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On a General Solution of Hill's Equation. By E. Lindsay Ince, M.A., B.Sc., Research Student in the Mathematical Laboratory, University of Edinburgh.

(*Communicated by E. T. Whittaker.*)

§ 1. *Introduction.*—The differential equation of G. W. Hill,

$$\frac{d^2y}{dz^2} + (\Theta_0 + 2\Theta_1 \cos 2z + 2\Theta_2 \cos 4z + \ldots)y = 0 \text{ *} \quad . \quad (1)$$

may be regarded as an extended case of Mathieu's equation,

$$\frac{d^2y}{dz^2} + (a + 16q \cos 2z)y = 0. \dagger \quad . \quad . \quad . \quad (2)$$

In it y may be taken to represent the extent of departure from the periodic orbit caused by the action of various perturbative forces, and z as the time-variable. The quantities Θ_0, Θ_1, Θ_2, etc., are merely known numerical constants.

From the general theory of linear differential equations with periodic coefficients, it is known ‡ that the general solution of both (1) and (2) is of the form

$$y = Ae^{\mu z}\phi(z) + Be^{-\mu z}\psi(z),$$

where A and B are arbitrary constants, μ is a constant depending solely on the Θ_0, Θ_1, Θ_2, etc., in the first, and on a and q in the second case, and ϕ and ψ are purely periodic functions of z, with period equal to that of the coefficients in the differential equation.

In general, μ is not zero, but it may so happen that the numerical constants (Θ_0, Θ_1, Θ_2, etc., or a and q) of the differential equation are such that μ does vanish, in which case ϕ and ψ become identical and y is expressed as a purely periodic function of z. Thus in this case, not the general solution, but a single solution is known, to which a second and quasi-periodic solution is related as in the "logarithmic case" of Bessel functions.

In the case of Mathieu's equation these purely periodic solutions are known ;§ in particular, if

$$a = 1 + 8q - 8q^2 - 8q^3 + \ldots$$

the equation (2) has a solution

$$y = \sin z + q \sin 3z + q^2(\sin 3z + \tfrac{1}{3} \sin 5z) + q^3(\tfrac{1}{3} \sin 3z + \tfrac{4}{9} \sin 5z + \tfrac{1}{18} \sin 7z) + \ldots \quad . \quad . \quad (3)$$

* Hill, Cambridge, U.S.A., 1877 ; *Acta Mathematica*, vol. viii. pp. 1–36 (1886) ; and *Collected Works*, i. pp. 243–270.

† *v.* Bruns, *Ast. Nach.*, No. 2533, S. 193–204 (1883), and No. 2553, S. 129–132 (1884).

‡ *v.* Floquet, *Annales de l'École Normale Supérieure*, (2), T. 12, pp. 47–88 (1883).

§ *v.* Whittaker, *Proceedings of the International Congress of Mathematicians*, vol. i., Cambridge, 1912.

and if $$a = 1 - 8q - 8q^2 + 8q^3 + \ldots$$

the equation (2) has a solution

$$y = \cos z + q \cos 3z + q^2(-\cos 3z + \tfrac{1}{3}\cos 5z) + q^3(\tfrac{1}{3}\cos 3z - \tfrac{4}{9}\cos 5z + \tfrac{1}{18}\cos 7z) + \ldots \quad (4)$$

The very close resemblance in form which exists between these two solutions suggested to Whittaker that they might be mere degenerate cases of a general solution of Mathieu's equation having the form

$$y = e^{\mu z}u,$$

where

$$u = \sin(z-\sigma) + a_3\cos(3z-\sigma) + b_3\sin(3z-\sigma) + a_5\cos(5z-\sigma) + b_5\sin(5z-\sigma) + \ldots$$

where σ was a new parameter depending solely on the constants a and q of the differential equation (2). The solution (3) above would then correspond to the case $\sigma = 0$, and the solution (4) to $\sigma = \frac{\pi}{2}$.* In this expression the coefficient of $\sin(z-\sigma)$ is taken to be unity, and no term whatsoever in $\cos(z-\sigma)$ appears. By means of this stipulation, σ is completely determined—it amounts, in fact, to a definition of σ. It would, of course, be equally possible to express u as a series beginning with $\cos(z-\sigma)$ in which no term in $\sin(z-\sigma)$ appears, in which case σ would be differently defined.

Acting on this suggestion, Whittaker found that Mathieu's equation was actually satisfied by an expression of the form $y = e^{\mu z}u(z)$, where $u(z)$ is a purely periodic function of z of the above form and μ is given by the power-series in q,

$$\mu = 4q\sin 2\sigma - 12q^3\sin 2\sigma - 12q^4\sin 4\sigma + \ldots \quad (5)$$

the parameter σ being connected with the constants a and q of the differential equation by the relation

$$a = 1 + 8q\cos 2\sigma + (-16 + 8\cos 4\sigma)q^2 - 8q^3\cos 2\sigma + \ldots \quad (6)$$

and the coefficients a_3, b_3, a_5, b_5, etc., of the Fourier series for u are themselves power-series in q with coefficients involving the parameter σ, viz.,

$$b_3 = q + q^2\cos 2\sigma + (-\tfrac{14}{3} + 5\cos 4\sigma)q^3 + \ldots$$
$$a_3 = 3q^2\sin 2\sigma + 3q^3\sin 4\sigma + \ldots$$
$$b_5 = \tfrac{1}{3}q^2 + \tfrac{4}{9}q^3\cos 2\sigma + \ldots$$
$$a_5 = \tfrac{14}{9}q^3\sin 2\sigma + \ldots$$

and so on.

In the astronomical problem of the mean motion of the Moon's

* Whittaker, *Proc. Edin. Math. Soc.*, xxxii. (1913–14), p. 76.

perigee which led Hill to the discussion of the differential equation known by his name, the principal aim is to evaluate the quantity μ on which depends the secular departure from the truly periodic orbit. This is a matter of considerable difficulty, because no simple relation can be found expressing μ directly in terms of the constants Θ_0, Θ_1, Θ_2 . . . Hill himself solved the problem by the use of an infinite determinant, a method previously unknown in analysis, and obtained μ numerically to fifteen places of decimals (sixteen significant figures). It is the object of the present paper to apply Whittaker's method of solving Mathieu's equation to Hill's equation, and thus to obtain literal expressions for μ in terms of σ, and for the parameter Θ_0 in terms of σ and the constants Θ_1, Θ_2, etc., of the equation, by means of which expressions μ might be evaluated. The process itself is really equivalent to introducing a new parameter σ in place of the old parameter Θ_0, and then obtaining a simple relation between μ and this new parameter and the constants of the equation, from which we may ultimately deduce for μ its numerical value.

§ 2. *The Method of Solution.*—As in the case of Mathieu's equation, we assume that the general solution of Hill's equation is

$$y = e^{\mu z}u,$$

where

$$\mu = p_1(\sigma)\Theta_1 + p_2(\sigma)\Theta_2 + \ldots + q_1(\sigma)\Theta_1{}^2 + q_2(\sigma)\Theta_2{}^2 + \ldots$$
$$+ q_{12}(\sigma)\Theta_1\Theta_2 + q_{13}(\sigma)\Theta_1\Theta_3 + q_{23}(\sigma)\Theta_2\Theta_3 + \ldots + r_1(\sigma)\Theta_1{}^3 + \ldots$$

and

$$u = \sin(z-\sigma) + A_1(z, \sigma)\Theta_1 + A_2(z, \sigma)\Theta_2 + \ldots + B_1(z, \sigma)\Theta_1{}^2$$
$$+ B_2(z, \sigma)\Theta_2{}^2 + \ldots + B_{12}(z, \sigma)\Theta_1\Theta_2 + \ldots + C_1(z, \sigma)\Theta_1{}^3 + \ldots$$

σ being determined by the relation

$$\Theta_0 = 1 + \lambda_1(\sigma)\Theta_1 + \lambda_2(\sigma)\Theta_2 + \ldots + \mu_1(\sigma)\Theta_1{}^2 + \mu_2(\sigma)\Theta_2{}^2 + \ldots$$
$$+ \mu_{12}(\sigma)\Theta_1\Theta_2 + \ldots + \nu_1(\sigma)\Theta_1{}^3 + \ldots$$

Substituting these expressions in the differential equation, we get

$$\begin{aligned}
&-\sin(z-\sigma) + A_1''\Theta_1 + A_2''\Theta_2 + \ldots + B_1''\Theta_1{}^2 + B_2''\Theta_2{}^2 + \ldots \\
&\qquad + B_{12}''\Theta_1\Theta_2 + B_{13}''\Theta_1\Theta_3 + \ldots + C_1''\Theta_1{}^3 + \ldots \\
&\qquad + C_{12}''\Theta_1{}^2\Theta_2 + \ldots + C_{21}''\Theta_1\Theta_2{}^2 + \ldots \\
&+ 2(p_1\Theta_1 + p_2\Theta_2 + \ldots + q_1\Theta_1{}^2 + q_2\Theta_2{}^2 + \ldots + q_{12}\Theta_1\Theta_2 + \ldots \\
&\qquad + r_1\Theta_1{}^3 + \ldots)\{\cos(z-\sigma) + A_1'\Theta_1 + A_2'\Theta_2 + \ldots + B_1'\Theta_1{}^2 \\
&\qquad + B_2'\Theta_2{}^2 + \ldots + B_{12}'\Theta_1\Theta_2 + \ldots\} \\
&+ \{(p_1\Theta_1 + p_2\Theta_2 + \ldots + q_1\Theta_1{}^2 + \ldots)^2 + 1 + \lambda_1\Theta_1 + \lambda_2\Theta_2 + \ldots \\
&\qquad + \mu_1\Theta_1{}^2 + \mu_2\Theta_2{}^2 + \ldots + \mu_{12}\Theta_1\Theta_2 + \ldots + \nu_1\Theta_1{}^3 + \ldots \\
&\qquad + 2\Theta_1\cos 2z + 2\Theta_2\cos 4z + \ldots\}\{\sin(z-\sigma) + A_1\Theta_1 + A_2\Theta_2 + \\
&\qquad \ldots + B_1\Theta_1{}^2 + B_2\Theta_2{}^2 + \ldots + B_{12}\Theta_1\Theta_2 + \ldots\} = 0.
\end{aligned}$$

The terms independent of Θ_1, Θ_2 . . . in this equation are identically zero. If we equate to zero the terms involving nothing but the first power of Θ_1, we get

$$A_1'' + A_1 + 2p_1 \cos(z-\sigma) + 2\sin(z-\sigma)\cos 2z + \lambda_1 \sin(z-\sigma) = 0$$

or

$$A_1'' + A_1 + 2p_1 \cos(z-\sigma) + \lambda_1 \sin(z-\sigma) + \sin(3z-\sigma)$$
$$- \sin(z-\sigma)\cos 2\sigma - \cos(z-\sigma)\sin 2\sigma = 0.$$

It has already been observed that the expression for u is to contain no term in $\cos(z-\sigma)$. We may conveniently take the coefficient of $\sin(z-\sigma)$ in it to be unity, which simply amounts to fixing the arbitrary constant by which the solution is multiplied. Such being the case, A_1, A_2 . . . B_1, B_2 . . . B_{12} . . . and so on, contain no terms in either $\cos(z-\sigma)$ or $\sin(z-\sigma)$.

We thus have, in the case of Θ_1,

$$p_1 = \tfrac{1}{2}\sin 2\sigma.$$
$$\lambda_1 = \cos 2\sigma.$$
$$A_1 = \tfrac{1}{8}\sin(3z-\sigma).$$

Proceeding in this way we may obtain as many coefficients of the terms in the expression for u as we may desire. We shall not obtain an expression for the general term of the series, but from the computing point of view this is a matter of no consequence.

§ 3. *The Solution in detail.*—The terms in the expansions, so far as they have been worked out, are entered below, under the power or product of the Θ's to which they correspond. The small Roman letters, p, q, r . . . refer to the coefficients in the expansion of μ; Greek letters, λ, μ, ν . . . refer to the expansion of Θ_0; and capital Roman letters, A, B, C . . . to the expansion of u.

Terms in Θ_1—

$$p_1 = \tfrac{1}{2}\sin 2\sigma.$$
$$\lambda_1 = \cos 2\sigma.$$
$$A_1 = \tfrac{1}{8}\sin(3z-\sigma).$$

Terms in Θ_r, *where* $r = 2, 3, 4$ . . . —

$$p_r = 0.$$
$$\lambda_r = 0.$$
$$A_r = \frac{1}{4r(r+1)}\sin\{(2r+1)z-\sigma\} - \frac{1}{4r(r-1)}\sin\{(2r-1)z+\sigma\}$$

Terms in Θ_1^2—

$$q_1 = 0.$$

$$\mu_1 = -\frac{1}{4} + \frac{1}{8}\cos 4\sigma.$$

$$B_1 = \frac{1}{192}\sin(5z-\sigma) + \frac{3}{64}\sin 2\sigma\cos(3z-\sigma)$$
$$+ \frac{1}{64}\cos 2\sigma\sin(3z-\sigma).$$

Terms in Θ_r^2, *where* $r=2, 3, 4 \ldots$—

$$q_r = 0.$$

$$\mu_r = -\frac{1}{2(r^2-1)}.$$

$$B_r = \frac{1}{32r^2(r+1)(2r+1)}\sin\{(4r+1)z-\sigma\} - \frac{1}{32r^2(r-1)(2r-1)}\sin\{(4r-1)z+\sigma\}.$$

Terms in $\Theta_1\Theta_2$—

$$q_{12} = \frac{1}{8}\sin 2\sigma.$$

$$\mu_{12} = \frac{1}{4}\cos 2\sigma.$$

$$B_{12} = \frac{1}{288}\sin(7z-\sigma) + \frac{1}{288}\sin 2\sigma\cos(5z-\sigma) - \frac{1}{288}\cos 2\sigma\sin(5z-\sigma) - \frac{1}{32}\sin 4\sigma\cos(3z-\sigma) + \left(\frac{1}{48} - \frac{1}{32}\cos 4\sigma\right)\sin(3z-\sigma).$$

Terms in $\Theta_1\Theta_r$, *where* $r=3, 4, 5 \ldots$—

$$q_{1r} = 0.$$

$$\mu_{1r} = 0.$$

$$\begin{aligned} B_{1r} = {} & \frac{r^2+r+2}{32r(r+1)^2(r+2)}\sin\{(2r+3)z-\sigma\} \\ & + \frac{r^2-r-1}{8r^2(r+1)(r^2-1)}\sin 2\sigma\cos\{(2r+1)z-\sigma\} \\ & - \frac{1}{8r^2(r+1)(r^2-1)}\cos 2\sigma\sin\{(2r+1)z-\sigma\} \\ & - \frac{1}{16r(r-1)^2}\sin 4\sigma\cos\{(2r-1)z-\sigma\} \\ & + \frac{1}{16r(r-1)}\left\{\frac{r+2}{r(r+1)} - \frac{\cos 4\sigma}{r-1}\right\}\sin\{(2r-1)z-\sigma\} \\ & - \frac{r^2-r+2}{32r(r-1)^2(r-2)}\sin\{(2r-3)z+\sigma\}. \end{aligned}$$

Terms in $\Theta_r\Theta_{r+1}$, *where* $r=2, 3, 4 \ldots$—

$$q_{r,\,r+1} = \frac{1}{4r(r+1)}\sin 2\sigma.$$

$$\mu_{r,\,r+1} = \frac{1}{2r(r+1)}\cos 2\sigma.$$

$$B_{r,\,r+1}=\frac{1}{16r(r+1)(r+2)(2r+1)}\sin\{(4r+3)z-\sigma\}$$
$$-\frac{1}{16r(r^2-1)(2r+1)}\sin\{(4r+1)z+\sigma\}$$
$$+\frac{r^2+r+1}{16r(r^2-1)(r+2)}\sin(3z-\sigma).$$

Terms in $\Theta_r\Theta_s$, *where* $r=2, 3, 4 \ldots$ *and* $s>r+1$—

$q_{rs}=0.$
$\mu_{rs}=0.$

$$B_{rs}=\frac{1}{16(s+r)(s+r+1)}\left\{\frac{1}{r(r+1)}+\frac{1}{s(s+1)}\right\}\sin\{(2r+2s+1)z-\sigma\}$$
$$-\frac{1}{16(s+r)(s+r-1)}\left\{\frac{1}{r(r-1)}+\frac{1}{s(s-1)}\right\}\sin\{(2r+2s-1)z+\sigma\}$$
$$+\frac{1}{16(s-r)(s-r+1)}\left\{\frac{1}{r(r-1)}+\frac{1}{s(s+1)}\right\}\sin\{(2s-2r+1)z-\sigma\}$$
$$-\frac{1}{16(s-r)(s-r-1)}\left\{\frac{1}{r(r+1)}+\frac{1}{s(s-1)}\right\}\sin\{(2s-2r-1)z+\sigma\}.$$

Terms in $\Theta_1{}^3$—

$$r_1=-\frac{3}{128}\sin 2\sigma.$$
$$\nu_1=-\frac{1}{64}\cos 2\sigma.$$
$$C_1=\frac{1}{9216}\sin(7z-\sigma)+\frac{7}{2304}\sin 2\sigma\cos(5z-\sigma)$$
$$+\frac{1}{1152}\cos 2\sigma\sin(5z-\sigma)+\frac{3}{512}\sin 4\sigma\cos(3z-\sigma)$$
$$+\left(\frac{5}{512}\cos 4\sigma-\frac{7}{768}\right)\sin(3z-\sigma).$$

Terms in $\Theta_r{}^3$, *where* $r=2, 3, 4 \ldots$—

$r_r=0.$
$\nu_r=0.$

$$C_r=\frac{1}{384r^3(r+1)(2r+1)(3r+1)}\sin\{(6r+1)z-\sigma\}$$
$$-\frac{1}{384r^3(r-1)(2r-1)(3r-1)}\sin\{(6r-1)z+\sigma\}$$
$$-\frac{7r^2+4r+1}{128r^3(r+1)^2(r^2-1)(2r+1)}\sin\{(2r+1)z-\sigma\}$$
$$+\frac{7r^2-4r+1}{128r^3(r-1)^2(r^2-1)(2r-1)}\sin\{(2r-1)z+\sigma\}.$$

Terms in $\Theta_1^2\Theta_2$—

$$r_{12} = \frac{1}{128}\sin 4\sigma.$$

$$\nu_{12} = -\frac{11}{192} + \frac{5}{64}\cos 4\sigma.$$

$$\begin{aligned} C_{12} = {} & \frac{1}{9216}\sin(9z-\sigma) + \frac{43\sin 2\sigma}{27648}\cos(7z-\sigma) + \frac{\cos 2\sigma}{3072}\sin(7z-\sigma) \\ & - \frac{11}{6912}\sin 4\sigma\cos(5z-\sigma) + \left(\frac{5}{13824} - \frac{7}{6912}\cos 4\sigma\right)\sin(5z-\sigma) \\ & + \left(\frac{119}{4608}\sin 2\sigma - \frac{1}{128}\sin 6\sigma\right)\cos(3z-\sigma) \\ & + \left(\frac{119}{9216}\cos 2\sigma - \frac{1}{128}\cos 6\sigma\right)\sin(3z-\sigma). \end{aligned}$$

Terms in $\Theta_1^2\Theta_3$—

$$r_{13} = \frac{5}{384}\sin 2\sigma.$$

$$\nu_{13} = \frac{5}{192}\cos 2\sigma.$$

Terms in $\Theta_1^2\Theta_r$, $r > 3$—

$$r_{1r} = 0.$$

$$\nu_{1r} = 0.$$

Terms in $\Theta_r^2\Theta_{2r-1}$—

$$r_{r,2r-1} = \frac{3r-2}{32r^2(r-1)^2(2r-1)}\sin 2\sigma.$$

$$\nu_{r,2r-1} = \frac{3r-2}{16r^2(r-1)^2(2r-1)}\cos 2\sigma.$$

Terms in $\Theta_r^2\Theta_{2r}$—

$$r_{r,2r} = 0.$$

$$\nu_{r,2r} = -\frac{3}{4(r^2-1)(4r^2-1)}.$$

Terms in $\Theta_r^2\Theta_{2r+1}$—

$$r_{r,2r+1} = \frac{3r+2}{32r^2(r+1)^2(2r+1)}\sin 2\sigma.$$

$$\nu_{r,2r+1} = \frac{3r+2}{16r^2(r+1)^2(2r+1)}\cos 2\sigma.$$

Terms in $\Theta_r^2\Theta_s$, except for $\begin{cases} s=2r-1 \\ s=2r \\ s=2r+1 \\ s=1 \end{cases}$

$$r_{rs}=0.$$
$$\nu_{rs}=0.$$

Terms in $\Theta_1\Theta_2^2$—

$$r_{21}=\frac{7}{288}\sin 2\sigma.$$

$$\nu_{21}=-\frac{1}{144}\cos 2\sigma.$$

$$\begin{aligned} C_{21}=\frac{23}{691200}\sin(11z-\sigma)+\frac{\sin 2\sigma}{14400}\cos(9z-\sigma) \\ -\frac{\cos 2\sigma}{14400}\sin(9z-\sigma)-\frac{1}{1152}\sin 4\sigma\cos(7z-\sigma) \\ +\left(\frac{7}{11520}-\frac{1}{1152}\cos 4\sigma\right)\sin(7z-\sigma) \\ +\frac{19}{9216}\sin 2\sigma\cos(5z-\sigma)+\frac{1}{3072}\cos 2\sigma\sin(5z-\sigma) \\ -\frac{1}{128}\sin 4\sigma\cos(3z-\sigma)+\left(\frac{1}{576}-\frac{1}{128}\cos 4\sigma\right)\sin(3z-\sigma). \end{aligned}$$

Terms in $\Theta_1\Theta_r^2$—

$$r_{r1}=\frac{2r^2-1}{8r^2(r-1)^2(r+1)^2}\sin 2\sigma.$$

$$\nu_{r1}=-\frac{1}{4r^2(r-1)^2(r+1)^2}\cos 2\sigma.$$

Terms in $\Theta_1\Theta_2\Theta_3$—

$$r_{123}=\frac{1}{1152}\sin 4\sigma.$$

$$\nu_{123}=\frac{13}{576}\cos 4\sigma-\frac{13}{192}.$$

Terms in $\Theta_1\Theta_{2\ 4}$—

$$r_{124}=\frac{1}{384}\sin 2\sigma.$$

$$\nu_{124}=\frac{1}{192}\cos 2\sigma.$$

Terms in Θ_1^4—

$$s_1=-\frac{3}{1024}\sin 4\sigma.$$

$$\varpi_1=\frac{1}{48}-\frac{11}{512}\cos 4\sigma.$$

$$D_1 = \frac{1}{737280}\sin(9z-\sigma) + \frac{35\sin 2\sigma}{442368}\cos(7z-\sigma) + \frac{\cos 2\sigma}{49152}\sin(7z-\sigma)$$
$$+\frac{11\sin 4\sigma}{27648}\cos(5z-\sigma) + \left(\frac{41\cos 4\sigma}{55296} - \frac{155}{221184}\right)\sin(5z-\sigma)$$
$$+\left(\frac{9\sin 6\sigma}{4096} - \frac{137\sin 2\sigma}{18432}\right)\cos(3z-\sigma)$$
$$+\left(\frac{7\cos 6\sigma}{4096} - \frac{37\cos 2\sigma}{18432}\right)\sin(3z-\sigma).$$

Terms in Θ_r^4—

$$s_r = 0.$$
$$\varpi_r = \frac{7r^2+5}{32(r^2-1)^3(4r^2-1)}.$$

Terms in $\Theta_1^3\Theta_2$—

$$s_{12} = -\frac{211}{9216}\sin 2\sigma + \frac{5}{1024}\sin 6\sigma$$
$$\varpi_{12} = -\frac{175}{9216}\cos 2\sigma + \frac{7}{512}\cos 6\sigma.$$

Terms in Θ_1^5—

$$t_1 = \frac{137}{36864}\sin 2\sigma - \frac{9}{8192}\sin 6\sigma$$
$$\rho_1 = \frac{1}{288}\cos 2\sigma - \frac{13}{4096}\cos 6\sigma.$$

Terms in Θ_1^6—

$$v_1 = \frac{337}{442368}\sin 4\sigma - \frac{15}{65536}\sin 8\sigma.$$
$$\sigma_1 = -\frac{893}{221184} + \frac{9181}{1769472}\cos 4\sigma - \frac{35}{32768}\cos 8\sigma.$$

§ 4. *Method of Verification.*—A simple check upon the correctness of the above results, and a means of obtaining further terms in the expansions for Θ_0 and μ, may easily be derived from the original differential equation.

As we have seen, Hill's equation has a solution $y = e^{\mu z}\phi(z)$, where $\phi(z)$ is periodic in z and of period 2π.

Now consider the more simple case of Hill's equation,

$$\frac{d^2y}{dz^2} + (\Theta_0 + 2\Theta_1\cos 2z + 2\Theta_2\cos 4z)y = 0,$$

and try to satisfy it by an expression of the form

$$y = e^{\mu z}\left\{\begin{array}{l}\sin(z-\sigma) + b_3\sin(3z-\sigma) + b_5\sin(5z-\sigma) + \ \ldots \\ \qquad\qquad\quad + a_3\cos(3z-\sigma) + a_5\cos(5z-\sigma) + \ \ldots\end{array}\right\}$$

the a's and b's being independent of z.

We thus find that the following relations must hold :—

$$\mu=\tfrac{1}{2}\Theta_1\cos 2\sigma-\tfrac{1}{2}a_3\Theta_1-\tfrac{1}{2}a_5\Theta_2-\tfrac{1}{2}a_3\Theta_2\cos 2\sigma+\tfrac{1}{2}b_3\Theta_2\sin 2\sigma,$$

and

$$\Theta_0=1+\Theta_1\cos 2\sigma-\mu^2-b_3\Theta_1+a_3\Theta_2\sin 2\sigma-b_5\Theta_2+b_3\Theta_2\cos 2\sigma$$

which represent μ and Θ_0 so far as they involve only Θ_1 and Θ_2. Knowing a_3, b_3, etc., in the general expansion, we may use the above relations as a verification of our results, or even in some cases to determine unknown coefficients in either or both of the expansions of μ and Θ_0.

§ 5. *Series for* Θ_0 *and* μ.—As the purpose of this paper is to indicate a possible method for computing μ, rather than to carry out such a computation in its fullest details, an accuracy of ten places of decimals only will be aimed at. Since in Hill's equation Θ_i is a quantity of the $2i$th order, the following expansions for Θ_0 and μ will be sufficient for our purpose, viz.,

$$\begin{aligned}\Theta_0=1&+\Theta_1\cos 2\sigma+\frac{1}{4}\Theta_1\Theta_2\cos 2\sigma+\frac{1}{12}\Theta_2\Theta_3\cos 2\sigma+\left(-\frac{1}{4}+\frac{1}{8}\cos 4\sigma\right)\Theta_1^2\\&-\frac{1}{6}\Theta_2^2-\frac{1}{64}\Theta_1^3\cos 2\sigma+\left(-\frac{11}{192}+\frac{5}{64}\cos 4\sigma\right)\Theta_1^2\Theta_2\\&+\frac{5}{192}\Theta_1^2\Theta_3\cos 2\sigma-\frac{1}{144}\Theta_1\Theta_2^2\cos 2\sigma+\left(\frac{13}{576}\cos 4\sigma-\frac{13}{192}\right)\Theta_1\Theta_2\Theta_3\\&+\left(\frac{1}{48}-\frac{11}{512}\cos 4\sigma\right)\Theta_1^4+\left(-\frac{175}{9612}\cos 2\sigma+\frac{7}{512}\cos 6\sigma\right)\Theta_1^3\Theta_2\\&+\left(\frac{1}{288}\cos 2\sigma-\frac{13}{4096}\cos 6\sigma\right)\Theta_1^5\\&+\left(-\frac{893}{221184}+\frac{9181}{1769472}\cos 4\sigma-\frac{35}{32768}\cos 8\sigma\right)\Theta_1^6.\end{aligned}$$

$$\begin{aligned}\mu=\frac{1}{2}\Theta_1\sin 2\sigma&+\frac{1}{8}\Theta_1\Theta_2\sin 2\sigma+\frac{1}{24}\Theta_2\Theta_3\sin 2\sigma-\frac{3}{128}\Theta_1^3\sin 2\sigma\\&+\frac{1}{128}\Theta_1^2\Theta_2\sin 4\sigma+\frac{5}{384}\Theta_1^2\Theta_3\sin 2\sigma+\frac{7}{288}\Theta_1\Theta_2^2\sin 2\sigma\\&+\frac{1}{1152}\Theta_1\Theta_2\Theta_3\sin 4\sigma-\frac{3}{1024}\Theta_1^4\sin 4\sigma\\&+\left(-\frac{211}{9216}\sin 2\sigma+\frac{5}{1024}\sin 6\sigma\right)\Theta_1^3\Theta_2\\&+\left(\frac{137}{36864}\sin 2\sigma-\frac{9}{8192}\sin 6\sigma\right)\Theta_1^5\\&+\left(\frac{337}{442368}\sin 4\sigma-\frac{15}{65536}\sin 8\sigma\right)\Theta_1^6.\end{aligned}$$

These, of course, reduce, when $\Theta_1 = 8q$ and $\Theta_2 = \Theta_3 = \Theta_4 = \ldots = 0$, to the expressions (6) and (5) previously given for the a and μ of Mathieu's equation respectively.

It is noteworthy that of terms of the first degree, that in Θ_1 alone appears; product terms of the second degree appear only in the form $\Theta_r\Theta_{r+1}$, and no terms whatsoever in Θ_r^2 exist in the expansion for μ. In addition, it may be remarked that there are no terms in either $\Theta_2\Theta_4$ or $\Theta_1^2\Theta_4$, which is important in view of the fact that the order of such a term is that of Θ_1^6, which is retained in the above expansions.

§ 6. *Numerical Computation of* μ.—The actual numerical values of the quantities Θ_0, Θ_1, Θ_2, etc., which enter into Hill's equation are as follows (to eleven places of decimals):—

$$\Theta_0 = 1{\cdot}15884\ 39396\ 0$$
$$\Theta_1 = -{\cdot}05704\ 40187\ 5$$
$$\Theta_2 = {\cdot}00038\ 32380\ 0$$
$$\Theta_3 = -{\cdot}00000\ 91732\ 9.$$

So far as the present work is concerned, Θ_4 and higher Θ's are not required, as they do not give rise to terms whose order is less than the twelfth.

Using then the above values, we find that the expression for Θ_0 becomes

$$1{\cdot}15884\ 39396\ 0 = 1 - {\cdot}00081\ 33804\ 9 - {\cdot}05704\ 65855\ 7 \cos 2\sigma$$
$$+ {\cdot}00040\ 66226\ 27 \cos 4\sigma + {\cdot}00000\ 00009\ 45 \cos 6\sigma$$
$$- {\cdot}00000\ 00000\ 368 \cos 8\sigma.$$

whence

$$\cos 2\sigma = -2{\cdot}79871\ 82492\ 1 + {\cdot}00712\ 79047\ 3 \cos 4\sigma$$
$$+ {\cdot}00000\ 00165\ 65 \cos 6\sigma - {\cdot}00000\ 00064\ 51 \cos 8\sigma.$$

As there are no tables available giving circular functions of a complex argument to a sufficient degree of accuracy, it is found best to treat the above equation as an algebraic equation in $\cos 2\sigma$. If, therefore, we make the substitution $x = \cos 2\sigma$, the equation becomes

$$1{\cdot}00000\ 00496\ 9x = -2{\cdot}80584\ 61603\ 9 + {\cdot}01425\ 58610\ 7x^2$$
$$+ {\cdot}00000\ 00662\ 60x^3 - {\cdot}00000\ 00516\ 08x^4,$$

whence we obtain

$$x = -2{\cdot}80584\ 60209\ 7 + {\cdot}01425\ 58603\ 6x^2$$
$$+ {\cdot}00000\ 00662\ 60x^3 - {\cdot}00000\ 00516\ 08x^4.$$

From this equation x is easily obtained by the usual methods of successive approximation. The value we find is

$$x = \cos 2\sigma = -2{\cdot}70178\ 48031\ 8.$$

We thus see that σ is a complex quantity; its numerical value may be obtained from the usual tables. We find, in fact, that to four places of decimals,

$$\sigma = \frac{\pi}{2} + 0{\cdot}8255i, \quad \text{where } i = \sqrt{-1}.$$

Knowing now the numerical value of $\cos 2\sigma$ we may compute the corresponding values of $\sin 2\sigma$, $\sin 4\sigma$, etc., by means of the usual relations:—

$$\sin 2\sigma = \pm i\sqrt{\cos^2 2\sigma - 1}, \quad \cos^2 2\sigma \text{ being greater than unity.}$$

$$\sin 4\sigma = 2 \sin 2\sigma \cos 2\sigma, \text{ etc.}$$

The ambiguous sign leads to two values of μ numerically equal but of different sign, as may be expected from the nature of the general solution of differential equations of the type considered. We shall see that the positive value of $\sin 2\sigma$ leads to the negative value of μ and *vice versa*.

The values we find are:—

$$\begin{aligned} \sin 2\sigma &= 2{\cdot}50990\ 86064\ 0\,i \\ \sin 4\sigma &= -13{\cdot}56247\,i \\ \sin 6\sigma &= 70{\cdot}7758\,i \\ \sin 8\sigma &= -368{\cdot}8\,i. \end{aligned}$$

The expression which we have obtained for μ in terms of σ becomes, when for the Θ's their corresponding numerical values are inserted, the following:—

$$\begin{aligned} \mu = &-{\cdot}02852\ 03942\ 5 \sin 2\sigma - {\cdot}00000\ 00212\ 52 \sin 4\sigma \\ &+ {\cdot}00000\ 00003\ 162 \sin 6\sigma - {\cdot}00000\ 00000\ 00026 \sin 8\sigma \\ = &-{\cdot}07158\ 35829\ 9\,i \\ &+ {\cdot}00000\ 02882\ 3\,i \\ &+ {\cdot}00000\ 00223\ 8\,i \\ = &-{\cdot}07158\ 32723\ 8\,i. \end{aligned}$$

Hence, to ten places of decimals (no consideration being taken of the sign),

$$\mu = {\cdot}07158\ 32724\,i.$$

In Hill's celebrated paper, already referred to, the corresponding constant c is given to fifteen places of decimals. To ten places of decimals its value is

$$c = 1{\cdot}07158\ 32774.$$

The discrepancy in the ninth place of decimals is probably due to our having neglected high order terms whose influence on the final result is appreciable. The connection between Hill's value of c and the value we have obtained for μ is easily seen. In Hill's case c is real, but in the present working μ was found to be a pure imaginary, which arises in the fact that Hill so changed his independent variable that the imaginary unit i was absorbed into it. It may also be noticed that the value we have obtained for μ is $\cdot 07158 \ldots i$ and not $1 \cdot 07158 \ldots i$, as might be expected by comparison with Hill's value of c. But if we write the factor $e^{\mu z}$ in the form $e^{\lambda i z}$ we see that its secular nature is unchanged by the addition of unity, thus $e^{(\lambda+1)iz}$, and that in this respect the non-appearance of the unity is a matter of no consequence. The general solution is now $e^{(\lambda+1)iz}e^{-iz}u(z)$, of which the factor $e^{-iz}u(z)$ has the period π, as is demanded by the general theory, whereas $u(z)$ itself has only the period 2π. This indicates that the other factor, which is in general secular as regards the period π, should be taken in the form $e^{(\lambda+1)iz}$ and not as it originally arose, and that therefore an initial unity should be supplied to our value of μ. The fact that μ is imaginary shows that the solution is stable, so that its corrected value gives us a measure of the period of small oscillations about the truly periodic orbit.

The method of dealing with Hill's equation indicated above seems to be quite satisfactory, and can be used to any desired degree of precision once the more advanced terms in the series for μ and Θ_0 have been obtained to a sufficiently high order. In the present paper they have only been taken to the tenth order, but to obtain higher terms is, though laborious, a matter of no great difficulty. The expansions for Θ_0 and μ which we have obtained are suitable only for the particular cases, such as that dealt with by Hill, in which Θ_0 is nearly unity. It is, however, equally possible to obtain similar expansions which are suited to the cases in which the quantity Θ_0 approaches the values 4, 9, 16, etc.

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NOTES ON THE GENERAL SOLUTION OF HILL'S EQUATION.

BY

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Notes on the General Solution of Hill's Equation. By E. Lindsay Ince, M.A., B.Sc.(Edin.), Carnegie Fellow.

(Communicated by Prof. H. F. Baker, F.R.S.)

Contents of Paper.—In § 1 the solution of Hill's equation appropriate to the case of Θ_0 nearly equal to unity is taken, and the variation of Θ_0 and μ as the auxiliary parameter σ varies is discussed numerically, special regard being taken as to the stability or instability of the solution. In § 2 the solution applicable to the case of Θ_0 nearly equal to 4 is obtained and discussed in a similar manner, and reference is incidentally made to the cases of Θ_0 nearly equal to 9, 16 . . . . Finally, in § 3, the solution which diverges in the vicinity of $\Theta_0 = 1, 4, 9, \ldots$ is taken, and its relation to the other solutions is demonstrated.

§ 1. In a paper "On a General Solution of Hill's Equation," which I communicated last year to this Society,* I showed that Hill's equation

$$\frac{d^2y}{dz^2} + (\Theta_0 + 2\Theta_1 \cos 2z + 2\Theta_2 \cos 4z + \ldots)y = 0 \quad . \quad (1)$$

was satisfied by a function $e^{\mu z}u(z)$, where $u(z)$ was a purely periodic function of z of the form

$$\sin(z-\sigma) + a_3 \cos(3z-\sigma) + b_3 \sin(3z-\sigma) + a_5 \cos(5z-\sigma) + b_5 \sin(5z-\sigma) + \ldots,$$

and μ appeared as a function of σ and of the parameters Θ_1, Θ_2, etc. (but not Θ_0), of the differential equation. The new parameter σ is not arbitrary, for the quantity Θ_0 has to satisfy a certain relation involving σ, so that if Θ_0 is fixed, σ is determined also. By this means it was possible to evaluate μ by first determining σ from the known constants of the equation, and then, knowing σ, to compute the numerical value of μ.

This method of attacking the problem appears, from a theoretical point of view, so artificial that it has been suggested to me that I should investigate its real significance and so explain its apparent artificiality. These notes are therefore devoted to an inquiry into the properties of the new parameter σ, and incidentally to the question of the stability or instability of the solution corresponding to any given value of the parameter Θ_0, when the constants Θ_1, Θ_2, Θ_3, etc., have definite values assigned to them.†

For the purposes of this paper, Hill's values of the constants Θ_1, Θ_2, Θ_3, etc., will be adopted, but Θ_0 will, of course, be left quite arbitrary. As the series for Θ_0 and μ are unsuited for accurate numerical work when large imaginary or complex values of σ are inserted, an accuracy of only four or five places of

* *M.N.*, lxxv. p. 436, March 1915.

† For a somewhat similar investigation applied to Mathieu's equation, *vide* A. W. Young, *Proc. Edin. Math. Soc.*, xxxii. (1913–14), p. 81.

decimals will be aimed at. This degree of accuracy will, however, be adequate to illustrate the phenomena with which this paper professes to deal. The series for Θ_0 and μ obtained in my previous paper are as follows:—

$$\Theta_0 = 1 + \Theta_1 \cos 2\sigma + (-\tfrac{1}{4} + \tfrac{1}{8}\cos 4\sigma)\Theta_1^2 + \tfrac{1}{4}\Theta_1\Theta_2 \cos 2\sigma - \tfrac{1}{64}\Theta_1^3 \cos 2\sigma + (-\tfrac{11}{192} + \tfrac{5}{64}\cos 4\sigma)\Theta_1^2\Theta_2 + (\tfrac{1}{48} - \tfrac{11}{512}\cos 4\sigma)\Theta_1^4 + \ldots \quad (2)$$

$$\mu = \tfrac{1}{2}\Theta_1 \sin 2\sigma + \tfrac{1}{8}\Theta_1\Theta_2 \sin 2\sigma - \tfrac{3}{128}\Theta_1^3 \sin 2\sigma + \tfrac{1}{128}\Theta_1^2\Theta_2 \sin 4\sigma - \tfrac{3}{1024}\Theta_1^4 \sin 4\sigma + \ldots \quad (3)$$

which become, when Hill's values of the constants are inserted,

$$\Theta_0 = \cdot99918\ 7 - \cdot05704\ 66 \cos 2\sigma + \cdot00040\ 662 \cos 4\sigma + \ldots \quad (4)$$

$$\mu = \cdot02852\ 04 \sin 2\sigma + \cdot00000\ 002 \sin 4\sigma - \ldots\text{*} \quad . \quad . \quad (5)$$

In the previous paper Hill's value of the quantity Θ_0 was assumed, and the corresponding value of σ deduced, but in the present notes Θ_0 will be regarded as a dependent variable whose behaviour, as σ varies, is to be investigated. For the present we see that if Θ_0 is real, then $\cos 2\sigma$ is real, and consequently σ is either (i.) purely real, or (ii.) purely imaginary, or (iii.) (as in Hill's case) a complex quantity of the form $\frac{\pi}{2} +$ a pure imaginary. In the first case μ also is purely real, but in the two other cases μ is a pure imaginary.

We will first consider the case when σ is purely real, in order that we may find out the limits within which Θ_0 lies, and also the extent of the variation of μ. For this purpose we may construct the table.

Table Ia.

σ.	Θ_0.	μ.
0	·94255	0
$\frac{\pi}{12}$	·94999	·01426
$\frac{\pi}{6}$	·97046	·02470
$\frac{\pi}{4}$	·99878	·02852
$\frac{\pi}{3}$	1·02791	·02470
$\frac{5\pi}{12}$	1·04879	·01426
$\frac{\pi}{2}$	1·05664	0
$\frac{7\pi}{12}$	1·04879	·01426
...	...	...

* For convenience the sign of μ has been changed, and in the following pages μ will always be regarded as positive.

We thus see that for values of Θ_0 lying between $\cdot 94255 \ldots$ and $1\cdot 05664 \ldots$, μ is real and the corresponding solution unstable, its maximum instability being that introduced by the factor $e^{\cdot 02852 \ldots z}$ which arises when $\Theta_0 = \cdot 99878 \ldots$.

When Θ_0 assumes either of the particular values $\cdot 94255 \ldots$ or $1\cdot 05664 \ldots$ μ becomes identically zero, and the corresponding solutions of Hill's equation are purely periodic with the period 2π. In general these purely periodic solutions are given by

$$\Theta_0 = 1 + \Theta_1 - \tfrac{1}{8}\Theta_1^2 + \tfrac{1}{4}\Theta_1\Theta_2 - \tfrac{1}{64}\Theta_1^3 + \tfrac{1}{48}\Theta_1^2\Theta_2 - \tfrac{1}{1536}\Theta_1^4 + \ldots \quad (6)$$

corresponding to $\sigma = 0$, and

$$\Theta_0 = 1 - \Theta_1 - \tfrac{1}{8}\Theta_1^2 - \tfrac{1}{4}\Theta_1\Theta_2 + \tfrac{1}{64}\Theta_1^3 + \tfrac{1}{48}\Theta_1^2\Theta_2 - \tfrac{1}{1536}\Theta_1^4 + \ldots \quad (7)$$

corresponding to $\sigma = \frac{\pi}{2}$.*

These two series define the unstable region which surrounds the value $\Theta_0 = 1$, and we see that the smaller the constants Θ_1, $\Theta_2 \ldots$, the more contracted is the region, and ultimately, when these constants vanish, no instability remains in the neighbourhood of $\Theta_0 = 1$.

The point of maximum instability in this region is given by $\sigma = \frac{\pi}{4}$, and so corresponds to

$$\Theta_0 = 1 - \tfrac{3}{8}\Theta_1^2 - \tfrac{13}{96}\Theta_1^2\Theta_2 + \tfrac{65}{1536}\Theta_1^4 + \ldots \quad (8)$$

the amount of this maximum instability being given by the exponent factor

$$\mu = \tfrac{1}{2}\Theta_1 + \tfrac{1}{8}\Theta_1\Theta_2 - \tfrac{3}{128}\Theta_1^3 + \ldots \quad (9)$$

Turning now to the imaginary and complex values of σ, we may draw up a similar table, exhibiting side by side the values of Θ_0 and μ, corresponding to values of σ of the form $i\rho$ and $\frac{\pi}{2} + i\rho$ respectively, where ρ is a purely real quantity and $i \equiv \sqrt{-1}$.

As has already been remarked, in these cases Θ_0 is purely real and μ is purely imaginary. We will first of all consider the last three columns of the following table, and observe that between the values $\frac{\pi}{2} + 2\cdot 1i$ and $\frac{\pi}{2} + 2\cdot 2i$ of σ, μ takes on the value i and the solution again becomes purely periodic, with a period π. Corresponding to this the quantity Θ_0 assumes a value very nearly equal to 4. But it will be shown in the next section that near $\Theta_0 = 4$ there are two such purely periodic solutions of the equation, viz. those corresponding to $\Theta_0 = 4\cdot 00011 \ldots$ and $\Theta_0 = 4\cdot 00091 \ldots$ respectively, and that to all values of Θ_0 intermediate to these

* These correspond to the cases in which Mathieu's equation gives rise to the elliptic cylinder functions $se_1(z)$ and $ce_1(z)$ respectively.

TABLE IB.

σ.	Θ_0.	μ.	σ.	Θ_0.	μ.
0	·9426	0	$\frac{\pi}{2}$	1·0566	0
·5i	·9106	·0334i	$\frac{\pi}{2}+$ ·5i	1·0908	·0334i
1·0i	·7957	·1034i	$\frac{\pi}{2}+$1·0i	1·2249	·1034i
1·5i	·5097	·2857i	$\frac{\pi}{2}+$1·5i	1·6527	·2857i
1·75i	·2767	·4718i	$\frac{\pi}{2}+$1·75i	2·1675	·4718i
2·0i	·0474	·7783i	$\frac{\pi}{2}+$2·0i	3·1631	·7783i
2·1i	·0006	·9508i	$\frac{\pi}{2}+$2·1i	3·8054	·9506i
2·2i	·0239	1·1615i	$\frac{\pi}{2}+$2·2i	4·6705	1·1610i
2·3i	·1721	1·4191i	$\frac{\pi}{2}+$2·3i	5·8470	1·4181i
2·4i	·5318	1·7334i	$\frac{\pi}{2}+$2·4i	7·4623	1·7313i
2·5i	1·2441	2·1189i	$\frac{\pi}{2}+$2·5i	9·7102	2·1136i

correspond real values of μ, and, in consequence, unstable solutions. But no real values of μ could possibly appear in our table (IB.), since every term in the series for μ is now a pure imaginary. The only explanation of this peculiarity which readily suggests itself is that the expansion of Θ_0 as a series in σ is discontinuous near the value 4; that is to say, as σ varies from $\frac{\pi}{2}$ to $\frac{\pi}{2}+2·125\ldots i$,* Θ_0 varies continuously from 1·0566 to 4·0001 . . . , and then, as σ is further increased, suddenly jumps to 4·0009 . . . and then continues for a time to increase steadily. Now as σ continues further to increase we see from the table that μ soon approaches a value $2i$, which again leads to a purely periodic solution corresponding to a value of Θ_0 very nearly equal to 9. But it will be shown in the next section that near $\Theta_0=9$ there are two purely periodic solutions, corresponding respectively to $\Theta_0=9·00019\ 689$ . . . and $\Theta_0=9·00021\ 015$ . . . . As before, the intermediate values of Θ_0 give rise to an unstable solution, and thus the series for Θ_0 in terms of σ would be expected to be discontinuous near $\Theta_0=9$. Likewise two values of Θ_0 near 16 can be found which give rise to purely periodic solutions, and here again the intermediate values give rise to instability. Thus in this region also the series for Θ_0 would naturally be expected to exhibit a discontinuity. Unfortunately, however, the convergence of that series

* This value is obtained by inverse interpolation in the table, and corresponds to $\mu=i$.

rapidly diminishes together with its applicability to numerical work, and for this reason the table has been stopped before any such discontinuity could be reached. Similarly, regions of instability can be shown to exist near $\Theta_0 = 25, 36, 49 \ldots$, and there the series for Θ_0, if convergent, would most probably become discontinuous.

Now let us examine the values of Θ_0 and μ, which correspond to a purely imaginary σ (Table IB.). As σ increases numerically so also does μ, but Θ_0 decreases at first until when $\sigma = 2{\cdot}12 \ldots i$, $\mu = i$, and Θ_0 has a small negative value. Thus we have a purely periodic solution.* But it will be shown in § 3 that no other periodic solutions exist for values of Θ_0 less than the present one ($\Theta_0 = -{\cdot}00163 \ldots$), and that for all such values of Θ_0 the solution is unstable. Thus the series in σ does not become discontinuous at this point, but has a minimum after which Θ_0 begins steadily to increase until between $\sigma = 2{\cdot}4i$ and $\sigma = 2{\cdot}5i$, μ attains the value i and Θ_0 (which is then nearly unity) has a discontinuity. Thereafter Θ_0 continues steadily to increase, and the range of values of Θ_0 and μ contained in the last three columns of the table are reproduced, so that the first three columns give us no further information than that already obtained. Thus we see that the series for Θ_0 in terms of σ would give the complete set of possible values of Θ_0 that give rise to stability and omit all those that produce instability. There seems at present to be no means of definitely proving the existence of discontinuities in the series considered, but there seems to be little doubt that they actually do exist.

§ 2. At the conclusion of my paper "On a General Solution of Hill's Equation" I remarked that though the solution I there gave was only of value in the case when Θ_0 was nearly unity, it was equally possible to obtain solutions appropriate to the cases when Θ_0 was approximately 4, 9, etc. I now propose to deal with such cases, and to link them up with the former case.

As before, we assume a solution of the form

$$y = e^{\mu z} u(z)$$

where

$$\mu = p_1(\sigma)\Theta_1 + p_2(\sigma)\Theta_2 + \ldots + q_1(\sigma)\Theta_1^2 + q_2(\sigma)\Theta_2^2 + \ldots$$
$$+ q_{12}(\sigma)\Theta_1\Theta_2 + \ldots + r_1(\sigma)\Theta_1^3 + \ldots,$$

but in this case

$$u(z) = \sin(2z - \sigma) + A_1(z, \sigma)\Theta_1 + A_2(z, \sigma)\Theta_2 + \ldots + B_1(z, \sigma)\Theta_1^2$$
$$+ B_2(z, \sigma)\Theta_2^2 + B_{12}(z, \sigma)\Theta_1\Theta_2 + \ldots C_1(z, \sigma)\Theta_1^3 + \ldots,$$

the functions A, B, C, . . . being conditioned to contain no terms in either $\sin(2z - \sigma)$ or $\cos(2z - \sigma)$.

The relation between Θ_0 and σ is in this case

$$\Theta_0 = 4 + \lambda_1(\sigma)\Theta_1 + \lambda_2(\sigma)\Theta_2 + \ldots + \mu_1(\sigma)\Theta_1^2 + \mu_2(\sigma)\Theta_2^2 + \ldots$$
$$+ \mu_{12}(\sigma)\Theta_1\Theta_2 + \ldots + \nu_1(\sigma)\Theta_1^3 + \ldots,$$

* In fact, that one which corresponds to the Mathieu function $ce_0(z)$.

giving us a solution applicable in particular to the case in which Θ_0 does not differ greatly from the value 4.

Proceeding exactly as before, we find that the terms in the general solution are as follows. (Only those terms whose order is the eighth or less than the eighth are investigated, as they will be sufficient for all present purposes.)

Terms in Θ_1—

$$p_1 = 0.$$
$$\lambda_1 = 0.$$
$$A_1 = \tfrac{1}{4}\sin\sigma + \tfrac{1}{12}\sin(4z-\sigma).$$

Terms in Θ_2—

$$p_2 = \tfrac{1}{4}\sin 2\sigma.$$
$$\lambda_2 = \cos 2\sigma.$$
$$A_2 = \tfrac{1}{32}\sin(6z-\sigma).$$

Terms in Θ_r, *where* $r > 2$—

$$p_r = 0.$$
$$\lambda_r = 0.$$
$$A_r = \frac{1}{4r(r+2)}\sin\{(2r+2)z-\sigma\} - \frac{1}{4r(r-2)}\sin\{(2r-2)z+\sigma\}.$$

Terms in Θ_1^2—

$$q_1 = -\tfrac{1}{16}\sin 2\sigma.$$
$$\mu_1 = \tfrac{1}{6} - \tfrac{1}{4}\cos 2\sigma.$$
$$B_1 = \tfrac{1}{384}\sin(6z-\sigma).$$

Terms in Θ_2^2—

$$q_2 = 0.$$
$$\mu_2 = -\tfrac{3}{64} + \tfrac{1}{32}\cos 4\sigma.$$
$$B_2 = \tfrac{1}{6144}\sin(10z-\sigma) + \tfrac{3}{1024}\sin 2\sigma\cos(6z-\sigma) + \tfrac{1}{1024}\cos 2\sigma\sin(6z-\sigma).$$

Terms in Θ_r^2, *where* $r > 2$—

$$q_r = 0.$$
$$\mu_r = -\frac{1}{2(r^2-4)}.$$

Terms in $\Theta_1\Theta_2$—

$$q_{12} = 0.$$
$$\mu_{12} = 0.$$
$$B_{12} = \tfrac{11}{5760}\sin(8z-\sigma) + \tfrac{5}{144}\sin 2\sigma\cos(4z-\sigma) + \left(\tfrac{1}{36}\cos 2\sigma - \tfrac{7}{384}\right)\sin(4z-\sigma) - \tfrac{1}{32}\sin 3\sigma + \tfrac{5}{96}\sin\sigma.$$

Terms in $\Theta_1\Theta_3$—

$$q_{13} = \tfrac{1}{24} \sin 2\sigma.$$
$$\mu_{13} = \tfrac{1}{6} \cos 2\sigma.$$

Terms in $\Theta_1\Theta_r$, *where* $r > 3$—

$$q_{1r} = 0.$$
$$\mu_{1r} = 0.$$

Terms in Θ_1^3—

$$r_1 = 0.$$
$$\nu_1 = 0.$$
$$\begin{aligned} C_1 = {} & \tfrac{1}{23040} \sin (8z - \sigma) - \tfrac{1}{288} \sin 2\sigma \cos (4z - \sigma) \\ & + (\tfrac{19}{13824} - \tfrac{1}{576} \cos 2\sigma) \sin (4z - \sigma) + \tfrac{1}{128} \sin 3\sigma - \tfrac{7}{384} \sin \sigma. \end{aligned}$$

Terms in $\Theta_1^2\Theta_2$—

$$r_{12} = -\tfrac{1}{72} \sin 2\sigma + \tfrac{1}{128} \sin 4\sigma.$$
$$\nu_{12} = \tfrac{1}{12} - \tfrac{1}{9} \cos 2\sigma + \tfrac{1}{64} \cos 4\sigma.$$

Terms in Θ_1^4—

$$s_1 = \tfrac{1}{288} \sin 2\sigma - \tfrac{1}{512} \sin 4\sigma.$$
$$\varpi_1 = -\tfrac{149}{6912} + \tfrac{1}{36} \cos 2\sigma - \tfrac{3}{512} \cos 4\sigma.$$
$$\begin{aligned} D_1 = {} & \tfrac{1}{2216064} \sin (10z - \sigma) - \tfrac{25}{147456} \sin 2\sigma \cos (6z - \sigma) \\ & + (\tfrac{1}{17280} - \tfrac{11}{147456} \cos 2\sigma) \sin (6z - \sigma). \end{aligned}$$

Terms in Θ_1^5—

$$t_1 = 0.$$
$$\rho_1 = 0.$$

We thus obtain, as the required expansions of Θ_0 and μ, the following:—

$$\begin{aligned} \Theta_0 = 4 + \Theta_2 \cos 2\sigma & + (\tfrac{1}{8} - \tfrac{1}{4} \cos 2\sigma)\Theta_1^2 + (-\tfrac{3}{64} + \tfrac{1}{32} \cos 4\sigma)\Theta_2^2 \\ & + (\tfrac{1}{12} - \tfrac{1}{9} \cos 2\sigma + \tfrac{1}{64} \cos 4\sigma)\Theta_1^2\Theta_2 \\ & + (-\tfrac{149}{6912} + \tfrac{1}{36} \cos 2\sigma - \tfrac{3}{512} \cos 4\sigma)\Theta_1^4 + \ldots \end{aligned} \tag{10}$$

$$\begin{aligned} \mu = \tfrac{1}{4}\Theta_2 \sin 2\sigma - \tfrac{1}{16}\Theta_1^2 \sin 2\sigma & + (-\tfrac{1}{72} \sin 2\sigma + \tfrac{1}{128} \sin 4\sigma)\Theta_1^2\Theta_2 \\ & + (\tfrac{1}{288} \sin 2\sigma - \tfrac{1}{512} \sin 4\sigma)\Theta_1^4 + \ldots \end{aligned} \tag{11}$$

As before, when σ is real μ also is real, and consequently the solution is unstable, and when σ is imaginary μ is imaginary, and the solution is stable. When σ assumes either of the values 0 or $\frac{\pi}{2}$, μ vanishes, and the solution becomes purely periodic with a period π. The corresponding values of Θ_0 in these cases are respectively given by the relations

$$\Theta_0 = 4 + \Theta_2 - \tfrac{1}{12}\Theta_1^2 - \tfrac{1}{64}\Theta_2^2 - \tfrac{7}{576}\Theta_1^2\Theta_2 + \tfrac{5}{13824}\Theta_1^4 + \ldots \quad (12)$$

and

$$\Theta_0 = 4 - \Theta_2 + \tfrac{5}{12}\Theta_1^2 - \tfrac{1}{64}\Theta_2^2 + \tfrac{121}{576}\Theta_1^2\Theta_2 - \tfrac{763}{13824}\Theta_1^4 + \ldots \quad (13)^*$$

Adopting now Hill's values of the constants $\Theta_1, \Theta_2 \ldots$, the expansions of Θ_0 and μ become

$$\Theta_0 = 4{\cdot}00054 - {\cdot}00043 \cos 2\sigma + \ldots \quad (14)$$

and

$$\mu = {\cdot}00010\ 7 \sin 2\sigma + \ldots \quad (15)$$

Thus when σ is purely real, Θ_0 and μ assume the values given in the following table.

TABLE IIA.

σ.	Θ_0.	μ.
0	4·00011	0
$\frac{\pi}{12}$	4·00017	·00005
$\frac{\pi}{6}$	4·00033	·00009
$\frac{\pi}{4}$	4·00054	·00011
$\frac{\pi}{3}$	4·00076	·00009
$\frac{5\pi}{12}$	4·00091	·00005
$\frac{\pi}{2}$	4·00097	0
$\frac{7\pi}{12}$	4·00091	·00005
...	...	...

All values of Θ_0 lying within the interval between $\Theta_0 = 4{\cdot}00011$ and $\Theta_0 = 4{\cdot}00097$ give rise to real values of μ, and consequently to an unstable solution, the maximum instability being that represented by the factor $e^{{\cdot}00010 \ldots z}$. It will at once be noticed that the interval within which instability would occur is considerably smaller than in the case in which Θ_0 is nearly unity, and further, that the secular factor is enormously less in this case than in the last. If we turn for a moment to the case in which Θ_0 is nearly 9, we find that we may obtain expansions of the form

$$\Theta_0 = 9 + \Theta_3 \cos 2\sigma + \tfrac{1}{16}\Theta_1^2 + \tfrac{1}{10}\Theta_2^2 - \tfrac{1}{4}\Theta_1\Theta_2 \cos 2\sigma + \tfrac{1}{64}\Theta_1^3 \cos 2\sigma + \tfrac{13}{20480}\Theta_1^4 + \ldots \quad (16)$$

* In the case of Mathieu's equation these would correspond to the functions $se_2(z)$ and $ce_2(z)$ respectively.

and $\mu = \frac{1}{8}\Theta_3 \sin 2\sigma - \frac{1}{24}\Theta_1\Theta_2 \sin 2\sigma + \frac{1}{384}\Theta_1^3 \sin 2\sigma + \ldots$ (17)

which, with Hill's numbers, become

$$\Theta_0 = 9\cdot00020\ 352 - \cdot00000\ 663 \cos 2\sigma + \ldots \quad (18)$$

and

$$\mu = \cdot00000\ 110 \sin 2\sigma + \ldots \quad (19)$$

Thus in this case the interval during which μ is real is extremely short, and the maximum real value of μ itself is only ·000001, and hence the region of instability near $\Theta_0 = 9$ is very limited, and the amount of instability is at most relatively minute. This applies generally to the higher expansions of Θ_0 and μ, that is to say, to those applicable in the neighbourhood of $\Theta_0 = 16, 25, 36 \ldots$. So that, as Θ_0 increases in magnitude, the possibility of its being associated with a markedly unstable solution becomes less and less.

Returning now to equations (12) and (13), and tabulating values of Θ_0 and μ corresponding to the two possible sets of non-real values of σ, we obtain the following:—

TABLE IIB.

σ.	Θ_0.	μ.	σ.	Θ_0.	μ.
0	4·00011	0	$\frac{\pi}{2}$	4·00097	0
$1\cdot0i$	3·99849	$\cdot00039i$	$\frac{\pi}{2}+1\cdot0i$	4·00173	$\cdot00039i$
$2\cdot0i$	5·9888	$\cdot0029i$	$\frac{\pi}{2}+2\cdot0i$	4·0123	$\cdot0029i$
$3\cdot0i$	3·914	$\cdot022i$	$\frac{\pi}{2}+3\cdot0i$	4·087	$\cdot022i$
$4\cdot0i$	3·35	$\cdot16i$	$\frac{\pi}{2}+4\cdot0i$	4·65	$\cdot16i$
...	...	...	...	...	...

The tendency in this case seems to be as in the previous one, but the series are not sufficiently convergent to show the phenomena in their full detail.

§ 3. The method of dealing with Hill's equation which has concerned us so far is suited to the cases in which Θ_0 is nearly 1, 4, 9, . . . but not to the case of Θ_0 approximately zero. In order to complete the discussion we will obtain a solution appropriate to this case. To do so we assume a solution

$$y = e^{cz}u$$

where c is a constant and

$$u = 1 + A_1(z)\Theta_1 + A_2(z)\Theta_2 + \ldots + B_1(z)\Theta_1^2 + B_2(z)\Theta_2^2 + \ldots$$
$$+ B_{12}(z)\Theta_1\Theta_2 + \ldots + C_1(z)\Theta_1^3 + \ldots$$

Θ_0 is now to be determined by the relation

$$\Theta_0 = a_0 + a_1\Theta_1 + a_2\Theta_2 + \ldots + b_1\Theta_1^2 + b_2\Theta_2^2 + \ldots + b_{12}\Theta_1\Theta_2 + \ldots + c_1\Theta_1^3 + \ldots$$

On substituting these expressions in Hill's equation and equating to zero the terms of the result which contain no Θ, we have

$$a_0 = -c^2.$$

We now proceed as usual, and obtain the following terms in the expansions of Θ_0 and u:—

Terms in Θ_n, where $n = 1, 2, 3 \ldots$—

$$a_n = 0.$$

$$A_n = \frac{1}{2(c^2+n^2)}\cos 2nz - \frac{c}{2n(c^2+n^2)}\sin 2nz.$$

Terms in Θ_n^2, where $n = 1, 2, 3$—

$$b_n = -\frac{1}{2(c^2+n^2)}.$$

$$B_n = \frac{2n^2-c^2}{16n^2(c^2+n^2)(c^2+4n^2)}\cos 4nz - \frac{3c}{16n(c^2+n^2)(c^2+4n^2)}\sin 4nz.$$

Terms in $\Theta_r\Theta_s$, where $r, s = 1, 2, 3 \ldots$ and $r \neq s$—

$$b_{rs} = 0.$$

Terms in Θ_n^3, where $n = 1, 2, 3 \ldots$—

$$c_n = 0.$$

Term in Θ_1^4—

$$d_1 = \frac{1-c^2}{16(1+c^2)^3} - \frac{1-2c^2}{32(1+c^2)^2(4+c^2)}.$$

Thus the following expression for Θ_0 in terms of Θ_1, Θ_2, and c is obtained, and will be sufficient for our purpose, viz.

$$\Theta_0 = -c^2 - \frac{1}{2(1+c^2)}\Theta_1^2 - \frac{1}{2(4+c^2)}\Theta_2^2 + \left[\frac{1-c^2}{16(1+c^2)^3} - \frac{(1-2c^2)}{32(1+c^2)^2(4+c^2)}\right]\Theta_1^4 + \ldots \quad (20)$$

which, with Hill's numbers, becomes

$$\Theta_0 = -c^2 - \frac{\cdot00162\ 70100}{1+c^2} - \frac{\cdot00000\ 00734}{4+c^2} + \frac{7-5c^2}{(1+c^2)^3(4+c^2)}(\cdot00000\ 03309) + \ldots \quad (21)$$

This expression is quite unsuited for dealing with the purely periodic solutions which occur at $c=i$, $2i$, $3i$ . . ., as these are precisely the values of c which render it divergent. It is, however, useful for dealing with those solutions which are intermediate between the periodic solutions, and also enables us to dispose of the whole range of negative values of Θ_0.

When c is zero we have a periodic solution given by

$$\Theta_0 = -\tfrac{1}{2}\Theta_1{}^2 - \tfrac{1}{8}\Theta_2{}^2 - \ldots + \tfrac{7}{128}\Theta_1{}^4 + \ldots \qquad (22)^*$$

In Hill's case the value of this expression is $-{\cdot}00162$. . . . This is the periodic solution of lowest order, for, corresponding to all real values of c, equation (21) gives us negative values of Θ_0 extending uninterruptedly down to negative infinity. We thus conclude that to all values of Θ_0 less than $-{\cdot}00162$ . . . correspond unstable solutions of Hill's equation, and the larger Θ_0 is numerically, the greater is the amount of instability.

We will now consider imaginary values of c, and to treat of them we construct the following table :—

Table III.

c.	Θ_0.	c.	Θ_0.
0	$-{\cdot}00163$		
$\cdot 1i$	$+{\cdot}00836$	$1{\cdot}1i$	1·21758
$\cdot 2i$	·03831	$1{\cdot}2i$	1·44364
$\cdot 3i$	·08821	$1{\cdot}3i$	1·69235
$\cdot 4i$	·15806	$1{\cdot}4i$	1·96169
$\cdot 5i$	·24783	$1{\cdot}5i$	2·25130
$\cdot 6i$	·35746	$1{\cdot}6i$	2·56104
$\cdot 7i$	·48682	$1{\cdot}7i$	2·89000
$\cdot 8i$	·63550	$1{\cdot}8i$	3·24072
$\cdot 9i$	·80160	$1{\cdot}9i$	3·61062
...	...	...	...

This table may be continued indefinitely, giving us the values of Θ_0 which correspond to all values of c except those near the critical points $c=i$, $2i$, $3i$ . . . . In fact, the formulæ (20) and (21), though useless near these critical values, are much more useful than those of §§ 1, 2, for values of c remote from the critical values. On the other hand, the formulæ of §§ 1, 2 are particularly adapted to investigations near the appropriate critical values of c (or μ), and (unless a large number of terms in the series are used) relatively useless away from the particular critical values with which they are respectively associated. Too much weight must therefore not be given to the figures in Table Ib., which was computed with series of only a limited number of terms, and which was constructed merely to show the tendency of the series in σ when σ was large and imaginary.

* This periodic solution corresponds to the Mathieu function $ce_0(z)$.

Reverting the series (20), we obtain (*cf.* Tisserand, *Méc. Cél.*, iii. p. 10)

$$c^2 = -\Theta_0 - \frac{\Theta_1{}^2}{2(1-\Theta_0)} - \frac{\Theta_2{}^2}{2(4-\Theta_0)}$$
$$-\left[\frac{3-\Theta_0}{16(1-\Theta_0)^3} + \frac{1+2\Theta_0}{32(1-\Theta_0)^2(4-\Theta_0)}\right]\Theta_1{}^4 - \ldots \quad (23)$$

which, when the solution is stable, may be used for determining c when the constants Θ_0, Θ_1, Θ_2 . . . are given. Thus, using Hill's numbers, we have

$$c^2 = -1{\cdot}158844 + {\cdot}010246 + {\cdot}000289.$$
$$= -1{\cdot}148309.$$

The correct value of c^2 to six places of decimals is $-1{\cdot}148281$. This formula, in fact, does not give so good an approximation as do the formulæ involving σ when terms of the same order in Θ_1, Θ_2 . . . are retained in the former as in the latter, so that for problems such as Hill's, in which Θ_0 is near a critical value, the method of introducing the parameter σ is to be preferred.

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FURTHER NOTES ON THE GENERAL SOLUTION OF HILL'S EQUATION.

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Further Notes on the General Solution of Hill's Equation.
By E. Lindsay Ince, M.A., B.Sc.

§ 1. *Introduction.*—This paper is a sequel to the two papers * I have previously published on the subject of Hill's equation. In the first paper I obtained an expression for the exponent-factor μ in terms of the constants $\Theta_1, \Theta_2, \Theta_3, \ldots$ of the equation and a new parameter σ which itself depends on the constant Θ_0 as well as on $\Theta_1, \Theta_2, \Theta_3, \ldots$ In the second paper I considered the question of the stability or instability of the solutions which might arise when $\Theta_1, \Theta_2, \Theta_3, \ldots$ had the values given in the case studied by Hill, and Θ_0 took all possible values. This is a more or less artificial problem, because, in the astronomical case at least, any variation in Θ_0 implies also variations in $\Theta_1, \Theta_2, \Theta_3, \ldots$ I now propose to consider the case of more interest to astronomers, viz. that in which $\Theta_0, \Theta_1, \Theta_2, \ldots$ are expressed as functions of the quantity m (the ratio of the synodic month to the sidereal year) and vary simultaneously. In the first place I shall obtain an expression for $\cos 2\sigma$ in the form of a series of ascending powers of m, from which I shall deduce a corresponding series for $\sin 2\sigma$. This last I shall make use of to obtain the first six terms of the literal expansion for μ in terms of m, and in conclusion I shall show how $\sin 2\sigma$ varies numerically for a range of values of m from 0 to ·25, and give a rough estimate of the value of m which gives rise to a periodic solution.

§ 2. *The Series for Θ_0 and μ transformed into Series in* m.—As far as terms of the eighth order the expressions which were found for Θ_0 and μ in terms of $\Theta_1, \Theta_2, \Theta_3, \ldots$ and σ are respectively

$$\Theta_0 = 1 - \frac{1}{2^2}\Theta_1^2 - \frac{1}{2 \cdot 3}\Theta_2^2 - \frac{11}{2^6 \cdot 3}\Theta_1^2\Theta_2 + \frac{1}{2^4 \cdot 3}\Theta_1^4$$
$$+ \cos 2\sigma \left\{ \Theta_1 + \frac{1}{2^2}\Theta_1\Theta_2 - \frac{1}{2^6}\Theta_1^3 \right\}$$
$$+ \cos 4\sigma \left\{ \frac{1}{2^3}\Theta_1^2 + \frac{5}{2^6}\Theta_1^2\Theta_2 - \frac{11}{2^9}\Theta_1^4 \right\} \quad . \quad . \quad (1)$$

and

$$\mu = \sin 2\sigma\left\{\frac{1}{2}\Theta_1 + \frac{1}{2^3}\Theta_1\Theta_2 - \frac{3}{2^7}\Theta_1^3\right\} + \sin 4\sigma\left\{\frac{1}{2^7}\Theta_1^2\Theta_2 - \frac{3}{2^{10}}\Theta_1^4\right\} \quad (2)$$

$\Theta_3, \Theta_4, \ldots$ only appear in terms of the tenth and higher orders.

To enable us to express these equations in terms of m, we avail

* *M.N.*, lxxv. p. 436 (1915 March); lxxvi. p. 431 (1916 March).

ourselves of the following relations,* which are correct to the terms in m^8 :—

$$\Theta_0 = 1 + 2m - \frac{1}{2}m^2 + \frac{255}{2^5}m^4 + 19m^5 + \frac{80}{3}m^6 + \frac{533}{2 \cdot 3^2}m^7 + \frac{11230225}{2^{13} \cdot 3^3}m^8,$$

$$\Theta_1 = -\frac{15}{2}m^2 - \frac{57}{2^2}m^3 - 11m^4 - \frac{23}{2 \cdot 3}m^5 - \frac{68803}{2^9 \cdot 3^2}m^6 - \frac{1792417}{2^{10} \cdot 3^3}m^7 - \frac{7172183}{2^7 \cdot 3^4 \cdot 5}m^8,$$

$$\Theta_1^2 = \frac{225}{2^2}m^4 + \frac{855}{2^2}m^5 + \frac{5889}{2^4}m^6 + 371m^7 + \frac{697679}{2^9 \cdot 3}m^8,$$

$$\Theta_1^3 = -\frac{3375}{2^3}m^6 - \frac{38475}{2^4}m^7 - \frac{205605}{2^5}m^8,$$

$$\Theta_1\Theta_2 = -\frac{1665}{2^5}m^6 - \frac{33609}{2^7}m^7 - \frac{169621}{2^8}m^8,$$

$$\Theta_1^4 = \frac{50625}{2^4}m^8,$$

$$\Theta_1^2\Theta_2 = \frac{24975}{2^6}m^8,$$

$$\Theta_2^2 = \frac{12321}{2^8}m^8.$$

With the aid of these relations we are able to express equation (1) in the form

$$\begin{aligned} 2m - \frac{1}{2}m^2 + \frac{465}{2^4}m^4 + \frac{3173}{2^5}m^5 + \frac{63241}{2^7 \cdot 3}m^6 \\ = \cos 2\sigma \left\{ -\frac{15}{2}m^2 - \frac{57}{2^2}m^3 - 11m^4 - \frac{23}{2 \cdot 3}m^5 - \frac{6023}{2^5 \cdot 3^2}m^6 - \frac{1284239}{2^9 \cdot 3^3}m^7 \right\} \\ + \cos^2 2\sigma \left\{ \frac{225}{2^4}m^4 + \frac{855}{2^4}m^5 + \frac{5889}{2^6}m^6 + \frac{371}{2^2}m^7 + \frac{493983}{2^{12} \cdot 3}m^8 \right\} \quad . \quad (3) \end{aligned}$$

while the series in μ takes the form

$$\begin{aligned} \mu = \sin 2\sigma \left\{ -\frac{15}{2^2}m^2 - \frac{57}{2^3}m^3 - \frac{11}{2}m^4 - \frac{23}{2^2 \cdot 3}m^5 - \frac{18809}{2^9 \cdot 3^2}m^6 - \frac{122707}{2^9 \cdot 3^3}m^7 \right\} \\ - \sin 4\sigma . \frac{101925}{2^{14}}m^8 . \quad . \quad . \quad (4) \end{aligned}$$

Here only those terms which will be required in the following work have been retained.

* Hill, *Annals of Mathematics*, ix. (1894) pp. 35, 37, and *Collected Works*, iv. pp. 44, 47.

§ 3. *Determination of* $\cos 2\sigma$ *as a Series of Powers of* m.—Equation (3) shows us that, as a first approximation, $\cos 2\sigma$ has the value $-\dfrac{2^2}{3 \cdot 5\mathrm{m}}$. Let us therefore assume a series of the form

$$\cos 2\sigma = -\frac{2^2}{3 \cdot 5\mathrm{m}}[1 + a_1\mathrm{m} + a_2\mathrm{m}^2 + a_3\mathrm{m}^3 + a_4\mathrm{m}^4 + a_5\mathrm{m}^5 + \ . \ . \ . \ ],$$

whence also

$$\cos^2 2\sigma = \frac{2^4}{3^2 \cdot 5^2\mathrm{m}^2}[1 + 2a_1\mathrm{m} + (a_1{}^2 + 2a_2)\mathrm{m}^2 + (2a_1a_2 + 2a_3)\mathrm{m}^3 + \ldots].$$

The coefficients $a_1, a_2, a_3,$ . . . have now to be determined, we therefore substitute these assumed values of $\cos 2\sigma$ and $\cos^2 2\sigma$ in equation (3) and compare the coefficients of successive powers of m on the two sides of the equation obtained. We find that the coefficient of m gives rise to

$$2 = \left(-\frac{15}{2}\right)\left(-\frac{2^2}{3 \cdot 5}\right),$$

an identity. Taking the coefficients of m^2, we find that

$$-\tfrac{1}{2} = 2a_1 + \frac{57}{3 \cdot 5} + 1,$$

whence

$$a_1 = -\frac{53}{2^2 \cdot 5}.$$

The coefficients of m^3 give rise to the equation

$$0 = 2a_2 + \frac{57}{3 \cdot 5}a_1 + \frac{44}{3 \cdot 5} + 2a_1 + \frac{19}{5},$$

whence

$$a_2 = \frac{2591}{2^3 \cdot 3 \cdot 5^2}.$$

Thus in succession we find that

$$a_3 = \frac{156073}{2^4 \cdot 3^2 \cdot 5^3},$$

$$a_4 = \frac{13328113}{2^6 \cdot 3^3 \cdot 5^4},$$

and

$$a_5 = \frac{2054674331}{2^9 \cdot 3^4 \cdot 5^5}.$$

The series for $\cos 2\sigma$ therefore is

$$\cos 2\sigma = -\frac{2^2}{3 \cdot 5\mathrm{m}}\left[1 - \frac{53}{2^2 \cdot 5}\mathrm{m} + \frac{2591}{2^3 \cdot 3 \cdot 5^2}\mathrm{m}^2 + \frac{156073}{2^4 \cdot 3^2 \cdot 5^3}\mathrm{m}^3 + \frac{13328113}{2^6 \cdot 3^3 \cdot 5^4}\mathrm{m}^4 + \frac{2054674331}{2^9 \cdot 3^4 \cdot 5^5}\mathrm{m}^5 + \ . \ . \ . \right] \quad . \quad (5)$$

§ 4. *The Series for sin 2σ.*—The next step is to obtain a similar series for $\sin 2\sigma$. Let us assume therefore the expansion

$$\sin 2\sigma = \pm \frac{2^2 i}{3 \cdot 5\mathrm{m}}[1 + b_1\mathrm{m} + b_2\mathrm{m}^2 + b_3\mathrm{m}^3 + b_4\mathrm{m}^4 + b_5\mathrm{m}^5 + \ldots],$$

$$(i = \sqrt{-1})$$

whence

$$\sin^2 2\sigma = -\frac{2^4}{3^2 \cdot 5^2\mathrm{m}^2}[1 + 2b_1\mathrm{m} + (b_1^2 + 2b_2)\mathrm{m}^2 + (2b_1b_2 + 2b_3)\mathrm{m}^3 + \ldots],$$

and use the identity

$$\cos^2 2\sigma + \sin^2 2\sigma = 1$$

to set up the relations between the a's and the b's.

From the coefficients of $\frac{1}{\mathrm{m}}$ in this identity we have

$$0 = a_1 - b_1,$$

whence

$$b_1 = -\frac{53}{2^2 \cdot 5}.$$

From the coefficients of m^0 we have

$$\frac{3^2 \cdot 5^2}{2^5} = a_2 - b_2 + \tfrac{1}{2}(a_1^2 - b_1^2),$$

whence

$$b_2 = -\frac{6511}{2^5 \cdot 3 \cdot 5^2}.$$

Likewise from the coefficients of m we have

$$0 = a_1a_2 + a_3 - b_1b_2 - b_3,$$

whence

$$b_3 = -\frac{1434541}{2^7 \cdot 3^2 \cdot 5^3},$$

and in succession from the coefficients of m^2 and m^3 we obtain

$$b_4 = -\frac{1084909759}{2^{11} \cdot 3^3 \cdot 5^4}$$

and

$$b_5 = -\frac{185839132579}{2^{13} \cdot 3^4 \cdot 5^5}.$$

Thus we obtain a series expression for $\sin 2\sigma$, viz.

$$\sin 2\sigma = \pm \frac{2^2 i}{3 \cdot 5\mathrm{m}}\Big[1 - \frac{53}{2^2 \cdot 5}\mathrm{m} - \frac{6511}{2^5 \cdot 3 \cdot 5^2}\mathrm{m}^2 - \frac{1434541}{2^7 \cdot 3^2 \cdot 5^3}\mathrm{m}^3 - \frac{1084909759}{2^{11} \cdot 3^3 \cdot 5^4}\mathrm{m}^4 - \frac{185839132579}{2^{13} \cdot 3^4 \cdot 5^5}\mathrm{m}^5\Big] \quad (6)$$

We also have

$$\sin 4\sigma = 2 \sin 2\sigma \cos 2\sigma = \pm \frac{2^5 i}{3^2 \cdot 5^2 m^2}\left[1 - \frac{53}{2 \cdot 5}m - \ . \ . \ . \right].$$

The sign of these expressions is of course ambiguous; we shall arbitrarily choose the positive sign.

§ 5. *The Literal Expression for* c.—The expression (4) which gives μ in terms of $\sin 2\sigma$ and m may be written in the form

$$i\mu = i \sin 2\sigma \left\{ -\frac{15}{2^2}m^2 - \frac{57}{2^3}m^3 - \frac{11}{2}m^4 - \frac{23}{2^3 \cdot 3}m^5 - \frac{18809}{2^9 \cdot 3^2}m^6 - \frac{122707}{2^9 \cdot 3^3}m^7 - \ . \ . \ . \right\}$$

$$+ i \sin 4\sigma \left\{ -\frac{101925}{2^{14}}m^8 - \ . \ . \ . \right\}.$$

If we use the series for $\sin 2\sigma$ and $\sin 4\sigma$ in terms of m, obtained in the preceding section, we may express μ as a series of ascending powers of m alone. We find, in fact, that on the right-hand side of the equation

the coefficient of $m = +1$,

" " $m^2 = b_1 + \frac{19}{2 \cdot 5} = -\frac{3}{2^2}$,

" " $m^3 = b_2 + \frac{19}{2 \cdot 5}b_1 + \frac{22}{3 \cdot 5} = -\frac{201}{2^5}$,

" " $m^4 = -\frac{2367}{2^7}$,

" " $m^5 = -\frac{111749}{2^{11}}$,

" " $m^6 = -\frac{4095991}{2^{13} \cdot 3}$.

The exponent c being defined by the relation

$$c = 1 + i\mu,$$

we thus arrive at the known literal expression for c, viz.

$$c = 1 + m - \frac{3}{2^2}m^2 - \frac{201}{2^5}m^3 - \frac{2367}{2^7}m^4 - \frac{111749}{2^{11}}m^5 - \frac{4095991}{2^{13} \cdot 3}m^6 - \ . \ . \ . \qquad (7)$$

§ 6. *Numerical Results—a Periodic Solution.*—In my previous papers on the subject of Hill's equation I pointed out that when $\sin 2\sigma$ vanishes, μ vanishes, and the solution of Hill's equation becomes purely periodic; when $\sin 2\sigma$ is purely imaginary, μ also is purely imaginary and the solution is stable. It is therefore interesting to examine how $\sin 2\sigma$ varies as m assumes different values in a convenient range.

The numerical values of the coefficients b in the expansion of $\sin 2\sigma$ are as follows:—

$$b_1 = -\frac{53}{20} = -2 \cdot 65000,$$

$$b_2 = -\frac{6511}{2400} = -2 \cdot 71292,$$

$$b_3 = -\frac{1434541}{144000} = -9 \cdot 96209,$$

$$b_4 = -\frac{1084909759}{34560000} = -31 \cdot 39206,$$

$$b_5 = -\frac{185839132579}{2073600000} = -89 \cdot 62150.$$

The series $1 + b_1\mathrm{m} + b_2\mathrm{m}^2 + b_3\mathrm{m}^3 + \ldots$ is therefore not very convergent for values of m greater than about 0·1. The ratios $\frac{b_3}{b_2}$, $\frac{b_4}{b_3}$, and $\frac{b_5}{b_4}$ are respectively 3·67, 3·15 and 2·85. On the assumptions that the b's always remain negative (so that the series has one zero) and that the series tends to become geometric, we may fairly assume the radius of convergence to be not less than ·4. If this is so we may take the remainder after the term in m^5 to be of the form

$$225\mathrm{m}^6[1 + 2 \cdot 5\mathrm{m} + (2 \cdot 5\mathrm{m})^2 + (2 \cdot 5\mathrm{m})^3 + \ldots]$$

or

$$\frac{225\mathrm{m}^6}{1 - 2 \cdot 5\mathrm{m}}.$$

In the following table, the column S_6 gives the values of $\sin 2\sigma$ when the first six terms of the series only are retained. The numbers in the column headed S take into account also the above assumed value for the remainder of the series. The last column gives the corresponding value of $i\mu$. As the series for $i\mu$ appears to diverge when m is at all greater than 0·2, little reliance can be put on the last entries in this column.

The series for sin 2σ seems to be more satisfactory than the series for $i\mu$ for investigating the question of periodicity.

m.	S_s.	S.	$i\mu$.
·00	∞i	∞i	0
·05	$4·5828i$	$4·5828i$	·0472
·08085	$2·5104i$	$2·5101i$	·0716
·10	$1·8503i$	$1·8495i$	·0835
·15	$·8625i$	$·8586i$	·107
·20	$·2705i$	$·2323i$	·04
·21	$·1700i$	$·1176i$	·02
·22	$·0726i$	$·0042i$	·00
·23	$-·0228i$	$-·1133i$	...
·24	$-·1173i$	$-·2362i$	...
25	$-·2111i$	$-·3666i$	...

m = ·08085 corresponds to the case of the lunar perigee discussed by Hill.

If, as we have supposed, all the coefficients b in the series $1 + b_1 m + b_2 m^2 + \ldots$ are negative, the series must have one zero for some positive value of m. For this value of m, sin 2σ vanishes, and consequently μ vanishes and the solution is purely periodic. The above table indicates that this zero of sin 2σ occurs at a value of m near 0·22. In other words, for values of m about $\frac{2}{9}$, the apse of the orbit considered has only a vanishingly small motion.

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On the Continued Fractions associated with the Hypergeometric Equation.

Preface.

This paper deals more or less completely with the continued fractions in terms of which the logarithmic derivate of a hypergeometric function may be expressed. So far as I know, the subject has not been treated in any detail except by Perron, to whom the theorem stated on p. 4 and its application are due. The contents of the paper, with this exception, I believe to be original. Some few of the results of the paper were previously given by myself, but in the main they have not been published.

E.L.I.

On the Continued Fractions connected with the Hypergeometric Equation.

§1. Introductory Remarks.

The infinite continued fractions dealt with in the present paper differ radically from those which most frequently occur in analysis. The latter are usually of the form

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots \text{ ad inf.}$$

or of the form

$$\frac{b_1 z}{c_1 z + d_1} + \frac{b_2 z^2}{c_2 z + d_2} + \frac{b_3 z^2}{c_3 z + d_3} + \cdots \text{ ad inf.}$$

or again of Stieltjes' form

$$\frac{1}{b_1 z} + \frac{1}{b_2} + \frac{1}{b_3 z} + \frac{1}{b_4} + \cdots \text{ ad inf.}$$

These three forms of continued fraction may readily be transformed into one another, it is therefore sufficient to consider only the first which is the form into which a power series is most easily converted*. It has been studied, from the point of view of function theory, by Pringsheim and by van Vleck † who found that if the coefficients $a_1, a_2, \ldots$ tend to a definite limit k, the continued fraction will converge and will represent an analytic function over the whole of the z-plane with the exception of (1) a rectilinear cut drawn from the point $z = -\frac{1}{4k}$ to infinity in the direction away from the origin, and (2) certain isolated points. The cut in question does not divide

* Heilermann, Crelle 33 (1845) p. 174
Muir Proc, Lond. Math. Soc

† see especially Trans. Amer. Math. Soc. 5 (1904) p. 253

the plane into two or more distinct regions, and the same holds for the other two continued fractions into which the first may be transformed.

The case of the continued fractions connected with the hypergeometric equation will as, we have said, be found to differ from the above. The important point of difference is that in their case there exists a <u>closed</u> cut or cuts which divide the z-plane into distinct regions. ~~The~~ any one continued fraction represents different analytic functions on ~~diff~~ opposite sides of a cut, in other words, the cuts form impassable barriers, beyond which the continued fraction ceases to represent ~~a certain~~ one particular function and begins to represent another.

There are, however, different forms of continued fraction associated with the hypergeometric equation, which may have different cuts. It results that when one continued fraction ceases to represent a certain function, it may be possible to find another continued fraction which will continue to do so. We have thus a kind of analytical continuation, by means of which we are able, to some extent, to carry the representation of a function over the barrier. We shall find, however, that in no case can a function be represented over the entire plane by continued fractions of the type considered.

§2. The twelve different continued fractions connected with the hypergeometric equation; the functions represented by them at different parts of the plane.

By successive differentiation of the hypergeometric equation

$$z(1-z)\frac{d^2y}{dz^2}+\{\gamma-(\alpha+\beta+1)z\}\frac{dy}{dz}-\alpha\beta y=0 \quad \text{---} \quad (1)$$

we obtain the set of equations

$$z(1-z)\frac{d^{r+2}y}{dz^{r+2}}+\{\gamma+r-(\alpha+\beta+2r+1)z\}\frac{d^{r+1}y}{dz^{r+1}}-(\alpha+r)(\beta+r)\frac{d^ry}{dz^r}=0$$
$$(r=1,2,3,\text{-----}).$$

This set of differential equations is equivalent to the infinite set of recurrence-relations

$$x_r=b_r x_{r+1}+a_{r+1}x_{r+2}, \qquad (r=0,1,2,\text{---}) \quad \text{---} \quad (2)$$

in which

$$x_r=\frac{1}{r!}\frac{d^ry}{dx^r}$$

$$b_r=\frac{(r+1)\{\gamma+r-(\alpha+\beta+2r+1)x\}}{(\alpha+r)(\beta+r)},$$

$$a_{r+1}=\frac{(r+1)(r+2)z(1-z)}{(\alpha+r)(\beta+r)}.$$

When r increases indefinitely b_r and a_{r+1} tend to definite limits which we shall denote by b and a respectively; we find, in fact, that

$$b=\lim_{r\to\infty}\frac{(r+1)\{\gamma+r-(\alpha+\beta+2r+1)z\}}{(\alpha+r)(\beta+r)}=1-2z$$

$$\text{and } a=\lim_{r\to\infty}\frac{(r+1)(r+2)z(1-z)}{(\alpha+r)(\beta+r)}=z(1-z)$$

limits which are independent of the constants α,β,γ of the original equation.

We now form the quadratic equation $\rho^2=b\rho+a$ and denote by ρ_1 its larger and by ρ_2 its smaller root in absolute magnitude (assuming for the moment z to be such that these roots are unequal in modulus).

We are now in a position to apply a theorem given by Perron,[*] viz. that provided $\overline{\lim_{r\to\infty}} \sqrt[r]{|x_r|} < \frac{1}{|\rho_2|}$ and further provided that all a_r's and at least one x_r be distinct from zero, the following formula certainly holds:

$$\frac{x_1}{x_0} = \frac{1}{b_0 +}\ \frac{a_1}{b_1 +}\ \frac{a_2}{b_2 +} \cdots$$

this infinite continued fraction being uniformly convergent except in the immediate neighbourhood of the zeros of x_0.

In the case we are considering, the solutions of the quadratic equation. $\rho^2 = b\rho + a$ are $\rho = -z$ and $\rho = 1-z$ so that when $R(z) < \frac{1}{2}$ the root ρ_2 is $1-z$, and when $R(z) > \frac{1}{2}$ the root ρ_2 is $-z$, where $R(z)$ denotes "the real part of z"

The above set of recurrence-relations (2) is satisfied by

$$x_r = \frac{1}{r!}\frac{d^r}{dz^r} F(\alpha, \beta, \gamma, z)$$

where $F(\alpha,\beta,\gamma,z)$ denotes the ordinary hypergeometric series $1 + \frac{\alpha\beta}{1\cdot\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}z^2 + \cdots$.

It is likewise satisfied by

$$x_r = \frac{1}{r!}\frac{d^r}{dz^r} F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-z).$$

We shall denote (with Perron) by $F_1(\alpha,\beta,\gamma,z)$ the function defined by the series $F(\alpha,\beta,\gamma,z)$ and its analytical continuation. Then the functions $x_r = \frac{1}{r!}\frac{d^r}{dz^r}F_1(\alpha,\beta,\gamma,z)$ and $x_r = \frac{1}{r!}\frac{d^r}{dz^r}F_1(\alpha,\beta,\alpha+\beta-\gamma+1,1-z)$, which are analytic at all points of the (appropriately cut) z-plane, satisfy one and the same set of recurrence relations (2).

Now the radius of convergence of the expression for the function $F_1(\alpha,\beta,\gamma,z)$ valid in the neighbourhood of a point ξ, that is to say,

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* Die Lehre von den Kettenbrüchen, p. 291

the radius of convergence of the series $\sum_{r=0}^{\infty} F^{(r)}(\alpha,\beta,\gamma,\xi)\,\frac{(z-\xi)^r}{r!}$ is $|1-\xi|$. Hence from the Cauchy-Hadamard theorem, we deduce that

$$\overline{\lim_{r\to\infty}}\ \sqrt[r]{\frac{1}{r!}\,|F_1^{(r)}(\alpha,\beta,\gamma,z)|} = \frac{1}{|1-z|}$$

whence the convergence criterion is $|1-z| > |\rho_2|$. But we have seen that when $R(z)<\frac{1}{2}$, $|\rho_2| \leq |z|$, and the criterion is then satisfied. Consequently the continued fraction

$$\frac{1}{b_0+}\ \frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\cdots \qquad (3)$$

converges, when $R(z)<\frac{1}{2}$, to $\dfrac{F_1'(\alpha,\beta,\gamma,z)}{F_1(\alpha,\beta,\gamma,z)} = \dfrac{d}{dz}\log F_1(\alpha,\beta,\gamma,z)$.

Applying the same reasoning to the function $F_1(\alpha,\beta,\alpha+\beta-\gamma+1,1-z)$ we deduce that

$$\overline{\lim_{r\to\infty}}\ \sqrt[r]{\frac{1}{r!}\,|F_1^{(r)}(\alpha,\beta,\alpha+\beta-\gamma+1,1-z)|} = \frac{1}{|z|}$$

whence the convergence criterion is $|z| > |\rho_2|$, and since when $R(z)>\frac{1}{2}$, $|\rho_2| = |1-z|$, the criterion is satisfied in the region $R(z)>\frac{1}{2}$, in which the continued fraction converges to $\frac{d}{dz}\log F_1(\alpha,\beta,\alpha+\beta-\gamma+1,1-z)$

By a simple reduction, the continued fraction (3) may be brought into the form

$$\frac{\alpha\beta}{\gamma-(\alpha+\beta+1)z+}\ \frac{(\alpha+1)(\beta+1)z(1-z)}{\gamma+1-(\alpha+\beta+3)z+}\ \cdots + \frac{(\alpha+r)(\beta+r)z(1-z)}{\gamma+r-(\alpha+\beta+2r+1)z+}\cdots$$

which we shall denote by S_1.

Thus we conclude that the continued fraction S_1 converges all over the z-plane with the exception firstly of the cut $R(z)=\frac{1}{2}$ and secondly of a certain number of isolated points which are the zeros of one or other of the functions $F_1(\alpha,\beta,\gamma,z)$ and $F_1(\alpha,\beta,\alpha+\beta-\gamma+1,1-z)$. Its limit is $\frac{d}{dz}\log F_1(\alpha,\beta,\gamma,z)$ when $R(z)<\frac{1}{2}$, and $\frac{d}{dz}F_1(\alpha,\beta,\alpha+\beta-\gamma+1,1-z)$ when $R(z)>\frac{1}{2}$ *.

* The significance of the continued fraction S_1 when $R(z)<\frac{1}{2}$ was known to Perron, to whom the above procedure is due. That it has an entirely different limit when $R(z)>\frac{1}{2}$ does not seem to have been so far noted.

There are 24 fundamental series-solutions of the hypergeometric differential equation, which we shall denote (with Forsyth[*]) by $y_1, y_2 \cdots y_{24}$. We shall, however, extend this notation to include, as well as the series solution, its analytical continuation in each case. Further we shall denote by $f_1, f_2, \cdots f_{24}$, the corresponding logarithmic derivates with respect to z. Thus we have so far found a continued fraction S_1 which represents f_1 or $\frac{d}{dz}\log F_1(\alpha,\beta,\gamma,z)$ when $R(z)<\frac{1}{2}$, and f_5 or $\frac{d}{dz}\log F_1(\alpha,\beta,\alpha+\beta-\gamma+1,1-z)$ when $R(z)>\frac{1}{2}$. We shall find that we may likewise group all the solutions in pairs such that to each pair there corresponds a single continued fraction.

If, for example, we take the solution $y_2=(1-z)^{\gamma-\alpha-\beta}F_1(\gamma-\alpha,\gamma-\beta,\gamma,z)$, we must associate with it the solution $y_7=(1-z)^{\gamma-\alpha-\beta}F_1(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-z)$[†]. For from y_2 we deduce $f_2=\frac{\alpha+\beta-\gamma}{1-z}+\frac{d}{dz}\log F_1(\gamma-\alpha,\gamma-\beta,\gamma,z)$, and by what has gone before we know that, when $R(z)<\frac{1}{2}$,

$$\frac{d}{dz}\log F_1(\gamma-\alpha,\gamma-\beta,\gamma,z) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma-(2\gamma-\alpha-\beta+1)z+}\,\frac{(\gamma-\alpha+1)(\gamma-\beta+1)z(1-z)}{\gamma+1-(2\gamma-\alpha-\beta+3)z+}\cdots$$

a continued fraction which, when $R(z)>\frac{1}{2}$ converges to $\frac{d}{dz}\log F_1(\gamma-\alpha,\gamma-\beta,\gamma-\beta-\alpha+1,1-z)$

We therefore conclude that the continued fraction

$$S_2 \equiv \frac{\alpha+\beta-\gamma}{1-z}+\frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma-(2\gamma-\alpha-\beta+1)z+}\,\frac{(\gamma-\alpha+1)(\gamma-\beta+1)z(1-z)}{\gamma+1-(2\gamma-\alpha-\beta+3)z}+\cdots+\frac{(\gamma-\alpha+r)(\gamma-\beta+r)z(1-z)}{\gamma+r-(2\gamma-\alpha-\beta+2r+1)z+}\cdots$$

represents f_2 when $R(z)<\frac{1}{2}$ and f_7 when $R(z)>\frac{1}{2}$.

Similarly, by considering the solution $y_3=z^{1-\gamma}F_1(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,z)$ with which we associate the solution $y_6=z^{1-\gamma}F_1(\alpha-\gamma+1,\beta-\gamma+1,\alpha+\beta-\gamma+1,1-z)$, we conclude

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[*] Treatise on Differential Equations Chap VI.

[†] The first two terms of the bracket in $F_1(\ \ )$, and also the outside factor are always the same in the two associated solutions

that the continued fraction

$$S_3 \equiv \frac{1-\gamma}{z} + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{2-\gamma-(\alpha+\beta-2\gamma+3)z+}\;\frac{(\alpha-\gamma+2)(\beta-\gamma+2)z(1-z)}{3-\gamma-(\alpha+\beta-2\gamma+5)z+}\cdots\frac{(\alpha-\gamma+r+1)(\beta-\gamma+r+1)z(1-z)}{r+1-\gamma-(\alpha+\beta-2\gamma+2r+3)z+\cdots}$$

represents f_3 when $R(z) < \frac{1}{2}$ and f_6 when $R(z) > \frac{1}{2}$, and again by considering the mutually-associated solutions $f_4 = z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}F_1(1-\alpha, 1-\beta, 2-\gamma, z)$ and $f_8 = z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}F_1(1-\alpha, 1-\beta, \gamma-\alpha-\beta+1, 1-z)$ we conclude that the continued fraction [x]

$$S_4 \equiv \frac{1-\gamma}{z} + \frac{(\alpha+\beta-\gamma)}{1-z} + \frac{(1-\alpha)(1-\beta)}{2-\gamma-(3-\alpha-\beta)z+}\;\frac{(2-\alpha)(2-\beta)z(1-z)}{3-\gamma-(5-\alpha-\beta)z} + \cdots + \frac{(r+1-\alpha)(r+1-\beta)z(1-z)}{r+1-\gamma-(2r+3-\alpha-\beta)z+\cdots}$$

represents f_4 when $R(z) < \frac{1}{2}$, and f_8 when $R(z) > \frac{1}{2}$.

We have thus disposed of the four solutions of the hypergeometric equation which are appropriate to the singularity $z=0$, and of the four which are appropriate to the singularity ~~at infinity~~ $z=1$, and turn to the eight solutions appropriate to the singularity at infinity.

Taking first of all $y_9 = z^{-\alpha}F_1(\alpha, \alpha-\gamma+1, \alpha-\beta+1, \frac{1}{z})$ we have

$$f_9 = -\frac{\alpha}{z} + \frac{d}{dz}\log F_1(\alpha, \alpha-\gamma+1, \alpha-\beta+1, \tfrac{1}{z})$$
$$= -\frac{\alpha}{z} - \frac{1}{z^2}\frac{d}{d\zeta}\log F_1(\alpha, \alpha-\gamma+1, \alpha-\beta+1, \zeta)$$

where $\zeta = \frac{1}{z}$. Now $\frac{d}{d\zeta}\log F_1(\alpha, \alpha-\gamma+1, \alpha-\beta+1, \zeta)$ is represented, when $R(\zeta) < \frac{1}{2}$ by the continued fraction $\frac{\alpha(\alpha-\gamma+1)}{\alpha-\beta+1-(2\alpha-\gamma+2)\zeta+}\;\frac{(\alpha+1)(\alpha-\gamma+2)\zeta(1-\zeta)}{(\alpha-\beta+2)-(2\alpha-\gamma+4)\zeta+\cdots}$
which, when $R(z) > \frac{1}{2}$ represents $\frac{d}{d\zeta}\log F_1(\alpha, \alpha-\gamma+1, \alpha+\beta-\gamma+1, 1-\zeta)$.
Further

$$-\frac{\alpha}{z} - \frac{1}{z^2}\frac{d}{d\zeta}\log F_1(\alpha, \alpha-\gamma+1, \alpha+\beta-\gamma+1, 1-\zeta)$$
$$= -\frac{\alpha}{z} + \frac{d}{dz}\log F_1(\alpha, \alpha-\gamma+1, \alpha+\beta-\gamma+1, \tfrac{z-1}{z}) = f_{21},$$

since $y_{21} = z^{-\alpha}F_1(\alpha, \alpha-\gamma+1, \alpha+\beta-\gamma+1, \frac{z-1}{z})$. Now when $R(\zeta) < \frac{1}{2}$, $|z-1| > 1$ and when $R(\zeta) > \frac{1}{2}$, $|z-1| < 1$. Hence on substituting $\frac{1}{z}$ for ζ in the above continued fraction, and simplifying, we arrive at the result that the

[x] This is the continued fraction which arises out of the set of recurrence-relations derived from the hypergeometric equation by repeated integration. See a paper by the present author, Proc. Edin. Math. Soc XXXIV.

continued fraction

$$S_5 \equiv -\frac{\alpha}{z} - \frac{1}{z}\cdot\frac{\alpha(\alpha-\gamma+1)}{(\alpha-\beta+1)z-(2\alpha-\gamma+2)+}\,\frac{(\alpha+1)(\alpha-\gamma+2)(z-1)}{(\alpha-\beta+2)z-(2\alpha-\gamma+4)+}\cdots+\frac{(\alpha+r)(\alpha-\gamma+r+1)(z-1)}{(\alpha-\beta+r+1)z-(2\alpha-\gamma+2r)+}\cdots$$

represents f_9 when $|z-1|>1$ and f_{21} when $|z-1|<1$.

By a similar process of reasoning we conclude that the continued fraction

$$S_6 \equiv -\frac{\beta}{z} - \frac{1}{z}\cdot\frac{\beta(\beta-\gamma+1)}{(\beta-\alpha+1)z-(2\beta-\gamma+2)+}\,\frac{(\beta+1)(\beta-\gamma+2)(z-1)}{(\beta-\alpha+2)z-(2\beta-\gamma+4)+}\cdots+\frac{(\beta+r)(\beta-\gamma+r+1)(z-1)}{(\beta-\alpha+r+1)z-(2\beta-\gamma+2r)+}\cdots$$

represents f_{10} or $\frac{d}{dz}\log z^{-\beta}F_1(\beta, \beta-\gamma+1, \beta-\alpha+1, \frac{1}{z})$ when $|z-1|>1$ and f_{22} or $\frac{d}{dz}\log z^{-\beta}F_1(\beta, \beta-\gamma+1, \alpha+\beta-\gamma+1, \frac{z-1}{z})$ when $|z-1|<1$; that the continued fraction

$$S_7 \equiv \frac{\alpha-\gamma}{z} + \frac{\alpha+\beta-\gamma}{1-z} - \frac{1}{z}\cdot\frac{(1-\alpha)(\gamma-\alpha)}{(\beta-\alpha+1)z-(\gamma-2\alpha+2)+}\,\frac{(2-\alpha)(\gamma-\alpha+1)(z-1)}{(\beta-\alpha+2)z-(\gamma-2\alpha+4)+}\cdots+\frac{(r+1-\alpha)(\gamma-\alpha+r)(z-1)}{(\beta-\alpha+r+1)z-(\gamma-2\alpha+2r+2)+}\cdots$$

represents f_{11} or $\frac{d}{dz}\log z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}F_1(1-\alpha, \gamma-\alpha, \beta-\alpha+1, \frac{1}{z})$ when $|z-1|>1$ and f_{23} or $\frac{d}{dz}\log z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}F_1(1-\alpha, \gamma-\alpha, \gamma-\alpha-\beta+1, \frac{z-1}{z})$ when $|z-1|<1$; and that the continued fraction

$$S_8 \equiv \frac{\beta-\gamma}{z} + \frac{\alpha+\beta-\gamma}{1-z} - \frac{1}{z}\,\frac{(1-\beta)(\gamma-\beta)}{(\alpha-\beta+1)z-(\gamma-2\beta+2)+}\,\frac{(2-\beta)(\gamma-\beta+1)(z-1)}{(\alpha-\beta+2)z-(\gamma-2\beta+4)+}\cdots+\frac{(r+1-\beta)(\gamma-\beta+r)(z-1)}{(\alpha-\beta+r+1)z-(\gamma-2\beta+2r+2)+}\cdots$$

represents f_{12} or $\frac{d}{dz}\log z^{\beta-\gamma}(1-z)^{\gamma-\alpha-\beta}F_1(1-\beta, \gamma-\beta, \alpha-\beta+1, \frac{1}{z})$ when $|z-1|>1$ and f_{24} or $\frac{d}{dz}\log z^{\beta-\gamma}(1-z)^{\gamma-\alpha-\beta}F_1(1-\beta, \gamma-\beta, \gamma-\alpha-\beta+1, \frac{z-1}{z})$ when $|z-1|<1$.

Taking now the case of $y_{13} = (1-z)^{-\alpha}F_1(\alpha, \gamma-\beta, \alpha-\beta+1, \frac{1}{1-z})$, we have

$$f_{13} = \frac{\alpha}{1-z} + \frac{d}{dz}\log F_1(\alpha, \gamma-\beta, \alpha-\beta+1, \frac{1}{1-z})$$
$$= \frac{\alpha}{1-z} + \frac{1}{(1-z)^2}\frac{d}{d\zeta}\log F_1(\alpha, \gamma-\beta, \alpha-\beta+1, \zeta)$$

where $\zeta = \frac{1}{1-z}$. Now the continued fraction

$$\frac{\alpha(\gamma-\beta)}{\alpha-\beta+1-(\alpha-\beta+\gamma-1)\zeta+}\,\frac{(\alpha+1)(\gamma-\beta+1)\zeta(1-\zeta)}{\alpha-\beta+2-(\alpha-\beta+\gamma-2)\zeta+}\cdots$$

which represents $\frac{d}{d\zeta}\log F_1(\alpha, \gamma-\beta, \alpha-\beta+1, \zeta)$ when $R(\zeta)<\frac{1}{2}$, represents also $\frac{d}{d\zeta}\log F_1(\alpha, \gamma-\beta, \alpha-\beta+1, 1-\zeta)$ when $R(\zeta)>\frac{1}{2}$, and moreover

$$\frac{\alpha}{1-z} + \frac{1}{(1-z)^2}\,\frac{d}{d\zeta}\log F_1(\alpha, \gamma-\beta, 2-\gamma, 1-\zeta)$$

$$= \frac{\alpha}{1-z} + \frac{d}{dz}\log F_1\left(\alpha, \gamma-\beta, 2-\gamma, \frac{z}{z-1}\right) = f_{17},$$

since $y_{17} = (1-z)^{-\alpha} F_1\left(\alpha, \gamma-\beta, 2-\gamma, \frac{z}{z-1}\right)$. When $R(\zeta) < \frac{1}{2}$, $|z| > 1$ and when $R(\zeta) > \frac{1}{2}$, $|z| < 1$. On substituting $\frac{1}{z-1}$ for ζ in the above continued fraction, and simplifying, we obtain the result that the continued fraction

$$S_9 \equiv \frac{\alpha}{1-z} + \frac{1}{1-z}\cdot\frac{\alpha(\gamma-\beta)}{-\gamma-(\alpha-\beta+1)z -}\;\frac{(\alpha+1)(\gamma-\beta+1)z}{-\gamma-1-(\alpha-\beta+3)z -}\cdots - \frac{(\alpha+r)(\gamma-\beta+r)z}{-\gamma-r-(\alpha-\beta+2r+1)z - \cdots}$$

represents f_{13} when $|z| > 1$ and f_{17} when $|z| < 1$.

In like manner we conclude that the continued fraction

$$S_{10} \equiv \frac{\beta}{1-z} + \frac{1}{1-z}\cdot\frac{\beta(\gamma-\alpha)}{-\gamma-(\beta-\alpha+1)z -}\;\frac{(\beta+1)(\gamma-\alpha+1)z}{-\gamma-1-(\beta-\alpha+3)z - \cdots} - \frac{(\beta+r)(\gamma-\alpha+r)z}{-\gamma-r-(\beta-\alpha+2r+1)z - \cdots}$$

represents f_{14} or $\frac{d}{dz}\log(1-z)^{-\beta}F_1\left(\beta, \gamma-\alpha, \beta-\alpha+1, \frac{1}{1-z}\right)$ when $|z| > 1$ and f_{18} or $\frac{d}{dz}\log(1-z)^{-\beta}F_1\left(\beta, \gamma-\alpha, 2-\gamma, \frac{z}{z-1}\right)$ when $|z| < 1$; that the continued fraction

$$S_{11} \equiv \frac{1-\gamma}{z} + \frac{\alpha-\gamma+1}{1-z} + \frac{1}{1-z}\cdot\frac{(\alpha-\gamma+1)(1-\beta)}{\gamma-2-(\alpha-\beta+1)z -}\;\frac{(\alpha-\gamma+2)(2-\beta)z}{\gamma-3-(\alpha-\beta+2)z - \cdots} - \frac{(\alpha-\gamma+r+1)(r+1-\beta)z}{\gamma-r-2-(\alpha-\beta+r+1)z - \cdots}$$

represents f_{15} or $\frac{d}{dz}\log z^{1-\gamma}(1-z)^{\gamma-\alpha-1}F_1\left(\alpha-\gamma+1, 1-\beta, \gamma-2, \frac{1}{1-z}\right)$ when $|z| > 1$ and f_{19} or $\frac{d}{dz}\log z^{1-\gamma}(1-z)^{\gamma-\alpha-1}F_1\left(\alpha-\gamma+1, 1-\beta, \alpha-\beta+1, \frac{z}{z-1}\right)$ when $|z| < 1$; and that the continued fraction

$$S_{12} \equiv \frac{1-\gamma}{z} + \frac{\beta-\gamma+1}{1-z} + \frac{1}{1-z}\cdot\frac{(\beta-\gamma+1)(1-\alpha)}{\gamma-2-(\beta-\alpha+1)z} - \frac{(\beta-\gamma+2)(2-\alpha)z}{\gamma-3-(\beta-\alpha+2)z - \cdots} - \frac{(\beta-\gamma+r+1)(r+1-\alpha)z}{\gamma-r-2-(\beta-\alpha+r+1)z - \cdots}$$

represents f_{16} or $\frac{d}{dz}\log z^{1-\gamma}(1-z)^{\gamma-\beta-1}F_1\left(\beta-\gamma+1, 1-\alpha, \gamma-2, \frac{1}{1-z}\right)$ when $|z| > 1$ and f_{20} or $\frac{d}{dz}\log z^{1-\gamma}(1-z)^{\gamma-\beta-1}F_1\left(\beta-\gamma+1, 1-\alpha, \beta-\alpha+1, \frac{z}{z-1}\right)$ when $|z| < 1$.

9. Relations existing between the twelve continued fractions at different parts of the z-plane.

It is well known that the following identities between solutions of the hypergeometric equation subsist throughout the z-plane, viz.

1. $y_1 \equiv y_2 \equiv y_{17} \equiv y_{18}$,
2. $y_3 \equiv y_4 \equiv y_{19} \equiv y_{20}$,
3. $y_5 \equiv y_6 \equiv y_{21} \equiv y_{22}$,
4. $y_7 \equiv y_8 \equiv y_{23} \equiv y_{24}$,
5. $(-1)^{\alpha} y_9 \equiv (-1)^{\beta-\gamma} y_{12} \equiv y_{13} \equiv (-1)^{1-\gamma} y_{15}$,
6. $(-1)^{\beta} y_{10} \equiv (-1)^{\alpha-\gamma} y_{11} \equiv y_{14} \equiv (-1)^{1-\gamma} y_{16}$,

where as before y_r denotes not only the series solution but also its analytical continuation. From these we at once deduce

1 $f_1 \equiv f_2 \equiv f_{17} \equiv f_{18} \equiv H_1$ say,
2 $f_3 \equiv f_4 \equiv f_{19} \equiv f_{20} \equiv H_2$ "
3 $f_5 \equiv f_6 \equiv f_{21} \equiv f_{22} \equiv H_3$ "
4 $f_7 \equiv f_8 \equiv f_{23} \equiv f_{24} \equiv H_4$ "
5 $f_9 \equiv f_{12} \equiv f_{13} \equiv f_{15} \equiv H_5$ "
6. $f_{10} \equiv f_{11} \equiv f_{14} \equiv f_{16} \equiv H_6$ "

The twelve continued fractions we have obtained converge over the whole of the z-plane with the exception of certain isolated points which represent the zeros of the solution whose logarithmic derivate is in question and with the exception of the three closed cuts $R(z) = \frac{1}{2}$, $|z-1| = 1$ and $|z| = 1$. Each continued fraction ceases to converge on one only of these cuts and represents on one side of that cut a certain one of the functions H and on the other side a different function H. The three cuts divide up the plane into six distinct regions A, B, C, D, E, F (Fig. 1.) within each of which every one of the twelve continued fractions represents a definite function, and in each of which the continued fractions can be grouped into four sets, the continued fractions in each set being ~~identical~~ equal within the particular region considered. Thus we have the following sets of identities, viz.

In region A, $S_1 \equiv S_2 \equiv H_1$, $S_3 \equiv S_4 \equiv H_2$, $S_5 \equiv S_8 \equiv S_9 \equiv S_{11} \equiv H_5$, $S_6 \equiv S_7 \equiv S_{10} \equiv S_{12} \equiv H_6$,

" B, $S_1 \equiv S_2 \equiv S_9 \equiv S_{10} \equiv H_1$, $S_3 \equiv S_4 \equiv S_{11} \equiv S_{12} \equiv H_2$, $S_5 \equiv S_8 \equiv H_5$, $S_6 \equiv S_7 \equiv H_6$,

" C, $S_1 \equiv S_2 \equiv S_9 \equiv S_{10} \equiv H_1$, $S_3 \equiv S_4 \equiv S_{11} \equiv S_{12} \equiv H_2$, $S_9 \equiv S_{10} \equiv H_3$, $S_7 \equiv S_8 \equiv H_4$,

" D, $S_1 \equiv S_3 \equiv S_5 \equiv S_6 \equiv H_3$, $S_2 \equiv S_4 \equiv S_7 \equiv S_8 \equiv H_4$, $S_9 \equiv S_{10} \equiv H_1$, $S_{11} \equiv S_{12} \equiv H_2$,

" E, $S_1 \equiv S_3 \equiv S_5 \equiv S_6 \equiv H_3$, $S_2 \equiv S_4 \equiv S_7 \equiv S_8 \equiv H_4$, $S_9 \equiv S_{11} \equiv H_5$, $S_{10} \equiv S_{12} \equiv H_6$,

" F, $S_1 \equiv S_3 \equiv H_3$, $S_2 \equiv S_4 \equiv H_4$, $S_5 \equiv S_8 \equiv S_9 \equiv S_{11} \equiv H_5$, $S_6 \equiv S_7 \equiv S_{10} \equiv S_{12} \equiv H_6$.

Here we notice that no equation between any two of the twelve continued fractions holds throughout the whole plane, but each continued fraction is equal, in two of the regions, to three other continued fractions and in the other four regions, to one other continued fraction. Further, no function H is represented all over the plane by means of the continued fractions, but each H is represented in four regions only.

§ Logarithmic Cases.

Exceptional cases arise in three distinct circumstances as follows:

(i) $\gamma = 1$,

(ii) $\gamma = \alpha+\beta$,

(iii) $\alpha = \beta$.

Consider first the case $\gamma = 1$. The two solutions $y_1 = F_1(\alpha, \beta, \gamma, x)$ and $y_2 = x^{1-\gamma} F_1(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$, which are distinct if $\gamma \neq 1$ now coincide and we are led to the "logarithmic case". The continued fraction S_1 now becomes

$$\frac{\alpha\beta}{1-(\alpha+\beta+1)z +} \quad \frac{(\alpha+1)(\beta+1)z(1-z)}{2-(\alpha+\beta+3)z + \cdots}$$

The previous discussion of the continued fraction is equally valid in the particular case $\gamma=1$, so that the continued fraction represents (when $R(z) < \frac{1}{2}$) the function $\frac{d}{dz}F_1(\alpha, \beta, 1, z)$. The continued fraction S_3 now degenerates into S_1, S_4 into S_2, S_{11} into S_9 and S_{12} into S_{10}. The function H_2 also becomes H_1. Thus in this case the C.F. obtained by repeated integration (S_4) gives us the same solution as that obtained by repeated differentiation (S_1), so that, in the logarithmic case we are not able completely to solve the equation by continued fractions. This occurs in the case of Legendre's equation.

The same kind of phenomenon arises in cases (ii) and (iii) which need not be discussed in full.

§. Riemann's Transformation.

An interesting set of equations connecting continued fractions arises out of the relation

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \alpha & 0 & x \\ \frac{1}{2} & \beta & \frac{1}{2}-\alpha-\beta & \end{matrix}\right\} = P\left\{\begin{matrix} -1 & \infty & 1 & \\ 0 & 2\alpha & 0 & z \\ \frac{1}{2}-\alpha-\beta & 2\beta & \frac{1}{2}-\alpha-\beta & \end{matrix}\right\}$$

where $x = z^2$; which is known as Riemann's transformation.

The differential equation whose solutions are represented by the P-function on the left-hand side of this equation is

$$x(x-1)\frac{d^2y}{dx^2} + \left\{-\tfrac{1}{2} + (\alpha+\beta+1)x\right\}\frac{dy}{dx} + \alpha\beta y = 0 \quad \text{- - - -} \quad (4)$$

which gives rise directly to the continued fraction

$$\frac{1}{y}\frac{dy}{dx} = \frac{\alpha\beta}{\frac{1}{2}-(\alpha+\beta+1)x -}\;\frac{(\alpha+1)(\beta+1)x(x-1)}{\frac{3}{2}-(\alpha+\beta+3)x - }\text{ - - - - }\frac{(\alpha+r)(\beta+r)x(x-1)}{\frac{2r+1}{2}-(\alpha+\beta+2r+1)x - }\cdots \quad (5)$$

or, writing z^2 for x,

$$\frac{1}{2z}\frac{1}{y}\frac{dy}{dz} = \frac{\alpha\beta}{\frac{1}{2}-(\alpha+\beta+1)z^2 -}\;\frac{(\alpha+1)(\beta+1)z^2(z^2-1)}{\frac{3}{2}-(\alpha+\beta+3)z^2 -}\text{ - - - - - - - -} \quad (6)$$

Now when $R(x) > \frac{1}{2}$, the continued fraction (5) corresponds to that particular solution $F_1(\alpha, \beta, \frac{1}{2}-\alpha-\beta, 1-x)$ of the differential equation (4) which is appropriate to the singularity $x=0$ and exponent 0. Consequently, when $R(z^2) > 1$, the continued fraction (6) corresponds to the solution of the transformed equation appropriate to the exponent zero at the singularity $z^2 = 1$, that is to say, at the singularities $z=1$ and $z=-1$.

Let $z = \xi + i\eta$,

then $R(z^2) = \xi^2 - \eta^2 > \frac{1}{2}$.

The region $R(z^2) > \frac{1}{2}$ is thus the region to the right of the right-hand branch and to the left of the left-hand branch of the rectangular hyperbola $\xi^2 - \eta^2 = \frac{1}{2}$ (Fig. 2).

The transformed differential equation, which corresponds to the P-function on the right-hand side of the above relation is

$$(z^2-1)\frac{d^2y}{dz^2} + (2\alpha+2\beta+1)z.\frac{dy}{dz} + 4\alpha\beta y = 0 \quad \text{- - - - -} \quad (7)$$

This is not, strictly speaking, a hypergeometric differential equation but it is of that type and can, in fact, be transformed by a simple linear transformation, into a hypergeometric equation. The continued fraction which arises directly out of it is.

$$\frac{1}{y}\frac{dy}{dz} = \frac{-4\alpha\beta}{(2\alpha+2\beta+1)z -}\;\frac{(2\alpha+1)(2\beta+1)(z^2-1)}{(2\alpha+2\beta+3)z - \cdots -}\;\frac{(2\alpha+r)(2\beta+r)(z^2-1)}{(2\alpha+2\beta+2r+1)z - \cdots} \quad (8)$$

When $R(z) > 0$, the continued fraction converges to the logarithmic derivate of the solution corresponding to the exponent 0 at the singularity $z = +1$, and when $R(z) < 0$, to the logarithmic derivate of the solution corresponding to the exponent 0 at the singularity $z = -1$. It so happens, on account of the symmetry of the P- function, that these two solutions are identical with one another, so that the continued fraction (8) represents a certain odd function over the whole of the z-plane with the exception of the imaginary. axis.

These results show us that in region I (Fig. 2.), ~~the~~ continued fraction (6) $= \frac{1}{2z} \times$ continued fraction (8). In region II, (8) retains the significance it had in I, but (6) loses this significance as it there represents the logarithmic derivate of a solution appropriate to the neighbourhood of the origin.

We might, as before, deduce from each of the two differential equations (4) and (7), twelve continued fractions, and set up relations between them.

On the Continued Fractions associated with the hypergeometric equation.

Fig. 1.

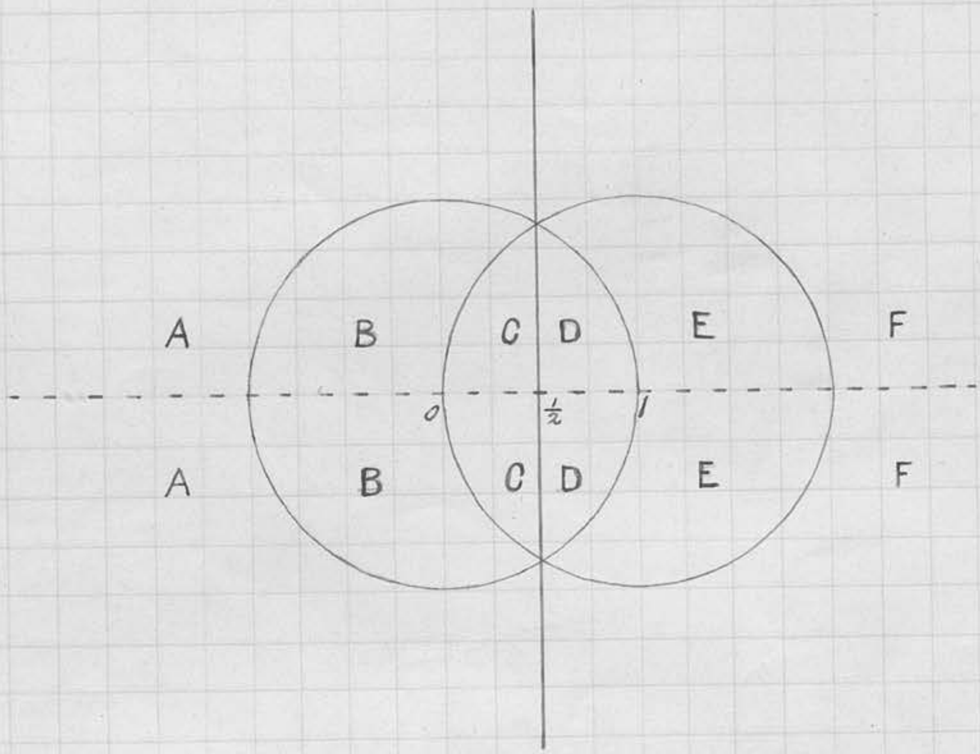
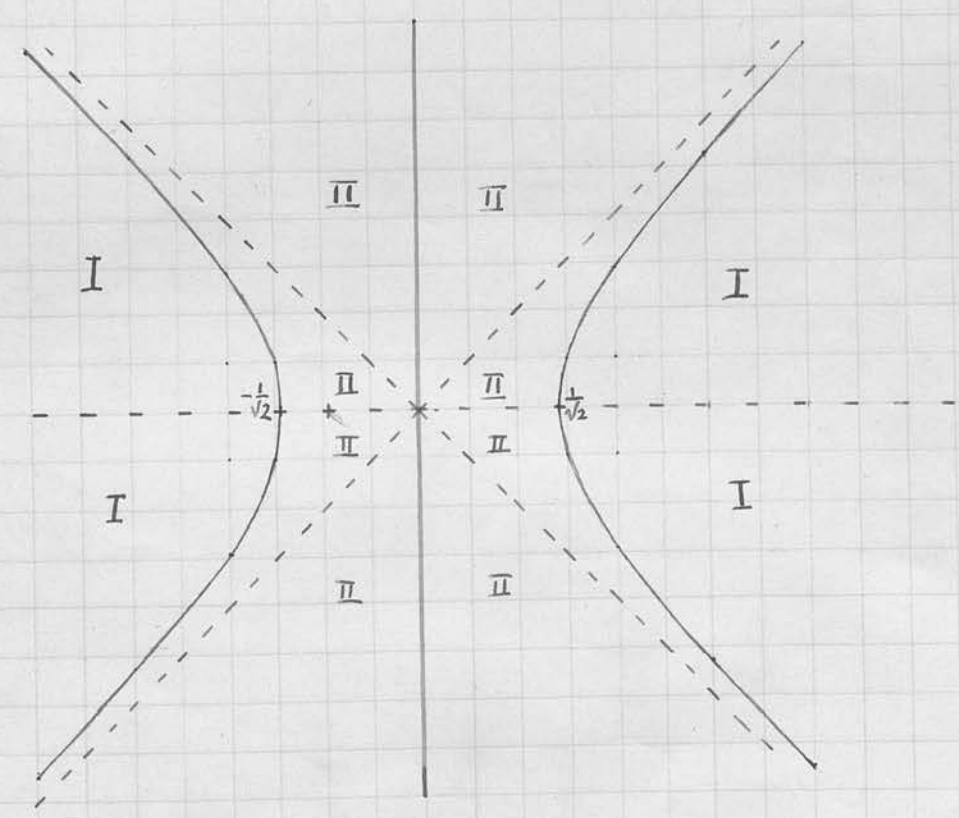


Fig. 2.



Suggestions for a systematic study of Linear Differential equations of the second order.

Preface.

This dissertation was originally intended to be an attempt to study linear differential equations (of the second order) from a systematic point of view, and to investigate as far as possible, the consequences of the system of classification which it proposes. Unfortunately, however, circumstances ~~[illegible]~~ forced me to lay aside this piece of work before I was able to bring it to completion. In its present, unfinished state it gives the basis of a systematic study, but does not enter, in any detail, into the consequences of the ideas which it proposes. As to its claim to originality, § 2 of Chapter I is due to Klein and Bôcher (as expounded in Whittaker & Watson's "Modern Analysis") and is introduced merely to exemplify the general ideas of § 1. The remainder of the paper I believe in general to be original.

E. L. I.

Chapter I. Classification of Linear Differential Equations into Types according to the number and nature of their singular points.

§1. Introductory remarks, the formula of a linear differential equation.

The following is an attempt to systematise the study of linear differential equations of the second order, with rational coefficients, by grouping them into types according to the number and the nature of their singular points. Such a classification, if carried out on sound general principles, would presumably simplify the study of these differential equations by indicating properties which are common to the members of a particular class, and relations subsisting between the individuals of any one class and the corresponding members of another class.

The attempt is founded on the well-known theorem due to Klein and Bôcher* that all the linear differential equations which arise out of the ordinary problems of mathematical physics can be derived from the equation with five regular singular points in which the difference between the two exponents or indices corresponding to each of the singular points is $\frac{1}{2}$. The positions of these five regular singular points in the plane of the independent variable are in the first place considered to be quite arbitrary. If any two of these points be allowed to approach each other and ultimately to coalesce, the resulting

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* Klein. Ueber lineare Differentialgleichungen der zweiten Ordnung. p. 40.
Bôcher. Über die Reihenentwickelungen der Potentialtheorie. p. 193

singular point will still be regular, but it will be of a more general character than either of the two original singular points in that the difference between its exponents is no longer $\frac{1}{2}$, but is quite arbitrary. If a third out of the original singular points were allowed to approach and coalesce with this new singular point; or what is the same thing, if three singular points in the original equation were allowed to coalesce and form a new singular point, this new singular point would no longer be regular, but would be an essential singularity. So also if a singular point be formed by the coalescence of four or of all the five singular points of the differential equation, the new singular point will be an essential singularity which would, generally speaking, be of greater complexity the larger the number of regular singular points with exponent-difference $\frac{1}{2}$ from whose coalescence it was derived.

These considerations at once indicate the lines on which any systematic classification of the proposed kind must proceed. Taking any particular linear differential equation of the second order, we consider

(α) the number of regular singular points with exponent-difference $\frac{1}{2}$ the equation considered possesses; this we denote by $\underline{a}$

(β) the number of regular singular points with exponent-difference other than $\frac{1}{2}$, this we denote by $\underline{b}$.

(γ) the number of irregular singular points; this we denote by $\underline{c}$, for the moment ignoring the possibility of the irregular singular points being of different complexity from one another.

The equation under consideration is then said to have the formula (a, b, c).

This formula does not indicate the positions of the singular points in the plane of the independent variable, but this is a matter of no consequence as the form and character of the differential equation depend more on the exponents relative to the singular points than on the positions of the singular points themselves. Any three of the singular points may, by a simple homographic transformation applied to the independent variable, be brought to any three points on the plane. In what follows, if an equation has only one singular point (which must then be an essential singularity), that point will be taken to be the point at infinity; if it has more than one, the origin and the point at infinity will usually be considered as the seats of two of them.

The formula tells us nothing as to the actual exponents relative to the regular singular points, it merely indicates the nature of the difference of the exponents. But by multiplying every solution by an appropriate factor of the form $(x-a_1)^{\alpha_1}(x-a_2)^{\alpha_2}\cdots(x-a_n)^{-\alpha_1-\alpha_2-\cdots-\alpha_{n-1}}$

where $a_1, a_2 \cdots a_n$ are the regular singular points and $\alpha_1, \alpha_2, \cdots \alpha_{n-1}$ the smaller exponent at the respective singularities) we may reduce one exponent to zero and leave the other positive. Such a transformation does not fundamentally alter the character of the equation considered.

The formula is defective in one respect, it tells us nothing about the nature of the c essential singularities, and if we were to consider equations having large numbers of essential singularities, the formula would have to be very considerably extended. In the present paper, however, we shall deal almost solely with equations having a very small number of essential singularities, and as will be seen in the next section, a very slight modification will then enable the formula to give us the information we require. We may here so far anticipate the results of the paper as to say that the formula (a, b, c), where a, b, c are definite fixed integers represents all those equations having a regular singular points with exponent-difference $\frac{1}{2}$, b regular singular points with arbitrary exponent-differences, and c essential singularities, nothing being said as to the complexity of the latter. Thus we have grouped under one formula a set (in fact an infinite set) of equations differing from one another only in the nature of their essential singularities.

§2. The equations considered by Klein and Bôcher.*

We shall illustrate the preceding remarks by a consideration of the equations derived from $(5,0,0)$ by the coalescence of the singular points. The equation $(5,0,0)$ when the singular points are taken to be $\zeta = a_1, a_2, a_3, a_4$ and infinity, the exponents at the four finite singularities being $\alpha_r, \alpha_r + \frac{1}{2}$ ($r = 1, 2, 3, 4$ respectively) is

$$\frac{d^2u}{d\zeta^2} + \left\{ \sum_{r=1}^{r=4} \frac{\frac{1}{2} - 2\alpha_r}{\zeta - a_r} \right\} \frac{du}{d\zeta} + \left\{ \sum_{r=1}^{r=4} \frac{\alpha_r(\alpha_r + \frac{1}{2})}{(\zeta - a_r)^2} + \frac{A\zeta^2 + 2B\zeta + C}{\prod_{r=1}^{r=4}(\zeta - a_r)} \right\} u = 0 \quad \ldots (1)$$

wherein B and C are arbitrary, but A is conditioned by the fact that the exponent difference relative to the singular point at infinity is $\frac{1}{2}$. This condition gives us

$$A = \left(\sum_{r=1}^{r=4} \alpha_r \right)^2 - \sum_{r=1}^{r=4} \alpha_r^2 - \frac{3}{2} \sum_{r=1}^{r=4} \alpha_r + \frac{3}{6} \quad \ldots \ldots (2)$$

so that α_r being finite, A is finite.

To derive the equation $(3,1,0)$, let the singularity $\zeta = a_4$ move off to infinity. For B write $\frac{n(n+1)}{8} a_4$, for C write $\frac{h}{4} a_4$ and let $\alpha_1 = \alpha_2 = \alpha_3 = 0$. The equation thus obtained is

$$\frac{d^2u}{d\zeta^2} + \left\{ \sum_{r=1}^{r=3} \frac{\frac{1}{2}}{\zeta - a_r} \right\} \frac{du}{d\zeta} - \frac{n(n+1)\zeta + h}{4 \prod_{r=1}^{r=4} (\zeta - a_r)} u = 0 \quad \ldots \quad (3)$$

an algebraic form of Lamé's equation.

To obtain the equation $(1,2,0)$ let $a_1 = a_2 = 0$, $a_3 = a_4 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = \frac{1}{4}$ in equation (1). The points 0 and 1 now become regular singularities with exponent-difference other than $\frac{1}{2}$, the point at infinity remains

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* This section follows closely Whittaker and Watson, Modern Analysis §10.6.

When the two regular singular points in $(2, 0, 1_3)$ coalesce in the origin we have a new singularity with arbitrary exponent-difference, and the equation has the formula $(0, 1, 1_3)$. To obtain it, in (1) let $a_1 = a_2 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and $a_3 = a_4 \to \infty$, and we arrive at the equation

$$\zeta^2 \frac{d^2u}{d\zeta^2} + \zeta \frac{du}{d\zeta} + \tfrac{1}{4}(\zeta - n^2)u = 0$$

which is transformed by the substitution $\zeta = z^2$ into

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{d\zeta} + (z^2 - n^2)u = 0 \qquad \text{- - - - - -} \qquad (6)$$

or Bessel's equation.

Now let the three singularities a_3, a_4 and a_5 move off to infinity, and let $a_1 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. The equation $(1, 0, 1_4)$ which is thus obtained is

$$\zeta \frac{d^2u}{d\zeta^2} + \frac{1}{2}\frac{du}{d\zeta} + \frac{1}{4}\left(n + \tfrac{1}{2} - \tfrac{1}{4}\zeta\right)u = 0$$

which is transformed by the substitution $\zeta = z^2$ into

$$\frac{d^2u}{dz^2} + \left(n + \tfrac{1}{2} - \tfrac{1}{4}z^2\right)u = 0 \qquad \text{- - - - —} \qquad (7)$$

or Weber's equation.

Lastly, if $\alpha_r = 0$ and $a_r \to \infty$ $(r = 1, 2, 3, 4)$ we obtain the equation $(0, 0, 1_5)$, which is

$$\frac{d^2u}{d\zeta^2} + (B_1\zeta + C_1)u = 0$$

where B_1 and C_1 are new arbitrary constants, an equation given by Stokes. The substitutions

$$u = (B_1\zeta + C_1)^{1/2} v,$$

$$B_1\zeta + C_1 = \left(\tfrac{9}{4}B_1\right)^{1/3} z^{2/3}$$

reduce it to

$$z^2\frac{d^2v}{dz^2} + z\frac{dv}{dz} + \left(z^2 - \frac{1}{9}\right)v = 0$$

a particular case of Bessel's equation.

To sum up, from the equation (5, 0, 0) can be derived six types of equations, and no more, viz.

I	(3, 1, 0)	of the type of	Lamé's	equation	(3)
II	(1, 2, 0)	" "	Legendre's	"	(4)
III	$(2, 0, 1_3)$	" "	Mathieu's	"	(5)
IV	$(0, 1, 1_3)$	" "	Bessel's	"	(6)
V	$(1, 0, 1_4)$	" "	Weber's	"	(7)
VI	$(0, 0, 1_5)$	" "	Stokes'	"	(8)

In a later section of the paper (§ 4) the equations deduced from (6, 0, 0) will be given and will be seen to exhibit important relations with the above.

§3. The number of distinct equations which can be derived from the equation possessing r singular points with exponent-difference $\frac{1}{2}$.

Let $\zeta = a_1, a_2 \dots a_{r-1}$ and the point ∞ be the r singular points, and let the algebraically smaller exponent at the point $\zeta = a_p$ $(p = 1, 2 \dots r-1)$ be α_p. Then the equation $(r, 0, 0)$ thus particularised is

$$\frac{d^2u}{d\zeta^2} + \left\{ \sum_{p=1}^{p=r-1} \frac{\frac{1}{2} - 2\alpha_p}{\zeta - a_p} \right\} \frac{du}{d\zeta} + \left\{ \sum_{p=1}^{p=r-1} \frac{\alpha_p(\alpha_p + \frac{1}{2})}{(\zeta - a_p)^2} + \frac{A_0 + A_1\zeta + \dots + A_{r-3}\zeta^{r-3}}{\prod_{p=1}^{p=r-1} (\zeta - a_p)} \right\} u = 0 \quad \text{----(9)}$$

where, on account of the condition that the exponents relative to the singular point $\zeta = \infty$ differ by $\frac{1}{2}$,

$$A_{r-3} = \Big(\sum_{p=1}^{p=r-1} \alpha_p\Big)^2 - \Big\{ \sum_{p=1}^{p=r-1} \alpha_p^2 - \frac{r-2}{2} \sum_{p=1}^{p=r-1} \alpha_p + \frac{(r-2)(r-4)}{16} \quad \text{---(10)}$$

The constant A_{r-3} is therefore finite (when all α_p are finite and r is limited) and definite, but the remaining constants $A_0, A_1, \dots A_{r-4}$ are all quite arbitrary and do not depend upon the quantities α_p.

Now suppose that two of the regular singular points be made to coalesce, for example, suppose $a_2 = a_1$. The indicial equation relative to this new singular point is

$$\rho(\rho - 1) + (1 - 2\alpha_1 - 2\alpha_2)\rho + \alpha_1(\alpha_1 + \tfrac{1}{2}) + \alpha_2(\alpha_2 + \tfrac{1}{2}) + A = 0 \quad \text{----(11)}$$

where $$A = \frac{A_0 + A_1 a_1 + A_2 a_1^2 + \dots\dots + A_{r-3} a_1^{r-3}}{(a_1 - a_3)(a_1 - a_4) \dots (a_1 - a_{r-3})} . \quad \text{----(12)}$$

The exponent-difference at the singularity $\zeta = a_1$ is thus dependent on A, that is to say, on the constants $A_0, \dots A_{r-3}$, and is therefore arbitrary. The singularity, however, is still regular.

Now suppose that the q (>2) singular points $a_1 \dots a_4$ be caused to approach one another and ultimately to coalesce in a_1.

The new singular point a_1 has no indicial equation so that this point is an essential singularity.

We thus see that, as in the case of Klein and Bôcher, so also in the general case, the coalescence of two singularities with exponent-difference $\frac{1}{2}$ generates a singularity with arbitrary exponent-difference, and the coalescence of more than two such singularities gives rise to an essential singularity.

If of the singularities of the equation $(r, 0, 0)$ some be caused to coalesce in pairs, the equation degenerates into one with a smaller number of singular points, but the singular points may have exponent differences other than $\frac{1}{2}$. If no more than two of the original number of singular points coalesce in any one point, the degenerate equation will have no irregular singularities. If one pair only of the singular points coalesces, $(r,0,0)$ degenerates into $(r-2,1,0)$; if two pairs coalesce, it degenerates into $(r-4,2,0)$ and so on down, when r is even, to $(0,\frac{r}{2},0)$ where $\frac{r}{2}$ pairs have coalesced, or when r is odd to $(1,\frac{r-1}{2},0)$ where $\frac{r-1}{2}$ pairs have coalesced. Thus from an equation with r singular points, each with exponent-difference $\frac{1}{2}$ (and no other singular points) can be derived when r is even $\frac{r}{2}$ new equations, and when r is odd $\frac{r-1}{2}$ new equations, which have only regular singular points.

We have here assumed that no two pairs of singular points coalesce in one single point, and that the singular points which are not made to coalesce with one another remain distinct from each other and from the new singular points which arise. These stipulations are necessary to ensure that no essential singularity is introduced. We shall now proceed to remove these restrictions, and to enumerate the equations which may then arise. These equations will evidently all have at least one irregular singular point, and in the first place we shall confine ourselves to the case in which the equation has no more than one irregular singularity.

Such a singular point arises through the coalescence of three or more regular singular points all with exponent-difference $\frac{1}{2}$, the simplest case being that in which three such singular points have coalesced. Thus from the original equation $(r, 0, 0)$ is derived in the first place the equation $(r-3, 0, 1_3)$ having one essential singularity. Now let the regular singular points of the last equation coalesce in pairs to form regular singular points with arbitrary exponent-differences; we obtain in turn equations of formulae $(r-5, 1, 1_3)$, $(r-7, 2, 1_3)$ and so on, the last possibility being $(1, \frac{r}{2}-2, 1_3)$ when r is even, and $(0, \frac{r+1}{2}-2, 1_3)$ when r is odd. Hence the number of equations containing one essential singularity of the type considered, which can be derived from a given equation $(r, 0, 0)$ is $\frac{r}{2}-1$ when r is even and $\frac{r-1}{2}$ when r is odd.

In general, an essential singularity may arise through the coalescence of p $(\geqq 3)$ regular singular points with exponent-difference $\frac{1}{2}$

The equation of formula $(r, 0, 0)$ thus gives rise to an equation of formula $(r-p, 0, 1/p)$. Further equations arise out of the latter, for by causing one or more pairs out of the $r-p$ regular singular points it contains to coalesce, we may obtain in turn equations of formulae $(r-p-2, 1, 1/p)$, $(r-p-4, 2, 1/p)$ and so on, the ultimate possibility being $(0, \frac{r-p}{2}, 1/p)$ when $r-p$ is even, in all $\frac{r-p}{2}+1$ equations and $(1, \frac{r-p-1}{2}, 1/p)$ when $r-p$ is odd, in all $\frac{r-p+1}{2}$ equations.

If we consider all values of p from $p=3$ to $p=r$ we arrive at the conclusion that the total number of possible equations which may be derived from the equation of formula $(r, 0, 0)$ and which have one irregular singularity and only one, is

$$\left(\tfrac{r}{2}-1\right) + \left(\tfrac{r}{2}-1\right) + \left(\tfrac{r}{2}-2\right) + \left(\tfrac{r}{2}-2\right) + \cdots + 2+2+1+1 = \tfrac{r}{2}\left(\tfrac{r}{2}-1\right)$$

when r is even, and

$$\tfrac{r-1}{2} + \left(\tfrac{r-1}{2}-1\right) + \left(\tfrac{r-1}{2}-1\right) + \cdots \quad +2+2+1+1 = \left(\tfrac{r-1}{2}\right)^2$$

when r is odd.

In the same way we might proceed to enumerate the equations which have two irregular singular points and which may be derived from the equation of formula $(r, 0, 0)$ and so on. Since an irregular singularity is generated through the coalescence of not less than three regular singular points with exponent-difference $\frac{1}{2}$, the largest number of irregular singular points which any equation derived from the equation of formula $(r, 0, 0)$ can possess will be the greatest integer in $\frac{r}{3}$.

For any particular value of r we may proceed to enumerate all the equations which can be derived from $(r, 0, 0)$, but it is not possible to write down a formula enumerating them for general values of r. Any given equation, derived from $(r, 0, 0)$ is, as we have seen, characterised by the number of its regular singularities of exponent-difference $\frac{1}{2}$, the number of its regular singularities, and the number of its irregular singularities. If we count each of the regular singularities once or twice according as the exponent-difference is $\frac{1}{2}$ or other than $\frac{1}{2}$, (for in the latter case it may be considered as generated through the coalescence of two singularities of exponent difference $\frac{1}{2}$) and each of the irregular singularities three or more times according to the number of regular singularities of exponent-difference $\frac{1}{2}$ which might be considered as having gone to form it, and add these numbers, the total will be r. Hence the number of equations which may be derived from the equation whose formula is $(r, 0, 0)$ this equation itself being included, is the same as the number of partitions of the number r into any number of parts. This quantity, which is the coefficient of x^r in the product $(1+x)(1+x^2) \ldots (1+x^r)$ cannot, so far as is known, be expressed as a general formula. Using Euler's enumeration of partitions we may discover the number N of <u>new</u> equations which may be derived from the equation of formula $(r, 0, 0)$ as in the table:

r	5	6	7	8	9	10	11	....
N	6	10	14	21	29	41	55	....

§4. The ten linear differential equations of the second order which can be derived from that possessing six regular singular points with exponent-difference $\frac{1}{2}$.

Let the six given singular points be taken to be the points $\zeta = a_1, a_2, a_3, a_4, a_5$ and ∞, and let the exponent-pairs relative to these singular points be respectively $(\alpha_1, \alpha_1+\frac{1}{2})$, $(\alpha_2, \alpha_2+\frac{1}{2})$, $(\alpha_3, \alpha_3+\frac{1}{2})$, $(\alpha_4, \alpha_4+\frac{1}{2})$, $(\alpha_5, \alpha_5+\frac{1}{2})$ and $(\mu, \mu+\frac{1}{2})$. The solutions u of the differential equation can then be represented by the scheme

$$u = P\left\{ \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & \infty \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \mu \\ \alpha_1+\frac{1}{2} & \alpha_2+\frac{1}{2} & \alpha_3+\frac{1}{2} & \alpha_4+\frac{1}{2} & \alpha_5+\frac{1}{2} & \mu+\frac{1}{2} \end{array} \quad \zeta \right\} \quad \text{---- (13)}$$

and the differential equation, which has the formula $(6,0,0)$, in its most general form is

$$\frac{d^2u}{d\zeta^2} + \left\{ \sum_{r=1}^{r=5} \frac{\frac{1}{2}-2\alpha_r}{\zeta - a_r} \right\} \frac{du}{d\zeta} + \left\{ \sum_{r=1}^{r=5} \frac{\alpha_r(\alpha_r+\frac{1}{2})}{(\zeta-a_r)^2} + \frac{A\zeta^3+3B\zeta^2+3C\zeta+D}{\prod_{r=1}^{r=5}(\zeta-a_r)} \right\} u = 0 \quad \text{--- (14)}$$

provided that A is such that the roots of the quadratic equation

$$\rho^2 + \rho\left\{ 2\sum_{r=1}^{r=5}\alpha_r - \frac{3}{2} \right\} + \sum_{r=1}^{r=5}\alpha_r(\alpha_r+\tfrac{1}{2}) + A = 0$$

are μ and $\mu+\frac{1}{2}$. This equation is simply the indicial equation corresponding to the singular point $\zeta = \infty$. The condition that the roots of this equation differ by $\frac{1}{2}$ gives us the value of A, viz.

$$A = \left(\sum_{r=1}^{r=5}\alpha_r\right)^2 - \sum_{r=1}^{r=5}\alpha_r^2 - 2\sum_{r=1}^{r=5}\alpha_r + \frac{1}{2} \quad \text{----- (15)}$$

by means of which we may reduce the indicial equation to

$$\rho^2 + \rho\left\{ 2\sum_{r=1}^{r=5}\alpha_r - \frac{3}{2} \right\} + \left(\sum_{r=1}^{r=5}\alpha_r\right)^2 - \frac{3}{2}\sum_{r=1}^{r=5}\alpha_r^2 + \frac{1}{2} = 0 \quad \text{--- (16)}$$

The quantities B, C and D are numerical constants of a perfectly arbitrary nature.

We shall now proceed to investigate the equations which may be derived from the above differential equation by the coalescence of its singularities. First let the singularity $\zeta = a_5$ move off up to and ultimately coalesce with the singularity $\zeta = \infty$. The coefficient of the member $\frac{du}{d\zeta}$ of the differential equation becomes simply $\sum_{r=1}^{r=4} \frac{\frac{1}{2} - 2\alpha_r}{\zeta - a_r}$, since the term $\frac{\frac{1}{2} - 2\alpha_5}{\zeta - a_5}$ ultimately vanishes. The coefficient of the member u is

$$\sum_{r=1}^{r=4} \frac{\alpha_r(\alpha_r + \frac{1}{2})}{\zeta - a_r} + \frac{\alpha_5(\alpha_5 + \frac{1}{2})}{\zeta - a_5} + \frac{A\zeta^3 + 3B\zeta^2 + 3C\zeta + D}{(\zeta - a_5)\prod_{r=1}^{r=4}(\zeta - a_r)}.$$

The first term here remains unaltered, the second term vanishes, and in the third term $\frac{A}{\zeta - a_5}$ vanishes since in virtue of (15) A necessarily remains finite whilst a_5 becomes indefinitely great. We can, however, assign such value to B, C and D that the ratios $\frac{3B}{\zeta - a_5}$, $\frac{3C}{\zeta - a_5}$ and $\frac{D}{\zeta - a_5}$ become in the limit C_2, $2C_1$ and C_0 respectively. The differential equation now assumes the form

$$\frac{d^2u}{d\zeta^2} + \left(\sum_{r=1}^{r=4} \frac{\frac{1}{2} - 2\alpha_r}{\zeta - a_r}\right)\frac{du}{d\zeta} + \left\{\sum_{r=1}^{r=4} \frac{\alpha_r(\alpha_r + \frac{1}{2})}{(\zeta - a_r)^2} + \frac{C_2\zeta^2 + 2C_1\zeta + C_0}{\prod_{r=1}^{r=4}(\zeta - a_r)}\right\} u = 0 \quad \text{--- (17)}$$

Let us suppose that $\alpha_r = 0$ $(r = 1, 2, 3, 4)$, which is equivalent to supposing all the solutions of the equation to be multiplied by a factor of the form $\prod_{r=1}^{r=4}(\zeta - \alpha_r)^{-\alpha_r}$, and proceed to construct the indicial equation relative to the singularity $\zeta = \infty$. We find that it is

$$\rho^2 + \rho + C_2 = 0,$$

so that the exponent-difference relative to the singular point at infinity is now arbitrary, and depends only on C_2, or on the constant B of the original equation. C_2 may conveniently be taken to be $-n(n+1)$ so that the exponent-difference is $2n+1$. The remaining constants C_0 and C_1

are still quite arbitrary. Our equation, which has the formula $(4,1,0)$, now reads

$$\frac{d^2u}{d\zeta^2}+\left(\sum_{r=1}^{r=4}\frac{\frac{1}{2}}{\zeta-a_r}\right)\frac{du}{d\zeta}+\left\{\frac{-n(n+1)\zeta^2+2C_1\zeta+C_0}{\prod_{r=1}^{r=4}(\zeta-a_r)}\right\}u=0 \quad \cdots \quad (18)$$

which is an extended form of Lamé's equation having one singular point more than the latter. It can be reduced to the form

$$\frac{d^2u}{dz^2}+p(z)u=0$$

where $p(z)$ is a transcendental function analogous to the elliptic function of Lamé's equation, but only through the use of an Abelian integral of the form

$$\int^z\frac{dt}{\prod_{r=1}^{r=4}(t-a_r)}$$

Leaving this case, we pass on to the next, in which we suppose that in the degenerate equation (17) two of the finite singular points coalesce to form ~~an~~ one singular point with arbitrary exponent-difference q. For this purpose we may take $a_1=0$, $a_2=a_3=b$, $a_4=c$ and let $\alpha_1=\alpha_4=0$ so that the exponents at both of the singular points $\zeta=0$ and $\zeta=c$ are 0 and $\frac{1}{2}$. The exponents at $\zeta=b$ may be taken to be 0 and q, the conditions for which are

$$q=2(\alpha_2+\alpha_3)$$
$$0=\alpha_2(\alpha_2+\tfrac{1}{2})+\alpha_3(\alpha_3+\tfrac{1}{2})+\frac{C_2b^2+2C_1b+C_0}{b(b-c)}$$

by means of which relations the differential equation may be reduced to

$$\frac{d^2u}{d\zeta^2}+\left\{\frac{\frac{1}{2}}{\zeta}+\frac{1-q}{\zeta-b}+\frac{\frac{1}{2}}{\zeta-c}\right\}\frac{du}{d\zeta}+\left\{\frac{C'\zeta+c_0}{\zeta(\zeta-b)(\zeta-c)}\right\}u=0$$

where $C'=-\dfrac{b^2C_2+2bC_1+C_0}{b(b-c)}$

$$c_0=-\frac{C_0}{b}$$

The indicial equation relative to the singularity $\zeta=\infty$ is

$$\rho^2-(q-1)\rho+C'=0$$

C' is arbitrary, and may conveniently be taken to be $-\frac{1}{2}n(\frac{1}{2}n+1-q)$, so that the differential equation, which has the formula $(2,2,0)$, becomes

$$\zeta(\zeta-b)(\zeta-c)\frac{d^2u}{d\zeta^2}+\left\{\tfrac{1}{2}(\zeta-b)(\zeta-c)+(1-q)\zeta(\zeta-c)+\tfrac{1}{2}\zeta(\zeta-b)\right\}\frac{du}{d\zeta}$$
$$+\left\{-\tfrac{1}{2}n\left(\tfrac{1}{2}n+1-q\right)\zeta+C_0\right\}u=0 \quad - - - - (19)$$

which is an equation first introduced into analysis and studied by Whittaker.* Its standard form is

$$(b-c\sin^2 z)\frac{d^2u}{dz^2}-2(1-q)c\sin z\cos z\frac{du}{dz}+\left\{-n(n+2-2q)c\sin^2 z+4C_0\right\}u=0 \quad - - - (20)$$

which is obtained through the substitution $\zeta=c\sin^2 z$.

We shall now return to equation (17) and suppose that the singularities $\zeta=a_1$ and $\zeta=a_2$ coalesce to form a new singular point and that the singularities $\zeta=a_3$ and $\zeta=a_4$ likewise coalesce. The differential equation now degenerates into one of formula $(0,3,0)$. The two finite singular points of this new equation may be taken to be the points $\zeta=0$ and $\zeta=1$ respectively, so that the equation becomes

$$\frac{d^2u}{d\zeta^2}+\left\{\frac{1-2\alpha_1-2\alpha_2}{\zeta}+\frac{1-2\alpha_3-2\alpha_4}{\zeta-1}\right\}\frac{du}{d\zeta}+\left\{\frac{\alpha_1(\alpha_1+\frac{1}{2})+\alpha_2(\alpha_2+\frac{1}{2})}{\zeta^2}\right.$$
$$\left.+\frac{\alpha_3(\alpha_3+\frac{1}{2})+\alpha_4(\alpha_4+\frac{1}{2})}{(\zeta-1)^2}+\frac{C_2\zeta^2+2C_1\zeta+C_0}{\zeta^2(\zeta-1)^2}\right\}u=0$$

Now suppose that the exponents relative to the singularity $\zeta=0$ are 0 and $1-\gamma$, and that those relative to the singularity $\zeta=1$ are 0 and $\gamma-\alpha-\beta$. We are thus led to the conditions

* Proc. Edin. Math. Soc XXXIII (1914-15), p. 17.

$$1-\gamma = 2(\alpha_1+\alpha_2)$$
$$0 = \alpha_1(\alpha_1+\tfrac{1}{2}) + \alpha_2(\alpha_2+\tfrac{1}{2}) + C_0$$
$$\gamma-\alpha-\beta = 2(\alpha_3+\alpha_4)$$
$$0 = \alpha_3(\alpha_3+\tfrac{1}{2}) + \alpha_4(\alpha_4+\tfrac{1}{2}) + C_2 + 2C_1 + C_0$$

by means of which we can reduce the differential equation to

$$\frac{d^2u}{d\zeta^2} + \left\{\frac{\gamma}{\zeta} + \frac{(\alpha+\beta-\gamma+1)}{\zeta-1}\right\}\frac{du}{d\zeta} - \frac{2(C_0+C_1)}{\zeta(\zeta-1)}u = 0$$

The indicial equation relative to the singularity at infinity is

$$\rho^2 + (\alpha+\beta)\rho - 2(C_0+C_1) = 0$$

in which we can conveniently, and without any loss of generality, take $-2(C_0+C_1)$ to be $\alpha\beta$. We thus arrive at the ordinary hypergeometric differential equation

$$\zeta(\zeta-1)\frac{d^2u}{d\zeta^2} + \left\{-\gamma+(\alpha+\beta+1)\zeta\right\}\frac{du}{d\zeta} + \alpha\beta u = 0 \quad \text{-----} (20)$$

Since equations of formulae $(4,1,0)$, $(2,2,0)$ or $(0,3,0)$ are the only types of equations, having only regular singular points, which can be derived from the equation of formula $(6,0,0)$, it follows that we have exhausted all possibilities of equations having no essential singularities. We now turn to equations which have one essential singularity at infinity and shall find that there are in all six distinct types of such equations.

Suppose then that in equation (14) the singular points $\zeta = a_4$ and $\zeta = a_5$ both move off to infinity. The equation then becomes

$$\frac{d^2u}{d\zeta^2} + \left\{\sum_{r=1}^{r=3}\frac{\tfrac{1}{2}-2\alpha_r}{\zeta-a_r}\right\}\frac{du}{d\zeta} + \left\{\sum_{r=1}^{r=3}\frac{\alpha_r(\alpha_r+\tfrac{1}{2})}{(\zeta-a_r)^2} + \frac{A\zeta^3+3B\zeta^2+3C\zeta+D}{a_4a_5\prod_{r=1}^{r=3}(\zeta-a_r)}\right\}u = 0 \quad \text{---}(21)$$

A is finite and definite, but B, C and D are arbitrary parameters, and may be chosen in such manner that $\frac{3B}{a_4 a_5}$, $\frac{3C}{a_4 a_5}$ and $\frac{D}{a_4 a_5}$ have limiting values which are finite and equal to the three arbitrary constants C_2, $2C_1$ and C_0 respectively. We may also choose α_1, α_2 and α_3 to be all zero, so that the equation now reads

$$\frac{d^2u}{d\zeta^2} + \left\{ \sum_{r=1}^{r=3} \frac{\frac{1}{2}}{\zeta - a_r} \right\} \frac{du}{d\zeta} + \left\{ \frac{C_2\zeta^2 + 2C_1\zeta + C_0}{(\zeta-a_1)(\zeta-a_2)(\zeta-a_3)} \right\} u = 0 \quad \text{---} (22)$$

This equation differs from Lamé's equation in that the point at infinity is an essential singularity, and from Mathieu's equation in that it has three singular points with exponent-difference $\frac{1}{2}$, instead of only two such points. In the degenerate case of $C_2 = 0$, it actually becomes Lamé's equation. If we make the substitution $\zeta = sn^2(z, k)$ after having applied the homographic transformation which moves the singular points a_1 and a_2 to the points 0 and 1 while the point a_3 becomes $\frac{1}{k^2}$, we transform the equation into

$$\frac{d^2u}{dz^2} + 4k^2(C_0' + 2C_1' sn^2 z + C_2' sn^4 z)\, u = 0 \quad \text{----} \quad (23)$$

which exhibits ~~est~~ clearly its relation with Lamé's equation. It has the formula $(3, 0, 1_3)$.

Now suppose that, in (21) the singular points $\zeta = a_4$ and $\zeta = a_5$ both move off to infinity as in the case just discussed, and further let the points $\zeta = a_2$ and $\zeta = a_3$ coalesce in the point $\zeta = 1$, and let $\zeta = a_1$ become $\zeta = 0$ with $\alpha_1 = 0$; we then have the equation

$$\frac{d^2u}{d\zeta^2} + \left\{ \frac{\frac{1}{2}}{\zeta} + \frac{1 - 2\alpha_2 - 2\alpha_3}{\zeta - 1} \right\} \frac{du}{d\zeta} + \left\{ \frac{\alpha_2(\alpha_2 + \frac{1}{2}) + \alpha_3(\alpha_3 + \frac{1}{2})}{(\zeta-1)^2} + \frac{C_2\zeta^2 + 2C_1\zeta + C_0}{\zeta(\zeta-1)^2} \right\} u = 0$$

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* k would depend on a_1 and a_2 as well as upon a_3; the point at infinity remains fixed.

If the exponents relative to the singular point $\zeta = 1$ be taken to be 0 and ν, we have

$$\nu = 2(\alpha_2 + \alpha_3)$$

$$0 = \alpha_2(\alpha_2 + \tfrac{1}{2}) + \alpha_3(\alpha_3 + \tfrac{1}{2}) + C_2 + 2C_1 + C_0$$

and the differential equation becomes

$$\frac{d^2u}{d\zeta^2} + \left\{ \frac{\frac{1}{2}}{\zeta} + \frac{1-\nu}{\zeta - 1} \right\} \frac{du}{d\zeta} + \frac{C_2\zeta + C_0}{\zeta(\zeta-1)}\, u = 0 \quad \text{----- (24)}$$

This equation has the formula $(1, 1, 1_3)$: it is of a type hitherto uninvestigated. When $\nu = \frac{1}{2}$ it degenerates into Mathieu's equation $(2, 0, 1_3)$. The substitution $\zeta = \cos^2 z$ transforms it into

$$\frac{d^2u}{d\zeta^2} + (1 - 2\nu)\frac{\cos z}{\sin z}\frac{du}{dz} + (A + k^2\cos^2 z)\, u = 0 \quad \text{----- (25)}$$

where A has been written for $4C_0$ and k^2 for $-4C_2$.

Removing the second term by the substitution

$$u = \sin^{\nu - \frac{1}{2}} z \cdot v$$

we get

$$\frac{d^2v}{dz^2} + \left(a + k^2\cos^2 z + \frac{(\nu - \frac{1}{2})(\nu - \frac{3}{2})}{\sin^2 z}\right) v = 0 \quad \text{----- (26)}$$

where $a = A - (\nu - \frac{1}{2})^2$.

The two latter forms exhibit the relation of the equations of the type considered to Mathieu's equation

$$\frac{d^2u}{dz^2} + (a + k^2\cos^2 z)\, u = 0.$$

The two types of equation represented respectively by the formulae $(3,0,1_3)$ and $(1,1,1_3)$ are the only two which can be derived from the original equation $(6,0,0)$ and which have the simplest form of essential singularity, namely that which is formed by the coalescence of three regular singular points all with exponent-difference $\frac{1}{2}$. We shall now investigate the cases in which the three singular points $\zeta = a_3$, $\zeta = a_4$ and $\zeta = a_5$ all move off to infinity thus producing an irregular singular point which may be regarded as arising out of the coalescence of four regular singular points, all of exponent-difference $\frac{1}{2}$. The differential equation (14) in these circumstances becomes

$$\frac{d^2u}{d\zeta^2} + \left\{\frac{\frac{1}{2}-2\alpha_1}{\zeta-a_1} + \frac{\frac{1}{2}-2\alpha_2}{\zeta-a_2}\right\}\frac{du}{d\zeta} + \left\{\frac{\alpha_1(\alpha_1+\frac{1}{2})}{(\zeta-a_1)^2} + \frac{\alpha_2(\alpha_2+\frac{1}{2})}{(\zeta-a_2)^2} - \frac{A\zeta^3+3B\zeta^2+3C\zeta+D}{a_3a_4a_5(\zeta-a_1)(\zeta-a_2)}\right\}u = 0$$

In the limit $\frac{A}{a_3a_4a_5}$ vanishes, but B, C and D are arbitrary and may be so chosen that the ratios $-\frac{3B}{a_3a_4a_5}$, $-\frac{3C}{a_3a_4a_5}$ and $\frac{D}{a_3a_4a_5}$ remain finite and equal to say C_2, $2C_1$ and C_0 respectively.

Now suppose, in the first place, that the finite singularities a_1 and a_2 are 0 and 1 respectively, and that $\alpha_1 = \alpha_2 = 0$. We thus obtain the equation

$$\frac{d^2u}{d\zeta^2} + \left\{\frac{\frac{1}{2}}{\zeta} + \frac{\frac{1}{2}}{\zeta-1}\right\}\frac{du}{d\zeta} + \left\{\frac{C_0+2C_1\zeta+C_2\zeta^2}{\zeta(\zeta-1)}\right\}u = 0 \quad \text{---} \quad (27)$$

which possesses the formula $(2,0,1_4)$. Making the substitution $\zeta = \cos^2 z$ we have

$$\frac{d^2u}{dz^2} - 4\left\{C_0 + 2C_1\cos^2 z + C_2\cos^4 z\right\}u = 0$$

which may be written in the form

$$\frac{d^2u}{dz^2} + \left\{A - (n+1)l\cos 2z + \frac{1}{8}l^2\cos 4z\right\}u = 0 \quad \text{----} \quad (28)$$

This equation was first studied by Whittaker[*], who considered it as derived from the equation (2, 2, 0) by coalescence of the two singular points with arbitrary exponent-difference which the latter equation possesses.

In the second place suppose that the two finite singular points $\zeta = a_1$ and $\zeta = a_2$ coalesce in the origin, forming there a singularity with exponents $\frac{1}{2} - m$ and $\frac{1}{2} + m$. The conditions for this are

$$1 = 2(\alpha_1 + \alpha_2)$$

$$\frac{1}{4} - m^2 = \alpha_1(\alpha_1 + \tfrac{1}{2}) + \alpha_2(\alpha_2 + \tfrac{1}{2}) + C_0$$

and the differential equation, which now is of formula $(0, 1, 1_4)$ may be written

$$\frac{d^2u}{d\zeta^2} + \left\{ \frac{\frac{1}{4} - m^2 - C_0}{\zeta^2} + \frac{C_2\zeta^2 + 2C_1\zeta + C_0}{\zeta^2} \right\} u = 0.$$

Without loss of generality we may take $C_2 = -\frac{1}{4}$ and $2C_1 = k$ and so reduce the equation to the form

$$\frac{d^2u}{d\zeta^2} + \left\{ -\frac{1}{4} + \frac{k}{\zeta} + \frac{\frac{1}{4} - m^2}{\zeta^2} \right\} u = 0 \qquad \text{-----} \qquad (29)$$

which is none other than the equation of the confluent hypergeometric functions $W_{k,m}(\zeta)$.[†]

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[*] Proc. Edin. Math. Soc. XXXIII. (1914–15) p. 22

[†] Whittaker. Bull. Amer. Math. Soc X. p. 125.

Now let us suppose that the four singular points a_2, a_3, a_4 and a_5 of the original equation (14) move off to infinity and that the remaining singular point is transferred to the origin, and let $\alpha_1 = 0$. The differential equation thus obtained is

$$\frac{d^2u}{d\zeta^2} + \frac{1}{2\zeta}\frac{du}{d\zeta} + \frac{C_0 + 2C_1\zeta + C_2\zeta^2}{\zeta}u = 0 \qquad (30)$$

and has the formula $(1, 0, 1_5)$. The substitution $\zeta = z^2$ transforms it into

$$\frac{d^2u}{d\zeta^2} + 4\left\{C_0 + 2C_1 z^2 + C_2 z^4\right\}u = 0,$$

the origin now becoming an ordinary point. This equation is an extension of Weber's equation into which it degenerates when C_2 becomes zero.

When all the five finite singular points of the original equation coalesce with the point at infinity to form an irregular singular point at infinity the equation assumes the form

$$\frac{d^2u}{d\zeta^2} + (C_0 + 2C_1\zeta + C_2\zeta^2)u = 0$$

and has the formula $(0, 0, 1_6)$. A simple linear transformation reduces it to the form

$$\frac{d^2u}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)u = 0 \qquad (31)$$

which is Weber's equation.

We have now enumerated all the possible types of equation, deducible from the equation of formula $(6, 0, 0)$ which have only one irregular singular point. Only one further possibility ~~arises~~ remains, namely that in the original equation two singular points, say a_4 and a_5 move off to and coalesce with the singular point at infinity, to form an irregular singular point at infinity, while the remaining singular points a_1, a_2 and a_3 coalesce to form a second irregular singularity in some point in the finite part of the ζ-plane, say in the origin. The equation then becomes

$$\frac{d^2u}{d\zeta^2} + \frac{3}{2\zeta}\frac{du}{d\zeta} + \frac{A\zeta^3 + 3B\zeta^2 + 3C\zeta + D}{a_4 a_5 \zeta^3} u = 0$$

which in the limit reduces to an equation of the form

$$\zeta^3 \frac{d^2u}{d\zeta^2} + \frac{3}{2}\zeta\frac{du}{d\zeta} + \{C_0 + 2C_1\zeta + C_2\zeta^2\} u = 0 \quad \text{------} (33)$$

This is the simplest linear differential equation of the second order having two irregular singular points; its formula is $(0, 0, 2_3)$.

Thus from the original equation of formula $(6, 0, 0)$ we have deduced the following ten types, viz.

I $(4, 1, 0)$ an extended Lamé equation,

II $(2, 2, 0)$, an equation given by Whittaker,

III $(0, 3, 0)$ the ordinary hypergeometric equation,

IV. $(3, 0, 1_3)$

V $(1, 1, 1_3)$

VI $(2, 0, 1_4)$ an equation given by Whittaker

VII $(0, 1, 1_4)$ the equation of the confluent hypergeometric functions

VIII $(1, 0, 1_5)$ an extended Weber's equation

IX $(0, 0, 1_6)$ Weber's equation

X $(0, 0, 2_3)$ the simplest equation with two irregular singular points

This enumeration is evidently exhaustive. The types numbered II, III, VI, VII and IX have already been studied to a greater or less extend, the remaining five types appear to be new

§5. Equations which may be regarded as generalisations of equations of simpler type.

Among the equations of the last section occurred one of formula $(1,1,1_3)$ This equation was seen to be a generalisation of Mathieu's equation, which it included as a particular case. It has two regular singular points, one with exponent-difference $\frac{1}{2}$, the other with arbitrary exponent-difference. If the latter exponent-difference is given the value $\frac{1}{2}$, the formula becomes $(2,0,1_3)$, which represents Mathieu's equation. If, on the other hand, the singularity with exponent-difference $\frac{1}{2}$ be generalised so that its exponent-difference becomes also arbitrary, we arrive at an equation which is more general than either Mathieu's equation or the equation $(1,1,1_3)$. This equation is of formula $(0,2,1_3)$ and is a coalescent case of $(7,0,0)$ just as $(1,1,1_3)$ is a coalescent case of $(6,0,0)$. No further generalisation can be arrived at without either altering the nature of the irregular singularity or causing one or other of the regular singularities to become irregular.

These considerations lead us to the conclusion that equations of formulae (a,b,c), $(a-1,b+1,c)$, $(a-2,b+2,c)$ --- $(a-r,b+r,c)$ are members of a particular class and increase in generality as r is increased, provided always that the c irregular singularities are the same in all cases. For a diminution of r by unity implies that the particular value $\frac{1}{2}$ has been given to an exponent-difference which was previously arbitrary, and that the equation has lost in generality. An increase of r by unity implies a corresponding gain in generality. When the second term of the formula is zero, the equation is the simplest of its class; when the first term is zero, the equation is the most general. As the simpler equations are particular cases of the more general, the equations of the class represented by the above set of formulae will have, for every assigned a, b and c properties in common which are typical of the class. As an example, take the class of equations represented by the set of formulae $(4,0,0)$, $(3,1,0)$, $(2,2,0)$, $(1,3,0)$ and $(0,4,0)$. Taking the four regular singular points of the last (and most general) of these equations to be $z=0$, $z=b$, $z=c$ and infinity; and the corresponding exponents at the first three of these points to be 0 and p, 0 and q, 0 and r respectively, we may represent its solutions by the scheme

$$P\left\{\begin{matrix} 0 & b & c & \infty & \\ 0 & 0 & 0 & -\tfrac{1}{2}n & z \\ p & q & r & \tfrac{1}{2}n+2-p-q-r & \end{matrix}\right\}$$

When $p=\frac{1}{2}$, this is $(1,3,0)$; when $p=r=\frac{1}{2}$ it is $(2,2,0)$ an equation we have already come across (19); when $p=q=r=\frac{1}{2}$ it is $(3,1,0)$ or Lamé's equation, and when, in addition $n=0$ it is $(4,0,0)$ or Lamé's equation of zero order. These equations are all of the same nature, but are of increasing simplicity. Their "distinguished solutions" are the solutions of homogeneous integral equations of very close resemblance to each other[x].

An equation having regular singular points of exponent-difference $\frac{1}{2}$ can thus be generalised into a higher member of the same class by making the exponent-difference arbitrary. The question of generalisation by increasing the complexity of irregular singular points will be dealt with in a later section of the paper (§9)

[x] $(4,0,0)$ is, in a way, an exception owing to its extreme simplicity.

§6. Relations existing between classes of equations; generalised Weber-Hermite equation.

In the preceding section we saw that the equations which have a definite number of irregular singularities, fixed as to their complexity, and a definite number of regular singularities, form a class whose members have properties in common. We shall now consider the classes formed by those equations which have, firstly, a definite number $\underline{a}$ of regular singularities with exponent-difference $\frac{1}{2}$, secondly, a definite number $\underline{b}$ of regular singular points with arbitrary exponent-difference, and, thirdly, a definite number $\underline{c}$ of irregular singular points of unassigned complexity. The classes considered in the last section each contained as many distinct types of equation as each each equation contained regular singular points, but the classes now considered all contain an indefinite number of types of equation, since the irregular singular points are all of unassigned complexity and may therefore each be produced by the coalescence of any number of regular singular points of exponent-difference $\frac{1}{2}$. Different equations are always produced by the coalescence of different numbers of such regular singular points to produce an irregular singularity, and hence each class contains an infinite number of equations. Such equations are all typified by the formula (a, b, c), which as it stands, tells us the number $\underline{c}$ but not the nature of the irregular singularities and thus may be taken to represent the whole class of equations with which we are concerned. Furthermore, if we consider a second class of equations (a', b', c), the number $\underline{c}$ being the same as before, each member of this class always bears the same relation to the corresponding member of ~~the~~ (a, b, c). By the term <u>corresponding members</u>, we here mean the equations whose irregular singularities have all been built up in exactly the same way in the one case as in the other. These considerations become self-evident when we regard each irregular singular point as a limiting case of a certain number of regular singular points with exponent-difference $\frac{1}{2}$, which have moved up close together and ultimately coalesced. Corresponding equations may then be regarded as degenerating either from a single equation with all its singularities regular, or from a pair of such equations the difference between the number of the singular points being constant for each set of corresponding members. When $a + 2b = a' + 2b'$, these relations become particularly clear as then one equation of the corresponding pair (a, b, c) and (a', b', c) where $\underline{c}$ is definite, may be regarded as produced from the other by the coalescence of one or more pairs of regular singular points with exponent-difference $\frac{1}{2}$ to produce as many single regular singular points with arbitrary exponent-difference. Take for example the cases $(2, 0, 1)$ and $(0, 1, 1)$. The simplest pair, viz. $(2, 0, 1_3)$ and $(0, 1, 1_3)$ are respectively Mathieu's and Bessel's equations and the latter may be regarded as a coalescent case of the former, in which the two simple singularities have united in a single

§6. Relations existing between classes of equations; generalised Weber-Hermite equation.

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$$\frac{d^2u}{d\zeta^2} + \frac{1}{2\zeta}\frac{du}{d\zeta} + \frac{A_0 + A_1\zeta + A_2\zeta^2 + \cdots + A_{r-3}\zeta^{r-3}}{\zeta}u = 0 \quad \text{---} (34)$$

which is transformed by the substitution $\zeta = z^2$ into

$$\frac{d^2u}{dz^2} + 4\left(A_0 + A_1 z^2 + A_2 z^4 + \cdots + A_{r-3} z^{2r-6}\right)u = 0 \quad \text{---} (35)$$

whereas the latter is, (no generality being lost in omitting the terms in z in the coefficient of u)

$$\frac{d^2u}{dz^2} + \left(B_0 + B_2 z^2 + B_3 z^3 + \cdots + B_{p-4} z^{p-4}\right)u = 0 \quad \text{---} (36)$$

The coefficient of u in the last equation contains odd as well as even powers of z, at least when $p > 6$. In order to bring it into correspondence with the equation $(1, 0, 1_r)$ in the latter's transformed form (35) we must assign to p the value $2r-2$. Then all we can conclude is that for $r > 3$ the transformation $\zeta = z^2$ transforms the equation $(1, 0, 1_r)$ with independent variable ζ into that particular (not degenerate) case of $(0, 0, 1_{2r-2})$ which has only even powers of the independent variable z in the coefficient of its independent variable. The exceptional case $r = 3$ is the case of the Weber-Hermite equation.

§7. Equations with one irregular singular point (at infinity); Hill's equation and the higher Lamé equation.

The Weber-Hermite equation, Bessel's equation and Mathieu's equation are the three simplest and best-known members of the very extensive and important class of equations having one irregular singular point. The equations of this class are all characterised by the general formula $(a, b, 1)$, where a and b may have any positive integral values; if a and b be fixed, the formula represents a particular class of equation which may be expected to have certain characteristics peculiar to itself. As an example let us consider the equations grouped under the formula $(2, 0, 1)$. The particular equation corresponding to $(2, 0, 1_r)$ when the two regular points are taken to be $\zeta = 0$ and $\zeta = 1$ is

$$\frac{d^2u}{d\zeta^2} + \left\{ \frac{\frac{1}{2}}{\zeta} + \frac{\frac{1}{2}}{\zeta - 1} \right\} \frac{du}{d\zeta} + \left\{ \frac{A_0 + A_1\zeta + \cdots + A_{r-2}\,\zeta^{r-2}}{\zeta(\zeta-1)} \right\} u = 0 \quad (r \geqq 3) \qquad (37)$$

The substitution $\zeta = \cos^2 z$ transforms this equation into

$$\frac{d^2u}{dz^2} + 4\left\{ A_0 + A_1 \cos^2 z + \cdots + A_{r-2} \cos^{2(r-2)} z \right\} u = 0$$

which may be written in the form

$$\frac{d^2u}{dz^2} + \left(\Theta_0 + 2\Theta_1 \cos 2z + \cdots + 2\,\Theta_{r-2} \cos 2(r-2)z \right) u = 0 \qquad (38)$$

in which the constants Θ_0 are linearly dependent upon the constants A.

These equations are, for all values of r, particular cases of Hill's equation; their solutions, for different values of r closely resemble each other, and either of the three known general methods of solution* may be applied to all such equations

* 1) Hill's original method Acta Mathematica VIII (1886)
2) The method of change of parameter, Whittaker Proc Edin Math Soc XXXIII (1914). Ince M.N., R.A.S. LXXVI (1915) LXXVII (1916)
3) The Lindemann-Stieltjes method, Lindemann Math. Ann. XXII (1883), Stieltjes Ast: Nach CIX (1884)

The equations for $r = 3$ and $r = 4$ have (for particular values of the constant Θ_0) periodic solutions with period 2π identical with the periodic solutions of certain linear homogeneous integral-equations. This probably holds for $r > 4$, but no such integral equations have yet been discovered. Asymptotically, the equations differ radically so that the equations comprised in the formula $(2, 0, 1)$ may be classified according to the asymptotic behaviour of their solutions. This classification we shall consider with more generality in the next section.

As we saw in the preceding section, if the two regular points in each of the equations included in the formula $(2, 0, 1)$ coalesce (say in the origin, there forming a singularity with exponents $\frac{1}{2} \pm m$), the set of equations so obtained is defined by the formula $(0, 1, 1)$. The equation $(0, 1, 1_3)$ is easily transformed into Bessel's equation; $(0, 1, 1_4)$ is the equation of the confluent hypergeometric functions. Generally, the equation of formula $(0, 1, 1_r)$ is

$$z^2 \frac{d^2 u}{dz^2} + \left\{ \frac{1}{4} - m^2 + A_1 z + A_2 z^2 + \cdots + A_{r-2} z^{r-2} \right\} u = 0 \qquad (39)$$

which may be called the generalised Weber-Hermite equation of rank $r-2$. This equation bears much the same relation to $(2, 0, 1_r)$ – of which it is a coalescent case – as Bessel's equation $(0, 1, 1_3)$ does to Mathieu's equation $(2, 0, 1_3)$. Take for example the equation of the functions $W_{k,m}(z)$ which corresponds to the case of $r = 4$; viz.

$$z^2 \frac{d^2 W}{dz^2} + \left\{ \tfrac{1}{4} - m^2 + kz - \tfrac{1}{4} z^2 \right\} W = 0.$$

The substitution $W = z^{\frac{1}{2}} u$, which leaves unaltered the exponent-difference at the singularity $z=0$, and for which therefore the formula $(0, 1, 1_4)$ is invariant, transforms it into

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + \left\{ -m^2 + kz - \tfrac{1}{4} z^2 \right\} u = 0 \quad \text{-----} \quad (40)$$

This equation may be compared with

$$\frac{d^2 u}{d\zeta^2} + \left\{ \frac{\frac{1}{2}}{\zeta} + \frac{\frac{1}{2}}{\zeta - \alpha} \right\} - \frac{C - (l^2 + 2\overline{n+1}\, l)\frac{\zeta}{\alpha} + l^2 \frac{\zeta^2}{\alpha^2}}{4\zeta(\zeta - \alpha)} u = 0 \quad \text{- - -} (41)$$

of formula $(2, 0, 1_4)$ which is the algebraic form of

$$\frac{d^2 u}{dx^2} + \left\{ A - (n+1) l \cos 2x + \tfrac{1}{8} l^2 \cos 4x \right\} u = 0$$

with $\zeta = \alpha \cos^2 x$; $\quad C = A + (n+1) l + \frac{l^2}{8}$

When α tends to zero, but n and l are such that

$$\lim \frac{l^2}{\alpha^2} = 1, \quad \lim \frac{l^2 + 2(n+1) l}{4\alpha} = \frac{n+1}{2} = k, \quad \frac{C}{4} = m^2$$

and ζ such that $\frac{\zeta}{\alpha}$ is finite in the limit, the equation degenerates into the equation in z where z is written for $\lim \frac{\zeta}{\alpha}$, just as in similar circumstances Mathieu's equation degenerates into Bessel's.

Now just as the generalised Weber-Hermite equation is a coalescent form of ~~whittaker's may~~ Hill's equation, so Hill's equation is itself a coalescent form of what we may call the higher Lamé equation. This equation is of formula $(3, 0, 1)$ and has in common with the ordinary Lamé equation the property of having three regular singular points with exponent-difference $\frac{1}{2}$.

It differs from the ordinary Lamé equation in that its fourth singular point is irregular, whereas in Lamé's case the fourth singularity is regular, though in general with arbitrary exponent difference. The higher equation can, as we shall see, be regarded as a generalisation of the ordinary Lamé equation. Let us consider the case $(3, 0, 1_{r-1})$ and let us suppose that the three regular singular points are $\zeta = a_1$, $\zeta = a_2$ and $\zeta = a_3$, and that to each of these singular points correspond exponent 0 and $\frac{1}{2}$. The equation then is

$$\frac{d^2u}{d\zeta^2} + \left\{ \sum_{i=1}^{i=3} \frac{\frac{1}{2}}{\zeta - a_i} \right\} \frac{du}{d\zeta} + \left\{ \frac{A_0 + A_1\zeta + \cdots + A_{r-2}\zeta^{r-2}}{(\zeta - a_1)(\zeta - a_2)(\zeta - a_3)} \right\} u = 0 \quad \text{---(42)}.$$

a particular case of which (for $r = 4$) has already occurred in §4. Let a_3 become infinite, then the equation becomes

$$\frac{d^2u}{d\zeta^2} + \left\{ \frac{\frac{1}{2}}{\zeta - a_1} + \frac{\frac{1}{2}}{\zeta - a_2} \right\} \frac{du}{d\zeta} + \left\{ \frac{A_0' + A_1'\zeta + \cdots + A_{r-2}'\zeta^{r-2}}{(\zeta - a_1)(\zeta - a_2)} \right\} u = 0 \quad \text{---(43)}$$

where $A_p' = -\lim \dfrac{A_p}{a_3}$.

If we now apply the substitution $\zeta - a_1 = a_2 \cos^2 z$ we shall transform this equation into a case of Hill's equation. Thus just as Mathieu's equation is a confluent case of the ordinary Lamé equation, so is Hill's equation a confluent case of the extended Lamé equation. There is, however, this difference, that in the first case an essential singularity (necessarily of the simplest kind) is produced through the coalescence of two regular singular points, the one with arbitrary exponent-difference, the other with exponent-difference $\frac{1}{2}$, whereas in the second ~~[illegible]~~ case an essential singularity (of higher kind) is produced through the coalescence of a regular singular point with exponent-difference $\frac{1}{2}$ and an essential singularity.

Let us transform the independent variable by the homographic substitution

$$\frac{x}{x-1} = \frac{a_3 - a_1}{a_1 - a_2} \frac{1}{k^2} \frac{\zeta - a_1}{\zeta - a_2}$$

This substitution moves $\zeta = a_1$ to the origin of the x-plane and $\zeta = a_2$ to the point $x=1$, whilst $\zeta = a_3$ becomes $x = 1/k^2$. If now we make the substitution $x = sn^2(z, k)$ we transform the equation into one of the form

$$\frac{d^2u}{dz^2} + \left\{ B_0 + B_1 sn^2 z + \cdots + B_{r-2}\, sn^{2(r-2)} z \right\} u = 0 \quad \text{---} (44)$$

which reduces to Lamé's equation when $B_2 = B_3 = \cdots = B_{r-2} = 0$.

If, instead of causing one of the regular singularities of the equation $(3, 0, 1_{r-1})$ to coalesce with the irregular singularity, and thus to produce a form of Hill's equation, we were to cause two of the regular singularities to coalesce, we should arrive at an equation typified by the formula $(1, 1, 1_{r-1})$, a case of which has already occurred (V, § 4).

§8. Equations having the point at infinity as an essential singularity of the simplest type. The order of the corresponding asymptotic expansion.

When, in the preceding section, we considered certain types of equations which have an irregular singularity at infinity, we grouped these equations into classes, each member of a particular class ~~being~~ having a certain definite number of regular singular points with exponent-difference $\frac{1}{2}$, and a definite number of regular singular points with arbitrary exponent-difference. The equations of each class differed from one another solely and simply in the varying complexity of the singularity at infinity. We shall now approach these same equations from another point of view, and this time proceed to classify them according to the complexity of the singular point at infinity. The simplest class of this nature comprises all those equations which have the point at infinity as an irregular singular point formed through the coalescence of three regular singular points all with exponent-difference $\frac{1}{2}$, when all the other singularities are regular, the equations of this class have all a formula (p, q, l_3) where p and q may be any positive integers whatever (or zero). Let us for simplicity first of all suppose q to be zero. If the p regular singular points are the points $z = a_1, a_2, \ldots a_p$, the corresponding equation is

$$\frac{d^2u}{dz^2} + \left\{ \sum_{i=1}^{i=p} \frac{\frac{1}{2}}{z-a_i} \right\} \frac{du}{dz} + \left\{ \frac{A_0 + A_1 z + \cdots + A_{p-1} z^{p-1}}{\prod_{i=1}^{i=p} (z-a_i)} \right\} u = 0 \quad \ldots (45)$$

The constants A are undetermined by the conditions so far imposed upon them, though in order that the type may not degenerate (so that the point at infinity becomes a regular point) we must stipulate that $A_{p-1} \neq 0$.

To examine the asymptotic behaviour of this equation in the immediate neighbourhood of the essential singularity at infinity, we may regard z as having very large values, and therefore discard all but the terms of lowest order of smallness in the coefficients of $\frac{du}{dz}$ and u. The equation is then replaced by the asymptotic relation

$$\frac{d^2u}{dz^2} + \frac{p}{2z}\frac{du}{dz} + \frac{m^2}{z}u \sim 0 \quad \text{-----} \quad (46)$$

where $m^2 = A_{p-1} \neq 0$.

From this we deduce the asymptotic form of u, viz.

$$u \sim z^{-\frac{1}{4}-\frac{p}{4}} e^{\pm 2mi\sqrt{z} + O(\frac{1}{z})} \quad \text{-----} \quad (47)$$

where the symbol $O(\frac{1}{z})$ represents terms which vanish to the order of $\frac{1}{z}$ at least at $z=\infty$

Now let us slightly increase the generality of our conclusion, by considering equations of type (q, r, l_3). Let the q regular singularities with exponent-difference $\frac{1}{2}$ be seated at the points $z = a_1, \dots a_q$, and let the other r singular points be $z = b_1, \dots b_r$, with corresponding exponent-differences $\beta_1, \dots \beta_r$. The typical equation is

$$\frac{d^2u}{dz^2} + \left\{\sum_{i=1}^{i=q}\frac{\frac{1}{2}}{z-a_i} + \sum_{j=1}^{j=r}\frac{1-\beta_j}{z-b_j}\right\}\frac{du}{dz} + \frac{B_0 + B_1 z + \dots + B_{q+r-1}z^{q+r-1}}{\prod_{i=1}^{i=q}(z-a_i)\prod_{j=1}^{j=r}(z-b_j)}u = 0$$

where the constants B are arbitrary, but $B_{q+r-1} \neq 0$. Asymptotically, this equation becomes

$$\frac{d^2u}{dz^2} + \frac{\frac{q}{2}+r-\sum_{j=1}^{j=r}\beta_j}{z}\frac{du}{dz} + \frac{B_{q+r-1}}{z}u \sim 0$$

or, if we write $\frac{p}{2} = \frac{q}{2} + r - \sum_{j=1}^{j=r}\beta_j$

and $m^2 = B_{q+r-1}$

it takes the form

$$\frac{d^2u}{dz^2} + \frac{p}{2z}\frac{du}{dz} + m^2 u \sim 0$$

as before.

In fact, all equations of the type (a, b, l_3) in which the essential singular point is the point at infinity, are of the same asymptotic type no matter what values are given to the positive integers a and b. The leading term of the asymptotic expansion is of the form

$$z^{\alpha} e^{\pm\beta\sqrt{z}}$$

where α and β depend upon the constants of the equation.

Now the asymptotic expansion of a simple integral function of order $\frac{1}{2}$ is of the form[*]

$$z^{-\frac{1}{2}} e^{\pi\sqrt{z}} + O\left(\frac{1}{z}\right)$$

The factor $z^{-\frac{1}{2}}$ is of little consequence compared with the exponential term, which is of the same type (viz. $e^{\beta\sqrt{z}}$) as that in the asymptotic solutions of the equations we have been considering. We may therefore say that the asymptotic expansions of equations of type (a, b, l_3) are all of order $\frac{1}{2}$, and in dealing with the asymptotic solutions of equations having the point at infinity as an essential singularity of higher kind we may define, as the order of the expansion, the order of the simple integral function which has an asymptotic expansion whose exponential term tends to infinity as does the exponential term in the asymptotic solution of the differential equation.

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[*] Barnes, Phil. Trans (A) 199 (1902). p 477.

§9. Equations having the point at infinity as an essential singularity of higher type.

Let us take, first of all, the equation $(p, 0, 1_4)$, the p regular singularities with exponent-difference $\frac{1}{2}$ being situated at the points $a_1 \dots a_p$. The equation then reads

$$\frac{d^2u}{dz^2} + \left\{ \sum_{i=1}^{i=p} \frac{\frac{1}{2}}{z-a_i} \right\} \frac{du}{dz} + \left\{ \frac{A_0 + A_1 z + \dots + A_p z^p}{\prod_{i=1}^{i=p} (z-a_i)} \right\} u = 0 \quad \dots (48)$$

where $A_p \neq 0$, and asymptotically,

$$\frac{d^2u}{dz^2} + \frac{p}{2z}\frac{du}{dz} + m^2 u \sim 0 \qquad (49)$$

m^2 being written for A_p.

From this we deduce the asymptotic value of v, viz.

$$u \sim z^{-\frac{p}{4}} e^{\pm miz + O(\frac{1}{z})} \qquad (50)$$

This is of order 1. A similar asymptotic expansion will correspond to the equation $(p, q, 1_4)$.

Now let us turn to the general case in which the point at infinity is an essential singularity produced through the coalescence of a finite number r of regular singular points with exponent difference $\frac{1}{2}$. Let the remaining singular points p in number, be all regular, with exponent-difference $\frac{1}{2}$, and as before be seated at $z = a_1 \dots a_p$. The corresponding equation $(p, 0, 1_r)$ then takes the form

$$\frac{d^2u}{dz^2} + \left\{ \sum_{i=1}^{i=p} \frac{\frac{1}{2}}{z-a_i} \right\} \frac{du}{dz} + \left\{ \frac{A_0 + A_1 z + \cdots + A_{p+r-4} z^{p+r-4}}{\prod_{i=1}^{i=p} (z-a_i)} \right\} u = 0 \quad \text{---- (51)}$$

where $A_{p+r-4} \neq 0$. Asymptotically, the equation reads

$$\frac{d^2u}{dz^2} + \frac{p}{2z}\frac{du}{dz} + m^2 z^{r-4} u = 0 \quad \text{------ (52)}$$

where $m^2 = A_{p+r-4}$; from this we deduce

$$u \sim z^{1-\frac{p+r}{4}} e^{\pm \frac{2mi}{r-2} z^{\frac{r-2}{2}} + O(z^{\alpha})} \quad \text{-------- (53)}$$

where $\alpha = \frac{r}{2} - \frac{5}{2}$ when r is odd,

$\alpha = \frac{r}{2} - 3$ when r is even.

The asymptotic expansion is of order $\frac{r-2}{2}$. More generally, this is the order of the asympotic expansions relative to all equations (p, q, l_r).

We thus see that the order of the asymptotic expansion relative to an essential singularity at infinity depends only on the number of regular singular points with exponent difference $\frac{1}{2}$ which went to form that essential singularity, and is independent of the number of regular singular points in the finite part of the z-plane provided (as is always the case) that number be finite.

Chapter II. On the Nature of the Solutions of certain Types of Linear Differential Equations of the second Order.

§1. Introductory Remarks.

The considerations which we developed in the preceding chapter show us that linear differential equations of the second order may be classified according to the number and the nature of their singular points. The most important class of such equations contains those which have three singular points all of which are regular; the most general equation it includes is the ordinary hypergeometric equation. When the three singularities and the corresponding exponents are assigned, the equation is completely determined; a fact of great importance in the theory of these equations. The solutions of these equations are very simple, they can be expressed in the form of a family of definite integrals.

When, however we turn to equations of a higher type, that is to say, to equations having four or more regular singular points, or to those which are confluent cases of such equations, we encounter new difficulties. Even if all the singularities, with the corresponding exponents, are known, the equation which corresponds is not completely determined, for it contains an arbitrary parameter or several arbitrary parameters; and the solution in the form of a definite integral cannot be obtained in the usual way.

The present chapter deals with such equations; in the first place it treats of the equation (4, 0, 0) having four regular singular points with arbitrary exponents, and shows that the corresponding solutions in certain cases satisfy a homogeneous integral-equation. A general theorem embracing all equations having one arbitrary parameter is given which generalises all the known cases in which the solution of an equation is expressible as the solution of a homogeneous integral-equation. In these cases the integral equation gives only one solution of the differential equation considered; the question of the second solution is touched on at the end of the chapter.

§2. The Equation with four regular singular points having arbitrary exponents.

Let the four regular singular points be $z = a, b, c, d$ and let the corresponding exponents be α and α'; β and β'; γ and γ'; δ and δ' respectively. These exponents are not all independent, but are subject to the condition $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'+\delta+\delta' = 2$. The corresponding differential equation is

$$\frac{d^2u}{dz^2} + \left\{\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} + \frac{1-\delta-\delta'}{z-d}\right\}\frac{du}{dz}$$
$$+ \frac{1}{(z-a)(z-b)(z-c)(z-d)}\left\{\frac{\alpha\alpha'(a-b)(a-c)(a-d)}{z-a} + \frac{\beta\beta'(b-a)(b-c)(b-d)}{z-b}\right.$$
$$\left. + \frac{\gamma\gamma'(c-a)(c-b)(c-d)}{z-c} + \frac{\delta\delta'(d-a)(d-b)(d-c)}{z-d} + C_0\right\}u = 0, \quad \text{---} (54)$$

C_0 being an arbitrary constant, and its solutions may be represented by the scheme

$$u = P\left\{\begin{matrix} a & b & c & d & \\ \alpha & \beta & \gamma & \delta & z \\ \alpha' & \beta' & \gamma' & \delta' & \end{matrix}\right\} \quad \text{-- -- -- --} (55)$$

If we make the substitution $y = (z-a)^{-\alpha}(z-b)^{-\beta}(z-c)^{-\gamma}(z-d)^{\alpha+\beta+\gamma}u$, we arrive at an equation in which one of the exponents at each of the three singular points a, b, c is zero, and whose solutions are therefore represented by the scheme

$$y = P\left\{\begin{matrix} a & b & c & d & \\ 0 & 0 & 0 & \delta & z \\ p & q & r & 2-p-q-r-\delta & \end{matrix}\right\}$$

where $p = \alpha'-\alpha$, $q = \beta'-\beta$, $r = \gamma'-\gamma$. It is convenient to take δ to be of the form $-\frac{1}{2}n$. By a simple homographic transformation we might transfer one of the singularities to the origin and another to infinity, so that there is really no loss of generality in taking $a=0$, $d=\infty$.

Let us therefore consider the equations whose solutions are contained in the scheme

$$y = P\left\{\begin{matrix} 0 & b & c & \infty & \\ 0 & 0 & 0 & -\frac{1}{2}n & z \\ p & q & r & +\frac{1}{2}n+2-p-q-r & \end{matrix}\right\} \quad \text{-- -- --} (56)$$

it is (cf. Ch I. §4)

$$z(z-b)(z-c)\frac{d^2y}{dz^2}+\left\{(1-p)(z-b)(z-c)+(1-q)z(z-c)+(1-r)z(z-b)\right\}\frac{dy}{dz}$$

$$+\left\{-\tfrac{1}{2}n\left(\tfrac{1}{2}n+2-p-q-r\right)z+c_0\right\}y=0 \quad \text{----} \quad (57)$$

This equation reduces, when $p=r=\frac{1}{2}$ to an equation given by Whittaker[x] and when $p=q=r=\frac{1}{2}$ to the algebraic form of Lamé's equation of order n. The substitution $z=c\sin^2 x$ transforms it into

$$(b-c\sin^2 x)\frac{d^2y}{dx^2}+\left\{(1-p)(b-c\sin^2 x)\frac{\cos x}{\sin x}-2c(1-q)\sin x\cos x-(1-2r)(b-c\sin^2 x)\frac{\sin x}{\cos x}\right\}\frac{dy}{dx}$$

$$+\left\{-n(n+4-2p-2q-2r)c\sin^2 x+4c_0\right\}y=0 \quad \text{----} \quad (58).$$

On analogy with Whittaker's solution of the equation with $p=r=\frac{1}{2}$ we shall investigate the possibility of the existence of a solution of the form

$$y(x)=\int_0^{2\pi}\left(\sqrt{b-c}\sin x\sin s+\sqrt{b}\cos x\cos s\right)^n w(s)\,ds \quad \text{----} \quad (59)$$

where $w(s)$ is periodic in s, with period 2π, but otherwise at present undetermined.

Substituting this expression for y in the above equation (58) and writing, for brevity, U to represent $(\sqrt{b-c}\sin x\sin s+\sqrt{b}\cos x\cos s)$, we find that the left-hand side of the equation becomes

$$\int_0^{2\pi}ds\;w(s)\,U^n\left[\frac{n(n-1)(b-c\sin^2 x)(b-c\sin^2 s)}{U^2}-n^2(b-c\sin^2 x)\right.$$

$$+\frac{\sqrt{b-c}\cos x\sin s-\sqrt{b}\sin x\cos s}{U}\left\{n(1-2p)(b-c\sin^2 x)\frac{\cos x}{\sin x}\right.$$

$$\left.-2nc(1-q)\sin x\cos x-n(1-2r)(b-c\sin^2 x)\frac{\sin x}{\cos x}\right\}$$

$$\left.-n(n+4-2p-2q-2r)c\sin^2 x+4c_0\right] \quad \text{----} \quad (60)$$

Now taking into consideration the periodicity of $w(s)$ we easily find that

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[x] Proc. Edin. Math. Soc XXXIII (1914-15) p. 17.

$$\int_0^{2\pi} ds\, w(s)\, U^n \left[\frac{n(n-1)(b-c\sin^2 x)(b-c\sin^2 s)}{U^2} - n^2(b-c\sin^2 x)\right]$$

$$= \int_0^{2\pi} ds\, w(s) \left[(b-c\sin^2 s)\frac{\partial^2}{\partial s^2}U^n + cn^2(\sin^2 x - \sin^2 s)\, U^n\right]$$

$$= \int_0^{2\pi} ds\, U^n \Big[(b-c\sin^2 s)\frac{d^2 w(s)}{ds^2} - 4c\sin s\cos\, \frac{d\, w(s)}{ds}$$

$$+ \left\{-2c(\cos^2 s - \sin^2 s) + cn^2(\sin^2 x - \sin^2 s)\right\} w(s)\Big] \text{---}(61)$$

$$\int_0^{2\pi} ds\, w(s)\; U^n n(b-c\sin^2 x)\frac{\cos x}{\sin x}\;\frac{\sqrt{b-c}\cos x\sin s - \sqrt{b}\sin x\cos s}{U}$$

$$= \int_0^{2\pi} ds\, w(s)\; U^n \left[n(b-c\sin^2 s)\frac{\cos s}{\sin s}\;\frac{\sqrt{b-c}\cos s\sin x - \sqrt{b}\sin s\cos x}{U} + nc(\sin^2 x - \sin^2 s)\right]$$

$$= \int_0^{2\pi} ds\, w(s)\left[(b-c\sin^2 s)\frac{\cos s}{\sin s}\frac{\partial}{\partial s}U^n + nc(\sin^2 x - \sin^2 s)\, U^n\right]$$

$$= \int_0^{2\pi} ds\, U^n \Big[-(b-c\sin^2 s)\frac{\cos s}{\sin s}\;\frac{d\,w(s)}{ds} + \Big\{ 2c\,\cancel{\sin}\cos^2 s + (b-c\sin^2 s)$$

$$+ (b-c\sin^2 s)\frac{\cos^2 s}{\sin^2 s} + nc(\sin^2 x - \sin^2 s)\Big\} w(s)\Big]$$

$$\text{-----}(62)$$

$$\int_0^{2\pi} ds\, w(s)\; U^n n\; \sin x\cos x\; \frac{\sqrt{b-c}\cos x\sin s - \sqrt{b}\sin x\cos s}{U}$$

$$= \int_0^{2\pi} ds\, w(s)\; U^n \left[n\sin s\cos\; \frac{\sqrt{b-c}\cos s\sin x - \sqrt{b}\sin s\cos x}{U} + n(\sin^2 s - \sin^2 x)\right]$$

$$= \int_0^{2\pi} ds\, w(s)\left[\sin s\cos \frac{\partial}{\partial s}U^n + n(\sin^2 s - \sin^2 x)\right]$$

$$= \int_0^{2\pi} ds\, \cancel{w}\, U^n\Big[-\sin s\cos\; \frac{d\,w(s)}{ds} + \left\{ (\sin^2 x - \cos^2 s) + n(\sin^2 s - \sin^2 x)\right\} w(s)\Big]$$

$$\text{-----}(63)$$

$$\int_0^{2\pi} ds\, w(s)\, U^n n(b-c\sin^2 x)\frac{\sin x}{\cos x}\;\frac{\sqrt{b-c}\cos x\sin s - \sqrt{b}\sin x\cos s}{U}$$

$$= \int_0^{2\pi} ds\, w(s)\; U^n \left[n(b-c\sin^2 s)\frac{\sin s}{\cos s}\;\frac{\sqrt{b-c}\cos s\sin x - \sqrt{b}\sin s\cos x}{U} + nc(\sin^2 s - \sin^2 x)\right]$$

$$= \int_0^{2\pi} ds\, w(s)\left[(b-c\sin^2 s)\frac{\sin s}{\cos s}\frac{\partial}{\partial s}U^n + nc(\sin^2 s - \sin^2 x)\, U^n\right]$$

$$= \int_0^{2\pi} ds\, U^n\Big[-(b-c\sin^2 s)\frac{\sin s}{\cos s}\;\frac{d\,w(s)}{ds} + \Big\{ 2c\sin^2 s - (b-c\sin^2 s)$$

$$- (b-c\sin^2 s)\frac{\sin^2 s}{\cos^2 s} + nc(\sin^2 s - \sin^2 x)\Big\} w(s)\Big]$$

$$\text{-----}(64)$$

By the use of these formulae we now reduce the integral (60) to

$$\int_0^{2\pi} U^n ds \Big[(b - c\sin^2 s)\frac{d^2 w(s)}{ds^2} - 4c \sin s \cos \frac{d\,w(s)}{ds}$$

$$+ \{-2c(\cos^2 s - \sin^2 s) + cn(\sin^2 x - \sin^2 s)\}\, w(s)$$

$$- (1-2p)(b - c\sin^2 s)\frac{\cos s}{\sin s}\frac{d\,w(s)}{ds} + (1-2p)\Big\{ 2c\cos^2 s + (b - c\sin^2 s) + (b - c\sin^2 s)\frac{\cos^2 s}{\sin^2 s}$$

$$+ nc(\sin^2 x - \sin^2 s)\Big\}\, w(s) \quad + 2c(1-q)\sin s \cos s \frac{d\,w(s)}{ds}$$

$$+ 2c(1-q)\{(\cos^2 s - \sin^2 s) + n(\sin^2 x - \sin^2 s)\}\, w(s)$$

$$+ (1-2r)(b - c\sin^2 s)\frac{\sin s}{\cos s}\frac{d\,w(s)}{ds} + (1-2r)\Big\{-2c\sin^2 s + (b - c\sin^2 s) + (b - c\sin^2 s)\frac{\sin^2 s}{\cos^2 s}$$

$$+ nc(\sin^2 x - \sin^2 s)\Big\}\, w(s) + \{-n(n+4-2p-2q-2r)c\sin^2 s + 4c_0\}\, w(s) \Big]$$

$$= \int_0^{2\pi} U^n ds \Big[(b - c\sin^2 s)\frac{d^2 w(s)}{ds^2} + \Big\{ -(1-2p)(b - c\sin^2 s)\frac{\cos s}{\sin s}$$

$$-2c(1+q)\sin s\cos s + (1-2r)(b - c\sin^2 s)\frac{\sin s}{\cos s}\Big\}\frac{d\,w(s)}{ds}$$

$$+ \Big\{ (1-2p)(b - c\sin^2 s)\frac{\cos^2 s}{\sin^2 s} + (1-2r)(b - c\sin^2 s)\frac{\sin^2 s}{\cos^2 s}$$

$$+ 2(1-p-r)(b - c\sin^2 s) + 2c(1-q-2p)\cos^2 s - 2c(1-q-2r)\sin^2 s$$

$$- n(n+4-2p-2q-2r)c\sin^2 s + 4c_0\Big\}\, w(s) \Big]$$

The integral (60) therefore vanishes identically provided $w(s)$ is a periodic solution of the differential equation

$$(b - c\sin^2 s)\frac{d^2 w}{ds^2} + \Big\{-(1-2p)(b - c\sin^2 s)\frac{\cos s}{\sin s} - 2c(1+q)\sin s\cos s + (1-2r)(b - c\sin^2 s)\frac{\sin s}{\cos s}\Big\}\frac{dw}{ds}$$

$$+ \Big\{(1-2p)(b - c\sin^2 s)\frac{\cos^2 s}{\sin^2 s} + (1-2r)(b - c\sin^2 s)\frac{\sin^2 s}{\cos^2 s}$$

$$+ 2(1-p-r)(b - c\sin^2 s) + 2c(1-q-2p)\cos^2 s - 2c(1-q-2r)\sin^2 s$$

$$- n(n+4-2p-2q-2r)c\sin^2 s + 4c_0\Big\}\, w = 0 \quad \text{- - - (65)}$$

In order to compare this equation with (58) we apply to it the substitution which transforms the coefficient of $\frac{dw}{ds}$ into the same form as the coefficient of $\frac{dy}{dx}$ in (58). This substitution is

$$w(s) = \sin^{1-2p} s\,(b - c\sin^2 s)^{-q} \cos^{1-2r} s \quad y(s)$$

which reduces the above equation to

$$(b - c\sin^2 s)\frac{d^2y}{ds^2} + \left\{(1-2p)(b-c\sin^2 s)\frac{\cos s}{\sin s} - 2c(1-q)\sin s\cos s - (1-2r)(b-c\sin^2 s)\frac{\sin s}{\cos s}\right\}\frac{dy}{ds}$$
$$+ \left\{-n(n+4-2p-2q-2r)\,c\sin^2 s + 4c_0\right\} y = 0$$

which, except for the difference of the variable, is exactly the same equation as (58)

Consequently, if $y(s)$ is a solution of this equation, then the integral

$$\int_0^{2\pi} (\sqrt{b-c}\,\sin x \sin s + \sqrt{b}\,\cos x \cos s)^n \sin^{1-2p} s\,(b - c\sin^2 s)^{-q} \cos^{1-2r} s \; ds$$

supposed convergent, is a solution of (58). This integral is evidently a periodic function of x with period 2π. But we know* that equation (58) cannot have more than one periodic solution for any one value of c_0, this integral must therefore be the same function of x (save for a constant multiplier λ) as $y(s)$ is of s.

We have therefore proved that <u>the periodic solutions of equation (58) are identical with the periodic solutions of the homogeneous integral equation</u>

$$y(x) = \lambda \int_0^{2\pi} (\sqrt{b-c}\,\sin x \sin s + \sqrt{b}\,\cos x \cos s)^n \sin^{1-2p} s\,(b - c\sin^2 s)^{-q} \cos^{1-2r} s \; y(s)\, ds \quad \text{---} \quad (66)$$

This integral equation is a generalised form of Whittaker's integral equation solution for the case of $p = r = \frac{1}{2}$. It is itself a particular case of the general theorem dealt with in the following section.

* Floquet; Ann. Ec. Norm (2) XIII.

§3. A general theorem expressing the solutions of certain differential equations involving arbitrary parameters as solutions of a homogeneous integral equation.

Consider the linear differential equation of the second order

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + \{A+q(x)\}y = 0 \quad \text{- - - - - - (67)}$$

where $p(x)$ and $q(x)$ are finite, continuous functions of the variable x for the domain of values to be considered, and A is a numerical constant which may be either arbitrary (as in Lamé's equation) or, less generally, definitely assigned (as in the equations of the Legendre functions of order n). Let $K(x,s)$ be a solution of the partial differential equation

$$\frac{\partial}{\partial x}\left\{p(x)\frac{\partial y}{\partial x}\right\} - \frac{\partial}{\partial s}\left\{p(s)\frac{\partial y}{\partial s}\right\} + \{q(x) - q(s)\}y = 0 \quad \text{- - - - (68)}$$

where p and q are as before, and let $I(x)$ denote the definite integral $\int_a^b \overline{K}(x,s)\,v(s)\,ds$ supposed convergent, where $v(s)$ is, as yet, an arbitrary function of s differentiable in (a,b). We then have

$$\frac{d}{dx}\left\{p(x)\frac{dI(x)}{dx}\right\} + \{A+q(x)\}I(x)$$

$$= \int_a^b ds\, v(s)\left[\frac{\partial}{\partial x}\left\{p(x)\frac{\partial K(x,s)}{\partial x}\right\} + A\,K(x,s) + q(x)\,K(x,s)\right]$$

$$= \int_a^b ds\, v(s)\left[\frac{\partial}{\partial s}\left\{p(s)\frac{\partial K(x,s)}{\partial s}\right\} + A\,K(x,s) + q(s)\,K(x,s)\right]$$

in virtue of (68)

$$= \int_a^b ds\left[v(s)\frac{\partial}{\partial s}\left\{p(s)\frac{\partial K(x,s)}{\partial s}\right\} - K(x,s)\frac{\partial}{\partial s}\left\{p(s)\frac{\partial v(s)}{\partial s}\right\}\right]$$

(where $v(s)$ has now been so chosen as to be a solution of the equation

$$\frac{d}{ds}\left\{p(s)\frac{dv}{ds}\right\} + \{A+q(s)\}v = 0.)$$

$$= \left[v(s)\, p(s) \frac{\partial K(x,s)}{\partial s} - K(x,s)\, p(s) \frac{\partial v(s)}{\partial s} \right]_a^b$$

If now the limits of integration are so chosen that this quantity vanishes identically, then we may at once conclude that $I(x)$ is a solution of equation (67). The most obvious case which arises is that wherein $v(s)$ and $p(s)$ are periodic functions of s, ~~though not~~ with period $(b-a)$ and $K(x,s)$ is similarly periodic in s, though not necessarily periodic in x. When $v(s)$ is the same function of s (save for a constant multiplier) as $I(x)$ is of x (for which ~~A~~ $K(x,s)$ must be appropriately chosen) we have the integral equation solutions of (67) of which the integral equation of the preceding section is an example.

Apart from this, we may note that differential equations which differ essentially from each other only in the parameter A have associated with them the <u>same</u> nucleus or set of nuclei K. Thus the equation of the elliptic cylinder function $ce_n(z,k)$ differs only in its parametric term from the equation of the Bessel function $J_n(ik\cos z)$ (k indefinitely small, but $k\cos z$ finite) and thus the nuclei in these two cases are essentially the same. We have in fact

$$ce_n(z) = \text{const} \times \int_0^{2\pi} \cosh(k\cos z\cos s)\; ce_n(s)\, ds \quad --- (69)$$

$$\text{and } J_n(ik\cos z) = \text{const} \times \int_0^{\pi} \cosh(k\cos z\cos s)\; \cos ns\, ds \quad --- (70),$$

$(n \text{ even})$

where the nucleus $\cosh(k\cos z\cos s)$ is the same in both cases.

§4. The Second solutions of the equations of the preceding section – the second solutions of Lamé's equation.

In the preceding section, $v(s)$ was taken to be the same function of s as $I(x)$ was of x. In this manner we arrive at a homogeneous integral equation whose characteristic solutions are the same as the distinguished solutions of the corresponding differential equation. Having now obtained $I(x)$ and therefore knowing $v(s)$ let us substitute for the nucleus $K(x,s)$ we have been considering, any other function $L(x,s)$ which satisfies the same partial differential equation (68).

The integral $\int_a^b L(x,s)\,v(s)\,ds$, if convergent will be a solution of the original differential equation (67) provided, as before, that

$$\left[v(s)\,p(s)\,\frac{\partial L(x,s)}{\partial s} - L(x,s)\,p(s)\,\frac{\partial v(s)}{\partial s}\right]_a^b$$

vanishes. We thus conclude that when a solution of the differential equation (67) is given by an integral equation with nucleus $K(x,s)$ as defined in §3, a second solution is given by the definite integral

$$\int_a^b L(x,s)\,v(s)\,ds \qquad (71)$$

where $L(x,s)$ satisfies the same partial differential equation as does $K(x,s)$.

For example, the differential equation of Lamé

$$\frac{d^2y}{dx^2} = \{n(n+1)k^2 sn^2 x + A\} y \quad \text{--- (72)}$$

is of the type considered; the corresponding partial differential equation leading to the functions $K(x,s)$ and $L(x,s)$ is

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial s^2} = n(n+1)\, k^2 (sn^2 x - sn^2 s) K \quad \text{--- (73)}$$

The solution $P_n(k\, sn\, x\, sn\, s)$ of this equation (P_n being the Legendre function) leads, as was shewn by Whittaker* to the doubly-periodic solutions of Lamé's equation, that is to say, to the Lamé functions of the first kind $E_n(x)$ which thus coincide with the characteristic solutions of the homogeneous integral equation

$$E_n(x) = \lambda \int_0^{4K} P_n(k\, sn\, x\, sn\, s)\; E_n(s)\, ds \quad \text{--- (74)}$$

Now the solution $Q_n(k\, sn\, x\, sn\, s)$ of (72) also exists and gives rise to a distinct set of Lamé functions. For since both $Q_n(k\, sn\, x\, sn\, s)$ and $E_n(s)$ are periodic in s with the real period $4K$ we may take the limits of integration in $\int_a^b L(x,s)\, v(s)\, ds$ to be 0 and $4K$, and deduce that the solutions of Lamé's equation which are not rational in either of $sn\, x$, $cn\, x$ or $dn\, x$ and which, together with the solution $E_n(x)$ make up the complete primitive are given by the definite integral

$$y(x) = \text{Const} \times \int_0^{4K} Q_n(k\, sn\, x\, sn\, s)\; E_n(s)\, ds \quad \text{--- (75)}$$

* Proc. L.M.S. (2) vol 14 (1914) p.262.

§ 5. Relations of the exceptional second-solutions to the general solutions involving an arbitrary parameter; the second solutions of Mathieu's equation.

In considering the equation

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + \{A + q(x)\}\,y = 0,$$

we occasionally find it convenient to express the arbitrary parameter A as a function $\lambda(\sigma)$ of an auxiliary parameter σ. If a solution of this equation is then $y = y(x,\sigma_1)$, a second solution will be $y = y(x,\sigma_2)$ where σ_1 and σ_2 are distinct roots of the equation $\lambda(\sigma) = A$, such that $y(x,\sigma_1)$ and $y(x,\sigma_2)$ are distinct functions. Generally speaking, two such distinct functions y can be obtained, but it may happen, for particular values of the parameter A, that all the roots σ of the equation $\lambda(\sigma) = A$ lead to the same function $y(x,\sigma)$. For example, the general solution of Mathieu's equation

$$\frac{d^2y}{dx^2} + (\lambda(\sigma) + k^2\cos^2 x)\,y = 0$$

is $A\,e^{\mu(\sigma)x}\,u(x,\sigma) + B\,e^{\mu(-\sigma).x}\,u(x,-\sigma)$

when $\sigma = 0$, $\mu(\sigma)$ vanishes and $u(x,0) = u(x,-0)$, so that the general solution reduces to a single solution.

We shall now deal with the second solution in these cases.

Let a solution of the equation

$$\frac{d}{dx}\left\{ p(x)\frac{dy}{dx}\right\} + \left\{\lambda(\sigma) + q(x)\right\} y = 0 \quad \text{-------} (76)$$

be $y = y(x,\sigma)$, and let σ_1 be a value of σ for which the general solution breaks down. Supposing that the root σ_1 of $\lambda(\sigma) = A$ is isolated, we consider the value $\sigma_1 + \delta\sigma$ of σ, to which corresponds a solution $y = y(x, \sigma_1 + \delta\sigma)$; we have

$$\frac{d}{dx}\left\{ p(x) \frac{dy(x,\sigma_1+\delta\sigma)}{dx}\right\} + \left\{\lambda(\sigma_1+\delta\sigma) + q(x)\right\} y(x,\sigma_1+\delta\sigma) = 0.$$

$\delta\sigma$ being supposed small and $y(x,\sigma_1+\delta\sigma)$ and $\lambda(\sigma+\delta\sigma)$ supposed developable as Taylor's series in $\delta\sigma$, we have

$$\frac{d}{dx}\left\{ p(x)\frac{d\,y(x,\sigma_1)}{dx}\right\} + \left\{\lambda(\sigma_1) + q(x)\right\} y(x,\sigma_1)$$
$$+\delta\sigma\left[\frac{d}{dx}\left\{p(x)\frac{d}{dx}\frac{\partial}{\partial\sigma}y(x,\sigma)\right\} + \left\{\lambda(\sigma)+q(x)\right\}\frac{\partial}{\partial\sigma}y(x,\sigma) + \lambda'(\sigma)\,y(x,\sigma)\right]_{\sigma=\sigma_1}$$
$$+ O(\delta\sigma^2) = 0$$

The terms of this equation independent of $\delta\sigma$ are identically zero, equating to zero the coefficient of $\delta\sigma$, we have

$$\left[\frac{d}{dx}\left\{p(x)\frac{d}{dx}\frac{\partial}{\partial\sigma}y(x,\sigma)\right\} + \left\{\lambda(\sigma)+q(x)\right\}\frac{\partial}{\partial\sigma}y(x,\sigma) + \lambda'(\sigma)\,y(x,\sigma)\right]_{\sigma=\sigma_1} = 0$$

Hence provided σ_1 is a root of the equation $\frac{d}{d\sigma}\lambda(\sigma) = 0$, the equation

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + \left\{\lambda(\sigma) + q(x)\right\} y = 0$$

is satisfied by $y = \frac{\partial}{\partial\sigma_1}(x,\sigma_1)$.

In both these cases $\frac{\partial}{\partial\sigma} a(\sigma) = 0$, so that $\frac{\partial}{\partial\sigma} \Lambda(x,\sigma,q)$ satisfies the differential equation. We have

$$\frac{\partial}{\partial\sigma} \Lambda(x,\sigma,q) = x\,\Lambda(x,\sigma,q)\frac{\partial\mu}{\partial\sigma} + e^{\mu x}\frac{\partial u(x,\sigma)}{\partial\sigma}.$$

The values of this function when $\sigma = 0$ and $\sigma = \frac{\pi}{2}$ give the required second solutions. Now

$$\left(\frac{\partial\mu}{\partial\sigma}\right)_{\sigma=0} = 8q - 24q^3 - 48q^4 + O(q^5) \quad ; \quad \left(e^{\mu x}\right)_{\sigma=0} = 1.$$

$$-\left[\frac{\partial}{\partial\sigma} u(x,\sigma)\right]_{\sigma=0} = \cos x + q\cos 3x + q^2\left(\tfrac{1}{3}\cos 5x - 5\cos 3x\right) + q^3\left(\tfrac{1}{18}\cos 7x - \tfrac{8}{3}\cos 5x - \tfrac{35}{3}\cos 3x\right)$$
$$+ q^4\left(\tfrac{1}{180}\cos 9x - \tfrac{61}{108}\cos 7x - \tfrac{343}{54}\cos 5x + \tfrac{17}{3}\cos 3x\right) + O(q^5).$$

so that the second solution corresponding to $se_1(x)$ is

$$-8q\left(1 - 3q^2 - 6q^3 + \cdots\right) x\, se_1(x,q) + \cos x + q\cos 3x + q^3\left(\tfrac{1}{3}\cos 5x - 5\cos 3x\right) + \cdots$$
as above

and likewise we find that the second solution corresponding to $ce_1(x)$ is

$$-8q\left(1 - 3q^2 + 6q^3 + \cdots\right) x\, ce_1(x,q) + \sin x + q\sin 3x + q^3\left(\tfrac{1}{3}\sin 5x - 5\sin 3x\right)$$
$$+ q^3\left(\tfrac{1}{18}\sin 7x + \tfrac{8}{3}\sin 5x - \tfrac{35}{3}\sin 3x\right)$$
$$+ q^4\left(\tfrac{1}{180}\sin 9x + \tfrac{61}{108}\sin 7x - \tfrac{343}{54}\sin 5x - \tfrac{17}{3}\sin 3x\right) + \cdots$$

These second solutions are identical with those previously obtained by the present author by employing quite different methods of procedure*.

* Proc. Edin. Math. Soc. XXXIII. (1914-15). p. 2.