THE TEACHING OF ELEMENTARY MATHEMATICS IN SCOTLAND IN THE 19th CENTURY

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by

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TABLE OF CONTENTS

Introduction 1

Part I. A. Organisation and Content of Mathematical Teaching.

(i) The Framework of Education 16

(ii) Systems of Mathematical Teaching.. .. 24

(iii) "Supply and Demand" 36

(iv) Individual Subjects 47

B. Factors influencing the Purpose, Content and Organisation of Mathematical Teaching.

(i) Social and Economical Factors 92

(ii) Contemporary Mathematical Thought .. 101

(iii) Contemporary Educational Thought.. .. 123

(iv) Administrative Control.. 128

Part II. Method: General 137

(i) Teaching Technique in individual Subjects 146

(ii) Social and Economic Factors 175

(iii) Contemporary Mathematical Thought .. 184

(iv) Contemporary Educational Thought .. 191

(v) Administrative Control.. 196

Review of the Century 201

Bibliography 206

INTRODUCTION

"Mathematics are studied either by gentlemen of birth and fortune as a necessary part of genteel education or by those in the middle rank of life, in order to qualify them for the professions." [1] These words were written in 1779 by Ewing, an Edinburgh teacher of Mathematics. He was successful, independent, respected and conducted his classes in his own house in Bishop's-Land-Close. He had no doubts as to the two-fold purpose of his teaching. Mathematics was to be taught as a necessary part of liberal education or else as part of the technical training for certain professions. The worthy master was not much concerned with why it had become part of a liberal education - it might trace its lineage to Plato through many vicissitudes or it might go back no further than the theory of mental discipline - but he knew why he personally was teaching it to a particular pupil and he adapted his teaching accordingly.

This clear recognition of the purpose of teaching Mathematics was typical of the end of the 18th and beginning of the 19th century when education was confined to comparatively few individuals. Later the dual nature of the teaching became obscured and lost sight of through the increasing demands of

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1 Ewing: "Practical Mathematics"

science, the growth of the examination system and the provision of educational facilities for all of school age. At the end of the century mathematics was still taught in the schools but its teachers had lost that intimate contact with the reason for its being taught which was so marked earlier.

The picture just given of the upper strata of society studying under teachers who were themselves independent professional men was typical of town life at the end of the 18th century. In the rural districts, the parish schoolmaster, that versatile pedagogue, professed mathematics along with the Classics, English and whatever else was required. Great social and economic changes took place, however, which by the end of the 19th century had completely altered this picture. Higher education was no longer the prerogative of the few. Mathematics for the many if not for 'the million' had come. Great new secondary schools had been founded, staffed by many masters under the authority of a headmaster. The older burgh schools and academies had been reorganised on similar lines. The parish schoolmaster was now the headmaster of a primary school though he frequently still retained pupils at the secondary stage. Lastly a co-ordinated system of education controlled by a

state department had come into being.

In this very brief preliminary survey we have indicated factors influencing the purpose of mathematical teaching, the reason for studying mathematics and the material aspects of teaching. As long as the pupil had freedom of choice there was usually no difference between the purpose of the teaching and the reason for the study but once a compulsory factor was introduced this agreement was destroyed. Except under a system of compulsion the reason for the study of any subject is the driving force in the teaching of that subject. It is only through an understanding of the driving force that one can appreciate the reaction of the other forces. Over a period of a hundred years the driving force did not remain unaltered and it is our intention now to state the main reasons for mathematical study as given by contemporary writers or illustrated by contemporary events and then later to discuss the changes.

In our opening quotation Ewing gives what he considers the main reasons for studying mathematics and therefore for the existence of mathematical teaching. By "genteel education" he meant the education of a gentleman or what was then held to be synonymous, a liberal education. It is obvious why mathematics was part of the professional training of

say, a surveyor or a civil engineer but an explanation is needed as to why it was part of the education of a gentleman. Professor Playfair of Edinburgh University wrote this concerning the study of Euclid: "The mind, especially when beginning to study the art of reasoning, cannot be employed to better advantage than in analysing those judgments, which though they appear simple, are in reality complex, and capable of being distinguished into parts." His successor, Sir John Leslie, claimed that "the study of mathematics holds forth two capital objects: while it traces the beautiful relations of figure and quantity it likewise accustoms the mind to the invaluable exercise of patient attention and accurate reasoning. Of these distinct objects, the last is perhaps the most important in a course of liberal education."

At St. Andrew's University, Professor Duncan advocated the study of mathematics as part of professional training and also "as a preparation for other studies; as a discipline to the mind; and as a source of elegant pleasure." In addition he raises certain objections which he does not regard too seriously: "It unfits for the finer pursuits of poetry and eloquence," and "it requires a peculiar turn of mind." A new reason to emerge here is the "preparation for other studies." This

grew in importance. For instance during the first year of the Edinburgh School of Arts (1821-22) a number of the students found that they could not follow the lectures in Natural Philosophy because of an inadequate knowledge of mathematics so on their own initiative they formed a "Mathematical Academy of Tradesmen" with one of their number, a joiner, acting as the instructor. A second class was formed almost immediately, this time under a cabinet maker, and both classes continued to the following year when the Directors provided an official course. Another instance is shown in the encouragement given by Professor J. Thomson of Glasgow to students of algebra to reflect "on the extreme value of the science. In itself, indeed, and in its application and extension in the differential and integral calculus and in other branches of pure mathematics, it is a most powerful, an indispensable instrument for prosecuting investigations in mechanics, astronomy and other subjects in physical science; and, without its aid it is impossible to understand, or duly to appreciate, the discoveries of Newton, Laplace, and the other great men who have done such wonders in extending the boundaries of modern science."

Next we shall give the views of a teacher not

a professor. George Lees was an exceedingly busy Edinburgh master of wide experience who at one time regularly conducted classes in the High School, the Scottish Military and Naval Academy and the School of Arts all in the same day. In addition at a later period he taught in the "Scottish Institution for the Education of Young Ladies" and in his report for 1835 for that school appears these words: "The study of mathematics....by its closeness of reasoning and clearness of demonstration, more than any other, corrects the wanderings of the imagination and by teaching us to connect our ideas in a chain of dependence, exercises our judgment and enables us to choose between truth and error." Thirty five years later after a lifetime spent in teaching the same man giving "An address in defence of Euclid's elements of Geometry as a classbook for students" commented on "the great excellence of its educating power.....as drawing out, exercising and developing those faculties by which a human being thinks, and reasons, and judges."

It is rather significant that in the early part of the century the authors of text books were accustomed to preface their works with "an enconium on the usefulness and excellency" of their particular subject but that at the end of the century this

practise was uncommon and instead bald statements were made such as "this book contains the material required for the - examination" or else "the present volume is up to the standard of - certificate." No higher reason was given for its study than its utility in obtaining examination passes.

For a re-statement of the reasons for mathematical teaching as it appeared to a writer at the end of the century we shall turn to a work[1] by Professor Smith of Edinburgh University who at one time held the chair of Evangelistic Theology, New College, Edinburgh. "The use of general education" wrote Professor Smith, "is not to fit any man for a specific calling or profession. It is to fit every man to be the best that his nature and abilities admit of, for any profession that he may adopt, or any position that he may have to occupy. If it is to do this it must be by cultivating all his powers, physical, mental, moral and spiritual." He continued thus: "One of the most important of human powers is reason...Every subject of study, languages, history, geography, poetry will aid in the culture of this faculty. But each of them seems to want an element indispensable for completeness, the element of continuity . . . . . We are convinced that there

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[1] Euclid: "His Life and System."

neither is nor can be any department of study which could supply the means of continuous exercise of the reasoning faculty to any extent approaching that to which they are afforded by geometry. It seems impossible, that with any materials other than geometrical, so long a chain could be forged. And then if the study is rightly pursued, every link of the chain must be tested . . . . ." Later he amplified his view further on the "importance of the study of mathematics as a means of cultivating the reasoning faculty," praised the study of geometry as an aid "in developing the faculty of attention" and last of all commented on the twofold use of mathematical study "as a mental discipline" and "as a means of acquiring important and useful knowledge."

From this review of the writings of leading men of the century the following observations can be made:-

1. Despite differences in the phraseology used such as "mental discipline", "cultivation of the reasoning faculties", it is clear that these writers held in common the view that the highest purpose of mathematical teaching was the training it gives in the handling of abstract ideas.

2. The traditional intellectual outlook was becoming too narrow with the development of the idea that mathematics should be studied as an aid to further studies in the sciences.

3. The value of mathematics in professional training was never questioned.

4. Towards the end of the century, for many teachers the purpose of teaching came to be the preparation of pupils for passing particular examinations.

Such then were the contemporary reasons for teaching and studying mathematics. These were the ideals set before its teachers. It is the purpose of this paper through a historical survey to show in what measure these ideals were achieved, to show how far material considerations limited their attainment, to show how the ideals led to differences in the teaching and to show why these ideals themselves changed. But as we are dealing with men with all their weaknesses and inability to live continuously at a high idealistic level it is also the purpose of this paper to examine the day-to-day details of teaching technique when ideals tend to be pushed aside under the immediate pressure of attaining some small local objective and to see if these ideals were realised or were just 'bait' to attract the customers.

In accordance with this view of the purpose of this essay we propose to consider our subject under two main headings: the first will treat of the position of mathematical teaching in the educational system, that is, mathematical teaching as viewed

objectively by someone not actively engaged in its teaching or study; and the second will treat of mathematical teaching at the classroom level when it has become a man-to-man issue between master and pupil. In a military operation these divisions would correspond to strategy and tactics. Under the first head we shall discuss such questions as the purpose of mathematical teaching, its organisation, the branches taught, the status and working conditions of the teachers, the standard of attainment of the pupils and other matters relevant to the broad, impersonal issues. Under the second head we will deal with the field of day-to-day teaching. Professor Kelland once told his students that a teacher "to succeed.....must build a bridge from his own attainments to theirs (his hearers) and he must himself be the first to cross it." It is with the building and crossing of these mathematical bridges that we are concerned under this second head: with technical details such as the way in which multiplication was taught and the 'settings' used; with the ordering of topics within a branch such as the 'placing' of the study of simple equations in the development of algebra; and last of all with the way in which mathematical ideas were developed, for example, the idea of ratio.

In both divisions, in so far as it is of advantage to our main purpose the plan of campaign will be to state first, without comment, the historical facts as far as these can be ascertained and then afterwards to discuss separately the various factors which influenced the situation. By thus gathering together all the variations and modifications attributable to the same cause it is hoped to come to some assessment concerning the relative importance of that cause in shaping the pattern of mathematical Teaching. This will be no hard and fast plan for in considering a text book, such as Leslie's "Philosophy of Arithmetic" it will give a better conception of the book and its significance if we review it as a whole and refer back when necessary rather than review it piecemeal throughout the essay.

We have dealt at some length with the two divisions into which we propose to divide this work for simple, well defined terms to describe them are not easy to find. Using everyday loosely-defined notions we might say that the first division comprises 'Organisation and Content' whereas the second deals with 'Method. But as regards mathematical teaching what is 'content'? And what is 'method'? If mathematics is taught for the sake of arriving at a knowledge of the so called 'truths'

(e.g. the result of the Theorem of Pythagoras) then a list of these truths defines the content. But if the purpose of the teaching is to "exercise and develope those faculties by which a human being thinks and reasons" then surely the method of proof and not the thing proved becomes the content of the study. We have already shown that mathematics was taught for both these reasons therefore a particular topic might be classified under one or the other or both of these headings according to the reason for teaching it. Thus the terms are not mutually exclusive and their use would lead to confusion unless they are more carefully defined. So far as the remainder of this paper is concerned the word 'content' will be used objectively in the sense in which it is used in the 'table of contents' at the start of a text book. It will be taken as comprising the truths established whether that was the purpose of the study or merely incidental to it. On the other hand we will restrict the use of the word method to the 'building and crossing of the mathematical bridges,' that is, to the actual way in which a master transmitted a knowledge of the truths to educate, to train and to inform his pupils. With these restrictions we shall use these terms but always without prejudice to the question that a

'method' may be taught as 'content'.

One last matter remains to be discussed in this introduction, namely the scope of the essay. In the title we have used the term "Elementary Mathematics" a somewhat vague phrase which demands a word of explanation. Hardy's "Pure Mathematics" a wellknown work studied by advanced students at our Universities is described by its author as "being really elementary" that is, dealing with the fundamental ideas which are the starting points for chains of deductive reasoning. On the other hand, in common speech, by association of ideas, 'elementary' is often used as synonymous with the initial stages of a study and even with the easy parts of the study. In the teaching of Latin it is customary to study the 'elements' of the language before proceeding to read the authors. As the study progresses less stress is laid on the 'elements'. Mathematics however is somewhat unusual in that the novice and the don both study its elements and perhaps the don gives far more attention to them. Leslie wrote that it was "the nature of mathematical science to advance in continual progression. Each step carries it to others still higher." Mathematics starts from certain hypotheses which may or may not be true and from these develops trains

of reasoning leading to conclusions which are true if the hypotheses are true. In this sense Mathematics may be held to 'advance'. But it does not advance in only one direction. Mathematicians are constantly examining their hypotheses to see if some more general hypothesis could be found of which these are particular cases. Thus Mathematicians try to extend their knowledge in the reverse direction. Indeed to use a metaphor from its own language, Mathematics is like a continuum, the origin can be chosen anywhere and progress made in either the positive or the negative direction. Or a rotation of the axis can be made and the whole investigation given a new direction altogether. Mathematics is most definitely a science in which a study of the 'elements' is not confined to the initial stages but is taking place all the time. Thus if our subject had been "The Teaching of the Elements of Mathematics" its scope would have included the whole field of mathematics. However it is "The Teaching of Elementary Mathematics", a change in phrase which permits of quite a different interpretation. "Elementary Mathematics" will be taken as referring to those parts of mathematics which are normally studied as the elements or basis of mathematical education not as the basis

of mathematics itself. The phrase 'normally studied' is used advisedly for we have known a student who had never studied trigonometry make excellent progress in the calculus and learn trigonometry as a by-product of his main study.

Our first task then is to ascertain what parts of mathematics were studied as the basis of mathematical education. Opinions differ on this question but for the purpose of this essay we will consider them as comprising those parts of mathematics which form part of a general education and which therefore serve as the basis of specialised mathematical education. In our present educational system even this is not a clearly-defined stage for some pupils start to specialise in their fourth year at a secondary school while others take the first university class as part of their general education. As a convenient dividing line and speaking of the present day we might say that the full primary and secondary school course constitutes the basis of mathematical education as developed and expanded in the universities and technical colleges. But were the elementary parts of mathematical education the same in 1800 as today?

The main topics taught at the present time in the schools, viz. arithmetic, algebra, geometry,

trigonometry were taught in both schools and universities in 1800. On occasion the professors developed their courses on the assumption that the students only knew arithmetic. On the other hand topics such as landsurveying, navigation and bookkeeping which were quite common in the schools then are very uncommon today though beginning to return to favour, particularly in the new junior secondary schools. What is regarded today as the basis of mathematical education was in 1800 regarded by many as the complete thing. It is one of the aims of this essay to show how this change came about. To do this it is necessary to discuss the teaching of subjects not thought elementary (on our definition) in 1800, to discuss some outwith the present school curriculum and to discuss University teaching where applicable.

Part I. Organisation and Content of Mathematical Teaching.

1. The Framework of Education:

A. Prior to 1872 At the start of the century the burden of elementary mathematical teaching was borne in the rural districts by the parish schools and in the towns by the burgh schools, grammar schools, academies, and private specialist teachers.

Much elementary teaching was also given in the universities. These were the principal agencies but outside this framework there existed many small private schools and further the custom of employing tutors was exceedingly common particularly in farming communities where during the winter the farmer might employ a tutor for his own and his workmen's children. The work of tutors and the private schools was in the main to prepare pupils for the parish or burgh schools. Thus the mathematics taught was very simple indeed; consisting of numeration, the four rules of arithmetic and little else.

As the century progressed the system of education was realised to be inadequate particularly in regard to the provision of schools for the rapidly developing industrial areas and also for the isolated rural parishes. Some improvements were made in the supply and accomodation of parish schools and efforts, largely successful, were made to improve the standard of scholarship and technique of the teachers. But the legal provision of one school per parish was hopelessly inadequate in the new industrial areas. Between 1801 and 1840 something like 350,000 people - nearly four times the 1801 population of Glasgow - settled in urban

conditions in the Clyde valley*. To meet the requirements of these areas innumerable small private schools sprang up. In addition new schools were founded by the Church of Scotland, the Free Church, charitable societies and factory owners. The Church founded many schools in isolated districts to which it wished to bring the 'blessings of education'. The other charitable schools and the 'adventure' schools were more concerned with the prevention of illiteracy amongst the lower social orders. Prior to 1872 there were over four thousand schools in existence.

The new schools founded by the Churches served as Parish schools. As regards the others, despite their numbers their role in the teaching of mathematics was the very minor one of teaching the first stages of arithmetic. There were isolated but distinguished exceptions mainly in the University towns where private schools, such as Edinburgh Academy, were founded to provide more secondary education. The instruction in these schools and their educational ideals were similar to those of the burgh schools; and indeed they were recognised as fulfilling the same function in the life of the community.

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* Mackenzie "Scotland in Modern Times" p. 108

Part of the drive to found schools came from would-be students themselves and this period saw the growth and decay of the Mechanic's Institutes.

Broadly speaking the organisation of mathematical teaching in Scotland prior to 1872 was as follows:

(I) Large numbers of private schools of various types teaching numeration and the four rules

(II) Parish schools providing a good grounding in arithmetic and frequently the equivalent of modern junior secondary education

(III) Higher Class Schools (i.e. Burgh Schools, Grammar Schools, Academies and a few private schools) providing secondary education and also in many cases primary education

(IV) Universities teaching the Principles of arithmetic and duplicating the mathematics of the higher class schools

(V) Miscellaneous organisations, such as the Mechanic's Institutes and Ladies' Institutes whose work did not dovetail with any other organisation even in this widely overlapping system.

B. The Period of Transition 1872-1892. After the Act of 1872 the pattern becomes less complex partly through the large reduction in the number of schools and partly through the separation of function of

school and university. The school boards used their powers with energy and enthusiasm and within a few years the country had an adequate and efficient system of primary schools. The instruction given was governed by the very rigorous provisions of the Revised Code and the newly instituted system of school inspection ensured that the required standards were in practice reached. The educational ideal underlying this code was a very simple one. The State desired the maximum of reading, writing and arithmetic for its money,

This Act however did nothing to coordinate school and university teaching. Indeed, as will be shown later, its very success jeopardised secondary education in districts dependent on the parish school. The State was forced to further action. Public funds were made available for the development of secondary education. In addition, as free primary education was now available to all, parliamentary sanction was given for the diversion of endowments from primary to secondary education. This enables the existing higher class schools to re-equip and rebuild where necessary and at the same time made possible the establishment of great new secondary schools such as George Heriot's School and George

Watson's College.

The overlapping of school and university teaching had long been deplored and only tolerated as a necessary evil because the necessary administrative machinery could not be set up. In 1886 government inspection of the higher class schools was undertaken and this was followed in 1888 by the establishment of the Leaving Certificate which created a common standard for these schools. The Universities were persuaded to accept this standard as the commencing point of their teaching and confirmed it by the establishment in 1892 of a common Entrance Examination of a similar standard.

C. The Co-ordinated System. 1892 onwards. During the last decade a simple pattern emerged: the primary schools taught arithmetic; the secondary schools or secondary departments of schools taught arithmetic, algebra, geometry and trigonometry; the universities devoted themselves to more advanced work. But the edges of the pattern still remained blurred. Some primary schools retained their pupils past the primary stage but only offered them one or two years of secondary instruction. Some of the best of the secondary schools had always taught to a standard higher than the new Leaving Certificate and continued

to do so. Lastly the evening continuation schools had no place in this scheme although they were left free to exploit the fields of commercial and technical education.

While this became the dominant educational pattern later an alternative pattern for mathematical education emerged: primary school: secondary school for three to six years: evening or day classes at commercial institutes or technical colleges.

Attention has already been drawn to the two and later three-fold purpose of mathematical teaching. Unfortunately, when the Leaving Certificate was established, this characteristic was either ignored or not thought to cause material differences in the teaching. The first brief given to the Scottish Education Department was that they should maintain contemporary school standards of attainment. There was no mention of developing a liberal outlook, developing a system suitable for further study in the science or developing professional training. The Department sidestepped the issue and compromised. It fixed its standards by 'averaging out' the examination results of a selected number of schools of different types. Within a few years time most

secondary schools were presenting candidates for the Leaving Certificate and "schemes of work leading to the Higher Leaving Certificate" began to make their appearance. A new reason for teaching had emerged: the preparation of pupils for a particular examination. These new schemes provided material suitable for a liberal study but insufficient time for it. They also formed a fairly useful foundation for future study. But they proved unsuitable for professional education. The fault did not lie in the content of the courses but in the fact that they were geared to a five or six year study whereas most pupils completed only two or three years. If this latter class of pupil wished further mathematical training they could only obtain it by attendance at Continuation Schools or Technical Colleges.

Although this second development belongs to the twentieth century for many years previously the Department of Science and Art had given grants to promote instruction in mathematical subjects. In addition when the Scottish Education Department issued in 1893 its Code for Evening Continuation Schools it recognised, among other subjects, navigation. Also although such classes did not qualify for government aid the technical colleges

and continuation schools ran well-attended courses in Practical Mathematics and commercial subjects. Thus the foundations for this second pattern were laid in the nineteenth century even if at the time these schools and colleges were regarded as auxiliaries and not as part of a co-ordinated system of education.

2. Systems of Mathematical Teaching

Throughout the century a wide variety of mathematical subjects was taught in the schools - some perhaps only to one or two pupils at a time but instruction was available to the pupil who was prepared to make the necessary effort to obtain it. Regular courses were run on Mathematics, Practical Mathematics and Mixed Mathematics or if preferred the student could sample individual subjects. This freedom of choice continued until the requirements of the Leaving Certificate canalised mathematical teaching. Mixed mathematics, despite its name, is outwith the scope of this essay as it referred to what is now known as Physics.

Although the terms were loosely used the distinction between Practical Mathematics and Mathematics appears, in the early decades at least, to have depended on the reason for teaching rather than on the subject matter which was frequently the same.

If mathematics was being taught as part of a liberal education where the object was the general development of the intellectual powers of the individual then the way of arriving at a particular result was the important thing and the study was named "Theoretical Mathematics" or, more often "Mathematics". On the other hand when mathematics was taught as part of a professional training where more stress was laid on the acquisition of certain techniques, on the knowledge and application of certain results then the study was termed "Practical Mathematics". The distinction rested on the reason for the teaching which in its turn dictated the manner of the teaching.

Later, the distinction became one of content and method rather than reason. There was no sudden shift of opinion, no clear exposition of principle but just a casual drift from one commonly held viewpoint to another. Practical Mathematics because it was less abstract came to be regarded as that part of mathematics which dealt with actual measurements and the use made of them in calculation for example mensuration, ruler and compass constructions, problems in navigation etc. It did not matter how the topic was taught, if it included measurement by the pupil it was "Practical"; or if it included

some technique without any understanding of the principle. For instance we have "practical arithmetic" where the pupil was taught the four fundamental rules without any understanding of the principles involved and as a result frequently could not apply his knowledge unless the problem was clearly marked "multiplication" or whatever operation was necessary. Then again navigation came to be regarded as solely belonging to Practical Mathematics yet in the early decades it had belonged to both divisions according to the way in which it was taught. The Scotch professors taught navigation not to produce navigators, that was the job of special schools, but to illustrate and enrich mathematical ideas and so develop and train the minds of their students. In like fashion Surveying came to be regarded as "practical"; yet cannot a study of the principles of triangulation be as educative as the comparison of triangles in an abstract problem? The same facts about a triangle must be known yet the former involves ideas of locus, position and proportion whereas the latter involves an abstraction of the idea of superposition. The answer lies in the manner in which surveying is taught. At the beginning of the century Practical Mathematics was taught in a liberal

way but by the end of the century pruned of professional subjects such as navigation it had degenerated into an unconnected set of rules and procedures.

At the end of the 18th century the word "practical" was used of any application of mathematical theory e.g. the use of calculus in solving a problem on velocity or the use of square root in finding the side of a square of known area. There was no question of not studying the theory first. The university professors following the lead given by Gregory in the 18th Century were accustomed to teach the full theoretical course first without interrupting it to give any practical applications and then to select a single subject say navigation or surveying to demonstrate principles already taught. From this logical, rather than phychological approach, it is easy to see how the name "Practical Mathematics" came to be attached to a certain group of mathematical topics. As far as the schools were concerned it was an unfortunate name for there this group of subjects was taught to prepare youths for entering on professional careers, but, as we have already mentioned, with the passage of time, the word "practical" came to be applied in its lowest sense and the course became debased and discredited as an educative

instrument. A contemporary English writer, Professor De Morgan contributed a paper on this subject in 1838 to the Central Society of Education wherein he suggested that the name be altered to "Professional Mathematics". We shall quote from two paragraphs in this article firstly to show the use of the word "practical" in mathematics (and incidentally to spotlight contemporary mathematical thought) and secondly to show what professions were then regarded as mathematical.

"The man whose thoughts are occupied by the development of the relations of magnitude, and who never stops to think whether the object of his meditations will help to find a planet or build an arch, contends that . . . . . the real and practical bearings of the subject are to be looked for in its effects upon the intellect. Next comes the follower of Newton, applying the most profound analysis to the development of the theory of gravitation; and by his side is the deducer of new laws of light from the undulatory theory, and the analyzer of electrical and magnetical phenomena: all mathematicians of a high class; all agreeing that mathematics by themselves are barren and must be applied to something practical . . . . . Go down a step lower, and we find the

friends we have just quitted laughed to scorn, under the title of 'great philosophers' by men who care little for laws of matter, and to whom the mention of practical mathematics suggests the builder's price-book and the sliding rule . . . . . The same downward process continues until at last we come to a class, any individual of which on being asked "Are you acquainted with fractions?" would answer disdainfully, "Sir, I am a practical man." Now of all the divisions which we have named we may safely say there are practical men in each."

De Morgan, having thus shown how useless the word 'practical' is as an exclusive term for indicating a particular section of mathematical study then discusses the function of the course on Practical Mathematics and arrives at much the same conclusion as we have from our researches a century later into Scottish sources, namely that it was a course for training professional mathematicians. He continues "We shall adopt the term 'professional mathematician' meaning by that term

1. The accountant, actuary, or merchantile mathematician.
2. The civil engineer, including every species of artificer who uses calculation.

3. The military mathematician.

4. The navigator.

5. The surveyor, including all whose occupation is to measure quantities.

6. The draughtsman."

It is very curious, that considering all the topics that might have been regarded as 'practical' there was apparently pretty close agreement among contemporary writers, in Scotland at least, as to what did constitute a course on Practical Mathematics. It consisted of some or all of the following

(I) Practical Geometry. (Mainly Euclid's Definitions and the 'ruler and compass' constructions)

(II) Plane Trigonometry (Usually to the solution of the right-angled triangle)

(III) Mensuration of heights and distances

(IV) Mensuration of surfaces (Rectilineal figures and the circle).

(V) Mensuration of solids (Duodecimals; Gauging).

(VI) Navigation

(VII) Logarithms

(VIII) Gunnery

(IX) Fortification

(X) Projectiles

(XI) Spherical Trigonometry

The first five topics were the basis of the course and often the full course but the others were also regarded as part of the course to be taught according to the wishes of the pupil. In passing, it may be observed that this Scottish course was quite different from that taught under the same name at the English Public Schools at the close of the century. Certainly the English course included mensuration but otherwise it was a course in physics along with some engineering workshop calculations.

There were few textbooks available solely devoted to Practical Mathematics. In the first decade Ewing's was still in use but it had been published in the previous century (Edinburgh 1779). Gregory's book, another 18th century text was still used at Glasgow University in 1837. One of the first acts of the Scottish School Book Association when it was formed in 1818 was to commission "a system of practical mathematics at five shillings". Davidson's was popular in the middle of the century, the sixth edition being published at Edinburgh in 1854. In the Universities it was more common to use the

relevant sections of larger works such as Hutton's "Mathematics" or Duncan's Supplement". Despite the efforts of the Scottish School Book Association to produce a full work on the subject at a popular price it became customary to make the texts on mensuration cover the basic topics and to publish individual texts for the remainder. Popular works conceived on this plan were Ingram's "Mensuration" (1822) and Elliot's "Practical Geometry and Mensuration (1845) the latter being used in the universities as well as the schools. Another plan, popular with writers on navigation, was to preface the work with a short treatise on those parts of mathematics necessary for a complete understanding of the subject under discussion. Pryde's "Navigation" (Edinburgh 1867) adopts this practice. The texts just mentioned covered in whole or in part the content given above and treated it 'theoretically' as well as 'practically'. Practical Mathematics was not a watered-down version of the traditional mathematics course but was a development of mathematical principles in which the Euclidean order was not followed.

In contrast to the practical or professional course, the content of the theoretical course, or more simply, mathematics, was less varied. In the

schools in the early decades it comprised Euclid, Algebra and Mensuration. In the second half of the century Trigonometry was added. Having completed this basic course a few pupils went on to study Conic Sections, Calculus, or Modern Geometry. In the universities in the early decades, the course of study of the first class usually included Euclid, Plane Trigonometry, Algebra and the Theory of Arithmetic. In the second class conic sections and the practical applications of mathematics were studied and in the third class (which very few students took), calculus. There were of course variations in this scheme of the placement of the topics mainly in the position of Trigonometry. Round about the forties, the universities reorganised their curriculum making the calculus their main objective instead of an additional target for the specialist. To prevent overcrowding the curriculum they were forced to abandon the teaching of 'practical mathematics' but they retained the other subjects.

There were several comprehensive texts available devoted to Mathematics but for the most part separate texts were published on each subject. Favourite authors throughout the century were Hutton, Gregory, Wood, Duncan, B. Smith, Bryce, Todhunter, Colenso,

Loney and of course "Euclid" but there was a wide range to choose from. More information on this point is given in the form of an appendix. One interesting Scottish text was the "Supplement" written by Professor Duncan of St. Andrews. It was written as a supplement to Playfair's 'Geometry' and Wood's 'Algebra'. In conjunction with these works it was designed to give a complete three years course of mathematics as taught at the university level. Its content comprised the Theory of Arithmetic, Plane Trigonometry, Mensuration, Landsurveying, Solid Geometry, Mathematical Principles of Geography, Navigation, Fortification, Conic Sections, Spherical Trigonometry and Fluxions. For such an exceedingly comprehensive work the title seems too modest!

Both courses, Mathematics and Practical Mathematics were taught in the parish schools, the higher class schools and by private teachers. After the Act of 1872 however the parish schoolmasters had to divert much of their energy to purely primary work and the remainder to the specific subjects where the course laid down was mainly theoretical. Thus Practical Mathematics disappeared from these schools except perhaps for a 'bit of surveying' at the end of the summer term. It also disappeared from the higher

class schools but here the process was one of slow attrition. After the reorganisation of the university curriculum those schools whose main function was to prepare boys for the university gradually gave up teaching the practical course. This reorganisation cut drastically the supply of men with a knowledge of this course and so as the older generation of teachers died out so did the course. By the time the Leaving Certificate was instituted most of the secondary schools taught only the 'theoretical' course which therefore became established as the only development permissible. There still existed a demand for professional courses. This demand was met by the Technical Colleges, Continuation Schools and private schools. Indeed the demand was such that the Technical Colleges were able to do this without the aid of a government grant.

It was unfortunate that in the last decades the importance of the professional course was overlooked when the secondary school mathematics was standardised and harnessed to the needs of the universities. Had this not been done the problem of the 'non-academic' pupil who required mathematics for a technical career would never have arisen. For the next half century all that the day schools could offer such a boy was

a watered-down version of a course leading to the University with the ideas spread over five years for the sake of vigorous proof where he required them packed into two. Trigonometry was not started till the fourth year of the secondary course whereas recent experiments have shown that boys can appreciate this subject at a much earlier age. There would have been no occasion for these experiments had we remembered the work done in Practical Mathematics a century ago. Not till recent years[1] was freedom given to produce suitable courses unrelated to university requirements. These new courses which are at present being tried out in our Junior Secondary schools bear a strong resemblance to the 'Practical Mathematics' course of the last century in their content and in the motive for teaching them - the desire that the study should be educative as well as of practical value.

3. "Supply and Demand"

In the early decades we find schoolmasters, both parish and burgh claiming to give instruction in such branches as navigation, bookkeeping, fortification, mensuration, landsurveying, gauging, astronomy, use of the globes and dialling.[2] There was a certain

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1 "Mathematics in Secondary Schools" 1950. Scottish Education Department.

2 Gnomonology

looseness in terminology however which allowed considerable individual freedom in subject classification. For instance the schoolmaster at Saline, Fife in reply to a Parliamentary questionaire in 1826 said that he taught "arithmetic" but added as an afterthought "including mensuration, bookkeeping, navigation and landsurveying". Gauging by some was regarded as a separate subject, by others as part of arithmetic and by yet others as part of Practical Mathematics. It was not unknown to include a few books of Euclid under the heading "arithmetic". Many such examples could be quoted and this elasticity of subject division makes impossible an exact analysis of even the answers to fact-finding Parliamentary questionaires. Despite these difficulties, the following summary of an analysis of the 1826 and 1834 Educational Returns[1] does give an accurate enough picture of the relative importance of the different subjects. These returns included all the parish schools and most of the burgh schools.

Subject	Percentage of Schools teaching each subject 1826	1834	Subject	Percentage of Schools teaching each subject 1826	1834
Arithmetic	100	100	Mensuration	15	8
Bookkeeping	66	45	Geometry	5	6
Mathematics	24	32	Land Surveying	8	4
Practical Mathematics	18	14	Algebra	4	2
Navigation	12	10	Trigonom etry	3	1
			Astronomy	-	1

[1] The full analysis is given in appendix

In interpreting these statistics it must be stressed that these percentages refer to schools and not to pupils. Indeed as the schools mostly had one master these figures can be taken as reflecting more the attainments of the masters than those of the pupils.

Thus all the masters could teach arithmetic and there is no reason to doubt that they did not. On the other hand in 1834 ten per cent of the masters could teach navigation but they did not always have pupils taking it. The instruction, however, was there for those who wanted it.

If the whole country be considered, the relative importance of the different subjects is well illustrated by this table but average figures do not do justice to the higher class schools of the larger towns. These were not run on the basis of one man teaching every subject but were run on a faculty basis. For instance from as early as 1732, Inverness Academy had an English school, a Classical school and a Mathematics school. While the senior master of the Classical school was called Rector, he had no authority over the other schools or their masters. This organisation into three schools was common at the beginning of the century though new schools such as Edinburgh Academy adopted the English system of

a headmaster with authority over the other masters. The three schools sometimes met in the same building, sometimes in different buildings but the masters were completely independent of each other and lived by collecting their own fees. Mathematics masters will note with interest that in 1873 the commission for Endowed Schools reported that the salary of the Senior Classics master of Glasgow High School was £512 but that of the Mathematics master was £901[1] This faculty system was gradually abandoned but the commission just referred to found that out of forty-eight endowed schools sixteen were run on these lines. These sixteen included such important schools as Glasgow High, Dundee High, and Perth Academy. At Perth, curiously enough, it was the Mathematics master who had the title of Rector.

This system of specialist teachers permitted intensive study of a particular subject. For instance at Perth there was always a strong Mathematical school and in 1867 the master with the aid of one assistant taught (1) Arithmetic (2) Algebra (3) Geometry (4) Mensuration and Surveying (5) Book-keeping

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[1] Commission for Endowed Schools and Hospitals. The Mathematics master employed two assistants but this probably only cost him £50 - £60 apiece.

(6) Astronomy (7) Geology (8) Physiology (9) Physics, including practical and experimental mechanics (10) Chemistry.[1] At the same school in 1800 the same branches in mathematics were studied.[2] This list is representative of the scope of the teaching of the specialist masters though in the first decades the stress was more on the practical side and such branches as fortification and gunnery were often professed as well.

If the specialist teachers are excepted, the table quoted above can be taken as representative of the range of mathematical teaching and the popularity of the different branches. Despite minor fluctuations comparison with the yearly statistics of the Church of Scotland shows that these figures are typical of at least the first half of the century. Later there was a growing tendency in the higher class schools to drop the 'practical' branches in favour of the 'theoretical' for which there was a steadily increasing demand. Prominence is sometimes given to the discovery by the Argyll Commission that Navigation was taught in only one burgh school,

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1 Report on Burgh and Middle Class Schools.

2 Wilson "History of Mathematical Teaching in Scotland".

Burntisland, and to a class of one pupil. This does not indicate a startling decrease in the teaching of navigation in Scotland, but merely shows that its omission from the curriculum of the higher class schools was one of the first steps in the evolution by these schools of a basic system of mathematics which would satisfy the new three-fold reason for mathematic teaching. In course of time the teaching of other practical branches was discontinued in the schools and passed into the hands of the new Technical Colleges, the Continuation Schools and specialist private schools. This new basic system comprised arithmetic, algebra, geometry and trigonometry.

The actual numbers of pupils studying mathematics ("theoretical mathematics") over the whole country are available in Church of Scotland Returns and in government returns. These show that about one to two per cent of the total school population studied mathematics. As this figure is based on total school population including primary and secondary pupils it is not very illuminating. Professor Laurie, in 1880, giving evidence before a Royal Commission stated that the proportion of pupils in the parish schools studying mathematics

was 2 - 3 per cent. but rose in the North East counties to 4 - 5 per cent. As might be expected the figures for the Burgh Schools were higher. In 1868 the figure was 11 per cent. Average figures have little meaning here as a few schools in the principal towns had many pupils and most schools had but one or two. A better idea of the numbers involved can be obtained by examining the figures for a particular year. The Church of Scotland returns for 1840 show that at 916 parish schools there was a total of 1,159 pupils studying mathematics while at 2,092 non-parish schools the number was 1,017. Out of a total school population of 117,436 there were 2,576 pupils studying mathematics at 3,008 schools.[1] These statistics are for the whole country and give some idea of the smallness of the numbers involved compared with to-day when a large secondary school

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[1] Statistics for most of the century must be interpreted very cautiously as returns were often inaccurate. In 1861, in its report to the General Assembly the Education Committee of the Church of Scotland remarked that "the total number of non-parochial schools in Scotland were reported as being 201 fewer than those examined." This discrepancy was regarded as being perhaps a little unfortunate and the presbyteries were asked to do better next time. The same report remarks with pride: "Registers are now kept by all teachers who are not either indifferent or slovenly." Even in the last decade, there were annually ten-twenty cases of disciplinary action by the Department against teachers for inaccurate registration.

in one of the principal towns will itself have over a thousand pupils studying mathematics. Even if the higher class schools be regarded separately the numbers were exceedingly small by modern standards. It was not till the last decades when assistance from public funds made expansion possible that there was any notable increase. As typical of the last decade the figures for 1895 - 6 will be quoted. In that year a total of 12,425 pupils were examined in the three stages of Mathematics under the Revised Code. This total corresponds roughly with that for parish schools given above. In the Leaving Certificate 3,137 pupils were presented on the Lower Grade and 1,069 on the Higher.

The standard of attainment aimed at by the schools was entrance to the Universities or the professions but until the last decade this was a very variable quantity and by modern standards often very low. Fourteen was reckoned an average age for entering the Universities[1] and although individual Universities experimented there was no common Entrance Examination until 1892. For more than half the century the Universities found themselves, in fairness

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1 Lord Kelvin matriculated at Glasgow when 10 years 3 months old and carried off two prizes in the Humanity class before he was eleven.

to all their students, compelled to teach Algebra and Geometry from the beginning and to assume only a knowledge of Arithmetic. This gave rise to considerable overlapping for even the parish schools taught a few books of Euclid, Algebra, often to quadratic equations and some Mensuration. In addition, even after the institution of the Leaving Certificate some of the higher class schools carried their pupils well beyond the first university class. Variations in the standards reached will be discussed later when the content of individual subjects is treated in detail but in a general survey it has to be noted that when the Leaving Certificate was instituted papers were set on three different standards corresponding to what was required by those (1) studying for the University but not taking mathematics there (2) studying for the University and taking Mathematics there (3) studying for some high professional examination such as the Indian Civil Service. These three standards were deliberately fixed as corresponding to contemporary practice in the Secondary Schools. In Appendix A. is given the regulations concerning these three grades. The parish schools also aimed at entrance to the Universities and their average attainment has been given

above. This attainment was little different from the "standard" for the third year of Mathematics which was fixed after the introduction in 1872 of the "specific" subjects. The "standards" for the three years are given in Appendix. B

An indirect consequence of the introduction of 'Codes' and 'Regulations' by the state was the increasing standardisation of terminology. The content of arithmetic, for instance, as therein laid down, came to be accepted as defining the limits of arithmetic. Hitherto no one text book had been in such common use that its table of contents had become accepted as defining 'arithmetic'. In actual practice most schools used several text books whose authors had widely different ideas on what constituted 'arithmetic'. For example at Kenmay in 1852 the Arithmetics of Hamilton, Hutton, Gray and Melrose were all studied. In the early decades communications were bad and except in the towns the schoolmasters were isolated and had little opportunity for comparing notes with their fellows. A text book well known in one area would be unheard of in another. For instance even in the case of the famous 'Gray'[1], De Morgan, commenting on a copy of the fortieth edition, wrote: "How many works on arithmetic there must be

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1 See P. 58 below for discussion of this book.

which I have never seen! I never met with any one of the forty." Admittedly De Morgan was an English professor but the "Gray"had reached America if not London and De Morgan, an ardent book-collector, shows familiarity with other Scottish writers. The point which we would make here is that the first table of content of arithmetic which would be circulated among the whole teaching profession was that of the different Codes. This universal circulation alone would have been sufficient to standardise the term but it was reinforced by the fact that schools accepting the government grant were compelled to work to this definition.

4. Individual Subjects.

1. Arithmetic.

Arithmetic was universal. All the various educational agencies, including the Universities taught it in some form or other. These agencies can be classified in three groups:- (i) the Universities, (ii) the higher class schools, (iii) the parish schools, the writing schools and the hundreds of small private adventure schools fulfilling the function of modern infant schools. In the middle group, the specialist teachers in the higher class schools covered the whole range of modern arithmetic and more besides. The top group, the Universities, taught what the professors thought had not been adequately taught by the other groups. This was a variable quantity, but always included the theory of the subject. The third group taught arithmetic from the beginning and prepared their pupils for entering the group above, which at its lowest meant teaching them to count and at its highest (the Parish Schools) preparing them for entrance to the Universities.

(i) The Universities. The arithmetical teaching undertaken by the Universities at times overlapped that of the best schools but was undertaken to meet the needs of students coming direct from the parish schools. The different professors taught what they

considered necessary to enable their students to follow the Mathematics and Physics courses. Normally they considered their students proficient in "practical" arithmetic but very weak on the theory. During the first half of the century the four universities taught arithmetic but after the institution of local entrance examinations Aberdeen and Glasgow dropped it. It was not till 1887 that Edinburgh ceased to teach the Principles of Arithmetic in the Junior Mathematics Class and in that year a question on Stocks and Shares appeared in the degree examination. St. Andrews was more conservative still. In 1900 Arithmetic was still given in the Calendar as part of the Mathematics course though possibly receiving little notice in the lectures as no questions were asked on it in the degree examinations and it disappeared from the Calendar in 1901.

During the first part of the century Leslie's[1] Arithmetic was the standard work on the theory but later Bernard Smith's took its place. Leslie's book followed closely the lectures delivered by him at Edinburgh University and is a work of such importance that we shall give a full account of it here even though certain aspects should perhaps be considered later. Its importance is due to the fact

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1 "The philosophy of Arithmetic - exhibiting a progressive view of the Theory and practice of Calculation"

that it was written as a deliberate attempt to put an educational ideal into practice. But 'good wine needs no bush' - Leslie needs no one to speak for him. The very first sentence of the Preface makes his intention clear: "I now discharge my promise, by the publication of a volume in which Arithmetic is deduced from its principles and treated as a branch of liberal education."

"The object proposed" he continued "was not merely to teach the rules of calculation but to train the young student to the invaluable habit of close and patient investigation. I have therefore preferred the analytical mode of advancing, and have pursued a route entirely different from that which is followed by the common treateses of arithmetic. In seeking to unfold the natural progress of discovery, I have traced the science of numbers through the succession of ages, from its early germs, till it acquired the strength and expansion of full maturity. This species of history, combining solid instruction with curious details, cannot fail to engage the attention of inquisitive readers.

"If the execution of this little work, which is the result of very considerable research, should at all correspond with the importance of its design, it

may supply a capital defect in our systems of public instruction."

He concludes his preface with these words: "The time required for the study of this treatise could scarcely be more beneficially employed, since it will not only rivet in the mind of the pupil the theory and enlarged practice of calculations, but invigorate, by proper exercise, his reasoning faculties and consequently prepare him either for entering the labyrinths of business, or engaging in the higher pursuits of science."

In the Introduction he starts with the idea of number tracing its growth among primitive hunting communities. He describes the use of pebbles or fine nuts or hard grains as 'portable emblems' representing numbers. (The cricket umpire of to-day with his 'six small objects' in the pocket of his coat uses the same idea.) From this he develops the idea of representing larger numbers by arranging the little objects or counters in regular rows of twos or threes or fours, etc. Leslie continues: "We may discern around us traces of the progress of numeration, through all its gradations.

"The earliest and simplest mode of reckoning was by 'pairs' arising naturally from the circumstance of

both hands being employed in it, for the sake of expedition. It is now familiar among sportsmen, who use the names of 'brace' and'couple', words that signify pairing or yoking. To count by threes was another step, though not practised to an equal extent. It has been preserved, however, by the same class of men, under the term 'leash', meaning the strings by which three dogs and no more can be held at once in the hand. The numbering by fours has had a more extensive application. It was evidently suggested by the custom of taking, in the rapid tale of objects, a pair in each hand. Our fishermen, who generally count in this way, call every double pair of herrings, for instance, a 'throw' or 'cast'; and the term warp, which from its German origin, has exactly the same import, is employed to denote four, in various articles of trade."

"These simple arrangements would, on their first application, carry the power of reckoning but a very little way. To express larger numbers, it became necessary to renew the process of classification; and the ordinary steps by which language ascends from particular to general objects, might point out the right path of proceeding. A collection of 'individuals' forms a 'species'; a cluster of 'species' makes

a 'genus'; a bundle of 'genera' composes an 'order'; a group of 'orders' constitutes a 'class'; and an aggregation of 'classes' may complete a 'kingdom'. Such is the method indispensably required in framing the successive distribution of the almost unbounded subjects of Natural History.

"In following out the classification of numbers, it seemed easy and natural after the first step had been made, to repeat the same procedure. If a heap of pebbles were disposed in certain rows, it would evidently facilitate their enumeration, to break down each of these rows into similar parcels, and thus carry forward the successive subdivision till it stopped. The heap, so analysed by a series of partitions, might then be expressed with a very few low numbers, capable of being distinctly retained. The particular system adopted for this decomposition would soon become clothed in terms borrowed from the vernacular idiom.

"Let us endeavour to trace the steps by which a child or a savage, prompted by native curiosity, would proceed in classing, for instance, twenty-three similar objects . . . . "

We have quoted at length to illustrate the character of this book and to show the clearness and

force of its author's style. Obviously Leslie was no narrow mathematician but an educated man in the widest sense of the term.

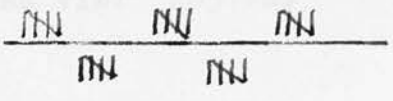
Leslie goes on to discuss the idea of place value and the use of symbols instead of small objects to denote number. Time is found to conjecture on the possible reasons for the shape of the numeral characters used by the Egyptians, the Romans and the Greeks. He concludes his introduction by outlining the work of Archimedes, Appolonius and Ptolemy in extending the system of numeration.

In the main part of the work Arithmetic is viewed "under two very distinct forms" which are called "Palpable Arithmetic" and Figurate Arithmetic. In the former, "numbers are exhibited by counters, or abbreviated representatives of the objects themselves. In the latter numbers are denoted by the "help of certain symbols, or artificial characters disposed after a particular order."

The topics discussed in Palpable Arithmetic are as follows: the Binary Scale, Ternary Scale, Quaternary Scale, ... up to the Duodenary Scale; historical notes; demonstration of fractions; the 'four rules'; extraction of square root; and a discussion of the abacus and its descendant the

chequered board. Each number is represented by a diagram showing the position of counters on a set of parallel lines and all the scales and operations are illustrated in full. Once the idea of a "palpable number" has been grasped the discussion of every scale and of every operation on most of the scales becomes rather tedious but it is enlivened by notes such as the following which may be of interest to modern statisticians:

"The first bar of the Quinary Scale is actually used in this country among wholesale traders. In reckoning articles delivered at the warehouse, the person who takes charge of the 'tale', having traced a long horizontal line, continues to draw, alternately above and below it, a warp or four vertical strokes, each set of which he crosses by an oblique score and calls out 'tally'! as often as the number five is completed."



The same topics are discussed in Figurate Arithmetic but of course the numbers are represented by symbols and not by diagrams showing the position of counters on a frame. This section commences with a long, historical discussion on the characters used to denote number in this and in other countries. Many fac simile copies of sets of numerals are shown.

There follows a full discussion on how the number 430685 on the denary scale would be written on the different number scales from Binary to Duodenary. Fractions come next and are regarded as extended powers in the descending order on any number scale. The 'four rules' and the extraction of roots are again discussed. The whole work is rounded off by copious historical notes.

Great scholar, philosopher and mathematician as Leslie undoubtedly was, yet he could be fascinated by the curious properties of numbers. At the very end of his book he apparently deliberately chooses to express $\frac{1}{7}$ as a decimal fraction in order that he may point out how the digits of the number 142857 are repeated in its first six multiples viz: 285714, 428571, 571428, 714285, 857142.

Because of the unique character of this work we have reviewed it at length. It was a work of great importance for Leslie was 'no prophet crying in the wilderness' but the leader of a popular movement in educational theory. Various teachers had previously tried to draw attention to the theoretical side of arithmetic by including 'reasons for the rules' with the rules. Here however was a book which without compromise was of so 'really elementary' a character

that it was useless to the calculator. It was a trumpet call to action. At St. Andress, Professor Duncan inaugurated a similar course. Aberdeen and Glasgow were more cautious but taught the principles as well as the processes with the stress on the former. The movement spread outwards to the schools where it became considerably modified.

Perhaps because the revolutionary zeal with which Leslie tackled the representation of 430685 on number scale after number scale began to prove wearisome or perhaps because it was realised that other subjects would also calim to be treated as part of a liberal education, after the first flush of enthusiasm there was a growing tendency to give a less-elaborate analysis of the principles and to use Bernard Smith's 'Arithmetic' instead of Leslie's. This tendency continued till eventually, as we have already mentioned, arithmetic was quietly dropped from the curriculum of the Universities. Probably the only institutions to-day where Leslie's views of arithmetic is studied is in Training Colleges for Teachers where lectures are given on number scales to help the students to appreciate the psychological approach to the teaching of arithmetic and to impress on them, say by performing calculations on the Duodenary Scale,

the ideas and difficulties involved in learning number processes.

Although the 'Philosophy of Arithmetic' had a brief career as a separate subject of study, it had a profound influence on the teaching of arithmetic. Because it was taught in the Universities, it was able to reach out over the whole country. It gave a great boost and much-needed prestige to the advocates of the view that reasons should be given and not just blind rules. It showed that there was obviously more to Arithmetic than a series of rules. In this way it produced a more liberal outlook in the teaching. In addition it prepared the way for the introduction of concrete material and new methods in the early teaching of number.

(ii) The Higher Class Schools. It was in the higher class schools that the whole range of arithmetic was taught. As usual, for this period there was a wide fluctuation in attainment but the best schools maintained a uniformly high standard throughout and a remarkable uniformity in content. The wide range can be gauged best by studying some popular text such as Mair's "Arithmetic: Rational and Practical" or "The Gray" which ran to over a hundred editions. The course of instuction given in these books covers all that is now taught in the secondary schools and in addition such topics as circulating decimals, tare and tret, cube roots, duodecimals, surveying and gauging. These, or similar works were widely used in the best of the higher class schools whose pupils received, a thorough grounding in the fundamental operations, in vulgar and decimal fractions and in the common applications of arithmetic to commercial problems. The teachers tried to get their pupils to understand the principles involved bu the stress was laid on ability to use arithmetic rather than to understand it.

"The Gray" had such a long and venerable career that it deserves a paragraph to itself. "There must," wrote the author of a popular work[1] on mathematics, "have lived somewhere and sometime - presumably in

[1] Smith: "Euclid. His Life and System."

upper Clydesdale or Tweeddale, and about the middle or towards the end of the eighteenth century, a Mr. Gray. He had written a book on arithmetic, and locally the name 'Gray' or 'the Gray' had been transferred from the man to the book, and from the book to the whole subject of which it treated. To 'begin the Gray' was to enter on the learning of arithmetic. To be 'through the Gray' was to attain a position of which the senior wranglership of Cambridge gives but a faint conception." This atmosphere of reverence is shattered by the next sentence: "From our recollection we should say that 'the Gray' was as bad a book as could have been written."! This criticism is too sweeping for 'the Gray' was no worse than many of its contemporaries and no better than its author's purpose which was "to initiate youth in the fundamental Rules of Arithmetic; and to be of easy purchase to the Scholar." No attempt was made to explain or justify any rule. The pupil had to have blind faith in it - even blinder faith than is usual in this type of teaching for the book contains no worked examples to help the learner. Despite its defects, the Arithmetic of James Gray "late of Peebles and Dundee," was used for over a century and ran to 101 editions. Like many a good Scot it emigrated

to Canada where it was still in use in the 1870s. De Morgan, a connoisseur of arithmetic books, described 'the Gray' as a "neat little work" but perhaps he was only thinking of the pocket size of the volume!

Compared to 'the Gray' Mair's Arithmetic had some pretensions to scholarship. He attempted to build up a science from a set of twelve "Principles and Axioms." He had not the insight of Leslie, and his twelve tenets of faith do nothing to foster a conception of the ideas, behind the processes, as distinct from ability to use the processes mechanically. If the postulates are accepted his book was as he claimed. "The Theory of the Science deduced from first Principles; the Methods of Operation demonstratively explained; and the whole reduced to Practice in a great variety of useful Rules." Again, the Rules! His book represents an attempt to place arithmetical teaching on a higher level. It weakness lies in the lack of appeal to concrete material to help the understanding of number. For instance to justify 'carrying' he has to resort to two of his axioms: "III. Ten in any inferior place makes one or an unit in the next higher place, and the reverse. IV. If the right hand figure of any number be cut off, the remaining

figure or figures are a just number of tens, and the right hand figure so cut off is the overplus." Mair was longwinded. "A cypher signifies nothing of itself, yet it serves to cupply vacant places and raises the value of significant figures on its left hand, by throwing them into higher places." That was his description of the zero as a 'place-holder.' Or consider this example: "Troy was taken and destroyed by the Greeks, two thousand eight hundred and twenty years after the Creation, and one thousand one hundred and sixty-four years after the Flood, and four hundred and thirty-six years before the building of Rome and eleven hundred and eighty-four years before the birth of Christ and two thousand nine hundred and forty-four years before the accession of King George III to the throne of Britain. How are these dates expressed in letters?" Our sympathies would be with any schoolboy who simply wrote 'C.C.L.' (Slang: "couldn't care less").

The two works just discussed are representative of the two types of text available at the beginning of the century; the one dogmatically stating rules; the other trying to arrive at some conception of Arithmetic higher than mere calculation. In the first category were many popular 18th century writers: Vyse, Dilworth, Walkingame, Cocker, Melrose and of course Gray. The

second category includes Hutton, Bonnycastle, Mair, Leslie, Lees and later Thomson, Smith De Morgan, Munn etc. As the century progressed the new works published were mainly of the second type wherein as Thomson said "the reasons of the rules and operations are explained," so that "what is peculiarly fitted to call forth and improve the reasoning powers" is not "degraded into a dry exercise of memory." De Morgan writing in 1846[1] on "the importance of establishing arithmetic in the young mind upon reason and demonstration" said that there "was no need to advocate a change which was already in progress." After the Revised Code came into operation a new type of text appeared divided into sections, each section corresponding to the work of a 'standard'. A simple explanation, if any, of the process was given, then one or two worked examples and last of all, many examples for the pupils to do. These texts were intended to be used as part of the arithmetic lesson, the class teacher explaining the process and the text supplying a model answer and plenty of examples for practice.

While it is easy to determine what was considered to be the extent of a full course, it is not quite so easy to determine how much of it was taught in actual practice. School prospectuses are unreliable as they contained what the master professed to teach, not always what he did teach. In the same way a knowledge

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[1] Elements of Arithmetic.

of the text books used is unreliable, but various Government commissions got over this difficulty by asking what parts of the book were actually studied by the top class. Old examination papers are not of much help for, prior to the 70s, written examinations were uncommon. The only material of this kind which we have been able to trace are degree papers for University examinations - papers set to teachers wishing to qualify for the Dick Bequest, and prize scripts for the Edinburgh School of Arts. Examples of these are given in the Appendix. The Argyll Commissioners comment again and again on the lack of experience of written examinations shown by school pupils. In some instances, information can be obtained from old copy books which have survived the ravages of time. For the last decades reliable information is easily obtained from the Revised Code and the regulations for the Leaving Certificate. From these various sources a fairly comprehensive picture can be formed of the standard and content of teaching over the century. To do this, we will consider particular cases representative of the best schools in the country.

At Perth, round about 1800, the arithmetic course was carried as far as cube root and problems in exchange.[1] It is interesting to note that Mair was at one time Rector there and that his arithmetic was still in use

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[1] Wilson: "History of Mathematical Teaching in Scotland."

in 1868.

At Edinburgh Academy in 1837, the curriculum of the top classes was as follows:-

Fourth class: Proportion, Practice, Interest, Profit and Loss, Barter, Vulgar Fractions.

Fifth class: Vulgar fractions, Decimal fractions, Square and Cube Roots, Duodecimals.

Sixth class: Same as fifth. This course was modelled on that of the High School of Edinburgh, where, in 1848, the later stages of instruction were as follows:-

Third class: Simple and Compound Proportion, Practice, Partnership, Loss and Gain, Interest, Discount, Equation of Payments.

Fourth class: Simple and Compound Proportion, Vulgar and Decimal Fractions, Extraction of Square and Cube Roots, Mensuration of Surfaces and Solids. Book-keeping.

Rector's class: Higher Arithmetic.

At Aberdeen Grammar School in 1867, the Argyle Commissioners found a very similar course followed, except that the commercial rules were taught in the fourth class not the third.

In 1873, the Commission for Endowed Schools obtained much detailed information which confirms that the cases quoted above are representative of the best schools.

In 1886, Government inspection of the higher class schools was begun and from the experience gained the papers were set for the first Leaving Certificate Examination in 1888. It was decided to have two grades but in addition, to award honours in the higher grade. The higher paper contained "ordinary arithmetical questions, with optional questions on logarithmic computations of an easy kind." The lower paper contained "questions of a general kind but somewhat simpler." The following year, however, more guidance was given as to the nature of the lower paper: Practice, proportion, percentage, square root, simple interest, vulgar and decimal fractions (omitting recurring decimals).

It must be understood clearly that the Department did not try to introduce new standards but that it attempted to set examinations that would maintain contemporary standards. The honours award was instituted because there existed a number of schools where arithmetic, as well as other subjects, was habitually taught to a standard higher than that of the new certificate. The character of the examination reflected current practice in mathematical teaching. The questions were designed to test two things: the competence of the pupil to carry out certain procedures in

which he had been trained i.e. "the bookwork"; and secondly, his resource in applying his knowledge and training to new problems. In the appendix is given a paper set by the Argyle Commission. It will be seen that this paper only tests the "bookwork."

With the institution of the Leaving Certificate, content and standard of attainment became stabilised at a level a little lower than the contemporary level of the best schools. It can be seen from the illustrations given that this level had been maintained over the century. The difference was that at the end of the century owing to the general increase in educational efficiency far more schools and pupils reached this level. Also pupils and teachers alike had more understanding of the nature of arithmetical processes.

(iii) The Parish Schools, Writing Schools, etc. So far only the best of the higher class schools have been considered. The others, usually small provincial schools taught what they could and in this respect must be classified along with the parish schools in our third group. Prior to 1872, there is no lack of detailed information about this group, but generalisations are impossible as the information is not representative. Each school was a law unto itself. Some got no further than counting and sewing a straight

seam. Others prepared pupils for the University, more or less successfully. The only common feature was that each started right from the beginning. Further progress depended more on the capabilities of the teacher than the capacity of the pupils. A regrettable feature of this period was the decay of successful schools as the master grew old and decrepit and could not retire owing to the absence of a pension.

In areas which were educationally backward, such as the Highlands and some of the Border Counties, arithmetic suffered along with the other subjects. In such areas, even the parish schools (if in existence) did not reach any great flights. It was only in the North East where anything approaching a common standard existed among the parish schools. There from the 30s onwards, because of the operation of the Dick Bequest certain common standards developed. It would appear from the Bequest inspector's report that proportion was the high-light of the young arithmetician's career. In a sample "excellent" school,[1] "14 pupils worked simple and compound proportion accurately and well," while "some 40 or 50 worked proportion and fractions exceedingly well." In the sample "good" school, 5 professed proportion but could only do it mechanically." To complete the picture it must be

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[1] Laurie: "Dick Bequest Report" 1865

added that Aberdeen trained pupils were held in great respect over the whole country!

In the sphere of arithmetic the writing schools have a place. It was extremely common, even in the higher class schools, for the writing master to teach the elements of arithmetic. Often he taught more than the elements by including book-keeping and commercial arithmetic in his repertoire. Indeed when state-subsidised schools put such large numbers of private teachers out of business, many of the writing masters maintained their independence by developing the commercial side of their activities. In the higher class schools the situation often arose of both the Mathematics master and the Writing master teaching the same arithmetic, but the latter charging a lower scale of fees. In the University towns the Mathematics masters had often to face similar competition from the Universities themselves! The reorganisation of the educational system did not completely break this traditional linkage of writing and arithmetic for even in the early twentieth century, private schools, such as Merchiston Castle, continued to employ visiting teachers of writing and arithmetic.

Some idea of the work of these writing masters can be gleaned from a study of Butterworth's "Young Arithmetician's Instructor" published in Edinburgh in

1805. It was "designed for the use of schools and private families." It contains many examples of beautiful penmanship. The content comprised numeration, the four rules, practice and the rule of three. No attempt was made to explain the processes involved, and in this respect it resembles 'the Gray'. The author, who was at one time writing master at the High School of Edinburgh, claimed that this was "sufficient for every computation," despite the total absence of fractions of any sort! He also stressed the need for accurate writing, correct figures and judicious arrangement. The contents of this book and the attitude of the writer illustrate very well the nature of the work accomplished in the writing schools.

In the hundreds of adventure schools, the instruction imparted was extremely rudimentary, and did not extend beyond the four rules. There were exceptions but these have been considered as belonging to either parish schools or higher class schools according to their functions.

After the reorganisation of primary education following the Act of 1872, there is little difficulty in ascertaining the content and standard of arithmetical teaching for the work of the primary school was governed by the exceedingly rigid "standards" of the Revised

Code. These Standards for the year 1873 are given below.

I. Notation and numeration up to 1,000. Simple addition and subtraction of numbers of not more three figures.

II. Notation and numeration up to 10,000. Multiplication table to 5 times 12. Simple addition and subtraction of sums of 10 figures.

III. Notation and numeration up to 1,000,000. Four simple rules. Money and time tables.

IV. Compound rules (money and common weights and measures.) Aliquot parts of a pound sterling.

V. Practice, bills of parcels, simple and compound proportion.

VI. Vulgar and decimal fractions.

In the following year compound proportion was moved to Standard VI. Otherwise with only minor changes this was the content of primary teaching for the remainder of the century. Although it only applied to the state-aided schools, the competition from these was now so strong that the private schools had to do at least as well if they wished to remain in business.

The Evening Continuation Schools. The "Standards" also applied to the Evening Continuation Schools. These schools were of two types - (a) schools which were really a continuation of the ordinary day schools

where the same subjects were taught, but often to a lower level; (b) schools for adults founded for the sole purpose of giving special instruction in the evenings.

The first type presents no new features in arithmetical teaching. These schools were simply day schools taught in the evening for the benefit of the many unfortunates who would not otherwise have received any schooling. Once the "standards" were introduced pupils from the evening classes earned the Government grant for passing the standards along with the day school pupils. As the "Standards" correspond to the present level attained by the primary schools, the contribution of these evening schools to education was of poor quality indeed. Until 1893, this type of evening school must simply be classified along with the primary schools. In that year, however, a separate Code of Regulations for Evening Schools came into force. This allowed a breakaway from purely primary teaching but developments in arithmetic did not take place till the next century.

The second type is of more interest as it includes Mechanics' Institutes, Literary and Philosophical Institutes, the Schools of Arts and the Technical Colleges. The instruction given was of the nature of

secondary education, yet often arithmetic found a place. In 1828, at the Edinburgh School of Arts, "the theory of Arithmetical computation was investigated and its results were displayed in the formation of rules for practical application."[1] (The last phrase would give a free hand to any examiner). This course was inspired by Leslie's "Philosophy of Arithmetic" but was later modified considerably.

2. Algebra.

In the early part of the century a veil of silence shrouds the purpose of teaching Algebra. Writers of the popular texts do not seem to have regarded it as necessary to give their views on this question. Wood simply states that his work 'is designed for the use of students in the University'. Colenso makes no statement whatever. Kelland's advocacy practically amounts to saying that by means of algebra, problems can be solved demonstratively which could not be solved thus using arithmetical procedures. In fairness to Kelland we would point out that this criticism is directed against a passage which may have been meant as introductory to the study and not as giving the reason for it, but even if Kelland is absolved, others were of this view. We find, however, that Horner when a lad of seventeen was recommended to study algebra "as

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1 7th Report of the Directors of the School of Arts.

affording an admirable exercise of his reasoning powers and the best means of cultivating that talent for analysis, close investigation and logical inference which he possessed at an early period." This advice was given to an exceptionally brilliant student so is not necessarily representative. Perhaps De Morgan lifts the veil when he wrote "the science of algebra, independently of any of its uses, has all the advantages which belong to mathematics in general as an object of study." This somewhat negative attitude is reflected in the small numbers studying it. Very few were going on to the abstract studies to which it led; most regarded the study of Euclid as sufficient exercise for their reasoning powers without studying a second subject for this reason; so for many there seemed little point in beginning the study.

Professor James Thomson of Glasgow was among the first writers to adopt a more positive attitude. In our introduction, we have already quoted his views on this question. Algebra was the key which would unlock many doors. A view of enchanting horizons was held out to the traveller beginning the long and tedious journey. Nor were these horizons lacking in material rewards as the citizens of Glasgow and elsewhere in Scotland were quick to perceive. There was

from this period onwards a steady increase in the numbers studying algebra and the study was carried to a higher level.

School prospectuses of the early period often advertised "Algebra to the Simple Equation." As the practice was to include the extraction of roots and the manipulation of complex and rather improbable fractions before studying the simple equation, this content appears to have satisfied the two early aims of teaching by providing some mental gymnastics and the means of solving awkward arithmetical problems. With the growth of the idea that a study of algebra was a necessary preliminary to a study of the physical sciences, there was no longer any point in stopping at the simple equation. A knowledge of quadratic equations gradually came to be the accepted target for school teaching. In Scotland the study was quickened and enlivened by this new idea, but in England the claims of mathematical physics were long denied, the 'vision splendid' was ignored and University or examination success was the only effective spur. Unfortunately for the rapid development of this movement most of the popular texts were by English authors and contained much material irrelevant to the development of the subject. The staleness which develops when there is no clear purpose in the teaching is apparent when

Chrystal wrote thus: "The English text-book of Algebra in vogue during the latter part of this century have tended to degenerate into a mere farrago of rules and artifices, directed to the solution of examination puzzles of a somewhat stereotyped character." The aim of Chrystal's teaching was not "the utmost rigour of accurate logical deduction" but "a gradual development of algebraic ideas." In the last decade appeared a number of excellent texts in which some, at least, of the 'pruning' was carried out and in which the subject was developed to a level sufficiently high to act as a basis for the study of the calculus.

3. Geometry.

A course in Geometry simply comprised a study of one or more of the books of Euclid. Schools of the standing of the High Schools of Edinburgh and Dundee studied the first six books but many of the smaller burgh schools were lucky to achieve Book I. The well-reputed Aberdeen schools seldom taught beyond Book I for the simple reason that the Aberdeen University Bursary Competition only required a knowledge of this Book. The parish schoolmasters also taught Euclid as the need arose. Prior to 1892 the Universities found it necessary to teach Euclid from the start. Even after 1892, Euclid was still taught in the Junior Classes.

Although the higher class schools clung to Euclid, the parish schoolmasters and the private teachers of Mathematics frequently conducted courses in Practical Geometry. Most of the teachers of navigation found it necessary to start with such a course. In the early decades the Universities followed the study of Euclid with a course in Practical Geometry wherein, among other things, the use of the protractor was taught.

The common course in Practical Geometry consisted of the following: the definitions the 'ruler and compass' constructions; the measurement of angles; the properties and construction of plane rectilineal figures; the properties of the circle including tangency. The University course was more comprehensive including much material now classified under Trigonometry.

By the end of the century Practical Geometry was a rarity, but it had not disappeared altogether as parts were absorbed into and enriched the school teaching of Euclid.

4. Trigonometry. In the first part of the century, in a period when a remarkably wide range of mathematical subjects was available in the schools, it is worthy of note that Trigonometry was seldom taught as a separate subject. Normally, part of the course in practical mathematics was portioned out among Mensuration, Land Surveying, Practical Geometry and Navigation. In the theoretical course it was placed after the study of Euclid and was often described as Practical Geometry. This position severely limited the number of pupils in the schools. After the first decades, however, the practice grew of studying it before the full geometry course was completed and in the 70s, the commission for Endowed Schools found that it was taught by all the schools without any pretensions to higher education. Latterly, the text book commonly used was Todhunter and the emergence of trigonometry as a separate subject was in no small measure due to the availability and excellence of the text.

The higher class schools and the specialist private teachers taught trigonometry as required but very few of the parish schoolmasters even professed it. As far as the schools were concerned it comprised the solution of the right-angled triangle and its application

to the mensuration of heights and distances. The gifted pupil of course was carried to a much higher standard. At Perth, the Argyll Commissioners were amazed at the performance of one boy in Spherical Trigonometry.

In the Universities, in the early part of the century, Trigonometry was taught from its foundations. In the first class in Mathematics, the ratios were defined, right-angled triangles resolved and the knowledge gained applied to the mensuration of heights and distances. The application of plane trigonometry to land surveying was next discussed and where circumstances permitted, field work was undertaken - usually on a Saturday morning. The study was continued to higher levels in the second and third classes. When the Universities stopped teaching Practical Mathematics, they discontinued only the practical applications of trigonometry and continued to teach the remainder until the end of the century despite the institution of the Leaving Certificate and the University Preliminary Examination. An exception was Aberdeen who in session 1893-94 stopped teaching the initial stages of trigonometry and also of algebra.

5. Mensuration. Mensuration, which was at first part

of the course in practical mathematics, was often professed under its own name. Later, parts of it became associated with the ordinary mathematics course (e.g. in Aberdeenshire) or were included in arithmetic (e.g. at Edinburgh Academy). When the Revised Code came into force among the specific subjects, the Elements of Mensuration found a place in the 3rd year of Mathematics.

At the end of the 18th century, several texts of repute, which gave a complete treatise on Mensuration were in use in Britain. Hutton's very comprehensive work was highly thought of but was mainly used for reference. Bonnycastle borrowed largely from Hutton but even his easier, shorter and cheaper work was not used much in Scotland. The Scottish practice in the early decades was to treat Mensuration as part of the course in Practical Mathematics and not to use texts devoted solely to it. In the Universities, the works mainly used were Gregory's 'Practical Geometry', Duncan's 'Supplement' and later Elliot's 'Practical Geometry and Mensuration'. In the schools it was frequently studied from Arithmetics such as Gray's or Mair's.

The University course covered the mensuration of lines and angles, of surfaces including the conic sections, and of volumes including the cylinder, cone and sphere. Field work in surveying was carried out and

the use of the plane table was demonstrated. Careful instruction was given in the theory and use of the instruments for finding heights and distances. These included the geometrical square, the plane mirror, the staffs, the quadrant, the theodolite and its forerunner, the graphometer. Even the use of the protractor was investigated with meticulous care.

In the schools, on the other hand, the stress was not on measurement and the handling of instruments but on calculation. The pupils very frequently did not go further than the rectangular solids and the man or boy who could gauge a cask was regarded as no mean arithmetician. A curious little sidelight is that Hutton, the acknowledged authority on Mensuration, on giving one of the rules in gauging, makes one of his exceedingly rare acknowledgments when he states that it was "hinted at by Mr. James Davidson, Teacher of Mathematics at Dundee, in North Britain." Most of the textbooks used in the schools in the early period provide many examples in estimating timber and on "casting up" tradesmen's work, such as joiner's, painter's, glazier's, mason's, paviour's etc. Duodecimals were used for these calculations and continued to be used and taught throughout the century. Not all schools neglected or ignored the more practical side of the subject; "mensuration

and land surveying" were frequently professed as a single subject. At Dumfries in 1866, there was a theodolite "for the use of the pupils under the direction of their master."[1] In the last decades, mensuration as a school subject became very limited in scope. Land surveying was dropped;[2] the calculation of heights and distances was transferred to trigonometry, and the measurement of angles to geometry. The teaching of the tradesmen's calculations became the prerogative of the evening schools though strangely enough day school pupils were still required to paper the walls of innumerable rooms probably because duodecimals were not essential for this calculation. Stripped of these, mensuration became an exercise in computation, a matter of applying the four rules in estimating (i) the area of the common rectilinear figures and the circle, (ii) the volumes of the rectangular solids, the prism, cylinder, cone and sphere.

For the best part of the century, Mensuration was a rather colourful subject of a dual character, sometimes arithmetical, sometimes geometrical. The name was rather loosely applied; the terms Mensuration, Practical Geometry and Land Surveying often applying to almost

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1 Report of Commission on Burgh and Middle Class Schools.

2 The Code for Evening Continuation Schools (1893) included in Mensuration the "use of the chain and field book in land surveying."

identical courses of study. The arithmetical side predominated, except in the Universities. Latterly, after the re-organisation of the whole educational system, only this side remained, but even then, shorn of much of its former glory.

6. Land Surveying. Because of the tangled skein of subject classification, much of what has been written above under various headings is applicable to a discussion on Land Surveying.

The term 'surveying' was frequently used in reference to the measurements necessary for the tradesmen's calculations, i.e. in reference to quantity surveying. The term Land Surveying, however, was used only when a piece of land, part of an estate or a farm, was actually surveyed. This subject included estimation of heights and distances, levelling, chain survey, chain and theodolite survey, offsets, the field book, the plane table and the preparation of plans. A background knowledge of trigonometry and geometry was required. In the Universities this presented no difficulties, for Land Surveying was treated as an application - an exceedingly interesting topical application - of the principles demonstrated in the practical mathematics course. In the schools, the commonest practice was to supply the theory as necessary with or without proof as circumstances permitted.

7. Navigation. Navigation was taught in the principal seaport towns from the beginning of the 18th century. References occur to it being taught at Dunbar in 1721, Ayr 1727, Dundee 1735 and so on. By the beginning of the 19th century it was being taught in Burgh, Parish and specialist private schools in all the coastal districts of Scotland. Up till the 40s, about ten per cent. of the Parish and Burgh schools taught it, but later it fell into disfavour in the burgh schools. In 1868, the Commissioner, reporting on burgh schools, found it being taught in only one school - Burntisland - and to a class of one pupil! This Commission, however, did not visit either the important endowed schools or the quite large number of private schools specialising in Navigation, such as St. Peter's Episcopal School, Fraserburgh which opened in 1854, was in its early years "an outstanding centre for sailors studying navigation.[1] After 1872, the parish schools gave up teaching it so that even though there was a slight swing back to favour in the higher class schools, the bulk of the teaching passed to the specialist private schools and to the evening continuation schools. It is remarkable that in the state re-organisation of the educational system, navigation went unnoticed till 1893 when in the Code for Evening Schools it was recognised as a grant-earning

[1] This quotation is given in "Education in Aberdeenshire before 1872" as from Cranna's "Fraserburgh: Past and Present."

subject for "Boys and Men." At the same time it was added to the day school code as an additional specific subject. Three years later, in 1895-6 a total of 397 pupils was presented for examination in the three stages and 321 passed. The last development was the founding of Nautical Colleges, the first of which was opened at Leith in 1903.

The instruction given was not merely a matter of pilotage or coastal sailing but dealt with navigation in the open seas out of sight of land and in all latitudes. Many manuals of instruction were available but the cheaper texts tended to be full of errors. In 1773, Wm. Wilson of Dysart published the first good text at a reasonable price which covered all aspects of the subject. He had many imitators who pirated his work freely, including the errors. One author, Mackay (1810) went to the trouble of pointing out the errors in rival texts and found many errors of over forty degrees and even on one occasion 10° W instead of 70° E! At the Universities, it was customary, in the early decades, to include navigation in the second or third year of the Mathematics course. Although the principles were discussed, the training was not intensive. The real burden of teaching was borne by private or public schoolmasters. The student learned

through the direct teaching of the master, through the study of text-books and through personal experience on board ship. The master's knowledge was not always theoretical for it is recorded that the navigator of the steamship Comet on her first run was the parish schoolmaster of Helensburgh.

A full course covered much ground. At Monifeith in 1826 there was taught "a complete course of navigation, including necessary previous preparation, trigonometry, logarithms, the sailings, the construction and use of Mercator's chart, several journals, nautical astronomy and the luners, etc., etc., three guineas and a half." The fee was per quarter and dear for the time. Normally all the sailings were taught, i.e. Plane, Traverse, Current, Parallel, Middle Latitude, Mercator's and Great Circle sailings; sailing to windward and Oblique sailings. Also normally taught were the different methods of fixing position at sea, including stellar observations. As a natural development, schools specialising in navigation, occasionally taught Astronomy.

The instruction in all the schools cannot have been carried as far as this for many of the masters professing navigation had no further experience than their University course in Mathematics. This did not

deal with the subject in the detail given above. For instance, at St. Andrews, Professor Duncan in a brief course covered the "Fundamental Principles" and "Plane and Globular Sailing." The former included rhumb lines; the magnetic compass; leeway; logline and halfminute glass; and the effect of currents. The latter was a demonstration of the mathematical principles underlying middle latitude, parallel and Mercator's sailings. The use of Astronomy was mentioned but left for the Natural Philosophy class. Although the masters might not have pursued their own studies further than this, yet they were able to lay a sound foundation on which their pupils could build. There is the well-authenticated case of the two young emigrants from the Isles, who, on the death of the captain and officers of their ship, took over the navigation and sailed the ship from Cape De Verde Islands to the Cape of Good Hope on what they had learned from the parish schoolmaster and picked up on coastal vessels visiting their island. Another tribute to the soundness of the instruction given was paid by Professor Menzies, the first Dick Bequest Inspector when he reported: "It has been interesting to find pupils, whose knowledge was all obtained at the parish school in a fishing village on the coast of Buchan, held qualified for the post of

mate, upon examination by the Board of Trade."[1] The pupils in the navigation classes were usually young men of 18-20 years or older. This was the case even in the parish schools where in the same class room would be tiny tots learning their ABC.

From the above considerations it is seen that the history of the organisation of the teaching of navigation presents few features essentially different from that of the course in Practical Mathematics of which it was at one time a part. Originally taught in the Universities, the higher class schools, the parish schools and specialist private schools, by the end of the century it had passed mainly to the evening schools and technical colleges, though the specialist private schools were still of importance. The teaching was sound. The content was very comprehensive as befitted the needs and maturity of its students. The age of the students was unique in contemporary education. The most striking feature, however, was the relatively unimportant position of the Universities. Admittedly, the teachers received their basic training there, but it had to be supplemented. It was the schools, particularly the private schools, which specialised in this branch of Mathematics.

8. Book-keeping. For most of the 19th century, Book-

[1] Simpson: "Education in Aberdeenshire before 1872."

keeping enjoyed an amazing popularity as a school subject. It was already well-established, for its development and spread belongs to the previous century even if one of the earliest references gives it as taught at Carstairs in 1696.[1] On the average it was professed by over 50 per cent. of the parish schools and was almost as popular in the higher class schools. Attention was drawn to this feature of Scottish Education by the Argyll Commission who, at the same time, dealt its prestige a damaging blow by stating that the results of the instruction were very indifferent. Partly as a result of this adverse criticism, it was not included in the list of specific subjects in 1872 and so virtually disappeared from the parish schools. In the last decade it was recognised but it never regained its old popularity in this class of school. The higher class schools continued to teach it but a somewhat similar situation developed. The first Leaving Certificate Examination did not include it. The following year, ungraded papers on Book-keeping and Commercial Arithmetic were added. It was not banned from the secondary schools. It was just discouraged and crowded out. Like other discarded subjects, it found a resting place in the evening schools and in private adventure schools. This was one field in which the

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[1] State of Parochial Education, 1826.

private unsubsidised schools were able to hold their own. Many modern successful private business colleges, such as Skerry's College (1878) were founded about this time.

It was usual to teach two systems of book-keeping: Single Entry, with use of the Day Book and Ledger; Double Entry with use of the Waste Book, Journal and Ledger. The text books of the period, for the most part, presented a set of commercial transactions and showed how to prepare the different books from them. As exercises they provided the student with similar sets of transactions from which to prepare his own books. There was at least one protest against this essentially practical way of proceeding. De Morgan, in his 'Elements of Arithmetic' devotes ten pages to a discussion "on the main principle of book-keeping." In the last decade, candidates for the Leaving Certificate had, in addition, to be able to frame balance sheets and profit and loss accounts.

9. Miscellaneous. The structure of the Scottish Educational system permitted the teaching in the schools of any subject which the master was prepared to teach, and for which pupils could be found.

There are records of Calculus (or Fluxions) having been taught throughout the century in schools in the

University towns and also in towns like Perth, Ayr and Dumfries. Even in the parish schools, during the first half of the century there are occasional records of schoolmasters professing Fluxions; but 'professing' and 'teaching' were not always the same thing. Papers on the Differential Calculus, Geometrical Conics, Analytical Geometry and Dynamics were set in the first Leaving Certificate examination since it was customary at that period to teach such subjects in the best mathematical schools. Dynamics was taught more or less throughout the century, but the other two only in the last decades with any regularity.

Natural Philosophy (or Mixed Mathematics) was in the early decades taught by the mathematics master.

The elements of Astronomy were often taught in schools specialising in Navigation. The Educational Returns for 1826 and 1834 show that even the parish schoolmasters occasionally had pupils in this subject.

A very occasional sideline of the mathematics master was "Architectural and Mechanical drawing." In the first part of the previous century this had been one of his provinces and mathematical text books of that period included chapters on it.[1]

These miscellaneous subjects have been mentioned for, although they are outwith the scope of this thesis,

[1] e.g. Wilson, J., "Trigonometry" published Edinburgh, 1724.

yet they do complete the picture of school mathematical activities and demonstrate further the capabilities and versatility of the teachers of mathematics.

Factors influencing the Purpose, Content and Organisation of Mathematical Teaching.

1. Social and Economic Factors. Because of the nature of the administrative control the traditional Scottish system was extremely sensitive to local demands. Its capacity for adaptation was limited only by the versatility of the schoolmaster. Not until the last decades were the masters bound by a code which could not be altered either by the master on his own authority or by local agreement within the town or parish. The influence exerted by social and economic factors illustrates this sensitivity very well.

In the 18th century the subjects normally taught in the parish and burgh schools were English, Arithmetic and Latin. During the 18th and beginning of the 19th century, the increasing industrialisation of the country created a demand for technical and commercial education. In mathematics, this demand was along practical lines. The practical applications of the previous century, Fortification, Gunnery and in some measure Navigation, belonged to the life of a gentleman! Of these, only Navigation was required to any great extent by an expanding industrialised state. As education ceased to be the prerogative of the upper social class and was extended to the

rising middle class so the demand for fortification and gunnery sank into relative obscurity. These subjects ceased to be professed generally and the demand for them was met by the founding of special Military Academies. The industrial need was for men with a knowledge of mathematical methods and the ability to apply them. This need was met by the courses in Practical Mathematics. The relationship between the growing commercialisation of the nation and education is clearly shown by the growth and popularity of Book-keeping as a school subject. When the masters had the ability and the time, the growing public demand for instruction was met within the legal provision. In the industrial towns this was not the case and a feature of late 18th - early 19th centuries was the foundation of schools by private and semi-private enterprise to provide the necessary quantity of such education. There is no doubt that it was pressure which forced the local communities to provide facilities for the teaching of mathematics, adequate to their own demands.

An analysis of the 1834 Education Return on a geographical basis shows some interesting features. Book-keeping, for instance, was taught in over 50 per cent. of the schools in the counties round the Clyde valley and the Firth of Clyde, that is, in the region

where industrial expansion was greatest. It was taught in less than 30 per cent. of the schools in the Lothians, the Borders and the East coast south of Aberdeen. The high percentage in the West is directly related to the economic situation. Highland economy was breaking down. The land was poor and could not support its population, except at almost starvation level. A young lad, to gain a decent livelihood had to go to the big towns or emigrate. In the towns, he was entering a capitalist economy. He had no capital. He could sell his brains or his brawn. A clerk earned more than a labourer, and had better opportunities for advancement, so economic necessity forced his choice of subject. On the other hand the Lothians and the East Coast were rich agricultural areas which could support a growing population. A young lad could reasonably expect to find a livelihood in his own community so that although industrialisation was taking place, there was not the same pressing necessity to prepare for a new life. In the Borders, the development of the tweed trade absorbed the surplus population locally. There was no economic urge towards a new life requiring higher standards of "education" and although schools were not lacking, the content of education was low even by

contemporary standards.

The geographical distribution of the teaching of Navigation shows clearly the influence of social and economic factors. It was rarely taught except in those districts from which the men followed sea-faring professions. It was taught in all the coastal districts, in Orkney and Shetland, and in the Western Isles. An analysis of the 1834 Education Return shows that, taking the country as a whole, 10 per cent. of the parish schools taught navigation. It was not taught in the parish schools of Lanark, Peebles, Kinross and Selkirk, the counties which were not dependent on water communications. In Fife, Argyll and Wigtown, the percentage was roughly 25 per cent., while in Orkney and Shetland it rose to 50 per cent. The actual figures for Orkney and Shetland show that it was taught by fourteen out of twenty-eight parish schools and also by a number of private schools. The strength of the economic factor in shaping content can be clearly seen when it is noted that only three out of these twenty-eight schools taught Mathematics.

The detailed study of the 1834 Return also shows that Mathematics (i.e. the theoretical course in mathematics) was studied to a much greater extent in the principal towns than anywhere else. This corresponds to the concentration of the professional classes in

the University towns, and the most important of the provincial towns. The needs of the professions did not dictate the contents of the mathematics course, but, because the professional classes wanted a liberal education of which mathematics was an accepted part, it ensured that mathematics was taught. Later the requirements of external examinations such as the Indian Civil Service, the Aberdeen Bursary Competition and the local University Entrance examinations, did condition the content of this course. For instance, the otherwise well-advanced Aberdeen-shire schools only taught the first book of Euclid because that was all that was required for the University Bursary Competition.

A change in social custom which altered considerably the content of arithmetic and mensuration was the reformation of the national system of Weights and Measures in 1824. As illustrating the variation in the 'pound' we would quote from the Statistical Account of Scotland which gives the following account of weights used in Glasgow and Lanarkshire (circa 1822)

"Meal is sold by Dutch Weight; the pound contains 17 oz. 7 dr. Fresh fish, Scotch cheese and fresh and salt butter is sold by tron weight; the pound contains 22 oz. 7 dr. Butcher meat is sold by

a pound which contains 22 oz. 8 dr." This reform simplified the training of an arithmetician and made more of his limited school time available for other topics. With the establishment of standard units, there was no longer a need to change, for instance, Scottish acres to Yorkshire acres. Those with a taste for such calculations could still find plenty of scope for their talents in changing British units to foreign units but this type of calculation ceased to be part of the common experience of everyday life, and so could be relegated to a much less important position in the scheme of arithmetic.

The 'social class' of the pupil often determined the subjects studied. Mathematics was an established part of the education of a gentleman. The rising merchant and industrial classes aspired to be on the same level as the landed aristocracy. So nineteenth century snobbishness ensured a steady flow of pupils to the mathematics masters.

Mathematics has frequently been called 'the mirror of civilisation' but perhaps nowhere is this quality more apparent than in the practical examples in the text books. Changes in social customs are faithfully reflected there. This shows in two ways - in the actual topics, e.g. Tare and Tret; and in material for examples on a topic, e.g. in the calculation of

prices.

The early 19th century was still the great age of the landed proprietor. Many were improving their estates by afforestration. Contemporary arithmetic texts included planting tables showing the number of trees required to plant a given area at given distances apart and devoted space to the estimation of the timber content of trees.

Another early 19th century feature was the inclusion of warping tables for use by the hand weavers. These tables do not occur later as the power looms had deprived the weavers of their livelihood.

The commercial activities of the nation were extensive then as now, so it is not surprising to find much space devoted to topics such as Loss and Gain, Brokerage, Exchange, etc. These have a very familiar look but the commodities bought and sold in the early texts are not all so familiar. These included wool, tobacco, snuff, coffee, sugar, coal, potatoes, brandy and French wines (probably smuggled) - a list representative of Scottish trade at that time. The writers of text books were very conservative. Once an example appeared in print it had a long life. They bought and sold tobacco and herring long after the tobacco lords of Glasgow had crashed and the herring trade with the continent was at a standstill. As an

instance of blind conservatism, we would quote the following futile, artificial example which appears with great persistence in text books throughout the century, viz. "calculate the number of barleycorns placed end-to-end required to encircle the world." Its only merit appears to be that it would keep the pupil from troubling the master for some time.

In Scotland itself, it is not always remembered that the Scots are as great a sea-faring people as the English. During the Napoleonic Wars, Scotland supplied many admirals to the Royal Navy. A strong nautical flavour is shown in the contemporary examples on proportion when not 'rations' but 'seaman's rations' were calculated and, this more frequently, in the division of prize money.

The strong religious background of the period is also mirrored in the arithmetic text books. The 19th century dominie thought nothing of using passages of the Bible for grammatical analysis and for material for arithmetical calculations. Some of these appear odd to modern eyes. Jacob labours for Laban eleven years, eleven months, eleven weeks, eleven days, eleven hours, eleven minutes, eleven seconds and wonders how long he still has to serve! The student labours to find how many hours since the Creation of the world! By the end of the century such examples were rare as changing public taste regarded this use of the Bible as lacking in

proper respect. Perhaps it was these examples that De Morgan had in mind when he wrote concerning the tenth edition of Vyse's Arithmetic (1799). "If a new edition were published some of the examples must be omitted as rather opposed to modern ideas of decency."

In these different ways, mathematical teaching was affected by social and economic factors which, however, were not as potent as they might have been. Economic pressure from the rising tide of industrialisation increased the demand for mathematical teaching, particularly for the course on Practical Mathematics. This course in its day, was quite adequate to meet the needs of the 'captains of industry.' Later when the increasing application of the sciences to industry made necessary a five years' secondary course with perhaps University training to meet the specialised needs of these leaders, very little action was taken to provide suitable mathematical training for the rank and file. Then another way in which the influence of environment was not as strong as it might have been is indicated by slowness of text books to discard applications which were no longer topical. We have already noted several instances of this, but we would like to mention the vinculom which is still

cautiously mentioned in text books on algebra but which was very little used even at the beginning of last century.

2. Contemporary Mathematical Thought. In order to determine the influence of contemporary mathematical thought, we must first find the answer to several questions. What was the training of the teacher of Mathematics? What journals or books were available to him? What were his personal contacts? To what learned societies did he belong? In other words, we must first find out whether new ideas in mathematics and in mathematical teaching ever reached the schoolmasters at all. The answers to these questions are different for the two main classes, the parish schoolmasters and the specialist teachers.

At the beginning of the century there were few University graduates settled in the parish schools, but their numbers steadily increased. The other teachers had no training whatever. The establishment of training colleges by the church improved very considerably the qualifications, both professional and academic, of the non-graduate teacher. By the middle of the century a man could not expect to be settled in a parish school unless he had a University or a College training. Because of the nature of his employment -

the wide range of subjects to be taught, the isolated position of the school, the poor emoluments - it is very unlikely that the parish schoolmaster had any contact with new mathematical ideas and it is reasonable to assume that he had no alternative but to put in practice what he had learned as a student, and to use the text books of his student days. This generalisation is a little dangerous for it cannot be denied that a single copy of a new book, borrowed perhaps, may bring a flood of new ideas to a teacher. Had this happened with any frequency, it seems hardly possible even on the score of expense that schools would, as they indeed did, continue to use the same text book for over fifty years. The longevity of text books, often inferior works, certainly does not suggest the continuous inspiration of new ideas, the resurgence of spring. No, it suggests the sere and yellow leaf, and even the barrenness of winter. It suggests teaching that was stereotyped and stale.

During the second half of the century, there was some improvement in the financial standing of the parish schoolmaster, but what was more important, the wall of isolation around him was broken down. The annual meetings of the Educational Institute became a great occasion and provided social, cultural and educational contacts with his fellows. There had been

earlier associations of teachers but apart from the Scottish School Book Association these had little effect in bringing about an interchange of ideas among schoolmasters. The wall of isolation was also broken down in a physical sense by the improvements in communications due to the development of roads, railways and postal services. The way was there for adequate contact with the outside world but the traffic was too great for group discussions of particular subjects. The Educational Institute were concerned with broader aspects of teaching and also with the ever-pressing question of salaries. Sectional meetings of the Institute arranged on a subject basis belong to the twentieth century. Nevertheless, it was possible for the parish schoolmaster at the end of the 19th century without a great deal of effort to keep up-to-date, whereas at the start of the century it was well-nigh impossible to do so in some areas.

The North East of Scotland was exceptional. Through the stimulus of the Aberdeen University Bursary Competition and the operations of the Dick Bequest, schoolmasters there were aware of the standard reached by their colleagues. They were also given personal help and advice by the Dick Bequest Inspectors on how to raise their standards by improved teaching methods.

The personal position of the teacher of mathematics was quite different. He was a graduate. He was much better off financially and he could afford to buy books. He taught in the towns and was not isolated from cultural influences. He could maintain contact with the Universities. He could keep abreast of the times whereas the parish schoolmaster might fall thirty or forty years behind.

Having considered the opportunities for outside contact, we must now consider the nature of these contacts. There were no professional journals till late in the century and contemporary writings on mathematics, apart from text books were exceedingly rare. In the first decade the operations of the Ordnance Survey were front page news in the newspapers despite their limited space. Otherwise the newspapers and journals ignored mathematics and the teaching of mathematics. An exceptional instance was a long article which appeared in the Edinburgh Review in 1830. This article was a review of a pamphlet by Whewell of Cambridge University entitled "Thoughts on the study of Mathematics as part of a liberal education," which merely echoed the current arguments for the teaching of mathematics. The review, not the pamphlet appears to have aroused considerable interest for even seventy years later, Smith is at pains to refute it in his book on Euclid.

The thunders of the Edinburgh Review were first unloosed in a virulent attack on "the utility of mathematical study, as an exercise of the mind." The author, Sir Wm. Hamilton contemptuously tossed aside "the modern analysis" as "a gymnastic of the mind; its formulae mechanically transporting the student with closed eyes to the conclusion, whereas the geometrical construction leads him to the end more circuitously indeed but by his own exertion, and with a clear consciousness of every step in the procedure." Despite this partial commendation, he proceeded to castigate geometry as a "tedious and complex operation." Finally he conjured up a host of quotations from Greek, Latin, German, French and Scottish authors to show the utter worthlessness of mathematical study "for the conduct of business or for the enjoyment of the leisure of after life." Sandwiched among these writers' opinions, for their arguments were not given, there was included a saying proverbial among the French then regarded as the most mathematical nation of Europe: "Lourd comme un geometre." (Dull as a mathematician).

The only credit which he conceded was that mathematics might be "beneficial in the correction of a certain vice and in the formation of its corresponding virtue. The vice is the habit of mental distraction

the virtue the habit of continuous attention. This is the single benefit to which the study of mathematics can justly pretend in the cultivation of the mind..... But mathematics are not the only subject which cultivates the attention...there is no science which does not equally require it." He then launched into a long, deeply philosophical but tedious argument which he himself summarised thus: "The first and principal condition of academical encouragement, is, that the study tends to cultivate a greater number of the nobler faculties in a higher degree. The study of mathematics effects this, at best, in the most inadequate and precarious manner, while its too exclusive cultivation tends positively to incapacitate and to deform the mind."

The writer's argument is greatly weakened by his closing paragraph wherein it appears that his real purpose was not to decry the study of mathematics but to show the superiority and excellence of Philosophy as part of a liberal education. For him the people of this world were divided into three distinct classes, philosophers, mathematicians and others, the greatest being the philosophers. This writer's argument was carried to such excess that few were willing to agree with him. Indeed the conclusions to which he forced his argument caused men to wonder if perhaps abstract

studies were as 'liberalising' as their adherents claimed. It was generally thought that a boy or a man only had to work hard enough to master mathematics but 'philosophy' - well that was different. The point which Hamilton ignored was that while philosophy might be suitable 'educative' material for some University students, it was quite beyond the mental capacity of others and too mature a study for schoolboys. Even at the end of the century this idea, when applied to the teaching of mathematics, particularly geometry, was only beginning to be appreciated. Not till compulsory education brought to the schools the complete spread of mental capacity, was it slowly realised that perhaps Euclidean geometry was too abstract and formal, too far advanced in thought and expression to serve as an 'educative' study for all. Gradually in the next century, it came to be realised that what might stimulate and inspire one person to mental activity and effort would have no effect on another because of its difficulty. The recent (1950) directive given by the Scottish Education Department on the teaching of mathematics tries to make allowances for this. This Note suggests five basic schemes of work 'designed for different streams of pupils in secondary schools.' "In the syllabus intended for the weakest pupils, mathematics covers little more than the simple arithmetic of

everyday life; in the syllabus for average pupils, a treatment is encouraged which would relate mathematics to the life of the adult world in its technical, commercial, domestic and civic aspects; and in the syllabus for the ablest pupils it is sought to combine this broader approach with the more vigourous treatment required as a foundation for advanced study." These schemes represent a praiseworthy attempt to find mathematical material suitable for the mental capacity of the pupils yet fulfilling the purpose of mathematical teaching as stated so clearly at the beginning of the 19th century for even of the scheme for the weakest pupils, it is laid down that "there should be scope for real, if simple, thinking."

The argument that philosophy was infinitely superior to mathematics in its 'educative worth' was one frequently advanced in the early decades. In practice, even for University students, it was a counsel of perfection, for many Scottish students started their University careers at the age of fourteen, and so could not have the necessary maturity to appreciate philosophy. One of the beauties of mathematics is that it provides suitable material for intellectual activities for all ages and all minds. Everyone loves a puzzle, particularly if they can solve it. We have seen boys in the

70-85 I.Q. range when faced with trying to explain 'mass hypnotism' puzzles of the type, "Think of a number; double it; add on six; half it; take away the number first thought of and the answer is three," grasp the idea of the use of 'x' and promptly try to use it in other problems. We have also seen Training College students who imagined they knew something about arithmetic tie themselves in intellectual knots with the greatest enjoyment while trying to explain just how many eggs one hen would lay in one day 'if a hen-and-a half laid an egg-and-a-half in a day-and-a half.' The point which we wish to stress here is that mathematics can be an exciting living study for all and not just an austere exercise for a few, provided the material for study is suitably chosen.

In discussing the implications of this article in the Edinburgh Review, we have strayed somewhat from our immediate purpose which was to show what contacts schoolmasters could have with new writings on mathematics and mthematical teaching. Two more articles are worthy of notice, written by B. Mackay and S. S. Laurie but as these deal with methods of teaching, they will be discussed later. To complete this meagre list we have a reprint of an address given to a gathering in the High School of Edinburgh in 1870 by George Lees. The title was "An Address in defence of Euclid's

elements of Geometry as a class book for students." It was apparently a reply to an address to a similar gathering delivered by one of the Mathematical masters at Rugby advocating a different approach to the teaching of geometry. We do not propose to give an account of this address as the title is self-explanatory and as we have quoted from it elsewhere in this essay. Last of all, throughout the century there were good text books Scottish and English available but the cost of these was often prohibitive.

At one period there was a very intimate contact between the professors of the Universities and the schools, particularly the higher class schools. Many of the professors had first-hand experience of the tasks of a schoolmaster having served as such before accepting 'chairs.' From 1861-1872 the Universities were responsible for the examination of schoolmasters before they were appointed to Burgh or Parish schools. Later a great deal of school inspection was carried out by the professors of all the Universities during the long vacation. This inspection was at the request of the appropriate school authorities and, unlike the annual presbytery inspection, was critical. It was on a series of inspections carried out by Professor Chrystal that the Scottish Education Department based its first Leaving Certificate

papers. The schools turned naturally to the Universities for inspiration and leadership. In the famous Kenmay dispute, when the local presbytery said the school was efficient and the Dick Bequest Trustees said it was not, a panel of University professors was called in to examine the school. Many professors joined the Educational Institute of Scotland when it was formed. In the evidence given before various Royal Commissions, many of the professors showed great knowledge of conditions in the schools and a deep understanding of the problems of the teacher. More than one described their work in the junior classes as being that of a schoolmaster! At the time of the Argyll Commission, the Glasgow Senatus appointed a committee to examine various aspects of education in the schools. Although there were no chairs in education till the end of the century, yet the professors as a whole took a keen interest in the schools right up to the last decade when the state re-organised and took over control of secondary education. Last, but by no means least, University graduates were appointed direct to the schools without, as is required to-day, any period of professional training.

The Training Colleges do not appear to have had any contacts with their students once these were placed

in the schools, but it might not be out of place here to discuss just how far these institutions did influence mathematical teaching. The 'pupil teacher' carried on his own studies in English, Mathematics etc. at the Training College, and if good enough, he might take a few University classes. His final standard of attainment was not high. In 1872 this comprised the first book of Euclid and the four rules of Algebra. In 1878, the standard had risen considerably. The new attainment was the first three books of Euclid and "algebra to quadratic equations." After admission to the Training College, he received a further two years training. The best students were able to win scholarships to enable them to attend certain University classes. Because of this low standard of attainment combined with the sending of the best students to the Universities, it is reasonable to assume that the College teaching was relatively unimportant and was an echo of University teaching. As regards mathematics and mathematical teaching, the Universities were the source of new ideas; the colleges diffused them to a wider public. In a memorandum on the pupil-teacher system, submitted to the Glasgow Senatus, Professor Dickson commended the purely technical part of the training, but added "the ideas of the pupil-teacher are narrow and his attainments are meagre, while his conceit is unbounded."

In considering the two classes of teachers we find that their sources of inspiration were books, the Training College and the University. As the first two were in the long run dependent on the third, we must examine the ideas contained in University teaching.

Let us consider first the teaching of Professor Leslie at Edinburgh (1805-1818) whose 'Philosophy of Arithmetic' we have already reviewed. His predecessors had laid great stress on the 'purity' of Euclid. He held the same view and there is in existence correspondence between Leslie and the great Frenchman, Legendre, on some of the subtler points of the Euclidean proofs. But in addition to the traditional course, Leslie in his lectures and in his 'Arithmetic' (published 1817) examined the foundations of number systems and analysed the four basic processes. Had he used modern jargon, he would have written of the 'number concept' the 'addition concept' and so on, but although the terminology is different the ideas are there. Because of his written work, Leslie is most prominent in this field yet at the other Scottish Universities, similar courses of lectures were delivered. Leslie lamented that the 'publication of abstract works in this country procures neither fame nor emolument' but his book was closely studied by those directly

connected with the teaching of arithmetic and his ideas greatly influenced the men of his time and later. As a consequence of his teaching, the 'Principles of Arithmetic' emerges as a school and a University subject.

An earlier writer of originality who attracted the attention of the connoisseurs was John West, lecturer in Mathematics at St. Andrews.[1] West attacked the place held by Euclid in the schools and in particular he attacked Euclid's treatment of proportion. This was rank rebellion. Simson and the previous generation of mathematicians had regarded Euclid as perfection - it might be possible to polish up the translation, but that was all. West assailed the very foundations of the Euclidean argument. So did Playfair and in a lesser degree, Leslie. In England the pre-eminence of Euclid was not seriously challenged till over a hundred years later at the British Association in 1901.[2] The mathematical schools of the Scottish Universities appreciated the value of Euclid as a mental discipline, but regarded it as too narrow a conception of mathematics. In the 18th century, they had implemented its study with Trigonometry and with the so-called Practical Geometry,[3]

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[1] Leslie and Dr. Chalmers were both pupils of West. Thomas Carlyle, the writer, studied West's 'Mathematics' while he was a student at Edinburgh

[2] Oddly enough this meeting was not in England but in Glasgow.

[3] Wilson: "History of Mathematical Teaching" gives the credit for this to David Gregory and Colin Maclaurin. P. 57

thus evolving the course on Practical Mathematics, popular in the schools in the early decades. Scottish teaching was thus partly in sympathy with the continental view that "Euclid's propositions were drawn out with a view to meeting all possible cavils, and not with a view of developing geometrical ideas in the most lucid and natural manner."[1]

The Universities also took the lead in dropping the course on Practical Mathematics. The value of the work of the French mathematical school in the field of mathematical physics was early appreciated by the Scottish professors. It was realised that to exploit this work, a much more intensive course on the Calculus must be given at as early a stage as possible. By the 50s the mathematical courses of all the Universities had as their objective a knowledge of the Differential and Integral Calculus. The earlier practical applications were crowded out. Schools whose sole function was to prepare pupils for the University, followed suit. Their example was followed by others and eventually only the former 'theoretical' course was taught in the schools. It was through the Universities that Scottish teaching came into contact with the important developments in France.

Any undergraduate of a Scottish University who

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[1] Schools Inquiry, 1867-68.

took a mathematics class, which Arts students could not avoid, was inculcated with a great respect, almost devotion for the French School. At St. Andrews, Professor Duncan in his reading list of 26 works, included 13 in French and 3 in Latin. Among the French authors were Legendre, Lacroix, Euler, Cagroli, Carnot, Biot, Cramer and Lagrange. In a reference to his early teachers at Glasgow, Lord Kelvin once said: "Dr. Meikleham taught his students reverence for the great French Mathematicians Legendre, Lagrange and Laplace. His immediate successor, Dr. Nichol added Fresnel and Fourier to this list of scientific nobles." The celebrated writer Thomas Carlyle who was a student at Edinburgh has to his credit a translation of Legendre's Geometry.

The connection with France was more than a mere study of textbooks. When Lord Kelvin as a young man went to study there, Professor Kelland of Edinburgh supplied him with a personal letter of introduction to Cauchy with whom he seems to have been on intimate terms. Sir John Leslie, Dr. Chalmers, Sir Wm. Hamilton and Professor Forbes were corresponding members of the National Institute of France. Other instances of the close connection between the two countries could be given, but it is sufficient here to note that French

methods were admired by the leading teachers of the Universities. This admiration was passed on to the students who became the teachers of the next generation and through them the French influence reached the class rooms.

The admiration for French Mathematics extended in some very influential quarters to admiration of the French school system as a whole. The Argyll Commission found that in some schools "the French system was adopted or something like it apparently "with great success." Professors Fleeming Jackson and Kelland, giving evidence before the Endowed Schools Commission (1873) strongly recommended the establishment of mathematical courses similar to those at the Ecole Centrale. This recommendation was based not on an admiration for French teaching methods so much as on an admiration for the amount of pure mathematics which was taught in the French schools before ever a boy reached the University.

The outstanding work of the century was the application by Kelvin, Maxwell and Rayleigh of mathematics to physical problems. The laying of the Atlantic cable and the technical problems involved in telephonic communication caught the public imagination. The public saw that pure mathematics could pay dividends - Kelvin died a rich man. The public also

saw that these striking results were obtained through a study of the so-called theoretical mathematics and that the popular practical course was really only for seamen or surveyors or other specialised jobs requiring some, but not much, mathematics. There was popular recognition of the view put forward a generation earlier by mathematicians that mathematics should be studied as a preliminary to studies in the sciences.

This development displaced the practical course from the schools and even threatened to overwhelm the study of Euclid. It offered instead mathematical analysis of similar elegance but with apparently unbounded possibilities. The whole balance of mathematical teaching in the schools was altered. The public wanted 'theoretical mathematics' and because of the responsiveness of the system to outside influences, it quickly got what it wanted. But whether from a lack of appreciation of the fact that all parts of the traditional theoretical course did not lead to the new analysis or whether from an adherence to the cultural theory, Euclid was retained as part of the course. Before its place in the theoretical course could be properly examined in the light of the new requirements, it was granted a reprieve. The State stabilised the school curriculum and the system lost its sensitivity.

Had the State action been delayed for another decade, it is unlikely that so much of Euclid would have remained.

It is, at first sight, a curious paradox that the practical application of pure mathematics to physics by Kelvin and his contemporaries should have led to the abandonment of the practical mathematics course in the Scottish schools, but it is not curious when one remembers that the 'practical' course was a misnomer for the professional course. As the work of Kelvin was a natural development of that of the great Frenchmen, it is interesting to note just how far-reaching the French influence was.

The teaching of the Scottish Universities spread downwards through the schools and it also spread outwards. There was collaboration with France, but strangely enough, a curious sort of one-sided liaison with England developed. English mathematicians had admired and accepted the work of Dr. Simson on Euclid. De Morgan was appreciative of the work of Leslie. Then comes a blank. There appears to have been no interchange of ideas. The same text books might be studied, English dons might be appointed to Scottish chairs, Government Commissions might visit both countries, but Mathematical Teaching developed independently in the two

countries. The word 'developed' can hardly be applied to England for this period was one of extreme dullness and lack of experiment. The English dons and teachers shut their eyes stubbornly to the new developments in mathematical physics at least as far as their educational system was concerned and did not waken up till the nineties when schools like Oundle introduced courses in Practical Mathematics. That the Scots borrowed freely from the English, there is no doubt. A glance at a list of text books used in Scottish schools at any time during the century shows a high percentage of English authors. Men such as Kelland and Thomson received their training in England and maintained their associations there. English ignorance of developments in Scotland is well illustrated by reference to the discussion on the teaching of Mathematics at the meeting of the British Association in 1901.

At this discussion, Professor Perry put forward what he, and most of the speakers thought were completely new and revolutionary proposals attacking Euclid and advocating teaching on more practical lines. There were only three Scottish contributors, but they quietly disposed of the notion that the proposals were revolutionary. Professor Jamieson paid tribute to

the schoolmasters of Banff and the effectiveness of the course on practical mathematics taught there in his youth. Lord Kelvin dismissed the whole matter in a sentence. The new syllabus was very good indeed, just 'like the teaching I had from my father.'[1] The two Scottish professors could not see what all the fuss was about. For a century Scottish mathematics had been free of the shackles of Euclid. Euclid was a part of Mathematics, not the whole of it. Scottish Mathematics was a living thing - not a dead scholastic study. They had spent their lives in a system which while it appreciated the intellectual character of mathematical training, did not despise the 'blessed result of the study.'

Although the older institutions ignored each other's work pretty successfully, the rising technical colleges were aware of each other's achievements. Dr. Sumpner[2] at the same discussion pointed out that long before Professor Perry had outlined his syllabus, many of the larger technical schools such as those at Glasgow, Manchester and Birmingham, had supplemented the orthodox mathematics classes with classes in practical mathematics which had proved more popular. This sidelight shows up again the ignorance in English educational circles of happenings outwith its immediate environment, though whether this was the deliberate ignoring of the

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[1] Professor James Thomson, Glasgow University.

[2] Technical College, Birmingham.

importance of the efforts of a socially inferior class of schools, is a point for discussion elsewhere.

To sum up, those responsible for the teaching of Mathematics drew their main inspiration directly or indirectly from the teaching of the Scottish Universities. University teaching stressed the logical basis of arithmetic as well as using processes. It taught Euclid as an exercise in reasoning, and in the handling and ordering of abstract ideas, but not as the whole sum of mathematics. Euclid was given a place in the course of instruction, but was not allowed to dominate. Practical applications of mathematics were demonstrated partly for their professional value and partly to illustrate mathematical ideas. Later these 'professional' applications were dropped and the study carried to a higher level so that the student would have sufficient knowledge of mathematics to tackle physical problems. The schools followed the lead of the Universities. The Theory of Arithmetic became a school subject. The course in Practical Mathematics was adopted at the beginning of the century, but later was passed on to the evening schools and technical colleges thus allowing the schools to adopt basic 'pure' courses preparing the way for a later study of the calculus and the wider field of mathematical physics.

3. Contemporary Educational Thought. We have already put forward contemporary views of mathematicians and laymen as to the particular reasons for teaching mathematics. It is our intention now to review quickly generally-held educational theories and to see how these directly or indirectly influenced mathematical teaching.

In the first half of the century, if not later, there was a strongly held view that people should be educated 'according to their station in life.' At Edinburgh, Professor Pillans in 1835, delivered a series of three lectures on 'The proper objects and methods of education in reference to the different orders of society.' The Central Society of Education which met in London had as its object 'to collect, to classify and to diffuse information concerning the education of all classes, in every department, in order to learn by what means individuals may be best fitted in health, in mind and in morals to fill the stations which they are destined to occupy in society.' In 1839 the Dalkeith Presbytery reported: 'it is lamentable to think that, by the lower orders, reading and writing are still considered education.' This class view on Education, though common enough among parents, was not held by all. In a memorandum on the High School, the Town Council of Edinburgh stated in 1823 that they

wished 'to preserve unimpaired the distinguishing character of Scotland above every other country, that of imparting knowledge by means of its public institutions, to all classes of society at a humble rate . . . . . excluding none from a liberal education.' Despite this noble statement, it is clear that the 'three R's' was considered sufficient for the labouring classes, higher class schools for the professional classes, and the Universities for the landed aristocracy and the intellectual aristocracy. There is no doubt the brilliant boy, the 'lad of parts' could and did go to the University. In terms of mathematics, this meant arithmetic for the lower orders, practical mathematics or mathematics for the middle class and University mathematics for the wealthy and for the clever.

The theory of mental discipline, though not a new educational theory, was one of the popular theories of the century. To those holding this view, the mathematicians pointed out the mental discipline involved in a study of Euclid or algebra and thus ensured their support for the teaching of mathematics. The material to be studied was mainly abstract. It seemed an ideal mental discipline. Even the philosophers saw more value in mathematics for this purpose than any other subject except philosophy. No one apparently questioned whether it really did achieve this aim.

Scottish education was run on a commercial basis so the law of supply and demand operated. The choice of subject lay entirely in the hands of the customer and not in those of a state department. The customer could choose what he considered best for himself as an individual whereas the state chooses what is best for the customer as a member of the community. In practice, these two choices are not always the same, even though on some theories they should be. The customers, or perhaps it would be more accurate to say the parents of the customers, had no use for theories. They held to two main viewpoints, the cultural and the utilitarian, with of course all sorts of intermediate shades of opinion.

Those holding the cultural view were quite content to let the theorists determine what subjects should be regarded as cultural. Farm labourers studied classical Greek which had no possible vocational value for them. There is no doubt they enjoyed and benefited from these studies, but had the parish minister or the local dominie held the view that Chinese was a subject of great cultural value, they would have studied it with equal diligence and enjoyment! The holders of the cultural view were clay in the hands of the mental disciplinists. For them, there was only one mathematical pattern, the 'theoretical' course.

Scotland was not a wealthy country. Of those desiring education there were very few so rich that they could ignore its commercial value and also very few so poor that they could do likewise. For the bulk of the people, education was a commercial asset and subjects had to be carefully chosen with this end in view. Again and again, foreign observers (including the English) commented on this utilitarian viewpoint of the Scots people. The way in which vocational value influenced the curriculum in favour of certain practical branches, has already been discussed under social and economic factors.

The two viewpoints have been given, but there were few extremists. Most Scots respected learning for its own sake, but at the same time appreciated the commercial value of learning. This was the view of the common people but by the end of the century, direct control had passed from their hands and the people could only speak through their leaders which was a very 'hit-and-miss' affair in those days before Gallup Polls, B.B.C. Listener research and such-like techniques for ascertaining the views of large groups of people. Administrators really had nothing to guide them except their own experience, and that of their immediate advisers. To discover the views of the

leaders in education, let us consider those men whom Dr. Morgan thought fit to include in 'makers of Scottish Education.' There is not a mathematician among them, although Dr. Bell showed some talent as an undergraduate at St. Andrews. Robert Owen was an amateur; so was David Stow. Professor Laurie distinguished himself in classics as a student and later his interests lay in the wider fields of philosophy and education. Professor Darroch distinguished himself in philosophy, education and economics. On the administrative side, we find that Sir Henry Craik, the first secretary of the Scottish Education Department, was a classical scholar and was also distinguished in law and history. His successor, Sir John Struthers graduated with honours in mental philosophy and classics.

These men had no personal prejudices with regard to the teaching of mathematics. They recognised its place in a balanced curriculum, but had no personal interest in what parts of it should or should not be taught. Their energies were mostly taken up with the battle for adequate educational facilities for all, which was the main educational problem of the century. Indeed, it was the only educational problem which aroused much public interest. The teaching of mathematics benefited or suffered according to the fortunes

of secondary education as a whole. The work of these men increased the facilities for mathematical teaching, but caused no alterations in its content.

The theory of mental discipline had a stabilising influence on the content of mathematical teaching. Although most of mathematics can be regarded as a mental discipline, the study of Euclid must rank exceedingly high as a cultural subject by this criterion. On the other hand, the utilitarian view caused great swings in the popularity of the different subjects, for it allowed the free play of economic factors. The practical branches were taught according as there was a demand for surveyors, navigators, soldiers or book-keepers. At the end of the century the curious position arose of the utilitarians supporting the theoretical course of the mental disciplinists in the schools because it eventually led to much wider practical applications.

4. Administrative Control. Under the provisions of the "Act for Settling of Schools," 1696, the Church of Scotland claimed that 'all schoolmasters and teachers of youth in schools are and shall be liable to the trial, judgment and censure of the presbytery of the bounds for their sufficiency, qualifications and deportment in the said office.' Accordingly, the Church

claimed and exercised the right to inspect annually all schools, not merely the Parish and the various church-sponsored schools. The annual visitation and inspection by the Presbytery was a great event in school life. Its Education Committee through the presbyteries yearly collected statistics concerning the subjects taught and the number studying each subject in every school in the country. Indeed, because of this work the Church was able to dispute the accuracy of the statistics collected by the Argyll Commission. The Disruption, however, lowered the prestige of the Church. It was no longer a representative national organisation. In 1861, the state divested the Church of the right of supervision of the Burgh Schools and in the sixties generally an increasing number of the independent schoolmasters refused admission to the Presbyteries and refused to give information concerning their schools. Despite the difficulties, the Church continued to collect educational statistics until the eighties.

Although the Church annually collected statistics to show the state of education in the country, although the Presbyteries conscientiously carried out the annual inspection of all schools within their bounds, yet in practice this supervision did not affect the conduct of

individual schools or the subjects taught in them. The knowledge of the state of education was used to determine the need for establishing additional schools in particular area or for closing redundant ones. Contemporary writers are agreed that the annual inspection was uncritical. The day of inspection was known in advance. The scholars attended in their best clothes and showed their paces before an audience already biassed in their favour as it included not only the representatives of the Presbytery, but also the children's parents. Such an inspection added a little spice to the work of a school, but had no effect on what was taught or how it was taught. Church control did have some influence on the subjects taught in the Parish Schools. The parish schoolmaster was appointed by the heritors convened by the parish minister. In practice this frequently meant that the master was appointed by the parish minister. The appointing body laid down what subjects were to be taught, and what fees were to be charged, but these were subject to revision. Since the tenure of office was 'ad vitam aut culpam' great care was taken to select a candidate favourable to the Established Church, and capable of teaching the subjects required by the local community, for the qualifications of the master

in these single-teacher schools fixed the range of subjects available during his period of office. The Church interpreted its duty to education in a broad sense. It devoted its energies almost completely to the provision of educational facilities. It saw that the heritors did their duty in establishing and maintaining the parish schools. As far as its own funds permitted it established schools in districts where the legal provision was inadequate. It did not order particular subjects to be taught. Occasionally local presbyteries pushed the claims of Latin and Greek as necessary in order to prepare pupils for the study of Divinity at the Universities, but the Church never made education subservient to its own needs. It advocated a balanced curriculum and in the schools completely under its control carried out this policy. Church-trained teachers received a thorough grounding in arithmetic, the elements of geometry and algebra. As far as the church was concerned, there were no fetters on mathematical teaching.

Under State control, limits were fixed sometimes by direct action more often by indirect action. Like the Church, the State saw to the provision of schools. Like the Church it claimed the right of inspection, but it made its inspection critical. Unlike the

church, it subsidised the teaching of certain chosen subjects. School attendance was made compulsory. Lastly, by the introduction of the 'standards' the 'specific subjects' and the Leaving Certificate, it introduced uniform standards of attainment in primary and secondary schools.

Careful consideration was given to the type of new school to be formed. The prevalent system of large numbers of single-teacher schools or, in the case of higher education, groups of specialised one-teacher schools was discarded in favour of a system of larger schools presided over by head masters. This change did not affect the curriculum directly but the new large schools subsidised by public funds put so many of the private schools out of business that they obtained a virtual monopoly which resulted in the subjects taught by the state-aided schools becoming the subjects generally available.

Through its inspectors, the State found out what was taught and what could be taught in the schools. After this experience of actual school practice, the State fixed the levels of attainment necessary for the Standards, the specific subjects and the Leaving Certificate on the assumption that these levels could reasonably be expected from an efficient school. This administrative action created targets for primary and

secondary teachers in state-aided schools. At the same time, indirectly, it created standards for the private schools since they had to do at least as well as the public schools if they wished to remain in business.

The method by which state aid was granted directly affected the curriculum. Only passes in certain subjects qualified. At first only the theoretical mathematics course was recognised and in consequence practical mathematics disappeared from the state-aided schools which meant, in view of their ever-increasing monopoly, from the schools generally. Later navigation and the commercial subjects were recognised as grant-earning and at once these reappeared in the schools but never regained their popularity.

Because of compulsory education, large numbers of pupils attended school without really knowing why they were required to do so or what benefits they might obtain. Previously each pupil (or at least his parents) knew exactly why he was at school and why he, personally, was studying a particular subject. The state favoured courses of instruction instead of instruction in individual subjects. The pupil was no longer free to choose subjects contained in his course. He now selected a course whose component subjects were already fixed. The educational 'meal' was now 'table d'hote' instead of 'a la carte'. The pupil and his parents

lost some of their personal responsibility through this combination of the two factors, compulsory education and predetermined courses. Mathematics was included in all these courses and we have the spectacle of the whole post primary school population starting courses, geared for the few who attained the Leaving Certificate, and leaving long before such courses could benefit them.

During the century, the central authority in education was vested first in the Church and then in the State. Under the Church almost complete authority was delegated to the local community - the parish, the town or even the schoolmaster himself. In consequence, a system of intricate local organisation grew up which was exceedingly sensitive to the wishes of each individual community, but which defies classification, into the broad divisions liked by administrators. As the masters were dependent on their fees, they were compelled to teach what was required at the moment by their local community or lose their livelihood. This system allowed free play to outside factors whose influence has been discussed above.

The change from Church to State was not achieved at one step. The transition took many years. Elsewhere in this study we have outlined the various changes in organisation which were involved. By the end of the century the new pattern had emerged though some of the outlines were still blurred.

The State produced a co-ordinated system of education with national standards. Only the Universities had complete freedom of action with regard to the choice of curriculum. The State did not introduce new standards of attainment but it did try through administrative action to ensure that all schools and not just some schools reached these standards. This increase in efficiency, in the sense of transmission of knowledge, deprived the Universities of their excuse for teaching the initial stages of mathematics, and they gladly abandoned it.

The new system did not have the sensitivity of the old to outside influences. Changes could be made in the curriculum, but not at the local level of the parish. Significant changes could only be made at the highest level after consultation with various outside bodies. The time factor involved had a very stabilising effect on the curriculum. 'Theoretical' mathematics was recognised at once as a grant-earning subject but only some of the professional branches were recognised, and these only after delay and hesitation.

The change in central authority in education caused the many changes in organisation outlined above. The main results of these were (i) the standardisation of the content of the teaching, (ii) the increase in

efficiency of many schools, (iii) the standardisation of levels of attainment, (iv) the abandonment by the Universities of elementary teaching.

Part II. Method.

General. Under the general heading "Method" we propose to discuss the day-to-day presentation of the subject when the problem of teaching has been reduced to the class room level, and the personal problem of how the master will train and educate his pupils. That is, we propose to deal with the 'tactics' of teaching as opposed to the 'strategy'. Under this heading we shall include the ordering of topics, classroom techniques and procedures, and the way in which the teachers tried to develop mathematical ideas. For the most part we shall discuss the points of difference between the 'tactics' used in the 19th century and modern 'tactics' and we shall also try to discuss the reasons for these differences.

In the 18th century, text books had been written mainly for the master and not the pupil. It was quite usual for the master to dictate rules and examples which the pupil slavishly reproduced in his copy book. In the middle of this century 'unwrought' examples were a novelty and even well on in the 19th century many masters followed the old methods. The late 18th and early 19th centuries saw many texts published suitable for pupils.

There were fashions in names then as now - 'The Tutor's Guide', 'The Schoolmaster's Assistant', 'The

Tutor's Assistant' and 'The Tutor's and Scholar's Assistant'. Those were most popular which contained the rules expressed in the briefest fashion and which contained large numbers of easy examples which could be worked more or less mechanically. In proportion, for example, all sorts of special rules were given for the different cases but no attempt was made to explain the underlying principle. Much time was spent in teaching clever tricks of computation of the most limited application. This mechanical method of teaching a rule and its applications was roundly attacked by the best teachers again and again over the two centuries. So strong was the practice that it became common to divide Arithmetic into two sections, 'rational and practical'. In spite of their views on the value of an understanding of the theory the authors of the best texts were careful to state the rules clearly and to give plenty of examples for experience had shown that this was the only way to get a ready sale.

In Grant's "History of the Burgh Schools of Scotland" 1876 appears this illuminating paragraph. "Even in the teaching of so elementary a branch as arithmetic, much progress has been made and it is less common now than formerly to consider this subject

merely in its commercial bearings. Dexterity in calculating is not the only object aimed at; the learning of rules or methods is always accompanied by some rational explanation, and along with facility in the manipulation of numbers a pupil gains also an insight into the principles of calculation. As evidence of this, one may mention the frequency with which "Theory of Arithmetic" appears in the statements of work done".

In dealing with Mensuration and Trigonometry the same writer maintains that "improvement has taken the same course as in the case of arithmetic, rote learning of rules . . . . . once so universal is falling into discredit".

The writer was unduly optimistic in his assessment of the improvement in teaching and his views must have been coloured by his own experience as a master of Edinburgh Academy where in 1867 the Argyll Commission found that the arithmetic was "probably second to none in Scotland". The higher class schools in the cities had always maintained a high standard of teaching. There had been some improvement in the provincial schools and the parish schools but the Universities still found it necessary to teach the principles of arithmetic. Except in the parish

schools in the northeast there is very little evidence of any marked improvement. Had the writer contented himself with saying that many more teachers now appreciated the value of teaching the principles he would have been nearer the mark. The teaching profession as a whole were much better qualified but because of lack of teaching time were not always able to use the best methods of presentation. The Argyll Commission found that in the smaller schools "teaching was in the old groove - Euclid learned off the book and said according to the letters in the figures given in the books; arithmetic learned by rule of thumb, without any explanation of the principles or application of them". Even at the end of the century the Examiner's report (1899) on the lower grade papers of the Leaving Certificate comments thus: "whole classes have been allowed to learn by rote mathematical book-work which they do not properly understand." No such criticism was made of the candidates on the higher grade.

The number of pupils to be handled by one master was often large by modern standards so the technique of class management must have presented some quite difficult problems. The ways in which the masters surmounted their difficulties are shown by the

following contemporary accounts. The first is taken from the annual report (1836) of the Mathematics master of Glasgow High School.

"Arithmetic. The pupils in this Department were arranged into (six) classes and, in rotation, examined by the Master on the rationale of the Rules. After this examination a competition took place in the Solution of Questions connected with any particular Rule under consideration, the other classes in the meantime being partially under the superintendence of monitors. The subjects embraced during the past Session, have extended from the simplest elements of the science to the most complex business of the counting house."

The second quotation is taken from Stevens "History of the High School of Edinburgh and is a rather flowery account of the life of a new pupil there (circa 1848).

"At two . . . . begins arithmetic . . . . Our youth must take his place with his arithmetical compeers. All are formed into classes. The various powers of the numerals are explained and developed on the blackboard; illustrative examples are given, and oral and written answers required from the pupil. The theory of the rules is carefully gone over and a

similar application is made. The slate is in constant requisition. According to combined accuracy and rapidity on the whole of the examples given out during the hour, places are assigned at the end. And exercises are prescribed on stated days, which must be solved at home, and the operation fairly written out is brought to the class and examined by the Master."

In the second and third year classes " . . . . his exercises to be done at home are both numerous and more difficult. Attention is paid to mental arithmetic . . . . ."

The mathematical teaching in the Universities was conducted on similar lines. Homework and weekly exercises were given. The students were questioned in class on their work and called on to demonstrate propositions or work out problems. It was the practice of Professor James Thomson, Glasgow, the father of Lord Kelvin, "to catechise his class at the beginning of each day on the work of the preceding day, viva voce questions being passed with energy and enthusiasm from bench to bench".[1]

In contrast there is a drier, more academic flavour about the following very interesting pen-picture of Professor Wallace of Edinburgh which was

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[1] "Life of Lord Kelvin": Sylvanus P. Thomson

written in retrospect over sixty years later by one of his students.[1]

"He was a man somewhat over middle age, with noble forehead and grave aspect; profound of knowledge, and ever seeking more. Accurate himself, and painfully extracting accuracy from others. Occasionally absent-minded as he listens to demonstrations which he has heard hundreds of times before, yet quick as lightning to be "down on" every slip. Patient - within limits - of ignorance and stupidity; and, when these limits are overpast, checking the expression of contempt and scorn which is ready to break forth, and substituting for it some snappish utterance in which humour and sarcasm are happily blended, the humour as genuine as if sarcasm were unknown, the sarcasm - bitter it may be but never sour - as sharp as if humour had no place in the composition. Such was (Wallace) as he stood from day to day in decent academic gown, ever liberally powdered with chalk, and rejoiced in the thought that in the course of his long occupancy of the chair he had turned out one or two who might possibly become mathematicians unconscious that every session he turned out a whole class of friends."

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[1] Smith: "Euclid: His Life and System"

Consideration of these quotations shows that the only modern feature missing from the system of instruction was the end-of-term written examination. The Argyll commissioners gave written examinations to the top classes of the schools they visited. Again and again they comment that the pupils did not do themselves justice because of their unfamiliarity with the system. For instance written examinations were not introduced at George Heriot's School until 1869. At the Universities they were introduced at a much earlier period in the century. In the appendix we give papers set at Edinburgh in the degree examinations in 1836. By the end of the century however the written examination was an established part of educational practice.

Our knowledge of the technique used in teaching individual subjects comes mainly from the study of contemporary text books. Those most commonly used can be ascertained from school prospectuses, and the evidence submitted to various Royal Commissions. Usually these books were written by teachers and contain the actual courses taught by the authors set out in the way in which they taught them. The later texts were written for pupils working under the immediate direction of a master but the earlier texts

were written to enable a student to study on his own. Their approach was very direct. Dilworth's "Schoolmaster's Assistant" was written as a catechism. In some cases the text was merely a printed edition of the manuscript rules and examples which succeeding generations of the author's pupils were accustomed to copy out. In other cases, such as Mair's "Arithmetic: Rational and Practical" a serious attempt was made to develop "the Theory of the Science . . . from first Principles" without neglecting the need for plenty of practice by the pupil and there is little doubt that this was the style of Mair's teaching. Mair was at one time Rector of Perth Academy. Extant notebooks of a slightly later period show the teaching there was in this tradition.

The texts which gave the rules were also careful to show the pupil how to set down his working. Gray was an exception. Normally he gave the rule without a worked example to show the setting. The 'settings' used for arithmetical calculations such as reduction, multiplication, division of money, etc. are shown in full. Then in case this help was not sufficient "keys" were published for many of the standard works.[1] These keys gave model solutions to every 'unwrought'

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[1] Board of Education. "Teaching of Mathematics in the United Kingdom" 1912.

example in the text. By comparison the modern custom of publishing the answers with occasional hints on the harder examples seems almost unhelpful.

That very interesting social document "Duncan Dewar's Memorandum Book" records that while Dewar was attending the Mathematics classes at St. Andrews he made the following purchases:

		£	s.	d.
1821 - 22	Ready Reckoner	0	1	0
	Keay to Gray's Arithmetic	0	2	0
	Gunter's Scale	0	2	6
	Mathematical Instruments	0	4	0
1822	Gray's Arithmetic	0	1	3

There may or may not be any significance in the fact that the "Keay" was bought some considerable time before the text but the use of keys was widespread and emphasised the very strong hold that 'copy' methods had at the beginning of the century. No wonder it was with a certain measure of pride that Professor Gibson wrote a century later "pupils can now do where formerly they could only say."

Teaching Technique in individual subjects

1. Arithmetic. Of the many text books published those most commonly used in the higher class schools and the Universities were by Walkingame, Mair, Gray, Thomson, Colenso and Barnard Smith. Mair's and Gray's

texts first appeared in the previous century but were still in use in the seventies. The only assumption made by these various authors was that the pupil could count. The common practice was to start with a brief discussion of notation followed by the four rules; the four rules applied to compound quantities; reduction; the rule of three; Vulgar fractions; Decimal fractions; lastly the various 'practical' applications of arithmetic such as Factorage, Fellowship, Tare and Tret, Profit and Loss, Stocks and Shares, etc. The extraction of square and cube roots was also included but had no fixed place in the order of development. There were certain points of difference with modern practice and these will be discussed below under the appropriate topic.

1a. Addition. In dealing with addition Mair gives the "Dot and Carry" method commenting that it was still in use but "is too low for any but a child". It was still used however in some of the remoter schools into the 19th century.

In "dot and carry" each time the sum was ten or more a 'dot' was put at the side and the units figure only was carried on. This process was repeated until the column was totalled, the final sum giving the units figure of the total. To get the tens

9·
3
5
6·
9·
8·
7
47

or carrying figure, the dots were then counted. In the example shown the addition would proceed thus: 7 and 8 is 15; dot and carry 5: 5 and 9 is 14; dot and carry 4: 4 and 6 is 10; dot and carry 0: 5 and 3 is 8; 8 and 9 is 17; dot and place 7 in answer: count the dots (4) and place 4 at the left of the 7 in the total.

Such a method was invaluable for persons who did not know their number combinations and were compelled to add by a process of counting on the fingers. During the course of the century this method became obsolete as teachers insisted more and more on the number facts being memorised and on an instantaneous response to questions on them. Many of the contemporary references to mental arithmetic in the early stages of the teaching simply refer to practice in the addition and subtraction 'bonds'.

In the addition of compound quantities as distinct from integers the "dot and carry" method retained its popularity longer. A variation in the addition of money mentioned by Mair as fairly common was the practice of "dotting" at 60 pence and at 60 shillings instead of at 12 and 20 though whether this was a relic of the growth of a number system based on the radix 60 one cannot say. More probably

it was just a convenient 'breathing place' in a long calculation but the coincidence is remarkable and the choice of 60 for the shillings group does not admit of an obvious explanation. Even in Mair's generation however the modern method of adding mixed units was in general use by "men of business".

1b. Subtraction. In subtraction the methods of "equal additions", "decomposition" and "borrowing and paying back" were all in use throughout the period. Gray favoured "equal additions". Mair gave both the other methods adding that "borrowing and paying back" seemed to be the most common.

1c. Multiplication. An interesting sidelight on 18th century methods is the frequent advice given to the pupil to learn the multiplication tables by heart. This was given as a recommendation but in the next century it became an order. Leslie expressed the feeling of his generation when he wrote that the multiplication tables "must be engraved on the memory of the arithmetician". Such advice seems odd today but steady drill in the tables and in the 'number bonds' did not become widespread until the second part of the century.

Perhaps the most astonishing feature of text books written about the end of the 18th - beginning of the

19th century is the partiality shown for smart tricks in computation of extremely limited application. The craze for these was carried to such an extent that many of the popular texts included no simple general method for long multiplication. This omission is very remarkable because the method of partial products had been in common use previously. The only method of general application offered to the pupil was an extension of continuous multiplication by factors.

For instance to multiply say 73 by 29 the procedure was to multiply by 4; multiply the product obtained by 7; and then add on one times 73. This, though laborious, is a comparatively simple method to understand when illustrated by a numerical example even if in practice it is not always easy to find the factors. Consider however the mental state of the poor child faced with the statement of this procedure as a rule which must be first memorised and then applied. According to Mair, "If your multiplier consists of two or more figures, multiply continually by its component parts, or by the component parts of the composite number that comes nearest it; and then multiply the given multiplicand by the difference of the multiplier and the nearest composite number: the sum or difference

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 73
  4
292
  7
2044
 73
2117
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of these two products is the answer".[1] No wonder multiplication was a vexation!

For the multiplication of compound quantities (e.g. money, or weight) this method was offered to the beginner in all its tediousness. The 'man of business' for such calculations used the method of Practice but this was not taught in the schools till after Vulgar Fractions.

The method of partial products was soon restored to favour but for long enough the practice of teaching a large number of so-called "short methods" instead of concentrating on a simple, straightforward general method cluttered up the teaching of multiplication.

1d. Division.

"Dic quot? Multiplica, subduc, transferque sequentem."

"First ask how oft? in quot the answer make;
Then multiply, subtract, and down a figure take."

The above lines were in vogue at the beginning of the century as aids to the beginner. Our present methods were in use then also. Division by factors was taught as well as long division. It was customary to express the remainder as a fraction of the division. As in multiplication however much time was wasted in teaching smart tricks of very

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[1] Mair: "Arithmetic; Rational and Practical"

limited use. The methods used in long division of money, weight etc. have not changed much as the present 'long column' and 'short column' procedure have been current throughout the period.

1e. Reduction. The teaching of 'reduction' shows little variation except in the type of problem set. For instance a hundred years ago it was common to ask the pupil to calculate "How many minutes since the birth of Christ?"[1] This type of example persisted well into the century for Colenso puts forward the problem of how long a cannon ball will take to travel to the moon at 1000 ft. per sec.

1f. "The Rule of Three it puzzles me"[2].

It was customary in the early part of the century to teach proportion in a very clumsy, mechanical fashion. Euclid's treatment was beyond the ordinary pupil's understanding, and relatively few pupils ever got the length of the algebraic demonstration. No attempt was made to explain the principles involved. The pupil simply learned a set of rules covering every possible arrangement of the three known and the one unknown quantity for

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1 Butterworth: "Arithmetic"

2 Old rhyme

both direct and inverse proportion. When some form of algebraic notation was used these rules were fairly short but this was not always the case. About the seventies the 'unitary method' was introduced and soon became the popular method. It was the first procedure which seemed to throw any light on the idea behind the process. The lack of any clear conception of 'ratio' and of 'proportion' was the stumbling block in the teaching as we shall show below.

Let us consider first the geometrical treatment. The fifth book of Euclid is a brilliant intellectual achievement when one considers the handicaps. The Greeks, by the use of compasses, could only tell whether two lines were equal or unequal; their number system was clumsy and gave no help; graphical representation of a function was unknown; yet through a long chain of abstract reasoning they did arrive at the ideas of ratio and proportion. It was not the easy way to arrive there and not all could travel the road. Many students simply memorised the fifth book. Many, even of the best teachers, had no clear appreciation of the idea or they would not have taught as they did. Many thought the study of the fifth book should be omitted as incomprehensible.

Those who had grasped the idea defended the book stoutly. West (1784) wrote "the doctrine of proportion is perhaps the most important in mathematics." "As to the defects, the prolixness and obscurity" wrote another defender, Playfair, "they seem to arise entirely from the nature of the language, which being no other than that of ordinary discourse, cannot express without much tediousness and circumlocution, the relations of mathematical quantities, when taken in their utmost generality and when no assistance can be received from diagrams."

The current definitions were as follows:

"Definition III. Ration is a mutual relation of two magnitudes, of the same kind to one another in respect of quantity.

Definition V. If there be four magnitudes, and if any equimultiples whatsoever be taken of the first and third, and any equimultiples whatsoever of the second and fourth, and if according as the multiple of the first is greater than the multiple of the second, equal to it, or less, the multiple of the third is also greater than the multiple of the fourth, equal to it, or less; then the first of the magnitudes is said to have to the second the same ratio that the third has to the fourth.

Definition VI. Magnitudes are said to be proportionals when the first has the same ratio to the second that the third has to the fourth; and the third to the fourth the same ratio which the fifth has to the sixth, and so on, whatever be their number."

Playfair admitted that the definition of ratio had little merit in the eyes of a geometer because nothing concerning the properties of ratios could be deduced from it. In the eyes of most schoolboys it had no merit whatsoever, for what even slender ray of light does it throw on the idea of ratio? West suggested modifications in the Euclidean definitions and simplification of the language used. He was more concerned with the question of framing definitions that would include commensurables and incommensurables than with the clarifying of the idea of ratio, but some of his remarks must have been helpful; "Every magnitude is measured by another of the same kind, called the measuring unit. Thus a line is measured by a line, an angle by an angle, a surface by a surface and a solid by a solid". This would put some flesh on the bare bones of Definition III by introducing through a comparison of measurements the idea of number. However, the geometers in their teaching did not pursue this idea. Playfair suggested the use

of algebraic symbols but while these simplified the writing out of the theorems they did nothing to illuminate the fundamental ideas. Throughout the century similar attempts were made to remove the language difficulty but with little success. At the end of the century Euclid's fifth book was little changed, was still to most budding mathematicians as incomprehensible as the catechism was to their young brothers and sisters and was still taught ostensibly for its "educative value". It is a good illustration of a study being too abstract to achieve its purpose except in the case of very mature minds.

A study of algebra might have helped to bring understanding, but algebra was not popular. Even of those who studied it, because of the order of the topics, there were few who reached the algebraic treatment of proportion. Even if they did reach this stage it was much later in their mathematical careers than their struggles with either the arithmetical or the geometrical treatment.

Bernard Smith in his "Arithmetic and Algebra" (1853) does not introduce the Rule of Three till after he has covered the ordinary course on algebra. In his explanation he uses the algebraic notation a:b::c:d and derives a very elaborate set of rules for use by the arithmetician. These could bring no

enlightenment to the arithmetician for they depended on the manipulation of algebraic quantities about which he was ignorant. The various attempts to simplify the Rule of Three by the use of algebraic methods all broke down on this apparently obvious point - the arithmetician had no knowledge of algebraic procedures.

Even in trigonometry the idea of ratio was lost sight of. Trigonometry was not recognised as the "method by which the main theorems on similarity are made suitable for use".[1] Indeed the sine, cosine and tangent were not defined as ratios till the second part of the century but were defined as particular lines in a circle of unit radius.

The arithmeticians laid no claim to any understanding of the conception. The very name "the Rule of Three" implies this. Unfortunately, for the learner it was not one rule but a whole series. Gray states them as follows: "Simple Proportion teaches from three given numbers to find a fourth. Of the three given numbers two are always of the same kind, the other is of the same kind as the fourth, or number required in the question.

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[1] Whitehead: "Aims of Education".

Rule

1. Write down that number, which is of the same kind or species with the number required, in the middle, with two points before it : and four after it ::

2. Consider from the sense of the question, whether the answer ought to be greater or less than this number; if greater, place the least of the other two numbers on the left hand, for the first, and the other on the right; but if less, place the greatest for the first.

3. If the first and third numbers are of different denominations, reduce them into the same, and the second number into the lowest denomination mentioned.

4. Multiply the second and third numbers together, and divide their product by the first, the quotient is the answer if there be no remainder and is always of the same denomination with the second number.

5. If there be a remainder, reduce it to the next lower denomination, and divide by the same divisor: Proceed thus with all the remainders till you have reduced them to the lowest denominator which the second number admits of, and the several quotients will be the answer required "

That was the Rule of Three as stated in a very popular text. What chance had the student of grasping the simple idea that if the change in the one quantity is caused by, say, doubling then the related quantity must also be doubled? Or if the change is caused by taking, say, the fifth part then one must take the fifth part of the related quantity? Mair, who had tried to develop arithmetic as a science,

cites the authority of geometry that "the product of the means is equal to the product of the extremes". This was a most unfortunate choice as this theorem obscures the whole idea that if one quantity increases in a certain way then the related quantity increases in a similar way. As a result the rules given by Mair are no more illuminating than those of Gray. It is a great pity that Leslie did not discuss this question in his Philosophy of Arithmetic. In the first half of the century the position was that only advanced mathematicians such as Playfair understood the conception of ratio; that even the teachers had little or no conception of what it was and so the method of teaching was by mechanical rules 'explained' at times by the use of algebraic notation.

The first writer to break away from this tradition was De Morgan. In his "Elements of Arithmetic" which was first published in 1832 his explanation of the rule of three comes very near to the 'unitary' method. He also points out quite bluntly that it "is no more than the process for finding the fourth term of a proportion from the other three". De Morgan's work does not seem to have borne much fruit, in England at least, for the Schools Inquiry Commission found the teaching of proportion to be very poor. Their commissioner who

visited France (circa 1867) reported thus: "Everyone who has watched a French teacher employing with his pupils the simple process called 'réduction à l'unité', and has also watched an English boy's bewildered dealing with a rule of three sum, and heard his questions about its 'statement', which to him is a mere trick, learnt mechanically, not understood and easily misapplied, has a good notion of the difference between the arithmetic of French and English schools".

The credit for the introduction of the unitary method to Scotland appears to belong to David Munn, one time of Dumfries and later of the High School of Edinburgh. Munn had visited France and studied the methods used in the schools there. The commissioner of the Schools Inquiry Commission who visited Scotland reported that "Mr. Munn's teaching of arithmetic was one of the most impressive things I saw in my Scotch tour." From Munn's book "Theory of Arithmetic" published in 1871 we can gather a good impression of this master's methods. He states what is a very common modern aim: "two ends should constantly be held in view - the power of reasoning out any problem proposed, and expertness and accuracy in handling figures". His treatment

of proportion is extremely good. He first gives a very clear explanation of what is now known as the 'ratio' method in which he demonstrates the idea that if the first quantity varies in a certain way then the second quantity varies in the same way. He then goes on to discuss the "Méthode de réduction a l'unité". This latter was the method which through the publicity and recommendation given by the Schools Inquiry Commission swept over England in the last decades and also over Scotland. Munn gave as the authorities on whom he had drawn for his book, Kelland, Thomson, De Morgan and M.M. Bertraud, Bourdon and Serret.

To the twentieth century belongs the development of the 'ratio' method and at the present time both methods are used in our schools.

In this discussion we have tried to show how the lack of understanding of the idea of ratio by teachers led to a number conception being illustrated first by geometrical methods, then algebraic methods and last of all by arithmetical methods. Even today, in our opinion, the idea of ratio could be used to a much greater extent in the teaching of proportion and would lead to a much better understanding of this important 'doctrine'.

1g. Vulgar Fractions.

The only change worth noting in the treatment of vulgar fractions concerns the common denominator used in addition. The modern practice is to use the lowest common denominator but in the early decades the product of all the denominators was used.

e.g. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{32}{64}+\frac{16}{64}+\frac{8}{64}$ etc.

One method used for a time for setting out the working in the reduction of a fraction (cancelling) was as follows:

$$5\overline{)}\,\frac{270}{375} = 3\overline{)}\,\frac{54}{75} = \frac{18}{25}$$

1h. Decimal Fractions.

In many otherwise good texts, great stress was laid on repeating and circulating decimals. The pupil was introduced to these straight away before even he was taught how to apply the four rules. This unfortunate placement caused many pupils to be discouraged and never to reach the much easier procedure of using these fractions in ordinary computation. There is no doubt that the tedious study of repeating and circulating decimals prejudiced a great many people against the use of the decimal system. Apart from this the treatment of decimals was on modern lines.

ii. Percentages and Percentage Profit and Loss

Modern school texts make a feature of the treatment of percentages and percentage profit and loss. This was an innovation in the last decades. Before that the term 'percentage' was scarcely used in a text. When it was used it was often tucked away in an odd example or two on proportion. It was seldom defined but was used in the sense "so many out of one hundred". Percentage profit and loss was treated by Gray as an exercise in proportion. He gave the following rule:

> "As the prime cost:
> Is to the profit or loss on it::
> So is 100:
> To the profit or loss on it."

For the common variations of this problem the rules were very complicated.

ij. Mental Arithmetic.

Round about the fifties and sixties there are many references to the teaching of mental arithmetic. Among the suggestions made by Professor Laurie in the Dick Bequest Report of 1865 was that "much greater prominence be given to mental arithmetic". Four years later the Church of Scotland instructed its schoolmasters that "Arithmetic . . . should be taught as mental arithmetic in the first stages". This simply meant that the basic number combinations

were to be known by heart so that no counting device was needed. This way of teaching quickly became widespread and its scope was increased till it took on its present form. Contemporary writers regarded it as an antidote to the mechanical 'copy' procedures so prevalent at the time and an excellent device for ensuring alertness in the pupils.

Summary.

During this era there was little change in the development of the subject. No new topics were introduced and the order was unaltered except for circulating decimals. In primary teaching much more time and attention came to be given to the teaching of notation and the four rules. The steady drilling of classes in the tables and the number 'bonds' was introduced along with mental arithmetic. The alteration in the order of topics in decimal fractions made possible the teaching of these fractions to the ordinary pupil. Some old methods died out. Two important ideas were put into practice; percentages were treated as vulgar fractions and the 'Rule of Three' was replaced by the 'unitary' method.

Teaching Technique in individual subjects

2. Algebra.

The methods used in the teaching of algebra showed little variation over the century. The treatment was traditional and was based on the order of topics followed in arithmetic. The almost universal development was: definitions, the four rules, fractions, involution and evolution, simple equations, simultaneous equations, problems, quadratic equations.

The approach was through the use of a letter, say x, for an unknown number or quantity. The rules for operating on these unknown numbers were then given. Much energy and time was spent in manipulating fractions and even in extracting square and cube roots before the pupil was introduced to the simple equation. Only Thomson used the simple equation to illumine the study but his example was not followed. Scarcely any work was done requiring the use and manipulation of formulae. The evaluation of long complicated expressions was often carried out but the connection between this exercise and the use of formulae was not shown. Nor was much time spent on factorisation. This feature of the modern treatment of algebra appeared in the last decades but never

approached the present extent. Graphs do not appear till the last decade.

Late in the century it became customary to discuss equations before the really hard work on fractions and extraction of roots but apart from this there was no real change in the development of the subject. Professor James Thomson wrote "much absolute novelty, unless in the mode of exposition, cannot now be expected in a work on algebra". This expectation was not proved false in his era. It was left to the twentieth century teachers to experiment in introducing algebra through generalised arithmetic, or through statistical graphs and to experiment in delaying the introduction of directed numbers. The development of Thomson's era with its close parallelism to arithmetic may have been sound deductive reasoning but it has yet to be shown whether it was as sound psychologically.

Throughout the century there was so little change in the treatment that authors observe a complete silence as to why certain procedures were carried out. The silence is broken by Chrystal in the last decade. "The utmost rigour of accurate logical deduction", he wrote, "has been less my aim than a gradual development of algebraic ideas". This he hoped to

do through generalised arithmetic, graphs and equations. "By the constant exercise of graph tracing the beginner acquires through his fingers three fundamental mathematical notions, viz. the Idea of a Continuously Varying Function, the Conception of a Limit, and the Method of Successive Approximation."

As regards particular procedures we have already remarked that factorisation was rarely practised. The 'common factor' was really the only method in use. Factorisation of the 'trinomial' was not taught. This meant that the easy, though not general, method of solving quadratic equations by factorisation could not be used. 'Completing the square' was the method always employed. There is one other point of difference from present practice, this time a small point in notation. During the first decade the practice was introduced of writing 'aa' as 'a^2' and 'aaa' as 'a^3' and by the third decade it was the common custom.

3. Geometry.

The Universities studied Euclid in a rather leisurely, scholastic way. Time was found to annotate the translation. The underlying assumptions of the Euclidean argument were analysed. Alternative axioms were suggested. The object of the course was

to demonstrate the Euclidean theorems from the traditional premises or to suggest new premises from which the same theorems could be deduced in a simpler way. The logical basis and the resulting logical development were all-important. The mathematical result was incidental. The use which could be made of the results was left to the course on practical geometry.

Such an approach was unsuitable for the schools where the method generally used was rote teaching. There were enthusiastic teachers who aspired to and attained high levels but the ordinary master was well content if his pupils could get off by heart one or two of the books.

Thus in the Universities the professed aim of the teaching, to cultivate the mind, was apparently achieved, but in the schools it was not. It was not till the end of the century that men questioned whether there was any 'carry over' of the training to other activities or whether the training was only of use in solving geometrical problems.

Later by the introduction of 'exercises' or 'deductions' Euclid was transformed from a completely static study. An English writer Cresswell in 1817 published a large collection of exercises but laid no

claim to them being original. They had been selected from other authors "either as exhibiting some remarkable property of lines or figures, omitted by Euclid, or as furnishing a mere exercise of ingenuity". It soon became customary to bind up collections of exercises at the end of Euclid's texts, though the first Scottish text to do so was Thomson's in 1833. Earlier Scottish texts had contained theorems and constructions not given by Euclid but these had been demonstrated in full and not left as an exercise for the student. In the 'thirties' and 'forties' the 'deduction' became an established feature of University teaching but its adoption by the schools was a much slower affair. The 'deduction' was something quite beyond the capacity of rote teaching. There are earlier references to it but probably it was the 'seventies' before its use was at all frequent. By the 'nineties' however its use had increased so much that the study of the theorem and the solution of riders were regarded as of equal importance. This trend in geometrical teaching is in some ways analogous to the introduction of "unwrought" examples in arithmetical teaching in the previous century. At the end of the century the school study of geometry was broadened still further by incorporating parts of

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'practical geometry' such as the 'ruler and compass' constructions.

In passing we might mention an innovation of Leslie's which gained some currency for a time. This was the introduction of 'geometrical analysis', a process in which the result was assumed to be true. From this assumption the necessary construction or proof was deduced. The proof was then 'composed' by the ordinary deductive method. Leslie gives a great many examples of this type of exercise but works each out in full. This procedure almost completely disappeared from text books in favour of the modern 'deduction'.

In France the drift from Euclid followed a different pattern tending more towards "modern geometry". In both countries at the beginning of the century the development was similar. Legendre's and Leslie's texts on Geometry were closely parallel:- first, the Elements of Euclid followed by a Commentary; second, a short treatise on Trigonometry. Towards the end of the century Comberousse outlined the following course of geometry in 'Le Cours de Mathématiques' which represented the experience of thirty years of teaching from 1852 to 1882.[1]

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[1] Comberousse taught at the Collège Chaptal which was one of the two great municipal schools of Paris. In 1866 it had 1000 scholars, 600 of them boarders. There was no Latin or Greek in the curriculum which included French, modern languages, mathematics and science as its principal subjects.

1. "Straight Line" : lines, angles, triangles, perpendiculars, parallels, polygons.

2. "Circumference of Circle" : arcs and chords, perpendiculars and parallels in a circle, mutual positions of two circles, angles, constructions.

3. "Proportion" : ratio theorems, similar triangles, mean proportionals, tangent and secant problems.

4. "Circumference of circle" : regular polygons, method of limits, .

5. "Surfaces" : (i)area of polygons, circles, Simpson's rule.
(ii)comparisons of areas, problems and exercises, Theorem of Pythagoras.

6. "Solids" : Planes, dihedral angles, polyhedral angles. Area and volume of prisms, cylinders, cones, spheres and truncated bodies.

7. "Conic Sections"

4. Trigonometry.

The little evidence available to illustrate the way in which trigonometry was taught indicates that it was regarded mainly as an aid to calculation, and not as a stimulus to the development of new ideas. West in 1784 wrote: "the first part of mensuration . . . . is commonly named trigonometry". The Mathematics master at the High School of Glasgow in preparing his report for 1836 stated that the trigonometry studied by the senior class was "one hundred and eight formulae with their application

to the solution of Plane and Spherical Triangles and Astronomical Problems". This acquisition of "one hundred and eight formulae" certainly does not appear consistent with a liberal view of the educative value of trigonometry. Even University teaching where a more liberal attitude might have been expected tended to become lost in the mastery of detail leading to calculation of use to the navigator, the surveyor and the civil engineer. In the early partof the century Trigonometry was not taught to illuminate the ideas of ratio, of proportionality, of similar figures or of the periodicity of the function. It was the science which "teaches the relations and calculations of the sides and angles of plane triangles".[1]

During the century there was a steady drift from the viewpoint that "plane trigonometry is that branch of mathematics by which we learn how to determine or compute three of the six parts of a plane triangle from the other three when that is possible"[2] and towards the view that the mutual relationships of the sine, cosine and tangent should

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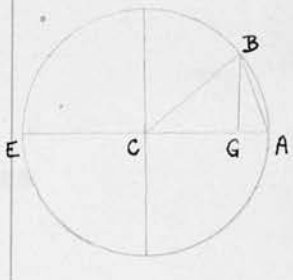
1 Hutton: 'Mensuration'.

2 Gregory: 'Mathematics for Practical Men'.

be the main aim of the teaching. This might have led to a more liberal treatment of the subject but in practice it did not; the study was reduced to the acquisition of innumerable formulae and their use in particular circumstances without any clear recognition of underlying principles.

During the first half of the century it was customary to define the trigonometrical functions as lines in a circle whose radius was unit length. For example:

> The sine BG of an arc AB, is a straight line drawn from B, one of its extremities, perpendicular to the diameter AE which passes through the other



Playfair, pointed out that the sine was not the line itself but the ratio of the line to the radius. He stressed that the sine was not a magnitude but the ratio of two magnitudes, in other words, a number. However, he did not use this idea when defining the functions as he regarded it as "a little more abstract than the common one . . . . though that which should in strictness be pursued". To Todhunter belongs the credit for popularising the modern method of first defining the functions with respect to the sides of a right-angled triangle and later extending the definition to include angles

greater than a right angle. This innovation quickly became the common practice. It did not lead immediately to any better appreciation of the idea of a ratio as it appears to have been adopted by most teachers simply to bypass a few theorems and so make a quicker start to the practical applications of the subject, but at least it was a step in the right direction.

A small change in teaching technique but one which must have helped the beginner very much was the introduction of symbols to clarify the statement of formulae. How many present day pupils would recognise the following formula given in Hutton's Mathematics?

> "As the sum of those two sides,
> Is to the difference of the same sides;
> So is the tangent of half the sum of their opposite angles
> To the tangent of half the difference of the same angles".

Or would they prefer this rule given by Ingram?

> "Rule. Oblique triangles: when the three sides are given. Add the three sides, and from $\frac{1}{2}$ the sum subtract the side opposite to the angle sought, then take the arithmetical complement of the logarithms of the two sides containing the angle sought, and the logarithms of the half sum, and of the remainder and add these four together, and $\frac{1}{2}$ the sum will be the log cosine of $\frac{1}{2}$ the angle sought."

5. Other Mathematical Subjects.

As the study of other mathematical subjects was usually on an individual and not a class basis it is impossible to give a detailed discussion of them. Some general observations can be made. Mensuration as taught by Hutton to the Gentlemen Cadets at Woolwich was an elegant exercise in mathematical reasoning, but as taught in the Scottish Schools was mainly an exercise in arithmetical computation. We see a similar contrast in the teaching of the professional subjects. The Universities planned their teaching of navigation, for instance, to illustrate the principles, whereas the schools planned their teaching to produce navigators, which meant that great stress was laid on accurate calculation and a knowledge of special techniques.

Social and Economic Factors

In the eighteenth century the social organisation of the nation had been into fairly well-defined divisions, the landed aristocracy, a small middle class and the common people. With the spread of industrialisation the lines of social demarcation became less definite; the middle class increased so much in size that terms such as "upper middle class" and "lower middle class" became current in

contemporary literature. Mathematics had been the prerogative of the upper classes and had been taught as a cultural subject. The steady growth of a middle class who were able and willing to pay for education created an increased demand for instruction in mathematics. The utilitarian viewpoint of this new middle class was reflected in the increased study of practical mathematics and also in the modifications introduced in the teaching of Geometry.

This increase in the public demand made it economically possible to produce cheap texts of good quality. Through the efforts of the Scottish School Book Association many of these were produced. The rising publishing houses of Edinburgh and Glasgow saw that such ventures were profitable and proceeded to exploit the educational market by commissioning and producing texts at their own risk. By the end of the century good texts were cheap and common instead of rare and expensive. This helped the spread of good teaching methods. It also shortened the life of text books by enabling outmoded texts to be replaced quickly and cheaply. No longer could poverty or isolation from cultural influences be offered as an excuse for the use of inferior books or inferior teaching methods.

The preference of many parents for one master having control of their children's education was a social custom which had different effects on teaching methods in the higher class schools and in the parish schools.

The specialist teacher of mathematics drew his pupils from the middle and upper classes. He commanded high fees and so could afford to teach his pupils in relatively small tutorial groups. He could suit the subject to the needs and capabilities of the pupil and had time for explanations. The parish schoolmaster on the other hand had in his room a large number of pupils of varying social classes of different ages and at different stages. In an overcrowded class room the mathematics pupil had to compete with all the others for a share of the master's attention. The master had little time to demonstrate, hence he made great use of the rules. The rule was copied into the child's copybook and learned by rote. The rule was then blindly applied without any attempt at understanding. We suspect that problems involving long, laborious calculations such as the reduction problems were used to keep one set of pupils occupied while the master gave his attention elsewhere and were not given for their

disciplinary value. Such practices were frowned at by the leading teachers and mathematicians yet only by such methods could one man keep a large heterogeneous collection of pupils busily employed and the parish school system be made to work. The personal economic position of the master undoubtedly affected teaching methods. The poor methods resultant from overcrowded classrooms have been mentioned above but the overcrowding was due in no small measure to financial reasons. Theparish schoolmaster could not afford to pick his pupils. His emoluments were poor and the scale of fees so low that he had to overcrowd his classroom in order to make a living. The specialist teacher in the large towns was in comfortable circumstances and could afford to employ assistance if his classes grew too big.

Various attempts were made during the century to improve the position of the ordinary schoolmaster. For instance the salary revision of 1803 improved the position greatly and, as it was intended to do, attracted better qualified men to the profession. Unfortunately all such salary improvements were on a fixed basis and were soon negatived by the steady rise of prices which continued throughout the century. In addition, although at the time they were sufficient

to attract better recruits, yet in the period of single teacher schools they were not sufficient to prevent the evil of overcrowding in the classrooms.

Many instances can be quoted of the connection between the financial position of the master and the style and quality of the teaching.

Mr. Wood of the Sessional school, a leading teacher of his time, commenting on an exceptionally good performance by a new pupil thought "he must be Aberdeen trained". This opinion of Aberdeenshire schooling was not confined to Mr. Wood. The chairman of the Royal Commission of 1880 assumed it when he asked Dr. John Kerr, one of His Majesty's Inspectors, the reason for the disparity of higher education in Aberdeenshire and the South of Scotland. Dr. Kerr had no doubt as to the answer. "The existence of the Dick Bequest and the large numbers of University bursaries open to competition." This Bequest granted additions to teachers' salaries but was administered in such a way as to raise the standard of teaching and maintain it at a high level. The better endowments attracted better men but these men had to be efficient or they did not receive the grant. The test of efficiency was not confined to a test of the knowledge acquired by the pupils, or the discipline and organisation of the school. The master taught in the presence of the inspector who

tried to assess how far the style of teaching employed was conducive to inspiring mental activity or to a mechanical acquisition of knowledge.

The Argyll Commission found that mathematics was best taught at Dumfries, Ayr, Perth, Madras College, Dundee High and Dollar. It is significant that since these were well-endowed schools the master was not entirely dependent on his fees. Also since these schools were situated at large centres of population or served large rural districts there was a steady supply of pupils wishing instruction and willing to pay for it.

The lack of pensions for teachers affected the efficiency of the schools. The masters when they grew old and infirm had still to carry on their profession or lose their means of livelihood. Royal Commissions repeatedly drew attention to this phenomenon. Parish schools often had no pupils because the master was too infirm to carry out his duties.

These examples illustrate the connection that existed between the financial standing of the master and the style and efficiency of his teaching. This connection was fully realised by the men who framed the Education Act of 1872 and they tried to take advantage of it in their method of paying teachers in the state-supported schools. This method has come to be known as "Payment by Results".

Each school received a small addition to its grant for every pupil who passed a 'standard' or a stage in the 'specific' subjects.

This scheme, in the Government's opinion, had the desired effect of improving primary schools but it had an adverse effect on the secondary instruction given in their post-primary departments. Masters found that it was profitable to take pupils up to the first stage in a specific subject, say Mathematics. The following year instead of proceeding to the second stage of Mathematics the same group of pupils would be presented for the first stage of a different subject, say, English. The second stage in Mathematics was avoided because more successes could be achieved in the first stage of a subject for the expenditure of the same time and energy.

The balance of teaching in the small schools was altered. The schoolmaster working on his own no longer concentrated on his best pupils in the hope that their brilliant successes would attract others and swell the fees. Instead he concentrated on pushing large numbers of pupils through the easy first few standards. While few teachers carried this to extremes it was quickly realised that the smallness of salaries made this a very real temptation

and might lead to abuses. After several attempts to tinker with this system by means of increased bonuses for more advanced work it was finally swept away in

Some idea of how "Payment by Results" operated can be gathered from a letter sent by a schoolmaster to the committee of the Lgasgow Senatus which investigated this question in 1869. This schoolmaster claimed to have been one of the few advocates of the system but after personal experience of it he wrote when dispensing with the services of his assistant; "I saw that under the Revised Code I could not employ an assistant except at a loss. I have allowed, therefore, the extra classes to drop, and have confined the pupil teachers to the subjects demanded of them, and confined myself mainly to the subjects of the Code." Later in the letter he writes, "Teachers find that the minimum of efficiency produced by the minimum of staff will pay best and bread must come before honour". He also gave it as his opinion that "the idea of any benefit resulting from capitation grants for advanced arithmetic, history, etc. is altogether visionary".

The finding of this committee was that the mere introduction of the examinations according to the

Revised Code would have effected the desired improvement in Scotland without the system of 'Payment by Results'.

During the period of its operation steady drilling in the tables was introduced in order that the pupils could be relied on to pass the appropriate standard. Another not so satisfactory feature affecting teaching technique in Arithmetic was the introduction of many methods and settings which while adequate to pass a pupil in a particular standard provided no basis on which to build for future progress. Even after the discontinuation of this system it was long enough before some of these sterile methods passed out of general use.

In the state-aided schools the teaching of Mathematics, as distinct from Arithmetic, suffered a setback but there were no alterations in method. The teachers gave up teaching Mathematics rather than change their methods. The higher class schools were not subject to this system of payment so it did not influence the methods of teaching Mathematics in the schools generally.

Contemporary Mathematical Thought.

The leading mathematicians of the late eighteenth and early nineteenth centuries had a great admiration for the works of Euclid. Simson, Playfair, Leslie and the others saw in Euclid beauty, elegance, and economy of word and thought. Leslie and his contemporaries recognised the limitation of the Euclidean methods of analysis but they regarded the study a sufficient end in itself. In like manner they admired the works of the great French Mathematicians and Physicists, Euler, Laplace, Lagrange, Fourier etc. The fact that their works were fruitful of great developments was beside the point. Euclid and the great Frenchmen were admired for the style and grace of their analysis. A similar admiration for analytical rather than creative thought is shown in Leslie's work on Arithmetic. Such influences cannot be measured quantitatively but part at least of the increasing demand for rational methods, of the demand for clear expression and for economy of language was due to French influence. It may not have been the direct cause of this demand but at least it strengthened teaching on those lines.

In Scotland such methods have a long history. Dr. Simson's work on Euclid was greatly admired and

held up as a shining example by his successors at the Universities. Euclid was retranslated and annotated in much the same way as the Bible was treated by the Divinity professors. The Scottish mathematician was trained to admire and to practice economy of word, clarity of thought and elegance of style. Thomson's Algebra includes an appendix "On Inaccuracies in Style which frequently occur in Mathematical Composition".

The Teacher of Mathematics made part of his living from his practical courses. In this sphere a study of the books written by university-trained men shew that they tried to carry out these ideals in their works on Navigation, Fortification etc. Even when they wrote on Arithmetic and were stating the rules they polished and restated the rules to be in as clear and succinct a form as possible. If teaching was to be by rule, then the rule had to be brief, unambiguous and logically sound.

So far we have attempted to analyse the influence of the Scottish and French mathematicians, but what was the influence of the leading English mathematicians? The century was graced by such men as Horner, Peacock, Babbage, Barlow, Hamilton, Salmon, De Morgan, Boole, Sylvester, Cayley, Smith, Clifford and Todhunter. The last-named is remembered

for the stream of text books that flowed from his pen. In pure mathematics it was a period of quiet consolidation and far-reaching development of existing ideas rather than of brilliant epoch making discoveries. Apart from De Morgan and Todhunter the work and influence of these men was not felt in the schools. Even Todhunter's fertile pen caused no great stir in Scotland as plenty of excellent texts were already available.

Mathematically speaking, the main influences moulding the general style of Scottish teaching were (i) the work of Simson, Gregory, Playfair etc. on Euclid and (ii) the work of the French mathematicians. Text books on the 'teaching' of mathematics belong to the twentieth century. There were however two important papers on the teaching of arithmetic. The earlier was a pamphlet by Benjamin Mackay published in 1834 on the "System of Education practised in the High School of Edinburgh" and recommended by him to the patrons. He laid down that "all teachers of arithmetic should explain the reasons of rules and operations". "In the usual mode of teaching arithmetic three errors are generally committed (1) The pupils are taught to perform the operations not by associating numbers with sensible objects, but by artificial rules

which they do not understand. (2) Theyare presented at the very threshhold of the science with numbers so large that the mind cannot form any correct conception of them, or comprehend the chain of reasoning; and (3) They are not systematically, progressively, or sufficiently drilled in the four fundamental rules." This criticism came not from a mathematical but from a classical teacher. Mackay was one of the leading private teachers in Edinburgh and in 1826 from his Greek class alone had an income of over £700. It is probable that this pamphlet had only a local circulation. Nevertheless it circulated in Edinburgh, which was the centre of the Scottish educational world, and so may have reached further afield. We have quoted it, not for its own importance, but because it brings together in one compass criticisms current among the leading teachers of the time.

The second paper was contributed by S.S. Laurie as a supplement to the Dick Bequest report for 1865. It is really a short treatise on school management and teaching method and includes all the subjects commonly taught. Laurie had not interpreted his duties as Dick Bequest inspector in any narrow sense. Officially his task was to assess the attainments of the children and also the style and character of the master's

teaching. Privately he gave the masters as much advice and encouragement as he could. This unofficial activity was so much appreciated that he was requested to draw up a memorandum on teaching methods which was published with his report for 1865. He suggested the following improvements in the methods of teaching arithmetic.

"1. The teaching to be begun earlier and to begin with the ball-frame.
2. Much greater prominence to be given to 'Mental Arithmetic'.
3. Principles to be taught as well as the mechanical side.
4. The master should not yield to indolence by reading figures instead of numbers.
5. Accuracy imperative.

This memorandum circulated freely in the North East counties, and also, because of Laurie's personal standing and position as secretary of the Church of Scotland Education Committee, had a wide circulation over the whole country.

There are certain points of similarity between the recommendations of both papers. Both uphold the general view that the attainment of mechanical proficiency alone is not enough and both suggest certain improvements in the initial stages of the teaching.

One curious little theory which led to the use of poor methods was the "all-of-all" theory. This meant that if one started teaching, say, addition, one had to

teach the whole of addition before beginning subtraction. This led to difficulties in the case of long multiplication when the multiplicand was a compound quantity. To effect this some use is usually made of division or reduction but on this theory these processes could not be used and hence it was necessary to use techniques such as we have already noted for multiplication by 29 where the quotients of any divisions necessary can be found from the common multiplication tables. Writers who tried to develop the 'Science of Arithmetic', such as Mair, were apt to fall victims to this theory. For instance in adding compound quantities it is necessary to divide to change to a higher denomination. Mair gets over this difficulty by employing the "dot and carry" method. For example to change ounces to pounds, he 'dots' each time the sum of the ounces is 16 and carries the 'overplus'. This was done to achieve what he considered to be a rigorous development of the science although he had previously written that "dot-and-carry" was "too low for any but a child". Even today traces of this "all-of-all" theory is discernible in the attitude of writers who maintain that if subtraction is taught by the method of 'equal additions' then the same method must be used for subtracting vulgar fractions and not 'decomposition'.

This theory may lead to pretty chains of deductive reasoning but psychologically it was a denial of the right to mature and develop and in practice led to extremely poor and cumbersome procedures.

In the early part of the century when the method of teaching by rules was so prevalent, contemporary text books contain a tremendous number of worked examples on the tables of compound quantities. For instance the writer would show how to add Money, Advoirdupois weight, Troyweight, Apothecaries weight, Wool weight, Dry measure, etc. Each time the process was worked in full because when changing, say, inches to feet the writer said "divide by 12 because of Rule II" and not "divide by 12 because there are 12 inches in a foot". In consequence a set of rules had to be given for each table.

We have already mentioned the practice of demonstrating many "short ways" of computation and the frequent accompanying failure to give a general method. In looking over Mair's 'Arithmetic' we find three pages devoted to "simple ways" of division by 9, 99, 999, 98, 997, 996, etc. i.e. division by numbers whose digits are all 9s except the units digit. Clutter like this was really due to the teaching by rules. Many of the rules were so long and so difficult to remember let

alone understand that people were constantly on the lookout for "short cuts" to bypass the rules.

Contemporary Educational Thought.

In Scotland, among the general public, there existed, side by side, two main view points on education, the cultural and the utilitarian. Those subscribing to the cultural viewpoint regarded mathematics, and in particular, Euclidean geometry, as a most valuable part of a liberal education. It was a science "admirably calculated to develop and exercise the reasoning powers and to habituate the mind to habits of close thinking".[1] On the other hand the advocates of the utilitarian viewpoint echoed the sentiments of the same writer when he said that the function of education was to prepare "youth for discharging honourably and efficiently the business of after life". The powerful and wealthy merchant classes translated this last phrase literally. If a boy was to be a seaman, he must learn navigation: if the boy was destined for a commercial house then he must learn bookkeeping. The aristocracy also interpreted it literally: if a boy was to be a 'gentleman' then he must learn 'culture'.

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1 Mackay 1834.

The working classes followed both points of view: they admired education as one of the marks of a gentleman; they saw its 'bread and butter' value; and they saw it as a ladder whereby a poor but gifted youth could climb to high position.

In accordance with the first view the 'demonstrative' method of teaching mathematics was followed, i.e. the method employed by Euclid. In the late eighteenth early nineteenth century some authors even tried to expound the principles of Natural Philosophy by the same method. In accordance with the second view, the utilitarian, the method of teaching was to transmit as much knowledge of mathematical facts and procedures as possible. The extreme application of this method was in the teaching of arithmetic by rules. At the beginning of the century, the methods of teaching required by these different views were kept separate, courses being run on mathematics and on practical mathematics.

If we compare developments in the method of teaching arithmetic with developments in the method of teaching geometry we observe a slow change leading to a somewhat similar result. In arithmetic, after attempts to introduce the 'philosophy' as a separate subject, the 'science', to use the phraseology of the time, was grafted

on to the 'art'. In geometry, on the other hand, by the increasing use of deductions throughout the course and by the absorption of 'practical' geometry, the 'art' was grafted on to the 'science'. By the end of the century the teaching of both subjects blended theory and practice. Teachers no longer paid lipservice, if not real service, to the theory that once 'a priori' reasoning was begun from then onwards any form of empiricism was taboo. Nor was it any longer thought necessary to go through the whole of the theory before attempting the practice of any bit of it, as had been done in the Universities. It had slowly become apparent through "the stress of experience" that rules alone, or the philosophy alone, were neither able to give a proper conception of mathematical ideas, but that the one enriched the other. Chrystal put his finger on the weakness of much early nineteenth century teaching when he wrote "a mathematical truth is not made part of the mental furniture of a pupil merely by furnishing him with an irrefragable demonstration; it is not until he has tried it in particular cases and seen not only where it succeeds, but where it fails to apply, that it becomes a sword loose in the scabbard and ready for emergencies".

The nineteenth century witnessed many attempts to place education on a psychological basis. Pestalozzi (1746 - 1826) had attempted to analyse knowledge and break it up into its simplest elements as these present themselves naturally to the child. These elements were further to be developed by a progressive series of exercises graded by almost imperceptible degrees into a continuous chain. Such exercises were to be based primarily upon the study of objects rather than upon the study of words.

The development of infant room work in the early decades appears to follow this pattern, but it did so from quite different causes. The work of Leslie has already been mentioned. The processes of counting, adding etc. had been broken up, not to meet the natural requirements of the child but to analyse the philosophical basis of arithmetic. The careful gradation of steps is implicit in mathematical reasoning where one step leads to the next. It was purely accidental if this start from a study of notation and the use of objects to illustrate the concepts of addition, subtraction etc. happened to fit in with the 'natural' growth of ideas concerning number in the young child. It was similarly accidental that the careful progression step by step, required by the

deductive reasoning of mathematics, should similarly fit in with the developing mental powers of the child. It was not till the twentieth century that experiments were carried out to see how far this logical development from first principles did fit in with the natural development of the child.

While we are of the opinion that developments in Scotland in the early approach to the teaching of number were a direct consequence of the teaching of the Scottish professors on the philosophy of arithmetic, yet it cannot be denied that the works of Pestalozzi were known to some extent. The only reference, however, we have come across of direct influence was at the New Lanark mills. Concerning the school there, Owen wrote in 1824 that "the elder classes are just beginning a course of mental arithmetic, similar to that adopted by M. Pestalozzi". There was no opportunity afforded the teaching profession for the systematic study of Pestalozzi's work, or indeed any work on educational theory. In 1875 the president of the Educational Institute of Scotland, in his address to the annual meeting, questioned how many present knew the works of Pestalozzi even at second-hand! The teacher who passed through the Normal Training College received lectures on Pedagogy and the Theory of Education.[1]

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[1] First Training College founded at Glasgow 1836.

The academically better qualified graduate took up teaching with only casual contacts with educational theory, as no chair in education existed in Scotland before 1876. From these considerations it would appear that chronologically the Scottish teacher had regular contact with the ideas underlying the philosophy of arithmetic long before he had any regular contact with the ideas of Pestalozzi.

Administrative Control

Prior to 1872, administrative control of the public schools lay, theoretically at least, in the hands of the church and the heritors or the patrons. These bodies might order that a certain subject be taught, but never did they say by what method. The schoolmaster was free to teach in the manner he thought best. This freedom of choice of method was still retained after the Scottish Education Department took over. Occasionally the Department might issue 'Notes' or directives of a general character, but responsibility for the method of teaching rested with the teacher.

In north east Scotland the Trustees of the Dick Bequest administered the fund in such a way as to improve the standard of teaching. A bonus was paid to each parochial teacher in the area, the amount

depending on how well or how badly he taught his school. The Trust was exceedingly fortunate in its choice of inspectors, among whom was S. S. Laurie, later Professor of Education at Edinburgh. This gentleman considered that his duties amounted to more than just grading schools. By precept and example he tried to show 'poor' masters how to become 'good' and 'good' masters how to become 'very good'. In his 1865 report there is included a special report which contains the advice and suggestions which he was accustomed to give. A resumé of the relevant part of this has already been given above. The point we would stress here is that this report was executed, printed, published and distributed at the express desire of this Trust and owed its large circulation to the activities of the Trust.

Today, in a quiet way, the government inspectors can influence teaching method through the style and character of their personal examinations of schools. This factor was missing in government inspection between 1840 and 1872, as the time and energy of the inspectors was directed to other tasks, and collectively during this period they contributed nothing to teaching method. Round about the 'sixties', in public schools managed by patrons or a board of governors, it became the custom to have an annual inspection of the work of the school

carried out by a panel of distinguished visitors. Many of the university professors devoted part of their long vacation to this work. Such an examination was not a purely formal occasion, like the annual visit of the presbytery to the parish schools. The panels questioned the pupils and prepared a critical report on the work of the school, which was presented privately to the governing body. Such examinations were designed to find out what the pupils knew. It was only the Dick Bequest Inspectors who examined the work of the teacher to see how he taught. The Dick Bequest were the only examining body who tried to use their examination to bring about the introduction of better teaching methods.

The part played by the Church must now be considered. For three-quarters of the century the Church claimed that the education of the young was its responsibility. It made no attempt to influence the methods of established teachers. Through the institution of Training Colleges it ensured a steady stream of new teachers, well trained in school management and sufficiently well educated to train boys up to the University entrance standards of the time. Students at these Colleges received lectures on the Theory of Education, but as the system of training was really that of apprenticeship the methods they saw and studied were mainly the

traditional methods. Because of their immature years little could be expected of the students beyond a copy of the master's methods. The contribution of the Training Colleges to education was the spread of sound day-to-day teaching methods in primary schools all over the country. Their aim was not to produce better teachers than already existed but to produce a sufficient supply of good teachers.

A discussion on the effects of administrative control would not be complete without mention of Dr. Bell's 'monitorial system'. On several occasions Bell stumped the country preaching the merits of his scheme, but Scottish parents preferred direct personal contact between the pupil and the master, and it was adopted in only a few schools. Modifications of this scheme had previously been used in Scotland. A case in point was the famous 'Rector's class' at the High School of Edinburgh where one man instructed upwards of two hundred boys. In the early stages in arithmetic where much drilling in the tables is customary, Bell's system met with a measure of success. Otherwise through its nature it stressed the worst features of learning by rote and the mechanical performance of operations. The most famous school to adopt Dr. Bell's system was Madras College. Dr. Bell had endowed

this institution most liberally on condition that the 'monitorial system' was used. This was done, and the system was operated for some years, but eventually government permission was sought to abandon it. The monitorial system is a passing phase of interest to the historian, but leaving no permanent mark on Scottish teaching methods.

The nature of the administrative control at times allowed economic factors to influence teaching methods adversely. In the early period the lack of proper provision for paying adequate salaries caused the teachers to enroll more pupils than they could handle. The lack of pensions often meant a master continuing in office long after he was unable to carry out his duties. The system of Payment by Results encouraged certain practices which were not conducive to good teaching methods.

This discussion on administrative control has been of a general character, as Mathematics was never singled out for special attention in any way. Through the operation of the Dick Bequest and the institution of Training Colleges there was a great increase in the use of good teaching methods. Otherwise administrative control made no attempt to influence teaching method directly. Unfortunately 'Payment

by Results' unexpectedly affected teaching method indirectly but after some years this was remedied.

Review of the Century.

A review of the century shows many outstanding features which cannot be separated from the development of education as a whole. Among these are the widespread general improvement in primary teaching, in the provision of better schools and the training of teachers. For about eighty years the energies of the administrators were spent in obtaining sufficient primary educational facilities and not till that had been achieved was any attention paid to secondary education. The institution of the Leaving Certificate brought about a rapid rise in the standard of attainment of the smaller secondary schools. It is a matter of considerable pride to Scotland that there was always available in different parts of the country teaching of a very high order. The quality was there but, until the last decade, the quantity was small.

The history of mathematical teaching also cannot be separated foom the slow co-ordination of the work of the different types of public school, the universities, the technical colleges and the private adventure schools. In the second part of the century the personal standing

of the teacher of mathematics was affected by the growth of many-teacher schools, teaching all subjects, with a headmaster after the English pattern, and by the decay of the separate schools for English, Latin, and Mathematics. While these features have been discussed in detail to give a balanced picture, yet they were part of the common educational experience of the time.

There were other features peculiar to the teaching of mathematics. In the early decades the organisation of the subject into branches was by no means uniform and the terms used then had frequently a different meaning from today e.g. 'arithmetic' was often synonymous with 'mathematics'. Two courses were taught - Mathematics and Practical Mathematics. The basic parts of the first course were arithmetic, algebra, geometry and mensuration. Later trigonometry was added and mensuration decreased in importance. Further subjects of study in this course were conic sections, calculus and modern geometry. The basic parts of the second course were practical geometry, trigonometry and mensuration which were followed by a wide choice of subjects, navigation, gunnery, fortification, spherical trigonometry, astronomy etc. For various reasons these two courses eventually became merged into one. To effect this the teaching of the more professional

branches, such as navigation, which had always been studied by maturer students, was stopped in the schools and passed on to specialist colleges. Bookkeeping in particular almost entirely disappeared from the schools in the last decade.

There was no lack of well-written text books for those who could afford to buy them. The best contemporary English texts were used as well as the Scottish. In the second part of the century plenty of cheap, reliable texts were available. The text-books, particularly in their choice of practical applications, reflect faithfully the changing manners of the times and the changes in the internal economy of the country.

The source of inspiration of the teachers was the Universities. These in their turn were greatly influenced by the French mathematicians and by the developments in mathematical physics. Current educational theories do not appear to have introduced any new movements in mathematical teaching, but accelerated or retarded existing tendencies. The teaching technique of the best schools or masters (in the first decades synonymous terms) was good, but of the others it was often poor. The battle against rote and mechanical methods was waged continuously, but because of economic factors success on a wide front was not achieved till the very end of the century.

In this essay we have attempted to show how the teaching of a particular subject, mathematics, was affected by a complete change in the organisation and administration of the whole educational system of the country; how it was modified by changing social customs; and how it was affected by contemporary theories in educational and mathematical thought. Because of the nature of these questions no wide generalisations are possible. Let it suffice to say that the best teaching maintained a high standard, was readily adaptable to local requirements and to the increasing demand for more and more mathematics and, as well, covered an exceedingly wide field.

Lastly, how far has the purpose of the teaching been fulfilled and to what extent have teachers been able to carry out their ideals? When we first contemplated this essay it was in a mood of cynicism. We believed that we would be able to show that the material factors, of which one hears so little in educational treatises, were at least as important as the spiritual. All we have succeeded in showing is that these material factors modified, sometimes helped, more often hampered, but never completely halted, the driving force of ideas conceived in the minds of men. Material factors were only of importance when they

influenced the reason for men thinking as they did, as for instance when teachers began to think that the passing of examinations was the reason for the teaching. Throughout, as seemed appropriate in this historical survey, we have tried to show the failures and successes of teachers in holding to their purpose. The details of these are scattered throughout the context. At the end we find ourselves not with the cynics but with the poets - "one man with a dream . . . shall go forth and conquer a crown".

We began with a quotation showing the clear two-fold purpose of mathematical teaching. We would close with a noble piece of advice given by a celebrated mathematics master at the end of a long teaching career: "unless the teacher illustrates largely from his own resources so as to meet the various gifts, and tastes and capabilities of his pupils, the teaching will certainly lack that quickening power, that stirring animation, that healthy vitality so indispensable to successful teaching in any subject."

BIBLIOGRAPHY

(1) Historical Background

- (a) General
- (b) Educational
- (c) Mathematical

(2) Contemporary 19th Century Works and Writings

- (a) Parliamentary Papers
- (b) University Calendars, College Calendars
- (c) Reports
- (d) Contemporary Histories
- (e) Miscellaneous Writings
- (f) Text Books

Historical Background

1.(a) General:

History of the Homeland - Hamilton

Scotland in Modern Times - Mackenzie

(b) Educational:

Secondary education in 19th Century - Archer

Centenary Handbook of the Educational Institute of Scotland - Belford

History of Aberdeen University - Bullock

Short Studies in Education in Scotland - Clarke,J.

Part-time Education in Great Britain - Dent

Makers of Scottish Education - Morgan

History of Aberdeen University - Rait

Education in Aberdeenshire - Simpson

History of Secondary Education in Scotland - Strong

Post-Primary Education in Primary Schools, 1872-1936 - Wade

History of Glasgow University -

(c) Mathematical:

Story of Arithmetic - Cunnington

Teaching of Elementary Mathematics Smith, D.

History of Mathematics - Smith, D.

Euclid: His Life and System - Smith, T.

History of Mathematical Teaching in Scotland up to 1800 - Wilson

Teaching of Arithmetic through four hundred years - Yeldham

Contemporary Writings

2.(a) Parliamentary Papers:

Parochial Education in Scotland - 1826

Education Return - 1834

University Commission - 1837

Argyll Commission - Elementary Schools - 1867
Burgh and Middle-class Schools - 1868

Endowed Schools and Hospitals - 1873-75

Endowed Institutions - 1880

Scottish Universities Commission - 1900

Teaching of Mathematics in the United Kingdom - 1912

Reports of Committee of Council

Codes for 1873 onwards.

Code for Evening Schools - 1893

Scottish Education Department Reports

(b) Calendars, Prospectuses, etc.

Calendars of Aberdeen, Glasgow, Edinburgh and St. Andrews Universities

Calendars of Heriot Watt and Glasgow & West of Scotland Technical Colleges

Prospectuses of the Church of Scotland Training Colleges

Prospectus of the Scottish Military & Naval Academy - 1826

(c) Reports:

Reports of the Education Committee of the Church of Scotland

Reports of the School of Arts, Edinburgh.

2.(c) Dick Bequest Report - 1865

Report of the British Association - 1901

(d) Histories:

History of the Burgh Schools of Scotland - 1876 Grant.

History of Schools in Scotland - 1880 Barclay

History of the High School - Steven

History of George Heriot's School - Bedford

History of Leith Academy

(e) Miscellaneous:

Edinburgh Review - 1830

The Schoolmaster in the Wynds - 1850 - Buchanan

State of our Educational Enterprises - 1858 - Fraser

Addresses to the Mathematics Class at Edinburgh, 1855, 1860, 1870 - Kelland

Address in defence of Euclid's elements as a class book - 1870 Lees.

Presidential Address to the Educational Institute of Scotland - 1875 Hodgson

Higher Subjects in Public and Elementary Schools - 1879 Laurie.

Life of the Scottish Universities - 1886 Ley

Edinburgh Sketches - 1892 Masson

Duncan Dewar's Accounts - 1926 Scott Lang

(f) Text Books:

Land Surveying - 1849 Ainslie

2.(f) The Scholar's Guide to Arithmetic - 6th edition - 1795 - Bonnycastle

Mensuration and Practical Geometry - 1782 - Bonnycastle

Young Arithmetician's Instructor - 1805 - Butterworth

Algebra - 3rd Edition 1905 Chrystal

Arithmetic - 1751 Cocker

Algebra - 3rd Edition 1842 Colenso

Arithmetic - 1850 Colenso

Le Cours de Mathematiques - 1884 Comberousse

Key to Walkingame's Tutor's Assistant Crosby

Practical Mathematics - 6th Edition 1854 Davidson

Arithmetical Books - 1847 De Morgan

Elements of Arithmetic 5th Edition 1858 De Morgan

The Schoolmaster's Assistant 18th edition 1777 Dilworth

Supplement - 1824 Duncan

Practical Geometry and Mensuration - 1845 Elliot

Elements of Geometry - various editors: Simson, Playfair, Thomson, Wallace, etc. Euclid.

Algebra - 2nd edition 1810 Euler

Navigation, Theory and Practice - 1873 Evers

System of Practical Mathematics - 1779 Ewing

Surveying, Levelling and Railway Engineering - 1842 Galbraith

Introduction to Arithmetic 21st edition - 1825 Gray

2.(f) Practical Geometry 10th edition - 1787 Gregory

Algebra 3rd edition - 1889 Hall & Knight

The Complete Measurer 16th edition - 1789 Hawney

Mathematics - 1798 Hutton

Mensuration - 1788 Hutton

Practical Arithmetic & Book-keeping - 1793 Hutton

Mensuration - 1822 Ingram

Key to Melrose's Arithmetic - 1849 Ingram

Principles of Demonstrative Mathematics - 1843 - Kelland

Algebra - 1839 Kelland

Elements of Arithmetic, Algebra and Geometry - 1826 Lees

Key to Ingram's Mathematics - 1843 Lees

Geometry 10th edition - 1817 Legendre

Geometry 2nd edition - 1811 Leslie

Philosophy of Arithmetic - 1817 Leslie

Navigation - 1810 Mackay

Algebra - 1748 MacLaurin

Arithmetic, Rational and Practical 6th edition - 1799 - Mair

Practical Land Surveying - 1810 Nesbit

Arithmetic - 1826 Nesbit

Navigation - 1867 Pryde

Navigation 3rd edition - 1849 Raper

The Tutor and Scholar's Assistant - 1797 Saul

2.(f) Epitome of Arithmetic - 1753 Scott

Arithmetic and Algebra - 1853 Smith, B.

Trigonometry 3rd edition - 1837 Snowball

Algebra - 1844 Thomson, J.

Arithmetic 29th edition - 1848 Thomson, J.

Algebra 5th edition - 1870 Todhunter

Trigonometry - 1859 Todhunter

Algebra - 1779 Trail

The Tutor's Guide 9th edition - 1793 Vyse

The Tutor's Assistant 27th edition - 1795
Walkingame

Mathematics - 1784 West

Trigonometry - 1724 Wilson, John

Navigation - 1773 Wilson, Wm.

Algebra - 1830 Wood

Arithmetic - 1767 Wright

APPENDICES

A. Leaving Certificate Examination - 1888 213

B. Standards for the 'specific subjects' - 1873.. 214

C. Dick Bequest - Examination of School-masters - 1864 215

D. Prize Questions: Edinburgh School of Arts - 1826 216

E. Examination questions for the Degree of Arts, Edinburgh - 1836.. 217

F. Specimen Examination Papers - Argyll Commission - 1865-68 218

G. List of Mathematics Professors at the Scottish Universities 222

H. Principal Elementary Textbooks used at the Scottish Universities 223

J. Analysis of Parliamentary Returns 224

APPENDIX A.

Leaving Certificate Examination - 1888

Note as to Mathematical Papers. The scope of the Mathematical Papers for the Senior Certificate that will be set is indicated in the following list:-

(1) Arithmetic, 1½ hrs. Ordinary arithmetic questions with optional questions on logarithmic computations of an easy kind.

(2) Geometry and Trigonometry, 2 hrs. 8 standard questions on the subject matter of Euclid, Books I, II, III, IV, VI, with easy riders and numerical examples; 4 optional questions, partly on the geometry and mensuration of solid figures, partly on modern propositions usually taught as a sequel to Euclid on trigonometry; properties and logarithmic solution of triangle.

(3) Algebra and Trigonometry, 2 hrs. 8 standard questions on algebra up to and including quadratic equations taking in the elementary theory of irrational forms, surds, proportion and the progressions; 6 optional questions on the higher parts of the subject usually taught in schools, and on the elementary theory of trigonometrical functions.

(4) Geometrical Conics, 2 hrs. 8 standard questions on the parabola and the ellipse with easy riders. 4 optional questions on the general conic, the hyperbola, and the sections of the cone.

(5) Analytical Geometry, 2 hrs. 8 standard questions on the straight line treated by Cartesian co-ordinates. 4 optional questions on the simpler abridged or trilinear methods as applied to the straight line; on polar co-ordinates, the circle and the easier parts of the conic sections.

(6) Dynamics, 2 hrs. 8 standard questions on velocity, acceleration, the laws of motion, composition of forces, equilibrium, centre of gravity, the lever and its modifications, pulleys and the inclined plane. 4 optional questions on work, energy, friction, machines in motion, and hydrostatics.

APPENDIX B.

Standards for the 'specific subjects' - 1873

First year: Algebra: notation, addition, subtraction. Euclid I, propositions 1-15.

Second year: Algebra: to simple equations.
Euclid I.

Third year: Algebra: to quadratic equations.
Euclid, I, II, III.
Elements of Mensuration.

APPENDIX C.

Dick Bequest - Examination of Schoolmasters - 1864

Papers were set on Arithmetic, Algebra, Geometry and Trigonometry. The Trigonometry Paper is given below.

(1) Prove the relations which exist between radius and (i) sine and cosine (ii) tangent and cotangent (iii) secant and cosecant.

(2) The three sides of a triangle are 148, 195 and 239; find its area.

(3) The three sides of a triangle are 2, $\sqrt{6}$, and $1 + \sqrt{3}$; find the least angle.

(4) Prove that R sin (A + B) = sin A cos B + cos A sin B. Find the numerical value of sin 75°.

(5) Determine the volume and convex surface of a cone, the area of whose base is 16 feet, and whose slant height is 3 feet.

APPENDIX D.

Questions given to the Competitors for the Cambridge Prizes at the Edinburgh School of Arts in 1826.

Senior Math. Class.

(1) Required the diameter of a circle, the sides of a square, and also the sides of an equilateral triangle, the areas of which shall be respectively equal to that of a triangle whose three sides are 312, 280 and 376.

(2) If C and c be the two circumferences, and D and d the diameters of two concentric circles, then it may be demonstrated that the area of the annulus will be expressed by $\frac{C + c}{2} \times \frac{D - d}{2}$

(3) A chest is 3 feet long, 25 inches broad and 20 inches deep; required the dimensions of a similar one that will hold three times as much.

(4) From the top of a tower 148 feet high, I measured the angles of depression of two objects in the horizontal plane below and found that of the nearer to be $56^{\circ}40'$ and that of the more remote $32^{\circ}12'$. Required the distance between these objects.

(5) Three observers, A,B and C in a horizontal line, take, at the same instant, the altitude of a balloon. A finds it 14°, B 20° and C 23°. The distance from A to B is the same with that of B to C and is 1560 yards. Required the perpendicular height of the balloon.

These questions and a selection of the actual solutions given by the competitors were attached as an appendix to the Fifth Report of the Directors. This custom was continued in later reports. The mathematics lecturer was George Lees, M.A. who at this time was the Teacher of Mathematics in the Scottish Naval and Military Academy and who later, from 1829-34 was also simultaneously the Teacher of Mathematics at the High School.

APPENDIX E.

University of Edinburgh

Examination questions for the Degree of Arts, 1836.

Mathematics:

(1) If a straight line be drawn from the centre of a circle to any point in a chord, the square of that line together with the rectangle contained by the segments of the chord is equal to the radius; required the proof.

(2) Is s be the side, b the base and A the vertical angle of a plane isosceles triangle, prove that

$$\text{Cos } A = \frac{2s^2 - b^2}{2s^2}$$

(3) From the equations $x + 2y = 8 + y$

$$xy^2 + x^2y = 120$$

Find the values of x and y.

(4) Prove by the binomial theorem that

$$\frac{\sqrt{1 \quad x}}{\sqrt{1 - x^2}} \quad 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \text{etc.}$$

(5) A point moves in a plane at equal distances from a straight line given in position, and the circumference of a given circle both in the plane. What is the nature of the line described by the point?

(6) A point moves in a plane at equal distances from a given point, and the circumference of a given circle, both in the plane. What is the nature of the line described by the point when it moves within the circle? Also when it moves without?

APPENDIX F.

Specimen papers set at Burgh and other Secondary Schools by the Assistant Commissioners of the Argyll Commission - 1865-68.

Arithmetic:

(1) Add together: seven hundred and three, eighty thousand and twenty-two, one hundred thousand and fifteen, twenty million and two, one thousand. three hundred and seven, nine, fifty thousand two hundred, twenty one, one million and one, two hundred and twenty six.

(2) Divide 9 lb 9 oz. 3 dwt. 12 gr. by 5 cwt 9 gr.

(3) Find (by practice) the dividend on £1710:14: 6d at 13s 4½d in the pound.

(4) If 18 men can dig a trench 30 yds. long in 24 days by working 8 hrs. a day, how many will dig a trench 60 yds. long in 64 days working 6 hours a day?

(5) Add together: $13\frac{4}{15}$, $\frac{37}{20}$, 300 and $\frac{59}{60}$

(6) Find the value of:- (a) £1:11: 6d divided by $\frac{1\frac{3}{9}}{3\frac{1}{7}}$. (b) $\frac{3\frac{7}{11}}{4\frac{2}{7}}$ of $\frac{10\frac{5}{7}}{7\frac{1}{2}}$ of $\frac{77}{540}$ of 27s.

(7) Reduce: $7\frac{1}{5}$ of £2: 3: $6\frac{1}{2}$d to the fraction of 7s 6d.

(8) Reduce: (a) £2:6:$10\frac{1}{2}$d to a decimal of a £

(b) 1 cwt. 3 qr. 7 lbs. to the decimal of $2\frac{1}{2}$ tons.

(9) In what time with £389 amount to £486:4:$3\frac{1}{2}$d at $4\frac{3}{8}$ per cent.?

(10) Define a circulating decimal. Divide $\cdot 10\dot{6}$ by $583\dot{}$ and multiply the quotient by $\cdot 91\dot{4}285\dot{7}$

(11) Find accurately the square root of $222\frac{169}{196}$.

(12) A person rows from A to B (a mile and a half) and back again in half an hour. How long would it have taken him if there had been a stream at the rate of a mile and a half per hour from A to B?

Euclid. Junior:

(1) What do you mean by the word 'geometry'; and to what use is it chiefly applied?

(2) Define a point, a line, a straight line, a plane superficies, according to Euclid; give also definitions of the same in your own words.

(3) What different angles and triangles does Euclid employ? How does he define a right angle, a square, a rhomboid, parallel straight lines?

(4) Explain the words postulate and axiom. Write down Euclid's postulates and his last axiom.

(5) If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

(6) If the square described upon one of the sides of a triangle be equal to the square described upon the other two sides of it, the angle contained by these two sides is a right angle.

(7) Through a given point, draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other.

Algebra:

(1) Multiply (i) $1 - ax + bx^2 - cx^3$ by $1 + x - x^2$

(ii) $a + mx - nx^2$ by $a - 2mx + nx^2$

(2) Find the square root of:-

$x^6 - 4x^5 + 10x^4 - 20x^3 + 25x^2 - 24x + 16.$

(3) Extract the cube root of:-

$a^6 - 3a^5b + 6a^4b^2 - 7a^3b^3 + 6a^2b^4 - 3ab^5 + b^6.$

(4) Solve the following equations:-

(i) $\frac{1}{7}(4x - 21) + 7\frac{5}{6} + \frac{7}{3}(x - 4) = x + 3\frac{3}{4} - \frac{1}{8}(9 - 7x) + \frac{11}{12}$

(ii) $\frac{(2x + 3)x}{2x + 1} + \frac{1}{3x} = x + 1.$

(iii) $\frac{6x - 7\frac{1}{3}}{13 - 2x} + 2x + \frac{1 + 16x}{24} = 4\frac{5}{12} - \frac{12\frac{5}{8} - 8x}{3}$

(5) Divide £607:1:8d into two sums, such that the simple interest of the greater sum for two years, at $3\frac{1}{2}$ per cent., shall exceed that of the less for $2\frac{1}{2}$ years at $3\frac{1}{4}$ per cent, by £18:16s.

(6) Solve the following:-

(i) $\frac{3x^2}{4} - \frac{15x^2 + 8}{6} = 2x^2 - 3$

(ii) $37x - 7x^2 = 47\frac{1}{7}$

(7) Bought two flocks of sheep for £15, in one of which there were five more than in the other; each sheep in each flock cost as many shillings as there were sheep in the other flock. How many were there in each?

APPENDIX G.

List of Mathematics Professors At Scottish Universities

Edinburgh

John Playfair:	1785
John Leslie:	1805
William Wallace:	1819
Philip Kelland:	1838
George Chrystal:	1879

Glasgow

James Millar:	1796
James Thomson:	1832
Hugh Blackburn:	1849
W. Jack:	1879

St. Andrews

Nicolas Vilant:	1765
Robert Haldane:	1807
Thomas Duncan:	1820
John C. Adams:	1858
William L. F. Fischer:	1859
George Chrystal:	1877
P. R. Scott Lang:	1879

Aberdeen - Marischal

P. Copland:	1779
R. Hamilton:)	
)	1817
T. Cruickshank:)	

Aberdeen - Kings

W. Duncan:	1800
J. Tulloch:	1811
F. Fuller:	1851
G. Pirie:	1878

APPENDIX H.

Principal Elementary Textbooks used at the Scottish Universities

(a) At the time of the Universities Commission - 1837

Simpson	"Euclid"
Gregory	"Practical Geometry"
Hutton	"Algebra"
Playfair	"Geometry"
Wood	"Algebra"
Duncan	"Supplement"

(b) During the Period 1860-1900

Todhunter..	"Algebra"
Todhunter..	"Trigonometry"
Smith	"Algebra"
Kelland	"Algebra"
Snowball	"Trigonometry"
Loney	"Trigonometry"
Elliot	"Mensuration"
Playfair	"Geometry"
Euclid	various editors

APPENDIX J.

The accompanying tables although compiled from Parliamentary Returns can only be interpreted in a general way because of the haphazard nature of their statistical basis and because of contemporary looseness in phraseology. These returns included all the parish schools but only "most" of the burgh and grammar schools. Subjects were included if they had been taught in recent years. The terms "arithmetic" and "mathematics" were often used to denote identical courses of study.

The numbers given in the tables refer to schools, not pupils.

APPENDIX J.

Education Return. Scotland 1826.

	Algebra	Arithmetic	Book-keeping	Geometry	Land Surveying	Mensuration	Mathematics	Practical Maths.	Navigation	Trigonometry	Astronomy
Aberdeen	5	85	54	2	8	8	32	9	6	1	·
Argyll	·	47	38	1	·	2	4	1	11	·	·
Ayr	·	44	31	4	2	2	6	13	4	·	·
Banff	1	23	13	-	·	·	9	2	3	-	·
Berwick	1	33	20	7	10	8	6	5	5	4	·
Bute	·	5	4	-	·	1	1	1	3	·	·
Caithness	1	10	9	-	·	·	4	·	1	·	·
Clackmannan	1	4	3	-	·	·	2	·	1	·	·
Dumbarton	·	13	7	-	1	2	2	4	2	·	·
Dumfries	3	52	38	5	4	26	10	4	4	5	·
Edinburgh	3	28	13	·	3	3	8	11	2	·	·
Elgin	·	20	17	3	1	4	11	1	4	1	·
Fife	3	62	33	7	9	9	10	18	16	3	1
Forfar	2	51	33	1	2	11	14	19	4	1	1
Haddington	·	25	14	·	2	2	9	5	1	·	·
Inverness	·	34	24	1	·	3	8	-	·	·	·
Kincardine	2	19	8	3	4	1	5	2	3	1	·
Kinross	·	5	3	·	·	6	2	·	-	·	·
Kirkcudbright	1	33	25	-	·	1	4	7	7	·	·
Lanark	1	47	30	·	1	6	3	9	-	·	·
Linlithgow	1	14	9	2	·	4	2	3	1	1	·
Nairn	·	4	3	-	·	·	2	1	·	·	·
Orkney & Shetland	·	29	21	·	2	1	3	·	13	1	·
Peebles	·	17	6	·	3	2	3	2	-	-	·
Perth	2	65	50	·	4	7	16	18	2	2	·
Renfrew	·	12	5	1	1	-	·	2	1	-	·
Ross & Cromarty	·	30	18	1	2	3	11	1	2	·	·
Roxburgh	3	31	21	1	7	11	10	9	2	4	·
Selkirk	·	5	3	1	·	1	3	·	-	·	·
Stirling	2	25	18	1	3	2	6	7	2	-	·
Sutherland	2	13	7	-	·	1	3	-	·	1	·
Wigtown	1	18	15	1	1	6	3	6	6	1	·
	35	903	593	42	71	133	212	160	106	26	2
	3·9%	100%	65·7%	4·7%	7·9%	14·7%	24%	17·7%	11·7%	2·9%	–

Education Return. Scotland 1834.

	Algebra	Arithmetic	Book-keeping	Geometry	Land Surveying	Mensuration	Mathematics	Practical Maths.	Navigation	Trigonometry	Astronomy
Aberdeen	6	93	44	13	10	5	48	5	5	2	-
Argyll	1	70	53	-	-	1	10	4	17	-	-
Ayr	-	46	30	6	4	3	7	11	3	·	·
Banff	1	25	13	1	·	·	17	2	3	·	·
Berwick	1	34	8	1	-	2	15	11	1	-	-
Bute	-	10	10	·	·	1	3	·	2	·	·
Caithness	·	10	6	1	·	·	3	1	1	·	·
Clackmannan	·	5	3	1	·	·	1	·	1	·	1
Dumbarton	·	13	10	·	·	·	3	1	1	·	·
Dumfries	8	65	29	12	5	18	12	7	7	4	·
Edinburgh	1	32	9	2	1	2	9	4	·	·	·
Elgin	·	21	8	·	1	1	13	1	2	·	1
Fife	1	55	15	4	2	4	23	7	12	1	-
Forfar	·	53	16	2	2	4	18	17	6	·	-
Haddington	·	30	4	1	·	2	5	5	-	-	·
Inverness	·	33	10	1	1	2	10	2	2	·	·
Kincardine	·	22	6	4	5	1	11	2	2	1	·
Kinross	-	5	3	-	1	1	3	-	-	-	·
Kirkcudbright	3	49	25	3	·	14	8	7	7	2	1
Lanark	-	72	23	·	·	2	18	3	·	·	·
Linlithgow	-	13	4	-	-	1	4	1	·	·	·
Nairn	·	4	1	·	·	·	1	·	·	·	·
Orkney & Shetland	·	28	15	3	·	·	3	1	14	·	·
Peebles	·	16	4	·	·	·	3	3	·	·	·
Perth	1	73	30	3	1	2	25	14	2	1	·
Renfrew	2	18	9	·	·	1	4	2	2	·	·
Ross & Cromarty	·	33	19	·	·	2	12	1	1	1	·
Roxburgh	·	44	15	2	4	6	17	20	3	·	3
Selkirk	·	5	1	·	·	2	2	1	·	·	·
Stirling	·	32	20	·	2	2	10	2	1	·	·
Sutherland	·	13	8	1	·	1	6	·	·	·	·
Wigtown	·	18	12	1	1	4	8	6	5	·	·
	25	1040	463	62	40	84	332	141	100	12	7
	2·4%	100%	45%	6·0%	3·8%	8·1%	32%	13·6%	9·6%	1·2%	0·7%