \mathcal{L}_X, \nabla_Y \bigr] = \bigl[ \nabla_X, \nabla_Y \bigr] + [\mathcal{L}_X, A_Y] = \nabla_{[X,Y]} + F(X, Y) + \mathcal{L}_X A_Y
$$

$$
= \nabla_{[X,Y]} + F(X, Y) + A_{[X,Y]}
$$

$$
= \mathcal{L}_{[X,Y]} + F(X, Y).
$$

(4.49)

The third equation follows directly from the Leibniz rule.

Covariant Cartan calculus

Recall that the choice of connection $\mathcal{A}$ on $\hat{P}$ induces not just a covariant derivative on all associated bundles, but a covariant exterior derivative (which we will denote by $\hat{d}$) on differential forms with values in an associated bundle. Regarding $\hat{R}$ as a 2-form with values in $\Omega^2(M; \text{ad}\hat{P})$ and $F$ as a 2-form with values in $\Omega^2(M; \text{ad}\hat{Q} \cong \hat{P} \times_{\text{Spin}^R(V)} \mathfrak{t})$, the Bianchi identities can be rendered $\hat{d}\hat{R} = 0$ and $\hat{d}F = 0$.

We recall that the ordinary Lie derivative, exterior derivative and interior derivative (contraction) on differential forms satisfy a number of identities collectively known as Cartan calculus which are extremely useful for computations. We will now show that our covariant derivatives satisfy a similar covariant Cartan calculus when acting on differential forms with values in bundles associated to the spin-$R$ structure.

**Proposition 4.7** (Covariant Cartan calculus). Let $W$ be a representation of $\text{Spin}^R(V)$ with associated bundle $\underline{W} = \hat{P} \times_{\text{Spin}^R(V)} W$. Then we have the following Cartan formula for the covariant Lie derivative along any Killing vector $X$ when acting on $\Omega^\bullet(M; \underline{W})$:

$$
\mathcal{L}_X = t_X \hat{d} + \hat{d} t_X.
$$

(4.50)

We also have the following identities of operators on $\Omega^\bullet(M; \underline{W})$:

$$
\bigl[ \mathcal{L}_X, \hat{d} \bigr] = t_X F \wedge,
$$

(4.51)

for all $X \in \text{iso}(M, g)^4$;

$$
\bigl[ \mathcal{L}_X, t_Y \bigr] = t_{[X,Y]},
$$

(4.52)

\[4\]Here and elsewhere, the wedge of a form with values in $\text{ad}\hat{Q}$ (or locally in $\mathfrak{t}$) with a form with values in $W$ is understood to include the action of the former on the latter via the representation of $\mathfrak{t}$ on $W$. 

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for all \( X \in \text{iso}(M, g) \) and \( Y \in \mathfrak{X}(M) \):

\[
\left[ \mathcal{L}_X, \mathcal{L}_Y \right] = \mathcal{L}_{[X,Y]} + F(X,Y),
\]

(4.53)

for all \( X, Y \in \text{iso}(M, g) \).

**Proof.** The covariant Cartan formula can be proven in a number of ways. We will show it locally, using the Cartan formula for the ordinary Lie derivative. For \( \omega \in \Omega^k(M; W) \), in a local trivialisation, we treat \( \omega \) as taking values in \( W \) and the local gauge 1-form \( \alpha \) as taking values in \( \mathfrak{g} \) and compute

\[
\mathcal{L}_X \omega = \mathcal{L}_X \omega + (\imath_X \alpha) \cdot \omega \\
= \imath_X d \omega + d \imath_X \omega + (\imath_X \alpha) \cdot \omega \\
= \imath_X d \omega + \hat{d} \imath_X \omega - \imath_X (\alpha \wedge \omega) + (\imath_X \alpha) \cdot \omega - \alpha \wedge \imath_X \omega,
\]

where the cancellation in the last line follows by a basic property (essentially a Leibniz identity) of the contraction. The next identity follows from the Cartan formula and the standard identity \( \hat{d}^2 = F \wedge (\text{which one can verify by locally writing } \hat{d} = d + \alpha \wedge) \):

\[
\left[ \mathcal{L}_X, \hat{d} \right] \omega = \imath_X \hat{d}^2 \omega + \hat{d} \imath_X d \omega - \hat{d} \imath_X \omega - \hat{d}^2 \imath_X \omega \\
= \imath_X (F \wedge \omega) - F \wedge \imath_X \omega \\
= (\imath_X F) \wedge \omega,
\]

(4.55)

once again using the Leibniz identity for the contraction. The next identity is nothing but a Leibniz identity for the covariant Lie derivative. We already saw the final identity as part of Proposition 4.6, but we provide an alternative proof when acting on associated bundle-valued forms. Indeed, using all of the previous identities, we have

\[
\left[ \mathcal{L}_X, \mathcal{L}_Y \right] \omega = \left[ \mathcal{L}_X, \imath_Y \hat{d} \right] \omega + \left[ \mathcal{L}_X, \hat{d} \imath_Y \right] \omega \\
= \imath_Y \left[ \mathcal{L}_X, \hat{d} \right] \omega + \left[ \mathcal{L}_X, \imath_Y \right] d \omega + \hat{d} \left[ \mathcal{L}_X, \imath_Y \right] \omega + \left[ \mathcal{L}_X, \hat{d} \right] \imath_Y \omega \\
= \imath_Y \left( (\imath_X F) \wedge \omega \right) + \imath_{[X,Y]} \hat{d} \omega + \hat{d} \imath_{[X,Y]} \omega + (\imath_X F) \wedge \imath_Y \omega \\
= \mathcal{L}_{[X,Y]} \omega + F(X,Y) \cdot \omega
\]

(4.56)

for all Killing vectors \( X, Y \). \( \Box \)

**Equivariant maps of representations and squaring maps**

Let \( \phi : W_1 \rightarrow W_2 \) be a \( \text{Spin}^{\mathbf{R}}(V) \)-equivariant map of representations; equivalently, \( \psi \) is an invariant element of \( W_1^+ \otimes W_2 \). This induces a map of associated bundles

\[
\underline{\phi} : \underline{W}_1 \rightarrow \underline{W}_2
\]

(4.57)

given by \( \phi([p, w]) = [p, \phi(w)] \). We also denote the corresponding section of the bundle \( \mathcal{H} \text{Hom}(\underline{W}_1, \underline{W}_2) = W_1^+ \otimes M \underline{W}_2 \) by \( \phi ; \) for any \( x \in M \), any element of the fibre \( (\underline{W}_1)_x \) is of the form \( [p, w] \) where \( p \in \mathcal{P} \) is a lift of \( x \) and \( w \in W \), and we have
\[ \phi([p, w]) = [p, \phi(w)]. \]

Recall that, as \( C^\infty(M) \)-modules,

\[
\Gamma(\text{Hom}(W_1, W_2)) \cong C^\infty_{\text{Spin}^R(V)}(P; \text{Hom}(W_1, W_2)) = C^\infty_{\text{Spin}^R(V)}(P; W_1^* \otimes W_2);
\]  

under this isomorphism, the section \( \phi \) corresponds to the constant map \( p \mapsto \phi \), so

\[
\nabla \phi = d\phi = d\phi + \mathcal{A} \cdot \phi = 0.
\]

By invariance of \( \phi \), for \( X \in \text{iso}(M, g) \) we also have \( A_X \cdot \phi = 0 \), and so

\[
\nabla_X \phi = \nabla_X \phi + A_X \cdot \phi = 0.
\]

Since the squaring map \( \kappa : \mathcal{B}^2 S \to V \) is \( \text{Spin}^R(V) \)-equivariant, it induces a bundle map \( \kappa : \mathcal{B}^2 \mathcal{S} \to TM \), and the corresponding section \( \kappa \) satisfies \( \nabla \kappa = 0 \), and \( \nabla_X \kappa = 0 \) for all \( X \in \text{iso}(M, g) \). For ease of notation, outside of this discussion we will omit the underline from the sections \( \kappa \) and \( \phi \); this should not cause confusion. We note that this discussion provides a generalisation of Lemma 3.4.

### 4.1.3 The symmetry algebra of a spin-\( R \) structure

The even parts of the Killing superalgebras described in Chapter 3 consists of a subset of the Killing vectors which is closed under the Lie bracket; that is, a subalgebra of the isometry algebra \( \text{iso}(M, g) \). In this chapter, we seek to find an appropriate analogue of \( \text{iso}(M, g) \) in the spin-\( R \) setting, and we will later define the even part of the Killing superalgebra to be a subalgebra of this structure. We will call this structure the symmetry algebra of the spin-\( R \) structure. We note that this symmetry algebra can be described quite independently of the existence of any Killing superalgebra, however, and even where the latter exists we will see that the full symmetry algebra has a richer structure than the subalgebra which will form the even part of the Killing superalgebra.

#### The symmetry algebra

We begin by identifying some Lie algebras of sections on \( M \) which (one might expect to) have natural actions on the space of spinor fields \( \mathcal{S} \). Two such algebras present themselves in our geometric setting: the Killing vectors \( X \in \text{iso}(M, g) \) (via the covariant Lie derivative) and sections of \( \text{ad} \bar{Q} = \bar{P} \times_{\text{Spin}^R(V)} \mathfrak{r} \), that is, the (infinite-dimensional) Lie algebra of infinitesimal gauge transformations on \( \bar{Q} \). Let us denote the space of such sections by \( \mathfrak{r} \); we will refer to them as “infinitesimal \( R \)-symmetries”, or “\( \mathfrak{r} \)-symmetries” for short, when convenient. Locally, these sections are functions with values in \( \mathfrak{r} \).

One might naively expect that the direct product Lie algebra \( \text{iso}(M, g) \oplus \mathfrak{r} \) or some subalgebra thereof is the algebra we seek, however we note that by (4.47), the action of \( \text{iso}(M, g) \) on \( \mathcal{S} \) via the covariant Lie derivative does not define a representation, so we must choose a modified bracket on \( \text{iso}(M, g) \oplus \mathfrak{r} \). We will denote this bracket by \( [[-,-]] \) in order to distinguish it from the ordinary Lie bracket of vector fields and other Lie brackets and commutators.
Let us first consider a natural non-trivial bracket between $\mathfrak{iso}(M, g)$ and $\mathfrak{R}$ given by the covariant Lie derivative. Such a derivative exists since $a \in \mathfrak{R}$ is a section of an associated bundle to $\hat{P}$. Since $A_X \cdot a = [A_X, a] = 0$ (where the latter commutator is taken pointwise in $\text{ad} \hat{P}$), we find

$$\hat{\mathcal{D}}_X a = \hat{\nabla}_X a + A_X \cdot a = \hat{\nabla}_X a + [A_X, a] = \hat{\nabla}_X a$$

(4.61)

for all Killing vectors $X$. Thus we will choose to set

$$[[X, a]] := \hat{\mathcal{D}}_X a = \hat{\nabla}_X a$$

(4.62)

for a Killing vector $X \in \mathfrak{iso}(M, g)$ and $\tau$-symmetry $a \in \mathfrak{R}$. Let us now use the Jacobi identity to motivate the definition of another component of the bracket; using (4.47) we find

$$[[X, [[Y, a]]]] - [[Y, [[X, a]]]] = \left[\hat{\mathcal{D}}_X, \hat{\mathcal{D}}_Y\right] a = \hat{\mathcal{D}}_{[X, Y]} a + [F(X, Y), a],$$

(4.63)

for all $X, Y \in \mathfrak{iso}(M, g)$ and $a \in \mathfrak{R}$, so the Jacobi identity is satisfied if we define

$$[[X, Y]] := [X, Y] + F(X, Y) = \mathcal{L}_XX + F(X, Y).$$

(4.64)

Now we need to check the Jacobi identity for three Killing vectors; we find that

$$[[X, [[Y, Z]]]] = \mathcal{L}_X \mathcal{L}_Y Z + F(X, \mathcal{L}_Y Z) + \hat{\mathcal{D}}_X(F(Y, Z)),$$

(4.65)

and, noting that $\hat{\mathcal{D}}_X(F(Y, Z)) = \hat{\nabla}_X(F(Y, Z))$, upon taking cyclic permutations of the above we find that this component of the Jacobi identity is equivalent to

$$\left(\hat{\nabla}_X F\right)(Y, Z) + \left(\hat{\nabla}_Y F\right)(Z, X) + \left(\hat{\nabla}_Z F\right)(X, Y) = 0$$

(4.66)

for all Killing vector fields $X, Y, Z$. But since $F = \hat{R} - R$, the above is identically zero by the Bianchi identities for $\hat{R}$ and $R$ – in fact, it is actually just a form of the Bianchi identity for $F$. Finally, for $a, b \in \mathfrak{R}$ let us take

$$[[a, b]] := [a, b]$$

(4.67)

where the latter is the natural bracket on $\mathfrak{R}$; that is, the one given pointwise by the bracket of $\tau$. Thus the Jacobi identity for three elements of $\mathfrak{R}$ is automatically satisfied.

**Proposition 4.8.** The space $\mathfrak{g} = \mathfrak{iso}(M, g) \oplus \mathfrak{R}$ equipped with the bracket $[[\cdot, \cdot]]$ given by (4.62), (4.64) and (4.67) is a Lie algebra. The space of spinor fields $\hat{\mathcal{S}}$ is a module of this algebra, where elements of $\mathfrak{iso}(M, g)$ act via $\mathcal{D}$ and those of $\mathfrak{R}$ act via the pointwise action of $\tau$ on $S$.

**Proof.** The first claim follows from the discussion above. Equation (4.47) gives us

$$[[X, Y]] \cdot \epsilon = \mathcal{D}_{[X, Y]} \epsilon + F(X, Y)\epsilon = \mathcal{D}_X \left(\mathcal{D}_Y \epsilon\right) - \mathcal{D}_Y \left(\mathcal{D}_X \epsilon\right)$$

(4.68)
for all $X, Y \in \mathfrak{iso}(M, g)$ and $e \in \mathcal{S}$, while the Leibniz rule for $\mathcal{L}_X$ gives

$$\left(\mathcal{L}_X a\right)e = \mathcal{L}_X (ae) - a \left(\mathcal{L}_X e\right)$$

(4.69)

for $X \in \mathfrak{iso}(M, g)$, $a \in \mathcal{R}$ and $e \in \mathcal{S}$. We also have

$$[a, b]e = a(be) - b(ae)$$

(4.70)

for all $a, b \in \mathcal{R}$ and $e \in \mathcal{S}$, since the same equation holds pointwise for the action of $\tau$ on $S$. This shows that $\mathcal{S}$ is a module for $\mathfrak{g}$. □

It is manifest in the definition of the bracket $[[-, -]]$ that $\mathfrak{g}$ is not the direct sum of Lie algebras $\mathfrak{iso}(M, g) \oplus \mathcal{R}$ in general, but rather an extension of $\mathfrak{iso}(M, g)$ by $\mathcal{R}$; we have a short exact sequence of Lie algebras

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{iso}(M, g) \longrightarrow 0$$

(4.71)

where the map $\mathcal{R} \rightarrow \mathfrak{g}$ is the inclusion and $\mathfrak{g} \rightarrow \mathfrak{iso}(M, g)$ is projection. Note that if $i_X F = 0$ for all $X \in \mathfrak{iso}(M, g)$ then $[[-, -]]$ restricted to $\mathfrak{iso}(M, g)$ is simply the Lie bracket, and we have $\mathfrak{g} = \mathfrak{iso}(M, g) \ltimes \mathcal{R}$.

Including spinors

It is a natural question whether it is possible to extend the bracket $[[-, -]]$ to $\mathfrak{g} \oplus \mathcal{S}$. However, even if we have a squaring map on the spinor bundle as in Lemma 3.4, giving us a map $\bigodot^2 \mathcal{S} \rightarrow \mathfrak{X}(M)$ or $\Lambda^2 \mathcal{S} \rightarrow \mathfrak{X}(M)$, the image of an arbitrary pair of spinors is not a Killing vector. In Chapter 3, we needed to introduce the notion of an admissible connection in order to define the Killing superalgebra. In the next section, we develop an analogous notion for a spin-$\mathcal{R}$ structure which will again lead us to a definition of a Killing superalgebra.

4.2 Killing spinors and Killing (super)algebras with gauged $\mathcal{R}$-symmetry

We now seek to find a notion of admissible connections and Killing spinors in the geometric setting of a spin-$\mathcal{R}$ structure, generalising the treatment of Chapter 3.

4.2.1 Killing spinors with gauged $\mathcal{R}$-symmetry

Connections on spinors

Let $D$ be a connection on $\mathcal{S}$. The difference between two connections on a bundle is a 1-form with values in the endomorphisms of the bundle, so there exists some $\beta \in \Omega^1 (M; \text{End} \mathcal{S})$ such that $D = \nabla - \beta$. We denote the contraction of $\beta$ with a vector field $X \in \mathfrak{X}$ by $\beta_X = i_X \beta$. 


Lemma 4.9. For all \( X \in \text{iso}(M, g) \),
\[
\left[ \mathfrak{L}_X, D \right] = \iota_X F - \mathfrak{L}_X \beta.
\] (4.72)

Proof. Using Proposition 4.6, we have
\[
\left[ \mathfrak{L}_X, D \right] = \left[ \mathfrak{L}_X, \nabla \right] - \left[ \mathfrak{L}_X, \beta \right] = \iota_X F - \mathfrak{L}_X \beta
\] (4.73)
for all \( X \in \text{iso}(M, g) \), \( Y \in \mathfrak{X}(M) \).

We denote the curvature 2-form \( R^D \in \Omega^2(M; \text{End} \hat{S}) \) of \( D \) by
\[
R^D(X, Y) = [D_X, D_Y] - D_{[X,Y]}
\] (4.74)
for all \( X, Y \in \mathfrak{X}(M) \). Recalling now the embedding of adjoint bundles into \( \text{End} \hat{S} \) (equation (4.35)), we have the following.

Proposition 4.10. Considering each term as a 2-form with values in the bundle \( \text{End} \hat{S} \), the curvature \( R^D \) is given by
\[
R^D = R + F + \frac{1}{2} [\beta \wedge \beta] - \hat{\alpha} \theta\beta
\] (4.75)
where \( R \) is the Riemann curvature, \( F \) is the field strength of the gauge connection \( \alpha \) on \( Q \), and \( \hat{\alpha} = d \hat{\psi} \) is the covariant exterior derivative.

Proof. For all \( X, Y \in \mathfrak{X}(M) \), using definitions and the Leibniz rule,
\[
R^D(X, Y) = \left[ \hat{\nabla}_X, \hat{\nabla}_Y \right] + [\beta_X, \beta_Y] - \left[ \hat{\nabla}_X, \beta_Y \right] - \left[ \beta_X, \hat{\nabla}_Y \right] - \hat{\nabla}_{[X,Y]} + \beta_{[X,Y]}
\] (4.76)
\[
= \hat{R}(X, Y) + [\beta_X, \beta_Y] - \hat{\nabla}_X(\beta_Y) + \hat{\nabla}_Y(\beta_X) + \beta_{[X,Y]}
\]
\[
= R(X, Y) + F(X, Y) + [\beta_X, \beta_Y] - \hat{\alpha} \theta(\beta, X, Y)
\]
The identity follows by abstracting the vector fields \( X, Y \).

This result is of course a generalisation of Proposition 3.3 where we have used the notation of forms with values in associated bundles to write the formula in a more compact form. The identity \( R^D \theta = 0 \) for \( \theta \in \hat{S}_D \) is an integrability condition for the existence of \( D \)-parallel sections of \( \hat{S} \).

Existence of the Killing (super)algebra

Following a similar approach to Section 3.2, we will now seek to find conditions under which there exists a (super-)algebra \( \hat{\mathfrak{g}}_D \) whose odd part \( (\hat{\mathfrak{g}}_D)_\mathbb{1} \) is the space of \( D \)-parallel spinor fields,
\[
\hat{\mathfrak{g}}_D = \{ \theta \in \hat{\mathfrak{g}} \mid D\theta = 0 \}. \tag{4.77}
\]
Since we have already seen that the Lie algebra \( \mathfrak{g} \) discussed in §4.1.3 acts on the space of spinor fields \( \hat{\mathfrak{g}} \), the natural starting point is to find a subalgebra of \( \mathfrak{g} \) which preserves \( \hat{\mathfrak{g}}_D \); the even part \( (\hat{\mathfrak{g}}_D)_\mathbb{1} \) of our (super-)algebra will be a subalgebra of this
space. We will demand that, as a vector subspace of \( g \), this subalgebra is of the form \( \mathfrak{W}_D \oplus \mathfrak{R}_D \), where \( \mathfrak{W}_D \) is a subspace of \( \mathfrak{iso}(M, g) \) and \( \mathfrak{R}_D \) is a subspace of \( \mathfrak{r} \). It follows that \( \mathfrak{W}_D \) must be closed under the Lie bracket and consist of Killing vectors which preserve \( \tilde{\mathfrak{g}}_D \), that \( \mathfrak{R}_D \) must be a subalgebra of \( \mathfrak{R}_D \) and also preserve \( \tilde{\mathfrak{g}}_D \), and that \( \mathfrak{W}_D \oplus \mathfrak{R}_D \) is closed under the bracket \([[-, -]]\).

We start by considering a condition on the Killing vector fields suggested by the formula (4.72).

**Lemma 4.11.** If a Killing vector \( X \) satisfies

\[
i_X F - \mathcal{L}_X \beta = 0. \tag{4.78}
\]

then the covariant Lie derivative along \( X \) preserves \( \mathfrak{g}_D \). Furthermore,

\[
\mathcal{L}_X R^D = 0. \tag{4.79}
\]

If \( X, Y \in \mathfrak{iso}(M, g) \) both satisfy equation (4.78) then

\[
i_{[X, Y]} F - \mathcal{L}_{[X, Y]} \beta = \tilde{\nabla}(F(X, Y)) - \left[ \beta, F(X, Y) \right]. \tag{4.80}
\]

**Proof.** The first claim follows immediately from equation (4.72). For the second claim, recall that \( \mathcal{L}_X R = \mathcal{L}_X R = 0 \) for Killing vectors, so using the formula for \( R^D \) in Proposition 4.10 and the covariant Cartan calculus (Proposition 4.7), we have

\[
\mathcal{L}_X R^D = \mathcal{L}_X R + \mathcal{L}_X F - \mathcal{L}_X \tilde{d} \beta + \left[ \mathcal{L}_X \beta \wedge \beta \right]
= \tilde{d} (i_X F + i_X \tilde{d} \beta - \mathcal{L}_X \beta) - [i_X F \wedge \beta] + \left[ \mathcal{L}_X \beta \wedge \beta \right] \tag{4.81}
\]

which vanishes if \( X \) satisfies (4.78). Using the Cartan calculus again as well as the Bianchi identity for \( F \), we can compute

\[
i_{[X, Y]} F = \mathcal{L}_X i_Y F - i_Y \mathcal{L}_X F
= \mathcal{L}_X i_Y F - i_Y \tilde{d} i_X F - i_Y \tau_X \tilde{d} F \tag{4.82}
= \mathcal{L}_X i_Y F - i_Y \tilde{d} i_X F
= \mathcal{L}_X i_Y F - \mathcal{L}_Y i_X F + \tilde{d} i_Y i_X F,
\]

for all Killing vectors \( X, Y \). Then, assuming \( X, Y \) satisfy equation (4.78) and using equation (4.53),

\[
i_{[X, Y]} F - \mathcal{L}_{[X, Y]} \beta = \mathcal{L}_X \mathcal{L}_Y \beta - \mathcal{L}_Y \mathcal{L}_X \beta + \tilde{\nabla}(F(X, Y)) - \mathcal{L}_{[X, Y]} \beta
= \tilde{\nabla}(F(X, Y)) + \left[ F(X, Y), \beta \right], \tag{4.83}
\]

hence the final claim.

Thus the space of Killing vectors satisfying (4.78) preserves \( \tilde{\mathfrak{g}}_D \) but is not closed under the Lie bracket; we must impose further conditions in addition to (4.78). Before deciding what these conditions should be, let us first examine the \( \tau \)-symmetries.
Recall that $\text{ad} \, \overline{Q} \rightarrow \text{End} \, \hat{S}$ (see (4.35)), so we can consider $a \in \mathcal{R}$ as a section of $\text{End} \, \hat{S}$. The connection $D$ induces a connection (also denoted $D$) on the endomorphism bundle via the Leibniz rule. Thus to an $\tau$-symmetry $a \in \mathcal{R}$ we can assign the element $D a \in \Omega^1(M; \text{End} \, \hat{S})$. Explicitly, this is given by $D_X a := [D_X, a] = \hat{\nabla}_X a - [\beta_X, a]$ for $X \in \mathfrak{X}(M)$. Note that equation (4.80) can now be rendered as

$$i_{[X,Y]} F - \mathcal{L}_{[X,Y]} \beta = D(F(X,Y)).$$

(4.84)

We also have the following.

**Lemma 4.12.** The space

$$\mathcal{R}_D := \{a \in \mathcal{R} | Da = 0\}$$

is a subalgebra of $\mathcal{R}$ and of $\mathfrak{g}$ with dimension at most $\dim \tau$ which preserves $\hat{S}_D$, and for any $X \in \text{iso}(M,g)$ satisfying (4.62) and $a \in \mathcal{R}$,

$$[[X,a]] \in \mathcal{R}_D.$$  

(4.86)

**Proof.** The Leibniz rule for $D$ shows that $\mathcal{R}_D$ is a subalgebra of $\mathcal{R}$ and thus of $\mathfrak{g}$. Since elements of $\mathcal{R}_D$ are sections which are parallel with respect to a connection, they are determined by their value at a point; these values lie in the fibre which is isomorphic to $\tau$, whence $\dim \mathcal{R}_D \leq \dim \tau$. The action of this space on $\hat{S}$ preserves $\hat{S}_D$ by construction; if $De = 0$ and $Da = 0$ then $D(\epsilon a) = (Da) \epsilon - D(\epsilon a) = 0$. For the last claim, one can check that (4.72) holds when acting on $\mathcal{R}$, so we have

$$D \mathcal{L}_X a = \mathcal{L}_X Da + [i_X F - \mathcal{L}_X \beta, a] = 0,$$

(4.87)

hence $[[X,a]] = \mathcal{L}_X a \in \mathcal{R}_D$ as claimed.

The $\mathcal{R}$-component of the bracket of two Killing vector fields $X, Y$ satisfying (4.78) lies in $\mathcal{R}_D$ if and only if

$$D(F(X,Y)) = D\left(i_Y \mathcal{L}_X \beta\right) = -D\left(i_X \mathcal{L}_Y \beta\right) = 0,$$

(4.88)

while by (4.84), the same condition is also necessary and sufficient for $[X, Y]$ to satisfy (4.78).

These results suggest that instead of using (4.78) as the condition on our Killing vector fields, we should take $i_X F = 0$ and $\mathcal{L}_X \beta = 0$ separately. Doing so gives us the following.

**Proposition 4.13.** The bracket $[-, -]$ restricted to the space

$$\hat{\mathfrak{g}}_D := \{X \in \mathfrak{X}(M) | \mathcal{L}_X g = 0, \mathcal{L}_X \beta = 0, \text{ and } i_X F = 0\}$$

(4.89)

is equal to the ordinary Lie bracket of vector fields; $\hat{\mathfrak{g}}_D$ is a subalgebra of both $\mathfrak{g}$ and $\text{iso}(M,g)$; the covariant Lie derivative gives a representation of $\hat{\mathfrak{g}}_D$ on the space of sections of any vector bundle associated to the spin-$\tau$ structure, and the action on $\hat{S}$ preserves $\hat{S}_D$.

$^5$Note that $D_X a$ need not be an $\tau$-symmetry.
Proof. Let \( X, Y \in \mathfrak{H}_D \). Since \( t_X F = t_Y F = 0 \), it is immediate that \( [[X, Y]] = [X, Y] \) – in particular, there is no \( \mathfrak{R} \)-component. From (4.47) we find
\[
\mathcal{L}_{[X,Y]} = \begin{bmatrix} \mathcal{L}_X \mathcal{L}_Y \end{bmatrix},
\]
(4.90)
hence \( \mathcal{L}_{[X,Y]} g = 0 \), \( \mathcal{L}_{[X,Y]} \beta = 0 \), and then from equation (4.80) of Lemma 4.11 we find that \( t_{[X,Y]} F = 0 \). Thus \( \mathfrak{H}_D \) is closed under \([[-,-]] = [-,-]\), making it a subalgebra of both \( \mathfrak{g} \) and \( \text{iso}(M, \mathfrak{g}) \). That \( \mathcal{L} \) defines representations on spaces of sections follows from (4.90), and \( \mathfrak{H}_D \) is preserved by this representation by Lemma 4.11.

The following is immediate from the preceding results.

**Corollary 4.14.** \( (\mathfrak{H}_D \times \mathfrak{R}_D, [[-,-]]) \) is a Lie subalgebra of \( \mathfrak{g} \) with \( \mathfrak{H}_D \) as a module.

We now attempt to extend the bracket \([[-,-]]\) to include the space of parallel spinors \( \mathfrak{H}_D \). We define a \( \mathbb{Z}_2 \) grading on \( \mathfrak{H}_D \oplus \mathfrak{R}_D \oplus \mathfrak{S}_D \) by declaring \( \mathfrak{H}_D \oplus \mathfrak{R}_D \) to be the even subspace and \( \mathfrak{S} \) the odd subspace, and we seek to extend \([[-,-]]\) to a bracket on this space which preserves this grading. We will of course take
\[
[[X, \epsilon]] := \mathcal{L}_X \epsilon, \quad [[a, \epsilon]] := a \epsilon,
\]
(4.91)
for \( X \in \mathfrak{H}_D \), \( a \in \mathfrak{R}_D \) and \( \epsilon \in \mathfrak{S}_D \). The bracket of two spinors is less straightforward. While the squaring map \( \kappa : \bigotimes^2 \mathfrak{S}^* \to T M \) constructed at the end of §4.1.2 gives us a natural way of producing a vector field from a pair of spinor fields, we have no obvious way of obtaining an \( \tau \)-symmetry, so let us define
\[
[[\epsilon, \zeta]] = \kappa(\epsilon, \zeta) + \rho(\epsilon, \zeta)
\]
(4.92)
for \( \epsilon, \zeta \in \mathfrak{S}_D \), where \( \rho \) is some section of the bundle \( \bigotimes^2 \mathfrak{S}^* \otimes \text{ad} \overline{Q} \) (if \( \kappa \) is symmetric) or \( \bigwedge^2 \mathfrak{S}^* \otimes \text{ad} \overline{Q} \) (if \( \kappa \) is skew-symmetric) which we leave undetermined for now. We could of course simply choose \( \rho = 0 \), but we will see that our notion of Killing spinors, Killing (super-)algebras and supersymmetry becomes much richer if we allow for \( \rho \neq 0 \). We have thus defined a pre-Lie (super-)algebra structure \([[-,-]]\) on \( \mathfrak{H}_D \oplus \mathfrak{R}_D \oplus \mathfrak{S}_D \); that is, a bracket satisfying the axioms of a Lie bracket except for the Jacobi identity.

The components of the Jacobi identity which are not satisfied by construction are those with two or three odd entries. Let us examine them now. First, for \( X \in \mathfrak{H}_D \) and \( \epsilon, \zeta \in \mathfrak{S}_D \) we have
\[
[[X, [[\epsilon, \zeta]]]] - [[[X, \epsilon], \zeta]] - [[\epsilon, [[X, \zeta]]]]
\]
\[
= \mathcal{L}_X \kappa(\epsilon, \zeta) + F(X, \kappa(\epsilon, \zeta)) + \mathcal{L}_X (\rho(\epsilon, \zeta))
\]
\[
- \kappa(\mathcal{L}_X \epsilon, \zeta) - \rho(\mathcal{L}_X \epsilon, \zeta) - \kappa(\epsilon, \mathcal{L}_X \zeta) - \rho(\epsilon, \mathcal{L}_X \zeta)
\]
\[
= \left( \mathcal{L}_X \rho \right)(\epsilon, \zeta),
\]
(4.93)
where we have used the fact that \( \mathcal{L}_X \kappa = 0 \) identically and \( t_X F = 0 \) for \( X \in \mathcal{X}(M) \). For
where we have used the r-invariance of \( \kappa \); for \( \epsilon, \zeta, \eta \in \tilde{\mathfrak{D}}_D \),
\[
[[[\epsilon, \zeta]], \eta]] + [[[\zeta, \eta]], \epsilon] + [[[\eta, \epsilon]], \zeta]] = \mathcal{L}_{\kappa(\epsilon, \zeta)} \eta + \mathcal{L}_{\kappa(\zeta, \eta)} \epsilon + \mathcal{L}_{\kappa(\eta, \epsilon)} \zeta + \rho(\epsilon, \zeta) \eta + \rho(\zeta, \eta) \epsilon + \rho(\eta, \epsilon) \zeta.
\]
(4.95)

Thus the Jacobi identity is satisfied if and only if
\[
\begin{align*}
\mathcal{L}_X \rho (\epsilon, \zeta) &= 0, \\
\mathcal{L}_{\kappa(\epsilon, \zeta)} a - (a \cdot \rho)(\epsilon, \zeta) &= 0,
\end{align*}
\]
(4.96, 4.97)

for all \( \epsilon, \zeta, \eta \in \tilde{\mathfrak{D}}_D, X \in \tilde{\mathfrak{D}}_D, a \in \mathfrak{R}_D \). The first two equations here suggest that we should further restrict our space of allowed Killing vectors and \( r \)-symmetries to those that preserve \( \rho \); clearly (4.96) is satisfied if \( \mathcal{L}_X \rho = 0 \), while since \( Da = 0 \),
\[
\mathcal{L}_X a = \nabla_X a = [\beta_X, a]
\]
(4.99)

for all Killing vectors \( X \in \mathfrak{iso}(M, g) \), so (4.97) is satisfied if \( a \cdot \beta = [a, \beta] = 0 \) and \( a \cdot \rho = 0 \). These conditions are natural if we consider \( \rho \), along with \( g, \beta \) and \( F \) (or the connection \( a \)), to be “background data” which we demand that our vector fields must preserve.

**Lemma 4.15.** Lemma 4.12, Proposition 4.13 and Corollary 4.14 still hold if we replace \( \tilde{\mathfrak{D}}_D \) and \( \mathfrak{R}_D \) by the subalgebras
\[
\begin{align*}
\tilde{\mathfrak{H}}_{(D, \rho)} &= \left\{ X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = 0, \mathcal{L}_X \beta = 0, \mathcal{L}_X \rho = 0 \text{ and } \iota_X F = 0 \right\}, \\
\mathfrak{R}_{(D, \rho)} &= \left\{ a \in \mathfrak{R} \mid \nabla a = 0, [a, \beta] = 0, a \cdot \rho = 0 \right\}.
\end{align*}
\]
(4.100)

Moreover, the action of \( \tilde{\mathfrak{H}}_{(D, \rho)} \) on \( \mathfrak{R}_{(D, \rho)} \) via the covariant Lie derivative is trivial, so \( \tilde{\mathfrak{H}}_{(D, \rho)} \oplus \mathfrak{R}_{(D, \rho)} \) is a subalgebra of both \( \mathfrak{g} \) and the direct sum algebra \( \mathfrak{iso}(M, g) \oplus \mathfrak{R} \).

**Proof.** For the first claim, it is sufficient to show that \( \tilde{\mathfrak{H}}_{(D, \rho)} \) and \( \mathfrak{R}_{(D, \rho)} \) are subalgebras of \( \tilde{\mathfrak{D}}_D \) and \( \mathfrak{R}_D \) respectively. For the vectorial part, we need only show that \( \mathcal{L}_{[X, Y]} \rho = 0 \) for \( X, Y \in \tilde{\mathfrak{D}}_D \). But this follows immediately from equation (4.90). For the \( \tau \)-symmetries, \( \mathfrak{R}_{(D, \rho)} \) is a subspace of \( \mathfrak{R}_D \) since \( Da = \nabla a = [\beta, a] \) for all \( a \in \mathfrak{R} \), and it is clearly closed under the bracket. For the second claim, we clearly have 
\[
[[X, a]] = \mathcal{L}_X a = \nabla a = 0 \text{ for all } X \in \tilde{\mathfrak{H}}_{(D, \rho)}, a \in \mathfrak{R}_{(D, \rho)}, \text{ whence } \tilde{\mathfrak{H}}_{(D, \rho)} \oplus \mathfrak{R}_{(D, \rho)} \text{ is a subalgebra of } \tilde{\mathfrak{D}}_D \times \mathfrak{R}_D.
\]
The claim follows immediately.

We can finally state the analogue of Theorem 3.6 for a gauged background. Analogously to the section defined by equation (3.24), we define \( \gamma \in \Gamma(\text{Hom}(\otimes^2 \tilde{S}, \text{End}(TM))) \)
we must have (Definition 3.7), we propose the following definition. We note that conditions (1), (2)

Theorem 4.16 (Existence of (super)algebra associated to $D$). Let $D$ be a connection on $\hat{\mathcal{S}}$, $\beta = \hat{\nabla} - D$ an End $\hat{\mathcal{S}}$-valued 1-form and $\hat{\mathfrak{R}}_{(D,\rho)} = \hat{\mathfrak{S}}_{(D,\rho)} \oplus \mathfrak{N}_{(D,\rho)} \oplus \hat{\mathfrak{S}}_{D}$ a $\mathbb{Z}_2$-graded vector space where the components of the even subspace are given by (4.100) and the odd subspace is $\hat{\mathfrak{S}}_{D} = \{ \xi \in \hat{\mathfrak{S}} \mid D\xi = 0 \}$. We equip $\hat{\mathfrak{R}}_{(D,\rho)}$ with the brackets

$$[X, Y] = \mathcal{L}_X Y, \quad [X, a] = \hat{\mathcal{D}}_X a = 0, \quad [a, b] = [a, b]_{\mathfrak{R}},$$

$$[X, \xi] = \hat{\mathcal{D}}_X \xi, \quad [a, \xi] = a\xi, \quad [\xi, \zeta] = \kappa(\xi, \zeta) + \rho(\xi, \zeta),$$

for $X, Y \in \hat{\mathfrak{S}}_{(D,\rho)}, \xi, \zeta \in \hat{\mathfrak{S}}_{D}, a, b \in \mathfrak{N}_{(D,\rho)}$. Then $\hat{\mathfrak{R}}_{(D,\rho)}$ is a Lie (super)algebra if and only if the following conditions are satisfied for all $\xi, \eta \in \hat{\mathfrak{S}}_{D}$:

$$\gamma(\xi, \eta) \in \mathfrak{so}(M),$$

$$\hat{\mathcal{D}}_{\kappa(\xi, \eta)} \beta = 0, \quad \hat{\mathcal{D}}_{\kappa(\xi, \eta)} \rho = 0, \quad \iota_{\kappa(\xi, \eta)} F = 0,$$

$$[\rho(\xi, \eta), \beta] = 0, \quad \hat{\nabla}(\rho(\xi, \eta)) = 0, \quad \rho(\xi, \eta) \cdot \rho = 0,$$

$$\beta_{\kappa(\xi, \eta)} \eta - \gamma(\xi, \eta) \cdot \eta + \rho(\xi, \eta) \eta + \text{cyclic perm's} = 0.$$ (4.105)

Proof. By Lemma 4.15, it remains only to check that the $\hat{\mathfrak{S}}_{D} \otimes \hat{\mathfrak{S}}_{D} = \hat{\mathfrak{S}}_{(D,\rho)} \oplus \mathfrak{N}_{(D,\rho)}$ component of the bracket closes and that the $[011], [111]$ components of the Jacobi identity are satisfied. Now let $\xi, \zeta \in \hat{\mathfrak{S}}_{D}$. Recall that $\kappa(\xi, \eta)$ is a Killing vector if and only if the endomorphism $A_{\kappa(\xi, \eta)} = \gamma(\xi, \eta)$ is skew-symmetric with respect to the metric, which is just (4.102), so $\kappa(\xi, \eta) \in \hat{\mathfrak{S}}_{(D,\rho)}$ if and only if (4.102) and (4.103) are all satisfied. On the other hand, $\rho(\xi, \eta) \in \mathfrak{N}_{(D,\rho)}$ if and only if (4.104) are satisfied.

The only component of the Jacobi identity which is not identically satisfied is given by equation (4.98). Letting $\xi, \eta, \zeta \in \hat{\mathfrak{S}}_{D}$ and using the fact that $\nabla_X \eta = \beta_X \eta$ for all $X \in \mathfrak{X}(M)$ and $A_{\kappa(\xi, \eta)} = \gamma(\xi, \eta)$, we can write

$$\hat{\mathcal{D}}_{\kappa(\xi, \eta)} \eta = \hat{\nabla}(\kappa(\xi, \eta) \eta) + A_{\kappa(\xi, \eta)} \eta = \beta_{\kappa(\xi, \eta)} \eta + \gamma(\xi, \eta) \cdot \eta,$$ (4.106)

so the second Jacobi identity is equivalent to (4.105).

Remark 9. As remarked at the beginning of the previous section, the structure of $\hat{\mathfrak{R}}_{(D,\rho)}|_{\mathfrak{N}}$ is much more trivial than that of the algebra $\mathfrak{g}$ in which it embeds, in the sense that $\mathfrak{g}$ is an extension of $\mathfrak{so}(M, g)$ by $\mathfrak{N}$, whereas $\hat{\mathfrak{R}}_{(D,\rho)}|_{\mathfrak{N}}$ is a direct sum of subalgebras of $\mathfrak{so}(M, g)$ and $\mathfrak{N}$. One might take this as motivation to try to relax some of the conditions in the definition of $\hat{\mathfrak{R}}_{(D,\rho)}$. Note also that if the algebra acts locally transitively, in the sense that the values of vectors in $\mathfrak{N}_{(D,\rho)}$ span the tangent space at every point (in particular in what we will later term the highly supersymmetric case), we must have $F = 0$, providing further motivation for a less restrictive definition.

Following our definition of admissible connections from the previous chapter (Definition 3.7), we propose the following definition. We note that conditions (1), (2)

---

*Whether $\hat{\mathfrak{R}}_{(D,\rho)}$ is an algebra or superalgebra depends on the symmetry of $\kappa$. 

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and (4) are direct generalisations of the conditions from the earlier definition, while (3) is new. Given the comments above, this condition especially is subject to change, along with \( \kappa(\epsilon, \zeta)F = 0 \) from (1).

**Definition 4.17** (Admissible pairs, Killing spinors and Killing (super)algebras with \( R \)-symmetry). An admissible pair \((D, \rho)\) consists of a connection \( D \) on \( \mathcal{S} \) and a section \( \rho \) of \( \bigotimes^2 \mathcal{S}^* \otimes \text{ad} \mathcal{Q} \) (if \( \kappa \) is symmetric) or \( \bigwedge^2 \mathcal{S}^* \otimes \text{ad} \mathcal{Q} \) (if \( \kappa \) is skew-symmetric) satisfying the following:

1. \( \gamma \) is skew-symmetric (it takes values \( \text{ad} \mathcal{S} \O \));
2. \( \overline{\mathcal{L}}_{\kappa(\epsilon, \zeta)} \beta = 0, \overline{\mathcal{L}}_{\kappa(\epsilon, \zeta)} \rho = 0, \kappa(\epsilon, \zeta)F = 0 \) for all \( \epsilon, \zeta \in \mathcal{S}_{(D, \rho)} \);
3. \( [\rho(\epsilon, \zeta), \beta] = 0, \rho(\epsilon, \zeta) \cdot \rho = 0 \), for all \( \epsilon, \zeta \in \mathcal{S} \) and \( \overline{\nabla}(\rho(\epsilon, \zeta)) = 0 \) for all \( \epsilon, \zeta \in \mathcal{S}_{(D, \rho)} \);
4. \( \beta(\kappa(\epsilon, \zeta)) \eta + \beta(\kappa(\zeta, \eta)) = \beta(\kappa(\eta, \epsilon)) + \gamma(\epsilon, \zeta) \cdot \eta + \gamma(\zeta, \eta) \cdot \epsilon + \gamma(\eta, \epsilon) \cdot \zeta = 0 \) for all \( \epsilon, \zeta, \eta \in \mathcal{S} \).

If \((D, \rho)\) is an admissible pair, we will refer to the the differential equation \( De = 0 \) (equivalently \( \overline{\nabla}e = \beta e \)) as the Killing spinor equation and \( \mathcal{S}_{(D, \rho)} \) the space of Killing spinors; \( \mathcal{H}_{(D, \rho)} \) is the space of restricted Killing vectors and \( \mathcal{H}_{(D, \rho)}^\prime \) is the Killing (super)algebra.

**4.2.2 Algebraic structure of Killing (super)algebras with \( R \)-symmetry**

The goal of this section is to prove the obvious generalisation of Theorem 3.10. We will simply lift much of the proof of that theorem.

**Theorem 4.18** (Structure of Killing (super)algebras with gauged \( R \)-symmetry). Let \((D, \rho)\) be an admissible pair. Then the associated Killing (super)algebra \( \mathcal{H}_{(D, \rho)} \) is a filtered subdeformation of the flat model \( \mathcal{S} = V \oplus \mathcal{S} \oplus (\mathfrak{so}(V) \oplus \mathfrak{r}) \).

**Killing transport**

Let us define \( \mathcal{E} := \mathcal{P} \times_{\text{Spin}^R(V)} \mathcal{S} \), the associated vector bundle with respect to \( \mathcal{S} = V \oplus \mathcal{S} \oplus \mathfrak{so}(V) \oplus \mathfrak{r} \) as a \( \text{Spin}^R(V) \)-module. Note that

\[
\mathcal{E} \cong TM \oplus \mathcal{S} \oplus \text{ad} \mathcal{P} \oplus \text{ad} \mathcal{Q}. \tag{4.107}
\]

We define a connection \( \mathcal{D} \) on \( \mathcal{E} \) as follows\(^7\):

\[
\mathcal{D}_Y (X, \epsilon, A, a) := (\nabla_Y X + AY, D_Y \epsilon, \nabla_Y A + R(Y, X), D_Y a) \tag{4.108}
\]

for \( X, Y \in \mathcal{X}(M), \epsilon \in \mathcal{E}, A \in \mathfrak{so}(M, g), a \in \mathfrak{R} \).

\(^7\)Strictly speaking, \( \mathcal{D} \) does not define a connection on \( \mathcal{E} \) since \( D \) does not preserve \( \mathfrak{R} \); we should really consider \( \mathcal{D} \) as a connection on \( TM \oplus \mathcal{S} \oplus \text{ad} \mathcal{P} \oplus \text{End} \mathcal{S} \). However, since we are really only interested in sections of \( \mathfrak{R}_{(D, \rho)} \), which are annihilated by \( D \), this distinction will not matter for our purposes, so we allow ourselves the abuse of notation.
Proposition 4.19. The \( \hat{\mathcal{R}} \)-parallel sections of \( \hat{\mathcal{E}} \) are triples \((X, \epsilon, A_X, a)\), where \( X \) is a Killing vector, \( \epsilon \in \hat{\mathcal{S}}_D \), \( a \in \mathfrak{R}_D \) and \( A_X = -\nabla X \).

**Proof.** Follows immediately from the analogous Proposition 3.9. \( \square \)

We immediately see that any element of \( \hat{\mathcal{R}}_{(D, \rho)} = \hat{\mathcal{S}}_{(D, \rho)} \oplus \mathfrak{R}_{(D, \rho)} \oplus \hat{\mathcal{S}}_D \) defines a parallel section of \( \hat{\mathcal{E}} \) by sending \((X, a, \epsilon) \mapsto (X, \epsilon, A_X, a)\). By parallel transport, the \( \hat{\mathcal{R}} \)-parallel sections of \( \hat{\mathcal{E}} \) (in particular, elements of \( \hat{\mathcal{R}}_{(D, \rho)} \)) are determined by their value \((X, \epsilon, A_X, a)\) at any point \( p \in M \). As we did in \$3.2.2\), we can localise the brackets of \( \hat{\mathcal{R}}_{(D, \rho)} \) at \( p \) – that is, we can express the transport data of a bracket in terms of the transport data of its inputs. We let \( \Phi_p : \hat{\mathcal{R}}_{(D, \rho)} \to \hat{\mathcal{E}}_p \) be the \( \mathbb{R} \)-linear map sending an element of \( \hat{\mathcal{R}}_{(D, \rho)} \) to its Killing transport data at \( p \):

\[
\Phi_p(X, \epsilon, a) := (X_p, \epsilon_p, (A_X)_p, a_p),
\]

and we find

\[
\begin{align*}
\Phi_p([X, Y]) &= ([A_X]_p Y_p - (A_Y)_p X_p, 0, [(A_X)_p, (A_Y)_p] - R_p(X_p, Y_p), 0), \\
\Phi_p([X, a]) &= (0, 0, 0, \left[ a_X, a_p \right]), \\
\Phi_p([a, b]) &= (0, 0, 0, \left[ a_p, b_p \right]), \\
\Phi_p((X, \epsilon)) &= (0, \beta_p(X_p) \epsilon_p + (A_X)_p \cdot \epsilon_p, 0, 0), \\
\Phi_p([a, \epsilon]) &= (0, 0, a_p \epsilon_p, 0), \\
\Phi_p([\epsilon, \zeta]) &= (\kappa_p(\epsilon_p, \zeta_p), 0, -\kappa_p(\beta_p \epsilon_p, \zeta_p) - \kappa_p(\epsilon_p, \beta_p \zeta_p), \rho_p(\epsilon_p, \zeta_p)).
\end{align*}
\]

**Proof of the theorem**

**Proof of Theorem 4.18.** We follow the proof of Theorem 3.10, spelling out details only in places where modifications are necessary. We identify \( \hat{\mathcal{E}}_p \) with \( \hat{s} \) and define the spaces

\[
\begin{align*}
V' &= \{ X_p \in V \mid X \in \hat{\mathcal{S}}_{(D, \rho)} \}, \\
\mathfrak{h} &= \{ (A_X)_p \in \text{so}(V) \mid X \in \hat{\mathcal{S}}_{(D, \rho)} : X_p = 0 \}, \\
S' &= \{ \epsilon_p \in S \mid \epsilon \in \hat{\mathcal{E}}_D \}, \\
\mathfrak{r}' &= \{ a_p \in \mathfrak{r} \mid a \in \mathfrak{R}_{(D, \rho)} \}.
\end{align*}
\]

Recall that \( \mathfrak{h} \) can be identified with the space of Killing vectors in \( \hat{\mathcal{S}}_{(D, \rho)} \) which vanish at \( p \); we denote the Killing vector corresponding to \( A \in \mathfrak{h} \) by \( X_A \). We can also identify \( S' \) with \( \hat{\mathcal{S}}_D \) and \( \mathfrak{r}' \) with \( \mathfrak{R}_{(D, \rho)} \), and will not distinguish in notation between \( D \)-parallel sections and their values at \( p \).

The evaluation at \( p \) map gives us a short exact sequence

\[
0 \longrightarrow \mathfrak{h} \longrightarrow \hat{\mathcal{S}}_{(D, \rho)} \longrightarrow V' \longrightarrow 0
\]

which can be split by a choice of linear map \( \Lambda : V' \to \hat{\mathcal{S}}_{(D, \rho)} \) with \( \Lambda(\nu)_p = \nu \), giving a vector space isomorphism \( V' \oplus \mathfrak{h} \to \hat{\mathcal{S}}_{(D, \rho)} \), with the map in one direction given by \((\nu, A) \mapsto \Lambda(\nu) + X_A\) and the other being \( X \mapsto \left( X_p, (A_X - \Lambda(X_p))_p \right) \). We also define the
map \( \lambda : V' \to so(V) \) by \( \lambda(v) = (A_{\Lambda(v)})_{\delta} \); this contains the same information as \( \Lambda \) since \( \Lambda(v) \) can be reconstructed from its Killing transport data \((v, \lambda(v))\). Recall that \( \lambda \) is unique up to addition of a map \( V' \to h \).

It is not difficult to see that \( a = V' \oplus S' \oplus h \oplus t' \) is a graded subalgebra of \( \mathfrak{h} \), and we can define a vector space isomorphism \( \hat{\Psi} : a \to \hat{\mathfrak{R}}(D, D, \rho) \) by

\[
\hat{\Psi}(v, \epsilon, A, a) = (\Lambda(v) + X_A, \epsilon, a),
\]
\[
\hat{\Psi}^{-1}(Y, \epsilon, a) = \left( Y_P, \epsilon, \left( A_{\Lambda(Y_p)} \right)_P, a \right)
\]

where \( v \in V', A \in h, \epsilon \in S' \cong \mathfrak{S}_D \) and \( a \in t' \cong \mathfrak{R}(D, D, \rho) \). We use \( \hat{\Psi} \) to define a new bracket \([\cdot, \cdot]'\) on \( a \):

\[
\hat{\Psi}\left( \left[ (v, \epsilon, A, a), (w, \zeta, B, b) \right] \right) = \left[ \hat{\Psi}(v, \epsilon, A, a), \hat{\Psi}(w, \zeta, B, b) \right] = \left[ (\Lambda(v) + X_A, \epsilon, a), (\Lambda(w) + X_B, \zeta, b) \right].
\]

By construction, \( \hat{\Psi} \) is a Lie algebra isomorphism \((a, [\cdot, \cdot]) \cong \hat{\mathfrak{R}}(D, D, \rho)\), and inverting it gives us the bracket \([\cdot, \cdot]'\) explicitly:

\[
[A, B]' = [A, B], \quad [a, b]' = [a, b], \quad [A, a]' = 0,
\]
\[
[A, v]' = Av + [A, \lambda(v)] - \lambda(Av), \quad [a, v]' = 0,
\]
\[
[A, \epsilon] = A \cdot \epsilon, \quad [a, \epsilon]' = a \cdot \epsilon,
\]
\[
[v, w]' = \lambda(v)w - \lambda(w)v + \theta_\lambda(v, w),
\]
\[
[v, \epsilon]' = \beta_p(v)\epsilon + \lambda(v) \cdot \epsilon,
\]
\[
[\epsilon, \zeta]' = \kappa_p(\epsilon, \zeta) + \gamma_p(\epsilon, \zeta) - \lambda(\kappa_p(\epsilon, \zeta)) + \rho_p(\epsilon, \zeta),
\]

where \( \theta_\lambda : \wedge^2 V' \to h \) is the map

\[
\theta_\lambda(v, w) = [\lambda(v), \lambda(w)] - R_p(v, w) - \lambda(\lambda(v)w - \lambda(w)v).
\]

We then have

\[
[\cdot, \cdot]' = [\cdot, \cdot] + \beta_p + \gamma_p + \rho_p + \partial \lambda + \theta_\lambda
\]

where \( \beta_p, \gamma_p, \rho_p \) and \( \theta_\lambda \) have been trivially extended to maps \( \wedge^2 \hat{\mathfrak{S}} \to \hat{\mathfrak{S}} \). The map \( \beta_p + \gamma_p + \partial \lambda \) has degree \(+2\) with respect to the grading on \( a \), while \( \theta \) has degree \(+4\), thus \( \hat{\mathfrak{R}}(D, D, \rho) \cong (a, [\cdot, \cdot]') \) is a filtered deformation of \((a, [\cdot, \cdot])\).

\[\square\]

### 4.3 Filtered subdeformations of the Poincaré superalgebra with R-symmetry

Now that we have established that Killing superalgebras with \( R \)-symmetry are filtered subdeformations of \( \hat{s} \), we will develop a theory of such deformations in much the same way as we did for filtered subdeformations of \( s \) in Section 3.3.

We consider graded subalgebras of the Poincaré superalgebra with \( R \)-symmetry \( \hat{s} \) of the form

\[
a = V' \oplus S' \oplus (h \oplus t')
\]
where \( a_{-1} = S' \) is a vector subspace of \( S \), \( a_{-2} = V' \) is a vector subspace of \( V \) such that \( \kappa_s \in V \) for all \( s \in S' \), and \( a_0 = \mathfrak{h} \oplus t' \) is a subalgebra of \( \hat{s}_0 = \mathfrak{so}(V) \oplus \mathfrak{r} \) such that \( \mathfrak{h} \) is a subalgebra of \( \mathfrak{so}(V) \) and \( \mathfrak{r} \) is a subalgebra of \( \mathfrak{r} \) and the action of \( a_0 \) preserves \( S' \) and \( V' \). Note that this is not the most general form of a graded subalgebra of \( \hat{s} \) since \( a_0 \) need not be of the form prescribed here; a general subalgebra of \( \hat{s} \) may be diagonally embedded with respect to the decomposition \( \hat{s}_0 = \mathfrak{so}(V) \oplus \mathfrak{r} \), but we will not discuss such cases here.

### 4.3.1 The Spencer (2,2)-cohomology

#### The complex

The degree-2 Spencer complex for a graded subalgebra \( a = V' \oplus S' \oplus (\mathfrak{h} \oplus t') \) of \( \hat{s} \) is

\[
0 \rightarrow C^{2,1}(a_{-}; a) = \text{Hom}(V', \mathfrak{h}) \oplus \text{Hom}(V', t') \\
\rightarrow C^{2,2}(a_{-}; a) = \text{Hom}(\wedge^2 V', V') \oplus \text{Hom}(V' \otimes S', S') \\
\hspace{2cm} \oplus \text{Hom}(\wedge^2 S', \mathfrak{h}) \oplus \text{Hom}(\wedge^2 S', t') \hspace{2cm} (4.124) \\
\rightarrow C^{2,3}(a_{-}; a) = \text{Hom}(V' \otimes \wedge^2 S', V') \oplus \text{Hom}(\wedge^3 S', S') \\
\rightarrow 0.
\]

We denote the projections to the components of \( C^{2,2}(a_{-}; a) \) by

\[
\begin{align*}
\pi_1 : C^{2,2}(a_{-}; a) & \rightarrow \text{Hom}(\wedge^2 V', V'), \\
\pi_2 : C^{2,2}(a_{-}; a) & \rightarrow \text{Hom}(V' \otimes S', S'), \\
\pi_3 : C^{2,2}(a_{-}; a) & \rightarrow \text{Hom}(\wedge^2 S', \mathfrak{h}), \\
\pi_4 : C^{2,2}(a_{-}; a) & \rightarrow \text{Hom}(\wedge^2 S', t').
\end{align*}
\]

(4.125)

The following two results provide an analogue to Corollary 3.17, but the presence of \( R \)-symmetry makes the conclusion that \( H^{2,1}(a_{-}; a) = Z^{2,1}(a_{-}; a) = 0 \) a little more technical. The first result is essentially the same as Corollary 3.17 without the homological claim.

**Corollary 4.20.** If \( V' = V \) then \( \pi_1 \circ \partial|_{\text{Hom}(V, \mathfrak{h})} : \text{Hom}(V, \mathfrak{h}) \rightarrow \text{Hom}(\wedge^2 V, V) \) is an injective map of \( \mathfrak{h} \oplus t' \)-modules. If \( \mathfrak{h} = \mathfrak{so}(V) \), it is an isomorphism.

**Lemma 4.21.** If \( t' \) acts faithfully on \( S' \) then

\[
\pi_2 \circ \partial|_{\text{Hom}(V, t')} : \text{Hom}(V, t') \rightarrow \text{Hom}(V' \otimes S', S')
\]

(4.126)

is an injective map of \( \mathfrak{h} \oplus t' \)-modules. If \( V' = [S', S'] \) then

\[
\pi_4 \circ \partial|_{\text{Hom}(V, t')} : \text{Hom}(V, t') \rightarrow \text{Hom}(\wedge^2 S', t')
\]

(4.127)

is an injective map of \( \mathfrak{h} \oplus t' \)-modules. In either case, if we also have \( V = V' \) then \( H^{2,1}(a_{-}; a) = Z^{2,1}(a_{-}; a) = 0 \).

**Proof.** Let \( \lambda \in \text{Hom}(V, t') \). Then if

\[
(\pi_2 \circ \partial)(\lambda)(\nu, s) = \lambda(\nu) \cdot s = 0
\]

(4.128)
for all \( v \in V' \) and \( s \in S' \), and \( \tau' \) acts faithfully on \( S' \), we must have \( \lambda = 0 \). On the other hand, if
\[
(\pi_4 \circ \partial)(s, s) = -\lambda(\kappa_s) = 0 \tag{4.129}
\]
for all \( s \in S' \), and \([S', S'] = V'\) then we must have \( \lambda = 0 \). The final claim follows from the first two and Corollary 4.20.

### The cohomology

Recall that \( \kappa \) is a surjective \( \tilde{s}_0 = \mathfrak{so}(V) \oplus \tau \)-invariant linear map \( \bigodot^2 S \to V \), whence we have an \( \tilde{s}_0 \)-module isomorphism \( V \cong \bigodot^2 S / \ker \kappa \). By standard theory on squares of spinor modules, we have a decomposition of \( \tilde{s}_0 \)-modules
\[
\bigodot^2 S \cong V \oplus \ker \kappa. \tag{4.130}
\]

We will denote the components of any map (linear map or \( \tilde{s}_0 \)-module map) with respect to this splitting \( \phi : \bigodot^2 S \to U \) (where \( U \) is some other vector space or module) by \( \phi_V \) and \( \phi_{\ker \kappa} \).

The proof of the following lemma is essentially the same as that of Lemma 3.18, except we use the extra normalisation condition that \( \rho_V = 0 \) to fix unique cocycle representatives of cohomology classes.

**Lemma 4.22.** Let \( \alpha + \beta + \gamma + \rho \) be a cocycle in \( C^{2,2}(\tilde{s}_-; \tilde{s}) \) where
\[
\begin{align*}
\alpha & \in \text{Hom}(\bigodot^2 V, V), \\
\beta & \in \text{Hom}(V \otimes S, S), \\
\gamma & \in \text{Hom}(\bigodot^2 S, \mathfrak{so}(V)), \\
\rho & \in \text{Hom}(\bigodot^2 S, \tau).
\end{align*}
\]

Then the cohomology class \( [\alpha + \beta + \gamma + \rho] \in H^{2,2}(\tilde{s}_-; \tilde{s}) \) has a unique representative \( \bar{\beta} + \bar{\gamma} + \bar{\rho} \) with zero \( \text{Hom}(\bigodot^2 V, V) \) component and for which \( \bar{\rho}_V = 0 \). We denote the space of such normalised cocycles by \( \overline{\mathcal{H}}^{2,2} \):
\[
\overline{\mathcal{H}}^{2,2} = \left\{ \begin{array}{l}
\beta + \gamma + \rho \in \mathbb{Z}^{2,2}(\tilde{s}_-; \tilde{s}) \\
\beta \in \text{Hom}(V \otimes S, S), \\
\gamma \in \text{Hom}(\bigodot^2 S, \mathfrak{so}(V)), \\
\rho \in \text{Hom}(\bigodot^2 S, \tau), \rho_V = 0
\end{array} \right\} \tag{4.132}
\]

and then we have \( H^{2,2}(\tilde{s}_-; \tilde{s}) \cong \overline{\mathcal{H}}^{2,2} \) as \( \tilde{s}_0 \)-modules.

**Proof.** As in the proof of the analogous Lemma 3.18, by Lemma 3.16 there is a unique map \( \lambda_1 : V \to \mathfrak{so}(V) \) such that \( \alpha = \partial \lambda_1 |_{\bigodot^2 V} \). Moreover, using the decomposition of \( \bigodot^2 S \) discussed above, we can write \( \rho_V = \partial \lambda_2 |_{\bigodot^2 S} \) for some \( \lambda_2 : V \to \tau \). Then we can define a cocycle \( \bar{\beta} + \bar{\gamma} + \bar{\rho} \in \mathbb{Z}^{2,2}(\tilde{s}_-; \tilde{s}) \) by
\[
\bar{\beta} + \bar{\gamma} + \bar{\rho} = \alpha + \beta + \gamma + \rho - \partial \lambda \tag{4.133}
\]

where \( \lambda = \lambda_1 + \lambda_2 \in \text{Hom}(V, \mathfrak{so}(V) \oplus \tau) = C^{2,1}(\tilde{s}_-; \tilde{s}) \) which by construction lies in \( \overline{\mathcal{H}}^{2,2} \).

It follows that \( \mathbb{Z}^{2,2} = \overline{\mathcal{H}}^{2,2} \oplus B^{2,2} \) as \( \tilde{s}_0 \)-modules, hence the final claim. \( \square \)
The normalised $(2, 2)$-cocycle condition is the following pair of equations:

\[2\kappa(s, \beta(v, s)) + \gamma(s, s)v = 0,\]
\[\beta(s, s) + \gamma(s, s) \cdot s + \rho(s, s)s = 0,\]

for \(v \in V', s \in S'\); we note that we must also separately impose the \(\rho_V\) condition. Similarly to the case of normalised cocycles for \(s\) (see Lemma 3.18 and the discussion following it), a normalised cocycle \(\beta + \gamma + \rho\) is invariant under the action of a subalgebra \(\mathfrak{h} \oplus \mathfrak{r}' \subseteq \mathfrak{s}_0(V) \oplus \mathfrak{r}\) if and only if \(\beta, \gamma\) and \(\rho\) are all separately invariant. Moreover, invariance of \(\beta\) implies invariance of \(\gamma\), whence invariance of the cocycle is equivalent to invariance of \(\beta\) and \(\rho\).

We also note that if we set \(\rho = 0\), the equations above are precisely the normalised cocycle conditions (3.118), (3.119) for \(s\). Thus we have \(\mathcal{H}^{2, 2} = \mathcal{H}^{2, 2} \cap \{\rho = 0\}\).

### 4.3.2 General filtered deformations and cohomology

#### Parametrising the filtered deformations

A filtered deformation \(\tilde{a}\) of a graded subalgebra \(a = V' \oplus S' \oplus (\mathfrak{h} \oplus \mathfrak{r}')\) of \(\hat{a}\) has the brackets

\[
\begin{align*}
[A, B] &= AB - BA, & [a, b] &= ab - ba, & [a, A] &= 0, \\
[A, v] &= Av + \delta(A, v), & [a, v] &= \delta(a, v), & [s, s] &= \kappa_s + \gamma(s, s) + \rho(s, s), \\
[A, s] &= A \cdot s, & [a, s] &= as, & [v, s] &= \beta(v, s), \\
[v, w] &= \alpha(v, w) + \theta(v, w)
\end{align*}
\]

where \(A, B \in \mathfrak{h}, v, w \in V', s \in S', a \in \mathcal{A}'.\) The deformation maps of degree 2 are

\[\alpha : \wedge^2 V' \to V', \quad \beta : V' \otimes S' \to S', \quad \gamma : \bigodot^2 S' \to \mathfrak{h}, \quad \rho : \bigodot^2 S' \to \mathfrak{r}',\]

as well as a map \(\delta : a_0 \otimes V' \to a_0\) which has the four components

\[\delta_1 : \mathfrak{h} \otimes V' \to \mathfrak{h}, \quad \delta_2 : \mathfrak{h} \otimes V' \to \mathfrak{r}', \quad \delta_3 : \mathfrak{r}' \otimes V' \to \mathfrak{h}, \quad \delta_4 : \mathfrak{r}' \otimes V' \to \mathfrak{r}'.\]

The degree-4 deformation map is

\[
\theta : \wedge^2 V' \to a_0
\]

with components

\[\theta_1 : \wedge^2 V' \to \mathfrak{h}, \quad \theta_2 : \wedge^2 V' \to \mathfrak{r}'.\]

We denote the full degree-2 deformation by

\[\mu = \alpha + \beta + \gamma + \rho + \delta : a \otimes a \to a\]
so the defining sequence for the deformation is \((\mu, \theta, 0, \ldots)\). Then from Proposition 2.6, we have the following:

\[
\begin{align*}
\mu & \in Z^{2,2}(a; a), \quad (4.142) \\
\mu & |_{a \otimes a} \in Z^{2,2}(a_-, a), \quad (4.143) \\
\left[ \mu |_{a \otimes a} \right] & \in H^{2,2}(a_-; a)^{a_0}. \quad (4.144)
\end{align*}
\]

The first condition implies the other two, as we will soon see explicitly. In the case without \(r\)-symmetry, we saw that there was a converse for subalgebras with \(V' = V\) (Lemma 3.19); in the present case, the situation is more complicated.

**Unpacking the cocycle conditions**

The condition (4.142) is equivalent to the following system of equations:

\[
\begin{align*}
\alpha & (\kappa, s, v) + 2\kappa(s, \beta(v, s)) + \gamma(s, s)v = 0, \quad (4.145) \\
\beta & (\kappa, s) + \gamma(s, s) \cdot s + \rho(s, s)s = 0, \quad (4.146)
\end{align*}
\]

for all \(v \in V', s \in S'\) (note that the condition (4.143) is equivalent to this pair of equations alone):

\[
\begin{align*}
A\alpha(v, w) - \alpha(Av, w) - \alpha(v, Aw) & - \delta_1(A, v)w + \delta_1(A, w)v = 0, \quad (4.147) \\
A \cdot (\beta(v, s)) - \beta(Av, s) - \beta(v, A \cdot s) & - \delta_1(A, v) \cdot s - \delta_2(A, v)s = 0, \quad (4.148) \\
\left[ A, \gamma(s, s) \right] - 2\gamma(A \cdot s, s) & + \delta_1(A, \kappa_s) = 0, \quad (4.149) \\
-2\rho(A \cdot s, s) + \delta_2(A, \kappa_s) & = 0, \quad (4.150)
\end{align*}
\]

for all \(A \in \mathfrak{h}, v, w \in V, s \in S'\);

\[
\begin{align*}
\delta_3(a, v)w - \delta_3(a, w)v & = 0, \quad (4.151) \\
\alpha(\beta(v, s)) - \beta(v, as) - \delta_3(a, v) \cdot s - \delta_4(a, v)s & = 0, \quad (4.152) \\
-2\gamma(as, s) + \delta_3(a, \kappa_s) & = 0, \quad (4.153) \\
\left[ a, \rho(s, s) \right] - 2\rho(as, s) + \delta_4(a, \kappa_s) & = 0, \quad (4.154)
\end{align*}
\]

for all \(a \in \mathfrak{t}', v, w \in V, s \in S'\);

\[
\begin{align*}
\delta_1([A, B], v) - [A, \delta_1(B, v)] + [B, \delta_1(A, v)] - \delta_1(A, Bv) & + \delta_1(B, Av) = 0, \quad (4.155) \\
\delta_2([A, B], v) - \delta_2(A, Bv) + \delta_2(B, Av) & = 0, \quad (4.156)
\end{align*}
\]

for all \(A, B \in \mathfrak{h}, v \in V'\);

\[
\begin{align*}
\delta_3([a, b], v) & = 0, \quad (4.157) \\
\delta_4([a, b], v) - [a, \delta_4(b, v)] + [b, \delta_4(a, v)] & = 0, \quad (4.158)
\end{align*}
\]

for all \(a, b \in \mathfrak{t}', v \in V'\); and

\[
\begin{align*}
[A, \delta_3(a, v)] - \delta_3(a, Av) & = 0, \quad (4.159)
\end{align*}
\]
for all $A \in \mathfrak{h}$, $a \in \mathfrak{r}'$, $\nu \in V$. The (4.145) and (4.146). Equations (4.147)-(4.150) are equivalent to

$$A \cdot (\alpha + \beta + \gamma + \rho) = \partial t_1(\delta_1 + \delta_2),$$

(4.161)

for all $A \in \mathfrak{h}$, and (4.151)-(4.154) are equivalent to

$$a \cdot (\alpha + \beta + \gamma + \rho) = \partial t_a(\delta_3 + \delta_4),$$

(4.162)

for all $a \in \mathfrak{r}'$, and this last pair of equations together imply the condition (4.144). The remaining equations (4.155)-(4.160) can be interpreted as yet another cocycle condition (using the isomorphism of $a_0$-modules $\text{Hom}(a_0 \otimes V', a_0) \cong \text{Hom}(a_0, (V')^* \otimes a_0)$):

$$\delta \in Z^1(a_0; (V')^* \otimes a_0).$$

(4.163)

Perhaps more usefully, we note that each of the equations (4.155)-(4.159) involves only one of the components of $\delta$, and these have the following homological interpretations:

\begin{align*}
(4.155) &\iff \delta_1 \in Z^1(\mathfrak{h}; (V')^* \otimes \mathfrak{h}), \\
(4.156) &\iff \delta_2 \in Z^1(\mathfrak{h}; (V')^* \otimes \mathfrak{r}') \cong Z^1(\mathfrak{h}; (V')^* \otimes \mathfrak{r'}), \\
(4.157) &\iff \delta_3 \in Z^1(\mathfrak{r}'; (V')^* \otimes \mathfrak{h}) \cong Z^1(\mathfrak{r}'; (V')^* \otimes \mathfrak{h}), \\
(4.158) &\iff \delta_4 \in Z^1(\mathfrak{r}'; (V')^* \otimes \mathfrak{r}') \cong Z^1(\mathfrak{r}'; (V')^* \otimes \mathfrak{r}'), \\
(4.159) &\iff \delta_3 \in Z^0(\mathfrak{h}; (\mathfrak{r'} \otimes (V')^* \otimes \mathfrak{h}) \cong Z^0(\mathfrak{h}; (V')^* \otimes \mathfrak{h}) \otimes (\mathfrak{r}')^*,
\end{align*}

where the isomorphisms are of $\mathfrak{h} \oplus \mathfrak{r}'$-modules and are due to the fact that $\mathfrak{h}$ and $\mathfrak{r}'$ act trivially on one another and $\mathfrak{r}'$ acts trivially on $(V')^*$. One must remember to supplement this interpretation with the mixed equation (4.160), however. Another alternative is as follows:

\begin{align*}
(4.155), (4.157), (4.159) &\iff \delta_1 + \delta_3 \in Z^1(\mathfrak{h} \oplus \mathfrak{r}'; (V')^* \otimes \mathfrak{h}), \\
(4.156), (4.158), (4.160) &\iff \delta_2 + \delta_4 \in Z^1(\mathfrak{h} \oplus \mathfrak{r}'; (V')^* \otimes \mathfrak{r}').
\end{align*}

Clearly, general filtered subdeformations of $\hat{s}$ are much more complicated than those of $s$, even a the infinitesimal level. Nonetheless, we do have the following analogue of Lemma 3.19. There are some more technicalities here, but the upshot is essentially that if $S'$ is sufficiently large, a Spencer cocycle with $\mathfrak{a}_0$-invariant cohomology class uniquely determines $\delta$ and therefore a cocycle in $Z^2(\mathfrak{a}; \mathfrak{a})$.

**Lemma 4.23.** If $\mu \in Z^2(\mathfrak{a}; \mathfrak{a})$ then $\mu|_{\mathfrak{a}_0 \mathfrak{a}_0} \in Z^{2,2}(\mathfrak{a}_0; \mathfrak{a}_0)$ and $|\mu|_{\mathfrak{a}_0 \mathfrak{a}_0} \in H^{2,2}(\mathfrak{a}_0; \mathfrak{a}_0)$ and $\mu|_{\mathfrak{a} \mathfrak{a}_0} \in Z^{2,2}(\mathfrak{a}; \mathfrak{a})$ such that $|\mu| \in H^{2,2}(\mathfrak{a}; \mathfrak{a})^{\mathfrak{a}_0}$ then there exists a unique $\delta \in Z^1(\mathfrak{a}_0; V^* \otimes \mathfrak{a}_0)$ such that $\mu = \mu_+ + \delta \in Z^2(\mathfrak{a}; \mathfrak{a})$.

**Proof.** The first claim is immediate from the discussion above. For the converse statement, we first note that by Lemma 4.21, we have $Z^{2,1}(\mathfrak{a}_0; \mathfrak{a}_0) = 0$; that is, $\partial : C^{2,1}(\mathfrak{a}_0; \mathfrak{a}) \rightarrow C^{2,2}(\mathfrak{a}_0; \mathfrak{a})$ is injective. The rest of the proof proceeds exactly as in the
discussion preceding Lemma 3.19. Indeed, invariance of the Spencer cohomology class means that for all \( X \in a_0 \),
\[
X \cdot \mu_- = \partial \chi_X; \tag{4.164}
\]
for some \( \chi_X \in C^2,1(a_-; a) = \text{Hom}(V; a_0) \), but by injectivity of \( \partial \), \( \chi_X \) is unique and the assignment \( X \mapsto \chi_X \) linear. Thus we can uniquely define a linear map \( \delta : a_0 \otimes V \rightarrow a_0 \) by \( \delta(X, \nu) = \chi_X(\nu) \). By construction, the components of \( \delta \) satisfy the equations (4.147)-(4.147) (where of course \( \mu = \alpha + \beta + \gamma + \delta \) as usual); it remains only to check that (4.155)-(4.160) are satisfied, i.e. that \( \delta \in Z^1(a_0; V^* \otimes a_0) \). Defining a map \( \delta_{X,Y} \in C^2,1(a_-; a) = \text{Hom}(V; a_0) \) by
\[
\delta_{X,Y} = X \cdot 1_Y \delta - Y \cdot 1_X \delta - i_{[X,Y]} \delta \tag{4.165}
\]
for all \( X, Y \in a_0 \), for \( \nu \in V \) we can easily show that \( (\partial \delta)(X, Y, \nu) = \delta_{X,Y}(\nu) \) identically, while on the other hand, using the defining relation for \( \chi_X \),
\[
(\partial \delta_{X,Y})(\nu) = X \cdot \partial \chi_Y - Y \cdot \partial \chi_X - \partial \chi_{[X,Y]} = 0, \tag{4.166}
\]
hence \( \delta_{X,Y} \) vanishes for all \( X, Y \in a_0 \) by injectivity of \( \delta \), thus \( \partial \delta = 0 \). \( \square \)

Of course, we can be a little more explicit about the form of the solution.

**Lemma 4.24.** Let \( V = V' \). Then we have the following:

- The general solution \((\alpha, \delta_1)\) to equation (4.147) is
\[
\alpha(v, w) = \lambda_1(v) w - \lambda_1(w) v, \tag{4.167}
\]
\[
\delta_1(A, v) = [A, \lambda_1(v)] - \lambda_1(Av), \tag{4.168}
\]
for all \( A \in \mathfrak{h}, v, w \in V \), where \( \lambda_1 : V \rightarrow \mathfrak{so}(V) \) is some linear map. Equation (4.155) is identically satisfied by \( \delta_1 \) in this form.

- Equation (4.151) implies
\[
\delta_3 = 0, \tag{4.169}
\]
and (4.157), (4.159) are trivially satisfied.

- If \( V = [S', S'] \) then equations (4.150) and (4.154) define \( \delta_2 \) and \( \delta_4 \) uniquely in terms of \( \rho \), and the equations (4.156), (4.158) and (4.160) are then identically satisfied.

- If \( \nu' \) acts faithfully on \( S' \) then equations (4.148) and (4.152) define \( \delta_2 \) and \( \delta_4 \) uniquely in terms of \( \beta \) and \( \lambda_1 \), and (4.158) is then identically satisfied.

**Proof.** We have essentially already seen that the first claim follows from Lemma 3.16 with some straightforward calculations (see the discussion following Lemma 3.19). The second claim follows immediately from Lemma 3.16. If \( V = [S', S'] \) then clearly \( \delta_2 \) and \( \delta_4 \) are determined by equations (4.150) and (4.154) respectively. We will verify the remaining claim for \( \nu = \kappa_s \), with \( s \in S' \); if \( V = [S', S'] \) then they must therefore hold for \( \nu \in V \). We can depolarise (4.150) to find
\[
\delta_2(A, \kappa(s_1, s_2)) = \rho(A \cdot s_1, s_2) + \rho(s_1, A \cdot s_2) \tag{4.170}
\]
for \( s_1, s_2 \in S \) and \( A \in \mathfrak{h} \). Using the above along with \( \mathfrak{so}(V) \)-equivariance of \( \kappa \), we have

\[
\delta_2([A, B], \kappa_s) - \delta_2(A, B \kappa_s) + \delta_2(B, A \kappa_s) \\
= \delta_2([A, B], \kappa_s) - 2\delta_2(A, \kappa(B \cdot s), s) + 2\delta_2(B, \kappa(A \cdot s), s) \\
= 2\rho([A, B] \cdot s, s) - 2\rho(A \cdot (B \cdot s), s) - 2\rho(B \cdot s, A \cdot s) \\
\quad + 2\rho(B \cdot (A \cdot s), s) + 2\rho(A \cdot s, B \cdot s) \\
= 2\rho([A, B] \cdot s - A \cdot (B \cdot s) + B \cdot (A \cdot s), s) = 0,
\]

which is (4.156). Similarly, depolarising (4.154) yields

\[
\delta_4(a, \kappa(s_1, s_2)) = \rho(as_1, s_2) + \rho(s_1, as_2) - [a, \rho(s_1, s_2)],
\]

for \( s_1, s_2 \in S \) and \( a \in \mathfrak{r} \), and we then have

\[
\delta_4([a, b], \kappa_s) - [a, \delta_4(b, \kappa_s)] + [b, \delta_4(a, \kappa_s)] \\
= 2\rho([a, b] s, s) - [([a, b], \rho(s, s)] - 2[a, \rho(bs, s)] \\
\quad + [a, [b, \rho(s, s)]] + 2[b, \rho(as, s)] - [b, [a, \rho(s, s)]] \\
= 2\rho(a(bs), s) - 2\rho(b(as), s) - 2[a, \rho(bs, s)] + 2[b, \rho(as, s)] \\
= -2\delta_4(a, \kappa(bs, s)) - 2\rho(bs, as) + 2\delta_4(b, \kappa(as, s)) + 2\rho(as, bs) \\
= 0
\]

where in the final line we have used the symmetry of \( \rho \) and \( \mathfrak{r} \)-invariance of \( \kappa \); this is (4.158). Using both depolarised equations as well as all of the same symmetry and invariance properties, we have

\[
[a, \delta_2(A, \kappa_s)] + \delta_4(a, A \kappa_s) = 2[a, \rho(A \cdot s, s)] + 2\delta_4(a, \kappa(A \cdot s, s)) \\
= 2\rho(a(A \cdot s), s) + 2\rho(A \cdot s, as) \\
= 2\rho(A \cdot (as), s) + 2\rho(as, A \cdot s) \\
= 2\delta_2(A, \kappa(as, s)) = 0,
\]

which is (4.160).

If \( \mathfrak{r}' \) acts faithfully on \( S' \), the first part of the final claim is immediate; for the second part, we act the left-hand side of (4.158) on \( s \in S' \):

\[
\delta_4([a, b], v)s - [a, \delta_4(b, v)]s + [b, \delta_4(a, v)]s \\
= [a, b] \beta(v, s) - \beta(v, [a, b]) \\
\quad - ab \beta(v, s) + a \beta(v, bs) + b \beta(v, as) - \beta(v, bs) \\
\quad + ba \beta(v, s) - b \beta(v, as) - a \beta(v, bs) + \beta(v, abs) \\
= ([a, b] - ab + ba) \beta(v, s) - \beta(v, ([a, b] - ab + ba)s) = 0,
\]

whence, by faithfulness of the action of \( \mathfrak{r}' \) on \( S' \), we can abstract \( s \) to obtain (4.158).  \( \Box \)
We note that, in the final two points of the lemma, the maps $\delta_2, \delta_4$ are not automatically well-defined for arbitrary $\beta, \rho, \lambda$; for example, one requires that $\rho(s, A \cdot s)$ depends only on $\kappa$, and that the endomorphism of $S'$ given by $s \mapsto \alpha \beta(v, s) - \beta(v, as)$ lies in $t'$. Furthermore, in the final point, the conditions (4.156) and (4.160) do not seem to be automatically satisfied by $\delta_2, \delta_4$ defined by (4.148) and (4.152). However, by Lemma 4.23, these obstructions are not present if we assume that $\alpha + \beta + \gamma + \rho$ is a cocycle with $\alpha_0$-invariant cohomology class.

**Jacobi identities**

The Jacobi identities are as follows. By “mixed” cocycle conditions, we mean the conditions on the Chevalley-Eilenberg cocycle $\alpha + \beta + \gamma + \rho + \delta$ which involve both the components of the Spencer cocycle $\alpha + \beta + \gamma + \rho$ and $\delta$.

- $[\tilde{a}_0, \tilde{a}_0, \tilde{a}_0]$: This is the Jacobi identity for the Lie algebra $\alpha_0 = h \oplus t'$.
- $[\tilde{a}_0, \tilde{a}_0, \tilde{a}_{-1}]$: Follows from $S'$ being a representation of $\tilde{a}_0$. In particular, it is a representation of both $h$ and $t'$, and these actions commute with one another.
- $[\alpha_0, \alpha_0, \tilde{a}_{-2}]$: Equivalent to the cocycle conditions (4.155)-(4.160) for $\delta$.
- $[h, \tilde{a}_{-1}, \tilde{a}_{-1}]$: Using the $so(V) \oplus t'$-equivariance of $\kappa$, this is equivalent to the mixed cocycle conditions (4.149), (4.150), (4.153) and (4.154).
- $[h, \tilde{a}_{-1}, \tilde{a}_{-2}]$: Equivalent to mixed cocycle condition (4.148).
- $[t', \tilde{a}_{-1}, \tilde{a}_{-2}]$: Equivalent to mixed cocycle condition (4.152).
- $[h, \tilde{a}_{-2}, \tilde{a}_{-2}]$: This has components in $V'$, $h$ and $t$; the first is equivalent to the mixed condition (4.147), and the the other two are

\[
(A \cdot \theta_1)(v, w) = \delta_1(\delta_1(A, v), w) + \delta_3(\delta_2(A, v), w) \\
- \delta_1(\delta_1(A, w), v) - \delta_3(\delta_2(A, w), v) - \delta_1(A, \alpha(v, w)),
\]

\[
(A \cdot \theta_2)(v, w) = \delta_2(\delta_1(A, v), w) + \delta_4(\delta_2(A, v), w) \\
- \delta_2(\delta_1(A, w), v) - \delta_4(\delta_2(A, w), v) - \delta_2(A, \alpha(v, w)),
\]

for all $A \in h$ and $v, w \in V'$.

- $[t', \tilde{a}_{-2}, \tilde{a}_{-2}]$: Again, there are components in $V'$, $h$ and $t$. The first is equivalent to mixed condition (4.151), and the others are

\[
\delta_1(\delta_3(a, v), w) + \delta_3(\delta_4(a, v), w) \\
- \delta_1(\delta_3(a, w), v) - \delta_3(\delta_4(a, w), v) - \delta_3(a, \alpha(v, w)) = 0,
\]

\[
[a, \theta_2(v, w)]_1 = \delta_2(\delta_3(a, v), w) + \delta_4(\delta_4(a, v), w) \\
- \delta_2(\delta_3(a, w), v) - \delta_4(\delta_4(a, w), v) - \delta_4(a, \alpha(v, w)),
\]

for all $a \in t'$ and $v, w \in V'$.

- $[\tilde{a}_{-1}, \tilde{a}_{-1}, \tilde{a}_{-1}]$: Equivalent to the second (2,2)-Spencer cocycle condition (4.146).
• $[\widetilde{a}_{-1}, \widetilde{a}_{-1}, \widetilde{a}_{-2}]$: This has components in $V'$, $\mathfrak{h}$ and $\mathfrak{r}$. The first is the first Spencer cocycle condition (4.145), and the others are

$$\begin{align*}
\theta_1(\kappa, s, v) + \delta_1(\gamma(s, s), v) + \delta_3(\rho(s, s), v) + 2\gamma(s, \beta(v, s)) &= 0, \quad (4.180) \\
\theta_2(\kappa, s, v) + \delta_2(\gamma(s, s), v) + \delta_4(\rho(s, s), v) + 2\rho(s, \beta(v, s)) &= 0, \quad (4.181)
\end{align*}$$

for all $s \in S'$, $v \in V'$.

• $[\widetilde{a}_{-1}, \widetilde{a}_{-2}, \widetilde{a}_{-2}]$: For $s \in S'$, $v, w \in V'$,

$$\begin{align*}
\theta_1(v, w) \cdot s + \theta_2(v, w)s \\
+ \beta(\alpha(v, w), s) - \beta(v, \beta(w, s)) + \beta(w, \beta(v, s)) &= 0. \quad (4.182)
\end{align*}$$

• $[\widetilde{a}_{-2}, \widetilde{a}_{-2}, \widetilde{a}_{-2}]$: This has components in $V'$, $\mathfrak{h}$ and $\mathfrak{r}$ which are as follows:

$$\begin{align*}
\theta_1(u, v) w + \alpha(\alpha(u, v), w) + \ldots &= 0, \quad (4.183) \\
\theta_1(\alpha(u, v), w) + \delta_1(\theta_1(u, v), w) + \delta_3(\theta_2(u, v), w) + \ldots &= 0, \quad (4.184) \\
\theta_2(\alpha(u, v), w) + \delta_2(\theta_1(u, v), w) + \delta_4(\theta_2(u, v), w) + \ldots &= 0, \quad (4.185)
\end{align*}$$

where by $\ldots$ we denote cyclic permutations of the explicitly given terms in $u, v, w \in V$.

Thus, as with filtered subdeformations of $s$, we find that there are two types of constraints on the deformation maps given by the Jacobi identities: those linear in the deformation maps, which are the cocycle conditions and involve only maps of degree 2, and those quadratic in the deformation maps, which involve the degree-4 map $\theta$. Determining a general filtered deformation of a graded subalgebra $\mathfrak{a} \subseteq \widehat{\mathfrak{s}}$ therefore consists of solving the cocycle conditions and then checking for consistency with the quadratic equations (4.176)-(4.185).

### 4.3.3 Highly supersymmetric and transitive graded subalgebras

As in §3.3.3, we work in Lorentzian signature with a symmetric and causal squaring map and can take advantage of the Homogeneity Theorem 3.21. We have already seen in various places that the assumptions that $V' = V = [S', S']$ and that $\mathfrak{r}'$ acts faithfully on $S'$ give significant simplifications.

We call a graded subalgebra $\mathfrak{a} = V \oplus S' \oplus (\mathfrak{h} \oplus \mathfrak{r}')$ of $\widehat{\mathfrak{s}}$ with $\dim S' > \frac{1}{2} \dim S$ *highly supersymmetric*, and it is *maximally supersymmetric* if $S' = S$, and likewise for their filtered deformations. The Homogeneity Theorem (Theorem 3.21) tells us that we must have $a_{-2} = V = [S', S']$ for such subalgebras, and the subalgebra $\mathfrak{h}$ of $\mathfrak{so}(V)$ acts faithfully on $S'$ by Corollary 3.22. In places, we will need to assume that $\mathfrak{r}'$ also acts faithfully. While this is somewhat restrictive, for most applications in Lorentzian signature it is not problematic, thanks to the following lemma.

**Lemma 4.25.** If $R$ is compact, $\mathfrak{r}'$ does not act faithfully on $S'$ then $\mathfrak{r}' = \mathfrak{r}'' \oplus \text{ann}_{\mathfrak{r}'}(S')$ for some subalgebra $\mathfrak{r}''$ of $\mathfrak{r}'$ which does act faithfully on $S'$.

**Proof.** If $R$ is compact then $\mathfrak{r}$ admits an ad-invariant inner product which restricts to an ad-invariant positive-definite inner product on $\mathfrak{r}'$. Now, since $\text{ann}_{\mathfrak{r}'}(S')$ is an ideal...
in \( \mathfrak{t}' \), its orthogonal complement \( \mathfrak{t}'' := \text{ann}_{\mathfrak{t}'}(S') \) in \( \mathfrak{t}' \) is an ideal which commutes with \( \text{ann}_{\mathfrak{t}'}(S') \); indeed, for all \( x \in \mathfrak{t}' \), \( y \in \mathfrak{t}'' \) and \( z \in \text{ann}_{\mathfrak{t}'}(S') \),

\[
\langle [x, y], z \rangle = -\langle y, [x, z] \rangle = 0 \tag{4.186}
\]

since \( \text{ann}_{\mathfrak{t}'}(S') \) is an ideal, and then since both \( \text{ann}_{\mathfrak{t}'}(S') \) and \( \mathfrak{t}'' \) are both ideals, we have \( \{ y, z \} \in \text{ann}_{\mathfrak{t}'}(S') \cap \mathfrak{t}'' \). But since the two ideals are orthogonal we have \( \text{ann}_{\mathfrak{t}'}(S') \cap \mathfrak{t}'' = 0 \), thus \( \{ y, z \} = 0 \) and \( \mathfrak{t}'' \) acts faithfully on \( S' \). \( \square \)

Recall from our discussion in §2.2.3 that in Lorentzian signature, it is always possible to choose a symmetric squaring map such that the \( R \)-symmetry group is compact (see [30, 58]). The upshot of this lemma is that if we have chosen such a map and \( \mathfrak{t}' \) does not act faithfully, we have \( a = a' \oplus \text{ann}_{\mathfrak{t}'}(S') \) as Lie algebras where \( a' = V \oplus S' \oplus \mathfrak{so}(V) \oplus \mathfrak{t}'' \) is a graded subalgebra of \( a \) (thus of \( \mathfrak{s} \)) with \( \mathfrak{t}'' \) acting faithfully on \( S' \), and we can simply choose to work with \( a' \) instead of \( a \). We will therefore freely assume whenever necessary that \( \mathfrak{t}' \) acts faithfully. If the subgroup \( R' \) of \( R \) generated by \( \mathfrak{t}' \) is closed, we can integrate this result to a statement about groups.

**Lemma 4.26.** Suppose \( R \) is compact and let \( \mathfrak{t}' \) be a subalgebra of \( \mathfrak{t} \) such that the connected subgroup \( R' \) of \( R \) generated by \( \mathfrak{t}' \) is closed. Then \( R' = R'' \times K \) where \( R'' \) and \( K \) are closed Lie subgroups which respectively act effectively and trivially on \( S' \).

**Proof.** Since \( R \) is compact, if \( R' \) is closed then it is also compact. Thus \( S \) admits an \( R' \)-invariant positive-definite inner product and hence an invariant \( R' \)-invariant decomposition \( S = S' \oplus (S')^{\perp} \). Since \( R' \) consists of invertible endomorphisms of \( S \), with an adapted choice of basis \( \{ s_{i} \}_{i=1}^{\dim S} \) where \( \{ s_{i} \}_{i=1}^{\dim S} \) is a basis for \( S \) and \( \{ s_{i} \}_{i=k+1}^{\dim S} \) is a basis for \( (S')^{\perp} \), we can write its elements as invertible block matrices

\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
\tag{4.187}
\]

where \( A, B \) are invertible matrices of size \( k \) and \( n-k \) respectively. Thus as a Lie group, \( R' = R'' \times K \) where

\[
R'' = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in R' \right\}, \quad K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \in R' \right\}; \tag{4.188}
\]

clearly both of these groups are closed and \( K \) is the kernel of the representation of \( R' \) on \( S' \), so it acts trivially on \( S' \) and \( R'' \) acts effectively on \( S' \). \( \square \)

Now that we have argued that we may choose \( \mathfrak{t}' \) to act faithfully on \( S' \) without loss of generality, we have the following result as a direct analogue of Lemma 3.24. By the second bullet point in the lemma, we will refer to \( a \) with faithfully-acting \( \mathfrak{t}' \) as *transitive* (recall Definition 2.7).

**Lemma 4.27.** Let \( a = V \oplus S' \oplus (\mathfrak{h} \oplus \mathfrak{t}') \) be a highly supersymmetric graded subalgebra of \( \mathfrak{s} \). Then

- \( a \) is fundamental,
- \( a \) is transitive if and only if \( \mathfrak{t}' \) acts faithfully on \( S' \),

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• $a$ is a full prolongation of degree 2.
• $H^{d,2}(a_-; a) = 0$ for all even $d > 2$.

Proof. Most of the proof of Lemma 3.24 goes through *mutatis mutandis*; we note however that the vanishing of $H^{2,1}(a_-; a)$ is somewhat less trivial but follows by Lemma 4.21. The only other condition to be more carefully checked is transitivity, which is equivalent to $\mathfrak{h} \oplus \mathfrak{r}'$ acting faithfully on $V \oplus S'$. Clearly then, if $a$ is transitive, $\mathfrak{r}'$ must act faithfully on $S'$. Conversely, let us assume that there exist $A \in \mathfrak{h}$ and $a \in \mathfrak{r}'$ such that $(A, a) \cdot s = A \cdot s + as = 0$ for all $s \in S'$. By $\mathfrak{so}(V)$- and $\mathfrak{r}$-invariance of $\kappa$, we find that $A \kappa_s = 0$ for all $s \in S'$, so by homogeneity $A$ acts trivially on $V$. But the action of $\mathfrak{h}$ on $V$ is faithful, so $A = 0$, thus $a \cdot s = 0$ for all $s \in S'$. If $\mathfrak{r}'$ acts faithfully on $S'$, we must have $a = 0$, showing that $a$ is transitive.

The following is the analogue of Proposition 3.25; note that we must once again assume that $a$ is transitive. The proof is identical.

**Proposition 4.28.** Let $a = V' \oplus S' \oplus (\mathfrak{h} \oplus \mathfrak{r}')$ be a graded subalgebra of $\widehat{a}$ and $\widehat{a}$ a filtered deformation of $a$ with defining sequence $(\mu, \theta, 0, \ldots)$ as above. Then

1. $\mu|_{a_- \hookrightarrow a}$ is a cocycle in $C^{2,2}(a_-; a)$, and
   \[ [\mu|_{a_- \hookrightarrow a}] \in H^{2,2}(a_-; a)^a. \quad (4.189) \]
   Furthermore, $\mu$ is a cocycle in $C^2(a; a)$.

2. Suppose that $\dim S' > \frac{1}{2} \dim S$ and $a$ is transitive ($\mathfrak{r}'$ acts faithfully on $S'$). Then if $\widehat{a}'$ is another filtered deformation of $a$ with degree-2 deformation map $\mu'$ such that $[\mu'|_{a_- \hookrightarrow a}] = [\mu|_{a_- \hookrightarrow a}]$ then $\widehat{a} \cong \widehat{a}'$ as filtered Lie superalgebras.

We are once again interested in calculating the “infinitesimal deformations” of highly supersymmetric $a$, that is the $a_0$-invariants in $H^{2,2}(a_-; a)^a_0$, and finding conditions for them to integrate to actual filtered deformations of $a$.

**Maps in cohomology**

Let $a$ be a graded subalgebra of $\widehat{s}$. The maps of cochains induced by the inclusion $i : a \hookrightarrow \widehat{s}$ are

\[
i_* : C^{*,*}(a_-; a) \rightarrow C^{*,*}(a_-; \widehat{s}), \quad \phi \mapsto i_* \phi = i \circ \phi, \quad (4.190)
\]

\[
i^* : C^{*,*}(\widehat{s}_-; \widehat{s}) \rightarrow C^{*,*}(a_-; \widehat{s}), \quad \phi \mapsto i^* \phi = \phi \circ i. \quad (4.191)
\]

**Lemma 4.29.** Let $a$ be a highly supersymmetric graded subalgebra of $\widehat{s}$. Then we have a diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \overline{H}^{2,2}(a_-) & \rightarrow & H^{2,2}(\widehat{s}_-; \widehat{s}) & \cong \overline{H}^{2,2} & \rightarrow & H^{2,2}(a_-; \widehat{s}) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \overline{H}^{2,2}(a_-) & \rightarrow & H^{2,2}(\widehat{s}_-; \widehat{s}) & \rightarrow & H^{2,2}(a_-; \widehat{s}) & \rightarrow & 0 \\
\end{array}
\]
where $H^{2,2}(\bar{s};\bar{s}) \cong \bar{H}^{2,2}$ is the space of normalised cocycles of $\bar{s}$ (Lemma 4.22), and

$$\bar{H}^{2,2}(a) = \left\{ \beta + \gamma + \rho \in \bar{H}^{2,2} \mid \beta|_{V \otimes S'} = 0, \rho|_{\Omega^2 S'} = 0 \right\}.$$  

(4.193)

**Proof.** For $\beta : V \otimes S \rightarrow S$, $\gamma : \Omega^2 S \rightarrow \mathfrak{s}\mathfrak{o}(V)$, $\rho : \Omega^2 V \rightarrow \mathfrak{t}$ such that $\beta + \gamma + \rho \in \bar{H}^{2,2}$ is a normalised cocycle in $C^{2,2}(\bar{s};\bar{s})$, $i^*[\beta + \gamma + \rho] = 0$ if and only if $\beta|_{V \otimes S'} + \gamma|_{\Omega^2 S'} + \rho|_{\Omega^2 S'} = \delta \alpha_1 + \delta \alpha_2$, where $\lambda_1 : V \rightarrow \mathfrak{s}\mathfrak{o}(V)$ and $\lambda_2 : V \rightarrow \mathfrak{t}$. We see that $\delta \lambda_1| \wedge^2 V = 0$, so $\lambda_1| = 0$ by Lemma 3.16. Taking the $r$-component, we find that $\rho|_{\Omega^2 S'} = \delta \lambda_2|_{\Omega^2 S'}$. But by the normalisation of $\rho$, this shows that $\rho|_{\Omega^2 S'} = 0$ and $\gamma|_{\Omega^2 S'} = 0$. The latter says that $\lambda_2(\kappa_s) = 0$ for all $s \in S'$, but then by homogeneity we have $\lambda_2 = 0$. Looking at the remaining components, we find that $\beta|_{V \otimes S'} = 0$ and $\gamma|_{\Omega^2 S'} = 0$, and since the former implies the latter by the cocycle condition for $\beta + \gamma + \rho$, we find that $i^* \equiv \bar{H}^{2,2}(a)$.

For $\alpha : \Lambda^2 V \rightarrow V$, $\beta : V \otimes S' \rightarrow S'$, $\gamma : \Omega^2 S' \rightarrow \mathfrak{h}$, $\rho : \Omega^2 S' \rightarrow \mathfrak{t}$, with $\alpha + \beta + \gamma + \rho$ a cocycle in $C^{2,2}(a;\alpha)$, $i_*[\alpha + \beta + \gamma + \rho] = 0$ if and only if $\alpha + \beta + \gamma + \rho = \delta \lambda$ where $\lambda : V \rightarrow \mathfrak{s}\mathfrak{o}(V) \oplus \mathfrak{t}$. Then $\gamma(s,s) + \rho(s,s) = -\lambda(\kappa_s)$, in particular $\lambda(\kappa_s) \subseteq q_0$, for all $s \in S'$. Thus by homogeneity, $\lambda$ takes values in $q_0$, in particular it lies in $C^{1,2}(a;\alpha)$, hence $[\alpha + \beta + \gamma + \rho] = 0$. This shows that $i_* : H^{2,2}(a;\alpha) \rightarrow H^{2,2}(a;\bar{s})$ is injective. 

**Remark 10.** Unfortunately, there is no direct analogue in $Z^{2,2}(a;\bar{s})$ of the space of normalised cocycles $\bar{H}^{2,2}(a)$ (equation (3.152)) for $a \neq \bar{s}$ (i.e. if the subalgebra in question is not maximally supersymmetric) in the present case. This is because the normalisation condition for the $\rho$ component in $\bar{H}^{2,2}$ relied on the $\mathfrak{s}\mathfrak{o}$-module splitting $\Omega^2 S' \cong V \oplus \ker \kappa$, and such a splitting does not exist in general for $\Omega^2 S'$; we only have the short exact sequence (3.191) associated to $\kappa|_{\Omega^2 S'}$ discussed in §3.3.5.

We could, however, construct partially normalise cocycles in $Z^{2,2}(a;\bar{s})$ by setting the $\alpha$ component to zero – the space of such cocycles is invariant under $q_0$ and surjects onto $H^{2,2}(a;\bar{s})$. By choosing a section $\Sigma : V \rightarrow \Omega^2 S'$ of the squaring map, we can fully normalise our cocycles with respect to $\Sigma$ by demanding that $\rho(\Sigma(V)) = 0$ – every cohology class has a unique representative satisfying both this and $\alpha = 0$ – but this space of cocycles is not $q_0$-invariant.

### 4.3.4 Realisable subdeformations

We will not discuss re-parametrising general (or even transitive) highly supersymmetric filtered subdeformations $\alpha$ of $\bar{s}$ as we did for those of $\bar{s}$ in Proposition 3.27 since, although they are determined (at least in the transitive case) by a Spencer cocycle $[\alpha + \beta + \gamma + \rho] \in H^{2,2}(a;\alpha)$ with $q_0$-invariant cohomology class, the more complicated form of $\delta : a_0 \otimes a_2 \rightarrow a_0$ and the fact that general cocycles in $Z^{2,2}(a;\bar{s})$ cannot be normalised in an invariant way means that their structure is much more complicated and the re-parametrisation not particularly enlightening. We will instead restrict our attention, as we did in §3.3.4, to a class of deformations corresponding to a restricted type of cocycle, which we will eventually see Killing superalgebras of highly supersymmetric backgrounds as a subclass of.
Admissibility

Definition 4.30. A cohomology class \([\alpha + \beta + \gamma + \rho] \in H^{2,2}(\mathfrak{a}; \mathfrak{a})\) for a transitive highly supersymmetric graded subalgebra \(\mathfrak{a}\) of \(\mathfrak{s}\) is admissible if there exists an \(a_0 = \mathfrak{h} \oplus \mathfrak{r}'\) invariant element \(\check{\beta} + \check{\gamma} + \check{\rho} \in H^{2,2}\) such that \(i_*[\alpha + \beta + \gamma + \rho] = i^* [\check{\beta} + \check{\gamma} + \check{\rho}]\).

We call a cocycle in \(Z^{2,2}(\mathfrak{a}; \mathfrak{a})\) admissible if its cohomology class is admissible.

Fixing an admissible cohomology class, the corresponding element in \((\check{H}^{2,2})^0\) is unique up to elements of \(\check{H}^{2,2}(\mathfrak{a}; \mathfrak{a})^0\). We have

\[
i_* (\alpha + \beta + \gamma + \delta) = i^* (\check{\beta} + \check{\gamma} + \check{\rho}) + \partial \lambda
\]

where \(\lambda \in C^{2,1}(\mathfrak{a}; s) = \text{Hom}(V; \mathfrak{so}(V) \oplus \mathfrak{r})\) is uniquely determined by the admissible cocycle and is determined up to elements of \(C^{2,1}(\mathfrak{a}; a) = \text{Hom}(V; \mathfrak{h} \oplus \mathfrak{r}')\) by the cohomology class. Denoting by \(\lambda_1 : V \to \mathfrak{so}(V)\) and \(\lambda_2 : V \to \mathfrak{r}\) the components of \(\lambda\), we can describe this even more explicitly:

\[
\begin{align*}
\alpha(v, w) &= \lambda_1(v)w - \lambda_1(w)v, \\
\beta(v, s) &= \check{\beta}(v, s) + \lambda_1(v) \cdot s + \lambda_2(v) s, \\
\gamma(s, s) &= \check{\gamma}(s, s) - \lambda_1(\kappa_s), \\
\rho(s, s) &= \check{\rho}(s, s) - \lambda_2(\kappa_s),
\end{align*}
\]

for \(v, w \in V, s \in S'\). Note that the first equation or the third equation (by homogeneity) uniquely determines \(\lambda_1\) and the fourth equation determines \(\lambda_2\) by homogeneity. Since \(i_*\) is injective, the \(a_0\)-invariance of \(\check{\beta} + \check{\gamma} + \check{\rho}\) implies that the cohomology class \([\alpha + \beta + \gamma + \rho]\) is also invariant, whence it is an infinitesimal filtered deformation. Explicitly, we have

\[
\begin{align*}
(A \cdot \beta)(v, s) &= [A, \lambda_1(v)] \cdot s - \lambda_1(Av) \cdot s - \lambda_2(Av) s, & (A \cdot \beta)(v, s) &= [a, \lambda_2(v)] s, \\
(A \cdot \gamma)(s, s) &= -[A, \lambda_1(\kappa_s)] + \lambda_1(A \kappa_s), & (A \cdot \gamma)(s, s) &= 0, \\
(A \cdot \rho)(s, s) &= \lambda_2(A \kappa_s), & (A \cdot \rho)(s, s) &= [a, \lambda_2(\kappa_s)],
\end{align*}
\]

for all \(v \in V, s \in S', A \in \mathfrak{h}\) and \(a \in \mathfrak{r}'\); the cocycle \(\delta \in Z^1(a_0; V^* \otimes a_0)\) making \(\alpha + \beta + \gamma + \rho + \delta\) into a cocycle in \(Z^{2,2}(\mathfrak{a}; \mathfrak{a})\) (which we expect to exist and be unique by Lemma 4.23) is given by \(\delta = \partial \lambda\) (where we note that \(\lambda \in C^0(a_0; V^* \otimes a_0)\)), or

\[
\begin{align*}
\delta_1(A, v) &= [A, \lambda_1(v)] - \lambda_1(Av), & \delta_2(A, v) &= -\lambda_2(Av), \\
\delta_3(a, v) &= 0, & \delta_4(a, v) &= [a, \lambda_2(v)].
\end{align*}
\]

The following lemma follows quite straightforwardly from the definition, but we make note of it nonetheless as it will be useful when working with the hatted maps.

Lemma 4.31. For an admissible cocycle,

- For all \(v \in V, s \in S'\), \(\check{\beta}(v, s) + \lambda_1(v) \cdot s + \lambda_2(v) s \in S'\);
- For all \(s \in S'\), \(\check{\gamma}(s, s) - \lambda_1(\kappa_s) \in \mathfrak{h}\) and \(\check{\rho}(s, s) - \lambda_2(\kappa_s) \in \mathfrak{h}\);
- For all \(A \in \mathfrak{h}, v \in V\), \([A, \lambda_1(v)] - \lambda_1(Av) \in \mathfrak{h}\) and \(\lambda_2(Av) \in \mathfrak{r'}\);
- For all \(a \in \mathfrak{r'}, v \in V\), \([a, \lambda_2(v)] \in \mathfrak{r'}\).
Integrability

We now consider obstructions to the integration of an admissible cocycle to a filtered
deformation by checking the Jacobi identities. The linear identities are automatically
satisfied by the discussion above, so we must consider the quadratic identities (4.176)-
(4.185). We first re-parametrise the degree-4 deformation map, defining \( \tilde{\theta} : \Lambda^2 V \to a_0 \)
with components \( \tilde{\theta}_1 : \Lambda^2 V \to \mathfrak{so}(V) \) and \( \tilde{\theta}_2 : \Lambda^2 V \to \tau' \) by

\[
\begin{align*}
\tilde{\theta}_1(v, w) & := \theta_1(v, w) + \lambda_1(\lambda_1(v)w - \lambda_1(w)v) - [\lambda_1(v), \lambda_1(w)], \\
\tilde{\theta}_2(v, w) & := \theta_2(v, w) + \lambda_2(\lambda_1(v)w - \lambda_1(w)v) - [\lambda_2(v), \lambda_2(w)].
\end{align*}
\]

(4.198)

(4.199)

for \( v, w \in V \). The quadratic Jacobi identities then take a much simpler form.

- Equations (4.176) and (4.177) become

\[
A \cdot \tilde{\theta}_1 = 0, \quad A \cdot \tilde{\theta}_2 = 0,
\]

(4.200)

for all \( A \in \mathfrak{h} \); that is, \( \tilde{\theta} \) is \( \mathfrak{h} \)-invariant.

- Equation (4.178) is trivial since \( \delta_3 = 0 \), (this corresponds to the trivial fact that

\( \tilde{\theta}_1 \) is \( \tau' \)-invariant) and (4.179) is equivalent to

\[
a \cdot \tilde{\theta}_2 = 0
\]

(4.201)

for all \( a \in \tau' \), i.e. \( \tau' \)-invariance of \( \tilde{\theta} \).

- Equations (4.180) and (4.181) become

\[
\begin{align*}
\tilde{\theta}_1(v, \kappa s) & = 2 \tilde{\gamma}(s, \tilde{\beta}(v, s)) - (\lambda_1(v) \cdot \tilde{\gamma})(s, s) - (\lambda_2(v) \cdot \tilde{\gamma})(s, s), \\
\tilde{\theta}_2(v, \kappa s) & = 2 \tilde{\rho}(s, \tilde{\beta}(v, s)) - (\lambda_1(v) \cdot \tilde{\rho})(s, s) - (\lambda_2(v) \cdot \tilde{\rho})(s, s),
\end{align*}
\]

(4.202)

(4.203)

and by the Homogeneity Theorem fully determine \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) in terms of the
maps \( \tilde{\beta}, \tilde{\gamma}, \tilde{\rho} \) and \( \lambda \). By \( a_0 \)-invariance of \( \tilde{\beta} \) and \( \tilde{\gamma} \), changing \( \lambda \) by addition of a
map \( V \to a_0 \) does not change \( \tilde{\theta}_1 \) or \( \tilde{\theta}_2 \), and by a slight rearrangement one can
show that changing \( \tilde{\beta} + \tilde{\gamma} + \tilde{\rho} \) by an element of \( \mathcal{K}^{2,2}(a_-) \) also does not change
\( \tilde{\theta} \). Thus these equations determine \( \tilde{\theta} \) uniquely for any admissible cohomology
class.

- Equation (4.182) becomes

\[
\begin{align*}
\tilde{\theta}_1(v, w) \cdot s + \tilde{\theta}_2(v, w) s & = \tilde{\beta}(v, \tilde{\beta}(w, s)) - \tilde{\beta}(w, \tilde{\beta}(v, s)) \\
& + (\lambda_1(v) \cdot \tilde{\beta})(w, s) - (\lambda_1(w) \cdot \tilde{\beta})(v, s) \\
& + (\lambda_2(v) \cdot \tilde{\beta})(w, s) - (\lambda_2(w) \cdot \tilde{\beta})(v, s).
\end{align*}
\]

(4.204)

- The remaining identities (4.183), (4.184), (4.185) are, respectively,

\[
\begin{align*}
\tilde{\theta}_1(u, v) w + \tilde{\theta}_1(v, w) u + \tilde{\theta}_1(w, u) v & = 0, \\
(\lambda_1(u) \cdot \tilde{\theta}_1)(v, w) + (\lambda_1(v) \cdot \tilde{\theta}_1)(w, u) + (\lambda_1(w) \cdot \tilde{\theta}_1)(u, v) & = 0,
\end{align*}
\]

(4.205)

(4.206)
\[
(\lambda_1(u) \cdot \tilde{\theta}_2)(v, w) + (\lambda_2(u) \cdot \tilde{\theta}_2)(v, w) + \cdots = 0,
\]

for all \( u, v, w \in V \), where the omitted terms in \( \ldots \) are of course cyclic permutations of the given terms.

Thus to integrate an admissible cocycle, we must first check whether the maps \( \tilde{\theta}_1, \tilde{\theta}_2 \) determined by (4.202), (4.203) are well-defined, \( a_0 \)-invariant maps such that

\[
\begin{align*}
\tilde{\theta}_1(v, w) &= \lambda_1(\lambda_1(v) w - \lambda_1(w) v) + [\lambda_1(v), \lambda_1(w)], \in \mathfrak{h}, \\
\tilde{\theta}_2(v, w) &= \lambda_2(\lambda_1(v) w - \lambda_1(w) v) + [\lambda_2(v), \lambda_2(w)], \in \mathfrak{k}'
\end{align*}
\]

and that they satisfy the remaining equations (4.204)-(4.207). If any of these fail, the cocycle does not define a filtered deformation.

We begin with a number of results which will reduce the number of equations we have to check.

**Lemma 4.32.** Any map \( \tilde{\theta}_1 : \wedge^2 V \to \mathfrak{so}(V) \) satisfying equation (4.202) also satisfies equation (4.208). Any map \( \tilde{\theta}_2 : \wedge^2 V \to \mathfrak{k}' \) satisfying equation (4.203) also satisfies equation (4.209).

**Proof.** The first claim follows in the same way as in the proof of Lemma 3.31. The second is similar; for all \( v, w \in V \) and \( s \in S' \) we have

\[
\begin{align*}
\tilde{\theta}_2(v, \kappa s) &= \lambda_2(\lambda_1(v) \kappa s - \lambda_1(\kappa s) v) + [\lambda_2(v), \lambda_2(\kappa s)] \\
&= 2(\rho - \lambda_2 \circ \kappa)(s, \beta(v, s) + \lambda_1(v) \cdot s + \lambda_2(v) s) - [\lambda_2(v), \rho(s, s) - \lambda_2(\kappa s)] \\
&\quad + 2\lambda_2(\kappa(s, \beta(v)) + \lambda_1(v) \cdot s + \lambda_2(v) s + \lambda_1(\kappa s) v - \lambda(v) \kappa s) \\
&\quad + 2(\rho - \lambda_2 \circ \kappa)(s, \beta(v, s) + \lambda_1(v) \cdot s + \lambda_2(v) s) - [\lambda_2(v), \rho(s, s) - \lambda_2(\kappa s)] \\
&\quad + 2\lambda_2(2\kappa(s, \beta(v)) + \lambda_1(v) \cdot s + \lambda_2(v) s + \lambda_1(\kappa s) v - \lambda_1(v) \kappa s) \\
&\quad + 2(\rho - \lambda_2 \circ \kappa)(s, \beta(v, s) + \lambda_1(v) \cdot s + \lambda_2(v) s) - [\lambda_2(v), \rho(s, s) - \lambda_2(\kappa s)] \\
&\quad + 2\lambda_2(\gamma(s, s) v + \lambda_1(\kappa s) v),
\end{align*}
\]

where in the last line we have made use of a Spencer cocycle condition as well as \( a_0 \)-invariance of \( \kappa \). By Lemma 4.31, each term in the last line lies in \( \mathfrak{k}' \).

The proof of the following result is nearly identical to that of Lemma 3.32, except that we must also use the fact that \( \kappa(\tilde{\theta}_2(v, w)s, s) = 0 \) since \( \kappa \) is \( \mathfrak{k} \)-invariant.

**Lemma 4.33.** Assume that we have a map \( \tilde{\theta}_1 : \wedge^2 V \to \mathfrak{so}(V) \). Then equation (4.204) gives the following expression for \( \tilde{\theta}_1 \):

\[
\begin{align*}
\tilde{\theta}_1(v, w) \kappa s &= 2\tilde{\gamma}(\tilde{\beta}(v, s), s) w - 2\tilde{\gamma}(\tilde{\beta}(w, s), s) v \\
&\quad - (\lambda_1(v) \cdot \tilde{\gamma})(s, s) w + (\lambda_1(w) \cdot \tilde{\gamma})(s, s) v \\
&\quad - (\lambda_2(v) \cdot \tilde{\gamma})(s, s) w + (\lambda_2(w) \cdot \tilde{\gamma})(s, s) v,
\end{align*}
\]

where \( v, w \in V \), \( s \in S' \).
Lemma 4.34. Let $\alpha + \beta + \gamma$ be an admissible Spencer (2,2)-cocycle for a highly supersymmetric graded subalgebra $\mathfrak{s}$ and assume that we have some maps $\tilde{\theta}_1 : \wedge^2 V \to \mathfrak{so}(V)$ and $\tilde{\theta}_2 : \wedge^2 V \to \mathfrak{v}'$. Then

- Equations (4.202) and (4.203) imply that $\tilde{\theta}$ is $a_0$-invariant (that is, they imply equations (4.200) and (4.201));

- Equation (4.204) implies that $\tilde{\theta}_1 + \tilde{\theta}_2 : \wedge^2 V \to a_0$ is $a_0$-invariant;

- Assuming equation (4.202), equation (4.211) is equivalent to the algebraic Bianchi identity (4.205) for $\tilde{\theta}_1$; in particular, equations (4.202) and (4.204) together imply the Bianchi identity;

- Equations (4.202) and (4.204) together imply (4.206);

- Equations (4.203) and (4.204) together imply (4.207).

Proof. The results follow by similar manipulations as in the proof of Lemma 3.33, using $\mathfrak{so}(V) \oplus \tau$-invariance of $\kappa$, the Spencer cocycle conditions and $a_0$-invariance of $\tilde{\beta}, \tilde{\gamma}$ and $\tilde{\rho}$ and the properties of Lemma 4.31; they are similar enough to the analogous parts of Lemma 3.33 that we will not repeat the calculations here. Note that equations (4.205)-(4.207) can be slightly more compactly presented as

\[
\begin{align*}
\tilde{\theta}(u, v) w + \tilde{\theta}(v, w) u + \tilde{\theta}(w, u) v &= 0, \quad (4.212) \\
(\lambda(u) \cdot \tilde{\theta})(v, w) + (\lambda(v) \cdot \tilde{\theta})(w, u) + (\lambda(w) \cdot \tilde{\theta})(u, v) &= 0, \quad (4.213)
\end{align*}
\]

where we re-collect $\lambda_i$ into a single map $\lambda : V \to a_0$, and similarly for $\tilde{\theta}$; other expressions can be re-written similarly, and the the calculations in the proof of Lemma 3.33 are formally identical to those required here.

Thus we have indeed significantly reduced the number of conditions to be checked: we need only check that equations (4.202) and (4.203) define a map $\tilde{\theta} : \wedge^2 V \to a_0$ and that this map satisfies (4.204).

We thus must tackle the question of whether the maps are well-defined. Let us now define maps $\Theta_1 : V \otimes \bigotimes^2 S \to \mathfrak{so}(V)$ and $\Theta_2 : V \otimes \bigotimes^2 S \to \mathfrak{v}'$ by

\[
\begin{align*}
\Theta_1(v, s, s) &= 2\tilde{\gamma}(s, \tilde{\beta}(v, s)) - (\lambda_1(v) \cdot \tilde{\gamma})(s, s) - (\lambda_2(v) \cdot \tilde{\gamma})(s, s), \quad (4.214) \\
\Theta_2(v, s, s) &= 2\tilde{\rho}(s, \tilde{\beta}(v, s)) - (\lambda_1(v) \cdot \tilde{\rho})(s, s) - (\lambda_2(v) \cdot \tilde{\rho})(s, s). \quad (4.215)
\end{align*}
\]

The maps $\tilde{\theta}_i$ are well-defined if the $\Theta_1$s annihilate the Dirac kernel $\mathcal{D} = \ker \kappa|_{\bigotimes^2 S'}$ (that is, if $\Theta_1(V \otimes \mathcal{D})$ and thus factor through $V \otimes V$ as follows:

\[
\begin{align*}
V \otimes \bigotimes^2 S' &\xrightarrow{\Theta_1} \mathfrak{so}(V) & V \otimes \bigotimes^2 S' &\xrightarrow{\Theta_2} \mathfrak{v}' \\
V \otimes V &\xrightarrow{\tilde{\theta}_1} V \otimes V & V \otimes V &\xrightarrow{\tilde{\theta}_2} V \otimes V
\end{align*}
\]

and we find that if this is the case, they are in fact alternating, as in the following lemma.
**Lemma 4.35.** If the maps $\Theta_1$ and $\Theta_2$ annihilate the Dirac kernel and thus factor as above, equations (4.202) and (4.203) define a maps $\tilde{\theta}_1 : \wedge^2 V \to \mathfrak{so}(V)$ and $\tilde{\theta}_2 : \wedge^2 V \to \tau$.

**Proof.** Once again, the proof is nearly identical to that of the analogous Lemma 3.34, with the only modification being that we need to use $\tilde{\gamma}(s, (\tilde{\rho}(s, s) - \tilde{\lambda}(k_s))s) = 0$ which holds by $t'$-invariance of $\tilde{\gamma}$. For the second, using a cocycle condition and $\mathfrak{a}_0$-invariance of $\tilde{\rho}$, we have

$$
\begin{align*}
\tilde{\theta}_2(k_s, k_s) &= 2\tilde{\rho}(s, \tilde{\rho}(k_s, s) + \lambda_1(k_s) \cdot s + \lambda_2(k_s) s) - [\lambda_2(k_s), \tilde{\rho}(s, s)] \\
&= -2\tilde{\rho}(s, [\tilde{\gamma}(s, s) - \lambda(k_s)] \cdot s + (\tilde{\rho}(s, s) - \tilde{\lambda}(k_s)) s) - [\lambda_2(k_s), \tilde{\rho}(s, s)] \\
&= -\lbrack \rho(s, s) - \lambda_2(k_s) + \lambda_2(k_s), \rho(s, s) \rbrack \\
&= 0
\end{align*}
$$

so by the Homogeneity Theorem, $\tilde{\theta}_2$ is alternating. \qed

We thus make the following definition

**Definition 4.36.** Let $\alpha + \beta + \gamma + \rho$ be an admissible cohomology class and let $\Theta_1 : V \otimes \bigotimes^2 S' \to \mathfrak{so}(V), \Theta_2 : V \otimes \bigotimes^2 S' \to \tau$ be the maps defined by equations (4.214), (4.215). We say that $\alpha + \beta + \gamma + \rho$ is integrable if

1. $\Theta_1$ and $\Theta_2$ annihilate the Dirac kernel $\mathfrak{D}$ of $\bigotimes^2 S'$
2. The maps $\tilde{\theta}_1 : \wedge^2 V \to \mathfrak{so}(V), \tilde{\theta}_2 : \wedge^2 V \to \mathfrak{so}(V)$ defined by $\tilde{\theta}_i \circ \kappa = \Theta_i$ satisfy (4.204).

An admissible cocycle is integrable if its cohomology class is.

The results above imply the following.

**Theorem 4.37** (Integration of admissible cocycles with $R$-symmetry). Let $\mathfrak{a}_0 = \mathfrak{a}_0 \otimes S$ be an admissible, integrable cohomology class for a highly supersymmetric graded subalgebra $\mathfrak{a}$ of $\mathfrak{s}$ with $i_*(\alpha + \beta + \gamma + \rho) = i_*(\tilde{\beta} + \tilde{\gamma} + \tilde{\rho}) + \partial \lambda$ for some map $\lambda : V \to \mathfrak{a}_0$ and let $\tilde{\theta}_1, \tilde{\theta}_2$ be the maps defined by equations (4.202) and (4.203). Then the brackets

$$
\begin{align*}
[A, B] &= \underbrace{AB - BA}_h, \\
[A, v] &= \underbrace{Av + [A, \lambda_1(v)] - \lambda_1(Av)}_V - \underbrace{\lambda_2(Av)}_{t'}, \\
[A, s] &= \underbrace{A \cdot s}_{S'}, \\
[a, b] &= \underbrace{ab - ba}_h, \\
[a, v] &= \underbrace{[a, \lambda_2(v)]}_{t'}, \\
[a, s] &= \underbrace{as}_{S'}
\end{align*}
$$

\[169\]
\[ [s, s] = \kappa_s + \gamma(s, s) - \lambda_1(\kappa_s) + \lambda_2(\kappa_s), \]
\[ [v, s] = \beta(v, s) + \lambda_1(\gamma + \rho)(v, s), \]
\[ [v, w] = \lambda(v, w) - \lambda(\gamma + \rho)(v, w), \]
(4.218)

and \([A, a] = 0\) define a filtered deformation of \(a\).

**Remark 11.** The brackets of the Killing superalgebra localised at a point given in equation (4.120) are clearly of the form above but with \(\lambda_2 = 0\) and \(\theta_2 = 0\). In Remark 9, we noted that it might be possible to relax the definition of admissible pairs (Definition 4.17) and to allow for Killing superalgebras with less trivial even parts, in particular to allow for infinitesimal \(R\)-symmetry with non-vanishing Lie derivatives and \(R\)-symmetry gauge field strengths \(F\) which do not vanish on contraction with the even symmetry generators. The form of \(\theta_2\) suggests that it may have a geometric interpretation as such a field strength, while the non-zero \([a, v]\) bracket may have an interpretation as the covariant Lie derivative of an infinitesimal \(R\)-symmetry if a better definition of admissible pairs could be found.

In view of the remark above, we make the following definition.

**Definition 4.38 (Geometric realisability).** A transitive highly supersymmetric filtered subdeformation of \(\mathfrak{s}\) is geometrically realisable if the associated Spencer cohomology class is admissible (and thus necessarily integrable) and if a representative cocycle can be chosen such that \(\lambda_2 = 0\) and \(\theta_2 = 0\).

This definition is not entirely satisfactory since it is rather unnatural to set some of the deformation maps to zero “by hand” rather than via some homological condition, but we choose this definition now for compatibility with Definition 4.17 but leave open the possibility for both to be modified in future work.

### 4.3.5 Towards a classification scheme

**Admissible and normalised cocycles**

We will not develop a full generalisation of the classification scheme for odd-generated realisable subdeformations of \(\mathfrak{s}\) in §3.3.5 here, but we will make some comments about which results carry over and which present challenges. First, we note that if \(a = V \oplus S' \oplus (\mathfrak{h} \oplus \mathfrak{r}')\) is a graded subalgebra of \(\mathfrak{s}\) and \(\beta + \gamma + \rho \in \mathcal{H}^{2,2}\), then we have

\[ i_* [\alpha + \beta + \gamma + \rho] = i^* \left[ \beta + \gamma + \rho \right], \]
(4.219)

for some \((2,2)\)-cocycle \(\alpha + \beta + \gamma + \rho\) of \(a\) if and only if
We finally note that Spin\(^{\mathcal{D}}\) \(\subseteq h\) and \(\hat{\rho}(\mathcal{D}) \subseteq \tau\)' (where \(\mathcal{D}\) is of course the Dirac kernel for \(S'\)),

(2) for all \(v \in V\), \(i_v \hat{\rho} + i_v \hat{\gamma}(\Sigma(v)) + \hat{\rho}(\Sigma(v))\) \(\subseteq S'\) for some (hence any) section \(\Sigma\) of \(\kappa\),

and \(\alpha + \beta + \gamma + \rho\) can be chosen such that

\[
i_*(\alpha + \beta + \gamma + \rho) = i^*(\hat{\beta} + \hat{\gamma} + \hat{\rho}) + \partial \lambda \text{ where } \lambda = (\hat{\gamma} + \hat{\rho}) \circ \Sigma;
\]

(4.220)

this is to be compared with Lemma 3.38.

**Lie pairs and envelopes**

Now, let us fix some subspace \(S' \subseteq S\) with \(\dim S' > \frac{1}{2} \dim S\). We also choose a normalised cocycle \(\beta + \gamma + \rho \in \mathcal{H}^{2,2}\), where we note that we have dropped the decorations from the notation. There are now two possible natural notions of an envelope \(\hat{h}(S', \beta + \gamma + \rho)\) generalising Definition 3.39, one of which is a subspace of the other:

\[
(\gamma + \rho)(\mathcal{D}) \subseteq (\gamma(\mathcal{D}) \oplus (\rho(\mathcal{D})) \subseteq s\theta(V) \oplus \tau.
\]

(4.221)

Whichever we choose, we call \((S', \beta + \gamma + \rho)\) a Lie pair if both \(\beta + \gamma + \rho\) and \(S'\) are preserved by the action of elements in \(\hat{h}(S', \beta + \gamma + \rho)\). With either definition, \(\hat{h}(S', \beta + \gamma + \rho)\) is a Lie algebra if \((S', \beta + \gamma + \rho)\) is a Lie pair. The choice matters when we consider the graded subalgebra \(a(S', \beta + \gamma + \rho) = V \oplus S' \oplus \hat{h}(S', \beta + \gamma + \rho)\) of \(\hat{s}\) associated to a Lie pair; the smaller space \((\gamma + \rho)(\mathcal{D})\) need not be of the form \(h \oplus \tau\)', which is the form we prefer to work with. On the other hand, if we have \(i_v \beta + \gamma(\Sigma(v)) + \rho(\Sigma(v))\) \(\subseteq S'\) for all \(v \in V\) then the \((2,2)-\)cocycle of \(a(S', \beta + \gamma + \rho)\) given by (4.220) (which is necessarily admissible) surjects onto \((\gamma + \rho)(\mathcal{D})\), not \(\gamma(\mathcal{D}) \oplus (\rho(\mathcal{D}))\) (if the latter properly contains the former).

Thus if we wish to discuss odd-generated subdeformations, we are forced to choose \(\hat{h}(S', \beta + \gamma + \rho) = (\gamma + \rho)(\mathcal{D})\), while if we prefer to work with subalgebras of the form \(h \oplus \tau\)', we must make the other choice.

If we wish to work with realisable subdeformations, we have the added complication that we must demand the deformation maps taking values in \(\tau\) (other than \(\rho\)) to be zero (see Definition 4.38).

**Equivalence relations**

We finally note that Spin\(^{\mathcal{D}}\) acts on the \(\hat{s}\) and on Lie pairs in an analogous way to how the group Spin\((V) \times R\) acted on \(s\) and Lie pairs in §3.3.5 – in fact, now that we have discussed Spin\(^{\mathcal{D}}\), we can easily see that all relevant actions of the product group factor through it, so the treatment in that section could also be stated in terms of the action of Spin\(^{\mathcal{D}}\). One should expect a similar correspondence between (an appropriate notion of) equivalence classes of Lie pairs and isomorphism classes of (appropriately constrained) subdeformations of \(\hat{s}\) to Proposition 3.42, where we define isomorphisms of subdeformations as follows, entirely analogously to Definition 3.37.

**Definition 4.39.** Two subdeformations \(\tilde{a}, \tilde{a}'\) of \(\hat{s}\) are isomorphic if there exists a (strict) filtered isomorphism of Lie superalgebras \(\Phi : \tilde{a} \to \tilde{a}'\) such that the associated graded morphism \(\text{Gr} \Phi : a \to a'\) is given, for all \(x \in a\), by \(\text{Gr} \Phi (x) = g \cdot x\) for some \(g \in \text{Spin}^{\mathcal{D}}(V)\).
An embedding of subdeformations $\tilde{\mathfrak{a}} \rightarrow \tilde{\mathfrak{a}}'$ of $\tilde{\mathfrak{a}}$ is an injective (strict) filtered morphism of Lie superalgebras which is an isomorphism of subdeformations onto its image.

4.4 Highly supersymmetric Lorentzian spin-$R$ manifolds

Here we generalise Section 3.4 to the spin-$R$ case. We once again rely heavily on the formalism of Section 2.4.

4.4.1 Homogeneous spin-$R$ structures

We first generalise our treatment of homogeneous spin structures to spin-$R$ structures.\(^9\) Let $(G, H, \eta)$ be a metric Klein pair, let $(M = G/H, g)$ be the associated homogeneous space and let $\mathcal{V} = T_oM$ where $o = H$.

Suppose that there exists a lift of the linear isotropy representation $\varphi : H \rightarrow \text{SO}(\mathcal{V})$ to $\text{Spin}^R(\mathcal{V})$; that is, a Lie group morphism $\tilde{\varphi} : H \rightarrow \text{Spin}^R(\mathcal{V})$ making the following diagram commute:

\[
\begin{array}{ccc}
\text{Spin}^R(\mathcal{V}) & \xrightarrow{\tilde{\varphi}} & H \\
\downarrow & & \downarrow \varphi \\
\text{SO}(\mathcal{V}) & \xrightarrow{\pi} & H
\end{array}
\]

(4.222)

Then the canonical map $\tilde{\pi} : \text{Spin}^R(\mathcal{V}) \rightarrow \text{SO}(\mathcal{V})$ induces a spin-$R$ structure on $M$,

\[
\tilde{P} := G \times_H \text{Spin}^R(\mathcal{V}) \rightarrow F_{\text{SO}} \cong G \times_H \text{SO}(\mathcal{V}).
\]

(4.223)

Note that $\tilde{P}$ admits a left action of $G$ compatible with the right action of $\text{Spin}^R(\mathcal{V})$ given by $g \cdot [g', A] = [gg', A]$ for $g, g' \in G$, $A \in \text{Spin}^R(\mathcal{V})$, and clearly this lifts the action on $F_{\text{SO}}$. We call this the homogeneous spin-$R$ structure associated to the lift $\tilde{\varphi} : H \rightarrow \text{Spin}^R(\mathcal{V})$. We call the manifold $M$ together with this spin structure the homogeneous Lorentzian spin-$R$ manifold associated to the metric Klein pair and the lift.

Of course, we always have the $G$-equivariant bundle isomorphisms

\[
TM \cong F_{\text{SO}} \times_{\text{SO}(\mathcal{V})} \mathcal{V} \cong G \times_H \mathcal{V},
\]

(4.224)

and similarly for even associated bundles (the cotangent bundle, exterior bundle etc.); if we have a homogeneous spin-$R$ structure, we have similar $G$-equivariant isomorphisms for odd associated bundles, in particular

\[
\tilde{S} := \tilde{P} \times_{\text{Spin}^R(\mathcal{V})} S \cong G \times_H S.
\]

(4.225)

We saw in Section 3.4.1 that, for $G$ simply connected and $\eta$ positive-definite, spin lifts of the isotropy group classify spin structures on a homogeneous $G$-space (see Lemma 3.44), while in the more general setting the result fails because there

9 For ease of notation and later application we speak of spin-$R$ here, but the results of this subsection do not rely on the assumption that $R$ is an $R$-symmetry group, only that it is a Lie group with a normal subgroup isomorphic to $\mathbb{Z}_2$. 
are obstructions to lifting the action of $G$ to a spin bundle. There are even more obstructions to the analogous result in the spin-$R$ setting since a spin-$R$ structure is not even a cover of the frame bundle.\footnote{It is a cover of the product bundle $F_{SO} \times_M \overline{Q}$, so one could consider the conditions under which an action of $G$ on this bundle covering the action on $M$ lifts to a spin-$R$ bundle, but we will not do this here.} We therefore work in a more restrictive context, essentially assuming the existence of a lift of the action.

**Definition 4.40.** A homogeneous spin-$R$ structure on a homogeneous Lorentzian $G$-space $(M, g)$ is a spin-$R$ structure $\tilde{\phi} : \tilde{P} \rightarrow F_{SO}$ on $M$ equipped with a left action of $G$ on $\tilde{P}$ such that $\tilde{\phi}$ is $G$-equivariant.

Two such structures $\tilde{\phi} : \tilde{P} \rightarrow M$ and $\tilde{\phi}' : \tilde{P}' \rightarrow M$ are equivalent if there exists a $G$-equivariant isomorphism of $\text{Spin}^R(V)$-principal bundles $\Phi : \tilde{P} \rightarrow \tilde{P}'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{P} & \xrightarrow{\Phi} & \tilde{P}' \\
\tilde{\phi} & & \tilde{\phi}' \\
F_{SO} & \xrightarrow{} & F_{SO}
\end{array}
$$

(4.226)

We note that to check that a bundle map $\Phi : \tilde{P} \rightarrow \tilde{P}'$ is an equivalence of homogeneous spin-$R$ structures, we must check that it is equivariant under the left action by $G$ and the right action by $\text{Spin}^R(V)$. We also recall that every morphism (equivariant bundle map) of principal bundles for the same group is an isomorphism.

If $\tilde{\phi} : H \rightarrow \text{Spin}^R(V)$ is a lift of the linear isotropy representation $\phi : H \rightarrow SO(V)$ as in the diagram (4.222), then $\gamma_r \circ \tilde{\phi}$, where $\gamma_r : \text{Spin}^R(V) \rightarrow \text{Spin}^R(V)$ is conjugation by $[1, r] \in \text{Spin}^R(V)$, is also a lift for any $r \in R$. We say that two lifts $\tilde{\phi}, \tilde{\phi}'$ are $R$-conjugate if there exists $r \in R$ such that $\tilde{\phi}'(h) = \gamma_r \circ \tilde{\phi}(h)$.

We can now state our analogue of Lemma 3.44 for homogeneous spin structures.

**Proposition 4.41.** Let $(G, H, \eta)$ be a metric Klein pair where $G$ is a connected and simply connected Lie group. Then there is a one-to-one correspondence between equivalence classes of homogeneous spin-$R$ structures on $(M = G/H, g)$ and $R$-conjugacy classes of lifts of the isotropy representation to $\text{Spin}^R(V)$ as in the diagram (4.222).

**Proof.** Throughout this proof, fix some $f \in (F_{SO})_G$ and use this frame to define the $SO(V)$-principal bundle structure on $F_{SO}$ as well as the $(G$-equivariant) isomorphism $G \times_H SO(V) \cong F_{SO}$.

We have already shown that a lift $\tilde{\phi}$ of the isotropy representation $\phi$ gives rise to a homogeneous spin-$R$ structure. Let $r \in R$ so that $\tilde{\phi}' := \gamma_r \circ \tilde{\phi}$ is a conjugate lift. We must show that there is an equivalence of the associated homogeneous spin-$R$ structures

$$
\Phi : G \times_{\tilde{\phi}} \text{Spin}^R(V) \rightarrow G \times_{\tilde{\phi}'} \text{Spin}^R(V),
$$

(4.227)

where we now explicitly write the morphisms $\tilde{\phi}, \tilde{\phi}' : H \rightarrow \text{Spin}^R(V)$ used to define the associated principal bundles. We define $\Phi([g, A]) = [g, rA]^\prime$, where the prime on the right-hand side indicates that the equivalence class is taken in $G \times_{\tilde{\phi}'} \text{Spin}^R(V)$. This is well-defined since if $[g', A'] = [g, A]$, there exists $h \in H$ such that $g' = gh^{-1}$.

\[ A' = \hat{\varphi}(h)A, \text{ so } \]
\[ [g', r A']' = [gh^{-1}, r\hat{\varphi}(h)A]' = [gh^{-1}, \hat{\varphi}'(h)r A]' = [g, r A]. \quad (4.228) \]

It is clear from the definition that \( \Phi \) is a \( G \)-equivariant morphism of \( \text{Spin}^R(V) \)-principal bundles covering \( F_{SO} \cong G \times_H SO(V) \), whence it is an isomorphism.

On the other hand, given a homogeneous spin-\( R \) structure \( \varrho : \hat{P} \to F_{SO} \), we must produce a conjugacy class of lifts \( \hat{\varphi} \) of the isotropy representation. Let us first fix a lift \( \hat{f} \in \hat{P}_o \) of the frame \( f \). This lift is unique only up to (right) multiplication by elements of \( R \); we will consider the effect of this shortly. Since the subgroup \( H \) fixes \( o \), it preserves the fibre \( \hat{P}_o \), upon which \( \text{Spin}^R(V) \) acts freely and transitively, so for each \( h \in H \) there exists a unique \( \hat{\varphi}(h) \in \text{Spin}^R(V) \) such that \( \hat{h} \cdot \hat{f} = \hat{f} \cdot \hat{\varphi}(h) \). This defines a homomorphism \( \hat{\varphi} : H \to \text{Spin}^R(V) \) since for all \( h, h' \in H \),
\[ h \cdot (h' \cdot \hat{f}) = h \cdot (\hat{f} \cdot \hat{\varphi}(h')) = (h \cdot \hat{f}) \cdot \hat{\varphi}(h') = (\hat{f} \cdot \hat{\varphi}(h)) \cdot \hat{\varphi}(h') = \hat{f} \cdot (\hat{\varphi}(h) \hat{\varphi}(h')) \quad (4.229) \]

while \( (hh') \cdot \hat{f} = \hat{f} \hat{\varphi}(hh') \), so \( \hat{\varphi}(hh') = \hat{\varphi}(h) \hat{\varphi}(h') \). Now we have
\[ h \cdot f = h \cdot \varrho(\hat{f}) = \varrho(h \cdot \hat{f}) = \varrho(\hat{f} \cdot \hat{\varphi}(h)) = \varrho(\hat{f}) \cdot \hat{\varphi}(h) = f \cdot (\hat{\varphi} \circ \hat{\varphi}(h)) \quad (4.230) \]

Note that for all \( A \in SO(V) \), representing \( A \) as a matrix \( [A^i_j] \) in the fixed frame \( f \), we have
\[ (f \cdot A)_i = f_j A^j_i = Af_i \quad (4.231) \]
where in the last expression we use the natural action of \( SO(V) \) on \( V \). Applying this to the isotropy representation \( \varphi : H \to SO(V) \), we have
\[ h \cdot f_i = \varphi(h) f_i = (f \cdot \varphi(h))_i, \quad (4.232) \]
and so we have \( h \cdot f = f \cdot \varphi(h) \), since the action of \( G \) (hence of \( H \)) on frames is given by (2.127). Then from (4.230) we have
\[ f \cdot \varphi(h) = f \cdot (\hat{\varphi} \circ \hat{\varphi}(h)) \quad (4.233) \]
whence \( \varphi(h) = \hat{\varphi} \circ \hat{\varphi}(h) \) for all \( h \in H ; \hat{\varphi} \) is a lift of \( \varphi \). Since the definition of \( \hat{\varphi} \) involved a choice of lift \( \hat{f} \) of \( f \), we must consider the effect of choosing a different lift \( \hat{f}' \) of \( f \) and defining a lift \( \hat{\varphi}' : H \to \text{Spin}^R(V) \) of \( \varphi \) by \( h \cdot \hat{f}' = \hat{f}' \cdot \hat{\varphi}'(h) \). But there exists a unique \( r \in R \) such that \( \hat{f}' = \hat{f} \cdot r \), so
\[ \hat{f}' \hat{\varphi}'(h) = h \cdot (\hat{f} \cdot r) = (h \cdot \hat{f}) \cdot r = (\hat{f} \cdot \hat{\varphi}(h)) r = \hat{f}' \cdot (r^{-1} \hat{\varphi}(h) r) \quad (4.234) \]
thus \( \hat{\varphi}' = \gamma_{r^{-1}} \cdot \hat{\varphi} \). Hence we have a well-defined assignment of \( R \)-conjugacy classes of lifts of the isotropy representation to homogeneous spin-\( R \) structures. We must still show that \textit{equivalent} homogeneous spin-\( R \) structures give rise to the same class of lifts. Suppose we have an equivalence \( \Phi : \hat{P} \to \hat{P}' \), define \( \hat{f} \) and \( \hat{\varphi} \) as before, define \( \hat{f}' := \Phi(\hat{f}) \in \hat{P}'_o \) (clearly this is a lift of \( f \)), and let \( h \cdot \hat{f}' = \hat{f}' \cdot \hat{\varphi}'(h) \) define \( \hat{\varphi}' : H \to \text{Spin}^R(V) \). Then
\[ \hat{f}' \cdot \hat{\varphi}'(h) = h \cdot \Phi(\hat{f}) = \Phi(h \cdot \hat{f}) = \Phi(\hat{f} \cdot \hat{\varphi}(h)) = \hat{f}' \cdot \hat{\varphi}(h) \quad (4.235) \]
whence $\tilde{\phi}' = \tilde{\phi}$.

It remains to show that the association of an equivariant spin-$R$ structure to a lift of the isotropy representation (up to the respective equivalences) and vice-versa are inverse to each other.

Suppose we have an equivariant spin-$R$ structure $\tilde{P}$ and define $\tilde{f}$ and $\tilde{\phi}$ as above. Let us define an equivalence of spin-$R$ structures

$$\Phi : G \times_H \text{Spin}^R(V) \xrightarrow{\cong} \tilde{P}$$

(4.236)

by $[g, A] \mapsto g \cdot \tilde{f} \cdot A$. This is well-defined since for all $h \in H$, $[gh, \tilde{\phi}(h^{-1})A]$ is mapped to

$$(gh) \cdot \tilde{f} \cdot (\tilde{\phi}(h^{-1})A) = g \cdot (h \cdot \tilde{f} \cdot \tilde{\phi}^{-1}(h)) \cdot A = g \cdot ((\tilde{f} \cdot \tilde{\phi}(h)) \cdot \tilde{\phi}^{-1}(h)) \cdot A = g \cdot \tilde{f} \cdot A$$

(4.237)

where we have used the definition of $\tilde{\phi}$; it is clear from the definition that it is equivariant and covers $F_{SO}$, whence it is indeed an equivalence.

Now suppose we have a lift $\tilde{\phi}$ and define $\tilde{P} := G \times_H \text{Spin}(V)$. Recall that the isomorphism $G \times_H \text{SO}(V) \cong F_{SO}$ which makes this a spin-$R$ structure requires a choice of frame $f \in (F_{SO})_o$ and is given by $[g, A] \mapsto g \cdot f \cdot A$. If $\hat{f} \in \tilde{P}_o$ is a lift of $f$ then $\hat{f} = [1_G, r]$ for some $r \in R$. Thus for all $h \in H$ we have

$$h \cdot \hat{f} = h \cdot [1_G, r] = [h, r] = [1_G, \tilde{\phi}(h)r] = [1_G, r] \cdot (r^{-1} \tilde{\phi}(h)r)$$

(4.238)

so the lift associated to $\tilde{P}$ is $\gamma_{r^{-1}} \circ \tilde{\phi}$, which completes the proof.

We conclude our discussion of homogeneous spin-$R$ structures by noting that given a lift $\tilde{\phi} : H \to \text{Spin}^R(V)$, there exists a unique lift $\tilde{\phi} : H \to \text{Spin}(V)$ completing the diagram

$$\begin{array}{ccc}
\text{Spin}(V) & \xrightarrow{\tilde{\phi}} & \text{Spin}^R(V) \\
\downarrow & & \downarrow \\
H & \xrightarrow{\phi} & \text{SO}(V)
\end{array}$$

(4.239)

if and only if the image of $\tilde{\phi}$ is contained in the image of $\text{Spin}(V)$ in $\text{Spin}^R(V)$. When this is the case, we of course have a homogeneous spin structure on $M$. This also allows us to reduce the spin-$R$ structure to the product of a spin structure with a trivial $R$-principal bundle. Indeed, we can trivially extend $\tilde{\phi}$ to a map $H \to \text{Spin}(V) \times R$, $h \mapsto (\tilde{\phi}(h), 1)$. The resulting action of $H$ on the left of $\text{Spin}(V) \times R$ makes the quotient $\tilde{\pi} : \text{Spin}(V) \times R \to \text{Spin}^R(V)$ $H$-equivariant, whence we have a $(G, \text{Spin}(V) \times R)$-birel-equivariant bundle map

$$G \times_H (\text{Spin}(V) \times R) \longrightarrow G \times_H \text{Spin}^R(V),$$

(4.240)

which, using the natural isomorphism $G \times_H (\text{Spin}(V) \times R) \cong (G \times_H \text{Spin}(V)) \times R$ is

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simply
\[ P \times_M M_R \longrightarrow \tilde{P}. \] (4.241)

where \( P := G \times_H \text{Spin}(V) \) is the homogeneous spin structure, \( \tilde{P} := G \times_H \text{Spin}^R(V) \) is the homogeneous spin-\( R \)-structure, and \( M_R \) is the trivial principal \( R \)-bundle. More generally, a lift of \( \tilde{\phi} \) to \( \text{Spin}(V) \times R \) gives a reduction of the homogeneous spin-\( R \)-structure to a product \( P \times_M Q \) where \( Q \) need not be trivial.

Note that if \( \tilde{\phi} \) reduces to \( \phi \) as above then since the image of \( R \) commutes with that of \( \text{Spin}(V) \), we have \( \gamma_r \circ \tilde{\phi} = \tilde{\phi} \) for all \( r \in R \). On the other hand, a lift \( \tilde{\phi} \) which is fixed by \( R \)-conjugation must take values in \( \text{Spin}(V) \times Z_2 \) where \( Z_2 \) is the centre of \( R \). Thus, if \( Z(R) = Z_2 \), \( R \)-conjugate spin-\( R \) lifts correspond to homogeneous spin-\( R \) structures which reduce to products of homogeneous spin structures with trivial \( R \)-principal bundles.

4.4.2 Reconstruction of highly supersymmetric backgrounds with \( R \)-symmetry

We finish this chapter by generalising the treatment of Section 3.4.1 to include \( R \)-symmetry. We say that a Lorentzian spin-\( R \) manifold \((M, g)\) with spinor bundle \( \hat{S} \) with fibre \( S \) and admissible pair \((D, \rho)\) is highly supersymmetric if \( \dim \hat{S}_D > \frac{1}{2} \dim S \).

Equivalently, \( \hat{S}_{(D, \rho)} \) is highly supersymmetric as a subdeformation of \( \hat{s} \).

Super Harish-Chandra pairs

Let \( g \) be a filtered deformation of a transitive highly supersymmetric graded subalgebra \( a = V \oplus S' \oplus (h \oplus r') \) of \( \hat{s} \) on \( S' \) and note that the zeroth filtered subspace of \( g \) is canonically isomorphic to \( a_0 = h \oplus r' \), with which we now identify it. Recalling Lemma 4.25, if the squaring map is chosen such that \( R \) is compact, we can assume that \( r' \) acts faithfully on \( S' \).

If \((G_0, g)\) is a super Harish-Chandra pair (see Definition 3.45 of §3.4.2) such that the connected Lie subgroup \( A \) of \( G_0 \) generated by \( a_0 \) is closed then \((G_0, A, \eta)\) is a metric Klein pair. By the following lemma, which is analogous to Lemma 3.46, if the connected subgroup \( R' \) of \( R \) corresponding to \( r' \) is closed, there is a homogeneous Lorentzian spin-\( R \) structure on \((G_0, A, \eta)\). The proof of the lemma is much the same as that of the analogous Lemma 3.46, with some additional technicalities due to the \( R \)-symmetry.
Lemma 4.42. Let $(G_\overline{\mathbb{G}}, g)$ be a super Harish-Chandra pair where $g$ is a highly supersymmetric filtered subdeformation of $\overline{s}$, and suppose that the connected subgroup $A$ of $G_\overline{\mathbb{G}}$ generated by $a_0 = h \oplus t'$ is closed. Then if $t'$ acts faithfully on $S'$ and the connected subgroup $R'$ of $R$ generated by $t'$ is closed, the inclusion $a_0 \hookrightarrow so(V) \oplus \tau$ integrates to a unique morphism $\overline{\phi} : A \to Spin^R(V)$ lifting the isotropy representation $\varphi : A \to SO(V)$.

Proof. We have isomorphisms of $a_0$-modules $V \cong g_\overline{\mathbb{G}}/a_0$ and $S' \cong a_{-1}$, the first of which makes $(g_\overline{\mathbb{G}}, a_0, \eta)$ into a metric Lie pair. Thus $(G_\overline{\mathbb{G}}, A, \eta)$ is a metric Klein pair and we have the isotropy representation $\varphi : A \to SO(V)$ integrating the map $a_0 \hookrightarrow so(V) \oplus \tau$.

Now note that the universal cover of $A = \tilde{A} \times \tilde{R}'$, where $\tilde{H}$ and $\tilde{R}'$ are the simply-connected Lie groups corresponding to $h$ and $t'$ respectively. The map $a_0 \hookrightarrow so(V) \oplus \tau$ integrates to maps $\phi_1 \times \phi_2 : \tilde{A} \to Spin(V) \times R$ (where $\phi_1 : \tilde{A} \to Spin(V)$, $\phi_2 : \tilde{A} \to R$) and $\overline{\phi} : \tilde{A} \to Spin^R(V)$ making the solid arrows of the diagram commute. We will show that $\pi_1(A) \subseteq \ker \overline{\phi}$ in $\tilde{A}$, hence there exists a unique map $\overline{\phi}$ completing the commutative diagram, which proves the lemma.

We first observe that the maps $\pi_A$ and $\overline{\phi}$ both induce an action of $\tilde{A}$ on $V$ (resp. $S'$), and these actions agree in both cases since they both integrate the same representation of $a_0$ on $V$ (resp. $S'$). Note that the fundamental group $\pi_1(A) \subseteq \tilde{A}$ acts trivially on both modules. Now let $(h, r) \in \pi_1(A)$; since this pair acts trivially on both modules, so must $(\phi_1(h), \phi_2(r))$. Trivial action on $V$ implies that $\phi_1(h) = \pm \mathbb{1}_{Spin(V)}$ (since $R$ acts trivially on $V$). But then trivial action on $S'$ implies that $\pm \phi_2(r)$ acts trivially on $S'$. By Lemma 4.26, $R'$ acts effectively on $S'$, so $\phi_2(r) = \pm \mathbb{1}_R$. But then

$$\overline{\phi}(h, r) = [(\phi_1(h), \phi_2(r))] = [(\pm \mathbb{1}_{Spin(V)}, \pm \mathbb{1}_R)] = \mathbb{1}_{Spin^a(V)},$$

(4.243)

so $\pi_1(A) \subseteq \ker \overline{\phi}$ as claimed.\hfill $\Box$

Connections and curvature

Recall now that on manifolds with a spin-$R$ structure, we do not have a canonical lift of the Levi-Civita connection but instead a connection $\tilde{\nabla}$ whose components in a local trivialisation are those of the Levi-Civita connection plus an $\tau$-valued connection. We must construct such a connection as well as maps $\beta \in \Omega^1(M; \text{End} \overline{S})$, $\rho \in \text{Hom}(\bigotimes^2 \overline{S}, \text{ad} \overline{Q})$ in order to define the pair $(D = \tilde{\nabla} - \beta, \rho)$ which we hope to be able to show is admissible.

Lemma 4.43. Let $(G_\overline{\mathbb{G}}, g)$ be the super Harish-Chandra pair, $A \subseteq G_\overline{\mathbb{G}}$ the closed subgroup and $\overline{\phi} : A \to Spin^R(V)$ the lift of the isotropy representation as in the lemma above. Suppose furthermore that the class $[\mu \_] \in H^{2,2}(a_{-1}; a)$ corresponding to $g$ is admissible.
Then there exists an $A$-equivariant map $\Phi : g_\mathbb{H} \to so(V) \oplus \tau$ such that

$$\Phi|_{a_0} = d_e \hat{\phi} = i : a_0 = \mathfrak{h} \oplus \tau \to so(V) \oplus \tau$$

and a corresponding $G_{\mathbb{H}}$-invariant principal connection on the spin-$R$ structure on $M = G/A$ which lifts the Levi-Civita connection. Moreover, the curvature of this connection is determined by the 4th-order deformation maps $\tilde{\theta}_i$ determined by $[\mu_-]$.

Proof. Let us fix a representative $\mu_-$ of the cohomology class corresponding to $g$. By the assumption of admissibility, there exists an $a_0$-invariant cocycle $\beta + \gamma + \rho \in H^{2,2}(\mathfrak{a}_-; s) = \text{Hom}(V, so(V) \oplus \tau)$ such that

$$i_* \mu_- = i^* (\beta + \gamma + \rho) + \partial \lambda \in H^{2,2}(\mathfrak{a}_-; s),$$

and we can write a presentation of $g$ as an explicit deformation of $a$ using this expression as in equation (4.218) of Theorem 4.37. Let $A_1 : V \to so(V), A_2 : V \to \tau$ be the components of $\lambda$. We then define $\Phi : g_\mathbb{H} \to so(V) \oplus \tau$, writing $g_0 = a_0 = V \oplus \mathfrak{h} \oplus \tau'$, as follows:

$$\Phi(v + A + a) = (A + A_1(v)) + (a + A_2(v)),$$

for $v \in V, A \in \mathfrak{h}, a \in \tau'$. We immediately see by putting $v = 0$ that $\Phi|_{a_0} = d_e \hat{\phi} = i$ as claimed. For $A$-invariance, since $A$ is connected, it is sufficient for us to show that $\Phi$ is $a_0 = \text{Lie} A$ is invariant. Indeed, for $A' \in \mathfrak{h}, a' \in \tau'$ we have

$$\Phi(\text{ad}^\mathbb{H}_{(A', a')}(v + A + a))$$

$$= \Phi\big(A'v + \big([A, a'] + [A', \lambda_1(v)] - \lambda_1(A'v)\big) + \big([a', a] - [a', \lambda_2(v)]\big)\big)$$

$$= \big[A', A + \lambda_1(v)\big] + [a', a + \lambda_2(v)]$$

$$= \text{ad}^\mathbb{H}_{(A', a')}(v + A + a).$$

Thus $\Phi$ satisfies the conditions required in Wang’s Theorem 2.14 to induce a $G_{\mathbb{H}}$-invariant principal connection $\mathcal{A}$ on $\hat{P} = G_{\mathbb{H}} \times_A \text{Spin}^R(V)$. We can show that the Nomizu map of the Levi-Civita connection on $F_{SO} = G \times_A SO(V), L : g \to so(V)$, is exactly the $so(V)$-component of $\Phi$, which shows that the connection induced by $\Phi$ lifts the Levi-Civita connection. The curvature of the connection $\mathcal{A}$ is given by

$$\hat{R}(\xi^M_{v + A + a}, \xi^M_{v' + A' + a'})_o$$

$$= \big[\Phi(v + A + a), \Phi(v' + A' + a')\big]_{so(V)\oplus \tau} - \Phi\big([v + A + a, v' + A' + a']\big)$$

$$= \big[\lambda_1(v) + A, \lambda_1(v') + A'\big] + \big[\lambda_2(v) + a, \lambda_2(v') + a'\big]$$

$$- \lambda_1(\lambda_1(v)v' - \lambda_1(v)v + Av' - A'v)$$

$$- \lambda_2(\lambda_1(v)v' - \lambda_1(v)v + Av' - A'v)$$

$$- \lambda_2(\lambda_1(v)v' - \lambda_1(v)v + Av' - A'v)$$

$$= -\tilde{\theta}_1(v, v' - [\lambda_1(v), \lambda_1(v')] + \lambda_1(\lambda_1(v)v' - \lambda_1(v)v)$$

$$- \lambda_2(\lambda_1(v)v' - \lambda_1(v)v + Av' - A'v)$$

$$- \lambda_2(\lambda_1(v)v' - \lambda_1(v)v + Av' - A'v)$$

$$- \lambda_2(\lambda_1(v)v' - \lambda_1(v)v + Av' - A'v)$$

$$= -\tilde{\theta}_1(v, v') - \tilde{\theta}_2(v, v'),$$
where the unlabelled brackets are those of \(a_0 = \mathfrak{so}(V) \oplus a\). The first equality is part of Wang’s Theorem, the second is simply and application of definitions and the last is the result of cancellation between terms. By homogeneity, this completely determines the curvature. 

We note that the final calculation in the proof above shows that the Riemann curvature and \(\tau\)-symmetry curvature are given by

\[
R(\xi^M_{\nu + \alpha + \alpha}, \xi^M_{\nu' + \alpha' + \alpha'}) = -\tilde{\Theta}_1(v, v'), \quad F(\xi^M_{\nu + \alpha + \alpha}, \xi^M_{\nu' + \alpha' + \alpha'}) = -\tilde{\Theta}_2(v, v'),
\]

so as previously mooted in Remark 11, the map \(\tilde{\Theta}_2\) corresponds to a gauge field strength. This is further evidence that, as suggested in Remarks 11 and 9, it should be possible to modify our notion of admissible pairs and geometric realisability to be less restrictive. Nevertheless, we will restrict ourselves below to deformations which are geometrically realisable in the sense of Definition 4.38 (in particular setting \(\tilde{\Theta}_2 = 0\)).

**Reconstruction theorem**

We finish this chapter by providing a generalisation of Theorem 3.47 to the case of geometrically realisable deformations of \(\widehat{s}\).

**Theorem 4.44** (Reconstruction of highly supersymmetric background with \(R\)-symmetry).

Let \(\mathfrak{g}\) be a geometrically realisable filtered deformation of a highly supersymmetric graded subalgebra \(a = V \oplus S' \oplus (h \oplus \mathfrak{r}')\) of the Poincaré superalgebra with \(R\)-symmetry such that \(\mathfrak{r}'\) acts faithfully on \(S'\). Let \(G_{\mathfrak{g}}\) be the connected and simply-connected Lie group corresponding to the Lie algebra \(\mathfrak{g}_{\mathfrak{g}}\) and suppose that the connected subgroups \(A \subseteq G_{\mathfrak{g}}\) and \(R' \subseteq R\) corresponding to the subalgebras \(a_0 \subseteq \mathfrak{g}_{\mathfrak{g}}\) and \(\mathfrak{r}' \subseteq \mathfrak{r}\) are closed. Then there exists a homogeneous spin-\(R\) structure on the Lorentzian homogeneous space \(M = G_{\mathfrak{g}}/A\) with a connection \(D\) on the spinor bundle \(\widehat{S} = G_{\mathfrak{g}} \times_A S\) and an injective \(\mathbb{Z}_2\)-graded linear map \(\Psi : \mathfrak{g} \hookrightarrow \widehat{\Omega}_{(D, \rho)} \oplus \mathfrak{N}_{(D, \rho)} \oplus \widehat{\mathcal{D}}\) which restricts to a Lie algebra embedding \(\mathfrak{g}_{\mathfrak{g}} \hookrightarrow \widehat{\Omega}_{(D, \rho)} \oplus \mathfrak{N}_{(D, \rho)}\) and which satisfies

\[
\Psi([(X, s)] = \mathcal{L}_{\Psi(X)} \Psi(s), \quad \Psi([a, s]) = \Psi(a) \cdot \Psi(s), \quad \Psi([s, s']) = \kappa(\Psi(s), \Psi(s'))
\]

for all \(X \in V \oplus h \subset \mathfrak{g}_{\mathfrak{g}}, a \in \mathfrak{r}' \subset \mathfrak{g}_{\mathfrak{g}}\) and \(s, s' \in \mathfrak{T}\).

Furthermore, if \((D, \rho)\) is an admissible pair, in particular if \(\Psi(\mathfrak{g}_{\mathfrak{g}}) = \widehat{S}_D\), then \(\mathfrak{g}\) embeds into the Killing superalgebra \(\widehat{\mathcal{K}}_{(D, \rho)}\).

**Proof.** Lemma 4.42 and Lemma 4.43 give us a homogeneous spin-\(R\) structure \(\widehat{\rho} : G_{\mathfrak{g}} \times_A \text{Spin}^R(V) \to M\) with \(G_{\mathfrak{g}}\)-invariant connection \(\omega'\) induced by the map \(\Phi : \mathfrak{g}_{\mathfrak{g}} \to \mathfrak{so}(V) \oplus \mathfrak{r}\). We will denote the Koszul connection induced on \(\widehat{S}\) by \(\widehat{\nabla}\), as earlier this chapter.

Now let \(\beta + \gamma + \rho \in (\widehat{\mathcal{H}}^{2,2})_{a_0}\) be the normalised Spencer cocycle used in the construction of \(\Phi\) and \(\omega'\). Since \(\beta\) and \(\rho\) are \(a_0\)-invariant, and hence \(A\)-invariant since \(A\) is connected, we can construct \(G_{\mathfrak{g}}\)-invariant bundle maps \(\beta \in \Omega^1(M; \text{End} \widehat{S})\) and \(\rho \in \text{Hom}(\widehat{\mathcal{O}}^2 \widehat{S}, \text{ad} \Omega)\) via Frobenius reciprocity. We then have a \(G_{\mathfrak{g}}\)-invariant connection \(D = \widehat{\nabla} - \beta\) on \(\widehat{S}\).
We will make use of the isomorphism

\[ \mathcal{X}(M) \oplus \mathcal{R} \oplus \widehat{\mathcal{S}} = \Gamma \left( G_0 \times_A (V \oplus \mathcal{r} \oplus S) \right) \cong C^\infty \left( G_0, V \oplus \mathcal{r} \oplus S \right) \]  

(4.251)

in defining the map \( \Psi \) and in calculations. For \( X \in V \oplus \mathcal{h} \subseteq g_0, a \in \mathcal{r}' \subseteq g_0 \) and \( s \in g_1 \), we define sections \( \Psi(X) \in \mathcal{X}(M), \Psi(a) \in \mathcal{R} \) and \( \Psi(s) \in \widehat{\mathcal{S}} \), corresponding to the \( A \)-invariant maps

\[ \overline{\Psi}(X) : G_0 \to g_0/\mathcal{a}_0 \cong V, \quad g \mapsto \text{Ad}_{g^{-1}} X \mod \mathcal{a}_0, \]  

(4.252)

\[ \overline{\Psi}(a) : G_0 \to \mathcal{r}' \subseteq \mathcal{r}, \quad g \mapsto \text{Ad}_{g^{-1}} a, \]  

(4.253)

\[ \overline{\Psi}(s) : G_0 \to g_1 \subseteq S, \quad g \mapsto \text{Ad}_{g^{-1}} s. \]  

(4.254)

Note that up to a sign, \( \Psi(X) \) is nothing but the fundamental vector field \( \xi^M_X \), a Killing vector. A computation using Wang's theorem gives us

\[ \hat{\nabla}_{\xi^M_X} \Psi(s) = \beta (\xi^M_X, \Psi(s)), \quad \text{and} \quad \hat{\nabla}_{\xi^M_X} \Psi(a) = 0, \]  

(4.255)

which since the fundamental vector fields span all tangent spaces shows that \( \Psi(s) \in \widehat{\mathcal{S}}_D \) and that \( \Psi(a) \) is covariantly constant. By the definition of geometric realisability, we have \( \tilde{\Theta}_2 = 0 \), whence \( F = 0 \); the \( r \)-symmetry component \( \alpha \) of \( \mathscr{A} \) is flat. It follows that there are local choices of gauge in which \( \alpha \) vanishes, so we can locally treat covariant derivatives \( \hat{\nabla} \) and \( \hat{\mathcal{L}} \) as their non-gauged counterparts (even if these do not exist globally). Then by \( G_0 \)-invariance of \( \beta \) and \( \rho \), we find \( \hat{\mathcal{L}}_{\Psi(X)} \beta = \hat{\mathcal{L}}_{\Psi(X)} \rho = 0 \) so \( \Psi(X) \in \mathfrak{H}_{(D, \rho)} \) and \( \Psi(a) \cdot \beta = \Psi(a) \cdot \rho = 0 \), whence since \( F = 0 \) we find that \( \Psi(a) \in \mathcal{R}_{(D, \rho)} \). It can now be checked that \( \Psi \) satisfies (4.250), for example by comparing the brackets of \( g \), which are given by (4.218), to the Killing transport data of its image as determined in §4.2.2. The final claim follows immediately. \( \square \)
Chapter 5

Spencer cohomology of Poincaré superalgebras in $D = 5$ and $D = 6$

The Spencer (2,2)-cohomology of the Poincaré superalgebra has been computed in the minimal 5-dimensional case without $R$-symmetry by the author and JMF [50] and in the 6-dimensional case with and without $R$-symmetry by PdM, JMF and AS [49]. In this chapter, we generalise the results of those works by calculating the Spencer cohomology of the Poincaré superalgebra in 5 and 6 dimensions for arbitrary $N$-extended supersymmetry, with and without $R$-symmetry. Dimensions 5 and 6 are grouped together here both because of similarities in the structure of their spinor modules (both possessing a symplectic Majorana structure) and because of similarities in the Spencer data. All results here, in particular Propositions 5.1 and 5.2, are new and not published elsewhere at time of writing.

We choose our sign conventions here to be consistent with previous works [49, 50]. Thus, we have a + sign in the defining relation (2.17) of the Clifford algebra in both dimensions, and the signature of $\eta$ is mostly-minus $(+, -, -, -, -)$ in 5 dimensions and mostly-plus $(-, +, +, +, +)$ in 6 dimensions.

5.1 Symplectic modules

Recall our definition of Majorana and symplectic Majorana structures on pinor (and spinor) representations 2.1.4. To impose the symplectic Majorana reality condition, we had to tensor the pinor representation with an auxiliary module $\Delta^N$ with a quaternionic structure (from now on, we will omit the $\mathbb{H}$ superscript). Here, we will consider $\Delta$ to be a $G = \text{Sp}(N)$-module and $\mathbb{H}^N$ for some $N \in \mathbb{N}$. This will also essentially turn out to be the $R$-symmetry group.

5.1.1 The general case

Auxiliary module conventions

Let $N \in \mathbb{N}$, and let $\Delta_N$ be the right quaternionic vector space $\mathbb{H}^N$. We consider $\mathbb{H}^N$ as a $\mathbb{C}$-vector space (under right multiplication by complex scalars) with a quaternionic structure $\mathcal{J}_{\Delta}$ given by multiplication on the right by $j$; $\mathcal{J}_{\Delta}(q) = qj$. We identify
this with $\mathbb{C}^{2N}$ by identifying $q$ with $z = (z_1, z_2)^T$ where $q = z_1 + jz_2$. Under this identification, we have

$$q^\dagger q' = (z_1^\dagger z'_1 + z_2^\dagger z'_2) + j(z_1^T z'_2 - z_2^T z_1) = z^\dagger z + jz^T \Omega z$$

(5.1)

where

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix};$$

(5.2)

that is, we have

$$\langle q, q' \rangle = (z, z') + j\Omega(z, z'),$$

(5.3)

where on the LHS we have the standard $\mathbb{H}$-Hermitian inner product on $\Delta_N$ and on the RHS we have the standard $\mathbb{C}$-Hermitian inner product $(-,-)$ and the standard $\mathbb{C}$-symplectic form $\Omega$. The standard $\mathbb{C}$-basis $\{e_A\}_{A=1,2,\ldots,2N}$ for $\mathbb{C}^{2N}$ is of course an orthonormal basis for the former and symplectic basis for the latter. We will work with component expressions in this basis so that, for example, the symplectic form has components

$$\Omega_{AB} := \Omega(e_A, e_B)$$

(5.4)

given by the matrix $\Omega$. This can also be used to represent $\mathcal{J}_{\Delta}$ in complex notation:

$$\mathcal{J}_{\Delta}(z) = -\Omega^* z.$$  

(5.5)

This symplectic product induces the musical isomorphism $\flat: \Delta_N \to \Delta^*_N$ sending $u \mapsto u^\flat$, where

$$u^\flat(v) = \Omega(v, u)$$

(5.6)

for $u, v \in V$. The inverse isomorphism $\sharp: \Delta^*_N \to \Delta_N$ sends $\alpha \mapsto \alpha^\sharp$. These musical isomorphisms induce various others, such as $\otimes^2 \Delta_N \cong \mathbb{C} \text{End}_\mathbb{C} \Delta_N$. We define the “inverse” symplectic product with raised indices by (note the sign convention)

$$\Omega^{AC} \Omega_{CB} = \Omega_{BC} \Omega^{CA} = -\delta^A_B.$$  

(5.7)

This convention ensures that the matrix representation of the inverse form is also $\Omega$. These symbols are used to implement the musical isomorphisms at the level of components by raising and lowering indices using the following convention:

$$u_A = \Omega_{AB} u^B, \quad \alpha^A = \alpha_B \Omega^{BA}.$$  

(5.8)

These conventions are consistently chosen such that

$$u^\flat(v) = \Omega(v, u) = \Omega_{AB} v^A u^B = u_A v^A.$$  

(5.9)

Finally, we can use the symplectic product to “trace” objects with a pair of $\Delta_N$-indices. With the conventions we have chosen, this is the same as the trace of endomorphisms of $\Delta_N$, and we use the notation $\langle - \rangle$ for this trace. We demonstrate these last few points by showing how the trace is implemented on some $\Theta \in \otimes^2 \Delta_N \cong \mathbb{C} \text{End}_\mathbb{C} \Delta_N$:

$$\langle \Theta \rangle := \Omega_{AB} \Theta^{AB} = \Omega^{AB} \Theta_{AB} = \Theta^A_A = -\Theta^A_A.$$  

(5.10)
Note that only the $\Lambda^2 C \Delta_N$ component of $\otimes^2 C \Delta_N$ contributes to the trace. We use the notation $\langle \Lambda^2 C \Delta_N \rangle$ for the trace part of $\Lambda^2 C \Delta_N$ and $(\Lambda^2 C \Delta_N)_0$ for the traceless part, giving a decomposition

$$\Lambda^2 C \Delta_N = \langle \Lambda^2 C \Delta_N \rangle \oplus (\Lambda^2 C \Delta_N)_0. \quad (5.11)$$

$\text{Sp}(N)$ and $\text{USp}(2N)$

Now, we let $\text{Sp}(N)$ be the quaternionic unitary group; that is, the group of $N \times N$ quaternionic matrices whose (left) action on $\mathbb{H}^N$ preserves the standard $\mathbb{H}$-Hermitian inner product above, i.e. the $\mathbb{H}$-unitary matrices. One can show that this group also preserves $(-,-)$ and $\Omega$, and clearly any $\mathbb{C}$-linear transformation preserving $(-,-)$ and $\Omega$ also preserves $(-,-)$, so representing elements of $\text{Sp}(N)$ as complex matrices, we find that $\text{Sp}(N) \cong \text{USp}(2N) := \text{SU}(2N) \cap \text{Sp}(2N, \mathbb{C})$, the latter group consisting of $2N \times 2N$ $\mathbb{C}$-matrices $X$ satisfying both $X^\dagger X = \mathbb{V}_{2N}$ and $X^T \Omega X = \mathbb{V}_{2N}$. The musical isomorphisms (and all induced isomorphisms) are isomorphisms of $\text{Sp}(N)$ modules.

We will actually work more directly with the Lie algebra $\mathfrak{sp}(N)$ of $\text{Sp}(N)$. This is the space of $N \times N$ skew-Hermitian $\mathbb{H}$-matrices, or in the complex representation, the $\mathbb{C}$-matrices $X$ such that

$$X^\dagger + X = X^T \Omega + \Omega X = 0; \quad (5.12)$$

in components, this is equivalent to

$$\begin{pmatrix} X^A_B \end{pmatrix}^* = -X^B_A \quad \text{and} \quad X_{AB} = X_{BA}, \quad (5.13)$$

where the apparently strange placement of indices on the first equation is because $(e_A, e_B) = \delta_{AB}$ but we use $\Omega_{AB}$ to raise lower indices; restoring the $\delta_{AB}$s gives us $(X^C_B)^* \delta_{CA} = -X^C_A \delta_{CB}$ where we use the Einstein summation convention as usual. By contracting the first equation with an $\Omega_{BC} = \Omega^{BC}$ and then relabelling indices, the equations can be combined into

$$\begin{pmatrix} X^{AB} \end{pmatrix}^* = X_{BA} = X_{AB}. \quad (5.14)$$

Of course, these conditions could also be written in terms of matrices with $N \times N$ blocks, but we will not use such expressions.

**Quaternionic endomorphisms**

More generally, we will need to work with the space $\text{End}_{\mathbb{H}}(\Delta_N)$ of $\mathbb{H}$-endomorphisms of $\Delta_N$; that is, those $\mathbb{C}$-endomorphisms which preserve the quaternionic structure $\mathfrak{J}_\Delta$. We can use (5.5) to write this condition as a matrix equation:

$$\Omega X^* = X \Omega \quad (5.15)$$

or as a component expression,

$$\begin{pmatrix} X_{AB} \end{pmatrix}^* = X^{AB}. \quad (5.16)$$

One can easily see that $\mathfrak{sp}(N)$ is a subspace of $\text{End}_{\mathbb{H}}(\Delta_N)$.
Another description of \( \mathfrak{sp}(N) \) that we will find useful is that under the musical isomorphisms \( \bigotimes_2^\ast \Delta_N \cong \text{End}_\mathbb{C} \Delta_N \), it is the intersection of the image of \( \bigotimes^2 \Delta_N \) with \( \text{End}_\mathbb{H} \Delta_N \); equivalently, it consists of the elements which are \textbf{anti-Hermitian} as \( \mathbb{C} \)-endomorphisms and whose musical image is in \( \bigotimes^2 \Delta_N \). It has a complementary \( \mathfrak{sp}(N) \)-submodule in \( \text{End}_\mathbb{H} \Delta_N \) consisting of the \textbf{Hermitian} \( \mathbb{C} \)-endomorphisms of \( \Delta_N \) with musical image in \( \bigwedge^2 \Delta_N \). For want of better notation, we denote this latter module by \( \bigwedge^2 \Delta_N |_H \).

### 5.1.2 The minimal case

We now make some brief comments about the case \( N = 1 \). This is a special case, both because it is the relevant \( R \)-symmetry module for minimal 5- and 6-dimensional Poincaré supersymmetry, and because it is simpler to work with than \( \Delta_N \) for larger \( N \). The main reason for this is that \( \Delta_1 \) is 2-dimensional over \( \mathbb{C} \), and so \( \bigwedge^2 \Delta_1 \) is 1-dimensional and spanned by (the inverse of) the symplectic form. In other words, \( (\bigwedge^2 \Delta_1)_0 = 0 \) and \( \bigwedge^2 \Delta_1 = \langle \bigwedge^2 \Delta_1 \rangle \cong \mathbb{C} \), and so any \( \Theta \in \bigwedge^2 \Delta_1 \) is proportional to its trace and (the inverse of) \( \Omega_A \):

\[
\Theta^{AB} = \frac{1}{2} \langle \Theta \rangle \Omega^{AB} \quad \text{or} \quad \Theta_{AB} = \frac{1}{2} \langle \Theta \rangle \Omega_{AB}
\]

(5.17)

It is also particularly convenient to work with a symplectic basis \( \{ e_A \}_{A=1,2} \) in this case, since then \( \Omega^{AB} = \epsilon^{AB} = \epsilon_{AB} \), where \( \epsilon \) is the Levi-Civita symbol with the convention \( \epsilon_{12} = e^{12} = 1 \). For the groups, we have a natural isomorphism \( \text{Sp}(1) \cong \text{SU}(2) \).

### 5.2 Spencer cohomology of Poincaré superalgebras in \( D = 5 \)

The computation of the Spencer \((2,2)\)-cohomology group of the Poincaré superalgebra in the minimal 5-dimensional case without \( R \)-symmetry was one of the main results of the author’s MSc thesis [51] and is also found in [50]. Here, we present the results of the Spencer cohomology calculation in 5 dimensions in full generality – with \( N \)-extended supersymmetry and with or without \( R \)-symmetry. The signature of the metric is \textbf{mostly-minus}.

#### 5.2.1 Spinorial conventions

**Basic structure**

With the conventions above, we have \( \text{Cl}(V) \cong \text{Cl}(4,1) \cong \mathbb{H}(2) \oplus \mathbb{H}(2) \), so there are two distinct quaternionic pinor modules given by the left and right fundamentals – note that these are isomorphic as modules of \( \text{Cl}_0(V) \cong \mathbb{H}(2) \) and thus as modules of \( \text{Spin}(V) \), and they are irreducible with respect to any of these actions. They are distinguished by the action of the volume element, which squares to \( 1 \) and acts as \( + \text{Id} \) on one module and as \( - \text{Id} \) on the other; we will work with the former, denoting it by \( \Sigma \) as in [50] rather than by \( \mathbb{P}_+ \) as in Section 2.1 of this work. We note that
Cl(V) \otimes \mathbb{C} \cong \text{Cl}(5; \mathbb{C}) \cong \mathbb{C}(4) \oplus \mathbb{C}(4), and we can also view \( \Sigma \) as the complex pinor representation \( \mathbb{P}_+^C \). As a vector space, \( \Sigma \cong \mathbb{H}^2 \cong \mathbb{C}^4 \).

We consider \( \Sigma \) to be a complex Spin(V)-module with invariant quaternionic structure \( \mathcal{J} \), so we will work with symplectic Majorana spinors. Recalling our discussion of such structures from Section 2.1, \( \Sigma \) has a \( \mathbb{C} \)-Hermitian Spin(V)-invariant inner product \( \langle \cdot, \cdot \rangle \) which in this case satisfies

\[
\langle \sigma_1, v \cdot \sigma_2 \rangle = \langle v \cdot \sigma_1, \sigma_2 \rangle,
\]

and a Spin(V)-invariant \( \mathbb{C} \)-symplectic form \( C \) given by

\[
C(\sigma_1, \sigma_2) = \langle \mathcal{J}(\sigma_1), \sigma_2 \rangle.
\]

For \( \sigma \in \Sigma \), will denote by \( \tilde{\sigma} : \Sigma \to \mathbb{C} \) the Dirac conjugate defined by \( \sigma' \mapsto \langle \sigma, \sigma' \rangle \), and by \( \overline{\sigma} : \Sigma \to \mathbb{C} \) the Majorana conjugate defined by \( \sigma' \mapsto C(\sigma, \sigma') \). Note that (using (5.19)) we have \( \overline{\sigma} = \mathcal{J}(\sigma) \) for all \( \sigma \in \Sigma \). By the discussion below, we will work with symplectic Majorana spinors, which will allow us to equate these two notions.

**Symplectic Majorana spinors**

Now, fixing some positive integer \( N \), we can form the Spin(V) \times \text{Sp}(N)-module \( \Sigma \otimes \mathbb{C} \Delta_N \), denoting by \( \mathcal{J}_ \sigma = \mathcal{J} \otimes \mathcal{J}_\Delta, C_\sigma = C \otimes \Omega \) the corresponding invariant real structure and \( \mathbb{C} \)-invariant bilinear. Any \( s \in \Sigma \otimes \mathbb{C} \Delta_N \) has a unique representation \( s = s^A \otimes e_A \) where \( e_A \) is the standard basis on \( \Delta_N \) and the \( 2N \) elements \( s^A \in \Sigma \) are the pinors uniquely defined by that expression. We can then write the symplectic Majorana condition as any of the following:

\[
\mathcal{J}_\sigma(s) = s, \quad \mathcal{J}(s^A) = \Omega_{AB} s^B, \quad \overline{s}^A = \Omega_{AB} \overline{s}^B, \quad (s^A)^* = \Omega_{AB} B s^B,
\]

where we sum over repeated indices. The second expression follows from the first by applying the definition of \( \mathcal{J}_\sigma \) and that of the components \( s^A \), and the third then immediately follows by taking the Dirac conjugate, while the final expression holds in any explicit representation of \( \Sigma \) as a complex vector space where we set \( B = -(CA^{-1})^T \), where \( A, C \) are the matrices representing \( \langle -, - \rangle \) and \( C \) respectively. In this notation, \( \mathcal{J}(\sigma) = -B \sigma^* \).

We denote the space of \( N \)-extended symplectic Majorana spinors, the real Spin(V) \times \text{Sp}(N)-submodule of \( \Sigma \otimes \mathbb{C} \Delta_N \) of elements satisfying (5.20), by \( S_N \). Note that \( S_N = \bigoplus_{N \text{times}} \Sigma \otimes \Sigma \otimes \cdots \otimes \Sigma \) as Spin(V)-modules.

**Endomorphism of the symplectic module and \( R \)-symmetry**

We will need to work with elements of \( \text{End}_{\mathbb{C}} \Delta_N \cong \mathbb{C}^2 \Delta_N \) whose action on \( \Sigma \otimes \mathbb{C} \Delta_N \) preserves \( S_N \), i.e. those which preserve the symplectic Majorana condition. These are precisely the elements of \( \text{End}_{\mathbb{R}} \Delta_N \), those endomorphisms which preserve the quaternionic form \( \mathcal{J}_\Delta \). Indeed, it be checked that a \( \mathbb{C} \)-endomorphism preserves the last condition (5.20) if and only if it satisfies (5.16). We now note that in this description, \( R = \text{Sp}(N) \) is the \( R \)-symmetry group; in particular, \( \text{Spin}^R(V) = \text{Spin}(V) \times \mathbb{Z}_2 \text{Sp}(N) \). For
$N = 1$, this is actually the spin-$h$ group.

**Bilinears**

It will be convenient for our calculations to associate to a spinor $s \in \Sigma \otimes C^\Delta \Delta_N$ the Dirac forms $\omega^{(p)} \in \Lambda^p V \otimes \Delta^2 \Delta_N$ defined by

$$\omega^{(p)}_{\mu_1 \ldots \mu_p}^{AB} := \bar{s}^A \Gamma_{\mu_1 \ldots \mu_p} s^B$$

(5.21)

for $p = 0, 1, 2$ and note that

$$\bar{s}^A s^B = -\bar{s}^B s^A, \quad \bar{s}^A \Gamma_{\mu} s^B = -\bar{s}^B \Gamma_{\mu} s^A, \quad \bar{s}^A \Gamma_{\mu \nu} s^B = \bar{s}^B \Gamma_{\mu \nu} s^A,$$

(5.22)

so that the forms have the $\Delta$-index symmetries

$$\omega^{(0)}_{AB} = -\omega^{(0)}_{BA}, \quad \omega^{(1)}_{AB} = -\omega^{(1)}_{BA}, \quad \omega^{(2)}_{AB} = \omega^{(2)}_{BA}.$$ (5.23)

We also define the Dirac scalar and current of $s$ as follows: $\mu := \langle \omega^{(0)} \rangle$ and $\kappa := \langle \omega^{(1)} \rangle \sharp$, where $\sharp$ is the musical isomorphism $V^* \cong V$ induced by $\eta$, or

$$\mu = \Omega_{AB} \bar{s}^A s^B, \quad \kappa^\mu = \Omega_{AB} \bar{s}^A \Gamma^\mu s^B.$$ (5.24)

These are real for $s \in S_N$.

We will also make use of the Fierz identity

$$s^A \bar{s}^B := -\frac{1}{4} \left[ \omega^{(0)}_{AB} + \omega^{(1)}_{AB} + \omega^{(2)}_{AB} \right].$$ (5.25)

### 5.2.2 Solving the cocycle conditions

In 5 dimensions, the $N$-extended Poincaré superalgebra with $R$-symmetry takes the form $\hat{s} = V \oplus S_N \oplus \mathfrak{so}(V) \oplus \tau$, with $S_N$ as described above and $\tau = \mathfrak{sp}(N)$. We take the squaring map to be the Dirac current defined above so that the bracket on the odd part is $[s, s] = \Omega_{AB} \bar{s}^A \Gamma_{\mu} s^B$. We can thus write the cocycle conditions (4.134) and (4.135) as

$$\bar{s}^A \Gamma_{(\mu} \beta_{\nu)}^{AB} s^B = 0,$$ (5.26)

$$\gamma(s, s)_{\mu \nu} = -2 \bar{s}^A \Gamma_{[\mu} \beta_{\nu]}^{AB} s^B,$$ (5.27)

$$\kappa^\mu \beta^A_{\mu \nu} + \frac{1}{4} \gamma(s, s)_{\mu \nu} \Gamma^\mu \nu s^A + \rho(s, s)^A s^B = 0,$$ (5.28)

where we recall that $\beta \in \text{Hom}(V, \text{End} S), \gamma \in \text{Hom}(\bigwedge^2 S_N, \mathfrak{so}(V)), \rho \in \text{Hom}(\bigwedge^2 S, \tau)$ are the components of a Spencer cochain.

The first of these equations provides a constraint on $\beta$ which can be solved independently of the other equations. The second simply expresses $\gamma$ in terms of $\beta$. Substitution of the expression for $\gamma$ and the solution $\beta$ of the first equation into the last allows it to be solved. We will show that this provides further constraints on $\beta$ and expresses $\rho$ in terms of components of $\beta$. 

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To tackle these equations, we first parametrise $\beta$ and $\rho$ as follows:

$$\beta_{\mu AB} = (A_{\mu AB} + B_{\mu AB}) + (C_{\mu AB} + D_{\mu AB})\Gamma^\nu + \frac{1}{2}(E_{\mu \nu AB} + F_{\mu \nu AB})\Gamma^{\nu \rho}, \quad (5.29)$$

$$\rho_{CD}(s,s) = \omega^{(0)\rho}_{ABCD} + \omega^{(1)\rho}_{ABCD} + \frac{1}{2}\omega^{(2)\rho}_{ABCD}, \quad (5.30)$$

where the Dirac bilinears $\omega^{(p)}$ are formed from the spinor $s$, and

$$A \in V \otimes \wedge^2 \Delta_N, \quad C \in V \otimes V \otimes \wedge^2 \Delta_N, \quad E \in V \otimes \wedge^2 V \otimes \wedge^2 \Delta_N,$$

$$B \in V \otimes \wedge^2 \Delta_N, \quad D \in V \otimes V \otimes \wedge^2 \Delta_N, \quad F \in V \otimes \wedge^2 V \otimes \wedge^2 \Delta_N, \quad (5.31)$$

We note that this parametrisation actually makes $\beta_\mu$ an endomorphism of $\Sigma \otimes \Omega \Delta_N$ rather than $S_N$; we will need to impose (5.16) on the components of $\beta$ to make sure that it preserves the Majorana condition, but we will do this after solving the cocycle conditions. This is the most general complexified form of $\beta$ because the $\Gamma$-matrices span End$_C \Sigma$ and $\wedge^2 \Delta_N \otimes V_2 \Delta_N \cong \otimes V_2 \Delta_N \cong \text{End}_C \Delta$ and $\text{End}_C \Sigma \otimes \Omega \Delta_N \cong \text{End}_C \Sigma \otimes \text{End}_C \Delta$ (indeed, this is at least in part the reason for the Majorana spinor construction). Likewise, the components of $\rho$ are implicitly complexified here, but after imposing the reality conditions on $\beta$, $\rho$ will also be of the correct form. We implicitly use the fact that the map $\wedge^2 \Delta_N \rightarrow \bigotimes_{p=0}^2 \wedge^p V \otimes \wedge^2 \Delta_N$ defined by the polarisation $(s,s) \rightarrow \omega^0 \otimes \omega^1 \otimes \omega^2$ is injective, whence maps from $\wedge^2 \Delta_N$ can be written in terms of the bilinears. For a more comprehensive treatment of these issues, at least in the $N = 1$ case, see [50].

**First cocycle condition**

We begin by computing the quantity $\bar{s}^A \Gamma_\mu \beta_{\nu AB} s^B$ which appears in equations (5.26) and (5.27) by substituting (5.29), evaluating products of $\Gamma$ matrices and identifying Dirac bilinears. The result is

$$\bar{s}^A \Gamma_\mu \beta_{\nu AB} s^B = \omega^{(1)AB}_{\mu} A_{\nu AB} + \omega^{(2)AB}_{\mu} A_{\nu AB} + \omega^{(0)AB}_{\mu} C_{\nu AB}$$

$$- \frac{1}{4} \epsilon^{\mu}_{\alpha \beta \gamma \delta} \omega^{(2)AB}_{\alpha \beta} F_{\nu \gamma \delta AB} + \omega^{(1)\alpha AB}_{\mu} E_{\nu \gamma \delta AB}$$

$$= \omega^{(0)AB}_{\mu} C_{\nu AB} + \omega^{(1)\alpha AB}_{\mu} \left( \eta_{\alpha \mu} A_{\nu AB} + E_{\nu \mu AB} \right)$$

$$+ \omega^{(2)AB}_{\alpha \beta} \left( \delta^\rho_{\mu} D_{\nu} D_{\beta}^{\rho} - \frac{1}{4} \epsilon^{\mu}_{\alpha \beta \gamma \delta} F_{\nu \gamma \delta AB} \right), \quad (5.32)$$

where we have used the symmetry properties (5.23) of the Dirac bilinears to eliminate some terms, and for our convenience in the next step, we have collected the coefficients of each Dirac bilinear in this expression. Since equation (5.26) must hold for all $s \in S$, the $\mu \nu$-symmetric part of each of these coefficients must vanish:

$$C_{(\mu \nu)AB} = 0, \quad (5.33)$$

$$\eta_{\alpha (\mu} A_{\nu) AB} + E_{(\mu \nu)\alpha AB} = 0, \quad (5.34)$$

$$\delta_{(\mu}^{[\alpha} D_{\nu) \beta]} - \frac{1}{4} \epsilon_{(\mu}^{\alpha \beta \gamma \delta} F_{\nu \gamma \delta AB} = 0. \quad (5.35)$$

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Equation (5.33) simply says $C \in \wedge^2 V \otimes \wedge^2 \Delta_N$. Taking cyclic permutations of equation (5.34), we find that $\eta_{(\alpha \mu A \nu)} AB = 0$, which upon contracting a pair of indices gives us $A = 0$. Substituting this back into (5.34), we have $E_{(\mu \nu) A B} = 0$, so $E \in \wedge^3 V \otimes \wedge^2 \Delta_N$. Contracting the indices $\mu$ and $\nu$ in (5.35) gives us

$$D^{[\alpha \beta]}_{AB} - \frac{1}{4} \epsilon_{\mu \nu \rho} F^{\mu \nu \rho}_{AB} = 0 \quad (5.36)$$

while on the other hand, contracting $\nu$ and $\beta$ yields

$$\delta^\alpha_\mu D^\nu v_{AB} - 5 D^\alpha_{AB} - \frac{1}{2} \epsilon_\mu \beta \gamma \delta F^\gamma \delta_{AB} = 0. \quad (5.37)$$

Now, lowering the $\alpha$ index and considering the $\mu \alpha$-symmetric part of this equation, we find that $D_{(\mu \alpha) AB} = \frac{1}{3} D^\mu_{AB}$, while the skew-symmetric part gives us

$$D_{[\mu \alpha] AB} + \frac{1}{10} \epsilon \mu \alpha \rho \sigma \tau F^\rho \sigma \tau_{AB} = 0. \quad (5.38)$$

Comparing this with equation (5.36), we see that we must have $D_{[\mu \nu]} = 0$ and $F^{[\mu \nu \rho \sigma]}_{AB} = 0$. We now set $d_{AB} := \frac{1}{5} D^\mu_{AB}$, so using (5.37), $D_{\mu \nu AB} = d_{AB} \eta_{\mu \nu}$. We can substitute this back into equation (5.35) to get

$$\epsilon_{\mu \alpha \beta \gamma \delta} F^\gamma \delta_{AB} + \epsilon_{\nu \alpha \beta \gamma \delta} F^\gamma \delta_{AB} = 0. \quad (5.39)$$

We contract this equation with $\epsilon^{\nu \alpha \beta \rho \sigma}$ to obtain

$$16 F^{\rho \sigma}_{\mu AB} + 8 \delta^{[\rho \sigma]}_{\mu} F^\nu_{v AB} = 0 \Rightarrow F_{\mu \nu AB} = \frac{1}{2} \eta_{\mu \nu} \phi_{\alpha AB} \quad (5.40)$$

where we define $\phi_{\mu AB} := F^\alpha_{\mu AB}$.

In summary, $\beta$ now takes the form

$$\beta_{\mu AB} = B_{\mu AB} + C_{\mu \nu AB} \Gamma^\nu_{AB} + d_{AB} \Gamma_{\mu AB} + \frac{1}{2} E_{\mu \nu AB} \Gamma^\nu_{AB} + \frac{1}{4} \phi^\nu_{AB} \Gamma_{\mu AB} \quad (5.41)$$

where $C \in \wedge^2 V \otimes \wedge^2 \Delta_N$, $E \in \wedge^3 \otimes \wedge^2 \Delta_N$, $d \in \wedge^0 V \otimes \wedge^2 \Delta_N$, and $\phi \in \wedge^1 V \otimes \wedge^2 \Delta_N$. Substituting this new form of $\beta$ into (5.27) now gives us

$$\gamma(s, s)_{\mu \nu} = 2 C_{\mu \nu AB} \omega_{(0) AB}^{(0)} + 2 E_{\mu \nu AB} \omega_{(1) AB}^{(1)} + \frac{1}{4} \epsilon_{\mu \nu \rho \sigma \tau} \phi^\rho_{AB} \omega_{(2) AB}^{(2)} \quad (5.42)$$

**Second cocycle condition**

We now turn to solving the second cocycle condition (5.28). We will substitute the expressions for $\beta$ and $\gamma$ above, but we must first arrange the equation into a different form, one which is quadratic in $s$ rather than cubic. This is necessary due to the Fierz identities, which introduce non-trivial cubic relations on spinors.

To make this rearrangement, we must first re-polarise the equation to obtain an (equivalent) symmetric trilinear equation in terms of three spinor arguments $s_1, s_2, s_3 \in S$. We then use the Fierz identity to move $s_3$ to the right of all products to obtain an equation of the form $W(s_1, s_2) s_3 = 0$, where $W(s_1, s_2) = W(s_2, s_1) \in \text{End}(S)$. 

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We can then abstract away the argument \( s_3 \) and depolarise to finally obtain the quadratic equation \( W(s, s) = 0 \).

The polarised equation (with the index \( A \) lowered with the symplectic form \( \Omega \)) is

\[
\kappa(s_1, s_2)^\mu \beta_{\mu AB} s_3^B + \frac{1}{4} \Omega_{AB} \gamma(s_1, s_2) \mu_{\nu} \Gamma^{\mu \nu} s_3^B + \rho(s_1, s_2)_{AB} s_3^B + \text{cyclic perm's of } (1,2,3) = 0.
\]

(5.43)

where we recall that \( \kappa(s_1, s_2)^\mu = [s_1, s_2]^\mu = \Omega_{CD} s_1^C s_2^D \). The LHS can be rearranged into the form

\[
\kappa(s_1, s_2)^\mu \beta_{\mu AB} s_3^B + \beta_{\mu AB} \{ s_1^B \kappa(s_2, s_3)^\mu + s_2^B \kappa(s_3, s_1)^\mu \} + \frac{1}{4} \Omega_{AB} [ \gamma(s_1, s_2)_{\mu \nu} \Gamma^{\mu \nu} s_3^B + \Gamma^{\mu \nu} \{ s_1^B \gamma(s_2, s_3)_{\mu \nu} + s_2^B \gamma(s_3, s_1)_{\mu \nu} \} ] + \rho(s_1, s_2)_{AB} s_3^B + (s_1^B \rho(s_2, s_3)_{AB} + s_2^B \rho(s_3, s_1)_{AB}).
\]

(5.44)

We have

\[
s_1^B \kappa(s_2, s_3)^\mu = \Omega_{CD} (s_1^B s_2^C) \Gamma^D s_3^3,
\]

(5.45)

\[
s_1^B \gamma(s_2, s_3)_{\mu \nu} = \frac{1}{2} s_1^B \{ \gamma(s_2 + s_3, s_2 + s_3)_{\mu \nu} - \gamma(s_2, s_2)_{\mu \nu} - \gamma(s_3, s_3)_{\mu \nu} \} = s_1^B \left( 2 C_{\mu \nu CD} (s_2^C s_3^D) + 2 E_{\mu \nu \rho CD} (s_2^C \Gamma^\rho s_3^D) \right) - 2 d_{CD} (s_2^C \Gamma^D s_3^D) + \frac{1}{4} \epsilon_{\mu \nu \rho \sigma \tau} \phi_{CD}^\rho (s_2^C \Gamma^\sigma s_3^D) \right),
\]

(5.46)

\[
s_1^B \rho(s_2, s_3)_{AB} = \frac{1}{2} s_1^B \left( \rho(s_2 + s_3, s_2 + s_3)_{AB} - \rho(s_2, s_2)_{AB} - \rho(s_3, s_3)_{AB} \right) = (s_1^B s_2^C) \rho_{CDAB}^0 + \frac{1}{2} \rho_{CDAB}^1 \Gamma^\rho + \frac{1}{2} \rho_{CDAB}^2 \Gamma_{\rho \sigma} s_3^D.
\]

(5.47)

The endomorphisms \((s_1^B s_2^C)\) could be evaluated using a Fierz identity at this stage, but we postpone that and write (5.43) as

\[
0 = \kappa(s_1, s_2)^\mu \beta_{\mu AB} s_3^B + \Omega_{CD} \beta_{\mu AB} (s_1^B s_2^C) \Gamma^D s_3^3 + \frac{1}{4} \Omega_{AB} \gamma(s_1, s_2)_{\mu \nu} \Gamma^{\mu \nu} s_3^B + \frac{1}{4} \Omega_{AB} \Gamma^{\mu \nu} (s_1^B s_2^C + s_2^B s_1^C) \left( 2 C_{\mu \nu CD} + 2 E_{\mu \nu \rho CD} \Gamma^\rho - 2 d_{CD} \Gamma_{\mu \nu} + \frac{1}{4} \epsilon_{\mu \nu \rho \sigma \tau} \phi_{CD}^\rho \Gamma_{\sigma \tau} s_3^D \right)
\]

(5.48)

This is of the form \( W(s_1, s_2)_{AD} s_3^D = 0 \) as desired; we may abstract \( s_3^D \) and write the equivalent equation \( W(s, s)_{AD} = 0 \):

\[
0 = \kappa^\mu \beta_{\mu AD} + 2 \Omega_{CD} \beta_{\mu AB} (s_1^B s_2^C) \Gamma^D + \frac{1}{4} \Omega_{AD} \gamma(s, s)_{\mu \nu} \Gamma^{\mu \nu} + \Omega_{AB} \Gamma^{\mu \nu} (s_1^B s_2^C) \left( C_{\mu \nu CD} + E_{\mu \nu \rho CD} \Gamma^\rho - d_{CD} \Gamma_{\mu \nu} + \frac{1}{4} \epsilon_{\mu \nu \rho \sigma \tau} \phi_{CD}^\rho \Gamma_{\sigma \tau} \right)
\]

(5.49)

This is the Fierz identity (5.25) can be rendered as

\[
s^{A}_{\mu} s^{B}_{D} = - \frac{1}{4} \left( \omega_{\mu}^{(0) BC}_{\sigma} + \omega_{\mu}^{(1) BC}_{\sigma} \Gamma^{\sigma} + \frac{1}{2} \omega_{\mu}^{(2) BC}_{\sigma \tau} \Gamma^{\sigma \tau} \right),
\]

(5.50)
and we must use this to replace each instance of $s^{AB}$ in (5.49). In particular, we have
\[
\beta_{AB}(s^{BC})C^{\mu}= -\frac{1}{4}\beta_{AB}\Gamma^{\mu}\omega(0)^{BC} \cdot \frac{1}{4}\beta_{AB}\Gamma^{\mu}\omega(1)^{BC} - \frac{1}{8}\beta_{AB}\Gamma^{\mu}\omega(2)^{BC},
\]
and we expand
\[
\gamma_{\mu\nu}(s,s)C^{\mu}= 2\epsilon_{\mu\nu\rho\sigma\tau}\phi^{\rho}\epsilon_{\tau}^{\sigma}BC
\]
Making these replacements, we note that the coefficient of each Dirac bilinear in (5.49) must vanish, giving us the equations
\[
0 = -\frac{1}{2}\beta_{AB}\Omega_{CD}d^{\mu} + \frac{1}{2}\Omega_{AD}C_{\mu
u}, \quad 0 = -\frac{1}{4}\Omega_{AB}\Gamma^{\mu}(C_{\mu}^{CD} - E_{\mu\nu\rho}D^{\rho}) + \frac{1}{8}\epsilon_{\mu\nu\rho\sigma\tau}\phi^{\rho}C_{\mu}^{CD} \Gamma^{\sigma},
\]
Starting with (5.53), we can evaluate all of the products of $\Gamma$-matrices:
\[
0 = -\frac{1}{2}(\beta_{AB} - 4\phi_{AB})\Gamma^{\mu} - C_{\mu
u}(\Gamma^{\mu} + 5d_{\mu} - \frac{1}{4}\epsilon_{\mu\nu\rho\sigma\tau}\epsilon_{\sigma\tau}^{\rho}E_{\mu\nu\rho}[\Gamma^{\sigma}D]_{\Omega}C_{\nu}^{CD} + \frac{1}{2}\Omega_{AD}C_{\mu\nu}^{BC}G^{\mu\nu} \cdot \frac{1}{4}\Omega_{AB}\Gamma^{\mu}(C_{\mu}^{CD} - E_{\mu\nu\rho}D^{\rho}) + \frac{1}{8}\epsilon_{\mu\nu\rho\sigma\tau}\phi^{\rho}C_{\mu}^{CD} \Gamma^{\sigma} + 20d_{\mu} + 3\phi_{\mu}C_{\nu}^{CD} \Gamma^{\mu} \cdot \frac{1}{8}\Omega_{AB}\Gamma^{\mu}C_{\mu}^{CD}d^{\mu} + \frac{1}{2}\Omega_{AD}C_{\mu
u}^{BC}G^{\mu\nu} + \frac{1}{2}\rho^{(0)}_{BC} + \frac{1}{2}\rho^{(1)}_{\rho\sigma} + \frac{1}{2}\rho^{(2)}_{\sigma\rho}.
\]
Since $1, \Gamma^{\mu}, \Gamma^{\mu
u}$ are independent, their coefficients must vanish, giving us
\[
-\frac{5}{2}d_{\mu}C_{\nu}^{CD} - 5\phi_{AB}d_{\mu}^{CD} - \frac{1}{2}\Omega_{AD}C_{\mu\nu}^{BC} + \frac{1}{2}\rho^{(0)}_{BC} = 0,
\]
We obtain further equations by similar manipulations of (5.54) and (5.55) which we
do not print explicitly. These equations are solved by taking various symmetrisation
and traces, the details of which we also omit. The full solution is given below.

Summary of results
The general solution to the cocycle equations is as follows. For all values of $N$, we have

$$C_{\mu \nu AB} = \frac{1}{2N} \langle C_{\mu \nu} \rangle \Omega_{AB},$$

$$\rho_{(0)AB}^{(0)} = \Omega_{AB} d_{CD} - 8\delta_{[A}^B d_{D]}^C,$$

$$\rho_{(1)\mu}^{AB} = -\Omega_{\mu CD} \left(B_{\mu CD} + \frac{1}{2} \phi_{\mu CD}\right),$$

$$\rho_{(2)\mu \nu}^{AB} = \frac{1}{N} (C_{\mu \nu}) \delta_{(A}^C \delta_{B)}^D - \frac{2}{3} \epsilon_{\mu \nu \rho \sigma} E^{\rho \sigma \tau (A C D)},$$

and furthermore,

$$3 \langle C_{\mu \nu} \rangle + \epsilon_{\mu \nu \rho \sigma} (E^{\rho \sigma \tau}) = 0, \quad \phi = 0, \quad \text{for } N > 1,$$

$$3C_{\mu \nu AB} + \epsilon_{\mu \nu \rho \sigma} E^{\rho \sigma \tau AB} = 0, \quad \phi = 0, \quad \text{for } N > 2.$$

We have not yet fully normalised the cocycle so that the $\kappa$ coefficient of $\rho$, the trace
$\rho_{(1)\mu}^{AB} A_{CD}$, vanishes. Inspecting the formula for $\rho_{(1)}$ in (5.60), we see that this trace
(and thus all of $\rho_{(1)}$) vanishes if and only if

$$B = -\frac{1}{2} \phi.$$

The parametrisation for cocycles without $R$-symmetry is obtained by setting $\rho = 0$:

$$d = 0, \quad \text{for all } N,$$

$$E_{\mu \nu \rho} = \frac{1}{3} \epsilon_{\mu \nu \rho \tau} C^{\sigma \tau}, \quad B = -\frac{1}{2} \phi, \quad \text{for } N = 1,$$

$$C = 0, \quad \langle E \rangle = 0, \quad B = \phi = 0, \quad \text{for } N = 2,$$

$$C = 0, \quad E = 0, \quad B = \phi = 0, \quad \text{for } N > 2.$$

As promised, we now impose the reality conditions (5.16) on the components of $\beta$ to obtain a parametrisation for the spaces of (real) normalised cocycles $\tilde{H}^{2,2}$
and $\tilde{N}^{2,2}$. The result is summarised in the proposition below. Note that we change
notation slightly; $C_{\mu \nu}$ below is what was previously denoted $\frac{1}{2N} \langle C_{\mu \nu} \rangle$, and we change
from a 3-form $E$ to its dual 2-form $e = *E$.

**Proposition 5.1.** The space of normalised cocycles $\tilde{H}^{2,2}$ for the $N$-extended Poincaré
superalgebra is parametrised as follows:

$$\beta_{\mu AB} = C_{\mu \nu} \Omega_{AB} \Gamma^\nu + \frac{1}{3} \epsilon_{\mu \nu \rho \tau} e^{\rho \tau AB} \Gamma^{\rho \sigma \tau} + d A_B \Gamma_{\mu} - \frac{1}{2} \phi_{\mu AB} + \frac{1}{2} \phi_{\mu AB},$$

$$\gamma(s, s)_{\mu \nu} = 2 \epsilon_{\mu \nu \rho \sigma} e^{\rho \sigma AB} \omega^{(1) T AB},$$

$$\rho(s, s)_{AB} = \frac{1}{2} d_{CD} - 8 \delta^A (C_{[A}^B d_{D]}^C) + C_{\mu \nu} \omega_{CD}^{(2)} - 2 \epsilon_{\mu \nu \rho \sigma} e^{\rho \sigma A (C \omega_{(2)}^{A C D})},$$

where $C \in \wedge^2 V$, $e \in \wedge^2 \otimes \wedge^2 \Delta N \otimes H$, $d \in \wedge^0 V \otimes \mathfrak{sp}(N)$, and $\phi \in \wedge^1 V \otimes \mathfrak{sp}(N)$, subject to
the additional constraints
\[ \phi = 0 \text{ for } N > 1, \]
\[ \langle e_{\mu\nu} \rangle = -\frac{1}{2} C_{\mu\nu} \text{ for } N = 2, \]
\[ e_{\mu\nu} = -\frac{1}{2} C_{\mu\nu} \Omega_{AB} \text{ for } N > 2. \]

The same parametrisation holds for \( H^{2,2} \) except that we must have \( d = 0 \) and
\[ e = \frac{1}{2} C \text{ for } N = 1, \]
\[ C = \langle e \rangle = 0 \text{ for } N = 2, \]
\[ \beta = 0 \text{ for } N > 2. \]

These results will be discussed in more detail in Chapter 8, where we will also provide an alternative summary by describing \( H^{2,2} \) and \( H^{2,2} \) as \( \text{Spin}^R(V) \)-modules for each integer \( N \).

### 5.3 Spencer cohomology of Poincaré superalgebras in \( D = 6 \)

The minimal 6-dimensional case with and without \( R \)-symmetry was published in [49]. In this section, those results are generalised to the case of \( N \)-extended supersymmetry. The signature of the metric is mostly-plus.

#### 5.3.1 Spinorial conventions

**Basic structure**

In 6 dimensions, \( \text{Cl}(V) \cong \text{Cl}(1,5) \cong \mathbb{H}(4) \), so there is a unique irreducible pinor module \( \Sigma \) of \( \mathbb{H} \)-dimension 4, which we of course think of as an 8-dimensional complex space with invariant quaternionic (i.e. symplectic Majorana) structure \( J \). The structure of the spinor modules is closely analogous to the 5-dimensional case discussed in §5.2.1, so we will focus on the differences here instead of reiterating everything. The main difference is the chirality of the pinor module.

The volume element of \( \text{Cl}(V) \) squares to 1, and so its image \( \Gamma_7 \in \text{End}(\Sigma) \) has eigenvalues \( \pm 1 \). The module splits into irreducible chiral (Weyl) \( \text{Spin}(V) \)-modules \( \Sigma = \Sigma_+ \oplus \Sigma_- \) where \( \Sigma_{\pm} \) are the \( \pm 1 \) eigenspaces of \( \Gamma_7 \), which are preserved by elements of even rank in \( \text{Cl}(V) \cong \Lambda^* V \) and interchanged by elements of odd rank. We define the chiral projection operator

\[ P_{\pm} := \frac{1}{2} (\text{Id} \pm \Gamma_7). \quad (5.65) \]

**Symplectic Majorana(-Weyl) spinors**

The invariant quaternionic structure \( J \) restricts to such a structure \( J_{\pm} \) on either submodule \( \Sigma_{\pm} \), just as the irreducible spinor module in 5 dimensions does. We can perform the same procedure of tensoring with symplectic modules \( \Delta_N \), except here we choose a pair \( (N_+, N_-) \in \mathbb{N}^2 \) and denote by \( J_{\Delta_{\pm}} \) the invariant quaternionic structure on \( \Delta_{N_{\pm}} \) and form three modules with invariant real structure as follows.

First, we have the complex \( \text{Spin}(V) \times \text{Sp}(N_{\pm}) \)-module \( \Sigma_+ \otimes_{\mathbb{C}} \Delta_{N_+} \) with real structure \( J_{\Delta_{\pm}} = J_{\pm} \otimes J_{\Delta_{\pm}} \) whose real subspace \( S_{N_{\pm}} \) is the space of symplectic Majorana-Weyl...
spinors; the sum of these spaces \( \bigoplus \Sigma_\pm \otimes_C \Delta_\pm \) is a Spin(\(V\)) \(\times (\text{Sp}(N_+) \times \text{Sp}(N_-))\)-module with invariant real structure \(J = J^+ + J^-\) and real subspace \(S_{N_+} \oplus S_{N_-}\) consisting of \((N_+, N_-)\)-extended symplectic Majorana-Weyl spinors.

Given the canonical basis \(e_A\) for \(\Delta_\pm\), we can write any element \(s \in \Sigma_\pm \otimes_C \Delta_\pm\) as \(s = s^A \otimes e_A\) for \(N_\pm\) elements \(s^A \in \Sigma_\pm\). In component expressions, we work with the spinors \(s^A\), but note that if \(s \in S_{N_\pm}\), these are subject to reality conditions. Since in general we will need to work simultaneously with \(\Delta_\pm\) and \(\Delta_\mp\), to distinguish them we will denote their symplectic products by \(\Omega_+\) and \(\Omega_-\) respectively. When working with expressions with indices in both modules, we will use \(N_\pm\) for one, say, the one labelled with \(\pm\), and \(I, J, K, \ldots\) for the other, labelled with \(\mp\).

There is a Spin(\(V\))-invariant \(\mathbb{C}\)-Hermitian product \(\langle \cdot, \cdot \rangle\) on \(\Sigma\) with respect to which the submodules \(\Sigma_\pm\) are maximally isotropic and which satisfies

\[
\langle \sigma_1, v \cdot \sigma_2 \rangle = -\langle \sigma_1, v \cdot \sigma_2 \rangle
\]

where \(v \in V\) and \(\sigma_1, \sigma_2 \in \Sigma\). The \(\mathbb{C}\)-symplectic form corresponding to the quaternionic structure \(J\) is given by

\[
C(\sigma_1, \sigma_2) = \langle J(\sigma_1), \sigma_2 \rangle.
\]

This does not restrict to symplectic structures on \(\Sigma_\pm\) since the two spaces are totally isotropic, but nonetheless it can be used to express the symplectic Majorana condition on chiral spinors by considering the (Dirac or Majorana) conjugates of an element \(\sigma_\pm \in \Sigma_\pm\) as maps \(\bar{\sigma}_\pm, \bar{\sigma}_\mp : \Sigma_\mp \to \mathbb{C}\). Doing so gives us similar Majorana conditions to those in (5.20), and we also have similar conditions to (5.16) for endomorphisms of \(\Delta_\pm\) which preserve the Majorana conditions. The \(R\)-symmetry group is of course \(R = \text{Sp}(N_+) \times \text{Sp}(N_-)\).

### Dirac bilinears

We will require Dirac bilinears formed from two of the same chiral spinor or from two spinors of opposite chirality. Note that for \(s_\pm \in \Sigma \otimes_C \Delta_\pm\), we have

\[
\begin{align*}
\bar{s}^A s^B_\pm &= 0, & \bar{s}^A_\pm s^B_\mp &= \mp i s^B_\mp s^A_\pm, \\
\bar{s}^A_\pm \Gamma_{\mu} s^B_\mp &= -\bar{s}^B_\mp \Gamma_{\mu} s^A_\pm, & \bar{s}^A_\pm I_{\mu} s^B_\mp &= 0, \\
\bar{s}^A_\pm \Gamma_{\mu \nu} s^B_\mp &= 0, & \bar{s}^A_\pm \Gamma_{\mu \nu \rho} s^B_\mp &= \mp \bar{s}^B_\mp \Gamma_{\mu \nu \rho} s^A_\pm.
\end{align*}
\]

and so the bilinears we work with are

\[
\omega^{(1)AB}_\pm = -\omega^{(1)BA}_\pm, \quad \omega^{(3)AB}_\pm = \omega^{(3)BA}_\pm, \quad \omega^{(0)IA}_\pm = \omega^{(0)AI}_\pm, \quad \omega^{(2)IA}_\mp = -\omega^{(2)AI}_\mp.
\]

We define the Dirac current \(\kappa_\pm\) of a chiral spinor \(s \in \Sigma \otimes_C \Delta_\pm\) as the musical dual of the \(\Delta_\pm\)-trace of the Dirac 1-form: \(\kappa_\pm := \langle \omega^{(1)}_\pm, s \rangle\), or

\[
\kappa^\mu_\pm = \Omega_{\pm AB} \bar{s}^A_\pm \Gamma^\mu s^B_\pm,
\]

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and we note that this real when $S$ is symplectic Majorana-Weyl, $s \in S_{N\pm}$.

We will require the Fierz identities

$$s^A s^B = -\frac{1}{4} \left[ \omega^{(1)AB}_{\pm} + \frac{1}{2} \omega^{(3)AB}_{\pm} \right] P_\pm, \quad (5.71)$$

$$s^A s^B = \frac{1}{4} \left[ \omega^{(0)AB}_{\pm} + \omega^{(2)AB}_{\pm} \right] P_\pm, \quad (5.72)$$

where $P_\pm : \Sigma \to \Sigma_\pm$ is the chiral projection operator.

### 5.3.2 Solving the cocycle conditions

In 6 dimensions, the $(N_+, N_-)$-extended Poincaré superalgebra with $R$-symmetry takes the form

$$b_s = V \oplus S_{(N_+, N_-)} \oplus \mathfrak{so}(V) \oplus \mathfrak{r},$$

with

$$S_{(N_+, N_-)} = S_{N_+} \oplus S_{N_-}$$

as described above and $\mathfrak{r} = \mathfrak{sp}(N_+) \oplus \mathfrak{sp}(N_-)$. The bracket on the odd part is given, for $s_\pm \in S_{N\pm}$, by

$$[s_\pm, s_\pm] = \kappa_\pm = \Omega_{\pm AB} s^A \Gamma_{\mu} s^B, \quad (5.73)$$

We will write the cocycle conditions (4.134) and (4.135) in components in a similar manner as in 5 dimensions. Here, however, the equations must be decomposed chirally. Recall that even-rank $\Gamma$-endomorphisms (and hence the Clifford action of even-rank forms) preserve chirality, thus so does $\gamma$ (which acts on spinors as a 2-form). The $R$-symmetry also preserves chirality, hence so does $\rho$. Since each $\beta_\mu$ is a general endomorphism, it can be split into the four components $\beta_{\mu} \in \text{End}(S_{N\pm})$, $\beta_{-\mu} \in \text{End}(S_{N\pm})$, $\beta_{-\mu} \in \text{Hom}(S_{N\pm}, S_{N\mp})$, and $\beta_{-\mu} \in \text{Hom}(S_{N\pm}, S_{N\mp})$. Similarly, the $\mathfrak{r}$-valued map $\rho : \bigotimes^2 S_{(N_+, N_-)} \to \mathfrak{r} = \mathfrak{sp}(N_+) \oplus \mathfrak{sp}(N_-)$ splits into the two components $\rho_\pm : \bigotimes^2 S_{(N_+, N_-)} \to \mathfrak{sp}(N_\pm)$.

Again, the first cocycle condition provides a constraint on $\beta$ and expresses $\gamma$ in terms of it, and then the second cocycle further constrains $\beta$ and, as we will see, determines $\rho$ in terms of it.

#### First cocycle condition

For the first cocycle condition (4.134), we first consider the equation for a spinor with fixed chirality, $s_\pm \in S_{N\pm}$:

$$2 \left[ s_\pm, \beta(v, s_\pm) \right] + \left[ \gamma(s_\pm, s_\pm), v \right] = 0. \quad (5.74)$$

This is given by the component expressions

$$\bar{s}^{\mu} A \Gamma_{(\mu |\beta_{\pm\pm v}| AB) s^B} = 0, \quad (5.75)$$

$$\gamma(s_\pm, s_\pm)_{\mu v} = -2 \bar{s}^{\mu} A \Gamma_{(\mu |\beta_{\pm\pm v}| AB) s^B}. \quad (5.76)$$

A spinor $s \in S$ of mixed chirality can be expressed as a sum $s = s_+ + s_-$ of chiral components $s_\pm \in S_{N\pm}$. We can substitute this expansion into the cocycle condition.
(4.134) and expand to obtain

\[
2[s_+, \beta(v, s_+)] + [\gamma(s_+, s_+), v] \\
+ 2[s_-, \beta(v, s_-)] + [\gamma(s_-, s_-), v] \\
+ 2[s_+, \beta(v, s_-)] + 2[s_-, \beta(v, s_+)] + 2[\gamma(s_+, s_-), v] = 0.
\]

(5.77)

The first two lines of this equation vanish separately by the fixed-chirality equation (5.74). In components, the remaining equation is

\[
\tilde{s}_I^\pi \Gamma_{[\mu} \beta_{-\nu]I} \mathbf{A}^\pi + \tilde{s}_I^\pi \Gamma_{[\mu} \beta_{+\nu]I} \mathbf{A}^\pi = 0,
\]

(5.78)

\[
[\gamma(s_+, s_-)]_{\mu \nu} = -[\tilde{s}_I^\pi \Gamma_{[\mu} \beta_{-\nu]I} \mathbf{A}^\pi + \tilde{s}_I^\pi \Gamma_{[\mu} \beta_{+\nu]I} \mathbf{A}^\pi].
\]

(5.79)

Equations (5.75), (5.76), (5.78) and (5.79) together are equivalent to cocycle condition (4.134), and we will now solve them in turn. We first expand the components of \( \beta \) as follows:

\[
\beta_{\pm\mu}^{\pm \mu A} = A_{\pm}^{\pm \mu A} + B_{\pm}^{\pm \mu A} + \frac{1}{2} F_{\pm}^{\pm \mu \rho A} \Gamma^{\rho \nu} + \frac{1}{2} F_{\pm}^{\pm \mu \nu A} \Gamma^{\rho \nu} + \frac{1}{2} F_{\pm}^{\pm \mu \nu A} \Gamma^{\rho \nu},
\]

(5.80)

\[
\beta_{\pm\mu}^{\pm \mu I} = C_{\pm}^{\pm \mu I} + \frac{1}{6} G_{\pm}^{\pm \mu \rho \sigma I} \Gamma^{\rho \sigma},
\]

(5.81)

where

\[
A_{\pm} \in V \otimes \Lambda^2 \Delta_{N_+}, \quad B_{\pm} \in V \otimes \Lambda^2 \Delta_{N_+}, \quad E_{\pm} \in V \otimes \Lambda^2 \Delta_{N_+}, \quad F_{\pm} \in V \otimes \Lambda^2 \Delta_{N_+}.
\]

(5.82)

Note that we assume that the 3-form parts of \( G_{\pm} \) are \( \mp \) self-dual; any \( \pm \) self-dual part does not contribute since it acts trivially on \( S_{N_+} \). We also note that as in the 5-dimensional case, we are implicitly working with complexified endomorphisms of the spinors and will have to impose reality conditions at the end of the calculations.

We first compute the following quantity:

\[
\tilde{s}_I^\pi \Gamma_{[\mu} \beta_{\pm \nu]A} \mathbf{A}^\pi + \tilde{s}_I^\pi \Gamma_{[\mu} \beta_{\pm \nu]A} \mathbf{A}^\pi = \omega_{\pm}^{(1) \alpha A} \left( \eta_{\alpha \mu} A_{\pm \nu}^{\alpha A} + E_{\pm \nu}^{\alpha \mu A} + F_{\pm \nu}^{(3) \mu} \beta_{\pm \nu}^{A AB} \right) + \frac{\omega_{\pm}^{(3) \mu}}{2} \Gamma_{[\mu} \beta_{\pm \nu]A} \mathbf{A}^\pi.
\]

(5.83)

By equation (5.110), the \( \mu \nu \)-symmetric part of this expression must vanish. In particular, the coefficients of each Dirac bilinear must vanish, and so we have the system of equations

\[
\eta_{\alpha \mu} A_{\pm \nu}^{\alpha A} + E_{\pm \nu}^{\alpha \mu A} = 0,
\]

(5.84)

\[
\Pi^T_{\mu \rho \sigma} \beta_{\gamma \nu}^{\gamma \nu \gamma} \beta_{\gamma \nu}^{A AB} = 0.
\]

(5.85)

where \( \Pi_{\pm} \) is a self-duality projector\(^1\) – since the Dirac 3-form of a chiral spinor \( s_{\pm} \) obeys \( \ast \omega_{\pm}^{(3)} = \pm \omega_{\pm}^{(3)} \), only the \( \mp \)-self-dual part of its coefficient contributes to \( \tilde{s}_I^\pi \Gamma_{[\mu} \beta_{\pm \nu]A} \mathbf{A}^\pi \).

\(^1\)The self-duality projectors \( \Pi_{\pm} : \Lambda^3 V \to \Lambda^3 V \) are defined as \( \Pi_{\pm} \Xi := \Xi \mp \frac{1}{2} (\Xi \mp *\Xi) \), or in component form,

\[
\Pi_{\pm}^{\lambda \mu \nu} = \frac{1}{2} \left( \delta_{\mu \nu}^{\lambda \mu \nu} - \frac{1}{6} \epsilon_{\mu \nu}^{\lambda \mu \nu} \right).
\]

(5.86)
Equation (5.84) is similar to an analogous equation in 5 dimensions and has the solution $A_\pm = 0$ and $E_\pm \in \Lambda^3 V \otimes \Lambda^2 \Delta N_\pm$. Equation (5.85) expands to give (we suppress $R$-symmetry indices)

$$F_{\pm [\rho \sigma] \mu} + F_{\pm \mu [\rho \sigma] \eta} + \frac{1}{6} \epsilon_{\rho \sigma \tau \mu} \beta^\gamma F_{\pm \nu \beta \gamma} + \frac{1}{6} \epsilon_{\rho \sigma \tau \nu} \beta^\gamma F_{\pm \mu \beta \gamma} = 0. \quad (5.87)$$

Contracting $\mu$ and $\nu$ gives

$$F_{\pm [\rho \sigma \tau]} = \pm \frac{1}{10} \epsilon_{\rho \sigma \tau} \beta^\gamma F_{\pm \alpha \beta \gamma}, \quad (5.88)$$

while contracting $\nu$ and $\tau$ gives

$$5F_{\pm \mu \rho \sigma} + 2F_{\pm \nu [\rho \sigma] \mu} \pm \frac{1}{2} \epsilon_{\rho \mu \sigma} F_{\pm \alpha \beta \gamma} = 0. \quad (5.89)$$

Fully skew-symmetrising in the free indices $\mu\rho\sigma$, this becomes

$$F_{\pm [\mu \rho \sigma]} = \pm \frac{1}{10} \epsilon_{\mu \rho \sigma} \beta^\gamma F_{\pm \alpha \beta \gamma}, \quad (5.90)$$

so comparing with equation (5.88), we have $F_{\pm [\mu \rho \sigma]}$. Setting $\phi = F_{\pm \nu \mu} \nu$, we are left with

$$F_{\pm \mu \nu \rho \sigma} = \eta_{\mu \nu} \phi_{\pm \rho \sigma}. \quad (5.91)$$

Turning now to equation (5.78), we first compute

$$\tilde{\omega}_{-}^{IA} \Gamma_{- \mu} \delta_{- \nu} \omega^{A}_{-} \eta_{IA} \delta_{- \nu I} = \omega^{(2)}_{- \mu} \nu_{\nu} C_{+ \nu} A_{IA} + \omega^{(0)}_{- \mu} \nu_{\nu} C_{+ \nu} IA$$

$$- \frac{1}{12} \epsilon_{\mu \rho \sigma} \beta^\gamma \omega^{(2)}_{- \sigma} \nu_{\nu} C_{+ \nu} \beta_{IA} + \frac{1}{2} \omega^{(2)}_{- \mu} \nu_{\nu} C_{+ \nu} \beta_{IA}$$

$$= \omega^{(0)}_{- \mu} \nu_{\nu} C_{+ \nu} IA + \omega^{(2)}_{- \mu} \nu_{\nu} C_{+ \nu} \beta_{IA}$$

and so, the symmetry properties (5.69), equation (5.78) becomes

$$\omega_{- \mu}^{IA} \Gamma_{- \mu} \delta_{- \nu} \omega^{A}_{-} \eta_{IA} \delta_{- \nu I} = \omega^{(2)}_{- \mu} \nu_{\nu} C_{+ \nu} A_{IA} - \omega^{(0)}_{- \mu} \nu_{\nu} C_{+ \nu} IA - \omega^{(2)}_{- \mu} \nu_{\nu} C_{+ \nu} \beta_{IA} = 0. \quad (5.93)$$

Stripping the coefficients of the Dirac bilinears, we find that this is equivalent to the following system of equations:

$$C_{+ (\nu \mu) IA} + C_{- (\nu \mu) AI} = 0, \quad (5.94)$$

$$\delta_{(\nu \mu)} A_{IA} - \delta_{(\nu \mu)} C_{A} = G_{+ (\nu \mu)} A_{IA} + G_{- (\nu \mu)} A_{IA} = 0. \quad (5.95)$$

Taking equation (5.95), we can contract $\mu$ and $\nu$ to obtain

$$C_{+ [\alpha \beta]} A_{IA} - C_{- [\alpha \beta]} A_{IA} + G_{+ [\alpha \beta]} A_{IA} - G_{- [\alpha \beta]} A_{IA} = 0, \quad (5.96)$$

while contracting $\nu$ and $\beta$ (and lowering $\alpha$) yields

$$\eta_{\mu \alpha} C_{+} v_{IA} - 5C_{+ \mu \alpha} A_{IA} - \eta_{\mu \alpha} C_{-} v_{IA} + 5C_{- \mu \alpha} A_{IA} + 2G_{+} v_{\mu \alpha} A_{IA} - 2G_{-} v_{\mu \alpha} A_{IA} = 0. \quad (5.97)$$
Symmetrising in \( \mu \alpha \), we find
\[
C_{+ (\mu \alpha) I A} - C_{- (\mu \alpha) I A} = \frac{1}{3} (C_+^\nu v_{I A} - C_-^\nu v_{I A}) \eta_{\mu \alpha},
\]
(5.98)
but then by equation (5.94), we have simply
\[
C^\pm_{(\mu \nu) I A} = \frac{1}{3} C^\pm_\nu v_{I A} \eta_{\mu \nu} \quad \text{and} \quad C^+_{(\mu \nu) I A} = - C^-_{(\mu \nu) I A}.
\]
(5.99)
The \( \mu \alpha \)-skew part is
\[
-5 C^+_{[\mu \alpha] I A} + 5 C^-_{[\mu \alpha] I A} + 2 G^+_{\nu \mu \alpha I A} - 2 G^-_{\nu \mu \alpha I A},
\]
(5.100)
so comparing with equation (5.96), we find that \( C^+_{[\mu \alpha] I A} = C^-_{[\mu \alpha] I A} \) and \( G^+_{\nu \mu \alpha I A} = G^-_{\nu \mu \alpha I A} \). We define \( c^\pm := C^\pm_\mu \) and \( \psi^\pm_{\mu \nu} := C^\pm_{[\mu \nu]} \), so that
\[
C^\pm_{\mu \nu I A} = c^\pm_{\mu I A} \eta_{\mu \nu} + \psi^\pm_{\mu \nu I A}, \quad \text{with} \quad c^\pm_{\mu I A} = - c^\mp_{\mu I A} \quad \text{and} \quad \psi^\pm_{\mu \nu I A} = \psi^\mp_{\mu \nu I A}.
\]
(5.101)
The \( C \)-terms now cancel in equation (5.95), leaving \( G^+_{(\mu \nu) \rho \sigma I A} = G^-_{(\mu \nu) \rho \sigma I A} \). In summary, we now have
\[
\begin{align*}
\beta^\pm_{\mp \mu AB} &= B^\pm_{\mp \mu AB} + \frac{1}{2} E^\pm_{\nu \mu \rho AB} \Gamma^{\nu \rho} + \frac{1}{2} \phi^\pm_{AB} \Gamma^{\nu \rho} \mu \nu, \\
\beta^\pm_{\mp \mu I A} &= c^\pm_{\mu I A} \Gamma_{\mu} + \psi^\pm_{\mu \nu I A} \Gamma^{\nu \rho} + \frac{1}{6} G^\pm_{\mu \nu \rho \sigma I A} \Gamma^{\nu \rho} \sigma,
\end{align*}
\]
(5.102, 5.103)
where \( E^\pm_\varepsilon \in \wedge^3 V \otimes \wedge^2 \Delta \mathcal{N}_\varepsilon \), \( B^\pm_\varepsilon \), \( \phi^\pm_\varepsilon \in V \otimes \Delta \mathcal{N}_\varepsilon \), \( c^\pm_\varepsilon \in \wedge^3 V \otimes \Delta \mathcal{N}_\varepsilon \), \( \psi^\pm_\varepsilon \in \wedge^2 V \otimes \Delta \mathcal{N}_\varepsilon \), and \( G^\pm_\varepsilon \in V \otimes \wedge^3 V \otimes \Delta \mathcal{N}_\varepsilon \), with
\[
c^\pm_{\mu I A} = - c^\mp_{\mu I A}, \quad \psi^\pm_{\mu \nu I A} = \psi^\mp_{\mu \nu I A}, \quad \text{and} \quad G^+_{(\mu \nu) \rho \sigma I A} = G^-_{(\mu \nu) \rho \sigma I A}.
\]
(5.104)
Finally, from equations (5.76) and (5.79), we find
\[
\begin{align*}
\gamma(s^\pm, s^\pm)_{\mu \nu} &= 2 E^\pm_{\nu \rho AB} \omega^{(1) AB}_{\pm} - \phi^\pm_{AB} \omega^{(2) \mu \rho}_{\pm} \rho \sigma, \\
\gamma(s^+, s^-)_{\mu \nu} &= 2 \psi^\pm_{\nu \rho I A} \omega^{(0) I A}_{\pm} - 2 c^\pm_{\nu I A} \omega^{(2) \mu \rho}_{\pm} \rho \sigma + \left( G^+_{[\mu \nu] \rho \sigma I A} - G^-_{[\mu \nu] \rho \sigma I A} \right) \omega^{(2) \rho \sigma I A}_{\pm}
\end{align*}
\]
(5.105, 5.106)
where the second formula holds with either choice of sign.

**Second cocycle condition**

We begin with the polarised version of the second cocycle condition:
\[
\beta([s_1, s_2], s_3) + \left[ \gamma(s_1, s_2), s_3 \right] + \left[ \rho(s_1, s_2), s_3 \right] + \text{cyclic perm's of (1,2,3)} = 0.
\]
(5.107)
By the trilinearity of this equation, it is sufficient to consider the case where all three spinor variables have definite chirality; since the equation is symmetric in the three spinors, it is in fact sufficient to consider only two cases: the case where all three variables are the same chiral spinor \( s_\pm \in S_\pm \), and the case where two are the same chiral spinor \( s_\pm \in S_\pm \) and the third has the opposite chirality \( s_\mp \in S_\mp \), giving us the two
The components contribute. Again, our parametrisation is complexified and we will need to impose cubic second cocycle condition in 5 dimensions. The latter pair are quadratic in \( s_\pm \), and linear in \( \pm \rho \). Let us parametrise the various components of the \( \rho \) map as follows:

\[
\begin{align*}
\rho_\pm(s_\pm, s_\pm)_{CD} &= \omega_\pm^{(1)\mu AB} \rho_\pm^{(1)} {}_{\mu ABCD} + \frac{1}{6} \omega_\pm^{(3)\mu \nu \rho AB} \rho_\pm^{(3)} {}_{\mu \nu \rho A B C D}, \\
\rho_\mp(s_\pm, s_\mp)_{IJ} &= \omega_\pm^{(1)\mu AB} \rho_\pm^{(1)} {}_{\mu A B I J} + \frac{1}{6} \omega_\pm^{(3)\mu \nu \rho AB} \rho_\pm^{(3)} {}_{\mu \nu \rho A B I J}, \\
\rho_\pm(s_\pm, s_\pm)_{CD} &= \omega_\pm^{(0) A I} \rho_\pm^{(0)} {}_{A I C D} + \frac{1}{6} \omega_\pm^{(2) \mu \nu A I} \rho_\pm^{(2)} {}_{\mu \nu A I C D},
\end{align*}
\]

where

\[
\begin{align*}
\rho_\pm^{(1)} &\in \Lambda^1 V \otimes \Lambda^2 \Delta N_\pm \otimes \text{sp}(N_\pm), & \rho_\pm^{(3)} &\in \Lambda^3 V \otimes \mathcal{O}^2 \Delta N_\pm \otimes \text{sp}(N_\pm), \\
\rho_\pm^{(1)'} &\in \Lambda^1 V \otimes \Lambda^2 \Delta N_\pm \otimes \text{sp}(N_\mp), & \rho_\pm^{(3)'} &\in \Lambda^3 V \otimes \mathcal{O}^2 \Delta N_\pm \otimes \text{sp}(N_\mp), \\
\rho_\pm^{(0)} &\in \Lambda^0 V \otimes (\Delta N_\pm \otimes \Delta N_\mp) \otimes \text{sp}(N_\mp), & \rho_\pm^{(2)} &\in \Lambda^2 V \otimes (\Delta N_\pm \otimes \Delta N_\mp) \otimes \text{sp}(N_\mp).
\end{align*}
\]

The components \( \rho_\pm^{(3)} \) and \( \rho_\pm^{(3)'} \) are both taken to be \( \mp \) self-dual, since they appear in (5.114) contracted with \( \omega_\pm^{(3)} \) which is \( \pm \) self-dual, so only their \( \mp \) self-dual parts contribute. Again, our parametrisation is complexified and we will need to impose reality conditions at the end.

We tackle the equations out of order, seeking to solve the easier equations first and thus simplify the more complicated equations.

### Solution to equation (5.113)

This equation is of the form \( W(s_\pm, s_\mp)_{s_\mp} = 0 \), where \( W : \mathcal{O}^2 S_\pm \rightarrow \End(S_\mp) \). Since the equation must hold for all \( s_\mp \in S_\mp \), that argument may be abstracted to leave \( W(s_\pm, s_\pm) = 0 \)

\[
\kappa^\mu_{\pm} \beta_{\mp \mp}^{\mu \nu} {_{\mu I J} + \frac{1}{4} \gamma(s_\pm, s_\pm)_{\mu \nu} \Gamma^{\mu \nu \Omega_{\mp I J} + \rho_\mp(s_\pm, s_\pm)_{I J} = 0},
\]

(5.116)
where we have also lowered the index $I$. Note that this equation is non-trivial only if both $N_\pm > 0$, so we now assume this to be the case. Taking the $IJ$-symmetric part of this equation, we have

$$
0 = \rho_\pm(s_\pm, s_\pm)_I + \kappa_\pm^\mu \beta_\pm \mu I J
= \omega^{(1)}_{\pm AB} \left( B_\pm \mu I J + \Omega_{\pm AB} \left( B_\pm \mu I J + \frac{1}{2} \phi_\pm \nu I J \Gamma_{\mu \nu} \right) \right) + \frac{1}{6} \omega^{(3)}_{\pm \mu \nu \rho AB} \rho^{(3)}_\pm \mu \nu \rho A B I J.
$$

(5.117)

This must hold for all $s_\pm \in S_\pm$, so the coefficients of $\omega^{(1)}_{\pm}$ and $\omega^{(3)}_{\pm}$ must vanish; this leaves us with

$$
\rho^{(1)}_{\pm AB I J} = -\Omega_{\pm AB} B_\pm \mu I J, \quad \phi_\pm = 0, \quad \rho^{(3)}_{\pm \mu \nu \rho A B I J} = 0, \quad \text{if both } N_+, N_- > 0. \quad (5.118)
$$

The $IJ$-skew part is

$$
0 = \kappa_\pm^\mu \beta_\pm \mu I J + \frac{1}{4} \gamma (s_\pm, s_\pm)_{\mu \nu} \Gamma_{\mu \nu} \Omega_{\pm I J}
= \frac{1}{2} \left( \kappa_\pm^\mu E_\pm \mu \nu \rho A B I J + \frac{1}{2} \gamma (s_\pm, s_\pm)_{\nu \rho} \Omega_{\pm I J} \right) \Gamma_{\nu \rho}
= \frac{1}{2} \left( \Omega_{\pm AB} E_\pm \mu \nu \rho A B I J + E_\pm \mu \nu \rho A B \Omega_{\pm I J} \right) \omega^{(1)}_{\pm AB} - \frac{1}{2} \phi_\pm^\rho \Omega_{\pm AB} \Omega_{\pm I J} \Gamma_{\nu \rho}. \quad (5.119)
$$

Again, the coefficients of the Dirac bilinears must vanish, so we have

$$
\Omega_{\pm AB} E_\pm \mu \nu \rho A B I J + E_\pm \mu \nu \rho A B \Omega_{\pm I J} = 0, \quad \phi_\pm = 0, \quad \text{if both } N_+, N_- > 0. \quad (5.120)
$$

Taking the first equation above, contracting with $\Omega_{\pm}^{AB}$ and $\Omega_{\pm I J}$, we have

$$
\Omega_{\pm AB} E_\pm \mu \nu \rho A B I J + E_\pm \mu \nu \rho A B \Omega_{\pm I J} = 0 \implies 2 N_\pm E_\pm \mu \nu \rho I J + \langle E_\pm \mu \nu \rho \rangle_\pm \Omega_{\pm I J} = 0 \implies N_\pm \langle E_\pm \mu \nu \rho \rangle + N_\pm \langle E_\pm \mu \nu \rho \rangle = 0, \quad (5.121)
$$

where $\langle E_\pm \rangle := \Omega_{\pm}^{AB} E_\pm^{AB}$. Thus, we can conclude that there exist $\Xi_\pm \in \wedge^3 V$ such that

$$
E_\pm^{AB} = \frac{1}{2N_\pm} \Omega_{\pm}^{AB} \Xi_\pm \quad \text{and} \quad N_+ \Xi_- + N_- \Xi_+ = 0, \quad \text{if both } N_+, N_- > 0. \quad (5.122)
$$

Like equation (5.113), equation (5.112) is quadratic in $s_\pm$ but unlike (5.113) it is not of the form $W(s_\pm, s_\pm)_{\pm} = 0$ for some $W : \bigwedge^2 S_\pm \rightarrow \text{Hom}(S_\pm, S_\pm)$; it must be manipulated into this form. We postpone doing this, since it will actually be easier to solve (5.111) first.

**Solution to equation (5.111)**

Equation (5.111) is cubic in $s_\pm$; it must be depolarised in order to convert it into a quadratic equation. Its polarised form is

$$
0 = \Omega_{\pm BC} \left( \Omega_{\pm}^A \Gamma_{\pm} \Omega_{\pm}^C \right) \beta_\pm \pm I A s_\pm^A + \text{cyclic perm's of (1,2,3)}
= \left[ \Omega_{\pm AB} \left( \Omega_{\pm}^A \Gamma_{\pm} \Omega_{\pm}^B \right) \beta_\pm \pm I C + \Omega_{\pm BC} \beta_\pm \pm I A \left( s_\pm^A \Gamma_{\pm}^B + s_\pm^A \Gamma_{\pm}^B \right) \Gamma_{\pm} \right] s_{\pm}^C. \quad (5.123)
$$

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so, abstracting \( s_{\pm 3} \), we find
\[
\Omega_{\pm AB}(s^A_{\pm 1} \Gamma^\mu s^B_{\pm 2}) \beta_{\mp \pm \mu IC} + \Omega_{\pm BC} \beta_{\mp \pm \mu IA}(s^A_{\pm 1} s^B_{\pm 2} + s^A_{\pm 1} s^B_{\pm 2}) \Gamma^\mu = 0. \tag{5.124}
\]

Since this is symmetric in \( s_{\pm 1}, s_{\pm 2} \), we can re-polarise and then use the Fierz identity (5.71) to get
\[
0 = \kappa_{\pm}^\mu \beta_{\mp \pm \mu IC} + 2\Omega_{\pm BC} \beta_{\mp \pm \mu IA}(s^A_{\pm 1} s^B_{\pm 2}) \Gamma^\mu
\]
\[
= \kappa_{\pm}^\mu \beta_{\mp \pm \mu IC} - \frac{1}{2} \Omega_{\pm BC}(\omega^{(1)}_{\mp AB} \beta_{\mp \pm \mu IA} \Gamma^V \Gamma^\mu + \frac{1}{12} \omega^{(3)}_{\pm v} AB \beta_{\mp \pm \mu IA} \Gamma^\mu \Gamma^\nu \Gamma^\rho \Gamma^\sigma \Gamma^\mu). \tag{5.125}
\]

As in our 5-dimensional calculations, we now require that the coefficient of each Dirac bilinear must vanish. Taking the \( \omega^{(1)} \) equation first, we evaluate the products of \( \Gamma \) matrices and then finally obtain in equations in the components of \( \beta_{\mp \pm} \) by noting that the coefficients of each basis \( \Gamma \)-matrix vanish. We find that the solution to this equation is
\[
G_{\pm \lambda \mu \nu \rho} = -3\Pi_{\mp \mu \nu \lambda \sigma} \gamma_{\pm \sigma}, \quad \text{for all } N_{\pm}, \quad c_{\pm} = 0, \quad \psi_{\pm} = 0, \quad \text{for } N_{\pm} > 1, \tag{5.126}
\]
where \( \Pi_{\pm} \) is a self-duality projector. Substituting this back into (5.103) gives
\[
\beta_{\mp \pm \mu IA} = c_{\pm} \Gamma^\mu + \psi_{\mp \mu IA} \Gamma^V - \frac{1}{2} \psi_{\mp \mu IA} \Gamma^\mu = (c_{\pm} - \frac{1}{2} \gamma_{\pm \mu IA} \Gamma^\mu). \tag{5.127}
\]

We then find that the coefficient of \( \omega^{(3)} \) in (5.125) vanishes identically thanks to the \( \Gamma \)-matrix identity \( \Gamma_\mu \Gamma^{\nu \rho \sigma} \Gamma^\mu = 0 \). Note that we now have
\[
\gamma(s_{\mp}, s_{\pm})_{\mu \nu} = 2\psi_{\mp \mu IA} \omega^{(0)}_{\pm \pm} + \frac{1}{2} \epsilon_{\mu \nu \rho \sigma \beta} \psi_{\pm}^{\beta} \omega^{(2)}_{\pm} - 2c_{\pm} \omega^{(2)}_{\pm} \psi_{\pm}. \tag{5.128}
\]

**Solution to (5.112)**

Now we can turn back to (5.112). First we manipulate \( s_{\mp} \) to the right-hand side (using the expression for \( \gamma(s_{\mp}, s_{\pm})_{\mu \nu} \) above)

\[
0 = \kappa_{\pm}^\mu \beta_{\mp \pm \mu IA} s^C_{\mp \pm} + \frac{1}{2} \gamma(s_{\mp}, s_{\pm})_{\mu \nu} \Omega_{\pm AB} \gamma_{\pm AB} s^B_{\pm} + 2\rho_{\pm} (s_{\mp}, s_{\pm})_{AB} s^B_{\pm}
\]
\[
= \kappa_{\pm}^\mu \beta_{\mp \pm \mu IA} s^C_{\mp \pm} + 2\rho_{\pm} (s_{\mp}, s_{\pm})_{AB} s^B_{\pm},
\]
\[
= \kappa_{\pm}^\mu \beta_{\mp \pm \mu IA} s^C_{\mp \pm} + 2\rho_{\pm} (s_{\mp}, s_{\pm})_{AB} s^B_{\pm},
\]
\[
= \frac{1}{2} \gamma(s_{\mp}, s_{\pm})_{\mu \nu} \Omega_{\pm AB} \gamma_{\pm AB} s^B_{\pm} + 2\rho_{\pm} (s_{\mp}, s_{\pm})_{AB} s^B_{\pm}.
\]
\[
= \kappa_{\pm}^\mu \beta_{\mp \pm \mu IA} s^C_{\mp \pm} + 2\rho_{\pm} (s_{\mp}, s_{\pm})_{AB} s^B_{\pm}.
\]
\[
\text{and this is of the required form; we now abstract } s_{\pm}^C, \text{ leaving}
\]
\[
\kappa_{\pm}^\mu \beta_{\mp \pm \mu IA} + 2\rho_{\pm} (s_{\mp}, s_{\pm})_{AB} s^B_{\pm} = 0. \tag{5.130}
\]
The LHS must be evaluated using the Fierz identity (5.71); as usual we then have to extract coefficients of Dirac bilinear, evaluate some products of $\Gamma$-matrices and then extract coefficients again. Note, however, that the equation is now identically satisfied for $N_\pm > 1$ due to equation (5.126), so we only have the simplest mixed-chirality case $N_\pm = 1$ to tackle; here we find

\[
\begin{align*}
\rho_\pm^{(0)}_{AICD} &= -6c_\pm t(C_\pm D)_A, \\
\rho_\pm^{(2)}_{\mu\nu BICA} &= 2\psi_\pm\mu\nu t(C_\pm D)_A,
\end{align*}
\]

where we have used $\Omega_{\pm AB} = \epsilon_{AB}$ for $N_\pm = 1$.

**Solution to equation (5.110)**

Equation (5.110) is also cubic in $S_\pm$ so must be depolarised, rearranged so that one spinor can be removed, re-polarised and then evaluated using the Fierz identity. We thus find that the solution to this equation is

\[
\begin{align*}
\rho_\pm^{(1)}_{AB \mu CD} &= \Omega_{\pm AB} \left( \frac{1}{2} \phi_\pm_{\mu CD} - B_\pm_{\mu CD} \right) - 4 \delta^{[A}_(C \phi_\pm_{\mu} B)]_D, \\
\rho_\pm^{(3)}_{\mu\nu \rho B CD} &= -4 \delta^{(A E^\pm}_{(C \mu\nu \rho \ D)} = 4 \\
\end{align*}
\]

and

\[
\begin{align*}
\langle E^\pm_\pm \rangle &= 0 \quad \text{for } N_\pm = 2, \\
E^\pm_\pm &= 0 \quad \text{for } N_\pm > 2,
\end{align*}
\]

where $\Xi^{\pm}$ denotes the $\pm$ self-dual part of a 3-form $\Xi \in \Lambda^3 V$. Like in the 5-dimensional case, we normalise the cocycle by setting $\rho_\pm^{(1)}_{\mu ACD} = 0$ to remove a degree of freedom. Doing so yields

\[
\begin{align*}
B_\pm &= \left( \frac{1}{2} - \frac{2}{N_\pm} \right) \phi_\pm, \\
\rho_\pm^{(1)}_{\mu AB CD} &= 4 \left( \frac{1}{2N_\pm} \Omega_{\pm AB} \phi_\pm_{\mu CD} - \delta^{[A}_(C \phi_\pm_{\mu} B)]_D \right).
\end{align*}
\]

**Summary of results**

We now summarise the results above, applying reality conditions (5.16) so that we are truly working with $S_{(N^+,N^-)}$ and not $\Sigma_+ \otimes \Sigma_+ \otimes \Sigma_- \otimes \Sigma_- \cong S_{(N^+,N^-)} \otimes \mathbb{C}$.

**Proposition 5.2.** The space of normalised cocycles $\widetilde{F^2}$ for the $(N^+,N^-)$-extended Poincaré superalgebra in 6 dimensions is parametrised as follows. The components which do not mix spinors of different chirality are

\[
\begin{align*}
\beta_{\pm \mu AB} &= \frac{1}{2} E_{\pm \mu \nu \rho AB} \Gamma^{\nu \rho} + \frac{1}{2} \phi_{\pm AB} \Gamma_{\mu \nu} + \left( \frac{1}{2} - \frac{2}{N_\pm} \right) \phi_{\pm AB}, \\
\gamma(s_\pm, s_\pm)_{\mu \nu} &= 2 E_{\pm \mu \nu \rho AB} \omega^{(1)\rho AB}_{\pm} - \phi_{\pm AB} \omega^{(3)\mu \nu \rho}_{\pm AB}, \\
\rho_{\pm} (s_\pm, s_\pm)_{CD} &= \frac{2}{N_\pm} \phi_{\pm CD} \kappa_\mu + 4 \phi_{\pm A} \omega^{(1)\mu}_{\pm |A|D} - \frac{2}{3} \mu_{\pm \mu \nu \rho} (C \omega^{(3)\mu \nu \rho}_{\pm |A|D}).
\end{align*}
\]

\[
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\]
where $E_{\pm} \in \wedge^3 V \otimes \wedge^2 \Delta_{N_{\pm}}|_{H}$, $\phi_{\pm} \in V \otimes \mathfrak{sp}(N_{\pm})$ satisfy the following constraints:

$$E_{\pm AB} = \frac{1}{2N_{\pm}^2} \Omega_{\pm AB} \Xi_{\pm}, \quad N_+ \Xi_- + N_- \Xi_+ = 0, \quad \phi_{\pm} = 0, \quad \text{if both } N_+, N_- > 0,$$

$$\langle E_{\pm} \rangle = 0, \quad \text{for } N_{\pm} = 2, \quad (5.136)$$

$$E_{\pm} = 0, \quad \text{for } N_{\pm} > 2.$$

The remaining components are all zero in all cases except $N_+ = N_- = 1$, where we have

$$\beta_{\mp \pm \mu I A} = \left( c_{\pm I A} - \frac{1}{2} \psi_{\pm \nu \rho I A} \Gamma_{\nu \rho} \right) \Gamma_{\mu},$$

$$\gamma(s_{\mp}, s_{\pm})_{\mu \nu} = 2\psi_{\pm \mu \nu I A} \omega_{\pm I A}^{(0)} + \frac{1}{2} \epsilon_{\mu \rho \sigma \tau I} \psi_{\pm}^{\rho \sigma} \omega_{\mp I A}^{(2)} \tau \omega_{\pm I A}^{(1)} - 2c_{\pm I A} \omega_{\mp I A}^{(2)} \mu \nu, \quad (5.137)$$

$$\rho_{\pm}(s_{\mp}, s_{\pm})_{CD} = -6c_{\pm I}(C \omega_{\pm I A}^{(0)})_{D} + \frac{1}{3} \psi_{\pm \mu \nu I}(C \omega_{\pm I A}^{(2)} \mu \nu),$$

for some $c_{\pm} \in \wedge^0 V \otimes \otimes^2 \Delta_1$, $\psi_{\pm} \in \wedge^2 V \otimes \otimes^2 \Delta_1$ satisfying

$$c_{\pm I A} = -c_{\mp A I}, \quad \psi_{\pm \mu \nu I A} = \psi_{\mp \mu \nu A I},$$

$$(c_{\pm I A})^* = c_{\pm I A}, \quad (\psi_{\pm \mu \nu I A})^* = \psi_{\pm \mu \nu I A}, \quad (5.138)$$

and $\rho_{\pm}(s_{\mp}, s_{\pm})_{IJ} = 0$ in all cases.

The same parametrisation holds for $H^{2,2}$ subject to the additional constraint that $\rho = 0$, or equivalently

$$\phi_{\pm} = 0 \quad \text{for } N_{\pm} > 1, \quad E_{\pm} = 0, \quad c_{\pm} = 0, \quad \psi_{\pm} = 0. \quad (5.139)$$

We leave discussion of these results to Chapter 8, where we also provide descriptions of $H^{2,2}$ and $H^{2,2}$ as Spin$_R(V)$-modules in a summary table along with the previous results from $D = 5$. 
Chapter 6

Spencer cohomology of the Type IIA Poincaré superalgebra

The Spencer (2,2)-cohomology group has been calculated for the Type I (that is, \( D = 10, (N_+, N_-) = (1, 0) \)) Poincaré superalgebra by JMF and PdM in unreleased work. Here we cover Type IIA ((\( N_+, N_-) = (1, 1) \)), which has also been done by AS in unreleased work. Our results agree; by the end of this chapter, we will show that \( \mathcal{H}^{2,2} \) is trivial (Theorem 6.1). We hope to publish these calculations in the near future.

The signature of the metric is mostly-plus\(^1\) and we take a + in the relation (2.17).

6.1 Spinorial conventions

We have \( \text{Cl}(V) \cong \mathbb{R}(32) \), so there is a unique irreducible Clifford module of dimension 32 we denote by \( \Sigma \). The "volume element" endomorphism \( \Gamma_{11} = \Gamma_{0123456789} \) has \( \Gamma^2_{11} = 1 \), so has eigenvalues \( \pm 1 \). This induces a splitting \( \Sigma = \Sigma_+ \oplus \Sigma_- \), where \( \Sigma_\pm \) is the eigenspace of \( \Gamma_{11} \) corresponding to eigenvalue \( \pm 1 \). The \( \Gamma \)-matrices of even rank preserve the eigenspaces, while the odd-rank matrices map \( \Sigma_\pm \to \Sigma_{\mp} \). The eigenspaces are, in particular, inequivalent irreducible \( \text{Spin}(V) \)-modules. We say that elements of \( \Sigma \) are (Majorana) spinors\(^1\), while elements of \( \Sigma_\pm \) are (Majorana-Weyl) spinors of chirality \( \pm \).

Bilinears

The Clifford module \( \Sigma \) admits a symplectic structure \( \langle - , - \rangle \) with respect to which the Clifford action of any \( \nu \in V \) (equivalently the action of \( \Gamma_\mu \)) is skew in the sense that for all \( s_1, s_2 \in \Sigma \) and all \( \nu \in V \),

\[
\langle s_1, s_2 \rangle = -\langle s_2, s_1 \rangle, \quad \langle s_1, \nu \cdot s_2 \rangle = -\langle \nu \cdot s_1, s_2 \rangle, \quad (6.1)
\]

and with respect to which \( \Sigma_\pm \) are both (maximally) isotropic; the symmetric form vanishes when restricted to either of them. Note that \( \langle - , - \rangle \) is \( \text{Spin}(V) \)-invariant:

\[
\langle s_1, \Gamma_\mu \nu s_2 \rangle = -\langle \Gamma_\mu \nu s_1, s_2 \rangle. \quad (6.2)
\]

\(^1\)These spinors are explicitly real ("really real") by construction, so it is not necessary to work with complexified spinors and reality conditions.
We will only form Dirac bilinears from two copies of the same chiral spinor or from two different spinors of opposite chirality. Furthermore, due to the Hodge duality relation between the $\Gamma$-matrices,

$$
\Gamma_{\mu_1 \mu_2 \ldots \mu_p} = (-1)^{\frac{p(p-1)}{2}} \frac{1}{(10-p)!} \epsilon_{\mu_1 \mu_2 \ldots \mu_p \nu_1 \nu_2 \ldots \nu_{10-p}} \Gamma^{\nu_1 \nu_2 \ldots \nu_{10-p}} \Gamma_{11}, \quad (6.3)
$$

we need only consider the bilinears of rank less than or equal to 5. Some of these vanish due to the isotropy of $\Sigma^\pm$ or symmetry; the non-zero bilinears are

$$
\omega^{(1)}_{\pm} = \pm \frac{1}{14} \Gamma_{\mu} \Gamma_{\nu} \Gamma_{\rho} \Gamma_{\sigma} \Gamma_{\tau} \Gamma_{\mu} \Gamma_{\nu} \Gamma_{\rho} \Gamma_{\sigma} \Gamma_{\tau},
$$

where $s_{\pm} \in \Sigma_{\pm}$. Again by Hodge duality relations on the $\Gamma$-matrices, we have

$$
\omega^{(5)}_{\pm} = \pm \frac{1}{16} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \Gamma^{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} \Gamma_{\mu_1} \Gamma_{\mu_2} \Gamma_{\mu_3} \Gamma_{\mu_4} \Gamma_{\mu_5}, \quad \quad \quad (6.6)
$$

The mixed-chirality bilinears have the following symmetries:

$$
\omega_{\pm}^{(0)} = -\omega_{\pm}^{(0)}, \quad \quad \omega_{\pm}^{(2)} = \omega_{\pm}^{(2)}, \quad \quad \omega_{\pm}^{(4)} = -\omega_{\pm}^{(4)}. \quad (6.7)
$$

The dual of $\omega^{(1)}_{\pm}$ under the musical isomorphism induced by the inner product $\eta$ is the Dirac current $\kappa_{\pm}^{\mu} := \tilde{s}_{\pm} \Gamma_{\mu} s_{\pm}$. We will use the Fierz identities

$$
s_{\pm} \tilde{s}_{\pm} = \frac{1}{16} (\omega^{(1)}_{\pm} + \frac{1}{2} \omega^{(5)}_{\pm}) P_{\pm}, \quad (6.8)
$$

$$
s_{\pm} \tilde{s}_{\pm} = -\frac{1}{16} (\omega^{(0)}_{\pm} + \omega^{(2)}_{\pm} + \omega^{(4)}_{\pm}) P_{\pm}, \quad (6.9)
$$

as endomorphisms of $\Sigma$, where $P_{\pm} = \frac{1}{2} (\mathbb{1} \pm \Gamma_{11}) : \Sigma \rightarrow \Sigma_{\pm}$ are the projections induced by the splitting $\Sigma = \Sigma_+ \oplus \Sigma_-$. 

### The Type IIA algebra

For applications in supergravity, the Poincaré superalgebra takes the form $\mathfrak{s} = V \oplus S \oplus \mathfrak{so}(V)$, where the odd subspace $S$ takes one of three forms; using the standard nomenclature from the supergravity literature, these are respectively

- **Type I**: $S = S_{(1,0)} = \Sigma_+$,
- **Type IIA**: $S = S_{(1,1)} = \Sigma$,
- **Type IIB**: $S = S_{(2,0)} = 2\Sigma_+$.

Here, we will treat Type IIA. In this case, the $R$-symmetry Lie algebra is trivial, so $\tilde{s} = s$. The odd-odd bracket is given (in polarised from) by the Dirac current:

$$
[s_{\pm}, s_{\mp}] = \kappa_{\pm} = \Omega_{\pm AB} \tilde{s}_{\pm} \Gamma_{\mu} s_{\pm}^{B}, \quad (6.10)
$$

$$
[s_{\pm}, s_{\mp}] = 0
$$

for $s_{\pm} \in S_{N_\pm}$. 

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6.2 Solving the cocycle conditions

We parametrise the cocycle $\beta \in \text{Hom}(V, \text{End} S)$ in two different ways which we spell out below; both will be useful, and we detail the correspondence between them. The first is straightforward:

$$\beta_\mu = \frac{1}{p!} \left( \beta^{(p)}_{\mu v_1 \ldots v_p} v_1^{\gamma_1} \cdots v_p^{\gamma_p} + \beta^{(10-p)}_{\mu v_1 \ldots v_p} v_1^{\gamma_1} \cdots v_p^{\gamma_p} \Gamma_{11} \right) + \frac{1}{5!} \beta^{(5)}_{\mu v_1 \ldots v_5} v_1^{\gamma_1} \cdots v_5^{\gamma_5}, \quad (6.11)$$

where $\beta^{(p)}, \beta^{(10-p)} \in V \otimes \Lambda^p V$ for $0 \leq p \leq 5$. Here, we use the fact that $S = \Sigma_+ \oplus \Sigma_-$ is the irreducible pinor module, so its real endomorphism algebra is $\text{Cl}(V) \cong \mathbb{R}(32)$; in particular, this is spanned by the $\Gamma$-matrices and the $\Gamma$-matrices of rank $p > 5$ can be written as a sum of products of rank-$(10-p)$ matrices with $\Gamma_{11}$ via $(6.3)$.

On the other hand, by considering chirality, we can write $\text{End}(S) = \text{End}(\Sigma_+) \oplus \text{End}(\Sigma_-) \oplus \text{Hom}(\Sigma_+, \Sigma_-) \oplus \text{Hom}(\Sigma_-, \Sigma_+)$, so $\beta$ can be composed with projections to these four subspaces to obtain the components $\beta_+, \beta_-, \beta_+, \beta_-; \beta_+$ equivalently,

$$\beta_{\pm \pm \mu} := P_\pm \beta_\mu P_\pm, \quad \beta_{\mp \pm \mu} := P_\mp \beta_\mu P_\mp. \quad (6.12)$$

The alternative parametrisation is then given by

$$\beta_{\pm \pm \mu} = \left( \beta^{(0)}_{\pm \mu} + \frac{1}{2!} \beta^{(2)}_{\pm \mu v_1 v_2} \Gamma_{v_1 v_2} + \frac{1}{4!} \beta^{(4)}_{\pm \mu v_1 v_2 v_3 v_4} \Gamma_{v_1 v_2 v_3 v_4} \right) P_\pm, \quad (6.13)$$

$$\beta_{\mp \pm \mu} = \left( \beta^{(1)}_{\pm \mu} \Gamma_{v_1} + \frac{1}{2!} \beta^{(3)}_{\pm \mu v_1 v_2 v_3} \Gamma_{v_1 v_2 v_3} + \frac{1}{4!} \beta^{(5)}_{\pm \mu v_1 v_2 v_3 v_4 v_5} \Gamma_{v_1 v_2 v_3 v_4 v_5} \right) P_\pm, \quad (6.14)$$

where $\beta^{(p)}_{\pm \mu} \in V \otimes \Lambda^p V$ for $0 \leq p \leq 4$, and $\beta^{(5)}_{\pm \mu} \in V \otimes \Lambda^5 V$, where we note that the Hodge operator squares to $+ \text{Id}$, so it is consistent to have real self-dual 5-forms. For $0 \leq p \leq 4$, we have

$$\beta^{(p)}_{\pm} = \beta^{(p)} \pm \beta^{(10-p)}, \quad (6.15)$$

or equivalently,

$$\beta^{(p)} = \frac{1}{2} \left( \beta^{(p)}_+ + \beta^{(p)}_- \right), \quad \beta^{(10-p)} = \frac{1}{2} \left( \beta^{(p)}_+ - \beta^{(p)}_- \right). \quad (6.16)$$

Finally, $\beta^{(5)}_{\mu}$ is the self-dual part of $\beta^{(5)}$, and $\beta^{(5)}_{\mu}$ its anti-self-dual part:

$$\beta^{(5)}_{\pm \mu} = \frac{1}{2} \left( \beta^{(5)}_{\mu} \pm * \beta^{(5)}_{\mu} \right), \quad (6.17)$$

which we also write as $\beta^{(5)}_{\pm} = \frac{1}{2} \left( \beta^{(5)} \pm * \beta^{(5)} \right)$, where $*$ is the Hodge star operator acting only on the last 5 indices of $\beta^{(5)}$, i.e. on its $\Lambda^5 V$ part. Thus we may also write

$$\beta^{(5)} = \beta^{(5)}_+ + \beta^{(5)}_-, \quad * \beta^{(5)} = \beta^{(5)}_+ - \beta^{(5)}_. \quad (6.18)$$
6.2.1 First cocycle condition

Similarly to the 6-dimensional case, we can use chirality to split the first cocycle condition (3.118) into the following two equations:

\[
2\left[s_\pm, \beta(v, s_\pm)\right] + \left[\gamma(s_\pm, s_\pm), v\right] = 0, \tag{6.19}
\]
\[
\left[s_+, \beta(v, s_-)\right] + \left[s_-, \beta(v, s_+)\right] + \left[\gamma(s_+, s_-), v\right] = 0, \tag{6.20}
\]

where \(s_\pm \in \Sigma_\pm\). As usual, we write these equations in components and extract the symmetric and skew-symmetric components:

\[
\bar{s}_\pm \Gamma_{(\mu \beta_{\pm} v)} s_\pm = 0, \tag{6.21}
\]
\[
\bar{s}_- \Gamma_{(\mu \beta_{-} - v)} s_+ + \bar{s}_+ \Gamma_{(\mu \beta_{-} + v)} s_- = 0, \tag{6.22}
\]
\[
\gamma(s_\pm, s_\pm)_{\mu v} = -2\bar{s}_\pm \Gamma_{(\mu \beta_{\pm} v)} s_\pm, \tag{6.23}
\]
\[
\gamma(s_+, s_-)_{\mu v} = -\left[\bar{s}_- \Gamma_{(\mu \beta_{-} + v)} s_+ + \bar{s}_+ \Gamma_{(\mu \beta_{-} - v)} s_-\right]_{\mu v}. \tag{6.24}
\]

Using our second parametrisation, we solve the first two equations for \(\beta\), and then use the second pair of equations to write \(\gamma\) in terms of the components of \(\beta\).

We first compute \(\bar{s}_\pm \Gamma_{\mu \beta_{\pm} v} s_\pm\), expressing it as a linear combination of Dirac bilinears:

\[
\bar{s}_\pm \Gamma_{\mu \beta_{\pm} v} s_\pm = \beta_{\pm}(0) v \bar{s}_\pm \Gamma_{\mu} s_\pm + \frac{1}{2} \beta_{\pm}(2)_{v_1 \rho_2} \bar{s}_\pm \Gamma_{\mu_1 \rho_2} s_\pm \\
+ \frac{1}{4} \beta_{\pm}(4)_{v_1 \rho_2 \rho_3 \rho_4} \bar{s}_\pm \Gamma_{\mu_1 \rho_2 \rho_3 \rho_4} s_\pm \\
= \beta_{\pm}(0) v \omega_{\pm}^{(1)} + \frac{1}{2} \beta_{\pm}(2)_{v_1 \rho_2} \left(\omega_{\pm}^{(3)} \rho_1 \rho_2 + 2\delta_{\mu_1}^{\rho_1} \omega_{\pm}^{(1) \rho_1}\right) \\
+ \frac{1}{4} \beta_{\pm}(4)_{v_1 \rho_2 \rho_3 \rho_4} \left(\omega_{\mu_1}^{(5)} \rho_1 \rho_2 \rho_3 \rho_4 + 4\delta_{\mu_1}^{\rho_1} \omega_{\pm}^{(3) \rho_1 \rho_3 \rho_4}\right) \\
= \left(\delta_{\mu}^{\rho_1} \beta_{\pm}(0)_{v} + \beta_{\pm}(2)_{v_1 \rho_2} \rho\right) \omega_{\pm}^{(1)} + \frac{1}{4} \delta_{\mu_1}^{\rho_1} \beta_{\pm}(4)_{v_1 \rho_2 \rho_3 \rho_4} \omega_{\pm}^{(5)} \\
\] 

where we have used the fact that \(\omega_{\pm}^{(3)} = 0\) identically. Equation (6.21) is then equivalent to

\[
\delta_{\mu_1}^{\rho_1} \beta_{\pm}(0)_{v} + \beta_{\pm}(2)_{v_1 \rho_2} \rho\ = 0, \tag{6.26}
\]
\[
\delta_{\mu}^{\rho_1} \beta_{\pm}(4)_{v_1 \rho_2 \rho_3 \rho_4} \rho\ = 0, \tag{6.27}
\]

where in the second equation we take the \(\mp\) self-dual part of the coefficient of \(\omega_{\pm}^{(5)}\), since that bilinear is \(\pm\) self-dual. Taking the \(\mu v\)-trace of the first equation yields

\[
\beta_{\pm}(0)_{\mu} + \beta_{\pm}(2)_{\mu} \rho = 0 \tag{6.28}
\]

while the \(\nu \rho\)-trace is

\[
\frac{1}{2} \beta_{\pm}(0)_{\nu} - \frac{1}{2} \beta_{\pm}(2)_{\nu} \rho = 0 \tag{6.29}
\]

Comparing these equations, we conclude that \(\beta_{\pm}(0) = 0\) and \(\beta_{\pm}(2)_{\mu} \rho = 0\), so substituting back into equation (6.26) gives

\[
\beta_{\pm}(0) = 0, \quad \beta_{\pm}(2)_{(\nu \rho)} = 0 \implies \beta_{\pm}(2) \in \wedge^3 V. \tag{6.30}
\]
Comparing with the Spencer cohomology and Killing superalgebras, we similarly compute

\[ \beta^{(4)}_{\pm} \sigma_1 \sigma_2 \sigma_3 \sigma_4 \delta = \frac{1}{5} \beta^{(4)}_{\pm} \rho_1 \rho_2 \rho_3 \rho_4 \epsilon_{\rho_1 \rho_2 \rho_3 \rho_4} \sigma_1 \sigma_2 \sigma_3 \sigma_4 = 0 \]  

(6.31)

while the \( \nu \sigma_5 \)-trace is

\[ \frac{1}{10} \left( 11 - 4 \right) \beta^{(4)}_{\pm} \mu \sigma_1 \sigma_2 \sigma_3 \sigma_4 \delta + 4 \delta_\mu^{(4)} \beta^{(4)}_{\pm} \nu \sigma_1 \sigma_2 \sigma_3 \sigma_4 \epsilon_{\rho_2 \rho_3 \rho_4 \rho_5} \sigma_1 \sigma_2 \sigma_3 \sigma_4 \nu = 0 \]

\[ \iff 7 \beta^{(4)}_{\pm} \mu \sigma_1 \sigma_2 \sigma_3 \sigma_4 + 4 \eta_{\mu \nu} \beta^{(4)}_{\pm} \sigma_2 \sigma_3 \sigma_4 \nu \pm \frac{1}{4} \beta^{(4)}_{\pm} \epsilon_{\rho_1 \rho_2 \rho_3 \mu \sigma_1 \sigma_2 \sigma_3 \sigma_4} = 0. \]  

(6.32)

The full skew-symmetrisation of this equation is

\[ 7 \beta^{(4)}_{\pm} |_{\mu \sigma_1 \sigma_2 \sigma_3 \sigma_4} \pm \frac{1}{4} \beta^{(4)}_{\pm} \epsilon_{\rho_1 \rho_2 \rho_3 \mu \sigma_1 \sigma_2 \sigma_3 \sigma_4} \epsilon_{\nu \sigma_1 \sigma_2 \sigma_3 \sigma_4}. \]  

(6.33)

Comparing with the \( \nu \sigma_5 \)-trace, we see that \( \beta^{(4)}_{\pm} \mid_{\mu \sigma_1 \sigma_2 \sigma_3 \sigma_4} = 0 \). Substituting this back into the \( \nu \sigma_5 \)-trace, we find

\[ \beta^{(4)}_{\pm} \mu \sigma_1 \sigma_2 \sigma_3 \sigma_4 \delta = -\frac{4}{7} \eta_{\mu \nu} \beta^{(4)}_{\pm} \sigma_2 \sigma_3 \sigma_4 \nu \epsilon_{\rho_1 \rho_2 \rho_3 \rho_4} \]  

(6.34)

or, defining \( \Xi_{\pm} \in \wedge^3 V \) by \( \Xi_{\pm} \mid_{\mu \rho_1 \rho_2} := -\frac{1}{7} \beta^{(4)}_{\pm} \mu \rho_1 \rho_2 \nu \epsilon_{\rho_1 \rho_2 \rho_3} \),

\[ \beta^{(4)}_{\pm} \mu \sigma_1 \sigma_2 \sigma_3 \sigma_4 \delta = 4 \eta_{\mu \nu} \Xi_{\pm} \sigma_2 \sigma_3 \sigma_4 \].  

(6.35)

Then

\[ \beta_{\pm \mu} = \left( \frac{1}{2} \beta^{(2)}_{\pm} \mu \nu \gamma_{\nu \gamma} \right) \left( \frac{1}{3} \Xi_{\pm} \mu \nu \rho \right) \epsilon_{\rho \sigma_1 \sigma_2 \sigma_3 \sigma_4}, \]

(6.36)

and, using equation (6.23),

\[ \gamma(s_{\pm} , s_{\pm})_{\nu \mu} = 2 \left( \beta^{(2)}_{\pm} \mu \nu \omega_{(1)} \rho - \frac{1}{3} \Xi_{\pm} \rho \sigma_1 \sigma_2 \sigma_3 \sigma_4 \right) \]  

(6.37)

where \( \beta^{(2)}_{\pm} , \Xi_{\pm} \in \wedge^3 V \).

We similarly compute \( \Xi_{\pm} \mu \beta_{\pm \nu \mu} s_{\pm} \):

\[ \Xi_{\pm} \mu \beta_{\pm \nu \mu} s_{\pm} = \beta^{(1)}_{\pm} \mu \nu_1 \Xi_{\pm} \mu \gamma_{\nu_1 \rho_1} s_{\pm} + \frac{1}{3} \beta^{(3)}_{\pm} \mu \nu_1 \rho_1 \rho_2 \rho_3 \Xi_{\pm} \mu \gamma_{\rho_1 \rho_2 \rho_3} s_{\pm} \]

\[ + \frac{1}{5} \beta^{(5)}_{\pm} \mu \nu_1 \rho_1 \rho_2 \rho_3 \rho_4 \rho_5 \Xi_{\pm} \mu \gamma_{\rho_1 \rho_2 \rho_3 \rho_4 \rho_5} s_{\pm} \]

\[ = \beta^{(1)}_{\pm} \mu \nu_1 \left( \omega_{(2)}_{\pm} \rho_1 + \delta_{\mu \rho_1} \omega_{(0)} \right) + \frac{1}{3} \beta^{(3)}_{\pm} \mu \nu_1 \rho_1 \rho_2 \rho_3 \left( \omega_{(4)}_{\pm} \rho_1 \rho_2 \rho_3 + 3 \delta_{\mu \rho_1 \rho_2 \rho_3} \right) \]

\[ + \frac{1}{5} \beta^{(5)}_{\pm} \mu \nu_1 \rho_1 \rho_2 \rho_3 \rho_4 \rho_5 \left( \pm \frac{1}{4} \epsilon_{\rho_1 \rho_2 \rho_3 \rho_4 \rho_5} \omega_{(4)}_{\pm} \sigma_1 \cdots \sigma_4 + 5 \delta_{\mu \rho_1 \rho_2 \rho_3 \rho_4 \rho_5} \right) \]

(6.38)

\[ = \beta^{(1)}_{\pm} \mu \omega_{(0)} + \left( \frac{1}{2} \beta^{(3)}_{\pm} \mu \nu \rho_2 + \eta_{\mu \rho_2} \right) \omega_{(2)}_{\pm} \rho_1 \rho_2 \]

\[ + \frac{1}{4} \left( 4 \eta_{\mu \rho_1} \beta^{(3)}_{\pm} \nu \sigma_2 \sigma_3 \sigma_4 + \beta^{(5)}_{\pm} \nu \sigma_1 \cdots \sigma_4 + \frac{1}{3} \beta^{(5)}_{\pm} \nu \sigma_1 \cdots \sigma_4 \epsilon_{\rho_1 \rho_2 \rho_3 \mu \sigma_1 \cdots \sigma_4} \omega_{(2)}_{\pm} \right) \]

\[ = \beta^{(1)}_{\pm} \mu \omega_{(0)} + \left( \frac{1}{2} \beta^{(3)}_{\pm} \mu \nu \rho_2 + \eta_{\mu \rho_2} \right) \omega_{(2)}_{\pm} \rho_1 \rho_2 \]

\[ + \frac{3}{4} \left( 2 \delta_{\mu \rho_1 \rho_2} \beta^{(5)}_{\pm} \nu \sigma_2 \sigma_3 \sigma_4 + \beta^{(5)}_{\pm} \nu \sigma_1 \cdots \sigma_4 \omega_{(2)}_{\pm} \right) \]

(6.39)
Now, we can use the symmetries (6.7) to write

$$\tilde{s}_- \Gamma_\mu \beta_{-\nu} s_+ + \tilde{s}_+ \Gamma_\mu \beta_{+\nu} s_-$$

$$= \left( \beta_{\nu \mu}^{\text{(9)}} + \beta_{\nu \mu}^{\text{(1)}} + \beta_{\nu \mu}^{\text{p1p2}} + \beta_{\nu \mu}^{\text{p1p2}} \right) \omega_{-0}^{(0)} + \frac{1}{2} \left[ \beta_{\nu \mu}^{\text{(3)}} \rho_1 \rho_2 + \beta_{\nu \mu}^{\text{(3)}} \rho_1 \rho_2 \right] + \delta_{\mu \nu} \left( \beta_{\nu \mu}^{\text{(1)}} \rho_2 + \beta_{\nu \mu}^{\text{(1)}} \rho_2 \right) \omega_{-0}^{(2)}$$

$$+ \frac{2}{4} \left( 2\delta_{\mu \nu} \left( \beta_{\nu \mu}^{\text{(3)}} \rho_2 \rho_3 \rho_4 \right) - \beta_{\nu \mu}^{\text{(3)}} \rho_2 \rho_3 \rho_4 \right) \omega_{-0}^{(2)}$$

$$= 2 \beta_{\nu \mu}^{\text{(9)}} \omega_{-0}^{(0)} + \left\{ \beta_{\nu \mu}^{\text{(3)}} \rho_1 \rho_2 + 2 \delta_{\mu \nu} \beta_{\nu \mu}^{\text{(1)}} \rho_2 \right\} \omega_{-0}^{(2)}$$

$$+ \frac{1}{12} \left\{ 4 \delta_{\mu \nu} \beta_{\nu \mu}^{(7)} \rho_2 \rho_3 \rho_4 \right\} - * \beta_{\nu \mu}^{(5)} \rho_1 \rho_2 \rho_4 \omega_{-0}^{(4)}$$

(6.39)

where in the last line we have used the relations (6.16) and (6.18) to change to the evidently more convenient parametrisation. Thus equation (6.22) is equivalent to

$$\beta_{\nu \mu}^{\text{(9)}} = 0,$$  \hspace{1cm} (6.40)

$$\beta_{\nu \mu}^{\text{(3)}} \rho_1 \rho_2 + 2 \delta_{\mu \nu} \beta_{\nu \mu}^{\text{(1)}} \rho_2 = 0,$$  \hspace{1cm} (6.41)

$$4 \delta_{\mu \nu} \beta_{\nu \mu}^{(7)} \rho_2 \rho_3 \rho_4 - * \beta_{\nu \mu}^{(5)} \rho_1 \rho_2 \rho_4 = 0.$$  \hspace{1cm} (6.42)

The first equation just says $\beta_{\nu \mu}^{\text{(9)}} \in \wedge^3 V$. The $\mu \nu$-trace of equation (6.40) is

$$\beta_{\mu \mu}^{\text{(3)}} \rho_1 \rho_2 + 2 \beta_{\mu \mu}^{\text{(1)}} \rho_1 \rho_2 = 0$$  \hspace{1cm} (6.43)

and the $\nu \rho_2$-trace is

$$\frac{1}{2} \beta_{\nu \mu}^{\text{(3)}} \rho_1 \rho_2 + \frac{1}{2} \delta_{\mu \nu} \beta_{\nu \mu}^{(1)} = 0,$$  \hspace{1cm} (6.44)

$$\Leftrightarrow \beta_{\mu \mu}^{\text{(3)}} \rho_1 \rho_2 + \eta_{\mu \rho_2} \beta_{\nu \mu}^{(1)} = 0.$$  \hspace{1cm} (6.45)

Comparing the skew-symmetrised part with the $\mu \nu$-trace, we find that $\beta_{\mu \mu}^{\text{(9)}} \rho_1 \rho_2 = 0$ and $\beta_{\mu \mu}^{\text{(3)}} \rho_1 \rho_2 = 0$, and then the symmetric part gives us

$$\beta_{\mu \mu}^{(1)} \rho_1 \rho_2 = \eta_{\mu \rho_2} a$$  \hspace{1cm} (6.46)

where we have defined $a = \frac{1}{10} \beta_{\nu \mu}^{(1)} \nu$. Substituting this back into (6.40) gives

$$\beta_{\mu \mu}^{\text{(3)}} \rho_1 \rho_2 = 0 \Rightarrow \beta_{\mu \mu}^{\text{(3)}} \in \wedge^4 V.$$  \hspace{1cm} (6.47)

The $\mu \nu$-trace of (6.42) is

$$4 \beta_{\nu \mu}^{(7)} \rho_1 \rho_2 \rho_3 \rho_4 - * \beta_{\nu \mu}^{(5)} \rho_1 \rho_2 \rho_4 = 0,$$  \hspace{1cm} (6.48)

while the $\nu \rho_4$-trace is

$$\frac{3}{2} \delta_{\mu \nu} \beta_{\nu \mu}^{(1)} \rho_2 \rho_3 \rho_4 \nu + \frac{1}{2} (3 - 10) \beta_{\nu \mu}^{(1)} \rho_2 \rho_3 \rho_4 - \frac{1}{2} \beta_{\nu \mu}^{(5)} \rho_1 \rho_2 \rho_3 \nu = 0$$

$$\Leftrightarrow 3 \eta_{\mu \rho_1} \beta_{\nu \mu}^{(7)} \rho_2 \rho_3 \nu - 8 \eta_{\mu \rho_1 \rho_2} \rho_3 \nu = 0.$$  \hspace{1cm} (6.49)

Comparing the fully skew-symmetrised part to the $\mu \nu$-trace, we see that $\beta_{\nu \mu}^{(7)} \in \wedge^4 V$. 

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and \( \hat{\beta}^{(5)}_{\mu p_1 p_2 p_3 \rho \nu} \) both vanish. The remaining part of the equation is

\[
\beta^{(7)}_{\mu p_1 p_2 p_3} = 3\eta_{\mu [p_1} \Theta_{p_2 p_3]} \tag{6.49}
\]

where \( \Theta_{\mu_1 \mu_2} := \frac{1}{8} \beta^{(7)}_{\nu \mu_1 \nu \mu_2} \). Finally, substituting this back into equation (6.42) gives

\[
\hat{\beta}^{(5)}_{(\mu \nu)} \rho^{1 \cdots p_4} = 0 \quad \Rightarrow \quad \Phi = \hat{\beta}^{(5)}_{\mu \nu} \in \wedge^6 V, \tag{6.50}
\]

where the equation following the implication arrow defines \( \Phi \in \wedge^6 \).

In summary, we can now use the relations (6.15) and (6.17) to write \( \beta_{\pm \pm} \) as follows (for the last term, we also require the duality relation \( \Gamma_{\mu_1 \cdots \mu_5} P_\pm = \mp \epsilon_{\mu_1 \cdots \mu_5 v_1 \cdots v_5} \Gamma^{v_1 \cdots v_5} P_\pm \)):

\[
\beta_{\pm \pm} = \left[ a \Gamma_\mu \pm \beta^{(9)}_{\mu \nu} \Gamma^\nu + \frac{1}{3!} \beta^{(3)}_{\mu \nu v_1 v_3} \Gamma^{v_1 v_2 v_3} \right. \\
\left. + \frac{1}{2} \Theta_{\mu v_1 v_2} \Gamma_{\nu 1 v_2} + \frac{1}{5!} \Phi_{\mu v_1 v_2} \Gamma^{v_1 \cdots v_5} \right] P_\pm \tag{6.51}
\]

where \( a \in \mathbb{R}, \beta^{(9)}, \Theta \in \wedge^2 V, \beta^{(3)} \in \wedge^4 V \) and \( \Phi \in \wedge^6 V \), and we can use (6.24) to find

\[
\gamma(s_+, s_-) = 2 \beta^{(9)}_{\mu \nu} \omega^{(0)}_{- \mp} + \left( \beta^{(3)}_{\mu \nu v_1 v_3} - 2 \delta^{(1)}_{[\mu} \delta^{(2)}_{\nu]} a \right) \omega^{(2)}_{\pm \mp} \\
- \left( \delta^{(2)}_{\mu} \delta^{(2)}_{\nu} \Theta_{\rho \sigma} \omega^{(4)}_{\rho \sigma} + 1 \right) \omega^{(4)}_{\mu \nu} \\
= 2 \left[ \beta^{(9)}_{\mu \nu} \omega^{(0)}_{- \mp} + \frac{1}{2} \beta^{(3)}_{\mu \nu \rho \sigma} \omega^{(2)}_{\pm \mp} - a \omega^{(2)}_{\mu \nu} \\
- \frac{1}{2} \Theta_{\rho \sigma} \omega^{(4)}_{\rho \sigma} - \frac{1}{4!} \Phi_{\rho \sigma} \omega^{(4)}_{\rho \sigma} \right]. \tag{6.52}
\]

### 6.2.2 Second cocycle condition

We also take an approach to cocycle condition (3.119) in a similar manner to the 6-dimensional case. Here, that condition is equivalent to the following system of equations:

\[
\kappa^\mu_\pm \beta_{\pm \mp} s_\pm + \frac{1}{4} \gamma(s_+, s_-) \Gamma^{\mu \nu} s_\pm = 0, \tag{6.53}
\]

\[
\kappa^\mu_\pm \beta_{\mp \pm} s_\pm = 0, \tag{6.54}
\]

\[
\kappa^\mu_\pm \beta_{\mp \mp} s_\pm + \frac{1}{4} \gamma(s_+, s_-) \Gamma^{\mu \nu} s_\pm = 0, \tag{6.55}
\]

\[
\kappa^\mu_\pm \beta_{\mp \mp} s_\pm + \frac{1}{4} \gamma(s_+, s_-) \Gamma^{\mu \nu} s_\pm = 0. \tag{6.56}
\]

We begin with equation (6.56), since it does not require use of the Fierz identity and is thus the simplest of these equations to solve. We abstract \( s_\pm \) from this equation, leaving

\[
0 = \kappa^\mu_\pm \beta_{\mp \mp} + \frac{1}{4} \gamma(s_+, s_-) \Gamma^{\mu \nu} P_\mp \\
= \omega^{(1)}_{\pm} \left[ \frac{1}{2} \rho^{(2)}_{\mp \nu v_1 v_2} \Gamma^{v_1 v_2} + \frac{1}{3!} \rho^{(4)}_{\mp v_1 v_2 v_3} \Gamma^{v_1 v_2 v_3} \right] P_\mp \\
+ \frac{1}{2} \left[ \beta^{(2)}_{\pm \nu v_1 v_2} + \rho^{(2)}_{\pm \nu v_1 v_2} \right] \omega^{(1)}_{\pm} - \frac{1}{3!} \rho^{(4)}_{\pm \rho \sigma} \omega^{(5)}_{\pm v_1 v_2 v_3} \Gamma^{v_1 v_2 v_3} P_\mp \\
+ \frac{1}{3!} \rho^{(4)}_{\pm \rho \sigma} v_1 v_2 v_3 \Gamma^{v_1 v_2 v_3} P_\mp \tag{6.57}
\]
Since this equation must hold for all \( s_\pm \in \Sigma_\pm \), we must have

\[
\beta^{(2)} = \frac{1}{2} \left( \rho^{(2)}_+ + \rho^{(2)}_- \right) = 0, \quad \Xi_+ = 0, \quad \Xi_- = 0. \tag{6.58}
\]

The next-simplest of the equations is (6.55). Here, we need to use the Fierz identity once in the second term to move \( s_\mp \) to the right, so that it can abstracted from the equation. Using our expression for \( \gamma \), one can show

\[
\frac{1}{2} \gamma(s_+, s_-) \mu \nu \Gamma^{\mu \nu} s_\pm = \Gamma^{\mu \nu} (s_+ s_-) X_{\pm \mu \nu} s_\pm
\tag{6.59}
\]

where

\[
X_{\pm \mu \nu} := \left[ \mp \beta^{(3)}_\mu \nu + \frac{1}{2} \beta^{(3)}_\mu \nu \rho_1 \rho_2 \Gamma^{\rho_1 \rho_2} - a \Gamma^{\mu \nu} \right] P_\mp.
\tag{6.60}
\]

Equation (6.55) can now be written as

\[
\kappa^\mu \beta_\pm \mp \mu s_\pm + \Gamma^{\mu \nu} (s_+ s_-) X_{\pm \mu \nu} s_\mp = 0.
\tag{6.61}
\]

Abstracting \( s_\mp \) and using the Fierz identity to write

\[
\Gamma^{\mu \nu} (s_+ s_-) X_{\pm \mu \nu} = \frac{1}{16} \Gamma^{\mu \nu} \left( \omega^{(1)} \pm \frac{1}{2} \omega^{(5)} \right) P_\mp
\tag{6.62}
\]

equation (6.55) becomes

\[
0 = 16 \omega^{(1)} \beta_\pm \mp \mu \nu \rho_1 \rho_2 \Gamma^{\rho_1 \rho_2} + \left( \omega^{(1)} \gamma^{\mu \nu} \Gamma_\sigma + \frac{1}{2} \omega^{(5)} \gamma^{\mu \nu} \Gamma_\sigma \right) X_{\pm \mu \nu}
\tag{6.63}
\]

Since this must hold for all \( s_\pm \), the coefficients of both Dirac bilinears must vanish:

\[
16 \beta_\pm \mp \mu \nu \rho_1 \rho_2 \Gamma^{\rho_1 \rho_2} = 0, \tag{6.64}
\]

\[
\Gamma^{\mu \nu} \Gamma_\sigma \pm \pm \mu \nu = 0. \tag{6.65}
\]

We can rewrite the first equation as follows:

\[
0 = 16 \beta_\pm \mp \mu \nu + \Gamma_\sigma \mp \mu \nu X_{\pm \mu \nu} + 4 \Gamma^{\mu \nu} X_{\pm \mu \nu},
\tag{6.66}
\]

and then it is straightforward to evaluate

\[
\Gamma_\sigma \Gamma^{\mu \nu} X_{\pm \mu \nu} = \left[ \mp \beta^{(3)}_\rho_1 \rho_2 \Gamma_\sigma \pm \pm \mu \nu \rho_1 \rho_2 \Gamma^{\rho_1 \rho_2} + \frac{1}{2} \beta^{(3)}_\rho_1 \rho_2 \Gamma_\sigma \pm \pm \mu \nu \rho_1 \rho_2 \Gamma^{\rho_1 \rho_2} + \frac{1}{2} \beta^{(3)}_\rho_1 \rho_2 \Gamma_\sigma \pm \pm \mu \nu \rho_1 \rho_2 \Gamma^{\rho_1 \rho_2} \right] P_\mp
\tag{6.67}
\]
We also have

\[\Gamma^\mu X_{\pm \mu \sigma} = \left[ \pm \beta^{(9)}_{\sigma \rho} \Gamma^\rho - \frac{1}{2} \beta^{(3)}_{\sigma \rho_1 \rho_2 \rho_3} \Gamma^{\rho_1 \rho_2 \rho_3} - 9a \Gamma_\sigma \right.\]

\[\left. \pm \frac{7}{2} \Theta_{\rho_1 \rho_2} \Gamma^\rho_{\rho_1 \rho_2} \pm \frac{1}{4} \Phi_{\sigma \rho_1 \ldots \rho_5} \Gamma^{\rho_1 \ldots \rho_5} \right] P_{\mp} \]

so

\[
\Gamma_\sigma \Gamma^{\mu \nu} X_{\pm \mu \nu} + 4 \Gamma^\mu X_{\pm \mu \sigma} = \left[ 54a \Gamma_\sigma \pm \left( 2 \beta^{(9)}_{\sigma \rho} - 56 \Theta_{\sigma \rho} \right) \Gamma^\rho \pm \left( \beta^{(9)}_{\rho_1 \rho_2} + 14 \Theta_{\rho_1 \rho_2} \right) \Gamma^\rho_{\rho_1 \rho_2} \right.
\]

\[-5(\Phi)_{\sigma \rho_1 \rho_2 \rho_3} \Gamma^{\rho_1 \rho_2 \rho_3} + \frac{1}{2} \beta^{(3)}_{\rho_1 \ldots \rho_4} \Gamma^\rho_{\rho_1 \ldots \rho_4} \pm \frac{1}{12} \Phi_{\sigma \rho_1 \ldots \rho_5} \Gamma^{\rho_1 \ldots \rho_5} \left. \right] P_{\mp}.
\]

We also have

\[\beta_{\pm \tau \sigma} = \left[ a \Gamma_\sigma \pm \beta^{(9)}_{\sigma \rho} \Gamma^\rho + \frac{1}{3} \beta^{(3)}_{\sigma \rho_1 \rho_2 \rho_3} \Gamma^{\rho_1 \rho_2 \rho_3} \right.
\]

\[\left. \pm \frac{1}{2} \Theta_{\rho_1 \rho_2} \Gamma^\rho_{\rho_1 \rho_2} \pm \frac{1}{4} \Phi_{\sigma \rho_1 \ldots \rho_5} \Gamma^{\rho_1 \ldots \rho_5} \right] P_{\mp},
\]

so (6.64) becomes

\[0 = \left[ 70a \Gamma_\sigma \pm 14 \left( \beta^{(9)}_{\sigma \rho} + 4 \Theta_{\sigma \rho} \right) \Gamma^\rho \pm \left( \beta^{(9)}_{\rho_1 \rho_2} + 22 \Theta_{\rho_1 \rho_2} \right) \Gamma^\rho_{\rho_1 \rho_2} \right.
\]

\[+ \left( \frac{8}{3} \beta^{(3)}_{\rho_1 \rho_2 \rho_3} - 5(\Phi)_{\rho_1 \rho_2 \rho_3} \right) \Gamma^{\rho_1 \rho_2 \rho_3} \left. \right]

\[\left. + \frac{1}{2} \beta^{(3)}_{\rho_1 \ldots \rho_4} \Gamma^\rho_{\rho_1 \ldots \rho_4} \pm \frac{13}{40} \Phi_{\sigma \rho_1 \ldots \rho_5} \Gamma^{\rho_1 \ldots \rho_5} \right] P_{\mp}.
\]

Equivalently,

\[70a \eta_{\sigma \rho} \pm 14 \left( \beta^{(9)}_{\sigma \rho} + 4 \Theta_{\sigma \rho} \right) = 0,
\]

\[\mp \left( \eta_{\sigma \rho_1} \beta^{(9)}_{\rho_2 \rho_3} + 22 \eta_{\sigma \rho_1} \Theta_{\rho_2 \rho_3} \right) + \left( \frac{8}{3} \beta^{(3)}_{\rho_1 \rho_2 \rho_3} - 5(\Phi)_{\rho_1 \rho_2 \rho_3} \right) = 0,
\]

\[\frac{1}{2} \eta_{\sigma \rho_1} \beta^{(3)}_{\rho_2 \ldots \rho_5} \pm \frac{13}{40} \Phi_{\sigma \rho_1 \ldots \rho_5} = 0.
\]

Taking symmetric and skew-symmetric parts of the first equation gives us

\[a = 0
\]

and \(\beta^{(9)}_{\sigma \rho} = -4 \Theta_{\sigma \rho}\). The skew-symmetric part of the second equation gives

\[\beta^{(3)}_{\sigma \rho_1 \rho_2 \rho_3} = \frac{15}{8} (\Phi)_{\sigma \rho_1 \rho_2 \rho_3}.
\]

while its \(\rho_2\)-trace gives \(\beta^{(9)}_{\sigma \rho} = -22 \Theta_{\sigma \rho}\), so comparing with the symmetric part of the first equation,

\[\beta^{(9)}_{\sigma \rho} = \Theta_{\sigma \rho} = 0.
\]

Let us rewrite the third equation as
\[ \eta_{\rho_{1}\beta^{(3)}_{\rho_{2}...\rho_{5}}} \pm \frac{1}{3!} e_{\sigma_{\rho_{1}...\rho_{5}}\sigma_{1}...\sigma_{4}} \beta^{(3)}_{\sigma_{1}...\sigma_{4}} \pm \frac{13}{30} (\Phi_{\sigma_{\rho_{1}...\rho_{5}}} \pm \frac{1}{3!} \Phi_{\sigma_{\sigma_{1}...\sigma_{5}}} e_{\sigma_{1}...\sigma_{5}}^{\rho_{1}...\rho_{5}}) = 0. \] (6.78)

The \( \sigma_{\rho_{1}} \)-trace of this is

\[ 6\beta^{(3)}_{\rho_{2}...\rho_{5}} - \frac{13}{5} (\Phi)_{\rho_{2}...\rho_{5}} = 0, \] (6.79)

thus, comparing with equation (6.76), we conclude that \( \beta^{(3)} = 0 \) and \( \Phi = 0 \).

In summary,

\[ \beta_{\pm \pm} = 0, \quad \gamma(s_{\pm}, s_{\pm}) = 0. \] (6.80)

Equation (6.54) now holds trivially, so only (6.53) remains to be solved. We now have

\[ \beta_{\pm \mu} = \frac{1}{2} \beta^{(2)}_{\mu \nu} \Gamma_{\nu} \Gamma_{\mu} \Gamma_{\nu} P_{\pm}, \quad \gamma(s_{\pm}, s_{\pm}) \mu \nu = 2\beta^{(2)}_{\mu \nu} \mu \nu \omega^{(1) \rho}_{\pm}, \] (6.81)

so this equation is simply

\[ \beta^{(2)}_{\mu \nu} \Gamma_{\mu} s_{\pm} \omega^{(1) \rho}_{\pm} = 0. \] (6.82)

Here, we must depolarise and use the Fierz identity. For ease of notation, we take \( s_{1}, s_{2}, s_{3} \in \Sigma_{\pm} \). Depolarising the equation, we write

\[ 0 = \beta^{(2)}_{\mu \nu} \Gamma_{\mu} s_{1} s_{2} \Gamma_{\rho} s_{3} + \text{cyclic perm's of (1, 2, 3)} \]

\[ = \beta^{(2)}_{\mu \nu} \Gamma_{\mu} \left[ \left( s_{1} s_{2} + s_{2} s_{1} \right) \Gamma_{\rho} + \left( s_{1} \Gamma_{\rho} s_{2} \right) \right] s_{3}, \]

so we can abstract \( s_{3} \) and repolarise the remaining equation, leaving

\[ 0 = \beta^{(2)}_{\mu \nu} \Gamma_{\mu} \left[ 2 \left( s_{\pm} s_{\pm} \right) \Gamma_{\rho} + \omega^{(1) \rho}_{\pm} \right] P_{\pm} \]

\[ = \beta^{(2)}_{\mu \nu} \Gamma_{\mu} \left[ \frac{1}{8} \left( \omega^{(1) \sigma}_{\pm} \Gamma_{\sigma} + \frac{1}{2 \sqrt{3}} \omega^{(5)}_{\pm \sigma_{1}...\sigma_{5}} \Gamma^{\sigma_{1}...\sigma_{5}} + \omega^{(1) \rho}_{\pm} \right) \right] P_{\pm} \] (6.84)

Since this must hold for all \( s_{\pm} \in \Sigma_{\pm} \), we must have

\[ \beta^{(2)}_{\rho_{1} \rho_{2} \rho_{3}} \left( \Gamma^{\rho_{1} \rho_{2} \rho_{3}} \Gamma_{\sigma} \Gamma^{\rho_{3}} + 8 \Gamma^{\rho_{1} \rho_{2} \gamma} \delta^{\rho_{3}}_{\sigma} \right) = 0, \] (6.85)

\[ \beta^{(2)}_{\rho_{1} \rho_{2} \rho_{3}} \Gamma^{\rho_{1} \rho_{2} \rho_{3}} \Gamma_{\sigma} \Gamma^{\rho_{3}} = 0. \] (6.86)

We compute

\[ \Gamma^{\rho_{1} \rho_{2} \Gamma_{\sigma} \Gamma^{\rho_{3}}} = \Gamma^{\rho_{1} \rho_{2} \rho_{3}} - \delta^{\rho_{1}}_{\sigma} \Gamma^{\rho_{2} \rho_{3}} \] (6.87)

and so the first equation becomes

\[ \beta^{(2)}_{\rho_{1} \rho_{2} \rho_{3}} \Gamma^{\rho_{1} \rho_{2} \rho_{3}} + 7 \beta^{(2)}_{\rho_{1} \rho_{2} \rho_{3}} \Gamma^{\rho_{1} \rho_{2} \rho_{3}} = 0, \] (6.88)

whence we conclude that \( \beta^{(2)}_{\pm} = 0 \), and then the second equation is trivially satisfied.

### 6.2.3 Final result

In conclusion, we have shown that \( \beta = 0 \), and so \( \gamma = 0 \) and

\[ H^{2,2} (s_{-}; s) \cong \mathcal{H}^{2,2} = 0 \] (6.89)

in Type IIA. That is, we have shown the following.
**Theorem 6.1.** The Type IIA Spencer $(2,2)$-cohomology group is trivial.

This is consistent with the fact that the only maximally supersymmetric type IIA background is the trivial one, i.e. maximal supersymmetry implies that all fluxes vanish and the background is locally isometric to Minkowski space [21]. On the other hand, this also implies that there are no highly supersymmetric geometrically realisable subdeformations of $s$. This is not inconsistent with the existence of highly supersymmetric backgrounds (see e.g. [108]), however, since there are algebraic constraints (algebraic equations arising from vanishing supersymmetry variations of spin-$\frac{1}{2}$ fields) in IIA which our analysis does not take account of. See §3.2.3 for some comments on such constraints. We will also discuss this further in Chapter 8.
Chapter 7

Killing superalgebras with $R$-symmetry in $D = 5$

We now go on to apply the results of the Spencer cohomology calculation in the $D = 5, N = 1$ case from §5.2.2. We determine the maximally supersymmetric filtered deformations and the associated geometries. We find that even this restricted case is extremely rich; it not only captures the maximally supersymmetric backgrounds of minimal supergravity both with and without gauged $R$-symmetry and matter couplings but also a much richer range of examples. The physical interpretation of the latter examples is not completely clear, but suggestions are made in Chapter 7.

We note that the minimal (without $R$-symmetry) case was treated in [50], a published version of work first appearing in the author’s Masters thesis [51]. We will occasionally make comments comparing work described here to that work; otherwise, the work described here is entirely original and at time of writing has not yet been published elsewhere.

Note that we use the same conventions as in Section 5.2 and [50, 51]; in particular, the metric signature is mostly-minus, $(+, -, -)$, and we use the notation defined for spinors laid out in §5.2.1.

7.1 Maximally supersymmetric subdeformations

Our starting point is the following result, which follows directly from the discussion of §5.2.2, with some slight changes in notation. We once again use an orthonormal basis for $V$ and a symplectic basis for the fundamental representation $\Delta$ of the $R$-symmetry algebra $\mathfrak{sp}(1)$.

**Theorem 7.1.** Let $\mathfrak{h} = V \oplus S \oplus (\mathfrak{so}(V) \oplus \mathfrak{sp}(1))$ be the minimal $R$-symmetry extended Poincaré superalgebra in 5 dimensions. As modules of $\mathfrak{so}(V) \oplus \mathfrak{sp}(1)$, we have isomorphism:

$$H^{2,2}(\mathfrak{h}^\perp; \mathfrak{h}) \cong \mathcal{H}^{2,2} \cong (2 \wedge^2 V) \oplus (V \otimes \mathfrak{sp}(1)) \oplus \mathfrak{sp}(1).$$

(7.1)

Explicitly, normalised cocycles $\beta + \gamma + \rho \in \mathcal{H}^{2,2}$ are parametrised by arbitrary elements
$C, e \in \wedge^2 V, F \in V \otimes \mathfrak{sp}(1), d \in \mathfrak{sp}(1)$ as follows:
\[
(\beta_{\mu}s)^A = \left( C_{\nu\rho}^{\mu} + \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} e^{\nu\rho \sigma \tau} \right) s^A + \left( d^A_B \Gamma_{\mu} + \frac{1}{4} F^{A}_{\nu} \Gamma_{\mu\nu} - \frac{1}{2} F_{\mu}^A B \right) s^B.
\]
\[
\gamma(s, s)_{\mu\nu} = 2\mu C_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} e^{\rho \sigma \tau} \kappa^\tau - 2d_{AB} \omega_{\mu\nu}^{AB} + \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \omega_{\rho\sigma}^{AB},
\]
\[
\rho(s, s)_{AB} = (C_{\nu\rho}^{\mu} - 2\epsilon_{\mu\nu}) \omega_{\mu\nu}^{AB} - 3\mu d_{AB},
\]
for $v \in V, s \in S$.
We also note that $\beta$ can be expressed in a basis-free way as
\[
\beta(v, s)^B = \frac{1}{2} (v \cdot C - C \cdot v) + (v \cdot (e + (e \cdot v))) \cdot s^B + d^B_C v \cdot s^C - \frac{1}{8} (v \cdot F^{B}_C + 3F^{B}_C \cdot v) \cdot s^C.
\]

### 7.1.1 Maximally supersymmetric subdeformations via admissible and integrable cocycles

Let us first recall Lemma 4.29 and the preceding discussion. If $\alpha$ is a maximally supersymmetric graded subalgebra of $\mathfrak{s}$, we have $\alpha_{\mu} = \mathfrak{s}_{\mu}$, so $i^*: \mathcal{H}^{2,2} \simeq H^{2,2}(\mathfrak{s}_{\mu}; \mathfrak{s}) \to H^{2,2}(\mathfrak{s}_{\mu}; \mathfrak{s})$ becomes the identity map, and we have an injective map $i_*: H^{2,2}(\mathfrak{s}_{\mu}; \mathfrak{a}) \to H^{2,2}(\mathfrak{s}_{\mu}; \mathfrak{s}) \cong \mathcal{H}^{2,2}$. It follows that any Spencer $(2,2)$ cocycle is admissible, and that the normalised cocycle $\beta + \gamma + \rho \in \mathcal{H}^{2,2}$ representing it is unique (we will not decorate this cocycle with hats as in parts of Section 4.3).

Let $\beta + \gamma + \rho$ be an arbitrary normalised cocycle parametrised as in (7.2). We let
\[
\mathfrak{h} := \text{stab}_{\mathfrak{s}_0(V)}(C, e, F), \quad \mathfrak{r} = \text{stab}_{\mathfrak{sp}(1)}(F, d)
\]
be the simultaneous stabilisers of the data in the algebras $\mathfrak{s}_0(V)$ and $\mathfrak{sp}(1)$ respectively – note that for notational simplicity we now explicitly write $\mathfrak{sp}(1)$ for the $\mathfrak{r}$-symmetry algebra and use $\mathfrak{r}$ rather than $\mathfrak{r}'$ for a subalgebra of it. Our goal here will be to describe the filtered deformations of the graded subalgebra $\alpha = V \oplus S \oplus (\mathfrak{h} \oplus \mathfrak{r})$. By construction, this is the unique maximal such subalgebra of $\mathfrak{s}$ with $\beta + \gamma + \rho \in (\mathcal{H}^{2,2})^{\mathfrak{s}_0}$. Any maximally supersymmetric filtered subdeformation of $\mathfrak{s}$ embeds in a deformation of a graded subalgebra of this type, so considering this case is sufficient to determine all maximally supersymmetric subdeformations of $\mathfrak{s}$.

Our first step is to find some $\lambda \in C^{2,1}(\mathfrak{s}_{\mu}; \mathfrak{s})$ such that $\beta + \gamma + \rho + \partial \lambda$ lies in the image of $i_*: Z^{2,2}(\mathfrak{s}_{\mu}; \mathfrak{a}) \to Z^{2,2}(\mathfrak{s}_{\mu}; \mathfrak{s})$ – that is, we require $\lambda: V \to \mathfrak{s}_0(V) \oplus \mathfrak{sp}(1)$ such that $\gamma(s, s) - \partial \lambda_1(\kappa_x) \in \mathfrak{h}$ and $\rho(s, s) - \partial \lambda_2(\kappa_x) \in \mathfrak{r}$ for all $s \in S$, where $\lambda_1, \lambda_2$ are the components of $\lambda$. However, since we have normalised the cocycle $\alpha + \beta + \gamma$ such that $\rho|_V = 0$ under the splitting $\mathfrak{g}^{(2)} S' \cong V \oplus \ker \kappa_x$, we must have $\partial \lambda_2(\kappa_x) = -\lambda_2(\kappa_x) \in \mathfrak{r}$ so thus $\lambda_2$ takes values in $\mathfrak{r}'$, thus $\rho$ also takes values in $\mathfrak{r}$, and $\lambda_2 = i_* \tilde{\lambda}_2$ for $\tilde{\lambda}_2 \in H^{2,1}(\mathfrak{s}_{\mu}; \mathfrak{a})$. It follows that we can without loss of generality take $\lambda_2 = 0$.

We first compute the actions of the images of $\gamma$ and $\rho$ on each of the forms. We have
\[
(\gamma(s, s) \cdot C)_{\mu\nu} = \gamma(s, s)_{\mu\rho} C_{\alpha\nu} + \gamma(s, s)_{\nu\rho} C_{\mu\alpha} - 2\epsilon_{\mu\nu\rho\sigma} C_{\nu\rho\sigma\sigma} - \left( 4d_{AB} C_{\mu\rho}^{\sigma} \delta_{v}^{\sigma} + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} C_{\nu\rho\sigma\sigma} F_{\rho\sigma} \right) ,
\]
\[
(\rho(s, s) \cdot F)_{\mu\nu} = \rho(s, s)_{\mu\rho} F_{\nu\rho} - \left( 4d_{AB} C_{\mu\rho}^{\sigma} \delta_{v}^{\sigma} + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} C_{\nu\rho\sigma\sigma} F_{\rho\sigma} \right) ,
\]
\[
(\beta_{\mu}s)^A = \left( C_{\nu\rho}^{\mu} + \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} e^{\nu\rho \sigma \tau} \right) s^A + \left( d^A_B \Gamma_{\mu} + \frac{1}{4} F^{A}_{\nu} \Gamma_{\mu\nu} - \frac{1}{2} F_{\mu}^A B \right) s^B,
\]
and similarly

\[
\left( \gamma(s, s) \cdot e \right)_{\mu \nu} = 4 C_{\alpha [\mu} e^{\alpha}_{\nu]} \mu + 2 e_{\mu}^{\beta \gamma \sigma} e_{\nu]} a e_{\gamma \beta} \kappa_{\sigma} + \left( 4 d_{AB} e^{\sigma}_{[\mu} \delta_{\nu]} + \frac{1}{2} e_{[\mu}^{\alpha \beta \sigma \tau} e_{\nu]} a F_{\beta AB} \right) \omega_{\sigma \tau}^{AB},
\]

(7.6)

\[
\left( \gamma(s, s) \cdot F \right)_{\mu AB} = 2 C_{\mu \alpha} F_{\alpha AB} + e_{\mu \alpha} e_{\beta \gamma} F_{\alpha AB} e_{\gamma \beta} \kappa_{\sigma} + \left( 2 d_{CD} F_{\alpha AB} \delta_{\mu}^{\sigma \nu} + \frac{1}{4} e_{\mu}^{\alpha \beta \sigma \tau} F_{\alpha AB} F_{\beta CD} \right) \omega_{\sigma \tau}^{CD}.
\]

(7.7)

Given a map \( \lambda : V \to \mathfrak{so}(V) \), the action of \( \lambda(\kappa_s) \) is given as follows:

\[
\begin{align*}
(\lambda(\kappa_s) \cdot C)_{\mu \nu} &= 2 C_{\alpha [\mu} \lambda_{\nu]}^{\alpha} \kappa^{\sigma}, \\
(\lambda(\kappa_s) \cdot e)_{\mu \nu} &= 2 e_{\alpha \mu} \lambda_{\nu]}^{\alpha} \kappa^{\sigma}, \\
(\lambda(\kappa_s) \cdot F)_{\mu AB} &= F_{\alpha AB}^{\alpha} \lambda_{\mu \alpha \sigma} \kappa^{\sigma},
\end{align*}
\]

(7.8)

where \( \lambda_{\mu \nu \rho} := \eta(e_{\mu}, \lambda(e_{\rho}) e_{\nu}) \). Recall that we have freedom to redefine \( \lambda \) by addition of an arbitrary map \( V \rightarrow \mathfrak{h} \). Thus, for the \( \mathfrak{so}(V) \oplus \mathfrak{r}^* \)-module decomposition \( \mathcal{O}^2 \mathfrak{S} \cong \mathfrak{V} \oplus \ker \kappa \), if \( \gamma(s, s) - \lambda(\kappa_s) \in \mathfrak{h} \) for all \( s \in S \), we can assume without loss of generality that \( \gamma|_V = \lambda \), where \( \gamma|_V \) is the restriction of \( \gamma \) to \( V \) under the decomposition \( \mathcal{O}^2 \mathfrak{S} \cong \mathbb{R} \oplus \mathfrak{V} \oplus (V \otimes \mathfrak{sp}(1)) \); more explicitly, this means that we assume that the coefficient of \( \kappa_s \) in the Dirac bilinear expansion of \( \gamma(s, s) - \lambda(\kappa_s) \) is zero, meaning we must have

\[
\lambda_{\mu \nu \rho} = e_{\mu \nu \rho \sigma} \delta_{\sigma \tau}.
\]

(7.9)

With this assumption, we find that \( \gamma(s, s) - \lambda(\kappa_s) \in \mathfrak{h} \) if and only if

\[
\begin{align*}
C_{\alpha [\mu} e^{\alpha}_{\nu]} &= 0, \\
2 C_{\mu \nu} F_{AB} &= 0,
\end{align*}
\]

(7.10)

(7.11)

\[
\begin{align*}
4 d_{AB} C_{[\sigma [\mu} e^{\sigma}_{\nu]} + \frac{1}{2} e_{[\mu}^{\alpha \beta \sigma \tau} e_{\nu]} a F_{\beta AB} &= 0, \\
4 d_{AB} e^{\sigma}_{[\mu} e^{\sigma}_{\nu]} + \frac{1}{2} e_{[\mu}^{\alpha \beta \sigma \tau} e_{\nu]} a F_{\beta AB} &= 0, \\
2 d_{CD} F_{[\sigma [\mu} e^{\sigma}_{\nu]} + \frac{1}{4} e_{[\mu}^{\alpha \beta \sigma \tau} F_{\alpha AB} F_{\beta CD} &= 0.
\end{align*}
\]

(7.12)

(7.13)

(7.14)

By taking various traces, one can show that equation (7.12) is equivalent to

\[
d_{AB} e_{\mu \nu \rho \sigma} C_{\alpha \beta} - C_{\mu \nu} F_{\rho AB} = 0,
\]

(7.15)

so (7.11) is redundant. Similarly, (7.13) is equivalent to

\[
d_{AB} e_{\mu \nu \rho \sigma} e_{\alpha \beta} - e_{\mu \nu} F_{\rho AB} = 0.
\]

(7.16)

By taking the trace and full skew-symmetrisation of (7.14) respectively, we find

\[
F_{\mu AB} d_{CD} = 0,
\]

(7.17)

\[
F_{\mu AB}^{[\mu} F_{\nu] CD} = 0.
\]

(7.18)

From (7.17), we must have either \( d = 0 \) or \( F = 0 \). In the former case, (7.15) and (7.16)
give us either \( F = 0 \) or \( C = e = 0 \). In the latter case, we similarly find that either \( d = 0 \) or \( C = e = 0 \), and moreover, (7.18) is equivalent to

\[
[F_\mu, F_\nu]_{AB} = 0.
\] (7.19)

In summary, with \( \lambda \) given by (7.9), \( \gamma(s, s) - \lambda(\kappa_s) \in \mathfrak{h} \) if and only if one of the following holds:

1. \( F = 0, d = 0, \) and \( C_{\alpha|\mu} e^{\alpha|\nu} = 0 \);
2. \( d = 0, C = e = 0, \) and \( [F_\mu, F_\nu]_{AB} = 0 \);
3. \( F = 0 \) and \( C = e = 0 \).

We can obtain some further equations by examining the action of \( \rho \) on \( d \) and \( F \):

\[
\begin{aligned}
\left( \rho(s, s) \cdot d \right)_{AB} &= (C_{\mu\nu} - 2e_{\mu\nu}) \omega^{\mu\nu}_{(A|C)} d^C_{B)}, \\
\left( \rho(s, s) \cdot F \right)_{\rho AB} &= (C_{\mu\nu} - 2e_{\mu\nu}) \omega^{\mu\nu}_{(A|C)} F^C_{B}) - 3\mu [d, F_\rho]_{AB},
\end{aligned}
\] (7.20)

which clearly vanish in any of the three cases detailed above; that is, \( \rho \) takes values in \( \tau \) in any of those cases. This shows that \( \alpha + \beta + \gamma + \delta \lambda \) defines an admissible cocycle for \( a \) if and only if one of those cases holds. We must now check whether it is integrable.

Recall that the degree-4 deformation maps \( \Theta_1: \wedge^2 V \to \mathfrak{so}(V) \) and \( \Theta_2: \wedge^2 V \to \tau \) exist if and only if the maps \( \Theta_1: V \otimes \bigotimes^2 S \to \mathfrak{so}(V), \Theta_2: V \otimes \bigotimes^2 S \to \mathfrak{sp}(1) \) defined by

\[
\begin{aligned}
\Theta_1(v, s, s) &= 2\gamma(s, \beta(v, s)) - (\lambda(v) \cdot \gamma)(s, s), \\
\Theta_2(v, s, s) &= 2\rho(s, \beta(v, s))
\end{aligned}
\] (7.22)

for \( v \in V, s \in S \), annihilate the Dirac kernel (this is the first integrability condition of Definition 4.36.). This means that when the RHSs of these equations are written in terms of Dirac bilinears, the coefficients of \( \mu \) and \( \omega \) vanish, so that the expressions are proportional to the Dirac current (see the diagrams (4.216) and the surrounding discussion).

Without imposing the conditions above, we compute the following by evaluating products of \( \Gamma \)-matrices:

\[
\gamma(s, \beta_\rho s)_{\mu\nu} = \mathcal{S}_{\mu\nu\rho} \mu + \mathcal{B}_{\mu\nu\rho\sigma} \kappa^\sigma + \frac{1}{2} \mathcal{E}_{\mu\nu\rho\sigma} \omega^{\sigma\tau CD},
\] (7.24)

\[
(\lambda \cdot \gamma)(s, s) = \mathcal{B}_{\mu\nu\rho} \mu + \mathcal{E}_{\mu\nu\rho\sigma} \kappa^\sigma + \frac{1}{2} \mathcal{F}_{\mu\nu\rho\sigma} \omega^{\sigma\tau CD},
\] (7.25)

\[
\rho(s, \beta_\rho s + \lambda \cdot s)_{AB} = \mathcal{G}_{\rho AB} \mu + \mathcal{H}_{\rho AB} \rho_\sigma + \frac{1}{2} \mathcal{J}_{\rho AB} \omega^{\sigma\tau CD},
\] (7.26)

where

\[
\mathcal{S}_{\mu\nu\rho} = \varepsilon_{\mu\nu\alpha\beta} \varepsilon_{\rho\gamma} C_{\rho} \cdot e^{\beta\gamma} + \frac{1}{2} d_{AB} E^C_{\mu\nu} \eta_{\nu|\rho},
\] (7.27)

\[
\mathcal{B}_{\mu\nu\rho\sigma} = 2(C_{\mu\nu} C_{\rho\sigma} - e_{\mu\nu} e_{\rho\sigma} - \eta_{[\mu|\rho]} \eta_{\nu|\sigma} e_{\alpha|\beta} e^{\alpha|\beta} + 2\eta_{[\mu|\rho]} e^{\alpha|\beta} e_\sigma + 2\eta_{[\nu|\sigma]} e^{\alpha|\beta} e_\rho) + \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} d_{AB} E^C_{\alpha AB} - 2d_{AB} d_{C} E^D_{\alpha AB} \eta_{[\mu|\rho]} \eta_{\nu|\sigma} + \frac{1}{8} \eta_{[\mu|\rho]} F^{C}_{\nu|AB} - \frac{1}{2} \eta_{[\mu|\rho]} (F^{C}_{\nu|AB} F_{\nu|AB} - \eta_{[\mu|\rho]} \eta_{\nu|\sigma} F^{C}_{\nu|AB} F_{\nu|AB}),
\] (7.28)
To reduce the number of indices in some expressions, we now introduce the notation $\epsilon_{\mu\nu\rho\sigma} = (C_{\mu\nu}\eta_{\rho\sigma} - 2C_{\mu\rho}\eta_{\nu\sigma})F_{\tau\nu\rho\sigma} + 2\epsilon_{\mu\nu\rho\sigma\tau}C^\rho_{\tau\nu\sigma}d_{\nu\tau\rho\sigma} + 3\eta_{\nu\rho}\eta_{\tau\sigma}\epsilon_{\mu\nu\rho\sigma}F_{\nu\rho\sigma\tau} + 2\epsilon_{\mu\nu\rho\sigma\tau}C^\rho_{\tau\nu\sigma}d_{\nu\tau\rho\sigma}$ and for convenience we have used $\eta_{\nu\rho} = 2e_{\nu\rho} + 2\epsilon_{\nu\rho\sigma\tau}C^\sigma_{\tau\nu\sigma}d_{\nu\tau\rho\sigma}$ in the last three expressions.

In any of the three maximally supersymmetric cases mentioned above, we find that most of the terms vanish, giving

$$\gamma(s,\beta s, s)_{\mu\nu} = \begin{cases} 2(C_{\mu\nu}C_{\rho\sigma} - \epsilon_{\mu\nu}e_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma}\epsilon_{\alpha\beta\alpha\beta}) + 2\eta_{\mu\rho}\eta_{\nu\sigma}\epsilon_{\alpha\beta\alpha\beta}k_{\alpha\beta} & \text{in case (1)}, \\ \epsilon_{\mu\nu\alpha\beta}\gamma_{\rho}C^\rho_{\alpha\beta}d_{\nu\rho\mu} + \frac{1}{8}\kappa_{\mu\nu}F_{AB}F_{\rho\sigma AB} + \frac{1}{8}\eta_{\mu\rho}\eta_{\nu\sigma}F_{AB}F_{\rho\sigma AB} & \text{in case (2)}, \\ -2d_{AB}d_{\mu\nu}\eta_{\mu\rho}\eta_{\nu\sigma} & \text{in case (3)} \end{cases}$$

$$\lambda_{\rho}(s, s)_{\mu\nu} = \begin{cases} 2\epsilon_{\mu\nu\alpha\beta}\gamma_{\rho}C^\rho_{\alpha\beta}d_{\nu\rho\mu} + 12\epsilon_{\mu\nu\rho\sigma}e_{\rho\sigma}k_{\sigma} & \text{in case (1)}, \\ 0 & \text{in cases (2),(3)} \end{cases}$$

And for convenience we have used $A = C - 2e$ in the last three expressions.

$$\rho(s, \beta s + \lambda_{\rho} s)_{AB} = \begin{cases} \frac{1}{2}\epsilon_{\lambda\mu\nu\sigma}\epsilon_{\mu\nu}C^\lambda_{\rho\gamma}C^{\rho\gamma} - 4\epsilon_{\rho\gamma}e^{\rho\gamma}d_{\gamma\rho\tau} & \text{in case (1)}, \\ 0 & \text{in cases (2),(3)} \end{cases}$$

To reduce the number of indices in some expressions, we now introduce the notation\(^1\)\(^{-}\) for traces over pairs of $R$-symmetry indices; for $a, b \in \mathfrak{sp}(1)$, we have

$$\langle ab \rangle := \text{tr}(ab) = (ab)^A_A = a^A_B b^B_A = -a_{AB}b^{AB},$$

and we note that this gives a negative-definite inner product on $\mathfrak{sp}(1)$. We also use

\(^1\)We use a different notation for this trace than in [50]; we have $\langle ab \rangle = -(a \cdot b)$ where the latter is the notation used in the older work.
the shorthand $\langle F^2 \rangle = F_{\mu}^{\nu} F_{\mu}^{\nu}.$

Using the calculations above, we now have

$$
\Theta_1(\rho, s) = \begin{cases}
4(C_{\mu\nu}C_{\rho\sigma} - e_{\mu\nu}e_{\rho\sigma} - 3e_{\{\mu\nu}e_{\rho\sigma\}})k^\sigma \\
+ 4(2\eta_{[\mu|\rho]}e_{\sigma]e_{\rho\sigma}} - 6\eta_{[\mu|\rho]}e_{\rho\sigma}]e_{\sigma\rho} - 2\eta_{[\mu|\rho]}k_{\sigma}e_{\rho\sigma} - 2\eta_{[\mu|\rho]}e_{\rho\sigma}e_{\sigma\rho}) & \text{in case (1)}, \\
\frac{1}{4}(\eta_{[\mu|\rho]}(F_{\nu}F_{\mu}k^\sigma) - k_{[\mu}(F_{\nu}F_{\rho]} - k_{\sigma]}F_{[\mu|\rho]}(F^2)) & \text{in case (2)}, \\
4(d^2)\eta_{[\mu|\rho]}k_{\sigma] & \text{in case (3)},
\end{cases}
$$

(7.40)

$$
\Theta_2(\rho, s) = \begin{cases}
\epsilon_{\lambda\mu\nu\sigma\tau}(C_{\mu}^{\lambda} C_{\nu}^{\sigma\tau} - 4e_{\mu}^{\lambda} e_{\nu}^{\nu\tau}) & \text{in case (1)}, \\
0 & \text{in cases (2), (3)}.
\end{cases}
$$

(7.41)

We see that $\Theta_1$ is proportional to $k$ in any of the three allowed cases, while $\Theta_2$ is proportional to $\omega$ in the first case and it vanishes in the other two cases. Thus in cases (2) and (3), there is no obstruction to the existence of $\theta_1$ or $\theta_2$ but $\theta_2 = 0$, while in case (1) we must impose a constraint which makes $\Theta_2$ vanish; we will return to this in more detail soon. For now, we will check the second integrability condition of Definition 4.36, which is

$$
\theta_1(v, w) = \beta(v, w) - \beta(w, v) - (\lambda(v) - \beta)(w, v, s)
$$

(7.42)

for all $v, w \in V$, $s \in S$, or with indices,

$$
\frac{1}{4}(\theta_1)_{\mu\nu\rho\sigma} \Gamma^{\rho\sigma} s^A = \left( [\beta_{(\mu}, \beta_{\nu)] s + (\lambda_{(\mu} \cdot \beta)_{\nu)] s - (\lambda_{\nu} \cdot \beta)_{(\mu} s \right)^A.
$$

(7.43)

By the expression above for $\Theta_1$, we have

$$
(\theta_1)_{\mu\nu\rho\sigma} = \begin{cases}
4(C_{\mu\nu}C_{\rho\sigma} - e_{\mu\nu}e_{\rho\sigma} - 3e_{\{\mu\nu}e_{\rho\sigma\}})k^\sigma \\
+ 2\eta_{[\mu|\rho]}e_{\sigma]e_{\rho\sigma}} - 6\eta_{[\mu|\rho]}e_{\rho\sigma}]e_{\sigma\rho} - 2\eta_{[\mu|\rho]}k_{\sigma}e_{\rho\sigma} - 2\eta_{[\mu|\rho]}e_{\rho\sigma}e_{\sigma\rho}) & \text{in case (1)}, \\
\frac{1}{4}(\eta_{[\mu|\rho]}(F_{\nu}F_{\mu}k^\sigma) - k_{[\mu}(F_{\nu}F_{\rho]} - k_{\sigma]}F_{[\mu|\rho]}(F^2)) & \text{in case (2)}, \\
4(d^2)\eta_{[\mu|\rho]}k_{\sigma] & \text{in case (3)}.
\end{cases}
$$

(7.44)

Without imposing any of the conditions we have thus far derived, we can compute

$$
\left( [\beta_{(\mu}, \beta_{\nu)] s - (\lambda_{\nu} \cdot \beta)_{(\mu} s \right)^A = \mathcal{J}_{\mu\nu\lambda\sigma} s^B + \mathcal{K}_{\mu\nu\sigma\tau} A^B \Gamma^{\sigma\tau} s^B + \mathcal{L}_{\mu\nu\sigma\tau} A^B \Gamma^{\sigma\tau} s^B,
\right.

(7.45)

$$
\left( [\lambda_{\nu} \cdot \beta)_{(\mu} s - (\lambda_{\nu} \cdot \beta)_{(\mu} s \right)^A = \mathcal{J}_{\mu\nu\lambda\sigma} s^B + \mathcal{K}_{\mu\nu\sigma\tau} A^B \Gamma^{\sigma\tau} s^B + \mathcal{L}_{\mu\nu\sigma\tau} A^B \Gamma^{\sigma\tau} s^B,
\right.

(7.46)

where

$$
\mathcal{J}_{\mu\nu\lambda\sigma} = \frac{3}{16} [F_{\mu\nu}, F_{\lambda\sigma}]_{AB},
$$

(7.47)

$$
\mathcal{K}_{\mu\nu\sigma\tau} = \left[ 2\epsilon_{\sigma\tau\lambda\nu\mu} C_{\nu\mu} e^{\mu\nu} + \frac{1}{2} \eta_{[\mu|\sigma]} \langle \phi \rangle_{d} \right] e_{AB} - \frac{1}{16} \epsilon_{\mu\nu\sigma\rho\beta} [F^{\sigma\tau}, F^{\rho\beta}]_{AB}
$$

(7.48)
Theorem 7.2. A normalised cocycle $\beta + \gamma + \rho$ for the Poincaré superalgebra $\mathfrak{s}$ in 5 dimensions parametrised as in (7.2) defines a maximally supersymmetric subdeformation of $\mathfrak{s}$ if and only if one of the following holds:

1. $F = 0$, $d = 0$, $C_{[\mu|\nu]}e_{\alpha|\nu} = 0$ and $C \wedge C = 4e \wedge e$; in this case, the infinitesimal deformation ((2,2) -Spencer cohomology class for $a) has a representative cocycle (whose image under $i_*$ is) $\beta + \gamma + \rho + \partial \lambda$, where $\lambda : V \to \mathfrak{s}o(V)$ is given by $\lambda_{\mu\nu\sigma} = e_{\mu\nu\sigma\tau}e^{\alpha|\tau}$. 

2. $d = 0$, $C = e = 0$, and $[F_{\mu}, F_{\nu}]_{AB} = 0$; $\beta + \gamma + \rho$ is the infinitesimal deformation;

3. $F = 0$ and $C = e = 0$; $\rho = 0$ and $\beta + \gamma$ is the infinitesimal deformation.

In all cases, the non-vanishing part of the degree-4 deformation map $\theta_1 : \wedge^2 V \to \mathfrak{s}o(V)$ is given by (7.44).
7.1.2 Explicit form of deformations

General form of maximally supersymmetric subdeformations

We now explicitly describe the maximally supersymmetric subdeformations in each of the three cases described above. We recall that we have chosen an orthonormal basis \(\{e_\mu\}\) for \(V\) and denote the corresponding bases of translations and Lorentz transformations by \(\{P_\mu\}\) and \(\{L_{\mu\nu}\}\). Given our symplectic basis \(\{e_A\}_{A=1,2}\) for \(\Delta\), we have a corresponding basis \(\{\tau_i\}_{i=1,2,3}\) for \(\mathfrak{sp}(1)\) where \((\tau_i)^A_B = -\frac{i}{2} \delta_{ij}^A\). Then \([\tau_i, \tau_j] = \epsilon_{ijk} \tau_k\) and \([\tau_i, \tau_j] = \text{tr} (\tau_i \tau_j) = -\frac{1}{2} \delta_{ij}\).

We write \(F = F^i \tau_i\) and \(\omega_s = \omega^i \tau_i\) for a triplet of 1-forms \(F^i\) and 2-forms \(\omega^i\) given by

\[
\omega_1 = \frac{1}{i} (\omega^{11} - \omega^{22}) = i (\omega_{11} - \omega_{22}),
\]

\[
\omega_2 = \omega^{11} + \omega^{22} = \omega_{11} + \omega_{22},
\]

\[
\omega_3 = 2i \omega^{12} = -2i \omega_{12},
\]

which are real by the Majorana condition\(^2\).

The brackets which remain undeformed are

\[
\begin{align*}
[L_{\mu\nu}, L_{\rho\sigma}] &= 2\theta_{\mu\nu\rho\sigma} L_{\sigma\nu} - 2\theta_{\nu\sigma\rho\mu} L_{\rho\mu}, \\
[L_{\mu\nu}, \tau_i] &= 0, \\
[L_{\mu\nu}, s] &= \frac{1}{2} \Gamma_{\mu\nu} s,
\end{align*}
\]

while the deformed brackets have the general form

\[
\begin{align*}
[P_\mu, P_\nu] &= -2\lambda_{\mu\nu|^\rho} P_\sigma + \frac{i}{2} \left((\theta_1)^{\rho\sigma}_{\mu\nu} + 2\lambda_{\mu\nu\alpha} \lambda^{|\rho|\sigma\alpha} - 2\lambda_{[\mu} |\rho| \lambda_{\sigma]\alpha] L_{\rho\sigma}\right), \\
[L_{\mu\nu}, P_\rho] &= 2P_\mu \eta_{\nu\rho} - 2\lambda_{\nu|^\rho} L_{\sigma\rho} - \lambda_{\mu|^\rho} L_{\sigma\tau}, \\
[P_\mu, s] &= \beta_\mu(s) + \frac{i}{4} \lambda_{\mu\nu\rho} \Gamma^{\nu\rho} s,
\end{align*}
\]

\[
[s, s] = \kappa^\mu P_\mu + \frac{1}{2} (\gamma(s, s) L^{\mu\nu} - \lambda^{\mu\nu\rho} \kappa_\rho) L_{\mu\nu} + \rho (s, s)^i \tau_i.
\]

Maximally supersymmetric subdeformations, type (1)

In this branch we have \(\mathfrak{h} = \text{stab}_{\mathfrak{so}(V)}(C) \cap \text{stab}_{\mathfrak{so}(V)}(e)\) and \(\mathfrak{t} = \mathfrak{sp}(1)\). The deformed brackets are

\[
\begin{align*}
[P_\mu, P_\nu] &= -2\epsilon_{\mu\nu\alpha\beta} e^{\alpha\beta} P_\sigma + \left(2C_{\mu\nu} C_{\rho\sigma} - C_{[\mu\nu} C_{\rho\sigma]}\right) L_{\rho\sigma}, \\
[L_{\mu\nu}, P_\rho] &= 2P_\mu \eta_{\nu\rho} + \frac{3}{2} \left(\eta_{\mu\rho} \epsilon_{\sigma\tau\nu\beta} - \eta_{\nu\rho} \epsilon_{\sigma\tau\mu\beta}\right) e^{\alpha\beta} L_{\sigma\tau}, \\
[P_\mu, s] &= \left(C_{\mu\nu} \Gamma^\nu + \frac{1}{2} \epsilon_{\mu\nu\rho\alpha} e^{\alpha\beta} \Gamma^{\nu\rho}\right) s,
\end{align*}
\]

\[
[s, s] = \kappa^\mu P_\mu + \mu C_{\mu\nu} L^{\mu\nu} + \left(C_{\mu\nu} - 2e_{\mu\nu}\right) \omega^{\mu\nu} \tau_i.
\]

This is by far the richest class of maximally supersymmetric deformations. It contains the Killing superalgebras of pure minimal supergravity (for \(C = 2e\)), and we will explore a range of examples in §7.5.1-§7.5.4.

\(^2\)This is a slightly different choice of real 2-forms to that made in [50] but it is necessary for \(\omega_s = \omega^i \tau_i\). Similar combinations allow us to write any \(X \in \mathfrak{sp}(1)\) (i.e. Majorana condition-preserving elements of \(\mathfrak{o}^2\Delta\)) in the \(\tau_i\) basis.
Maximally supersymmetric subdeformations, type (2)

This case already appeared in [50], but we will describe the maximally supersymmetric deformations more explicitly. Since $[F_\mu, F_\nu] = 0$ and $\mathfrak{sp}(1)$ is simple, $F_\mu$ and $F_\nu$ must be collinear as elements of $\mathfrak{sp}(1)$; we have $F = \phi \otimes r$ for some $\phi \in V^*$ and $r \in \mathfrak{sp}(1)$. Moreover, we can choose an adapted basis $\{e_a\}$ such that $r$ is proportional to $\tau_3$; by rescaling $\phi$ we can arrange this such that $F = 4\phi \tau_3$. Then we have $\mathfrak{h} = \text{stab}_\mathfrak{so}(V)(F) = \text{stab}_\mathfrak{so}(V)(\phi)$ and $r = \text{stab}_{\mathfrak{sp}(1)}(F) = \mathbb{R} \tau_3$ and

$$
[P_\mu, P_\nu] = \phi^\sigma [\phi^{\sigma} L_{\nu}]_\sigma + \frac{1}{2} \phi^2 L_{\mu\nu},
\left[ L_{\mu\nu}, P_\rho \right] = 2 P_\mu [\eta_{\nu}]_\rho,
[P_\mu, s] = (\phi^\rho \Gamma_\mu - 2 \phi_\mu \Gamma_3)(\tau_3 s),
[s, s] = \kappa^\mu P_\mu - a^2 \omega^{\mu\nu}_3 L_{\mu\nu} - 3 a \mu \tau_3.
$$

(7.59)

We could now go on to classify the algebras of this form by analysing the cases in which $\phi$ is spacelike, timelike or null, but we postpone this treatment to §7.5.5 where we will also give a geometric interpretation of the results.

Maximally supersymmetric subdeformations, type (3)

We can choose an adapted basis $\{e_a\}$ such that $d$ is proportional to $\tau_3$; let us take $d = a \tau_3$ for $a > 0$. Then $\mathfrak{h} = \mathfrak{so}(V)$ and $r = \mathbb{R} \tau_3$ and the deformed brackets are

$$
[P_\mu, P_\nu] = -a^2 L_{\mu\nu},
\left[ L_{\mu\nu}, P_\rho \right] = 2 P_\mu [\eta_{\nu}]_\rho,
[P_\mu, s] = a \Gamma_\mu \tau_3 s,
[s, s] = \kappa^\mu P_\mu - a^2 \omega^{\mu\nu}_3 L_{\mu\nu} - 3 a \mu \tau_3.
$$

(7.60)

Note that the even part of the algebra here is nothing but the $AdS_5$ algebra (in our mostly-minus signature convention), thus, up to local isometry, the maximally supersymmetric geometry is locally isomorphic to $AdS_5$. We will give a more geometrical argument for this in §7.5.6.

7.2 The Killing spinor equation

7.2.1 Geometric setup

Bundles and conventions

Let $M$ be a 5-dimensional Lorentzian manifold with a spin-$R$ structure $\tilde{\Omega} : \tilde{P} \to F_{SO}$ as described in Section 4.1. Note that we have $R = \text{Sp}(1) \cong \text{SU}(2)$, whence a spin-$R$ structure in this setting is nothing but the more classical notion of spin-$h$, the quaternionic analogue of spin-$c$ [100–102]. This is in fact the most general setting in which a Lorentzian 5-manifold admits a real pinor bundle (which here is the same as a spinor bundle) [104, 105].

We will make use of the conventions for Majorana spinors established in Chapter 5. The associated bundle $\tilde{P} \times_{\text{Spin}^R(V)} \Sigma \otimes \mathbb{C} \Delta$ has a real structure with respect to which
we can consider as the real subspace the bundle \( \tilde{S} = \tilde{P} \times_{\text{Spin}^R(V)} S \) associated to the space of symplectic Majorana spinors \( S \).

Recall that we can obtain a principal \( \text{Sp}(1)/\mathbb{Z}_2 \cong \text{SO}(3) \)-bundle \( \tilde{Q} = \tilde{P}/\mathbb{Z}_2 \) associated to the spin-\( R \) structure; we will not need to work with this bundle explicitly but it will be useful to denote the algebra of sections of its adjoint bundle\(^3\) (the infinitesimal gauge transformations) by \( \mathfrak{sp}(M) = \text{ad} \tilde{Q} \). We will refer to connections \( \alpha \) on \( \tilde{Q} \) as \textit{gauge connections}, and we recall that pulling back \( \alpha \) and the Levi-Civita connection to the spin-\( h \) structure allows us to define a connection \( \mathcal{A} \) on that structure. We denote the field strength of \( \alpha \) by \( \mathcal{F} \in \Omega^2(M; \text{ad} \tilde{Q}) \). Note that for symplectic Majorana spinor fields in \( \mathcal{S} \), \( \omega^2 \in \Omega^2(M; \text{ad} \tilde{Q}) \). The \( \text{Sp}(1) \)-invariant inner product on \( \mathfrak{sp}(1) \) given by \( (a, b) \mapsto \langle ab \rangle = \text{tr}(ab) \) induces an inner product on \( \tilde{Q} \).

Where necessary, we choose a local orthonormal frame \( \{ e_\mu \}_{\mu = 0, \ldots, A} \) for \( TM \) and a local symplectic frame \( \{ e_A \}_{A = 1, 2} \) for \( \Delta \). Thus local expressions can be written using the same conventions as in Chapter 5.

**Gauged backgrounds**

Let us define a \textit{(gauged, bosonic) background} \( (M, g, \alpha, C, e, d, F) \) to be a Lorentzian spin-\( h \) 5-manifold \( (M, g) \) equipped with a gauge connection \( \alpha \) as well as the background fields

\[
C, e \in \Omega^2(M), \quad F \in \Omega^1(M; \text{ad} \tilde{Q}), \quad d \in \mathfrak{sp}(M) = \Omega^0(M; \text{ad} \tilde{Q});
\]

(7.61)
in the following we will omit the background fields from the notation and simply say that \( M \) is a background. Note that \( F \) is a 1-form with values in the adjoint \( R \)-symmetry bundle and is independent of the field strength \( \mathcal{F} \), which is a 2-form. Recalling (7.2), we define a 1-form with values in spinor endomorphisms \( \beta \) by

\[
\beta_X s = \frac{1}{2}((X \cdot C - C \cdot X) + (X \cdot e + e \cdot X)) \cdot s + (dX) \cdot s - \frac{1}{8} (X \cdot F + 3F \cdot X) \cdot s,
\]

(7.62)
where \( s \) is a symplectic Majorana spinor field, \( X \) is a vector field, \( \cdot \) denotes both the Clifford multiplication of forms and the Clifford action of forms on spinor fields, and \( F \) and \( d \) also act via the \( \mathfrak{sp}(M) \) action on spinor fields to the right. With our earlier conventions, in components we have

\[
(\beta_\mu s)^A = (C_{\mu \nu} \Gamma^\nu + \frac{1}{4} \epsilon_{\mu \nu \rho \sigma \tau} e_{\rho} \Gamma^{\sigma \tau}) s^A + \left( \Gamma^A_B \Gamma_{\mu \nu} + \frac{1}{4} F^V_B \Gamma_{\mu \nu} - \frac{1}{2} F_{\mu B} \right) s^B.
\]

(7.63)
As discussed in Section 4.1, the structures described here allow us to define the covariant connection \( \nabla \) and covariant Lie derivative \( \mathcal{L} \). We can locally abuse notation slightly to write the action of \( \nabla \) on spinors \( s \in \mathcal{S} \) as

\[
\nabla_\mu s^A = \nabla_{\mu} s^A + \alpha_{\mu} B s^B
\]

(7.64)
for \( X \in \mathcal{X}(M) \), where \( \nabla \) is the (local spin lift of) the Levi-Civita connection. We define another connection \( D \) on \( \mathcal{S} \) by

\[
D_X s = \hat{\mathcal{V}}_X s - \beta_X s,
\]

(7.65)
\(^3\)This was denoted by \( \mathfrak{g} \) in Chapter 4, but we give it a more evocative name here.
for $s \in \mathbb{S}$. In components,
\[
D_\mu s^A = \bar{\nabla}_\mu s^A - (C_{\mu\nu} \Gamma^\nu + \frac{1}{4} \epsilon_{\mu\rho\sigma\tau} \epsilon^{\rho\sigma\tau} \Gamma_{\rho\sigma\tau}) s^A - \left( d^A_{\ B} \Gamma_{\mu} + \frac{1}{4} F^A_{\ B\ \nu} \Gamma^{\nu \mu} - \frac{1}{2} F_{\mu\ A} \right) s^B.
\]

(7.66)

**Definition 7.3.** A Killing spinor (field) on a background is a spinor field $s \in \mathbb{S}$ which is parallel with respect to the connection $D$; that is, if it satisfies the Killing spinor equation
\[
\bar{\nabla} s = \beta s.
\]

(7.67)

The space of Killing spinors is denoted $\mathbb{S}_D$. A background is supersymmetric if it admits a Killing spinor, and maximally supersymmetric if its space of Killing spinors has maximal dimension.

### 7.2.2 Comparison with gauged supergravity

“Gauged” supergravity has a number of different meanings in the supergravity literature, but we will use it in the strict sense of a supergravity theory in which the full $R$-symmetry group (a global symmetry in the pure supergravity theory) is “gauged” using vector fields which possibly belonging to a number of different supermultiplets. In particular, the gravitino is charged under the gauge symmetry. The full $R$-symmetry gauging in $D = 5$ supergravity was first tackled, contemporaneously, by Günaydin and Zagermann in [109] and by Ceresole and Dall’Agata [110]. Supersymmetric solutions and Killing spinors were treated by [111, 112].

We will not provide a full account of the theory or its field content here, but we will highlight the part we are interested in for the purposes of studying Killing spinor equations: the bosonic sector and the infinitesimal gravitino supersymmetry variation. These can be read off from either of the above references. We use conventions closer to those of [109]. The theory is built by first considering minimal $D = 5$ supergravity coupled to $n$ vector supermultiplets and $m$ tensor supermultiplets. Coupling to so-called hypermultiplets is also allowed in [110–112], but doing so does not change the features we are interested in, so for simplicity we ignore this possibility here. Thus the bosonic fields are as follows: $n + 1$ gauge fields $A^I_\mu$, $I = 0, 1, \ldots, n$, with $A^0_\mu$ (the “graviphoton”) coming from the gravity supermultiplet and the $n$ fields $A^x_\mu$, $x = 1, \ldots, n$ coming from vector supermultiplets; $2m$ self-dual 2-forms $B^M_{\mu\nu}$, $M = 1, \ldots, 2m$ coming from tensor supermultiplets; $n$ scalars $\phi^2$ and $2m$ further scalars $\phi^M$ coming from the vector and tensor supermultiplets respectively.

One notes that the scalars (along with some fermions which we ignore) in this theory participate in a sigma model with Riemannian target whose geometric structure is constrained by supersymmetry. If this target manifold has isometry group $G$, the ungauged theory has a global symmetry group $R \times G$ where $R = \text{Sp}(1)$ is the $R$-symmetry group which acts on the theory by rotating the fermions within each supermultiplet, including the gravitini. Since $R$ acts trivially on the vector fields, we cannot simply gauge it on its own using these fields. Instead, we assume that $G$ contains an Sp(1) subgroup, consider the obvious diagonal embedding $\text{Sp}(1) \hookrightarrow R \times G$ and gauge the diagonal subgroup which is the image of this map. This subgroup is also isomorphic to Sp(1) and rotates both the gravitini and (some of) the vector fields.
Taking \( n \geq 3 \) and possibly using global symmetries to rearrange the multiplets, we can take the vector fields \( A^i_\mu, i = 1, 2, 3 \) coming from the first three vector multiplets to be transformed into each other under the adjoint representation of this subgroup. We use these fields to gauge our chosen subgroup – the remaining vectors can be used to gauge further “directions” in \( G \), but this will not explicitly play a role for us. The gauging process involves modifying the covariant derivatives, action and supersymmetry variations of the theory, as is standard when one “promotes” global symmetries to gauge symmetries. See the references for more details.

In [109], the supersymmetry variations for the gauged theory are not given in a single formula, but the following can be deduced from equations (2.4), (2.8) and (2.13) in that work\(^4\). One can then read off that the gravitino variation \( \delta_\epsilon \Psi^A_\mu \) (on bosonic solutions) is

\[
\hat{\nabla}_\mu \epsilon^A + \frac{1}{4\sqrt{6}} \left( h_I(\phi)_\mu \mathcal{F}^I + h_M(\phi)_\mu B^M_{\nu \rho} \right) \left( \Gamma^\nu_\mu - 4\delta^\nu_\mu \Gamma^\rho \right) \epsilon^A + \frac{1}{\sqrt{6}} g_R h^i(\phi) \Sigma^A_i B^A B \epsilon^B,
\]

(7.68)

where \( \mathcal{F}^I \) are the (non-abelian) field strengths of the \( A^I \), the \( h_I, h_M \) and \( h^i \) are functions on the target manifold of the sigma model (hence the \((\phi)\) arguments), \( g_R \) is a coupling constant, \( \Sigma^A_i \) are a basis for \( \mathfrak{sp}(1) \) and \( \hat{\nabla} \) is the \( \text{Sp}(1) \) covariant derivative which acts on spinors as \( \hat{\nabla}_\mu \epsilon^A = \nabla_\mu \epsilon^A + g_R A^I_\mu \Sigma^A_i A_i B \epsilon^B \).

We now note that by setting

\[
C_{\mu \nu}(\mathcal{F}, B, \phi) = 2\epsilon_{\mu \nu}(\mathcal{F}, B, \phi) = \frac{1}{\sqrt{6}} \left( h_I(\phi)_\mu \mathcal{F}^I + h_M(\phi)_\mu B^M_{\nu \rho} \right),
\]

\[
d^A B(\phi) = \frac{1}{\sqrt{6}} g_R h^i \Sigma^A_i B,
\]

\[
F^A_\mu B = 0,
\]

(7.69)

we have

\[
\delta_\epsilon \Psi = D\epsilon,
\]

(7.70)

where \( D \) is the spinor connection defined in the previous section. That is, the connection appearing in the Killing spinor equation of (generic matter-coupled, minimally supersymmetric) gauged supergravity in \( D = 5 \) is a special case of that recovered by our general theory. The Spencer cohomology calculation for \( \hat{s} \) therefore captures the Killing spinors of gauged \( D = 5 \) supergravity. Unlike in the pure ungauged theory [50], however, the fields appearing in the Spencer calculation are not simply the bosonic fields of the theory, but some particular functions of those fields given by (7.69). Another feature which must be noted here is that since other fermions appear in the theory, for a bosonic solution to be supersymmetric, there must exist a supersymmetry parameter which is not only \( D \)-parallel but for which the variations of the other fermions also vanish, and we do not yet have a way to account for these additional constraints via Spencer cohomology.

To the author’s knowledge, it has not been shown in the literature that the Killing spinors of this theory generate a Killing superalgebra. Two of the three conditions

\[\text{We have changed notation and some other conventions slightly for consistency with the rest of the present work; in particular we have relabelled indices } i, j, k, \ldots \rightarrow A, B, C, \ldots \text{ and replaced } \mathcal{D} \rightarrow \hat{\nabla}, \Sigma_A \rightarrow -\tau_i, \text{ while apparently missing factors of } i \text{ account for a change in signature.}\]
for the pair \((D, \rho)\) to be admissible (which implies that a Killing superalgebra of the type constructed in \$sec:killing-spinors-gauged\) exists) are automatically satisfied by the Spencer cocycle conditions; it remains to be shown that the Killing vectors \(\kappa_\epsilon\) (where \(\epsilon\) is a Killing spinor) preserve the background fields. One expects that this should be the case on general grounds by the closure of the supersymmetry algebra of the theory, but one should be able to show it directly by using the bosonic equations of motion for the theory, possibly together with the algebraic constraints arising from the supersymmetry variations of the matter fermions. There was no time to investigate this for the present work, but we hope to include it in future work. We note that to fully incorporate the Killing superalgebras of this theory into our theoretical framework, the discussion of the inclusion of algebraic constraints in \$3.2.3\) must be expanded upon and extended to the gauged case.

### 7.3 Curvature of the connection \(D\)

#### 7.3.1 Determination of the curvature

We can use Proposition 4.10 to write the curvature of the connection \(R^D\) in terms of the Riemann curvature, gauge field strength and \(\beta\); for the present purposes the local expression

\[
R^D_{\mu \nu}{}^{A}{}_{B} = \frac{1}{4} R_{\mu \nu \sigma \tau} \Gamma^{\sigma \tau} \delta^{A}{}_{B} + \mathbb{F}_{\mu \nu}{}^{A}{}_{B} - [\beta_{\mu}, \beta_{\nu}]^{A}{}_{B} + 2 \hat{\nabla}_{[\mu} \beta_{\nu]}^{A} \tag{7.71}
\]

will be more useful. Note that some of the signs here differ from those in the aforementioned proposition because we follow the sign conventions of [50] here; for example, the curvature of the connection \(D\) is given by

\[
R^D(X, Y) = D_{[X, Y]} - [D_X, D_Y].
\]

The components \(\beta^{A}{}_{B}\) of \(\beta\) are defined by \((\beta_{\mu}s)^{A} = \beta^{A}{}_{B} s^{B}\). The differential term is

\[
\hat{\nabla}_{[\mu} \beta_{\nu]}^{A} = \left(\nabla_{[\mu} C_{\nu]}\right) \Gamma^{\sigma} + \frac{1}{4} \nabla_{[\mu} e^{ab} \epsilon_{\nu]} a \sigma a \beta \Gamma^{\sigma \tau} \delta^{A}{}_{B} - \eta_{[\mu[a]} \hat{\nabla}_{\nu]} d^{A}{}_{B} \Gamma^{\sigma} s^{B} - \frac{1}{4} \eta_{[\mu[a]} \hat{\nabla}_{\nu]} F^{A}{}_{B} \Gamma^{\sigma} - \frac{1}{2} \hat{\nabla}_{[\mu} F_{\nu]}^{A} \tag{7.72}
\]

We have essentially already computed the commutator \([\beta_{\mu}, \beta_{\nu}] = 2(\beta_{[\mu} \beta_{\nu]}\), as we can interpret the equation (7.45) as holding pointwise. We can express \(R^D\) in components as follows:

\[
R^D_{\mu \nu A B} = \frac{1}{2} L_{\mu \nu \sigma \tau A B} \Gamma^{\sigma \tau} + M_{\mu \nu \sigma A B} \Gamma^{\sigma} + N_{\mu \nu A B}, \tag{7.73}
\]

where

\[
L_{A B} = L^{A} \epsilon_{A B} + L^{A} \epsilon_{A B}, \quad M_{A B} = M^{A} \epsilon_{A B} + M^{A} \epsilon_{A B}, \quad N_{A B} = N^{A} \epsilon_{A B} + N^{A} \epsilon_{A B}, \tag{7.74}
\]
and, now suppressing $AB$ indices to lighten the notation,

\[
L^\Lambda_{\mu
u\rho\sigma} = \frac{1}{2} \left[ -8C_{[\mu|\rho}C_{\nu]\sigma} + 8\varepsilon_{[\mu|\rho}e_{\nu]\sigma} - 8\eta_{[\mu|\rho}e_{\nu]\sigma} - 8\eta_{[\mu|\rho}\eta_{\nu]\sigma}e_{\sigma}^a + 8\eta_{[\mu|\rho}e_{\nu]\sigma}e_{\sigma}^a + 8\eta_{[\mu|\rho}\eta_{\nu]\sigma}e_{\sigma}^a + 8\eta_{[\mu|\rho}e_{\nu]\sigma}e_{\sigma}^a \right] + 4\eta_{[\mu|\rho}\eta_{\nu]\sigma}e_{\sigma}^a + 2\nabla_\mu e^{\alpha\beta}e_{\nu\sigma\rho\alpha\beta} - 4\langle d^2 \rangle \eta_{[\mu|\rho}\eta_{\nu]\sigma} (F^2) + R_{\mu\nu\rho\sigma} \right]
\]

\[
M^\Lambda_{\mu\nu\rho} = 2\epsilon_{\alpha\beta\gamma\rho}[\mu C_{\nu}]^a e^\beta_\nu - \frac{1}{2} \eta_{[\mu|\rho}\langle F_{\nu}\rangle d + 2\nabla_\mu C_{\nu}\rho \]

\[
N^\Lambda_{\mu\nu} = 0
\]

\[
L^\Lambda_{\mu\nu\rho\sigma} = -\frac{1}{4} \left[ \eta_{[\mu|\rho}[F_{\nu]}, F_{\sigma]} - \eta_{[\lambda|\nu]}[F_{[\mu|\rho}, F_{\sigma]}] \right] + \epsilon_{\alpha\beta\mu\nu}\eta_{[\rho|\sigma]e^{\alpha\beta}e_{\nu\sigma\rho\alpha\beta} - \frac{1}{2} \eta_{[\mu|\rho}\eta_{\nu]\sigma}e_{\sigma}^a + 2\epsilon_{\mu\nu\rho\sigma} e_{\sigma}^a F^\gamma
\]

\[
M^\Lambda_{\mu\nu\rho} = \frac{1}{16} \epsilon_{\mu\nu\rho\sigma} [F^a, F^\beta] - \eta_{[\mu|\rho][F_{\nu}], d} - 2\eta_{[\mu|\rho}\tilde{\nabla}_{\nu}d
\]

\[
N^\Lambda_{\mu\nu} = -\frac{3}{16} [F_{[\mu|\rho}], F_{\nu}] - \tilde{\nabla}_{[\mu} e_{\nu]} + \mathcal{F}_{\mu\nu}
\]

We recall that $\langle \cdots \rangle$ is the trace over products in $\mathfrak{sp}(M)$ and that we have the shorthand $\langle F^2 \rangle = F^\mu_A F^\mu_B$.

### 7.3.2 Clifford-trace of connection curvature

In [50] it is shown that on backgrounds without $R$-symmetry (the case $\rho = 0$ here), the vanishing of the Clifford-trace of $R^D_{\mu\nu}$ is sufficient for $D$ to be an admissible connection, in the language of Chapter 3 (see Definition 3.7). We will eventually show that the analogous result does not hold in the $\rho \neq 0$ case; indeed, we will see that not even the condition $R^D_{\mu\nu}$ will entirely suffice in this case. We have the following.

**Proposition 7.4.** Let $M$ be a 5-dimensional background with connection $D$ given by equation (7.65). The Clifford-trace of the curvature $\Gamma^\nu R^D_{\mu\nu}$ vanishes if and only if the following equations hold:

\[
R_{\mu\nu} + 4C_{\mu\alpha} C_{\nu}^\alpha + 8\varepsilon_{\mu\alpha} e_{\nu}^\alpha - 4\eta_{\mu\nu} e_{\alpha\beta} e^{\alpha\beta} + 8\langle d^2 \rangle \eta_{\mu\nu} + \frac{3}{8} (\langle F_{\mu} F_{\nu} \rangle - \langle F^2 \rangle \eta_{\mu\nu}) = 0,
\]

\[
\nabla^\alpha e_{\alpha\mu} - \frac{1}{3} \epsilon_{\mu\alpha\beta\gamma\delta} \left( C^{\alpha\beta} C_{\gamma\delta} - e^{\alpha\beta} e_{\gamma\delta} \right) = 0,
\]

\[
\nabla_{\mu} e_{\nu\rho} = 0,
\]

\[
\nabla_{\mu} C_{\nu\rho} - 2\nabla^e_{\nu\rho} + \frac{1}{2} \eta_{\mu|\nu} (F_{\rho}) d + \epsilon_{\nu\rho\alpha\beta\gamma} C_{\mu}^{\alpha} e_{\gamma\delta} + \frac{1}{3} \eta_{\mu|\nu} \varepsilon_{\rho|\alpha\beta\gamma\delta} \left( C^{\alpha\beta} C_{\gamma\delta} - e^{\alpha\beta} e_{\gamma\delta} \right) = 0,
\]

and, with values in $\mathfrak{sp}(M)$ (omitting indices),

\[
[F_{\mu}, d] + e_{\mu\alpha} F^\alpha = 0,
\]
\[ \nabla_\mu F_\nu + 8(C_{\mu\nu} - 2e_{\mu\nu})d + \frac{1}{3}\epsilon_{\mu\nu\alpha\beta\gamma}(2C^{\alpha\beta} - e^{\alpha\beta})F_\gamma = 0, \]  
(7.86)
\[
\frac{1}{2}\epsilon_{\mu\nu\rho\delta}\left(\mathcal{F}^{\alpha\beta} + \frac{3}{16}\left[F^\alpha, F^\beta\right] + 8C^{\alpha\beta}d - 4e^{\alpha\beta}d\right) + (C_{\mu\nu} - 2e_{\mu\nu})F_\rho - 2\eta_{\rho\mu}\hat{\nabla}_\nu d = 0. \]  
(7.87)

We omit the details of the calculation; the method is similar to the analogous calculation in [50]. Indeed, by setting \( C = 2e \) and \( d = 0 \) (in other words, \( \rho = 0 \) as well as \( \mathcal{F} = 0 \), we recover the same set of equations as in loc. cit., and as observed there, if we also have \( F = 0 \) then we recover the bosonic equations of motion for minimal \( D = 5 \) supergravity, where we identify \( C \) as a rescaling of the 2-form field strength along with the Bianchi identity \( dC = 0 \).

Note that equations (7.82)-(7.84) together imply
\[
\nabla^\alpha C_{\alpha\mu} - \epsilon_{\alpha\mu\beta\gamma\delta}C^{\alpha\beta}e^{\gamma\delta} + \langle F_\mu, d \rangle = 0, \]  
(7.88)
\[
\nabla_{[\mu} C_{\nu\rho]} - \frac{2}{3}\epsilon_{\mu\nu\rho\alpha\beta}C_\gamma^{\alpha\beta}e^{\beta\gamma} = 0, \]  
(7.89)
and equations (7.85)-(7.87) imply
\[
4\hat{\nabla}_\mu d + C_{\mu\alpha}F^\alpha + 2[F_\mu, d] = 0, \]  
(7.90)
\[
\hat{\nabla}_\mu F_\nu - 4\mathcal{F}_{\mu\nu} - 24C_{\mu\nu}d + \epsilon_{\mu\nu\rho\alpha\beta}e^{\alpha\beta}F_\gamma - \frac{3}{4}[F_\mu, F_\nu] = 0. \]  
(7.91)

Using the contracted differential Bianchi identity \( \nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla_\nu R = 0 \) (where \( R \) is the Ricci scalar), equation (7.81) implies an additional constraint. Tracing (7.81) gives
\[
R + 8\|C\|^2 - 24\|e\|^2 + 40\langle d^2 \rangle - \frac{3}{2}\langle F^2 \rangle = 0 \]  
(7.92)
where \( \| - \|^2 \) denotes the standard norm on forms induced by the metric; locally, we have \( \|C\|^2 = \frac{1}{2}C_{\alpha\beta}C^{\alpha\beta} \) and \( \|e\|^2 = \frac{1}{2}e_{\alpha\beta}e^{\alpha\beta} \). We can then use some index combinatorics to write
\[
\nabla^\mu\left(4C_{\mu\alpha}C_\nu^\alpha + 8e_{\mu\alpha}e_\alpha^\nu\right) = 2\nabla_\nu(\|C\|^2 + 2\|e\|^2) + 4C_\nu^\alpha\nabla^\mu C_{\mu\alpha} + 8e_\nu^\alpha\nabla^\mu e_{\mu\alpha} - 6C^{\alpha\beta}\nabla_\nu C_{\alpha\beta} - 12e^{\alpha\beta}\nabla_\nu e_{\alpha\beta} \]  
(7.93)
so the contracted differential Bianchi identity is
\[
2\nabla_\nu(\|C\|^2 - 4\|e\|^2) + 4C_\nu^\alpha\nabla^\mu C_{\mu\alpha} + 8e_\nu^\alpha\nabla^\mu e_{\mu\alpha} - 6C^{\alpha\beta}\nabla_\nu C_{\alpha\beta} - 12e^{\alpha\beta}\nabla_\nu e_{\alpha\beta} - 12\nabla_\nu(\langle d^2 \rangle + \frac{3}{8}\langle \nabla^\mu (F_\mu F_\nu) + \nabla_\nu (F^2) \rangle) = 0. \]  
(7.94)

Using our expressions for derivatives of the background fields above and some combinatorial identities, the last expression above can be rendered as
\[
2\nabla_\nu(\|C\|^2 - 4\|e\|^2) + \frac{1}{3}e^{\alpha\beta\gamma\delta}e_{\nu\alpha}C_{\beta\gamma}C_{\delta\epsilon} + 3C_{\sigma\alpha}\langle F^\alpha d \rangle = 0. \]  
(7.95)

Note that in the \( \rho = 0 \) case, \( C = 2e \) and \( d = 0 \), so this is trivially satisfied.
7.3.3 Flatness of the connection and maximal supersymmetry

Flatness of the connection $D$ is a necessary condition for maximal supersymmetry; if the maximal number of Killing spinors is attained, the values of $D$-parallel spinors span the fibre of $\mathcal{S}$ at every point, and since $R^D$ annihilates Killing spinors, it must annihilate all spinors at every point, thus it must vanish. The following is obtained by setting the expressions (7.75)-(7.80) to zero and then performing some simplifications.

**Proposition 7.5.** Let $M$ be a 5-dimensional background with connection $D$ given by Equation (7.65). Then $R^D = 0$ if and only if the following equations hold.

\[
\begin{align*}
\nabla_\mu C_{\nu\rho} + \frac{1}{2} \eta_{[\nu} C_{\rho]}^\alpha F_{\alpha}^\mu d + \epsilon_{\nu\rho\alpha} b_\gamma C^\alpha_{\mu \gamma} e^{\beta\gamma} &= 0, \\
\nabla_\mu e_{\nu\rho} - \frac{1}{6} \eta_{[\nu} C_{\rho]}^\alpha C_\mu^{\gamma\delta} e^{\alpha\beta} C^{\gamma\delta} &= 0, \\
C_{\mu\nu} F_\rho - 2\eta_{\rho[\mu} F_{\nu]} d - \eta_{\rho[\nu} F_{\mu]} d - \epsilon_{\nu\rho\mu\alpha} [2 e^{\alpha\beta} d + \frac{1}{10} \left[ F^\alpha, F^\beta \right]] &= 0, \\
4 e_{\nu\mu} F_\rho - 2\eta_{\rho[\mu} [F_{\nu]} d] - \epsilon_{\nu\mu\rho\alpha} \left[ \mathcal{F}^\alpha_{\mu} + \frac{5}{18} \left[ F^\alpha, F^\beta \right] + 8 C^{\alpha\beta} d \right] &= 0, \\
R^{\mu
u\rho\sigma} &= \frac{8}{3} \left( C^{\mu\nu} C_{\rho\sigma} + C_\mu^{[\rho} C_\nu^{\sigma]} - e^{\mu\nu} e_{\rho\sigma} - e_{\rho[\mu} e^{\sigma]} \right) \\
&+ 16 \delta^{[\mu}_{\rho} e^{\nu]\sigma] e_{\sigma\alpha} - 4 e^{[\mu}_{\rho} e^{\nu\sigma]} e_{\alpha\beta} e^{\sigma]) \\
&+ \frac{1}{4} \left( 2 \delta^{[\mu}_{\rho} (F^{\nu}] e_{\sigma]} - (F^2) \delta^{[\mu}_{\rho} e^{\nu\sigma]} e_{\sigma\alpha} + 4 (d^2) \delta^{[\mu}_{\rho} e^{\nu\sigma]} e_{\sigma\alpha} \right),
\end{align*}
\]

where $R_{\mu\nu\rho\sigma}$ are the components of the Riemann tensor.

As in the discussion of the Clifford-trace conditions above, there is an additional condition implicit in the compatibility of the equations above with the differential Bianchi identity. Applying the covariant derivative to equation (7.101) side of the equation and replacing derivatives of the background fields using the first four equations above, one obtains an algebraic equation which is cubic in the background fields. Indeed, we find that

\[
\nabla_\lambda R^{\rho\sigma}_{\mu\nu} = -\frac{8}{3} \left( \epsilon_{\alpha\beta} C_{\mu\nu}^{\rho\sigma} C_{\lambda}^{\alpha\beta} + \epsilon_{\alpha\beta} C_{\mu\nu}^{\rho\sigma} C_{\lambda}^{\alpha\beta} - 2 \epsilon_{\alpha\beta} C_{\mu\nu}^{\rho\sigma} C_{\lambda}^{\alpha\beta} \right) e^{\rho\sigma}
\]

\[
+ \frac{2}{3} \left( \epsilon_{\alpha\beta} C_{\mu\nu}^{\rho\sigma} C_{\lambda}^{\alpha\beta} + \epsilon_{\alpha\beta} C_{\mu\nu}^{\rho\sigma} C_{\lambda}^{\alpha\beta} \right) \left( C_{\gamma\delta} - e^{\alpha\beta} e^{\gamma\delta} \right)
\]

\[
+ \frac{8}{3} \left( \delta^{[\rho}_{\lambda} C_{\mu\nu]} (F^{\sigma]} d) + \delta^{[\rho}_{\lambda} C_{\mu\nu]} (F^{\sigma]} d) \right)
\]

\[
- \frac{1}{4} \left( \delta^{[\rho}_{\lambda} C_{\mu\nu]} (F^{\sigma]} d) + \delta^{[\rho}_{\lambda} C_{\mu\nu]} (F^{\sigma]} d) \right)
\]

Clearly this is not in its simplest form. Let us first contract this with $\delta^{[\rho}_{\mu} e^{\sigma]}$ to obtain the constraint coming from the contracted Bianchi identity $\nabla_\alpha R_{\alpha\lambda} - \frac{1}{2} \nabla_\lambda R = 0$; doing so gives us simply

\[
C_{\lambda\alpha} (F^{\rho\sigma} d) = 0.
\]

Alternatively, one can show that $\nabla_\lambda (\|C\|^2 - 4 \|e\|^2) = -\frac{1}{6} \epsilon_{\alpha\beta} C_{\gamma\delta} e_{\lambda\alpha} C_{\gamma\delta} C_{\lambda\alpha}$, so our previous expression for the contracted Bianchi identity (7.95) reduces to the above. Note
that in obtaining this expression, we have used the fact that certain terms such as \( e^\alpha \beta \delta \epsilon \alpha \epsilon \beta \epsilon \delta \) and \( \langle \{ F_\lambda, d \} d \rangle \) vanish identically. Pairing equation (7.99) with \( \langle -d \rangle \) and contracting two indices, the contracted Bianchi identity above gives

\[
\nabla_\lambda \langle d^2 \rangle = 0,
\]

whence (7.99) gives us \( C_{\mu \nu} \langle F_\rho d \rangle = C_{[\mu \nu} \langle F_\rho] d \rangle \), so the expression in the second set of square brackets in (7.102) now vanishes identically. Equation (7.100) allows us to write

\[
\epsilon_{\mu \nu \alpha \beta} \epsilon^{\alpha \beta} \langle F^\gamma F_\rho \rangle = 12 \left( \langle \mathcal{F}_{\mu \nu} F_\rho \rangle + \frac{5}{16} \left( \left[ F^\alpha, F^\beta \right] F_\rho \right) + 8 C^{\alpha \beta} \langle F_\rho d \rangle \right). \tag{7.105}
\]

The second term on the right-hand side vanishes identically, so using \( C_{\mu \nu} \langle F_\rho d \rangle = C_{[\mu \nu} \langle F_\rho] d \rangle \) again we find

\[
\epsilon_{\mu \nu \alpha \beta} \epsilon^{\alpha \beta} \langle F^\gamma F_\rho \rangle + \epsilon_{\alpha \beta \gamma \rho \mu} \epsilon^{\alpha \beta} \langle F^\gamma F_\nu \rangle = \frac{3}{2} \left( \epsilon_{\mu \nu \alpha \beta} \epsilon^{\alpha \beta} \langle F^\gamma F_\rho \rangle - \epsilon_{\alpha \beta \gamma \rho \mu} \epsilon^{\alpha \beta} \langle F^\gamma F_\nu \rangle \right) \tag{7.106}
\]

where we have used equation (7.98) in the last line.

To make further progress, we contract the full expression (7.102) with a Levi-Civita symbol and obtain, after a somewhat lengthy calculation using combinatorial identities from Appendix 7.A which we will omit the details of,

\[
e^{\mu \nu \alpha \beta \gamma} \nabla_\alpha R_{\beta \gamma \sigma \tau} = \frac{54}{3} \left[ C_{\sigma \tau} C_{[\mu} [\alpha \epsilon \nu] e^{\nu] \alpha} + C_{[\sigma} [\mu C_{\tau] \alpha} e^{\nu] \alpha} - 4 e_{[\sigma} [\mu e_{\nu]} e^{\nu] \alpha} - 4 e_{[\sigma} [\mu e_{\nu]} e^{\nu] \alpha}
\]

\[
+ \frac{1}{3} \delta^{[\mu}_{[\sigma} C^{\nu] [\alpha \epsilon \nu] e^{\nu] \alpha} + 3 e^{\nu] \alpha} C_{\tau] \beta \epsilon \alpha \beta}
\]

\[
+ 8 e^{\nu] \alpha} e_{\tau]} e_{\alpha \beta} + 2 C^{[\mu}_{\tau] \langle e, C \rangle - 8 e_{[\mu} [\nu \parallel e \parallel^2] \right) \tag{7.107}
\]

where \( (e, C) = \frac{1}{2} e^{\alpha \beta} C_{\alpha \beta} \) is the pairing on differential forms induced by the metric and \( \parallel e \parallel^2 \) the induced square-norm of \( e \). There is no clear way to further simplify the expression above. However, by imposing some quadratic constraints suggested by the integrability conditions which arose in our analysis of maximally supersymmetric filtered subformations of the Poincaré superalgebra in §7.1 (see Theorem 7.2), the condition simplifies significantly. We will also find that the same condition is necessary for the existence of a maximally supersymmetric Killing superalgebra in §7.4.2.

**Lemma 7.6.** If \( C \wedge C = 4 e \wedge e \) and \( C_{[\mu} e^{\nu] \alpha} = 0 \) then the differential Bianchi constraint is \( \hat{\nabla}_\lambda \langle F_\mu F_\nu \rangle = 0 \).

**Proof.** The terms involving \( C \) and \( E \) in equation (7.107) can be written in the follow-
We recall that the Dirac bilinears are covariantly constant, invariant under the action we also have
\[ b^{C} \frac{\partial}{\partial x} \] which manifestly vanishes if \( C \) and \( e \) satisfy the conditions given. The expression computed in (7.106) must therefore vanish, i.e. \( \langle (\widehat{V}_{\mu} F_{\nu}) F_{\rho} \rangle \) is totally skew-symmetric in \( \mu, \nu, \rho \), so
\[ \nabla_{\lambda} \langle F_{\mu} F_{\nu} \rangle = \langle (\widehat{V}_{\lambda} F_{\mu}) F_{\nu} \rangle + \langle F_{\mu} (\widehat{V}_{\lambda} F_{\nu}) \rangle = \langle (\widehat{V}_{\lambda} F_{\mu}) F_{\nu} \rangle + \langle (\widehat{V}_{\lambda} F_{\nu}) F_{\mu} \rangle = 0. \] Conversely, \( \nabla_{\lambda} \langle F_{\mu} F_{\nu} \rangle = 0 \) implies that \( \langle (\widehat{V}_{\lambda} F_{\mu}) F_{\nu} \rangle = -\langle (\widehat{V}_{\lambda} F_{\nu}) F_{\mu} \rangle \), and recalling that we also have \( \widehat{V}_{\lambda} F_{\mu} = -\widehat{V}_{\lambda} F_{\mu} \), we see that \( \langle (\widehat{V}_{\lambda} F_{\mu}) F_{\nu} \rangle \) is totally skew-symmetric. □

Note that by the lemma, the differential Bianchi constraint is automatically satisfied in the gauged supergravity theory discussed in §7.2.2 since there we have \( C = 2e \) and \( F = 0 \).

7.4 The Killing superalgebra

7.4.1 Properties of Killing spinors

Dirac bilinears

We recall that the Dirac bilinears are covariantly constant, invariant under the action of \( \mathfrak{so}(M) \) and \( \mathfrak{sp}(M) \) and thus also constant with respect to the covariant Lie derivative. For symplectic Majorana spinors, the norm \( \mu \) and Dirac current \( \kappa \) are real and 2-form \( \omega \) takes values in \( \text{ad} Q \).

We briefly note that for the Dirac current, \( \widehat{V} \kappa = 0 \) is equivalent to
\[ \widehat{V}_{X} (\kappa(\epsilon, \zeta)) = \kappa (\widehat{V}_{X} \epsilon, \zeta) + \kappa (\epsilon, \widehat{V}_{X} \zeta) \] for all \( \epsilon, \zeta \in \widehat{S} \) and \( X \in \mathfrak{X} \). Now note that for Killing spinors \( \epsilon, \zeta \in \widehat{S}_{D} \), we have
\[ \widehat{V}_{X} (\kappa(\epsilon, \zeta)) = \kappa (\beta_{X} \epsilon, \zeta) + \kappa (\epsilon, \beta_{X} \zeta) \] and this and similar expressions allow us to calculate the derivatives of the Dirac bilinears as follows. Once again, we omit the details of the calculations since they are similar to those in [50].

**Proposition 7.7.** Let \( s \in \widehat{S} \) be a Killing spinor field on a background, and for notational convenience fix \( \mu, \kappa, \omega \) to be the Dirac bilinears of \( s \). Then the following identities hold:
\[ \nabla_{\mu} \mu = 2C_{\mu \alpha} \kappa^{\alpha} + \frac{1}{2} F_{\alpha}^{\alpha} \omega_{\mu\alpha}^{AB}, \] \[ \nabla_{\mu} \kappa_{\nu} = \gamma_{\mu\nu}(s, s) \] \[ = 2C_{\mu \nu} \mu + \epsilon_{\mu \nu \alpha \beta} \gamma^{\alpha \beta} \kappa^{\gamma} + \frac{1}{4} \epsilon_{\mu \nu \alpha \beta} F_{AB}^{\alpha \beta} \omega^{\gamma AB} - 2d_{AB} \omega^{AB}_{\mu \nu}, \]
We now consider the conditions for the Killing superalgebra of the supersymmetric background data. For such a background, the components spaces of the Killing superalgebra to exists; that is, we ask whether (\(D, \rho\)) is an admissible pair (Definition 4.17). For such a background, the components spaces of the Killing superalgebra \(T_D\) are as follows. The odd part is of course the space of \(D\)-parallel (symplectic Majorana) spinors

\[
\widehat{\mathfrak{p}}(\omega^n_{\nu\rho}) = \frac{1}{2}\eta_{\nu\rho}[F_{\nu\rho}]^A_{BC} \mu + 2\eta_{\nu\rho} d^{AB}\kappa_\rho - \frac{1}{4}\epsilon_{\nu\rho\alpha\beta} F^{\alpha AB}\kappa^\beta
\]

(7.114)

and in particular

\[
\widehat{\mathfrak{n}}^\nu \omega^n_{\nu\rho} = \epsilon_{\nu\rho\alpha\beta}(\mathfrak{c}^{\alpha\beta} - e^{\alpha\beta}) \omega^{\nu\rho\alpha\beta} + F_{\nu\rho} \mu - \frac{1}{2} F^{\nu\rho} (A C, \omega^B_{\rho\alpha\beta} - \epsilon_{\nu\rho\alpha\beta} d(A C, \omega^B_{\rho\alpha\beta} + 4 d^{\nu AB}\kappa_\rho),
\]

(7.115)

\[
\widehat{\mathfrak{m}}(\omega^n_{\nu\rho}) = -\frac{1}{3}\epsilon_{\nu\rho\alpha\beta} F^{\alpha AB}\kappa^\beta.
\]

(7.116)

The maps \(\gamma\) and \(\rho\)

We define the maps \(\gamma : \circ^2 \mathcal{S} \to \text{ad} F_{SO}\) and \(\rho : \circ^2 \mathcal{S} \to \text{ad} Q\) appearing in the construction of the Killing superalgebra by

\[
\gamma(s, s) = 2\mu C + \frac{1}{2} (e \wedge \kappa) - 2\omega d + \frac{1}{2} (e \wedge F),
\]

(7.117)

\[
\rho(s, s) = 2(C - 2e, \omega) - 3\mu d,
\]

(7.118)

or, more practically, via the local expressions

\[
\gamma(s, s)_{\mu\nu} = 2\mu C_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} e^{\rho\sigma} \kappa^\tau - 2d_{AB} \omega^A_{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu} \omega^A_{\rho\sigma} T_{AB} \omega^B_{\rho\sigma},
\]

(7.119)

\[
\rho(s, s)_{AB} = (C_{\mu\nu} - 2e_{\mu\nu}) \omega^A_{\mu\nu} - 3\mu d_{AB}.
\]

(7.120)

### 7.4.2 Admissibility of the connection

We now consider the conditions for the Killing superalgebra of the supersymmetric background \(M\) to exists; that is, we ask whether \((D, \rho)\) is an admissible pair (Definition 4.17). For such a background, the components spaces of the Killing superalgebra \(T_D\) are as follows. The odd part is of course the space of \(D\)-parallel (symplectic Majorana) spinors

\[
\widehat{\mathfrak{g}}_D = \{e \in \widehat{\mathfrak{g}} | D e = 0\}.
\]

(7.121)

The even part consists of the space of Killing spinors which preserve all the background data

\[
\widehat{\mathfrak{h}}_D = \{X \in \mathfrak{X}(M) | \mathcal{L}_X g = \mathcal{L}_X C = \mathcal{L}_X e = \mathcal{L}_X d = \mathcal{L}_X F = 0 \text{ and } \iota_X \mathcal{F} = 0\},
\]

(7.122)

and the space of infinitesimal \(R\)-symmetries which preserve the background,

\[
\mathfrak{R}_D = \{a \in \mathfrak{sp}(M) \mid \mathcal{L}_a = 0, [a, d] = 0 \text{ and } [a, F] = 0\},
\]

(7.123)

and we note that \(\mathcal{L}_X \beta = 0\) and \(\mathcal{L}_X \rho = 0\) for all \(X \in \widehat{\mathfrak{h}}_D\) and that \([a, \beta] = 0\) and \([a, \rho] = 0\) for all \(a \in \mathfrak{R}_D\). The conditions for admissibility of \((D, \rho)\) which are not identically satisfied by \((\beta, \gamma, \rho)\) being a Spencer cocycle are as follows: for all \(e \in \widehat{\mathfrak{g}}_D\),

\[
\mathcal{L}_e C = 0, \quad \mathcal{L}_e e = 0, \quad \mathcal{L}_e d = 0, \quad \mathcal{L}_e F = 0, \quad \iota_e \mathcal{F} = 0,
\]

(7.124)
\( \hat{\nabla}(\rho(\epsilon,\epsilon)) = 0, \quad [\rho(\epsilon,\epsilon), d] = 0, \quad [\rho(\epsilon,\epsilon), F] = 0. \) (7.125)

By definition of the covariant derivative \( \hat{\nabla} \) and \( \gamma \), for all \( \epsilon \in \mathfrak{S}_D \),

\[
\hat{\nabla}_{\kappa} \rho = \hat{\nabla}_{\kappa} + \gamma(\epsilon,\epsilon)
\]

(7.126)
as endomorphisms of differential forms with values in \( R \)-symmetry representations, so we can compute the Lie derivatives of the background fields as follows:

\[
\mathcal{L}_\kappa C_{\mu\nu} = \kappa^\lambda \nabla_\lambda C_{\mu\nu} - 2\gamma(\epsilon,\epsilon) \epsilon_{[\mu} C_{\nu]\alpha} a^\alpha C_{\nu]a}
\]

\[
= \left( \nabla_\lambda C_{\mu\nu} - 2\epsilon_{\alpha\beta\gamma[\mu} C_{\nu]\alpha} a^\beta C_{\nu]a} \right) \kappa^\lambda
\]

\[
- \frac{1}{2} \epsilon_{\alpha\beta\gamma[\mu} e_{\nu]\alpha} e_{\beta\gamma} e_{\nu]\alpha} a^\alpha C_{\nu]a} + 4d_{AB} \omega_{[\mu} a^{AB} C_{\nu]a}, \quad \gamma(\epsilon,\epsilon) \mu \epsilon_{\nu]\alpha} a^\alpha C_{\nu]a} ,
\]

(7.127)

\[
\mathcal{L}_\kappa e_{\nu\alpha} = \kappa^\lambda \nabla_\lambda e_{\nu\alpha} - 2\gamma(\epsilon,\epsilon) \epsilon_{\mu\nu}\alpha \epsilon_{\nu\alpha} e_{\mu\nu}\alpha}
\]

\[
= - 4C_{[\mu} a^\alpha e_{\nu]\alpha} \mu \mu + \left( \nabla_\lambda e_{\nu\alpha} - 2\epsilon_{\alpha\beta\gamma[\mu} C_{\nu]\alpha} a^\beta C_{\nu]a} \right) \kappa^\lambda
\]

\[
- \frac{1}{2} \epsilon_{\alpha\beta\gamma[\mu} e_{\nu]\alpha} e_{\beta\gamma} e_{\nu]\alpha} a^\alpha C_{\nu]a} + 4d_{AB} \omega_{[\mu} a^{AB} C_{\nu]a},
\]

(7.128)

\[
\hat{\nabla}_\kappa d_{AB} = \kappa^\lambda \hat{\nabla}_\lambda d_{AB},
\]

(7.129)

\[
\hat{\nabla}_\kappa F_{\mu AB} = \kappa^\lambda \hat{\nabla}_\lambda F_{\mu AB} + \gamma(\epsilon,\epsilon) \mu F_{\mu AB}
\]

\[
= 2C_{[\mu} a^{AB} \mu + \left( \hat{\nabla}_\lambda F_{\mu AB} + \epsilon_{\lambda\alpha\beta\gamma} a^\alpha B_{\gamma} F_{\mu AB} \right) \kappa^\lambda
\]

\[
+ \frac{1}{4} \epsilon_{\mu\nu\alpha\beta\gamma} F_{\mu AB} a^{CD} F_{\gamma}^{\alpha} d_{CD} \omega_{[\mu} a^{CD},
\]

(7.130)
The commutators of the \( \epsilon \)-valued background fields with \( \rho \) are

\[
[\rho(\epsilon,\epsilon), d] = (C_{\mu\nu} - 2\epsilon_{\mu\nu}) [\omega^{\mu\nu}, d],
\]

(7.131)

\[
[\rho(\epsilon,\epsilon), F] = (C_{\mu\nu} - 2\epsilon_{\mu\nu}) [\omega^{\mu\nu}, F] - 3\mu [d, F],
\]

(7.132)
and the covariant derivative of \( \rho \) can be computed directly as

\[
\hat{\nabla}_\lambda \rho(\epsilon,\epsilon)
\]

\[
= (\hat{\nabla}_\lambda C_{\mu\nu} - 2\epsilon_{\mu\nu}) \omega^{\mu\nu} + (C_{\mu\nu} - 2\epsilon_{\mu\nu}) \hat{\nabla}_\lambda \omega^{\mu\nu} - 3(\hat{\nabla}_\lambda \mu) d - 3\mu \hat{\nabla}_\lambda d
\]

\[
= (\nabla_\lambda C_{\mu\nu} - 2\epsilon_{\mu\nu}) \omega^{\mu\nu} - \frac{3}{2} F_{\alpha}^{\alpha} a^{\alpha} \omega_{\alpha}^{\alpha} A B d
\]

\[
+ (C_{\mu\nu} - 2\epsilon_{\mu\nu}) \left[ \epsilon_{\mu\nu\alpha\beta\gamma} C_{\alpha}^{\lambda} \omega^{\beta\gamma} + 2\epsilon_{\beta\gamma\alpha\mu\lambda} e_{\beta\gamma} e_{\mu\nu} + \frac{3}{4} \epsilon_{\lambda\alpha\beta\gamma} a^{\beta} \left[ d, \omega_{\alpha\beta} \right]
\]

\[
+ \frac{3}{4} \left( F_{\lambda\mu}, \omega_{\nu\mu} \right) - \frac{3}{4} \left( F_{\lambda\mu}, \omega_{\nu\mu} \right) + \frac{1}{2} \eta_{\lambda\mu} \left( F_{\alpha}^{\alpha}, \omega_{[\nu]} \right)
\]

\[
+ \left( -6C_{\lambda\alpha} d + 2(C_{\lambda\alpha} - 2\epsilon_{\lambda\alpha}) d + \frac{1}{4} \epsilon_{\lambda\alpha\beta\gamma} (C^{\beta\gamma} - 2\epsilon^{\beta\gamma}) F_{\gamma}^{\alpha} \right) \kappa^{\alpha}
\]

\[
+ (-3 \hat{\nabla}_\lambda d + \frac{1}{2} \left( C_{\lambda\alpha} - 2\epsilon_{\lambda\alpha} \right) F_{\alpha}^{\lambda}) \mu.
\]

(7.133)

For admissibility of \((D,\rho)\), all of the above must vanish. It is not feasible to completely classify all backgrounds admitting superalgebras from these expressions directly, but in the maximally supersymmetric case (where \( \text{dim} \hat{\mathfrak{S}}_D = \text{dim} S = 8 \)) it is possible to reduce the conditions to a more manageable set of constraints on the background fields which can then be solved.
Proposition 7.8. In the maximally supersymmetric case, the conditions (7.125) hold if and only if \( \mathcal{F} = 0 \) and one of the following holds:

1. \( F = 0, \ d = 0, \) and \( C \) and \( e \) satisfy the algebraic equations

\[
[A_C, A_e] = 0, \quad (7.134)
\]
\[
C \land C - 4 e \land e = 0, \quad (7.135)
\]

where \( A_C, A_e \in \mathfrak{so}(M) \) are the skew-symmetric endomorphisms corresponding to the 2-forms \( C, e, \) and the differential equations

\[
\nabla_X C = -2 * (e \land i_X C), \quad (7.136)
\]
\[
\nabla_X e = X^\flat \land * (e \land e), \quad (7.137)
\]

for all \( X \in \mathfrak{X}(M); \)

2. \( d = 0, \ C = e = 0, \) \( [F_\mu, F_\nu] = 0 \) and \( \hat{\nabla} F = 0; \)

3. \( F = 0, \ C = e = 0 \) and \( \hat{\nabla} d = 0. \)

Maximal supersymmetry also requires as an integrability condition that the connection \( D \) is flat. The first five equations listed in Proposition 7.5 are satisfied in any of the three cases listed in the lemma above (for \( \mathcal{F} = 0 \)), leaving only the final equation which gives us

\[
R_{\mu\nu\rho\sigma} = \begin{cases} 
\frac{5}{2} (C_{\mu\nu}C_{\rho\sigma} + C_{\mu|\rho}C_{|\nu|\sigma} - e_{\mu\nu}e_{\rho\sigma} - e_{\mu|\rho}e_{|\nu|\sigma}) \\
+ 8 \left( \eta_{\mu|\rho}e_{|\nu|}\alpha e_{\sigma|\alpha} - \eta_{\nu|\rho}e_{|\mu|}\alpha e_{\sigma|\alpha} \right) - 4 \eta_{\mu|\rho}\eta_{|\nu|\sigma}e_{\alpha\beta}e_{\alpha\beta} & \text{in case (1)}, \\
\frac{1}{4} \left( \eta_{\mu|\rho}\langle F_{|\nu|}F_{\sigma|} \rangle - \eta_{\nu|\rho}\langle F_{|\sigma|}F_{\rho|} \rangle - \langle F^2 \rangle \eta_{\mu|\rho}\eta_{|\nu|\sigma} \right) & \text{in case (2)}, \\
4 \langle d^2 \rangle \eta_{\mu|\rho}\eta_{|\nu|\sigma} & \text{in case (3)}. 
\end{cases}
\]

Furthermore, by Lemma 7.6 there is no additional constraint arising from the differential Bianchi identity in any of the three cases.

In Case (1), we note that since \( C \land C = 4 e \land e \), we have

\[
C_{\mu\nu}C_{\rho\sigma} + C_{\mu|\rho}C_{|\nu|\sigma} - e_{\mu\nu}e_{\rho\sigma} - e_{\mu|\rho}e_{|\nu|\sigma} = \frac{3}{2} (C_{\mu\nu}C_{\rho\sigma} - e_{\mu\nu}e_{\rho\sigma} - 3 e_{\mu\nu}e_{\rho\sigma}), \quad (7.139)
\]

so the formula for the Riemann curvature above is the same as that for \( \theta_1 \) in our discussion of filtered deformations (equation (7.44)). We also note that the algebraic conditions on the background data derived in that discussion are exactly the ones which appear here; the only extra conditions here are the differential ones.

Remark 12. In the minimal 4-dimensional case, there is no obstruction to the existence of the Killing superalgebra for any choice of background fields and connection defined using a Spencer cocycle [48]. In the 11-dimensional [45] (at least in the case of high supersymmetry) and minimal 5-dimensional cases without \( R \)-symmetry [50], there is an obstruction but the condition \( \Gamma^\mu R^D_\mu = 0 \) suffices to remove it; indeed, that condition is much stronger than required. However, comparing Propositions 7.4 and 7.8, we see that it is not sufficient here; indeed, it appears that even flatness of \( D \) is
not sufficient (compare Propositions 7.4 and Proposition 7.5), although we note that
the integrability condition arising from the Bianchi identity (7.107) has not been fully
analysed.

**Proof of Proposition 7.8.** We must check the conditions (7.125) in the case where
$D$ has the maximal number of parallel spinors. The Homogeneity Theorem (The-
orem 3.21) implies that the Dirac currents of Killing spinors span the tangent space
at every point, hence $\kappa, \mathcal{F} = 0$ for all Killing spinors $\epsilon$ if and only if $\mathcal{F} = 0$. For the
expressions (7.127)-(7.133) – all of which are linear combinations of Dirac bilinears
of Killing spinors – to vanish for all Killing spinors $\epsilon$, the coefficient of each bilinear
in each expression must vanish. Starting with the Lie derivatives, the $\mu$ terms vanish if and only if
\[ C_{[\mu}^a e_{\nu]a} = 0, \quad \text{and} \quad C_{\mu a} F_{AB}^a = 0; \] (7.140)
the first of which is just a local expression for (7.134). The $\kappa$ terms vanish if and
only if all of the following hold (using some of the combinatorial identities from
Appendix 7.A)
\[ \nabla_\lambda C_{\mu \nu} + \epsilon_{\mu \nu a \beta \gamma} C^a_{\lambda} e^{\beta \gamma} + 2\epsilon_{\lambda \mu \nu a \beta \gamma} C^a_{\tau} e^{\beta \gamma} = 0, \] (7.141)
\[ \nabla_\lambda e_{\mu \nu} - \frac{1}{2} \eta_{\lambda \mu} e_{\nu a \beta \gamma} e^{a \beta \gamma} e^{\rho \delta} = 0, \] (7.142)
\[ \nabla_\lambda d = 0, \] (7.143)
\[ \nabla_\lambda F_{\mu A B} + \epsilon_{\lambda \mu a \beta \gamma} e^{a \beta \gamma} F_{AB}^\gamma = 0, \] (7.144)
where the crossed-out term vanishes by the first of the $\mu$ equations. The first two of
these are local expressions for (7.136) and (7.137) respectively. The $\omega$ terms are more
complicated, but we will see that they all vanish identically after imposing some of
the other conditions.

We have two cases where $[\rho(\epsilon, \epsilon), d] = [\rho(\epsilon, \epsilon), F] = 0$:
\[ C = 2e \quad \text{and} \quad [d, F] = 0 \quad \text{or} \quad d = F = 0. \] (7.145)
In the former case, many of the terms in $\nabla_\lambda \rho(\epsilon, \epsilon)$ vanish identically, and the remain-
ing terms vanish if and only if
\[ d = 0 \quad \text{or} \quad C = e = F = 0 \text{ and } \nabla_\lambda d = 0. \] (7.146)
On the $C \neq 2e$, $F = d = 0$ branch, once again most of the terms in $\nabla_\lambda \rho(\epsilon, \epsilon)$ vanish identically leaving only
\[ \nabla_\lambda C_{\rho a} - 2\nabla_\lambda e_{\rho a} + \epsilon_{\rho a \beta \gamma} C_{\lambda}^a (C_{\beta \gamma} - 2e^{\beta \gamma}) + 2\epsilon_{\lambda \alpha \beta \gamma} (C_{\alpha \gamma} - 2e^{\alpha \gamma})(C_{\beta \gamma} - 2e_{\beta \gamma}) = 0. \] (7.147)
which, using identities from Appendix 7.A and some convenient relabelling of indices
and grouping of terms, can be rewritten as
\[ \nabla_\lambda C_{\mu \nu} + \epsilon_{\mu \nu a \beta \gamma} C_{\lambda}^a e^{\beta \gamma} - 2\nabla_\lambda e_{\mu \nu} \\
+ \frac{1}{2} \eta_{\lambda \mu} e_{\nu a \beta \gamma} (C_{\alpha \beta} C_{\gamma \delta} - 2e^{\alpha \beta} e^{\gamma \delta}) - 2\epsilon_{\lambda \mu a \beta \gamma} C_{\gamma \sigma} e^{\beta \gamma} = 0; \] (7.148)
then, using and equations (7.140), (7.141) and (7.142), we are left with

\[ \eta_{\lambda\mu} \epsilon_{\nu[\alpha\beta\gamma\delta} (C^{\alpha\beta} C^{\gamma\delta} - 4 e^{\alpha\beta} e^{\gamma\delta}) = 0 \] (7.149)

which is a local expression equivalent to (7.135).

We now return finally to the \( \omega \) terms in the Lie derivatives. We must have either \( d = 0 \) or \( C = e = F = 0 \); in the latter case all \( \omega \) terms now vanish identically, while in the former the remaining terms vanish if and only if

\[ \epsilon_{\alpha\beta\gamma\delta} [\mu \epsilon_{\nu]} a F_{AB}^{\beta} = 0, \] (7.150)

\[ \epsilon_{\alpha\beta\gamma\delta} [\mu \epsilon_{\nu]} a F_{AB}^{\beta} = 0, \] (7.151)

\[ \epsilon_{\mu\alpha\beta\gamma\delta} F^{\alpha}_{AB} F^{\beta}_{CD} = 0. \] (7.152)

Clearly these are satisfied if \( F = 0 \). If \( F \neq 0 \), one can show that the final equation above is equivalent to \( [F_a, F_\beta]_{AB} = 0 \) and we must have \( C = 2e \) so the first and second equations are equivalent. Contracting indices in the first equation above gives \( C_{[\alpha\beta} F_{\gamma]} = 0 \); on the other hand, contracting with \( \epsilon^{\mu\nu\rho\sigma\tau} \), taking more traces and using \( C_{\mu\alpha} F_{AB}^{\alpha} = 0 \) from (7.140) gives \( C_{[\alpha\beta} F_{\gamma]} = 0 \); it then follows that \( C_{[\alpha\beta} F_{\gamma]} = 0 \), so either \( C = 2e = 0 \) or \( F = 0 \), and (7.144) then becomes \( \nabla F = 0 \). \( \square \)

### 7.5 Maximally supersymmetric backgrounds and their Killing superalgebras

We now have a set of necessary local conditions for a background \( M \) to be maximally supersymmetric – that is, for it to have a Killing superalgebra with the maximum number of Killing spinors – namely, the conditions of Proposition 7.8 and the formula (7.138) for the Riemann tensor. Note that if \( M \) is simply connected, these conditions are also sufficient, since then \( R^D = 0 \) implies that the maximum number of Killing spinors exist. We now seek to explore the space of such geometries up to local isometry. By the homogeneity theorem, all of these geometries are locally homogeneous.

The supersymmetric geometries for which \( \rho = 0 \) were already treated in [50]; these correspond to the sub-branch of branch (1) with \( C = 2e \) (this is the pure ungauged supergravity case already fully classified by [19]) and to branch (2). We saw in §7.1.2 that on branch (3), one can deduce from the explicit form of maximally supersymmetric deformations of the Poincaré superalgebra \( \hat{\mathfrak{g}} \) (equations (7.56) and (7.60)) that the corresponding maximally supersymmetric background must be AdS\(_5\). This leaves only the backgrounds in the \( C \neq 2e \) sub-branch of branch (1) to be determined. We will not attempt to classify the maximally supersymmetric backgrounds in that case in this work, although we will make some general remarks about them and explore the remarkably rich sub-class of solutions where both \( C \) and \( e \) are decomposable. We will also give explicit descriptions of the Killing superalgebras of the solutions we discuss in this branch as well as all solutions in branches (2) and (3).

In any local expressions, we work in an orthonormal frame unless it is explicitly stated that a particular expression is in a coordinate frame.
7.5.1 General remarks on solutions with $C \neq 2e$

This is by far the richest branch of solutions and we will not attempt to fully classify all solutions here. We will however make some general remarks about the geometric structure of solutions and then treat some special cases.

For the following result, we recall that a conformal Killing form is a generalisation of the notion of conformal Killing vector; a differential form $\psi \in \Omega^k(N)$ on a pseudo-Riemannian manifold $N$ of dimension $n$ is conformal Killing (–Yano) if

$$\nabla_X \psi - \frac{1}{k+1} i_X d\psi + \frac{1}{n-k+1} X^\flat \wedge \delta \psi = 0$$  \hspace{1cm} (7.153)

for all $X \in \mathfrak{X}(N)$, where $\delta$ is the Hodge co-differential to $d$ ($\delta = * d *$ up to a sign). A conformal Killing form which is co-closed is called Killing (a generalisation of Killing vectors), while a closed conformal Killing form is $^*\text{-Killing}$ since its Hodge dual is Killing. Conformal Killing forms are closely related to twistor spinors in Riemannian geometry (for example, the Dirac bilinears of twistor spinors are conformal Killing); see [113] for more details.

We denote by $(-,-)$ the pairing on differential forms induced by the metric and by $\| - \|^2$ its square norm – note that neither is positive-definite.

Lemma 7.9. The 2-form $C$ is closed, $e$ is a closed conformal Killing form, and the metric norms and inner product $\|C\|^2$, $\|e\|^2$ and $(C,e)$ are constant.

Proof. We use equations (7.136) and (7.137) for all claims. One can show that $C$ and $e$ are closed using (7.134) (in a local expression, this can be done by skew-symmetrising indices on either differential equation and then using (7.134) along with some identities from Appendix 7.A). Taking traces of the two equations gives

$$d * C = -4C \wedge e,$$  \hspace{1cm} (7.154)

$$d * e = -4e \wedge e,$$  \hspace{1cm} (7.155)

hence $e$ is conformal Killing:

$$\nabla_X e = -\frac{1}{4} X^\flat \wedge * d * e.$$  \hspace{1cm} (7.156)

We note note that (7.137) is equivalent to $\nabla_X e = -2 * (e \wedge i_X e)$, so using the Leibniz rule, for every $X \in \mathfrak{X}(M)$ we have

$$\nabla_X \|e\|^2 = 2(e, \nabla_X e) = -4 * (e \wedge e \wedge i_X e) = -\frac{4}{3} * i_X (e \wedge e \wedge e) = 0.$$  \hspace{1cm} (7.157)

Similarly, using (7.135)

$$\nabla_X \|C\|^2 = -4 * (C \wedge e \wedge i_X C) = -2 * i_X(C \wedge C \wedge e) + 2 * (C \wedge C \wedge i_X e) = -8 * i_X (e \wedge e \wedge e) + 8 * (e \wedge e \wedge i_X e)$$  \hspace{1cm} (7.158)

$$= 0,$$

and

$$\nabla_X (C,e) = -2 * (e \wedge e \wedge i_X C) - 2 * (e \wedge e \wedge i_X e) = -\frac{1}{2} * (i_X C \wedge C \wedge C) = 0.$$  \hspace{1cm} (7.159)
Lemma 7.10. The vector fields $K, K'$ defined by

$$K^o = * (C \wedge e), \quad K'^o = * (e \wedge e) = \frac{1}{4} * (C \wedge C),$$

(7.160)

are Killing and satisfy

$$t_K C = t_K e = t_{K'} C = t_{K'} e = 0,$$

(7.161)

$$\mathcal{L}_K C = \mathcal{L}_K e = \mathcal{L}_{K'} C = \mathcal{L}_{K'} e = 0,$$

(7.162)

$$K^o \wedge K'^o = 0,$$

(7.163)

$$\|K\|^2 - 4\|K'\|^2 = \|C\|^2 \|e\|^2 - (C, e)^2,$$

(7.164)

$$\nabla_K C = \nabla_K e = \nabla_{K'} C = \nabla_{K'} e = 0.$$

(7.165)

In particular, $\|K\|^2 - 4\|K'\|^2$ is constant.

**Proof.** We locally compute (using (7.134) in the first computation)

$$\nabla_\mu K_\nu = \frac{1}{4} \varepsilon_{\nu\alpha\beta\gamma} \left( \nabla_\mu C^{\alpha\beta} \varepsilon_{\gamma\delta} + C^{\alpha\beta} \nabla_\mu \varepsilon_{\gamma\delta} \right),$$

(7.166)

$$= -\|e\|^2 C_{\mu\nu} - \|C\|^2 e_{\mu\nu} + 4C_{\alpha\beta} e_{\mu}^\alpha e_{\nu}^\beta,$$

$$\nabla_\mu K'_\nu = \frac{1}{2} \varepsilon_{\nu\alpha\beta\gamma} e^{\alpha\beta} \nabla_\mu \varepsilon_{\gamma\delta} = -2\|e\|^2 e_{\mu\nu} + 4e_{\alpha\beta} e_{\mu}^\alpha e_{\nu}^\beta,$$

(7.167)

which are clearly skew-symmetric in $\mu, \nu$, hence $K, K'$ are Killing. For all $X \in \mathfrak{X}(M)$

$$(t_K C)(X) = -(t_X C, K^o) = -(t_X C, * (C \wedge e)) = * (t_X C \wedge C \wedge e) = 0,$$

(7.168)

where we previously showed the final equality in the computation of $\nabla \|C\|^2$. Similar computations show that the other contractions of $K, K'$ with $C, e$ vanish. Since $C, e$ are both closed, their Lie derivatives along $K, K'$ vanish by the Cartan formula. We next have

$$K^o \wedge K'^o = -* t_K (e \wedge e) = -2 t_K e \wedge e = 0$$

(7.169)

and

$$\|K\|^2 = \|C \wedge e\|^2 = \|C\|^2 \|e\|^2 - (C, e)^2 + (C \wedge C, e \wedge e),$$

(7.170)

$$\|K'\|^2 = \|e \wedge e\|^2,$$

(7.171)

and since $C \wedge C = 4e \wedge e$, we have $(C \wedge C, e \wedge e) = 4 \|e \wedge e\|^2 = 4 \|K'\|$, hence the equation for the norms of $K, K'$. Now, for all $X \in \mathfrak{X}(M)$ we have

$$\nabla_X e = X^o \wedge K'^o, \quad \nabla_X C = -2 X^o \wedge K'^o + 2 * (t_X e \wedge C),$$

(7.172)

so by the previous results either expression above vanishes for $X = K, K'$. □

This concludes our general remarks. Over the next few subsections, we will investigate some particular sets of solutions in which $K' = 0$, so that $e \wedge e = C \wedge C = 0$. Indeed, for $B \in \Omega^2(M)$, the equation $B \wedge B = 0$ is a Plücker identity which implies
that $B$ is locally decomposable; locally, we can write $B = \phi^{(1)} \wedge \phi^{(2)}$ for some 1-forms $\phi^{(1)}, \phi^{(2)}$ and we have

$$\|B\|^2 = \|\phi^{(1)}\|^2 \|\phi^{(2)}\|^2 - (\phi^{(1)}, \phi^{(2)})^2.$$  

(7.173)

We denote by $T^*_B M$ the distribution in $T^* M$ spanned by these 1-forms. Their metrically-dual vector fields $\phi^{(1)}\#$, $\phi^{(2)}\#$ span a dual distribution of $T M$ denoted by $T^*_B M$. If $B$ is nowhere vanishing then $\phi^{(1)}, \phi^{(2)}$ are everywhere linearly independent, so these distributions are regular and of rank 2. In this case, by the equation above, the sign of $\|B\|^2$ at a point determines the causal type of the fibre plane (with respect to the induced metric) of each distribution at that point:

$$\|B\|^2 > 0 : \quad T^*_B M, T^*_B M \text{ Euclidean},$$
$$\|B\|^2 = 0 : \quad T^*_B M, T^*_B M \text{ degenerate},$$
$$\|B\|^2 < 0 : \quad T^*_B M, T^*_B M \text{ Lorentzian.}$$

Now assume that $\|B\|^2$ is constant, as this is the case for $B = C$ or $B = e$. Then if $B$ is not null, it is nowhere vanishing and the distributions $T_B M$ and $T^*_B M$ have constant causal type. If $B$ is null then it may vanish on some locus, but away from this locus the distributions are regular and of degenerate causal type. Note that $B$ is also nowhere vanishing if it happens to be parallel and non-zero.

### 7.5.2 Solutions with either $C = 0$ or $e = 0$

Let us first treat the simplest non-trivial cases, in which $C = 0$ or $e = 0$; note that this trivially implies that $\nabla C = \nabla e = 0$ and $K = K' = 0$. We will treat these cases in parallel; for convenience, in the $C = 0$ case we set $B = e$ and in the $e = 0$ case we set $B = C$. In either case, the equations for maximal supersymmetry reduce to

$$\nabla B = 0, \quad B \wedge B = 0,$$

(7.174)

and an expression for the Riemann tensor,

$$R_{\mu\nu\rho\sigma} = \begin{cases} 4B_{\mu\nu}B_{\rho\sigma} & \text{for } e = 0, \\ -4B_{\mu\nu}B_{\rho\sigma} + 8\left(\eta_{\mu\rho}B_{\nu|\alpha}B_{\sigma|\alpha} - \eta_{\nu\sigma}B_{\mu|\alpha}B_{\rho|\alpha}\right) & \text{for } C = 0. \end{cases}$$

(7.175)

The Ricci scalar is $-\|B\|^2$ in the $e = 0$ case and $\|B\|^2$ in the $C = 0$ case. Note that since $\nabla B = 0$, the Riemann tensor is covariantly constant, whence the geometry is locally symmetric. Since we work only up to local isometry, we thus take $(M, g)$ to be symmetric. We will first divide up the solutions by the causal type of $B$ and then use the Cahen-Wallach classification of Lorentzian symmetric spaces (Theorem 2.16) to identify them.
Non-degenerate causal type ($\|B\|^2 \neq 0$)

Since $B$ is parallel, $T^*_B M$ and $T_B M$ are holonomy invariant. Thus, in the non-degenerate cases, the de Rham-Wu Decomposition Theorem [78–80] gives us a holonomy invariant decomposition of the tangent bundle $TM = T_B M \oplus T^*_B M$, where $T^*_B M$ is the perpendicular distribution to $T_B M$, and an integral decompostion the base manifold $TM = M_1 \times M_2$ where for $B < 0$ (resp. $B > 0$) $M_1$ is a 2-dimensional Lorentzian (resp. (negative) Riemannian) manifold and $M_2$ is a 3-dimensional (negative) Riemannian (resp. Lorentzian) manifold. Analysis of the Riemann tensor under this decomposition gives us the following, where $R$ denote the scalar curvature$^5$:

\[
\begin{align*}
e 0, \quad \|C\|^2 > 0 : & \quad M = S^2 \times \mathbb{R}^{1,2}, \quad R_{S^3} = -4\|B\|^2, \\
e 0, \quad \|C\|^2 < 0 : & \quad M = AdS_2 \times \mathbb{R}^3, \quad R_{AdS_2} = -4\|B\|^2, \\
C = 0, \quad \|e\|^2 > 0 : & \quad M = \mathbb{R}^2 \times AdS_3, \quad R_{AdS_3} = 24\|B\|^2, \\
C = 0, \quad \|e\|^2 < 0 : & \quad M = \mathbb{R}^{1,1} \times S_3, \quad R_{S^3} = 24\|B\|^2. 
\end{align*}
\] (7.176)

Degenerate causal type ($\|B\|^2 = 0$)

In the degenerate case, the Decomposition Theorem does not apply but we can exploit some different causal properties.

**Lemma 7.11.** Let $B \in \Omega^2(M)$ be a non-vanishing, decomposable 2-form of degenerate causal type on $(M, g)$. Then there locally exist 1-forms $\phi^{(1)}, \phi^{(2)}$ such that

\[
B = \phi^{(1)} \wedge \phi^{(2)}, \quad \|\phi^{(1)}\|^2 = 0, \quad \|\phi^{(2)}\|^2 = -1, \quad (\phi^{(1)}, \phi^{(2)}) = 0.
\] (7.177)

If $B$ is also parallel, so is $\phi^{(1)}$; in particular, $(M, g)$ is (locally) a pp-wave and it can also be arranged that $\phi^{(2)}$ is parallel.

**Proof.** Since $B$ is decomposable we locally have $B = \phi^{(1)} \wedge \phi^{(2)}$ for some 1-forms $\phi^{(1)}, \phi^{(2)}$, and we since $B$ is non-vanishing, these are everywhere linearly independent. Since $\|B\|=0$, the distribution $T^*_B M$ defined by $\phi^{(1)}, \phi^{(2)}$ is degenerate; it follows that they cannot both be null at any point; at least one must be timelike. Fixing an arbitrary point, we without loss of generality let us assume that $\|\phi^{(2)}\|^2 < 0$ at this point and thus on some neighbourhood of it. On this neighbourhood we define

\[
\tilde{\phi}^{(1)} = \sqrt{-\|\phi^{(2)}\|^2} \phi^{(1)} + \frac{(\phi^{(1)}, \phi^{(2)})}{\sqrt{-\|\phi^{(2)}\|^2}} \phi^{(2)}, \quad \tilde{\phi}^{(2)} = \frac{1}{\sqrt{-\|\phi^{(2)}\|^2}} \phi^{(2)};
\] (7.178)

we then have $B = \tilde{\phi}^{(1)} \wedge \tilde{\phi}^{(2)}, \quad \|\tilde{\phi}^{(2)}\|^2 = -1$, and $(\tilde{\phi}^{(1)}, \tilde{\phi}^{(2)}) = 0$, and then by equation (7.173), $\|\tilde{\phi}^{(1)}\|^2 = 0$. This proves the first statement. We now drop the tildes from the notation and, letting $X \in \mathfrak{X}(M)$, note that $\nabla_X B = 0$ gives

\[
\nabla_X \phi^{(1)} \wedge \phi^{(2)} + \phi^{(1)} \wedge \nabla_X \phi^{(2)} = 0
\] (7.179)

$^5$Recall that we work in mostly-negative signature, so the scalar curvature apparently has the “wrong” sign.
for all \( X \in \mathfrak{X}(M) \). Contracting with \( \phi^{(1)\sharp} \) and \( \phi^{(2)\sharp} \) respectively, we have

\[
\begin{align*}
(\phi^{(1)}, \nabla_X \phi^{(1)}) \phi^{(2)} - (\phi^{(1)}, \nabla_X \phi^{(2)}) \phi^{(1)} &= 0, \\
(\phi^{(2)}, \nabla_X \phi^{(1)}) \phi^{(2)} + \nabla_X \phi^{(1)} - (\phi^{(2)}, \nabla_X \phi^{(2)}) \phi^{(1)} &= 0.
\end{align*}
\]

(7.180, 7.181)

The first term of the first equation vanishes since \( (\phi^{(1)}, \nabla_X \phi^{(1)}) = \frac{1}{2} \nabla_X \| \phi^{(1)} \|^2 = 0 \), and similarly for the last term of the second equation. Since \( B \) is nowhere vanishing, so is \( \phi^{(1)} \), thus the first equation gives \( (\phi^{(1)}, \nabla_X \phi^{(2)}) = 0 \). Then the second equation gives us

\[
\nabla_X \phi^{(1)} = - (\phi^{(2)}, \nabla_X \phi^{(1)}) \phi^{(2)} = ([\nabla_X \phi^{(2)}, \phi^{(1)}]) - \nabla_X (\phi^{(2)}, \phi^{(1)}) \phi^{(2)} = 0,
\]

(7.182)

thus \( \phi^{(1)} \) is parallel since \( X \) was arbitrary. In particular, \( \phi^{(1)} \) is a (nowhere-vanishing) parallel null vector, so we have a pp-wave geometry. Returning now to (7.179), we find that \( \nabla_X \phi^{(2)} = \alpha(X) \phi^{(1)} \) for some 1-form \( \alpha \). By the Poincaré lemma, we locally have \( \alpha = d f \) for some smooth function \( f \); thus, defining \( \tilde{g}^{(2)} = \phi^{(2)} - f \phi^{(1)} \), the pair \( (\phi^{(1)}, \tilde{g}^{(2)}) \) satisfies the properties in the first part of the lemma and we also have

\[
\nabla_X \tilde{g}^{(2)} = (\alpha(X) - \nabla_X f) \phi^{(1)} = 0,
\]

(7.183)

hence the final claim.

Writing the 2-form \( B \) on our maximally supersymmetric background as in the lemma above, we can locally complete the pair of 1-forms \( \{\phi^{(1)}, \phi^{(2)}\} \) to a Witt coframe \( \{\theta^+, \theta^- = 2\phi^{(1)}, \theta^1 = \phi^{(2)}, \theta^2, \theta^3\} \) so that

\[
B = \frac{1}{2} \theta^- \wedge \theta^1.
\]

(7.184)

The only non-zero component of \( B \) in this frame is \( B_{-1} = 1 \), and it follows that the only non-zero component of \( B_{\mu \alpha} B_{\nu}^\alpha \) is \( B_{-a} B_{-}^\alpha = g^{11} (B_{-1})^2 = -1 \). Substituting this into our expression for the Riemann tensor, in either case we find that the only possible non-zero components are \( R_{-i-} \) (no summation), with

\[
\begin{align*}
R_{-1-1} &= 1, & R_{-2-2} &= 0, & R_{-3-3} &= 0, & \text{for } e = 0, \\
R_{-1-1} &= 0, & R_{-2-2} &= 1, & R_{-3-3} &= 1, & \text{for } C = 0.
\end{align*}
\]

(7.185, 7.186)

Thus \( M \) is locally symmetric, scalar flat and possesses a parallel null vector field \( \theta^1 \); by the Cahen-Wallach classification, it must be a Cahen–Wallach pp-wave \( CW_5(A) \). Comparison with the components of the Riemann tensor for \( CW_5(A) \) given in the discussion in §2.4.5 tells us

\[
\begin{align*}
e = 0 \quad \| C \|^2 = 0 \quad M = CW_3(1) \times \mathbb{R}^2, \\
C = 0 \quad \| e \|^2 = 0 \quad M = CW_4(\mathbb{H}_2) \times \mathbb{R}.
\end{align*}
\]

(7.187)

Explicitly, the metrics are

\[
\begin{align*}
g &= 2 d x^+ d x^- + (x^1 d x^-)^2 - d x^i d x^i \quad \text{for } e = 0, \\
g &= 2 d x^+ d x^- + (x^2)^2 + (x^3)^2 ) (d x^-)^2 - d x^i d x^i \quad \text{for } C = 0.
\end{align*}
\]

(7.188, 7.189)
Killing superalgebras

We can now describe the Killing superalgebra \( \mathfrak{R}_D \) of each solution by localising to a point \( p \) on the manifold. First, we choose an adapted coframe \( \{ \theta^\mu \} \) for each solution. We have already chosen a Witt coframe for the degenerate solutions. For the non-degenerate solutions, if \( B \) is Lorentzian then we can find an orthonormal coframe such that \( B = \frac{a}{2}\theta^0 \wedge \theta^1 \) where \( a = 2\sqrt{-\langle B \rangle} \) by choosing an orthonormal basis for the \( B \)-plane at \( p \), parallel transporting (since \( B \) is parallel) to obtain \( \theta^0, \theta^1 \) an then orthogonally completing to a coframe by Gram–Schmidt; it is then not difficult to see that. Similarly, if \( B \) is Euclidean, we can find a coframe such that \( B = \frac{a}{2}\theta^1 \wedge \theta^2 \) where \( a = 2\sqrt{\langle B \rangle} \). The factors of 2 appearing here will reduce the number of numerical factors in our presentation of the Killing superalgebras.

We then choose a basis \( \{ P_\mu \} \) of \( V = T_pM \) dual to \( \{ \theta^\mu \} \) which induces a basis \( \{ L_{\mu\nu} \} \) of \( \mathfrak{so}(V) \), and we also locally choose a frame \( \tau_i \) for the \( \tau \)-symmetry bundle \( \Pi \) satisfying the standard relation \( \theta = \sqrt{\langle B \rangle} \).

In each case, the Killing superalgebra \( \mathfrak{K} \) is a deformation of the graded subalgebra \( V \oplus S \oplus \mathfrak{so}(1) \) of \( \mathfrak{s} \), where \( \mathfrak{h} = \text{stab}_{\mathfrak{so}(V)}(B) \) and its brackets are given by (7.56) and (7.58). In all of our examples, we have \( C \land C = e \land e = 0 \), and one can show that if \( e \) is proportional to the wedge of two orthonormal coframe forms then the terms proportional to \( e \) in the deformation of the \( [L, P] \) bracket vanish, so the deformed brackets (7.58) become

\[
[P_{\mu}, P_{\nu}] = -2\epsilon_{\mu\nu\alpha\beta} e^{a\beta} P_{\sigma} + 2C_{\mu\nu} C_{\rho\sigma} L^{\rho\sigma},
\]

\[
[L_{\mu\nu}, P_{\rho}] = 2\epsilon_{\mu\nu\rho\sigma} e^{a\beta} P_{\sigma},
\]

\[
[P_{\mu}, e] = \left(C_{\mu\nu} \Gamma^\nu + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} e^{a\beta} \Gamma^{\nu\rho} \right)e,
\]

(7.190)

\[
[e, e] = \kappa^\mu_e P_{\mu} + \mu_e C_{\mu\nu} L^{\mu\nu} + (C_{\mu\nu} - 2e_{\mu\nu}) \omega_e^{\mu\nu} \tau_i.
\]

Thus the \([L, P]\) bracket is actually not deformed in these cases. Note that this means that the vectorial part of the algebra always splits reductively as \( \mathfrak{U}_D = \mathfrak{h} \oplus V \).

For the solutions with Lorentzian \( B \), we have \( \mathfrak{h} = \text{stab}_{\mathfrak{so}(V)}(B) = \langle L_{01}, L_{23}, L_{24}, L_{34} \rangle \), but the \( C = 0 \) and \( e = 0 \) cases give different deformations. In the \( e = 0 \) case the non-zero deformed brackets are

\[
[P_0, P_1] = -a^2 L_{01}, \quad [P_0, e] = \frac{a}{2} \Gamma_1 e, \quad [P_1, e] = -\frac{a}{2} \Gamma_0 e, \quad [e, e] = \kappa^\mu_e P_{\mu} - a\mu_e L_{01} - a\omega^{i}_{01} \tau_i,
\]

(7.191)

while for \( C = 0 \),

\[
[P_2, P_3] = -a P_4, \quad [P_2, P_4] = a P_3, \quad [P_3, P_4] = -a P_2,
\]

\[
[P_2, e] = -a \Gamma_{34} e, \quad [P_3, e] = a \Gamma_{24} e, \quad [P_4, e] = -a \Gamma_{23} e, \quad [e, e] = \kappa^\mu_e P_{\mu} + 2a\omega^{i}_{01} \tau_i,
\]

(7.192)

For those with Euclidean \( B \), \( \mathfrak{h} = \text{stab}_{\mathfrak{so}(V)}(B) = \langle L_{12}, L_{02}, L_{03}, L_{34} \rangle \). For the \( e = 0 \) family we have

\[
[P_1, P_2] = a^2 L_{12}, \quad [P_1, e] = -\frac{a}{2} \Gamma_2 e, \quad [P_2, e] = \frac{a}{2} \Gamma_1 e, \quad [e, e] = \kappa^\mu_e P_{\mu} + a\mu_e L_{12} + a\omega^{i}_{12} \tau_i,
\]

(7.193)

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and for \( C = 0 \),
\[
[P_0, P_3] = aP_4, \quad [P_0, P_4] = -aP_3, \quad [P_3, P_4] = -aP_0, \\
[P_0, e] = a\Gamma_{34} e, \quad [P_3, e] = a\Gamma_{04} e, \quad [P_4, e] = -a\Gamma_{03} e, \\
\langle e, e \rangle = \kappa^\mu_\rho P_\mu - 2a\omega^i_1 \tau_i. 
\] (7.194)

For either of the pp-wave solutions (degenerate \( B \)), \( h = \text{stab_\( g_0 (V) \)} (B) = \langle L_{+1}, L_{+2}, L_{+3}, L_{23} \rangle \). For the \( e = 0 \) solution, the non-zero deformed brackets are
\[
[P_-, P_1] = -L_{+1}, \quad [P_-, e] = -\frac{1}{2} \Gamma_1 e, \quad [P_1, e] = -\frac{1}{2} \Gamma_4 e, \\
\langle e, e \rangle = \kappa^\mu_\rho P_\mu - \mu_e L_{+1} - \omega^j_1 \tau_i. 
\] (7.195)

along with the \( \{ L, P \} \) brackets, which are undeformed. Note that the 2,3 directions are undeformed; this reflects the Riemannian product decomposition \( M = CW_3(1) \times \mathbb{R}^2 \). Moreover, there is an even subalgebra \( \langle L_{+1}, L_{23}, P_\mu \rangle \) which acts locally transitively on \( M \) and decomposes into a direct sum
\[
\langle L_{+1}, L_{23}, P_\mu \rangle = \langle L_{+1}, P_+, P_-, P_1 \rangle \oplus \langle L_{23}, P_2, P_3 \rangle. 
\] (7.196)

where the second factor is \( \langle L_{23}, P_2, P_3 \rangle \cong \text{iso}(2) \).

In the \( C = 0 \) case, the deformation is
\[
[P_-, P_2] = 2P_3, \quad [P_-, P_3] = -2P_2, \\
[P_-, e] = \Gamma_{23} e, \quad [P_2, e] = \Gamma_{+3} e, \quad [P_3, e] = -\Gamma_{+2} e, \\
\langle e, e \rangle = \kappa^\mu_\rho P_\mu - 2\omega^j_1 \tau_i. 
\] (7.197)

One again the \( \{ L, P \} \) brackets are undeformed, and we have a decomposition of a transitive algebra
\[
\langle L_{+2}, L_{+3}, L_{23}, P_\mu \rangle = \langle L_{+2}, L_{+3}, L_{23}, P_+, P_-, P_2, P_3 \rangle \oplus \langle P_1 \rangle, 
\] (7.198)
corresponding to the decomposition \( M = CW_4(\mathbb{R}^2) \times \mathbb{R} \).

### 7.5.3 Solutions with \( K = K' = 0 \)

We now extend our analysis to a slightly more general class of solutions. We will use essentially the same geometrical techniques as before.

**Solutions with \( C \parallel e \)**

Let us first consider the case where \( C \) and \( e \) are collinear. We can treat this case using essentially the same techniques as in the previous section. The following lemma demonstrates how strong this assumption is. The upshot is that the only case in which \( K = K' = 0 \) and \( C \) and \( e \) are not collinear is where \( \| C \|^2 = \| e \|^2 = 0 \).

**Lemma 7.12.** For a maximally supersymmetric background \((M, g, C, e)\), we have the following.

1. The 2-form \( e \) is parallel if and only if \( K = 0 \);
(2) If \( C \) is parallel then \( K' = 0 \);

(3) If \( C \) and \( e \) are \( \mathbb{R} \)-collinear and \( C \neq \pm 2e \) then both are parallel and \( K = K' = 0 \);

(4) If \( K = K' = 0 \) and either \( \|C\|^2 \neq 0 \) or \( \|e\|^2 \neq 0 \) then \( C \) and \( e \) are \( C^\infty(M) \)-collinear;

(5) If \( K' = 0 \) and \( C \) and \( e \) are \( C^\infty(M) \)-collinear then \( C \) and \( e \) are \( \mathbb{R} \)-collinear.

**Proof.** Recall the differential equations (7.137), (7.136) which we can render as

\[
\nabla_X C = -2 * (e \wedge \iota_X C),
\]

(7.199)

\[
\nabla_X e = X^g \wedge K',
\]

(7.200)

for all \( X \in \mathfrak{X}(M) \).

(1) and (2): Clearly \( \nabla e = 0 \) if and only if \( K' = 0 \), and if \( \nabla C = 0 \) then in particular \( K = -\frac{1}{4} * d * C = 0 \).

(3): If \( e = 0 \) then we trivially have \( K = K' = 0 \) and \( \nabla e = \nabla C = 0 \), so let us assume \( e \neq 0 \). Thus if \( C \) and \( e \) are collinear, \( C = ae \) for some \( a \in \mathbb{R} \) and then we have \( C \wedge C = a^2 e \wedge e \); by equation (7.135) if \( a \neq \pm 2 \), we must have \( C \wedge C = e \wedge e = 0 \), and then clearly \( C \wedge e = ae \wedge e = 0 \) and we also have \( \iota_X C \wedge e = a e X \wedge e = \frac{a}{2} \iota_X (e \wedge e) = 0 \). Thus \( e \) and \( C \) are parallel and so \( K = K' = 0 \).

(4): We have \( K = 0 \) if and only if \( C \wedge C = 4e \wedge e = 0 \), so \( C \) and \( e \) are both wedges of a pair of 1-forms, but since \( C \wedge e = * K' = 0 \) this set of four 1-forms is not linearly independent. This implies that it is possible to find three 1-forms \( \phi^{(1)}, \phi^{(2)}, \phi^{(3)} \) such that \( C = \phi^{(1)} \wedge \phi^{(2)} \) and \( e = \phi^{(1)} \wedge \phi^{(3)} \); we may assume without loss of generality that \( (\phi^{(1)}, \phi^{(2)}) = (\phi^{(1)}, \phi^{(3)}) = 0 \). Note that we have

\[
\|C\|^2 = \|\phi^{(1)}\|^2 \|\phi^{(2)}\|^2,
\]

(7.201)

\[
\|e\|^2 = \|\phi^{(1)}\|^2 \|\phi^{(3)}\|^2,
\]

(7.202)

and that these are constant by Lemma 7.9. Thus if either is non-zero, \( \|\phi^{(1)}\|^2 \) must be nowhere vanishing. Now recall that, in a local frame, \( C |_{\mu} e |_{\nu} = 0 \) by equation (7.134); we have

\[
C |_{\mu} e |_{\nu} = (\phi^{(2)}, \phi^{(3)}) \phi^{(1)}_{\mu} \phi^{(1)}_{\nu} - (\phi^{(1)}, \phi^{(2)}) \phi^{(1)}_{\mu} \phi^{(3)}_{\nu} - (\phi^{(1)}, \phi^{(3)}) \phi^{(2)}_{\mu} \phi^{(3)}_{\nu} + \|\phi^{(1)}\|^2 \phi^{(2)}_{\mu} \phi^{(3)}_{\nu}
\]

(7.203)

so equation (7.134) holds if and only if

\[
\|\phi^{(1)}\|^2 \phi^{(2)} \wedge \phi^{(3)} = 0.
\]

(7.204)

Since \( \|\phi^{(1)}\|^2 \) is nowhere vanishing, \( \phi^{(2)} \) and \( \phi^{(3)} \) are not \( C^\infty(M) \)-linearly independent, hence neither are \( C \) and \( e \).

(5): If \( K' = 0 \) (hence \( \nabla e = 0 \)) and \( C, e \) are \( C^\infty(M) \)-collinear. The statement is trivial if \( e = 0 \). Since \( e \) is parallel then if is not zero then it is nowhere vanishing, so \( C = fe \) for some \( f \in C^\infty(M) \). But then

\[
\iota_X C \wedge e = f \iota_X e \wedge e = \frac{1}{2} f \iota_X (e \wedge e) = \frac{1}{2} f \iota_X K' = 0,
\]

(7.205)
whence $\nabla C = 0$. But then taking the covariant derivative of $C = fe$ gives us $(\nabla f)e = 0$. Since $e$ is nowhere vanishing, $\nabla f = 0$. \qed

Since we have already treated the $e = 0$ case, let us now assume that $e \neq 0$ and $C$ is a multiple of $e$. For later convenience, we take $C = ae$ and $B = 2\sqrt{2}e = \frac{2\sqrt{2}}{a}C$ for $a > 0$. The maximal supersymmetry equations are satisfied if and only if $\nabla B = 0$ and

$$R_{\mu\nu\rho\sigma} = \left(\frac{a^2}{2} - 1\right)B_{\mu\nu}B_{\rho\sigma} + \left(\eta_{[\mu}B_{\nu]}{^a}B_{\sigma]a} - \eta_{\nu[\mu}B_{\rho]a}B_{\sigma]a}\right) - \frac{3}{2}\eta_{[\mu}\eta_{\nu]a}B_{a\rho\sigma}B^{\rho\sigma}.$$  

(7.206)

We record that the Ricci and scalar curvatures are then

$$R_{\mu\nu} = -\left(\frac{a^2}{2} + 1\right)B_{\mu\nu}B^{\alpha\beta} + \eta_{\mu\nu}\|B\|^2,$$

$$R = (3 - a^2)\|B\|^2,$$  

(7.207)

while the Weyl tensor vanishes if and only if $a = 1$, i.e. $C = e$. Following almost exactly the same argument as in the $C = 0$ and $e = 0$ cases, we find that the geometry of $(M, g)$ is determined by $\|B\|^2$ and $a$ as follows:

- $\|B\|^2 > 0$: $\ M = AdS_3 \times S^2$, $\ R_{\text{AdS}_3} = 3\|B\|^2$, $\ R_{S^2} = -a^2\|B\|^2$, $\ R_{\text{AdS}_3} = -a^2\|B\|^2$,

- $\|B\|^2 < 0$: $\ M = AdS_2 \times S^3$, $\ R_{\text{AdS}_2} = -a^2\|B\|^2$, $\ R_{S^2} = 3\|B\|^2$, $\ R_{\text{AdS}_2} = -a^2\|B\|^2$,

(7.208)

- $\|B\|^2 = 0$: $\ M = CW_5(A)$, $\ A = \text{diag}(a^2, 1, 1)$.

where we recall that we work in mostly-minus signature. In the final line, we have chosen a particular scaling of the coframe to fix the numerical prefactor of the matrix $A$. We note that $CW_5(A)$ is conformally flat if and only if $a = 1$ and we recover maximally supersymmetric solutions to minimal 5-dimensional supergravity for $a = 2$. We recover the solutions from the previous section in the limits $a \to 0$ and $a \to \infty$ (keeping $ae$ and thus $C$ and $a^2\|B\|^2 = 4\|C\|^2$ fixed in the latter case).

As in the cases with a vanishing 2-form, the Killing superalgebras are deformations of a graded subalgebra $V \oplus S \oplus \mathfrak{h} \oplus \mathfrak{sp}(1)$ of $\mathfrak{g}$ with $\mathfrak{h} = \mathfrak{stab}_{g_0}(V)$ and deformed brackets given by (7.190). We will not explicitly give the Killing superalgebras in this case, but note that we will (essentially) recover the superalgebra of the $\|B\|^2 = 0$ solution as a limit of the next case.

**Solutions with $K = K' = 0$ but $C \nparallel e$**

We can complete our analysis of the case $K = K' = 0$ by taking now $\|e\|^2 = \|C\|^2 = 0$ and $C, e$ not collinear; that is, they are everywhere linearly independent.

**Lemma 7.13.** Suppose $K = K' = 0$, $\|e\|^2 = \|C\|^2 = 0$ and $C, e$ are not collinear. Then there exists a Witt coframe $\{\theta^+, \theta^-, \theta^1, \theta^2, \theta^3\}$ such that $\theta^-$ and $\theta^1$ are parallel and

$$C = \frac{1}{2}\theta^- \wedge (a\theta^1 + b\theta^2), \quad e = \frac{1}{2}\theta^- \wedge \theta^1,$$  

(7.209)

for some $a, b \in \mathbb{R}$, with at least one non-zero. In particular, $(M, g)$ is a pp-wave.

**Proof.** Since $C$ and $e$ are decomposable and $C \wedge e = 0$, we have $C = \phi^{(1)} \wedge \phi^{(2)}$, $e = \phi^{(1)} \wedge \phi^{(3)}$ and we can take $(\phi^{(1)}, \phi^{(2)}) = (\phi^{(1)}, \phi^{(3)}) = 0$. Since $C$ and $e$ are everywhere linearly independent, so are $\phi^{(2)}, \phi^{(3)}$; in particular $\phi^{(2)} \wedge \phi^{(3)}$ is nowhere vanishing,
so the constraint equation $[A_C, A_e] = 0$, which again takes the form (7.204), implies that $\phi^{(1)}$ is null. By Lemma 7.11, we can assume that $\|\phi^{(3)}\|^2 = -1$, and since $\nabla e = 0$, $\nabla \phi^{(1)} = \nabla \phi^{(3)} = 0$. Let us now define

$$\theta^{-} := 2\phi^{(1)}, \quad \theta^{1} := \phi^{(3)}.$$  \hspace{1cm} (7.210)

and complete to a Witt coframe $\{\theta^{+}, \theta^{-}, \theta^{1}, \theta^{2}, \theta^{3}\}$. Note that $e_+ = (\theta^{-})^\perp$ is a parallel null vector, so the geometry is a pp-wave. In general, $\phi^{(2)}, \phi^{(3)}$ are not necessarily orthogonal, so we cannot arrange for one of the coframe fields to be collinear to $\phi^{(2)}$. However, since $\phi^{(2)}$ is orthogonal to $\theta^{-}$, it has no $\theta^{+}$ component, and any $\theta^{-}$ component does not contribute to $C$, so we can choose $\phi^{(2)}$ to be in the $C^\infty(M)$-span of $\{\theta^{1}\}$. But then by rotating the 2,3 part of the coframe, we can arrange for the 3-component of $\phi^{(2)}$ to vanish. We thus have (7.209) where $a = - (\phi^{(2)}, \theta^{1}), \ b = - (\phi^{(2)}, \theta^{2})$ with at least one of these non-zero at any point on $M$. It remains to show that $a$ and $b$ are constant. Let $X \in \mathfrak{x}(M)$. Then

$$\left(\nabla_X \theta^{2}, \theta^{2}\right) = \frac{1}{2} \nabla_X \|\theta^{2}\|^2 = 0,$$  \hspace{1cm} (7.211)

and since $\theta^{1}$ is parallel,

$$\left(\nabla_X \theta^{2}, \theta^{1}\right) = \frac{1}{2} \nabla_X \left(\theta^{2}, \theta^{1}\right) - \left(\theta^{2}, \nabla_X \theta^{1}\right) = 0,$$  \hspace{1cm} (7.212)

so $\nabla_X \theta^{2}$ is in the $C^\infty(M)$-span of $\{\theta^{+}, \theta^{-}, \theta^{3}\}$. Using that $\theta^{-}$ and $\theta^{1}$ are parallel, we have

$$\nabla_X C = \frac{1}{2} \theta^{-} \wedge (\nabla_X a \theta^{1} + \nabla_X b \theta^{2} + a \nabla_X \theta^{1} + b \nabla_X \theta^{2}),$$  \hspace{1cm} (7.213)

$$\ast (e \wedge \iota_X C) = - \frac{1}{4} a X^{-} \ast (\theta^{-} \wedge \theta^{1} \wedge \theta^{2}) = - \frac{1}{4} a X^{-} \theta^{-} \wedge \theta^{3},$$  \hspace{1cm} (7.214)

whence by $\nabla_X C = - 2 \ast (e \wedge \iota_X C),$

$$\frac{1}{2} \theta^{-} \wedge (\nabla_X a) \theta^{1} + (\nabla_X b) \theta^{2} + b \nabla_X \theta^{2} - a \nabla_X \theta^{3} = 0.$$  \hspace{1cm} (7.215)

Since $\nabla_X \theta^{2}$ has no $\theta^{1}$ or $\theta^{2}$ component, this implies that $\nabla_X a = \nabla_X b = 0$, thus $a, b$ are constant since $X$ was arbitrary.

The non-zero components of the Riemann and Ricci tensors are then

$$R_{-1,-1} = a^2, \quad R_{-2,-2} = b^2 + 1, \quad R_{-3,-3} = 1, \quad R_{-1,-2} = ab, \quad R_{-} = a^2 + b^2 + 2,$$  \hspace{1cm} (7.216)

and the Ricci scalar is zero. This is the Riemann tensor of a Cahen–Wallach pp-wave $CW_5(A)$ with

$$A = \begin{pmatrix} a^2 & ab & 0 \\ ab & b^2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  \hspace{1cm} (7.217)

corresponding to a metric of the form

$$g = 2 dx^+ dx^- + \left( a^2 (x^1)^2 + (b^2 + 1)(x^2)^2 + (x^3)^2 + 2abx^1 x^2 \right) (dx^-)^2 - dx^i dx^i.$$  \hspace{1cm} (7.218)
We can diagonalise $A$ to
\[ A' = \begin{pmatrix} \lambda_+ & 0 & 0 \\ 0 & \lambda_- & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ where } \lambda_{\pm} = \frac{1}{2} \left( a^2 + b^2 + 1 \pm \sqrt{(a^2 + b^2)^2 + 2(b^2 - a^2) + 1} \right). \quad (7.219) \]
so there is a change of the transverse coordinates which allows the metric to be expressed in the form
\[ g = 2dx^+dx^- + \left( \lambda_+(x^1)^2 + \lambda_-(x^2)^2 + (x^3)^2 \right)(dx^-)^2 - dx^i dx^i. \quad (7.220) \]

Note that we can obtain the degenerate $C \parallel e$ solution by taking the limit $b \to 0$. The Killing superalgebra of this background is a deformation of the maximally supersymmetric subalgebra $V \oplus S \oplus \mathfrak{h} \oplus \mathfrak{sp}(1)$ of $\mathfrak{D}$ with $\mathfrak{h} = \langle L_{+1}, L_{+2}, L_{+3} \rangle$. The deformed brackets are given by (7.190):
\[ [P_-, P_2] = 2P_3 - abL_{+1} - b^2L_{+2}, \quad [P_-, P_1] = -a^2L_{+1} - abL_{+2}, \]
\[ [P_-, P_3] = -2P_2, \quad [P_2, P_3] = -2P_+, \]
\[ [P_1, e] = -\frac{2}{3}\Gamma_{+}\epsilon, \quad [P_2, e] = \left( -\frac{4}{3}\Gamma_{+} + \Gamma_{+1} \right)\epsilon, \]
\[ [P_3, e] = -\Gamma_{+2}\epsilon, \quad [P_-, e] = \left( -\frac{4}{3}\Gamma_{1} - \frac{b}{2}\Gamma_{2} + \Gamma_{23} \right)\epsilon, \]
\[ [e, e] = \kappa_{\mu\nu}P_{\mu} - \mu_{\nu}(aL_{+1} + bL_{+2}) + \left( 2 - a \right)\omega^{A\bar{B}}_{+1} + -b\omega^{i\bar{j}}_{+2}T_{i}; \quad (7.221) \]

the $[L, P]$ brackets remain undeformed, so we have a reductive split of the vectorial part of the algebra $\mathfrak{D} = \mathfrak{h} \oplus V$. Taking the $b \to 0$ limit gives us a subalgebra of the Killing superalgebra of the degenerate $C \parallel e$. But the only additional generator the whole algebra has in that case is $L_{23}$, and one can check that none of the brackets involving this generator are deformed, hence we recover the whole Killing superalgebra for that case as well.

### 7.5.4 Other solutions with $K' = 0$

Here we further generalise the previous class of solutions and classify all solutions with locally decomposable $e$ and $C$. Note that $K' = \ast(e \wedge e) = \frac{1}{4} \ast(C \wedge C)$, so $C$ and $e$ are locally decomposable if and only if $K' = 0$.

The following result will essentially determine the structure of the solutions. As we will see, the solutions will split into three branches depending on the causal type of $K$.

**Proposition 7.14.** If $K' = 0$ and $K \neq 0$ then $C$ and $e$ are locally decomposable, $e$ is parallel, $(C, e) = 0$ and $\|K\|^2 = \|C\|^2 \|e\|^2$; in particular the latter is constant. Moreover, wherever $K$ is non-vanishing, there locally exist 1-forms $\phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \phi^{(4)}$ which are everywhere linearly independent such that
\[ C = \phi^{(1)} \wedge \phi^{(2)}, \quad e = \phi^{(3)} \wedge \phi^{(4)} \quad (7.222) \]
and
\[
\left( \phi^{(i)}, K^\circ \right) = 0 \text{ for all } i.
\] (7.223)

Moreover, the \( \phi^{(1)} \)s can be chosen so that
\[
\left( \phi^{(i)}, \phi^{(j)} \right) = 0 \text{ for } i \neq j,
\] (7.224)

and at least three of these \( \phi^{(i)} \)s are spacelike; in particular, at least one of the 2-forms has Euclidean causal type.

Proof. By an equation from Lemma 7.10, since \( K' = 0 \) we have \( \|K\|^2 = \|C\|^2 \|e\|^2 - \langle C, e \rangle^2 \). Recall from Lemma 7.9 that \( \|C\|^2, \|e\|^2 \) and \( \langle C, e \rangle \) are constant, so in particular \( \|K\|^2 \) is constant. The vanishing of \( K' \) also implies that \( C, e \) are locally decomposable and \( \nabla e = 0 \). It only remains to show that \( \langle C, e \rangle = 0 \) and the final claim. Since \( \langle C, e \rangle \) is constant, it suffices to show that it vanishes at some point where \( K \) is not zero, hence let us now assume without loss of generality that \( K \) is nowhere vanishing.

Local decomposability implies that \( C = \phi^{(1)} \wedge \phi^{(2)} \) and \( e = \phi^{(3)} \wedge \phi^{(4)} \) for some 1-forms \( \phi^{(i)} \). Then we have
\[
K^\circ = *(C \wedge e) = *(\phi^{(1)} \wedge \phi^{(2)} \wedge \phi^{(3)} \wedge \phi^{(4)}),
\] (7.225)
so since \( K \) is nowhere vanishing, \( \phi^{(i)} \)s are everywhere linearly independent, and
\[
\left( K^\circ, \phi^{(i)} \right) = \iota_{\phi^{(i)}} K^\circ * (\phi^{(1)} \wedge \phi^{(2)} \wedge \phi^{(3)} \wedge \phi^{(4)}) = 0.
\] (7.226)

By a Gram–Schmidt argument or by Lemma 7.11 depending on the causal types of \( e \) and \( C \), we may further assume without loss of generality that \( (\phi^{(1)}, \phi^{(2)}) = 0 \), \( (\phi^{(3)}, \phi^{(4)}) = 0 \) and that \( \phi^{(2)} \) and \( \phi^{(4)} \) are spacelike. The constraint \([A_C, A_e] = 0 \) (equation (7.134)) can be expressed as
\[
(\phi^{(1)}, \phi^{(3)}) \phi^{(2)} \wedge \phi^{(4)} + (\phi^{(2)}, \phi^{(4)}) \phi^{(1)} \wedge \phi^{(3)} - (\phi^{(1)}, \phi^{(4)}) \phi^{(2)} \wedge \phi^{(3)} - (\phi^{(2)}, \phi^{(3)}) \phi^{(1)} \wedge \phi^{(4)} = 0,
\] (7.227)
which can be rearranged to give either of the two equivalent expressions
\[
\phi^{(2)} \wedge \iota_{\phi^{(1)}} e - \phi^{(1)} \wedge \iota_{\phi^{(2)}} e = 0, \quad \phi^{(3)} \wedge \iota_{\phi^{(4)}} C - \phi^{(4)} \wedge \iota_{\phi^{(3)}} C = 0.
\] (7.228, 7.229)

Contracting the first equation with \( \phi^{(2)} \) gives us
\[
\|\phi^{(2)}\|^2 (\phi^{(1)}, \phi^{(3)}) \phi^{(4)} - \|\phi^{(2)}\|^2 (\phi^{(1)}, \phi^{(4)}) \phi^{(3)} = e \left( \phi^{(1)}, \phi^{(2)} \right) \phi^{(1)} = 0;
\] (7.230)
contracting the second with \( \phi^{(4)} \) gives
\[
- \|\phi^{(4)}\|^2 (\phi^{(1)}, \phi^{(3)}) \phi^{(2)} + \|\phi^{(4)}\|^2 (\phi^{(2)}, \phi^{(3)}) \phi^{(1)} = C \left( \phi^{(1)}, \phi^{(4)} \right) \phi^{(4)} = 0.
\] (7.231)

But now, since \( \phi^{(2)} \) and \( \phi^{(4)} \) are spacelike, by linear independence we must have \( (C, e) = e \left( \phi^{(1)}, \phi^{(2)} \right) = C \left( \phi^{(3)}, \phi^{(4)} \right) = 0 \), and \( \phi^{(1)}, \phi^{(3)} \) are orthogonal to each of the
\(\phi^{(i)}\)'s other than themselves. We then see from (7.227) that \((\phi^{(2)}, \phi^{(4)}) = 0\), hence the \(\phi^{(i)}\)'s are pairwise orthogonal. It follows that, at any point, at most one of \(\phi^{(1)}, \phi^{(3)}\) are timelike or null, hence the other must be spacelike at that point and thus on a neighbourhood of that point, whence on this neighbourhood at least three of the \(\phi^{(i)}\)'s are spacelike.

We first consider the consequences of the above in the case \(\|K\|^2 \neq 0\).

**Corollary 7.15.** If \(K' = 0\) and \(\|K\|^2 \neq 0\), then \(K^0\), \(e\) and \(C\) generate an orthogonal decomposition of the cotangent bundle \(T^*M = T^*_C M \oplus T^*_e M\) and dual orthogonal decomposition of the tangent bundle \(TM = T_K M \oplus T_C M \oplus T_e M\). Furthermore, there is a corresponding decomposition of the base manifold into a Riemannian product which takes one of the following forms.

- If \(\|e\|^2 > 0\) then \(M = \mathbb{L}^3 \times \mathbb{N}^2\), with \(\mathbb{N}^2\) a Riemannian 2-manifold with \(TN^2 = T_e M\) and \(\mathbb{L}^3\) a Lorentzian 3-manifold with \(TL^3 = T_K M \oplus T_C M\); moreover we have either \(\|C\|^2 > 0\) and \(\|K\|^2 > 0\) (timelike) or \(\|C\|^2 < 0\) and \(\|K\|^2 < 0\) (spacelike).

- If \(\|e\|^2 < 0\) then \(M = \mathbb{L}^2 \times \mathbb{N}^3\), with \(\mathbb{L}^2\) a Lorentzian 2-manifold with \(TL^2 = T_e M\) and \(\mathbb{N}^3\) a Riemannian 3-manifold with \(TN^3 = T_K M \oplus T_C M\); moreover we must have \(\|C\|^2 > 0\) and \(\|K\|^2 < 0\) (spacelike).

**Proof.** By the lemma above, \(\|K\|^2 = \|C\|^2 \|e\|^2\) is nowhere vanishing, so \(K\) is nowhere vanishing, \(C\) and \(e\) are of non-degenerate causal type, and we locally have \(C = \phi^{(1)} \wedge \phi^{(2)}\) and \(e = \phi^{(3)} \wedge \phi^{(4)}\) where \(\{\phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \phi^{(4)}, K^0\}\) is an orthogonal set; since the whole set is non-vanishing it must be everywhere linearly independent, which we can also check by noting that

\[
\ast \left( \phi^{(1)} \wedge \phi^{(2)} \wedge \phi^{(3)} \wedge \phi^{(4)} \wedge K^0 \right) = \|K\|^2 \neq 0.
\] (7.232)

The claimed orthogonal decomposition of the cotangent bundle (hence also the tangent bundle) follows, and since \(V e = 0\), the \(T_e M\) distribution (spanned by \(\phi^{(1)}\), \(\phi^{(2)}\)) is holonomy invariant, hence the claimed product decomposition of the base manifold by the de Rham–Wu Theorem. The remaining statements follow from the equation \(\|K\|^2 = \|C\|^2 \|e\|^2\) and the fact that at least one of the 2-forms is Euclidean.

Let us now consider the null case. Unlike the \(\|K\|^2 \neq 0\) case, for \(\|K\|^2 = 0\), \(K\) may vanish on some locus even if it is not zero everywhere. If the vanishing locus is not measure zero then on its interior the solution is locally one of the pp-waves discussed in §7.5.3. By local homogeneity, the solution must be of the same form everywhere\(^6\), so without loss of generality we will assume from now on that \(K\) is non-vanishing.

**Corollary 7.16.** Let \(K' = 0\) and \(\|K\|^2 = 0\) but \(K \neq 0\) everywhere. Then one of the 2-forms \(C, e\) is degenerate and the other Euclidean, and the degenerate form can be written as \(K^0 \wedge \varphi\) for some spacelike \(\varphi \in \Omega^1(M)\) whose contraction with the Euclidean form vanishes. Moreover,

\(^6\)It would be preferable to show this directly using the local data, but it is not clear if this is possible since local homogeneity follows by maximal supersymmetry, which is really a global condition.
• If \( \|C\|^2 = 0 \) and \( \|e\|^2 > 0 \) then \( M = L^3 \times N^2 \), where \( N^2 \) is a (negative) Riemannian 2-manifold with \( TN^2 = T_eM \) and \( L^3 \) is a Lorentzian 3-manifold along which \( C \) and \( K \) both lie.

• If \( \|C\|^2 > 0 \) and \( \|e\|^2 = 0 \) then \( M = L^4 \times N^1 \), where \( L^4 \) is a pp-wave spacetime along which \( C \) and \( K \) both lie and \( N^1 \) is a 1-dimensional (negative) Riemannian manifold along which \( \varphi \) lies.

Proof. We again use the decomposition provided by Proposition 7.14. We know that one of the 2-forms must be Euclidean, so since \( \|C\|^2 \|e\|^2 = \|K\|^2 = 0 \) then the other must be degenerate; for concreteness, let us now assume that \( C \) is the degenerate one – the argument for degenerate \( e \) is of course entirely analogous. Since at least three of the \( \phi^{(i)} \)'s are spacelike but \( C \) is degenerate, either \( \phi^{(1)} \) or \( \phi^{(2)} \) is null; without loss of generality we assume \( \phi^{(1)} \) to be null. Then since

\[
\ast \left( \phi^{(1)} \wedge \phi^{(2)} \wedge \phi^{(3)} \wedge \phi^{(4)} \wedge K^3 \right) = \|K\|^2 = 0
\]

and the \( \phi^{(i)} \)'s are linearly independent, \( K \) must be a linear combination of the \( \phi^{(i)} \)'s; let us set \( K^\varphi = \sum_{i=1}^4 k_i \phi^{(i)} \) for some local functions \( k_i \). Since \( \{K^\varphi, \phi^{(i)}\} = 0 \), contracting this linear combination with \( \phi^{(i)\ast} \) gives us \( k_i \|\phi^{(i)}\|^2 = 0 \) for each \( i \). But this implies that \( k_2 = k_3 = k_4 = 0 \), so \( K^\varphi = k_1 \phi^{(1)} \). Since \( K \) is non-vanishing, so is \( k_1 \), whence \( C = K^\varphi \wedge 1/k_1 \phi^{(2)} \). Clearly \( t_{\phi^{(2)}} e = 0 \), so we have proven the existence of \( \varphi \) locally. By a partition of unity, we can construct a global 1-form \( \varphi \) with the claimed properties.

For the second part of the corollary, we recall that \( e \) is parallel. In the first case, we apply the de Rham–Wu Theorem to the Euclidean holonomy invariant distribution \( T_eM \) generated by \( e \). In the second case, by Lemma 7.11 we can adjust \( \phi^{(3)}, \phi^{(4)} \) in our local expressions such that both are parallel, \( \phi^{(3)} \) is null and \( \|\phi^{(4)}\| = -1 \). Then by the first part of the corollary, \( K^3 \) is collinear to \( \phi^{(3)} \) and we can write \( e = K \wedge \varphi \) where \( \varphi \) is everywhere collinear to a parallel spacelike vector, the distribution \( T_eM \) it generates is a non-degenerate holonomy invariant distribution, hence we can apply the de Rham–Wu theorem to obtain the decomposition. □

We are now ready to determine the geometries and their associated Killing superalgebras. As already mentioned, the solutions belong to three branches based on the causal type of \( K \). They will then branch further based on the causal types of \( C \) and \( e \).

Timelike Killing vector

Suppose \( K \) is timelike. From Corollary 7.15, \( C \) and \( e \) have Euclidean causal type and (at least locally) there is an orthonormal coframe \( \{\theta^\mu\} \) (obtained by normalising the orthogonal set of 1-forms constructed in the proof of the proposition) such that

\[
C = \frac{a}{2} \theta^1 \wedge \theta^2, \quad e = \frac{b}{2} \theta^3 \wedge \theta^4, \quad K = \frac{ab}{4} \theta^0,
\]  

(7.234)
for some \( a, b \in \mathbb{R}^\times \); without loss of generality we may take \( a, b > 0 \). Note that \( \|C\|^2 = \frac{a^2}{4}, \|e\|^2 = \frac{b^2}{4} \) (whence \( a \) and \( b \) are constant) and \( \|K\|^2 = \frac{1}{16} a^2 b^2 \). In the given frame, the equation (7.138) for the Riemann tensor gives

\[
R_{0010} = b^2, \quad R_{0202} = b^2, \quad R_{1212} = a^2 - b^2, \tag{7.235}
\]

with all other components vanishing. The de Rham–Wu decomposition is thus (locally) \( M = L^3 \times \mathbb{R}^2 \) (i.e. the Riemannian factor is flat), with \( \{\theta^0, \theta^1, \theta^2\} \) being an orthonormal coframe for \( L^3 \) in which the Riemann tensor is given above. The non-zero components of the Ricci tensor and scalar curvature of \( L^3 \) are

\[
R_{00} = 2b^2, \quad R_{11} = a^2 - 2b^2, \quad R_{22} = a^2 - 2b^2, \quad R = 6b^2 - 2a^2. \tag{7.236}
\]

Note that by tuning \( a^2 \) and \( b^2 \), the scalar curvature can take any real value.

As for the previous solutions, we determine the Killing superalgebra \( \mathfrak{K}_D \) by localising at a point and using the general form of the brackets (7.56) and (7.58). Here we find that \( \mathfrak{K}_D \) is a deformation of the graded subalgebra \( V \oplus S \oplus \mathfrak{h} \oplus \mathfrak{sp}(1) \) of \( \mathfrak{g} \), where \( \mathfrak{h} = \langle L_{12}, L_{34} \rangle \), and the non-zero deformed brackets (again using (7.190)) are

\[
[P_0, P_1] = 2bP_2, \quad [P_0, P_2] = -2bP_1, \quad [P_1, P_2] = -2bP_0 + a^2 L_{12}, \quad [P_0, \epsilon] = 2b\Gamma_{12}\epsilon, \quad [P_1, \epsilon] = (\frac{b}{2} \Gamma_1 + b\Gamma_{02})\epsilon, \quad [P_2, \epsilon] = (\frac{b}{2} \Gamma_1 - b\Gamma_{01})\epsilon, \tag{7.237}
\]

\[
[\epsilon, \epsilon] = \kappa^\mu_\nu P_\mu + am_\nu L_{12} + \left(a\omega^A_{12} - 2b\omega^i_{34}\right)\tau_i.
\]

As in previous examples, the \( \{L, P\} \) brackets remain undeformed. Note that the deformation is trivial along the 3,4 directions, corresponding to the flat geometry, and we have a decomposition of the vector part of the algebra as a direct sum of algebras acting transitively on each factor of the product \( M = L^3 \times \mathbb{R}^2 \):

\[
Q_D = \langle L_{12}, L_{34}, P_\mu \rangle = \langle L_{12}, P_0, P_1, P_2 \rangle \oplus \langle L_{34}, P_3, P_4 \rangle \tag{7.238}
\]

with the second summand being \( \langle L_{34}, P_3, P_4 \rangle \cong \text{iso}(2) \). Note that this shows that both \( L^3 \) and \( \mathbb{R}^2 \) are (locally) reductive homogeneous spaces.

We will now deduce the geometry of \( L^3 \) using the Komrakov classification discussed in §2.4.5. We first identify the metric Lie pair \((\bar{\mathfrak{g}}, \mathfrak{g}, B)\), where \( \bar{\mathfrak{g}} = \langle L_{12}, P_0, P_1, P_2 \rangle \), \( \mathfrak{g} = \langle L_{12} \rangle \) and \( B \) is Lorentzian inner product induced on \( \bar{\mathfrak{g}}/\mathfrak{g} \) by the metric on \( M \) (explicitly we have \( B(P_a, P_b) = \eta_{ab} \) where \( a, b = 0, 1, 2 \), with an entry in the local classification in [89] and then follow the global classification to identify \( L^3 \) as a Lorentzian homogeneous space. Our notation here is chosen to make comparison with Komrakov as straightforward as possible.

We can reduce the number of parameters and numerical factors in our presentation of \( \bar{\mathfrak{g}} \) by making the change of basis \( e_a = -\eta_{aa}\frac{1}{2b^2}P_a, \ell = -L_{12} \) and introducing the new parameter \( c = \frac{a^2}{4b^2} \). The bracket of \( \bar{\mathfrak{g}} \) is then given by

\[
[e_0, e_1] = -e_2, \quad [e_0, e_2] = e_1, \quad [e_1, e_2] = e_0 - c\ell, \quad [\ell, e_0] = 0, \quad [\ell, e_1] = -e_2, \quad [\ell, e_2] = e_1, \tag{7.239}
\]
and the pairing on $\mathfrak{g}/\mathfrak{g}$ is
\[ \langle e_a, e_b \rangle = \frac{1}{4b^2} \eta_{ab}. \quad (7.240) \]

Now note that $z = e_0 - \ell$ is a central element, so let us change basis:
\[ [z, e_1] = 0, \quad [z, e_2] = 0, \quad [e_1, e_2] = z + (1 - c)\ell, \]
\[ [\ell, z] = 0, \quad [\ell, e_1] = -e_2, \quad [\ell, e_2] = e_1. \quad (7.241) \]

We now have 3 cases:

- For $c = 1$, $\langle e_1, e_2, e_3 \rangle$ is a subalgebra isomorphic to the Heisenberg algebra $\mathfrak{n}_3$ expressed as a central extension of an abelian algebra $\mathbb{R}^2$ spanned by $e_1, e_2$ by a central charge $z$. Then $\ell$ spans an $\mathfrak{so}(2)$ acting on $\mathbb{R}^2$ via the fundamental representation, giving us $\mathfrak{g} = \mathfrak{so}(2) \ltimes \mathfrak{n}_3$, $\mathfrak{g} = \mathfrak{so}(2)$. This is Komrakov type 2.7 with
\[ B = \frac{1}{4b^2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7.242) \]

The simply connected Lorentzian homogeneous space $(M, g)$ can thus be described as follows: $M = \mathbb{R}^3 = (\mathfrak{so}(2) \ltimes \mathfrak{n}_3)/\mathfrak{so}(2)$ where $\mathfrak{n}_3$ is the 3-dimensional Heisenberg group, which we consider as $\mathfrak{n}_3$ with $\mathfrak{so}(2)$ acting via the fundamental representation on its first two coordinates, and
\[ g = \frac{1}{4b^2} \left( (dx^0 + \frac{1}{2}(x^2 dx^1 - x^1 dx^2))^2 - (dx^1)^2 - (dx^2)^2 \right) \quad (7.243) \]

- For $0 < c < 1$ we can take $u_i = \frac{1}{\sqrt{1-c}} e_i$ for $i = 1, 2$ and $u_3 = \frac{1}{1-c} z$ to get
\[ [u_1, u_2] = u_3 + \ell, \quad [u_1, u_3] = 0, \quad [u_2, u_3] = 0, \]
\[ [\ell, u_1] = -u_2, \quad [\ell, u_2] = u_1, \quad [\ell, u_3] = 0, \quad (7.244) \]

this is Komrakov type 2.3, $\mathfrak{g} = \mathfrak{sl}(2) \ltimes \mathbb{R}$, $\mathfrak{g} = \mathbb{R} \left( E - F = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), 1 \right)$, with
\[ B = \frac{1}{4b^2(1-c)} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{1-c} \end{pmatrix} = \frac{1}{4b^2 - a^2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{4b^2}{4b^2 - a^2} \end{pmatrix}. \quad (7.245) \]

- For $c > 1$ we can take $u_i = \frac{1}{\sqrt{c-1}} e_i$ for $i = 1, 2$ and $u_3 = \frac{1}{c-1} z$ to get
\[ [u_1, u_2] = u_3 - \ell, \quad [u_1, u_3] = 0, \quad [u_2, u_3] = 0, \]
\[ [\ell, u_1] = -u_2, \quad [\ell, u_2] = u_1, \quad [\ell, u_3] = 0, \quad (7.246) \]

this is Komrakov type 2.4, $\mathfrak{g} = \mathfrak{su}(2) \ltimes \mathbb{R}$, $\mathfrak{g} = \mathbb{R} \left( r_3, 1 \right)$ with
\[ B = \frac{1}{4b^2(1-c)} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{1-c} \end{pmatrix} = \frac{1}{a^2 - 4b^2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{4b^2}{a^2 - 4b^2} \end{pmatrix}. \quad (7.247) \]
The inner product also takes this form in the basis \( \{ t_1 = u_1, t_2 = u_2, t_3 = u_3 - \ell, \ell \} \), in which \( [ t_i, t_j ] = \epsilon_{ijk} t_k \) (no sum), whence we can explicitly realise the claimed form for \( (\mathfrak{g}, \mathfrak{g}) \) by mapping \( t_i \mapsto (\tau_i, 0) \) and \( \ell \mapsto (-\tau_3, \frac{1}{2}) \).

**Spacelike Killing vector, type 1**

If \( K \) is spacelike then since \( \| C \|_2 \| e \|_2 = \| K \|_2 < 0 \), the causal types of \( e \) and \( C \) must be different and non-degenerate. If \( C \) is Lorentzian and \( e \) Euclidean, there exits a frame in which

\[
C = \frac{\alpha}{2} \theta^0 \wedge \theta^1, \quad e = \frac{\beta}{2} \theta^3 \wedge \theta^4, \quad K = \frac{\alpha \beta}{4} \theta^2, \tag{7.248}
\]

and now \( \| C \|^2 = -\frac{\alpha^2}{4}, \| e \|^2 = \frac{\beta^2}{4} \) and \( \| K \|^2 = -\frac{1}{16} \alpha^2 \beta^2 \); again \( a, b \) are constants. The non-zero components of the Riemann tensor are

\[
R_{0101} = a^2 + b^2, \quad R_{0202} = b^2, \quad R_{1212} = -b^2. \tag{7.249}
\]

We then have a different family of solutions of the form \( M = L^3 \times \mathbb{R}^2 \), with the Ricci tensor and scalar of \( L^3 \) given by

\[
R_{00} = a^2 + 2b^2, \quad R_{11} = -a^2 - 2b^2, \quad R_{22} = -2b^2, \quad R = 2a^2 + 6b^2. \tag{7.250}
\]

The scalar curvature here must be positive. In our mostly-minus signature, this is \( AdS \)-type.

Using the same method as above, we determine the Killing superalgebra here to be a deformation of a graded subalgebra of \( \mathfrak{h} \) of similar form, except this time \( h = \langle L_0, L_{34} \rangle \), and the deformed brackets are given by

\[
\begin{align*}
[P_0, P_1] &= 2bP_2 - a^2 L_{01}, & [P_0, P_2] &= -2bP_1, & [P_1, P_2] &= -2bP_0, \\
[P_0, e] &= \left(-\frac{\alpha}{2} \Gamma_1 + b \Gamma_{12}\right)e, & [P_1, e] &= \left(-\frac{\alpha}{2} \Gamma_0 + b \Gamma_{02}\right)e, & [P_2, e] &= -b \Gamma_{01} e, \tag{7.251}
\end{align*}
\]

\[
[e, e] = \kappa_c \mu^P_\mu - a \mu_c L_{01} - \left(aw_{01}^{AB} + 2bu_{34}^j\right) \tau_i.
\]

Again, there is no deformation along the flat directions, and there is a decomposition of the vectorial part of the algebra:

\[
\mathcal{Q}_D = \langle L_{01}, L_{34}, P_\mu \rangle = \langle L_{01}, P_0, P_1, P_2 \rangle \oplus \langle L_{34}, P_3, P_4 \rangle, \tag{7.252}
\]

with \( \langle L_{34}, P_3, P_4 \rangle \cong \text{iso}(2) \). Similarly to the \( \| K \|^2 > 0 \), we identify the 3-dimensional Lorentzian factor by expressing it as a homogeneous space for the metric Lie pair \((\mathfrak{g} = \langle L_{01}, P_0, P_1, P_2 \rangle, \mathfrak{g} = \langle L_{01}, \langle -, -, \rangle \rangle) \). Setting \( e_a = -\eta_{aa} \frac{1}{2b} P_a, \ell = L_{01} \) and \( c = \frac{\alpha^2}{4b^2} \), we have

\[
\begin{align*}
[e_0, e_1] &= -e_2 + c \ell, & \quad [e_0, e_2] &= e_1, & \quad [e_1, e_2] &= e_0, \\
[\ell, e_0] &= e_1, & \quad [\ell, e_1] &= e_0, & \quad [\ell, e_2] &= 0, \tag{7.253}
\end{align*}
\]

and the pairing on \( \mathfrak{g} / \mathfrak{g} \) is

\[
\langle e_a, e_b \rangle = \frac{1}{4b^2} \eta_{ab}. \tag{7.254}
\]

We have a central element \( e_2 + \ell \), so let us set \( z = -(e_2 + \ell) \). We will also set \( e_\pm = \frac{1}{2}(e_1 \pm e_2) \).
\[ \frac{1}{2} (e_0 \pm e_1). \] Then we have

\[
\begin{align*}
[e_+, e_-] &= z + (c + 1) \ell, & [e_+, z] &= 0, & [e_-, z] &= 0, \\
[\ell, e_+] &= e_+, & [\ell, e_-] &= -e_-, & [\ell, z] &= 0.
\end{align*}
\] (7.255)

Now setting \( u_1 = \frac{1}{\sqrt{c+1}} e_+, u_2 = \frac{1}{\sqrt{c+1}} e_-, u_3 = \frac{1}{c+1} z, \) we have

\[
\begin{align*}
[u_1, u_2] &= u_3 + \ell, & [u_1, u_3] &= 0, & [u_2, u_3] &= 0, \\
[\ell, u_1] &= u_1, & [\ell, u_2] &= -u_2, & [\ell, u_3] &= 0.
\end{align*}
\] (7.256)

This is Komrakov type 1.5, \( \bar{g} = s\mathfrak{l}(2, \mathbb{R}) \times \mathbb{R}, \mathfrak{g} = \mathbb{R} \left( H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \) with

\[
B = \frac{1}{4b^2(c+1)} \left( \begin{array}{ccc} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{c+1} \end{array} \right) = \frac{1}{a^2 + 4b^2} \left( \begin{array}{ccc} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{4b^2}{a^2 + 4b^2} \end{array} \right).
\] (7.257)

**Spacelike Killing vector, type 2**

If \( K \) is spacelike, \( e \) Lorentzian and \( C \) Euclidean, we can write

\[
C = \frac{a}{2} \theta^1 \wedge \theta^2, \quad e = \frac{b}{2} \theta^0 \wedge \theta^4, \quad K = \frac{ab}{4} \theta^3,
\]

whence \( \|C\|^2 = \frac{a^2}{4}, \|e\|^2 = -\frac{b^2}{4} \) and \( \|K\|^2 = -\frac{1}{16} a^2 b^2 \) with \( a \) and \( b \) constant, and the Riemann tensor is given by

\[
R_{1212} = a^2 + b^2, \quad R_{1313} = b^2, \quad R_{2323} = b^2.
\] (7.259)

The decomposition this time is \( M = \mathbb{R}^{1,1} \times N^3 \), with \( N^3 \) (negative) Riemannian with coframe \( \{\theta^1, \theta^2, \theta^3\} \). The Ricci tensor and scalar here are given by

\[
R_{11} = a^2 + 2b^2, \quad R_{22} = a^2 + 2b^2, \quad R_{33} = 2b^2, \quad R = -2a^2 - 6b^2.
\] (7.260)

Here the scalar curvature is negative. In negative-definite signature, this is of spherical type.

The Killing superalgebra here is a deformation of a graded subalgebra of \( \hat{s} \) with \( \mathfrak{h} = \langle L_{12}, L_{04} \rangle \) and deformed brackets

\[
\begin{align*}
[P_1, P_2] &= -2bP_3 + a^2 L_{12}, & [P_1, P_3] &= 2bP_2, & [P_2, P_3] &= -2bP_1, \\
[P_1, \epsilon] &= \left(-\frac{a}{2} \Gamma_2 - b \Gamma_{23}\right) \epsilon, & [P_2, \epsilon] &= \left(\frac{a}{2} \Gamma_1 + b \Gamma_{13}\right) \epsilon, & [P_3, \epsilon] &= -b \Gamma_{12} \epsilon, \\
[\epsilon, \epsilon] &= \kappa_\mu^\rho P_\mu + a \mu \epsilon L_{12} + \left(a \omega_{12}^{AB} + 2b \omega_{04}^i \right) \tau_i,
\end{align*}
\] (7.261)

and we see that once again that the flat directions \((0,4)\) here are undeformed. There is a corresponding decomposition of the vectorial part of the algebra:

\[
\mathfrak{V}_D = \langle L_{12}, L_{04}, P_\mu \rangle = \langle L_{04}, P_0, P_4 \rangle \oplus \langle L_{12}, P_1, P_2, P_3 \rangle.
\] (7.262)
with \( \{L_{04},P_0,P_4\} \equiv \mathfrak{iso}(1,1) \). This time our metric Lie pair \((\mathfrak{g} = \{L_{12},P_1,P_2,P_3\},\mathfrak{g} = \langle L_{12},(-,-) \rangle)\) for the 3-dimensional factor is (negative) Riemannian; setting \( e_a = -\frac{1}{2b} P_a \) where the index \( a \) runs over 1,2,3, \( \ell = -L_{12} \) and \( c = \frac{a^2}{4b^2} \), we have

\[
[e_1,e_2] = e_3 - c\ell, \quad [e_1,e_3] = -e_2, \quad [e_2,e_3] = e_1, \\
[\ell,e_1] = -e_2, \quad [\ell,e_2] = e_1, \quad [\ell,e_3] = 0,
\]

and

\[
\langle e_a, e_b \rangle = \frac{1}{4b^2} \eta_{ab} = -\frac{1}{4b^2} \delta_{ab}.
\]

We once again have a central element \( z = e_3 + \ell \). Let us set \( u_1 = \frac{1}{\sqrt{c+1}} e_1, u_2 = \frac{1}{\sqrt{c+1}} e_2, u_3 = \frac{1}{c+1} z = \frac{1}{c+1} (e_3 - \ell) \); this gives us

\[
[u_1,u_2] = u_3 - \ell, \quad [u_1,u_3] = 0, \quad [u_2,u_3] = 0, \\
[\ell,u_1] = -u_2, \quad [\ell,u_2] = u_1, \quad [\ell,u_3] = 0
\]

This is Komrakov type 2.4, \( \mathfrak{g} = \mathfrak{su}(2) \times \mathbb{R}, \mathfrak{g} = \mathbb{R}(\tau_3,1) \), with

\[
B = \frac{1}{4b^2(c+1)} \begin{pmatrix} -1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -\frac{1}{c+1} \end{pmatrix} = \frac{1}{a^2 + 4b^2} \begin{pmatrix} -1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -\frac{4b^2}{a^2 + 4b^2} \end{pmatrix}.
\]

**Null Killing vector, type 1**

By Corollary 7.16, if \( \|K\|^2 = 0 \) then exactly one of \( C \) or \( e \) is degenerate and the other Euclidean. We first consider the case where \( \|C\|^2 = 0, \|e\|^2 > 0 \), thus by the corollary we have a decomposition \( M = L^3 \times N^2 \) with \( L^3 \) a 3-dimensional Lorentzian manifold and \( N^2 \) a 2-dimensional (negative) Riemannian one; note that \( K \) lies tangent to \( L^3 \).

Using the notation of Proposition 7.14, we have \( C = \phi^{(1)} \wedge \phi^{(2)} \) and \( e = \phi^{(3)} \wedge \phi^{(4)} \) where the \( \phi^{(i)} \)'s are linearly independent and pairwise orthogonal, \( \phi^{(1)} \) is null and the others are spacelike, and by Corollary 7.16, \( K \) is proportional to \( \phi^{(1)} \). It will be useful to rescale \( \phi^{(1)} \) and \( \phi^{(2)} \) now so that \( \|\phi^{(2)}\| = -1 \). We can then form a Witt coframe \( \{\theta^+,\theta^-,\theta^1,\theta^2,\theta^3\} \) where

\[
\theta^- := 2\phi^{(1)}, \quad \theta^1 := \phi^{(2)}, \quad \theta^2 := \frac{\phi^{(3)}}{\sqrt{-\|\phi^{(3)}\|^2}}, \quad \theta^3 := \frac{\phi^{(4)}}{\sqrt{-\|\phi^{(4)}\|^2}};
\]

the normalisation of \( \theta^- \) is chosen for later convenience. Note that \( \{\theta^+,\theta^-,\theta^1\} \) is a Witt coframe for \( L^3 \) and \( \{\theta^2,\theta^3\} \) is an orthonormal coframe for \( N^2 \). We have

\[
C = \frac{1}{2} \theta^- \wedge \theta^1, \quad e = \frac{b}{2} \theta^2 \wedge \theta^3, \quad K^b = \frac{b}{4} \theta^-,
\]

where \( b = 2\sqrt{\|\phi^{(3)}\|^2 \|\phi^{(4)}\|^2} \). Note that \( \|e\|^2 = \frac{b^2}{4} \); in particular \( b \) is constant, and \( K = \frac{b}{4} e_+ \) in the canonical Witt frame \( \{e_+,e_-,e_1,e_2,e_3\} \) defined by \( \theta^\mu(e_\nu) = \delta^\mu_\nu \). The non-zero components of the Riemann tensor are

\[
R_{+++} = b^2, \quad R_{--} = 1, \quad R_{++} = b^2, \quad R_{2323} = -b^2,
\]

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and the Ricci tensors and scalars for the two factors are

\[
R^L_{--} = 1, \quad R^L_{+-} = 2b^2, \quad R^L_{-+} = 4b^2, \quad R^L_{++} = R^N_{22} = -b^2, \quad R^N = 2b^2.
\]  

(7.270)

The Killing superalgebra is deformation of the maximally supersymmetric graded subalgebra of \( \mathfrak{s} \) with \( \mathfrak{h} = \langle L_{+1}, L_{23} \rangle \) and deformed brackets

\[
\begin{align*}
[P_+, P_-] &= 2bP_1, \quad [P_+, P_1] = 2bP_+, \quad [P_-, P_1] = -2bP_1 - L_{+1}, \quad [P_2, P_3] = -b^2 L_{23}, \\
[P_+, e] &= -b\Gamma_{+1} e, \quad [P_-, e] = (\frac{1}{2}\Gamma_{+1} + b\Gamma_{-1}) e, \quad [P_1, e] = (\frac{1}{2}\Gamma_{+1} - b\Gamma_{-1}) e, \\
[e, e] &= \kappa^\mu_\ell P_\mu - a\mu_\ell L_{+1} - \left(aw^{AB}_{\mu_1} + 2bw^{i}_{23}\right) \tau_i.
\end{align*}
\]  

(7.271)

Once again, the vectorial part of the algebra decomposes into algebras acting transitively on \( L^3 \) and \( N^2 \):

\[
\mathfrak{so}_D = \langle L_{+1}, L_{23}, P_\mu \rangle = \langle L_{+1}, P_+, P_-, P_1 \rangle \oplus \langle L_{23}, P_2, P_3 \rangle.
\]  

(7.272)

This time, however, the 2-dimensional factor is not flat; we have \( \langle L_{23}, P_2, P_3 \rangle \cong \text{iso}(H^2) \), the isometry algebra of the maximally symmetric 2-dimensional hyperbolic space, thus \( N^2 = H^2 \).

We once again identify the 3-dimensional factor as a homogeneous space. Our metric Lie pair is \( (\mathfrak{g} = \langle L_{+1}, P_+, P_-, P_1 \rangle, \mathfrak{g} = \langle L_{+1}, \langle -, - \rangle \rangle) \); we set \( e_\pm = -\frac{1}{2b}e_\pm, e_1 = \frac{1}{2b} P_1 \) and \( \ell = -L_{+1} \), finding

\[
\begin{align*}
[e_1, e_+] &= -e_+, \quad [e_1, e_-] = e_--\ell, \quad [e_+, e_-] = e_1, \\
[\ell, e_+] &= e_+, \quad [\ell, e_-] = 0, \quad [\ell, e_+] = e_1.
\end{align*}
\]  

(7.273)

and

\[
\langle e_a, e_b \rangle = \frac{1}{4b^2} \eta_{ab}
\]  

(7.274)

where the indices \( a, b \) run over \( +, -, 1 \). Note that this time there is no deformation parameter remaining in the brackets; there is a single metric Lie pair \( (\mathfrak{g}, \mathfrak{g}, B) \) rather than a 1-parameter family here.

We have a central element \( z = e_+ - \ell \), however, transforming to a basis containing \( z \) will not put this into a Komrakov form; let us instead set \( u_1 = e_+, u_2 = e_1, u_3 = e_- - \ell - \frac{1}{2}e_+ \); this gives us

\[
\begin{align*}
[u_1, u_2] &= u_1, \quad [u_1, u_3] = u_2, \quad [u_2, u_3] = u_3, \\
[\ell, u_1] &= 0, \quad [\ell, u_2] = u_1, \quad [\ell, u_3] = u_2.
\end{align*}
\]  

(7.275)

This is of Komrakov type 3.2 \( \mathfrak{g} = \text{sl}(2) \times \mathbb{R}, \mathfrak{g} = \mathbb{R}\left( E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 1 \right) \), with

\[
B = \frac{1}{4b^2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
\]  

(7.276)
Null Killing vector, type 2

Now let $\|C\|^2 \neq 0, \|e\|^2 = 0$. We again start with $C = \phi^{(1)} \wedge \phi^{(2)}$ and $e = \phi^{(3)} \wedge \phi^{(4)}$ except that now $\phi^{(3)}$ is the null 1-form (hence also collinear with $K$) and since $\nabla e = 0$, by Lemma 7.11 we may assume that $\|\phi^{(4)}\|^2 = -1$ and that $\phi^{(3)}, \phi^{(4)}$ are parallel. We can form a Witt coframe with

$$\theta^- := 2\phi^{(3)}, \quad \theta^1 := \frac{\phi^{(1)}}{\sqrt{-\|\phi^{(1)}\|^2}}, \quad \theta^2 := \frac{\phi^{(2)}}{\sqrt{-\|\phi^{(2)}\|^2}}, \quad \theta^3 := \phi^{(4)}. \quad (7.277)$$

We then have

$$C = \frac{a}{2} \theta^1 \wedge \theta^2, \quad e = \frac{1}{2} \theta^- \wedge \theta^3, \quad K^3 = \frac{a}{4} \theta^-, \quad (7.278)$$

where $a = 2\sqrt{\|\phi^{(1)}\|^2 \|\phi^{(2)}\|^2}$ and $\|C\|^2 = \frac{a^2}{4}$, so $a$ is constant, and $K = \frac{a}{4} e_+$ in the canonical Witt frame dual to our coframe. Note that $K$ is actually parallel.

Corollary 7.16 tells us that $M = L^4 \times N$, where $L^4$ is a pp-wave with parallel null vector $K$, and $N$ is a spacelike one-dimensional manifold which, since we work locally, we can assume for simplicity to be $\mathbb{R}$ (with negative-definite metric). Note that $\theta^3$ lies along the spacelike $\mathbb{R}$ factor and $\{\theta^+, \theta^-, \theta^1, \theta^2\}$ is a Witt frame for the Lorentzian factor $L^4$.

The non-zero components of the Riemann and Ricci tensors and the Ricci scalar (of $M$ and of $L^4$) are

$$R_{1-1} = R_{2-2} = 1, \quad R_{1212} = a^2, \quad R_{-} = 2, \quad R_{11} = R_{22} = a^2, \quad R = -2a^2. \quad (7.279)$$

For the Killing superalgebra, here we have $\mathfrak{h} = \langle L_{12}, L_{+3} \rangle$ and deformed brackets

$$[P_+ , P_1 ] = 2P_2, \quad [P_+ , P_2 ] = -2P_1, \quad [P_1 , P_2 ] = -2P_+ + a^2 L_{12}, \quad [P_- , \epsilon ] = \Gamma_{12} \epsilon, \quad [P_1 , \epsilon ] = (-\frac{9}{4} \Gamma_2 + \Gamma_{+2}) \epsilon, \quad [P_2 , \epsilon ] = (\frac{9}{4} \Gamma_1 - \Gamma_{+1}) \epsilon, \quad \{\epsilon, \epsilon \} = \kappa^\mu \epsilon \Gamma_{+} P_{+} + a \mu \epsilon L_{12} + \left( a \omega_{12} + 2 b w_{+3} \right) \tau_1. \quad (7.280)$$

The vectorial part of the algebra does not decompose here, but we do have a decomposition of a transitive subalgebra into a direct sum of subalgebras acting transitively on the two factors:

$$\langle L_{12}, P_{\mu} \rangle = \langle L_{12}, P_+, P_- , P_1 , P_2 \rangle \oplus \langle P_3 \rangle. \quad (7.281)$$

The Riemannian factor $L^4$ is thus a Lorentzian homogeneous space locally described by the metric Lie pair $(\mathfrak{g} = \langle L_{12}, P_+, P_- , P_1 , P_2 \rangle, \mathfrak{g} = \langle L_{12} \rangle, B)$ where $B$ is the bilinear form induced on $\mathfrak{g}/\mathfrak{g}$, with $B(P_a, P_b) = \eta_{ab}$ for $a = +, -, 1, 2$.

Let us choose a new basis $e_\pm = -\frac{1}{2} P_2$, $e_1 = \frac{1}{2} P_1$, $e_2 = \frac{1}{2} P_2$, $\ell = L_{12}$ for $\mathfrak{g}$ and introduce the new parameter $c = \frac{1}{4 \ell^2}$. The element $e_+$ is central, and the brackets of the other basis elements are

$$[e_-, e_1] = e_2, \quad [e_-, e_2] = e_1, \quad [e_1, e_2] = e_+ + \ell, \quad [\ell , e_1 ] = e_2, \quad [\ell , e_2 ] = -e_1, \quad [\ell , e_- ] = 0. \quad (7.282)$$
Note that we have another central element \( z = e_- + \ell \). Letting \( u_1 = e_1, u_2 = e_+ \), \( u_3 = e_2 \), \( u_4 = z = e_- + \ell \), we have

\[
[u_1, u_3] = u_2 + \ell, \quad [\ell, u_1] = u_3, \quad [\ell, u_3] = -u_1,
\]

(7.283)

with all other brackets vanishing. In the local classification of 4-dimensional pseudo-Riemannian homogeneous spaces by B. Komrakov Jnr. [92, 93] this is of type \( 1.1^2.3 \); \( \tilde{\mathfrak{g}} = \mathfrak{su}(2) \oplus \mathbb{R} \oplus \mathbb{R} \) and \( \mathfrak{g} = \mathbb{R}(\tau_2, 1, 0) \) with

\[
B = \frac{1}{4}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

(7.284)

We can more explicitly realise \( \tilde{\mathfrak{g}} \) as \( \mathfrak{su}(2) \oplus \mathbb{R} \oplus \mathbb{R} \) by making the basis change \( t_1 = u_1 \), \( t_2 = -u_2 - \ell, t_3 = u_3, z_1 = 2u_2, z_2 = u_4 \); we then have

\[
[t_i, t_j] = \varepsilon_{ijk} t_k,
\]

(7.285)

with \( z_1, z_2 \) central. Note that \( \mathfrak{g} \) is then spanned by \( \ell = -t_2 + \frac{1}{2} z_1 \). Now we can represent the \( t_i \) by \( \tau_i \), whence \( 2\ell = (i\sigma, 1, 0) \) in \( \mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R} z_1 \oplus \mathbb{R} z_2 \). As a homogeneous space, then, we have \( M = \tilde{G}/G \) where \( G = \mathbb{SU}(2) \times \mathbb{R}^+ \times \mathbb{R}^+ \) and \( G = \{(R_t, e^t, 1) | t \in \mathbb{R}\} \) where \( R_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \).

### 7.5.5 Maximally supersymmetric backgrounds with \( F \neq 0 \)

For completeness, we recall the results of [50]. Since \( F \) is parallel with respect to \( \nabla \) and \( [F \wedge F] = 0 \) then at least locally we can find a parallel 1-form \( \phi \) and a parallel section \( r \) of the \( \tau \)-symmetry bundle \( \mathfrak{sp}(M) \) such that \( F = \phi \otimes r \). Moreover, we can choose a frame for the \( R \)-symmetry bundle (i.e. fix a gauge) such that in this basis \( r \) is proportional to \( \tau_3 \); we can arrange for \( F = 4\phi \otimes \tau_3 \).

The Killing superalgebra for a background in this class is a deformation of the maximally supersymmetric subalgebra of \( \tilde{\mathfrak{g}} \) with \( h = \mathfrak{stab}(\mathfrak{g}_0(\nabla)) = \mathfrak{stab}(\mathfrak{sp}(1)(\tau_3)) \). Recall that the brackets (7.56) remain undeformed; the \( [L, P] \) bracket also remains undeformed in this case. With our chosen description of \( F \), the deformed brackets given in (7.59) become

\[
\begin{align*}
[P_{\mu}, P_{\nu}] &= 2\phi_{[\mu}\phi^{\sigma}L_{\nu]3} + \|\phi\|^2 L_{\mu\nu}, \\
[P_{\mu}, s] &= (\phi^{\nu} \Gamma_{\mu\nu} - 2\phi_{\mu})(\tau_3 s), \\
[s, s] &= \kappa^{\mu} P_{\mu} + \frac{1}{2} e^{\mu\nu\rho\sigma} \phi_{\mu} \omega^3_{\rho\sigma} L_{\mu\nu}.
\end{align*}
\]

(7.286)

We now consider the three sub-cases which are based on the causal type of \( \phi \).

#### Timelike \( \phi \)

If \( \|\phi\|^2 > 0 \) then \( M = \mathbb{R}^1 \times S^4 \) as a pseudo-Riemannian space, with \( \phi \) tangent to the first factor. Let us choose an adapted orthonormal coframe \( \{\theta^\mu\} \) in which \( \phi = a\theta^0 \).
and the spacelike coframe forms \( \theta^a \) for \( a = 1,2,3,4 \) are tangent to the \( S^4 \) factor. In this case, \( \mathfrak{h} = \langle L_{ab} \rangle \) and the deformed brackets are

\[
[P_a, P_b] = a^2 L_{ab}, \quad [P_0, P_a] = 0, \quad [P_0, \epsilon] = -2a \tau_3 \epsilon, \quad [P_a, \epsilon] = -a \Gamma_0 a \tau_3 \epsilon,
\]

\[
[\epsilon, \epsilon] = \kappa^\mu P_\mu + 2a (\omega_{34}^2 L_{12} - \omega_{24}^3 L_{13} + \omega_{23}^3 L_{14} + \omega_{14}^3 L_{23} - \omega_{13}^3 L_{24} + \omega_{12}^3 L_{34}).
\]

**Spacelike \( \phi \)**

If \( \| \phi \|^2 < 0 \) then \( M = AdS_4 \times \mathbb{R} \) as a pseudo-Riemannian space, with \( \phi \) tangent to the second factor. We choose an adapted orthonormal coframe \{ \( \theta^\mu \) \} in which \( \phi = a \theta^4 \) and the remaining coframe forms are an orthonormal coframe for the \( AdS_4 \) factor; we use spacetime indices \( a, b, c, \cdots = 0,1,2,3 \) and spacial indices \( i, j, k, \cdots = 1,2,3 \) for this factor. In this case, \( \mathfrak{h} = \langle L_{ab} \rangle = \langle L_{ij}, L_{0i} \rangle \) and the deformed brackets are

\[
[P_i, P_j] = -a^2 L_{ij}, \quad [P_0, P_i] = [P_4, P_i] = [P_0, P_4] = 0, \quad [P_0, \epsilon] = -a \Gamma_0 a \tau_3 \epsilon, \quad [P_i, \epsilon] = -a \Gamma_i a \tau_3 \epsilon, \quad [P_4, \epsilon] = -2a \tau_3 \epsilon,
\]

\[
[\epsilon, \epsilon] = \kappa^\mu P_\mu + 2a (\omega_{34}^3 L_{01} - \omega_{13}^3 L_{02} + \omega_{12}^3 L_{03} + \omega_{03}^3 L_{12} - \omega_{02}^3 L_{13} + \omega_{01}^3 L_{23}).
\]

**Null \( \phi \)**

If \( \| \phi \| = 0 \) and \( \phi \neq 0 \) then \( M = \mathbb{R}^{1,4} \) and the Killing superalgebra is \( \mathfrak{g} \). If \( \| \phi \| = 0 \) and \( \phi \neq 0 \) then \( M = CW_6(1) \), with the distinguished parallel null vector \( \phi^i \). In this case, we choose an adapted Witt coframe \{ \( \theta^+ = \theta^-, \theta^1, \theta^2, \theta^3 \) \} in which \( \phi = a \theta^- \) (so \( \phi^i = a \epsilon_+ \)) ; we use the index \( i, j, k, \cdots = 1,2,3 \) for the transverse spacelike directions.

Now, \( \mathfrak{h} = \langle L_{ij}, L_{+i} \rangle \) and the deformed brackets are

\[
[P_+, P_-] = 0, \quad [P_+, P_i] = 0, \quad [P_-, P_i] = -a^2 L_{+i},
\]

\[
[P_+, \epsilon] = 0, \quad [P_-, \epsilon] = -a (\Gamma_+ + 2\epsilon) \tau_3 \epsilon, \quad [P_i, \epsilon] = -a \Gamma_i a \tau_3 \epsilon,
\]

\[
[\epsilon, \epsilon] = \kappa^\mu P_\mu - 2a (\omega_{34}^3 L_{+1} - \omega_{13}^3 L_{+2} + \omega_{12}^3 L_{+3} + \omega_{+3}^3 L_{12} - \omega_{+2}^3 L_{13} + \omega_{+1}^3 L_{23}).
\]

In all of these cases, a change of basis \( e_\mu = \frac{1}{a} P_\mu \) gives us a presentation of \( \hat{\mathfrak{g}}_D \) in which the even-even and even-odd brackets are independent of the deformation parameter but the odd-odd bracket is not. In such a presentation, the \( a \rightarrow 0 \) contraction trivialises the odd-odd bracket.

### 7.5.6 Maximally supersymmetric backgrounds with \( d \neq 0 \)

We have already argued from the form of the filtered deformations of the Poincaré superalgebra \( \hat{\mathfrak{g}} \) that the maximally supersymmetric geometry must be \( AdS_5 \), but we can also show this geometrically as follows. Since \( \hat{V} d = 0 \), then the norm \( \langle d^2 \rangle \) is constant, and since the square bracket is just the trace in \( \mathfrak{sp}(1) \cong \mathfrak{su}(2) \), it is negative-definite. Let us fix a point \( p \) on \( M \) and choose an adapted frame for the \( R \)-symmetry bundle such that at \( p \), the value of \( d \) in that frame is proportional to \( \tau_3 \); for later convenience, we let \( d_\mu = a \tau_3 \) with \( a > 0 \) without loss of generality. Then by equation (7.138), the
curvature tensors are as follows:

\[ R_{\mu \nu \rho \sigma} = -2a^2 g_{\mu |\rho} g_{\nu |\sigma}, \quad R_{\mu \nu} = 4a^2 g_{\mu \nu}, \quad R = 20a^2. \]  

(7.290)

We see that \((M, g)\) has constant positive scalar curvature and also constant sectional curvature. Thus, in our sign conventions, \((M, g)\) is locally isomorphic to AdS\(_5\) with radius \(\alpha = \frac{1}{\sqrt{2}a}\) and cosmological constant \(\Lambda = -\frac{1}{2}a^2\).

Localising at \(p\) in any orthonormal frame, the Killing superalgebra is a deformation of a maximally supersymmetric graded subalgebra of \(\mathfrak{h}\) with \(h = \mathfrak{so}(V)\) and \(\tau' = \mathbb{R}\tau_3\). The brackets are given by (7.56) and (7.60); with our choices here the deformed brackets are

\[
\begin{align*}
[P_\mu, P_\nu] &= -a^2 L_{\mu \nu}, \\
[P_\mu, \epsilon] &= a \Gamma_\mu \tau_3 \epsilon, \\
[\epsilon, \epsilon] &= \kappa^\mu P_\mu - \frac{a}{2} \omega_3^{\mu \nu} L_{\mu \nu} - 3a \mu \tau_3.
\end{align*}
\]

(7.291)

### 7.A Tensorial identities in 5 spacetime dimensions

We collect here some identities which are required for various calculations in the main body of this chapter. Let \((M, g)\) be a 5-dimensional Lorentzian manifold and let \(A, B \in \Omega^2(M)\). We work in a local orthonormal frame, starting with the identity

\[
\epsilon_{\alpha \beta \gamma \mu \nu} A^\alpha_{\rho} B^\beta_{\gamma} + \frac{2}{3} \epsilon_{\mu \nu \rho \sigma} A^{\alpha \delta} B^\beta_{\delta} = 0,
\]

(7.292)

which can be verified by contracting the left hand side with \(\epsilon_{\nu \rho \sigma \tau \chi}\). It then follows that

\[
\epsilon_{\alpha \beta \gamma \mu \nu} A^\alpha_{\rho} B^\beta_{\gamma} + \epsilon_{\alpha \beta \gamma \mu \nu} A^{\alpha \beta} B^\gamma_{\rho} = 0.
\]

(7.293)

We can then show that

\[
\epsilon_{\nu \rho \sigma \beta \gamma} (A^\mu_{\alpha} B^{\beta \gamma} + A^{\alpha \beta} B^\gamma_{\mu}) + \eta_{\mu (\nu} \epsilon_{\rho \sigma) \beta \gamma} A^{\alpha \beta} B^\gamma_{\delta} = 0
\]

(7.294)

by contracting the left hand side with \(\epsilon^{\nu \rho \sigma \tau \chi}\) and using equation (7.293). In particular, we have

\[
\begin{align*}
\epsilon_{\alpha \beta \gamma \mu \nu} A^\alpha_{\rho} B^\beta_{\gamma} &= 0, \\
\epsilon_{\alpha \beta \gamma \mu \nu} A^\alpha_{\rho} A^{\beta \gamma} &= 0.
\end{align*}
\]

(7.295)

(7.296)

Now let us consider some identities relating expressions involving \(A\) and the dual 3-form \(*A\). The defining expression is

\[
(*A)_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma \tau} A^{\rho \sigma \tau} \quad \text{or} \quad A_{\mu \nu} = \frac{1}{6} \epsilon_{\mu \nu \rho \sigma \tau} (*A)^{\rho \sigma \tau}
\]

(7.297)

whence we can show

\[
(*A)_{\mu \nu \alpha} (*A)_{\rho \sigma}^{\alpha} = A_{\mu \nu} A_{\rho \sigma} + \eta_{\mu \rho} \eta_{\nu \sigma} A_{\alpha \beta} A^{\alpha \beta}
\]

\[
-2 \eta_{\mu \rho} A_{\sigma |\alpha} A_{\nu}^{\alpha} + 2 \eta_{\nu |\rho} A_{\sigma |\alpha} A_{\mu}^{\alpha}
\]

(7.298)
and from here we can easily derive the following identities:

\[(\ast A)_{\mu\alpha\beta}(\ast A)_{\rho}^{\alpha\beta} = \eta_{\mu\rho} A_{\alpha\beta} A_{\alpha\beta}^{\rho} - 2 A_{\mu\alpha} A_{\rho}^{\alpha}, \tag{7.299}\]

\[(\ast A)_{\alpha\beta\gamma}(\ast A)^{\alpha\beta\gamma} = 3 A_{\alpha\beta} A_{\alpha\beta}^{\gamma}, \tag{7.300}\]

\[(\ast A)_{\mu|\rho} A^{\alpha}(\ast A)_{|\nu|\alpha} = A_{\mu|\rho} A_{|\nu|\alpha} - \frac{1}{2} \eta_{\mu|\rho} \eta_{|\nu|\alpha} A_{\alpha\beta} A_{\alpha\beta}^{\rho} + \eta_{\mu|\rho} A_{|\sigma|\alpha} A_{\nu}^{\alpha} - \eta_{\nu|\rho} A_{|\sigma|\alpha} A_{\mu}^{\alpha}, \tag{7.301}\]

\[(\ast A)_{\mu|\nu}(\ast A)_{\rho|\sigma}^{\alpha} = A_{|\mu\nu} A_{\rho\sigma}. \tag{7.302}\]

Finally, we have a useful basis-free identity involving the Hodge operator. For all \(p\)-forms \(\Omega\) and vectors \(v\), we have

\[i_{v} \ast \Omega = \ast (\Omega \wedge v), \quad \ast i_{v} \Omega = (\ast \Omega) \wedge v^{p}. \tag{7.303}\]
Chapter 8

Conclusions and Discussion

We conclude with some a summary and discussion of the results and an outlook for future work.

8.1 Summary of results

8.1.1 Theoretical results

In Chapters 3 and 4, we have proven a number of general theoretical results on Killing superalgebras and algebras and on filtered deformations of flat model superalgebras (that is, generalisations of the Poincaré superalgebra), particularly in the highly supersymmetric Lorentzian case.

The main results of Chapters 3 are as follows. Theorem 3.6 determines the conditions under which a connection $D$ on a spinor bundle equipped with a squaring map over a spin manifold gives rise to a (super)algebra on the space of $D$-parallel spinors (Killing spinors) summed with a subspace of the space of Killing vectors, and this result prompted us to introduce the definition of an admissible connection (Definition 3.7). To the author’s knowledge, this is the first time this precise result has appeared in the literature, but closely-related considerations can be found in the thesis of Noel Hustler [33]. In Theorem 4.16, we showed that the Killing (super)algebra associated to any admissible connection is a filtered subdeformation of a flat model algebra $s$, generalising a result of JMF and AS [45] in signature $(10, 1)$.

We note that this result is expected on general grounds from a theorem in Cartan geometry [62] since in principle Killing (super)algebras can be described as the symmetry algebras of a (super)Cartan geometry [99], but an explicit proof has not previously appeared. These results hold for arbitrary dimension, signature, choice of squaring map and degree of extension $N$ of the spinors. We then shifted to a purely algebraic perspective; generalising further results from the $(10, 1)$ case [15, 16, 45] using theory from Cheng and Kac’s work on filtered deformations [46], we described the parametrisation of filtered subdeformations of $s$ in terms of Spencer cohomology and used this to prove a number of technical results. The Homogeneity Theorem provided numerous technical simplifications in the highly supersymmetric Lorentzian case, making it particularly amenable to study via Spencer cohomology. In this restricted setting, we identified a particular class of Spencer cocycles of graded
subalgebras of the Poincaré superalgebra which could be parametrised in terms of elements of $\mathcal{H}^{2,2}$, the space of normalised Spencer cocycles of the parent algebra. We determined the conditions under which these admissible cocycles integrate to full filtered deformations and explicitly described the deformations in terms of the cocycles (Theorem 3.36). Similar such conditions for the $(10,1)$ case appear in [47]. We used admissible cocycles to identify a class of geometrically realisable subdeformations to which Killing superalgebras belong. Our notion of admissibility and geometric realisability differs somewhat from that found in [45], which uses some properties particular to the space of normalised Spencer cocycles $\mathcal{H}^{2,2} \cong \mathcal{H}^{2,2}(\mathfrak{s}_{-};\mathfrak{s})$ of the 11-dimensional Poincaré superalgebra. In the more general setting, there is an issue of non-uniqueness in that distinct normalised cocycles in $\mathcal{H}^{2,2}$ might give rise to the same admissible cohomology class. We also proposed a generalisation of a classification scheme introduced in [45, 47] in which the non-uniqueness issue required us to modify the equivalence relation on the Lie pair data. The main result in this regard is Proposition 3.42. Finally, we built a bridge back to the geometric perspective, showing in Theorem 3.47 that our “realisable” highly supersymmetric filtered subdeformations of the Poincaré superalgebra can indeed be realised as (a subalgebra of) the Killing superalgebra on a homogeneous spin manifold, subject to some conditions on the global geometry. The aforementioned non-uniqueness issue manifested here in the fact that a given homogeneous Lorentzian spin manifold might admit two different admissible connections which have isomorphic Killing superalgebras, or at least the same Killing ideal (odd-generated ideal subalgebra). Interpreted in terms of supergravity backgrounds, this means that two backgrounds with different flux configurations may nonetheless be isometric and share a Killing ideal. More work is needed to establish how this issue might be resolved in terms of gauge-equivalence of flux configurations in general.

In Chapter 4, we proposed a more general notion of Killing superalgebras to correspond to those appearing in supergravity theories with gauged $R$-symmetry and generalised the results of Chapter 3 to this context. The same notion was first introduced in the minimal 6-dimensional Lorentzian case found in [49] from two different perspectives; our treatment unified these approaches and also allowed for non-flat $R$-symmetry connections (although ultimately not in the highly supersymmetric Lorentzian case). Such algebras were defined for pseudo-Riemannian manifolds with what we dubbed spin-$R$ structures (Definition 4.1), where we essentially have a spin structure twisted by an $R$-symmetry group. After developing a notion of symmetry algebra for such a structure equipped with a connection $\mathcal{A}$ which (non-uniquely) lifts the Levi-Civita connection, we investigated how a connection $D$ on an associated spinor bundle equipped with a squaring map could be used to define a Killing (super)algebra, as summarised in Theorem 4.16, and developed a notion of admissible pairs $(D, \rho)$ where $D$ is a connection on spinors and $\rho$ is an extension of the squaring map to the $R$-symmetry algebra (Definition 4.17). We then showed in Theorem 4.18 that such an algebra is a filtered deformation of $\mathfrak{s}$, the $R$-symmetry extended flat model (super)algebra. Turning to the algebraic side, we once again used Spencer cohomology to describe filtered subdeformations of $\mathfrak{s}$, with particular focus on the highly supersymmetric Lorentzian case, developed notions of admissible and integrable cocycles and realisable deformations, once again stated in terms of normalised cocycles $\mathcal{H}^{2,2} \cong \mathcal{H}^{2,2}(\mathfrak{s}_{-};\mathfrak{s})$, culminating in Theorem 4.37. We did not fully
develop a classification scheme in this case but made some comments and noted potential issues in §4.3.5. We also showed once again that the “realisable” filtered deformations are (subalgebras of) Killing superalgebras on highly supersymmetric backgrounds, again subject to some technicalities. Further technicalities relating to the $R$-symmetry group had to be introduced for some of the results in this chapter, but these were justified wherever the $R$-symmetry group is compact, a choice which can be (and for typical purposes always is) made in Lorentzian signature.

As noted in Remarks 9 and 11 and elsewhere, there is clear scope for definitions in these chapters to be refined, especially those of admissible connections and pairs. In particular, it is clear from supergravity theory and in the technicalities around the geometric reconstruction results (Theorem 3.47, Theorem 4.44) that development of the discussion of (algebraically) constrained Killing spinors in §3.2.3 should be expanded upon. There is particular scope for improvement in the spin-$R$ treatment, where our approach led us to a complicated set of conditions for the existence of a Killing superalgebra which forced (gauged field strength) to vanish in the highly supersymmetric case and forced us to adopt a rather unnatural notion of geometric realisability.

### 8.1.2 2-dimensional toy examples

To illustrate some background theory on flat model superalgebras and Dirac currents from §2.2.2 and our theoretical treatment discussed above, we carried out some calculations in various signatures in 2 dimensions. In §2.2.4, we explicitly constructed all possible Dirac currents on the real pinor representations $\mathbb{P}$ in all three possible signatures, checking the calculations against the classification [63] summarised in Table 2.6. For the remainder of the calculations, we considered only signatures $(1,1)$ and $(0,2)$ in which the Clifford algebra is the $2 \times 2$ real matrix algebra, making the analysis particularly simple. All Dirac currents were symmetric in these cases, and there were two choices in each signature which could be distinguished by the symmetry of the associated bilinear $\sigma_B = \pm 1$. All results are summarised in Table 8.1.

In §3.2.4, we defined connections $D$ on 2-dimensional spin manifolds for the different Dirac currents and showed using integrability conditions that they were admissible, so that Killing superalgebras exist. In each signature, for one choice of Dirac current the Killing spinors were (generalised) geometric Killing spinors with imaginary Killing function, while for the other choice they were a certain dualised version thereof, which we will call \text{*geometric} Killing. An integrability condition related the scalar curvature (hence also the Riemann tensor) to the square the Killing function and the norm of its first derivative. We found that maximally supersymmetric geometries had constant Killing function, and in this case the curvature condition reduced to a version of a well-known integrability condition for geometric Killing spinors.

In §3.3.6, we computed the Spencer cohomology of the flat model superalgebra $s$ in signatures $(1,1)$ and $(0,2)$ with $S = \mathbb{P}$ and with $\sigma_B = \pm 1$. We found that $H^{2,2} \cong \mathbb{R}$ in each case. In signature $(1,1)$, $\mathbb{P}$ is reducible, $\mathbb{P} = \mathbb{S}_+ \oplus \mathbb{S}_-$, and both Dirac currents restrict to the same current on $\mathbb{S}_\pm$. We showed that Poincaré superalgebra with $S = \mathbb{S}_\pm$ has $H^{2,2} = 0$. For the non-trivial cases, we also considered the maximally supersymmetric filtered subdeformations of $s$, concluding that the only non-trivial
subdeformations are deformations \( \tilde{s} \) of the whole algebra. There is a single one-
parameter family of deformations in each case and each of these families could be
identified with a maximally supersymmetric geometry, with the parameter being
identified as the Killing number.

<table>
<thead>
<tr>
<th>((p, q))</th>
<th>(S)</th>
<th>Dirac curr.</th>
<th>(\mathcal{H}^{2,2})</th>
<th>Killing spinors</th>
<th>Max. SUSY geom.</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 2))</td>
<td>(P = S)</td>
<td>(\sigma_B = +)</td>
<td>(\mathbb{R})</td>
<td>*geometric</td>
<td>(H^2)</td>
</tr>
<tr>
<td>(P = S)</td>
<td>(\sigma_B = -)</td>
<td>(\mathbb{R})</td>
<td>geometric</td>
<td>(H^2)</td>
<td></td>
</tr>
<tr>
<td>((1, 1))</td>
<td>(P = S_+ \oplus S_-)</td>
<td>(\sigma_B = +)</td>
<td>(\mathbb{R})</td>
<td>*geometric</td>
<td>(dS_2)</td>
</tr>
<tr>
<td>(P = S_+ \oplus S_-)</td>
<td>(\sigma_B = -)</td>
<td>(\mathbb{R})</td>
<td>geometric</td>
<td>(AdS_2)</td>
<td></td>
</tr>
<tr>
<td>(S_\mp)</td>
<td>(-)</td>
<td>(0)</td>
<td>covar. const.</td>
<td>–</td>
<td></td>
</tr>
</tbody>
</table>

It is interesting to compare the geometric results above to another case in the liter-
ature of a Killing algebra arising from geometric Killing spinors. In [42], it was shown
that geometric Killing spinors do not in general give rise to a Killing (super)algebra
but that they do on \(S^7\), \(S^8\) and \(S^{15}\). These cases are different from the results above in
two ways. First, the Killing number in those cases is real, while in the 2-dimensional
example it is imaginary (hence the curvature is negative), and second, the Dirac cur-
rent is skew-symmetric for the construction on the spheres, hence the Killing spinors
generate a Killing algebra, whereas here it is symmetric, giving us a superalgebra.

8.1.3 **Extended \(D = 5, 6\) and Type IIA Spencer calculations**

In Chapter 5 and 6, we computed the spaces of normalised cocycles \(\mathcal{H}^{2,2}\) and
\(\tilde{\mathcal{H}}^{2,2}\) in 5 and 6 dimensions in Lorentzian signature with arbitrary \(N\)-extension
and with \(R\)-symmetry (for the typical choice of squaring map in the physics liter-
ature), generalising analogous calculations in the minimal cases [49, 50], and also
for Type IIA. The results are summarised in Table 8.2. Each normalised cocycle
space is described as a module of \(\text{Spin}^R(V) = \text{Spin}(4,1) \times_{\mathbb{Z}_2} \text{Sp}(N)\) or \(\text{Spin}^R(V) = \text{Spin}(5,1) \times_{\mathbb{Z}_2} (\text{Sp}(N_+) \times \text{Sp}(N_-))\) for \(D = 5\) or 6 respectively and as a module of
\(\text{Spin}(V) = \text{Spin}(9,1)\) for Type IIA.

For ease of understanding the \(D = 5, 6\) results, we recall that \(\Lambda^2 \Lambda_N[H]\) is the real
subspace of \(\Lambda^2 \Lambda_N\) consisting of elements whose image (under the musical isomorph-
ism induced by the symplectic product on the auxiliary module \(\Lambda_N\)) in \(\text{End}_\mathbb{C}(\Lambda_N)\)
preserves the quaternionic structure. The analogous subspace of \(\mathcal{O}^2 \Lambda_N\) corresponds
to \(\mathfrak{sp}(N)\). Elements of \(\Lambda^2 \Lambda_N[H]\) can be traced using the symplectic form \(\Omega\), and we
denote by \((\Lambda^2 \Lambda_N[H])_0\) the \(\Omega\)-traceless part. Finally, the space of all quaternionic
structure-preserving endomorphisms of \(\Delta_1\) can be identified as an \(\text{Sp}(1)\)-module
with the quaternions, \(\text{End}_\mathbb{H} \Delta_1 \cong \mathbb{H}\).
Table 8.2: Table of Spencer cohomology groups in 5, 6 and 10 dimensions with Lorentzian signature and standard Dirac current, described via isomorphisms as \( \text{Spin}^R(V) \)- or \( \text{Spin}(V) \)-modules. In each case, we have \( \mathcal{N} = N \) or \( \mathcal{N} = (N_+, N_-) \) as appropriate.

<table>
<thead>
<tr>
<th>( D )</th>
<th>( \mathcal{N} )</th>
<th>( \mathcal{H}^{2,2} \cong \mathbb{H}^{2,2}(\mathbb{R}; \mathbb{S}) )</th>
<th>( \mathcal{H}^{2,2} \cong \mathbb{H}^{2,2}(\mathbb{R}; \mathbb{S}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>((\Lambda^2 V \otimes \mathbb{R}) \oplus (\Lambda^1 V \otimes \text{sp}(1)))</td>
<td>( \mathcal{H}^{2,2} \cong (\Lambda^2 V \otimes \mathbb{R}) \oplus (\Lambda^0 V \otimes \text{sp}(1)) )</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(\Lambda^2 V \otimes (\Lambda^2 \Delta_2</td>
<td>_{10})_0)</td>
</tr>
<tr>
<td></td>
<td>( \geq 3 )</td>
<td>0</td>
<td>(\mathcal{H}^{2,2} \cong (\Lambda^2 V \otimes \mathbb{R}) \oplus (\Lambda^0 V \otimes \text{sp}(N)) )</td>
</tr>
<tr>
<td>6</td>
<td>((1, 0))</td>
<td>((\Lambda^3 V \otimes \mathbb{R}) \oplus (\Lambda^1 V \otimes \text{sp}(1)))</td>
<td>(\mathcal{H}^{2,2} \cong (\Lambda^3 V \otimes \mathbb{R}))</td>
</tr>
<tr>
<td></td>
<td>((2, 0))</td>
<td>(\Lambda^3 V \otimes (\Lambda^2 \Delta_2</td>
<td>_{10})_0)</td>
</tr>
<tr>
<td></td>
<td>((\geq 3, 0))</td>
<td>0</td>
<td>(\mathcal{H}^{2,2} \cong (\Lambda^3 V \otimes \mathbb{R}) \oplus (\Lambda^1 V \otimes \text{sp}(N_+)) )</td>
</tr>
<tr>
<td></td>
<td>((1, 1))</td>
<td>0</td>
<td>(\mathcal{H}^{2,2} \cong (\Lambda^3 V \otimes \mathbb{R}) \oplus (\Lambda^2 V \otimes \mathbb{H}) )</td>
</tr>
<tr>
<td></td>
<td>((\geq 2, 1))</td>
<td>0</td>
<td>(\mathcal{H}^{2,2} \cong (\Lambda^3 V \otimes \mathbb{R}))</td>
</tr>
<tr>
<td></td>
<td>((\geq 2, \geq 2))</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>((1, 1) = \text{IIA})</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

8.1.4 Killing spinors and superalgebras with \( R \)-symmetry in \( D = 5 \)

In Chapter 7, we explored some consequences of the \( D = 5, N = 1 \) calculation of \( \mathcal{H}^{2,2} \) discussed above, essentially generalising previous work of the author [50, 51] to the spin-\( R \) case. We showed that there are obstructions to integrating infinitesimal subdeformations of the Poincaré superalgebra, even in the maximally supersymmetric case (which has not been previously explicitly observed) and characterised the obstruction as a quadratic expression in coefficients of the cocycle component \( \beta \). We also showed that the Clifford-trace condition is not sufficient for the closure of the Killing superalgebra, nor, apparently, is the much stronger condition of vanishing curvature of the spinor connection. On the other hand, at least the latter sufficed if we also imposed a condition corresponding to the vanishing of the obstruction to deformation on the algebraic side.

We compared the results of the Spencer calculation to gauged (and arbitrary matter-coupled) supergravity, showing that they are consistent but in this case the Spencer calculation is not enough to recover the field content; it only detects fields in the combinations which appear as coefficients of the Killing spinor equation. We also noted that the algebraic integrability condition mentioned above obtains in particular in the supergravity case. We discuss further comparison of our results to supergravity below.

We also explicitly described some supersymmetric solutions and their Killing superalgebras, demonstrating that the space of such solutions is very rich; even a highly degenerate subspace contains a number of 1- and 2-parameter families of which solutions of gauged supergravity constitute a measure-zero subset. This is due to the fact that while the algebraic integrability condition is automatically satisfied in both of these cases, it can still be satisfied under much weaker conditions.
The upshot is that even with the strong constraint of maximal supersymmetry and the restrictiveness of our definition of Killing superalgebras with $R$-symmetry discussed above (in particular that the gauge field strength must vanish), we obtain a far richer class of geometries equipped with superalgebras than one could expect from supergravity theory. We note that this is a significant improvement over even e.g. [50], where three new one-parameter families of highly supersymmetric geometries apparently not accessible via supergravity were found.

8.2 Comparison to supergravity

In previous works on Spencer cohomology applied to supergravity, it was found that the calculation of $H_{2,2}$ (or $\tilde{H}_{2,2}$) recovers the field content (in the form of the fluxes) and Killing spinor equation of the $D = 11$ and pure minimal $D = 4, 5, 6$ theories, and often one can also recover the Bianchi identities and bosonic equations of motion [15, 16, 45, 48–50]. In this work, we have seen that the situation is more complicated outside of the pure minimal case; for example, in $D = 10$ Type IIA, we found that $H_{2,2} = 0$, suggesting a Killing spinor equation (KSE) of the form $D\epsilon = \nabla\epsilon = 0$, but this is not correct for Type IIA supergravity. This is consistent with the fact that Type IIA has no maximally supersymmetric solutions with non-zero fluxes [21], but a full account of Type IIA and the other 10-dimensional theories requires an understanding of the algebraic constraints arising from the variation of fermions other than the gravitino which have not been properly brought into our present framework, although we made some comments about how this might done in §3.2.3. The main issue is that unlike the differential Killing spinor equation, these algebraic constraints do not have an obvious cohomological description, but it may nonetheless still be possible to modify our notion of admissibility to account for them. We leave this for future work.

It should also be noted that in the cases in which essentially the entire bosonic sector can be recovered from the $H_{2,2}$ computation as mentioned above, the gravitino is the only bosonic field, so there are no algebraic constraints to impose, thus it is perhaps not surprising that the Spencer method works well here. Another family of theories with algebraic constraints which are nonetheless at least somewhat amenable to our methods are the minimal $D = 5$ gauged and matter-coupled theories discussed in §7.2.2. We found there that the KSE suggested by the Spencer calculation had the supergravity KSE as a special case, although the Spencer data did not correspond to the field content, whence one cannot hope to recover all information about the bosonic sector as in pure minimal $D = 5$.

Let us now expand on this discussion and compare the full results of Chapter 5 to $N$-extended supergravity. The $N$-extended supergravity theories in $D = 5$ were first described in [114] and some supersymmetric solutions were considered in [115]. The $D = 5$ pure theories are $N = 2, 3, 4$ (or 4, 6, 8 according to other conventions) and only the $N = 2$ theory allows for matter coupling. The $D = 6$ theories are $N = (2, 0), (3, 0), (4, 0)$ and $D = 2$ with matter couplings allowed for $(2, 0)$. As in the $N = 1$ case, the coefficients of $\beta$ in the KSE for the extended cases are not the fundamental fields themselves but functions of them, so one cannot expect to recover the precise field content via Spencer cohomology. As such, we will ignore the details of the field
content and consider the general form of $\beta$. In the conventions which we established in Chapter 5, in $N$-extended $D = 5$ theories we have \cite{114, 115}\footnote{There appear to be some typos in \cite{114} which are fixed in \cite{115}, which also uses a different scaling for the fields.}

$$\beta_{\mu AB} = \Psi_{\mu AB} + (\tilde{F}_{\nu p AB} + \tilde{G}_{\nu p} \Omega_{AB}) \left( \Gamma_{\mu}^{\nu} - 4 \delta_{\mu}^{\nu} \Gamma_{\nu} \right)$$

where it is to be understood that the indices $A, B, \ldots = 1, \ldots, 2N$ label components in the fundamental representation of $\text{Sp}(N)$, that $\Psi_{\mu AB} = \Psi_{\mu[AB]}$ and $\tilde{F}_{\mu \nu AB} = \tilde{F}_{[\mu \nu][AB]}$ with both of these coefficients being $\Omega$-traceless, and $\Psi = 0$ for $N = 2$ (and both satisfy reality conditions which do not concern us here). By the results of Chapter 5, this appears to show that $D = \nabla + \beta$ is not admissible unless $A, \tilde{F}, \tilde{G}$ all vanish; however, it may be the case that the algebraic constraints can repair the situation. Indeed, in the pure $N = 2$ case, the constraint is $A \epsilon = 0$, where

$$\mathcal{A} = \Phi_{\nu} \Omega_{AB} \Gamma_{\nu} + (\tilde{F}_{\nu p AB} - 2 \tilde{G}_{\nu p} \Omega_{AB}) \Gamma_{\nu p}.$$ \hspace{1cm} (8.2)

Let us define

$$\beta'_{\mu AB} = \beta'_{\mu AB} + a \Gamma_{\mu} \mathcal{A}_{AB}$$

$$= a \Phi_{\rho} \Omega_{AB} (\eta_{\mu \rho} + \Gamma_{\mu \rho}) + [(1 + a) \tilde{F}_{\nu p AB} + (1 - 2 a) \tilde{G}_{\nu p} \Omega_{AB}] \Gamma_{\mu}^{\nu p}$$

$$+ [(2 a - 4) \tilde{F}_{\mu p AB} - 4 (1 + a) \tilde{G}_{\mu p} \Omega_{AB}] \Gamma^{\rho}$$

for $a \in \mathbb{R}$. Then a spinor field which satisfies the constraint is $D$-parallel if and only if it is $D' = \nabla - \beta'$-parallel. For $D'$ to be admissible, the $1$-term must vanish, so $\Phi_{\rho} = 0$, and so must the rank-$1$ term and the $\Omega$-trace part of the rank-$3$ term, so we must have $\tilde{G} = 0$ and $a = 2$, leaving an admissible solution $\beta'_{\mu AB} + 2 \Gamma_{\mu} \mathcal{A}_{AB} = 3 \tilde{F}_{\nu p AB} \Gamma_{\mu}^{\nu p}$. Thus, at least at this level of analysis, our results here are consistent with supersymmetric solutions in which at least some fluxes need not vanish. Similar comments hold for gauged and extended $D = 6$ theories.

This makes it clear once again that a proper understanding of algebraic constraints from the perspective advocated for in this work is imperative if we wish to use it to study supergravity backgrounds beyond those of the pure minimal theories.

### 8.3 Symmetries of solutions and higher structures

As mentioned above, even when the Spencer calculation can recover the form of the Killing spinor equation for a given supergravity theory, it does not necessarily recover the full field content. Moreover, even when it does recover the correct field content, it only does so at the level of fluxes, not gauge fields. In some sense this is to be expected; the coefficients appearing in $\beta$ should be globally-defined fields (in particular, sections of associated bundles for the spin- or spin-$R$ structure), while gauge connections are not globally defined on the base manifold; for ordinary gauge $1$-forms they are connections on principal bundles, and for higher forms they are sections of higher structures such as gerbes. This is a reflection of the fact that the Killing superalgebra of a bosonic supergravity solution does not detect any of this
“internal” structure; it describes only the spacetime supersymmetries (in the gauged case including some information about the R-symmetry gauge configuration). Since the solutions themselves generally involve higher gauge data, Lie superalgebras are not sufficient to fully encode the (super)symmetry of these solutions; one must pass to their higher analogues, $L_\infty$-algebras.

While higher gauge fields have been present in supergravity theories since the earliest days of the subject, for a long time they were only treated locally and they have only been given a true global description in terms of higher bundles in recent years – see [116] for an extensive recent review. Likewise, $L_\infty$-algebras have at least implicitly been present in the physics literature since the early 1980s including as the “FDAs” of the D’Auria–Regge–Fré formalism of supergravity [117–119] and were first rigorously described some years later – see the historical review by Stasheff [120] as well as the nLab page on $L_\infty$-algebras [121] and related pages for an overview.

All of this higher structure is far beyond the scope of this work, but it provides an intriguing possibility for future work since it is very amenable to the perspective taken here. Indeed, the central philosophy of this thesis and the work that it builds upon is to use homological algebra to understand and classify solutions via their (super)symmetries, so $L_\infty$-algebras, which essentially have homology built into their definitions, are a natural next step. Moreover, as we have already mentioned repeatedly, many of our results can be understood through the lens of Cartan geometry, and there is already a well-developed literature on the generalisation of Cartan geometry to allow for connections with values in $L_\infty$-algebras as well as the application of such structures to supergravity and string theories [122, 123], making precise the ideas already present in the D’Auria–Regge–Fré formalism since the early 1980s. This effort is part of a wider push in recent years to put supergravity theories on a firmer mathematical footing by finding global descriptions of the locally-defined objects in the familiar formulations, a project to which the author hopes the present work also represents a significant contribution.

### 8.4 Outlook

The issues already raised in the discussion above naturally lead to future research questions which build on this work. In particular, further work needs to be done to incorporate algebraic constraints into the Spencer framework in order to apply it to, for example, the 10-dimensional theories. Moreover, it seems evident that along with proper consideration of (higher) gauge symmetries, this should allow some of the issues around the definition of Killing superalgebra with $R$-symmetry and the reconstruction of highly supersymmetric backgrounds from superalgebras to be resolved. There are also numerous other avenues for further enquiry which we detail below.

**Further Spencer calculations and development of the method**

Calculation of $\mathcal{H}^{2,2}$ and $\tilde{\mathcal{H}}^{2,2}$ in more cases could reveal further novel possibilities for supersymmetric geometries, as well as insights into supergravity theory. As is evident from Chapters 5 and 6, these calculations can become very cumbersome,
so it is worth investigating new methods. Beyond obvious possibilities such as computer algebra systems, these calculations could be made more efficient if one could dimensionally reduce the Spencer data. The only published work in this direction is [52], which only considered Kaluza–Klein reduction of some maximally supersymmetric 5-dimensional non-supergravity backgrounds and not the Spencer cohomology itself. JMF, AS and the author have investigated the possibility of Kaluza-Klein reduction of $\mathcal{H}^{2,2}$ in unpublished work but encountered difficulties which have not been resolved. The author, JMF, AS and PdM all also have unpublished brute-force calculations which we hope to publish in the near future.

**Less-than-maximally supersymmetric backgrounds**

All novel supersymmetric backgrounds explicitly obtained via the Spencer method so far, apart from the 4-dimensional ones found in [48–50, 52] are *maximally* supersymmetric (notwithstanding the novel construction of known sub-maximal examples in 11-dimensions [47]) so in addition to the above it remains to develop better technology for explicitly constructing sub-maximal examples from the algebraic data and using it to find new solutions, with the ultimate goal of using it to move closer to a classification of highly supersymmetric supergravity solutions (and possibly of the wider class of supersymmetric backgrounds which we have discussed in this work). A particularly interesting problem to consider would be whether our generalised notion of Killing spinors and supersymmetry in 5 dimensions might allow us to construct some novel 5-dimensional supersymmetric black holes, generalising the treatment of e.g. [6] and related work.

**Rigid supersymmetry**

Finally, the full consequences of the existence of even the maximally supersymmetric backgrounds we have already explicitly constructed have not yet been fully explored. A possibility already mentioned in the introduction is that these backgrounds might admit rigid, non-Poincaré supersymmetric field theories which cannot be obtained by freezing out supergravity á la Festuccia–Seiberg [31]. There is a challenge here in that, unlike in the Festuccia–Seiberg method, we do not know the field content or supersymmetry transformations a priori, so one must start by understanding the representation theory of the Killing superalgebras. One expects that homological methods should help to understand this through deformation of the representation of the Killing superalgebra, but the details remain to be investigated.
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Is there an elegant proof of the existence of Majorana spinors?  


