# Option pricing techniques under stochastic delay models

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# Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. The work has not been submitted for any other degree or professional qualication.

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# Abstract

The Black-Scholes model and corresponding option pricing formula has led to a wide and extensive industry, used by financial institutions and investors to speculate on market trends or to control their level of risk from other investments. From the formation of the Chicago Board Options Exchange in 1973, the nature of options contracts available today has grown dramatically from the single-date contracts considered by Black and Scholes (1973) to a wider and more exotic range of derivatives. These include American options, which can be exercised at any time up to maturity, as well as options based on the weighted sums of assets, such as the Asian and basket options which we consider.

Moreover, the underlying models considered have also grown in number and in this work we are primarily motivated by the increasing interest in past-dependent asset pricing models, shown in recent years by market practitioners and prominent authors. These models provide a natural framework that considers past history and behaviour, as well as present information, in the determination of the future evolution of an underlying process.

In our studies, we explore option pricing techniques for arithmetic Asian and basket options under a Stochastic Delay Differential Equation (SDDE) approach. We obtain explicit closed-form expressions for a number of lower and upper bounds before giving a practical, numerical analysis of our result. In addition, we also consider the properties of the approximate numerical integration methods used and state the conditions for which numerical stability and convergence can be achieved.

# Acknowledgements

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# Contents

# Chapter 1

# Introduction

Over the course of the most recent decades, the trading of derivatives has grown to represent an important area in the world of finance. The growth of this industry can be attributed to the contribution of Black and Scholes (1973) and Merton (1973) in the formulation of the Black-Scholes model. Its results provided simple methods for the valuation of vanilla European options. Furthermore, the assumptions of the Black-Scholes model provided a framework for the pricing of additional derivatives. Black (1976) provided an extension of the Black-Scholes model in order to value bond and swap options. Later work demonstrated the pricing of more exotic derivatives. Rogers and Shi (1995) for example provided methods for the valuation of Asian options under this model. Such options, however, present difficulty in obtaining explicit solutions. This opened up the approach of valuing financial derivatives using Monte Carlo simulation, and Glasserman (2004) provides methods covering a wide variety of contracts and models.

Although the Black-Scholes model provides a benchmark for the valuation of financial derivatives, empirical studies have since outlined significant weaknesses caused as a result of its underlying assumptions. Even early work such as Blattberg and Gonedes (1974) demonstrated differences in the distribution of asset price returns: the property of "fat-tails". Moreover, this paper along with Scott (1987) outlined the importance of the volatility parameter of such assets. Their work outlined that volatility is highly unlikely to be constant.

A number of alternative approaches have been considered for the underlying asset model, in place of the geometric Brownian motion, which attempt to overcome the assumption of constant volatility. These include the use of local volatility (see, for example, Carr et al. (2004); Dupire (1994)) and stochastic volatility models (see, for example, Heston (1993); Hull and White (1987); Sabanis (2003)). In addition, Lévy models have also been considered and Albrecher et al. (2005) demonstrate the use of such models in the hedging of Asian options. However, it can be shown that such models remain Markovian in their nature and that stochastic processes under such assumptions are only influenced by the immediate present data. This results in a similar setting to that of Black and Scholes (1973).

Instead of assuming that the underlying asset price process follows a Markovian model, we cite work by authors such Hobson and Rogers (1998) and Arriojas et al. (2007) and present an alternative approach, where future asset price evolutions are additionally based on historical data. Our approach, based on stochastic delay models, combines both deterministic delay differential equations and ordinary stochastic differential equations, as outlined by Buckwar (2000).

The idea of modelling future asset price movements on historical properties is well-founded through the use of moving average and autoregression models in time series analysis. This idea has since been extended through prominent studies by Engle (1982) and Bollerslev (1986), which introduced ARCH and GARCH models respectively.

The past-dependent models that we consider are shown to exhibit a number of desirable characteristics. Of particular relevance, they display volatility 'smile' and 'skew' patterns similar to those observed in historical data (see Hobson and Rogers (1998)). Figure 1.2 show the results of simulating a daily implied volatility curve, as a function of its exercise price, of 3-month European call options written on four different companies, Tesco, Barclays, Lloyds and Vodafone, whose asset price processes follow 1-week fixed-delay models. The strike is quoted as a percentage of the initial value, using closing prices up to the end of Friday 1st July 2011 (with initial values of 401.15, 265.55, 50.81 and 164.5 respectively). These models use daily volatility functions, where $g(x)$ satisfies the respective equation in Table 1.1. These local volatility functions are obtained by fitting a curve to the estimated stock price volatilities over a period for which data was available. Figure 1.3 repeats this procedure for 1-month European call options, where in this case the respective assets follow 1-day fixed-delay models. We compare the two graphs to Figure 1.4, taken from Derman and Kani (1994), which provides the implied volatility surface of S&P 500 options on the 31st January 1994.

Secondly, it is shown that these models admit a unique equivalent martingale measure that leads to a complete market framework and thus preference-independent prices, see Arriojas et al. (2007). Finally, they exhibit a level of robustness with regards to delay parameter estimation, see Mao and Sabanis (2009), which makes them quite attractive and suitable for pricing contingent claims.

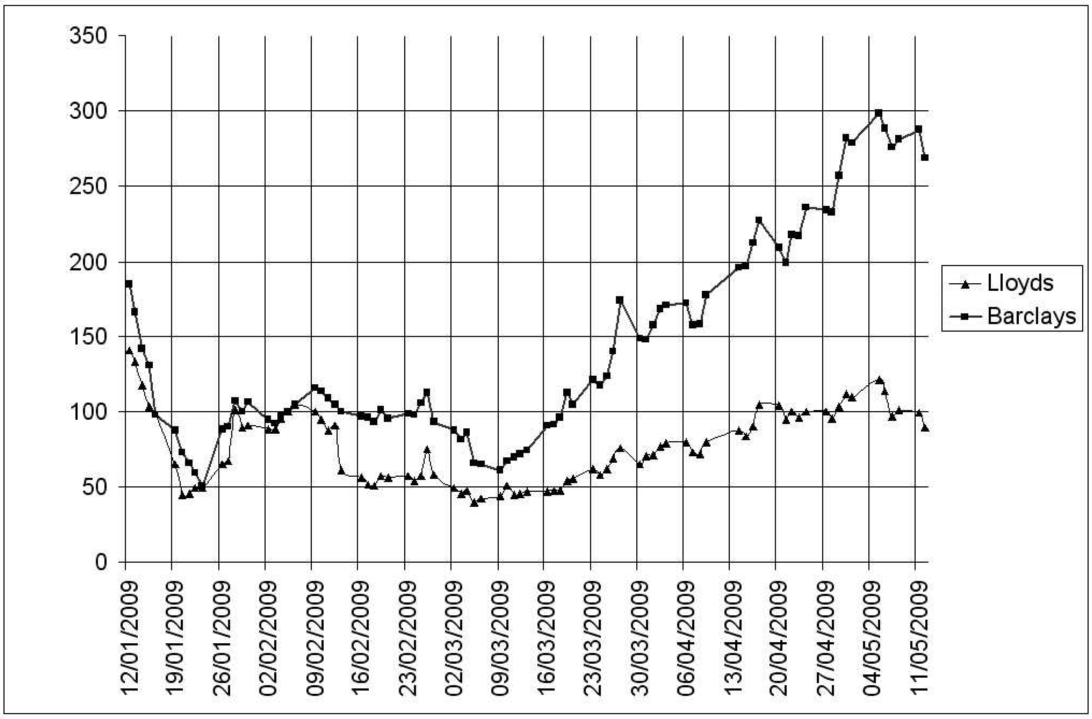


Figure 1.1: Barclays vs Lloyds, January-May 2009

The importance of considering delay models is that they provide a framework under which one can deviate somewhat from the statement of the efficient market hypothesis. That is, all publicly available information is fully reflected in current asset prices. Studies under such scenarios is demonstrated by Stoica (2004) and Back (1992). This is based on the idea that it is hard, for example, to believe that news published on the annual accounts of large corporations, such as investment banks, can be absorbed and held by all investors simultaneously and immediately (or even one hour) after their announcement.

As an example, we present Figure 1.1, from McWilliams and Sabanis (2011), which vividly highlights the importance played by historical information. In this example, one could observe the presence of a feedback effect after the announcement of the 2008 profit/losses results for two major UK financial institutions. Prior to these announcements, Barclays Bank and Lloyds Banking Group shares were traded at about the same price (see for example 4–6 February 2009). Then, on Monday, 9th February 2009,

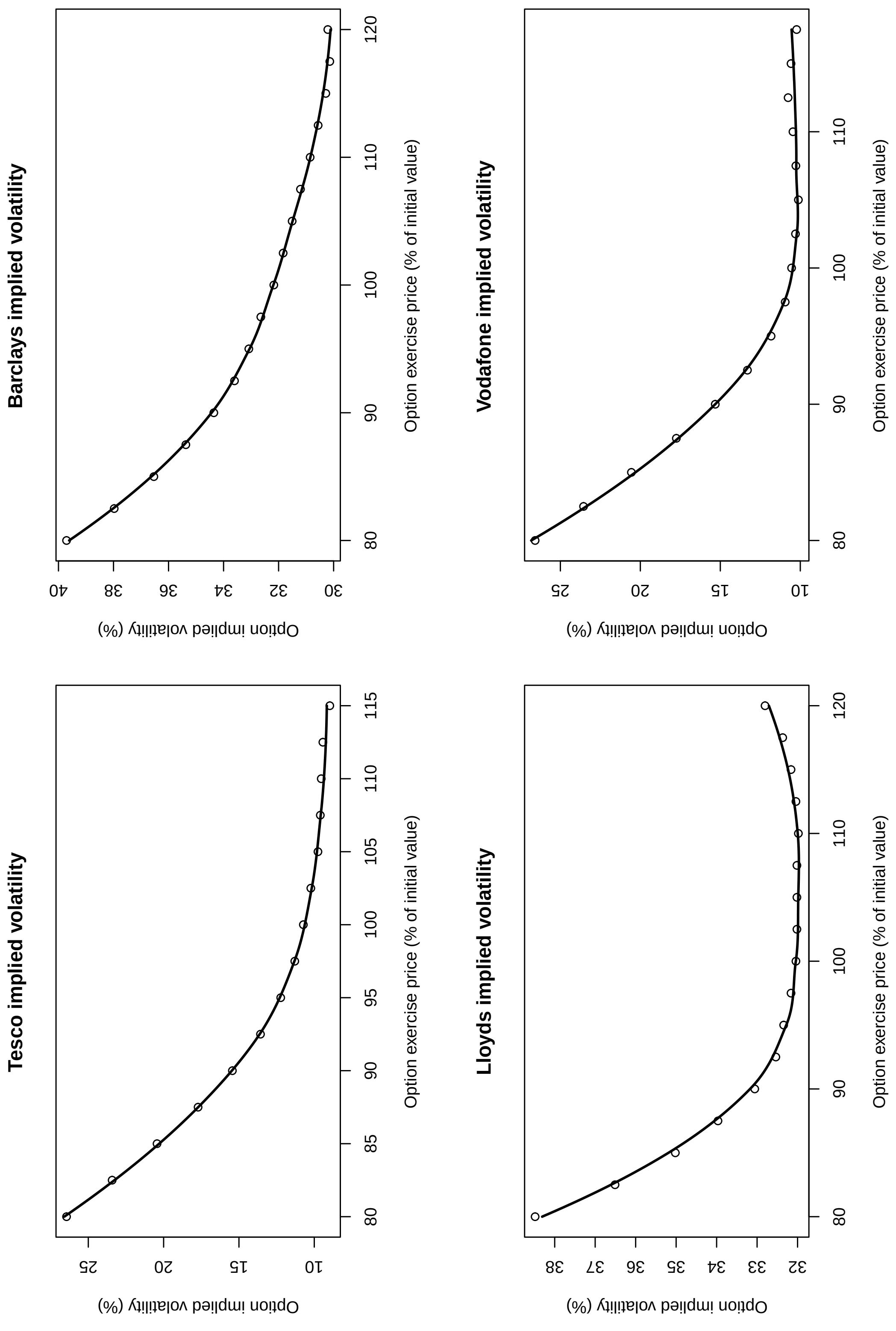


Figure 1.2: Implied volatility surface of 1-week fixed-delay assets with option maturity of 3 months.

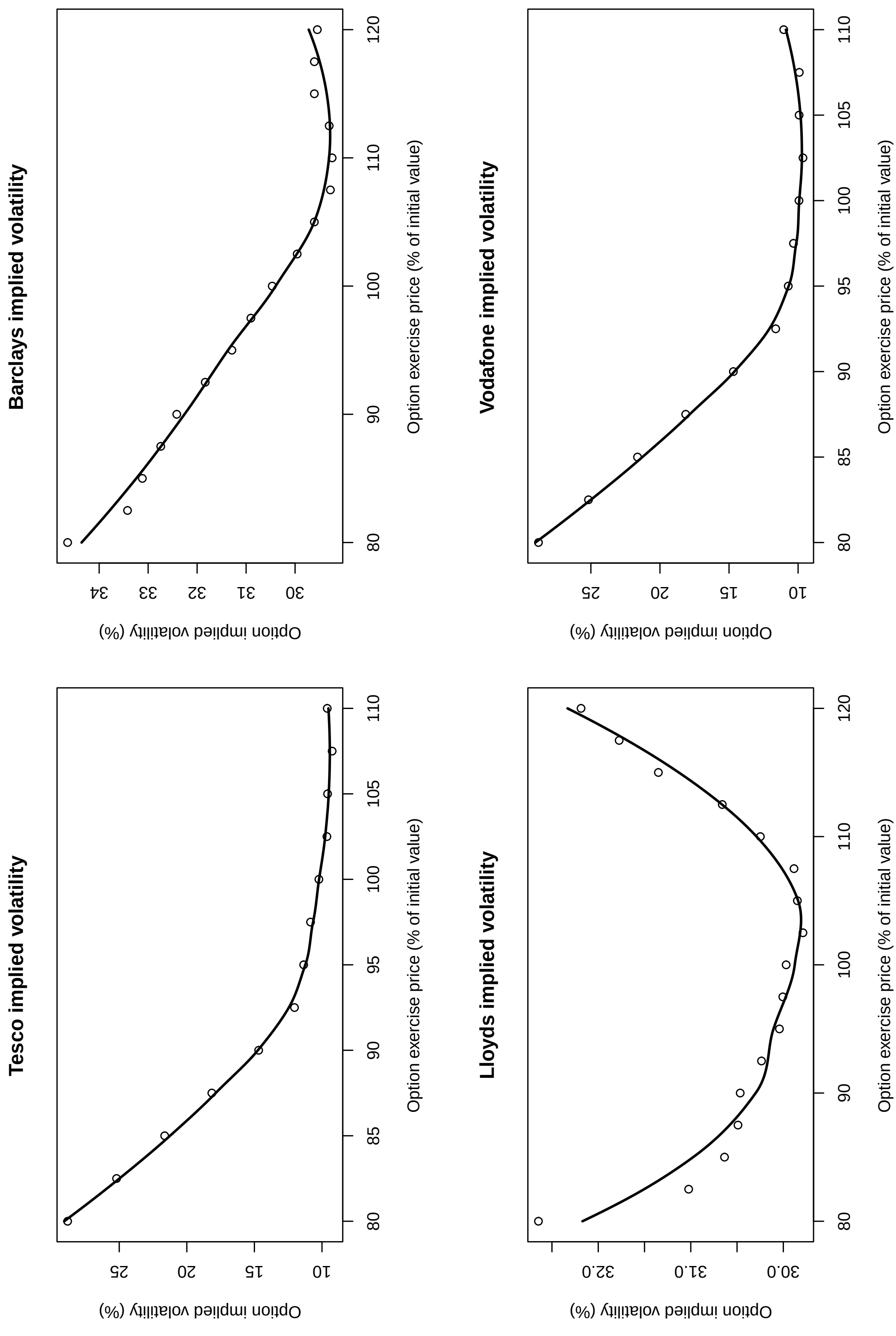


Figure 1.3: Implied volatility surface of 1-day fixed-delay assets with option maturity of 1 month.

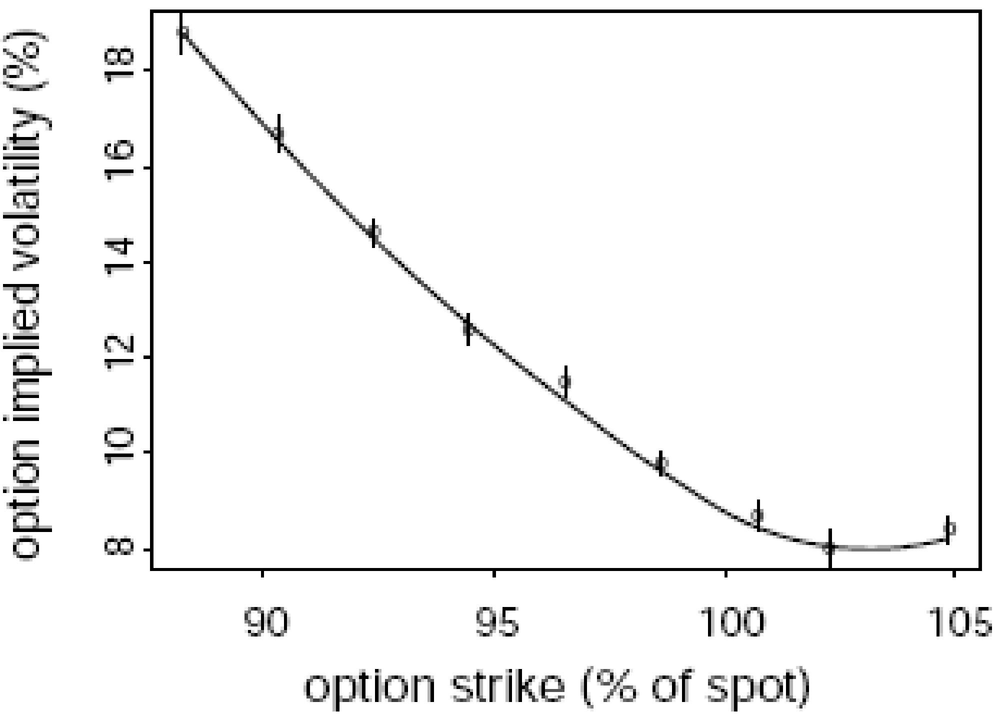


Figure 1.4: Implied volatility surface of S&P 500 options on 31st January 1994.

Barclays Bank reported profits before tax of £6.08bn for the full year of 2008 and its share price closed on that day 16% higher than Lloyds. On Friday, 13th February 2009, Lloyds Banking Group announced that it expected its subsidiary HBOS to report a pre-tax loss for the whole of 2008 of £10bn, which was £1.6bn more than it predicted in November 2008. As a result, shares in Lloyds Banking Group tumbled after this announcement and Barclays share price closed on that day 63% higher than Lloyds (i.e., 100.5 pence per share vs 61.4 pence per share). However, this gap continued to grow in the following weeks and months and reached levels of 150% and above. In retrospect, one could say that it seems natural that when some relative calmness returned to the markets in Spring 2009, investors chose to invest more in companies that announced profits rather than losses. Clearly the feedback effect is evident in this case.

Therefore, we believe it is important to further explore the application of a modelling approach based on historical, as well as current, information. This extends much further than the investigation of plain European option contracts conducted by Arriojas et al. (2007).

In this work, we are principally concerned with the valuation of exotic options based on arithmetic sums under stochastic delay models which maintain a lognormal distribution during during a subset of their corresponding contract length. The study undertaken by Arriojas et al. (2007) in the pricing of vanilla European options under similar models provides a solid framework for the research covered in this thesis. The valuation of Asian and basket options under similar models is a natural extension; such contracts are both popular and widely traded on today's markets. One particular benefit of Asian and basket options to investors is the reduced risk of market manipulation of the underlying assets close to maturity. Moreover, Vanmaele et al. (2006) discuss the application of Asian options in pricing more complex financial instruments, such as retirement plans and catastrophe insurance derivatives. Also cited is Nielsen and Sandmann (2003), who demonstrate the use of periodic investments in forming a retirement scheme.

The analysis of these derivatives reduces to the understanding of the underlying distribution involving sums of random variables, which are not necessarily mutually independent. An immediate problem arises due to the complexity of such distribution functions. As a result, even under the simpler Black-Scholes hypotheses, no closed-form formula exists for options contracts based upon arithmetic sums. Therefore, a large section of our work relies on the concept of comonotonicity applied to mathematical finance, as described by Dhaene et al. (2002a,b). In Chapter 2, we state the concepts given by these two papers before describing bounds for arithmetic Asian options, drawing upon the work of Albrecher et al. (2008) and Hobson et al. (2005).

In Chapter 3, we introduce the stochastic delay differential equation (SDDE) models that will be studied throughout. We consider general delay parameters which are not necessarily fixed. This allows us to emphasise specific dates, such as the issuing of annual or quarterly reports, the relevance of which is described above in conjunction with the behaviour observed in Figure 1.1.

We extend the techniques of Arriojas et al. (2007) in deriving explicit solutions for the underlying SDDEs, as well as their approach to the valuation of vanilla European options. One can demonstrate the environments for which it is possible to price asset-based derivatives and we then see the results one can gain from a simple change of probability measure. In Chapter 4, we draw upon various techniques in an attempt to value both arithmetic Asian and basket call options.

In this text, an extensive study has been undertaken on the numerical issues surrounding SDDE models. The valuation of Asian and basket options under such models is a novel approach and so an indication of the practical accuracy of the bounds we obtain to their true value is important. One area we are concerned with is in obtaining a statistical, Monte Carlo estimate of the actual option price. In order to achieve this, we need to consider appropriate numerical schemes. In particular, it is important to consider whether both a stochastic model and its corresponding numerical scheme remains stable as $t \to \infty$, as well as determining the convergence of such a numerical scheme to its true solution. Extensive study of this topic has been provided by authors such as Wu et al. (2010); Mao et al. (2008); Mao and Sabanis (2003); Mao (2002, 1999) in the case where delay is fixed. However, we also consider general-delay models which can place emphasis on specific time points regardless of their relation to the present. It is therefore important to consider stability and convergence of these models. In Chapter 5, we then discuss the additional conditions required for the relevant Euler-Maruyama discretisation of stochastic process (see, for example, Kloeden and Platen (1995)) to achieve numerical stability (demonstrated by Higham et al. (2007) for ordinary stochastic differential equations). Furthermore, we explore the requirements under which a discretised scheme for a stochastic delay differential equation model converges to its true value (see Mao (2003) for stochastic functional differential equations, for which we consider a special case). Finally, in Chapter 6, we demonstrate the numerical behaviour of arithmetic option values for the models we consider.

## 1.1 Valuation of put options

A primary goal for this report concerns the valuation of options written on differing choices of model for the underlying asset price processes. In this work, we study arithmetic Asian and basket call options: contracts for which the long-position holder has the right (but not the obligation) to *buy* a weighted portfolio of assets at a predetermined maturity date and exercise price. Put options, which allow their contract holders to *sell* such assets are not explicitly discussed in this work. However, with upper and lower bounds derived for call options it becomes immediately possible to write lower and upper bounds, respectively, for put options.

To see this in practice, let $\mathbb{S}$ denote the value of a pre-specified portfolio written on a selection of assets (for example, for a European option, this would be the value of a single underlying at maturity). Then, under the assumption of a constant rate of interest $r > 0$, the value of the call option on this portfolio is given by

$$C(K,T) = e^{-rT}\mathbf{E}^{\mathbf{Q}}\left[(\mathbb{S} - K)^{+}\right], \tag{1.1.1}$$

where $K > 0$ denotes the exercise (or strike) price. The corresponding put option value is then given by

$$P(K,T) = e^{-rT}\mathbf{E}^{\mathbf{Q}}\left[(K - \mathbb{S})^{+}\right]. \tag{1.1.2}$$

By taking the difference between call and put options on the same selection of assets, with identical strike values and maturity times, one then achieves the *put-call parity* formula, which we write in the following way.

$$C(K,T) - P(K,T) = e^{-rT}(\mathbf{E}^{\mathbf{Q}}\left[\mathbb{S}\right] - K). \tag{1.1.3}$$

We show in Chapters 3 and 4 that the expectation of $\mathbb{S}$ can be easily determined, when $\mathbb{S}$ defines an Asian or basket option in equations (1.1.1) and (1.1.2), as an $\mathcal{F}_0$-measurable variable. Therefore, by rearranging equation (1.1.3) and substituting the appropriate bounds for call options below, we are able to obtain the corresponding bounds for put options as well.

| Company Name | Local Volatility Function |
|---|---|
| Tesco | $\exp(-2376.434 - 9.283106x + 0.02323698x^2 - 0.00003322052x^3 + 0.00000001939124x^4 + 667.0202 \log x)$ |
| Barclays | $\exp(6.420131 + 0.03410825x - 0.00006539761x^2 + 0.00000004303758x^3 - 2.797781 \log x)$ |
| Lloyds | $\exp(-4.726255 + 0.02035911x - 0.0001145587x^2 + 0.0000002020148x^3 - 0.0000000001182479x^4)$ |
| Vodafone | $\exp(-98.877552 + 2.557498x - 0.02546654x^2 + 0.0001105752x^3 - 0.000000177512x^4)$ |

Table 1.1: Local volatility functions for the assets given in Figures 1.2 and 1.3.

# Chapter 2

# Comonotonicity and Stochastic Ordering

In this chapter, we introduce the concept of comonotonicity explored by Dhaene et al. (2002b). We focus on supplying a self-contained set of results which will provide a framework for valuing stop-loss transforms based upon arithmetic sums. We then demonstrate the methods for which one can obtain bounds for the value of Asian options under standard models, for example Black and Scholes (1973).

## 2.1 Stochastic ordering

In this report, the fundamental problem concerns the valuation of the following expectations, given by Definitions 1 and 2. This involves the arithmetic sum of a number of real-valued random variables, $X_i : \Omega \to \mathbb{R}$.

**Definition 1.** *Let $X$ be a real-valued random variable and let $d \in \mathbb{R}$ be fixed. Then, the* stop-loss transform *of $X$ with respect to $d$ is given by*

$$\Psi(X, d) = \mathbf{E}\left[(X - d)^+\right], \tag{2.1.1}$$

*where $f^+$ denotes the operation that returns $f$ when $f$ is positive and zero otherwise.*

**Definition 2.** *Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a random variable in $\mathbb{R}^n$ whose individual components are not necessarily mutually independent. Let $d \in \mathbb{R}$ be fixed. Then, the* stop-loss transform *of the arithmetic sum* $\sum_{i=1}^{n} X_i$ *with* retention $d$ *is given by*

$$\Psi\left(\sum_{i=1}^{n} X_i, d\right) = \mathbf{E}\left[\left(\sum_{i=1}^{n} X_i - d\right)^{+}\right]. \tag{2.1.2}$$

In these equations, the retention parameter $d$ is typically positive. For simplicity of notation, we shall assume throughout that the marginal distribution functions $F_{X_i}$ are injective and thus strictly increasing. We assume that these marginals are known, but are faced with the problem where the joint distribution involving $\mathbf{X}$ is either unknown or too difficult to work with.

An approach to estimating equation (2.1.2) is to replace $\mathbf{X}$ by another random vector $\mathbf{Y} = (Y_1, \dots, Y_n)$ such that $F_{\sum_{i=1}^{n} Y_i}$ is easier to work with. It is hoped that we can make a choice of $\mathbf{Y}$ such that the sum of its components forms a stochastic bound on $X := \sum_{i=1}^{n} X_i$.

Many of the results from this section are given by Dhaene et al. (2002b); Vyncke (2003), including more detailed theory on the case where the marginal distributions are not injective, which we do not require here. We cite the relevant sections where appropriate.

The first form of stochastic bound we need is taken from Definition 1 of Dhaene et al. (2002b).

**Definition 3.** *Consider two random variables $X$ and $Y$. Then $X$ is said to precede $Y$ in the* stop-loss order *sense, $X \leq_{\text{sl}} Y$, if and only if*

$$\mathbf{E}\left[(X-d)^{+}\right] \leq \mathbf{E}\left[(Y-d)^{+}\right], \quad \forall d \in \mathbb{R}. \tag{2.1.3}$$

A stronger relation between $X$ and $Y$ is the notion of convex order.

**Definition 4.** *$X$ is said to precede $Y$ in terms of* convex order*, $X \leq_{\text{cx}} Y$, if and only if $X \leq_{\text{sl}} Y$ and $\mathbf{E}[X] = \mathbf{E}[Y]$.*

Dhaene et al. (2002b) show that an equivalent condition for the same convex ordering is that $\mathbf{E}[X] = \mathbf{E}[Y]$ and $\mathbf{E}\left[(d-X)^{+}\right] \leq \mathbf{E}\left[(d-Y)^{+}\right]$.

## 2.2 Comonotonicity

Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ be any two vectors in $\mathbb{R}^n$. As in Dhaene et al. (2002b); Vyncke (2003), we linearly order $\mathbf{x}$, $\mathbf{y}$ such that $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i$

for every $i = 1, \ldots, n$.

**Definition 5.** *Any set $A \subset \mathbb{R}^n$ such that $\mathbf{x} \leq \mathbf{y}$ or $\mathbf{x} \geq \mathbf{y}$ for every $\mathbf{x}, \mathbf{y} \in A$ is said to be* comonotonic*.*

For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in A$, we see that if there exists an $i$ such that $x_i < y_i$, then it is necessarily the case that $\mathbf{x} \leq \mathbf{y}$. Hence, "a comonotonic set is simultaneously monotone in each component." As a result, it cannot have dimension greater than 1. A useful result of this is Lemma 1 of Dhaene et al. (2002b), which states that $A$ is comonotonic if and only if $\{(x_i, x_j) : \mathbf{x} \in A\}$ is for every $i \not\equiv j$. Indeed, any subset of a comonotonic set is also comonotonic.

Now let $\mathbf{X}$ be an $n$-dimensional random vector. Any set $A \subset \mathbb{R}^n$ is referred to as the *support* of $\mathbf{X}$ if $\mathbf{X} \in A$ almost surely (the smallest support can be thought of as the image of $\mathbf{X} : \Omega \to \mathbb{R}^n$). By Definition 4 of Dhaene et al. (2002b), $\mathbf{X}$ is said to be a *comonotonic random vector* if it has a comonotonic support: in other words, $\boldsymbol{X}(\omega_1) \leq \boldsymbol{X}(\omega_2)$ or $\boldsymbol{X}(\omega_1) \geq \boldsymbol{X}(\omega_2)$ for every $\omega_1, \omega_2 \in \Omega$.

A number of important results are given by Theorem 2 of Dhaene et al. (2002b) or, equivalently, Theorem 1.3.4 of Vyncke (2003), which we repeat below. Note that (3) is used as the definition of comonotonicity given by Goovaerts and Dhaene (1999).

**Theorem 6.** *A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is comonotonic if and only if one of the following equivalent conditions holds.*

1. *$\mathbf{X}$ has a comonotonic support.*

2. *For all $\mathbf{x} = (x_1, \ldots, x_n)$, we have*

$$F_{\mathbf{X}}(\mathbf{x}) = \min_{j=1,\ldots,n} F_{X_j}(x_j). \tag{2.2.1}$$

3. *For any standard uniform random variable $U \sim U(0, 1)$, we have*

$$\mathbf{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)). \tag{2.2.2}$$

4. *There exists a random variable $Z$ and non-decreasing functions $f_i$, $i = 1, \ldots, n$, such that*

$$\mathbf{X} \stackrel{d}{=} (f_1(Z), \ldots, f_n(Z)). \tag{2.2.3}$$

*Proof.* We give a proof of (1)⇒(2). Proofs of the other required relationships can be found in Dhaene et al. (2002b); Vyncke (2003). Let $\mathbf{X}$ have a comonotonic support $B$. Let $\mathbf{x} \in \mathbb{R}^n$ and define $A_j \subset B$ by

$$A_j = \{\mathbf{y} \in B : y_j \leq x_j\}, \quad j = 1, \dots, n.$$

Since the $A_j$ are comonotonic, they form ordered, 1-dimensional sets. Therefore, for each $j$, it is possible to use the total ordering relationship on $A_j$ to obtain its maximum element $\mathbf{y}_j = \{\mathbf{y} \in A_j : \mathbf{a} \leq \mathbf{y}, \mathbf{a} \in A_j\}$. Furthermore, the set $Y = \{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ is also comonotonic and so $\mathbf{z} = \{\mathbf{y} \in Y : \mathbf{y} \leq \mathbf{y}_j, j = 1, \dots, n\}$ defines the minimum such vector in $Y$. Suppose that $\mathbf{z} = \mathbf{y}_i$ for a given $i$. Then, for every $\mathbf{y} \in A_i$, we have $\mathbf{y} \leq \mathbf{z} \leq \mathbf{y}_j$. Hence, $\mathbf{y} \in A_j$ for every $j$ and so

$$A_i = \bigcap_{j=1}^{n} A_j.$$

Therefore, for the same value of $\boldsymbol{x}$ above, we have

$$F_{\mathbf{X}}(\mathbf{x}) = \mathbf{P}\left(X_j \leq x_j, \forall j\right) = \mathbf{P}\left(\mathbf{X} \in \bigcap_{j=1}^{n} A_j\right) = \mathbf{P}\left(\mathbf{X} \in A_i\right) = F_{X_i}(x_i).$$

Since $A_i \subseteq A_j$ and the probability measure $\mathbf{P}$ is monotonic,

$$F_{X_i}(x_i) = \mathbf{P}\left(\mathbf{X} \in A_i\right) \leq \mathbf{P}\left(\mathbf{X} \in A_j\right) = F_{X_j}(x_j).$$

In other words, $F_{\mathbf{X}}(\mathbf{x}) = \min\{F_{X_1}(x_1), \dots, F_{X_n}(x_n)\}$. □

Using this theorem, we see that an analogue of Lemma 1 of Dhaene et al. (2002b) in the case of comonotonic random vectors is given by Theorem 3 of Dhaene et al. (2002b). Therefore, in forming a comonotonic random variable $\mathbf{X^c} = (X_1^c, \dots, X_n^c)$, it suffices to check comonotonicity by showing that $(X_i^c, X_j^c)$ is comonotonic for every $i \not\equiv j$. If the components of the comonotonic $\mathbf{X^c}$ have the same marginal distribution as that of $\mathbf{X}$, then $\mathbf{X^c}$ is referred to as a *comonotonic counterpart* of $\mathbf{X}$. Such $\mathbf{X^c}$ will be very important in forming stochastic bounds for $X$.

## 2.3 Main results in comonotonicity

For any comonotonic counterpart $\mathbf{X^c}$ of $\mathbf{X}$, set $X^c = \sum_{i=1}^n X_i^c$. A simplification of Theorem 5 of Dhaene et al. (2002b) tells us the following.

**Theorem 7.** *The inverse distribution function $F_{X^c}^{-1}$ of the sum, $X^c$, of the components of the comonotonic random variable $\mathbf{X^c}$ is given by*

$$F_{X^c}^{-1}(p) = \sum_{i=1}^n F_{X_i^c}^{-1}(p), \quad \forall p \in (0,1). \tag{2.3.1}$$

*Proof.* Since $\mathbf{X^c}$ is a comonotonic counterpart of $\mathbf{X}$, we have $F_{X_i^c}(x_i) = F_{X_i}(x_i)$ and so $F_{X_i^c}^{-1}(p) = F_{X_i}^{-1}(p)$ for every $i$. Theorem 6(3) then gives us the following.

$$X^c = \sum_{i=1}^n X_i^c \stackrel{d}{=} \sum_{i=1}^n F_{X_i}^{-1}(U) =: g(U), \tag{2.3.2}$$

where $U \sim U(0,1)$ is a standard uniform random variable. We see that $g : (0,1) \to \mathbb{R}$ is an increasing function and so Theorem 1(a) of Dhaene et al. (2002b) states

$$F_{X^c}^{-1}(p) = F_{g(U)}^{-1}(p) = g(F_U^{-1}(p)) = g(p), \quad p \in (0,1),$$

which proves the desired result. □

This leads to the very important result, Theorem 6 of Dhaene et al. (2002b), below. This will allow us to split the stop-loss premium of a sum into a sum of relevant stop-loss premiums that are easier to evaluate.

**Theorem 8.** *The stop-loss premiums of the sum $X^c$ of comonotonic random variables $X_i^c$ are given by*

$$\mathbf{E}\left[(X^c - d)^+\right] = \sum_{i=1}^n \mathbf{E}\left[\left(X_i^c - F_{X_i^c}^{-1}(F_{X^c}(d))\right)^+\right], \quad d \in \mathbb{R}. \tag{2.3.3}$$

*Proof.* Since the support $B$ of $\mathbf{X^c}$ is comonotonic, it can have at most one point of intersection with the hyperplane $H = \{\mathbf{x} : x_1 + \ldots + x_n = d\}$. This is because decreasing any one of the $x_i$ of $\mathbf{x} \in H$ requires increasing an $x_j$, where $i \neq j$, contradicting the comonotonicity assumption. Hence, let $\mathbf{d} = B \cap H$ and let $\mathbf{x} \in B$. Then

$$\left(\sum_{i=1}^n x_i - d\right)^+ = \sum_{i=1}^n (x_i - d_i)^+.$$

This is because, if $x_i > d_i$ for any $i$ then $x_j \geq d_j$ for all $j$ and so the above equation is equal since $\sum_{i=1}^{n} d_i = d$. Otherwise, we have $x_i \leq d_i$ for all $i$ and so both sides of the equation are zero. Let $p = F_{X^c}(d)$. From Theorem 7, we hence have that

$$d = F_{X^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i^c}^{-1}(F_{X^c}(d)) = \sum_{i=1}^{n} d_i.$$

By replacing the $x_i$ by the random variables $X_i$ and taking expectations, we achieve the desired result. □

## 2.4 Stochastic bounds for sums of dependent risks

With the results from Section 2.3, we can derive stochastic bounds for the sum, of random variables for which the marginal distributions are given. As stated by Dhaene et al. (2002b), "the reason we will resort to convex bounds is that the joint distribution of $(X_1, \ldots, X_n)$ is either unspecified or too cumbersome to work with."

Let us define $X$ as the sum of the coordinates of the random vector $\boldsymbol{X} \in \mathbb{R}^n$.

$$X = \sum_{i=1}^{n} X_i.$$

We then proceed by describing upper and lower bounds for the stop-loss transform of an arithmetic sum of random variables whose marginal distributions are known.

### 2.4.1 Upper bounds for stop-loss transforms

The first upper bound arises as a result of Theorem 7 of Dhaene et al. (2002b); that is, for the comonotonic counterpart $\mathbf{X^c}$ of $\mathbf{X}$ that is defined by

$$\mathbf{X^c} = (F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)),$$

we have $X \leq_{\text{cx}} X^c$. In other words,

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} X_i - d\right)^+\right] \leq \mathbf{E}\left[\left(\sum_{i=1}^{n} X_i^c - d\right)^+\right] = \sum_{i=1}^{n} \mathbf{E}\left[\left(X_i - F_{X_i}^{-1}(F_{X^c}(d))\right)^+\right]. \tag{2.4.1}$$

Furthermore, Theorems 6 and 7 of Dhaene et al. (2002b) demonstrate that the comonotonic upper bound on the right hand side of equation (2.4.1) is the smallest upper bound of the form $\sum_{i=1}^{n} \mathbf{E}\left[(X_i - d_i)^+\right]$ where $\sum_{i=1}^{n} d_i = d$.

We can improve on this upper bound further if we assume "some additional information available concerning the stochastic nature of $(X_1, \ldots, X_n)$." That is, if we can find a random variable $\Lambda$, with a known distribution, such that the individual conditional distributions of $X_i$ given the event $\Lambda = \lambda$ are known for all $i$ and all possible values of $\lambda$. If this is the case, then equations 85 and 86 of Dhaene et al. (2002b) tell us

$$X \leq_{\mathrm{cx}} X^u \leq_{\mathrm{cx}} X^c, \tag{2.4.2}$$

where we define $X^u = \sum_{i=1}^n F^{-1}_{X_i|\Lambda}(U) =: \sum_{i=1}^n X_i^u$.

Now let $\mathbf{X^u} = (X_1^u, \ldots, X_n^u)$. We see, given the event $\Lambda = \lambda$, $\mathbf{X^u}$ is a vector of strictly increasing functions dependent on a single random variable $U$. Therefore, the conditional random variable $\mathbf{X^u}|(\Lambda = \lambda)$ is also comonotonic. Hence, from equation 89 of Dhaene et al. (2002b),

$$F^{-1}_{X^u|\Lambda=\lambda}(p) = \sum_{i=1}^n F^{-1}_{X_i|\Lambda=\lambda}(p), \quad p \in (0,1). \tag{2.4.3}$$

It follows that, in this case, $d \in \mathbb{R}$ solves

$$d = \sum_{i=1}^n F^{-1}_{X_i|\Lambda=\lambda}(F_{X^u|\Lambda=\lambda}(d)),$$

and so we have

$$\mathbf{E}\left[\left(\sum_{i=1}^n X_i^u - d\right)^+ \middle| \Lambda = \lambda\right] = \sum_{i=1}^n \mathbf{E}\left[\left(X_i - F^{-1}_{X_i|\Lambda=\lambda}(F_{X^u|\Lambda=\lambda}(d))\right)^+ \middle| \Lambda = \lambda\right]. \tag{2.4.4}$$

By taking the expectation of both sides and applying the tower property (see, for example, Williams (1991)), we obtain an improved upper bound for $X$.

$$\begin{aligned}
\mathbf{E}\left[(X-d)^+\right] &\leq \mathbf{E}\left[(X^u - d)^+\right] \\
&= \mathbf{E}\left[\mathbf{E}\left(\left(\sum_{i=1}^n X_i^u - d\right)^+ \middle| \Lambda\right)\right] \\
&= \sum_{i=1}^n \mathbf{E}\left[\mathbf{E}\left(\left(X_i - F^{-1}_{X_i|\Lambda=\lambda}(F_{X^u|\Lambda=\lambda}(d))\right)^+ \middle| \Lambda\right)\right]. \qquad (2.4.5)
\end{aligned}$$

### 2.4.2 A lower bound for stop-loss transforms

With this conditioning variable $\Lambda$ above, we can also find a lower bound for $X$, by making use of Theorem 10 of Dhaene et al. (2002b). Let $X^l = \sum_{i=1}^n \mathbf{E}\left(X_i|\,\Lambda\right)$. Then, by Jensen's inequality we have, for any convex function $v$,

$$\mathbf{E}\left[v(\mathbf{X})\right] = \mathbf{E}\left[\mathbf{E}\left(v(\mathbf{X})|\,\Lambda\right)\right] \geq \mathbf{E}\left[v(\mathbf{E}\left(\mathbf{X}|\,\Lambda\right))\right].$$

In other words, by setting $v(\mathbf{x}) = \left(\sum_{i=1}^n x_i - d\right)^+$, we have

$$\mathbf{E}\left[\left(\sum_{i=1}^n X_i - d\right)^+\right] \geq \mathbf{E}\left[\left(\sum_{i=1}^n \mathbf{E}\left(X_i|\,\Lambda\right) - d\right)^+\right]. \tag{2.4.6}$$

By combining this result with the tower property, it then follows that $X^l \leq_{\text{cx}} X$. An expression for the stop-loss transform of $X^l$, as a lower bound for the stop-loss transform of $X$, is given below.

**Lemma 9.** *The random vector* $\mathbf{X^l} = \left(\mathbf{E}\left(X_1|\,\Lambda\right), \ldots, \mathbf{E}\left(X_n|\,\Lambda\right)\right)$ *is comonotonic if* $\Lambda$ *is chosen such that the* $\mathbf{E}\left[X_i|\,\Lambda = \lambda\right]$ *are mutually nonincreasing or nondecreasing functions of* $\lambda$*, for every* $i$*. Under this choice of* $\Lambda$*, a lower bound for* $\Psi(X,d)$ *can be expressed using a sum of stop-loss transforms in the following way.*

$$\begin{aligned}\mathbf{E}\left[\left(\sum_{i=1}^n X_i - d\right)^+\right] &\geq \mathbf{E}\left[\left(\sum_{i=1}^n \mathbf{E}\left(X_i|\,\Lambda\right) - d\right)^+\right] \\ &= \sum_{i=1}^n \mathbf{E}\left[\left(\mathbf{E}\left(X_i|\,\Lambda\right) - F^{-1}_{\mathbf{E}(X_i|\Lambda)}(F_{X^l}(d))\right)^+\right].\end{aligned} \tag{2.4.7}$$

*Proof.* Under the condition that $\mathbf{X^l}$ is comonotonic, one can immediately apply Theorem 7, with $p = F_{X^l}(d)$, to demonstrate that $d$ solves

$$d = F^{-1}_{X^l}(F_{X^l}(d)) = \sum_{i=1}^n F^{-1}_{\mathbf{E}(X_i|\Lambda)}(F_{X^l}(d)).$$

Therefore, by using Theorem 8, we immediately see that the right-hand side of equation (2.4.6) satisfies equation (2.4.7). □

## 2.5 Bounds for exotic options

The above material and Dhaene et al. (2002b) demonstrate how to obtain convex order stochastic bounds for the sum of dependent risks $X = X_1 + \ldots X_n$. The subsequent paper, Dhaene et al. (2002a), give examples of how the results above may be used in actuarial science and finance. Further work by Albrecher et al. (2008); Hobson et al. (2005) show how to approximate the value of Asian and basket options respectively. We shall proceed by refining this work through the examples below.

### 2.5.1 Asian call options

Let us define $\{S(t)\}_{t\geq 0}$ to be the process of a single underlying asset. We wish to estimate the value of any Asian call option written on this process, observed at the monitoring times $0 \leq t_0 < t_1 < \ldots < t_m \leq T$. Using the notation from above, set $X_i = S(t_i)$ and let $d = mK$ be the scaled exercise price of the corresponding option. Then, we would like to bound the value of the following stop-loss premium under the risk neutral measure $\mathbf{Q}$:

$$\Psi(X, d) = \mathbf{E}^{\mathbf{Q}}\left[\left(\sum_{i=1}^{m} S(t_i) - mK\right)^+\right] = me^{rT}A(K, T, m). \tag{2.5.1}$$

### 2.5.2 A model-independent lower bound

Albrecher et al. (2008) attempt to estimate $\Psi(X, d)$, and the corresponding Asian option price, by deriving three lower bounds, $LB_1, LB_t^{(1)}$ and $LB_t^{(2)}$, each of which are successively closer to its actual value. We see that a convex order lower bound for $\mathbb{S} := \sum_{i=1}^{m} S(t_i)$ is given by the following, for any random variable $\Lambda$.

$$S^l := \sum_{i=1}^{m} \mathbf{E}^{\mathbf{Q}}\left(S(t_i)|\,\Lambda\right) \leq_{\text{cx}} \mathbb{S}.$$

This translates into the following lower bound.

$$A(K, T, m) \geq \frac{e^{-rT}}{m}\mathbf{E}^{\mathbf{Q}}\left[\left(\sum_{i=1}^{m} \mathbf{E}^{\mathbf{Q}}\left(S(t_i)|\,\Lambda\right) - mK\right)^+\right].$$

If $\Lambda$ is chosen so that the random vector $\left(\mathbf{E}^{\mathbf{Q}}\left(S(t_1)|\,\Lambda\right),\ldots,\mathbf{E}^{\mathbf{Q}}\left(S(t_m)|\,\Lambda\right)\right)$ is comonotonic, then we can apply Theorem 8 above, rewriting this lower bound as

$$A(K,T,m) \geq \frac{e^{-rT}}{m}\sum_{i=1}^{m}\mathbf{E}^{\mathbf{Q}}\left[\left(\mathbf{E}^{\mathbf{Q}}\left(S(t_i)|\,\Lambda\right) - F^{-1}_{\mathbf{E}^{\mathbf{Q}}(S(t_i)|\Lambda)}(F_{S^l}(mK))\right)^+\right]. \quad (2.5.2)$$

Albrecher et al. (2008) use this relationship to find a lower bound for Asian options under the assumption that $S(t) = S(0)e^{X(t)}$, where $\{X(t)\}_{t\geq 0}$ is any Lévy process. Indeed, under this assumption, we have $\mathbf{E}^{\mathbf{Q}}\left(X(t_i)|\,X(t)\right) = \frac{t_i}{t}X(t)$ for any $t_i < t$. Hence, for such $t_i$, using Jensen's inequality gives the following.

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}}\left[S(t_i)|\,S(t)=s\right] &= S(0)\mathbf{E}^{\mathbf{Q}}\left[e^{X(t_i)}\middle|\,X(t) = \log(s/S(0))\right]\\ &\geq S(0)\exp\left(\mathbf{E}^{\mathbf{Q}}\left[X(t_i)|\,X(t) = \log(s/S(0))\right]\right)\\ &= S(0)\exp\left(\frac{t_i}{t}\log\left(\frac{s}{S(0)}\right)\right).\end{aligned}$$

In other words, for all $t_i < t$,

$$\mathbf{E}^{\mathbf{Q}}\left(S(t_i)|\,S(t)\right) \geq S(0)\left(\frac{S(t)}{S(0)}\right)^{t_i/t}. \quad (2.5.3)$$

For $t_i \geq t$ we note that, under the risk neutral measure $\mathbf{Q}$, we have $\mathbf{E}^{\mathbf{Q}}\left(S(t_i)|\,S(t)\right) = S(t)e^{r(t_i-t)}$. As defined by Albrecher et al. (2008), let $j = \min\{i : t_i \geq t\}$ and set $\mathbf{Y} = (Y_1,\ldots,Y_m)$, where

$$Y_i = \begin{cases} S(0)\left(\frac{S(t)}{S(0)}\right)^{t_i/t} & i < j,\\ S(t)e^{r(t_i-t)} & i \geq j.\end{cases} \quad (2.5.4)$$

We see, as stated by Albrecher et al. (2008), that $\mathbf{Y}$ is comonotonic since its components all mutually increase if and only if $S(t)$ increases. Furthermore, equation (2.5.3) gives the following stop-loss order relationship between $\sum_{i=1}^m Y_i$ and $\mathbb{S}$.

$$S^{n_2} := \sum_{i=1}^{j-1} S(0)\left(\frac{S(t)}{S(0)}\right)^{t_i/t} + \sum_{i=j}^{m}\mathbf{E}^{\mathbf{Q}}\left(S(t_i)|\,S(t)\right) \leq \sum_{i=1}^{m}\mathbf{E}^{\mathbf{Q}}\left(S(t_i)|\,S(t)\right) \leq_{\text{sl}} \mathbb{S}.$$

Therefore, we see that the stop-loss transform of $\mathbb{S}$ is bounded below by equation 15 of Albrecher et al. (2008), given below, as a result of Theorem 8.

$$\Psi(\mathbb{S}, mK) \geq \mathbf{E}^{\mathbf{Q}}\left[(S^{n_2} - mK)^+\right] = \sum_{i=1}^{m} \mathbf{E}^{\mathbf{Q}}\left[\left(Y_i - F_{Y_i}^{-1}(F_{S^{n_2}}(mK))\right)^+\right], \quad (2.5.5)$$

where $F_{S^{n_2}}(mK)$ is the distribution function of $S^{n_2}$ evaluated at $mK$, satisfying

$$\begin{aligned} F_{S^{n_2}}(mK) &= \mathbf{Q}\left(\sum_{i=1}^{j-1} S(0)\left(\frac{S(t)}{S(0)}\right)^{t_i/t} + \sum_{i=j}^{m} S(t)e^{r(t_i-t)} \leq mK\right) \\ &= \mathbf{Q}\left(\sum_{i=1}^{j-1}\left(\frac{S(t)}{S(0)}\right)^{t_i/t} + \left(\frac{S(t)}{S(0)}\right)\sum_{i=j}^{m} e^{r(t_i-t)} \leq \frac{mK}{S(0)}\right). \end{aligned} \quad (2.5.6)$$

Let us define the function $H : \mathbb{R}_+ \to \mathbb{R}_+$ in the following way.

$$H(x) = \sum_{i=1}^{j-1} x^{t_i/t} + x\sum_{i=j}^{m} e^{r(t_i-t)}. \quad (2.5.7)$$

We can immediately observe that $H$ is a continuous, strictly increasing function of $x$. Therefore, in order to simplify the probability given by equation (2.5.6), we observe, as in Albrecher et al. (2008), that it is computationally straightforward to solve the equation

$$H(x) = \frac{mK}{S(0)},$$

which has a unique positive solution. Hence, we then see, from the monotonicity of $H$, that

$$S^{n_2} = S(0)H\left(\frac{S(t)}{S(0)}\right) \leq mK = S(0)H(x)$$

if and only if $S(t) \leq xS(0)$. Therefore, under $\mathbf{Q}$, we see that $F_{S^{n_2}}(mK) = F_{S(t)}(xS(0))$. By applying this result to the inverse distribution function of $Y_i$, using equation (2.5.4), we obtain the following result.

$$F_{Y_i}^{-1}(F_{S^{n_2}}(mK)) = F_{Y_i}^{-1}(F_{S(t)}(xS(0))) = \begin{cases} S(0)x^{t_i/t} & i < j, \\ S(0)xe^{r(t_i-t)} & i \geq j. \end{cases}$$

Replacing this into equation (2.5.5) and multiplying by the averaged discount factor

$e^{-rT}/m$, we realise the lower bound $LB_t^{(2)}$ from Albrecher et al. (2008):

$$A(K,T,m) \geq \frac{e^{-rT}}{m}\left(\sum_{i=1}^{j-1} S(0)^{1-t_i/t}\mathbf{E}^{\mathbf{Q}}\left[\left(S(t)^{t_i/t} - (xS(0))^{t_i/t}\right)^+\right] + C(xS(0),t)\sum_{i=j}^{m} e^{rt_i}\right) =: LB_t^{(2)}. \quad (2.5.8)$$

### 2.5.3 An improved lower bound under Black-Scholes conditions

In Section 2.5.2, we consider lower bounds for Asian options with limited restrictions on the model used for the underlying asset price. In this section, we attempt to improve the lower bound above under the well-documented assumption that $\{S(t)\}_{t\in[0,T]}$ follows the Black-Scholes model. We use the following stochastic differential equation under the risk neutral measure $\mathbf{Q}$.

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t), \qquad t \in [0,T]. \quad (2.5.9)$$

Using the notation from above, define $X(t) = \log S(t)$, the solution of which is provided in the next equation. As usual, $\{W(t)\}_{t\geq 0}$ denotes the process of a standard Brownian motion. We then make use of the subsequent two Propositions; the first of which is a well-known result from probability theory.

$$X(t) = \log S(0) + \left(r - \frac{\sigma^2}{2}\right)t + \sigma W(t).$$

**Proposition 10.** *Let $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$ be two normal random variables with correlation $\rho$. Then, the conditional distribution function of $X$, given the event $Y = y$, satisfies*

$$F_{X|Y=y}(x) = \Phi\left(\frac{x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)}{\sigma_X\sqrt{1-\rho^2}}\right). \quad (2.5.10)$$

*As usual, we use $\Phi$ to denote the standard normal distribution function, with the corresponding density function given by $\varphi$.*

**Proposition 11.** *Let $X$, $Y$ and $\rho$ be defined as in Proposition 10. Then, the conditional* *distribution of the lognormal random variable $e^X$, given the event $e^Y = y$ satisfies*

$$F_{e^X|e^Y=y}(x) = \Phi\left(\frac{\log x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(\log y - \mu_Y)\right)}{\sigma_X\sqrt{1-\rho^2}}\right). \tag{2.5.11}$$

*Proof.* This is immediate from Proposition 10. Indeed, we have

$$\begin{aligned} F_{e^X|e^Y=y}(x) &= \mathbf{P}\left(e^X \leq x \middle| e^Y = y\right) \\ &= \mathbf{P}\left(X \leq \log x \middle| Y = \log y\right) \\ &= F_{X|Y=\log y}(\log x). \end{aligned}$$

□

Given the time point $t_i$ for each $i$, fix $t$ and let $\rho$ denote the correlation between $X(t_i)$ and $X(t)$. Then, conditional on the event $S(t) = s_t$, we see from equation (2.5.11), that the distribution function of $S(t)$ satisfies $F_{S(t_i)|S(t)=s_t}(x) = \Phi(y(x))$, where $y(x)$ is given by

$$y(x) = \frac{\log x - \left(\log\left(S(0)\left(\frac{s_t}{S(0)}\right)^{\rho\sqrt{\frac{t_i}{t}}}\right) + \left(r - \frac{\sigma^2}{2}\right)(t_i - \rho\sqrt{t_i t})\right)}{\sigma\sqrt{t_i(1-\rho^2)}}. \tag{2.5.12}$$

Therefore, the conditional density function of $S(t_i)$ given the event $S(t) = s_t$ satisfies the following equation, which we can use in the subsequent statement. This then leads to a further lower bound for Asian options in Theorem 13, which we subsequently state.

$$f_{S(t_i)|S(t)=s_t}(x) = \frac{1}{x\sigma\sqrt{t_i(1-\rho^2)}}\varphi(y(x)).$$

**Proposition 12.** *Define the process $\{S(t)\}_{t\geq 0}$ as above. Then, the conditional expectation of $S(t_i)$ given $S(t)$ satisfies*

$$\mathbf{E}^{\mathbf{Q}}\left(S(t_i)\middle| S(t)\right) = \begin{cases} S(0)\left(\frac{S(t)}{S(0)}\right)^{\frac{t_i}{t}} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}, & t_i < t, \\ S(t)e^{r(t_i-t)}, & t_i \geq t. \end{cases} \tag{2.5.13}$$

*Proof.* The conditional expectation of $S(t_i)$ given the event $S(t) = s_t$ is given by

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}}\left[S(t_i)\right|S(t) = s_t] &= \int_0^\infty x f_{S(t_i)|S(t)=s_t}(x)\,dx \\ &= \int_0^\infty \frac{1}{\sigma\sqrt{t_i(1-\rho^2)}}\varphi(y(x))\,dx.\end{aligned}$$

With $y$ given by equation (2.5.12), we can write $x$ in terms of $y$.

$$x = S(0)\left(\frac{s_t}{S(0)}\right)^{\rho\sqrt{\frac{t_i}{t}}}\exp\left(\left(r-\frac{\sigma^2}{2}\right)(t_i - \rho\sqrt{t_i t}) + \sigma\sqrt{t_i(1-\rho^2)}y\right).$$

Let $Z \sim N(0,1)$ denote a standard normally distributed random variable. Then, the integral above becomes

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}}\left[S(t_i)\right|S(t) = s_t] &= S(0)\left(\frac{s_t}{S(0)}\right)^{\rho\sqrt{\frac{t_i}{t}}} e^{\left(r-\frac{\sigma^2}{2}\right)(t_i-\rho\sqrt{t_i t})}\int_{-\infty}^{\infty} e^{\sigma\sqrt{t_i(1-\rho^2)}y}\varphi(y)\,dy \\ &= S(0)\left(\frac{s_t}{S(0)}\right)^{\rho\sqrt{\frac{t_i}{t}}} e^{\left(r-\frac{\sigma^2}{2}\right)(t_i-\rho\sqrt{t_i t})}\mathbf{E}^{\mathbf{Q}}\left[e^{\sigma\sqrt{t_i(1-\rho^2)}Z}\right] \\ &= S(0)\left(\frac{s_t}{S(0)}\right)^{\rho\sqrt{\frac{t_i}{t}}} e^{\left(r-\frac{\sigma^2}{2}\right)(t_i-\rho\sqrt{t_i t})+\frac{\sigma^2}{2}(t_i(1-\rho^2))}. \qquad (2.5.14)\end{aligned}$$

We see that the correlation coefficient $\rho \in [0,1]$ satisfies

$$\rho = \begin{cases}\sqrt{\dfrac{t_i}{t}}, & t_i < t, \\ \sqrt{\dfrac{t}{t_i}}, & t_i \geq t,\end{cases},$$

which leads to the following expression for $1-\rho^2 \in [0,1]$.

$$1-\rho^2 = \begin{cases}\dfrac{t-t_i}{t}, & t_i < t, \\ \dfrac{t_i - t}{t_i}, & t_i \geq t.\end{cases}$$

By using this in equation (2.5.14) and replacing $s_t$ by the random variable $S(t)$ we obtain the result given by equation (2.5.13). □

**Theorem 13.** *Let $\{S(t)\}_{t\in[0,T]}$ be a stochastic process that satisfies the Black-Scholes SDE given by equation (2.5.9) under the risk-neutral measure $\mathbf{Q}$. Choose a set of monitoring times $0 \leq t_0 < t_1 < \cdots < t_m \leq T$. Fix $t \in [0,T]$ and set $j$ to be the index* *of the first such monitoring time that is greater than or equal to t.*

$$j = \min\{i : t_i \geq t\}. \tag{2.5.15}$$

*Then, the value of an arithmetic Asian call option written upon* $\{S(t)\}_{t\in[0,T]}$*, with exercise price* $K$ *and maturity time* $T$ *is given by*

$$A(K,T,m) \geq \frac{e^{-rT}}{m}\left(\sum_{i=1}^{j-1} S(0)^{1-t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} \mathbf{E}^{\mathbf{Q}}\left[\left(S(t)^{t_i/t} - (xS(0))^{t_i/t}\right)^+\right] + C(S(0)x,t)\sum_{i=j}^{m} e^{rt_i}\right) =: LB_t^{(3)}, \tag{2.5.16}$$

*where* $C(z_1, z_2)$ *denotes the value of a vanilla European call option with strike price* $z_1$ *and maturity time* $z_2$*, and* $x$ *satisfies the following equation.*

$$\sum_{i=1}^{j-1} x^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} + x\sum_{i=j}^{m} e^{r(t_i-t)} = \frac{mK}{S(0)}. \tag{2.5.17}$$

*Proof.* We can obtain a proof of Theorem 13 by reapplying the methods of Albrecher et al. (2008) used in Section 2.5.2. Let $S^{n_3} = \sum_{i=1}^{m} Y_i$, where this time $Y_i$ is given by

$$Y_i = \begin{cases} S(0)^{1-t_i/t} S(t)^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} & i < j, \\ S(t)e^{r(t_i-t)} & i \geq j. \end{cases} \tag{2.5.18}$$

Then, the distribution function of $S^{n_3}$ at $nK$ satisfies

$$\begin{aligned} F_{S^{n_3}}(mK) &= \mathbf{Q}\left(\sum_{i=1}^{m} Y_i \leq mK\right) \\ &= \mathbf{Q}\left(\sum_{i=1}^{j-1} S(0)\left(\frac{S(t)}{S(0)}\right)^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} + \left(\frac{S(t)}{S(0)}\right)\sum_{i=j}^{m} S(0)e^{r(t_i-t)} \leq mK\right). \end{aligned}$$

By setting $x = S(t)/S(0)$, it then immediately follows that we achieve equality within the probability given above when $x$ satisfies equation (2.5.17). As in the case given by Section 2.5.2, solving this equation for $x$ is computationally straightforward since the left hand side of equation (2.5.17) is also strictly increasing. We then obtain

$$F_{Y_i}^{-1}(F_{S^{n_3}}(mK)) = F_{Y_i}^{-1}(F_{S(t)}(S(0)x)) = \begin{cases} S(0)x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} & i < j, \\ S(0)xe^{r(t_i-t)} & i \geq j. \end{cases}$$

It is clear that $\mathbf{Y} = (Y_1, \ldots, Y_m)$ is comonotonic. Hence, as a result, we obtain

$$\begin{aligned} \mathbf{E}^{\mathbf{Q}}\left[(S^{n_3} - mK)^+\right] &= \sum_{i=1}^{m} \mathbf{E}^{\mathbf{Q}}\left[\left(Y_i - F_{Y_i}^{-1}(F_{S^{n_3}}(mK))\right)^+\right] \\ &= \sum_{i=1}^{j-1} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}\mathbf{E}^{\mathbf{Q}}\left[\left(S(0)^{1-t_i/t}S(t)^{t_i/t} - S(0)x^{t_i/t}\right)^+\right] \\ &\quad + \sum_{i=j}^{m} \mathbf{E}^{\mathbf{Q}}\left[\left(S(t)e^{r(t_i-t)} - S(0)xe^{r(t_i-t)}\right)^+\right] \\ &= \sum_{i=1}^{j-1} S(0)^{1-t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}\mathbf{E}^{\mathbf{Q}}\left[\left(S(t)^{t_i/t} - (xS(0))^{t_i/t}\right)^+\right] \\ &\quad + \sum_{i=j}^{m} e^{rt_i}C(S(0)x, t). \end{aligned}$$

Where $C$ denotes the European call option price above. By taking the discount factor at time $T$ and averaging, we obtain equation (2.5.16). □

As in the case with $LB_t^{(2)}$, this is a lower bound for all $t \geq 0$ and so we can obtain a lower bound for $A(K, T, m)$ by maximising $LB_t^{(3)}$ with respect to $t$. Furthermore, because we calculate $\mathbf{E}^{\mathbf{Q}}\left(S(t_i)|\, S(t)\right)$ explicitly, rather than finding a lower bound for it, it immediately follows that $LB_t^{(3)}$ is an improved lower bound compared to $LB_t^{(2)}$, in the case where the underlying asset price follows the Black-Scholes model.

### 2.5.4 An upper bound for Asian options

The results from Section 2.4 show that we can also find upper bounds for the value of Asian options. Hobson et al. (2005) uses the method of Lagrange multipliers to find an upper bound for basket options and we can use this approach to find an upper bound for Asian options. We can also demonstrate the link with this approach and comonotonicity. Indeed, let us rewrite the European call value as

$$C(K, T) = e^{-rT}\mathbf{E}^{\mathbf{Q}}[(S(T) - K)^+] = e^{-rT}\int_K^\infty 1 - F_{S(T)}(x)\, dx.$$

The derivative of $C$ with respect to $K$ is then

$$\frac{\partial C}{\partial K} = -e^{-rT}(1 - F_{S(T)}(K)) = -e^{-rT}\mathbf{Q}(S(T) \geq K).$$

Given any vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ under which $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$, we can find an upper bound for $A$, written as the sum of corresponding European calls.

$$\begin{aligned} A(K,T,m) &= \frac{e^{-rT}}{m}\mathbf{E}^{\mathbf{Q}}\left[\left(\sum_{i=1}^{m}(S(t_i) - \lambda_i mK)\right)^{+}\right] \\ &\leq \frac{e^{-rT}}{m}\sum_{i=1}^{m} e^{rt_i}C(\lambda_i mK, t_i). \end{aligned}$$

Since the $\lambda_i$ are arbitrary, the task is then to minimise this bound over all possible $\boldsymbol{\lambda}$. This is equivalent to minimising

$$\sum_{i=1}^{m} e^{rt_i}C(\lambda_i nK, t_i) = \sum_{i=1}^{m} \Psi(S(t_i), \lambda_i mK). \tag{2.5.19}$$

We assume that $C(K,T) > 0$ for every $K, T$ and that $C(K,T) \downarrow 0$ as $K \to \infty$. Then $C$ is a convex, strictly decreasing function of $K$ with a continuous, strictly increasing derivative $\partial C/\partial K < 0$. Let $L$ define the Lagrangian function given by

$$L(\boldsymbol{\lambda}, \phi) = \sum_{i=1}^{m} e^{rt_i}C(\lambda_i mK, t_i) + \phi\left(\sum_{i=1}^{m}\lambda_i - 1\right).$$

We wish to find the $\lambda_i$, for each $i$, that minimises $L$. Differentiating, we obtain

$$\frac{\partial L}{\partial \lambda_i} = -mK\mathbf{Q}(S(t_i) \geq \lambda_i mK) + \phi.$$

Therefore, $L$ has a stationary point when $\lambda_i$ solves the following for every $i$.

$$\lambda_i(\phi) = \frac{1}{mK}F^{-1}_{S(t_i)}\left(1 - \frac{\phi}{mK}\right).$$

Since $S(t_i) \geq 0$ for every $i$, it follows that $\lambda_i$ is nonnegative. To satisfy the constraint that $\sum_{i=1}^{m} \lambda_i = 1$, define $H$ by

$$H(\phi) = \sum_{i=1}^{m}\lambda_i(\phi) - 1 = \frac{1}{mK}\sum_{i=1}^{m} F^{-1}_{S(t_i)}\left(1 - \frac{\phi}{mK}\right) - 1.$$

Under the above assumptions, $H$ is a continuous function of $\phi$. Moreover, since we assume that $F_{S(t)}$ is injective for all $t \in [0,T]$, it follows that $H$ is strictly decreasing in $\phi$. Hence, a solution to $H(\phi) = 0$ exists if it can be shown that $\inf H(\phi) < 0 < \sup H(\phi)$. For $\phi = mK$, it follows that $H(\phi) = -1$. To see that there exists a $\phi$ such that $H(\phi) > 0$, note that $F_{S(t_i)}(K) = 1$ only in the limit $K \to \infty$. Hence, $\lim_{\phi \downarrow 0} H(\phi) = \infty$ and so, by the Intermediate Value Theorem, we can find $\phi^*$ that solves $H(\phi^*) = 0$. Furthermore, $\phi^*$ is unique since $H$ is strictly decreasing.

Since $\partial C/\partial K$ is strictly increasing, it follows that $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}(\phi^*)$ minimises the upper bound given in equation (2.5.19). Therefore, we see that an upper bound for the value of an Asian call option is given by

$$A(K,T,m) \leq \frac{e^{-rT}}{m} \sum_{i=1}^{m} e^{rt_i} C\left(F^{-1}_{S(t_i)}\left(1 - \frac{\phi^*}{mK}\right), t_i\right).$$

Since the parameter of $F^{-1}_{S(t_i)}$ in this equation is the same for that of $H(\phi)$, we can rewrite this upper bound as

$$A(K,T,m) \leq \frac{e^{-rT}}{m} \sum_{i=1}^{m} e^{rt_i} C(F^{-1}_{S(t_i)}(x), t_i) =: UB_1, \tag{2.5.20}$$

where $x \in (0,1)$ is the solution to

$$\sum_{i=1}^{m} F^{-1}_{S(t_i)}(x) = mK. \tag{2.5.21}$$

Finally, let $S^c = \sum_{i=1}^{m} F^{-1}_{S(t_i)}(U)$ and set $d_i = F^{-1}_{S(t_i)}(F_{S^c}(mK))$. Then, from Theorem 7, we see that $\sum_{i=1}^{m} d_i = mK$ and so $x$ is in fact the distribution of the sum $S^c$, of comonotonic components, evaluated at $mK$. We also have, from Theorem 8,

$$\left(\sum_{i=1}^{m}(s_i - d_i)\right)^+ = \sum_{i=1}^{m}(s_i - d_i)^+$$

where $\mathbf{s} = (s_1, \ldots, s_m)$ is in the comonotonic support of $S^c$ (that is, the connected comonotonic support of $\sum_{i=1}^{m} S(t_i)$). However, for any other $d_i$ such that $\sum_{i=1}^{m} d_i = 1$ we see, from Theorem 7 of Dhaene et al. (2002b),

$$\left(\sum_{i=1}^{m}(s_i - d_i)\right)^+ \leq \left(\sum_{i=1}^{m}(s_i - d_i)^+\right)^+ = \sum_{i=1}^{m}(s_i - d_i)^+.$$

By taking the expectation of both of the above two equations, we see that equation (2.5.20) with $x$ satisfying (2.5.21) is the smallest upper bound for $A(K, T, m)$ that can be written as the sum of unconditional European calls. Setting $x$ in this way is also discussed briefly by Albrecher et al. (2008) and the above method gives an alternative way of obtaining the upper bound derived in Section 4 of Simon et al. (2000).

### 2.5.5 An improved upper bound by conditioning

In Section 2.4, it is demonstrated that we can improve upon the upper bound of the stop-loss transform of $X$ given by $X^c$ by assuming there exists a random variable $\Lambda$ such that $\mathrm{Cov}\,(X_i, \Lambda) \not\equiv 0$ for all $i$. Suppose this is true here. Furthermore, suppose that $\{S(t)\}_{t\geq 0}$ depends on an underlying standard Brownian motion $\{W(t)\}_{t\geq 0}$. Then, let $S^u$ denote the sum of the inverse distribution functions of $S(t_i)$ conditional on $W(t)$ at time $t \in [0, T]$. Given the event $W(t) = w_t$, let $x$ be the solution to the following equation.

$$\sum_{i=1}^{m} F^{-1}_{S(t_i)|W(t)=w_t}(x) = mK. \tag{2.5.22}$$

Then we see, from equation 92 of Dhaene et al. (2002b), that $x = F_{S^u|W(t)=w_t}(mK)$. It therefore follows, as a result of equation 93 of Dhaene et al. (2002b) and equation (2.5.22), that an upper bound for $AC(K, T)$ is given by

$$\begin{aligned} &A(K, T, m) \\ &\leq \frac{e^{-rT}}{m} \sum_{i=1}^{m} \int_{-\infty}^{\infty} \mathbf{E}^{\mathbf{Q}} \left[ \left( S(t_i) - F^{-1}_{S(t_i)|W(t)=w_t}(x) \right)^+ \middle| W(t) = w_t \right] d\Phi\left(\frac{w_t}{\sqrt{t}}\right) =: UB^{(1)}_t. \end{aligned} \tag{2.5.23}$$

Since this is an upper bound for all $t$, it follows that we can find the optimal upper bound by minimising equation (2.5.23) over $t \in [0, T]$. As before, $x$ solves equation (2.5.22). We see from the results of Section 2.4 that this bound improves on the unconditional bound given by (2.5.19).

An explicit formula for the conditional inverse distribution function of $S(t_i)$ given the event $W(t) = w_t$, used in equation (2.5.23), is provided by the following result.

**Proposition 14.** *Under the assumptions of the Black-Scholes model, conditional on* *the event $W(t) = w_t$, the inverse distribution function of $S(t_i)$ is given by*

$$F^{-1}_{S(t_i)|W(t)=w_t}(x) = \begin{cases} S(0)e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma\frac{t_i}{t}w_t+\sigma\sqrt{\frac{t_i}{t}(t-t_i)}\Phi^{-1}(x)} & i < j, \\ S(0)e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma w_t+\sigma\sqrt{t_i-t}\Phi^{-1}(x)} & i \geq j, \end{cases} \quad (2.5.24)$$

*where $j = \min\{i : t_i \geq t\}$.*

*Proof.* Let us set $X = \sigma W(t_i), Y = W(t)$ and $y = e^{w_t}$. Then, from Proposition 11, we obtain the following expression for the conditional distribution function of $e^{\sigma W(t_i)}$ given the event $W(t) = w_t$.

$$F_{e^{\sigma W(t_i)}|W(t)=w_t}(s) = \Phi\left(\frac{\log s - \rho\sigma\sqrt{\frac{t_i}{t}}w_t}{\sigma\sqrt{t_i(1-\rho^2)}}\right).$$

It then follows that $F_{e^{\sigma W(t_i)}|W(t)=w_t}(s) = x$ if and only if

$$s = F^{-1}_{e^{\sigma W(t_i)}|W(t)=w_t}(x) = e^{\rho\sigma\sqrt{\frac{t_i}{t}}w_t+\sigma\sqrt{t_i(1-\rho^2)}\Phi^{-1}(x)}.$$

We can then obtain equation (2.5.24) by noting that $\rho = \sqrt{(t_i \wedge t)/(t_i \vee t)}$ and the following expression for the inverse conditional distribution function of $S(t_i)$ given $W(t) = w_t$.

$$F^{-1}_{S(t_i)|W(t)=w_t}(x) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)t_i}F^{-1}_{e^{\sigma W(t_i)}|W(t)=w_t}(x).$$

This completes the proof. □

It is of note that $F^{-1}_{S(t_i)|W(t)=w_t}$ is continuous when $t = t_i$ (that is if, for some $i$, we have $i = j$). From equation (2.5.22), we then wish to solve the following for $x$.

$$\sum_{i=1}^{j-1} e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma\frac{t_i}{t}w_t+\sigma\sqrt{\frac{t_i}{t}(t-t_i)}\Phi^{-1}(x)} + \sum_{i=j}^{m} e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma w_t+\sigma\sqrt{t_i-t}\Phi^{-1}(x)} = \frac{mK}{S(0)}. \quad (2.5.25)$$

Therefore, an upper bound in this case, which improves on simply taking the sum of unconditional European calls at times $t_i$ is given by the following set of equations, where $x \in (0,1)$ solves equation (2.5.25).

$$\begin{aligned} A(K,T,m) &\leq \frac{e^{-rT}}{m} \mathbf{E}^{\mathbf{Q}} \left[ (S^u - mK)^+ \right] \\ &= \frac{e^{-rT}}{m} \int_{-\infty}^{\infty} \left( \sum_{i=1}^{m} S(0) e^{\left(r - \frac{\sigma^2 (t_i \wedge t)^2}{2 t_i t}\right) t_i + \sigma \frac{t_i \wedge t}{t} w_t} \Phi(c_1^{(i)}) \right. \\ &\qquad \left. - mK(1-x) \right) d\Phi \left( \frac{w_t}{\sqrt{t}} \right) =: UB_t^{(1)}, \end{aligned} \tag{2.5.26}$$

$$c_1^{(i)} = \begin{cases} \sigma \sqrt{\frac{t_i}{t}(t - t_i)} - \Phi^{-1}(x) & i < j, \\ \\ \sigma \sqrt{t_i - t} - \Phi^{-1}(x) & i > j, \end{cases} \tag{2.5.27}$$

The optimal upper bound in this case is then given by minimising equation (2.5.26) over $t \in [0,T]$. We attempt to evaluate this in Chapter 6.

# Chapter 3

# Stochastic Delay Differential Equations

In Chapter 2, we provide an overview of the concept of comonotonicity. We use the techniques discussed to demonstrate upper and lower bound approximations for the value of a particular class of arithmetic derivative: Asian options. We then show how the upper bound can be improved under the assumption of a specific underlying model for the asset prices concerned.

As outlined in Chapter 1, the Black-Scholes model has been extensively analysed and applied to the valuation of option prices, but empirical studies have highlighted weaknesses in the Black-Scholes assumptions. One particular and significant flaw in such a model is the requirement of constant volatility, which is shown not to be the case in practice.

Alternative approaches have since been considered for the underlying asset model, in place of the geometric Brownian motion, which attempt to overcome the assumption of constant volatility. We underline the use of local and stochastic volatility models as an attempt to overcome the assumption of constant volatility in Chapter 1. Lévy models are also cited as another method for achieving a similar purpose. However, such models remain Markovian and are only influenced by the immediate present data. This results in a similar setting to that of Black and Scholes (1973).

In this chapter, we extend the studies of Hobson and Rogers (1998) and Arriojas et al. (2007). We consider stochastic delay models, which incorporate a feedback effect in the modelling of asset prices and thus take past data into consideration. We begin by introducing the models that we will consider throughout the remainder of this report.

In Chapter 4, we will apply the relevant processes to the valuation of arithmetic options.

## 3.1 A past-dependent approach

Throughout this paper, let $r > 0$ denote the rate of return on a risk-less asset, which is compounded continuously. Let the stochastic process $\{S(t)\}_{t\in[0,T]}$ describe the evolution of a stock price, whose past data is also provided over a fixed interval $[-\tau, 0]$. We shall refer to $\tau$ as the *delay parameter* and will assume that any option written on $\{S(t)\}_{t\in[0,T]}$ matures at a fixed time $T \in (0, \infty)$.

Let $\mathcal{F}_t$ denote the $\sigma$-algebra generated by $\{S(u) : u \leq t\}$, under the usual conditions of completeness and right-continuity, for all $t \geq 0$. Consider the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$, such that $\{B(t)\}_{t\in[0,T]}$ is a standard Brownian motion under the probability measure $\mathbf{P}$. We introduce the most general delay model that we shall consider, below.

**Definition 15.** *Let $\mu, \sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be two continuous, deterministic functions. Let $\bar{\delta}, \delta : [0,T] \to [-\tau, T]$ be $\mathcal{F}_0$-measurable functions such that, $\bar{\delta}(t) \leq t$ and $\delta(t) \leq t$ for all $t \in [0,T]$. The process $\{X(t)\}_{t\in[0,T]}$ is said to follow the following* stochastic delay differential equation (SDDE)*.*

$$\left.\begin{array}{rcl} dX(t) &=& \mu(t, X(t), X(\bar{\delta}(t)))\, dt + \sigma(t, X(t), X(\delta(t)))\, dB(t),\ t \in [0,T], \\ X(t) &=& \phi(t), \quad t \in [-\tau, 0]. \end{array}\right\} \tag{3.1.1}$$

*In this equation, $\phi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ denotes the* initial data *of $X$ and is a continuous, bounded, $\mathcal{F}_0$-measurable function in $\mathbb{R}^n$.*

*Remark.* This model can be extended to one which considers a large number of delay functions in a straightforward way. Let us define $2q$ delay functions $\bar{\delta}_1(t), \ldots, \bar{\delta}_q(t)$ and $\delta_1(t), \ldots, \delta_q(t)$. Then, define the random vector $\boldsymbol{X}(\bar{\delta}(t))$ as

$$\boldsymbol{X} = (X(\bar{\delta}_1(t)), \ldots, X(\bar{\delta}_q(t)))',$$

and similarly for $\boldsymbol{X}(\delta(t))$. Then, we can specify the following stochastic delay differential equation, where $\mu, \sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^{nq} \to \mathbb{R}^n$.

$$\left.\begin{array}{rcl} dX(t) & = & \mu(t, X(t), \boldsymbol{X}(\bar{\delta}(t)))\, dt + \sigma(t, X(t), \boldsymbol{X}(\delta(t)))\, dB(t),\ t \in [0,T], \\ X(t) & = & \phi(t), \quad t \in [-\tau, 0]. \end{array}\right\} \tag{3.1.2}$$

Examples for $\mu$ and $\sigma$ in this case include a moving average of $X$ at past time points as used in time series analysis, such as $\mu = \frac{1}{q}(X(\bar{\delta}_1(t)) + \cdots + X(\bar{\delta}_q(t)))$ and similarly for $\sigma$. It is possible to value arithmetic options under such models using the same assumptions stated below. For instance, assuming that $\bar{\delta}_i(t), \delta_i(t) \le k_i h$ when $k_i h \le t < (k_i + 1)h$ and the $k_i$ are increasing, we see that $\boldsymbol{X}(\bar{\delta}(t))$ and $\boldsymbol{X}(\delta(t))$ are $\mathcal{F}_0$-measurable when $t \le k_1 h$. We can then adapt the approach of Proposition 16 to obtain an explicit solution. We employ a large number of delay parameters for the valuation of basket options in Section 4.3.

In the practical analysis given later in Chapters 5–6, we will assume that the drift and delay functions of any underlying SDDE are time-homogeneous: that is, $\mu(t, x, y) = \mu(x, y)$ for all $t \in [0,T]$ and similarly for $\sigma$. In the valuation of arithmetic options, we will typically consider one-dimensional SDDEs with $n = 1$. We make use of the latter condition for the remainder of this chapter.

The most widely assumed case for the underlying asset is that it follows the geometric Brownian motion, popularised by results such as the Black-Scholes formula. Instead of making this assumption, we use the approach given by Arriojas et al. (2007). Of particular importance to us will be the function $g : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$, under which we make the following assumptions.

A1. $g(t, x)$ is a continuous function of $x$.

A2. $g(t, x)$ is strictly positive whenever $t$ or $x$ are strictly positive.

With $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$ and $\{B(t)\}_{t\in[0,T]}$ defined as above, let us consider the process $\{S(t)\}_{t\in[0,T]}$, which is defined on this probability space, follow the SDDE given by

$$\left.\begin{array}{rcl} dS(t) & = & f(t, S(\bar{\delta}(t)))S(t)\, dt + g(t, S(\delta(t)))S(t)\, dB(t),\ t \in [0,T], \\ S(t) & = & \phi(t), \quad t \in [-\tau, 0]. \end{array}\right\} \tag{3.1.3}$$

In this model, $\bar{\delta}$ and $\delta$ are $\mathcal{F}_0$-measurable functions. In order for (3.1.3) to represent a delay model, we assume that there exists a constant $h \in (0, T]$ such that $\bar{\delta}(t), \delta(t) \leq kh$ whenever $t \in [kh \wedge T, (k+1)h \wedge T)$, for any $k \in \mathbb{N} \cup \{0\}$, where the operator $\wedge$ takes the minimum value of two real numbers. In practice, we would also require $\bar{\delta}$ and $\delta$ to be bounded from below to reflect the level of past data available on $\phi$. This is reflected in our choice of $\tau > 0$. Good examples for $\bar{\delta}$ and $\delta$ include fixed-delay models, where $\delta(t) = t - \tau$, or variable step-function delay models, such as

$$\delta(t) = \left\lfloor \frac{t}{\tau} \right\rfloor \tau, \tag{3.1.4}$$

and similarly for $\bar{\delta}$. Both these models are discussed by Arriojas et al. (2007). It is also possible to use a mixture of fixed and variable delay functions, provided $\bar{\delta}$ and $\delta$ satisfy the conditions above.

The bounded process $\phi : \Omega \to C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$ is assumed to be measurable with respect to the Borel $\sigma$-algebra of $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$ and $f : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. The introduction of delay parameters is natural since such a delay would exist in practice. It is not possible to react to news instantaneously, for instance.

## 3.2 Stochastic delay models

In this section, we initially consider three important results from Arriojas et al. (2007), which are provided here for completeness. Throughout this report, we shall suppose that $S(0) > 0$. Then, we have the following expression for the solution to (3.1.3). From this result and Arriojas et al. (2007), we see that $S(t)$ is almost surely continuous and positive for all $t \in [0, T]$.

**Proposition 16.** *Let* $\{S(t)\}_{t \in [0,T]}$ *be a process that satisfies the stochastic variable delay model given by equation* (3.1.3)*. Let $k$ be a nonnegative integer. Then, for all* $t \in [kh \wedge T, (k+1)h \wedge T]$*, it follows that $S(t)$ can be written in the following way.*

$$S(t) = S(kh \wedge T) \exp\left( \int_{kh \wedge T}^{t} f(u, S(\bar{\delta}(u))) - \frac{1}{2} g^2(u, S(\delta(u)))\, du \right.$$
$$\left. + \int_{kh \wedge T}^{t} g(u, S(\delta(u)))\, dB(u) \right). \tag{3.2.1}$$

*Proof.* Let us first consider the case where $t \in [0, h)$. Then, $\bar{\delta}(t), \delta(t) \leq 0$ and so $f(t, S(\bar{\delta}(t))) = f(t, \phi(\bar{\delta}(t)))$ and $g(t, S(\bar{\delta}(t))) = f(t, \phi(\bar{\delta}(t)))$ are $\mathcal{F}_0$-measurable. We can then apply Itô's formula to equation (3.1.3) to obtain

$$S(t) = S(0)\exp\bigg(\int_0^t f(u,\phi(\bar{\delta}(u))) - \frac{1}{2}g^2(u,\phi(\delta(u)))\,du + \int_0^t g(u,\phi(\delta(u)))\,dB(u)\bigg).$$

Let us define $S(h)$ according to this equation with $t = h$. Then, it is straightforward to show by induction that (3.2.1) holds for all $t \in [0,T]$. □

### 3.2.1 Change of measure

Let $\{\mathcal{F}_t\}_{t\in[0,T]}$ define a filtration on the probability space $(\Omega,\mathcal{F},\mathbf{P})$ such that $\mathcal{F}_t = \sigma(S(u) : u \le t)$ is a complete $\sigma$-algebra for all $t \in [0,T]$. Since we make the assumption that $g(t,x) > 0$ for all $x > 0$, we can use Girsanov's Theorem, taking the approach of Arriojas et al. (2007), in order to find an equivalent measure, $\mathbf{Q}$, under which the discounted stock price process becomes a martingale. This can be done in the following way.

**Lemma 17.** *Let $h \in (0,T]$ be a fixed, positive constant. Then, let $\{S(t)\}_{t\in[0,T]}$ follow the stochastic delay differential equation given by equation* (3.1.3).

*Define the stochastic processes $\{b(t)\}_{t\in[0,T]}$ and $\{W(t)\}_{t\in[0,T]}$ by*

$$b(t) = \frac{r - f(t,S(\bar{\delta}(t)))}{g(t,S(\delta(t)))}, \quad t \in [0,T], \tag{3.2.2}$$

$$W(t) = B(t) - \int_0^t b(u)\,du. \tag{3.2.3}$$

*Then $\{W(t)\}_{t\in[0,T]}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbf{Q}$, where the Radon-Nikodym derivative of $\mathbf{Q}$ with respect to $\mathbf{P}$, restricted to the maturity-time $\sigma$-algebra, satisfies*

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_T} = \exp\bigg(\int_0^T b(u)\,dB(u) - \frac{1}{2}\int_0^T b^2(u)\,du\bigg). \tag{3.2.4}$$

*Under $\mathbf{Q}$, it follows that $\{S(t)\}_{t\in[0,T]}$ satisfies the following stochastic delay differential equation.*

$$\left.\begin{array}{rcll} dS(t) &=& rS(t)\,dt + g(t,S(\delta(t)))S(t)\,dW(t), & t \in [0,T], \\ S(t) &=& \phi(t), & t \in [-\tau,0]. \end{array}\right\} \tag{3.2.5}$$

*Its solution is given by, for all $t \in [kh \wedge T, (k+1)h \wedge T]$,*

$$S(t) = S(kh \wedge T) e^{r(t-(kh\wedge T)) - \frac{1}{2}\int_{kh\wedge T}^{t} g^2(u,S(\delta(u)))\,du + \int_{kh\wedge T}^{t} g(u,S(\delta(u)))\,dW(u)}. \tag{3.2.6}$$

*Proof.* This is a simple expansion of Arriojas et al. (2007). By using Proposition 16 with equation (3.1.3) replaced by (3.2.5), we easily obtain equation (3.2.6). □

*Remark.* This result holds since $\delta(t) < t$ almost everywhere and similarly for $\bar{\delta}$. Therefore, $b(t)$ is a predictable process almost everywhere, at all but a countable set of points.

Since the stochastic model under $\mathbf{Q}$ no longer depends on the function $\bar{\delta}$, we can now assume, without loss of generality, that $h$ is the largest value in $(0, T]$ such that $\delta(t) \leq kh$ throughout, for all $t \in [kh \wedge T, (k+1)h \wedge T]$.

**Proposition 18.** *Let $\mathbf{Q}$ define the risk-neutral probability measure given by Lemma 17, under which $S(t)$ satisfies the SDDE (3.2.5). Then, the discounted stock price process $\{\widetilde{S}(t)\}_{t\in[0,T]}$ satisfying*

$$d\widetilde{S}(t) = g(t, \widetilde{S}(\delta(t)))\widetilde{S}(t)\,dW(t) \tag{3.2.7}$$

*is a martingale under $\mathbf{Q}$.*

*Proof.* Assume, without any loss of generality, that $kh \leq T$ and let $t \in [kh, (k+1)h \wedge T)$. Using Lemma 17 with $r = 0$, we see that $\widetilde{S}(t)$ satisfies, for any $v \in [kh, t)$,

$$\widetilde{S}(t) = \widetilde{S}(v) e^{-\frac{1}{2}\int_{v}^{t} g^2(u,\widetilde{S}(\delta(u)))\,du + \int_{v}^{t} g(u,\widetilde{S}(\delta(u)))\,dW(u)}. \tag{3.2.8}$$

By definition of the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ and $\delta$, it is clear that $\widetilde{S}(t)$ is $\mathcal{F}_t$-adapted. Moreover, $\delta(u) \leq kh \leq v$ for all $u \in [v, t]$. Therefore, $g(u, \widetilde{S}(\delta(u)))$ is $\mathcal{F}_{kh}$-measurable and so, conditional on the $\sigma$-algebras $\mathcal{F}_{kh} \subseteq \mathcal{F}_v$, $\widetilde{S}(t)$ is lognormally distributed. Hence, for every $v \in [kh, t)$, one obtains

$$\mathbf{E}^{\mathbf{Q}}\left(\widetilde{S}(t)\middle|\,\mathcal{F}_v\right) = \widetilde{S}(v).$$

On the other hand, if we assume $v \leq kh$, then we can use the same approach above, with repeated uses of the tower property (see, for example, Williams (1991)), to obtain

$$\mathbf{E}^{\mathbf{Q}}\left(\widetilde{S}(t)\middle|\,\mathcal{F}_v\right) = \mathbf{E}^{\mathbf{Q}}\left(\ldots\mathbf{E}^{\mathbf{Q}}\left(\widetilde{S}(t)\middle|\,\mathcal{F}_{kh}\right)\ldots\middle|\,\mathcal{F}_v\right) = \widetilde{S}(v).$$

In particular, by setting $v = 0$, we see from the above equation that $\mathbf{E}^{\mathbf{Q}}\left[|\widetilde{S}(t)|\right] = 1$. This shows that $\{\widetilde{S}(t)\}_{t\in[0,T]}$ is a martingale under $\mathbf{Q}$. □

*Remark.* The only assumption that we need to make for this to hold is A1: that $g(t, x)$ is continuous in $x$.

### 3.2.2 Conditional expectation

In the next chapter, we aim to use comonotonicity techniques to obtain convex bounds for Asian option prices. As discussed in Dhaene et al. (2002b), conditioning arithmetic sums involving the stochastic process $\{S(t)\}_{t\in[0,T]}$ on a dependent random variable $\Lambda$ is important for finding stop-loss bounds using sums of comonotonic random variables. This idea is introduced by Rogers and Shi (1995) in the continuous case, concerning integrals involving $\{S(t)\}_{t\in[0,T]}$ and is put into practice in the arithmetic case by Dhaene et al. (2002a). A study highlighting the effects of choosing the conditioning variable appropriately is undertaken by Vanduffel et al. (2008) when considering the Black-Scholes model for $\{S(t)\}_{t\in[0,T]}$. This paper provides both locally and globally optimal choices for $\Lambda$, showing that better choices are possible under these circumstances.

Returning to the more general case, Albrecher et al. (2008) conditions on the value of $S(v)$ for a fixed, particular value of $v \in [0, T]$. A principal advantage of this approach is that, by conditioning in this way for our SDDE model, we obtain explicit values for call option bounds stated below. This is of interest to us and allows us to demonstrate the performance of our bounds numerically in Chapter 6. However, this results in a trade-off in accuracy described in our computational results. Additionally, Nielsen and Sandmann (2003), and Vanmaele et al. (2006) use a normalised version of the logarithm of the geometric average of $S(t)$ viewed at various monitoring times.

As before, let $S(t)$ satisfy equation (3.2.5) under the risk-neutral equivalent martingale measure $\mathbf{Q}$. Further, we take $\mathcal{F}_t$ to be the $\sigma$-algebra generated by $\{S(u) : u \le t\}$, for all $t \in [0, T]$. Let $v \in [0, h]$ be fixed. By applying the conditional form of Jensen's inequality (see, for example, Williams (1991)), we see that

$$\mathbf{E}\left(c(S(t))|\, S(v)\right) \ge c(\mathbf{E}\left(S(t)|\, S(v)\right)),$$

for any convex function $c : \mathbb{R} \to \mathbb{R}$. Setting $c(x) = (x - d)^+$, for any $d \in \mathbb{R}$, and using the tower property, we observe a convex-order lower bound for $S(t)$ under the risk neutral measure $\mathbf{Q}$, given by

$$S(t) \geq_{\text{cx}} \mathbf{E}^{\mathbf{Q}}\left( S(t) \middle| S(v) \right). \tag{3.2.9}$$

In order to determine the conditional expectation given in the above equation, we will make frequent use of the following well-known result from probability theory.

**Proposition 19.** *Let $X$ and $Y$ be two normally distributed random variables with means $\mu_X, \mu_Y$ and variances $\sigma_X, \sigma_Y$ respectively. Assume further that $X$ and $Y$ share a bivariate normal distribution with correlation $\rho$. Then, the conditional expectation of $e^X$ given $e^Y$ (equivalently, given $Y$) satisfies*

$$\mathbf{E}\left( e^X \middle| e^Y \right) = \exp\left( \mu_X + \rho \frac{\sigma_X}{\sigma_Y}(Y - \mu_Y) + \frac{\sigma_X^2}{2}(1 - \rho^2) \right). \tag{3.2.10}$$

**Corollary 20.** *Suppose that $S(t)$ satisfies the SDDE (3.2.5) and that $0 \leq v \leq t$. Then, the conditional expectation of $S(t)$ given $\mathcal{F}_v$ satisfies*

$$\mathbf{E}^{\mathbf{Q}}\left( S(t) \middle| \mathcal{F}_v \right) = S(v)e^{r(t-v)}. \tag{3.2.11}$$

*Proof.* This is immediate from Proposition 18 and the definition of $\widetilde{S}(t)$. □

# Chapter 4

# Option Pricing under Delay Models

Stochastic delay differential equations provide a more realistic model for the evolution of a stock price than the historical Black-Scholes model, by taking past events into account when considering the asset's future performance. In addition, our approach given by equation (3.2.5) treats the underlying asset price process in a similar way to a local volatility model.

The goal of this chapter is to price arithmetic options using the SDDE model. A common approach to this problem is to bound options by a linear combination of vanilla European options whose solutions can be obtained analytically. We make use of this approach, along with the results obtained by Arriojas et al. (2007), to find bounds for Asian options under the stochastic delay model given above.

As discussed in Dhaene et al. (2002b), conditioning arithmetic sums involving the stochastic process $\{S(t)\}_{t\in[0,T]}$ on a dependent random variable $\Lambda$ is important for finding stop-loss bounds. This is put into practice by Dhaene et al. (2002a). In particular, Albrecher et al. (2008) condition on the value of $S(v)$ for a fixed, particular value of $v \in [0, T]$. Nielsen and Sandmann (2003); Vanmaele et al. (2006) use a normalised version of the logarithm of the geometric average of $S(t)$ viewed at various monitoring times. Conditioning in the continuous case, concerning integrals involving $\{S(t)\}_{t\in[0,T]}$, is also covered by Rogers and Shi (1995).

We begin by studying upper and lower bounds for arithmetic Asian options. Basket options have a similar structure in terms of their stop-loss transform but require a small number of extra considerations. We will subsequently consider basket options in

Section 4.3.

## 4.1 Bounds for arithmetic Asian options

Recall the explanation of the stop-loss transform given by Definition 1. Let us begin by defining the value of an Asian option formally.

**Definition 21.** *Let $m \in \mathbb{N}$ and $\{t_i : i = 1, \ldots, m\}$ denote a set of monitoring times such that $0 \leq t_1 < \ldots < t_m \leq T$. Given any stock price process $\{S(t)\}_{t\in[0,T]}$, let $\mathbb{S}$ denote the arithmetic sum of the process observed at the above monitoring times:*

$$\mathbb{S} := \sum_{i=1}^{m} S(t_i). \tag{4.1.1}$$

*Let $r$ denote the risk-free rate of interest. Then, the value of an Asian call option written on $\{S(t)\}_{t\in[0,T]}$, with exercise price $K$, maturity time $T$ and observed at the $m$ monitoring times above, is given by*

$$A(K, T, m) = \frac{e^{-rT}}{m} \Psi(\mathbb{S}, mK). \tag{4.1.2}$$

**Corollary 22.** *Let $\Psi(X, x)$ be the stop-loss transform given by Definition 1. Then, $\Psi$ is a convex function of $x$.*

As before, let $S(t)$ satisfy equation (3.2.5) for all $t \in [-\tau, T]$ and let $\mathcal{F}_t$ be the $\sigma$-algebra generated by $\{S(u) : u \leq t\}$. Using the approach given in Chapter 2, we introduce a lognormally distributed conditioning variable $\Lambda$. In Chapter 6, we shall consider two different forms for $\Lambda$. In each case, we shall use the following expressions, involving stop-loss transforms of $\Lambda$.

**Definition 23.** *Let $\Lambda$ be a lognormally distributed random variable, let $L_\Lambda$ be given by*

$$L_\Lambda = \log\left(\frac{\Lambda}{S(0)}\right), \tag{4.1.3}$$

*and define the values $M_\Lambda$ and $V_\Lambda$ in the following way.*

$$M_\Lambda = \mathbf{E}^{\mathbf{Q}}\left[L_\Lambda\right], \quad V_\Lambda = \mathrm{Var}^{\mathbf{Q}}\left(L_\Lambda\right). \tag{4.1.4}$$

*In particular, when $\Lambda = S(t)$ for $t \in [0, h]$ and $S(t)$ is any solution to the SDDE* (3.2.5),

*$M_\Lambda$ and $V_\Lambda$ satisfy the following expressions:*

$$M_\Lambda = \mu_t := rt - \frac{1}{2}\int_0^t g^2(u, \phi(\delta(u)))\, du, \tag{4.1.5}$$

$$V_\Lambda = \sigma_t^2 := \int_0^t g^2(u, \phi(\delta(u)))\, du. \tag{4.1.6}$$

**Proposition 24.** *Let $\Lambda$ and $L_\Lambda$ be given by Definition 23. Then, the stop-loss transform of $\Lambda^x$ with respect to $K^x$ satisfies, for all $x, K > 0$,*

$$\Psi\left(\Lambda^x, K^x\right) = S^x(0)e^{x\left(M_\Lambda + \frac{1}{2}xV_\Lambda\right)}\Phi(d_2 + x\sqrt{V_\Lambda}) - K^x\Phi(d_2), \tag{4.1.7}$$

*where $d_2$ is given by*

$$d_2 = \frac{\log\left(\frac{S(0)}{K}\right) + M_\Lambda}{\sqrt{V_\Lambda}}, \tag{4.1.8}$$

*and $\Phi$ denotes the standard normal distribution function.*

It is important to note that $S(t)$ is lognormally distributed whenever $t \in [0, h]$. This can be observed using the solution given in Proposition 16. We can therefore make use of the result above when $t$ is in this range. However, this is not necessarily true when $t > h$. In this case, the distribution of $S(t)$ is usually unknown or very cumbersome to work with. We will attempt to combat this problem in the following sections.

### 4.1.1 A lower bound

A simple application of Jensen's inequality yields a convex-order lower bound for $\mathbb{S}$, given by

$$\mathbb{S} \geq_{\text{cx}} \sum_{i=1}^{m} \mathbf{E}^{\mathbf{Q}}\left( S(t_i) | \, \Lambda\right) =: S^l. \tag{4.1.9}$$

**Proposition 25.** *Fix $T, h > 0$ and let $S(t)$ satisfy the SDDE (3.2.5). Let $X(t)$ be such that $S(t) = S(0)e^{X(t)}$ and let $\Lambda$ be a lognormally distributed random variable with $L_\Lambda$ given by equation (4.1.3). Let us define the two functions $A_\Lambda : [0, h] \to [0, \infty)$ and $B_\Lambda : [0, h] \to \mathbb{R}$ in the following way.*

$$A_\Lambda(t) = \frac{\text{Cov}\left(X(t), L_\Lambda\right)}{\text{Var}^{\mathbf{Q}}\left(L_\Lambda\right)}, \quad B_\Lambda(t) = A_\Lambda(t)\left(M_\Lambda + \frac{1}{2}A_\Lambda(t)V_\Lambda\right). \tag{4.1.10}$$

*Then, the conditional expectation of $S(t)$ given $\Lambda$ satisfies*

$$\mathbf{E}^{\mathbf{Q}}\left(S(t)\middle|\,\Lambda\right) = S(0)\left(\frac{\Lambda}{S(0)}\right)^{A_\Lambda(t\wedge h)} e^{rt - B_\Lambda(t\wedge h)}. \tag{4.1.11}$$

*Proof.* Let us first assume that $t \leq h$. Then, $X(t)$ is normally distributed. Hence, by using Proposition 19 with $X = X(t)$ and $Y = L_\Lambda$, we see that the conditional expectation of $S(t)$ given $G$ satisfies

$$\mathbf{E}^{\mathbf{Q}}\left(S(t)\middle|\,\Lambda\right) = S(0)\exp\left(\mu_t + A_\Lambda(t)(L_\Lambda - M_\Lambda) + \frac{1}{2}\sigma_t^2\left(1 - A_\Lambda^2(t)\frac{V_\Lambda}{\sigma_t^2}\right)\right), \tag{4.1.12}$$

where $\mu_t$ and $\sigma_t$ are defined by equations (4.1.5) and (4.1.6) respectively. By rearranging this equation, noting that $\mu_t + \frac{1}{2}\sigma_t^2 = rt$ and $\Lambda = S(0)e^{L_\Lambda}$, we obtain equation (4.1.11) in this case.

Now let us assume that $t > h$. Then since $\{e^{-rt}S(t)\}_{t\in[0,T]}$ is a martingale, from Proposition 18, it follows that the conditional expectation of $S(t)$ given $\Lambda$ satisfies

$$\mathbf{E}^{\mathbf{Q}}\left(S(t)\middle|\,\Lambda\right) = \mathbf{E}^{\mathbf{Q}}\left(\mathbf{E}^{\mathbf{Q}}\left(S(t)\middle|\,\mathcal{F}_h\right)\middle|\,\Lambda\right) = \mathbf{E}^{\mathbf{Q}}\left(S(h)\middle|\,\Lambda\right)e^{r(t-h)}.$$

Applying this result to (4.1.12) and rearranging, we again obtain equation (4.1.11). □

We can obtain the components of $S^l$ using the result above. This then leads to the following lower bound for the value of an arithmetic Asian option. By applying Proposition 24, we can obtain an explicit form for this expression.

**Theorem 26.** *Let $\{S(t)\}_{t\in[0,T]}$ define a stock price process that satisfies the stochastic delay model given by equation (3.2.5). Let $h \in (0,T]$ be fixed and let $r > 0$ define the risk-free rate of interest. Let $0 \leq t_1 < \cdots < t_m \leq T$ denote $m \in \mathbb{N}$ monitoring times over the interval $[0,T]$. Then, the Asian call option written on $\{S(t)\}_{t\in[0,T]}$ with exercise price $K$, maturity time $T$ and the $m$ monitoring times above, given by equation (4.1.2), is bounded from below by the following equation.*

$$A(K,T,m) \geq \frac{e^{-rT}}{m}\sum_{i=1}^{m} S(0)^{1-A_\Lambda(t_i\wedge h)} e^{rt_i - B_\Lambda(t_i\wedge h)}\Psi\left(\Lambda^{A_\Lambda(t_i\wedge h)}, \lambda^{A_\Lambda(t_i\wedge h)}\right) =: LB, \tag{4.1.13}$$

*where $\lambda \in (0,\infty)$ solves*

$$\sum_{i=1}^{m}\left(\frac{\lambda}{S(0)}\right)^{A_\Lambda(t_i\wedge h)} e^{rt_i - B_\Lambda(t_i\wedge h)} = \frac{mK}{S(0)}. \tag{4.1.14}$$

*The values $A_\Lambda$ and $B_\Lambda$ are given in Proposition 25 and the explicit expression for the stop-loss transform $\Psi$ is given in Proposition 24.*

*Proof.* Since the components of this sum consist of increasing functions of a single random variable, we see that the vector $\left(\mathbf{E}^{\mathbf{Q}}\left(S(t_1)|\,\Lambda\right),\ldots,\mathbf{E}^{\mathbf{Q}}\left(S(t_m)|\,\Lambda\right)\right)$ is comonotonic. From Dhaene et al. (2002b), it then follows that the stop-loss transform of $S^l$ satisfies

$$\Psi\left(S^l, mK\right) = \sum_{i=1}^{m} \Psi\left(\mathbf{E}^{\mathbf{Q}}\left(S(t_i)|\,\Lambda\right), \mathbf{E}^{\mathbf{Q}}\left[S(t_i)|\,\Lambda = \lambda\right]\right),$$

where $\lambda \in (0, \infty)$ solves the strictly increasing equation

$$\sum_{i=1}^{m} \mathbf{E}^{\mathbf{Q}}\left[S(t_i)|\,\Lambda = \lambda\right] = mK.$$

By applying Proposition 25 to these two equations we obtain (4.1.13) and (4.1.14). □

### 4.1.2 A first upper bound

In Section 4.1.1, we consider a lower bound for arithmetic Asian options. We will see in Chapter 6 how the resulting value $LB$ approximates the true value for $A(K, T, m)$. A tight lower bound will be particularly useful in practice when combined with an upper bound for the same Asian option under the model given by equation (3.2.5), using the risk-neutral probability measure $\mathbf{Q}$.

In order to proceed with this section, we must extend the assumptions made on the volatility function $g$. From now on, let us suppose that $g$ satisfies the following condition, in addition to those set by A1 and A2.

A3. There exists a positive constant $\overline{G}$ such that $g(t, x) < \overline{G}$ for all $(t, x) \in [0, T] \times \mathbb{R}_+$.

Such an approach is proposed in Section 4.2 of Hobson and Rogers (1998) and this assumption does not seem unrealistic since it is unlikely that volatility is unbounded. Empirical studies are undertaken by Dumas et al. (1998) and the assumption of "uniformly bounded" local volatility is used in a number of sources, for example Berestycki et al. (2002); Coleman et al. (1999).

In this section, we consider a method for finding upper bounds for $A(K, T, m)$. This makes use of the following inequality for the stop-loss transform of $\mathbb{S}$.

**Proposition 27.** *Let $\{S(t)\}_{t\in[0,T]}$ follow the SDDE (3.2.5). Then, the arithmetic Asian option $A(K,T,m)$ is bounded above by the following equation*

$$A(K,T,m) \leq LB + \frac{e^{-rT}}{2m}\sum_{i=1}^{m}\mathbf{E}^{\mathbf{Q}}\left[\sqrt{\mathrm{Var}^{\mathbf{Q}}\left(S(t_i)|\,\Lambda\right)}\right]. \tag{4.1.15}$$

*Proof.* From Rogers and Shi (1995), we have the following upper bound for the stop-loss transform of any random variable $X$ and any conditioning variable $Y$.

$$0 \leq \mathbf{E}^{\mathbf{Q}}\left((X-x)^{+}\middle|\,Y\right) - \mathbf{E}^{\mathbf{Q}}\left(X-x|\,Y\right)^{+}$$

By taking the expectation of the above equation and using the tower property, we obtain

$$0 \leq \mathbf{E}^{\mathbf{Q}}\left[(X-x)^{+}\right] - \mathbf{E}^{\mathbf{Q}}\left[\left(\mathbf{E}^{\mathbf{Q}}\left(X|\,Y\right)-x\right)^{+}\right] \leq \frac{1}{2}\mathbf{E}^{\mathbf{Q}}\left[\sqrt{\mathrm{Var}^{\mathbf{Q}}\left(X|\,Y\right)}\right].$$

Therefore, by replacing $X$ with $\mathbb{S}$, $Y$ with $\Lambda$ and working under $\mathbf{Q}$, we obtain the following upper bound for the stop-loss transform of $\mathbb{S}$.

$$\Psi(\mathbb{S}, mK) \leq \Psi(S^{l}, mK) + \frac{1}{2}\mathbf{E}^{\mathbf{Q}}\left[\sqrt{\mathrm{Var}^{\mathbf{Q}}\left(\mathbb{S}|\,\Lambda\right)}\right]. \tag{4.1.16}$$

Applying the conditional form of Minkowski's inequality (see, for example Doob (1994)), we see that equation (4.1.16) is bounded above by the right-hand side of

$$\Psi(\mathbb{S}, mK) \leq \Psi(S^{l}, mK) + \frac{1}{2}\sum_{i=1}^{m}\mathbf{E}^{\mathbf{Q}}\left[\sqrt{\mathrm{Var}^{\mathbf{Q}}\left(S(t_i)|\,\Lambda\right)}\right].$$

Taking into account the discount factor and averaging, we obtain equation (4.1.15) □

*Remark.* Although equation (4.1.16) leads to a theoretically improved upper bound for $A(K,T,m)$, it requires knowledge of $\mathbf{E}^{\mathbf{Q}}\left[\sqrt{\mathrm{Var}^{\mathbf{Q}}\left(\mathbb{S}|\,\Lambda\right)}\right]$. In practice, it is difficult to find a closed-form expression, or even an upper bound, for this term. Therefore, we do not make use of it here.

To find an explicit form for equation (4.1.15), we need to obtain an expression for the conditional variance of $S(t_i)$ given $\Lambda$. An upper bound for this is given by the next statement. This then leads to the upper bound for $A(K,T,m)$ given by Theorem 29.

**Proposition 28.** *Let $\{S(t)\}_{t\in[0,T]}$ follow the SDDE (3.2.5). Then, the conditional* *variance of $S(t)$ given $\Lambda$ is given by the following, with equality when $t \leq h$.*

$$\mathrm{Var}^{\mathbf{Q}}\left(S(t)|\,\Lambda\right) \leq S^2(0)\left(\frac{\Lambda}{S(0)}\right)^{2A_\Lambda(t\wedge h)} e^{2(rt-B_\Lambda(t\wedge h))}\left(e^{\sigma^2_{t\wedge h}-A^2_\Lambda(t\wedge h)V_\Lambda+\overline{G}^2(t\vee h-h)}-1\right). \tag{4.1.17}$$

*Proof.* Let us rewrite the conditional variance as

$$\mathrm{Var}^{\mathbf{Q}}\left(S(t)|\,\Lambda\right) = \mathbf{E}^{\mathbf{Q}}\left(S^2(t)\middle|\,\Lambda\right) - \mathbf{E}^{\mathbf{Q}}\left(S(t)|\,\Lambda\right)^2.$$

Then, the case where $t \leq h$ can be shown by using Proposition 25. To obtain the conditional second moment, we simply replace $X(t)$ by $2X(t)$ and multiply by $S(0)$.

Now let us consider the case $t > h$. In order to find an expression for the conditional second moment of $S(t)$, let us first suppose that $t \in (h, 2h]$. Then, using the tower property, we have

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}}\left(S^2(t)\middle|\,\Lambda\right) &= \mathbf{E}^{\mathbf{Q}}\left(\mathbf{E}^{\mathbf{Q}}\left(S^2(t)\middle|\,\mathcal{F}_h\right)\middle|\,\Lambda\right) \\
&= \mathbf{E}^{\mathbf{Q}}\left(S^2(h)e^{2r(t-h)+\int_h^t g^2(u,S(\delta(u)))\,du}\middle|\,\Lambda\right) \\
&\leq \mathbf{E}^{\mathbf{Q}}\left(S^2(h)\middle|\,\Lambda\right)e^{2r(t-h)+\overline{G}^2(t-h)} \\
&= S^2(0)\left(\frac{\Lambda}{S(0)}\right)^{2A_\Lambda(h)} e^{2rt+\sigma^2_h-2B_\Lambda(h)-A^2_\Lambda(h)V_\Lambda+\overline{G}^2(t-h)},
\end{aligned}$$

where the final equation follows using the case $t \leq h$. By combining this with Proposition 25, we obtain equation (4.1.17). By continuing in this way and using induction, we see that this holds for all $t > h$. □

**Theorem 29.** *Let $\{S(t)\}_{t\in[0,T]}$ follow the SDDE (3.2.5). Then, the arithmetic Asian option $A(K,T,m)$ is bounded above by the following closed-form expression.*

$$A(K,T,m) \leq UB_1 := LB + \frac{S(0)e^{-rT}}{2m}\sum_{i=1}^{m} e^{rt_i}\left(e^{\sigma^2_{t_i\wedge h}-A^2_\Lambda(t_i\wedge h)V_\Lambda+\overline{G}^2(t_i\vee h-h)}-1\right)^{1/2}. \tag{4.1.18}$$

*Proof.* This result is obtained by replacing Proposition 27 with the upper bounds for conditional variance given by Proposition 28. □

### 4.1.3 A second upper bound

In Section 4.1.2, we obtain an upper bound for arithmetic Asian call options by using the lower bound obtained in Section 4.1.1 and adopting the approach taken in Rogers and Shi (1995). In this section, we use a similar approach to Hobson et al. (2005). We note that the stop-loss transform of $\mathbb{S}$ is bounded above by the following equation.

$$\Psi(\mathbb{S}, mK) \leq \sum_{i=1}^{m} \Psi(S(t_i), K_i), \tag{4.1.19}$$

where $\mathbf{K} = (K_1, \ldots, K_m) \in [0, mK]^m$ is any vector satisfying the sum $\sum_{i=1}^{m} K_i = mK$. This approach is first considered by Simon et al. (2000), who apply this result to the well-known Black and Scholes setting. This result is also described in detail by Dhaene et al. (2002b) and interpretation of this upper bound as the price of the cheapest static super-replicating strategy in terms of plain vanilla options is presented by Albrecher et al. (2005) under a Lévy market model.

Indeed, Hobson et al. (2005) finds the least upper bound for basket options subject to a similar constraint on $\mathbf{K}$, which can be adapted to arithmetic Asian options. We note in McWilliams and Sabanis (2011) that this only works in practice if the distribution of $S(t_i)$ is known for every $i$, which is not true for any $i$ satisfying $t_i > h$. This means that we are restricted to the case where $t_i \in [0, h]$ for every $i$, which is highly undesirable. For example, $h$ may only be a few hours.

Let us define the value $k_h$ in the following way.

$$k_h = \min\{i : t_i > h\}. \tag{4.1.20}$$

We then propose to make use of the following inequality for $\Psi(\mathbb{S}, mK)$.

$$\Psi(\mathbb{S}, mK) \leq \sum_{i=1}^{k_h - 1} \Psi(S(t_i), K_i) + \Psi\left(\sum_{i=k_h}^{m} S(t_i), \bar{K}\right). \tag{4.1.21}$$

In this case, we apply the constraint $\sum_{i=1}^{k_h-1} K_i + \bar{K} = mK$. This then leads to the following upper bound for $A(K, T, m)$.

**Proposition 30.** *Let $\{S(t)\}_{t \in [0,T]}$ follow the SDDE* (3.2.5) *and let $Y_i = \mathbf{E}^{\mathbf{Q}}\left(S(t_i) \middle| \Lambda\right)$, for all $i \geq k_h$. Then, the arithmetic Asian option $A(K, T, m)$ is bounded above by the* *following equation.*

$$A(K,T,m) \le \frac{e^{-rT}}{m}\left(\sum_{i=1}^{k_h-1}\Psi(S(t_i),F^{-1}_{S(t_i)}(x)) + \sum_{i=k_h}^{m}\Psi(Y_i,F^{-1}_{Y_i}(x)) + \frac{1}{2}\mathbf{E}^{\mathbf{Q}}\left[\sqrt{\mathrm{Var}^{\mathbf{Q}}\left(\sum_{i=k_h}^{m}S(t_i)\middle|\Lambda\right)}\right]\right), \quad (4.1.22)$$

*where $x \in (0,1)$ is the solution to*

$$\sum_{i=1}^{k_h-1}F^{-1}_{S(t_i)}(x) + \sum_{i=k_h}^{m}F^{-1}_{Y_i}(x) = mK. \quad (4.1.23)$$

*Proof.* By applying Rogers and Shi (1995) to equation (4.1.21), we see the following is an upper bound for the stop-loss transform of $\mathbb{S}$.

$$\Psi(\mathbb{S},mK) \le \sum_{i=1}^{k_h-1}\mathbf{E}^{\mathbf{Q}}\left[(S(t_i)-K_i)^+\right] + \mathbf{E}^{\mathbf{Q}}\left[\left(\sum_{i=k_h}^{m}\mathbf{E}^{\mathbf{Q}}\left(S(t_i)\middle|\,\Lambda\right) - \bar{K}\right)^+\right] + \frac{1}{2}\mathbf{E}^{\mathbf{Q}}\left[\sqrt{\mathrm{Var}^{\mathbf{Q}}\left(\sum_{i=k_h}^{m}S(t_i)\middle|\Lambda\right)}\right].$$

Noting that the final term in the above equation is constant with respect to the $K_i$ and $\bar{K}$, let $\bar{Y} = Y_{k_h} + \cdots + Y_m$ and define the Lagrangian $L : [0,mK]^{k_h} \times \mathbb{R} \to \mathbb{R}$ by

$$L(\mathbf{K},\lambda) = \sum_{i=1}^{k_h-1}\Psi(S(t_i),K_i) + \Psi(\bar{Y},\bar{K}) + \lambda\left(\sum_{i=1}^{k_h-1}K_i + \bar{K} - mK\right).$$

Note that the stop-loss transforms in this equation only involve lognormal random variables, since $\Lambda$ is lognormal from Definition 23. Since, from Proposition 16, $S(t)$ is almost surely continuous in $t$, we see that $L$ is differentiable for all $K_i$. Using, for example, equation (2) of Dhaene et al. (2002b), we see that the partial derivative of $\Psi$ with respect to $K$ satisfies, for all $t \in [0,h]$,

$$\frac{\partial\Psi}{\partial K} = \frac{\partial}{\partial K}\int_K^{\infty} 1 - F_{S(t)}(x)\,dx = -\mathbf{P}\left(S(t) > K\right), \quad (4.1.24)$$

and similarly for $\partial\Psi/\partial\bar{K}$. Hence, we obtain the following critical values for every $i \in \{1,\ldots,k_h-1\}$, which minimise $L$ since, from Corollary 22, $\Psi$ is a convex function of its second variable.

$$K_i = F_{S(t_i)}^{-1}(1-\lambda), \quad \bar{K} = F_{\bar{Y}}^{-1}(1-\lambda). \tag{4.1.25}$$

Since the vector $(Y_{k_h}, \ldots, Y_m)$ is comonotonic, we can rewrite this upper bound to obtain equation (4.1.22). Using Theorem 7, with $X^c$ replaced by $\bar{Y}$ and $p = F_{\bar{Y}}(\bar{K})$, then setting $x = 1-\lambda$, we see that this choice of $\mathbf{K}$ solves equation (4.1.23). □

**Corollary 31.** *An upper bound for the conditional variance of* $\sum_{i=k_h}^m S(t_i)$ *given* $\Lambda$ *is given by*

$$\begin{aligned}\mathrm{Var}^{\mathbf{Q}}\left(\sum_{i=k_h}^m S(t_i)\middle|\Lambda\right) &\leq S^2(0)\left(\frac{\Lambda}{S(0)}\right)^{2A_\Lambda(h)} e^{-2B_\Lambda(h)} \\ &\times\left(\sum_{i=k_h}^m e^{rt_i}\left(e^{\sigma_h^2 - A_\Lambda^2(h)V_\Lambda + \overline{G}^2(t_i-h)} - 1\right)\left(e^{rt_i} + 2\sum_{j=i+1}^m e^{rt_j}\right)\right). \end{aligned} \tag{4.1.26}$$

*Proof.* We can rewrite the above conditional variance as

$$\mathrm{Var}^{\mathbf{Q}}\left(\sum_{i=k_h}^m S(t_i)\middle|\Lambda\right) = \sum_{i=k_h}^m \mathrm{Var}^{\mathbf{Q}}\left(S(t_i)|\Lambda\right) + 2\sum_{i=k_h}^m\sum_{j>i}\mathrm{Cov}^{\mathbf{Q}}\left(S(t_i), S(t_j)|\Lambda\right)$$

By using the tower property along with Proposition 28 we have, for any $j \geq i \geq k_h$.

$$\begin{aligned} &\mathbf{E}^{\mathbf{Q}}\left(S(t_i)S(t_j)|\Lambda\right) \\ = \;& \mathbf{E}^{\mathbf{Q}}\left(S(t_i)\mathbf{E}^{\mathbf{Q}}\left(S(t_j)|\mathcal{F}_{t_i}\right)\middle|\Lambda\right) \\ \leq \;& S^2(0)\left(\frac{\Lambda}{S(0)}\right)^{2A_\Lambda(h)} e^{r(t_i+t_j)+\sigma_h^2-2B_\Lambda(h)-A_\Lambda^2(h)V_\Lambda+\overline{G}^2(t_i-h)}. \end{aligned}$$

We can then obtain the upper bound above for the conditional covariance of $S(t_i)$ and $S(t_j)$ by using Proposition 25. The conditional variance is achieved by taking $i = j$. This then leads to equation (4.1.26). □

**Theorem 32.** *Let* $\{S(t)\}_{t\in[0,T]}$ *satisfy the SDDE (3.2.5), where* $g$ *satisfies conditions A1–A3. Then, the Asian call option* $A(K,T,m)$*, written on* $\{S(t)\}_{t\in[0,T]}$*, is* *bounded from above by the following equation.*

$$
\begin{aligned}
A(K,T,m) \leq UB_2 := \frac{e^{-rT}}{m} \Bigg( &\sum_{i=1}^{k_h-1} \Psi\left(S(t_i), S(0)e^{\mu_{t_i}+\sigma_{t_i}y}\right) \\
&+ S(0)^{1-A_\Lambda(h)} e^{-B_\Lambda(h)} \Psi\left(\Lambda^{A_\Lambda(h)}, \left(S(0)e^{M_\Lambda+\sqrt{V_\Lambda}y}\right)^{A_\Lambda(h)}\right) \sum_{i=k_h}^{m} e^{rt_i} \\
+ \frac{S(0)}{2} \left( \sum_{i=k_h}^{m} e^{rt_i} \left(e^{\sigma_h^2 - A_\Lambda^2(h)V_\Lambda + \overline{G}^2(t_i-h)} - 1\right) \left(e^{rt_i} + 2\sum_{j=i+1}^{m} e^{rt_j}\right) \right)^{1/2} &\Bigg), \qquad (4.1.27)
\end{aligned}
$$

*where* $k_h = \min\{i : t_i > h\}$ *and* $y \in \mathbb{R}$ *solves*

$$
\sum_{i=1}^{k_h-1} e^{\mu_{t_i}+\sigma_{t_i}y} + e^{-\frac{1}{2}A_\Lambda^2(h)V_\Lambda + A_\Lambda(h)\sqrt{V_\Lambda}y} \sum_{i=k_h}^{m} e^{rt_i} = \frac{mK}{S(0)}. \qquad (4.1.28)
$$

*Proof.* We see, for all $t \in [0,h]$, that the inverse distribution function of $S(t)$ satisfies, for all $x \in (0,1)$,

$$
F_{S(t)}^{-1}(x) = S(0)e^{\mu_t + \sigma_t \Phi^{-1}(x)}.
$$

where $\Phi^{-1}$ denotes the standard inverse normal distribution function. On the other hand, from (4.1.11) we see that the inverse distribution of $Y_i$ satisfies, for all $i \geq k_h$,

$$
\begin{aligned}
F_{Y_i}^{-1}(x) &= S(0)\left(\frac{F_\Lambda^{-1}(x)}{S(0)}\right)^{A_\Lambda(h)} e^{rt_i - B_\Lambda(h)} \\
&= S(0)e^{rt_i - \frac{1}{2}A_\Lambda^2(h)V_\Lambda + A_\Lambda(h)\sqrt{V_\Lambda}\Phi^{-1}(x)}.
\end{aligned}
$$

By setting $y = \Phi^{-1}(x)$ in equation (4.1.23), we obtain equation (4.1.28). By appropriately substituting the components of this sum into equation (4.1.22) and using Corollary 31, we obtain equation (4.1.27). □

## 4.2 Asian bounds under a given conditioning variable

In Section 4.1, we obtain upper and lower bounds for arithmetic Asian options by considering the behaviour of an underlying stochastic process that models the evolution of an asset price, when conditioned upon by a lognormal random variable Λ. In McWilliams and Sabanis (2011), we investigate such bounds by using a specific conditioning variable: the value of the underlying asset at a fixed point in time. A similar approach is undertaken by Albrecher et al. (2008) in the determination of a lower bound, as discussed in Chapter 2. In this section, we investigate the effect of choosing such a random variable and explore the link between the results obtained above and those achieved in our paper.

Let us begin by fixing $v \in [0, h]$. Set $\Lambda = S(v) = S(0)e^{X(v)}$. Then by definition, we see that $S(v)$ is a lognormal random variable. Moreover, we immediately obtain the expectation and variance of $L_\Lambda$ as given by equations (4.1.5) and (4.1.6).

$$\begin{aligned} M_\Lambda &= \mu_v = rv - \frac{1}{2}\int_0^v g^2(u, \phi(\delta(u)))\, du, \\ V_\Lambda &= \sigma_v^2 = \int_0^v g^2(u, \phi(\delta(u)))\, du. \end{aligned}$$

Moreover, one can immediately determine the covariance between the log-returns at times $v \in [0, h]$ and $t \in [0, T]$. This leads to an expression for the conditional expectation. We provide an alternative proof to the one given by McWilliams and Sabanis (2011) to achieve the same result.

**Proposition 33.** *Fix $T, h > 0$ and let $S(t)$ satisfy the SDDE (3.2.5) under $\mathbf{Q}$. Let $v \leq h$ be fixed and define the two functions $\alpha : [0, T] \times [0, h] \to [0, 1]$ and $\beta : [0, T] \times [0, h] \to \mathbb{R}$ in the following way.*

$$\alpha(t, v) = \begin{cases} \dfrac{\int_0^t g^2(u, \phi(\delta(u)))\, du}{\int_0^v g^2(u, \phi(\delta(u)))\, du}, & t < v, \\ 1 & t \geq v, \end{cases} \tag{4.2.1}$$

$$\beta(t, v) = r(t - v\alpha(t, v)) + \frac{1}{2}(1 - \alpha(t, v)) \int_0^t g^2(u, \phi(\delta(u)))\, du. \tag{4.2.2}$$

*Then, the conditional expectation of $S(t)$ given $S(v)$ satisfies*

$$\mathbf{E}^{\mathbf{Q}}\left(S(t) | S(v)\right) = S(0)\left(\frac{S(v)}{S(0)}\right)^{\alpha(t,v)} e^{\beta(t,v)}. \tag{4.2.3}$$

*In particular, if $t \geq v$, then $\mathbf{E}^{\mathbf{Q}}\left(S(t) | S(v)\right) = S(v)e^{r(t-v)}$.*

*Proof.* Let us first assume that $t \in [0, v)$ and write $S(t) = S(0)e^{X(t)}$, where

$$X(t) \sim N\left(rt - \frac{1}{2}\int_0^t g^2(u, \phi(\delta(u)))\, du, \int_0^t g^2(u, \phi(\delta(u)))\, du\right).$$

Since $t < v$, we can use the independence of $\{W(u) : u \in [0, v)\}$ and $\{W(u) : u \geq v\}$ to obtain the following expression for the covariance between $X(t)$ and $X(v)$.

$$\mathrm{Cov}^{\mathbf{Q}}\left(X(t), X(v)\right) = \mathrm{Var}^{\mathbf{Q}}\left(X(t)\right) = \int_0^t g^2(u, \phi(\delta(u)))\, du.$$

On the other hand, if $t \geq v$ then $\mathrm{Cov}^{\mathbf{Q}}\left(X(t), X(v)\right) = V_\Lambda$. By considering equation (4.1.10) above, we immediately see that $A_\Lambda(t) = \alpha(t, v)$ for our particular choice of $\Lambda$. Substituting this into the expression for $B_\Lambda$, we obtain

$$B_\Lambda(t) = rv\alpha(t, v) - \frac{1}{2}\alpha(t, v)(1 - \alpha(t, v)) \int_0^v g^2(u, \phi(\delta(u)))\, du.$$

By definition of $\alpha$, we see the following expression is true when $t < v$.

$$\alpha(t, v) \int_0^v g^2(u, \phi(\delta(u)))\, du = \int_0^t g^2(u, \phi(\delta(u)))\, du.$$

Conversely, if $t \geq v$ then $1 - \alpha(t, v) = 0$. Hence,

$$rt - B_\Lambda(t) = r(t - v\alpha(t, v)) + \frac{1}{2}(1 - \alpha(t, v)) \int_0^t g^2(u, \phi(\delta(u)))\, du = \beta(t, v).$$

Therefore, by substituting the above results into equation (4.1.11), we obtain equation (4.2.3). $\square$

### 4.2.1 Lower bound under specific conditioning

Define $k_1$ in the following way.

$$k_1 = \min\{i : t_i \geq v\} \wedge (m + 1). \tag{4.2.4}$$

Let us then consider the set of $m$ random variables $\{Y_i : i = 1, \ldots, m\}$ given by

$$Y_i = \mathbf{E}^{\mathbf{Q}}\left( S(t_i) \middle| S(v)\right) = \begin{cases} S(0)\left(\dfrac{S(v)}{S(0)}\right)^{\alpha(t_i, v)} e^{\beta(t_i, v)} & i < k_1, \\ S(v)e^{r(t_i - v)} & i \geq k_1. \end{cases} \tag{4.2.5}$$

Then, we realise the convex-order lower bound $Y_1 + \cdots + Y_m = S^l(v)$ as a function of $v$. Since, for all $i$, the $Y_i$ depend only on a single random variable $S(v)$, where $v$ is fixed, and an $\mathcal{F}_0$-measurable process $\phi$ defined for $t \leq 0$, we observe that $S^l(v)$ leads to a comonotonic lower bound for $A(K, T, m)$. In order to obtain it, we will make use of the following statement.

**Corollary 34.** *With* $\{S(t)\}_{t\in[0,T]}$ *following the SDDE (3.2.5), let the function* $\gamma : [0,h] \to \mathbb{R}_+$ *denote a nonnegative,* $\mathcal{F}_0$*-measurable process. Then, for any* $t, v \in [0,h]$*, the stop-loss transform of* $S(t)^{\gamma(v)}$ *with respect to* $K^{\gamma(v)}$ *satisfies*

$$\Psi(S(t)^{\gamma(v)}, K^{\gamma(v)}) \\ = S(0)^{\gamma(v)} e^{\gamma(v)(rt - \frac{1}{2}(1-\gamma(v)) \int_0^t g^2(u,\phi(\delta(u)))\,du)} \Phi(d_1(t,\gamma(v))) - K^{\gamma(v)} \Phi(d_2(t)), \quad (4.2.6)$$

*where* $d_1$ *and* $d_2$ *satisfy*

$$d_2(t) = \frac{\log\left(\frac{S(0)}{K}\right) + rt - \frac{1}{2}\int_0^t g^2(u,\phi(\delta(u)))\,du}{\sqrt{\int_0^t g^2(u,\phi(\delta(u)))\,du}}, \quad (4.2.7)$$

$$d_1(t,\gamma(v)) = d_2(t) + \gamma(v)\sqrt{\int_0^t g^2(u,\phi(\delta(u)))\,du}, \quad (4.2.8)$$

*and* $\Phi : \mathbb{R} \to [0,1]$ *denotes the standard normal distribution function.*

*Proof.* Since $\gamma$ is nonnegative, we see that $S(t)^{\gamma(v)} > K^{\gamma(v)}$ if and only if $Z > -d_2(t)$, where

$$Z = \frac{\int_0^t g(u,\phi(\delta(u)))\,dW(u)}{\sqrt{\int_0^t g^2(u,\phi(\delta(u)))\,du}}$$

is a standard normally distributed random variable associated with $S(v)$ and $d_2(t)$ satisfies equation (4.2.7). By using standard techniques, noting that $\gamma(v)\log(S(t)/S(0))$ is normally distributed, we obtain equation (4.2.6). □

**Theorem 35.** *Let* $\{S(t)\}_{t\in[0,T]}$ *define a stock price process that satisfies the stochastic delay model given by equation (3.2.5). Let* $h \in (0,T]$ *be fixed and let* $r > 0$ *define the risk-free rate of interest under the risk-neutral measure* $\mathbf{Q}$*. Let* $0 \le t_1 < \cdots < t_m \le T$ *denote* $m \in \mathbb{N}$ *monitoring times over the interval* $[0,T]$*. Then, the Asian call option written on* $\{S(t)\}_{t\in[0,T]}$ *with exercise price* $K$*, maturity time* $T$ *and the* $m$ *monitoring times above, given by*

$$A(K,T,m) = \frac{e^{-rT}}{m}\Psi(\mathbb{S}, mK),$$

*is bounded from below by the following equation.*

$$
\begin{aligned}
A(K,T,m) &\geq LB(v) \\
&:= \frac{e^{-rT}}{m}\Bigg(\sum_{i=1}^{k_1-1} S(0)^{1-\alpha(t_i,v)} e^{\beta(t_i,v)} \Psi(S(v)^{\alpha(t_i,v)}, (xS(0))^{\alpha(t_i,v)}) \\
&\qquad\qquad + \sum_{i=k_1}^{m} e^{r(t_i-v)} \Psi(S(v), xS(0))\Bigg) \\
&= \frac{S(0)e^{-rT}}{m}\Bigg(\sum_{i=1}^{k_1-1} e^{\beta(t_i,v)}\big(e^{(rv-\frac{1}{2}(1-\alpha(t_i,v))\int_0^v g^2(u,\phi(\delta(u)))\,du)\alpha(t_i,v)}\Phi(d_{1,i}) \\
&\qquad\qquad - x^{\alpha(t_i,v,b)}\Phi(d_2)\big) + (\Phi(d_3) - xe^{-rv}\Phi(d_2))\sum_{i=k_1}^{m} e^{rt_i}\Bigg), \quad (4.2.9)
\end{aligned}
$$

*where* $x \in (0,\infty)$ *solves*

$$
\sum_{i=1}^{k_1-1} x^{\alpha(t_i,v)} e^{\beta(t_i,v)} + x\sum_{i=k_1}^{m} e^{r(t_i-v)} = \frac{mK}{S(0)}, \quad (4.2.10)
$$

$d_{1,i}, d_2$ *and* $d_3$ *are given by, for* $i \in \{1,\dots,k_1-1\}$,

$$
\begin{aligned}
d_2 &= \frac{-\log x + rv - \frac{1}{2}\int_0^v g^2(u,\phi(\delta(u)))\,du}{\sqrt{\int_0^v g^2(u,\phi(\delta(u)))}}, \quad (4.2.11) \\
d_{1,i} &= \frac{-\log x + rv + \big(\alpha(t_i,v) - \frac{1}{2}\big)\int_0^v g^2(u,\phi(\delta(u)))\,du}{\sqrt{\int_0^v g^2(u,\phi(\delta(u)))\,du}}, \\
&= d_2 + \alpha(t_i,v)\sqrt{\int_0^v g^2(u,\phi(\delta(u)))\,du}, \quad (4.2.12) \\
d_3 &= \frac{-\log x + rv + \frac{1}{2}\int_0^v g^2(u,\phi(\delta(u)))\,du}{\sqrt{\int_0^v g^2(u,\phi(\delta(u)))\,du}} \\
&= d_2 + \sqrt{\int_0^v g^2(u,\phi(\delta(u)))\,du}, \quad (4.2.13)
\end{aligned}
$$

*and* $\alpha$ *and* $\beta$ *are given in Proposition 33.*

*Proof.* Since the vector $\mathbf{Y} = (Y_1,\dots,Y_m)$, whose components are given by equation (4.2.5), is comonotonic, we see from Dhaene et al. (2002b) that the stop-loss transform of $S^l(v)$ is given by

$$
\Psi(S^l(v), mK) = \sum_{i=1}^{m} \mathbf{E}^{\mathbf{Q}}\left[\left(Y_i - F_{Y_i}^{-1}(F_{S^l(v)}(mK))\right)^+\right], \quad (4.2.14)
$$

where the distribution function of $S^l(v)$ is given by

$$F_{S^l(v)}(mK) = \mathbf{Q}\left(S^l(v) \leq mK\right) = \mathbf{Q}\left(\sum_{i=1}^{m} Y_i \leq mK\right), \tag{4.2.15}$$

and $F_{Y_i}^{-1} : [0,1] \to [0,\infty)$ is a continuous, injective, non-decreasing function given by

$$F_{Y_i}^{-1}(p) = \inf\{x : F_{Y_i}(x) \geq p\},$$

for each $i$ and $p \in (0,1)$. Therefore, by using the same approach as Albrecher et al. (2008), let $x \in (0,\infty)$ solve equation (4.2.10). Then, equation (4.2.15) can be rewritten in then following way.

$$\begin{aligned} F_{S^l(v)}(mK) = \mathbf{Q}\Bigg( \sum_{i=1}^{k_1-1} \left(\frac{S(v)}{S(0)}\right)^{\alpha(t_i,v)} e^{\beta(t_i,v)} + \left(\frac{S(v)}{S(0)}\right) \sum_{i=k_2}^{m} e^{r(t_i-v)} \\ \leq \sum_{i=1}^{k_1-1} x^{\alpha(t_i,v)} e^{\beta(t_i,v)} + x \sum_{i=k_2}^{m} e^{r(t_i-v)} \Bigg). \end{aligned}$$

It is computationally straightforward to solve (4.2.10) for $x$ since the left-hand side of this equation is strictly increasing. We can therefore see, from the above result, that $F_{S^l(v)}(mK) = F_{S(v)}(xS(0))$. Hence, $F_{S^l(v)}$ satisfies the following for every $v \in [0,h]$ and $K > 0$.

$$F_{S^l(v)}(mK) = F_{S(v)}(xS(0)) = \begin{cases} F_{Y_i}(S(0)x^{\alpha(t_i,v)}e^{\beta(t_i,v)}) & i < k_1, \\ F_{Y_i}(S(0)xe^{r(t_i-v)}) & i \geq k_1. \end{cases}$$

By substituting this equation into (4.2.14) appropriately and rearranging, we obtain the first expression for equation (4.2.9). The second expression follows by using Corollary 34, with $\gamma(v)$ replaced with $\alpha(t_i, v)$ for every $i \in \{1, \ldots, k_1 - 1\}$. In this case, we see that $d_2 = d_2(v)$, $d_{1,i} = d_1(v, \alpha(t_i, v))$ and $d_3 = d_1(v, 1)$, where $d_2(t)$ and $d_1(t, \gamma(v))$ are given by equations (4.2.7) and (4.2.8) respectively. □

### 4.2.2 First upper bound

The above results provide a lower bound for the value of an arithmetic Asian call option as a function of $v$. We numerically report on the behaviour of this lower bound in Chapter 6. To achieve the lower bound, we impose Assumptions A1 and A2 on the function $g$. In order to obtain an upper bound under our choice of conditioning variable, we additionally require Assumption A3 to hold. With this in mind, the expressions for $A_\Lambda$ and $B_\Lambda$ obtained in Proposition 33 remain unchanged. Given a set of monitoring times $0 \leq t_1 < \cdots < t_m \leq T$ and a fixed constant $h \in (0, T]$, let us define the constant $k_2$ by

$$k_2 = \min\{i : t_i > h\} \wedge (m+1). \tag{4.2.16}$$

We can then make a direct substitution into Proposition 28 and Theorem 29 to obtain an upper bound for $A(K, T, m)$ as a function of $v$.

**Proposition 36.** *Let $\{S(t)\}_{t\in[0,T]}$ follow the SDDE (3.2.5) and let $v \in [0, h]$. Then, the conditional variance of $S(t)$ given $S(v)$ is given by the following when $t \in [0, v)$.*

$$\begin{aligned}\mathrm{Var}^{\mathbf{Q}}\left(S(t)|\,S(v)\right) &= S^2(0)\left(\frac{S(v)}{S(0)}\right)^{2\alpha(t,v)} \\ &\times e^{2r(t-v\alpha(t,v))+(1-\alpha(t,v))\int_0^t g^2(u,\phi(\delta(u)))\,du}\left(e^{(1-\alpha(t,v))\int_0^t g^2(u,\phi(\delta(u)))\,du}-1\right).\end{aligned} \tag{4.2.17}$$

*On the other hand, if $t \geq v$, then the conditional variance of $S(t)$ given $S(v)$ is bounded above by the following, with equality whenever $t \leq h$.*

$$\mathrm{Var}^{\mathbf{Q}}\left(S(t)|\,S(v)\right) \leq S^2(v)e^{2r(t-v)}\left(e^{\int_v^{t\wedge h} g^2(u,\phi(\delta(u)))\,du+G^2(t\vee h-h)}-1\right). \tag{4.2.18}$$

*Proof.* Let us rewrite the conditional variance as

$$\mathrm{Var}^{\mathbf{Q}}\left(S(t)|\,S(v)\right) = \mathbf{E}^{\mathbf{Q}}\left(S^2(t)\big|\,S(v)\right) - \mathbf{E}^{\mathbf{Q}}\left(S(t)|\,S(v)\right)^2.$$

Then, the case where $t < v$ can be shown by using Proposition 33. To obtain the conditional second moment, we simply replace $X(t)$ by $2X(t)$.

Now let us consider the case $t \geq v$. In order to find an expression for the conditional second moment of $S(t)$, let us first suppose that $v \leq t \leq h$. Then, using the tower property and the fact that $S(t)$ is lognormally distributed, we have

$$\mathbf{E}^{\mathbf{Q}}\left(S^2(t)\big|\,S(v)\right) = S^2(v)e^{2r(t-v)+\int_v^t g^2(u,\phi(\delta(u)))\,du}.$$

Now suppose that $t \in [kh \wedge T, (k+1)h \wedge T]$. Then, we have the following upper bound for the conditional expectation of $S^2(t)$ given $S(v)$, under the assumption that $g$ is bounded above by $G$.

$$\mathbf{E}^{\mathbf{Q}}\left(S^2(t)\middle|\,S(v)\right) \leq S^2(v)e^{2r(t-v)+\int_v^h g^2(u,\phi(\delta(u)))\,du+G^2(t-h)}, \tag{4.2.19}$$

It follows that this holds for all $t \geq h$. By combining this with Proposition 33, we obtain equation (4.2.18). $\square$

**Theorem 37.** *Let $\{S(t)\}_{t\in[0,T]}$ follow the SDDE* (3.2.5)*. Then, the arithmetic Asian option $A(K,T,m)$ is bounded above by the following closed-form expression.*

$$\begin{aligned} A(K,T,m) &\leq LB(v) \\ &+ \frac{S(0)e^{-rT}}{2m}\left(\sum_{i=1}^{k_1-1} e^{rt_i}\left(e^{(1-\alpha(t_i,v))\int_0^{t_i} g^2(u,\phi(\delta(u)))\,du}-1\right)^{1/2}\right. \\ &+ \sum_{i=k_1}^{k_2-1} e^{rt_i}\left(e^{\int_v^{t_i} g^2(u,\phi(\delta(u)))\,du}-1\right)^{1/2} \\ &\left.+ \sum_{i=k_2}^{m} e^{rt_i}\left(e^{\int_v^{h} g^2(u,\phi(\delta(u)))\,du+G^2(t_i-h)}-1\right)^{1/2}\right) =: UB_1(v) \end{aligned} \tag{4.2.20}$$

### 4.2.3 Second upper bound

In order to achieve the second upper bound, we note that Proposition 14 of McWilliams and Sabanis (2011) is identical to Proposition 30 with $\Lambda = S(v)$. We are then left with the following two statements for the second upper bound for $A(K,T,m)$.

**Corollary 38.** *An upper bound for the conditional variance of $\sum_{i=k_2}^{m} S(t_i)$ given $S(v)$ is given by*

$$\begin{aligned} &\mathrm{Var}^{\mathbf{Q}}\left(\sum_{i=k_2}^{m} S(t_i)\middle|\,S(v)\right) \\ &\leq S^2(v)e^{-2rv}\left(\sum_{i=k_2}^{m} e^{2rt_i}\left(e^{\int_v^h g^2(u,\phi(\delta(u)))\,du+G^2(t_i-h)}-1\right)\right. \\ &\left.+ 2\sum_{i=k_2}^{m}\sum_{j=i+1}^{m} e^{r(t_i+t_j)}\left(e^{\int_v^h g^2(u,\phi(\delta(u)))\,du+G^2(t_i-h)}-1\right)\right). \end{aligned} \tag{4.2.21}$$

*Proof.* We can rewrite the above conditional variance as

$$\mathrm{Var}^{\mathbf{Q}}\left(\sum_{i=k_2}^{m} S(t_i)\middle| S(v)\right) = \sum_{i=k_2}^{m} \mathrm{Var}^{\mathbf{Q}}\left(S(t_i)| S(v)\right) + 2\sum_{i=k_2}^{m}\sum_{j>i} \mathrm{Cov}^{\mathbf{Q}}\left(S(t_i), S(t_j)| S(v)\right)$$

By using the tower property along with Proposition 36 we have, for any $t_j > t_i \geq h$.

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}}\left(S(t_i)S(t_j)| S(v)\right) &= \mathbf{E}^{\mathbf{Q}}\left(S(t_i)\mathbf{E}^{\mathbf{Q}}\left(S(t_j)| \mathcal{F}_{t_i}\right)\middle| S(v)\right) \\ &\leq S^2(v)e^{r(t_i+t_j-2v)+\int_v^h g^2(u,\phi(\delta(u)))\,du+G^2(t_i-h)}.\end{aligned}$$

We can then obtain the conditional covariance of $S(t_i)$ and $S(t_j)$ by using Proposition 33. This gives the second term in the right-hand side of equation (4.2.21). The first occurs by taking $i = j$. □

**Theorem 39.** *Let $\{S(t)\}_{t\in[0,T]}$ satisfy the SDDE (3.2.5), where $g$ satisfies conditions A1–A3. Then, the Asian call option $A(K,T,m)$, written on $\{S(t)\}_{t\in[0,T]}$, is bounded from above by the following equation.*

$$\begin{aligned}A(K,T,m) &\leq UB_2(v) \\ &:= \frac{S(0)e^{-rT}}{m}\left(\sum_{i=1}^{k_2-1} e^{rt_i}\Phi\left(\sqrt{\int_0^{t_i} g^2(u,\phi(\delta(u)))\,du} - y\right)\right. \\ &- \Phi(-y)\sum_{i=1}^{k_2-1} e^{rt_i-\frac{1}{2}\int_0^{t_i} g^2(u,\phi(\delta(u)))\,du+\left(\int_0^{t_i} g^2(u,\phi(\delta(u)))\,du\right)^{1/2}y} \\ &+ \left(\Phi\left(\sqrt{\int_0^{v} g^2(u,\phi(\delta(u)))\,du} - y\right)\right. \\ &\left.- e^{-\frac{1}{2}\int_0^{v} g^2(u,\phi(\delta(u)))\,du+\left(\int_0^{v} g^2(u,\phi(\delta(u)))\,du\right)^{1/2}y}\Phi(-y)\right)\sum_{i=k_2}^{m} e^{rt_i} \\ &\left.+\frac{1}{2}\left(\sum_{i=k_2}^{m} e^{rt_i}\left(e^{\int_v^h g^2(u,\phi(\delta(u)))\,du+G^2(t_i-h)}-1\right)\left(e^{rt_i}+2\sum_{j=i+1}^{m} e^{rt_j}\right)\right)^{1/2}\right),\end{aligned} \quad (4.2.22)$$

*where* $y \in \mathbb{R}$ *solves*

$$\sum_{i=1}^{k_2-1} e^{rt_i-\frac{1}{2}\int_0^{t_i} g^2(u,\phi(\delta(u)))\,du+\left(\int_0^{t_i} g^2(u,\phi(\delta(u)))\,du\right)^{1/2} y}$$

$$+ e^{-\frac{1}{2}\int_0^{v} g^2(u,\phi(\delta(u)))\,du+\left(\int_0^{v} g^2(u,\phi(\delta(u)))\,du\right)^{1/2} y} \sum_{i=k_2}^{m} e^{rt_i} = \frac{mK}{S(0)}, \quad (4.2.23)$$

*and* $k_2 = \min\{i : t_i > h\}$.

*Proof.* We see, for all $t \in [0, h]$, that the inverse distribution function of $S(t)$ satisfies, for all $u \in (0, 1)$,

$$F^{-1}_{S(t)}(u) = S(0)e^{rt-\frac{1}{2}\int_0^t g^2(u,\phi(\delta(u)))\,du+(\int_0^t g^2(u,\phi(\delta(u)))\,du)^{1/2}\Phi^{-1}(u)},$$

where $\Phi^{-1}$ denotes the standard inverse normal distribution function. By setting $y = \Phi^{-1}(1 - \lambda)$ in equation (4.1.25), we obtain equation (4.2.23). By appropriately substituting the components of this sum into equation (4.1.22) and using Corollary 34, we see that $d_1$ and $d_2$ there satisfy, for $t \in [0, h]$,

$$d_2(t) = -y$$
$$d_1(t, 1) = \sqrt{\int_0^t g^2(u, \phi(\delta(u)))\,du} - y.$$

Finally, we replace the unknown conditional variance in equation (4.1.22) with its upper bound (4.2.21), taking the square root and expectation. This results in equation (4.2.22). □

It then remains to compare the behaviour of $UB_2$ verses the upper bound obtained in Section 4.2.2. This turns out to be a fairly trivial result and is demonstrated by the following. This shows that $UB_2$ improves upon $UB_1$.

**Corollary 40.** *Let* $UB_1(v)$ *and* $UB_2(v)$ *describe upper bounds for an Asian call option written on* $\{S(t)\}_{t\in[0,T]}$, *given by equations (4.2.20) and* (4.2.22) *respectively, for any fixed* $v \in [0, h]$. *Then,*

$$UB_2(v) \le UB_1(v). \quad (4.2.24)$$

*Proof.* Since the value of $y$ in both equations is the same, and because both upper bounds must be positive, it is sufficient to observe

$$
\begin{aligned}
&2(UB_1^2(v) - UB_2^2(v)) \\
&\qquad = \sum_{i=k_2}^{m} \sum_{j=i+1}^{m} \left( e^{r(t_i+t_j)} \left( e^{\int_v^h g^2(u,\phi(\delta(u)))\,du + \overline{G}^2(t_i-h)} - 1 \right)^{1/2} \right. \\
&\qquad\qquad \left. \times \left( e^{\int_v^h g^2(u,\phi(\delta(u)))\,du + \overline{G}^2(t_j-h)} - 1 \right)^{1/2} \right) \\
&\qquad\qquad - \sum_{i=k_2}^{m} \sum_{j=i+1}^{m} e^{2rt_i} \left( e^{\int_v^h g^2(u,\phi(\delta(u)))\,du + \overline{G}^2(t_i-h)} - 1 \right) \geq 0, \quad (4.2.25)
\end{aligned}
$$

since $t_j > t_i$ for all $j > i$. □

*Remark.* We can use Minkowski's inequality (see, for example, Williams (1991)), applied to conditional expectation, to show that the theoretical upper bound (4.1.22) improves upon equation (4.1.15). However, the conditional variance in both equations is unknown. We must use Corollary 31 and Proposition 28 respectively to find upper bounds for both. It then cannot be said that equation (4.1.27), even in the specific case where $\Lambda = S(v)$ presented by equation (4.2.22), necessarily improves upon (4.1.15). We observe the improvement by $UB_2(v)$ over $UB_1(v)$ in Chapter 6

## 4.3 Bounds for basket options

In Section 4.1, we explore bounds for arithmetic Asian options under the stochastic delay model presented by equation (3.2.5). For the remainder of this chapter, we shall consider the value of a European-style basket call option written on $N \in \mathbb{N}$ underlying assets. This follows on from other work undertaken on basket options, in particular by Hobson et al. (2005) and Deelstra et al. (2008). However, we work under the assumption that each asset price process follows a stochastic delay differential equation. This approach was undertaken by Arriojas et al. (2007) in valuing European options. This goes beyond the assumption that the underlying asset follows the Black-Scholes model, Black and Scholes (1973), by introducing a non-deterministic volatility function that also depends on past data.

The aim of this section is to obtain bounds for basket call options under the assumption that each underlying asset follows a SDDE model under $\mathbf{Q}$. Using the notation provided by Definition 1, we proceed within this part by introducing basket options.

**Definition 41.** *Given $N$ stock price processes $\{S_i(t)\}_{t\in[0,T]}$, let $\mathbf{w} = (w_1, \ldots, w_N) \in (0,\infty)^N$ denote a vector of weights and define $\mathbb{S}$ to be the weighted sum given by*

$$\mathbb{S} = \sum_{i=1}^{N} w_i S_i(T). \tag{4.3.1}$$

*Let $r$ denote the common risk-free rate of interest. Then, the value of a European-style basket call option written on the $\{S_i(t)\}_{t\in[0,T]}$, with exercise price $K$, maturity time $T$ and with weights given by $\mathbf{w}$ satisfies*

$$B(K, T, \mathbf{w}) = e^{-rT}\Psi(\mathbb{S}, K). \tag{4.3.2}$$

Throughout and for simplification, we will work under the risk-neutral measure $\mathbf{Q}$ and assume that each asset shares a common risk-free rate of interest and fixed drift parameter $r \geq 0$. Let us then assume that $\{S_i(t)\}_{t\in[0,T]}$ satisfies the following SDDE.

$$\left.\begin{aligned} dS_i(t) &= rS_i(t)\,dt + g_i(t, S_i(\delta_i(t)))S_i(t)\,dW_i(t), & t &\in [0,T], \\ S_i(t) &= \phi_i(t), & t &\in [-\tau_i, 0]. \end{aligned}\right\} \tag{4.3.3}$$

In this model, $\mathbf{W}(t) = (W_1(t), \ldots, W_N(t))$ is a standard Brownian motion in $\mathbb{R}^N$. We shall make use of the following two types of $\sigma$-algebras.

$$\mathcal{F}_t^{(i)} = \sigma(\{S_i(u) : u \leq t\}), \quad \mathcal{F}_t = \sigma(\{S_i(u) : u \leq t, i = 1, \ldots, N\}). \tag{4.3.4}$$

The delay functions $\delta_i : [0,T] \to (-\infty, T]$ are deterministic and defined so that $\delta_i(t) \leq kh_i$ whenever $t \in [kh_i, (k+1)h_i)$ for any suitable $h_i \in (0,T]$ and $k \in \mathbb{N} \cup \{0\}$. We will use the notation $m_i = T \wedge h_i$, where the operator $\wedge$ takes the minimum value of two real numbers. In practice, we would also require $\delta_i$ to be bounded from below to reflect the level of past data available on $\phi_i$, which is given the respective delay parameter. As discussed in Chapter 3, good examples for $\delta_i$ include fixed-delay models, for example $\delta_i(t) = t - h_i$, as well as variable, step-function delay models, such as equation (3.1.4). Both types of models are discussed by Arriojas et al. (2007). It is possible to use a mixture of fixed and variable delay functions, provided $\delta_i$ satisfies the conditions above.

Of particular importance to this report are the functions $g_i : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$. These follow on from the single volatility function $g$ introduced in Section 4.1. For completeness, we restate the complete set of assumptions for each $g_i$ as follows:

A1. $g_i(t,x)$ is a continuous function of $x$.

A2. $g_i(t,x)$ is strictly positive whenever $t$ or $x$ are strictly positive.

A3. There exists a positive constant $\overline{G}_i$ such that $g_i(t,x) \leq \overline{G}_i$.

The processes $\phi_i : \Omega \to C^b_{\mathcal{F}_0}([-\tau_i, 0], \mathbb{R})$ are assumed to be continuous, bounded, $\mathcal{F}_0$-measurable functions. We are then presented with a set of models for underlying assets that are similar to those given by equation (3.2.5). A consequence is, in our view, a more natural set of processes compared to the geometric Brownian motion due to the existence of local volatility. See, for example, Hobson and Rogers (1998). Once again, this model incorporates delay parameters which would exist in practice, due to the inability to react to information instantaneously.

### 4.3.1 Stochastic delay models for basket options

Suppose that $S_i(0) > 0$ for every $i$. By considering each stochastic process separately, we can use the same approach given in Section 3.2 for individual stochastic delay models. For completeness, we present the following two results concerning any solution to (4.3.3). Both can be proven using the results given by Proposition 16, Lemma 17 and Proposition 18. From these results and Arriojas et al. (2007), we see that the $S_i(t)$ are almost surely continuous and positive for all $t \in [0,T]$ and $i \in \{1, \ldots, N\}$.

**Lemma 42.** *Let $\{S_i(t)\}_{t\in[0,T]}$ be a process that satisfies the stochastic variable delay model given by equation (4.3.3) under the risk-neutral measure $\mathbf{Q}$. Let $k \in \mathbb{N} \cup \{0\}$ be chosen so that $kh_i \leq T$. Then, for all $t \in [kh_i, (k+1)h_i \wedge T]$, it follows that $S_i(t)$ can be written in the following way.*

$$S_i(t) = S_i(kh_i) e^{r(t-(kh_i)) - \frac{1}{2}\int_{kh_i}^t g_i^2(u, S_i(\delta_i(u)))\,du + \int_{kh_i}^t g_i(u, S_i(\delta_i(u)))\,dW_i(u)}. \tag{4.3.5}$$

**Lemma 43.** *Let $S_i(t)$ satisfy the SDDE (4.3.3). Then, the discounted stock price process $\{\widetilde{S}_i(t)\}_{t\in[0,T]}$ satisfying*

$$d\widetilde{S}_i(t) = g_i(t, \widetilde{S}_i(\delta_i(t)))\,dW_i(t) \tag{4.3.6}$$

*is a martingale under $\mathbf{Q}$.*

*Remark.* The only assumption that we need to make for this to hold is A1: that, for every $i$, the function $g_i(t,x)$ is continuous in $x$.

#### Conditional expectation

As discussed in Dhaene et al. (2002b), conditioning arithmetic sums involving the stochastic process $\{S(t)\}_{t\in[0,T]}$ on a dependent random variable $\Lambda$ is important for finding stop-loss bounds. This is put into practice in Deelstra et al. (2008); Dhaene et al. (2002a); Vanmaele et al. (2006) when the underlying stochastic process follows a lognormal model. We wish to extend this approach to the set of stochastic delay processes given above.

As before, let $S_i(t)$ satisfy equation (4.3.3). Further, we take $\mathcal{F}_t^{(i)}$ to be the $\sigma$-algebra generated by $\{S_i(u) : u \leq t\}$, for all $t \in [0, T]$. A simple application of Jensen's inequality yields the following convex-order lower bound for $S_i(t)$, given any random variable $\Lambda$.

$$S_i(t) \geq_{\text{cx}} \mathbf{E}\left( S_i(t) \middle| \Lambda \right). \tag{4.3.7}$$

In order to determine the conditional expectation given by (4.3.7), we will consider $N$ lognormally distributed random variables, $\Lambda_i$. We then adapt Definition 23 in order to take the $\Lambda_i$ in account, in the following way.

**Definition 44.** *For every $i = 1, \dots, N$, let $\Lambda_i$ be a lognormally distributed random variable and let $L_{\Lambda_i}$, $M_{\Lambda_i}$ and $V_{\Lambda_i}$ be defined in the following way.*

$$L_{\Lambda_i} = \log \Lambda_i, \quad M_{\Lambda_i} = \mathbf{E}^{\mathbf{Q}}\left[L_{\Lambda_i}\right], \quad V_{\Lambda_i} = \mathrm{Var}^{\mathbf{Q}}\left(L_{\Lambda_i}\right). \tag{4.3.8}$$

*In particular, when $\Lambda_i = S_i(t)/S_i(0)$ for $t \in [0, h_i]$, $M_{\Lambda_i}$ and $V_{\Lambda_i}$ satisfy the following expressions.*

$$M_{\Lambda_i} = \mu_{t,i} := rt - \frac{1}{2}\int_0^t g_i^2(u, \phi_i(\delta_i(u)))\, du, \tag{4.3.9}$$

$$V_{\Lambda_i} = \sigma_{t,i}^2 := \int_0^t g_i^2(u, \phi_i(\delta_i(u)))\, du. \tag{4.3.10}$$

Let us assume, without any loss of generality, that the underlying stochastic processes given by (4.3.3) are ordered so that their corresponding $h_i$ are arranged in decreasing order:

$$\infty > h_1 \geq h_2 \geq \cdots \geq h_N > 0. \tag{4.3.11}$$

Further, we shall define $I \in \{0, 1, \dots, N\}$ as follows, using the convention that $\max \emptyset =$

0.

$$I = \max\{i : h_i \geq T\}. \tag{4.3.12}$$

The consequence of this is that $S_i(t)$ is remains lognormal for all $t \in [0, T]$ if $i \leq I$. This is not necessarily the case when $i > I$. Therefore, one candidate for each conditioning variable $\Lambda_i$ is contained within functions of the processes $\{S_1(t_1), \ldots, S_N(t_N) : t_j \in [0, m_j]\}$.

### 4.3.2 An upper bound

In order to find a suitable upper bound for basket options, we will adapt the approach given by Hobson et al. (2005), which we make use of in valuing arithmetic Asian options in Section 4.1.2. Due to the more complex dependency structure imposed by considering multiple asset price processes, the upper bound for Asian options considered in Section 4.1.3 that improves upon Section 4.1.2 becomes intractable and may not yield an explicit closed-form solution when considered here. Therefore, we solely study the methods underlined in Section 4.1.2 for basket options and not adapt Section 4.1.3 in this case.

We note that if $I < N$, then the distribution of $S_i(T)$ will be unknown whenever $i > I$. However, we can extend the results of this paper, making use of Rogers and Shi (1995), in order to obtain an upper bound for basket options. This bound in particular relies on Assumption A3, that the local volatility functions $g_i$ are bounded from above. Such an approach is proposed in Section 4.2 of Hobson and Rogers (1998) and this assumption does not seem unrealistic since it is unlikely that volatility is unbounded. Empirical studies are undertaken by Dumas et al. (1998) and the assumption of "uniformly bounded" local volatility is used in a number of sources, for example Berestycki et al. (2002); Coleman et al. (1999).

From Rogers and Shi (1995), we see that the following is an upper bound for a basket call option.

$$B(K, T, \mathbf{w}) \leq e^{-rT} \sum_{i=1}^{N} w_i \Psi\left(S_i(T), \frac{K_i}{w_i}\right), \tag{4.3.13}$$

for any vector $\mathbf{K} \in [0, K]^N$ satisfying $\sum_{i=1}^{N} K_i = K$. In the case where $I = N$, we see that $S_i(T)$ has a known, lognormal distribution for all $i$. We can then easily obtain an explicit form for (4.3.13) using the methods of Hobson et al. (2005).

To deal with the case where $I < N$, let us introduce a lognormal random vector $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_N)$, where $\Lambda_i$ is a function of $\{S_i(t) : t \leq h_i\}$ for all $i$. We can then bound (4.3.13) from above using the following result.

**Proposition 45.** *Let $\{S_i(t)\}_{t\in[0,T]}$ follow the SDDE (4.3.3) for every $i$. Then, the basket call option $B(K,T,\mathbf{w})$ is bounded above by the following equation, for any $K_i \in [0,K]$ satisfying $\sum_{i=1}^N K_i = K$.*

$$B(K,T,\mathbf{w}) \leq e^{-rT}\left(\sum_{i=1}^{I} w_i \Psi\left(S_i(T), \frac{K_i}{w_i}\right)\right.$$
$$\left. + \sum_{i=I+1}^{N} w_i \left(\Psi\left(\mathbf{E}^{\mathbf{Q}}\left(S_i(T)|\Lambda_i\right), \frac{K_i}{w_i}\right) + \frac{1}{2}\mathbf{E}^{\mathbf{Q}}\left[\sqrt{\mathrm{Var}^{\mathbf{Q}}\left(S_i(T)|\Lambda_i\right)}\right]\right)\right) \quad (4.3.14)$$

*Proof.* From Rogers and Shi (1995), we have the following upper bound for the stop-loss transform of any random variable $X$ and any conditioning variable $Y$.

$$0 \leq \mathbf{E}^{\mathbf{Q}}\left((X-x)^+\middle| Y\right) - \mathbf{E}^{\mathbf{Q}}\left(X - x| Y\right)^+$$

Therefore, by taking the expectation of the right-hand side of this equation and using the tower property, we obtain the following result.

$$\mathbf{E}^{\mathbf{Q}}\left[(X-x)^+\right] - \mathbf{E}^{\mathbf{Q}}\left[\left(\mathbf{E}^{\mathbf{Q}}\left(X|Y\right) - x\right)^+\right] \leq \frac{1}{2}\mathbf{E}^{\mathbf{Q}}\left[\sqrt{\mathrm{Var}^{\mathbf{Q}}\left(X|Y\right)}\right]. \quad (4.3.15)$$

By using (4.3.15) in equation (4.3.13) when $i > I$, replacing $X$ with $S_i(T)$, $x$ with $K_i/w_i$ and $Y$ with $\Lambda_i$, we obtain equation (4.3.14). □

**Proposition 46.** *Fix $h_i > 0$ and let $\{S_i(t)\}_{t\in[0,T]}$ satisfy the SDDE (4.3.3). Let $Y_i(t)$ be such that $S_i(t) = S_i(0)e^{Y_i(t)}$ and let $\Lambda_i$ be a lognormally distributed random variable with $L_{\Lambda_i}, M_{\Lambda_i}$ and $V_{\Lambda_i}$ given by equation (4.3.8). Let us define the following two functions, $A_{\Lambda_i} : [0,h_i] \to [0,\infty)$ and $B_{\Lambda_i} : [0,h_i] \to \mathbb{R}$.*

$$A_{\Lambda_i}(t) = \frac{\mathrm{Cov}\left(Y_i(t), L_{\Lambda_i}\right)}{\mathrm{Var}^{\mathbf{Q}}\left(L_{\Lambda_i}\right)}, \quad B_{\Lambda_i}(t) = A_{\Lambda_i}(t)\left(M_{\Lambda_i} + \frac{1}{2}A_{\Lambda_i}(t)V_{\Lambda_i}\right). \quad (4.3.16)$$

*Then for every $i$, the conditional expectation of $S_i(t)$ given $\Lambda_i$ satisfies*

$$\mathbf{E}^{\mathbf{Q}}\left(S_i(t)|\Lambda_i\right) = S_i(0)\Lambda_i^{A_{\Lambda_i}(t\wedge h_i)} e^{rt - B_{\Lambda_i}(t\wedge h_i)}. \quad (4.3.17)$$

*Proof.* Let us first assume that $t \in [0, h_i]$. Then, from Definition 44, we see that $Y_i(t)$ is normally distributed. Hence, by using Proposition 19 with $X = Y_i(t)$ and $Y = L_{\Lambda_i}$ noting that $rt = \mu_{t,i} + \frac{1}{2}\sigma_{t,i}^2$ and $\Lambda_i = e^{L_{\Lambda_i}}$, we obtain the first part of equation (4.3.17).

Now suppose that $t > h_i$. Since, from Lemma 43, the discounted asset price process $\{\widetilde{S}_i(t)\}_{t \in [0,T]}$ is a martingale, it follows that the conditional expectation of $S_i(t)$ given $\Lambda_i$ satisfies

$$\mathbf{E}^{\mathbf{Q}}\left(S_i(t)\middle|\,\Lambda_i\right) = \mathbf{E}^{\mathbf{Q}}\left(\mathbf{E}^{\mathbf{Q}}\left(S_i(t)\middle|\,\mathcal{F}_{h_i}^{(i)}\right)\middle|\,\Lambda_i\right) = \mathbf{E}^{\mathbf{Q}}\left(S_i(h_i)\middle|\,\Lambda_i\right)e^{r(t-h_i)}.$$

By applying this result to the first part of equation (4.3.17) and rearranging, we see that (4.3.17) holds for all $t \in [0, T]$. □

Since we assume $\Lambda_i$ is lognormally distributed, Proposition 46 allows us to compute the stop loss transform of $\mathbf{E}^{\mathbf{Q}}\left(S_i(T)\middle|\,\Lambda_i\right)$ for every $i$, including the cases where $i > I$. In order to work with the additional 'error' term given by Proposition 45, we make use of the following result.

**Proposition 47.** *Let* $\{S_i(t)\}_{t\in[0,T]}$ *follow the SDDE* (4.3.3). *Then,*

$$\mathbf{E}^{\mathbf{Q}}\left[\sqrt{\mathrm{Var}^{\mathbf{Q}}\left(S_i(T)\middle|\,\Lambda_i\right)}\right] \leq S_i(0)e^{rT}\sqrt{e^{\sigma_{m_i,i}^2 - A_{\Lambda_i}^2(m_i)V_{\Lambda_i} + \overline{G}_i^2(T-m_i)} - 1}, \qquad (4.3.18)$$

*where* $m_i = T \wedge h_i$.

*Proof.* Let us rewrite the conditional variance as

$$\mathrm{Var}^{\mathbf{Q}}\left(S_i(T)\middle|\,\Lambda_i\right) = \mathbf{E}^{\mathbf{Q}}\left(S_i^2(T)\middle|\,\Lambda_i\right) - \mathbf{E}^{\mathbf{Q}}\left(S_i(T)\middle|\,\Lambda_i\right)^2.$$

Then, the case where $i \leq I$ can be shown by using Proposition 19. To obtain the conditional second moment, we simply replace $Y_i(T)$ by $2Y_i(T)$ and multiply by $S_i(0)$.

Now suppose $h_i < T$. In order to find an expression for the conditional second moment of $S_i(T)$, let us first suppose that $T \in (h_i, 2h_i]$. Then, using the tower property, we have

$$
\begin{aligned}
\mathbf{E}^{\mathbf{Q}}\left(S_i^2(T)\middle|\,\Lambda_i\right) &= \mathbf{E}^{\mathbf{Q}}\left(\mathbf{E}^{\mathbf{Q}}\left(S_i^2(T)\middle|\,\mathcal{F}_{h_i}^{(i)}\right)\middle|\,\Lambda_i\right) \\
&= \mathbf{E}^{\mathbf{Q}}\left(S_i^2(h_i)e^{2r(T-h_i)+\int_{h_i}^T g_i^2(u,S_i(\delta_i(u)))\,du}\middle|\,\Lambda_i\right) \\
&\leq \mathbf{E}^{\mathbf{Q}}\left(S_i^2(h_i)\middle|\,\Lambda_i\right)e^{2r(T-h_i)+\overline{G}_i^2(T-h_i)} \\
&= S_i^2(0)\Lambda_i^{2A_{\Lambda_i}(h_i)}e^{2rT+\sigma_{h_i,i}^2-2B_{\Lambda_i}(h_i)-A_{\Lambda_i}^2(h_i)V_{\Lambda_i}+\overline{G}_i^2(T-h_i)},
\end{aligned}
$$

where the final equation follows using the case $T \leq h_i$. By combining this with Proposition 19 and taking the square root and expectation, we obtain equation (4.3.18). By continuing in this way and using induction, we see that this holds for all $T > 0$ and $i \in \{I+1,\ldots,N\}$. □

**Theorem 48.** *Let $\{S_i(t)\}_{t\in[0,T]}$ follow the SDDE (4.3.3) for all $i \in \{1,\ldots,N\}$. Assume that $h_i \geq h_{i+1}$ and set $I = \max\{i : h_i \geq T\}$. Then, the basket call option $B(K,T,\mathbf{w})$ is bounded from above by the following closed-form expression.*

$$
\begin{aligned}
B(K,T,\mathbf{w}) \leq UB := e^{-rT}\Bigg(&\sum_{i=1}^{I} w_i\Psi\left(S_i(T), S_i(0)e^{\mu_{T,i}+\sigma_{T,i}y}\right) \\
&+ \sum_{i=I+1}^{N} w_iS_i(0)e^{rT-B_{\Lambda_i}(h_i)}\Psi\left(\Lambda^{A_{\Lambda_i}(h_i)},\left(e^{M_{\Lambda_i}+\sqrt{V_{\Lambda_i}}y}\right)^{A_{\Lambda_i}(h_i)}\right) \\
&+ \frac{1}{2}\sum_{i=I+1}^{N} w_iS_i(0)e^{rT}\sqrt{e^{\sigma_{h_i,i}^2-A_{\Lambda_i}^2(h_i)V_{\Lambda_i}+\overline{G}_i^2(T-h_i)}-1}\Bigg), \quad (4.3.19)
\end{aligned}
$$

*where $y \in \mathbb{R}$ is the solution to*

$$
\sum_{i=1}^{I} w_iS_i(0)e^{\mu_{T,i}+\sigma_{T,i}y} + \sum_{i=I+1}^{N} w_iS_i(0)e^{rT-B_{\Lambda_i}(h_i)+A_{\Lambda_i}(h_i)(M_{\Lambda_i}+\sqrt{V_{\Lambda_i}}y)} = K. \quad (4.3.20)
$$

*Proof.* Equation (4.3.19) follows from combining equation (4.3.14) with the upper bound given by (4.3.18). In order to find the $K_i$ that minimises this upper bound, we use the method of Lagrangian multipliers described by Hobson et al. (2005). Let the function $L : [0,K]^N \times \mathbb{R} \to \mathbb{R}$ be defined in the following way.

$$
L(\mathbf{K},\eta) = \sum_{i=1}^{I} w_i\Psi\left(S_i(T),\frac{K_i}{w_i}\right) + \sum_{i=I+1}^{N} w_i\Psi\left(\mathbf{E}^{\mathbf{Q}}\left(S_i(T)\middle|\,\Lambda_i\right),\frac{K_i}{w_i}\right) + \eta\left(\sum_{i=1}^{N} K_i - K\right).
$$

By taking the derivative of $L$ with respect to each $K_i$ and using Corollary 22, we see that $L$ is minimised by the following equation, noting that $\Psi$ is a convex function of its second variable.

$$K_i = \begin{cases} w_i F^{-1}_{S_i(T)}(1-\eta) & i \leq I, \\ w_i F^{-1}_{\mathbf{E}^{\mathbf{Q}}(\,S_i(T)|\,\Lambda_i)}(1-\eta) & i > I. \end{cases}$$

We see, for all $i \leq I$, that the inverse distribution function of $S_i(T)$ satisfies, for all $\eta \in (0,1)$:

$$F^{-1}_{S_i(T)}(1-\eta) = S_i(0)e^{\mu_{T,i}+\sigma_{T,i}\Phi^{-1}(1-\eta)},$$

where $\Phi$ denotes the standard normal distribution function with inverse $\Phi^{-1}$. On the other hand, if $i > I$, then $m_i = h_i$ and so, by using Proposition 46, we see that the inverse distribution function of $\mathbf{E}^{\mathbf{Q}}\left(\,S_i(T)|\,\Lambda_i\right)$ is given by:

$$F^{-1}_{\mathbf{E}^{\mathbf{Q}}(\,S_i(T)|\,\Lambda_i)}(1-\eta) = S_i(0)e^{rT-B_{\Lambda_i}(h_i)+A_{\Lambda,i}(h_i)\left(M_{\Lambda_i}+\sqrt{V_{\Lambda_i}}\Phi^{-1}(1-\eta)\right)}.$$

By setting $y = \Phi^{-1}(1-\eta)$, we obtain equations (4.3.19) and (4.3.20) above. □

### 4.3.3 A lower bound

In Section 4.3.2, we obtain an upper bound for basket call options. For arithmetic Asian options, we are able to use results from Dhaene et al. (2002b) to obtain a comonotonic vector whose sum is less than $\mathbb{S}$ in convex order. Indeed, this work is undertaken in McWilliams and Sabanis (2011) for stochastic delay differential equations. For this approach to work with basket options we would be restricted to the case where all assets are positively correlated. Instead, we use the approach undertaken by Deelstra et al. (2008), which considers a non-comonotonic sum based on the approach of Rogers and Shi (1995) under the assumption of the Black-Scholes model, and extend it to our case.

Let us define $Y_i(t)$ in the same way as Proposition 46 and choose a single random variable $\Lambda$ such that $(Y_i(t), \Lambda)$ is bivariate normally distributed for all $t \in [0, h_i]$ and $i \in \{1, \ldots, N\}$. We see, from equation (4.3.7), that

$$\mathbb{S} \geq_{\text{cx}} S^l := \sum_{i=1}^{N} w_i \mathbf{E}^{\mathbf{Q}}\left(\,S_i(T)|\,\Lambda\right). \tag{4.3.21}$$

Moreover, we make use of the following relationship concerning the conditional expectation of $S_i(T)$.

**Proposition 49.** *Let* $\{S_i(t)\}_{t\in[0,T]}$ *be a process that satisfies the SDDE* (4.3.3)*, with delay parameter given by* $\tau_i$*. For each* $i$*, let* $h_i > 0$ *be chosen such that the corresponding delay function* $\delta_i(t) \le kh_i$ *whenever* $t \in [kh_i, (k+1)h_i \wedge T)$*. Let* $\Lambda \sim N(\mu_\Lambda, \sigma_\Lambda)$ *be a random variable under which* $(Y_i(t), \Lambda)$ *is bivariate normally distributed with correlation* $\rho_i(t)$*. Then, the conditional expectation of* $S_i(T)$ *given* $\Lambda$ *satisfies the following, where* $m_i = T \wedge h_i$*.*

$$\mathbf{E}^{\mathbf{Q}}\left(S_i(T)|\,\Lambda\right) = S_i(0)\exp\Bigg(rT - \frac{1}{2}\rho_i^2(m_i)\int_0^{m_i} g_i^2(u,\phi_i(\delta_i(u)))\,du + \rho_i(m_i)\sqrt{\int_0^{m_i} g_i^2(u,\phi_i(\delta_i(u)))\,du}\left(\frac{\Lambda-\mu_\Lambda}{\sigma_\Lambda}\right)\Bigg), \quad (4.3.22)$$

*Proof.* Let us first assume that $i \le I$. Then, $S_i(T)$ follows a lognormal distribution. We can then use Proposition 19 appropriately to obtain equation (4.3.22).

Now suppose $i > I$. Then, we can use the tower property and the fact that the discounted process $\{\widetilde{S}_i(t)\}_{t\in[0,T]}$ is a martingale, to obtain

$$\mathbf{E}^{\mathbf{Q}}\left(\mathbf{E}^{\mathbf{Q}}\left(S_i(t)|\,\mathcal{F}_{h_i}^{(i)}\right)\Big|\,\Lambda\right) = \mathbf{E}^{\mathbf{Q}}\left(S_i(h_i)|\,\Lambda\right)e^{r(T-h_i)}.$$

By combining this equation with the case where $h_i \le T$, we once again obtain equation (4.3.22). This completes the proof. $\square$

We can observe that, if the correlation vector $(\rho_1(T),\ldots,\rho_N(T)) \ge \mathbf{0}$, then the conditional expectation vector $(\mathbf{E}^{\mathbf{Q}}\left(S_1(T)|\,\Lambda\right),\ldots,\mathbf{E}^{\mathbf{Q}}\left(S_N(T)|\,\Lambda\right))$ is comonotonic. We can then use standard comonotonicity techniques (see, for example, Deelstra et al. (2004); Dhaene et al. (2002a,b)) to find a lower bound for basket options. However, we cannot assume that this will necessary be the case. Indeed, this is a very strong assumption to make and heavily restricts the types of basket options we can consider.

Instead, let us define the function $f:(0,1)\to\mathbb{R}$ as follows.

$$f(v) = \sum_{i=1}^N w_i S_i(0)\exp\Bigg(rT - \frac{1}{2}\rho_i^2(m_i)\int_0^{m_i} g_i^2(u,\phi_i(\delta_i(u)))\,du + \rho_i(m_i)\sqrt{\int_0^{m_i} g_i^2(u,\phi_i(\delta_i(u)))\,du\Phi^{-1}(v)}\Bigg) - K. \quad (4.3.23)$$

where $v = \Phi((\lambda - \mu_\Lambda)/\sigma_\Lambda)$. Then, $\mathbf{E}^{\mathbf{Q}}\left[\left(S^l - K\right)^+\right] = \mathbf{E}^{\mathbf{Q}}\left[f(V)^+\right]$, where $V$ is uniformly distributed. An important consideration in the valuation of $\mathbf{E}^{\mathbf{Q}}\left[f(V)^+\right]$ will be the interval upon which $f$ is positive. This can be obtained by using the following result.

**Proposition 50.** *If $\rho_i \geq 0$ for every $i$, then $f$ has a unique root in $(0,1)$. Otherwise, $f(v)$ has two solutions if and only if $\inf_{v\in(0,1)} f(v) < 0$.*

*Proof.* Let us first assume that $\rho_i \geq 0$ for every $i$. Then, $f$ is a continuous, strictly increasing function of $v$. Furthermore, we see that $f$ tends to $-K < 0$ as $v \downarrow 0$ and $\infty$ as $v \uparrow 1$. Therefore, by applying the Intermediate Value Theorem, we see that $f$ has a single root in $(0,1)$.

On the other hand, if $\rho_i$ and $\rho_j$ are of opposite sign for some $i \neq j$, then observe that the derivative of $f$ with respect to $v$ satisfies

$$f'(v) = \frac{1}{\varphi(\Phi^{-1}(v))} \sum_{i=1}^{N} \left\{ w_i S_i(0) \rho_i(m_i) \sqrt{\int_0^{m_i} g_i^2(u, \phi_i(\delta_i(u)))\, du} \right.$$
$$\times \exp\left( rT - \frac{1}{2}\rho_i^2(m_i) \int_0^{m_i} g_i^2(u, \phi_i(\delta_i(u)))\, du \right.$$
$$\left.\left. + \rho_i(m_i) \sqrt{\int_0^{m_i} g_i^2(u, \phi_i(\delta_i(u)))\, du}\, \Phi^{-1}(v) \right) \right\}, \quad (4.3.24)$$

where $\varphi$ denotes the standard normal density function. We see, from Deelstra et al. (2008), the denominator of $f'(v)$ is strictly positive, whereas the numerator is a non-decreasing function of $v$ since its derivative is positive. Moreover, if there exists $\rho_i, \rho_j$ of opposite sign for some $i \neq j$, then the numerator tends to $-\infty$ as $v \downarrow 0$ and $\infty$ as $v \uparrow 1$. Therefore, there exists a unique $v^* \in (0,1)$ such that $f'(v^*) = 0$.

We therefore obtain the following result concerning the infimum of $f$.

$$\inf_{v\in(0,1)} f(v) = f(v^*).$$

If $f(v^*) < 0$, then $f$ is a continuous, strictly decreasing function over the interval $(0, v^*)$, which tends to $\infty$ as $v \downarrow 0$. Hence, there exists a unique $v_1 \in (0, v^*)$ such that $f(v_1) = 0$. Moreover, $f$ is a continuous, strictly increasing function on $(v^*, 1)$, which tends to $\infty$ as $v \uparrow 1$. Therefore, from the Intermediate Value Theorem, we obtain an additional $v_2 \in (v^*, 1)$ such that $f(v_2) = 0$.

If $\inf_{v\in(0,1)} f(v) \geq 0$, then is immediate that $f$ can only have at most one root. The proof is therefore complete. □

We see from Proposition 50 that either $f(v) > 0$ for all $v \in (0,1)$ or there exist $v_1 < v_2$ such that $f(v) \leq 0$ for all $v \in [v_1, v_2]$, with $f(v)$ positive otherwise. This then leads to the following lower bound for basket call options.

**Theorem 51.** *Let $\{S_i(t)\}_{t\in[0,T]}$ be a process satisfying the SDDE (4.3.3) for any $i \in \{1,\dots,N\}$ and let $\Lambda$ be a normally distributed random variable such that $(Y_i(t),\Lambda)$ is bivariate normally distributed with correlation given by the deterministic function $\rho_i(t)$. Let $f(v)$ be defined according to equation (4.3.23). Then, a lower bound for the value of the basket call option written on $\mathbb{S}$, with exercise price $K$ and weights $\mathbf{w} \in \mathbb{R}_+^N$ is given by*

$$BC(K,T,\mathbf{w}) = LB,$$

*where*

$$LB := \sum_{i=1}^{N} w_i S_i(0) - Ke^{-rT} \tag{4.3.25}$$

*if $f(v) \geq 0$ for all $v \in (0,1)$. Otherwise,*

$$LB := \sum_{i=1}^{N} w_i S_i(0)\Phi(r_i(m_i) - z_2) - Ke^{-rT}\Phi(-z_2) \tag{4.3.26}$$

*if the $\rho_i$ are all of positive sign and*

$$LB := \sum_{i=1}^{N} w_i S_i(0)\left(\Phi(z_1 - r_i(m_i)) + \Phi(r_i(m_i) - z_2)\right) - Ke^{-rT}(\Phi(z_1) + \Phi(-z_2)) \tag{4.3.27}$$

*otherwise where, for every $i \in \{1,\dots,N\}$, $r_i : [0,h_i] \to \mathbb{R}$ satisfies*

$$r_i(t) = \rho_i(t)\sqrt{\int_0^t g_i^2(u,\phi_i(\delta_i(u)))\,du} \tag{4.3.28}$$

*and $z_1 \leq z_2$ solve the following equation in $z$.*

$$\sum_{i=1}^{N} w_i S_i(0) e^{rT - \frac{1}{2}r_i^2(m_i) + r_i(m_i)z} - K = 0. \tag{4.3.29}$$

*Proof.* The case where $f \geq 0$ is trivial. In the case where $f(v) < 0$ for some $v$, we see from Proposition 50 that $f(v) = 0$ has one solution in $(0,1)$ if the $\rho_i$ are of the same sign and two otherwise. By setting $z_i = \Phi^{-1}(v_i)$ for each $i$, we obtain the solutions to equation (4.3.29) (where the case with $\rho_i > 0$ for every $i$ is analogous to setting $z_1 = -\infty$). Let $z_1$ and $z_2$ solve equation (4.3.29) and set $v = \Phi(z)$. Then, we can write the stop-loss transform of $S^l$ in the following way.

$$\begin{aligned}\Psi(S^l, K) &= \mathbf{E}^{\mathbf{Q}}\left[\left(\sum_{i=1}^{N} w_i S_i(0) e^{rT-\frac{1}{2}r_i^2(m_i)+r_i(m_i)Z} - K\right)\mathbf{1}_{\{Z\in(-\infty,z_1)\cup(z_2,\infty)\}}\right] \\ &= \sum_{i=1}^{N} w_i S_i(0) e^{rT-\frac{1}{2}r_i^2(m_i)}\left(\int_{-\infty}^{z_1} e^{r_i(m_i)z}\varphi(z)\,dz + \int_{z_2}^{\infty} e^{r_i(m_i)z}\varphi(z)\,dz\right) \\ &\qquad - K(\Phi(z_1) + \Phi(-z_2))\end{aligned}$$

Therefore, we obtain equation (4.3.27). □

# Chapter 5

# Stability and Convergence of Numerical Schemes

In Chapters 3–4, we introduce stochastic delay differential equations and subsequently demonstrate how one obtains upper and lower bounds for arithmetic options. In Chapter 6, we will explore how these results are used in practice by comparing a selection of Asian and basket options to their corresponding Monte Carlo estimates. In order to do this, we need to consider relevant numerical schemes for the underlying SDDEs. We attempt to address two important issues concerning SDDEs and their numerical approximations.

A1. Under what conditions will solutions to a given SDDE remain stable: that is, a solution to the SDDE will remain finite for all $t \in [0, \infty)$?

A2. What additional conditions are required for the corresponding numerical scheme to achieve stability?

A3. In what sense and under which conditions will the numerical scheme for a SDDE converge to its true value as the step size $\Delta$ tends to zero?

In Section 5.1 we address Questions A1–A2 in terms of almost sure exponential stability. In other words, we explore the conditions for which any solution to a SDDE is almost surely bounded by an exponentially decaying function. This ensures that such systems tend to zero as $t \to \infty$ and thus preclude the output of infinite values. Figure 5.1 demonstrates this phenomenon for the stochastic process $X(t) = e^{-t}(\sin t + e^{B(t)})$, using two sample paths. Both paths tend to zero exponentially as $t \to \infty$ and can indeed be bounded from above by an exponentially decaying function.

In Section 5.2, we discuss mean-square convergence (that is, convergence in $L^2$) of a numerical approximation of an SDDE to its actual solution. One result of using such an approach is that this permits us to use a much larger class of functions for our delay parameter $\delta$. We observe in Section 5.1 that $\delta$ must be strictly increasing, which precludes use of step-functions for $\delta$, as considered in the previous chapters. This is shown not to be an issue in Section 5.2.

As before, let us consider the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ and assume that the stochastic process $\{B(t)\}_{t\geq 0}$ is a one-dimensional standard Brownian motion under the probability measure $\mathbf{P}$. Throughout this chapter, we are primarily concerned with the following types of stochastic process.

**Definition 52.** *Define the functions $f, g : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Then, we assume that the $n$-dimensional process $\{X(t)\}_{t\geq 0}$ satisfies the following general stochastic delay differential equation.*

$$dX(t) = f(t, X(t), X(\delta(t)))\,dt + g(t, X(t), X(\delta(t)))\,dB(t). \tag{5.0.1}$$

*Throughout this chapter, $|\cdot|$ denotes the standard Euclidean $L^2$-norm in $\mathbb{R}^n$.*

Let $\mathcal{F}_t$ denote the $\sigma$-algebra generated by $\{X(u) : u \leq t\}$, under the usual conditions of completeness and right-continuity, for all $t \geq 0$. Any solution $X$ to the SDDE (5.0.1) is also said to satisfy $X(t) = \phi(t)$, for a given bounded, continuous $\mathcal{F}_0$-measurable function $\phi$, whenever $t \in [-\tau, 0]$. We assume throughout that the deterministic delay function $\delta$ is piecewise continuous and, for all $t \geq 0$, there exists a positive constant $\underline{k}$ such that,

$$-\tau \wedge (t - \underline{k}\tau) \leq \delta(t) \leq t. \tag{5.0.2}$$

We will observe, in Section 5.2, that the above conditions on $\delta$ are sufficient to ensure mean-square convergence of any solution of the SDDE (5.0.1) to a corresponding Euler-Maruyama numerical approximation. On the other hand, we strengthen the condition on $\delta$ slightly to demonstrate the version of stability, both for any solution to (5.0.1) and its numerical scheme, in Section 5.1 below.

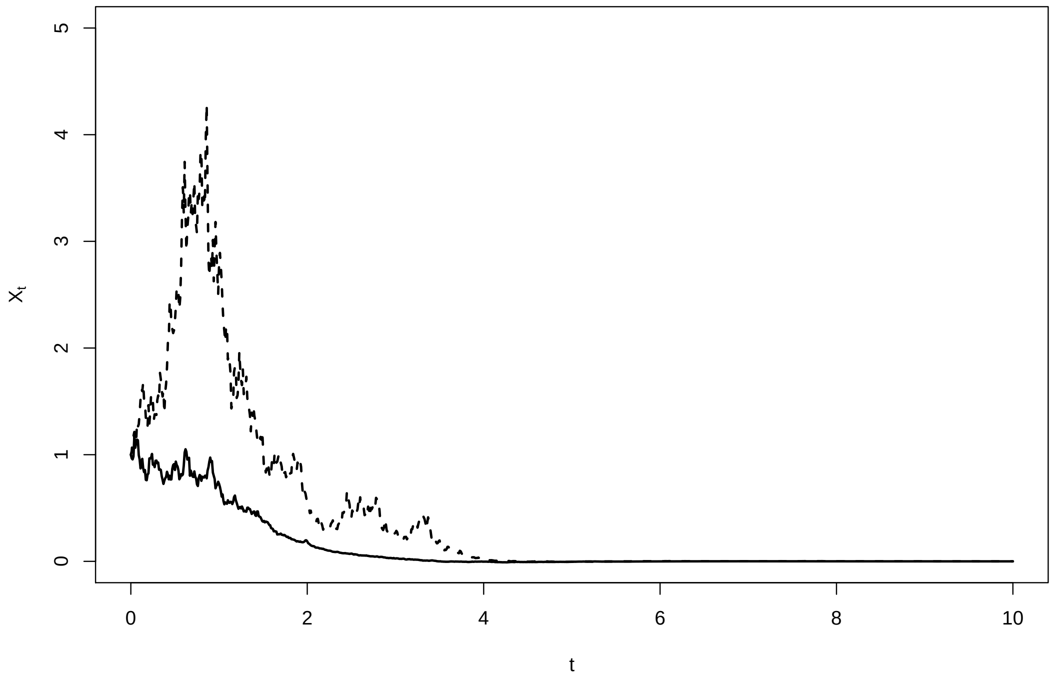


Figure 5.1: Exponential stability of the process $X(t) = e^{-t}(\sin t + e^{B(t)})$.

## 5.1 Exponential stability of SDDEs

In order to continue within this section, we impose an additional condition upon the delay function used in our SDDEs. This is defined in the following way

**Definition 53.** *Let* $\{X(t)\}_{t\geq 0}$ *be a process that satisfies the SDDE* (5.0.1) *under the assumption that the delay function* $\delta : \mathbb{R}_+ \to \mathbb{R}$ *is a strictly increasing, differentiable function at all but a countable set of points, satisfying*

$$\underline{\Lambda} := \inf_{t\geq 0} \delta'(t) > 0, \tag{5.1.1}$$

*where at any nondifferentiable point a, we take* $\delta'(a)$ *to mean the minimum of the two single-sided derivatives at the point a:*

$$\delta'(a) = \min\left\{\lim_{x\uparrow a} \delta'(x), \lim_{x\downarrow a} \delta'(x)\right\}, \tag{5.1.2}$$

*with* $\delta'(a) = \infty$ *if a is a singular point: that is,* $\delta$ *is discontinuous at a on both sides.*

Using the approach of Wu et al. (2010), we assume that the drift and diffusion functions $f$ and $g$ satisfy $f(t, 0, 0) = g(t, 0, 0) = 0$. Furthermore, we impose the local

Lipschitz condition on both $f$ and $g$ (see Mao (1994, 1997)): for every integer $j \geq 0$, there exists constant $c_j \geq 0$ such that

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \vee |g(t,x,y) - g(t,\bar{x},\bar{y})| \leq c_j(|x-\bar{x}| + |y-\bar{y}|), \tag{5.1.3}$$

for all $t \geq 0$ and $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ such that $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq j$.

Exponential stability of the above model has been discussed extensively for non-delay stochastic differential equations, where $f(t,x,y) \equiv f(t,x)$ and similarly for $g$. An extensive treatment for SDEs is given by Mao (1994), with analysis of the corresponding Euler-Maruyama and backward Euler numerical schemes undertaken by Higham et al. (2007). In addition, Mao (1999, 2002) demonstrates almost sure exponential stability of fixed-delay models, where $\delta(t) = t - \tau$. These results are adapted by Wu et al. (2010) to cover stability of the analogous numerical schemes, and by Mao et al. (2008) in the case of SDDEs with Markovian switching.

In our case, we relax the restriction of fixed delay to a more general case. This allows us to consider a greater emphasis on particular events within the process' life-cycle. The requirement of sufficient smoothness of $\delta$ precludes analysis of discontinuous, step-function models directly. However, these can still be reasonably well approximated in our case. Including a linear lower bound for $\delta$ is also necessary for our results, but can be seen as a minor restriction as, for example, information from a company's annual report is replaced by that of a subsequent year's.

### 5.1.1 Stability of the exact solution

In order to proceed in this section, let us state the definition of almost sure exponential stability for SDDEs. We shall also quote the continuous Semimartingale Convergence Theorem from Liptser and Shiryayev (1989); we shall make particular use of this result.

**Definition 54.** *Let $X(t)$ solve the stochastic delay differential equation (5.0.1). Then $X$ is said to be* almost surely exponentially stable *if there exists a positive constant $\eta$ such that*

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \leq -\eta \ a.s., \tag{5.1.4}$$

*for any initial data $\phi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$.*

**Lemma 55** (Semimartingale Convergence Theorem)**.** *Let $\{A(t)\}_{t\geq 0}, \{U(t)\}_{t\geq 0}$ be two $\{\mathcal{F}_t\}_{t\geq 0}$-adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$ almost surely. Let*

*$\{M(t)\}_{t\geq 0}$ be a real-valued local martingale with $M(0) = 0$ almost surely and let $\zeta$ be a nonnegative $\mathcal{F}_0$-measurable random variable. Assume that $X(t)$ is nonnegative and satisfies*

$$X(t) = \zeta + A(t) - U(t) + M(t), \quad t \geq 0.$$

*If $\lim_{t\to\infty} A(t) < \infty$ almost surely, then the following results are both true almost surely.*

$$\lim_{t\to\infty} X(t) < \infty \text{ and } \lim_{t\to\infty} U(t) < \infty.$$

*That is, if $A(t)$ converges to a finite random variable, then $X(t)$ and $U(t)$ also converge to finite random variables.*

We now state the following theorem: this provides the conditions under which a solution to the SDDE (5.0.1) is almost surely exponentially stable.

**Theorem 56.** *Let $f, g$ be locally Lipschitz functions satisfying equation (5.1.3). Suppose that the delay function $\delta$ for the SDDE* (5.0.1) *is a strictly increasing function that is differentiable except at a countable set of points, according to Definition 53. Furthermore, suppose $\delta$ satisfies $t - \underline{k}\tau \leq \delta(t) \leq t$, given a constant $\underline{k} \geq 0$, for all $t \geq 0$. Let $\underline{\Lambda}$ be defined by equation (5.1.1). Assume that there exist four nonnegative constants $\lambda_1, \dots, \lambda_4$ such that*

$$2 < x, f(t,x,0) > \leq -\lambda_1 |x|^2, \tag{5.1.5}$$

$$|f(t,x,y) - f(t,x,0)| \leq \lambda_2 |y|, \tag{5.1.6}$$

$$|g(t,x,y)|^2 \leq \lambda_3 |x|^2 + \lambda_4 |y|^2, \tag{5.1.7}$$

*for all $x, y \in \mathbb{R}^n$ and $t \geq 0$. Suppose that $\lambda_1, \dots \lambda_4$ also satisfy the following.*

$$\lambda_1 > \left(1 + \frac{1}{\underline{\Lambda}}\right) \lambda_2 + \lambda_3 + \frac{\lambda_4}{\underline{\Lambda}}. \tag{5.1.8}$$

*Then any solution $X(t)$ to the SDDE (5.0.1), with $\phi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, is almost surely exponentially stable, with the property*

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \leq -\frac{\gamma}{2}, \text{ a.s.}, \tag{5.1.9}$$

*where $\gamma$ is the unique positive root of*

$$\lambda_1 - \lambda_2 - \lambda_3 - \gamma = \frac{\lambda_2 + \lambda_4}{\underline{\Lambda}} e^{\gamma k \tau}. \tag{5.1.10}$$

*Proof.* Without any loss of generality, we consider the case where $\delta \in C^1(\mathbb{R}_+; [-\tau, \infty))$ is differentiable everywhere. The following remark explains why generality is maintained in this case. Define the function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ by

$$V(x,t) = e^{\gamma t}|x|^2.$$

Then, by applying Itô's formula, we obtain

$$dV(X(t),t) = e^{\gamma t}[\gamma|X(t)|^2 + 2X(t)' f(t, X(t), X(\delta(t))) + |g(t, X(t), X(\delta(t)))|^2]\, dt + dM(t),$$

where the process $\{M(t)\}_{t \geq 0}$ is a local martingale with $M(0) = 0$ almost surely. By applying equations (5.1.5) to (5.1.7) on $\lambda_1, \ldots, \lambda_4$, we realise the following upper bound for $V(X(t),t)$.

$$\begin{aligned} V(X(t),t) \leq V(X(0),0) + (\gamma - \lambda_1 + \lambda_2 + \lambda_3) \int_0^t e^{\gamma u}|X(u)|^2 \, du \\ + (\lambda_2 + \lambda_4) \int_0^t e^{\gamma u}|X(\delta(u))|^2 \, du + M(t). \end{aligned} \tag{5.1.11}$$

Let us make the substitution $v = \delta(u)$. Recall that $\delta$ is a differentiable, strictly increasing function, and therefore invertible. We can then rewrite the above equation as

$$\begin{aligned} V(X(t),t) \leq V(X(0),0) + (\gamma - \lambda_1 + \lambda_2 + \lambda_3) \int_0^t e^{\gamma u}|X(u)|^2 \, du \\ + (\lambda_2 + \lambda_4) \int_{\delta(0)}^{\delta(t)} \frac{e^{\gamma \delta^{-1}(u)}|X(u)|^2}{\delta'(\delta^{-1}(u))} \, du + M(t). \end{aligned}$$

We can then use the fact that $t - \underline{k}\tau \leq \delta(t) \leq t$ if and only if $t \leq \delta^{-1}(t) \leq t + \underline{k}\tau$. Using this property along with the fact that $\underline{\Lambda} \leq \delta'$, we obtain

$$\begin{aligned} V(X(t),t) &\leq V(X(0),0) + \zeta \\ &+ \left(\gamma - \lambda_1 + \lambda_2 + \lambda_3 + (\lambda_2 + \lambda_4)\frac{e^{\gamma \underline{k}\tau}}{\underline{\Lambda}}\right) \int_0^t e^{\gamma u}|X(u)|^2 \, du + M(t), \end{aligned} \tag{5.1.12}$$

where, given a constant $B > 0$, $\zeta = B \int_{\delta(0)}^{0} e^{\gamma u} |X(u)|^2 \, du$ is $\mathcal{F}_0$-measurable. Define the continuous function $h : \mathbb{R}_+ \to \mathbb{R}$ by

$$h(\gamma) = \gamma - \lambda_1 + \lambda_2 + \lambda_3 + (\lambda_2 + \lambda_4) \frac{e^{\gamma \underline{k} \tau}}{\underline{\Lambda}}.$$

It is straightforward to check that $h'(\gamma) > 0$ for all $\gamma \geq 0$. Furthermore, from the requirement given by equation (5.1.8), we see that $h(0) < 0$. Therefore, there exists a constant $\gamma^* > 0$ such that $h(\gamma^*) = 0$. By setting $\gamma = \gamma^*$, which satisfies the root of equation (5.1.10), we obtain the following upper bound for $V(X(t), t)$:

$$V(X(t), t) = e^{\gamma t} |X(t)|^2 \leq |X(0)|^2 + \zeta + M(t).$$

From the Semimartingale Convergence Theorem, it is clear that $t \mapsto V(X(t), t)$ converges to a finite random variable. By rearrangement, we then obtain

$$\limsup_{t \to \infty} \frac{1}{t} \log |X(t)| = \limsup_{t \to \infty} \frac{1}{2t} (\log V(X(t), t) - \gamma t) \leq -\frac{\gamma}{2}$$

almost surely. This completes the proof. □

*Remark.* The proof for Theorem 56 can be extended to the scenario where $\delta$ is not everywhere differentiable in a straightforward way, with only a minor change needed. Suppose that $\delta$ is differentiable except within the countable set $\Gamma := \{0, t_1, t_2, \dots\} \subset [0, \infty)$. Set $t_0 = 0$ and let $\Gamma_t = \Gamma \cap [0, t]$ for a given $t \geq 0$. Then, noting that $[0, t] = \bigcup_{i=1}^{\infty} [t_{i-1}, t_i \wedge t) \cup \{t\}$ consists of a union of disjoint intervals, we can rewrite the second integral in equation (5.1.11) as

$$\int_0^t e^{\gamma u} |X(\delta(u))|^2 \, du = \sum_{i=1}^{\infty} \int_{t_{i-1}}^{t_i \wedge t} e^{\gamma u} |X(\delta(u))|^2 \, du.$$

By definition of $\Gamma$, it follows that $\delta(u)$ is differentiable for all $u \in (t_{i-1}, t_i)$. We therefore have

$$\sum_{i=1}^{\infty} \int_{t_{i-1}}^{t_i \wedge t} e^{\gamma u} |X(\delta(u))|^2 \, du = \sum_{i=1}^{\infty} \int_{\delta(t_{i-1})}^{\delta(t_i \wedge t)} \frac{e^{\gamma \delta^{-1}(u)} |X(u)|^2}{\delta'(\delta^{-1}(u))} \, du.$$

As before, we make use of the fact that $u - \underline{k} \tau \leq \delta(u) \leq u$ if and only if $u \leq \delta^{-1}(u) \leq u + \underline{k} \tau$, since $\delta$ is strictly increasing. Furthermore, by definition of $\underline{\Lambda}$ and using the disjoint interval decomposition above, we realise the following.

$$\sum_{i=1}^{\infty}\int_{\delta(t_{i-1})}^{\delta(t_i\wedge t)}\frac{e^{\gamma\delta^{-1}(u)}|X(u)|^2}{\delta'(\delta^{-1}(u))}\,du \le \sum_{i=1}^{\infty}\int_{\delta(t_{i-1})}^{\delta(t_i\wedge t)}\frac{e^{\gamma\underline{k}\tau+\gamma u}|X(u)|^2}{\underline{\Lambda}}\,du$$
$$\le \frac{e^{\gamma\underline{k}\tau}}{\underline{\Lambda}}\int_{\delta(0)}^{\delta(t)} e^{\gamma u}|X(u)|^2\,du.$$

The right-hand side of this equation is then incorporated in equation (5.1.12). We can then continue the proof of Theorem 56 as before.

### 5.1.2 Stability of the Euler-Maruyama approximation

In Section 5.1.1, we investigate the conditions under which a solution to the SDDE (5.0.1) achieves exponential stability. In practice, an explicit closed-form equation for any such solution may not be possible and so the use of numerical methods become necessary. We therefore attempt to address the following important issue concerning numerical solutions to SDDEs. This follows from the analogous question posed by Wu et al. (2010) for fixed-delay models and by Higham et al. (2003) in the case of mean-square stability of SDEs.

- Suppose a solution exists to the SDDE (5.0.1) and is exponentially stable. Will a corresponding numerical method be able to reproduce exponential stability? What extra conditions are required in this case?

Let us choose a step size $\Delta > 0$ such that the delay parameter satisfies $\tau = m\Delta$ for some integer $m \in \mathbb{N}$. Furthermore, let us define the function $h : \mathbb{N}\cup\{0\} \to \mathbb{R}$ by

$$h(i) = \frac{\delta(i\Delta)}{\Delta}. \tag{5.1.13}$$

By applying the Euler-Maruyama (EM) method to equation (5.0.1), we achieve the following sequence of random variables as an approximation of $\{X(t)\}_{t\ge 0}$.

$$\begin{cases} X_k = \phi(k\Delta), & k<0,\\ X_{k+1} = X_k + f(k\Delta, X_k, X_{h(k)})\Delta + g(k\Delta, X_k, X_{h(k)})\Delta B_k, & k\in\mathbb{N}. \end{cases} \tag{5.1.14}$$

In this model, $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$ denotes the Brownian increment under the probability measure $\mathbf{P}$. In the case where $h(k)$ is not an integer, we use the following crude approximation for $X_{h(k)}$, for simplicity.

$$X_{h(k)} \simeq X_{\lfloor h(k) \rfloor}. \tag{5.1.15}$$

We state the following, Proposition 57. We demonstrate two examples under which this result holds, the first of which is typically used for fixed-delay SDDEs. These examples will help give an outline of the proof, which we then subsequently provide. We will make use of this result later on.

**Proposition 57.** *Let $C > 0$ be a positive constant. Let the delay function $\delta$ satisfy the conditions outlined within Section 5.1.1, where $\underline{\Lambda} > 0$ is defined in the following way. As in Definition 53, $\delta'(a)$ denotes the minimum single-sided derivative of $\delta$ at any non-differentiable $a \in \mathbb{R}_+$ (and is set to $\infty$ if this is undefined).*

$$\underline{\Lambda} := \inf_{t \geq 0} \delta'(t) \leq \min_{i \geq 0} h'(i). \tag{5.1.16}$$

*Set $L = (\lceil 1/\underline{\Lambda} \rceil)^{-1}$. Then, we obtain the following upper bound, where $\xi$ is an $\mathcal{F}_0$-measurable random variable.*

$$\sum_{i=0}^{k-1} C^{(i+1)\Delta} |X_{h(i)}|^2 \leq \xi + \frac{C^{\underline{k}\tau+\Delta}}{L} \sum_{i=1}^{k-1} C^{(i+1)\Delta} |X_i|^2. \tag{5.1.17}$$

**Example 58.** *Let $\delta : \mathbb{R}_+ \to [-\tau, \infty)$ be the fixed-delay equation given by*

$$\delta(t) = t - \tau. \tag{5.1.18}$$

*By definition of $h$ and $m$, we see that $h(i) = i - m$. Further, $\delta$ is a continuously differentiable function with $\Lambda = L = 1$ and $\underline{k} = 1$. It therefore follows that we have*

$$\begin{aligned}
\sum_{i=0}^{k-1} C^{(i+1)\Delta} |X_{h(i)}|^2 &= \sum_{i=0}^{m} C^{(i+1)\Delta} |X_{i-m}|^2 + \sum_{i=m+1}^{k-1} C^{(i+1)\Delta} |X_{i-m}|^2 \\
&= \xi + \sum_{i=1}^{k-m-1} C^{(i+m+1)\Delta} |X_i|^2 \\
&= \xi + C^{\tau} \sum_{i=1}^{k-m-1} C^{(i+1)\Delta} |X_i|^2 \\
&\leq \xi + C^{\tau} \sum_{i=1}^{k-1} C^{(i+1)\Delta} |X_i|^2,
\end{aligned}$$

*which satisfies the right-hand side of equation* (5.1.17).

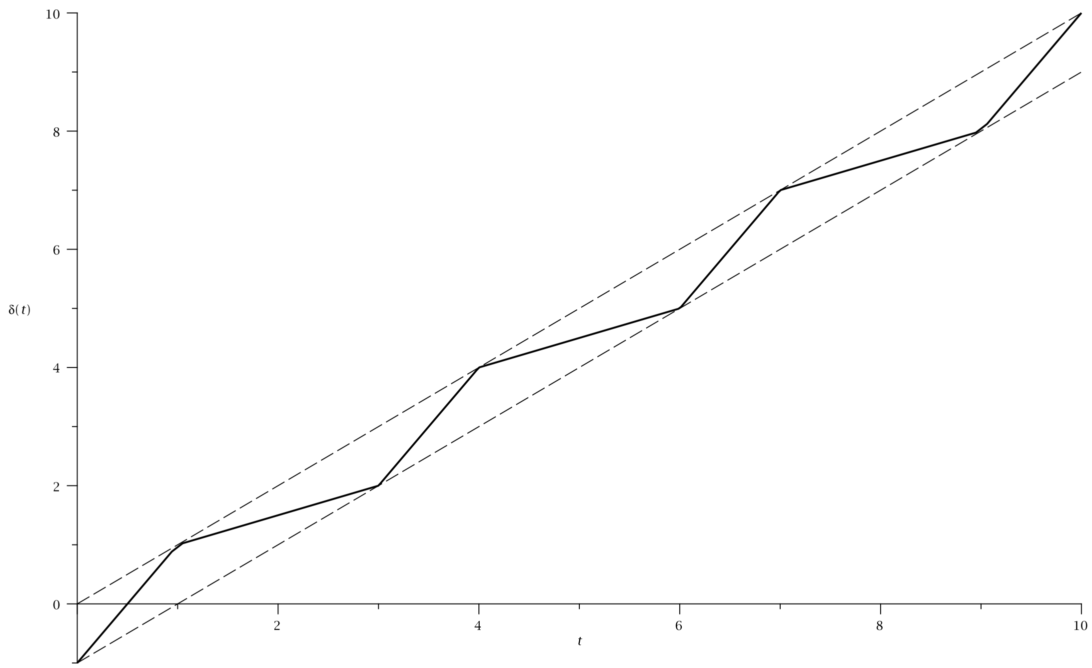


Figure 5.2: Graph of variable-delay function used in Example 59 over time.

**Example 59.** *Consider the following delay function.*

$$\delta(t) = \sum_{j=0}^{\infty} \left( (2t - (3j+1)\tau)\mathbf{1}_{[3j\tau,(3j+1)\tau)}(t) + \frac{1}{2}(t + (3j+1)\tau)\mathbf{1}_{[(3j+1)\tau,3(j+1)\tau)}(t) \right). \tag{5.1.19}$$

*With a model implementing the above $\delta$, we vary between a rapidly increasing delay function and one that increases more slowly. A consequence of such a model is an increased emphasis over the value of the corresponding stochastic process much further back in time when $t \in [(3j+1)\tau, 3(j+1)\tau)$, where $j$ is a nonnegative integer. Figure 5.2 illustrates how this particular delay function behaves over $t$, with $\tau = 1$ for simplicity. In this plot, $\delta$ is represented by the bold line and one clearly sees that $t - \tau \le \delta(t) \le t$. It then follows that $\delta$ is differentiable at all but a countable set of points $\{0, \tau, 3\tau, 4\tau, 6\tau, 7\tau, \dots\}$, that $\Lambda = L = \frac{1}{2}$ and $\underline{k} = 1$.*

*In order to consider the behaviour of the left-hand side of equation* (5.1.17)*, choose $\alpha \in \mathbb{N}$ to be such that $\alpha \bmod 3 = 1$. Then, for any $t \in [(\alpha-1)\tau, \alpha\tau)$, $\delta(t) = 2t - \alpha\tau$ and so $h$ satisfies*

$$h(i) = 2i - \alpha m,$$

*where* $i \in \{(\alpha-1)m, (\alpha-1)m+1, \ldots, \alpha m - 1\}$. *Therefore,*

$$\begin{aligned}
&\sum_{i=(\alpha-1)m}^{\alpha m-1} C^{(i+1)\Delta}|X_{h(i)}|^2 \\
&= C^{\Delta}\sum_{i=(\alpha-1)m}^{\alpha m-1} C^{i\Delta}|X_{2i-\alpha m}|^2 \\
&= C^{\Delta}\Bigg(C^{(\alpha-1)m\Delta}|X_{(\alpha-2)m}|^2 + C^{((\alpha-1)m+1)\Delta}|X_{(\alpha-2)m+2}|^2 \\
&\qquad + C^{((\alpha-1)m+2)\Delta}|X_{(\alpha-2)m+4}|^2 + \cdots + C^{(\alpha m-1)\Delta}|X_{\alpha m-2}|^2\Bigg) \\
&= C^{\tau+\Delta}\Bigg(C^{(\alpha-2)m\Delta}|X_{(\alpha-2)m}|^2 + C^{((\alpha-2)m+1)\Delta}|X_{(\alpha-2)m+2}|^2 \\
&\qquad + C^{((\alpha-2)m+2)\Delta}|X_{(\alpha-2)m+4}|^2 + \cdots + C^{((\alpha-1)m-1)\Delta}|X_{\alpha m-2}|^2\Bigg) \\
&\le C^{\tau+\Delta}\Bigg(C^{(\alpha-2)m\Delta}|X_{(\alpha-2)m}|^2 + C^{((\alpha-2)m+2)\Delta}|X_{(\alpha-2)m+2}|^2 \\
&\qquad + C^{((\alpha-2)m+4)\Delta}|X_{(\alpha-2)m+4}|^2 + \cdots + C^{(\alpha m-2)\Delta}|X_{\alpha m-2}|^2\Bigg) \\
&= C^{\tau}\sum_{i=(\alpha-2)m}^{\alpha m-1} C^{(i+1)\Delta}|X_i|^2 \mathbf{1}_{\{i-(\alpha-2)m \bmod 2 = 0\}} \\
&\le C^{\tau}\sum_{i=(\alpha-2)m}^{\alpha m-1} C^{(i+1)\Delta}|X_i|^2.
\end{aligned}$$

*If* $t$ *does not fall within the domain specified above, then it must be the case that* $t \in [\alpha\tau, (\alpha+2)\tau)$ *where* $\alpha - 1$ *is an integer multiple of 3. For such* $t$, $\delta(t) = (t+\alpha\tau)/2$ *and so*

$$h(i) = \frac{i+\alpha m}{2},$$

*where* $i \in \{\alpha m, \alpha m+1, \ldots, (\alpha+2)m-1\}$. *Using the numerical scheme rule described by* *equation* (5.1.15), *we realise the following outcome, making note that* $\lfloor \alpha m + \frac{1}{2} \rfloor = \alpha m$.

$$\begin{aligned}
&\sum_{i=\alpha m}^{(\alpha+2)m-1} C^{(i+1)\Delta}|X_{h(i)}|^2 \\
&= C^{\Delta}\Bigg( C^{\alpha m\Delta}|X_{\alpha m}|^2 + C^{(\alpha m+1)\Delta}|X_{\alpha m+\frac{1}{2}}|^2 + C^{(\alpha m+2)\Delta}|X_{\alpha m+1}|^2 \\
&\qquad + C^{(\alpha m+3)\Delta}|X_{\alpha m+\frac{3}{2}}|^2 + \cdots + C^{((\alpha+2)m-2)\Delta}|X_{(\alpha+1)m-1}|^2 \\
&\qquad + C^{((\alpha+2)m-1)\Delta}|X_{(\alpha+1)m-\frac{1}{2}}|^2 \Bigg) \\
&= C^{\tau+\Delta}\Bigg( C^{(\alpha-1)m\Delta}|X_{\alpha m}|^2 + C^{((\alpha-1)m+1)\Delta}|X_{\alpha m}|^2 \\
&\qquad + C^{((\alpha-1)m+2)\Delta}|X_{\alpha m+1}|^2 + C^{((\alpha-1)m+3)\Delta}|X_{\alpha m+1}|^2 \\
&\qquad + \cdots + C^{((\alpha+1)m-2)\Delta}|X_{(\alpha+1)m-1}|^2 \\
&\qquad + C^{((\alpha+1)m-1)\Delta}|X_{(\alpha+1)m-1}|^2 \Bigg) \\
&\le C^{\tau+\Delta}\Bigg( 2C^{\alpha m\Delta}|X_{\alpha m}|^2 + 2C^{(\alpha m+1)\Delta}|X_{\alpha m+1}|^2 \\
&\qquad + \cdots + 2C^{((\alpha+1)m-1)\Delta}|X_{(\alpha+1)m-1}|^2 \Bigg) \\
&= 2C^{\tau} \sum_{i=\alpha m}^{(\alpha+2)m-1} C^{(i+1)\Delta}|X_i|^2.
\end{aligned}$$

*We can easily check that the positive half line* $\mathbb{R}_+$ *is made up of disjoint unions of the two disjoint sets* $[(\alpha-1)\tau, \alpha\tau) \cup [\alpha\tau, (\alpha+2)\tau)$. *By combining the two cases above, we then obtain the following inequality*

$$\sum_{i=0}^{k-1} C^{(i+1)\Delta}|X_{h(i)}|^2 \le \xi + 2C^{\tau} \sum_{i=1}^{k-1} C^{(i+1)\Delta}|X_i|^2,$$

*which agrees precisely with equation* (5.1.17).

*Proof of Proposition 57.* Choose a step size $\Delta \in (0,1)$. For every $p \in \mathbb{N}$, define the integers $j_p$ and $k_p$ in the following way.

$$j_p = \min\{i : h(i) \ge p\}, \qquad k_p = j_p \wedge k.$$

Let $I_p = \{j_p, j_p+1, \ldots, j_{p+1}-1\}$. Then, for every $i \in I_p$, we immediately see that $\lfloor h(i) \rfloor = p$. Using the approximation given by equation (5.1.15), we can then write the left-hand side of equation (5.1.17) as

$$\sum_{i=0}^{k-1} C^{(i+1)\Delta}|X_{h(i)}|^2 = \xi + \sum_{p=1}^{k-1} \sum_{i=k_p}^{k_{p+1}-1} C^{(i+1)\Delta}|X_p|^2. \tag{5.1.20}$$

This is true since, for any $i < j_1$, $h(i) < 1$ and so $\xi$ is $\mathcal{F}_0$-measurable. For each $p \in \mathbb{N}$, define $\underline{\Lambda}_p$ in the following way.

$$\underline{\Lambda}_p = \inf_{t \in [j_p\Delta, j_{p+1}\Delta)} \delta'(t) \geq \underline{\Lambda}.$$

Since $\delta$ and $h$ are strictly increasing functions, it is immediately clear that $h(i) \geq h_p(i)$, for every $i \geq j_p$, where

$$h_p(i) = \underline{\Lambda}_p(i - j_p) + p.$$

Furthermore, for every $i \geq j_p$, $h_p(i) \geq p$ with $h_p(i) \geq p+1$ if and only if

$$i \geq \frac{1}{\underline{\Lambda}_p} + j_p.$$

Define $L_p = (\lceil 1/\underline{\Lambda}_p \rceil)^{-1}$. Then, if $1/\underline{\Lambda}_p \in \mathbb{N}$, it follows that $1/\underline{\Lambda}_p + j_p \in \mathbb{N}$ and so $1/\underline{\Lambda}_p + j_p - 1 = 1/L_p + j_p - 1$ is the largest integer $i$ such that $\lfloor h_p(i) \rfloor = p$. Indeed, if $1/\underline{\Lambda}_p \notin \mathbb{N}$, then

$$\left\lceil \frac{1}{\underline{\Lambda}_p} \right\rceil + j_p > \frac{1}{\underline{\Lambda}_p} + j_p > \left\lceil \frac{1}{\underline{\Lambda}_p} \right\rceil + j_p - 1,$$

and so $1/L_p + j_p - 1$ is also the largest integer $i$ such that $\lfloor h_p(i) \rfloor = p$. Since $h_p(i) \leq h(i)$ for all $i \geq j_p$, it follows that we have

$$|\{j_p, \ldots, j_{p+1}-1\}| \leq \left| \left\{ j_p, \ldots, \frac{1}{L_p} + j_p - 1 \right\} \right| = \frac{1}{L_p} \leq \frac{1}{L},$$

by definition of $L$, where $|\{a, \ldots, b\}|$ denotes the cardinality of the set $\{a, \ldots, b\}$, given by

$$|\{a, \ldots, b\}| = a \vee b - a \wedge b + 1.$$

In addition, recall that $i - \underline{k}m \leq h(i) \leq i$ and $p \leq h(i) < p+1$ for all $i \in I_p$. Therefore, we obtain

$$\begin{aligned}\sum_{i=k_p}^{k_{p+1}-1} C^{(i+1)\Delta}|X_p|^2 &= |X_p|^2 C^{\underline{k}m\Delta} \sum_{i=k_p}^{k_{p+1}-1} C^{(i-\underline{k}m+1)\Delta} \\ &\le |X_p|^2 C^{\underline{k}m\Delta} \sum_{i=k_p}^{k_{p+1}-1} C^{(h(i)+1)\Delta} \\ &< |X_p|^2 C^{\underline{k}m\Delta} C^{(p+2)\Delta} \sum_{i=k_p}^{k_{p+1}-1} 1 \\ &\le \frac{C^{\underline{k}\tau+\Delta}}{L} C^{(p+1)\Delta}|X_p|^2.\end{aligned}$$

By substituting both sides of the above statement into equation (5.1.20), we obtain

$$\sum_{i=0}^{k-1} C^{(i+1)\Delta}|X_{h(i)}|^2 \le \xi + \frac{C^{\underline{k}\tau+\Delta}}{L} \sum_{p=1}^{k-1} C^{(p+1)\Delta}|X_p|^2,$$

which coincides with equation (5.1.17). This completes the proof. □

In order to continue, we state the definition of almost sure exponential stability of numerical approximations of SDDEs. We also provide the discrete version of the Semimartingale Convergence Theorem. This is quoted in a number of papers; see, for example Wu et al. (2010).

**Definition 60.** *The approximate solution $X_k$ to equation (5.1.14) is said to be* almost surely exponentially stable *if there exists a positive constant $\bar{\eta}$ such that*

$$\limsup_{k\to\infty} \frac{1}{k\Delta} \log |X_k| \le -\bar{\eta} \ a.s., \tag{5.1.21}$$

*for any bounded variables $\phi(k\Delta), k < 0$.*

**Lemma 61** (Discrete Semimartingale Convergence Theorem)**.** *Let $\{A_i\}$ and $\{U_i\}$ be two sequences of nonnegative random variables such that $A_i$ and $U_i$ are $\mathcal{F}_{k-1}$-measurable for all $i \in \mathbb{N}$ and $A_0 = U_0 = 0$ almost surely. Let $\{M_i\}$ be a local martingale in $\mathbb{R}$ with $M_0 = 0$ almost surely. Let $\zeta$ be a nonnegative $\mathcal{F}_0$-measurable random variable. Assume that $\{X_i\}$ is a nonnegative semimartingale satisfying*

$$X_i = \zeta + A_i - U_i + M_i.$$

*If* $\lim_{i\to\infty} A_i < \infty$ *almost surely then the following results are both true almost surely.*

$$\lim_{i\to\infty} X_i < \infty \text{ and } \lim_{i\to\infty} U_i < \infty.$$

*In other words,* $X_i$ *and* $U_i$ *both converge to finite random variables.*

In this section, we aim to show that the EM method reproduces almost sure exponential stability of the exact solution given by equation (5.0.1), under the possible requirement of extra conditions. Wu et al. (2010) state that any solution satisfying the requirements on $\lambda_1, \ldots, \lambda_4$ in that paper exhibits an exponentially stable fixed-delay SDDE under the additional constraint that $f$ satisfies the linear growth condition. Our next result demonstrates a similar result in the general-delay case, but with a slight strengthening of the condition given by equation (5.1.8). This latter requirement is made necessary by the result given in Proposition 57.

**Theorem 62.** *Suppose that* $f$ *and* $g$ *are locally Lipschitz continuous functions which also satisfy the conditions given by equations (5.1.5) to (5.1.7). Let L be as defined in Proposition 57 and assume that the nonnegative constants* $\lambda_1, \ldots, \lambda_4$ *also meet the following condition.*

$$\lambda_1 > \left(1 + \frac{1}{L}\right)\lambda_2 + \lambda_3 + \frac{\lambda_4}{L}. \tag{5.1.22}$$

*Furthermore, assume that the drift function* $f$ *satisfies the linear growth condition: there exists a constant* $K > 0$ *such that, for any* $x, y \in \mathbb{R}^n$ *and* $t \geq 0$*, we have*

$$|f(t,x,y)|^2 \leq K(|x|^2 + |y|^2). \tag{5.1.23}$$

*Let* $\gamma > 0$ *be the root of the following equation*

$$\lambda_1 - \lambda_2 - \lambda_3 - \gamma = \frac{\lambda_2 + \lambda_4}{L} e^{\gamma \underline{k} \tau}. \tag{5.1.24}$$

*Choose any* $\varepsilon \in (0, \gamma/2)$*. Then, there exists a* $\Delta^* > 0$ *such that if* $\Delta < \Delta^*$*, then for any given bounded* $\mathcal{F}_0$*-measurable random variables,* $\phi(k\Delta)$*,* $k < 0$*, the EM numerical scheme given by equation (5.1.14) satisfies*

$$\limsup_{k\to\infty} \frac{1}{k\Delta} \log |X_k| \leq -\frac{\gamma}{2} + \varepsilon, \ \ a.s. \tag{5.1.25}$$

*Proof.* Analogous to the equivalent proof given by Wu et al. (2010), we obtain

$$
\begin{aligned}
|X_{k+1}|^2 =& < X_k + f(k\Delta, X_k, X_{h(k)})\Delta + g(k\Delta, X_k, X_{h(k)})\Delta B_k, \\
& X_k + f(k\Delta, X_k, X_{h(k)})\Delta + g(k\Delta, X_k, X_{h(k)})\Delta B_k > \\
=& |X_k|^2 + |f(k\Delta, X_k, X_{h(k)})|^2\Delta^2 + |g(k\Delta, X_k, X_{h(k)})|^2\Delta \\
& + 2 < X_k, f(k\Delta, X_k, X_{h(k)}) > \Delta + M_k \\
\leq& |X_k|^2 + K|X_k|^2\Delta^2 + K|X_{h(k)}|^2\Delta^2 - \lambda_1|X_k|^2\Delta \\
& + \lambda_2(|X_k|^2 + |X_{h(k)}|^2)\Delta + \lambda_3|X_k|^2\Delta + \lambda_4|X_{h(k)}|^2\Delta + M_k^{(1)},
\end{aligned}
$$

where

$$
\begin{aligned}
M_k^{(1)} =& |g(k\Delta, X_k, X_{h(k)})|^2(\Delta B_k^2 - \Delta) \\
& + 2 < X_k + f(k\Delta, X_k, X_{h(k)})\Delta, g(k\Delta, X_k, X_{h(k)})\Delta B_k >
\end{aligned}
$$

is a local martingale difference under $\mathbf{P}$. Making use of the following expression, for any $C > 1$,

$$
C^{(k+1)\Delta}|X_{k+1}|^2 - C^{k\Delta}|X_k|^2 = C^{(k+1)\Delta}(|X_{k+1}|^2 - |X_k|^2) + (C^{(k+1)\Delta} - C^{k\Delta})|X_k|^2,
$$

we realise the following result.

$$
\begin{aligned}
C^{(k+1)\Delta}|X_{k+1}|^2 - C^{k\Delta}|X_k|^2 \leq& [-\lambda_1\Delta + \lambda_2\Delta + \lambda_3\Delta + (1 - C^{-\Delta}) \\
& + K\Delta^2]C^{(k+1)\Delta}|X_k|^2 + (\lambda_2\Delta + \lambda_4\Delta + K\Delta^2)C^{(k+1)\Delta}|X_{h(k)}|^2 + M_k^{(1)}.
\end{aligned}
$$

By summing both sides of this inequality over $i = 0, 1, \ldots, k-1$ and making use of Proposition 57, one obtains

$$
\begin{aligned}
C^{k\Delta}|X_k|^2 \leq |X_0|^2 + \zeta + \Bigg[& -\lambda_1\Delta + \lambda_2\Delta + \lambda_3\Delta + (1 - C^{-\Delta}) + K\Delta^2 \\
& + (\lambda_2\Delta + \lambda_4\Delta + K\Delta^2)\frac{C^{\underline{k}\tau+\Delta}}{L}\Bigg] \sum_{i=0}^{k-1} C^{(i+1)\Delta}|X_i|^2 + M_k^{(2)},
\end{aligned}
$$

where $M_k^{(2)}$ is another local martingale difference and $\zeta$ is $\mathcal{F}_0$-measurable.

Let us define the continuous function $\psi : [1, \infty) \to \mathbb{R}$ in the following way.

$$\psi(C) = (\lambda_2 + \lambda_4 + K\Delta)\frac{\Delta C^{\underline{k}\tau+2\Delta}}{L} + (1 - (\lambda_1 - \lambda_2 - \lambda_3)\Delta + K\Delta^2)C^{\Delta} - 1.$$

From equation (5.1.22), we see that $\lambda_1 - \lambda_2 - \lambda_3 > 0$. Hence, we can choose $\Delta_1^* > 0$ such that $1 - (\lambda_1 - \lambda_2 - \lambda_3)\Delta + K\Delta^2 > 0$, for all $\Delta < \Delta_1^*$. For such $\Delta$, we see that $\psi'(C) > 0$ for all $C \geq 1$. Furthermore, we also obtain

$$\psi(1) = -\left(\lambda_1 - \left(1 + \frac{1}{L}\right)\lambda_2 - \lambda_3 - \frac{\lambda_4}{L} - K\left(1 + \frac{1}{L}\right)\Delta\right)\Delta.$$

By making the following setting

$$\Delta_2^* = \frac{\lambda_1 - (1 + \frac{1}{L})\lambda_2 - \lambda_3 - \frac{\lambda_4}{L}}{K(1 + \frac{1}{L})} > 0,$$

It follows that $\psi(1) < 0$ for all $\Delta < \Delta_2^*$. Hence, for $\Delta < \Delta_1^* \wedge \Delta_2^*$, there exists a unique $C_\Delta^* > 1$ such that $\psi(C_\Delta^*) = 0$. By setting $C = C_\Delta^*$, we obtain

$$(C_\Delta^*)^{k\Delta}|X_k|^2 \leq |X_0|^2 + \zeta + M_k^{(2)},$$

which by the discrete Semimartingale Convergence Theorem implies

$$\limsup_{k\to\infty}(C_\Delta^*)^{k\Delta}|X_k|^2 < \infty \text{ a.s.}$$

From the equations above, it is immediately apparent that

$$(\lambda_2 + \lambda_4 + K\Delta)\frac{C^{\underline{k}\tau+\Delta}}{L} - \lambda_1 + \lambda_2 + \lambda_3 + K\Delta + \frac{(1 - C^{-\Delta})}{\Delta} = 0,$$

when $C = C_\Delta^*$. Let $\mu = \log C$ and define $\bar{\psi}$ by

$$\bar{\psi}(\mu) = (\lambda_2 + \lambda_4 + K\Delta)\frac{e^{\mu(\underline{k}\tau+\Delta)}}{L} - \lambda_1 + \lambda_2 + \lambda_3 + K\Delta + \frac{(1 - e^{-\mu\Delta})}{\Delta}.$$

Set $\mu_\Delta^* = \log C_\Delta^*$. Then, for all $\Delta < \Delta_1^* \wedge \Delta_2^*$, $\bar{\psi}(\mu_\Delta^*) = 0$. Moreover, we see the following limit as $\Delta$ tends towards zero.

$$\lim_{\Delta\downarrow 0}\bar{\psi}(\mu) = (\lambda_2 + \lambda_4)\frac{e^{\mu\underline{k}\tau}}{L} - \lambda_1 + \lambda_2 + \lambda_3 + \mu.$$

The right-hand side of this equation is also zero when $\mu = \mu^*_\Delta$. This is achieved by the value $\gamma$ given by equation (5.1.24). Hence, we see that

$$\lim_{\Delta \downarrow 0} \bar{\psi}(\gamma) = \psi(\mu^*_\Delta) = 0,$$

for all $0 \leq \Delta \leq \Delta^*_1 \wedge \Delta^*_2$. Therefore,

$$\lim_{\Delta \downarrow 0} \mu^*_\Delta = \gamma.$$

Hence, for all $\varepsilon \in (0, \gamma/2)$, there exists $\Delta^*_3 > 0$ such that, for all $\Delta < \Delta^*_3$, $\mu^*_\Delta > \gamma - 2\varepsilon$. By rewriting the results above, we have

$$\limsup_{k \to \infty} e^{\mu^*_\Delta k\Delta} |X_k|^2 < \infty \text{ a.s.}$$

Therefore, by rearranging appropriately and choosing $\Delta^* = \Delta^*_1 \wedge \Delta^*_2 \wedge \Delta^*_3$, we see that $X_k$ following the result below.

$$\limsup_{k \to \infty} \frac{1}{k\Delta} \log |X_k| \leq -\frac{\gamma}{2} + \varepsilon \text{ a.s.},$$

as required. □

### 5.1.3 Comparing rates of convergence

In Theorem 56, we describe the conditions for which a solution to an SDDE (5.0.1) is exponentially stable. This concerns the value $\underline{\Lambda}$ given by equation (5.1.1), which is determined by the minimum gradient of the delay function $\delta$. In Theorem 62, we state the additional conditions under which the corresponding Euler-Maruyama numerical scheme given by equation (5.1.14) also achieves exponential stability. In both cases, the rate of convergence to zero is bounded by $\gamma$ solving equations (5.1.10) and (5.1.24) respectively.

Intuitively, one would expect the continous solution to the SDDE (5.0.1) to converge more rapidly than its corresponding Euler-Maruyama scheme. That is, a necessary condition for the discretised scheme to exhibit exponential stability is that its actual solution is exponentially stable. Indeed, one can observe that $\underline{\Lambda} \geq L$, where $L = (\lceil 1/\underline{\Lambda} \rceil)^{-1}$ is defined above. Therefore, the following equation demonstrates that the restriction on $\lambda_1$ given by equation (5.1.22), assuming that $\lambda_2, \lambda_3$ and $\lambda_4$ are fixed, satisfies

$$\lambda_1 > \left(1 + \frac{1}{L}\right)\lambda_2 + \lambda_3 + \frac{\lambda_4}{L} \geq \left(1 + \frac{1}{\underline{\Lambda}}\right)\lambda_2 + \lambda_3 + \frac{\lambda_4}{\underline{\Lambda}}, \tag{5.1.26}$$

where the right-hand side is given by equation (5.1.8). This then suggests an additional restriction on the parameter region described by equation (5.1.22) (as well as the additional constraint on the numerical scheme imposed by the linear growth condition on $f$, given by equation (5.1.23)), when moving from the continuous to the discrete case.

In order to compare the rate of convergence of an exponentially stable system more formally, let us consider the solution and corresponding Euler-Maruyama numerical scheme for a single SDDE (5.0.1). Then, we realise the following result concerning the rate of convergence of the two methods.

**Proposition 63.** *Let $\gamma_c$ and $\gamma_d$ solve the respective equations* (5.1.10) *and* (5.1.24) *in the continuous and discrete case respectively. Then,*

$$\gamma_c \geq \gamma_d, \tag{5.1.27}$$

*and so the solution to an exponentially stable SDDE* (5.0.1) *converges to zero more rapidly than its corresponding Euler-Maruyama numerical scheme.*

*Proof.* Suppose $\gamma_c$ and $\gamma_d$ solve equations (5.1.10) and (5.1.24). Then, upon rearrangement, we immediately obtain the following equation.

$$\lambda_1 - \lambda_2 - \lambda_3 = \gamma_c + \frac{\lambda_2 + \lambda_4}{\underline{\Lambda}} e^{\gamma_c \underline{k}\tau} = \gamma_d + \frac{\lambda_2 + \lambda_4}{L} e^{\gamma_d \underline{k}\tau}.$$

Since we observe that $\underline{\Lambda} \geq L$, it then follows that we have

$$\gamma_c + \frac{\lambda_2 + \lambda_4}{\underline{\Lambda}} e^{\gamma_c \underline{k}\tau} \geq \gamma_d + \frac{\lambda_2 + \lambda_4}{\underline{\Lambda}} e^{\gamma_d \underline{k}\tau}.$$

Upon further rearrangement, we therefore obtain the following lower bound:

$$\gamma_c - \gamma_d \geq \frac{\lambda_2 + \lambda_4}{\underline{\Lambda}} (e^{\gamma_d \underline{k}\tau} - e^{\gamma_c \underline{k}\tau}).$$

Let us assume that $\gamma_d > \gamma_c$. Then, we see that $e^{\gamma_d \underline{k}\tau} - e^{\gamma_c \underline{k}\tau} > 0$ and so the right hand side of the above equation is positive since the $\lambda_i$ and $\underline{\Lambda}$ are positive. However, $\gamma_c - \gamma_d < 0$, which contradicts our assumption. Therefore, equation (5.1.27) must hold. □

*Remark.* As $\Delta \downarrow 0$, the Euler-Maruyama numerical scheme tends to the actual solution of the SDDE (5.0.1). Therefore, a final remark on the rate of convergence of the numerical scheme is that $\gamma_d$ solving equation (5.1.24) becomes close to $\gamma_c$ solving equation (5.1.10). Moreover, the parameter region described by equation (5.1.8) is the limit of (5.1.22) as $\Delta \downarrow 0$.

## 5.2 Convergence of numerical schemes for SDDEs

Within this part, we are motivated by results obtained by Mao (2003) for stochastic functional differential equations. The proofs obtained in that paper are adapted for our case involving general-delay SDDEs, considering statements and techniques covered by Mao (1997). In order to proceed, we restate the definitions of the properties of locally Lipschitz continuous and linearly growing functions, which $f, g : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ both follow. The conditions below are equivalent to equations (5.1.3) and (5.1.23) respectively, but are stated in the form given by Mao (1997) for convenient comparison to the results obtained by Mao (2003).

**Definition 64.** *The functions $f$ and $g$ are said to satisfy the* local Lipschitz condition *if, for every $j \in \mathbb{N}$, there exists a positive constant $L_j$ such that*

$$|f(t,x,y) - f(t,\bar{x},\bar{y})|^2 \vee |g(t,x,y) - g(t,\bar{x},\bar{y})|^2 \leq L_j(|x-\bar{x}|^2 + |y-\bar{y}|^2), \tag{5.2.1}$$

*for all $t \geq 0$ and $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ such that $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq j$.*

**Definition 65.** *The functions $f$ and $g$ satisfy the* linear growth condition *if there exists a constant $K > 0$ such that*

$$|f(t,x,y)|^2 \vee |g(t,x,y)|^2 \leq K(1 + |x|^2 + |y|^2). \tag{5.2.2}$$

In addition to the above statements, we assume that the initial data $\phi$ is contained within the space $L^p_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$ for some $p > 2$. In other words, $\phi(t)$ satisfies $\mathbf{E}[\|\phi\|^p] < \infty$, where

$$\|\phi\| = \sup_{-\tau \leq t \leq 0} |\phi(t)|. \tag{5.2.3}$$

Under these conditions, we can easily see that any solution to the SDDE (5.0.1) is almost surely right-continuous. From this point, we begin by establishing a $p$th order exponential upper bound for any such solution: that is, an estimate for the maximum value of $|X(t)|$ dependent on an exponentially increasing function of $t$. This result follows Theorem 5.4.1 of Mao (1997) and is stated and used by Mao (2003) for solutions to stochastic functional differential equations.

**Theorem 66.** *Let $X(t)$ be any solution to the SDDE (5.0.1) with initial data satisfying $\phi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ for some $p > 2$. Then, under the conditions of Lipschitz continuity and local growth stated above we have, for any $T > 0$,*

$$\mathbf{E}\left[\sup_{-\tau \le t \le T} |X(t)|^p\right] \le \frac{3}{2} 2^{p/2}(1 + 2^{p/2}\mathbf{E}\left[\|\phi\|^p\right])e^{CT}, \tag{5.2.4}$$

*where $C = 2p(2\sqrt{K} + (65p-1)K)$.*

*Proof.* Using Itô's formula on the function $V(x,t) = (1+2|x|^2)^{p/2}$, for any $x \in \mathbb{R}^n$, provides the following identity when $t \in [0,T]$:

$$\begin{aligned}(1+2|X(t)|^2)^{p/2} &= (1+2|\phi(0)|^2)^{p/2} \\ &\quad + 2p\int_0^t (1+2|X(u)|^2)^{(p-2)/2} X(u)' f(u, X(u), X(\delta(u)))\, du \\ &\quad + p\int_0^t (1+2|X(u)|^2)^{(p-2)/2} |g(u, X(u), X(\delta(u)))|^2\, du \\ &\quad + 2p(p-2)\int_0^t (1+2|X(u)|^2)^{(p-4)/2} |X(u)' g(u, X(u), X(\delta(u)))|^2\, du \\ &\quad + 2p\int_0^t (1+2|X(u)|^2)^{(p-2)/2} X(u)' g(u, X(u), X(\delta(u)))\, dB(u),\end{aligned}$$

where $A'$ refers to the transpose of the matrix $A$. On the other hand, note for any $K > 0$, $x, y \in \mathbb{R}^n$ and $t \ge 0$, we have

$$x' f(t,x,y) \le |x||f(t,x,y)| \le \left(\frac{\sqrt{K}}{2}|x|^2 + \frac{1}{2\sqrt{K}}|f(t,x,y)|^2\right),$$

and

$$\begin{aligned}|g(t,x,y)|^2 + \frac{2(p-2)|x' g(t,x,y)|^2}{(1+2|x|^2)} &\le \left(1 + \frac{2(p-2)|x|^2}{(1+2|x|^2)}\right)|g(t,x,y)|^2 \\ &\le (p-1)|g(t,x,y)|^2.\end{aligned}$$

We can make use of the two above statements, along with the linear growth condition on $f$ and $g$, in order to write the following upper bound.

$$
\begin{aligned}
&(1+2|X(t)|^2)^{p/2} \\
&\le 2^{(p-2)/2}(1+2^{p/2}|\phi(0)|^p) \\
&\quad + p\int_0^t (1+2|X(u)|^2)^{(p-2)/2}\Bigg(\sqrt{K}\left(1+2|X(u)|^2+|X(\delta(u))|^2\right) \\
&\qquad\qquad + (p-1)K(1+|X(u)|^2+|X(\delta(u))|^2)\Bigg)\,du \\
&\quad + 2p\int_0^t (1+2|X(u)|^2)^{(p-2)/2}X(u)'g(u,X(u),X(\delta(u)))\,dB(u).
\end{aligned}
$$

Hence, for any $s \in [0,t]$, we have

$$
\begin{aligned}
\mathbf{E}\left[\sup_{0\le s\le t}(1+2|X(s)|^2)^{p/2}\right] &\le 2^{(p-2)/2}(1+2^{p/2}\mathbf{E}\left[\|\phi\|^p\right]) \\
&\quad + C_1\int_0^t \mathbf{E}\left[\sup_{-\tau\le r\le u}(1+2|X(r)|^2)^{p/2}\right]du \\
&+ 2p\mathbf{E}\left[\sup_{0\le s\le t}\int_0^s (1+2|X(u)|^2)^{(p-2)/2}X(u)'g(u,X(u),X(\delta(u)))\,dB(u)\right], \quad (5.2.5)
\end{aligned}
$$

where $C_1 = p(2\sqrt{K}+(p-1)K)$ By applying the Burkholder-Davis-Gundy inequality on the final term in the above equation (see, for example, Theorem 1.7.3 of Mao (1997)), along with the Hölder inequality and the fact that $|g'x| \le |g||x|$, we obtain

$$
\begin{aligned}
&2p\mathbf{E}\left[\sup_{0\le s\le t}\int_0^s (1+2|X(u)|^2)^{(p-2)/2}X(u)'g(u,X(u),X(\delta(u)))\,dB(u)\right] \\
&\le 8\sqrt{2}p\mathbf{E}\left[\left(\int_0^t (1+2|X(u)|^2)^{p-2}|X(u)'g(u,X(u),X(\delta(u)))|^2\,du\right)^{1/2}\right] \\
&\le 8\sqrt{2}p\mathbf{E}\Bigg[\left(\sup_{0\le s\le t}(1+2|X(s)|^2)^{p/2}\right)^{1/2} \\
&\qquad\qquad \times\left(K\int_0^t \sup_{-\tau\le r\le u}(1+2|X(r)|^2)^{p/2}\,du\right)^{1/2}\Bigg] \\
&\le \frac{1}{2}\mathbf{E}\left[\sup_{0\le s\le t}(1+2|X(s)|^2)^{p/2}\right] \\
&\quad + 64p^2K\mathbf{E}\left[\int_0^t \sup_{-\tau\le r\le u}(1+2|X(r)|^2)^{p/2}\,du\right]. \qquad (5.2.6)
\end{aligned}
$$

By substituting the upper bound in equation (5.2.6) into equation (5.2.5) and rearranging, we realise the following.

$$\mathbf{E}\left[\sup_{0\le s\le t}(1+2|X(s)|^2)^{p/2}\right] \le 2^{p/2}(1+2^{p/2}\mathbf{E}\left[\|\phi\|^p\right])$$
$$+C\int_0^t \mathbf{E}\left[\sup_{-\tau\le r\le u}(1+2|X(r)|^2)^{p/2}\right]du,$$

where $C = 2(C_1 + 64p^2K) = 2p(2\sqrt{K} + (65p-1)K)$. Therefore, by considering the behaviour of $\phi$ over $[-\tau, 0]$,

$$\mathbf{E}\left[\sup_{-\tau\le s\le 0}(1+2|X(s)|^2)^{p/2}\right] \le 2^{(p-2)/2}(1+2^{p/2}\mathbf{E}\left[\|\phi\|^p\right]),$$

we obtain

$$\mathbf{E}\left[\sup_{-\tau\le s\le t}(1+2|X(s)|^2)^{p/2}\right] \le \frac{3}{2}2^{p/2}(1+2^{p/2}\mathbf{E}\left[\|\phi\|^p\right])$$
$$+C\int_0^t \mathbf{E}\left[\sup_{-\tau\le r\le u}(1+2|X(r)|^2)^{p/2}\right]du.$$

Therefore, by applying Gronwall's inequality, the following result is true for any $t > 0$.

$$\mathbf{E}\left[\sup_{-\tau\le s\le t}(1+2|X(s)|^2)^{p/2}\right] \le \frac{3}{2}2^{p/2}(1+2^{p/2}\mathbf{E}\left[\|\phi\|^p\right])e^{Ct}. \tag{5.2.7}$$

Note that, for any $p > 2$ and $x \in \mathbb{R}^n$, we have $|x|^p = (|x|^2)^{p/2} \le (1+2|x|^2)^{p/2}$. Therefore, by considering $s$ over the entire interval $[-\tau, T]$, we obtain equation (5.2.4). The proof is then complete. □

Let us now introduce the following Euler-Maruyama numerical scheme for the SDDE (5.0.1). Choose $\Delta \in (0, 1\wedge\tau)$ to be the step size of the numerical scheme. For simplicity, we assume that $\Delta = \tau/N$ is a fraction of the delay parameter $\tau$ for some integer $N > \tau$. Given our piecewise continuous function $\delta$, let us define an equivalent function for our numerical scheme, $h : \mathbb{N}\cup\{0\} \to \mathbb{R}$, by

$$h(i) = \frac{\delta(i\Delta)}{\Delta}. \tag{5.2.8}$$

We can then define the discrete Euler-Maruyama numerical scheme in the following way. In this equation, $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$ denotes the Brownian increment whenever $k \geq 0$.

$$\begin{cases} \bar{Y}(k\Delta) = \phi(k\Delta), & -N \leq k \leq 0, \\ \bar{Y}((k+1)\Delta) = \bar{Y}(k\Delta) + f(k\Delta, \bar{Y}(k\Delta), \bar{Y}(h(k)\Delta))\Delta & \\ \qquad\qquad + g(k\Delta, \bar{Y}(k\Delta), \bar{Y}(h(k)\Delta))\Delta B_k, & k \geq 0. \end{cases} \tag{5.2.9}$$

In the above scheme, we define $\bar{Y}(u)$ as

$$\bar{Y}(u) = \bar{Y}(\bar{u}\Delta) \tag{5.2.10}$$

whenever $u \in [\bar{u}\Delta, (\bar{u}+1)\Delta)$ for a given integer $\bar{u}$, with $\eta = u/\Delta - \bar{u}$. It is easy to check that $\bar{Y}(h(k)\Delta) = \bar{Y}(\delta(k\Delta))$. Therefore, the continuous analogue of the above Euler-Maruyama approximate solution of the SDDE (5.0.1) can be defined using (5.2.9) in a straightforward way.

$$\begin{aligned} Y(t) &= \phi(0) + \int_0^t f(u, \bar{Y}(u), \bar{Y}(\delta(u)))\, du + \int_0^t g(u, \bar{Y}(u), \bar{Y}(\delta(u)))\, dB(u) \\ &= \bar{Y}(\bar{t}\Delta) + \int_{\bar{t}\Delta}^t f(u, \bar{Y}(u), \bar{Y}(\delta(u)))\, du + \int_{\bar{t}\Delta}^t g(u, \bar{Y}(u), \bar{Y}(\delta(u)))\, dB(u). \end{aligned} \tag{5.2.11}$$

As in the discrete case, we set $Y(t) = \phi(t)$ for all $t \in [-\tau, 0]$. From above, it follows that

$$|\bar{Y}(t)| \leq \sup_{-\tau \leq r \leq t} |Y(r)|. \tag{5.2.12}$$

In order to proceed, we state the following result. This is precisely Lemma 3.1 of Mao (2003) and the proof is identical in our case. We then follow this with the analogous proof of Lemma 3.2 of the same paper, applied under the SDDE (5.0.1), which provides an upper bound for the continuous Euler-Maruyama scheme that is independent of the step size $\Delta$.

**Lemma 67.** *Assume the initial data $\phi \in L^p_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$ for some $p > 2$. Define the function $\alpha : (0, \tau] \to \mathbb{R}_+$ by*

$$\alpha(u) = \sup_{t,s \in [-\tau,0], |t-s|<u} \mathbf{E}\left[|\phi(s) - \phi(t)|^2\right]. \tag{5.2.13}$$

*Then, $\alpha$ is nondecreasing and $\alpha(u) \to 0$ as $u \to 0$. Moreover,*

$$\mathbf{E}\left[|\phi(s) - \phi(t)|^2\right] \leq \alpha(t-s), \qquad -\tau \leq s \leq t \leq 0. \tag{5.2.14}$$

**Lemma 68.** *Let $\phi \in L^p_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$. For any $T > 0$ and under the assumptions of linear growth given by equation (5.2.2),*

$$\mathbf{E}\left[\sup_{-\tau \leq t \leq T} |Y(t)|^p\right] \leq H, \tag{5.2.15}$$

*where $H$ is a positive value dependent on $\phi, K, p$ and $T$, but not $\Delta$.*

*Proof.* On application of Hölder's inequality, we realise the following upper bound.

$$|Y(t)|^p \leq 3^{p-1}\Bigg(|\phi(0)|^p + t^{p-1}\int_0^t |f(u, \bar{Y}(u), \bar{Y}(\delta(u)))|^p \, du + \left|\int_0^t g(u, \bar{Y}(u), \bar{Y}(\delta(u))) \, dB(u)\right|^p\Bigg).$$

Hence, for any $t_1 \in [0, T]$,

$$\mathbf{E}\left[\sup_{0 \leq t \leq t_1} |Y(t)|^p\right] \leq 3^{p-1}\Bigg(\mathbf{E}\left[|\phi(0)|^p\right] + T^{p-1}\mathbf{E}\left[\int_0^{t_1} |f(u, \bar{Y}(u), \bar{Y}(\delta(u)))|^p \, du\right] + \mathbf{E}\left[\sup_{0 \leq t \leq t_1}\left|\int_0^t g(u, \bar{Y}(u), \bar{Y}(\delta(u))) \, dB(u)\right|^p\right]\Bigg).$$

Under the linear growth condition on $f$, we have

$$\begin{aligned}
&\mathbf{E}\left[\int_0^{t_1} |f(u, \bar{Y}(u), \bar{Y}(\delta(u)))|^p \, du\right] \\
&\leq K^{p/2}\mathbf{E}\left[\int_0^{t_1} (1 + |\bar{Y}(u)|^2 + |\bar{Y}(\delta(u))|^2)^{p/2} \, du\right] \\
&\leq 3^{(p-2)/2}K^{p/2}\mathbf{E}\left[\int_0^{t_1} 1 + |\bar{Y}(u)|^p + |\bar{Y}(\delta(u))|^p \, du\right] \\
&\leq 3^{(p-2)/2}K^{p/2}\left(T + 2\int_0^{t_1} \mathbf{E}\left[\sup_{-\tau \leq t \leq u} |Y(t)|^p\right] du\right).
\end{aligned}$$

From the Burkholder-Davis-Gundy and Hölder inequalities, we also obtain

$$
\begin{aligned}
&\mathbf{E}\left[\sup_{0\le t\le t_1}\left|\int_0^t g(u,\bar{Y}(u),\bar{Y}(\delta(u)))\,dB(u)\right|^p\right] \\
&\le C_p\mathbf{E}\left[\left|\int_0^{t_1}|g(u,\bar{Y}(u),\bar{Y}(\delta(u)))|^2\,du\right|^{p/2}\right] \\
&\le C_pT^{(p-2)/2}\mathbf{E}\left[\int_0^{t_1}|g(u,\bar{Y}(u),\bar{Y}(\delta(u)))|^p\,du\right] \\
&\le C_p(3T)^{(p-2)/2}K^{p/2}\left(T+2\int_0^{t_1}\mathbf{E}\left[\sup_{-\tau\le t\le u}|Y(t)|^p\right]du\right),
\end{aligned}
$$

where the final statement takes advantage of the linear growth condition on $g$ and uses the same approach as before. Combining the previous two results, we realise the following.

$$
\mathbf{E}\left[\sup_{0\le t\le t_1}|Y(t)|^p\right]\le 3^{p-1}\mathbf{E}\left[|\phi(0)|^p\right]+C_1+C_2\int_0^{t_1}\mathbf{E}\left[\sup_{-\tau\le t\le u}|Y(t)|^p\right]du,
$$

where $C_1$ and $C_2$ are positive values that depend only on $K, p$ and $T$. In the same way as equation 3.8 of Mao (2003), we obtain

$$
\mathbf{E}\left[\sup_{-\tau\le t\le t_1}|Y(t)|^p\right]\le (1+3^{p-1})\mathbf{E}\left[\|\phi\|^p\right]+C_1+C_2\int_0^{t_1}\mathbf{E}\left[\sup_{-\tau\le t\le u}|Y(t)|^p\right]du,
$$

which is finite by the requirement that $\phi\in L^p$. Hence, by applying Gronwall's inequality and letting $t_1=T$, we achieve our desired result:

$$
\mathbf{E}\left[\sup_{-\tau\le t\le T}|Y(t)|^p\right]\le ((1+3^{p-1})\mathbf{E}\left[\|\phi\|^p\right]+C_1)e^{C_2T}=:H.
$$

□

We now discuss numerical convergence of both the discrete and continuous Euler-Maruyama schemes as $\Delta\to 0$. This follows Lemma 3.3 of Mao (2003) but is adapted from stochastic functional differential equations to the general-delay SDDE case considered here. A consequence is a simplification of its proof, as we see below.

**Lemma 69.** *Let $f$ and $g$ satisfy the conditions of local Lipschitz continuity and linear growth described by Definitions 64 and 65 respectively. Then, for any $T>0$, there is a nondecreasing function $\beta:(0,\tau]\to\mathbb{R}_+$ which tends to zero as $u\to 0$ such that,*

$$
\mathbf{E}\left[|Y(s+\theta)-\bar{Y}(s+\theta)|^2\right]\le\beta(\Delta),\quad s\in[0,T],\theta\in[-\tau,0]. \tag{5.2.16}
$$

*Proof.* Let $s \in [0,T]$ and $\theta \in [-\tau, 0]$. Let $k_s, k_\theta \in \mathbb{Z}$ be chosen so that $s \in [k_s\Delta, (k_s + 1)\Delta)$ and $\theta \in [k_\theta, (k_\theta + 1)\Delta)$. It is easy to see from Mao (2003) that

$$0 \leq s + \theta - (k_s + k_\theta)\Delta < 2\Delta.$$

Let $k \in \mathbb{Z}$ be such that $s + \theta \in [k\Delta, (k+1)\Delta)$. Then, $\bar{Y}(s+\theta) = \bar{Y}(k\Delta)$. Therefore, we obtain

$$\mathbf{E}\left[|Y(s+\theta) - \bar{Y}(s+\theta)|^2\right] = \mathbf{E}\left[|Y(s+\theta) - \bar{Y}(k\Delta)|^2\right].$$

In order to bound this equation, we need to consider two possible cases (case 2 of Mao (2003) has no analogous scenario here as $s + \theta - k\Delta < \Delta$, whereas we can combine cases 3 and 4).

Let us first suppose that $k \geq 0$. Then, from equation (5.2.9), noting that $Y(k\Delta) = \bar{Y}(k\Delta)$,

$$Y(s+\theta) - \bar{Y}(k\Delta) = \int_{k\Delta}^{s+\theta} f(u, \bar{Y}(u), \bar{Y}(\delta(u)))\, du + \int_{k\Delta}^{s+\theta} g(u, \bar{Y}(u), \bar{Y}(\delta(u)))\, dB(u).$$

Using the linear growth condition, Hölder's inequality and Lemma 68, it then follows that we have

$$\begin{aligned}
\mathbf{E}\left[|Y(s+\theta) - \bar{Y}(k\Delta)|^2\right] &\leq 2\Bigg(\mathbf{E}\left[\left|\int_{k\Delta}^{s+\theta} f(u, \bar{Y}(u), \bar{Y}(\delta(u)))\, du\right|^2\right] \\
&\qquad + \mathbf{E}\left[\left|\int_{k\Delta}^{s+\theta} g(u, \bar{Y}(u), \bar{Y}(\delta(u)))\, dB(u)\right|^2\right]\Bigg) \\
&\leq 2\Bigg(2\Delta \int_{k\Delta}^{s+\theta} \mathbf{E}\left[|f(u, \bar{Y}(u), \bar{Y}(\delta(u)))|^2\right] du \\
&\qquad + \int_{k\Delta}^{s+\theta} \mathbf{E}\left[|g(u, \bar{Y}(u), \bar{Y}(\delta(u)))|^2\right] du\Bigg) \\
&\leq 6K \int_{k\Delta}^{s+\theta} \mathbf{E}\left[(1 + |\bar{Y}(u)|^2 + |\bar{Y}(\delta(u))|^2)\right] du \\
&\leq 6K \int_{k\Delta}^{s+\theta} \left(1 + 2\mathbf{E}\left[\sup_{-\tau \leq t \leq u} |Y(t)|^2\right]\right) du \\
&\leq 6K(1 + 2H^{2/p})\Delta.
\end{aligned}$$

On the other hand, if $k \leq -1$ then,

$$k\Delta \leq s + \theta < (k+1)\Delta \leq 0.$$

Therefore, from Lemma 67, we see that

$$\mathbf{E}\left[|Y(s+\theta) - \bar{Y}(k\Delta)|^2\right] = \mathbf{E}\left[|\phi(s+\theta) - \phi(k\Delta)|^2\right] \leq \alpha(\Delta).$$

By combining both cases, we obtain the following.

$$\mathbf{E}\left[|Y(s+\theta) - \bar{Y}(s+\theta)|^2\right] \leq 6K(1+2H^{2/p})\Delta + \alpha(\Delta) =: \beta(\Delta).$$

Since $H$ is independent of $\Delta$ and $\alpha$ is a nondecreasing function that tends to zero, from Lemma 67, it follows that $\beta$ is nondecreasing and satisfies

$$\lim_{\Delta \downarrow 0} \beta(\Delta) = 0.$$

The proof is therefore complete. □

The above results demonstrate boundedness of any solution to the SDDE (5.0.1) along with its corresponding Euler-Maruyama numerical scheme by a value that is constant with respect to $\Delta$. Furthermore, we observe mean-square convergence of any discrete numerical scheme (5.2.9) to its continuous counterpart given by equation (5.2.11). As in the case given by Mao (2003), we are now able to show mean-square convergence of the numerical scheme to its actual solution. Our proof of the following result draws upon the conclusions made above.

**Theorem 70.** *Let $\{X(t)\}_{t\in[-\tau,T]}$ be a stochastic process that satisfies the SDDE* (5.0.1) *for any $T > 0$, where the drift and diffusion functions $f, g : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the local Lipschitz and linear growth conditions specified by equations* (5.2.1) *and* (5.2.2) *respectively, and the initial data $\phi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ for a given $p > 2$. Let $\{Y(t)\}_{t\in[-\tau,T]}$ denote the corresponding Euler-Maruyama numerical scheme defined by equation (5.2.11) with step size given by $\Delta \in (0,1)$. Then,*

$$\lim_{\Delta \downarrow 0} \mathbf{E}\left[\sup_{0\leq t\leq T} |X(t) - Y(t)|^2\right] = 0. \tag{5.2.17}$$

*Proof.* Let $\bar{H} = \max(H, \frac{3}{2}2^{p/2}(1 + 2^{p/2}\mathbf{E}[\|\phi\|^p])e^{CT})$, where $C$ and $H$ are defined by Theorem 66 and Lemma 68 respectively. Then from these two results, we see that $\bar{H} > 0$ and

$$\mathbf{E}\left[\sup_{-\tau\leq t\leq T} |X(t)|^p\right] \vee \mathbf{E}\left[\sup_{-\tau\leq t\leq T} |Y(t)|^p\right] \leq \bar{H}.$$

Choose a sufficiently large integer $j \in \mathbb{N}$ and define following stopping times.

$$u_j = \inf\{t \geq 0 : |X(t)| \geq j\},$$
$$v_j = \inf\{t \geq 0 : |Y(t)| \geq j\},$$
$$\rho_j = u_j \wedge v_j.$$

As is conventional, we define $\inf \emptyset = \infty$. Set the function $e : [0,T] \to \mathbb{R}$ by

$$e(t) = X(t) - Y(t).$$

The following statement clearly holds.

$$\mathbf{E}\left[\sup_{0\leq t\leq T} |e(t)|^2\right] = \mathbf{E}\left[\sup_{0\leq t\leq T} |e(t)|^2 \mathbf{1}_{\{\rho_j > T\}}\right] + \mathbf{E}\left[\sup_{0\leq t\leq T} |e(t)|^2 \mathbf{1}_{\{u_j \leq T \text{ or } v_j \leq T\}}\right].$$

As stated by Mao (2003), we also make use of the inequality whenever $a, b > 0$ and $\gamma \in [0,1]$.

$$a^\gamma b^{1-\gamma} \leq \gamma a + (1-\gamma)b.$$

By applying this inequality, we see that the following is true for any $\kappa > 0$.

$$\mathbf{E}\left[\sup_{0\leq t\leq T} |e(t)|^2 \mathbf{1}_{\{\rho_j \leq T\}}\right] \leq \frac{2}{p}\kappa \mathbf{E}\left[\sup_{0\leq t\leq T} |e(t)|^p\right] + \frac{p-2}{p}\kappa^{-\frac{2}{p-2}} \mathbf{P}\left(\rho_j \leq T\right). \quad (5.2.18)$$

We see, using the same approach as Mao (2003) that

$$\mathbf{P}\left(\rho_j \leq T\right) \leq \frac{2\bar{H}}{j^p}. \quad (5.2.19)$$

Furthermore, by definition of $\bar{H}$,

$$\mathbf{E}\left[\sup_{0\leq t\leq T} |e(t)|^p\right] \leq 2^{p-1}\mathbf{E}\left[\sup_{0\leq t\leq T} (|X(t)|^p + |Y(t)|^p)\right] \leq 2^p \bar{H}. \quad (5.2.20)$$

Choose $\omega \in \Omega$. If $\rho_j(\omega) > T$, then $|e(t \wedge \rho_j, \omega)|^2 \mathbf{1}_{\{\rho_j > T\}}(\omega) = |e(t,\omega)|^2$. Otherwise,

$\rho_j(\omega) \le T$ and so $\mathbf{1}_{\{\rho_j > T\}}(\omega) = 0$. Hence, we have

$$
\begin{aligned}
\mathbf{E}\left[\sup_{0\le t\le T} |e(t)|^2 \mathbf{1}_{\{\rho_j>T\}}\right] &= \mathbf{E}\left[\sup_{0\le t\le T} |e(t\wedge\rho_j)|^2 \mathbf{1}_{\{\rho_j>T\}}\right] \\
&\le \mathbf{E}\left[\sup_{0\le t\le T} |e(t\wedge\rho_j)|^2\right]
\end{aligned}
$$

We see that the following statement is true

$$
\begin{aligned}
|e(t\wedge\rho_j)|^2 = \bigg| &\int_0^{t\wedge\rho_j} f(u,X(u),X(\delta(u))) - f(u,\bar{Y}(u),\bar{Y}(\delta(u)))\,du \\
&+ \int_0^{t\wedge\rho_j} g(u,X(u),X(\delta(u))) - g(u,\bar{Y}(u),\bar{Y}(\delta(u)))\,dB(u)\bigg|^2 \\
\le 2\bigg(&T\int_0^{t\wedge\rho_j} |f(u,X(u),X(\delta(u))) - f(u,\bar{Y}(u),\bar{Y}(\delta(u)))|^2\,du \\
&\bigg|\int_0^{t\wedge\rho_j} g(u,X(u),X(\delta(u))) - g(u,\bar{Y}(u),\bar{Y}(\delta(u)))\,dB(u)\bigg|^2\bigg).
\end{aligned}
$$

It then follows, for all $t_1 \in [0,T]$, by the Doob martingale inequality,

$$
\begin{aligned}
&\mathbf{E}\left[\sup_{0\le t\le t_1} |e(t\wedge\rho_j)|^2\right] \\
&\quad\le 4(T+4)\mathbf{E}\bigg[\bigg(\int_0^{t_1\wedge\rho_j} |f(u,X(u),X(\delta(u))) - f(u,\bar{Y}(u),\bar{Y}(\delta(u)))|^2\,du \\
&\qquad\qquad \vee |g(u,X(u),X(\delta(u))) - g(u,\bar{Y}(u),\bar{Y}(\delta(u)))|^2\,du\bigg)\bigg].
\end{aligned}
$$

Under the conditions of local Lipschitz continuity on $f$, for any $u \in (0, t_1\wedge\rho_j]$ we obtain

$$
\begin{aligned}
&|f(u,X(u),X(\delta(u))) - f(u,\bar{Y}(u),\bar{Y}(\delta(u)))|^2 \\
&\le 2|f(u,X(u),X(\delta(u))) - f(u,Y(u),Y(\delta(u)))|^2 \\
&\quad + 2|f(u,Y(u),Y(\delta(u))) - f(u,\bar{Y}(u),\bar{Y}(\delta(u)))|^2 \\
&\le 2L_j(|X(u)-Y(u)|^2 + |X(\delta(u)) - Y(\delta(u))|^2) \\
&\quad + 2L_j(|Y(u)-\bar{Y}(u)|^2 + |Y(\delta(u)) - \bar{Y}(\delta(u))|^2) \\
&\le 4L_j \sup_{-\tau\le t\le u} |X(t)-Y(t)|^2 \\
&\quad + 2L_j(|Y(u)-\bar{Y}(u)|^2 + |Y(\delta(u)) - \bar{Y}(\delta(u))|^2).
\end{aligned}
$$

We achieve a similar upper bound for $g$ under the same Lipschitz condition. Therefore,

$$\mathbf{E}\left[\sup_{0\leq t\leq t_1}|e(t\wedge\rho_j)|^2\right]\leq 16(T+4)L_j\int_0^{t_1}\mathbf{E}\left[\sup_{-\tau\leq t\leq u}|e(t\wedge\rho_j)|^2\right]du$$
$$+8(T+4)L_j\int_0^{T}\mathbf{E}\left[|Y(u)-\bar{Y}(u)|^2+|Y(\delta(u))-\bar{Y}(\delta(u))|^2\right]du.$$

Recall, from Lemma 69 that for any $u\in[-\tau,T]$, we have

$$\mathbf{E}\left[|Y(u)-\bar{Y}(u)|^2\right]\leq\beta(\Delta).$$

Since $u-\tau\leq\delta(u)\leq u$ for all positive $u$ and $e(u)=0$ for all $u\leq 0$, it follows that we have

$$\mathbf{E}\left[\sup_{0\leq t\leq t_1}|e(t\wedge\rho_j)|^2\right]\leq 16(T+4)L_j\int_0^{t_1}\mathbf{E}\left[\sup_{0\leq t\leq u}|e(t\wedge\rho_j)|^2\right]du$$
$$+16(T+4)L_jT\beta(\Delta).$$

Therefore, by applying Gronwall's inequality we obtain the following result, where $C_j=(16(T+4)L_jT)e^{16(T+4)L_jT}$ is independent of $\Delta$.

$$\mathbf{E}\left[\sup_{0\leq t\leq t_1}|e(t\wedge\rho_j)|^2\right]\leq C_j\beta(\Delta).$$

By combining this result appropriately with equations (5.2.18), (5.2.19) and (5.2.20), we achieve the following upper bound for the supremum of $|e(t)|^2$.

$$\mathbf{E}\left[\sup_{0\leq t\leq T}|e(t)|^2\right]\leq C_j\beta(\Delta)+\frac{2^{p+1}\kappa\bar{H}}{p}+\frac{2(p-2)\bar{H}}{p\kappa^{2/(p-2)}j^p}.$$

Let $\varepsilon>0$. Then, choose $\kappa>0$ sufficiently small such that $(2^{p+1}\kappa\bar{H})/p<\varepsilon/3$ and $j$ sufficiently large such that

$$\frac{2(p-2)\bar{H}}{p\kappa^{2/(p-2)}j^p}<\frac{\varepsilon}{3}.$$

Finally, set $\Delta>0$ small enough so that $C_j\beta(\Delta)<\varepsilon/3$. Then, we conclude that

$$\mathbf{E}\left[\sup_{0\leq t\leq T}|e(t)|^2\right]<\varepsilon.$$

This demonstrates that equation (5.2.17) holds. □

# Chapter 6

# Numerical Results

This report details the methods under which one can value arithmetic options under a wide variety of underlying models. In this chapter, we attempt to describe how our bounds from above behave by numerically comparing each to an approximate true value. We begin by considering the results obtained in Chapter 2 before studying the effect of choosing the delay models described by Chapters 3 and 4.

## 6.1 Asian options under non-delay models

In Chapter 2, we obtain stochastic bounds for sums of dependent risks, which we apply to Asian options in Section 2.5. To see how these results can be used in practice, we compute the appropriate bounds, comparing them to a brute force Monte Carlo estimate of the actual option price.

In this chapter, we assume that the process $\{S(t)\}_{t\in[0,T]}$ follows a model that is dependent on an underlying standard Brownian motion. Glasserman (2004) gives methods for estimating the value of an Asian option written on a single asset. In computing bounds for Asian options, we let $LB_t^{(2)}$ and $LB_t^{(3)}$ be defined by equations (2.5.8) and (2.5.16) respectively. We set $UB_1$ as the upper bound given by equation (2.5.20) and $UB_t^{(1)}$ as that given by equation (2.5.23). We also note the optimal monitoring time $t$ that achieves the bound value where relevant.

In Tables 6.1–6.4, we assume that the single asset $\{S(t)\}_{t\in[0,T]}$ follows the Black-Scholes model, given by the following stochastic differential equation (SDE).

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t).$$

We compute Asian call option upper and lower bounds in this case for differing values of the constant volatility parameter $\sigma$. Moreover, we use the following parameters in Tables 6.1–6.4, with $t$ given in months.

$$S(0) = 100, \quad r = 0.05, \quad T = \frac{4}{12}, \quad t_0 = 0, \quad m = 4.$$

In Table 6.5, we compute the relevant bounds for a stock price process following the model of Heston (1993), given by the system of SDEs below.

$$\begin{aligned} dS(t) &= rS(t)\,dt + \sqrt{v(t)}S(t)\,dW_1(t), \\ dv(t) &= \kappa(\theta - v(t))\,dt + \sigma_v\sqrt{v(t)}\,dW_2(t). \end{aligned}$$

To compute the vector of standard Brownian motions $\boldsymbol{W}(t) = (W_1(t), \ldots, W_m(t))'$, we assume that the covariance between the each underlying process is mutually fixed and let $\Sigma$ represent the correlation matrix of the $W_i(t)$ at time $t$. This is just the covariance matrix of a multinormal random variable $\boldsymbol{Z} \sim N_m(\boldsymbol{0}, \Sigma)$. We can therefore set $\boldsymbol{W}(t) \stackrel{d}{=} \sqrt{t}A'\boldsymbol{Z}$, where $A$ is an $m \times m$ matrix such that $A'A = \Sigma$. Shaked and Shanthikumar (2007) suggest that we take $A$ as the Cholesky factor of $\Sigma$, which we use in implementation. It is then straightforward to simulate multivariate Brownian increments and also possible to simulate basket options as well.

For the parameters in the Heston system above, we make use of the following settings, mostly agreeing with those used for Tables 5 and 6 of Albrecher et al. (2008).

$$S(0) = 100, \quad r = 0.03, \quad v_0 = 0.0175, \quad \kappa = 1.5768, \quad \theta = 0.0398,$$

$$\sigma_v = 0.5751, \quad T = \frac{4}{12}, \quad t_0 = 0, \quad m = 4.$$

We set the correlation between $W_1(t)$ and $W_2(t)$ to $-0.5711$. In this case $UB_t^{(1)}$ is defined to be conditional on $W_1(t)$. Here, $t$ is given in months.

Note that the lower bound $LB_t^{(2)}$ uses the assumption that $S(t) = S(0)e^{X(t)}$ for some Lévy process $\{X(t)\}_{t\in[0,T]}$. Under the Heston model, we see that $X(t)$ satisfies

$$X(t) = rt - \frac{1}{2}\int_0^t v(s)\,ds + \int_0^t \sqrt{v(s)}\,dW_1(s).$$

To satisfy the requirement that $\{X(t)\}_{t\in[0,T]}$ is stationary, we require the distribution of $X(t) - X(u)$ to depend only on the time difference $t - u$. We can see from the above equation that this is not necessarily the case. It follows that $\{X(t)\}_{t\in[0,T]}$ is in fact not a Lévy process and so we should not assume that $LB_t^{(2)}$ will work here. However, Albrecher et al. (2008) computes $LB_t^{(2)}$ under this model in Tables 5 and 6 and we do the same here for comparison. We do not compute $LB_t^{(3)}$ in this case, since it specifically assumes that $\{S(t)\}_{t\in[0,T]}$ follows the Black-Scholes model.

Finally, in each table we provide a Monte Carlo estimate of the actual option price using 1,000,000 sample paths. Since the models considered above place no reliance on past data, as opposed to the SDDEs we will soon examine, we are not faced with possible memory intensity issues when using a high number of sample paths.

| $\sigma$ | $K$ | $LB_t^{(2)}$ | $t$ | $LB_t^{(3)}$ | $t$ | $MC$ |
|---|---|---|---|---|---|---|
| 0.15 | 60 | 40.36974 | 1 | 40.36974 | 1 | 40.36938 |
| | 70 | 30.53503 | 1 | 30.53503 | 3 | 30.53467 |
| | 80 | 20.70031 | 1 | 20.70032 | 3 | 20.69999 |
| | 90 | 10.88110 | 2 | 10.90223 | 3 | 10.91783 |
| | 100 | 2.75571 | 3 | 2.77347 | 3 | 2.88831 |
| | 110 | 0.17073 | 3 | 0.17281 | 3 | 0.21462 |
| | 120 | 0.00218 | 3 | 0.00222 | 3 | 0.00395 |
| 0.20 | 60 | 40.36974 | 1 | 40.36974 | 1 | 40.36911 |
| | 70 | 30.53503 | 1 | 30.53503 | 3 | 30.53439 |
| | 80 | 20.70034 | 1 | 20.70171 | 3 | 20.70255 |
| | 90 | 10.99475 | 3 | 11.04642 | 3 | 11.09579 |
| | 100 | 3.47293 | 3 | 3.50353 | 3 | 3.65769 |
| | 110 | 0.50523 | 3 | 0.51245 | 3 | 0.59999 |
| | 120 | 0.03289 | 3 | 0.03355 | 3 | 0.04857 |
| 0.25 | 60 | 40.36974 | 1 | 40.36974 | 3 | 40.36875 |
| | 70 | 30.53503 | 1 | 30.53508 | 3 | 30.53420 |
| | 80 | 20.70134 | 1 | 20.71525 | 3 | 20.72418 |
| | 90 | 11.23667 | 3 | 11.31340 | 3 | 11.40561 |
| | 100 | 4.19005 | 3 | 4.23692 | 3 | 4.43021 |
| | 110 | 0.96823 | 3 | 0.98416 | 3 | 1.11939 |
| | 120 | 0.13980 | 3 | 0.14285 | 3 | 0.18736 |
| 0.30 | 60 | 40.36974 | 1 | 40.36974 | 3 | 40.36832 |
| | 70 | 30.53504 | 1 | 30.53599 | 3 | 30.53584 |
| | 80 | 20.70850 | 1 | 20.76225 | 3 | 20.79032 |
| | 90 | 11.57434 | 3 | 11.67950 | 3 | 11.81745 |
| | 100 | 4.90505 | 3 | 4.97166 | 3 | 5.20402 |
| | 110 | 1.51286 | 3 | 1.54111 | 3 | 1.72336 |
| | 120 | 0.34571 | 3 | 0.35393 | 3 | 0.43789 |

Table 6.1: Lower bounds for an Asian call with $S(0) = 100$, $r = 0.05$ and averaging months $t_i = i$, with $t_4 = T$.

| $\sigma$ | $K$ | $LB_t^{(2)}$ | $t$ | $LB_t^{(3)}$ | $t$ | $MC$ |
|---|---|---|---|---|---|---|
| 0.35 | 60 | 40.36974 | 1 | 40.36978 | 3 | 40.36796 |
| | 70 | 30.53531 | 1 | 30.54140 | 3 | 30.54541 |
| | 80 | 20.76671 | 2 | 20.86038 | 3 | 20.91742 |
| | 90 | 11.98160 | 3 | 12.11850 | 3 | 12.30261 |
| | 100 | 5.61704 | 3 | 5.70684 | 3 | 5.97822 |
| | 110 | 2.10937 | 3 | 2.15351 | 3 | 2.38153 |
| | 120 | 0.64673 | 3 | 0.66340 | 3 | 0.79254 |
| 0.40 | 60 | 40.36975 | 1 | 40.37016 | 3 | 40.36838 |
| | 70 | 30.53693 | 1 | 30.55846 | 3 | 30.57253 |
| | 80 | 20.88344 | 2 | 21.01684 | 3 | 21.10950 |
| | 90 | 12.43830 | 3 | 12.61023 | 3 | 12.83974 |
| | 100 | 6.32551 | 3 | 6.44195 | 3 | 6.75232 |
| | 110 | 2.73953 | 3 | 2.80310 | 3 | 3.07574 |
| | 120 | 1.02840 | 3 | 1.05705 | 3 | 1.23393 |
| 0.45 | 60 | 40.36981 | 1 | 40.37198 | 3 | 40.37178 |
| | 70 | 30.54251 | 1 | 30.59558 | 3 | 30.62679 |
| | 80 | 21.05277 | 2 | 21.23076 | 3 | 21.36382 |
| | 90 | 12.93012 | 3 | 13.14042 | 3 | 13.41459 |
| | 100 | 7.03011 | 3 | 7.17664 | 3 | 7.52602 |
| | 110 | 3.39191 | 3 | 3.47843 | 3 | 3.79484 |
| | 120 | 1.47500 | 3 | 1.51923 | 3 | 1.74454 |
| 0.50 | 60 | 40.37015 | 1 | 40.37754 | 3 | 40.38183 |
| | 70 | 30.55588 | 1 | 30.65990 | 3 | 30.71499 |
| | 80 | 21.27126 | 2 | 21.49713 | 3 | 21.67330 |
| | 90 | 13.44704 | 3 | 13.69904 | 3 | 14.01715 |
| | 100 | 7.73062 | 3 | 7.91069 | 3 | 8.29911 |
| | 110 | 4.05907 | 3 | 4.17205 | 3 | 4.53158 |
| | 120 | 1.97279 | 3 | 2.03622 | 3 | 2.31016 |

Table 6.2: Lower bounds for an Asian call with $S(0) = 100$, $r = 0.05$ and averaging months $t_i = i$, with $t_4 = T$.

## 6.2 Numerical results for delay models

### 6.2.1 Asian options

In the previous section, we demonstrate the numerical behaviour for Asian option bounds, on an underlying asset whose distribution is known for all $t \in [0, T]$. In this part, we study bounds for arithmetic Asian options under the assumption that $\{S(t)\}_{t\in[0,T]}$ follows the SDDE given by equation (3.2.5). To see how these results can be used in practice, we will compute these bounds, $LB$, $UB_1$ and $UB_2$, for differing exercise values $K$, and compare them to a brute force Monte Carlo estimate $MC$, of the actual option price $A(K, T, m)$.

For each of these bounds, we use a lognormal conditioning variable $\Lambda$. We introduce

| $\sigma$ | $K$ | $UB_t^{(1)}$ | $t$ | $UB_1$ | $MC$ |
|---|---|---|---|---|---|
| 0.15 | 60 | 40.37027 | 1 | 40.52303 | 40.36938 |
| | 70 | 30.53565 | 1 | 30.53503 | 30.53467 |
| | 80 | 20.70113 | 1 | 20.70061 | 20.69999 |
| | 90 | 10.93329 | 4 | 10.96766 | 10.91783 |
| | 100 | 2.98475 | 4 | 3.17324 | 2.88831 |
| | 110 | 0.25087 | 4 | 0.33484 | 0.21462 |
| | 120 | 0.00564 | 4 | 0.01174 | 0.00395 |
| 0.20 | 60 | 40.37057 | 1 | 40.36974 | 40.36911 |
| | 70 | 30.53596 | 1 | 30.53505 | 30.53439 |
| | 80 | 20.70609 | 4 | 20.71007 | 20.70255 |
| | 90 | 11.14146 | 4 | 11.23659 | 11.09579 |
| | 100 | 3.78717 | 4 | 4.03979 | 3.65769 |
| | 110 | 0.67549 | 4 | 0.83912 | 0.59999 |
| | 120 | 0.06181 | 4 | 0.09975 | 0.04857 |
| 0.25 | 60 | 40.37087 | 1 | 40.36974 | 40.36875 |
| | 70 | 30.53658 | 1 | 30.53600 | 30.53420 |
| | 80 | 20.73619 | 4 | 20.76006 | 20.72418 |
| | 90 | 11.48872 | 4 | 11.65497 | 11.40561 |
| | 100 | 4.59262 | 4 | 4.90895 | 4.43021 |
| | 110 | 1.23511 | 4 | 1.47876 | 1.11939 |
| | 120 | 0.22498 | 4 | 0.32195 | 0.18736 |
| 0.30 | 60 | 40.37118 | 1 | 40.36982 | 40.36832 |
| | 70 | 30.54041 | 1 | 30.54368 | 30.53584 |
| | 80 | 20.81991 | 4 | 20.88183 | 20.79032 |
| | 90 | 11.93965 | 4 | 12.17873 | 11.81745 |
| | 100 | 5.39930 | 4 | 5.77901 | 5.20402 |
| | 110 | 1.87834 | 4 | 2.19966 | 1.72336 |
| | 120 | 0.50983 | 4 | 0.68142 | 0.43789 |

Table 6.3: Upper bounds for an Asian call with $S(0) = 100$, $r = 0.05$ and averaging months $t_i = i$, with $t_4 = T$.

two choices for this random variable. The first of which is the value $S(v)$, of the stock price at a fixed time $v \in [0, h]$, for which we provide explicit results in Section 4.2. We also use as a candidate for $\Lambda$, the geometric average $G_n$ of $S(t)$ over a set of $n$ time points, $0 \leq u_1 < \cdots < u_n \leq h$, given by

$$G_n = \left( \prod_{j=1}^{n} S(u_j) \right)^{\frac{1}{n}}. \tag{6.2.1}$$

Using the value $k_h$ given by equation (4.1.20), taking $n = k_h - 1$ and setting $u_i = t_i$, we specifically condition on the geometric average, $G$, of the $S(t_i)$ when $t_i \leq h$. We can obtain the values $M_\Lambda, V_\Lambda, A_\Lambda$ and $B_\Lambda$ introduced in Chapter 4 by consulting Tables 6.6–

| $\sigma$ | $K$ | $UB_t^{(1)}$ | $t$ | $UB_1$ | $MC$ |
|---|---|---|---|---|---|
| 0.35 | 60 | 40.37175 | 1 | 40.37072 | 40.36796 |
| | 70 | 30.55559 | 4 | 30.57069 | 30.54541 |
| | 80 | 20.97306 | 4 | 21.08691 | 20.91742 |
| | 90 | 12.46356 | 4 | 12.77442 | 12.30261 |
| | 100 | 6.20636 | 4 | 6.64912 | 5.97822 |
| | 110 | 2.57480 | 4 | 2.97134 | 2.38153 |
| | 120 | 0.90331 | 4 | 1.15619 | 0.79254 |
| 0.40 | 60 | 40.37384 | 1 | 40.37516 | 40.36838 |
| | 70 | 30.59320 | 4 | 30.63117 | 30.57253 |
| | 80 | 21.19750 | 4 | 21.37183 | 21.10950 |
| | 90 | 13.03891 | 4 | 13.41980 | 12.83974 |
| | 100 | 7.01333 | 4 | 7.51881 | 6.75232 |
| | 110 | 3.30649 | 4 | 3.77618 | 3.07574 |
| | 120 | 1.38526 | 4 | 1.72137 | 1.23393 |
| 0.45 | 60 | 40.38052 | 1 | 40.38835 | 40.37178 |
| | 70 | 30.66372 | 4 | 30.73532 | 30.62679 |
| | 80 | 21.48753 | 4 | 21.72677 | 21.36382 |
| | 90 | 13.65114 | 4 | 14.10025 | 13.41459 |
| | 100 | 7.81990 | 4 | 8.38775 | 7.52602 |
| | 110 | 4.06236 | 4 | 4.60347 | 3.79484 |
| | 120 | 1.93700 | 4 | 2.35605 | 1.74454 |
| 0.50 | 60 | 40.39650 | 1 | 40.41687 | 40.38183 |
| | 70 | 30.77368 | 4 | 30.88799 | 30.71499 |
| | 80 | 21.83434 | 4 | 22.14070 | 21.67330 |
| | 90 | 14.29034 | 4 | 14.80602 | 14.01715 |
| | 100 | 8.62585 | 4 | 9.25567 | 8.29911 |
| | 110 | 4.83530 | 4 | 5.44639 | 4.53158 |
| | 120 | 2.54335 | 4 | 3.04411 | 2.31016 |

Table 6.4: Upper bounds for an Asian call with $S(0) = 100$, $r = 0.05$ and averaging months $t_i = i$, with $t_4 = T$.

6.7. As before, $X(t) = \log(S(t)/S(0))$. We also define the value $k_t = \min\{i : u_i > t\}$ in Table 6.7.

In the examples given by Tables 6.8–6.11, we use the conditioning variable $\Lambda = S(v)$ and let $LB(v)$, $UB_1(v)$ and $UB_2(v)$ be defined as before. Since these bounds depend on $v \in [0, h]$, we shall also note the optimal value of $v$ that maximises the lower bound (respectively, minimises the upper bounds). Since there is no known way of doing this analytically, we do this by computing each $LB(v)$ for each value of $v \in \{i/1000 : 0 \leq i \leq 1000h\}$.

In Tables 6.12–6.15, we set $\Lambda = G$. The first two tables in each case assumes a maturity time of 21 days, whereas the final two use a maturity of 84 days. In both sets of tables, we consider a fixed-delay model with $\delta(t) = t - h$, with $h$ set to 1 and 21 days

| $K$ | $LB_t^{(2)}$ | $t$ | $UB_t^{(1)}$ | $t$ | $UB_1$ | $MC$ |
|---|---|---|---|---|---|---|
| 60 | 40.22279 | 1 | 40.39262 | 1 | 40.22104 | 40.24168 |
| 70 | 30.32229 | 1 | 30.48909 | 1 | 30.32028 | 30.34118 |
| 80 | 20.42179 | 1 | 20.59014 | 1 | 20.42020 | 20.44246 |
| 90 | 10.53606 | 2 | 10.72978 | 1 | 10.60023 | 10.60754 |
| 100 | 2.28793 | 3 | 2.58637 | 1 | 2.63412 | 1.93965 |
| 110 | 0.05214 | 3 | 0.10239 | 4 | 0.13393 | 0.02952 |
| 120 | 0.00002 | 4 | 0.00027 | 4 | 0.00088 | 0.00010 |
| 130 | 0.00000 | 4 | 0.00000 | 4 | 0.00000 | 0.00000 |
| 140 | 0.00000 | 4 | 0.00000 | 4 | 0.00000 | 0.00000 |
| 150 | 0.00000 | 4 | 0.00000 | 4 | 0.00000 | 0.00000 |

Table 6.5: Lower and upper bounds for an Asian call following the Heston model, with $S(0) = 100$, $r = 0.03$ and averaging months $t_i = i$, with $t_4 = T$.

| $\Lambda$ | $S(v)$ |
|---|---|
| $M_\Lambda$ | $rv - \frac{1}{2}\int_0^v g^2(u, \phi(\delta(u)))\,du$ |
| $V_\Lambda$ | $\int_0^v g^2(u, \phi(\delta(u)))\,du$ |
| $\mathrm{Cov}\,(X(t), L_\Lambda)$ | $\int_0^{t\wedge v} g^2(u, \phi(\delta(u)))\,du$ |

Table 6.6: Values used to compute bounds when conditioning on $\Lambda = S(v)$.

respectively. This produces similar results (up to five decimal places) to the variable delay case, $\delta(t) = \lfloor t/h \rfloor h$.

In all cases, we will assume that the initial data $S(-t) = S_1(-t)$ is based on the closing values of Vodafone's share price up until the 10th June 2009, as can be observed from Table 6.16. We assume that the asset is monitored daily and will take a daily interest rate $r = 0.05/252$ and volatility function $g(t, x) = 0.00438x^{0.31572}$. This is achieved by taking a least-squares estimate of the natural logarithm of the function

$$g(t, x) = x_t^\beta e^{\alpha + e_t}, \tag{6.2.2}$$

where $x_t = \log(S(t)/S(t-1))$ denotes the daily log-returns over a 10 year period. We bound this volatility function from above by $\overline{G} = 0.4/\sqrt{252}$.[1]

In these results, we see that the lower bound for the Asian call option closely

[1] We also attempt to produce results using a linear volatility function, given by $g(x) = 0.0001266x$, as well as the nonnegative part of a logarithmic function, satisfying $g(x) = (0.0081696 \log x - 0.0179599)^+$. However, our numerical values are very similar to those described by here. Therefore, these results are omitted from this chapter.

| $\Lambda$ | $G_n$ |
|---|---|
| $M_\Lambda$ | $\frac{1}{n}\sum_{i=1}^{n} ru_i - \frac{1}{2}\int_0^{u_i} g^2(u,\phi(\delta(u)))\,du$ |
| $V_\Lambda$ | $\frac{1}{n^2}\sum_{i=1}^{n}(1+2(n-i))\int_0^{u_i} g^2(u,\phi(\delta(u)))\,du$ |
| $\mathrm{Cov}\,(X(t),L_\Lambda)$ | $\frac{1}{n}\left(\sum_{i=1}^{k_t}\int_0^{u_i} g^2(u,\phi(\delta(u)))\,du + (n-k_t)\int_0^{t} g^2(u,\phi(\delta(u)))\,du\right)$ |

Table 6.7: Values used to compute bounds when conditioning on $\Lambda = G_n$.

shadows the Monte Carlo estimate in this case, particularly if $T$ is small or the exercise price $K$ is close to being at or in the money. On the other hand, as $K$ increases, the relative difference between $LB$ and the Monte Carlo estimate increases substantially. For example, in Table 6.8, the estimated value for $A(K,T,m)$ is approximately 1,950.6 times greater than our lower bound when $K = 120$. The relative error in this case is also seen in Albrecher et al. (2008) and is a consequence of conditioning on $S(v)$ in order to find a straightforward, tractable solution in any given case. Further, we see that it is not necessarily the case that the lower bound is maximised at at the monitoring time $v = 1$, as demonstrated by Table 6.8 when $80 \le K \le 100$. This contrasts the examples considered in Albrecher et al. (2008).

The $UB_i$ each provide values that are indeed upper bounds for $A(K,T,m)$. They are not particularly close to the estimated values, given by $MC$, in absolute terms when $K$ is large. However, we note that the estimated relative error for the upper bound increases at a reduced rate for large $K$ compared to the lower bound. For instance, in Table 6.8, we see that $UB_2(v)$ is roughly 10.3 times larger than the Monte Carlo estimate when $K = 120$. In McWilliams and Sabanis (2011), we demonstrate a ten-fold increase in the relative error when $K$ is raised to 130 for a similar set of parameters. However, this error remains closer in relative terms compared to the equivalent $LB(v)$.

Moreover, further investigation shows that the two upper bounds tend to a limit that is significantly greater than zero as $K \to \infty$. This is demonstrated by McWilliams and Sabanis (2011) and is due to the addition of a term, in equations (4.1.18) and (4.1.27), that is constant with respect to $K$ but depends on the upper bound for the volatility function $\overline{G}$. As a result, the bounds perform considerably better for in-the-money options. This is true for the lower bound as well. By reducing $\overline{G}$ we can significantly reduce the upper bound and thus the error between it and the Monte Carlo estimate.

This is particularly true in the case where $h$ and $T$ are large. Indeed, by increasing $h$, we require fewer terms of the form given by equations (4.1.17) and (4.1.26) being used and $S(t)$ assuming a lognormal model over a much larger portion of the interval $[0, T]$. As an example, we see an immediate improvement when considering a 21-day fixed-delay SDDE in contrast to a 1-day delay model. In practice, having such a large value for $h$ is still worthy of consideration, as practitioners may still be interested in a stock price's long-term performance before investing in it.

When considering the choice of conditioning variable $\Lambda$, we observe that the bounds achieve similar values, whether considering the value of the underlying asset at a fixed-point in time, $S(v)$, or the geometric average of the underlying process at points where the distribution remains lognormal. One can consider beyond the face value presented by these numerical values to then deduce that using $G_n$ is more suitable as a conditioning variable than $S(v)$. This is because the extra overhead involved by comparing each bound for differing choices of $v$ is significant when the number of different choices made for $v$ is very large. From these results, we can observe that it is not necessarily the case that each bound is optimised at a monitoring time, meaning that it is necessary to consider a large number of choices for $v$. By using $\Lambda = G_n$, we eliminate this problem.

| $K$ | $LB$ | $v$ | $UB_1$ | $v$ | $UB_2$ | $v$ | $MC$ |
|---|---|---|---|---|---|---|---|
| 80 | 32.11072 | 0.08 | 36.25611 | 1.00 | 35.71119 | 1.00 | 32.34201 |
| 90 | 22.15230 | 0.59 | 26.29769 | 1.00 | 25.75277 | 1.00 | 22.38359 |
| 100 | 12.19388 | 0.62 | 16.33927 | 1.00 | 15.79435 | 1.00 | 12.45520 |
| 110 | 2.40305 | 1.00 | 6.53952 | 0.40 | 5.95620 | 0.31 | 3.80425 |
| 120 | 0.00018 | 1.00 | 4.41557 | 1.00 | 3.60065 | 1.00 | 0.35110 |

Table 6.8: 1-day fixed-delay SDDE with $\Lambda = S(v)$ and maturity of 21 days.

| $K$ | $LB$ | $v$ | $UB_1$ | $v$ | $UB_2$ | $v$ | $MC$ |
|---|---|---|---|---|---|---|---|
| 80 | 32.11072 | 0.08 | 33.82843 | 16.00 | 32.11072 | 0.01 | 32.34659 |
| 90 | 22.15230 | 0.59 | 23.87002 | 16.00 | 22.15258 | 0.01 | 22.38817 |
| 100 | 12.19302 | 9.00 | 13.92350 | 16.00 | 12.27639 | 0.01 | 12.46974 |
| 110 | 3.37949 | 12.00 | 4.97606 | 19.00 | 4.02817 | 0.01 | 3.86997 |
| 120 | 0.21745 | 12.00 | 1.93014 | 17.00 | 0.53954 | 0.01 | 0.36957 |

Table 6.9: 21-day fixed-delay SDDE with $\Lambda = S(v)$ and maturity of 21 days.

| $K$ | $LB$ | $v$ | $UB_1$ | $v$ | $UB_2$ | $v$ | $MC$ |
|---|---|---|---|---|---|---|---|
| 80 | 32.40513 | 0.34 | 40.95921 | 1.00 | 39.80929 | 1.00 | 32.45318 |
| 90 | 22.57042 | 0.92 | 31.12450 | 1.00 | 29.97458 | 1.00 | 22.64745 |
| 100 | 12.73570 | 0.86 | 21.28978 | 1.00 | 20.13987 | 1.00 | 13.33795 |
| 110 | 2.98696 | 1.00 | 11.52197 | 0.49 | 10.36012 | 0.43 | 6.08898 |
| 120 | 0.00058 | 1.00 | 8.55466 | 1.00 | 7.40474 | 1.00 | 2.10468 |

Table 6.10: 1-day fixed-delay SDDE with $\Lambda = S(v)$ and maturity of 84 days.

| $K$ | $LB$ | $v$ | $UB_1$ | $v$ | $UB_2$ | $v$ | $MC$ |
|---|---|---|---|---|---|---|---|
| 80 | 32.40513 | 0.34 | 38.55032 | 21.00 | 37.31479 | 21.00 | 32.45972 |
| 90 | 22.57042 | 0.92 | 28.71980 | 21.00 | 27.48807 | 21.00 | 22.66273 |
| 100 | 12.93386 | 21.00 | 18.08917 | 21.00 | 17.92582 | 21.00 | 13.39260 |
| 110 | 5.03555 | 21.00 | 9.68020 | 0.52 | 9.15868 | 0.01 | 6.15034 |
| 120 | 1.12724 | 21.00 | 6.28254 | 21.00 | 6.19202 | 1.32 | 2.12080 |

Table 6.11: 21-day fixed-delay SDDE with $\Lambda = S(v)$ and maturity of 84 days.

### 6.2.2 Basket options

As in Section 6.2.1 for Asian options, we will use this part to compute basket options under the assumption that each of the $\{S_i(t)\}_{t\in[0,T]}$ follow the SDDE given by (4.3.3). To see how these results can be used in practice, we will compute these bounds for differing exercise values $K$, and compare them to the respective Monte Carlo estimate of the actual option price.

We will let $UB$ denote the upper bound given by the right-hand side of equation (4.3.19) and use $LB$ for the lower bound given appropriately by equations (4.3.25) and (4.3.27). We will also let $MC$ denote the Monte Carlo estimate for $B(K, T, \mathbf{w})$.

In this example, we assume $N = 3$ asset price processes that each follow the SDDE (4.3.3) with a common daily interest rate $r = 0.05/252$. We will assume that each process follows a fixed-delay model (that is, $\delta_i(t) = t - h_i$), where the delay parameters are given by Table 6.18. Further, we assume that the daily volatility functions $g_i$ are time-homogeneous and follow the functions and upper bounds also given in Table 6.18. We assume that the initial data, $S_i(-t)$, corresponds to the adjusted closing values of Vodafone's, Tesco's and BP's share price on the 21 days leading up to the 10th June 2009 respectively. Moreover, using the least-squares approach applied separately to the function $g$ given by equation (6.2.2) for each respective asset's initial data, we realise the functions $g_i$ given by Table 6.18.

We also assume that the weights used for this basket option satisfy the following

| $K$ | $LB$ | $UB_1$ | $UB_2$ | $MC$ |
|---|---|---|---|---|
| 80 | 32.11072 | 36.25611 | 35.71119 | 32.34201 |
| 90 | 22.15230 | 26.29769 | 25.75277 | 22.38359 |
| 100 | 12.19388 | 16.33927 | 15.79435 | 12.45520 |
| 110 | 2.40305 | 6.54844 | 6.00352 | 3.80425 |
| 120 | 0.00018 | 4.14557 | 3.60065 | 0.35110 |

Table 6.12: 1-day fixed-delay SDDE with $\Lambda = G$ and maturity of 21 days.

| $K$ | $LB$ | $UB_1$ | $UB_2$ | $MC$ |
|---|---|---|---|---|
| 80 | 32.11072 | 33.71666 | 32.11072 | 32.34659 |
| 90 | 22.15230 | 23.75826 | 22.15258 | 22.38817 |
| 100 | 12.22733 | 13.83327 | 12.27639 | 12.46974 |
| 110 | 3.68869 | 5.29464 | 4.02817 | 3.86997 |
| 120 | 0.34471 | 1.95065 | 0.53954 | 0.36957 |

Table 6.13: 21-day fixed-delay SDDE with $\Lambda = G$ and maturity of 21 days.

equation.

$$w_i = \frac{S_1(0)}{NS_i(0)}, \quad i = 1, \ldots, N. \tag{6.2.3}$$

The effect of this is that $w_1S_1(0) + \cdots + w_NS_N(0) = S_1(0) = 112$. Finally, we assume a constant correlation between the log-returns of each asset price process, given by the following matrix,

$$\{\rho_{ij}\}_{1\leq i,j\leq N} = \begin{pmatrix} 1 & -0.02646 & -0.01987 \\ -0.02646 & 1 & 0.32719 \\ -0.01987 & 0.32719 & 1 \end{pmatrix}. \tag{6.2.4}$$

In computing the upper bound for this basket option, we will create a set of monitoring times $t_j = j$, where $j = 1, \ldots, n_i$ and $n_i$ satisfies

$$n_i = \max\{j : t_j \leq h_i\}. \tag{6.2.5}$$

Then, for each $i$ we will use a conditioning variable based on the geometric average over the $n_i$ monitoring times, given by equation (6.2.6). We solely focus on this conditioning variable for the sake of computational efficiency, described at the end of Section 6.2.1.

$$\Lambda_i = \left(\prod_{j=1}^{n_i} S_i(t_j)\right)^{\frac{1}{n_i}}. \tag{6.2.6}$$

| $K$ | $LB$ | $UB_1$ | $UB_2$ | $MC$ |
|---|---|---|---|---|
| 80 | 32.40513 | 40.95921 | 39.80929 | 32.45318 |
| 90 | 22.57042 | 31.12450 | 29.97458 | 22.64745 |
| 100 | 12.73570 | 21.28978 | 20.13987 | 13.33795 |
| 110 | 2.98696 | 11.54104 | 10.39112 | 6.08898 |
| 120 | 0.00058 | 8.55466 | 7.40474 | 2.10468 |

Table 6.14: 1-day fixed-delay SDDE with $\Lambda = G$ and maturity of 84 days.

| $K$ | $LB$ | $UB_1$ | $UB_2$ | $MC$ |
|---|---|---|---|---|
| 80 | 32.40513 | 38.80494 | 37.64179 | 32.45972 |
| 90 | 22.57226 | 28.97206 | 27.80969 | 22.66273 |
| 100 | 12.88779 | 19.28759 | 18.14722 | 13.39260 |
| 110 | 4.86215 | 11.26195 | 10.18245 | 6.15034 |
| 120 | 0.96755 | 7.36735 | 6.26928 | 2.12080 |

Table 6.15: 21-day fixed-delay SDDE with $\Lambda = G$ and maturity of 84 days.

The result of this is that $\Lambda_i$ is a lognormally distributed random variable. We can then obtain the values $M_{\Lambda_i}, V_{\Lambda_i}, A_{\Lambda_i}$ and $B_{\Lambda_i}$ used in Section 4.3 by consulting Table 6.17. In this case, $Y_i(t)$ is given by Proposition 46 and $k_{i,t}$ satisfies

$$k_{i,t} = \max\{j : t_j < t\}. \tag{6.2.7}$$

In order to compute the lower bound, we use a single conditioning variable for all processes in the basket. To do this, we will introduce the following standard normally distributed random variable for all $i$, recalling that the $m_i$ are placed in decreasing order, as required by equation (4.3.11).

$$X_i(t) = \frac{Y_i(t) - \mathbf{E}^{\mathbf{Q}}\left[Y_i(t)\right]}{\sqrt{\operatorname{Var}^{\mathbf{Q}}\left(Y_i(t)\right)}}, \quad t \in [0, m_N], \tag{6.2.8}$$

We then set $\Lambda = X_1(t) + \cdots + X_N(t)$. If the correlations between the log-returns are assumed to be constant, as given by equation (6.2.4), then the value $\rho_i$ given in Section 4.3.3 is also constant and satisfies

$$\rho_i \equiv \frac{\sum_{j=1}^{N} \rho_{ij}}{N + 2\sum_{j=1}^{N}\sum_{k=j+1}^{N} \rho_{jk}}. \tag{6.2.9}$$

In Table 6.19, we assume that maturity is 20 days. As a result, we see that the set $\{i : h_i \geq T\}$ is nonempty in this case. On the other hand, Tables 6.20 and 6.21 use

| $t$ | $S_1(t)$ | $S_2(t)$ | $S_3(t)$ |
|---|---|---|---|
| 0 | 112 | 357.6 | 506.19 |
| -1 | 114.35 | 360.5 | 503.11 |
| -2 | 113.1 | 361.5 | 498.83 |
| -3 | 112.6 | 363.5 | 497.41 |
| -4 | 112.5 | 360.8 | 497.17 |
| -5 | 113 | 353.3 | 492.42 |
| -6 | 118.8 | 352.4 | 503.35 |
| -7 | 117.7 | 359.3 | 498.36 |
| -8 | 116 | 364.9 | 485.3 |
| -9 | 117.2 | 368.4 | 482.69 |
| -10 | 118.5 | 373.6 | 479.6 |
| -11 | 117.8 | 363.3 | 479.84 |
| -12 | 115.5 | 350.9 | 475.57 |
| -13 | 113.8 | 348.3 | 472.95 |
| -14 | 118.9 | 356.1 | 487.2 |
| -15 | 122.4 | 355.8 | 483.64 |
| -16 | 127.45 | 359.8 | 487.44 |
| -17 | 123.2 | 355.4 | 475.8 |
| -18 | 126.1 | 355 | 478.89 |
| -19 | 124.3 | 352.9 | 484.83 |
| -20 | 123.5 | 352.5 | 477.7 |
| -21 | 119.65 | 340 | 475.39 |

Table 6.16: Initial data used for valuing options under delay models.

a maturity of 40 and 80 days respectively, reflecting the fact that delay is likely to be small compared to $T$ in practice.

Similar to the examples given by McWilliams and Sabanis (2011), we see that the lower bound for the basket call option closely shadows the Monte Carlo estimate in this case when $K$ is in or at the money. As in Section 6.2.1, we see that the relative value increases substantially with the exercise price and is very large when $K \geq 120$. However, the absolute difference between the lower bound and its Monte Carlo estimate remains small throughout.

In contrast, whilst $UB$ provides values that are indeed upper bounds for $B(K, T, \mathbf{w})$, these are only close to the estimated values when $T$ is small. This is due to fewer terms of the form given by equation (4.3.18) being used and $S_i(t)$ assuming a lognormal model over a much larger portion of the interval $[0, T]$ for each $i$. However, the relative difference increases at a slower rate compared to $LB$. We see that $UB$ is 684.3 times larger than $MC$ when $K = 120$ and $T = 80$, for instance, in Table 6.21. Whilst this is a large difference, it is not as large as $MC/LB$ for the same exercise and maturity

| | |
|---|---|
| $M_{\Lambda_i}$ | $\frac{1}{n_i}\sum_{j=1}^{n_i}\left(rt_j - \frac{1}{2}\int_0^{t_j} g_i^2(u,\phi_i(\delta_i(u)))\,du\right)$ |
| $V_{\Lambda_i}$ | $\frac{1}{n_i^2}\sum_{j=1}^{n_i}(1+2(n_i-j))\int_0^{t_j} g_i^2(u,\phi_i(\delta_i(u)))\,du$ |
| $\mathrm{Cov}\,(Y_i(t), L_{\Lambda_i})$ | $\frac{1}{n_i}\sum_{j=1}^{k_{i,t}}\left(\int_0^{t_j} g_i^2(u,\phi_i(\delta_i(u)))\,du + (n_i - k_{i,t})\int_0^{t} g_i^2(u,\phi_i(\delta_i(u)))\,du\right)$ |

Table 6.17: Values used for the upper bound with $\Lambda_i$ given by equation (6.2.6).

| $i$ | $h_i$ | $g(x)$ | $\overline{G}_i$ |
|---|---|---|---|
| 1 | 21 | $0.00438x^{0.31572}$ | 0.03794 |
| 2 | 10 | $0.00877x^{0.06563}$ | 0.02707 |
| 3 | 1 | $0.45739x^{-0.57060}$ | 0.02941 |

Table 6.18: Volatility functions and upper bounds used for basket options.

values.

The addition of these terms when $h_i < T$ means that the upper bound tends to a limit that is significantly greater than zero as $K \to \infty$. This is because the terms given as upper bounds to equation (4.3.18) are constant with respect to $K$. Like the lower bound, we see that the upper bound performs considerably better for in-the-money options as a result.

| $K$ | $LB$ | $UB$ | $MC$ |
|---|---|---|---|
| 80 | 32.31683 | 36.37225 | 35.73029 |
| 90 | 22.35644 | 26.40749 | 25.77606 |
| 100 | 12.39604 | 16.44933 | 15.81900 |
| 110 | 2.37170 | 7.50882 | 5.84167 |
| 120 | 0.00000 | 4.24210 | 0.00152 |
| 130 | 0.00000 | 4.04416 | 0.00000 |

Table 6.19: Bounds for a basket option with maturity of 20 days.

| $K$ | $LB$ | $UB$ | $MC$ |
|---|---|---|---|
| 80 | 32.63241 | 38.76823 | 35.44307 |
| 90 | 22.71146 | 28.84296 | 25.48781 |
| 100 | 12.79051 | 18.92383 | 15.58637 |
| 110 | 2.82524 | 9.95285 | 5.67007 |
| 120 | 0.00000 | 6.50200 | 0.00081 |
| 130 | 0.00000 | 6.26583 | 0.00000 |

Table 6.20: Bounds for a basket option with maturity of 40 days.

| $K$ | $LB$ | $UB$ | $MC$ |
|---|---|---|---|
| 80 | 33.25982 | 42.10047 | 35.06367 |
| 90 | 23.41729 | 32.25363 | 25.24299 |
| 100 | 13.57477 | 22.41097 | 15.42048 |
| 110 | 3.71691 | 13.35012 | 5.55052 |
| 120 | 0.00000 | 9.56029 | 0.01397 |
| 130 | 0.00000 | 9.26548 | 0.00000 |

Table 6.21: Bounds for a basket option with maturity of 80 days.

# Chapter 7

# Conclusion

In Chapter 2, we explain the important concepts of comonotonicity, undertaken by Dhaene et al. (2002a,b), that are relevant to the valuation of arithmetic options. We then discuss a lower bound achieved by Albrecher et al. (2008) before demonstrating how this can be approved under the Black-Scholes assumptions. Moreover, we compute an upper bound using the techniques of Hobson et al. (2005) which, when combined appropriately with a corresponding lower bound can be used to provide an estimate of the true arithmetic option price.

The results obtained in Chapter 2 provide close upper and lower bounds for Asian options, as demonstrated in Section 6.1. However, these results set a strong requirement that the underlying marginal distributions for the corresponding sums of random variables used are known throughout. Furthermore, the models considered in this part are Markovian in their nature and do not take an asset's past history into consideration when one analyses their future values. In their stead, we introduce stochastic delay differential equations which relax this restriction. In order to price arithmetic options, we provide a new set of techniques that only require a known distribution for the underlying asset for a small subset of its lifespan.

The numerical results obtained in Chapter 6 demonstrate how Asian and basket options behave under the stochastic delay models presented in Chapter 3. We see that the lower bounds given for such derivatives perform very well in absolute terms. However, the absolute and relative difference of both upper and lower bounds are more useful when the corresponding exercise price $K$ is small or close to the initial value of the stock (or weighted set of stocks) considered.

In addition to option pricing under delay models, we also considered the requirements for exponential stability of both a solution to the defined SDDE and of its corresponding Euler-Maruyama numerical scheme. We extended the results obtained for fixed-delay models (where $\delta(t) = t - \tau$) to a more general case, setting a small requirement that the delay function is strictly increasing throughout. Moreover, we outline the conditions for which mean-square numerical convergence is achieved, which does not require $\delta$ to be increasing. An ideal progression for this area would then be to find conditions under which more general delay functions can be used and yet still achieve exponential stability: an example in particular includes the step function $\delta(t) = \lfloor t/\tau \rfloor \tau$ introduced earlier in this report.

## 7.1 Further considerations

Exploring and understanding the behaviour of stochastic differential delay models has the potential for the study of a wide variety of fields, in mathematical finance and elsewhere, of which we have only considered a small area. The study of comonotonicity given by Dhaene et al. (2002a,b) can be immediately applied to the valuation of stop-loss premiums for stochastic annuities (see, for example, Goovaerts and Dhaene (1999); Goovaerts et al. (2000); Vyncke et al. (2001)). One can immediately apply Lévy models to these instruments (using Albrecher et al. (2008) and Hobson et al. (2005) along with Bertoin (1996)) and it may also be practical to apply delay models in this case as well. For instance, the effect of a change in a person's lifestyle cumulated over a number of years may result in a marked change in their longevity.

This report considers arithmetic options and focuses primarily on Asian options with basket options also studied, with vanilla European options under SDDEs presented by Arriojas et al. (2007). One can also attempt to apply delay processes to a much wider range of models and derivatives. For example, we could immediately introduce SDDEs, based respectively on the Vasicek and Cox-Ingersoll-Ross models (see Vasicek (1977) and Cox et al. (1985)), for the term-structure of interest rates as shown below.

$$dr_1(t) = \kappa_1(\hat{r_1} - r_1(t))\, dt + g_1(r_1(\delta(t)))\, dW_1(t), \tag{7.1.1}$$

$$dr_2(t) = \kappa_2(\hat{r_2} - r_2(t))\, dt + g_2(r_2(\delta(t)))\sqrt{r_2(t)}\, dW_2(t). \tag{7.1.2}$$

Under the study of such models, we can study further derivatives based on our results. For example, interest rate caps and floors are based on vanilla call and put options respectively.

For both of the models proposed above, we assume that the underlying interest rates satisfy local volatility models based on historical data. In extension of the material followed, one can also consider the valuation of options on assets and other derivatives under a probability measure that is not risk-neutral. This involves the study of SDDEs where the drift parameter can also be a function of past and present information, as presented by equation (3.1.1). As we assume a positive local volatility function $g$ in this equation and are considering the value of asset price options, we did not need to consider such valuation in this report.

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