THE ANALYTIC PROPERTIES OF THE SCATTERING AMPLITUDE IN NON-RELATIVISTIC QUANTUM MECHANICS

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CONTENTS

PREFACE

I am indebted to Professor N. Kemmer for the hospitality of the Tait Institute; I wish also to thank Dr. D.J. Candlin, who suggested the field of the present enquiry, for his interest and guidance in the work described in this thesis. Thanks are also due to Dr. G.R. Screaton for several valuable suggestions during the final stages of the work, and to my fellow research students for many useful discussions.

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I assert that the material presented in this thesis is original, except in so far as explicit reference is made to the work of others.

INTRODUCTION

Within the past few years the study of quantum field theory has been centred round the description of scattering phenomena by means of single and double dispersion relations. That is to say that the scattering amplitude, which for real scattering processes is a function of two real variables, is considered instead as a function of one or two complex variables, and the analytic properties in the complex plane are utilised by means of a Cauchy integration to obtain an integral representation. Of course, in performing the integration round a circle which is allowed to recede to infinity we need to know the behaviour of the function for large values of the argument so that we know how to treat the contribution from the integral round the infinite circle. Thus we see that the essential properties are the analytic properties of the scattering amplitude together with its asymptotic behaviour for large values of the (complex) variables.

There are two possible approaches to these integral representations. The first is to take the usual postulates of axiomatic quantum field theory and deduce from them the required analytic and asymptotic properties. The second is to replace the postulates of field theory, more or less by the postulate of a double dispersion relation or Mandelstam representation. More accurately the postulates involved are those of: (i) unitarity, (ii) maximal analyticity consistent with unitarity and

(iii) crossing symmetry. The first approach cannot be said to have been attended with any great success. No proof of a double dispersion relation starting from the axioms of quantum field theory has as yet been put forward, and indeed the proof of a single dispersion relation in the energy variable for the scattering of two particles of masses μ and M with the corresponding lowest mass intermediate states of masses m_1 and m_2 is only possible provided the following condition holds:

If t is the momentum transfer variable, and s the energy variable, then for the dispersion relation to hold we must have

$$t \le t_{max} = 4\,\mathrm{Min}_s \left\{ \frac{[s-(M+\mu)^2][s-(M-\mu)^2]}{4s} + \frac{(m_1^2-\mu^2)(m_2^2-M^2)}{s-(m_1-m_2)^2} \right\}$$

This will of course depend on the process we consider, i.e. on m_1, m_2 and the range of s. If t_{max} turns out to be negative, then for this particular process a dispersion relation cannot be proved at all. Application of the above general formula to special cases leads to the fact that single dispersion relations in the energy variable may be proved for the following processes:

(i) Pion nucleon scattering (with the momentum transfer in the range $0 < t < \frac{32}{3}\,\frac{2M+\mu}{2M-\mu}\,\mu^2$ where M = mass of nucleon and μ = mass of pion).

(ii) Pion-pion scattering (provided $0 < t < 28\mu^2$).

(iii) In general the scattering of pions or photons off any 'elementary' particle.

(iv) The photoproduction of pions.

(v) The double Compton effect.

The second approach has been attended with some success inasmuch as that the use of the Mandelstam representation together with unitarity has led to a large body of results in close agreement with experiment. This represents a considerable advance over previous work, since the infinite constants, previously necessary to obtain sensible results, are now dispensed with. But the question is whether the representation should be taken purely as a postulate, or whether some attempt at justification should be made. As mentioned above, no proof has yet been offered in the full quantum field theory, but much work has been done in both perturbation theory and in the theory of scattering by a potential in non-relativistic quantum mechanics. This work may be looked upon as a sort of guide as to what might be expected to occur in the full field theory. In other words, if the representation holds good in perturbation theory, which we know to give results consistent with experiment at least for the case of quantum electrodynamics, then it is at least plausible that it might hold good in the full field theory. On the other hand, if the representation were not valid in perturbation theory, then it would be extremely doubtful if it would be true in the full theory. Much progress has been made towards a proof of the Mandelstam representation in perturbation theory, and, indeed at one time it was thought that a complete proof had been given. However, an unexpected complication arose, and the proof remains in an incomplete state.

Similarly, if we can obtain a proof in non-relativistic quantum mechanics, then this will make more plausible the corresponding result in field theory. A further impulse for the study of potential theory, is to see whether the methods adopted in this case, where the state of our knowledge is at a relatively high level, will throw any light on the procedure to be used in the full field theory where, at the moment, we are comparatively in the dark. The analytic properties of the scattering amplitude in potential theory have indeed already been studied by a number of authors[1-9]. The various proofs to date of these analytic properties have not, however, indicated clearly any method to be adopted in field theory, and in addition they have been singularly unsuccessful in deriving the behaviour of the scattering amplitude for large values of the momentum transfer, this last property being necessary to derive the required integral representation from the known analytic properties. One proof, that of Regge, does in fact yield information concerning this point. It is unfortunate, however, that of all the methods so far adopted, that of Regge is the least amenable for any analogy to be made with field theory, since his investigation has as its starting point the Schrodinger equation, and proceeds via the techniques of complex angular momenta. It was with the hope of providing a method which would give some help to field theory and at the same time would yield information on the behaviour of the scattering amplitude for large (complex) values of the momentum transfer that the work described in this thesis was initiated.

Essentially what is involved in the method to be adopted in this thesis is an iteration of the Low equation instead of the Lippmann-Schwinger equation. Both these iterations lead to the Born series together with a remainder term. Now it is known that the Born series does not necessarily converge in the presence of bound states, so we should expect the remainder term to offer some information concerning this. From the Lippmann-Schwinger iteration no explicit information is conveyed concerning the bound states, but from the Low iteration the bound state poles appear explicitly in the remainder term so that it was to be hoped that with their appearance the remainder term would be amenable to treatment in order to obtain information concerning the amplitude as a whole.

In Section I we set up our notation and discuss some of the general methods of the dispersion relation approach and the unitarity relationship. We also introduce here the Lippmann-Schwinger and Low equations. In Section II we show how an iteration procedure may be set up for the Low equation, leading essentially to a series of Born terms together with a remainder term. In Section III we investigate the analytic properties of the Born terms using some of the methods devised for the study of perturbation theory. Section IV is devoted to the determination of the asymptotic behaviour of the Born terms, necessary before an integral representation may be deduced. In Section V we determine the analytic properties of the remainder term, and in Section VI we discuss the asymptotic behaviour of the scattering amplitude. In Section VII we determine the properties of partial wave amplitudes, and in Section VIII we obtain the analytic properties of the scattering amplitude on the unphysical sheet.

I. Notation and Preliminary Considerations

We consider the scattering of a particle of mass M by the spherically symmetric potential $V(r)$, or the interaction of two particles with masses m_1 and m_2 and reduced mass $M = \frac{m_1 m_2}{m_1+m_2}$ coupled through the potential $V(r_{12})$ where r_{12} is the distance of separation of the two particles. In the latter case the motion described is in the centre of mass frame of the colliding particles. We choose units such that $\hbar = c = 2M = 1$.

The potentials to be considered are restricted to those that may be written in the form

$$V(r) = \int_{\mu}^{\infty} \sigma(m) \frac{e^{-mr}}{r} dm \tag{1.1}$$

i.e. those which consist of a superposition of Yukawa potentials. The conditions that $V(r)$ must satisfy in order to have this representation, with $\sigma(m)$ bounded almost everywhere except possibly for δ-function singularities (necessary to allow the pure Yukawa potential) are(10)

(i) All orders of derivatives of $V(r)$ exist.

(ii) $\frac{d^k}{dr^k}(rV) < \frac{Ak!}{r^{k+1}}$

for all integral k, where A is some constant.

For many purposes it will suffice for us to consider the case of the pure Yukawa potential only, the extension of the results obtained to the superposition of Yukawa potentials being trivial.

We consider scattering from an initial state in which the particle has momentum $\underline{k}_i$ (relative to the centre of mass in the case of two particles) to a final state with momentum $\underline{k}_f$. If we denote, in the customary manner, a plane wave state of momentum $\underline{k}$ by $|\underline{k}\rangle$ and the corresponding incoming and outgoing scattering states by $|\underline{k}^+\rangle$ and $|\underline{k}^-\rangle$, then it is well known that the T-matrix element for the above process is given by

$$T(\underline{k}_i, \underline{k}_f) = \langle \underline{k}_f | V | \underline{k}_i^+ \rangle \tag{1.2}$$

where a suitable choice of normalisation has been made. This matrix element is the scattering amplitude as usually considered, due to its intimate connection with the differential cross-section. It is a function of only two scalar variables s and t related to the momenta $\underline{k}_i$ and $\underline{k}_f$ by

$$s(=k^2) = k_i^2 = k_f^2$$

$$t = (\underline{k}_i - \underline{k}_f)^2 = 2s(1-\cos\theta) \tag{1.3}$$

s is the energy, t the momentum transfer and θ the scattering angle (i.e. the angle between the momenta $\underline{k}_i$ and $\underline{k}_f$). It is in the variables s and t that we desire to obtain an integral representation, since they are the closest analogues to the field theoretic variables as usually chosen.

The scattering states and the plane wave states are connected by the Lippmann-Schwinger equations, the derivation of of which is on a fairly sound basis in non-relativistic quantum mechanics:-

$$|\underline{k}^{\pm}\rangle = |\underline{k}\rangle + \frac{1}{k^2 - H_0 \pm i\epsilon} V |\underline{k}^{\pm}\rangle \tag{1.4}$$

H_0 is the free Hamiltonian and

$$G_{0k} \equiv \frac{1}{k^2 - H_0 + i\epsilon} \tag{1.5}$$

is the free Green's function.

It has the momentum representation

$$G_{0k}(\underline{\ell},\underline{m}) = \langle \underline{\ell} | \frac{1}{k^2 - H_0 + i\epsilon} | \underline{m} \rangle$$

$$= \frac{1}{k^2 - \ell^2 + i\epsilon} \delta_3 (\underline{\ell} - \underline{m}) \tag{1.6}$$

and the coordinate representation

$$G_{0k}(\underline{x},\underline{y}) = \langle \underline{x} | \frac{1}{k^2 - H_0 + i\epsilon} | \underline{y} \rangle$$

$$= \frac{e^{ik|\underline{x}-\underline{y}|}}{4\pi |\underline{x}-\underline{y}|} \tag{1.7}$$

The matrix element of the potential between two plane wave states is written as

$$\langle \underline{\ell} | V | \underline{m} \rangle = V(\underline{m}, \underline{\ell}) \tag{1.8}$$

This is just the Fourier transform of the potential:-

$$\langle \underline{\ell} | V | \underline{m} \rangle = \int e^{-i \underline{\ell} . \underline{x}} V(x) e^{i \underline{m} . \underline{x}} d^3 \underline{x}$$

$$= \int e^{i (\underline{m} - \underline{\ell}) . \underline{x}} V(x) d^3 \underline{x} \tag{1.9}$$

so that $V(\underline{m}, \underline{\ell})$ is a function only of the difference between $\underline{\ell}$ and $\underline{m}$, and so may be written

$$V(\underline{m}, \underline{\ell}) \equiv V(\underline{m} - \underline{\ell}) \tag{1.10}$$

From the Lippmann-Schwinger equation (1.4) may be deduced immediately a corresponding equation for the scattering amplitude:-

$$\langle \underline{\ell} | V | \underline{k}^+ \rangle = \langle \underline{\ell} | V | \underline{k} \rangle + \langle \underline{\ell} | V \frac{1}{k^2 - H_0 + i\epsilon} V | \underline{k}^+ \rangle$$

which, on inserting a complete set of plane wave states, which are in fact complete for H_o, since H_o has no bound states, and changing to initial momentum $\underline{k}_i$ and final momentum $\underline{k}_f$ gives

$$T(\underline{k}_i, \underline{k}_f) = V(\underline{k}_i - \underline{k}_f) + \int \frac{T(\underline{k}_i, \underline{m}) V(\underline{m} - \underline{k}_f)}{k_i^2 - m^2 + i\epsilon} d^3 \underline{m} \tag{1.11}$$

It will be convenient to employ a suffix notation $T_{\underline{k}_f\underline{k}_i} = T(\underline{k}_i, \underline{k}_f)$ and to use a generalised summation convention in which a repeated suffix implies a three-dimensional integration. Thus equation (1.11) may be written

$$T_{\underline{k}_f\underline{k}_i} = V_{\underline{k}_f\underline{k}_i} + \frac{V_{\underline{k}_f\underline{q}}\, T_{\underline{q}\underline{k}_i}}{k_i^2 - q^2 + i\epsilon} \tag{1.12}$$

Because of the reality of the potential we also have the complex conjugate equation

$$T^*_{\underline{k}_f\underline{k}_i} = V_{\underline{k}_f\underline{k}_i} + \frac{V_{\underline{k}_f\underline{q}}\, T^*_{\underline{q}\underline{k}_i}}{k_i^2 - q^2 - i\epsilon} \tag{1.13}$$

We also have the Low equation which is very easy to derive formally from the Lippmann-Schwinger equations (1.4):-

$$|\underline{k}^\pm\rangle = |\underline{k}\rangle + \frac{1}{E - H \pm i\epsilon} V|\underline{k}\rangle \tag{1.14}$$

giving an equation for the scattering amplitude

$$\langle\underline{q}|V|\underline{k}^+\rangle = \langle\underline{q}|V|\underline{k}\rangle + \langle\underline{q}|V\frac{1}{k^2 - H + i\epsilon}V|\underline{k}\rangle \tag{1.15}$$

which on changing the initial momentum to $\underline{k}_i$ and the final momentum to $\underline{k}_f$, and inserting a complete set of states of H, namely the scattering states $\underline{k}^+$ together with the bound states B gives the equation for the scattering amplitude

$$T_{\underline{k}_f \underline{k}_i} = V_{\underline{k}_f \underline{k}_i} + \frac{T_{\underline{k}_f \underline{e}} T^*_{\underline{k}_i \underline{e}}}{k_i^2 - e^2 + i\epsilon} + \frac{T_{\underline{k}_f B} T^*_{\underline{k}_i B}}{k_i^2 + B} \tag{1.16}$$

which we shall henceforth call the Low equation. Here $T_{\underline{k}B} = T_B(\underline{k})$ is the matrix element of the potential taken between a plane wave state and a bound state: $T_{\underline{k}B} = \langle \underline{k} | V | B \rangle$. B, as well as labelling the bound state also denotes its energy, and the summation convention implies a sum over all the bound states.

We shall have occasion to employ the Born series for the scattering amplitude. This series is simply obtained by iteration of the Lippmann-Schwinger equation (1.12). We denote by $T^{(n)}(\underline{k}_i, \underline{k}_f)$ the n-th term in the iterated series (usually known as the n-th Born term):-

$$T^{(n)}(\underline{k}_i, \underline{k}_f) = \langle \underline{k}_f | V G_{0k} V G_{0k} \text{------} G_{0k} V | \underline{k}_i \rangle \tag{1.17}$$

(in which there appear n V's)

There are two explicit forms of the n-th Born term which prove useful. The first is in the coordinate representation:-

$$T^{(n)}(\underline{k}_i, \underline{k}_f) = \int d^3\underline{y}\, d^3\underline{z}\, d^3\underline{x}_1 \text{ ----- } d^3\underline{x}_{n-2} \times$$

$$\times\, e^{-i\underline{k}_f\cdot\underline{y}} V(y) \frac{e^{ik|\underline{y}-\underline{x}_1|}}{4\pi|\underline{y}-\underline{x}_1|} V(x_1) \frac{e^{ik|\underline{x}_1-\underline{x}_2|}}{4\pi|\underline{x}_1-\underline{x}_2|} \text{------} \frac{e^{ik|\underline{x}_{n-2}-\underline{z}|}}{4\pi|\underline{x}_{n-2}-\underline{z}|} V(z)\, e^{i\underline{k}_i\cdot\underline{z}} \tag{1.18}$$

In the momentum representation, with the potential of the form given by (1.1) we have

$$T^{(n)}(\underline{k}_i,\underline{k}_f) = \int d^3\underline{k}_1 \cdots d^3\underline{k}_{n-1}\int_\mu^\infty dm_1 \cdots dm_n \times$$

$$\times \frac{\sigma(m_1)}{m_1^2+(\underline{k}_f-\underline{k}_1)^2}\cdot\frac{1}{k^2-k_1^2+i\epsilon}\;\frac{\sigma(m_2)}{m_2^2+(\underline{k}_1-\underline{k}_2)^2}\cdots\cdots\frac{\sigma(m_n)}{m_n^2+(\underline{k}_{n-1}-\underline{k}_i)^2} \tag{1.19}$$

It is readily seen that the first Born term is just the Fourier transform of the potential, or, in other words that

$$T^{(1)}(\underline{k}_i,\underline{k}_f) = V(\underline{k}_f-\underline{k}_i)$$

In the energy conserving case $(k_i^2 = k_f^2)$ it is immediately apparent that the n-th Born term is a function only of the two scalar variables $s = k_i^2 = k_f^2$ (the energy) and $t = (\underline{k}_i - \underline{k}_f)^2$ (the momentum transfer). We shall have occasion to consider, however, also the non energy conserving terms (with $k_i^2 \neq k_f^2$); in this case the Born term is a function of the three scalar variables $s = k_i^2$, $\bar{s} = k_f^2$ (the two energies) and $t = (\underline{k}_i - \underline{k}_f)^2$ (the momentum transfer).

We now proceed to give a summary of the basic ideas involved in deriving a double dispersion relation for a scattering amplitude $T(s,t)$.

Suppose we know that $T(s,t)$ is an analytic function of s (for fixed real t) apart from the branch cut $0 \leq s < \infty$. Then we may write down the relationship

$$T(s,t) = \frac{1}{2\pi i}\int_C \frac{T(s',t)}{s-s'}\,ds' \tag{1.20}$$

which comes directly from Cauchy's Theorem with C as the contour shown in Fig. 1 and s a point inside that contour.

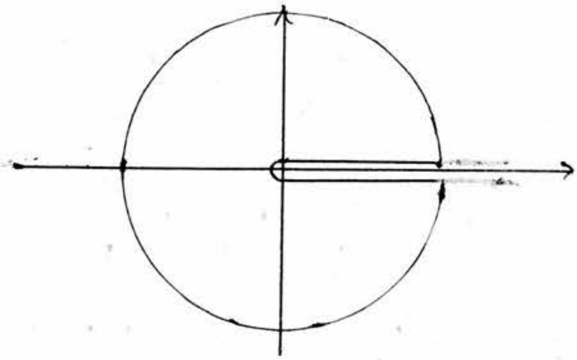

Fig. 1.

If $T(s',t) \to 0$ as $|s'| \to \infty$ and if we expand the contour C so that the circular part recedes to infinity, then the contribution to the integral from the circle will vanish, and we shall be left with the contributions from the paths above and below the branch cut:-

$$T(s,t) = \frac{1}{2\pi i}\left\{ \int_0^\infty \frac{T(s'+i\epsilon,t)}{s'+i\epsilon-s}\,ds' - \int_0^\infty \frac{T(s'-i\epsilon,t)}{s'-i\epsilon-s}\,ds' \right\} \tag{1.21}$$

$$= \frac{1}{2\pi i}\int_0^\infty \frac{T(s'+i\epsilon,t) - T(s'-i\epsilon,t)}{s'-s}\,ds' \tag{1.22}$$

since $s'-s+i\epsilon$ and $s'-s-i\epsilon$ both tend unambiguously to $s' - s$ as ε tends to zero when s is complex.

We now make use of the fact (which we shall justify later) that $T(s, t)$ is a real function of s, i.e. that

$T(s^*, t) = [T(s,t)]^*$

so that $$T(s' - i\varepsilon, t) = [T(s' + i\varepsilon, t)]^* \tag{1.23}$$

and thus

$$T(s'+i\epsilon, t) - T(s'-i\epsilon, t) = 2i \, \mathcal{I}m \, T(s'+i\epsilon, t) \tag{1.24}$$

and so from (1.22)

$$T(s,t) = \frac{1}{\pi}\int_0^\infty \frac{\mathcal{I}m \, T(s',t)}{s'-s} ds' \tag{1.25}$$

where we define $T(s,t)$ for real s as $\lim_{\epsilon \to 0} T(s + i\varepsilon, t)$.
Hence, from (1.25) we obtain the result that if s is real we shall have

$$T(s,t) = \frac{1}{\pi}\int_0^\infty \frac{\mathcal{I}m \, T(s',t)}{s'-s-i\epsilon} ds' \tag{1.26}$$

So we see that whatever value s has we shall have the relationship

$$T(s,t) = \frac{1}{\pi}\int_0^\infty \frac{\mathcal{I}m \, T(s',t)}{s'-s-i\epsilon} ds' \tag{1.27}$$

and thus to further exhibit the analytic properties in t and so obtain a double integral representation, it is only necessary to determine the analytic properties of the integrand in (1.27), Im $T(s',t)$ (or, if one so desires, $T(s',t)$ itself) as a function of t, for fixed real positive s' .

If the integral round the infinite circle had not vanished, i.e. if $T(s,t) \not\to 0$ as $|s| \to \infty$, then a procedure of the following type must be adopted:-

Suppose we take the relationship (1.18)

$$T(s,t) = \frac{1}{2\pi i}\int_C \frac{T(s',t)}{s'-s}\,ds'$$

and consider it also at the fixed point s_0:

$$T(s_0,t) = \frac{1}{2\pi i}\int_C \frac{T(s',t)}{s'-s_0}\,ds' \tag{1.28}$$

so that subtraction of these two relationships gives

$$T(s,t) - T(s_0,t) = \frac{s-s_0}{2\pi i}\int_C \frac{T(s',t)}{(s'-s)(s'-s_0)}\,ds' \tag{1.29}$$

so that in this case the integral round the infinite circle will vanish if $\frac{T(s',t)}{s'} \to 0$ as $|s'| \to \infty$ and an analogous dispersion relation to (1.27) may again be deduced.

This procedure may be generalised so that we may obtain by performing n 'subtractions' a dispersion relation with n arbitrary parameters for the case in which $\frac{T(s',t)}{s'^n} \to 0$ as

$s' \to \infty$.

In general if n subtractions are necessary, one of the forms in which the analogous dispersion relation to (1.27) may be written is

$$T(s,t) = \frac{s^{n+1}}{\pi} \int_0^\infty \frac{\operatorname{Im} T(s',t)}{s'^{n+1}(s'-s-i\epsilon)}\, ds' + \sum_{i=0}^{n} s^i f_i(t) \tag{1.30}$$

We shall have occasion to make use of the unitarity condition. In terms of the S-matrix this is usually in the form

$$S^+ S = 1 \tag{1.31}$$

But the S and T matrix elements are related by

$$S(\underline{k}_i, \underline{k}_f) = \delta_3(\underline{k}_i - \underline{k}_f) + 2\pi i\, \delta(k_i^2 - k_f^2)\, T(\underline{k}_i, \underline{k}_f) \tag{1.32}$$

The relationship (1.31) written in integral form is

$$\int S^*(\underline{k}_f, \underline{k}) S(\underline{k}, \underline{k}_i)\, d^3\underline{k} = \delta_3(\underline{k}_i - \underline{k}_f) \tag{1.33}$$

which on using (1.32) becomes

$$\int \left[\delta_3(\underline{k}_f - \underline{k}) - 2\pi i\, \delta(k_f^2 - k^2)\, T^*(\underline{k}_f, \underline{k})\right]\left[\delta_3(\underline{k} - \underline{k}_i) + 2\pi i\, \delta(k^2 - k_i^2)\, T(\underline{k}, \underline{k}_i)\right] d^3\underline{k}$$

$$= \delta_3(\underline{k}_i - \underline{k}_f) \tag{1.34}$$

i.e. $\delta_3(\underline{k}_i - \underline{k}_f) + 2\pi i\,\delta(k_i^2 - k_f^2)\left[T(\underline{k}_f, \underline{k}_i) - T^*(\underline{k}_f, \underline{k}_i)\right]$

$$+ 4\pi^2 \int \delta(k_i^2 - k_f^2)\,\delta(k_i^2 - k^2)\,T^*(\underline{k}_f, \underline{k})\,T(\underline{k}, \underline{k}_i)\,d^3\underline{k} = \delta_3(\underline{k}_i - \underline{k}_f)$$

i.e. $T(\underline{k}_f, \underline{k}_i) - T^*(\underline{k}_f, \underline{k}_i) = 2\pi i \int \delta(k^2 - s)\,T^*(\underline{k}_f, \underline{k})\,T(\underline{k}, \underline{k}_i)\,d^3\underline{k}$ (1.35)

where $k_f^2 = k_i^2 = s$

i.e. $\mathcal{I}m\,T(\underline{k}_f, \underline{k}_i) = \pi \int \delta(k^2 - s)\,T^*(\underline{k}_f, \underline{k})\,T(\underline{k}, \underline{k}_i)\,d^3\underline{k}$

$$= \pi \int \frac{1}{2\sqrt{s}}\left[\delta(k - \sqrt{s}) + \delta(k + \sqrt{s})\right] T^*(\underline{k}_f, \underline{k})\,T(\underline{k}, \underline{k}_i)\,k^2\,dk\,d\Omega_k$$

i.e. $\mathcal{I}m\,T(\underline{k}_f, \underline{k}_i) = \dfrac{\pi\sqrt{s}}{2}\displaystyle\int T^*(\underline{k}_f, \underline{k})\,T(\underline{k}, \underline{k}_i)\,d\Omega_k$ (1.36)

with $k_f^2 = k_i^2 = k^2 = s$

which is the form of the unitarity condition we shall use.

II. The Iteration Procedure

Using the generalised summation convention we have the Lippmann-Schwinger and Low equations:-

$$T_{\underline{k}_f \underline{k}_i} = V_{\underline{k}_f \underline{k}_i} + \frac{V_{\underline{k}_f \underline{l}}\, T_{\underline{l}\, \underline{k}_i}}{k_i^2 - l^2 + i\epsilon} \tag{2.1}$$

$$T^*_{\underline{k}_f \underline{k}_i} = V_{\underline{k}_f \underline{k}_i} + \frac{V_{\underline{k}_f \underline{l}}\, T^*_{\underline{l}\, \underline{k}_i}}{k_i^2 - l^2 - i\epsilon} \tag{2.2}$$

$$T_{\underline{k}_f \underline{k}_i} = V_{\underline{k}_f \underline{k}_i} + \frac{T_{\underline{k}_f \underline{l}}\, T^*_{\underline{k}_i \underline{l}}}{k_i^2 - l^2 + i\epsilon} + \frac{T_{\underline{k}_f B}\, T^*_{\underline{k}_i B}}{k_i^2 + B} \tag{2.3}$$

where B as well as labelling the bound state also represents the magnitude of its energy.

Using equations (2.1) and (2.2) in the second term of equation (2.3) gives us

$$\frac{T_{\underline{k}_f \underline{l}}\, T^*_{\underline{k}_i \underline{l}}}{k_i^2 - l^2 + i\epsilon} = \frac{1}{k_i^2 - l^2 + i\epsilon}\left[V_{\underline{k}_f \underline{l}} + \frac{V_{\underline{k}_f \underline{m}}\, T_{\underline{m}\, \underline{l}}}{l^2 - m^2 + i\epsilon}\right]\left[V_{\underline{k}_i \underline{l}} + \frac{V_{\underline{k}_i \underline{n}}\, T^*_{\underline{n}\, \underline{l}}}{l^2 - n^2 - i\epsilon}\right]$$

$$\begin{aligned} &= \frac{V_{\underline{k}_f \underline{l}}\, V_{\underline{k}_i \underline{l}}}{k_i^2 - l^2 + i\epsilon} + \frac{V_{\underline{k}_f \underline{l}}\, V_{\underline{k}_i \underline{n}}\, T^*_{\underline{n}\, \underline{l}}}{(k_i^2 - l^2 + i\epsilon)(l^2 - n^2 - i\epsilon)} + \frac{V_{\underline{k}_f \underline{m}}\, T_{\underline{m}\, \underline{l}}\, V_{\underline{k}_i \underline{l}}}{(k_i^2 - l^2 + i\epsilon)(l^2 - m^2 + i\epsilon)} \\ &\quad + \frac{V_{\underline{k}_f \underline{m}}\, T_{\underline{m}\, \underline{l}}\, V_{\underline{k}_i \underline{n}}\, T^*_{\underline{n}\, \underline{l}}}{(k_i^2 - l^2 + i\epsilon)(l^2 - m^2 + i\epsilon)(l^2 - n^2 - i\epsilon)} \end{aligned} \tag{2.4}$$

$$= \frac{V_{\mathbf{k}_f \mathbf{l}} V_{\mathbf{l} \mathbf{k}_i}}{k_i^2 - l^2 + i\epsilon} + \frac{V_{\mathbf{k}_f \mathbf{l}} T^*_{\mathbf{n} \mathbf{l}} V_{\mathbf{n} \mathbf{k}_i}}{(k_i^2 - l^2 + i\epsilon)(l^2 - n^2 - i\epsilon)} + \frac{V_{\mathbf{k}_f \mathbf{m}} T_{\mathbf{m} \mathbf{l}} V_{\mathbf{l} \mathbf{k}_i}}{(k_i^2 - l^2 + i\epsilon)(l^2 - m^2 + i\epsilon)}$$

$$+ \frac{V_{\mathbf{k}_f \mathbf{m}} T_{\mathbf{m} \mathbf{l}} T^*_{\mathbf{n} \mathbf{l}} V_{\mathbf{n} \mathbf{k}_i}}{(k_i^2 - l^2 + i\epsilon)(l^2 - m^2 + i\epsilon)(l^2 - n^2 - i\epsilon)}$$

$$= \frac{V_{\mathbf{k}_f \mathbf{l}} V_{\mathbf{l} \mathbf{k}_i}}{k_i^2 - l^2 + i\epsilon} + \frac{V_{\mathbf{k}_f \mathbf{l}} T^*_{\mathbf{n} \mathbf{l}} V_{\mathbf{n} \mathbf{k}_i}}{(k_i^2 - l^2 + i\epsilon)(l^2 - n^2 - i\epsilon)} + \frac{V_{\mathbf{k}_f \mathbf{l}} T_{\mathbf{l} \mathbf{n}} V_{\mathbf{n} \mathbf{k}_i}}{(k_i^2 - n^2 + i\epsilon)(n^2 - l^2 + i\epsilon)}$$

$$+ \frac{V_{\mathbf{k}_f \mathbf{l}} T_{\mathbf{l} \mathbf{m}} T^*_{\mathbf{n} \mathbf{m}} V_{\mathbf{n} \mathbf{k}_i}}{(k_i^2 - m^2 + i\epsilon)(m^2 - l^2 + i\epsilon)(m^2 - n^2 - i\epsilon)} \qquad (2.5)$$

where some of the dummy suffices have been rearranged.

If we now use the Low equation in the second and third terms of the above expression we obtain

$$\frac{T_{\mathbf{k}_f \mathbf{l}} T^*_{\mathbf{k}_i \mathbf{l}}}{k_i^2 - l^2 + i\epsilon} = \frac{V_{\mathbf{k}_f \mathbf{l}} V_{\mathbf{l} \mathbf{k}_i}}{k_i^2 - l^2 + i\epsilon} + \frac{V_{\mathbf{k}_f \mathbf{l}} V_{\mathbf{l} \mathbf{n}} V_{\mathbf{n} \mathbf{k}_i}}{(k_i^2 - l^2 + i\epsilon)(l^2 - n^2 - i\epsilon)} +$$

$$+ \frac{V_{\underline{k}_f \underline{\ell}} T^*_{\underline{n}\underline{m}} T_{\underline{\ell}\underline{m}} V_{\underline{n}\underline{k}_i}}{(k_i^2 - \ell^2 + i\epsilon)(\ell^2 - n^2 - i\epsilon)(\ell^2 - m^2 - i\epsilon)} + \frac{V_{\underline{k}_f \underline{\ell}} T^*_{B\underline{n}} T_{B\underline{\ell}} V_{\underline{n}\underline{k}_i}}{(k_i^2 - \ell^2 + i\epsilon)(\ell^2 - n^2 - i\epsilon)(\ell^2 + B)}$$

$$+ \frac{V_{\underline{k}_f \underline{\ell}} V_{\underline{\ell}\underline{n}} V_{\underline{n}\underline{k}_i}}{(k_i^2 - n^2 + i\epsilon)(n^2 - \ell^2 + i\epsilon)} + \frac{V_{\underline{k}_f \underline{\ell}} T_{\underline{\ell}\underline{m}} T^*_{\underline{n}\underline{m}} V_{\underline{n}\underline{k}_i}}{(k_i^2 - n^2 + i\epsilon)(n^2 - \ell^2 + i\epsilon)(n^2 - m^2 + i\epsilon)}$$

$$+ \frac{V_{\underline{k}_f \underline{\ell}} T_{B\underline{\ell}} T^*_{B\underline{n}} V_{\underline{n}\underline{k}_i}}{(k_i^2 - n^2 + i\epsilon)(n^2 - \ell^2 + i\epsilon)(n^2 + B)} + \frac{V_{\underline{k}_f \underline{\ell}} T_{\underline{\ell}\underline{m}} T^*_{\underline{n}\underline{m}} V_{\underline{n}\underline{k}_i}}{(k_i^2 - m^2 + i\epsilon)(m^2 - \ell^2 + i\epsilon)(m^2 - n^2 - i\epsilon)}$$

(2.6)

$$= \frac{V_{\underline{k}_f \underline{\ell}} V_{\underline{\ell}\underline{k}_i}}{k_i^2 - \ell^2 + i\epsilon} + \alpha(k_i^2, \ell^2, n^2) V_{\underline{k}_f \underline{\ell}} V_{\underline{\ell}\underline{n}} V_{\underline{n}\underline{k}_i}$$

$$+ \beta(k_i^2, \ell^2, n^2, B) V_{\underline{k}_f \underline{\ell}} T^*_{B\underline{n}} T_{B\underline{\ell}} V_{\underline{n}\underline{k}_i} + \gamma(k_i^2, \ell^2, m^2, n^2) V_{\underline{k}_f \underline{\ell}} T^*_{\underline{n}\underline{m}} T_{\underline{\ell}\underline{m}} V_{\underline{n}\underline{k}_i}$$

(2.7)

with

$$\alpha(k_i^2, \ell^2, n^2) = \frac{1}{(k_i^2 - \ell^2 + i\epsilon)(\ell^2 - n^2 - i\epsilon)}$$

$$+ \frac{1}{(k_i^2 - n^2 + i\epsilon)(n^2 - \ell^2 + i\epsilon)} \qquad (2.8)$$

$$\beta(k_i^2, \ell^2, n^2, B) = \frac{1}{(k_i^2-\ell^2+i\epsilon)(\ell^2-n^2-i\epsilon)(\ell^2+B)}$$

$$+ \frac{1}{(k_i^2-n^2+i\epsilon)(n^2-\ell^2+i\epsilon)(n^2+B)} \tag{2.9}$$

$$\gamma(k_i^2, \ell^2, m^2, n^2) = \frac{1}{(k_i^2-\ell^2+i\epsilon)(\ell^2-n^2-i\epsilon)(\ell^2-m^2-i\epsilon)}$$

$$+ \frac{1}{(k_i^2-n^2+i\epsilon)(n^2-\ell^2+i\epsilon)(n^2-m^2+i\epsilon)}$$

$$+ \frac{1}{(k_i^2-m^2+i\epsilon)(m^2-\ell^2+i\epsilon)(m^2-n^2-i\epsilon)} \tag{2.10}$$

Then we have

$$\alpha(k_i^2, \ell^2, n^2) = \frac{1}{(n^2-\ell^2+i\epsilon)}\left[\frac{1}{k_i^2-n^2+i\epsilon} - \frac{1}{k_i^2-\ell^2+i\epsilon}\right]$$

$$= \frac{k_i^2-\ell^2-k_i^2+n^2}{(n^2-\ell^2+i\epsilon)(k_i^2-n^2+i\epsilon)(k_i^2-\ell^2+i\epsilon)}$$

$$= \frac{1}{(k_i^2-n^2+i\epsilon)(k_i^2-\ell^2+i\epsilon)} \tag{2.11}$$

using the fact that $\dfrac{x}{x + i\varepsilon} = 1$

Hence the term arising from this part of the iteration procedure is

$$\frac{V_{\underline{k}_f \underline{l}}\, V_{\underline{l}\,\underline{n}}\, V_{\underline{n}\, \underline{k}_i}}{(k_i^2 - l^2 + i\epsilon)(k_i^2 - n^2 + i\epsilon)} \tag{2.12}$$

Next let us consider $\beta(k_i^2, l^2, n^2, B)$

$$\beta(k_i^2, l^2, n^2, B) = \frac{1}{n^2 - l^2 + i\epsilon}\left[\frac{1}{(k_i^2 - n^2 + i\epsilon)(n^2 + B)} - \frac{1}{(k_i^2 - l^2 + i\epsilon)(l^2 + B)}\right]$$

$$= \frac{l^2(k_i^2 - l^2) + B(k_i^2 - l^2) - n^2(k_i^2 - n^2) - B(k_i^2 - n^2)}{(n^2 - l^2 + i\epsilon)(k_i^2 - n^2 + i\epsilon)(n^2 + B)(k_i^2 - l^2 + i\epsilon)(l^2 + B)}$$

$$= \frac{k_i^2(l^2 - n^2) + (n^2 - l^2)(n^2 + l^2) + B(n^2 - l^2)}{(n^2 - l^2 + i\epsilon)(k_i^2 - n^2 + i\epsilon)(n^2 + B)(k_i^2 - l^2 + i\epsilon)(l^2 + B)}$$

$$= \frac{B + n^2 + l^2 - k_i^2}{(k_i^2 - n^2 + i\epsilon)(k_i^2 - l^2 + i\epsilon)(n^2 + B)(l^2 + B)} \tag{2.13}$$

Now, in equation (2.7) which gives an expression for the second term on the right hand side of equation (2.3), $\beta(k_i^2, \ell^2, n^2, B)$ goes with the factor $V_{\underline{k}_f\underline{\ell}}\, T_{\underline{n}B}\, T_{\underline{\ell}B}\, V_{\underline{n}\underline{k}_i}$. But the third term on the right hand side of equation (2.3) is $\dfrac{T_{B\underline{k}_f} T^*_{B\underline{k}_i}}{k_i^2 + B}$

which if we note that *

$$T_{B\underline{k}} = - \frac{V_{\underline{k}\underline{\ell}}\, T_{B\underline{\ell}}}{\ell^2 + B} \tag{2.14}$$

---

* We know that $H|B\rangle = -B|B\rangle$

so that $(H_0 + V)|B\rangle = -B|B\rangle$

i.e. $(H_0 + B)|B\rangle = -V|B\rangle$

i.e. $|B\rangle = -\dfrac{1}{H_0 + B} V|B\rangle$

and hence

$$T_{B\underline{k}} = \langle \underline{k}|V|B\rangle$$

$$= -\langle \underline{k}|V \frac{1}{H_0 + B} V|B\rangle$$

$$= -\langle \underline{k}|V|\underline{\ell}\rangle\langle\underline{\ell}| \frac{1}{H_0 + B} V|B\rangle$$

inserting a complete set of states

$$= -\langle \underline{k}|V|\underline{\ell}\rangle \frac{\langle\underline{\ell}|V|B\rangle}{\ell^2 + B}$$

$$= -\frac{V_{\underline{k}\underline{\ell}}\, T_{B\underline{\ell}}}{\ell^2 + B}$$

can be written in the form

$$\frac{T_{B\underline{k}_f} T^*_{B\underline{k}_i}}{k_i^2 + B} = \frac{V_{\underline{k}_f \underline{\ell}} T_{B\underline{\ell}} T^*_{B\underline{n}} V_{\underline{n}\underline{k}_i}}{(k_i^2+B)(\ell^2+B)(n^2+B)} \tag{2.15}$$

Hence the total coefficient of $V_{\underline{k}_f\underline{\ell}} T_{B\underline{\ell}} T_{B\underline{n}} V_{\underline{n}\underline{k}_i}$ in the iterated Low equation will be

$$\frac{B+n^2+\ell^2-k_i^2}{(k_i^2-n^2+i\epsilon)(k_i^2-\ell^2+i\epsilon)(n^2+B)(\ell^2+B)} + \frac{1}{(k_i^2+B)(\ell^2+B)(n^2+B)} \tag{2.16}$$

which reduces to

$$\frac{1}{(k_i^2-\ell^2+i\epsilon)(k_i^2+B)(k_i^2-n^2+i\epsilon)}$$

and so the iteration procedure gives for this term

$$\frac{V_{\underline{k}_f\underline{\ell}} T_{B\underline{\ell}} T^*_{B\underline{n}} V_{\underline{n}\underline{k}_i}}{(k_i^2-\ell^2+i\epsilon)(k_i^2+B)(k_i^2-n^2+i\epsilon)} \tag{2.17}$$

Let us now turn our attention to $\gamma(k_i^2, \ell^2, m^2, n^2)$

We have, from (2.10)

$$\gamma(k_i^2, \ell^2, m^2, n^2) = \frac{1}{(k_i^2-\ell^2+i\epsilon)(\ell^2-n^2-i\epsilon)(\ell^2-m^2-i\epsilon)}$$

$$+ \frac{1}{(k_i^2-n^2+i\epsilon)(n^2-\ell^2+i\epsilon)(n^2-m^2+i\epsilon)} + \frac{1}{(k_i^2-m^2+i\epsilon)(m^2-\ell^2+i\epsilon)(m^2-n^2-i\epsilon)}$$

$$= \frac{1}{(k_i^2-\ell^2+i\epsilon)(\ell^2-n^2-i\epsilon)(\ell^2-m^2-i\epsilon)} + \frac{1}{(k_i^2-n^2+i\epsilon)(n^2-\ell^2+i\epsilon)(n^2-m^2+i\epsilon)}$$

$$+ \frac{1}{(k_i^2-m^2+i\epsilon)(\ell^2-n^2-i\epsilon)}\left[\frac{1}{m^2-\ell^2+i\epsilon} - \frac{1}{m^2-n^2-i\epsilon}\right]$$

$$= \frac{1}{(m^2-\ell^2+i\epsilon)(n^2-\ell^2+i\epsilon)}\left[\frac{1}{k_i^2-\ell^2+i\epsilon} - \frac{1}{k_i^2-m^2+i\epsilon}\right]$$

$$+ \frac{1}{(n^2-\ell^2+i\epsilon)(n^2-m^2+i\epsilon)}\left[\frac{1}{k_i^2-n^2+i\epsilon} - \frac{1}{k_i^2-m^2+i\epsilon}\right]$$

$$= -\frac{1}{(n^2-\ell^2+i\epsilon)(k_i^2-\ell^2+i\epsilon)(k_i^2-m^2+i\epsilon)}$$

$$+ \frac{1}{(n^2-\ell^2+i\epsilon)(k_i^2-n^2+i\epsilon)(k_i^2-m^2+i\epsilon)}$$

$$= \frac{1}{(k_i^2-m^2+i\epsilon)(n^2-\ell^2+i\epsilon)}\left[\frac{1}{k_i^2-n^2+i\epsilon} - \frac{1}{k_i^2-\ell^2+i\epsilon}\right]$$

$$= \frac{1}{(k_i^2-m^2+i\epsilon)(k_i^2-n^2+i\epsilon)(k_i^2-\ell^2+i\epsilon)} \qquad (2.18)$$

So this term in the iteration gives

$$\frac{V_{\underline{k}_f\underline{\ell}}\, T_{\underline{\ell}\underline{m}}\, T^{*}_{\underline{n}\underline{m}}\, V_{\underline{n}\underline{k}_i}}{(k_i^2-\ell^2+i\epsilon)(k_i^2-m^2+i\epsilon)(k_i^2-n^2+i\epsilon)} \tag{2.19}$$

and thus we arrive at the final result

$$\frac{T_{\underline{k}_f\underline{\ell}}\, T^{*}_{\underline{k}_i\underline{\ell}}}{k_i^2-\ell^2+i\epsilon} + \frac{T_{B\underline{k}_f}\, T^{*}_{B\underline{k}_i}}{k_i^2+B} = \frac{V_{\underline{k}_f\underline{\ell}}\, V_{\underline{\ell}\underline{k}_i}}{k_i^2-\ell^2+i\epsilon}$$

$$+ \frac{V_{\underline{k}_f\underline{\ell}}\, V_{\underline{\ell}\underline{n}}\, V_{\underline{n}\underline{k}_i}}{(k_i^2-\ell^2+i\epsilon)(k_i^2-n^2+i\epsilon)} + \frac{V_{\underline{k}_f\underline{\ell}}\, T_{\underline{\ell}\underline{m}}\, T^{*}_{\underline{n}\underline{m}}\, V_{\underline{n}\underline{k}_i}}{(k_i^2-\ell^2+i\epsilon)(k_i^2-m^2+i\epsilon)(k_i^2-n^2+i\epsilon)}$$

$$+ \frac{V_{\underline{k}_f\underline{\ell}}\, T_{B\underline{\ell}}\, T^{*}_{B\underline{n}}\, V_{\underline{n}\underline{k}_i}}{(k_i^2-\ell^2+i\epsilon)(k_i^2+B)(k_i^2-n^2+i\epsilon)} \tag{2.20}$$

Now if we note that the first and second terms on the right hand side of equation (2.20) are just respectively $T^{(2)}(\underline{k}_f,\underline{k}_i)$ and $T^{(3)}(\underline{k}_f,\underline{k}_i)$ then we obtain formally

$$\frac{T_{\underline{k}_f\underline{\ell}}\, T^{*}_{\underline{k}_i\underline{\ell}}}{k_i^2-\ell^2+i\epsilon} = T^{(2)}(\underline{k}_f,\underline{k}_i) + T^{(3)}(\underline{k}_f,\underline{k}_i)$$

$$+ \frac{V_{\underline{k}_f\underline{\ell}}\, T_{\underline{\ell}\underline{m}}\, T^{*}_{\underline{n}\underline{m}}\, V_{\underline{n}\underline{k}_i}}{(k_i^2-\ell^2+i\epsilon)(k_i^2-m^2+i\epsilon)(k_i^2-n^2+i\epsilon)} \tag{2.21}$$

where now summations like $T_{\underline{k}_f \ell}\, T_{\underline{k}_i \ell}$ include the bound state terms $T_{\underline{k}_f B}\, T_{\underline{k}_i B}$.

Carrying out this iteration procedure n times we obtain

$$\frac{T_{\underline{k}_f \underline{\ell}}\, T^*_{\underline{k}_i \underline{\ell}}}{k_i^2 - \ell^2 + i\epsilon} = \sum_{s=2}^{2n+1} T^{(s)}(\underline{k}_i, \underline{k}_f) + R^{(2n+1)}(\underline{k}_i, \underline{k}_f) \tag{2.22}$$

where

$$R^{(2n+1)}(\underline{k}_i, \underline{k}_f) = \frac{T^{(n)}_{\underline{k}_f \underline{\ell}}\, T_{\underline{\ell}\underline{m}}\, T^*_{\underline{n}\underline{m}}\, T^{(n)}_{\underline{n}\underline{k}_i}}{(k_i^2 - \ell^2 + i\epsilon)(k_i^2 - m^2 + i\epsilon)(k_i^2 - n^2 + i\epsilon)} \tag{2.23}$$

remembering that integration over $\underline{m}$ includes summation over the bound states.

If we now go back to equation (2.3) we see that what we have arrived at is that

$$T(\underline{k}_i, \underline{k}_f) = \sum_{s=1}^{2n+1} T^{(s)}(\underline{k}_i, \underline{k}_f) + R^{(2n+1)}(\underline{k}_i, \underline{k}_f) \tag{2.24}$$

Thus an investigation of the analytic properties of the Born terms, together with the analytic properties of the remainder term $R^{(2n+1)}(\underline{k}_i, \underline{k}_f)$ will give us the analytic properties of the whole scattering amplitude. In addition if we can obtain information on the appropriate asymptotic behaviour, then this will give us the information necessary for the writing down of a double spectral representation.

III Analytic Properties of the Born Terms

These properties have already been obtained, by Bowcock and Walecka[5] in the case of analytic behaviour in the energy plane, and Bowcock and Martin[4] for the behaviour in the momentum transfer plane. However, we redetermine these analytic properties here by powerful new methods evolved for the purpose of locating the singularities associated with particular Feynman diagrams in perturbation theory. These methods were first introduced by Landau[11] and were developed by Polkinghorne and Screaton[12]; their application to potential theory was indicated by Fonda, Radicati and Regge.[13]

The method to be adopted may be illustrated by the example of the location of the singularities of a double integral. For the case of multiple integrals an obvious extension may be made.

The function $\phi(\zeta)$ defined by

$$\phi(\zeta) = \int_B d\beta \int_A d\alpha \, \frac{1}{[F(\alpha, \beta \zeta)]^n} \tag{3.1}$$

<u>may</u> have singularities at the points ζ^* given by either

(i) $F(\alpha_0, \beta_0, \zeta^*) = 0$

or

(ii) $F(\alpha, \beta_0, \zeta^*) = \frac{\partial}{\partial \alpha} F(\alpha, \beta_0, \zeta^*)$

or

(iii) $F(\alpha_0, \beta, \zeta^*) = \frac{\partial}{\partial \beta} F(\alpha_0, \beta, \zeta^*)$

or

(iv) $$F(\alpha,\beta,\zeta^*) = \frac{\partial}{\partial\alpha}F(\alpha,\beta,\zeta^*) = \frac{\partial}{\partial\beta}F(\alpha,\beta,\zeta^*) = 0 \tag{3.2}$$

where α_o and β_o are end-points of the paths A and B in the complex α and β planes respectively. These four cases arise for the following reasons:-

As α and β trace out the paths A and B in the respective planes, the integral (3.1) remains well defined unless $F(\alpha,\beta,\zeta)$ should happen to be zero. Even if $F(\alpha,\beta,\zeta)$ should be zero, this difficulty may be avoided by deforming the paths A and B away from these zeros, provided that none of the contingencies (i) - (iv) arise. These correspond to the following happenings which cannot be avoided by deformation of the α and β paths:-

(i) $F(\alpha,\beta,\zeta)$ is zero for some ζ with $\alpha = \alpha_o$ and $\beta = \beta_o$, i.e. at the end-point of the paths A and B, which end-points of course cannot be deformed. This is called an end-point singularity in α and β.

(ii) $F(\alpha,\beta,\zeta)$ is zero for some ζ with β at an end-point of the B path, and a coincident singularity in α, which coincidence has approached the path in the α-plane from opposite sides of the path, thus prohibiting any further deformation in the α-plane. This type of singularity is called an end-point singularity in β and a pinch singularity in α.

(iii) An end-point singularity in α together with a pinch singularity in β.

(iv) Pinch singularities in both α and β.

We now apply the generalisation of the above results to the n-th Born term. This is given by

$$T^{(n)}(\underline{k}_i, \underline{k}_f) = \lim_{\epsilon \to 0} \int d^3\underline{k}_1 \cdots d^3\underline{k}_{n-1} \int_{\mu}^{\infty} d\mu_1 \cdots d\mu_n \, \sigma(\mu_1) \cdots \sigma(\mu_n) \times$$

$$\times \frac{1}{\mu_1^2 + (\underline{k}_f - \underline{k}_1)^2} \, \frac{1}{k^2 - k_1^2 + i\epsilon} \, \frac{1}{\mu_2^2 + (\underline{k}_1 - \underline{k}_2)^2} \cdots \frac{1}{k^2 - k_{n-1}^2 + i\epsilon} \, \frac{1}{\mu_n^2 + (\underline{k}_{n-1} - \underline{k}_i)^2}$$

(3.3)

with $k_i^2 = k_f^2 = k^2 = s$

By means of the well known Feynman techniques* this integral may be written in the form

$$T^{(n)}(\underline{k}_i, \underline{k}_f) = \int_0^1 d\lambda_1 \cdots d\lambda_{2n-1} \int d^3\underline{k}_1 \cdots d^3\underline{k}_{n-1} \int_{\mu}^{\infty} d\mu_1 \cdots d\mu_n \, \sigma(\mu_1) \cdots \sigma(\mu_n) \times$$

$$\times \, \delta(\lambda_1 + \lambda_2 + \cdots + \lambda_{2n-1} - 1) \frac{1}{[F(\lambda_i, \underline{k}_j, \mu_\ell)]^{2n-1}}$$

(3.4)

(the suffix i on λ runs from 1 to 2n-1, the suffix j on $\underline{k}$ from 0 to n where $\underline{k}_0 = \underline{k}_f$ and $\underline{k}_n = \underline{k}_i$, and the suffix ℓ on μ from 1 to n), where $F(\lambda_i, \underline{k}_j, \mu_\ell)$ is given by

$$F(\lambda_i, \underline{k}_j, \mu_\ell) = \sum_{r=1}^{2n-1} \lambda_r Q_r$$

(3.5)

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* First introduced by Feynman in reference 15. A summary of the results may be found in reference 16.

with $$Q_{2r-1} = (\underline{k}_{r-1} - \underline{k}_r)^2 + \mu_r^2$$

$$Q_{2r} = k^2 - k_r^2 + i\epsilon \qquad (3.6)$$

It is now possible to perform the three-dimensional integrations over the variables $\underline{k}_1 \ \ldots . \ \underline{k}_{n-1}$. This will give

$$T^{(n)}(s,t) = \int_0^1 d\lambda_1 - d\lambda_{2n-1} \int_\mu^\infty d\mu_1 - d\mu_n \, \bar{\sigma}(\mu_1) - \sigma(\mu_n) \times$$

$$\times \quad \frac{\delta(\lambda_1 + --- + \lambda_{2n-1} - 1)\,\rho(\lambda_i)}{[F_1(\lambda_i, s, t)]^n} \qquad (3.7)$$

with $$s = k^2 = k_i^2 = k_f^2$$

$$t = (\underline{k}_i - \underline{k}_f)^2$$

where we have not explicitly written the form of the function $\rho(\lambda_i)$ since it contains neither s nor t and so at first sight would not appear to affect the location of singularities. However it has been shown in general perturbation theory that $\rho(\lambda_i)$ may give rise to a new type of singularity -- the so-called non-Landau or second type singularities. The discussion of these will not concern us here. The function F_1 may be obtained from the function F merely by eliminating the variables of integration $\underline{k}_1, \ldots, \underline{k}_{n-1}$ by means of the equations

$$\frac{\partial F}{\partial \underline{k}_j} = 0 \quad (j = 1 - - - n-1) \qquad (3.8)$$

This fact was particularly noted by Mathews[17]. Then using the results of (3.2) it is possible to show that the singularities of $T^{(n)}(s,t)$ occur for those values of s and t such that

$$\begin{aligned} &\text{either} \qquad \lambda_i = 0 \\ &\text{or} \qquad \frac{\partial F_1}{\partial \lambda_i} = 0 \;, \quad i = 1, \ldots\ldots (2n-1) \end{aligned} \tag{3.9}$$

From equation (3.8) it follows that

$$\begin{aligned} \frac{\partial F_1}{\partial \lambda_i} &= \frac{\partial F}{\partial \lambda_i} + \sum_{j=1}^{n-1} \frac{\partial F}{\partial \underline{k}_j} \frac{\partial \underline{k}_j}{\partial \lambda_i} \\ &= \frac{\partial F}{\partial \lambda_i} \\ &= Q_i \end{aligned} \tag{3.10}$$

using equation (3.5).

Hence, concisely, the singularities may occur for those values of s and t such that

$$\lambda_r Q_r = 0 \qquad (1 \leq r \leq 2n-1) \tag{3.11}$$

together with the relationship

$$\frac{\partial}{\partial \underline{k}_j} \sum_{r=1}^{2n-1} \lambda_r Q_r = 0 \tag{3.12}$$

ie. $$\lambda_r = 0$$

or $$Q_r = 0 \tag{3.13}$$

where $$Q_{2r} = s - k_r^2$$

$$Q_{2r+1} = (\underline{k}_{r+1} - \underline{k}_r)^2 + \mu_{r+1}^2 \tag{3.14}$$

together with

$$\lambda_{2j}\underline{k}_j = \lambda_{2j-1}(\underline{k}_{j-1} - \underline{k}_j) + \lambda_{2j+1}(\underline{k}_{j+1} - \underline{k}_j) \tag{3.15}$$

We may use these conditions to eliminate the λ's and the $\underline{k}$'s and so obtain a relationship between s and t. This relationship, plotted as a curve in the real s, t plane is known as a Landau curve.

To obtain our desired results it is necessary to utilise a powerful general theorem from the perturbation theoretic approach to analytic properties. For our purpose the theorem may be stated as follows:

Suppose we know some region of the real s, t plane in which all the denominators appearing in the expression (3.3) for $T^{(n)}(s,t)$ are non-zero. (This region is known as the Symanzik region). Then we may continue s and t out of this region without encountering any singularity of $T^{(n)}(s,t)$ until we come to a Landau curve with corresponding Feynman parameters lying between 0 and 1, and moreover, this Landau curve must necessarily be singular.

Thus our first task is to determine the Symanzik region, and our second is to determine the nearest Landau curves to this region, with Feynman parameters between 0 and 1.

To determine the Symanzik region we look for those values of s and t which will certainly make all denominators in equation (3.3) non-zero. First of all the propagator terms $s - k_r^2 + i\varepsilon$ will all certainly be non-zero in the limit only if $s \leq 0$. So the Symanzik region must certainly lie in the half-plane $s \leq 0$. The intermediate potential terms $(\underline{k}_r - \underline{k}_{r+1})^2 + \mu_{r+1}^2$ do not depend on t or s, and are all certainly non-zero. So we are left with the first and last potential terms:

$$(\underline{k}_f - \underline{k}_1)^2 + \mu^2$$

and $(\underline{k}_{n-1} - \underline{k}_i)^2 + \mu^2$ * (3.16)

which we shall rewrite for this purpose as

$$(\underline{k}_i' - \underline{k}')^2 + \mu^2$$

and $(\underline{k}_f - \underline{k}'')^2 + \mu^2$ (3.17)

---

* We utilise μ instead of μ_1 and μ_n in these equations, since μ is the minimum value of both μ_1 and μ_n, and it is on this minimum value that the region is found to depend.

So we require to look for the region of the s t plane with s < 0 for which the expressions (3.17) do not vanish. Let us find the region in which they do vanish; to this end we choose the special representation

$$\underline{k}_i = \left(\sqrt{s-\tfrac{t}{4}},\ \tfrac{1}{2}\sqrt{t},\ 0\right)$$

$$\underline{k}_f = \left(\sqrt{s-\tfrac{t}{4}},\ -\tfrac{1}{2}\sqrt{t},\ 0\right) \qquad (3.18)$$

$$\underline{k}' = (k_1', k_2', k_3')$$

$$\underline{k}'' = (k_1'', k_2'', k_3'') \qquad (3.19)$$

so that the regions of vanishing are just

$$\left(k_1' - \sqrt{s-t/4}\right)^2 + \left(k_2' - \tfrac{1}{2}\sqrt{t}\right)^2 + k_3'^2 + \mu^2 = 0 \qquad (3.20)$$

$$\left(k_1'' - \sqrt{s-t/4}\right)^2 + \left(k_2'' + \tfrac{1}{2}\sqrt{t}\right)^2 + k_3''^2 + \mu^2 = 0 \qquad (3.21)$$

We now note that the regions (3.20) and (3.21) are identical. This arises due to the fact that k_1', k_2', k_3', k_1'', k_2'', and k_3'' all take on values between $-\infty$ and $+\infty$, so that in (3.21) we may replace k_2'' by $-k_2''$, which will then make it identical with (3.20).

We remember that we are considering $s < 0$, so that we may write $s = -|s|$. Let us first consider the case where $t > 0$ so that $t = |t|$. Then (3.20) reduces to

$$\left(k_1' - i\sqrt{|s| + \tfrac{1}{4}|t|}\right)^2 + \left(k_2' - \tfrac{1}{2}\sqrt{|t|}\right)^2 + k_3'^2 + \mu^2 = 0 \qquad (3.22)$$

The left-hand side here will always have an imaginary part, and so be non-zero, unless $k_1' = 0$. In this eventuality we will have

$$-\left(|s| + \tfrac{1}{4}|t|\right) + \left(k_2' - \tfrac{1}{2}\sqrt{|t|}\right)^2 + k_3'^2 + \mu^2 = 0$$

$$|s| + \frac{1}{4}|t| = \theta^2 \qquad (3.23)$$

where θ^2 can take on all positive values greater than μ^2. Thus the Symanzik region for $s < 0$, $t > 0$ will be bounded by the line

$t = 4(s+\mu^2)$ and will in fact lie below this line, as shown in Fig. 2.

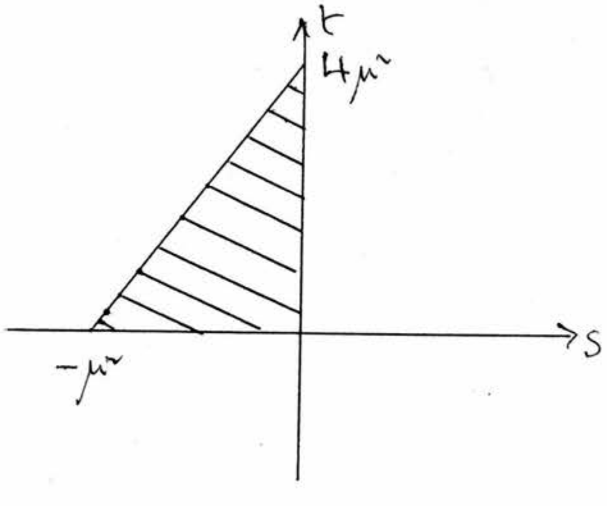

Fig. 2.

Let us now examine the case where $s < 0$ and $t < 0$, so that $s = -|s|$ and $t = -|t|$ Then equation (3.20) reduces to

$$\left(k_1' - \sqrt{\tfrac{1}{4}|t| - |s|}\right)^2 + \left(k_2' - \tfrac{1}{2} i \sqrt{|t|}\right)^2 + k_3'^2 + \mu^2 = 0 \qquad (3.23)$$

If first of all we consider $\frac{1}{4}|t| - |s| > 0$ then this equation will always have an imaginary part unless $k_2' = 0$ in which case $-\frac{1}{4}|t| + k_3'^2 + \mu^2 = 0$

$$\text{i.e.} \qquad |t| = 4(k_3'^2 + \mu^2) \qquad (3.24)$$

so that for the denominator to be zero we must have

$$|t| \geq 4\mu^2$$

$$\text{i.e.} \qquad t \leq -4\mu^2 \qquad (3.25)$$

so that we know the region shown in Fig. 3 will be added to the Symanzik region.

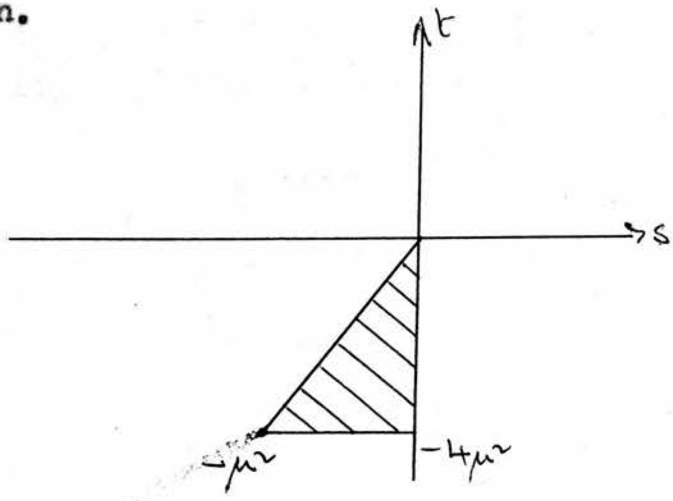

Fig. 3.

If now $\frac{1}{4}|t| - |s| < 0$ equation (3.23) must be written in the form

$$\left(k_1' - i\sqrt{|s| - \tfrac{1}{4}|t|}\right)^2 + \left(k_2' - \tfrac{1}{2} i \sqrt{|t|}\right)^2 + k_3'^2 + \mu^2 = 0 \qquad (3.26)$$

i.e. $$k'^2 - |s| + \mu^2 + i\left[2k_1'\sqrt{|s| - \tfrac{1}{4}|t|} + k_2'\sqrt{|t|}\right] = 0 \qquad (3.27)$$

The left hand side will always have an imaginary part and so be non-zero, unless

$$\frac{k_1'}{\sqrt{|t|}} = -\frac{k_2'}{2\sqrt{|s| - \frac{1}{4}|t|}} = \lambda, \text{ say} \qquad (3.28)$$

in which case (3.27) reduces to

$$|s| = \mu^2 + k_3'^2 + \lambda^2(4|s| - |t|) + \lambda^2|t| \qquad (3.29)$$

i.e. $$|s| = \frac{\mu^2 + k_3'^2}{1 - 4\lambda^2}$$

i.e. $$|s| \geq \mu^2$$

so that we have the addition to the Symanzik region of the area shown in Fig. 4.

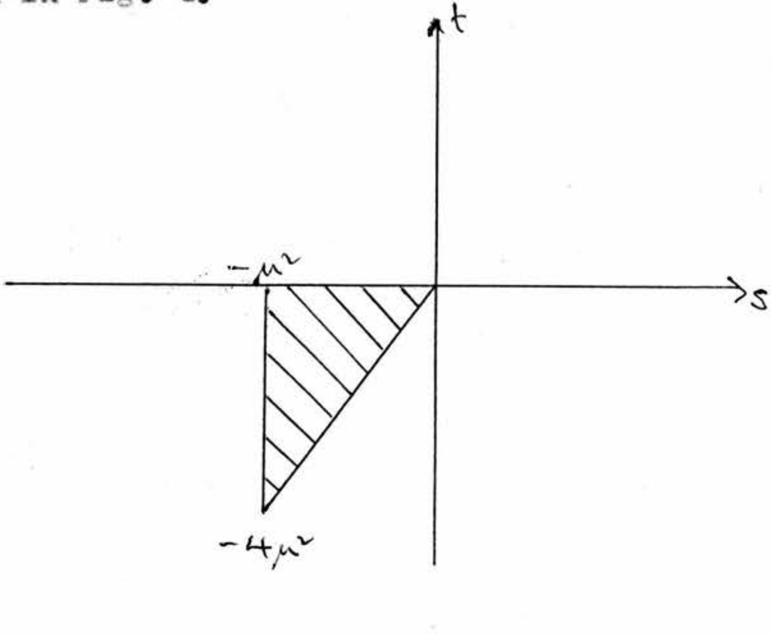

Fig. 4.

and thus the whole Symanzik region is of the form shown in Fig. 5.

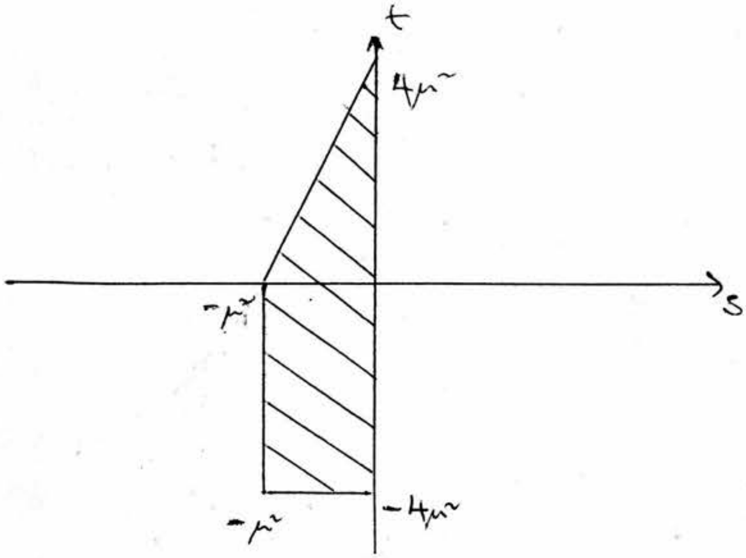

Fig. 5.

We now look for the Landau curves nearest this region with Feynman parameters between 0 and 1, i.e. we look for solutions of the equations (3.13), (3.14) and (3.15) with $0 \leq \lambda_r \leq 1$.

We consider several cases:-

(i) all λ's non-zero

If in this case we take equation (3.15) and take its scalar product with $\underline{k}_j$, then we shall obtain

$$\lambda_{2j} k_j^2 = \lambda_{2j-1}(\underline{k}_j.\underline{k}_{j-1} - k_j^2) + \lambda_{2j+1}(\underline{k}_j.\underline{k}_{j+1} - k_j^2) \qquad (3.30)$$

i.e. $$\lambda_{2j} s = \lambda_{2j-1}(s+\tfrac{1}{2}\mu_j^2 - s) + \lambda_{2j+1}(s+\tfrac{1}{2}\mu_{j+1}^2 - s)$$

using equations (3.13) and (3.14).

i.e. $$s = \frac{\lambda_{2j-1}}{2\lambda_{2j}}\mu_j^2 + \frac{\lambda_{2j+1}}{2\lambda_{2j}}\mu_{j+1}^2 \qquad (3.31)$$

since $\lambda_{2j} \neq 0$.

i.e. $$s > 0 \qquad (3.32)$$

because of the condition we have imposed on the λ's.

We shall show that there is a Landau curve (viz. $s = 0$) nearer the Symanzik region than any curve satisfying this condition, and so all types of curves arising from all λ's non-zero may be discarded from the argument.

(ii) Some λ_{2i}'s are zero, but all λ_{2i+1}'s are non-zero.

Provided all the λ_{2i} are not zero, equation (3.31) will hold for those λ_{2i} which are not zero and hence we may still deduce that the Landau curves lie in the region s 0. If, however, all the λ_{2i}'s are zero we cannot make the deduction that s is positive. In this case, then, we have

$$\lambda_{2j-1}\underline{\Delta}_{j-1,j} + \lambda_{2j+1}\underline{\Delta}_{j+1,j} = 0 \qquad (3.33)$$

(all j) $\qquad (\underline{\Delta}_{ab} = \underline{k}_b - \underline{k}_a)$

i.e. $$\lambda_{2j-1}\underline{\Delta}_{j-1,j} = \lambda_{2j+1}\underline{\Delta}_{j,j+1} \qquad (3.34)$$

Hence we may deduce that $\underline{\Delta}_{j-1,j}$ and $\underline{\Delta}_{j,j+1}$ are parallel, and, since λ_{2j-1} and λ_{2j+1} are both positive, that they also point in the same direction. Thus we can say that, since

$$
\begin{aligned}
(\underline{k}_f - \underline{k}_i) &= (\underline{k}_0 - \underline{k}_n) \\
&= \underline{\Delta}_{0n} \\
&= \underline{\Delta}_{01} + \underline{\Delta}_{12} + \underline{\Delta}_{23} + \text{-----} + \underline{\Delta}_{n-1,n} \qquad (3.35)
\end{aligned}
$$

that $|\underline{\Delta}_{0n}| = |\underline{\Delta}_{01}| + |\underline{\Delta}_{12}| + |\underline{\Delta}_{23}| + \text{---} + |\underline{\Delta}_{n-1,n}|$

$$= i\mu_1 + i\mu_2 + i\mu_3 + \text{---} + i\mu_n \qquad (3.36)$$

by virtue of the fact that $Q_{2r+1} = 0$ since all λ_{2r+1} are non zero.

Thus we have

$$
\begin{aligned}
t &= \Delta_{0n}^2 \\
&= -(\mu_1 + \mu_2 + \mu_3 + \text{---} + \mu_n)^2 \qquad (3.37)
\end{aligned}
$$

which has as its maximum value

$$t = -n^2\mu^2 \qquad (3.38)$$

and thus this line is a Landau curve, and so we may certainly not extend the Symanzik region below this line.

(iii) Some of the λ_{2i+1} are zero

First of all suppose λ_{2j-1} is zero, but λ_{2j}, λ_{2j+1} and λ_{2j+2} are non-zero. Then we shall have

$$\lambda_{2j}\,\underline{k}_j = \lambda_{2j+1}(\underline{k}_{j+1} - \underline{k}_j) \qquad (3.39)$$

together with $$(\underline{k}_j - \underline{k}_{j+1})^2 = -\mu_{j+1}^2 \tag{3.40}$$

and $$s = k_j^2 \qquad s = k_{j+1}^2 \tag{3.41}$$

(3.39) gives $$\underline{k}_{j+1} = \frac{\lambda_{2j} + \lambda_{2j+1}}{\lambda_{2j+1}} \underline{k}_j \tag{3.42}$$

and thus from (3.41)

$$s = \left(\frac{\lambda_{2j} + \lambda_{2j+1}}{\lambda_{2j+1}} \right)^2 s \tag{3.43}$$

which since the λ's are positive can only be satisfied for $s = 0$. But if $s = 0$, then we shall have

$$k_j^2 = k_{j+1}^2 = \underline{k}_j \cdot \underline{k}_{j+1} = 0 \tag{3.44}$$

and then it is impossible to satisfy (3.40). Thus we are led to no singularities.

In a similar manner it may be shown that there are no solutions of the Landau equations unless <u>all</u> the λ_{2i-1} are zero. Then we are left with the equations

$$\begin{aligned} \lambda_{2j} \underline{k}_j &= 0 \\ k_j^2 &= s \end{aligned} \tag{3.45}$$

for some j, which obviously have the consistent solution $s = 0$. Thus $s = 0$ is a Landau curve, corresponding to real positive Feynman parameters lying between 0 and 1.

Thus the lines $s = 0$ and $t = -n^2\mu^2$ are the nearest Landau curves (with Feynman parameters in the appropriate range) to the Symanzik region found above, and so this region may be extended to the whole horizontally shaded region shown in Fig. 6.

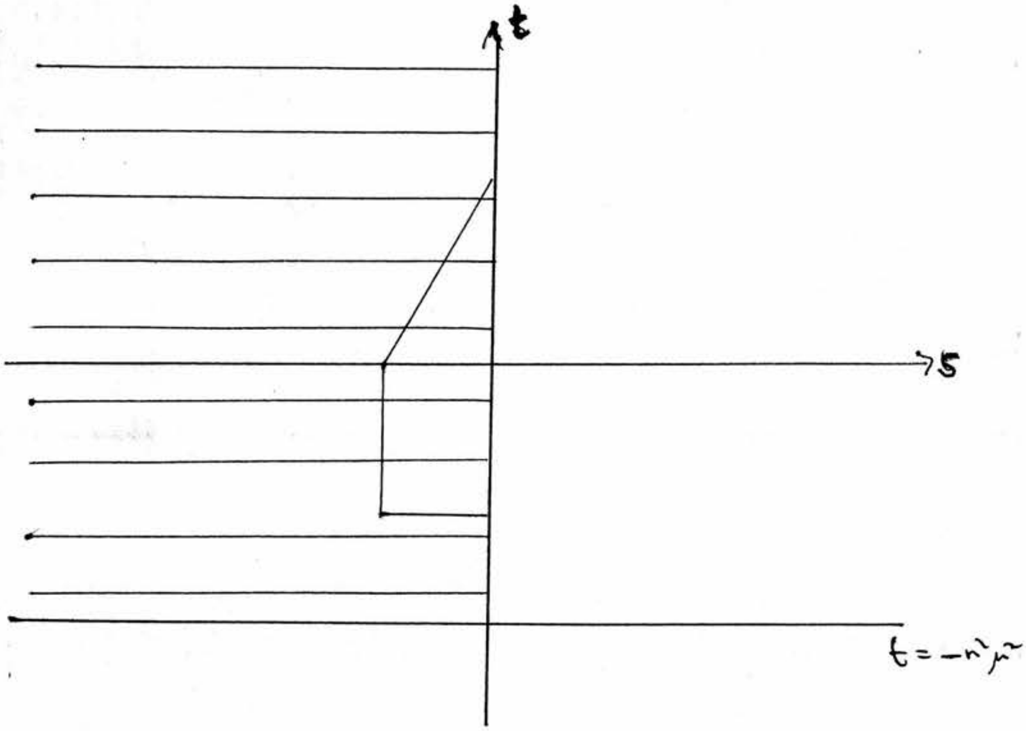

Fig. 6.

We shall now determine the form of the Landau curves when the Feynman parameters are allowed to take on any real or complex values consistent with the Landau equations. We again consider separately the different combinations of vanishing λ's. But first we note a general theorem which says that any singularities found for some of the λ's zero will actually occur in the Born term under consideration. This is a special case of a general theorem in perturbation theory which states that singularities appearing in a contracted graph on the physical sheet will also appear on the physical sheet of the uncontracted graph.

(i) <u>All $\lambda_i \neq 0$</u>

We shall first of all carry through the work with all μ_j equal (the so-called equal mass case) and then indicate the differences which arise when the μ_j are all different (the unequal mass case).

Since all the λ_i are non-zero we shall have

$$Q_r = 0 \qquad r = 1 \text{ ----- } 2n-1 \tag{3.46}$$

so that

$$(\underline{k}_{r-1} - \underline{k}_r)^2 + \mu^2 = 0$$
$$s - k_r^2 = 0$$
$$(r = 1 \text{ ---- } n, \ \underline{k}_0 = \underline{k}_f, \ \underline{k}_n = \underline{k}_i) \tag{3.47}$$

The first of the equations (3.47) may be written in the form

$$k_{r-1}^2 + k_r^2 - 2\underline{k}_r . \underline{k}_{r-1} + \mu^2 = 0 \tag{3.48}$$

which on using the second of the equations (3.47) becomes

$$2s - 2s\cos\theta_{r,r-1} + \mu^2 = 0 \tag{3.49}$$

(where θ_{ij} is the angle between $\underline{k}_i$ and $\underline{k}_j$) which may be written in the form

$$\cos\theta_{r,r-1} = 1 + \frac{\mu^2}{2s}$$

$$= 1 + y \quad \text{say} \tag{3.50}$$

The circuit relationship (3.15) still holds.

$$\lambda_{2j}\underline{k}_j = \lambda_{2j-1}(\underline{k}_{j-1} - \underline{k}_j) + \lambda_{2j+1}(\underline{k}_{j+1} - \underline{k}_j) \tag{3.51}$$

so that all the vectors $\underline{k}_r$ are coplanar, and hence we may write

$$\theta_{0n} = \theta_{01} + \theta_{12} + \theta_{23} + \text{———} + \theta_{n-1,n}$$

$$= n\theta_{01} \tag{3.52}$$

because of equation (3.50).

It is important to note that we do not have the ambiguity of minus signs in equation (3.52), even though equation (3.50) allows both the solutions $\theta_{r,\ r-1}$ and $-\theta_{r,\ r-1}$. This is because if we had the successive solutions $\theta_{r,\ r-1}$ and $-\theta_{r,\ r+1}$ we should then have $\underline{k}_{r-1} = \underline{k}_{r+1}$ and thus the circuit relation would become

$$(\lambda_{2r-1} + \lambda_{2r} + \lambda_{2r+1})\underline{k}_r = (\lambda_{2r-1} + \lambda_{2r+1})\underline{k}_{r+1} \tag{3.53}$$

so that, since $k_r^2 = s$ for all r, we must have

$$\lambda_{2r}^2 + 2\lambda_{2r}(\lambda_{2r-1} + \lambda_{2r+1}) = 0 \tag{3.54}$$

so that $\lambda_{2r} = 0$ (which is inadmissible since we have presupposed all λ's to be non-zero)

or else

$$\lambda_{2r} + 2(\lambda_{2r-1} + \lambda_{2r+1}) = 0 \tag{3.55}$$

in which case we should have

$$\lambda_{2r} \underline{k}_r = -\lambda_{2r} \underline{k}_{r+1} \tag{3.56}$$

or in other words $\lambda_{2r} = 0$ (inadmissible) or else $\theta_{r+1,r} = \pi$ in which particular case $-\theta_{r+1}$, r is <u>exactly</u> the same as $\theta_{r+1,r}$. Thus unless $\theta_{r+1,r} = \pi$ we are forced to the conclusion that we must measure all $\theta_{r+1,r}$ in the same direction.

Thus we now have

$$\begin{aligned} 1 - \frac{t}{2s} &= \cos \theta_{0n} \\ &= \cos nx \end{aligned} \tag{3.57}$$

where

$$\begin{aligned} \cos x &= \cos \theta_{01} \\ &= 1 + \frac{\mu^2}{2s} \\ &= 1 + y \end{aligned} \tag{3.58}$$

Hence we now wish to evaluate $\cos nx$ in terms of $\cos x$ and we shall then have the equation of the required Landau curve. This evaluation is carried out as follows:-

$$\cos nx = \mathrm{Re}\,(\cos x + i \sin x)^n$$

$$= \sum_{r\,\text{even}} {}^nC_r\,(i\sin x)^r(\cos x)^{n-r}$$

$$= \sum_{r\,\text{even}} {}^nC_r\,i^r(1-\cos^2 x)^{\frac{r}{2}}(\cos x)^{n-r}$$

$$= \sum_{r\,\text{even}} {}^nC_r\,i^r(1-\cos x)^{\frac{r}{2}}(1+\cos x)^{\frac{r}{2}}(\cos x)^{n-r}$$

$$= \sum_{r\,\text{even}} {}^nC_r\,i^r(-y)^{r/2}(2+y)^{r/2}(1+y)^{n-r}$$

using (3.58)

$$= \sum_{r\,\text{even},\,\le n} {}^nC_r\,i^r(-1)^{\frac{r}{2}}y^{\frac{r}{2}}\Big(\sum_{m\le\frac{r}{2}} {}^{\frac{r}{2}}C_m\,2^m y^{\frac{r}{2}-m}\Big)\Big(\sum_{l\le n-r} {}^{n-r}C_l\,y^l\Big)$$

$$= \sum_{r(\text{even})=0}^{n}\ \sum_{m=0}^{r/2}\ \sum_{l=0}^{n-r} {}^nC_r\,{}^{\frac{r}{2}}C_m\,{}^{n-r}C_l\,2^m y^{r-m+l} \qquad (3.59)$$

$$= \sum_{s=0}^{n} a_s y^s \qquad (3.60)$$

i.e. we have obtained

$$1-\frac{t}{2s} = a_0+a_1y+a_2y^2+\cdots+a_ny^n \qquad (3.61)$$

It is of interest to evaluate a_o and a_1.

From (3.59) we see that a_o may only be obtained by taking $r = 0$, $m = 0$ and $\ell = 0$, so that

$$\begin{aligned} a_0 &= {}^0C_0\,{}^0C_0\,{}^nC_0\,2^0 \\ &= 1 \end{aligned} \tag{3.62}$$

a_1, however may be obtained in two ways. We can have either $r = 0$, $m = 0$ and $\ell = 1$, or else $r = 2$, $m = 1$ and $\ell = 0$.

Thus

$$\begin{aligned} a_1 &= {}^nC_0\,{}^0C_0\,{}^nC_1\,2^0 + {}^nC_2\,{}^1C_1\,{}^{n-2}C_0\,2^1 \\ &= n + \tfrac{1}{2}n(n-1).2 \\ &= n^2 \end{aligned} \tag{3.63}$$

Concerning the other coefficients, all we do is to note that they are positive. This, however, gives us quite a lot of information concerning the character of the Landau curve. For (3.61), (3.62) and (3.63) give (remembering that $y = \mu^2/2s$)

$$t = -n^2\mu^2 - \frac{b_1}{s} - \frac{b_2}{s^2} - \cdots\cdots - \frac{b_{n-1}}{s^{n-1}} \tag{3.64}$$

with all $b_i > 0$

This is easily seen to be a curve with asymptotes $s = 0$ and $t = -n^2\mu^2$. For $s > 0$ it has positive gradient and thus the part of the curve with $s > 0$ is of the form shown in Fig. 7.

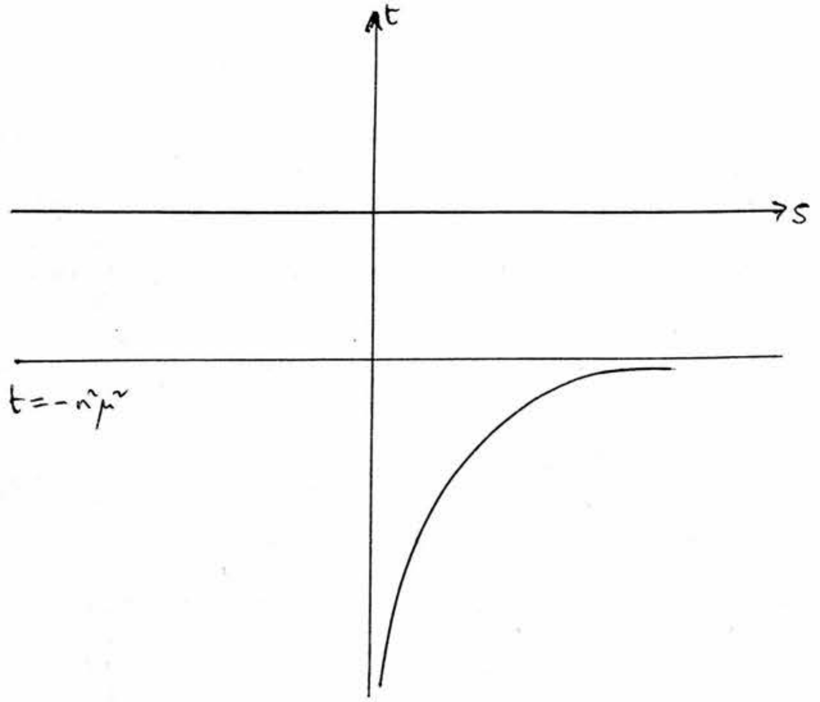

Fig. 7.

The part of the curve for $s < 0$ is much more complicated (and its complexity increases with increasing n); for example for the second Born term the entire Landau curve is just the rectangular hyperbola

$$s(t+4\mu^2) = -\mu^2 \tag{3.65}$$

but the form becomes much more complicated for higher Born terms, as may be seen from equation (3.64).

The form of the curve for the first Born term is specially important, and does not conform to the above pattern; it is just the straight line $t = -\mu^2$.

So much for the equal mass case. What now of the unequal mass case? Then we shall have instead of equation (3.50)

$$\cos\theta_{r,r-1} = 1 + \frac{\mu_r^2}{2s} \tag{3.66}$$

The coplanar condition for the vectors $\underline{k}_r$ still holds, but we can no longer assert that all angles $\theta_{r,r-1}$ must be of the same sign so that we can now only say that we must have

$$\theta_{0n} = \theta_{01} \pm \theta_{12} \pm \theta_{23} \pm - - - \pm \theta_{n-1,n} \tag{3.67}$$

We shall show by an induction procedure that these all lead to curves of the same type as before, with asymptotes s = 0 and

$$t = -(\mu_1 \pm \mu_2 \pm \mu_3 \pm \mu_4 \pm \ldots\ldots \pm \mu_n)^2 \quad (3.6?)$$

The curve we are interested in is

$$t = 2s(1 - \cos\theta_{0n}) \tag{3.68}$$

Let us consider the curve

$$t_{ij} = 2s(1 - \cos\theta_{ij}) \tag{3.69}$$

What we shall do is to show that if the curve $t_{o,j}$ has the desired behaviour, then so has $t_{o,j+1}$.

Now

$$t_{o,j+1} = 2s(1 - \cos\theta_{o,j+1})$$

$$= 2s\left[1 - \cos(\theta_{oj} \pm \theta_{j,j+1})\right]$$

$$= 2s\left[1 - (\cos\theta_{oj}\cos\theta_{j,j+1} \mp \sin\theta_{oj}\sin\theta_{j,j+1})\right] \tag{3.70}$$

$$= 2s\left[1-\left(1-\frac{t_{oj}}{2s}\right)\left(1+\frac{\mu_{j+1}^2}{2s}\right) \pm \sqrt{\left\{1-\left(1-\frac{t_{oj}}{2s}\right)^2\right\}\left\{1-\left(1+\frac{\mu_{j+1}^2}{2s}\right)^2\right\}}\right]$$

$$= 2s\left[1-1-\frac{\mu_{j+1}^2}{2s} + \frac{t_{oj}}{2s}\left(1+\frac{\mu_{j+1}^2}{2s}\right) \pm \sqrt{-\frac{t_{oj}\mu_{j+1}^2}{2s.2s}\left(2-\frac{t_{oj}}{2s}\right)\left(2+\frac{\mu_{j+1}^2}{2s}\right)}\right]$$

$$= -\mu_{j+1}^2 + t_{oj}\left(1+\frac{\mu_{j+1}^2}{2s}\right) \pm \sqrt{-\mu_{j+1}^2 t_{oj}\left(2-\frac{t_{oj}}{2s}\right)\left(2+\frac{\mu_{j+1}^2}{2s}\right)}$$

(3.71)

This obviously has the asymptotes $s = 0$, for as $s \to 0$ $t_{o,j+1} \to \infty$. Also as $s \to \infty$, if we assume that $t_{o,j} \to -(\mu_1 \pm \mu_2 \pm \cdots \pm \mu_j)^2$ then we shall have

$$t_{o,j+1} \to -\mu_{j+1}^2 - (\mu_1 \pm \mu_2 \pm \cdots \pm \mu_j)^2 \pm \sqrt{\mu_{j+1}^2 (\mu_1 \pm \cdots \pm \mu_j)^2 \cdot 2.2}$$

$$= -\mu_{j+1}^2 - (\mu_1 \pm \mu_2 \pm \cdots \pm \mu_j)^2 \pm 2\mu_{j+1}(\mu_1 \pm \cdots \pm \mu_j)$$

$$= -(\mu_1 \pm \mu_2 \pm \cdots \pm \mu_j \pm \mu_{j+1})^2 \qquad (3.72)$$

so that we get the asymptotes

$$t_{o,j+1} = -(\mu_1 \pm \mu_2 \pm \cdots \pm \mu_{j+1})^2 \qquad (3.73)$$

But this condition may be proved directly for t_{o2} and thus the induction procedure will lead to the required behaviour for t_{on} (which is just t). We note that we cannot readily deduce the fact that the curve is for $s > 0$ of positive gradient and so stays below its asymptote, but this will turn out to be unimportant.

We thus see from this part of the analysis that in the n-th Born term we have the possibility of singularities associated with the above described Landau curves. The character of these singularities we shall discuss presently.

(ii) <u>$\lambda_{2i} = 0$ for some i, but all $\lambda_{2i-1} \neq 0$</u>

We now discuss this case, corresponding in perturbation theory parlance to the singularities arising from a contracted graph (which by the general theorem mentioned above will also be singularities of the uncontracted graph). If we so wish we may represent the n-th Born term by means of the graph in Fig. 8, in which vertical lines represent interactions with the potential, and internal horizontal lines represent

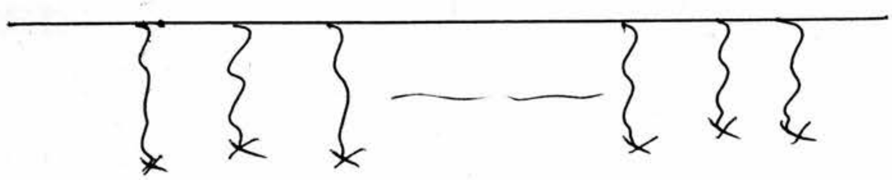

Fig. 8.

propagators. Then the case in which some of the λ_{2l}'s are chosen zero represents the case in which some of the propagator lines are contracted.

We first of all show that the asymptotes $t = -(\mu_1 \pm \mu_2 \pm \ldots\ldots \pm \mu_n)^2$ of the unequal mass Landau curves, are in fact Landau curves themselves, corresponding to all propagator lines being contracted. For in this case we shall have the circuit relation reducing to

$$\lambda_{2r-1}\underline{\Delta}_{r-1,r} = \lambda_{2r+1}\underline{\Delta}_{r,r+1} \tag{3.74}$$

so that $\underline{\Delta}_{r-1,r}$ and $\underline{\Delta}_{r,r+1}$ are linearly dependent and thus either parallel or antiparallel for all r. Thus we may write

$$|\underline{\Delta}_{0n}| = |\underline{\Delta}_{01}| \pm |\underline{\Delta}_{12}| \pm |\underline{\Delta}_{23}| \pm \text{---} \pm |\underline{\Delta}_{n-1,n}| \tag{3.75}$$

But since
$$\Delta^2_{r-1,r} = -\mu_r^2 \tag{3.76}$$

for all r, since all of the λ_{2r-1} are non-zero we will have

$$|\underline{\Delta}_{r-1,r}| = i\mu_r \tag{3.77}$$

and thus

$$|\underline{\Delta}_{0n}| = i(\mu_1 \pm \mu_2 \pm \mu_3 \pm \text{---} \pm \mu_n) \tag{3.78}$$

so that

$$\begin{aligned} t &= \Delta^2_{0n} \\ &= -(\mu_1 \pm \mu_2 \pm \text{———} \pm \mu_n)^2 \end{aligned} \tag{3.79}$$

and thus this is a Landau curve.

We now show that the integral corresponding to the contracted graph may be written in the form of an uncontracted Born term or graph, of order n - 1. It will be found that one of the potentials in this new Born term will have been altered from those appearing previously. Suppose that $\lambda_{2i} = 0$ so that the propagator $\frac{1}{s - k_i^2 + i\epsilon}$ is missing from the Born term, and we are left with the terms involving $\underline{k}_i$:-

$$\int_{\mu}^{\infty} \sigma(m)\,\sigma(m')\,dm\,dm' \int d^3\underline{k}_i \frac{1}{[(\underline{k}_{i-1} - \underline{k}_i)^2 + m^2][(\underline{k}_i - \underline{k}_{i+1})^2 + m'^2]} \qquad (3.80)$$

Now, it is well known that

$$\frac{1}{m^2 + \Delta^2} = \frac{1}{4\pi}\int d^3\underline{r}\; e^{-i\underline{\Delta}.\underline{r}}\;\frac{e^{-mr}}{r} \qquad (3.81)$$

so that (3.80) may be written in the form

$$\frac{1}{16\pi^2}\int_{\mu}^{\infty} \sigma(m)\,\sigma(m')\,dm\,dm' \int d^3\underline{k}_i\, d^3\underline{r}\, d^3\underline{r}' \times$$

$$\times\; e^{-i(\underline{k}_i - \underline{k}_{i-1}).\underline{r}}\;\frac{e^{-mr}}{r}\; e^{-i(\underline{k}_{i+1} - \underline{k}_i).\underline{r}'}\;\frac{e^{-m'r'}}{r'} \qquad (3.82)$$

$$= \frac{1}{16\pi^2} \int_{\mu}^{\infty} \sigma(m)\,\sigma(m')\,dm\,dm' \int d^3\underline{k}_i\, d^3\underline{r}\, d^3\underline{r}' \times$$

$$\times\, e^{i\underline{k}_{i-1}\cdot\underline{r} - i\underline{k}_{i+1}\cdot\underline{r}'}\ \frac{e^{-(mr+m'r')}}{rr'}\, e^{-i\underline{k}_i\cdot(\underline{r}-\underline{r}')}$$

$$= \frac{\bar{\lambda}}{2} \int_{\mu}^{\infty} \sigma(m)\sigma(m')dm\,dm' \int d^3\underline{r}\, d^3\underline{r}'\, e^{-i(\underline{k}_{i+1}\cdot\underline{r}' - \underline{k}_{i-1}\cdot\underline{r})}\ \frac{e^{-(mr+m'r')}}{rr'}\, \delta_3(\underline{r}-\underline{r}')$$

(on using the integral representation of the

-function)

$$= \frac{\bar{\lambda}}{2} \int_{\mu}^{\infty} \sigma(m)\sigma(m')\,dm\,dm' \int d^3\underline{r}\, e^{-i(\underline{k}_{i+1}-\underline{k}_{i-1})\cdot\underline{r}}\ \frac{e^{-(m+m')r}}{r^2} \tag{3.83}$$

Now, if we consider the factor in (3.83)

$$\int_{\mu}^{\infty} \sigma(m)\,\sigma(m')\,dm\,dm'\ \frac{e^{-(m+m')r}}{r^2} \tag{3.84}$$

and change variables to λ and λ' defined by

$$\lambda = m + m'$$
$$\lambda' = m \tag{3.85}$$

we find that the above expression (3.84) is equal to

$$\int_{2\mu}^{\infty} \rho(\lambda)\, \frac{e^{\lambda r}}{r^2}\, d\lambda \tag{3.86}$$

$$\text{with} \quad \rho(\lambda) = \int_{\mu}^{\cancel{\infty}\,\lambda-\mu} \sigma(\lambda')\, \sigma(\lambda-\lambda')\, d\lambda' \tag{3.87}$$

We now note that

$$\int_{2\mu}^{\infty} \rho(\lambda)\, \frac{e^{\lambda r}}{r^2}\, d\lambda = \int_{2\mu}^{\infty} dm \left[\int_{2\mu}^{m} \rho(\lambda)\, d\lambda \right] \frac{e^{-mr}}{r} \tag{3.88}$$

as may easily be verified on integration by parts.

Hence we have

$$\int_{\mu}^{\infty} \sigma(m)\, \sigma(m')\, dm\, dm' \;\; \frac{e^{-(m+m')r}}{r^2} = \int_{2\mu} \sigma'(m)\, dm\, \frac{e^{-mr}}{r} \tag{3.89}$$

with

$$\sigma'(m) = \int_{2\mu}^{\cancel{\infty}\,m} \rho(\lambda)\, d\lambda \tag{3.90}$$

and $\rho(\lambda)$ given by equation (3.87).

If we now put this expression given by equation (3.89) back into equation (3.83) we find that we finally obtain

$$\int_{\mu}^{\infty}\sigma(m)\sigma(m')\,dm\,dm'\int d^3\underline{k}_i\,\frac{1}{[(\underline{k}_{i-1}-\underline{k}_i)^2+m^2][(\underline{k}_i-\underline{k}_{i+1})^2+m'^2]}$$

$$=\int_{2\mu}^{\infty}\frac{\sigma'(m)}{(\underline{k}_{i-1}-\underline{k}_{i+1})^2+m^2}\,dm \qquad (3.91)$$

This means that the singularities corresponding to the omission of a propagator from the n-th Born term, can be discussed in terms of singularities of the (n-1)th; provided one of the potential terms is altered, both in respect of its range (the lower limit of the integral being changed from μ to 2μ) and its weight function (being altered in accordance with equations (3.87) and (3.90).

This procedure may now be repeated. In order, however to pass from one of these modified potential Born terms with a propagator missing to a Born term of one order lower, it is necessary to have a slight generalisation of equation (3.91). This is:-

$$\int_{n_1\mu}^{\infty}\sigma_1(m)\,dm\int_{n_2\mu}^{\infty}\sigma_2(m')\,dm'\int d^3\underline{k}\,\frac{1}{[(\underline{p}-\underline{k})^2+m^2][(\underline{k}-\underline{q})^2+m'^2]}$$

$$=\int_{(n_1+n_2)\mu}^{\infty}\sigma'(m)\,\frac{1}{(\underline{p}-\underline{q})^2+m^2}\,dm \qquad (3.92)$$

with
$$\sigma'(m)=\int_{(n_1+n_2)\mu}^{m}\rho(\lambda)\,d\lambda \qquad (3.93)$$

and
$$\rho(\lambda)=\int_{n_1\mu}\sigma_1(\lambda')\,\sigma_2(\lambda-\lambda')\,d\lambda' \qquad (3.94)$$

This generalisation presents absolutely no difficulties, the steps involved being exactly the same as those in passing from equation (3.80) to equation (3.91).

Thus we have finally arrived at the result that the singularities of the n-th Born term corresponding to the Feynman parameters multiplying the propagators being zero can all be expressed in terms of the singularities of lower order Born terms, with all Feynman parameters non-vanishing (and thus in terms of the Landau curves discussed above) providing the potential is suitably altered.

We must now consider the effect of this alteration in potential on the analysis above for non-zero Feynman parameters. Although the weight function and lower limits are altered in performing the contraction of one propagator, the actual sum of all the lower limits is left unaltered so that all the singular curves for the lower orders are bounded by the same asymptotes as in the case of the uncontracted graph. Indeed, the Landau curve corresponding to all propagators contracted will just be the straight line $t = -(\mu_1 + \dots + \mu_n)^2$, as is obvious from the fact that the above analysis leads us to a first Born term with modified potential.*

(iii) λ = 0 for some i

* The fact that we only obtain $t = -(\mu_1 + \dots + \mu_n)^2$ from from this analysis and not $t = -(\mu_1 \pm \mu_2 \pm \dots \pm \mu_n)^2$ would seem to indicate that the Landau curves with the asymptotes $t = -(\mu_1 \pm \mu_2 \pm \dots \pm \mu_n)^2$ are not valid solutions of the Landau equations, and that some argument could be devised to eliminate them from consideration.

$$k_i^2 = s$$
$$k_{i+1}^2 = s \tag{3.96}$$

we must have

$$\underline{k}_i = \pm \underline{k}_{i+1} \tag{3.97}$$

Inserting this into the relationship

$$(\underline{k}_i - \underline{k}_{i+1})^2 = -\mu_{i+1}^2 \tag{3.98}$$

gives us either a contradiction, or else

$$s = -\frac{\mu_{i+1}^2}{4} \tag{3.99}$$

It is not difficult to show that no more Landau curves arise unless **all** the λ_{2i-1} are zero in which case we have

$$\lambda_{2i}\underline{k}_i = 0$$
$$k_i^2 = s \qquad (\text{some } i) \tag{3.100}$$

with the solution $s = 0$.

We are now in a position to discuss the singularities corresponding to the Landau curves that we have found. First of all we note that by the general theorem quoted on page 34 that the curves $s = 0$ and $t = -n^2\mu^2$ (the minimum value of $t = -(\mu_1 + \mu_2 + \ldots + \mu_n)^2$) being the Landau curves nearest the Symanzik region with Feynman parameters in the range $0 < \lambda_i < 1$ are in fact singular curves. They correspond to the cuts $0 \leq s < \infty$ and $-n^2\mu^2 \geq t > -\infty$, and have obviously no complex shoots.

We next examine the curves arising from all λ's non-zero, and we consider first that curve with asymptotes $s = 0$ and $t = -(\mu_1 + \mu_2 + \ldots + \mu_n)^2$ (which, indeed, is the only curve for the equal mass case. Its form we know to be of the type shown in Fig. 9.

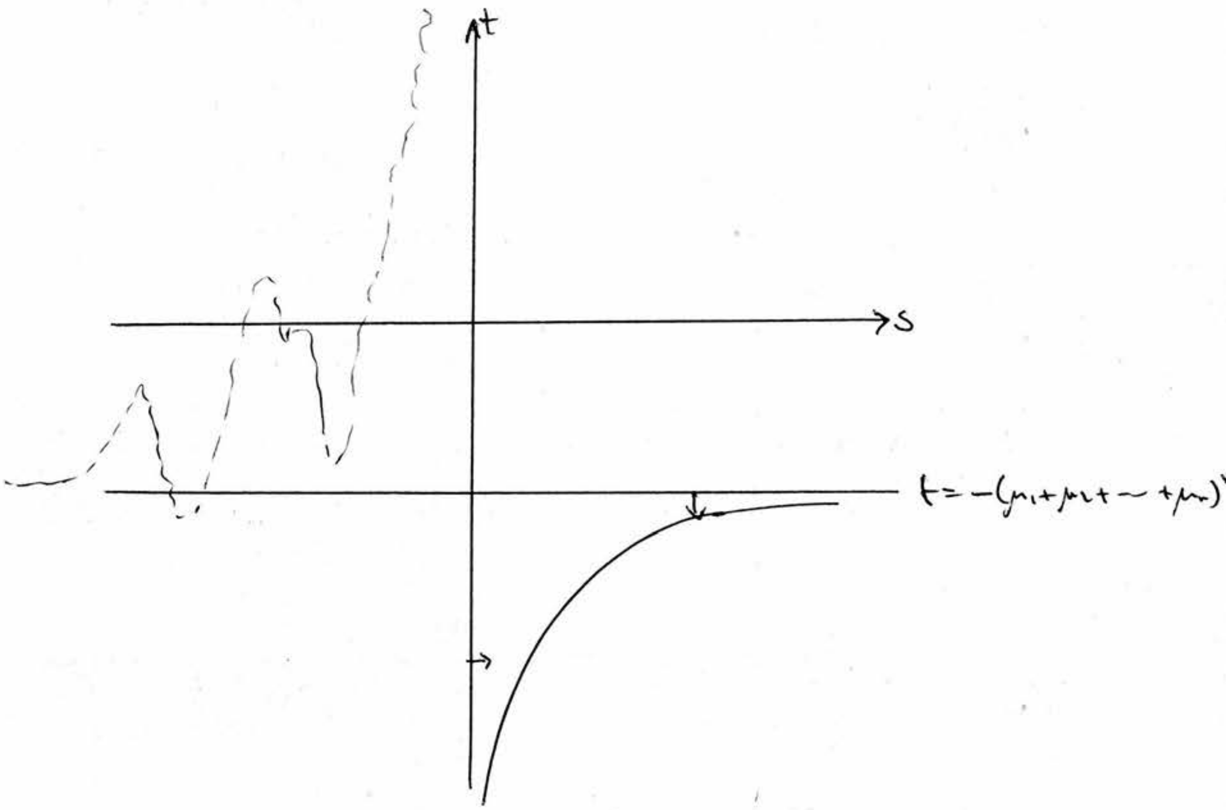

Fig. 9.

For $s > 0$ we know the curve to lie in the region of the crossed cuts ($s > 0$, $t < -(\mu_1 + \mu_2 + \ldots + \mu_n)^2$) and to have positive gradient. For $s < 0$, the behaviour is less well known, but it will have the same asymptotes, is continuous and is one-valued in t for any given s. Hence by continuity, since it certainly enters the extended Symanzik region and does not enter the cross cut region we may say that this branch of the curve is entirely non-singular, and thus the complex surface connecting the two branches of the curve is also non-singular. However, the search-line technique, as propounded by Tarski, enables us to leave the left-hand branch, travel over the complex surface and arrive on the right hand branch, which since it has positive gradient, we are able to assert must be non-singular in the appropriate sense (i.e. approaching the cuts either both from above or both from below). If we now look at the curves with asymptotes $t = -(\mu_1 \pm \mu_2 \pm \cdots \pm \mu_n)^2$ all that we know concerning these is that they are of the form shown in Fig. 10.

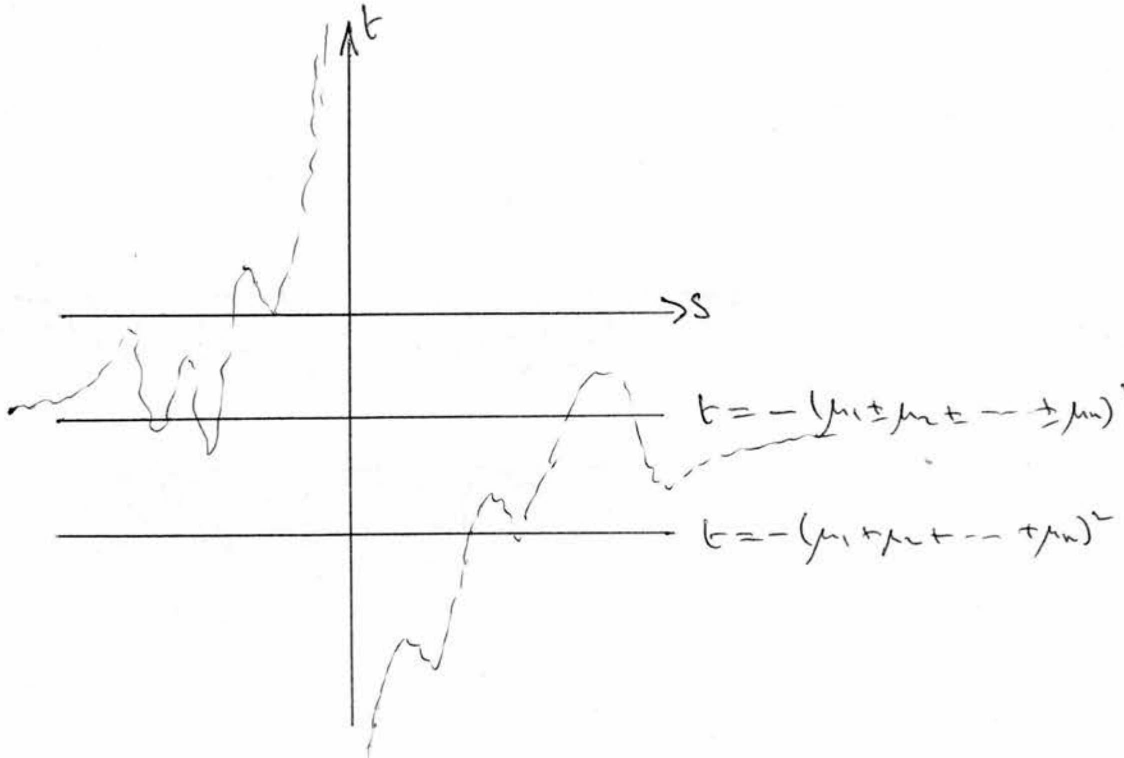

Fig. 10.

The difference now is that the right hand branch does not lie entirely in the cross cut region, and is not known to have positive gradient. However on both branches, since we have no effective intersection with a lower order singular curve the nature will be the same all along the curve. The left hand branch is entirely non-singular and so by continuity through infinity so must the right hand branch. (Note that this argument is possible in this case since the asymptote $t = -(\mu_1 \pm \mu_2 \pm \dots \pm \mu_n)^2$ with which there is an intersection at infinity is a non-singular lower order curve, whereas in the previous case the asymptote $t = -(\mu_1 + \mu_2 + \dots + \mu_n)^2$ was a singular lower order curve).

The Landau curves arising from propagator contractions are now treated in an exactly similar manner, and obviously lead to no new singularities.

We are thus only left with the straight line $s = -\frac{\mu^2}{4}$ which, since it enters the extended Symanzik region will certainly be non-singular.

We have thus come to the conclusion that $T^{(n)}(s,t)$ is an analytic function of the two complex variables s and t apart from the cuts $s \geqslant 0$, $t \leqslant -n^2\mu^2$.

Practically nothing in this section is altered if we consider instead of the energy conserving Born terms the non-energy conserving ones, i.e. $T^{(n)}(\underline{k}_i, \underline{k}_f)$ with $k_i^2 \neq k_f^2$. Then we have

$$T^{(n)}(\underline{k}_i, \underline{k}_f) = T^{(n)}(s, t, \bar{s}) \tag{3.101}$$

with

$$s = k_i^2$$

$$\bar{s} = k_f^2$$

$$t = (k_i - k_f)^2 \tag{3.102}$$

and the methods of this section lead us to the result that $T^{(n)}(s,t,\bar{s})$ is, for fixed $\bar{s}$, an analytic function of the complex variables s and t apart from the cuts $s \geqslant 0$ and $t \leqslant -n^2\mu^2$.

IV. Asymptotic Behaviour of the Born Terms

Before we can obtain an integral representation for $T^{(n)}(s,t)$ [or $T^{(n)}(s, t\ \bar{s})$] as a consequence of the analytic properties derived in the present section, it is necessary to know the behaviour of $T^{(n)}(s, t)$ as $|s| \to \infty$ and $|t| \to \infty$ in any direction in the complex plane.

(i) Behaviour as $|s| \to \infty$

For the determination of this asymptotic behaviour, it is convenient to use the coordinate integral representation for the n-th Born term, as given by equation (1.16):

$$T^{(n)}(s,t) = \int d^3\underline{y}\, d^3\underline{z}\, d^3\underline{x}_1 \text{———} d^3\underline{x}_{n-2} \times$$

$$\times e^{-i\underline{k}_f\cdot\underline{y}} V(y)\, G_{0k}(\underline{y},\underline{x}_1) V(x_1) - ---- V(x_{n-2}) G_{0k}(\underline{x}_{n-2},\underline{z}) V(z)\, e^{i\underline{k}_i\cdot\underline{z}} \tag{4.1}$$

with

$$G_{0k}(\underline{x},\underline{y}) = \frac{1}{4\pi} \frac{e^{ik|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|} \tag{4.2}$$

and the Yukawa potential

$$V(x) = \frac{1}{x} e^{-\mu x} \tag{4.3}$$

Then we have

$$|G_{0k}(\underline{x},\underline{y})| = \frac{1}{4\pi} \frac{e^{-\mathrm{Im}\,k|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|}$$

$$= \frac{1}{4\pi} \frac{e^{-b|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|} \tag{4.4}$$

where $k = a + ib$. (4.5)

It will always be possible to consider b as positive, since by the transformation $s = k^2$, the upper (or lower) half k plane is mapped on to the whole s plane. So we may consider our s plane as being that obtained by the mapping of the upper half k plane. This is essentially the definition of the physical sheet.

From (4.1) it is immediately obvious that

$$|T^{(n)}(s,t)| \leq \int d^3\underline{y}\, d^3\underline{z}\, d^3\underline{x}_1 \text{———} d^3\underline{x}_{n-2} \times \tag{4.6}$$

$$\times \left|e^{-i\underline{k}_f\cdot\underline{y}}\right| |V(y)| \left|G_{0R}(\underline{y},\underline{x}_1)\right| |V(x_1)| \text{———} \left|G_{0R}(\underline{x}_{n-2},\underline{z})\right| |V(z)| \left|e^{i\underline{k}_i\cdot\underline{z}}\right|$$

so that we are interested in the properties of integrals of the type

$$\int |G_{0R}(\underline{x},\underline{r})|\,|V(r)|\,|G_{0R}(\underline{r},\underline{y})|\, d^3\underline{r} \tag{4.7}$$

corresponding to performing one of the $\underline{x}_i$ integrations.

But

$$\int |G_{0R}(\underline{x},\underline{r})|\,|V(r)|\,|G_{0R}(\underline{r},\underline{y})|\, d^3\underline{r}$$

$$= \int \frac{e^{-b|\underline{x}-\underline{r}|}}{4\pi|\underline{x}-\underline{r}|}\;\frac{e^{-\mu r}}{r}\;\frac{e^{-b|\underline{r}-\underline{y}|}}{4\pi|\underline{r}-\underline{y}|}\, d^3\underline{r}$$

$$\leq \frac{1}{(4\pi)^2}\int \frac{e^{-b|\underline{x}-\underline{r}+\underline{r}-\underline{y}|}}{r|\underline{x}-\underline{r}||\underline{r}-\underline{y}|}\, e^{-\mu r} d^3\underline{r}$$

(using the facts that $|\underline{p}| + |\underline{q}| > |\underline{p}+\underline{q}|$ and $b > 0$)

$$= \frac{1}{(4\pi)^2} e^{-b|\underline{x}-\underline{y}|}\int \frac{e^{-\mu r}}{r|\underline{x}-\underline{r}||\underline{r}-\underline{y}|}\, d^3\underline{r} \tag{4.8}$$

Now consider the denominator appearing in this integral

$$\frac{1}{|\underline{x}-\underline{r}||\underline{r}-\underline{y}|}$$

We know that

$$\frac{1}{|\underline{x}-\underline{r}|} + \frac{1}{|\underline{r}-\underline{y}|} = \frac{|\underline{x}-\underline{r}| + |\underline{r}-\underline{y}|}{|\underline{x}-\underline{r}||\underline{r}-\underline{y}|} \tag{4.9}$$

so that

$$\frac{1}{|\underline{x}-\underline{r}||\underline{r}-\underline{y}|} = \frac{1}{(|\underline{x}-\underline{r}| + |\underline{r}-\underline{y}|)}\left[\frac{1}{|\underline{x}-\underline{r}|} + \frac{1}{|\underline{r}-\underline{y}|}\right]$$

$$\leq \frac{1}{|\underline{x}-\underline{y}|}\left[\frac{1}{|\underline{x}-\underline{r}|} + \frac{1}{|\underline{r}-\underline{y}|}\right] \tag{4.10}$$

again using the triangle inequality.

Hence we have from (4.8) and (4.10) that

$$\int |G_{0k}(\underline{x},\underline{r})||V(r)||G_{0k}(\underline{r},\underline{y})|\, d^3\underline{r}$$

$$\leq \frac{1}{(4\pi)^2}\frac{e^{-b|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|}\left[\int \frac{1}{|\underline{x}-\underline{r}|}\frac{e^{-\mu r}}{r}\, d^3\underline{r} + \int \frac{1}{|\underline{y}-\underline{r}|}\frac{e^{-\mu r}}{r}\, d^3\underline{r}\right]$$

$$\leq \frac{1}{(4\pi)^2}\frac{e^{-b|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|}\cdot 8\pi\int_0^\infty r\,dr\,\frac{e^{-\mu r}}{r}$$

(using the result proved in Appendix 2 of reference 5

that

$$\int |f(\underline{z})|\frac{d^3\underline{z}}{|\underline{x}-\underline{z}|} \leq 4\pi\int_0^\infty z\,|f(z)|\,dz \tag{4.11}$$)

$$= \frac{1}{4\pi} \frac{e^{-b|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|} \frac{2}{\mu} \tag{4.12}$$

Thus we have proved that

$$\int |G_{0k}(\underline{x},\underline{r})||V(r)||G_{0k}(\underline{r},\underline{y})| d^3\underline{r} \leq \frac{2}{\mu} |G_{0k}(\underline{x},\underline{y})| \tag{4.13}$$

Thus if we perform all the $\underline{x}_1$ integrations in (4.1) we shall arrive at the result that

$$|T^{(n)}(s,t)| \leq \left(\frac{2}{\mu}\right)^{n-2} \int d^3\underline{y}\, d^3\underline{z}\, |e^{-i\underline{k}_f\cdot\underline{y}}|\,|V(y)|\,|G_{0k}(\underline{y},\underline{z})|\,|V(z)|\,|e^{i\underline{k}_i\cdot\underline{z}}| \tag{4.14}$$

for $n \geqslant 2$

(If the potential, instead of being a pure Yukawa potential had been a superposition of Yukawa potentials,

$$V(r) = \int_\mu^\infty \sigma(m) \frac{e^{-mr}}{r} dm$$

then instead of the factor $(\frac{2}{\mu})^{n-2}$ in equation (4.14), we should have the factor $F(\mu)^{n-2}$ where

$$F(\mu) = 2\int_\mu^\infty \frac{|\sigma(m)|}{m} dm \tag{4.15}$$

It is of interest to note that if the integral in (4.14) exists, the Born series will be absolutely convergent if $F(\mu) < 1$, and hence analytic properties proved for individual terms of the series will also be true for their sum, the total amplitude. However, it has been shown by Bargmann(19) that under this condition there will be no bound states, and so this case is by no means sufficient to cover what we require for the total scattering amplitude.

However, to investigate the asymptotic behaviour of any particular Born term we see that what we require to do is to investigate the behaviour of the integral.

$$\underline{I} = \int d^3\underline{y}\, d^3\underline{z}\, \left|e^{-i\underline{k}_f\cdot\underline{y}}\right| \left|V(y)\right| \left|G_{0k}(\underline{y},\underline{z})\right| \left|V(z)\right| \left|e^{i\underline{k}_i\cdot\underline{z}}\right|$$

$$= \int d^3\underline{y}\, d^3\underline{z}\, \left|e^{-i\underline{k}_f\cdot\underline{y}}\right| \frac{e^{-\mu y}}{y} \frac{e^{-b|\underline{y}-\underline{z}|}}{4\pi|\underline{y}-\underline{z}|} \frac{e^{-\mu z}}{z} \left|e^{i\underline{k}_i\cdot\underline{z}}\right| \qquad (4.16)$$

It is convenient to change the variables by means of the transformation

$$\underline{y}-\underline{z} = \underline{r} \qquad\qquad \underline{y} = \tfrac{1}{2}(\underline{R}+\underline{r})$$

$$\underline{y}+\underline{z} = \underline{R} \qquad\qquad \underline{z} = \tfrac{1}{2}(\underline{R}-\underline{r}) \qquad (4.17)$$

so that we obtain

$$\underline{I} = \frac{1}{32\pi}\int d^3\underline{R}\, d^3\underline{r}\, \left|e^{-i\underline{k}_f\cdot\frac{\underline{R}+\underline{r}}{2}}\right| \left|e^{i\underline{k}_i\cdot\frac{\underline{R}-\underline{r}}{2}}\right| \frac{e^{-br}}{r} \frac{e^{-\frac{\mu}{2}\left(|\underline{R}+\underline{r}|+|\underline{R}-\underline{r}|\right)}}{|\underline{R}+\underline{r}||\underline{R}-\underline{r}|}$$

(the factor of $\frac{1}{8}$ coming from the Jacobian of the transformation (4.17)).

$$= \frac{1}{32\pi}\int d^3\underline{R}\, d^3\underline{r} \left| e^{\frac{1}{2}i(\underline{k}_i-\underline{k}_f)\cdot\underline{R}} \right| \left| e^{-\frac{1}{2}i(\underline{k}_i+\underline{k}_f)\cdot\underline{r}} \right| \frac{e^{-br}}{r} \frac{e^{-\frac{\mu}{2}(|\underline{R}+\underline{r}|+|\underline{R}-\underline{r}|)}}{|\underline{R}+\underline{r}||\underline{R}-\underline{r}|}$$

$$\leq \frac{1}{32\pi}\int d^3\underline{R}\, d^3\underline{r} \left| e^{\frac{1}{2}i(\underline{k}_i-\underline{k}_f)\cdot\underline{R}} \right| \left| e^{-\frac{1}{2}i(\underline{k}_i+\underline{k}_f)\cdot\underline{r}} \right| \frac{e^{-br}}{r} \frac{e^{-\frac{\mu}{2}(R+r)}}{|\underline{R}+\underline{r}||\underline{R}-\underline{r}|} \qquad (4.18)$$

(using the fact that

$$|\underline{R}+\underline{r}| + |\underline{R}-\underline{r}| \geqslant R+r \qquad (4.19)$$

which is a simple extension of the triangle rule.)

$$= \frac{1}{32\pi}\int d^3\underline{R}\, d^3\underline{r}\; e^{-\mathrm{Im}\frac{\underline{k}_i-\underline{k}_f}{2}\cdot\underline{R}}\; e^{\mathrm{Im}\frac{\underline{k}_i+\underline{k}_f}{2}\cdot\underline{r}}\; \frac{e^{-br}}{r} \frac{e^{-\frac{\mu}{2}r}\, e^{-\frac{\mu}{2}R}}{|\underline{R}+\underline{r}||\underline{R}-\underline{r}|} \qquad (4.20)$$

Let us now make the usual representation for the vectors $\underline{k}_i$ and $\underline{k}_f$:

$$\underline{k}_i = k(1, 0, 0)$$

$$\underline{k}_f = k(\cos\theta, \sin\theta, 0)$$

$$= \left(k - \frac{t}{2k}, \sqrt{t - \frac{t^2}{4k^2}}, 0\right) \qquad (4.21)$$

in terms of the square root of the energy k, and the scattering angle θ, or as we prefer to deal with, the momentum transfer t.

Then we shall have

$$\underline{k}_i - \underline{k}_f = \left(\frac{t}{2k} , -\sqrt{t - \frac{t^2}{4k^2}} , 0 \right)$$

$$\underline{k}_i + \underline{k}_f = \left(2k - \frac{t}{2k} , \sqrt{t - \frac{t^2}{4k^2}} , 0 \right) \tag{4.22}$$

We also have, using spherical polars for the $\underline{R}$ and $\underline{r}$ integration variables,

$$\underline{r} = r \left(\cos\alpha, \sin\alpha \cos\beta , \sin\alpha \sin\beta \right)$$

$$\underline{R} = R \left(\cos\alpha', \sin\alpha' \cos\beta', \sin\alpha' \sin\beta' \right) \tag{4.23}$$

Then we shall have

$$I \leq \frac{1}{32\pi} \int d^3\underline{R}\, d^3\underline{r}\; e^{-\mathfrak{Im}\frac{t}{2k}\frac{1}{2}R\cos\alpha' + \mathfrak{Im}\frac{1}{2}\sqrt{t - \frac{t^2}{4k^2}}\sin\alpha'\cos\beta' R} \times \tag{4.24}$$

$$\times\, e^{\mathfrak{Im}\, kr\cos\alpha - \mathfrak{Im}\frac{1}{2}\frac{t}{2k}r\cos\alpha + \mathfrak{Im}\frac{1}{2}\sqrt{t - \frac{t^2}{4k^2}}\sin\alpha\cos\beta\, r} \quad \frac{e^{-br}}{r} \; \frac{e^{-\frac{\mu}{2}R}\, e^{-\frac{\mu}{2}r}}{|\underline{R}+\underline{r}|\,|\underline{R}-\underline{r}|}$$

$$\leq \int d^3\underline{R}\, d^3\underline{r}\;\; e^{\left[\left|\mathfrak{Im}\frac{t}{2k}\right| + \left|\mathfrak{Im}\sqrt{t - \frac{t^2}{4k^2}}\right| \right] \frac{R+r}{2}}\; e^{br\cos\alpha} \times$$

$$\times\; \frac{e^{-br}}{r} \; \frac{e^{-\frac{\mu}{2}R}\, e^{-\frac{\mu}{2}r}}{|\underline{R}+\underline{r}|\,|\underline{R}-\underline{r}|} \tag{4.25}$$

and we are interested in what happens to this integral when $b \to \infty$ (i.e. $|k| \to \infty$ in the upper half k plane). t is considered physical, i.e. real and positive.

Let us consider first of all $\operatorname{Im}\frac{t}{2k}$ and $\operatorname{Im}\sqrt{t-\frac{t^2}{4k^2}}$ under these conditions.

$$\left|\operatorname{Im}\frac{t}{2k}\right| = \left|\operatorname{Im}\frac{t(a-ib)}{2(a^2+b^2)}\right|$$

$$= \frac{tb}{a^2+b^2}$$

$$\to 0 \quad \text{as} \quad b \to \infty$$

so that we can take

$$\left|\operatorname{Im}\frac{t}{2k}\right| < \epsilon \tag{4.26}$$

with ε arbitrarily small.

Now consider $\operatorname{Im}\sqrt{t-\frac{t^2}{4k^2}}$

If we put $t - \frac{t^2}{4k^2} = z = x + iy$ (4.27)

and $k = Re^{i\theta}$

so that

$$x = t - \frac{t^2}{4R^2}\cos 2\theta$$

$$y = \frac{t^2}{4R^2}\sin 2\theta \tag{4.28}$$

then as $R \to \infty$ we shall have

$$y \to 0$$

and $$x \to t \tag{4.29}$$

But we are interested in $\sqrt{z} = \xi + i\eta$, say, which relationship means that ξ and η are connected with x and y by

$$\xi^2 - \eta^2 = x$$

$$2\xi\eta = y$$

(4.30)

But since $y \to 0$ as $R \to \infty$, this means that as $R \to \infty$ either $\xi \to 0$ or $\eta \to 0$. But since $x \to t$ (real and positive) the first of equations (4.30) tells us that it must be η that tends to zero (and that ξ tends to $\sqrt{t}$) as R approaches infinity. Hence for large b (and thus large R) we may take $|\eta| < \varepsilon$ where ε is arbitrarily small: thus we have

$$\left| \mathrm{Im} \sqrt{t - \frac{t^2}{4k^2}} \right| < \epsilon \tag{4.31}$$

Hence equation (4.25) reduces to (if b is large enough)

$$I \leq \frac{1}{32\pi} \int d^3\underline{R}\, d^3\underline{r}\; e^{\epsilon(R+r)}\, e^{br\cos\alpha}\, \frac{e^{-br}}{r}\, \frac{e^{-\frac{\mu}{2}R}\, e^{-\frac{\mu}{2}r}}{|\underline{B}+\underline{r}||\underline{B}-\underline{r}|} \tag{4.32}$$

We now make use of the fact that

$$\frac{1}{|\underline{R}+\underline{r}||\underline{R}-\underline{r}|} = \frac{1}{|\underline{R}+\underline{r}|+|\underline{R}-\underline{r}|}\left[\frac{1}{|\underline{R}+\underline{r}|}+\frac{1}{|\underline{R}-\underline{r}|}\right]$$

$$\leq \frac{1}{2R}\left[\frac{1}{|\underline{R}+\underline{r}|}+\frac{1}{|\underline{R}-\underline{r}|}\right] \qquad (4.33)$$

so that

$$I \leq \frac{1}{32\pi}\int d^3\underline{r}\,\frac{1}{r}\,e^{\epsilon r}\,e^{br\cos\alpha}\,e^{-br}\,e^{-\frac{\mu}{2}r}\,\times$$

$$\times\int d^3\underline{R}\left[\frac{1}{|\underline{R}+\underline{r}|}+\frac{1}{|\underline{R}-\underline{r}|}\right]\frac{1}{2R}\,e^{-(\frac{\mu}{2}-\epsilon)R}$$

$$\leq \frac{1}{8}\int d^3\underline{r}\,e^{-(b+\frac{\mu}{2}-\epsilon)r}\,\frac{e^{br\cos\alpha}}{r}\int d^3\underline{R}\,e^{-(\frac{\mu}{2}-\epsilon)R} \qquad (4.34)$$

(on again using equation (4.11))

$$= \frac{1}{8(\frac{\mu}{2}-\epsilon)}\int_0^\infty dr\int_0^\pi d\alpha\,2\pi r^2\sin\alpha\,\frac{1}{r}\,e^{-(b+\frac{\mu}{2}-\epsilon)r}\,e^{br\cos\alpha}$$

$$= \frac{1}{b}\,\frac{\pi}{4(\frac{\mu}{2}-\epsilon)}\left[\frac{1}{\frac{\mu}{2}-\epsilon}-\frac{1}{2b+\frac{\mu}{2}-\epsilon}\right] \qquad (4.35)$$

which expression tends to zero as b tends to infinity, and thus we know that

$$|T^{(n)}(s,t)| \to 0 \quad \text{as } |s| \to \infty \qquad (4.36)$$
$$(t \text{ real}, > 0)$$

(ii) Behaviour as $|t| \to \infty$.

We now wish to consider what happens when $|t| \to \infty$, and s is kept real and constant. Since the momentum transfer t and the scattering angle θ are related by $t = 2s(1 - \cos\theta)$ an exactly equivalent problem is to find out what happens when $|\cos\theta| \to \infty$ with s kept fixed.

For this case we consider the momentum integral representation for the n-th Born term given by equation (1.17) with a pure Yukawa potential:-

$$T^{(n)}(s,t) = \int d^3\underline{k}_1\, d^3\underline{k}_2 \cdots d^3\underline{k}_{n-1} \times$$

$$\times \frac{1}{\mu^2+(\underline{k}_f-\underline{k}_1)^2}\,\frac{1}{s-k_1^2+i\epsilon}\,\frac{1}{\mu^2+(\underline{k}_1-\underline{k}_2)^2} \cdots \cdots \frac{1}{\mu^2+(\underline{k}_{n-1}-\underline{k}_i)^2} \qquad (4.37)$$

together with the specific choice of coordinate system characterized by

$$\underline{k}_i = k(1,0,0)$$

$$\underline{k}_f = k(\cos\theta, \sin\theta, 0) \qquad (4.38)$$

(where we note that k is real)

We then note that θ appears only in the denominator $\mu^2+(\underline{k}_f - \underline{k}_1)^2$ where it appears in the form

$$\mu^2+k^2+k_1^2-2kk_1(\cos\theta\cos\psi_1+\sin\theta\sin\psi_1\cos\phi_1) \qquad (4.39)$$

where we have the specific choice of spherical polar coordinates for the vectors $\underline{k}_r$:-

$$\underline{k}_r = k_r(\cos\psi_r,\ \sin\psi_r\cos\varphi_r,\ \sin\psi_r\sin\varphi_r) \qquad (4.40)$$

so that the integration $\int d^3\underline{k}_r$ appearing in equation (4.37) will be replaced by the integration

$$\int_0^\infty k_r^2\,dk_r \int_{-1}^{+1} d(\cos\psi_r) \int_0^{2\pi} d\varphi_r \qquad (4.41)$$

so that we may write

$$T^{(n)}(s,t) = \int p_1^2\,dp_1\, d(\cos\psi)\, d\varphi\, \frac{1}{\mu^2 + (\underline{k}_f - \underline{p}_1)^2}\, \frac{1}{s - p_1^2 + i\epsilon}\, T^{(n-1)}(\underline{p}_1, \underline{k}_i) \qquad (4.42)$$

But now we remember that $T^{(n)}(\underline{k},\underline{\ell})$ is just a function of the three variables k^2, ℓ^2 and $\underline{k}.\underline{\ell}$, so that we may write

$$T^{(n-1)}(\underline{p}_1, \underline{k}_i) = T^{(n-1)}(p_1^2,\ k_i^2,\ \underline{p}_1.\underline{k}_i)$$

$$= T^{(n-1)}(p_1^2, s, \psi) \qquad (4.43)$$

and thus equation (4.42) reduces to

$$T^{(n)}(s,t) = \int_0^\infty p^2\,dp \int_{-1}^{+1} d(\cos\psi) \int_0^{2\pi} d\varphi\, \frac{1}{\mu^2 + (\underline{k}_f - \underline{p}_1)^2}\, \frac{1}{s - p_1^2 + i\epsilon}\, T^{(n-1)}(p_1^2, s, \psi) \qquad (4.44)$$

We now perform the φ integration :-

$$\int_0^{2\pi} \frac{d\varphi_1}{(\mu^2 + k^2 + p_1^2 - 2kp_1\cos\theta\cos\psi_1) - 2kp_1\sin\theta\sin\psi_1\cos\varphi_1}$$

$$= \frac{\pi}{\sqrt{(\mu^2+k^2+p_1^2-2kp_1\cos\theta\cos\psi_1)^2 - 4k^2p_1^2\sin^2\theta\sin^2\psi}} \tag{4.45}$$

$$= \frac{\pi}{\sqrt{D}} \quad \text{, say} \tag{4.46}$$

Then we shall have

$$|T^{(n)}(s,t)| \leq \pi\int p_1^2 dp_1\, d(\cos\psi_1)\, \frac{1}{|D|^{\frac{1}{2}}} \left|\frac{1}{s-p_1^2+i\epsilon}\right| |T^{(n-1)}(\underline{p}_1, \underline{k}_i)| \tag{4.47}$$

But it is a simple matter to show that $T^{(n-1)}(\underline{p}_1,\underline{k}_j)$ is bounded when both p_1 and $\underline{k}_1$ are real,* so that we may write

---

* From equations (4.14) and (4.16) we see that for real vectors $\underline{p}_1$ and $\underline{k}_1$

$$|T^{(n)}(p_1, \underline{k}_i)| \leq \left(\frac{2}{\mu}\right)^{n-2}\int d^3\underline{y}\, d^3\underline{z}\; \frac{e^{-\mu y}}{y}\, \frac{1}{4\pi|\underline{y}-\underline{z}|}\, \frac{e^{-\mu z}}{z}$$

$$\leq \left(\frac{2}{\mu}\right)^{n-2}\int d^3\underline{y}\; \frac{e^{-\mu y}}{y}\int_0^\infty e^{-\mu z}dz$$

$$= \frac{4\pi}{\mu}\left(\frac{2}{\mu}\right)^{n-2}\int_0^\infty y\, e^{-\mu y}dy$$

$$= \frac{\pi}{\mu}\left(\frac{2}{\mu}\right)^{n}$$

$$|T^{(n-1)}(\underline{p}_1, \underline{k}_i)| \leq M \tag{4.48}$$

and thus

$$|T^{(n)}(s,t)| \leq \pi M \int p_1^2 dp_1 \, d(\cos\psi_1) \frac{1}{|D|^{\frac{1}{2}}} \left| \frac{1}{s - p_1^2 + i\epsilon} \right| \tag{4.49}$$

$$\leq \pi M \int p_1^2 dp_1 \, d(\cos\psi_1) \frac{1}{|D|^{\frac{1}{2}}} \left[P \frac{1}{|s-p_1^2|} + \pi \delta(s - p_1^2) \right]$$

on using the fact that (4.50)

$$\frac{1}{s - p_1^2 + i\epsilon} = P \frac{1}{s - p_1^2} - i\pi \delta(s - p_1^2)$$

and the triangle inequality.

Let us now consider $|D|$.

We have, from (4.50) and (4.49) (on writing p for p_1 and ψ for ψ_1)

$$D = (p^2 + k^2 + \mu^2)^2 - 4pk\cos\theta\cos\psi\,(p^2 + k^2 + \mu^2) + 4p^2k^2\cos^2\theta\cos^2\psi$$
$$- 4p^2k^2\sin^2\theta\sin^2\psi$$

$$= (p^2 + k^2 + \mu^2)^2 - 4pk\cos\theta\cos\psi\,(p^2 + k^2 + \mu^2) + 4p^2k^2\cos^2\theta$$
$$- 4p^2k^2\sin^2\psi \tag{4.51}$$

If we now let $\cos\theta$ be complex, say $\cos\theta = Re^{i\alpha}$, where we are eventually going to let $R \to \infty$, then we shall have

$$\mathrm{Re}\,D = (p^2+k^2+\mu^2)^2 - 4pkR\cos\alpha\cos\psi\,(p^2+k^2+\mu^2) + 4p^2k^2R^2\cos 2\alpha$$
$$-4p^2k^2\sin^2\psi \tag{4.52}$$

$$\mathrm{Im}\,D = -4kpR\sin\alpha\cos\psi\,(p^2+k^2+\mu^2)$$
$$+4p^2k^2R^2\sin 2\alpha \tag{4.53}$$

and hence

$$|D|^2 = (p^2+k^2+\mu^2)^4 + 16p^2k^2R^2\cos^2\psi\,(p^2+k^2+\mu^2) + 16p^4k^4R^4$$
$$-32k^3p^3R^3\cos\alpha\cos\psi\,(p^2+k^2+\mu^2) + 16p^4k^4\sin^4\psi$$
$$-8pkR\cos\alpha\cos\psi\,(p^2+k^2+\mu^2)^3 - 8p^2k^2\sin^2\psi\,(p^2+k^2+\mu^2)^2$$
$$+8p^2k^2R^2\cos 2\alpha\,(p^2+k^2+\mu^2)^2 + 32p^3k^3R\cos\alpha\cos\psi\sin^2\psi\,(p^2+k^2+\mu^2)$$
$$-32p^4k^4R^2\cos 2\alpha\sin^2\psi \tag{4.54}$$

The important thing to note in this complicated expression for $|D|^2$, is that the highest power of R, viz R^4, occurs as might be expected, with a positive definite coefficient, so that by choosing R large enough we can certainly say that

$$|D|^2 > 8p^2k^2R^2\cos 2\alpha\,(p^2+k^2+\mu^2)^2 \tag{4.55}$$

and hence that

$$\frac{1}{|D|^{\frac{1}{2}}} < \frac{1}{(8\cos 2\alpha)^{\frac{1}{4}}\sqrt{kR}\,\sqrt{p(p^2+k^2+\mu^2)}} \tag{4.56}$$

Hence we have that

$$|T^{(n)}(s,t)| < \frac{2\pi M}{(\delta\cos 2\alpha)^{1/4}\sqrt{kR}} \int \frac{p^2 dp}{\sqrt{p}\,(p^2+k^2+\mu^2)} \left[P\frac{1}{|s-p^2|} + \pi\delta(s-p^2) \right] \tag{4.57}$$

which integral certainly converges, so that we may write

$$|T^{(n)}(s,t)| < \frac{1}{\sqrt{R}} \quad \frac{2\pi M N(k^2,\mu^2)}{\sqrt{k}\,(\delta\cos 2\alpha)^{1/4}} \tag{4.58}$$

and thus $|T^{(n)}(s, t)| \to 0$ as $R \to \infty$.

This proof depends on the fact that $\cos 2\alpha \neq 0$. If $\cos 2\alpha = 0$, then the steps from (4.58) onwards can be slightly altered to lead to the same result.

Thus we are led to the results that for fixed real positive t $|T^{(n)}(s,t)| \to 0$ as $|s| \to \infty$, and that for fixed real positive s, $|T^{(n)}(s,t)| \to 0$ as $|t| \to \infty$.

It is important to note that the above results also hold good for the non-energy conserving Born terms $T^{(n)}(s, t, \bar{s}) = T^{(n)}(\underline{k}_i, \underline{k}_f)$ with $s = k_i^2$, $\bar{s} = k_f^2$ $t = (\underline{k}_i - \underline{k}_f)^2$ as far as the s and t variables are concerned.

If we now combine these results with the analytic properties deduced in the previous section by performing Cauchy integrations in the manner indicated in Section I, then we are led to the integral representation for $T^{(n)}(s,t)$ and $T^{(n)}(s, t, \bar{s})$:-

$$T^{(n)}(s,t) = \int_0^\infty ds' \int_{n^2\mu^2}^\infty dt' \frac{\rho_n(s',t')}{(s-s'+i\epsilon)(t+t')} \tag{4.59}$$

and

$$T^{(n)}(s,t,\bar{s}) = \int_0^\infty ds' \int_{n^2\mu^2}^\infty dt' \frac{\rho_n(s',t',\bar{s})}{(s-s'+i\epsilon)(t+t')} \qquad (4.60)$$

V. Analytic Properties of the Remainder Term

We have, from equation (2.23)

$$R^{(2n+1)}(s,t) =$$

$$\int T^{(n)}(\underline{k}_f,\underline{p}) \frac{1}{s-p^2+i\epsilon} \frac{T(\underline{p},\underline{q})T^*(\underline{r},\underline{q})}{s-q^2+i\epsilon} \frac{1}{s-r^2+i\epsilon} T^{(n)}(\underline{r},\underline{k}_i)\, d^3\underline{p}\, d^3\underline{q}\, d^3\underline{r}$$

$$+\sum_B \int T^{(n)}(\underline{k}_f,\underline{p}) \frac{1}{s-p^2+i\epsilon} \frac{T_B(\underline{p})\, T_B^*(\underline{r})}{s+B} \frac{1}{s-r^2+i\epsilon} T^{(n)}(\underline{r},\underline{k}_i)\, d^3\underline{p}\, d^3\underline{r}$$

(5.1)

which, if we make use of the integral representation (4.64) may be written in the form

$$R^{(2n+1)}(s,t) = \int_0^\infty ds'\,ds'' \int_{n^2\mu^2}^\infty dt'\,dt'' \int d^3\underline{p}\, d^3\underline{q}\, d^3\underline{r} \;\times$$

$$\times \frac{\rho_{\sim}(s',t',p^2)\,\rho_{\sim}^{*}(s'',t'',r^2)\,f(p^2,q^2,r^2,\underline{p}\cdot\underline{r})}{(s-s'+i\epsilon)\left[(\underline{k}_f-\underline{p})^2+t'\right](s-s''+i\epsilon)\left[(\underline{k}_i-\underline{r})^2+t''\right](s-p^2+i\epsilon)(s-q^2+i\epsilon)(s-r^2+i\epsilon)}$$

$$+\sum_{B}\int_{0}^{\infty}ds'\,ds''\int_{\mu^2}^{\infty}dt'\,dt''\int d^3\underline{p}\,d^3\underline{r}\ \times$$

$$\times \frac{\rho_{\sim}(s',t',p^2)\,\rho_{\sim}^{*}(s'',t'',r^2)\,g(B,p^2,r^2,\underline{p}\cdot\underline{r})}{(s-s'+i\epsilon)\left[(\underline{k}_f-\underline{p})^2+t'\right](s-s''+i\epsilon)\left[(\underline{k}_i-\underline{r})^2+t''\right](s-p^2+i\epsilon)(s+B)(s-r^2+i\epsilon)}$$

(5.2)

We now look at the denominator, on the right hand side of equation (5.2) to find the regions in the s and t planes for which they vanish; these will be the regions in which singularities may lie. The only difference in the denominator in the two expressions on the right hand side of (5.2) comes from the terms $s - q^2 + i\varepsilon$ and $s + B$. Zeros of the first will give rise to a positive real axis cut in the s plane from $s = 0$ to $s = \infty$, and zeros of the second lead to poles in the s plane at $s = - B$. Zeros of all the other energy denominators, i.e. the denominators

$$s - s' + i\varepsilon$$
$$s - s'' + i\varepsilon$$
$$s - p^2 + i\varepsilon$$
$$s - q^2 + i\varepsilon$$
$$s - r^2 + i\varepsilon \tag{5.3}$$

will likewise lead to a cut in the s-plane from $s = 0$ to $s = \infty$.

So we are left to consider the regions in which the terms

$$(\underline{k}_i - \underline{r})^2 + t'' \qquad \text{and} \qquad (\underline{k}_f - \underline{p})^2 + t'$$

vanish, where we must remember that t'' and t' lie between $n^2\mu^2$ and ∞, and $\underline{r}$ and $\underline{p}$ take on all real values.

So we wish to investigate the regions

$$(\underline{k}_i - \underline{r})^2 + t'' = 0 \tag{5.4}$$

and
$$(\underline{k}_f - \underline{p})^2 + t' = 0 \tag{5.5}$$

We first of all consider s fixed real and positive and look for the regions of possible singularities in t (or equivalently $\cos\theta$). It is useful to use the representations

$$\underline{k}_i = k(\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}, 0) \tag{5.6}$$

$$\underline{k}_f = k(\cos\tfrac{\theta}{2}, -\sin\tfrac{\theta}{2}, 0) \tag{5.7}$$

$$\underline{r} = (r_1,\ r_2,\ r_3) \tag{5.8}$$

$$\underline{p} = (p_1,\ p_2,\ p_3) \tag{5.9}$$

Then with this representation it is easy to see that equations (5.4) and (5.5) both lead to the same region, which is given by the equation

$$k^2 - 2k\left(r_1 \cos\frac{\theta}{2} + r_2 \sin\frac{\theta}{2}\right) + r^2 + t'' = 0 \tag{5.10}$$

giving

$$r_1 \cos\frac{\theta}{2} + r_2 \sin\frac{\theta}{2} = \frac{k^2 + r^2 + t''}{2k} \tag{5.11}$$

which, on putting $\theta = \alpha + i\beta$ yields

$$\cosh\frac{\beta}{2} = \frac{k^2 + r^2 + t''}{2k\sqrt{r_1^2 + r_2^2}} \tag{5.14}$$

which as $\underline{r}$ and t'' take on all their allowed values, will take on all values between $\sqrt{1 + n^2\mu^2/k^2}$ and $+\infty$.

Now, in the $\cos\theta$ plane the curve β = constant is an ellipse with semi major and semi minor axes $\cosh\beta$ and $\sinh\beta$ along the real and imaginary axes respectively, and with centre at $\cos\theta = 0$. Thus we know that all singularities lie outside an ellipse in the $\cos\theta$ plane with centre at $\cos\theta = 0$, semi major axis given by

$$\cosh\beta = 2\cosh^2\frac{\beta}{2} - 1$$

$$= 2\left(1 + \frac{n^2\mu^2}{k^2}\right) - 1$$

$$= 1 + \frac{2n^2\mu^2}{k^2}$$

semi minor axis given by

$$\sinh\beta = \sqrt{\cosh^2\beta - 1}$$

$$= \frac{2n\mu}{k}\sqrt{1 + \frac{n^2\mu^2}{k^2}} \tag{5.13}$$

Since we know that $t = 2s(1 - \cos\theta)$ it follows that we shall obtain analyticity inside the ellipse in the t-plane shown in Fig. 11.

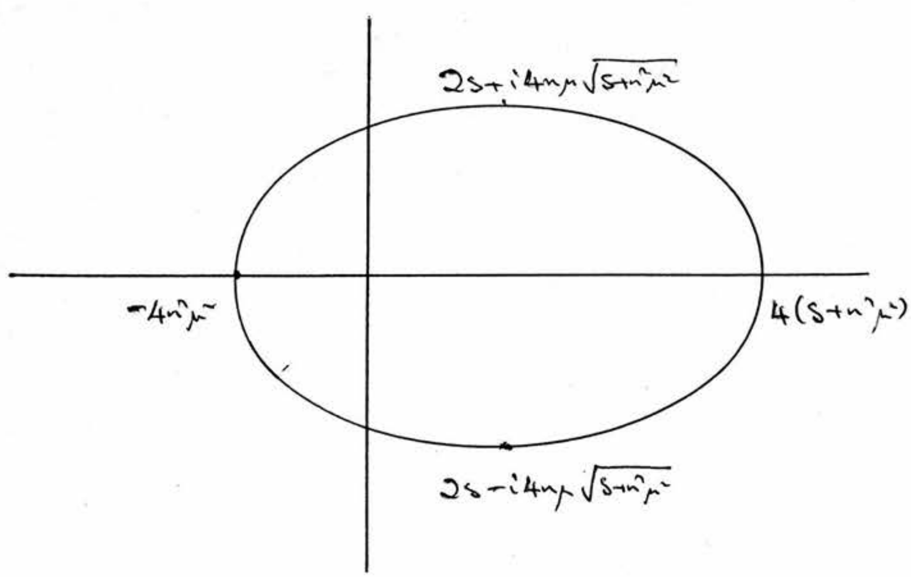

Fig. 11

It is immediately seen that this ellipse spreads out to cover the whole t-plane as n tends to infinity, and hence we may obtain analyticity inside an arbitrarily large domain merely by increasing the number of iterations to obtain a remainder term $R^{(2n+1)}(s,t)$ of sufficiently high order, and thus analyticity inside a sufficiently large ellipse.

Let us now consider analyticity in the s plane when t is kept fixed real and positive.

We first consider the region of possible singularities given by equation (5.4). This equation may be written as

$$k^2 - 2kr_1 + r^2 + t'' = 0 \qquad (5.14)$$

which on writing $k = x + iy$ gives

$$xy - r_1 y = 0 \qquad (5.15)$$

and

$$x^2 - 2r_1 x - y^2 + r^2 + t'' = 0 \qquad (5.16)$$

Equation (5.15) gives $y = 0$ or $x = r_1$. If $y = 0$ then equation (5.16) gives

$$(x-r_1)^2 + r_2^2 + r_3^2 + t'' = 0 \qquad (5.17)$$

which is impossible for real x and positive t''. Hence we must have instead $x = r_1$, which will then imply from equation (5.16)

$$y^2 = r_2^2 + r_3^2 + t'' \qquad (5.18)$$

Hence the values of x and y are restricted only by (5.18) which says that we must have $y^2 \geqslant n^2\mu^2$

Thus we have the possibility of singularities only in the shaded regions of the k plane as shown in Fig. 12.

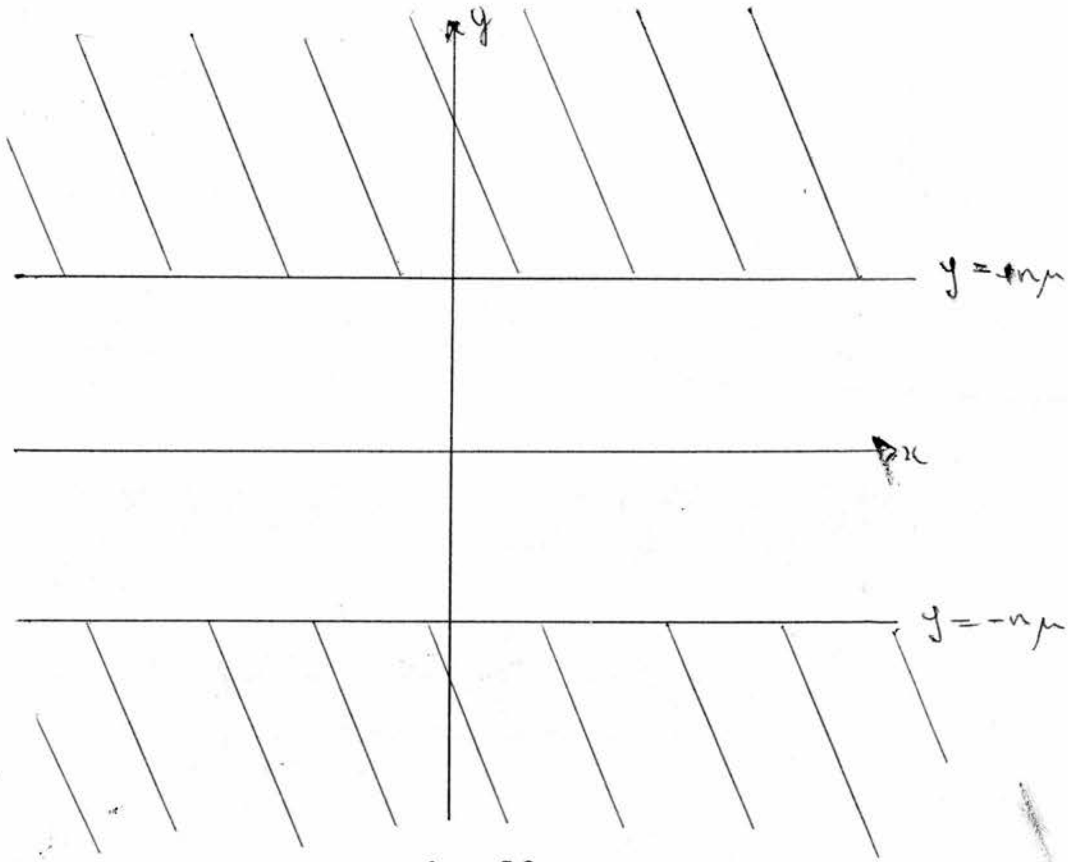

Fig. 12.

On performing the transformation to the s plane by means of the relationship $s = k^2$ we obtain the region inside the parabola

$$\eta^2 = 4n^2\mu^2(\xi + n^2\mu^2) \tag{5.19}$$

$$[s = \xi + i\eta]$$

to be free of singularities.

This region is shown as unshaded in Fig. 13.

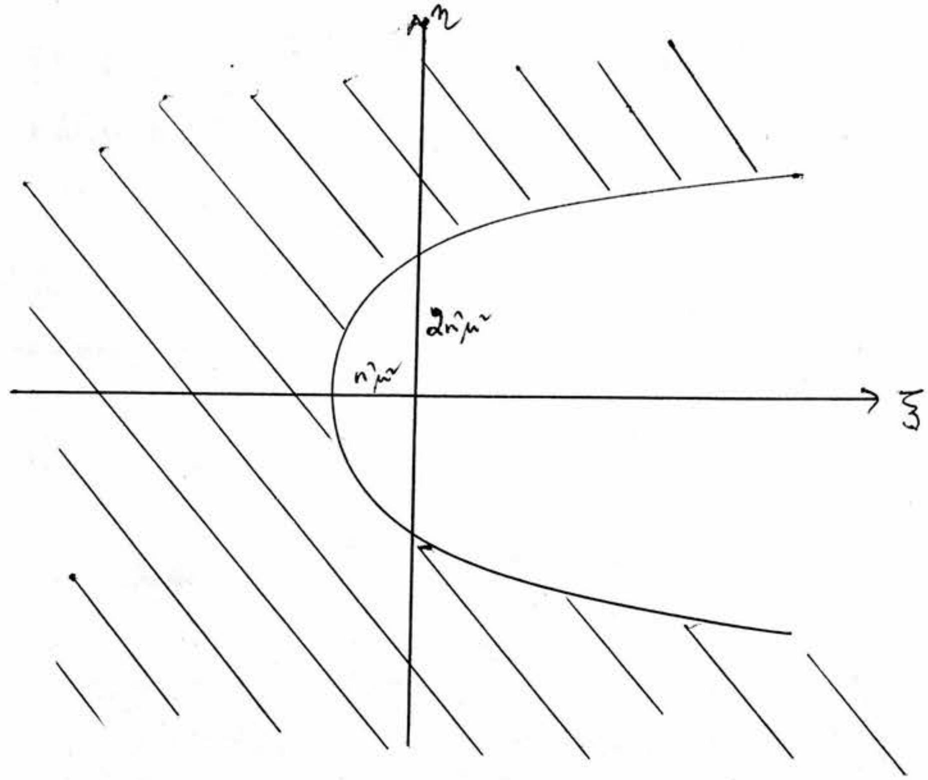

Fig. 13

From the points of intersection with the real and imaginary axes it is obvious that we have analyticity inside a region which, as n increases, spreads out to fill an arbitrarily large portion of the s plane.

Now consider the domain given by equation (5.5). We want to consider the case with t fixed and real, in closer analogue with field theory than the usual considerations which take cos θ as fixed.

Equation (5.5) with the representation

$$\underline{k}_i = k(\cos\theta,\ \sin\theta,\ 0)$$

$$\underline{p} = (p_1\ p_2\ p_3)$$

may be written as

$$k^2+p^2-2k\,(p_1\cos\theta+p_2\sin\theta+t') = 0 \tag{5.20}$$

In addition we have

$$k = x+iy \tag{5.21}$$

$$\theta = \alpha+i\beta \tag{5.22}$$

$$t = 2k^2(1-\cos\theta) \tag{5.23}$$

where t will be considered fixed and real.

Equation (5.23) may be written in the form

$$t = 4k^2\sin^2\frac{\theta}{2} \tag{5.24}$$

which on use of equations (5.22) and (5.21) will, on taking real and imaginary parts, yield the equations

$$\frac{1}{2}\sqrt{t} = x\sin\frac{\alpha}{2}\cosh\frac{\beta}{2} - y\cos\frac{\alpha}{2}\sinh\frac{\beta}{2} \tag{5.25}$$

$$0 = x\cos\frac{\alpha}{2}\sinh\frac{\beta}{2} + y\sin\frac{\alpha}{2}\cosh\frac{\beta}{2} \tag{5.26}$$

which give on elimination of α

$$\frac{t}{2} = \frac{(x^2+y^2)\sinh^2\beta}{\left[(y^2-x^2)+(y^2+x^2)\cosh\beta\right]} \tag{5.27}$$

Equation (5.20) together with (5.21) and (5.22) on taking real and imaginary parts gives the equations

$$x^2 - y^2 + p^2 - 2x(p_1 \cos\alpha \cosh\beta + p_2 \sin\alpha \cosh\beta)$$

$$- 2y(p_1 \sin\alpha \sinh\beta - p_2 \cos\alpha \sinh\beta) + t'' = 0 \tag{5.28}$$

and

$$2xy - 2y(p_1 \cos\alpha \cosh\beta + p_2 \sin\alpha \cosh\beta)$$

$$- 2x(-p_1 \sin\alpha \sinh\beta + p_2 \cos\alpha \sinh\beta) = 0 \tag{5.29}$$

If we now put

$$q_1 = p_1 \cos\alpha + p_2 \sin\alpha$$

$$q_2 = p_1 \sin\alpha - p_2 \cos\alpha$$

$$q_3 = p_3 \tag{5.30}$$

then the equations (5.28) and (5.29) reduce to

$$x^2 - y^2 + q^2 - 2xq_1 \cosh\beta - 2yq_2 \sinh\beta + t'' = 0 \tag{5.31}$$

$$2xy - 2yq_1 \cosh\beta + 2xq_2 \sinh\beta = 0 \tag{5.32}$$

i.e. $$x^2 - y^2 + q^2 - 2Ax - 2By + t'' = 0 \tag{5.33}$$

$$xy - Ay + Bx = 0 \tag{5.34}$$

where

$$A = q_1 \cosh\beta$$

$$B = q_2 \sinh\beta \tag{5.35}$$

Elimination of B gives

$$x^2 - y^2 + q^2 - 2Ax - \frac{2y^2}{x}(A-x) + t'' = 0 \tag{5.36}$$

But we have that

$$q^2 = q_1^2 - q_2^2 + q_3^2$$

$$= \frac{A^2}{\cosh^2\beta} + \frac{B^2}{\sinh^2\beta} + q_3^2$$

$$= \frac{A^2}{\cosh^2\beta} + \frac{y^2(A-x)^2}{x^2\sinh^2\beta} + q_3^2 \tag{5.37}$$

Hence substituting (5.37) in (5.36) gives

$$x^2 - y^2 + \frac{A^2}{\cosh^2\beta} + \frac{y^2(A-x)^2}{x^2\sinh^2\beta} + q_3^2 - 2Ax - \frac{2y^2}{x}(A-x) + t'' = 0 \tag{5.38}$$

If we now write $\quad t'' + q_3^2 = C^2 \qquad (5.39)$

and use the necessary condition that A is real, we obtain after some elementary but tedious algebra that this condition implies that we must have

$$\cosh^2\beta \geqslant \frac{c^2+x^2}{y^2+x^2} \tag{5.40}$$

which has as its bounding curve in the x-y plane (i.e. complex k plane) the curve

$$\cosh^2\beta = \frac{c^2+x^2}{y^2+x^2} \tag{5.41}$$

Elimination of β between the equations (5.41) and (5.27) gives almost immediately

$$\frac{t}{2} = \frac{(x^2+y^2)(c^2-y^2)}{(y^2-x^2)+\sqrt{(c^2+x^2)(y^2+x^2)}} \tag{5.42}$$

which is the bounding curve in the k plane of the region of analyticity, and the region of possible non-analyticity will occur for $c^2 \geqslant n^2\mu^2$.

We now investigate the form of this curve. It is immediately obvious that the curve lies between the lines $y = \pm c$ which it has for asymptotes and that it cuts the axes at the points

$$\begin{aligned} y &= 0 \qquad x^2 = \frac{t^2}{4(c^2+t)} \\ x &= 0 \qquad y = \pm\frac{c}{2} \pm \frac{\sqrt{c^2-2t}}{2} \end{aligned} \tag{5.43}$$

We shall now show that it is confined to the regions

$0 < y^2 < t^2/c^2$ and $c^2 - t < y^2 < c^2$, or in other words that it does not appear in the strip $t^2/c^2 \leq y^2 \leq c^2 - t$ within which we shall therefore have analyticity. (The proof will depend on our being able to choose c^2 large enough, which of course we shall be able to do, since we can take the number of iterations, and hence n, and hence c^2, to be as large as we please).

If we write equation (5.42) in the form of a quadratic in x^2 we obtain

$$x^4\left[t(c^2-y^2)+(c^2-y^2)^2\right] - x^2\left[\frac{3y^2t^2}{4}+\frac{t^2c^2}{4}-2y^2(c^2-y^2)^2\right]$$

$$-\left[\frac{t^2}{4}y^2(c^2-y^2)+ty^4(c^2-y^2)-y^4(c^2-y^2)^2\right] = 0 \qquad (5.44)$$

We want to find the values of y which give negative x^2, and the curve will then certainly not enter these regions. The conditions for equation (5.44) to have two negative roots is that the sum of the roots should be negative and the product positive, which conditions reduce to

$$y^2 > \frac{-(c^2-t)+\sqrt{c^4-2c^2t+2t^2}}{2} \qquad (5.45)$$

and

$$f(y^2) \equiv -2y^6 + 4y^4c^2 + y^2\left(\frac{3t^2}{4} - 2c^4\right) + \frac{t^2}{4}c^2$$

$$< 0 \tag{5.46}$$

First let us note that (5.45) is satisfied if $y^2 > \frac{t^2}{c^2}$ since

$$\frac{t^2}{c^2} > \frac{-(c^2-t) + \sqrt{c^4 - 2c^2t + 2t^2}}{2}$$

if $$\frac{2t^2}{c^2} + c^2 - t > \sqrt{c^4 - 2c^2t + 2t^2}$$

i.e. if $$\frac{4t^4}{c^4} - \frac{4t^3}{c^2} + 3t^2 > 0 \tag{5.47}$$

which may be trivially verified to be so.

Let us now look at condition (5.46). We are interested in the region where $f(y^2) < 0$. $f(y^2)$ is a cubic in y^2 and has the asymptotic behaviour

$$f(y^2) \to \infty \quad \text{as} \quad y^2 \to -\infty$$
$$f(y^2) \to -\infty \quad \text{as} \quad y^2 \to +\infty \tag{5.48}$$

When $y^2 = 0$, $$f(y^2) = \frac{t^2c^2}{4} > 0 \tag{5.49}$$

$f(y^2)$ has maximum and minimum values at

$$y^2 = \frac{8c^2 \pm \sqrt{16c^4 + 18t^2}}{12} \tag{5.50}$$

both of which are positive if c^2 is large enough. Under this condition the cubic will have the general form as shown in Fig. 14.

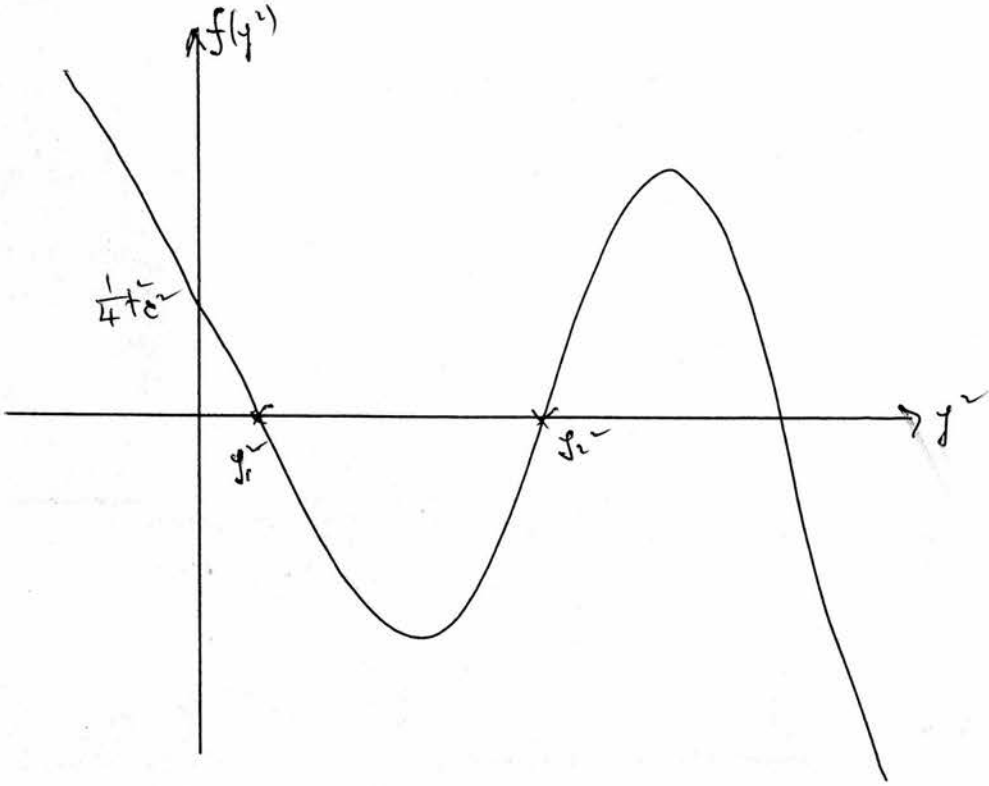

Fig. 14

Thus we know that we must have $f(y^2) < 0$ if $y_1^2 < y^2 < y_2^2$

where $$f(y_1^2) = f(y_2^2) = 0$$

and $$y_1^2 < y_2^2 < \frac{1}{12}\left[8c^2 + \sqrt{16c^4 + 18t^2}\right] \qquad (5.51)$$

It is now a matter of simple algebra to show that

$$f\left(\frac{t^2}{c^2}\right) < 0$$

$$f(c^2 - t) < 0$$

and $$c^2 - t < \frac{1}{12}\left[8c^2 + \sqrt{16c^4 + 18t^2}\right] \qquad (5.52)$$

and so we may deduce that $f(y^2) < 0$ if

$$\frac{t^2}{c^2} \leq y^2 \leq c^2 - t \qquad (5.53)$$

and hence that the curve never enters these strips. Thus we have analyticity in the k plane inside the unshaded regions in Fig. 15.

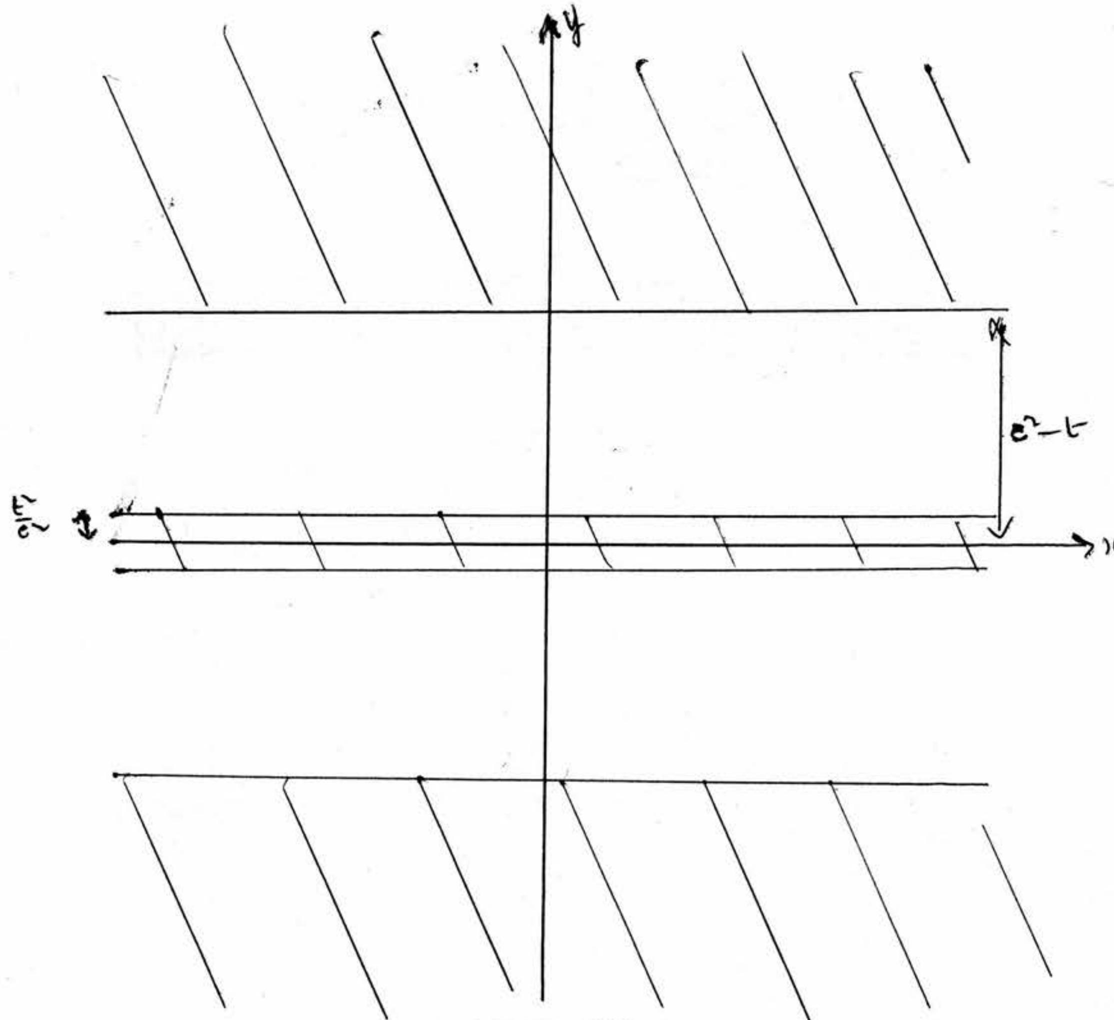

Fig. 15.

When the transformation from the k plane to the s plane is carried out, we will have analyticity between two parabolae as shown in Fig. 16.

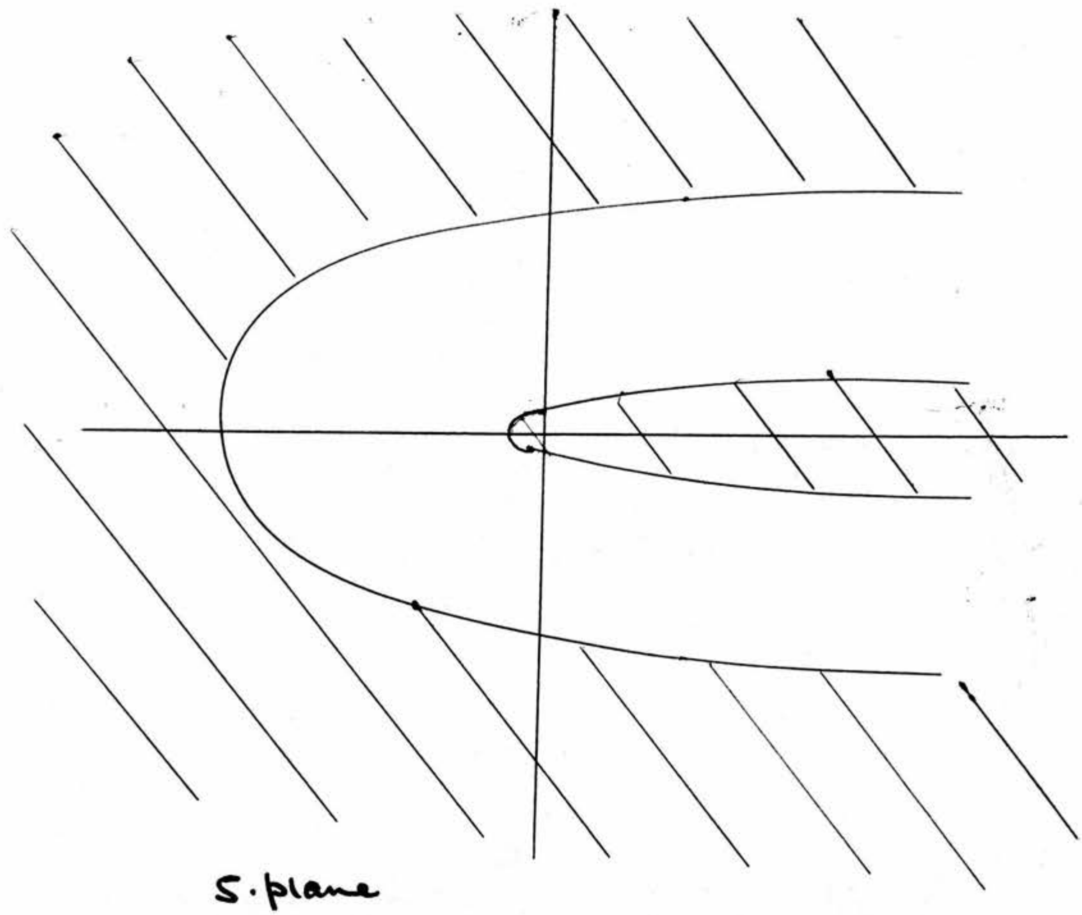

Fig. 16.

As n approaches infinity one parabola recedes to infinity, while the other collapses on to the positive real x-axis. Thus we can obtain analyticity inside an arbitrarily large domain of the s-plane (excluding the positive real axis) by taking a large enough number of iterations.

This result has been seen to hold for each term in the denominator for which singularities may arise and thus holds for the whole remainder term.

We have thus arrived at the result that by taking a large enough number of iterations of the Low equation, $T(s,t)$ may be made analytic in arbitrarily large regions of the s and t planes when the other variable is held fixed real and positive, with the exception of the cuts $0 \leq s < \infty$ and $-\infty < t \leq -\mu^2$ and the bound state poles.

We now proceed to verify the statement made earlier that $T(s,t)$ is a real function of the complex variable s, in other words that

$$T(s^*,t) = \left[T(s,t)\right]^* \tag{5.54}$$

First of all we note that the n-th Born term is, for complex s defined by

$$T^{(n)}(\underline{k}_i;\underline{k}_f) = \int d^3\underline{k}_1\, d^3\underline{k}_2 \cdots d^3\underline{k}_{n-1} \times$$

$$\times \frac{1}{\mu^2+(\underline{k}_f-\underline{k}_1)^2}\,\frac{1}{s-k_1^2}\,\frac{1}{\mu^2+(\underline{k}_1-\underline{k}_2)^2}\,\frac{1}{s-k_2^2} \cdots \frac{1}{\mu^2+(\underline{k}_{n-1}-\underline{k}_i)^2} \tag{5.55}$$

which from its very form we see is a real function of s, namely

$$T^{(n)}(s^*,t) = \left[T^{(n)}(s,t)\right]^* \tag{5.56}$$

This also obviously holds good for the non-energy conserving case

$$T^{(n)}(s^*,t,\bar{s}) = \left[T^{(n)}(s,t,\bar{s})\right]^* \tag{5.57}$$

If we next consider the remainder term we have, for complex s, from (5.1)

$$R^{(2n+1)}(s,t) = \int d^3p\, d^3q\, d^3r\, T^{(n)}(k_f, p) \frac{1}{s-p^2} \frac{T(p,q)\, T^*(r,q)}{s-q^2} \frac{1}{s-r^2} T^{(n)}(r, k_i)$$

$$+ \sum_B \int d^3p\, d^3r\, T^{(n)}(k_f, p) \frac{1}{s-p^2} \frac{T_B(p)\, T_B^*(r)}{s+B} \frac{1}{s-r^2} T^{(n)}(r, k_i) \tag{5.58}$$

and this, because of (5.57) may again be immediately seen to be a real function of s :

$$R^{(2n+1)}(s^*, t) = \left[R^{(2n+1)}(s, t) \right]^* \tag{5.59}$$

Thus, since we may write

$$T(s,t) = \sum_{i=1}^{2n+1} T^{(i)}(s,t) + R^{(2n+1)}(s,t) \tag{5.60}$$

we will have, by virtue of (5.56) and (5.59) the reality of T(s, t):

$$T(s^*, t) = \left[T(s,t) \right]^* \tag{5.61}$$

VI. <u>Asymptotic Behaviour of the Scattering Amplitude</u>.

Before we can write down a dispersion relation embodying the analytic properties deduced in the previous sections it is necessary to know the behaviour of the scattering amplitude as s and t approach infinity, so that the contribution from the integration over the infinite circle in the Cauchy integral may be dealt with in an appropriate manner.

(1) Behaviour as $|s| \to \infty$.

From equation (4.14) it is seen that the Born series is absolutely and uniformly convergent if $\mu > 2$. In this case since each Born term (apart from the first) tends to zero as $|s| \to \infty$, we may say that

$$T(s,t) \to T^{(1)}(t) \qquad \text{as} \quad |s| \to \infty \tag{6.1}$$

However, the condition that $\mu > 2$ is rather restrictive for it certainly excludes the possibility of bound states since it was shown by Bargmann[19] that if $I = \int_0^\infty r|V(r)|\,dr$ and n_ℓ are the number of bound states with angular momentum ℓ, then

$$(2\ell+1)n_\ell < I \tag{6.2}$$

and in the case of the pure Yukawa potential $I = \frac{1}{\mu}$.

However, even if $\mu \leqslant 2$ it may be shown, as was done by Klein and Zemach[3] that the Born series is in fact absolutely and uniformly convergent provided $s > B_{max}$ where B_{max} is the greatest binding energy of a bound state, and thus we still get the result that $T(s, t) \to T^{(1)}(t)$ as $|s| \to \infty$ Thus the scattering amplitude behaves like a constant for large values of the energy, and so one subtraction is necessary before a dispersion relation may be written down.

(2) Behaviour as $|t| \to \infty$.

If $\mu > 2$ again the problem is trivial, since the Born series is uniformly and absolutely convergent, and each term tends to zero as $|t|$ tends to infinity, and thus the scattering amplitude itself must tend to zero as $|t|$ approaches infinity.

However, the matter is very much more complicated if $\mu < 2$ and there is the possibility of bound states. As mentioned in the introduction what we need to write down a double dispersion relation in s and t after having a single dispersion relation in s, is not necessarily information concerning the behaviour in t of $T(s,t)$ but only of $\mathrm{Im}\, T(s,t)$. Now, the analytic and asymptotic behaviour in s leads to the dispersion relation

$$T(s,t) = T^{(1)}(t) + \sum \frac{A_B(t)}{s+B} + \int_0^\infty \frac{\mathrm{Im}\, T(s',t)}{s-s'+i\epsilon}\, ds' \qquad (6.3)$$

Hence **if** $\mathrm{Im}\, T(s',t) \to 0$ as $|t| \to \infty$ we can write

$$\mathrm{Im}\, T(s',t) = \int_{\mu^2}^\infty \frac{\rho(s',t')}{t'+t}\, dt' \qquad (6.4)$$

But the unitarity relationship as stated in the introduction tells us that

$$\mathrm{Im}\, T(s,t) = \frac{\pi\sqrt{s}}{2} \int T^*(s, (\underline{k}_f - \underline{k}')^2)\, T(s, (\underline{k}' - \underline{k}_i)^2)\, d\Omega_{k'}$$

Then the contribution of the bound state terms on the right hand side do not vanish for large $|t|$ while the left hand side does. Thus we see that the assumption that $\mathrm{Im}\, T(s,t) \to 0$ as $|t| \to \infty$

is inconsistent with the unitarity condition if there are bound states present. What we should like to do is to determine the behaviour of $\mathrm{Im}\, T(s,t)$ or $T(s, t)$ as $|t| \longrightarrow \infty$, knowing that at the best it will behave like a polynomial in t. It should be admitted immediately that the attempt to determine this behaviour, using the ideas of this thesis, has not been attended with any success, but since a good deal of time and effort were spent on this project, perhaps an outline of the method by which the attempt was made would not be out of place.

We have the iterated form of the Low equation

$$T(s,t) = \sum_{r=1}^{2n} T^{(r)}(s,t) + R^{(2n+1)}(s,t) \tag{6.6}$$

where for fixed real positive s we know that $R^{(2n+1)}(s,t)$ is analytic inside an ellipse in the t plane with centre at the point $t = 2s$ and semi-major and semi-minor axes $2(s + 2n^2\mu^2)$ and $4n\mu\sqrt{s + n^2\mu^2}$ respectively (6.7)

This ellipse may readily be shown to include the circle

$$|t| = 4n^2\mu^2 \tag{6.8}$$

We are interested in what happens to $T(s,t)$ when $|t| \longrightarrow \infty$. If we try to make $|t| \rightarrow \infty$ in the right hand side of equation (6.6) however we land in trouble when t reaches the edge of the ellipse inside which $R^{(2n+1)}(s,t)$ is certainly analytic. If we then want to increase t further, we must perform one more iteration to include two more Born terms in the series and to have the remainder term $R^{(2n+3)}(s,t)$ instead of $R^{(2n+1)}(s,t)$. Thus to determine the asymptotic behaviour of the scattering amplitude, what we should do is to consider the limit as n tends towards infinity of the functions

$$\sum_{r=1}^{2n} T^{(r)}(s, 4n^2\mu^2 e^{i\theta}) \tag{6.9}$$

and

$$R^{(2n+1)}(s, 4n^2\mu^2 e^{i\theta}) \tag{6.10}$$

A lengthy investigation of these functions was carried out, but without achieving any useful results. It is known that the expression in (6.9) will tend to zero as $n \rightarrow \infty$, since it can be shown that $T^{(n)}(s,t) \rightarrow 0$ as $|t| \longrightarrow \infty$ as fast as $\frac{1}{|t|}$. Hence we have $2n$ terms each tending to zero like $\frac{1}{4n^2\mu^2}$ and so the whole sum should also tend to zero. However, the treatment of $R^{(2n+1)}(s, 4n^2\mu^2 e^{i\theta})$ is rather more intractable and unfortunately, despite a thorough attempt to determine its asymptotic behaviour we have not been able to deduce these results.

Thus we are only able to write down a double dispersion relation with an arbitrary number of subtractions in the momentum transfer variable. One possible way of performing these subtractions leads to the integral representation:

$$T(s,t) = T^{(1)}(t) + \sum_B \frac{f_B(t)}{s+B}$$

$$+ t^{n+1}\int_0^\infty ds' \int_{\mu^2}^\infty dt' \frac{\rho(s',t')}{t'^{n+1}(t'+t)(s-s'+i\epsilon)}$$

$$+ \sum_{i=0}^{n} t^i \int_0^\infty ds' \frac{g_i(s')}{s-s'+i\epsilon} \qquad (6.11)$$

The $f_B(t)$ may be readily shown to be polynomials in t, the degree of the polynomial being ℓ_B the angular momentum of the bound state B.

VII. Partial Wave Amplitudes.

We now leave the consideration of the complete scattering amplitude and turn our attention to the analytic properties of the partial wave amplitudes. The motivation for this is derived from our inability to obtain an unambiguous integral representation for the scattering amplitude. Since the partial wave amplitudes are functions of the energy only, their behaviour both as regards analytic properties and as regards asymptotic properties should be able to be readily determined.

If we write the scattering amplitude $T(\underline{k}, \underline{\ell})$ as a function of the energy s and the scattering angle $\cos\theta$ ($k^2 = \ell^2 = s$, $\underline{k}.\underline{\ell} = s\cos\theta$) in the form $T(s, \cos\theta)$, then the ℓ-th partial wave amplitude $T_\ell(s)$ is defined by

$$T_\ell(s) = \frac{1}{2}\int_{-1}^{+1} d(\cos\theta)\, T(s, \cos\theta)\, P_\ell(\cos\theta) \tag{7.1}$$

where $P_\ell(\cos\theta)$ is the usual Legendre polynomial.

Then the total scattering amplitude can be expanded in terms of partial wave amplitudes

$$T(s, \cos\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, T_\ell(s)\, P_\ell(\cos\theta) \tag{7.2}$$

provided the series on the right converges. By general theorems on Legendre polynomials (see for example reference 20), it is known that the series will converge inside any ellipse in the $\cos\theta$ plane with foci ± 1 within which $T(s, \cos\theta)$ is analytic. Since this plane is cut from $1 + \frac{\mu^2}{2s}$ to $+\infty$ the largest ellipse for which this will hold must have major axis equal to $1 + \mu^2/2s$. Within this ellipse therefore the partial wave expansion is valid.

We now deduce the analytic properties of the partial wave amplitudes from the general Mandelstam representation (6.11) for the total amplitude.

If there are no subtractions, then in taking the partial wave projection we are interested in a term of the type

$$\int_{-1}^{+1} d(\cos\theta)\, \frac{1}{t'+t}\, P_\ell(\cos\theta)$$

$$= \int_{-1}^{+1} d(\cos\theta)\, \frac{1}{t' + 2s(1-\cos\theta)}\, P_\ell(\cos\theta)$$

$$= \frac{1}{2s}\int_{-1}^{+1} d(\cos\theta)\ \frac{1}{\left(1+\frac{t'}{2s}\right)-\cos\theta}\, P_\ell(\cos\theta)$$

$$= \frac{1}{2s}\int_{-1}^{+1} d(\cos\theta)\ \sum_{n=0}^{\infty}(2n+1)\,P_n(\cos\theta)\,Q_n\left(1+\frac{t'}{2s}\right)P_\ell(\cos\theta)$$

(see page 322 of reference 20)

$$= \sum_{n=0}^{\infty}\frac{1}{2s}(2n+1)\frac{2}{2n+1}\,\delta_{n\ell}\,Q_n\left(1+\frac{t'}{2s}\right)$$

$$= Q_\ell\left(1+\frac{t'}{2s}\right) \tag{7.3}$$

which has singularities for

$$-1 \le 1+\frac{t'}{2s} \le +1 \tag{7.4}$$

i.e. $\quad -2 \le \dfrac{t'}{2s} \le 0$

But $\quad \mu^2 \le t' < \infty \qquad$ (7.5)

So we may deduce that we must have $s < 0$ also $-2 \le \frac{t'}{2s}$ and since we have $s = -|s|$ then

$$-4 \le -\frac{t'}{|s|}$$

i.e. $\quad 4 \geqslant \dfrac{t'}{|s|}$

i.e.

$$|s| \geq t'/4$$

$$\geq \mu^2/4 \tag{7.6}$$

So we have the range of singularities (the cut)

$$-\infty < s \leq -\mu^2/4 \tag{7.7}$$

If there are subtractions and bound state terms in the general Mandelstam representation (6.11), it may be easily shown that this introduces no more singularities in s for the partial wave amplitudes. Thus we obtain the result that $T_\ell(s)$ is an analytic function of s apart from the cuts $0 \leq s < \infty$ and $-\infty < s \leq -\frac{\mu^2}{4}$.

We are also interested in the nature of the cut $0 \leq s < \infty$ for the total scattering amplitude, and the analytic properties of the amplitude if we continue through this cut. We use the methods of Zimmermann(21) to show that the cut is, for the partial wave amplitudes, a square root branch cut, and hence is the same type of cut for the total amplitude, at least for the region in which the series is convergent. We then know that there are only two sheets of the function associated with this cut, so we shall determine the analytic behaviour of the function on its second (unphysical) sheet.

To apply the methods of Zimmermann we need to make use of the unitarity relationship. This is, from equation (1.36)

$$\mathrm{Im}\, T(\underline{k},\underline{\ell}) = \frac{\pi\sqrt{s}}{2}\int T(\underline{k},\underline{k}')\,T^*(\underline{\ell},\underline{k}')\,d\Omega_{k'} \tag{7.8}$$

with $k^2 = \ell^2 = k'^2 = s$.

i.e.

$$\mathrm{Im}\, T(s, \cos(\underline{k}, \underline{\ell})) = \frac{\pi\sqrt{s}}{2}\int T(s, \cos(\underline{k}, \underline{k}'))\, T^{*}(s, \cos(\underline{\ell}, \underline{k}'))\, d\Omega_{k'} \qquad (7.9)$$

or, on putting in the partial wave expansion (7.2)

$$\sum (2\ell+1)\, T_\ell(s)\, P_\ell(\cos(\underline{k}, \underline{\ell})) =$$

$$= \frac{\pi\sqrt{s}}{2}\int \sum_{\ell, j} (2\ell+1)\, T_\ell(s)\, P_\ell(\cos(\underline{k}, \underline{k}'))\,(2j+1)\, T_j^{*}(s)\, P_j(\cos(\underline{\ell}, \underline{k}'))\, d\Omega_{k'}$$

(7.10)

$$= \frac{\pi\sqrt{s}}{2}\sum_{\ell, j} (2\ell+1)(2j+1)\, T_\ell(s)\, T_j^{*}(s)\, \delta_{\ell j}\, P_\ell(\cos(\underline{k}, \underline{\ell}))\, \frac{4\pi}{2\ell+1}$$

$$= 2\pi^2\sqrt{s}\sum_{\ell} (2\ell+1)\, |T_\ell(s)|^2\, P_\ell(\cos(\underline{k}, \underline{\ell}))$$

(7.11)

so that we may deduce that

$$\mathrm{Im}\, T_\ell(s) = 2\pi^2\sqrt{s}\, |T_\ell(s)|^2 \qquad (7.12)$$

Now we ~~may~~ know that $T_\ell(s)$ with s real is the boundary value of an analytic function $T_\ell(s)$ of a complex variable apart from the cuts $s \leq -\frac{1}{4}\mu^2$ and $s \geq 0$. Along the real axis we have from reality conditions the fact that we must have

$$\lim_{\epsilon \to 0} T_\ell^{*}(s - i\epsilon) = \lim_{\epsilon \to 0} T_\ell(s + i\epsilon) \qquad (7.13)$$

If we now define $T_e'(s)$ by

$$T_e'(s) = \frac{T_e(s)}{1 + i\,2\pi^2\sqrt{s}\,T_e(s)} \tag{7.14}$$

where $\sqrt{s}$ has a cut from $s = 0$ to $s = \infty$ we see that $T_e(s)$ is certainly analytic in the whole s plane apart from the cuts $s \geqslant 0$ and $s \leqslant -\mu^2/4$ (and possible poles at zeros of the denominator). Also $T_e'^{*}(s - i\varepsilon)$ =

$$= \frac{T_e^{*}(s-i\epsilon)}{1 - 2\pi^2 i\sqrt{s-i\epsilon}\,T_e^{*}(s-i\epsilon)}$$

$$= \frac{T_e(s+i\epsilon)}{1 + 2\pi^2 i\sqrt{s+i\epsilon}\,T_e(s+i\epsilon)}$$

$$= T_e'(s+i\epsilon) \tag{7.15}$$

so that $T_e(s)$ is also a real function of the complex variable s.

Now we have that

$$\mathrm{Im}\,T_e'(s) = \mathrm{Im}\,\frac{T_e(s)\left[1 - i2\pi^2\sqrt{s}\,T_e^{*}(s)\right]}{\left|1 + i2\pi^2\sqrt{s}\,T_e(s)\right|^2}$$

$$= \frac{\mathrm{Im}\,T_e(s) - 2\pi^2\sqrt{s}\,|T_e(s)|^2}{\left|1 + i2\pi^2\sqrt{s}\,T_e(s)\right|^2} \tag{7.16}$$

$$= 0$$

by the unitarity condition (7.12) which holds for real $s \geq 0$.

Thus the discontinuity of $T_\ell(s)$ across the cut $s \geq 0$ is zero and so $T_\ell(s)$ does not have this singularity. But now we may write equation (7.14) in the form

$$T_\ell(s) = \frac{T_\ell'(s)}{1 - i\,2\pi^2\sqrt{s}\,T_\ell'(s)} \tag{7.17}$$

so that we have

$$T_\ell(s) = F_\ell(s) + i\,2\pi^2\sqrt{s}\,G_\ell(s) \tag{7.18}$$

with

$$F_\ell(s) = \frac{T_\ell'(s)}{1 + 4\pi^4 s\,T_\ell'^2(s)}$$

(7.19)

$$G_\ell(s) = \frac{T_\ell'^2(s)}{1 + 4\pi^4 s\,T_\ell'^2(s)}$$

(7.20)

so that $G_\ell(s)$ and $F_\ell(s)$ are analytic (apart from poles) in the s-plane cut from $-\infty$ to $-\frac{\mu^2}{4}$ and are real on the positive real axis.

We hence deduce that $T_\ell(s)$ has, from the form (7.18) a square root branch point at the origin and thus the singularity $0 \leq s < \infty$ must be a square root branch cut, and hence this cut must connect two and only two sheets. The same result may be deduced for the total scattering amplitude, at least for the region in which the partial wave expansion is valid. This certainly includes the physical range of the scattering angle: $-1 \leq \cos\theta \leq +1$. The actual domain of convergence may be found as follows.

We know that $T(s,t)$ is an analytic function of t apart from the cut $t = -t'$ where $t' \geq \mu^2$. Hence we must have $T(s, \cos\theta)$ an analytic function of $\cos\theta$ apart from the cut

$$2s(1-\cos\theta) = -t'$$

$$\cos\theta = 1 + \frac{t'}{2s} \tag{7.21}$$

where $t' \geq \mu^2$.

Thus the cut starts at the point $\cos\theta = 1 + \frac{\mu^2}{2s}$ and runs off to infinity. So we know by a general theorem on Legendre polynomials that the partial wave expansion will be valid inside an ellipse in the $\cos\theta$ plane which has foci at the points ± 1 and passes through the point $1 + \frac{\mu^2}{2s}$ (where we remember that s may be complex). The calculation of the major semi axis of this ellipse is trivial but tedious. It turns out to be of length

$$a(s) = \sqrt{1 + \frac{\mu^2}{|s|^2}\,\mathrm{Re}\,s + \frac{\mu^4}{4|s|^2} + \frac{\mu^2}{8|s|^2}\sqrt{16|s|^2 + 8\,\mathrm{Re}\,s\,\mu^2 + \mu^4}} \tag{7.22}$$

VIII. Continuation Through the Square Root Branch Cut

This continuation of the total amplitude is effected by means of unitarity and reality. Unitarity says that we must have

$$\mathrm{Im}\, T(\underline{k}_i, \underline{k}_f) = \pi \int T^*(\underline{k}_f, \underline{k}')\, T(\underline{k}', \underline{k}_i)\, \delta(k'^2 - s)\, d^3\underline{k}' \tag{8.1}$$

which can be written in the form

$$\mathrm{Im}\, T(s,t) = \int K(s,t,t',t'')\, T^*(s,t')\, T(s,t'')\, dt'\, dt'' \tag{8.2}$$

where K is a function we shall determine later. But now if we continue through the cut and denote the function T(s,t) on the second sheet of s by $T^{II}(s,t)$ we will have (with s real in what immediately follows)

$$\begin{aligned} T^{II}(s-i\epsilon,t) &= T(s+i\epsilon,t) \\ &= T(s-i\epsilon,t) + \mathrm{Im}\, T(s-i\epsilon,t) \\ &= T(s-i\epsilon,t) + \int K(s-i\epsilon,t,t',t'')\, T^*(s-i\epsilon,t')\, T(s-i\epsilon,t'')\, dt'\, dt'' \\ &= T(s-i\epsilon,t) + \int K(s-i\epsilon,t,t',t'')\, T(s-i\epsilon,t'')\, T^{II}(s-i\epsilon,t')\, dt'\, dt'' \end{aligned} \tag{8.3}$$

using the fact that

$$\begin{aligned} T^*(s-i\epsilon,t') &= T(s+i\epsilon,t') \\ &= T^{II}(s-i\epsilon,t') \end{aligned} \tag{8.4}$$

Equation (8.3) written for general complex s takes the form

$$T^{II}(s,t) = T(s,t) + \int K(s,t,t',t'')\, T(s,t'')\, T^{II}(s,t')\, dt'dt'' \qquad (8.5)$$

This equation (8.5) is made the basis of the continuation procedure. It is actually an equation of the Fredholm type and thus by a general theorem due to Candlin and Screaton[22] $T^{II}(s,t)$ will be analytic in the region in which both $T(s,t)$ and the kernel are analytic. We thus look for the singularities due to the kernel.

Now the unitarity relationship may be written in the form

$$\operatorname{Im} T(s,t) = \frac{\pi\sqrt{s}}{2}\int T^{*}(\underline{k}_f,\underline{k}')\, T(\underline{k}_i,\underline{k}')\, d\Omega_{k'} \qquad (8.6)$$

$$(k_f^2 = k_i^2 = k'^2 = s)$$

so that we want to put the integral appearing on the right hand side of this relationship into the form

$$\int K(s,t,t',t'')\, T^{*}(s,t')\, T(s,t'')\, dt'dt'' \qquad (8.7)$$

Now, if we choose the representations

$$\underline{k}' = k(\cos\varphi,\ \sin\varphi\cos\psi,\ \sin\varphi\sin\psi)$$

$$\underline{k}_f = k\left(\cos\tfrac{\theta}{2},\ \sin\tfrac{\theta}{2},\ 0\right)$$

$$\underline{k}_i = k\left(\cos\tfrac{\theta}{2},\ -\sin\tfrac{\theta}{2},\ 0\right) \qquad (8.8)$$

we shall have

$$d\Omega_{k'} = d(\cos\varphi)\,d\psi \qquad (8.9)$$

and

$$t' = (\underline{k}_f - \underline{k}')^2 = 2s(1 - \hat{\underline{k}}_f \cdot \hat{\underline{k}}') = 2s\left(1 - \cos\frac{\theta}{2}\cos\varphi - \sin\frac{\theta}{2}\sin\varphi\cos\psi\right)$$

and $$t'' = (\underline{k}_i - \underline{k}')^2 = 2s(1 - \hat{\underline{k}}_i \cdot \hat{\underline{k}}') = 2s\left(1 - \cos\frac{\theta}{2}\cos\varphi + \sin\frac{\theta}{2}\sin\varphi\cos\psi\right) \tag{8.10}$$

Then we shall have

$$\begin{aligned} d\Omega_{k'} &= d(\cos\varphi)\, d\psi \\ &= \frac{\partial(\cos\varphi, \psi)}{\partial(t', t'')}\, dt' dt'' \end{aligned} \tag{8.11}$$

where $J = \dfrac{\partial(\cos\varphi, \psi)}{\partial(t', t'')}$ is the Jacobian of the transformation

But

$$\frac{\partial(t', t'')}{\partial(\cos\varphi, \psi)} = \begin{vmatrix} \dfrac{\partial t'}{\partial\cos\varphi} & \dfrac{\partial t'}{\partial\psi} \\[2ex] \dfrac{\partial t''}{\partial\cos\varphi} & \dfrac{\partial t''}{\partial\psi} \end{vmatrix}$$

$$= \begin{vmatrix} -2s\cos\frac{\theta}{2} + 2s\sin\frac{\theta}{2}\cot\varphi\cos\psi & 2s\sin\frac{\theta}{2}\sin\varphi\sin\psi \\[2ex] -2s\cos\frac{\theta}{2} - 2s\sin\frac{\theta}{2}\cot\varphi\cos\psi & -2s\sin\frac{\theta}{2}\sin\varphi\sin\psi \end{vmatrix}$$

which reduces to $$8s^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\varphi\sin\psi \tag{8.12}$$

so that $$\frac{1}{J} = 8s^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\varphi\sin\psi$$

But now

$$4s\left(1-\cos\frac{\theta}{2}\cos\phi\right) = t''+t'$$

and $$4s\sin\frac{\theta}{2}\sin\phi\cos\psi = t''-t' \tag{8.13}$$

so that $$\frac{1}{J} = 2s\cos\frac{\theta}{2}\tan\psi\,(t''-t') \tag{8.14}$$

and so we want to evaluate tan ψ in terms of t', t" the energy and the scattering angle. If we use (8.13) together with $\cos^2\phi + \sin^2\phi = 1$ we obtain

$$\frac{1}{\cos^2\frac{\theta}{2}}\left(1-\frac{t'+t''}{4s}\right)^2 + \frac{1}{(4s)^2}\,\frac{1}{\sin^2\frac{\theta}{2}}\,\frac{1}{\cos^2\psi}\,(t''-t')^2 = 1 \tag{8.15}$$

so that

$$\sec^2\psi = \left[1-\frac{1}{\cos^2\frac{\theta}{2}}\left(1-\frac{t'+t''}{4s}\right)^2\right]\frac{(4s)^2\sin^2\frac{\theta}{2}}{(t''-t')^2} \tag{8.16}$$

and hence, using the fact that $\tan^2\psi = \sec^2\psi - 1$

$$\tan^2\psi = \frac{1}{(t''-t')^2}\left[16s^2\sin^2\frac{\theta}{2} - \tan^2\frac{\theta}{2}(4s-t'-t'')^2\right] - 1 \tag{8.17}$$

so that eventually we obtain

$$\frac{1}{J} = 2s\cos\frac{\theta}{2}\sqrt{16s^2\sin^2\frac{\theta}{2} - \tan^2\frac{\theta}{2}(4s-t'-t'')^2 - (t'-t'')^2} \tag{8.18}$$

We must now determine the region in the t' - t" plane over which the integration has to be taken. This region will correspond to the region shown in the ϕ - ψ plane in Fig. 17.

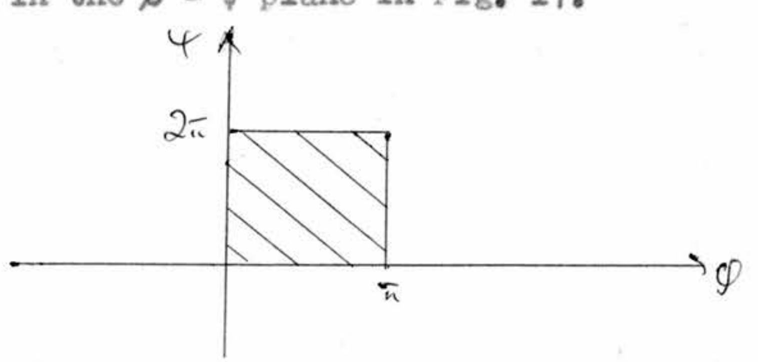

Fig. 17.

i.e. we have

$$-1 \leq \cos\varphi \leq 1$$

$$0 \leq \psi \leq 2\pi \tag{8.19}$$

so that we have the conditions

$$\sec^2\psi \geqslant 1 \tag{8.20}$$

and
$$\cos^2\varphi \leq 1 \tag{8.21}$$

(8.20) implies with the help of (8.16) that we have

$$\left[1 - \frac{1}{\cos^2\frac{\theta}{2}}\left(1 - \frac{t''+t'}{4s}\right)^2\right]\frac{(4s)^2\sin^2\frac{\theta}{2}}{(t''-t')^2} \geqslant 1 \tag{8.22}$$

while (8.21) yields on the use of (8.13) the fact that

$$\left(1 - \frac{t'+t''}{4s}\right)^2\frac{1}{\cos^2\frac{\theta}{2}} \leq 1 \tag{8.23}$$

Equation (8.22) rearranged gives the condition

$$\frac{(t''-t')^2}{(4s)^2\sin^2\frac{\theta}{2}} + \frac{1}{\cos^2\frac{\theta}{2}}\left(1 - \frac{t'+t''}{4s}\right)^2 \leq 1 \tag{8.24}$$

So we see that if this condition holds, then condition (8.23) will also hold.

The condition (8.24) reduces to

$$t''^2 - 2t''t'\cos\theta + t'^2 - 8s\sin^2\frac{\theta}{2}(t''+t') + 16s^2\sin^4\frac{\theta}{2} \leq 0 \tag{8.25}$$

For this condition to be true t' and t'' must lie inside the ellipse

$$t''^2 - 2t''t'\cos\theta + t'^2 - 8s\sin^2\frac{\theta}{2}(t''+t') + 16s^2\sin^4\frac{\theta}{2} = 0 \tag{8.26}$$

This is an ellipse with centre at the point (2s, 2s), major axis lying on the line $t'' = t'$ and of length $4\sqrt{2}\,s\cos\frac{\theta}{2}$, minor axis along the line $t'' + t' = 4s$ as shown in the Fig. 18.

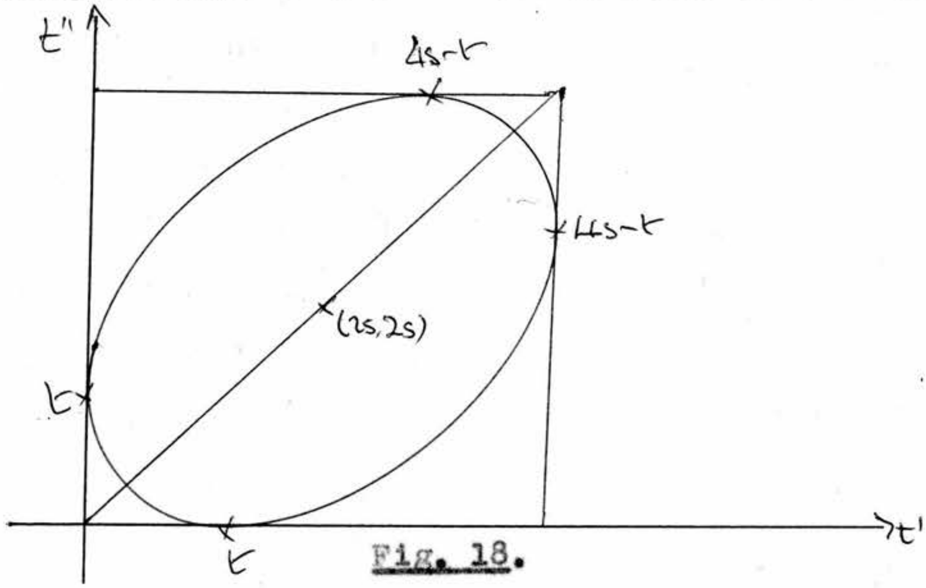

Fig. 18.

Thus the double integration in the integral (8.7) is only over the interior of this ellipse. This will be automatically ensured if we introduce a step function, the usual θ function, defined by

$$\theta(x) = 1, \quad x > 0$$

$$\theta(x) = 0, \quad x < 0 \qquad (8.27)$$

into the integral (8.7) namely

$$\theta\left(2t't''\cos\theta - t'^2 - t''^2 + 8s\sin^2\frac{\theta}{2}(t'+t'') - 16s^2\sin^4\frac{\theta}{2}\right) \qquad (8.28)$$

If we write this in terms of the momentum transfer instead of the scattering angle it turns out to be

$$\theta\left(2t't'' - t'^2 - t''^2 - tt't''/s + 2t(t'+t'') - t^2\right)$$
$$= \theta\left(-t'^2 - t''^2 - t^2 + 2tt'' + 2tt' + 2t't'' - tt't''/s\right) \qquad (8.29)$$

If we also write the Jacobian in terms of t we obtain

$$\frac{1}{J(s,t,t',t'')} = 2s\left[-t'^2-t''^2-t^2+2tt''+2tt'+2t''t'-t''t't/s\right]^{\frac{1}{2}} \tag{8.30}$$

So that the step function automatically ensures that the function under the square root is positive.

We now have the result that

$$T^{II}(s,t) = T(s,t) + \int_{\mathcal{E}} J(s,t,t',t'')\,T(s,t')\,T^{II}(s,t'')\,dt'dt'' \tag{8.31}$$

where $\mathcal{E}$ is the shaded region shown in Fig. 19.

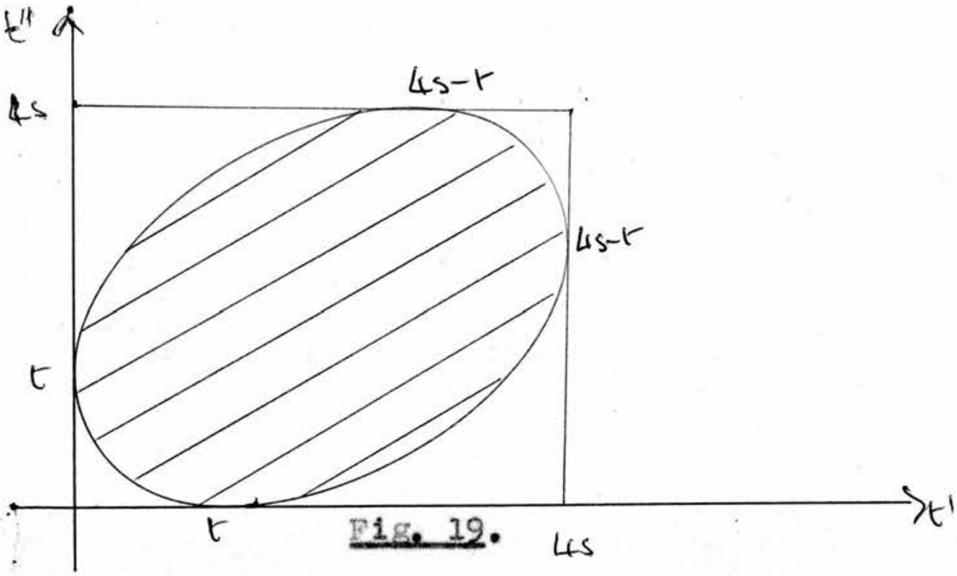

Fig. 19.

For $T(s,t')$ we can write the dispersion relation (if we assume no subtractions in the momentum transfer are necessary)

$$T(s,t') = \int_{-\infty}^{-\mu^2} \frac{\rho(s,t_1)}{t'-t_1}\,dt_1 \tag{8.32}$$

and let us perform the t' integration.

The Jacobian may be written in the form

$$\frac{1}{J} = 2s\sqrt{-(t'-\alpha)(t'-\beta)} \tag{8.33}$$

and then the equation of the ellipse $\mathcal{E}$ is just

$$(t'-\alpha)(t'-\beta) = 0 \tag{8.34}$$

so that the range of the t' integration (for fixed t") is just from α to β.

Thus we are interested in the integral

$$\int_{\alpha}^{\beta} \frac{1}{\sqrt{-(t'-\alpha)(t'-\beta)}} \frac{1}{t'-t_1} dt' \tag{8.35}$$

This is evaluated in the appendix where it is shown to be equal to

$$\frac{-i\pi}{\sqrt{-(t_1-\alpha)(t_1-\beta)}} \tag{8.36}$$

We thus have the possibility of singularities at

$$(t_1-\alpha)(t_1-\beta) = 0$$

i.e. $$t_1^2+t^2+t''^2 -2tt'' -2t_1t''-2t_1t + \frac{t_1tt''}{s} = 0 \tag{8.37}$$

where we remember that t_1 takes on values between $-\infty$ and $-\mu^2$.

This relationship, if we put

$$t = 2s(1-\cos\theta)$$

$$t_1 = 2s(1-\cos\varphi)$$

$$t'' = 2s(1-\cos\psi) \tag{8.38}$$

becomes

$$\cos^2\theta + \cos^2\varphi + \cos^2\psi - 2\cos\theta\cos\varphi\cos\psi - 1 = 0 \tag{8.39}$$

which is easily shown to be equivalent to

$$\cos\varphi = \cos(\theta \pm \psi) \tag{8.40}$$

Now since t" has to lie between 0 and 4s, ψ must be a real angle, and if we consider a physical scattering process likewise so must θ be a real angle. Hence we must have

$$-1 \leq \cos(\theta \pm \psi) \leq +1 \tag{8.41}$$

and so we shall have the possibility of singularities

if $$-1 \leq \cos\varphi \leq +1$$

i.e. $$-1 \leq 1 - \frac{t_1}{2s} \leq +1$$

i.e. $$-2 \leq -\frac{t_1}{2s} \leq 0 \tag{8.42}$$

But $$-\infty < t_1 \leq -\mu^2 \tag{8.43}$$

so that $-\frac{t_1}{2s} \leq 0$ implies that

$$2s \leq 0 \tag{8.44}$$

and so $$s = -|s| \tag{8.45}$$

so that $-2 \leq -\frac{t_1}{2s}$ is equivalent to

$$-2 \leq \frac{t_1}{2|s|}$$

$$= -\frac{|t_1|}{2|s|}$$

i.e.
$$|s| \geq \frac{|t_1|}{4}$$

$$\geq \frac{\mu^2}{4}$$

so that we have the possible range of singularities (cut) on the second sheet

$$-\infty < s \leq -\mu^2/4 \qquad (8.46)$$

It has been shown by Freund and Karplus[23] in the relativistic case that the corresponding cut on an unphysical sheet is a natural boundary of the scattering amplitude and their argument applies equally well in this case to show that the cut (8.46) is a natural boundary of the non-relativistic scattering amplitude.

This concludes our study of the analytic properties of the scattering amplitude in non-relativistic theory.

Appendix

We wish to evaluate the integral

$$I = \int_{\alpha}^{\beta} \frac{1}{\sqrt{-(x-\alpha)(x-\beta)}} \cdot \frac{1}{x-c}\, dx \tag{A.1}$$

If we make the substitution

$$u = \frac{1}{x-c} \tag{A.2}$$

we obtain

$$\begin{aligned} I &= \int_{1/\alpha-c}^{1/\beta-c} \frac{1}{\sqrt{-\left(\frac{1}{u}+c-\alpha\right)\left(\frac{1}{u}+c-\beta\right)}}\, u \cdot \left(-\frac{1}{u^2}\right) du \\ &= \int_{1/\alpha-c}^{1/\beta-c} \frac{-1}{\sqrt{-[1+(c-\alpha)u][1+(c-\beta)u]}}\, du \\ &= \int_{1/\alpha-c}^{1/\beta-c} \frac{i}{\sqrt{1+u(2c-\alpha-\beta)+u^2(c-\alpha)(c-\beta)}}\, du \end{aligned} \tag{A.3}$$

where we note that the quadratic form is positive definite within the range of integration

$$= \frac{i}{\sqrt{(c-\alpha)(c-\beta)}} \int_a^b \frac{du}{\sqrt{u^2 + \frac{2c-\alpha-\beta}{(c-\alpha)(c-\beta)} + \frac{1}{(c-\alpha)(c-\beta)}}} \tag{A.4}$$

where we have written

$$a = \frac{1}{\alpha-c}, \qquad b = \frac{1}{\beta-c} \tag{A.5}$$

$$= \frac{i}{\sqrt{(c-\alpha)(c-\beta)}} \int_a^b \frac{du}{\sqrt{\left[u + \frac{c-\frac{1}{2}(\alpha+\beta)}{(c-\alpha)(c-\beta)}\right]^2 + \frac{(c-\frac{\alpha+\beta}{2})^2}{(c-\alpha)^2(c-\beta)^2} + \frac{1}{(c-\alpha)(c-\beta)}}}$$

$$= \frac{i}{\sqrt{(c-\alpha)(c-\beta)}} \int_a^b \frac{du}{\sqrt{(u+u_0)^2 + k^2}} \tag{A.6}$$

on writing

$$u_0 = \frac{c - \frac{1}{2}(\alpha+\beta)}{(c-\alpha)(c-\beta)}$$

$$k^2 = \frac{1}{(c-\alpha)(c-\beta)} - \frac{[c - \frac{1}{2}(\alpha+\beta)]^2}{(c-\alpha)^2(c-\beta)^2}$$

$$= - \frac{(\alpha-\beta)^2}{4(c-\alpha)^2(c-\beta)^2} \tag{A.7}$$

If we then make the substitution

$$u + u_0 = k \sinh z \tag{A.8}$$

we obtain

$$I = \frac{i}{\sqrt{(c-\alpha)(c-\beta)}} \int_{u=a}^{b} dz$$

$$= \frac{i}{\sqrt{(c-\alpha)(c-\beta)}} \left[\sinh^{-1} \frac{u+u_0}{k} \right]_a^b$$

$$= \frac{i}{\sqrt{(c-\alpha)(c-\beta)}} \log \frac{z_2 + \sqrt{z_2^2 + k^2}}{z_1 + \sqrt{z_1^2 + k^2}} \qquad (A.9)$$

where

$$z_2 = b + u_0$$

$$z_1 = a + u_0 \qquad (A.10)$$

so that

$$z_2 = \frac{1}{2} \frac{(\alpha-\beta)}{(\beta-c)(\alpha-c)}$$

$$z_1 = -\frac{1}{2} \frac{(\alpha-\beta)}{(\beta-c)(\alpha-c)} \qquad (A.11)$$

using equations (A.5) and (A.7).

Hence we see that

$$z_1^2 + k^2 = z_2^2 + k^2 = 0 \qquad (A.12)$$

so that we have

$$\Sigma = \frac{i}{\sqrt{(c-\alpha)(c-\beta)}} \log \frac{z_2}{z_1}$$

$$= \frac{i}{\sqrt{(c-\alpha)(c-\beta)}} \log(-1)$$

$$= \frac{-i\pi}{\sqrt{-(c-\alpha)(c-\beta)}}$$

(A.13)

so that we have deduced the result that

$$\int_\alpha^\beta \frac{1}{\sqrt{-(x-\alpha)(x-\beta)}} \cdot \frac{1}{x-c}\, dx = \frac{-i\pi}{\sqrt{-(c-\alpha)(c-\beta)}}$$

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