Vacuum Stability of the Standard Model and BSM Extensions

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Abstract

The Standard Model scalar potential contains a minimum at the Electroweak scale, responsible for the masses of the weak gauge bosons through the Higgs mechanism. However, if the Electroweak minimum is only a local minimum, and there exists a global minimum at a higher energy in the Higgs potential, then in a sufficiently old universe we would expect the vacuum expectation value to be at the global minimum. The absence of a global minimum at higher energy is related to the condition that the Higgs self coupling is greater than or equal to zero for all energies. For any model that fails this, we expect new physics to enter before the energy at which the coupling becomes negative. We developed tools to automate the derivation of beta functions for renormalisable gauge theories, and used these to carry out evolution of the renormalisation group equations for the Standard Model and three extensions to the Standard Model — the Standard Model with a fourth generation, the Standard Model with right-handed neutrinos and a Left-Right Symmetric Model. We conclude that of these four models, the Standard Model is the only one in which all the couplings remain perturbative, and in which the Electroweak minimum is a global minimum.

Lay Summary

When fundamental particles interact with one another, they do so with a characteristic strength, called a coupling strength. This is, in general, a measure of how likely those particles are to interact. Although it might be expected that such a value is a basic property of the particles in question, the coupling strength in fact varies with the energy of the interacting particles. Beta functions, equations that describe how these values change with the energy scale, can be calculated, and then solved numerically in order to produce a picture of how we expect the coupling strengths to behave at higher energies than we can accurately measure them. Since large, or in some cases, negative couplings are forbidden, we can use these results to exclude models whose coupling strengths vary outside of these allowed regions. We calculated and integrated the beta functions for four different models of particle physics, of which only the Standard Model couplings remained in the allowed region.

Declaration

This dissertation is the result of my own work, except where explicit reference is made to the work of others, and has not been submitted for another qualification to this or any other university. This dissertation does not exceed the word limit for the respective Degree Committee.

James Carrington

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Contents

Chapter 1.

The Standard Model

The Standard Model of particle physics is one of the best tested, and most accurate theories in physics today. Although there are some observations of effects that are not described by the Standard Model, for example the existence of dark matter, or flavor-changing in the neutrino sector that requires neutrino masses, that offer tantalising hints of physics beyond the Standard Model, the vast majority of experimental evidence is in good agreement with theoretical predictions, to current theoretical and experimental accuracy.

The Standard Model is based on the gauge group $\mathrm{SU(3)_C \otimes SU(2)_L \otimes U(1)}_Y$, which are commonly separated into the color sector of $\mathrm{SU(3)_C}$, and the electroweak sector of $\mathrm{SU(2)_L \otimes U(1)}_Y$.

1.1. Electroweak Sector

For the electroweak sector of the Standard Model, we begin with the $\mathrm{SU(2)_L \otimes U(1)_Y}$ symmetry group, with coupling constants g and g' and 3+1 associated vector bosons, the $W_\mu^{1..3}$ and B respectively, and a complex doublet scalar Higgs [2]

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \sigma + i\phi_3 \end{pmatrix}. \tag{1.1}$$

The pure gauge sector of the Lagrangian contains kinetic terms for the gauge bosons of $\mathrm{SU(2)_L}\otimes\mathrm{U}(1)_Y$, as well as a non-Abelian coupling term for the W_μ^i

$$\begin{aligned} \mathcal{L}_{\text{Gauge}} &= -\frac{1}{4}\left(F^{\mu\nu\, i}F^i_{\mu\nu} + F'_{\mu\nu}F'^{\mu\nu}\right), \\ F^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\varepsilon^{ijk}W^j_\mu W^k_\nu, \\ F'_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu. \end{aligned} \tag{1.2}$$

The scalar part is

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} &= \left(D_\mu\Phi\right)^\dagger D^\mu\Phi - V\left(\Phi^\dagger\Phi\right), \\ D_\mu &= \partial_\mu - igW_\mu\cdot\tau - ig'B_\mu, \\ V(x) &= -\mu^2 x + \frac{\lambda}{4}x^2, \end{aligned} \tag{1.3}$$

where τ^i are the Pauli sigma matrices σ^i divided by 2.

When we solve to find the minimum of the potential of the Higgs doublet we discover that although the potential is symmetric under the electroweak symmetry group, any particular choice of minimum for the potential breaks the symmetry to a lower symmetry group, $\mathrm{SU(2)_L}\otimes\mathrm{U(1)_Y}\rightarrow\mathrm{U(1)_{EM}}$. Due to gauge invariance, we can, without loss of generality, choose the symmetry breaking minimum to be in the σ direction. We can then expand around this minimum [3],

$$\sigma \to v + H, \qquad \widehat{\Phi} = \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad v^2 = -\frac{2\mu^2}{\lambda}, \tag{1.4}$$

generating mass terms for the vector bosons and the H [4, 5]. When examining the vector boson mass matrix to find the mass eigenstates, we see that since $W^{1,2}$ have the same mass eigenvalue, $M_W^2 = \frac{g^2 v^2}{4}$, we can consider instead the superposition which gives charge eigenstates,

$$W^{\pm} = \frac{1}{\sqrt{2}} \left(W^1 \mp i W^2 \right) \tag{1.5}$$

while W^3 and B mix as

$$\begin{pmatrix} W^3 & B \end{pmatrix} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^3 \\ B \end{pmatrix} \tag{1.6}$$

with eigenvectors and eigenvalues

$$Z = \frac{1}{\sqrt{g^2 + g'^2}} \left(g W^3 - g' B \right), \qquad m_Z = \frac{v\sqrt{g^2 + g'^2}}{2} \tag{1.7}$$

$$\gamma = \frac{1}{\sqrt{g^2 + g'^2}} \left(g' W^3 + g B \right), \qquad m_\gamma = 0 \tag{1.8}$$

Two of the conserved quantities of the original symmetry group, Y, the hypercharge, and I_3, the third component of isospin, are related to the electromagnetic charge Q associated with the remaining conserved symmetry as

$$Q = I_3 + \frac{Y}{2}. \tag{1.9}$$

Additionally, the components of the Higgs doublet in the broken directions can be gauge-transformed such that they are absorbed into the massive vector bosons, leaving us with three massive vector bosons, one massless vector boson and a single massive real scalar.

1.2. QCD

Aside from the electroweak sector, the Standard Model also contains an SU(3) gauge symmetry that describes the color interactions of the quarks, mediated by eight vector bosons (gluons) [6, 7, 8]. This symmetry was first proposed with the observation of the Δ^{++}, which is composed of three spin-aligned up-type quarks. Without a new quantum number to differentiate between the constituents, the Pauli exclusion principle would prevent the formation of such a composite particle. This picture also provided a more fundamental understanding of the strong nuclear force, previously understood in terms of the exchange of massive pions, as a long-range effective theory. The six quarks, and their antiparticles, transforming under the fundamental representation of SU(3), and the gluons, in the adjoint representation, are the only particles in the Standard Model to carry a color charge.

1.3. Fermions

So far, we have covered the symmetry structure, and thus the gauge interactions of the Standard Model, and the Higgs mechanism by which the Electroweak symmetry group is broken to the familiar electromagnetic (EM) symmetry. However, there remains an important aspect of the Standard Model which has so far only been touched on: the fermionic particles that compose matter. These are spin-$\frac{1}{2}$ fields whose dynamics are governed by the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\gamma^{\mu} D_{\mu} - m\right)\psi, \tag{1.10}$$

where D_{μ} is the covariant derivative that encodes interactions with the gauge fields [1]. In a theory without explicit mass terms for the fermions, we can rewrite the Lagrangian above in terms of the left- and right-handed Weyl spinors composing the Dirac spinor [2]

$$\psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \\ \psi_{\mathrm{R}} \end{pmatrix}, \qquad \mathcal{L} = i\overline{\psi_{\mathrm{L}}}\overline{\sigma}_\mu D_\mu \psi_{\mathrm{L}} + i\overline{\psi_{\mathrm{R}}}\sigma_\mu D'_\mu \psi_{\mathrm{R}} \tag{1.11}$$

where the four-vector-indexed sigmas are

$$\sigma_\mu = \left(\mathbb{1}, \sigma^i\right), \qquad \overline{\sigma}_\mu = \left(\mathbb{1}, -\sigma^i\right). \tag{1.12}$$

Without mass terms, we now have a larger symmetry group, since we can transform the left- and right-handed fermions independently. This has the added benefit of allowing us to write evidently differing covariant derivatives for the left- and right-handed fermions, without the need for projection operators. For the Standard Model, the derivatives are given by

$$D_\mu = \partial_\mu - igW^a_\mu \tau^a - ig'YB_\mu, \qquad D'_\mu = \partial_\mu - ig'YB_\mu \tag{1.13}$$

since the right-handed fermions have no coupling to $\mathrm{SU}(2)_{\mathrm{L}}$. Y and T^a take the relevant value for the representation that the particular fermion belongs to. As a result in the standard model we have left-handed fermions in $\mathrm{SU}(2)_{\mathrm{L}}$ doublets, while right-handed fermions are singlets.

The fermions are further subdivided into the quarks, which carry a color charge, and thus interact via the strong force, and the leptons, which are color-neutral. Both of these groups contain six flavors, which are arranged as two families that compose the components of the $\mathrm{SU}(2)_{\mathrm{L}}$ doublet, and three generations. For the quarks, we have up-type, with EM charge $+\frac{2}{3}$ and down-type with EM charge $-\frac{1}{3}$. The generations are named, in order of increasing mass, up, charm and top, and down, strange and bottom respectively. Similarly for the leptons, we have electron, charge -1, and neutrino, charge

0 families, with the generations named electron, muon and tau, and ν_e, ν_μ and ν_τ. This is summarised in Table 1.1.

Table 1.1.: Particle charges

Family	Left-handed	$\mathrm{U}(1)_Y$	$\mathrm{SU}(2)_\mathrm{L}$	Right-handed	$\mathrm{U}(1)_Y$
up-type quark	u_L, c_L, t_L	$\frac{1}{3}$	$\frac{1}{2}$	u_R, c_R, t_R	$\frac{4}{3}$
down-type quark	d_L, s_L, b_L	$\frac{1}{3}$	$-\frac{1}{2}$	d_R, s_R, b_R	-$\frac{2}{3}$
neutrino	ν_e, ν_μ, ν_τ	-1	$\frac{1}{2}$	—	—
lepton	e_L, μ_L, τ_L	-1	$-\frac{1}{2}$	e_R, μ_R, τ_R	-2

Descriptive text!

We now have to consider the problem of fermion masses, since all of the fermions of the Standard Model, bar the neutrinos, have directly observed masses. We can use a method similar to that used above to obtain massive gauge bosons, via the vacuum expectation value of the scalar sector. Although we are forbidden from writing a direct $\psi_\mathrm{L}^\dagger \psi_\mathrm{R}$ term due to gauge invariance, we can, remembering that the scalar doublet has SU(2)⊗U(1) charge, write a set of Yukawa interactions of the form

$$\mathcal{L}_\mathrm{Yukawa} = -g_l^{ij}\overline{e}_i\Phi^\dagger \cdot l_j - g_d^{ij}\overline{d}_i\Phi^\dagger \cdot q_j - g_u^{ij}\overline{u}_i\Phi^{\dagger c} \cdot q_j + h.c. \tag{1.14}$$

The indices i and j here are generational indices, and the fermion representations are

$$e_i = (e, \mu, \tau)_\mathrm{R}\,, \qquad u_i = (u, c, t)_\mathrm{R}\,, \qquad d_i = (d, s, b)_\mathrm{R}$$

$$l_i = \left(\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}\right)_\mathrm{L}, \qquad q_i = \left(\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} b \\ t \end{pmatrix}\right)_\mathrm{L}$$

Here and elsewhere, we implicitly sum over color indices. When we substitute the scalar doublet for its vacuum expectation value, we obtain, for example in the case of the electron

$$\mathcal{L}_{\text{e Yukawa}} = -g_e \frac{v}{\sqrt{2}} \bar{e}_{\text{R}} e_{\text{L}} + h.c., \tag{1.15}$$

which is the missing fermion mass term, with

$$m_e = \frac{g_e v}{\sqrt{2}}. \tag{1.16}$$

The gauge interactions of the fermions result from changing the partial derivative in the free fermion theory to a covariant derivative, and as such we cannot have generation-changing gauge couplings. The Yukawa couplings are not based on gauge principles though, and without the existence of a new symmetry preventing it, we would expect couplings to the scalar sector to be able to mix generations. This is why in equation (1.14) above we have couplings with flavor indices. These matrices have no restrictions on them from symmetry and can thus be entirely general, 3×3 complex matrices. However we can reduce the possible space of generation-changing by making a change of basis. If we consider diagonalising the Hermitian matrices given by squaring the coupling matrices we can write

$$g_u g_u^\dagger = U_u D_u^2 U_u^\dagger, \qquad g_u^\dagger g_u = W_u D_u^2 W_u^\dagger \tag{1.17}$$

with U_u and W_u unitary and D_u the diagonal positive matrix giving the masses of the up-type quarks in this new basis. This gives us

$$g_u = U_u D_u W_u^\dagger, \tag{1.18}$$

and equivalently for the down-type quarks we get

$$g_d = U_d D_d W_d^\dagger. \tag{1.19}$$

We can now make a change of quark basis by making the transformation

$$u_{\mathrm{R}}^i \to W_u^{ij} u_{\mathrm{R}}^j, \qquad d_{\mathrm{R}}^i \to W_d^{ij} d_{\mathrm{R}}^j \tag{1.20}$$

in order to cancel the $W_{u,d}$ from the Yukawa terms above. There are no gauge interactions that allow right-handed up- and down-type quarks to mix and therefore the covariant derivative for the right-handed quarks commutes with the change of basis.

We can make a similar transformation on the left-handed quark basis using $U_{u,d}$

$$u_{\mathrm{L}}^i \to U_u^{ij} u_{\mathrm{L}}^j, \qquad d_{\mathrm{L}}^i \to U_d^{ij} d_{\mathrm{L}}^j \tag{1.21}$$

which reduces the Yukawa terms above to diagonal interactions. The covariant derivative for left-handed quarks is also invariant under this basis transformation, for those terms diagonal in isospin, that is the terms coupling W^3 and B. However, the $W^{1,2}$ interactions mix up- and down-type quarks and this gives us a unitary mixing of the down-quark mass eigenstates with those for the up quarks, for example

$$\overline{u}_{\mathrm{L}}^i \overline{\sigma}^\mu \left(W_\mu^1 - W_\mu^2\right) d_{\mathrm{L}}^i \to \overline{u}_{\mathrm{L}}^i \overline{\sigma}^\mu \left(W_\mu^1 - W_\mu^2\right) \left(U_u^\dagger U_d\right) d_{\mathrm{L}}^j. \tag{1.22}$$

Although this unitary matrix $U_u^\dagger U_d = V_{\mathrm{CKM}}$, called the CKM, or Cabibbo-Kobayashi-Maskawa matrix [9, 10], cannot be entirely removed, it can be shown that the number of relevant degrees of freedom is smaller than it might initially appear. As a 3×3 unitary matrix, V can be parameterised with 9 values — 3 rotation angles and 6 complex, CP-violating phases. The number of observable phases in V_{CKM} is reduced by our freedom to make phase rotations of the quark fields

$$q_{\mathrm{L}}^{i} \rightarrow e^{i\theta_i} q_{\mathrm{L}}^{i}. \qquad (1.23)$$

These rotations have no effect on any other terms in the Lagrangian, and can therefore be used to cancel the phases in the CKM matrix. However, a net rotation of all of the fields also cancels out of the weak interaction, leaving us with five usable phases. Our CKM matrix is therefore completely described by the 3 rotations and a single phase, the only source of CP-violation in the Standard Model.

Chapter 2.

Renormalisation Group Equations

As the central objects with which this thesis is concerned, we now turn our attention to renormalisation group equations. We begin with a summary of the physical interpretation of the renormalisation equations, and a brief description of the background field method used in reference [11]. We then write explicit expressions for the beta functions of both a generic gauge theory, to two-loop accuracy, and of the Standard Model in particular, to three loops in all couplings except the top Yukawa, in the $\overline{MS}$ scheme. The top coupling is excluded since at the time of working, the three-loop coupling was not calculated. There are a wealth of papers describing the two loop Standard Model result, using varying conventions, and not all entirely accurate. In order to have confidence in our results, we explain our conventions and how they differ from those used in our references, as well as comparing our calculation of the beta functions for the Standard Model couplings from the generic case to the published expressions.

2.1. Basics

When calculating loop diagrams in quantum field theories we often encounter infinities as a result of integrating over all possible momenta for the loop. Superficially, if the Feynman rules for a diagram contain 4 or fewer inverse powers of the loop momentum then the integral will be divergent for momenta tending to infinity. Deriving a functional form for these infinities is described as regularisation, and cancelling, or absorbing them is termed renormalisation. The basic series of steps to calculate a finite, observable quantity is to first calculate an expression that depends on the bare (unrenormalised) charges $(g_i)_0$, the bare masses $(m_i)_0$ and on an ultraviolet cutoff scale Λ. The physical charges g_i, masses m_i and field-strength renormalisations Z_i can then be calculated in terms of the bare parameters and the cutoff Λ and used to eliminate the dependence on the bare parameters in the original expression. The formula obtained will always have a finite limit as $\Lambda \to \infty$, in a renormalisable field theory.

In order to get a more physical appreciation for the process of renormalisation, we shall follow the method used by Wilson [12] and outlined in [1]. In seeking to understand the effect of the high-momentum parts of the integral that are omitted by introducing a cutoff scale, we can separate the integral into two momentum regions — those momenta that are close to the cutoff scale, and the rest of momentum space. For this calculation we will simply use a sharp momentum cutoff, and only include those momenta with absolute size less than or equal to the cutoff. However, since we can have momenta k with large components that nonetheless have small k^2, in order to have a well-defined cutoff we impose the cutoff condition on the Euclideanised momenta, $|k_E| \leq \Lambda$.

We start by working in ϕ^4 theory, with external sources J set to zero and in a Euclidean, d-dimensional space.

$$Z = \int [\mathcal{D}\phi]_\Lambda \exp\Bigg(-\int \mathrm{d}^d x \left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right]\Bigg), \tag{2.1}$$

$$[\mathcal{D}\phi]_\Lambda = \prod_{|k|<\Lambda} \mathrm{d}\phi(k)\,.$$

If we now introduce a parameter $b < 1$, we can split the integration variables as

$$\begin{aligned}
&\phi(k) \to \phi(k) + \widehat{\phi}(k)\,,\\
&\phi(k) = \begin{cases} \phi(k) & |k| \le b\Lambda \\ 0 & b\Lambda < |k| \end{cases},\\
&\widehat{\phi}(k) = \begin{cases} 0 & |k| \le b\Lambda \\ \phi(k) & b\Lambda < |k| < \Lambda \end{cases}.
\end{aligned}$$

With this substitution we can rewrite (2.1), with integrals over the two momentum domains $\mathcal{D}\phi$ and $\mathcal{D}\widehat{\phi}$, as

$$\begin{aligned}
Z &= \int \mathcal{D}\phi \int \mathcal{D}\widehat{\phi}\exp\Bigg(-\int \mathrm{d}^d x\left[\frac{1}{2}\left(\partial_\mu\phi + \partial_\mu\widehat{\phi}\right)^2 + \frac{m^2}{2}\left(\phi+\widehat{\phi}\right)^2 + \frac{\lambda}{4!}\left(\phi+\widehat{\phi}\right)^4\right]\Bigg)\\
&= \int \mathcal{D}\phi e^{-\int \mathcal{L}(\phi)} \int \mathcal{D}\widehat{\phi}\exp\Bigg(-\int \mathrm{d}^d x\left[\frac{1}{2}\left(\partial_\mu\widehat{\phi}\right)^2 + \frac{m^2}{2}\widehat{\phi}^2\right.\\
&\qquad \left. +\lambda\left(\frac{1}{6}\phi^3\widehat{\phi} + \frac{1}{4}\phi^2\widehat{\phi}^2 + \frac{1}{6}\phi\widehat{\phi}^3 + \frac{1}{4!}\widehat{\phi}^4\right)\right]\Bigg).
\end{aligned}$$

Quadratic terms of the form $\phi\widehat{\phi}$ vanish due to the orthogonality of Fourier components of different momentum. Treating the two variables m^2 and λ as being perturbations — $\lambda \ll 1$, $m^2 \ll \Lambda^2$ — the leading-order term is given by

$$\int \mathcal{L}_0 = \frac{1}{2}\int_{b\Lambda \le |k| < \Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d}\widehat{\phi}^*(k)\,k^2\widehat{\phi}(k)\,.$$

Using this definition we can then write a propagator

$$\widehat{\phi}(k)\,\widehat{\phi}(p) = \frac{\int \mathcal{D}\widehat{\phi} e^{-\int \mathcal{L}_0} \widehat{\phi}(k)\,\widehat{\phi}(p)}{\int \mathcal{D}\widehat{\phi} e^{-\int \mathcal{L}_0}} = \frac{1}{k^2}\,(2\pi)^d\,\delta^{(d)}(k+p)\,\Theta(k)$$

with

$$\Theta(k) = \begin{cases} 1 & b\Lambda \leq |k| < \Lambda \\ 0 & \text{otherwise} \end{cases}$$

We can now use this contraction to integrate out the dependence on the $\widehat{\phi}$ perturbations in the form of corrections to the Lagrangian in ϕ. For example if we look at the leading order term for $\phi^2\widehat{\phi}^2$ we get

$$-\int \mathrm{d}^d x \frac{\lambda}{4}\phi^2\,\widehat{\phi}\widehat{\phi} = -\frac{1}{2}\int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\mu^2\phi(k_1)\,\phi(-k_1)\,.$$

Although the coefficient μ in the above equation originates from the contraction of two $\widehat{\phi}$ fields it could also be written as an additional term in the low-momentum Lagrangian $\mathcal{L}(\phi)$

$$\exp\left(-\int \mathrm{d}^d x \frac{1}{2}\mu^2\phi^2 + \dots\right).$$

As such, this term would be a correction to the mass term $m^2\phi^2$, which we could rewrite as

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\left(m^2+\mu^2\right)\phi^2 = -\frac{m^2}{2}\left(1+\delta m^2\right)\phi^2. \tag{2.2}$$

These corrections can also be written in diagrammatic form, here with solid lines for ϕ and dashed lines for $\widehat{\phi}$. With this form, the above correction is given in Figure 2.1

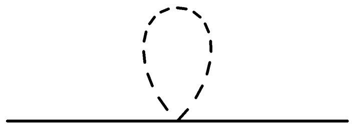


Figure 2.1.: 1-loop correction to m^2.

Higher order terms in the expansion of the exponential give higher loop corrections to m^2, as well as corrections to λ — at order λ^2 we get three corrections to m^2, shown in Figure 2.2

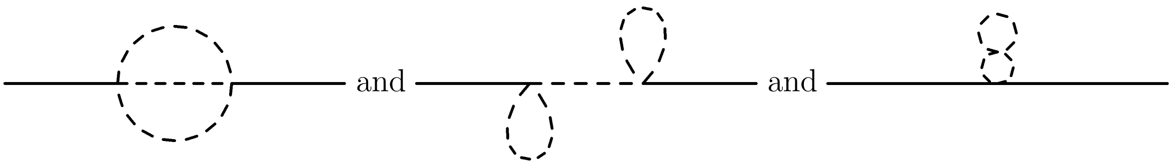


Figure 2.2.: 2-loop corrections to m^2.

as well as three corrections to the coupling λ, in Figure 2.3

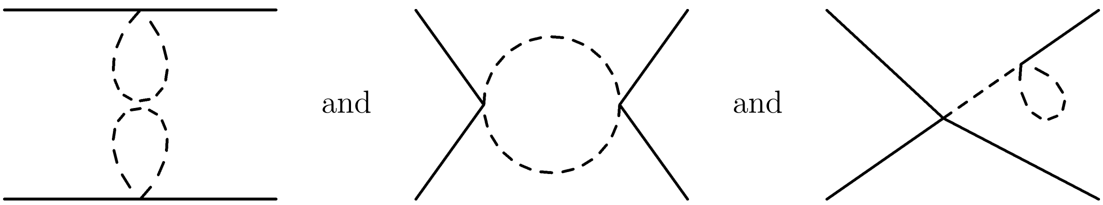


Figure 2.3.: 2-loop corrections to the quartic coupling, λ.

and a final diagram that creates a contribution to a new ϕ^6 interaction, given in Figure 2.4.

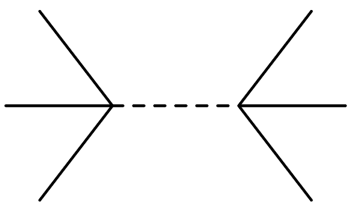


Figure 2.4.: 2-loop correction to a new ϕ^6 interaction.

We can now rewrite the generating functional z as

$$Z = \int [\mathcal{D}\phi]_{b\Lambda} \exp\Big(- \int \mathrm{d}^d x \mathcal{L}_{eff} \Big)$$

with the effective Lagrangian defined by our original Lagrangian with the addition of all connected diagrams resulting from contractions over the internal fields. These terms can then be rewritten in the style of the mass term addition in (2.2)

$$\int \mathrm{d}^d x \mathcal{L}_{eff} = \int \mathrm{d}^d x \left[\frac{1}{2} \left(1 + \Delta Z\right) \left(\partial_\mu \phi\right)^2 + \frac{1}{2} \left(m^2 + \Delta m^2\right) \phi^2 + \frac{1}{4!} \left(\lambda + \Delta\lambda\right) \phi^4 \right.$$
$$\left. + \left(C + \Delta C\right) \left(\partial_\mu \phi\right)^2 \phi^2 + \left(D + \Delta D\right) \phi^6 + ... \right],$$

where $\Delta m^2 = m^2 \delta m^2$, and in our original Lagrangian, C, D and all subsequent bare terms are equal to zero.

There is a slight clarification to be made concerning our choice of description of the process of renormalisation. If we are discussing renormalisation group equations, we should expect the process to satisfy the group axioms. However, there is no well-defined inverse procedure since we have integrated out the high-energy behaviour and truncated at a finite loop order, losing information. Since we are interested in taking the couplings as measured at low energies and then using the beta functions to evaluate them at large energies, we need to renormalisation group equations do in fact form a group. To do so, we could define the Wilson procedure to operate in the opposite direction, integrating out low-energy behaviour, which would give a consistent method to describe how the couplings change moving from low to high energies, and obtaining an inverse for the renormalisation already discussed. In the limit of the integration steps becoming continuous, we would expect our high-to-low and low-to-high beta functions to be the same.

We might now worry about the presence of unrenormalisable interactions — terms whose coefficient has a negative mass dimension — in our Lagrangian. We will see that these terms are in fact negligible when we look at the theory from a point sufficiently far from the cutoff scale. We start by making a redefinition of the positions, momenta, and the scalar field, as

$$
\begin{aligned}
k' &= \frac{k}{b}, \\
x' &= xb, \\
\phi' &= \left[b^{2-d}\left(1+\Delta Z\right)\right]^{\frac{1}{2}}\phi,
\end{aligned}
$$

with the field redefinition chosen so that the propagator remains unchanged under the transformation, and k' is integrated over values $|k'| < \Lambda$. If we look at the coefficients of the various terms in our Lagrangian close to the free field point, keeping only those terms linear in the perturbations, we see that we have

$$
\begin{aligned}
m'^2 &= \left(m^2+\Delta m^2\right)\left(1+\Delta Z\right)^{-1} b^{-2} \to m^2 b^{-2}, \\
\lambda' &= \left(\lambda+\Delta\lambda\right)\left(1+\Delta Z\right)^{-2} b^{d-4} \to \lambda b^{d-4}, \\
C' &= \left(C+\Delta C\right)\left(1+\Delta Z\right)^{-2} b^{d-2} \to C b^{d-2}, \\
D' &= \left(D+\Delta D\right)\left(1+\Delta Z\right)^{-3} b^{2d-6} \to D b^{2d-6}.
\end{aligned}
$$

Because we chose $b < 1$, terms with positive powers of b are less relevant, while those with negative powers are more relevant at low energies. In the case that the number of dimensions $d = 4$, we see that both unrenormalisable terms are less relevant at energies much smaller than the cutoff.

2.1.1. The Callan-Symanzik Equation

In order to completely define our renormalised theory, we must impose conditions on the values of the renormalised terms at an arbitrary renormalisation scale μ^2. However, since the bare Green's functions of our theory are independent of μ, the renormalised gauge functions can only depend on it via the renormalised field and coupling. We can then consider making an infinitesimal change in μ, which leads to corresponding changes in the renormalised coupling and field strength, since the bare Green's function is unchanged

$$\mu \to \mu + \delta\mu, \tag{2.3}$$

$$\lambda \to \lambda + \delta\lambda, \tag{2.4}$$

$$\phi \to (1 + \delta\eta)\,\phi. \tag{2.5}$$

The n-point Green's function is only changed due to the field rescaling

$$G^{(n)} \to (1 + n\delta\eta)\,G^{(n)}, \tag{2.6}$$

so that, if we consider the Green's function as a function of the renormalisation scale and the couplings — in this case, only λ — the infinitesimal transformation can be written

$$n\delta\eta G^{(n)} = \frac{\partial G^{(n)}}{\partial\mu}\delta\mu + \frac{\partial G^{(n)}}{\partial\lambda}\delta\lambda. \tag{2.7}$$

If we make the definitions

$$\beta_\lambda \equiv \frac{\mu}{\delta\mu}\delta\lambda, \qquad \gamma \equiv -\frac{\mu}{\delta\mu}\delta\eta \tag{2.8}$$

, we can rewrite the previous equation as

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + n\gamma\right)G^{(n)} = 0. \tag{2.9}$$

The two functions β_λ and γ must be independent of n, and also of the cutoff scale. As they are dimensionless, they must therefore also be independent of the only other dimensionful parameter, μ, and depend only on the coupling λ. For more complicated theories involving multiple interactions $\{g_x\}$ and fields $\{f_i\}$, this equation generalises as

$$\left(\mu\frac{\partial}{\partial\mu} + \sum_X \beta_X(\{g_x\})\,\frac{\partial}{\partial g_X} + \sum_i n_i\gamma_i(\{g_x\})\right) G^{\{n_i\}} = 0, \tag{2.10}$$

where n_i is the number of fields f_i associated with the Green's function, and γ_i is the anomalous dimension of f_i.

2.2. Generic Beta Functions to Two Loops

The two-loop beta functions presented in the main body of references [11, 13, 14] are given in the generic form of the beta functions for a gauge theory of a simple gauge group G containing two-component fermions ψ_j which transform under the representation F, scalars ϕ_a with representation S and gauge-mediating vector bosons V_μ^A described by the Lagrangian

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} + \frac{1}{2}D^\mu\phi_a D_\mu\phi_a + i\psi_j^\dagger\sigma^\mu D_\mu\psi_j - \left(Y^a_{jk}\psi_j\zeta\psi_k\phi_a + \text{h.c.}\right) - \frac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d \\ &+ (\text{mass terms}) + (\text{gauge-fixing and ghost terms})\,,\end{aligned} \tag{2.11}$$

where ζ is the spinor metric $\pm i\sigma_2$. All latin indices are flavor indices, and so can be raised or lowered with the Kronecker delta, while greek indices are raised or lowered through the metric with signature $(+,-,-,-)$. Knowledge of the mass terms is not necessary in order to calculate the beta functions, and the gauge fixing and ghost terms are those suitable for either an R_ξ or background field gauge. We can neglect the mass terms as we are working in the $\overline{\text{MS}}$ scheme, which is mass-independent. We define

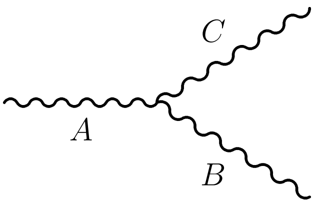
$$F^A_{\mu\nu} = \partial_\mu V^A_\nu - \partial_\nu V^A_\mu + g f^{ABC} V^B_\mu V^C_\nu\,, \tag{2.12}$$

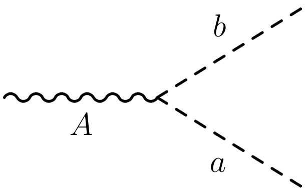
where f^{ABC} are the structure constants, and g the coupling constant of the gauge group. The covariant derivatives are

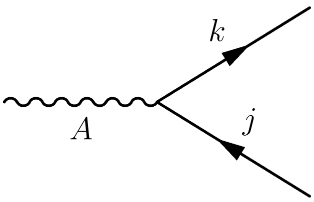
$$D_\mu \phi_a = \partial_\mu \phi_a + ig\theta^A_{ab} V^A_\mu \phi_b, \tag{2.13}$$

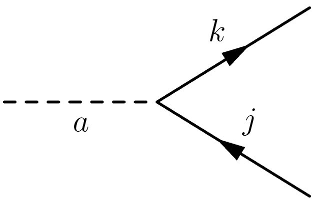
$$D_\mu \psi_j = \partial_\mu \psi_j + igt^A_{jk} V^a_\mu \psi_k, \tag{2.14}$$

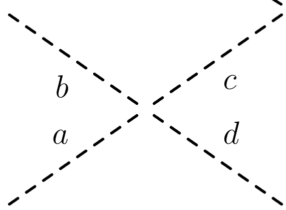
with θ^A_{ab} and t^A_{jk} the Hermitian generators of G acting on the scalar and fermionic fields respectively. By taking a real representation for the scalars, we make the θ^A imaginary and antisymmetric. Y^a_{jk} are the Yukawa matrices. Our set of generic interactions is thus

$gf^{ABC} =$  (2.15)

$g\theta^A_{ab} =$  (2.16)

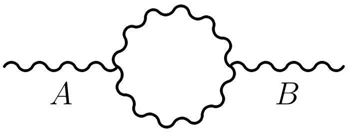
$gt^A_{jk} =$  (2.17)

$Y^a_{jk} =$  (2.18)

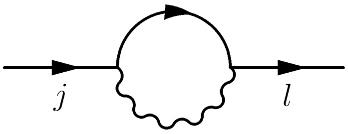
$\lambda_{abcd} =$  (2.19)

Much of the following work will be presented in terms of the Casimir invariants C_2 and Dynkin indices S_2 defined by

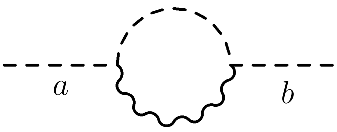
$$g^2\delta^{AB}C_2(G) \equiv g^2 f^{ACD} f^{BCD} = \tag{2.20}$$



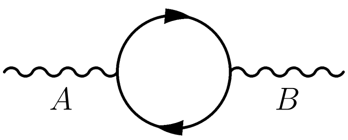
$$g^2\delta_{jl}C_2(F) \equiv g^2 t^A_{jk} t^A_{kl} = \tag{2.21}$$



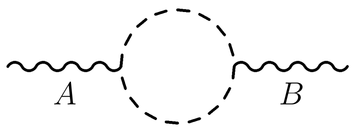
$$g^2 C_2^{ab}(S) \equiv g^2\delta_{ab}C_2(S) \equiv g^2\theta^A_{ac}\theta^A_{cb} = \tag{2.22}$$



$$g^2\delta^{AB}S_2(F) \equiv g^2 t^A_{jk} t^B_{kj} = \mathrm{Tr}\big(t^A t^B\big) = \tag{2.23}$$

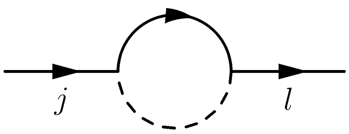


$$g^2\delta^{AB}S_2(S) \equiv g^2\theta^A_{ab}\theta^B_{ba} = \mathrm{Tr}\big(\theta^A\theta^B\big) = \tag{2.24}$$

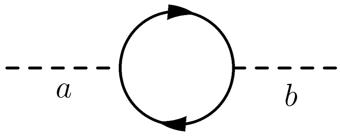


The δ terms here refer only to particles within the given representation, G for gauge bosons, S for scalars and F for fermions. There are additionally two quadratic invariants defined in terms of the Yukawa matrices

$$\delta_{jl}Y_2(F) \equiv Y^{\dagger a}_{jk} Y^a_{kl} = \tag{2.25}$$

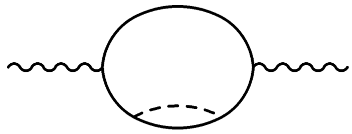


$$\delta^{ab} Y_2(S) \equiv Y^{\dagger a}_{jk} Y^{b}_{kj} = \mathrm{Tr}\big(Y^{\dagger a} Y^{b}\big) = \tag{2.26}$$

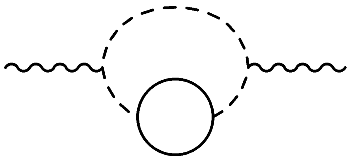


and the two quartic invariants,

$$Y_4(F) \equiv \frac{1}{d(G)} C_2(F)_{jk}\, Y^{a}_{kl} Y^{\dagger a}_{lj} = \frac{1}{d(G)}\, \mathrm{Tr}\big(C_2(F)\, Y^{a} Y^{\dagger a}\big) = \tag{2.27}$$



$$Y_4(S) \equiv \frac{1}{d(G)} C_2(S)\, Y^{a}_{jk} Y^{\dagger a}_{kj} = \frac{1}{d(G)}\, \mathrm{Tr}\big(C_2(S)\, Y^{a} Y^{\dagger a}\big) = \tag{2.28}$$

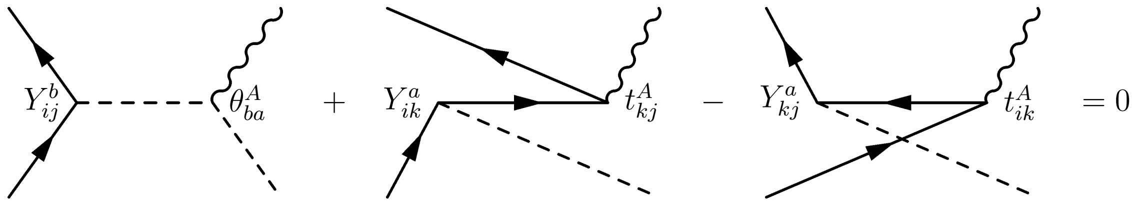


where $d(G)$ is the dimension of the gauge group, originating from effectively cutting the gauge boson in $C_2(F)$ and $C_2(S)$ respectively, and therefore not summing over all of the possible gauge bosons.

It is important to note that although it might appear that some classes of diagrams have been omitted from the beta functions, a lot of the possible diagrams have been reformulated using the Yukawa gauge invariance condition

$$\big[t^{g,A}, Y_a\big] = Y_b \theta^{g,A}_{ba}, \tag{2.29}$$

equivalent to the statement that the sum of the diagrams obtained by the emission of a vector boson from the Yukawa vertex is equal to zero.



(2.30)

$$Y^b_{ij}\theta^A_{ba} + Y^a_{ik}t^A_{kj} - t^A_{ik}Y^a_{kj} = 0. \tag{2.31}$$

We can now write the generic beta functions, defined as

$$\beta_{g_X} = \frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_X = \beta_{g_X}|_{\text{1-loop}} + \beta_{g_X}|_{\text{2-loop}} + \beta_{g_X}|_{\text{3-loop}} + \ldots \tag{2.32}$$

For gauge couplings, the beta functions are scalar functions, while for the Yukawa couplings and the scalar couplings, the beta functions have the same dimension as the Yukawa-coupling tensor and the scalar-coupling tensor.

The gauge-coupling beta function is given to one-loop accuracy by [11]

$$\left(16\pi^2\right)\beta_g|_{\text{1-loop}} = \left(\frac{11}{3}C_2(G) - \frac{4\kappa}{3}S_2(F) - \frac{1}{6}S_2(S)\right)g^3 \tag{2.33}$$

while the two-loop part of the gauge-coupling beta function is given by [11]

$$\begin{aligned}\left(16\pi^2\right)^2\beta_g|_{\text{2-loop}} = &-\frac{34}{3}g^5\left[C_2(G)\right]^2 + \kappa g^5\left[4C_2(F) + \frac{20}{3}C_2(G)\right]S_2(F)\\ &+ g^5\left[2C_2(S) + \frac{1}{3}C_2(G)\right]S_2(S) - 2\kappa g^3 Y_4(F)\end{aligned} \tag{2.34}$$

The constant κ appears in terms originating from fermion loops and takes a value of either 1 or $\frac{1}{2}$ depending on whether the fermion representation is Dirac or Weyl respectively. These two expressions are scheme invariant, although higher-order terms are not.

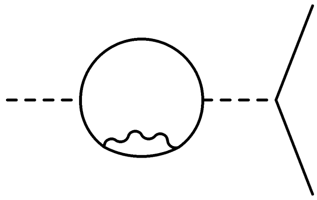
The Yukawa-coupling beta function is given at one loop by [13]

$$\begin{aligned}\left(16\pi^2\right)\beta_{Y^a}\big|_{\text{1-loop}} =& \frac{1}{2}\left[Y_2^\dagger(F)\,Y^a + Y^a Y_2(F)\right] + 2Y^b Y^{\dagger a} Y^b \\ &+ 2\kappa Y^b \operatorname{Tr}\left(Y^{\dagger b} Y^a\right) - 3g^2\left\{C_2(F)\,, Y^a\right\}\end{aligned} \tag{2.35}$$

and at two loops [13]

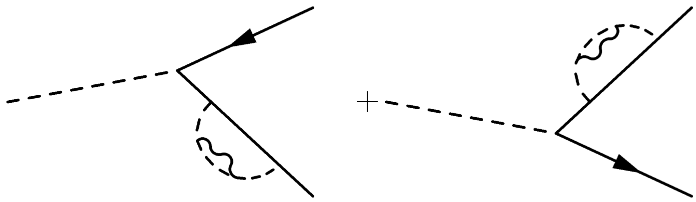
$$\begin{aligned}\left(16\pi^2\right)^2\beta_{Y^a}\big|_{\text{2-loop}} =& 2Y^c Y^{\dagger b} Y^a\left(Y^{\dagger c} Y^b - Y^{\dagger b} Y^c\right) - Y^b\left[Y_2(F)\,Y^{\dagger a} + Y^{\dagger a} Y_2^\dagger(F)\right] Y^b \\ &- \frac{1}{8}\left[Y^b Y_2(F)\,Y^{\dagger b} Y^a + Y^a Y^{\dagger b} Y_2^\dagger(F)\,Y^b\right] \\ &- 4\kappa Y_2^{ac}(S)\,Y^b Y^{\dagger c} Y^b - \frac{3}{2}\kappa Y_2^{bc}(S)\left[Y^b Y^{\dagger c} Y^a + Y^a Y^{\dagger c} Y^b\right] \\ &- \kappa Y^b \operatorname{Tr}\left(\frac{3}{2}\left[Y_2(F)\,Y^{\dagger b} + Y^{\dagger b} Y_2^\dagger(F)\right] Y^a + 2Y^{\dagger b} Y^c Y^{\dagger a} Y^c\right) \\ &- 2\lambda_{abcd} Y^b Y^{\dagger c} Y^d + \frac{1}{12}\lambda_{acde}\lambda_{bcde} Y^b + 3g^2\left\{C_2(F)\,, Y^b Y^{\dagger a} Y^b\right\} \\ &+ 5g^2 Y^b\left\{C_2(F)\,, Y^{\dagger a}\right\} Y^b - \frac{7}{4}g^2\left[C_2(F)\,Y_2^\dagger(F)\,Y^a + Y^a Y_2(F)\,C_2(F)\right] \\ &- \frac{1}{4}g^2\left[Y^b C_2(F)\,Y^{\dagger b} Y^a + Y^a Y^{\dagger b} C_2(F)\,Y^b\right] \\ &+ 6g^2\left[t^A Y^a Y^{\dagger b} t^A Y^b + Y^b t^A Y^{\dagger b} Y^a t^A\right] \\ &+ 5\kappa g^2 Y^b \operatorname{Tr}\left[Y^{\dagger b}\left\{C_2(F)\,, Y^a\right\}\right] \\ &+ 6g^2\left[C_2^{bc}(S)\,Y^b Y^{\dagger a} Y^c - 2C_2^{ac}(S)\,Y^b Y^{\dagger c} Y^b\right] \\ &+ \frac{9}{2}g^2 C_2^{bc}(S)\left[Y^b Y^{\dagger c} Y^a + Y^a Y^{\dagger c} Y^b\right] - \frac{3}{2}g^4\left\{\left[C_2(F)\right]^2, Y^a\right\} \\ &+ g^4\left[6C_2(S) - \frac{97}{6}C_2(G) + \frac{10}{3}\kappa S_2(F) + \frac{11}{12}S_2(S)\right]\left\{C_2(F)\,, Y^a\right\} \\ &- g^4 C_2(S)\left[\frac{21}{2}C_2(S) - \frac{49}{4}C_2(G) + 2\kappa S_2(F) + \frac{1}{4}S_2(S)\right] Y^a\end{aligned} \tag{2.36}$$

Two adjustments have been made here from the equation given in equation (3.3) of [13]. In the term $5\kappa g^2 Y^b \operatorname{Tr}\left[Y^{\dagger b}\left\{C_2(F)\,, Y^a\right\}\right]$, describing the diagram



(2.37)

the original omits the brackets for the trace, which we have included for clarity, and has the Yukawa matrix $Y^{\dagger b}$ as the final tensor. The subject of the trace is deducible from the requirement that the term describe a connected diagram. It is clear that, since the Yukawa matrix appears in a trace, it can be permuted with the anticommutator, due to the cyclic property of the trace. This is in order to more explicitly preserve the rule that Yukawa matrices should alternate as Y and $Y^\dagger$. The term $\frac{9}{2} g^2 C_2^{bc}(S)\left[Y^b Y^{\dagger c} Y^a + Y^a Y^{\dagger c} Y^b\right]$, corresponding to the diagrams

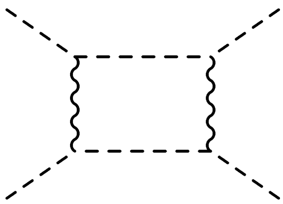


(2.38)

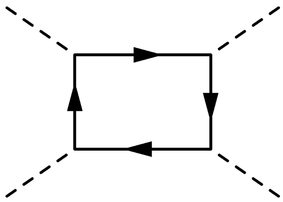
appears in the original without the factor of $C_2^{bc}(S)$. The missing invariant factor is obvious from the indices that are consequently not summed over. That the Yukawa matrix Y^a appears as the initial or final Yukawa matrix indicates that this term is due to the fermion wave function renormalisation, the factor of g^2 means that we must have a gauge coupling in the missing factor, and when checking the terms given in [11] we find the relevant term, with the correct numerical factor in equation (4.4). Although as noted above $C_2^{ab}(S) \propto \delta^{ab}$, we have not summed over the implicit delta function, for ease of comparison.

In order to write down the scalar beta functions concisely, we introduce the following set of terms for the one loop equation, together with representative diagrams. For concision, only example graphs are given for these, and the following terms. In addition, we omit diagrams for those terms that differ from previous ones only by the addition of a wave function renormalisation to an external leg of a previously described diagram.

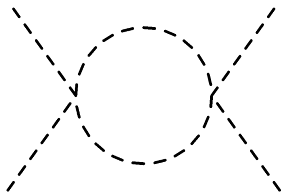
$$g^4 A_{abcd} \equiv \frac{1}{8} g^4 \sum_{\text{perms}} \left\{\theta^A, \theta^B\right\}_{ab} \left\{\theta^A, \theta^B\right\}_{cd} \sim \tag{2.39}$$



$$H_{abcd} \equiv \frac{1}{4} \sum_{\text{perms}} \mathrm{Tr}\big(Y^a Y^{\dagger b} Y^c Y^{\dagger d}\big) \sim \tag{2.40}$$



$$\Lambda^2_{abcd} \equiv \frac{1}{8} \sum_{\text{perms}} \lambda_{abef} \lambda_{cdef} \sim \tag{2.41}$$



The combination of the leading numerical factor and the sum on 4! permutations of the external legs $a, \ldots, d$ gives the complete set of diagrams giving a distinct result. For example, the Λ^2_{abcd} term given in (2.41) only gives a different result depending on whether the virtual particles are in the s, t or u channels. As there are only 3 distinct of permuting the legs, the leading factor for this term is $\frac{1}{8}$. Terms resulting from the addition of a 1-loop scalar wave-function renormalisation term to the bare scalar vertex are

$$\Lambda^S_{abcd} \equiv \sum_k C_2(k)\, \lambda_{abcd}, \tag{2.42}$$

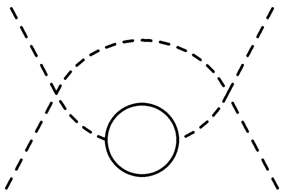
$$\Lambda^Y_{abcd} \equiv \sum_k Y_2(k)\, \lambda_{abcd}, \tag{2.43}$$

with the sum on k being over the external legs, and $C_2(k)$ and $Y_2(k)$ being the eigenvalues of $C_2(S)$ and $S_2(S)$ with respect to the scalar line represented by k. The one loop scalar coupling beta function is then

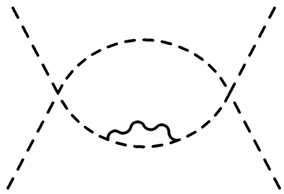
$$\left(16\pi^2\right) \beta_{abcd}|_{\text{1-loop}} = \Lambda^2_{abcd} - 8\kappa H_{abcd} + 2\kappa \Lambda^Y_{abcd} - 3g^2 \Lambda^S_{abcd} + 3g^4 A_{abcd}. \tag{2.44}$$

For the two loop equation, we introduce four further sets of terms, those based primarily on the scalar self-interaction

$$\overline{\Lambda}^{2Y}_{abcd} \equiv \frac{1}{8} \sum_{\text{perms}} Y_2^{fg}(S)\, \lambda_{abef}\lambda_{cdeg} \sim \tag{2.45}$$



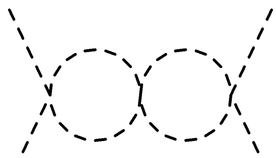
$$g^2\overline{\Lambda}^{2S}_{abcd} \equiv \frac{1}{8} g^2 \sum_{\text{perms}} C_2^{fg}(S)\, \lambda_{abef}\lambda_{cdeg} \sim \tag{2.46}$$



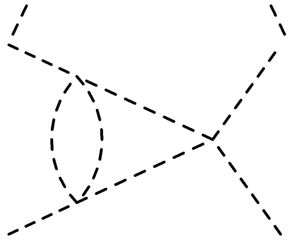
$$\Lambda^{YS}_{abcd} \equiv \sum_k C_2(k)\, Y_2(k)\, \lambda_{abcd} \tag{2.47}$$

$$\Lambda^{SS}_{abcd} \equiv \sum_k \left[C_2(k)\right]^2 \lambda_{abcd} \tag{2.48}$$

$$\Lambda^{3}_{abcd} \equiv \frac{1}{8} \sum_{\text{perms}} \lambda_{abef}\lambda_{efgh}\lambda_{ghcd} \sim \tag{2.49}$$

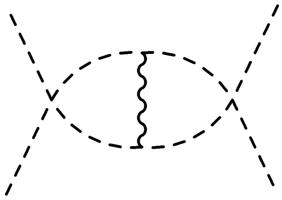


$$\overline{\Lambda}^{3}_{abcd} \equiv \frac{1}{4} \sum_{\text{perms}} \lambda_{abef}\lambda_{cegh}\lambda_{dfgh} \sim \tag{2.50}$$



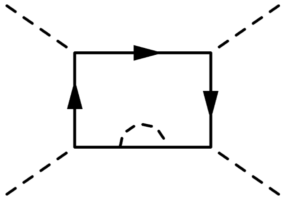
$$\Lambda^{2S}_{abcd} \equiv \sum_k C_2(k)\, \Lambda^2_{abcd} \tag{2.51}$$

$$g^2\Lambda^{2g}_{abcd} \equiv \frac{1}{8}g^2\sum_{\text{perms}}\lambda_{abef}\lambda_{cdgh}\theta^A_{eg}\theta^A_{fh} \sim \tag{2.52}$$

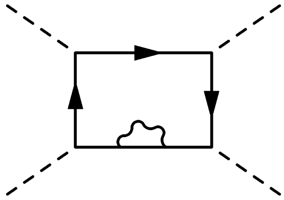


terms based primarily on a fermion loop

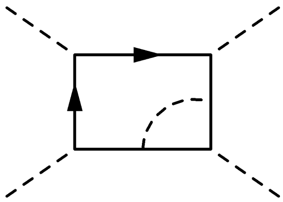
$$H^Y_{abcd} \equiv \sum_{\text{perms}}\mathrm{Tr}\big(Y_2(F)\,Y^{\dagger a}Y^bY^{\dagger c}Y^d\big) \sim \tag{2.53}$$



$$g^2H^F_{abcd} \equiv g^2\sum_{\text{perms}}\mathrm{Tr}\left[\{C_2(F)\,,Y^a\}\,Y^{\dagger b}Y^cY^{\dagger d}\right] \sim \tag{2.54}$$

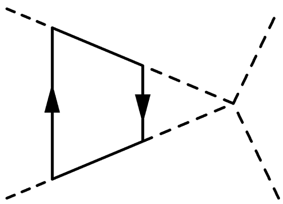


$$\overline{H}^Y_{abcd} \equiv \sum_{\text{perms}}\mathrm{Tr}\big(Y^eY^{\dagger a}Y^eY^{\dagger b}Y^cY^{\dagger d}\big) \sim \tag{2.55}$$

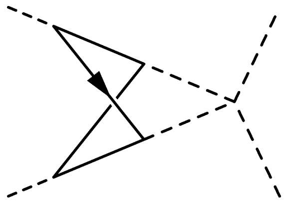


$$H^S_{abcd} \equiv \sum_k C_2(k)\,H_{abcd} \tag{2.56}$$

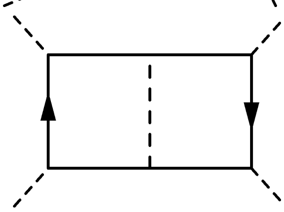
$$H^\lambda_{abcd} \equiv \frac{1}{2}\sum_{\text{perms}}\lambda_{abef}\,\mathrm{Tr}\big(Y^cY^{\dagger d}Y^eY^{\dagger f}\big) \sim \tag{2.57}$$



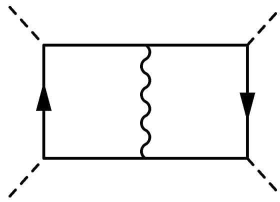
$$\overline{H}^\lambda_{abcd} \equiv \frac{1}{4}\sum_{\text{Perms}}\lambda_{abef}\,\mathrm{Tr}\big(Y^cY^{\dagger e}Y^dY^{\dagger f}\big) \sim \tag{2.58}$$



$$H^3_{abcd} \equiv \frac{1}{2}\sum_{\text{perms}}\mathrm{Tr}\big(Y^aY^{\dagger b}Y^eY^{\dagger c}Y^dY^{\dagger e}\big) \sim \tag{2.59}$$



$$g^2 H^g_{abcd} \equiv g^2 \sum_{\text{perms}} \mathrm{Tr}\big(t^A Y^a Y^{\dagger b} t^A Y^c Y^{\dagger b}\big) \sim \tag{2.60}$$

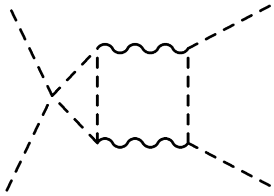


terms largely involving gauge interactions

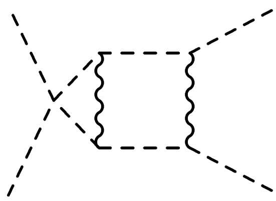
$$A^Y_{abcd} \equiv \sum_k Y_2(k)\, A_{abcd} \tag{2.61}$$

$$A^S_{abcd} \equiv \sum_k C_2(k)\, A_{abcd} \tag{2.62}$$

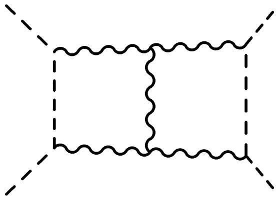
$$g^4 A^\lambda_{abcd} \equiv \frac{1}{4} g^4 \sum_{\text{perms}} \lambda_{abef} \left\{\theta^A, \theta^B\right\}_{ef} \left\{\theta^A, \theta^B\right\}_{cd} \sim \tag{2.63}$$



$$g^4 \overline{A}^\lambda_{abcd} \equiv \frac{1}{4} g^4 \sum_{\text{perms}} \lambda_{abef} \left\{\theta^A, \theta^B\right\}_{ce} \left\{\theta^A, \theta^B\right\}_{df} \sim \tag{2.64}$$

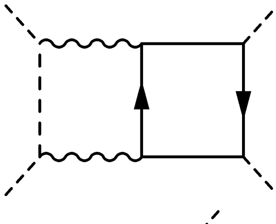


$$g^6 A^g_{abcd} \equiv \frac{1}{8} g^6 f^{ACE} f^{BDE} \sum_{\text{perms}} \left\{\theta^A, \theta^B\right\}_{ab} \left\{\theta^C, \theta^D\right\}_{cd} \sim \tag{2.65}$$

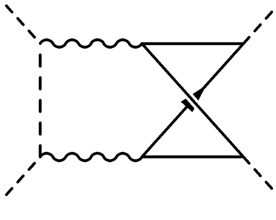


and based on a combination of gauge and Yukawa interactions

$$g^4 B^Y_{abcd} \equiv \frac{1}{4} g^4 \sum_{\text{perms}} \left\{\theta^A, \theta^B\right\}_{ab} \mathrm{Tr}\big[\left\{t^A t^B, Y^c\right\} Y^{\dagger d}\big] \sim \tag{2.66}$$



$$g^4 \overline{B}^Y_{abcd} \equiv \frac{1}{4} g^4 \sum_{\text{perms}} \left\{\theta^A, \theta^B\right\}_{ab} \mathrm{Tr}\big(t^A Y^c t^B Y^{\dagger d}\big) \sim \tag{2.67}$$



while the two loop result is

$$
\begin{aligned}
\left(16\pi^2\right)^2 \beta_{abcd}|_{\text{2-loop}} =& \frac{1}{2}\sum_k \Lambda^2(k)\,\lambda_{abcd} - \overline{\Lambda}^3_{abcd} + 4\kappa\left(H^Y_{abcd} + 2\overline{H}^Y_{abcd} + 2H^3_{abcd}\right) \\
&+ \kappa\left(8\overline{H}^\lambda_{abcd} - \sum_k\left[3H^2(k) + 2\overline{H}^2(k)\right]\lambda_{abcd}\right) - 4\kappa\overline{\Lambda}^{2Y}_{abcd} \\
&+ g^2\left[2\overline{\Lambda}^{2S}_{abcd} - 6\Lambda^{2g}_{abcd} + 4\kappa\left(H^S_{abcd} - H^F_{abcd}\right) + 5\kappa\sum_k Y^{2F}(k)\,\lambda_{abcd}\right] \\
&- g^4\left(\left[\frac{35}{3}C_2(G) - \frac{10}{3}\kappa S_2(F) - \frac{11}{12}S_2(S)\right]\Lambda^S_{abcd} - \frac{3}{2}\Lambda^{SS}_{abcd}\right. \\
&\left.-\frac{5}{2}A^\lambda_{abcd} - \frac{1}{2}\overline{A}^\lambda_{abcd} + 4\kappa\left(B^Y_{abcd} + 10\overline{B}^Y_{abcd}\right)\right) \\
&+ g^6\left(\left[\frac{161}{6}C_2(G) - \frac{32}{3}\kappa S_2(F) - \frac{7}{3}S_2(S)\right]A_{abcd}\right. \\
&\left.-\frac{15}{2}A^S_{abcd} + 27A^g_{abcd}\right).
\end{aligned} \tag{2.68}
$$

So far all these results apply only to the gauge theories of a single simple gauge group. However, we know that the Standard Model and any extensions to it are theories on a product of simple gauge groups, $\mathrm{SU}(3)_\mathrm{C}\otimes\mathrm{SU}(2)_\mathrm{L}\otimes\mathrm{U}(1)_Y$ as a minimal group. Although we might consider a unified gauge theory which contains the group structure of the Standard Model, we must also consider it at a scale where the larger, simple group breaks to a product of groups similar to the Standard model. Therefore, it is vital to specify how to generalise the above equations to deal with multiple groups. If we take our gauge theory to instead consist of a direct product of n simple groups, $G \to G_1\otimes\cdots\otimes G_n$ with corresponding couplings $g_1,\ldots,g_n$. Each of the subgroups G_p will have its own set of interaction tensors $\left(t^A\right)^p$ and $\left(\theta^A\right)^p$, and structure constants $\left(f^{ABC}\right)^p$, and so we will also have an equivalent number of Casimir operators and Dynkin indices, (2.20) through (2.24). With each of these invariants we can associate a factor of g_p^2, and by making the following substitutions

$$g^2 C_2(R) \to \sum_p g_p^2 C_2^p(R) \tag{2.69}$$

$$g^4 C_2(G)\, C_2(R) \to \sum_p g_p^4 C_2(G_p)\, C_2^p(R) \tag{2.70}$$

$$g^4 C_2(R)\, S_2(R') \to \sum_p g_p^4 C_2^p(R)\, S_2^p(R') \tag{2.71}$$

$$g^4 C_2(R)\, C_2(R') \to \sum_{p,q} g_p^2 g_q^2 C_2^p(R)\, C_2^q(R') \tag{2.72}$$

We also need to consider the tensor $\Theta_{ab,cd}$, called $\Lambda_{ab,cd}$ in [14], but renamed here to distinguish it from the scalar interaction terms $\Lambda^{\cdots}_{abcd}$. This is defined according to

$$\Theta_{ab,cd} \equiv \left(\theta^A\right)_{ac} \left(\theta^A\right)_{bd} \tag{2.73}$$

and for our product of groups, we make the substitution

$$g^2 \Theta_{ab,cd} \to \sum_p g_p^2 \Theta^p_{ab,cd}. \tag{2.74}$$

As the θ^A are antisymmetric, and all indices are specified explicitly, we have that

$$\Theta_{ab,cd} = \Theta_{ba,dc} = \Theta_{cd,ab} = \Theta_{dc,ba}. \tag{2.75}$$

We can then see that

$$\begin{aligned} g^4 A_{abcd} &= \frac{1}{4} g^4 \sum_{\text{perms}} \left(\Theta_{ac,ef}\Theta_{ef,bd} + \Theta_{ae,fd}\Theta_{eb,cf}\right) \\ &\to \frac{1}{4} \sum_{p,g} g_p^2 g_q^2 \sum_{\text{perms}} \left(\Theta^p_{ac,ef}\Theta^q_{ef,bd} + \Theta^p_{ae,fd}\Theta^q_{eb,cf}\right) \qquad (2.76) \\ g^4 A^\lambda_{abcd} &= g^4 \frac{1}{2} \sum_{\text{perms}} \lambda_{abef} \left(\Theta_{eg,ch}\Theta_{gf,hd} + \Theta_{gf,ch}\Theta_{eg,hd}\right) \end{aligned}$$

$$\rightarrow \frac{1}{2}\sum_{p,q} g_p^2 g_q^2 \sum_{\text{perms}} \lambda_{abef}\left(\Theta^p_{eg,ch}\Theta^q_{gf,hd} + \Theta^p_{gf,ch}\Theta^q_{eg,hd}\right) \tag{2.77}$$

$$g^4\overline{A}^{\lambda}_{abcd} = \frac{1}{2}g^4 \sum_{\text{perms}} \lambda_{abef}\left(\Theta_{cd,gh}\Theta_{gh,ef} + \Theta_{ch,gf}\Theta_{gd,eh}\right)$$

$$\rightarrow \frac{1}{2}\sum_{p,q} g_p^2 g_q^2 \sum_{\text{perms}} \lambda_{abef}\left(\Theta^p_{cd,gh}\Theta^q_{gh,ef} + \Theta^p_{ch,gf}\Theta^q_{gd,eh}\right). \tag{2.78}$$

We also have

$$g^6 S_2(R)\, A_{abcd} \rightarrow \frac{1}{4}\sum_{p,q} g_p^4 g_q^2 S_2^p(R) \sum_{\text{perms}} \left(\Theta_{ac,ef}\Theta_{ef,bd} + \Theta_{ae,fd}\Theta_{eb,cf}\right), \tag{2.79}$$

and since there is no gauge interaction between vectors of different simple gauge groups,

$$g^6 A^g_{abcd} \rightarrow \sum_p g_p^6 A^{g_p}_{abcd}. \tag{2.80}$$

2.3. Beta Functions for the Standard Model

In this section we present the beta functions for the Standard Model, due to their central use in this work, as well as the possibility for confusion due to various differing normalisations and occasional typographical errors between publications. The original source for these is the series of three papers by Machacek and Vaughn [11, 13, 14], with corrections largely from Appendix A of [15], as well as from [16]. For the three loop corrections in the $\overline{\text{MS}}$ scheme, we use [17] for the gauge couplings, [18] for the partial top Yukawa and [19] for the complete Higgs self-coupling beta functions. To the best of our knowledge, the three loop terms have not been calculated in any other papers, which prevents the same level of cross-checking as for the two loop beta functions. However, reference [18] includes the partial strong coupling three loop term, the included terms for which are in agreement with the full result given in reference [17]. In our integrations we have the three loop corrections for all calculations in the Standard Model at energies

$> M_t$. Although the full three loop result for the top Yukawa is not known, we anticipate that as with the scalar self-coupling, the effect of adding g_1 and g_2 is minimal.

In order to ensure clarity, it is also worthwhile to mention the normalisation used for the five relevant Standard Model couplings. The gauge couplings for SU(2) and SU(3) are the simplest, being

$$g_2(\mu) = g(\mu)\,, \tag{2.81}$$

$$g_3(\mu) = g_s(\mu)\,, \tag{2.82}$$

respectively. The top Yukawa coupling is given to leading order by

$$g_t(\mu) = \frac{\sqrt{2}M_t(\mu)}{v}. \tag{2.83}$$

The two couplings which most commonly vary between publications are the U(1) gauge coupling and the Higgs self coupling. Here we use

$$g_1(\mu) = g'(\mu)\,, \tag{2.84}$$

$$g_\lambda(\mu) = \frac{M_H^2(\mu)}{2v^2}, \tag{2.85}$$

the former in comparison to the commonly used normalisation $g'(\mu) = \sqrt{\frac{3}{5}}g_1(\mu)$, which derives from assuming SU(3)⊗SU(2)⊗SU(1) is a subset of some larger group $\mathcal{G}$, although it is commonly referred to as "SU(5) normalisation", used in the majority of formulae in references [11, 13, 14]. The Higgs self-coupling constant used here is related to that in Machacek and Vaughn as $\lambda = 2g_\lambda$, and when mentioning corrections to the beta functions presented by them, we shall use their coupling constant, λ.

The beta functions themselves we write order by order, as

$$\mu\frac{\mathrm{d}g_x}{\mathrm{d}\mu} = \beta_x = \frac{1}{16\pi^2}\beta_x^{(1)} + \frac{1}{(16\pi^2)^2}\beta_x^{(2)} + \frac{1}{(16\pi^2)^3}\beta_x^{(3)} + ... \tag{2.86}$$

We shall also write them with the number of generations, n_G, left explicit, as the extension to a fourth generation of fermions is the focus of a subsequent chapter. Similarly, we shall write these beta functions both in terms of the fermion mixing matrices F_L, F_D, F_U defined as

$$\mathcal{L}_{\text{Yukawa}} = -\overline{e}F_L\phi^\dagger l - \overline{d}F_D\phi^\dagger q - \overline{u}F_U\phi^{\dagger c}q + \text{h.c.} \tag{2.87}$$

and after expansion whilst neglecting all Yukawa couplings except for the top, with $F_L = F_D = 0$ and

$$F_U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g_t \end{pmatrix}. \tag{2.88}$$

This approximation is valid due to the large difference in scale between m_t and the other fermion masses in the Standard Model, so that the other Yukawa couplings all have a negligible effect on the running. Additionally, the third generation mixing elements V_{td} and V_{ts} are close to zero, with $V_{tb} \approx 1$. This notation for the up quark mixing matrix differs from that used in a number of papers, including the three Machacek and Vaughn papers, which instead denote the matrix as H. This should facilitate comparison with other sources, and make the extension to the fourth generation more straightforward. However, the three loop terms have not been calculated in terms of fermion mixing matrices, and consequently, those terms are omitted in the unsimplified form. For the three loop terms presented in [18], we substitute the Standard Model values for the factors C_A, C_F, T_F and d_R, which depend on the gauge group, giving the beta functions given in equations (26) and (29) of that paper. We have also made the substitution $n_f = 2n_G$ for consistency with other sources.

The gauge beta functions of the Standard Model are

$$\beta_1^{(1)} = g_1^3 \left(\frac{1}{6} + \frac{20n_G}{9} \right), \tag{2.89}$$

$$\begin{aligned}\beta_1^{(2)} = g_1^3 \Bigg[&\left(\frac{1}{2} + \frac{95n_G}{27} \right) g_1^2 + \left(\frac{3}{2} + n_G \right) g_2^2 + \frac{44n_G}{9} g_3^2 \\ &+ \mathrm{Tr}\Big(-\frac{17}{6} F_U^\dagger F_U - \frac{5}{6} F_D^\dagger F_D - \frac{5}{2} F_L^\dagger F_L \Big) \Bigg], \end{aligned} \tag{2.90}$$

the beta function for g_2

$$\beta_2^{(1)} = g_2^3 \left(-\frac{43}{6} + \frac{4n_G}{3} \right), \tag{2.91}$$

$$\begin{aligned}\beta_2^{(2)} = g_2^3 \Bigg[&\left(\frac{1}{2} + \frac{n_G}{3} \right) g_1^2 + \left(-\frac{259}{6} + \frac{49n_G}{3} \right) g_2^2 + \frac{4n_G}{3} g_3^2 \\ &+ \mathrm{Tr}\left(-\frac{3}{2} F_U^\dagger F_U - \frac{3}{2} F_D^\dagger F_D - \frac{1}{2} F_L^\dagger F_L \right) \Bigg], \end{aligned} \tag{2.92}$$

and the beta function for g_3

$$\beta_3^{(1)} = g_3^3 \left(-11 + \frac{4n_G}{3} \right), \tag{2.93}$$

$$\begin{aligned}\beta_3^{(2)} = g_3^3 \Bigg[&\frac{11n_G}{18} g_1^2 + \frac{3n_G}{2} g_2^2 + \left(-102 + \frac{76n_G}{3} \right) g_3^2 \\ &+ \mathrm{Tr}\Big(-2F_U^\dagger F_U - 2F_D^\dagger F_D \Big) \Bigg]. \end{aligned} \tag{2.94}$$

For these three beta functions, the only corrections from the original [11] are to rectify a series of mislabelled indices, where $C_k^{U,D,L}$ in Eq. (B.2) should read $C_l^{U,D,L}$ respectively, and an inverted factor in the definition of g_1, where the coefficient in $g_1 = \sqrt{\frac{5}{3}} g'$ is mistakenly rendered as $\sqrt{\frac{3}{5}}$ [15]. The up-type Yukawa coupling beta function, is defined as

$$\mu \frac{\mathrm{d}F_U}{\mathrm{d}\mu} = \beta_U = \frac{1}{16\pi^2}\beta_U^{(1)} + \frac{1}{(16\pi^2)^2}\beta_U^{(2)} + \frac{1}{(16\pi^2)^3}\beta_U^{(3)} + ... \tag{2.95}$$

and in our approximation contains the top Yukawa beta function as its only non-zero element. It is given by

$$\beta_U^{(1)} = F_U\left[-\frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + \frac{3}{2}F_U^\dagger F_U + \frac{3}{2}F_D^\dagger F_D + Y_2\right] \tag{2.96}$$

$$\begin{aligned}\beta_U^{(2)} = F_U\Bigg[&\left(\frac{9}{200} + \frac{29n_G}{45}\right)g_1^4 - \frac{9}{20}g_1^2g_2^2 + \frac{19}{15}g_1^2g_3^2 + \left(-\frac{35}{4} + n_G\right)g_2^4 \\ &+ 9g_2^2g_3^2 + \left(-\frac{404}{3} + \frac{80n_G}{9}\right)g_3^4 + 6g_\lambda^2 - 12F_U^\dagger F_U g_\lambda - X_4 + \frac{5}{2}Y_4 \\ &+ \left(-\frac{9}{4}F_U^\dagger F_U + \frac{5}{4}F_D^\dagger F_D\right)Y_2 + F_U^\dagger F_U\left(\frac{223}{80}g_1^2 + \frac{135}{16}g_2^2 + 16g_3^2\right) \\ &+ F_D^\dagger F_D\left(\frac{43}{80}g_1^2 - \frac{9}{16}g_2^2 + 16g_3^2\right) + \frac{3}{2}\left(F_U^\dagger F_U\right)^2 - F_U^\dagger F_U F_D^\dagger F_D \\ &- \frac{1}{4}F_D^\dagger F_D F_U^\dagger F_U + \frac{11}{4}\left(F_D^\dagger F_D\right)^2\Bigg].\end{aligned} \tag{2.97}$$

Here, there is a difference due to the omission of a term $-2F_U F_D^\dagger F_D \lambda$ from the original, Eq. (B.8) of [13], as the two diagrams that would contribute to this term cancel [20].

Finally for the Standard Model, we have the beta function for the scalar self coupling

$$\mu \frac{\mathrm{d}g_\lambda}{\mathrm{d}\mu} = \beta_\lambda = \frac{1}{16\pi^2}\beta_\lambda^{(1)} + \frac{1}{(16\pi^2)^2}\beta_\lambda^{(2)} + \frac{1}{(16\pi^2)^3}\beta_\lambda^{(3)} + ... \tag{2.98}$$

given by

$$\beta_\lambda^{(1)} = -\frac{27}{200}g_1^4 + \frac{9}{20}g_1^2g_2^2 - \frac{9}{5}g_1^2g_\lambda + \frac{9}{8}g_2^4 - 9g_2^2g_\lambda + 24g_\lambda^2 + 4g_\lambda Y_2 - 2Z_4 \tag{2.99}$$

$$\begin{aligned}\beta_\lambda^{(2)} = &\left(-\frac{59}{48} - \frac{20n_G}{9}\right)g_1^6 + \left(-\frac{239}{48} - \frac{20n_G}{9}\right)g_1^4g_2^2 + \left(\frac{229}{24} + \frac{50n_G}{9}\right)g_1^4g_\lambda \\ &+ \left[-\frac{171}{100}\,\mathrm{Tr}\left(F_U^\dagger F_U\right) + \frac{9}{20}\,\mathrm{Tr}\left(F_D^\dagger F_D\right) - \frac{9}{4}\,\mathrm{Tr}\left(F_L^\dagger F_L\right)\right]g_1^4 + \left(-\frac{97}{80} - \frac{4n_G}{5}\right)g_1^2g_2^4\end{aligned}$$

$$
\begin{aligned}
&+\frac{39}{4}g_1^2g_2^2g_\lambda+\left[\frac{63}{10}\operatorname{Tr}\Big(F_U^\dagger F_U\Big)+\frac{27}{10}\operatorname{Tr}\Big(F_D^\dagger F_D\Big)+\frac{33}{10}\operatorname{Tr}\Big(F_L^\dagger F_L\Big)\right]g_1^2g_2^2+36g_1^2g_\lambda^2\\
&+\left[\frac{17}{2}\operatorname{Tr}\Big(F_U^\dagger F_U\Big)+\frac{5}{2}\operatorname{Tr}\Big(F_D^\dagger F_D\Big)+\frac{15}{2}\operatorname{Tr}\Big(F_L^\dagger F_L\Big)\right]g_1^2g_\lambda\\
&+\operatorname{Tr}\left[-\frac{8}{5}\left(F_U^\dagger F_U\right)^2+\frac{4}{5}\left(F_D^\dagger F_D\right)^2-\frac{12}{5}\left(F_L^\dagger F_L\right)^2\right]g_1^2+\left(\frac{497}{16}-4n_G\right)g_2^6\\
&+\left(10n_G-\frac{313}{8}\right)g_2^4g_\lambda-\frac{3}{4}g_2^4Y_2+108g_2^2g_\lambda^2\\
&+\left[\frac{45}{2}\operatorname{Tr}\Big(F_U^\dagger F_U\Big)+\frac{45}{2}\operatorname{Tr}\Big(F_D^\dagger F_D\Big)+\frac{15}{2}\operatorname{Tr}\Big(F_L^\dagger F_L\Big)\right]g_2^2g_\lambda\\
&+\left[80\operatorname{Tr}\Big(F_U^\dagger F_U\Big)+80\operatorname{Tr}\Big(F_D^\dagger F_D\Big)\right]g_3^2g_\lambda-32\operatorname{Tr}\left[\left(F_U^\dagger F_U\right)^2+\left(F_D^\dagger F_D\right)^2\right]g_3^2-312g_\lambda^3\\
&-48g_\lambda^2Y_2-g_\lambda Z_4-42\operatorname{Tr}\Big(F_U^\dagger F_UF_D^\dagger F_D\Big)g_\lambda\\
&+10\operatorname{Tr}\left[3\left(F_U^\dagger F_U\right)^3+3\left(F_D^\dagger F_D\right)^3+\left(F_L^\dagger F_L\right)^3\right]-6\operatorname{Tr}\left[F_U^\dagger F_U\left(F_U^\dagger F_U+F_D^\dagger F_D\right)F_D^\dagger F_D\right].
\end{aligned}
\tag{2.100}
$$

The corrections for this beta function, as compared to the original in reference [14] are the most numerous.

1. In Eq. (B.3) of [14], the term $g_2^2\lambda$ should instead be $9g_2^2\lambda$. Eq. (B.4) omits a dagger from a matrix product; $3F_DF_D$ should read $3F_D^\dagger F_D$.

2. For Eq. (B.8), the coefficient of $g'^4\lambda$ is given as $-\left(\frac{229}{24}+2n_G\right)$, but should instead read $\frac{229}{24}+\frac{50}{9}n_G$, the magnitude agreeing with [21] for the specific case $n_G=3$, and the change of sign as pointed out in [16].

3. The same authors note that the sign of $-\frac{39}{4}g'^2g^2g_\lambda$ is also incorrect and the coefficient should be $\frac{39}{4}$. We calculated the reduction from the general case given in Eq. (4.3) of [14] to verify the sign, as these sign errors are, as noted in [16], commonly repeated.

4. In the same equation, the term $6\lambda\operatorname{Tr}\Big(H^\dagger HF_D^\dagger F_D\Big)$ should instead read $-42\lambda\operatorname{Tr}\Big(H^\dagger HF_D^\dagger F_D\Big)$, or $-42g_\lambda\operatorname{Tr}\Big(F_U^\dagger F_UF_D^\dagger F_D\Big)$ in our notation — the factor of two difference in the Higgs self-coupling constant cancels with the left hand side of the equation.

There are four commonly recurring expressions in the preceding beta functions for which shorthands are used; these are

$$X_4 = \frac{9}{4}\,\mathrm{Tr}\left[3\left(F_U^\dagger F_U\right)^2 + 3\left(F_D^\dagger F_D\right)^2 + \left(F_L^\dagger F_L\right)^2 - \frac{2}{3}F_U^\dagger F_U F_D^\dagger F_D\right], \tag{2.101}$$

$$Y_2 = \mathrm{Tr}\left(3F_U^\dagger F_U + 3F_D^\dagger F_D + F_L^\dagger F_L\right), \tag{2.102}$$

$$Y_4 = \left(\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right)F_U^\dagger F_U + \left(\frac{5}{12}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right)F_D^\dagger F_D + \left(\frac{3}{4}g_1^2 + \frac{3}{4}g_2^2\right)F_L^\dagger F_L, \tag{2.103}$$

$$Z_4 = \mathrm{Tr}\left[3\left(F_U^\dagger F_U\right)^2 + 3\left(F_D^\dagger F_D\right)^2 + \left(F_L^\dagger F_L\right)^2\right]. \tag{2.104}$$

Y_2, Y_4 and X_4 are defined in equations (B.7), (B.12) and (B.14) respectively of [13], while Z_4 is defined in [14], Eq. (B.5), but is referred to in that paper as $H(S)$.

In addition, we include the beta functions for the down-type quarks and the leptons, which will necessarily be included when extending to a fourth generation, as the masses of all fourth generation particles will be assumed to be of roughly the same order as M_t. However we shall not take the top-only limit here.

$$\beta_D^{(1)} = F_D\left(-\frac{1}{4}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + Y_2 + \frac{3}{2}\left(-F_U^\dagger F_U + F_D^\dagger F_D\right)\right), \tag{2.105}$$

$$\begin{aligned}\beta_D^{(2)} = F_D\Bigg[&\left(-\frac{29}{200} - \frac{n_G}{45}\right)g_1^4 - \frac{27}{20}g_1^2 g_2^2 + \frac{31}{15}g_1^2 g_3^2 + \left(-\frac{35}{4} + n_G\right)g_2^4 + 9g_2^2 g_3^2 \\ &+ \left(-\frac{404}{3} + \frac{80n_G}{9}\right)g_3^4 + 6g_\lambda^2 - 12F_D^\dagger F_D g_\lambda - X_4 + \frac{5}{2}Y_4 \\ &+ \left(-\frac{79}{80}g_1^2 + \frac{9}{16}g_2^2 - 16\right)F_U^\dagger F_U + \left(\frac{187}{80}g_1^2 + \frac{135}{16}g_2^2 + 16g_3^2\right)F_D^\dagger F_D \\ &+ \left(\frac{5}{4}F_U^\dagger F_U - \frac{9}{4}F_D^\dagger F_D\right)Y_2 + \frac{11}{4}\left(F_U^\dagger F_U\right)^2 - \frac{1}{4}F_U^\dagger F_U F_D^\dagger F_D \\ &- F_D^\dagger F_D F_U^\dagger F_U + \frac{3}{2}\left(F_D^\dagger F_D\right)^2\Bigg].\end{aligned} \tag{2.106}$$

Similarly to the expression for $\beta_U^{(2)}$ given above, this differs from that given in [13] by the omission of a term $-2F_U^\dagger F_U \lambda$,

$$\beta_L^{(1)} = F_L \left(-\frac{9}{4} g_1^2 - \frac{9}{4} g_2^2 + Y_2 + \frac{3}{2} F_L^\dagger F_L \right), \tag{2.107}$$

$$\begin{aligned}\beta_L^{(2)} = F_L \Bigg[&\left(\frac{51}{200} + \frac{11 n_G}{5} \right) g_1^4 + \frac{27}{20} g_1^2 g_2^2 + \left(-\frac{35}{4} + n_G \right) g_2^4 + 6 g_\lambda^2 - 12 F_L^\dagger F_L g_\lambda - X_4 \\ &+ \frac{5}{2} Y_4 - \frac{9}{4} F_L^\dagger F_L Y_2 + \left(\frac{387}{80} g_1^2 + \frac{135}{16} g_2^2 \right) F_L^\dagger F_L + \frac{3}{2} \left(F_L^\dagger F_L \right)^2 \Bigg].\end{aligned} \tag{2.108}$$

The following are the relevant three-loop beta functions for the couplings of the Standard Model with expanded Yukawa coupling matrices. However, we do not include the one-loop beta functions for the gauge couplings, which are unchanged. The three-loop terms are taken from [17, 22], which expresses the beta functions in terms of $\alpha_X = \frac{g_X^2}{4\pi}$ and defines the beta function as

$$\beta_i = \mu^2 \frac{\partial}{\partial \mu^2} \frac{\alpha_i}{\pi}.$$

However α_1 is defined with an additional factor of $\frac{3}{5}$, as with the majority of the Machacek and Vaughn papers.

The two papers factorise the possible contribution from more generic fermionic sectors differently — [17] includes a factor n_t for the number of top-type quarks, i.e. up-type quarks with a mass $\geq M_t$, while [22] uses generalised coupling matrices $\widehat{T}$, $\widehat{B}$ and $\widehat{L}$ for the top and bottom quarks, and the leptons respectively. In the case of the Standard Model we have $n_t = 1$, and neglecting all but the top quark Yukawa coupling, we can reduce the notation of the second paper as $\operatorname{Tr} \widehat{T} = g_t^2$ and $\operatorname{Tr} \widehat{T}^2 = \left(\operatorname{Tr} \widehat{T} \right)^2 = g_t^4$, with all other traces being zero.

The beta function, at three loop accuracy, for g_1 is given by

$$\beta_1^{(2)} = g_1^3\left[\left(\frac{1}{2}+\frac{95n_G}{27}\right)g_1^2+\left(\frac{3}{2}+n_G\right)g_2^2+\frac{44n_G}{9}g_3^2-\frac{17}{6}g_t^2\right] \tag{2.109}$$

$$\begin{aligned}\beta_1^{(3)} = g_1^3\Bigg[&\left(\frac{163}{576}-\frac{290n_G}{81}-\frac{5225n_G^2}{729}\right)g_1^4+\left(\frac{87}{32}-\frac{7n_G}{36}\right)g_1^2g_2^2-\frac{137}{81}g_1^2g_3^2-\frac{2827}{288}g_1^2g_t^2\\ &+\frac{3}{2}g_1^2g_\lambda+\left(-\frac{3401}{288}+\frac{83n_G}{18}-\frac{11n_G^2}{9}\right)g_2^4-\frac{n_G}{3}g_2^2g_3^2-\frac{785}{32}g_2^2g_t^2+\frac{3}{2}g_2^2g_\lambda\\ &+\left(\frac{1375n_G}{27}-\frac{484n_G^2}{81}\right)g_3^4-\frac{29}{3}g_3^2g_t^2+\frac{315}{16}g_t^4-3g_\lambda^2\Bigg]\end{aligned} \tag{2.110}$$

The complete 3-loop beta function for g_2 is given by

$$\beta_2^{(2)} = g_2^3\left[\left(\frac{1}{2}+\frac{n_G}{3}\right)g_1^2+\left(-\frac{259}{6}+\frac{49n_G}{3}\right)g_2^2+\frac{4n_G}{3}g_3^2+\frac{3}{2}g_t^2\right] \tag{2.111}$$

$$\begin{aligned}\beta_2^{(3)} = g_2^3\Bigg[&\left(\frac{163}{576}-\frac{35n_G}{27}-\frac{55n_G^2}{81}\right)g_1^4+\left(\frac{187}{32}+\frac{13n_G}{12}\right)g_1^2g_2^2-\frac{n_G}{9}g_1^2g_3^2-\frac{593}{96}g_1^2g_t^2\\ &+\frac{1}{2}g_1^2g_\lambda+\left(-\frac{667111}{1728}+\frac{6412n_G}{27}-\frac{415n_G^2}{27}\right)g_2^4+13n_Gg_2^2g_3^2-\frac{729}{32}g_2^2g_t^2+\frac{3}{2}g_2^2g_\lambda\end{aligned} \tag{2.112}$$

$$+\left(\frac{125n_G}{3}-\frac{44n_G^2}{9}\right)g_3^4-7g_3^2g_t^2+\frac{147}{16}g_t^4-3g_\lambda^2\Bigg] \tag{2.113}$$

and the complete 3-loop beta function for g_3 is given by

$$\beta_3^{(2)} = g_3^3\left[\frac{11n_G}{18}g_1^2+\frac{3n_G}{2}g_2^2+\left(-102+\frac{76n_G}{3}\right)g_3^2-2g_t^2\right] \tag{2.114}$$

$$\begin{aligned}\beta_3^{(3)} = g_3^3\Bigg[&\left(-\frac{65n_G}{216}-\frac{605n_G^2}{486}\right)g_1^4-\frac{n_G}{24}g_1^2g_2^2+\frac{77n_G}{27}g_1^2g_3^2-\frac{101}{24}g_1^2g_t^2+\left(\frac{241n_G}{24}-\frac{11n_G^2}{6}\right)g_2^4\\ &+7n_Gg_2^2g_3^2-\frac{93}{8}g_2^2g_t^2+\left(-\frac{2857}{2}+\frac{5033n_G}{9}-\frac{650n_G}{27}\right)g_3^4-40g_3^2g_t^2+15g_t^4\Bigg]\end{aligned} \tag{2.115}$$

The three-loop term for the top Yukawa coupling is given in [18], but the third-order term includes only terms with no dependence on g_1 or g_2. The expressions given here differ from theirs by an overall factor of two due to a difference in the definition of the beta function. ζ_3 is Apéry's constant, given by the Riemann zeta function evaluated at 3.

$$\beta_t^{(1)} = g_t\left(-\frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + \frac{9}{2}g_t^2\right) \tag{2.116}$$

$$\begin{aligned}\beta_t^{(2)} = g_t\Bigg[&\left(\frac{1}{8} + \frac{145n_g}{81}\right)g_1^4 - \frac{3}{4}g_1^2g_2^2 + \frac{76}{9}g_1^2g_3^2 + \frac{131}{16}g_1^2g_t^2 + \left(-\frac{35}{4} + 3n_G\right)g_2^4 + 9g_2^2g_3^2\\ &+\frac{225}{16}g_2^2g_t^2 + \left(-\frac{404}{3} + \frac{80n_G}{9}\right)g_3^4 + 36g_3^2g_t^2 - 12g_t^4 - 12g_t^2g_\lambda + 6g_\lambda^2\Bigg]\end{aligned} \tag{2.117}$$

$$\begin{aligned}\beta_t^{(3)} = g_t\Bigg[&\left(-2498 + \frac{8864n_G}{27} + \frac{1120n_G^2}{81} + \frac{640n_G\zeta_3}{3}\right)g_3^6 + \left(\frac{4799}{6} - 54n_G - 228\zeta_3\right)g_3^4g_t^2\\ &-157g_3^2g_t^4 + 16_3^2g_t^2g_\lambda + \left(\frac{339}{8} + \frac{27\zeta_3}{2}\right)g_t^6 + 198g_t^4g_\lambda + \frac{15}{4}g_t^2g_\lambda^2 - 36g_\lambda^3\Bigg]\end{aligned} \tag{2.118}$$

The three-loop term is given in a similar format to the top Yukawa term in [18], but is given in complete form in [19]. The latter paper also includes dependence on g_b and g_τ, which we have omitted here since we did not use those terms in our calculations.

$$\beta_\lambda^{(1)} = \frac{3}{8}g_1^4 + \frac{3}{4}g_1^2g_2^2 - 3g_1^2g_\lambda + \frac{9}{8}g_2^4 - 9g_2^2g_\lambda - 6g_t^4 + 12g_t^2g_\lambda + 24g_\lambda^2, \tag{2.119}$$

$$\begin{aligned}\beta_\lambda^{(2)} = &\left(-\frac{59}{48} - \frac{20n_G}{9}\right)g_1^6 + \left(-\frac{239}{48} - \frac{20n_G}{9}\right)g_1^4g_2^2 - \frac{19}{4}g_1^4g_t^2 + \left(\frac{229}{24} + \frac{50n_G}{9}\right)g_1^4g_\lambda\\ &+\left(-\frac{97}{48} - \frac{4n_G}{3}\right)g_1^2g_2^4 + \frac{21}{2}g_1^2g_2^2g_t^2 + \frac{39}{4}g_1^2g_2^2g_\lambda - \frac{8}{3}g_1^2g_t^4 + \frac{85}{6}g_1^2g_t^2g_\lambda + 36g_1^2g_\lambda^2\\ &+\left(\frac{497}{16} - 4n_G\right)g_2^6 - \frac{9}{4}g_2^4g_t^2 + \left(-\frac{313}{8} + 10n_G\right)g_2^4g_\lambda + \frac{45}{2}g_2^2g_t^2g_\lambda + 108g_2^2g_\lambda^2\\ &-32g_3^2g_t^4 + 80g_3^2g_t^2g_\lambda + 30g_t^6 - 3g_t^4g_\lambda - 144g_t^2g_\lambda^2 - 312g_\lambda^3,\end{aligned} \tag{2.120}$$

$$\begin{aligned}\beta_\lambda^{(3)} = &\left(-\frac{6845}{4608} + \frac{99\zeta_3}{64} - \frac{20735n_G}{864} + \frac{190n_G\zeta_3}{9} - \frac{250n_G^2}{81}\right)g_1^8 + \left(-\frac{29779}{3456} + \frac{75\zeta_3}{16}\right.\\ &\left.-\frac{18001n_G}{1296} + \frac{122n_G\zeta_3}{9} - \frac{500n_G^2}{243}\right)g_1^6g_2^2 + \left(-\frac{187n_G}{12} + \frac{44n_G\zeta_3}{3}\right)g_1^6g_2^2 + \left(\frac{42943}{1152}\right.\\ &\left.-5\zeta_3 + \frac{215n_G}{9}\right)g_1^6g_t^2 + \left(\frac{12679}{216} + \frac{9\zeta_3}{4} + \frac{5995n_G}{81} - \frac{380n_G\zeta_3}{9} + \frac{3500n_G^2}{243}\right)g_1^6g_\lambda\\ &+\left(-\frac{64693}{1728} + \frac{873n_G}{32} + \frac{149n_G}{324} + 14n_G\zeta_3 - \frac{100n_G^2}{81}\right)g_1^4g_2^4 + \left(-\frac{187n_G}{12}\right.\\ &\left.+\frac{44n_G\zeta_3}{9}\right)g_1^4g_2^2g_3^2 + \left(\frac{23521}{384} - 6\zeta_3 + \frac{5n_G}{3}\right)g_1^4g_2^2g_t^2 + \left(\frac{979}{4} - \frac{3\zeta_3}{4} + \frac{95n_G}{2}\right.\\ &\left.-12n_G\zeta_3\right)g_1^4g_2^2g_\lambda + \left(\frac{587}{12} - 36\zeta_3\right)g_1^4g_3^2g_t^2 + \left(55n_G - \frac{176n_G\zeta_3}{3}\right)g_1^4g_3^2g_\lambda\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{100913}{1728}+\frac{2957\zeta_3}{72}-\frac{115n_G}{18}\right)g_1^4g_t^4+\left(-\frac{112447}{864}-\frac{449\zeta_3}{3}-\frac{635n_G}{18}\right)g_1^4g_t^2g_\lambda\\
&+\left(-366-162\zeta_3-\frac{470n_G}{3}\right)g_1^4g_\lambda^2+\left(-\frac{54053}{1728}-\frac{405\zeta_3}{16}-\frac{8341n_G}{432}-\frac{20n_G^2}{27}\right)g_1^2g_2^6\\
&+\left(-\frac{51n_G}{4}+12n_G\zeta_3\right)g_1^2g_2^4g_3^2+\left(\frac{3103}{128}+\frac{27\zeta_3}{2}+n_G\right)g_1^2g_2^4g_t^2+\left(\frac{4553}{16}-\frac{249\zeta_3}{4}\right.\\
&\left.+33n_G-12n_G\zeta_3\right)g_1^2g_2^4g_\lambda+\left(\frac{249}{2}-72\zeta_3\right)g_1^2g_2^2g_3^2g_t^2+\left(-\frac{1079}{96}-\frac{748\zeta_3}{4}\right)g_1^2g_2^2g_t^4\\
&+\left(-\frac{6509}{16}+354\zeta_3\right)g_1^2g_2^2g_t^2g_\lambda+(-666-324\zeta_3)\,g_1^2g_2^2g_\lambda^2+\left(\frac{931}{9}-\frac{112\zeta_3}{3}\right)g_1^2g_3^2g_t^4\\
&+\left(-\frac{2419}{9}+272\zeta_3\right)g_1^2g_3^2g_t^2g_\lambda+\left(\frac{3467}{48}+34\zeta_3\right)g_1^2g_t^6+\left(-\frac{2485}{12}+114\zeta_3\right)g_1^2g_t^4g_\lambda\\
&+\left(-\frac{195}{2}-96\zeta_3\right)g_1^2g_t^2g_\lambda^2+(-316+48\zeta_3)\,g_1^2g_\lambda^3+\left(\frac{982291}{1536}-\frac{2781\zeta_3}{64}-\frac{14749n_G}{96}\right.\\
&\left.-90n_G\zeta_3-\frac{10n_G^2}{3}\right)g_2^8+\left(-\frac{153n_G}{4}+36n_G\zeta_3\right)g_2^6g_3^2+\left(-\frac{17217}{128}+\frac{297\zeta_3}{2}+27n_G\right)g_2^6g_t^2\\
&+\left(-\frac{46489}{144}+\frac{2259\zeta_3}{4}+\frac{3515n_G}{18}+180n_G\zeta_3+\frac{140n_G^2}{9}\right)g_2^6g_\lambda+\left(\frac{651}{4}-108\zeta_3\right)g_2^4g_3^2g_t^2\\
&+(135n_G-144n_G\zeta_3)\,g_2^4g_3^2g_\lambda+\left(\frac{13653}{64}-\frac{819\zeta_3}{8}-\frac{39n_G}{2}\right)g_2^4g_t^4+\left(-\frac{3933}{32}-351\zeta_3\right.\\
&\left.-\frac{63n_G}{2}\right)g_2^4g_t^2g_\lambda+\left(\frac{1995}{4}-1026\zeta_3-282n_G\right)g_2^4g_\lambda^2+(-31+48\zeta_3)\,g_2^2g_3^2g_t^4\\
&+(-489+432\zeta_3)\,g_2^2g_3^2g_t^2g_\lambda+\left(\frac{3411}{16}-54\zeta_3\right)g_2^2g_t^6+\left(-\frac{4977}{4}+1026\zeta_3\right)g_2^2g_t^4g_\lambda\\
&+\left(\frac{639}{2}-864\zeta_3\right)g_2^2g_t^2g_\lambda^2+(-948+144\zeta_3)\,g_2^2g_\lambda^3+\left(-\frac{1252}{3}+64\zeta_3+80n_G\right)g_3^4g_t^4\\
&+\left(\frac{3640}{3}-96\zeta_3-128n_G\right)g_3^4g_t^2g_\lambda+(-76+480\zeta_3)\,g_3^2g_t^6+(1790-2592\zeta_3)\,g_3^2g_t^4g_\lambda\\
&+(-2448+2304\zeta_3)\,g_3^2g_t^2g_\lambda^2+\left(-\frac{1599}{4}-72\zeta_3\right)g_t^8+\left(\frac{117}{4}-396\zeta_3\right)g_t^6g_\lambda+(1719\\
&+1512\zeta_3)\,g_t^4g_\lambda^2+1746g_t^2g_\lambda^3+(7176+4032\zeta_3)\,g_\lambda^4. \qquad (2.121)
\end{aligned}
$$

An important point, key to the possible problem of the positivity of the scalar self-coupling, is the difference between multiplicative and additive renormalisation. From the beta functions presented in this chapter, we can see that every term in the gauge coupling beta functions contains a factor of at least the gauge coupling cubed. This must be the case, as each contributing diagram necessarily includes three external gauge bosons. Similarly, every term in the Yukawa coupling beta functions contains at least a single factor of the relevant Yukawa coupling. As all the diagrams that contribute to the Yukawa coupling have a single external scalar, and Yukawa couplings are the only allowed couplings with an odd number of scalar particles, we can deduce that there must be at least a single Yukawa coupling in every diagram. However, it would seem possible to generate diagrams in which the single required Yukawa coupling is different from that of the external particle, such as that shown in Figure 2.5.

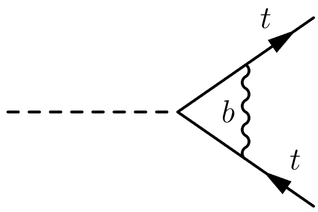
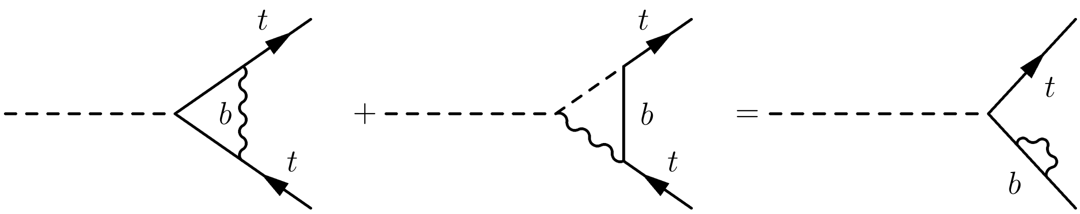


Figure 2.5.: Contributing diagram to the 1-loop top-Yukawa beta function, proportional to g_b.

That these diagrams do not contribute additive corrections to the beta function can be seen by using the result of gauge invariance expressed in (2.29).



(2.122)

Here, the first two diagrams are independent of the top Yukawa coupling, depending instead on the bottom Yukawa coupling. However, gauge invariance allows us to rewrite the sum of these two diagrams as the third diagram, which is proportional to the top Yukawa coupling, and independent of the bottom Yukawa coupling. Therefore, the net contribution of these three graphs must zero.

Having described the beta functions that we have used in this thesis, we can now discuss the details of the integration process in the next chapter.

Chapter 3.

Integration Method

In order to study the evolution of the couplings of a gauge theory, and in particular to check the positivity of the scalar self-coupling in the Standard Model, we first need to calculate its beta functions, and then numerically integrate them to high energy. That the scalar self-coupling should remain positive is a condition imposed by the requirement that the scalar potential must have a minimum at a finite energy in order for the theory to be well defined. A number of recent papers have discussed the vacuum stability of the Standard Model, including [23, 18, 24, 25, 26, 27], not all of whose results are in agreement, as well as studying the effect of adding right-handed neutrinos [28, 29] or a fourth generation of fermions [30, 15]. We have attempted to replicate this work both to verify that our beta function calculations are reliable before moving on to a more complex theory, and to try and understand the discrepancy between the published results. In this chapter we discuss the relevant details in the development of our Mathematica code, and then describe the derivation of the initial conditions used for the numerical integration.

3.1. Implementation

In order to automate the process of calculating beta functions from a given Lagrangian, and numerically evaluating the resultant set of coupled differential equations to obtain the coupling strengths at higher energy scales, a set of tools was developed in Mathematica, taking as input only the particle content, gauge structure and Lagrangian.

The key step in implementing the generalised framework for beta functions was making the transformation from the standard, more human-readable Lagrangians, to the more easily generalisable format introduced in Machacek and Vaughn [11, 13, 14]. Here we take arrays of all real scalars, ϕ_a, all distinguishable fermions, ψ_j and an array for the vector bosons of each simple gauge group, $V_{g,A}$. In addition to these general particle arrays, we require mass coupling tensors which describe all interactions between particles of given spin. This gives us 5 tensors; the triple gauge interaction tensor f^g_{ABC}, the gauge-fermion-fermion $t^{g,A}_{jk}$, gauge-scalar-scalar $\theta^{g,A}_{ab}$, the scalar-fermion-fermion Yukawa tensor Y^a_{jk} and the quartic scalar self-interaction λ_{abcd}.

Although, for the Standard Model, the number of nonzero entries in each tensor is relatively small, in general constructing these tensors by hand is time-consuming and error-prone — and for large numbers of particles, impractical. We therefore need a way to automate the generation of the interaction tensors. As an example, we can take the scalar self-interaction tensor for the Standard Model. At the Lagrangian level, we can write the 4-scalar interaction as simply $g_\lambda \left(\phi^\dagger\phi\right)^2$, which we can expand into real scalars with

$$\phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ H + i\phi_3 \end{pmatrix}. \tag{3.1}$$

Then, we can map the standard scalar names to indexed variables $\phi_1, \phi_2, \ldots$ — although this procedure is almost trivial in the case of the Standard Model, for models with multiple irreducible scalar representations, it allows for much easier manipulation of the resulting potential. Then, taking each term of the interaction Lagrangian individually, reading off the indices gives us a tensor entry, so $2g_\lambda \phi_1\phi_1\phi_2\phi_2$ corresponds to Λ_{1122}. Finally, remembering that the scalar self-interaction should be totally symmetric, we add a tensor entry for each permutation of the indices, with a value of the coupling divided by the number of permutations. So for our example we would end up with

$$\Lambda_{1122} = \Lambda_{1212} = \Lambda_{1221} = \Lambda_{2112} = \Lambda_{2121} = \Lambda_{2211} = \frac{g_\lambda}{3}. \tag{3.2}$$

Where possible, in writing the code for generating beta functions from the Lagrangian, we have tried to maintain the same form as the original generalisable equations given in the papers of Machacek and Vaughn, apart from the corrections already noted in the previous chapter. However, in two cases in the 2-loop gauge coupling, terms written in terms of the Casimir operators and Dynkin indices have been expanded to their full coupling matrix forms. These are the terms proportional to $C_2(F)\,S_2(F)$ and $C_2(S)\,S_2(S)$, which it is noted must be summed over irreducible representations of fermions and scalars respectively [11].

The first of these is the term describing diagrams of the form give in Figure 3.1

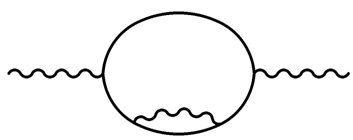


Figure 3.1.: A diagram contributing to the term proportional to $C(2)_F\,S(2)_F$.

which is given in the original paper [11] as $4g^4\kappa C_2(F)\,S_2(F)$. Writing the full product, with the tensor products over the suppressed fermion indices

$$4\kappa \sum_{i,j} \sum_{A_i,B_j} \frac{g_i^2 g_j^2}{d(G_i)} \operatorname{Tr}\left(t^{i,A} \cdot t^{i,A} \cdot t^{j,B} \cdot t^{j,B}\right) = 4\kappa \sum_{i,j} \sum_{A_i} g_i^2 g_j^2 \operatorname{Tr}\left(t^{i,A} \cdot t^{i,A} \cdot t^{j,1} \cdot t^{j,1}\right) \quad (3.3)$$

with the first sum over simple gauge groups, the second over vector bosons and $d(G_i)$ the dimension of the simple gauge group G_i, the number of gauge bosons associated with the group. Using this form we verified that we obtain the correct result for the Standard Model, and a consistent result for the Left-Right Symmetric Model — incautiously using the implicit summation gave a result which was asymmetric in left- and right-handed weak couplings, which obviously breaks the conserved left-right symmetry.

The second of these gives diagrams analogous to the first term, but with the fermion loop replaced with a scalar, given as $2C_2(F)\, S_2(F)$.

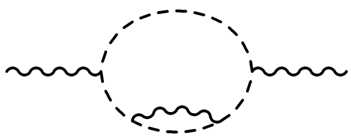


Figure 3.2.: A diagram contributing to the term proportional to $C(2)_S\, S(2)_S$.

Here the expansion gives us

$$2 \sum_{i,j} \sum_{A_i,B_j} \frac{g_i^2 g_j^2}{d(G_i)} \operatorname{Tr}\left(\theta_i^A \cdot \theta_i^A \cdot \theta_j^B \cdot \theta_j^B\right) = 2 \sum_{i,j} \sum_{A_i} g_i^2 g_j^2 \operatorname{Tr}\left(\theta_i^A \cdot \theta_i^A \cdot \theta_j^1 \cdot \theta_j^1\right). \quad (3.4)$$

3.2. Initial Conditions

The definition of consistent and accurate initial conditions is a key part of the process of numerical integration, and exploring the possible error based on the choice of initial conditions is an important guide to checking the reliability of your result. Indeed, with integration over such a wide range of energy values, the end result is potentially very sensitive to small variations in the initial conditions. Added to this is the complication that not all of the couplings have well-measured values at the same energy scale, and we therefore need to evolve a subset of the values from their initial energy to the energy at which other initial conditions are defined.

A further issue is that the renormalisation group equations for the coupling constants are calculated in $\overline{MS}$ scheme, whilst the Higgs and top couplings have initial conditions related to their masses, which are most commonly determined experimentally using their respective pole masses. To relate the $\overline{MS}$ coupling constants to the observed or postulated pole masses, we must make use of the matching equations [23]

$$g_t(M_t) = \frac{\sqrt{2}M_t}{v}\left(1+\delta_t(M_t)\right), \qquad g_\lambda(M_t) = \frac{M_H^2}{2v^2}\left(1+\delta_H(M_t)\right), \tag{3.5}$$

where the corrections are given, for the top mass, by [23]

$$\delta_t = \delta_t^{\rm QCD} + \delta_t^{\rm QED} + \delta_t^{\rm W}, \tag{3.6}$$

$$\delta_t^{\rm QCD} = -\frac{g_3^2(M_t)}{3\pi^2} - 9.1253\left(\frac{g_3^2(M_t)}{4\pi^2}\right)^2 - 80.4046\left(\frac{g_3^2(M_t)}{4\pi^2}\right)^3, \tag{3.7}$$

$$\begin{aligned}\delta_t^{\rm QED} + \delta_t^{\rm W} = {}& -4\frac{\widehat{\alpha}(M_t)}{9\pi} + \frac{M_t^2}{16\pi^2 v^2}\left[\frac{11}{2} - \frac{M_H^2}{4M_t^2} - 8\frac{M_H^4}{16M_t^4}\left(\frac{4M_t^2}{M_H^2}-1\right)^{\frac{3}{2}}\arccos\left(\frac{M_H}{2M_t}\right)\right.\\ &\left.+2\frac{M_H^2}{4M_t^2}\left(2\frac{M_H^2}{4M_T^2}-3\right)\log\frac{M_H^2}{M_t^2}\right] - 6.9\times 10^{-3} + 1.73\times 10^{-3}\log\frac{M_H}{300\ \text{GeV}}\\ &- 5.82\times 10^{-3}\log\frac{M_t}{175\ \text{GeV}},\end{aligned} \tag{3.8}$$

$$\widehat{\alpha}^{-1}(M_t) = 4\pi\frac{g_1^2(M_t)+g_2^2(M_t)}{g_1^2(M_t)\,g_2^2(M_t)}, \tag{3.9}$$

and for the Higgs mass by [23]

$$\delta_H(M_t) = \frac{M_Z^2}{32\pi^2 v^2}\left[\xi f_1(\xi) + f_0(\xi) + \xi^{-1}f_{-1}(\xi)\right], \tag{3.10}$$

$$\xi = \frac{M_H^2}{M_Z^2}, \tag{3.11}$$

$$f_1(\xi) = 6\log\frac{M_t^2}{M_H^2} + \frac{3}{2}\log\xi - \frac{1}{2}Z\left(\frac{1}{\xi}\right) - Z\left(c_W^2\right)\xi - \log c_W^2 + \frac{9}{2}\left(\frac{25}{9} - \frac{\pi}{\sqrt{3}}\right), \quad (3.12)$$

$$\begin{aligned} f_0(\xi) = & -6\log\frac{M_t^2}{M_Z^2}\left[1 + 2c_W^2 - 2\frac{M_t^2}{M_Z^2}\right] + \frac{3c_W^2\xi}{\xi - c_W^2}\log\frac{\xi}{c_W^2} + 2Z\left(\frac{1}{\xi}\right) \\ & + \left(\frac{3c_W^2}{s_W^2} + 12c_W^2\right)\log c_W^2 - \frac{15}{2}\left(1 + 2c_W^2\right) - 3\frac{M_t^2}{M_Z^2}\left[2Z\left(\frac{M_t^2}{M_Z^2\xi}\right)\right. \\ & \left. + 4\log\frac{M_t^2}{M_Z^2} - 5\right] + 4c_W^2 Z\left(\frac{c_W^2}{\xi}\right), \end{aligned} \quad (3.13)$$

$$\begin{aligned} f_{-1}(\xi) = & \ 6\log\frac{M_t^2}{M_Z^2}\left[1 + 2c_W^4 - 4\frac{M_t^4}{M_Z^4}\right] - 6Z\left(\frac{1}{\xi}\right) - 12c_W^4 Z\left(c_W^2\right)\xi - 12c_W^4\log c_W^2 \\ & + 24\frac{M_t^4}{M_Z^4}\left[\log\frac{M_t^2}{M_Z^2} - 2 + Z\left(\frac{M_t^2}{M_Z^2\xi}\right)\right] + 8\left(1 + 2c_W^4\right), \end{aligned} \quad (3.14)$$

$$Z(z) = \begin{cases} 2A\arctan\left(\dfrac{1}{A}\right) & \text{if } z > \dfrac{1}{4} \\ A\log\left(\dfrac{1 + A}{1 - A}\right) & \text{if } z < \dfrac{1}{4} \end{cases} \qquad A = \sqrt{|1 - 4z|} \quad (3.15)$$

where c_W and s_W are shorthands for the cosine and sine of the Weinberg angle respectively.

It is important to note here that the matching for the Higgs self-coupling is from the Higgs pole mass M_H to the $\overline{MS}$ value evaluated at the top pole mass, so that we may have as few as two energy scales at which initial conditions are defined for our integration. However, the decision of where to define the Higgs coupling is slightly more complicated. The condition we are interested in, that the Higgs coupling $g_\lambda \geqslant 0$ for all energies up to the Planck scale, can be bounded by setting $g_\lambda = 0$ at the Planck scale, and evaluating from here back to M_t in order to derive the pole mass of the Higgs. However, this requires the prior integration of all the other couplings of interest without knowing how g_λ varies with energy. For a two-loop calculation, this only affects the running of the Yukawa couplings directly, and so by first evaluating the remaining couplings up to M_{Planck} neglecting g_λ, and then using the resultant functions to integrate g_λ back down to M_t we can acquire an approximation to the true functions. Iterating the process allows us to refine the estimate until we obtain a sufficiently stable answer.

In practice, however, it is faster and simpler to perform the integration over all the coupling constants from M_t to M_{Planck}, with a range of fixed Higgs masses. With a range of top masses considered as well, the resultant values of the Higgs coupling at M_{Planck} can be interpolated to find the boundary of the region of viable parameter space. Using both methods to find allowed Higgs masses for the Standard Model at two loops with a top mass of 173.2 GeV gives identical results, confirming the validity of both methods.

Another possibility for determining the $\overline{MS}$ top mass is discussed in [27], which proposes using the inclusive cross-sections $\sigma(p\overline{p} \to t + X)$ and $\sigma(pp \to t + X)$ available from the Tevatron and the LHC respectively to determine the $\overline{MS}$ top mass by calculating the expected cross-section directly in $\overline{MS}$ scheme. Although the PDG includes an $\overline{MS}$ top mass, the errors on the measurement are much more significant than those of directly measured top mass. It should be noted that although using the directly measured value does introduce a second source of error via the perturbative matching equation, this correction is small.

In our basic case, the Standard Model, we have five coupling constants; the three gauge couplings $g_{1,2,3}$, the top Yukawa coupling g_t and the Higgs self-coupling g_λ. The remaining Yukawa couplings are much smaller than these five couplings, and so we can neglect them, as the leading order corrections they introduce are $\mathcal{O}\left(\frac{M_b^2}{M_t^2}\right)$. The three gauge constants are well-measured in $\overline{MS}$ at the Z pole, M_Z, while the top and Higgs masses that would be measured are easier to relate to the value of the $\overline{MS}$ coupling at the top pole mass M_t. In order to define a set of initial conditions at a single energy, we will evolve the gauge couplings from the Z mass to the top mass with a couple of simplifying assumptions.

The first of these is that below the top pole mass, we neglect the existence of the top quark. The most obvious consequence of this is that the top Yukawa coupling, which appears from two-loop order, is set to zero. In addition to this, we need to adjust the value of n_G, the number of generations, since it is neither 2 or 3, and indeed the variation of the effective number of generations depends on the bosons coupling to the fermion loop. We can compensate for this by multiplying each n_G-dependent term by a factor R whose value depends on the gauge couplings involved, and leaving the number of generations as 3.

These R values are quoted in Table 3.1 below, where the columns refer to the couplings to the fermion loop present in the diagram, and the rows to the specific pair of couplings we are considering. Their derivation is given in equation (3.16), with an example in (3.17).

Table 3.1.: Ratios for five-quark renormalisation group equations

R	g_1 only	g_2, not g_3	g_3, not g_2	g_2 and g_3
g_1^2	$\frac{23}{24}$	$\frac{11}{12}$	$\frac{5}{6}$	—
g_2^2	—	$\frac{3}{4}$	—	$\frac{2}{3}$
g_3^2	—	—	$\frac{5}{6}$	$\frac{2}{3}$

Ratios, R, that should be combined multiplicatively with factors of the gauge couplings squared to give renormalisation group equations for a model with only five quark flavors. The first column of ratios gives values for terms that include only g_1, the second for terms including g_2 but not g_3, the third for terms including g_3 but not g_2, and the final column gives the ratios when a term includes powers of both g_2 and g_3.

This is viable to two-loop order as no diagram will involve more than a single fermion loop, and we only need concern ourselves with adjusting the gauge beta functions, since we are not evolving the top Yukawa or the Higgs self-coupling at this point. Three loops are a problem because the same term in the beta function can describe several different graphs that should be treated differently — for example if we look at the term $g_1^3 g_2^4$ with a single factor of n_G, two of the graphs contributing to this are given in Figure 3.3.

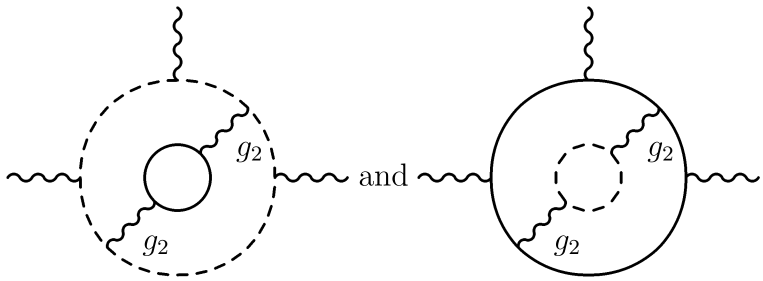


Figure 3.3.: 3-loop diagrams contributing to the same term, but receiving different corrections with only 5 quarks.

For the first of these diagrams, we have only a single pair of SU(2) interactions with the fermion loop, whereas the second has an additional pair of U(1) interactions. The two different diagrams would need to be adjusted by different amounts, but since we do not know in what proportion these diagrams contribute, we cannot be certain of the correct value without recalculating the term including only the first 5 quarks. So in the cases of the two diagrams above, we have only g_2 couplings in the first diagram, and g_1 and g_2 couplings in the second diagram. We therefore take values from the second column, as the presence of a g_1 coupling is less restrictive than either a g_2 or g_3 coupling. The first diagram has only a single pair of g_2 couplings so we would multiply the term it describes by $\frac{3}{4}$. For the second diagram, we have an additional factor of $\frac{11}{12}$ from the pair of g_1 couplings, and so the total multiplicative factor would be $\frac{11}{16}$. We do not need to concern ourselves with the additional single g_1 coupling since, as shown in [31], the gauge beta function is simply the gauge wavefunction renormalisation equation multiplied by the relevant gauge coupling.

The further approximation has been made here that the CKM matrix element for top and bottom is 1, while the mixing with the other quarks is 0, so that the bottom quark is excluded from all fermion loops coupled to an SU(2) boson. The ratios, R are obtained via

$$R(g_i) = \frac{\sum\limits_{\text{particles}} Q_i^2 n_C - \sum\limits_{\text{exclusions}} Q_i^2 n_C}{\sum\limits_{\text{particles}} Q_i^2 n_C}, \tag{3.16}$$

where the Q_i are the charges of the particle associated with the coupling in question, and n_C is the color factor. So in the case of the factor of $\frac{3}{4}$ stated earlier for the coupling g_2^2 in the absence of g_3 interactions, we have

$$R(g_2) = \frac{3\sum\limits_{\text{quarks}} \left(\frac{1}{2}\right)^2 + \sum\limits_{\text{leptons}} \left(\frac{1}{2}\right)^2 - 3\sum\limits_{t,b} \left(\frac{1}{2}\right)^2}{3\sum\limits_{\text{quarks}} \left(\frac{1}{2}\right)^2 + \sum\limits_{\text{leptons}} \left(\frac{1}{2}\right)^2} = \frac{3\frac{6}{4} + \frac{6}{4} - 3\frac{2}{4}}{3\frac{6}{4} + \frac{6}{4}} = \frac{3}{4}. \tag{3.17}$$

A further factor to consider is the Higgs self-coupling strength between M_Z and M_t. However, the coupling only makes an appearance at the 3-loop level in the g_1 and g_2 renormalisation group equations, and not until the 4-loop level for g_3, which correction has not yet been calculated. The first approximation is taken that the self-coupling does not vary between the two energies, and is therefore equal to the value at M_t. We can then use the resultant set of initial conditions to integrate the beta functions from M_t to M_Z iteratively until the values converge. In practice, the convergence is very swift, with only a couple of iterations needed to converge on a result with smaller difference than the error in the relevant observable.

In Table 3.2 we summarise the initial conditions used, for three different cases: the values we used at M_Z and M_t, and the values used at M_t in reference [18].

Table 3.2.: Initial conditions

Variable	M_Z	M_t	M_t for ref [18]
g_1	0.357458	0.358705	0.358729
g_2	0.651908	0.647267	0.648382
g_3	1.21978	1.16506	1.16471
g_t	—	0.937708	0.937936
g_λ	—	0.130451	0.129876

Initial conditions for the five couplings of the Standard Model. The first row of values is for the gauge couplings at the Z pole mass, the second column is for all the couplings at the top pole mass, and the third column is for the values used in [18] at the top pole mass.

The values of $g_1(M_Z)$ and $g_2(M_Z)$ are derived from the values for $\widehat{\alpha}(M_Z)$ and $\sin^2 \widehat{\theta_W}(M_Z)$ given in reference [23], while $g_3(M_Z) = \sqrt{4\pi\alpha_s(M_Z)}$, also given in reference [23]. g_t and g_λ are calculated assuming pole masses of $M_t = 172.9$ GeV and $M_H = 126$ GeV.

For additional physics beyond the Standard Model, similar approximations can be made, evaluating the relevant Standard Model couplings from M_t up to a new mass scale, M_{New}, at which the new physics effects become relevant. Examples of BSM models where such a simplified approach would be useful include massive (seesaw) right-handed neutrinos, or a fourth generation of fermions. Clearly, additions such as these are easiest to implement where the new particles are roughly mass-degenerate, so that only a single additional mass scale is needed, and where the new couplings are analogous to existing couplings in the Standard Model, simplifying the calculation of new renormalisation group equations.

With these difficulties considered, we can now perform the integration of the beta functions for a variety of models. We start with the Standard Model, as both the most straightforward test, and also with the widest range of papers with which to compare our results. We also look at the possible variation in results due to the choice of initial conditions.

Chapter 4.

Evolving the Standard Model to High Energy

The most immediate, and simplest test to implement, is to calculate the beta functions for the Standard Model, and compare the results with those from [23, 18, 29, 24] under the assumption that there is no new physics before the Planck scale. In addition, we look at the effects of varying the initial conditions, to study whether it is possible for the scalar self-coupling to remain positive all the way to the Planck scale.

4.1. Results

Presented in figure 4.1 are the results for the running with three loops in the gauge couplings in $\overline{\text{MS}}$, and the partial three loop results for the top Yukawa and Higgs self coupling, as discussed earlier. The Higgs pole mass is given as 126 GeV, the top pole mass as 173.2 GeV and $\alpha_S(M_Z)$ as 0.1184. The generic features are familiar, in particular, the near convergence of the three gauge couplings around the GUT scale. Clearly, the only coupling to cross zero is the Higgs self coupling, and so the majority of the plots in

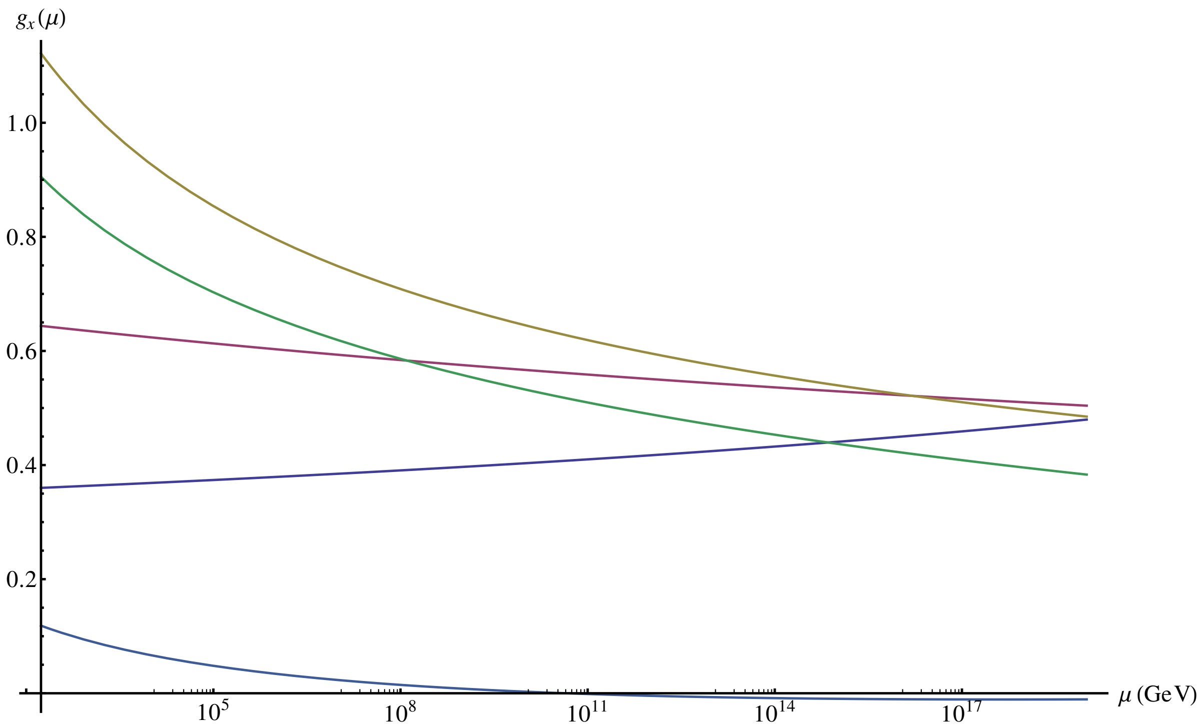


Figure 4.1.: The five coupling constants of interest in the Standard Model - g_1 (blue), g_2 (purple), g_3 (yellow), g_t (green) and g_λ (cyan). The vertical axis is at $\mu = M_t$

this section will focus on its behaviour. We also plot the value of the scalar self coupling, with the beta functions taken at one, two and three loops in figure **??** to show that the inclusion of additional loops does cause the result to converge.

The five figures 4.3-4.7 plot the difference in the Higgs self coupling with the variation of the five most relevant quantities with significant errors, the masses of the Z, the top quark and the Higgs, the strength of the strong coupling α_S and the weak mixing angle $\sin\theta_W$.

Clearly, the errors on $\sin\theta_W$ and m_Z are far less significant to the evolution of the Higgs self coupling, whilst the greatest variation is with the value of M_t. Figure 4.8 takes the maximum variation from the central value with 1σ variations in M_H, M_t and α_S, demonstrating that the Higgs coupling can remain stable up to the Planck scale within $\approx 2\sigma$ of the best fit to the Standard Model.

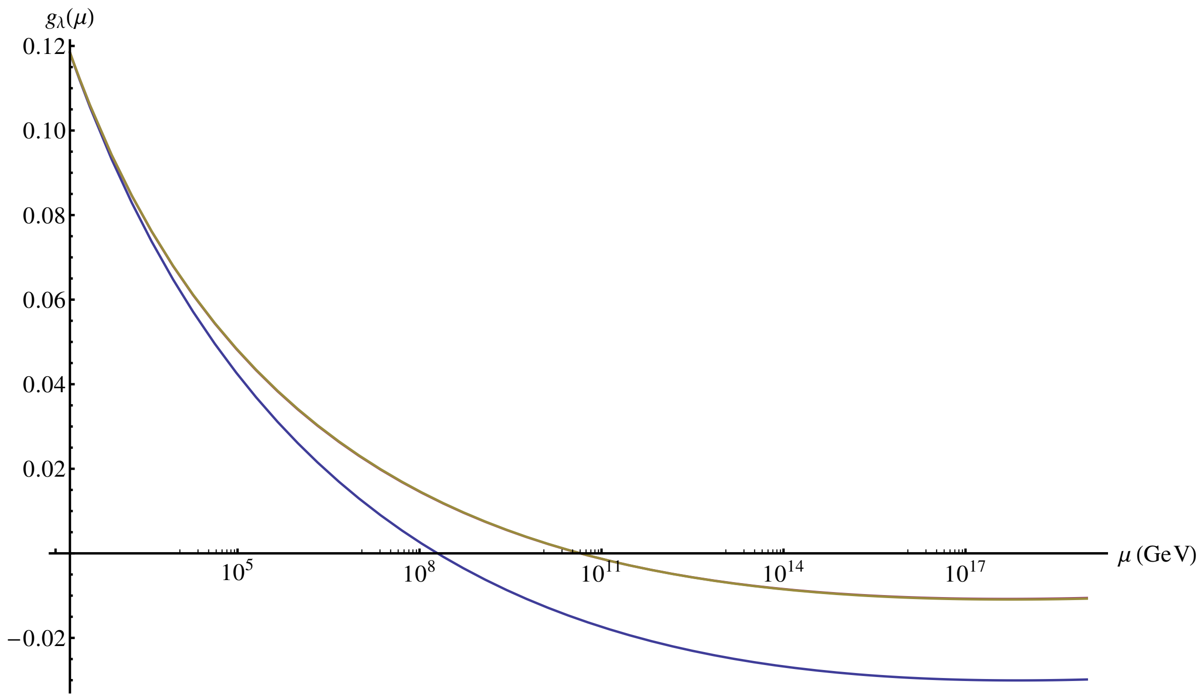


Figure 4.2.: The evolution of g_λ with beta functions truncated at one (blue), two (magenta) and three (yellow) loops.

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As a means of direct comparison, the authors of reference [18], Chetyrkin and Zoller, shared their integration code with us, so that a more precise analysis of results could be obtained. Taking their calculated result, our calculated result and the result of using our integration procedure combined with their initial conditions, which were derived from a Mathematica module developed by the authors of [25], allowed us to identify a number of minor errors in our own code, as a result of the inconsistencies in the literature mentioned in Chapter 2. Correcting those differences and recalculating confirmed that the two calculations were identical, and that the largest discrepancy in our final results was due to the choice of initial conditions. In Figure 4.9 we plot the results of integrating using initial conditions derived in the two methods for both $M_H = 124$ and 126 GeV. The plain lines are the results using initial conditions matching those in [18], while the dashed lines give the results of our own initial conditions.

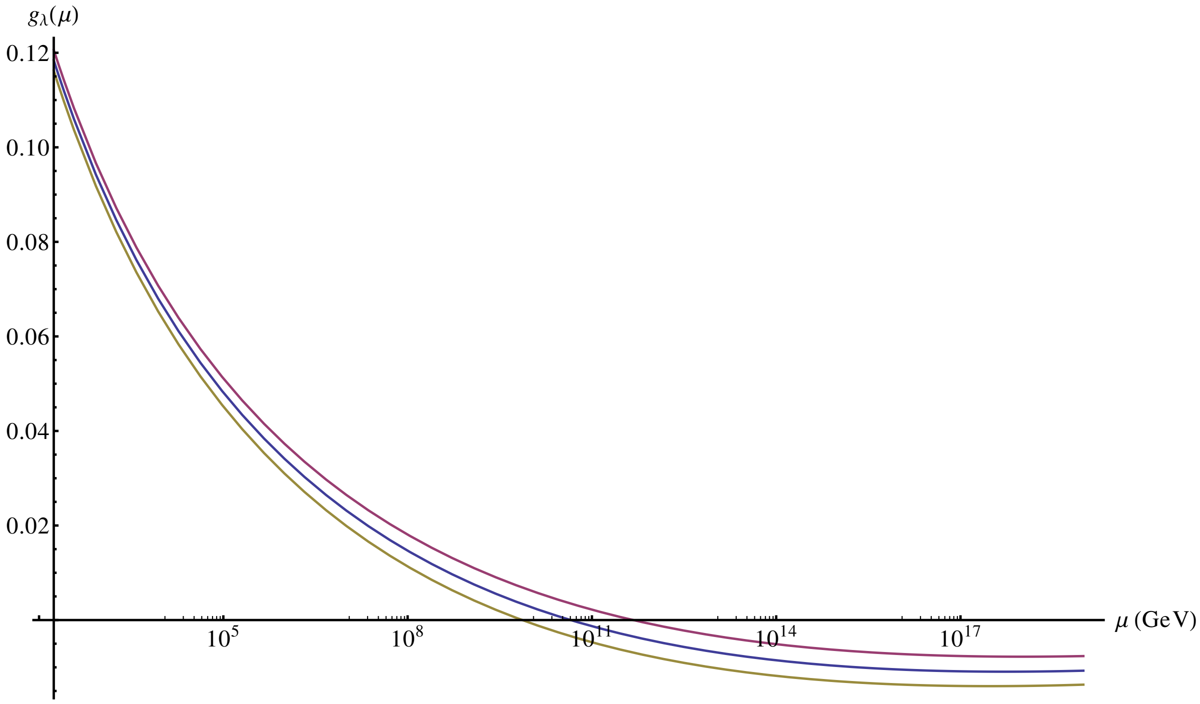


Figure 4.3.: Variation of g_λ with $M_H \pm 1$ GeV.

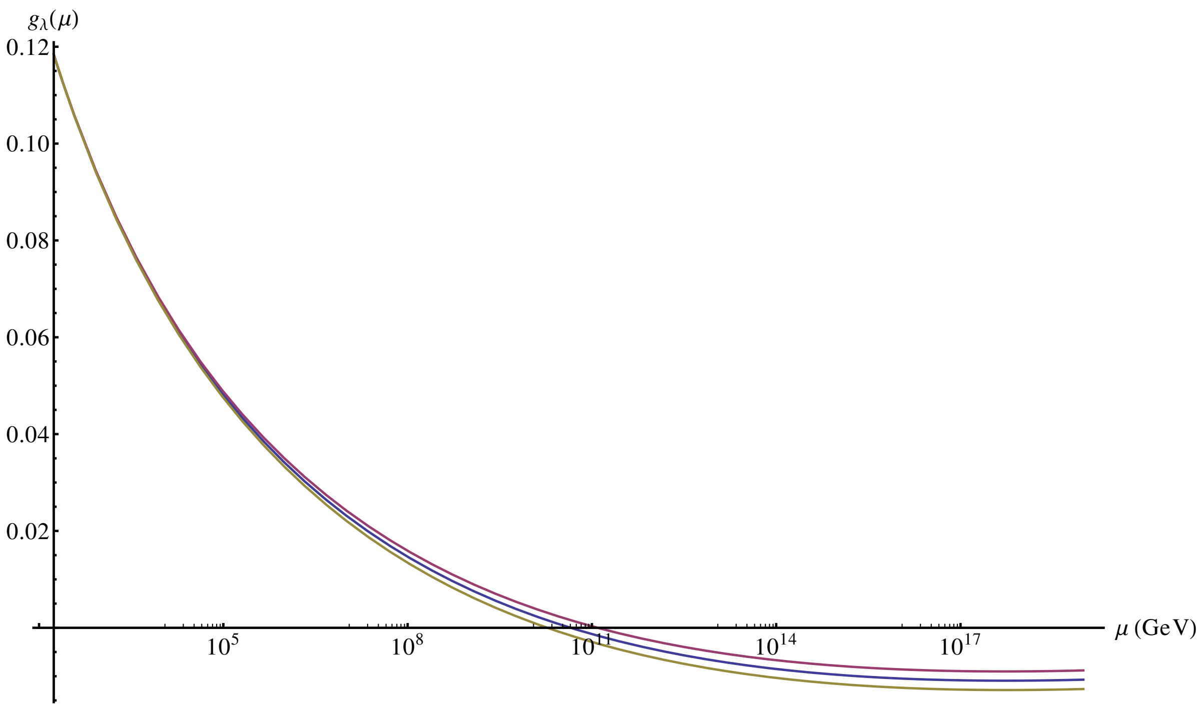


Figure 4.4.: Variation of g_λ with $\alpha_S \pm 1\sigma$.

With regards to other sources, we were in agreement with [23] from a very early stage, as their paper was the model for the development of our code, while we suspect that

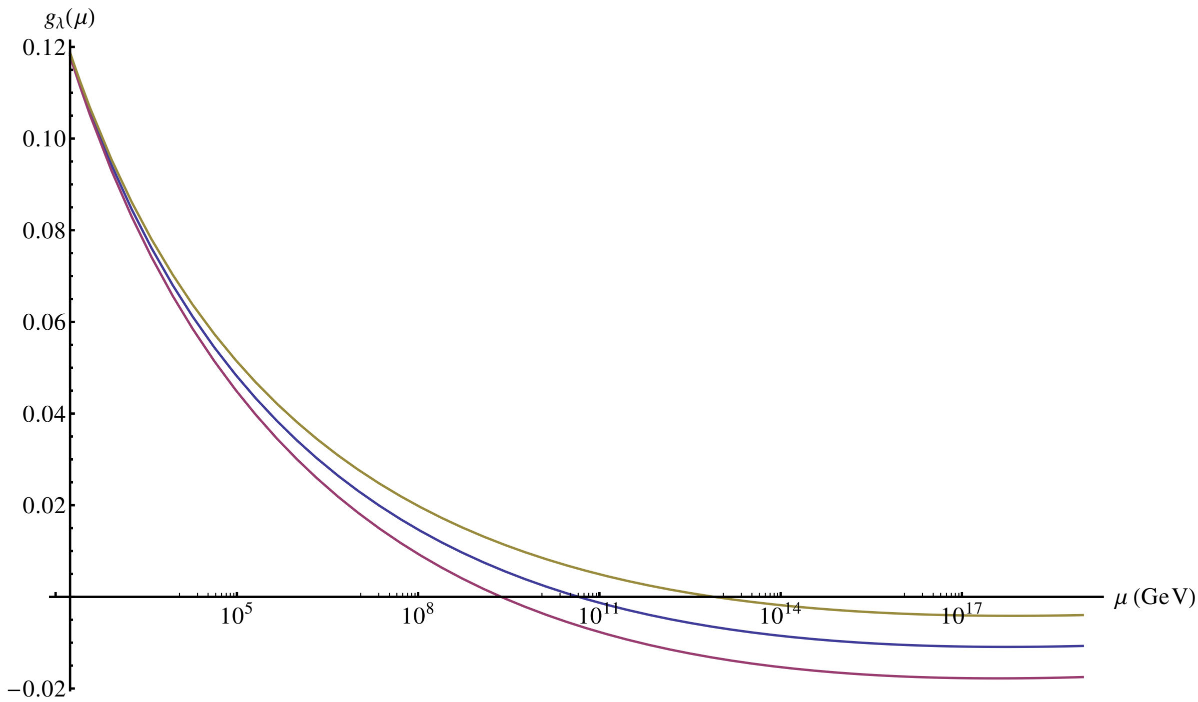


Figure 4.5.: Variation of g_λ with $M_t \pm 1\sigma$.

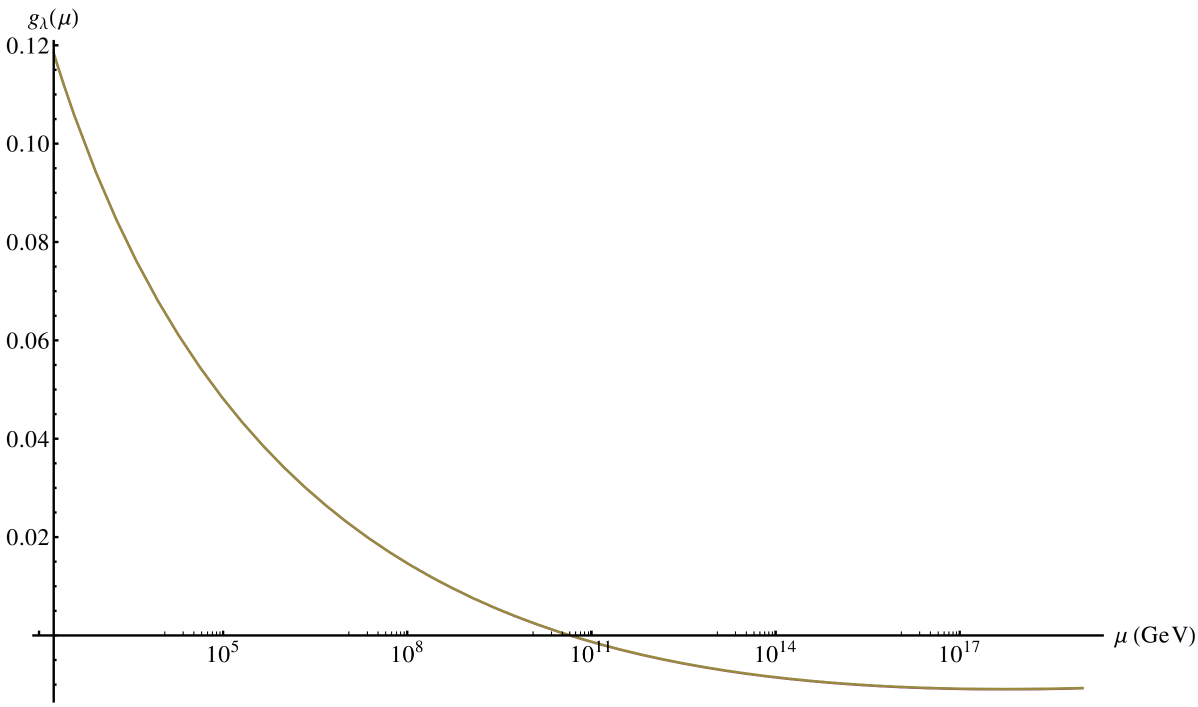


Figure 4.6.: Variation of g_λ with $\sin\theta_W \pm 1\sigma$.

the discrepancies with [24] are due to the choice of initial conditions in that paper. The results in [32] are presented in terms of the scale at which the Higgs coupling becomes

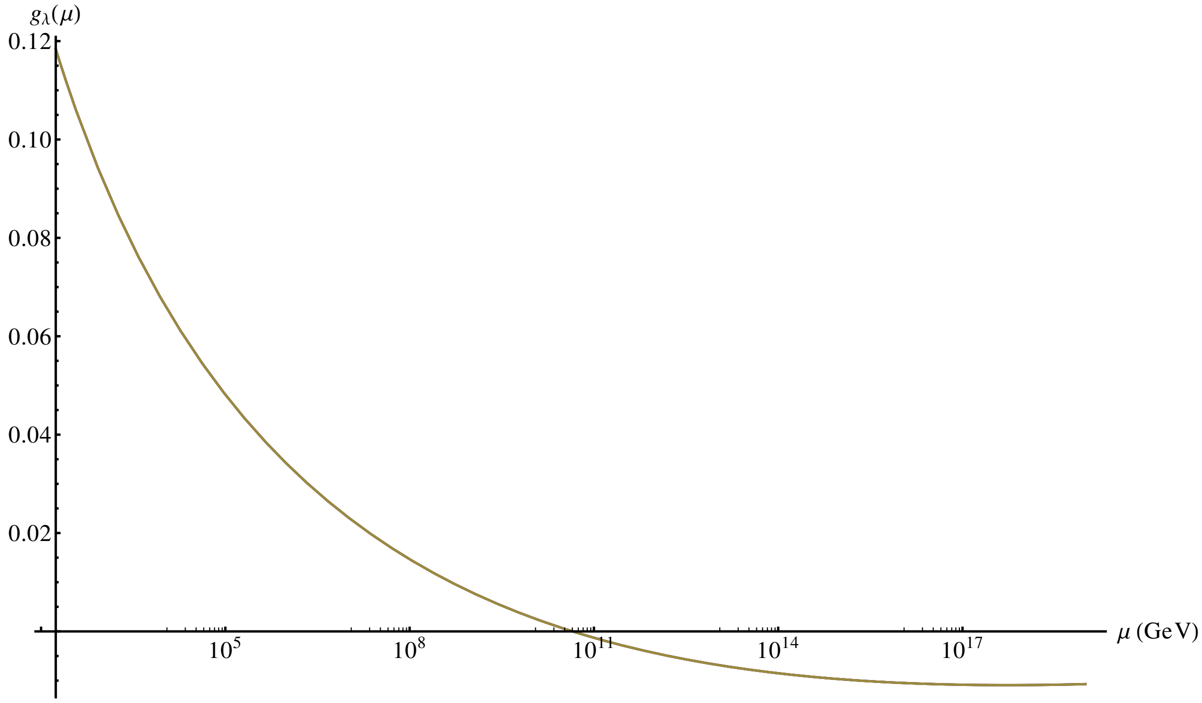


Figure 4.7.: Variation of g_λ with $M_Z \pm 1\sigma$.

negative, with varying M_H and M_t, making a comparison of their evolution difficult. However, their conclusions seem to be in agreement with those of [24] and [18] and within the variation expected from a differing choice of initial conditions.

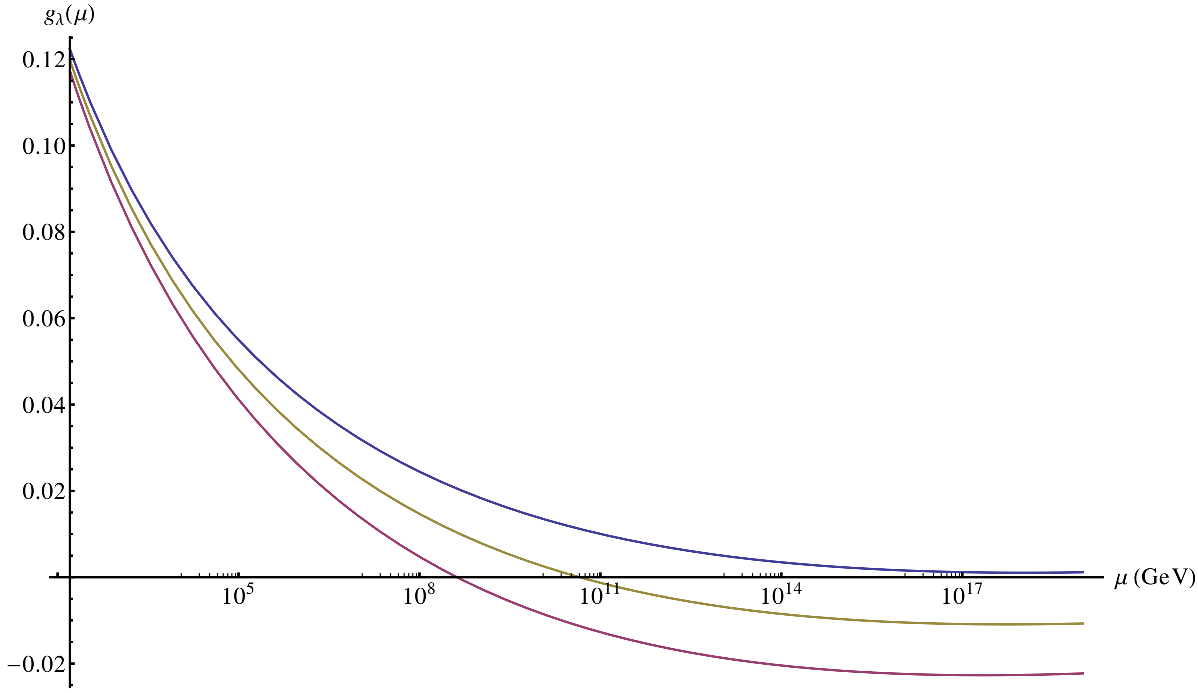


Figure 4.8.: Maximum deviation from central values with 1σ errors. Errors on $\sin\theta_W$ and M_Z are neglected due to their minimal effect.

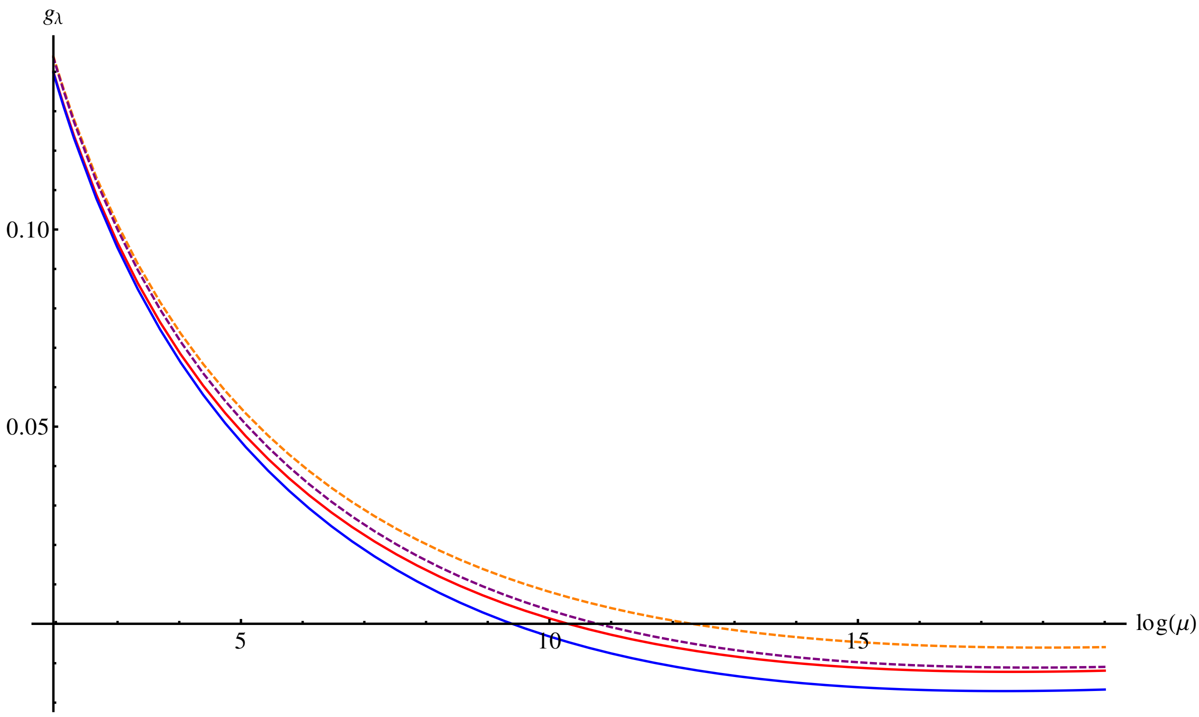


Figure 4.9.: g_λ for 4 sets of initial conditions — red and orange lines are for 126 GeV Higgs mass, while purple and blue are for 124 GeV. Dashed lines give the results of our initial conditions, plain lines the results of

Chapter 5.

Right-Handed Neutrinos

One of the few areas in which the effects of physics not described by the Standard Model is observed is in the neutrino sector. In the Standard Model, neutrinos are only left-handed, with no right-handed counterpart, and consequently no mass term or mixing of generations. However, clear evidence is seen for the existence of neutrino generation-changing processes, in both the observed solar neutrino flux as well as neutrino experiments using relatively short baseline sources. As the transition amplitude for neutrino generation-changing is proportional to the difference in mass of the generations, at least 2 of the neutrinos must have non-zero masses, though it is generally assumed that all three are massive. Adding a right-handed neutrino sector allows for both neutrino masses and generation-changing interactions, via a PMNS matrix.

The limits on neutrino masses from beta decay and cosmological measurements place them at a mass scale several orders of magnitude lower than any other fermion. If the mass of observed neutrinos is due solely to Yukawa couplings to the Higgs, we would expect their masses to be of similar size to those of the leptons. The Seesaw mechanism proposes a method where, by including Majorana mixing terms, the observed left-handed neutrino masses can be much lower, with correspondingly more massive right-handed neutrinos.

5.1. Theory

The simplest possible extension to the Standard Model which provides an explanation for the observed flavour changing among neutrinos is to simply add three SU(3)⊗SU(2)⊗U(1) singlet right-handed neutrinos, whose only interaction with the rest of the Standard Model is to the Higgs via Yukawa terms. As these particles are uncharged, we can also write a Majorana mass term — which requires the neutrino to be its own antiparticle, and hence must be uncharged with respect to all of the gauge groups. Taking all of this into consideration, the additional terms in our Lagrangian are

$$\mathcal{L}_\nu = -\frac{(M_N)^{ij}}{2}\overline{N}^i_{\mathrm{R}} N^{Cj}_{\mathrm{R}} - Y^{ia}_N \overline{N}^i_{\mathrm{R}} \phi^\dagger l^a_{\mathrm{L}} + h.c., \tag{5.1}$$

with N_{R} the right-handed neutrino spinor fields, and M_N the Majorana mass. As with the other Standard Model fermions, the neutrinos acquire a Dirac mass term from the vacuum expectation value v of the Higgs in the second term of (5.1). In the absence of the Majorana mass term, the Dirac mass $m_D = Y_N v$ would have to be significantly smaller than the other fermion masses in order to conform to experimental mass limits, giving us a theoretically unmotivated hierarchy. On the other hand, if we include a Majorana mass term, the mass matrix for the neutrinos in the basis $(\nu_{\mathrm{L}}, N_{\mathrm{R}})$ is

$$\begin{pmatrix} 0 & m_D^T \\ m_D & M_N \end{pmatrix}, \tag{5.2}$$

which we can diagonalise to a light neutrino - heavy neutrino basis as

$$U_N^T \begin{pmatrix} 0 & m_D^T \\ m_D & M_N \end{pmatrix} U_N = \begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix}, \tag{5.3}$$

where m and M are diagonal 3x3 matrices in neutrino flavor [33]. The rotation matrix U_N can be written in terms of two separate transformations,

$$U_N = \left[\exp \begin{pmatrix} 0 & \Theta \\ -\Theta^\dagger & 0 \end{pmatrix} \right] \begin{pmatrix} U & 0 \\ 0 & U' \end{pmatrix}, \tag{5.4}$$

of which the first, exponential part, block-diagonalises the masses, and U and U' are unitary matrices diagonalising the mass matrices for the light and heavy neutrinos respectively. By taking the seesaw limit, $M_N \gg m_D$, we obtain the relation $\Theta \approx m_D^\dagger M_N^{-1}$ and consequently $M_N \approx M$. We then see that this implies that

$$U^* m U^\dagger = -m_D^T M^{-1} m_D, \tag{5.5}$$

and that we do, as stated above, have three light and three heavy neutrinos as a result, without assuming $m_D \ll m_e$.

For the neutrino Yukawa coupling matrix we use the definition given in [34], where

$$Y_\nu = \frac{1}{v} \sqrt{M_\nu} R \sqrt{m_\nu} U^\dagger, \tag{5.6}$$

with M_ν and m_ν the diagonal mass matrices for light and heavy neutrinos respectively, U a unitary PMNS matrix [35, 36] diagonalising the light masses, and R a complex orthogonal matrix written as

$$R = Oe^{iA}, \qquad A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \tag{5.7}$$

where both O and A are real matrices.

With such a minimal interaction with the rest of the Standard Model, it is not surprising that the changes to the beta functions other than the scalar self-coupling are also small. We summarise here the corrections to the beta functions of the Standard Model due to right-handed neutrinos as $(\beta_X)_{\text{Full}} = (\beta_X)_{\text{SM}} + \beta_{X;\nu}$. The U(1) and SU(2) gauge couplings receive corrections only at the two-loop level, and the SU(3) gauge coupling is unchanged until the three-loop level [30].

$$\beta_{1,2;\nu}|_{\text{2-loop}} = -\frac{1}{2} g_{1,2}^3 \operatorname{Tr} Y_\nu^\dagger Y_\nu. \tag{5.8}$$

The beta functions for the quark Yukawa couplings receive somewhat more substantial modifications

$$\beta_{t,b;\nu}|_{\text{1-loop}} = Y_{t,b} \operatorname{Tr}\left(Y_\nu^\dagger Y_\nu\right), \tag{5.9}$$

$$\begin{aligned} \beta_{t,b;\nu}|_{\text{2-loop}} = Y_{t,b} \Big[& -\frac{9}{4} Y_{t,b}^\dagger Y_{t,b} \operatorname{Tr}\left(Y_\nu^\dagger Y_\nu\right) + \frac{5}{4} Y_{b,t}^\dagger Y_{b,t} \operatorname{Tr}\left(Y_\nu^\dagger Y_\nu\right) - \frac{9}{4} \operatorname{Tr}\left(Y_\nu^\dagger Y_\nu Y_\nu^\dagger Y_\nu\right) \\ & -\frac{1}{2} \operatorname{Tr}\left(Y_\nu^\dagger Y_\nu Y_\tau^\dagger Y_\tau\right) + \frac{5}{8} g_1^2 \operatorname{Tr}\left(Y_\nu^\dagger Y_\nu\right) + \frac{15}{8} g_2^2 \operatorname{Tr}\left(Y_\nu^\dagger Y_\nu\right) \Big], \end{aligned} \tag{5.10}$$

while the lepton Yukawa coupling receives additional changes

$$\beta_{\tau;\nu}|_{\text{1-loop}} = Y_\tau \left[-\frac{3}{2} Y_\nu^\dagger Y_\nu + \operatorname{Tr}\left(Y_\nu^\dagger Y_\nu\right) \right], \tag{5.11}$$

$$\beta_{\tau;\nu}|_{\text{2-loop}} = Y_\tau \Big[-Y_\tau^\dagger Y_\tau Y_\nu^\dagger Y_\nu - \frac{1}{4} Y_\nu^\dagger Y_\nu Y_\tau^\dagger Y_\tau + \frac{11}{4} \left(Y_\nu^\dagger Y_\nu\right)^2$$

$$+\frac{5}{4}Y_2(S)\,Y_\nu^\dagger Y_\nu - 4g_\lambda Y_\nu^\dagger Y_\nu - \frac{45}{16}g_1^2 Y_\nu^\dagger Y_\nu + \frac{9}{16}g_2^2 Y_\nu^\dagger Y_\nu\Big]\,. \tag{5.12}$$

The scalar self-coupling is changed with the addition of two terms due to Dirac neutrino loops

$$\beta_{\lambda;\nu}|_{\text{1-loop}} = 4g_\lambda \operatorname{Tr}\big(Y_\nu^\dagger Y_\nu\big) - 2\operatorname{Tr}\big(Y_\nu Y_\nu^\dagger Y_\nu Y_\nu^\dagger\big)\,, \tag{5.13}$$

$$\begin{aligned}\beta_{\lambda;\nu}|_{\text{2-loop}} = &-\frac{1}{4}g_1^4 \operatorname{Tr}\big(Y_\nu^\dagger Y_\nu\big) - \frac{1}{2}g_1^2 g_2^2 \operatorname{Tr}\big(Y_\nu^\dagger Y_\nu\big) - \frac{3}{4}\operatorname{Tr}\big(Y_\nu^\dagger Y_\nu\big) + \frac{5}{2}g_1^2 g_\lambda \operatorname{Tr}\big(Y_\nu^\dagger Y_\nu\big)\\ &+ \frac{15}{2}g_2^2 g_\lambda \operatorname{Tr}\big(Y_\nu^\dagger Y_\nu\big) - 48g_\lambda^2 \operatorname{Tr}\big(Y_\nu^\dagger Y_\nu\big) - g_\lambda \operatorname{Tr}\big(Y_\nu^\dagger Y_\nu Y_\nu^\dagger Y_\nu\big) + 2g_\lambda \operatorname{Tr}\big(Y_\nu^\dagger Y_\nu Y_\tau^\dagger Y_\tau\big)\\ &+ 10\operatorname{Tr}\big(Y_\nu^\dagger Y_\nu Y_\nu^\dagger Y_\nu Y_\nu^\dagger Y_\nu\big) - 2\operatorname{Tr}\big(Y_\nu^\dagger Y_\nu\,\big(Y_\nu^\dagger Y_\nu + Y_\tau^\dagger Y_\tau\big)\,Y_\tau^\dagger Y_\tau\big)\,.\end{aligned} \tag{5.14}$$

We note that although the authors of [28] state that the induced changes in the invariants $Y_2(S)$, $Y_4(S)$ and $X_4(S)$ are omitted, it appears that they are referring only to those places where the invariants appear in their given beta functions, which do not include all the instances in which they appear in the beta functions given in Chapter 2 — for example the terms proportional to $g_1^2 \operatorname{Tr}\big(Y_\nu^\dagger Y_\nu\big)$ and $g_2^2 \operatorname{Tr}\big(Y_\nu^\dagger Y_\nu\big)$ in the quark Yukawa beta functions are exactly those that would be due to $Y_4(S)$. To reduce the risk of transcription errors, we have retained their usage for these additional terms. We have also made some necessary changes due to their differing definition of $g'_\lambda \to 2g_\lambda$ and $g'_1 \to \sqrt{\frac{5}{3}}g_1$. We should also consider the evolution of the neutrino Yukawa coupling, given by

$$\beta_\nu|_{\text{1-loop}} = Y_\nu\left[\frac{3}{2}Y_\nu^\dagger Y_\nu - \frac{3}{2}Y_\tau^\dagger Y_\tau + Y_2(S) - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2\right], \tag{5.15}$$

$$\begin{aligned}\beta_\nu|_{\text{2-loop}} = Y_\nu\Big[&\frac{3}{2}Y_\nu^\dagger Y_\nu Y_\nu^\dagger Y_\nu - Y_\nu^\dagger Y_\nu Y_\tau^\dagger Y_\tau - \frac{1}{4}Y_\tau^\dagger Y_\tau Y_\nu^\dagger Y_\nu + \frac{11}{4}Y_\tau^\dagger Y_\tau Y_\tau^\dagger Y_\tau\\ &- \frac{9}{4}Y_2(S)\,Y_\nu^\dagger Y_\nu + \frac{5}{4}Y_2(S)\,Y_\tau^\dagger Y_\tau - X_4(S) + 6g_\lambda^2 - 12g_\lambda Y_\nu^\dagger Y_\nu - 4g_\lambda Y_\tau^\dagger Y_\tau\\ &+ \frac{93}{16}g_1^2 Y_\nu^\dagger Y_\nu + \frac{135}{16}g_2^2 Y_\nu^\dagger Y_\nu - \frac{81}{16}g_1^2 Y_\tau^\dagger Y_\tau + \frac{9}{16}g_2^2 Y_\tau^\dagger Y_\tau + \frac{5}{2}Y_4(S) + \frac{35}{24}g_1^4\\ &- \frac{9}{4}g_1^2 g_2^2 - \frac{23}{4}g_2^4\Big]\,.\end{aligned} \tag{5.16}$$

Finally, the new values of $Y_2(S)$, $Y_4(S)$ and $X_4(S)$ are given by

$$\begin{aligned}
Y_2(S) &= \mathrm{Tr}\Big(3Y_t^\dagger Y_t + 3Y_b^\dagger Y_b + Y_\tau^\dagger Y_\tau + Y_\nu^\dagger Y_\nu\Big)\,, && (5.17)\\
Y_4(S) &= \frac{9}{4}\,\mathrm{Tr}\Bigg(3\left(Y_t^\dagger Y_t\right)^2 + 3\left(Y_b^\dagger Y_b\right)^2 + \left(Y_\nu^\dagger Y_\nu\right)^2 + \left(Y_\tau^\dagger Y_\tau\right)^2 - \frac{2}{3}Y_t^\dagger Y_t Y_b^\dagger Y_b \\
&\quad -\frac{2}{9}Y_\nu^\dagger Y_\nu Y_\tau^\dagger Y_\tau\Bigg)\,, && (5.18)\\
X_4(S) &= \mathrm{Tr}\Bigg(\left(\frac{17}{12}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right)Y_t^\dagger Y_t + \left(\frac{5}{12}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right)Y_b^\dagger Y_b \\
&\quad + \left(\frac{1}{4}g_1^2 + \frac{3}{4}g_2^2\right)Y_\nu^\dagger Y_\nu + \left(\frac{3}{4}g_1^2 + \frac{3}{4}g_2^2\right)Y_\tau^\dagger Y_\tau\Bigg)\,. && (5.19)
\end{aligned}$$

5.2. Results

In the following, we neglect the effect of the bottom and τ Yukawa couplings, as their effect is negligible in comparison with the much larger neutrino and top Yukawa couplings. We are now only left with the matter of the initial conditions to use. In the interest of simplicity, we make the assumption that the heavy-neutrino and light-neutrino masses are approximately degenerate

$$M_1 \approx M_2 \approx M_3 \approx M_0, \tag{5.20}$$

$$m_1 \approx m_2 \approx m_3 \approx m_0. \tag{5.21}$$

This allows us to consistently introduce all of the new neutrino terms at a single new physics scale, as well as only requiring us to calculate the running of a single new Yukawa coupling. We can also reduce the common expression

$$\mathrm{Tr}\Big(\left(Y_\nu^\dagger Y_\nu\right)^n\Big) \to 3g_\nu^{2n}. \tag{5.22}$$

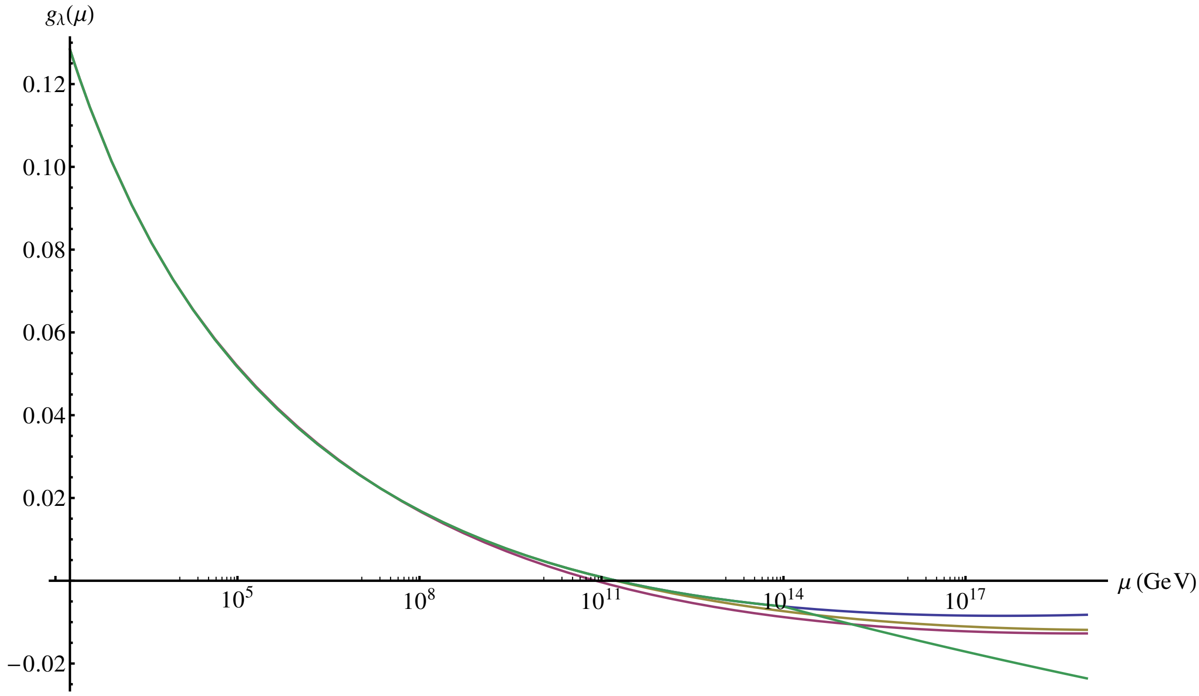


Figure 5.1.: The Higgs self coupling g_λ with the introduction of massive right handed neutrinos at $M_0 = 10^4, 10^{11}, 10^{14}$ GeV and $a = 6.5$, 2 and 0 in magenta, yellow and green respectively. The Standard Model result is plotted in blue. The vertical axis is at $\mu = M_t$

The neutrino Yukawa coupling can also be given a simpler form in this case, and is related to the light and heavy neutrino masses by

$$\mathrm{Tr}\left(Y_\nu^\dagger Y_\nu\right) = \frac{M_0 m_0}{v^2}\left(1 + 2\cosh(r)\right), \tag{5.23}$$

with r related to the real constants in the matrix A (5.7) as $r = 2\sqrt{a^2 + b^2 + c^2}$. We shall make a further simplification by taking the three constants a, b and c to be approximately the same size, so that $r = 2\sqrt{3}a$. We also neglect all other Yukawa couplings bar the top Yukawa, and adopt the values for a given in [28], $a = 6.5$, 2 and 0 for $M_0 = 10^4$, 10^{11} and 10^{14} respectively, as well as taking $m_0 = 0.1$ eV $= 10^{-10}$ GeV, in an attempt to reproduce their results. As a side note, the initial values of the neutrino Yukawa coupling g_ν are 0.182, 0.237 and 0.407 at their respective values of M_0.

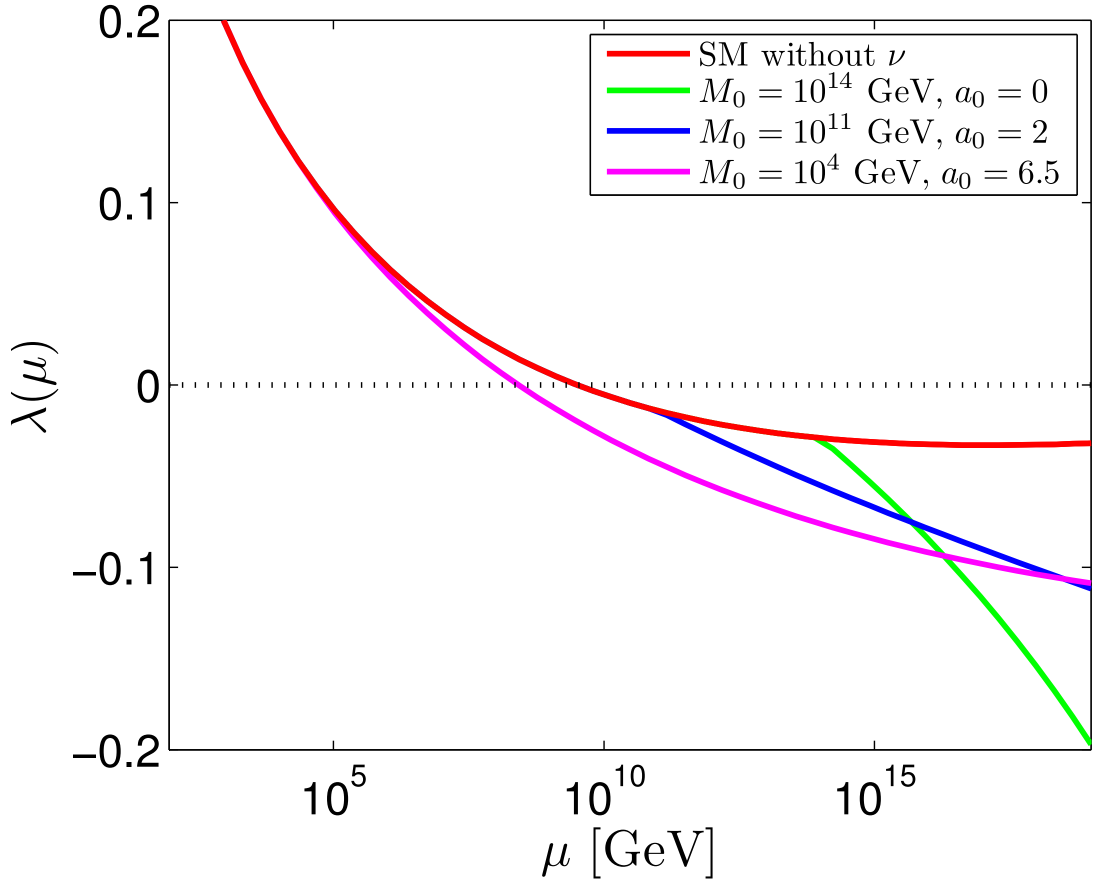


Figure 5.2.: The Higgs self coupling g_λ as computed by reference [28], with the introduction of massive right handed neutrinos at $M_0 = 10^4, 10^{11}, 10^{14}$ GeV and $a = 6.5$, 2 and 0.

We can see that, if we compare Figure 5.1 with 5.2, although the broad effect of the addition of massive right-handed neutrinos is consistent with [28], the magnitude of the result is quite different. The inclusion of additional fermions with significant Yukawa couplings generically makes the scalar coupling beta function more negative, as we shall see again in the following chapter. The differences in where the scalar self-coupling crosses zero are well within the bounds of variation in initial conditions as considered in the previous chapter, and as the paper does not give any detailed information on how they derive their initial couplings at $M_t = 172.9$ GeV, we are unable to make a direct comparison.

We made an attempt to find the values of a needed to reproduce the results of [28] was made, and the results are plotted in Figure 5.3. The values needed to reproduce these results for $M_0 = 10^4$, 10^{11} and 10^{14} were $a = 6.97$, 2.47 and 0.6 respectively. This resulted in initial neutrino Yukawa coupling values of 0.41, 0.53 and 0.71 respectively. Although these chosen parameters resulted in a striking similarity to the results of [28], we can see no obvious correlation between the two sets of neutrino Yukawa couplings.

It is worth noting however that for smaller values of a, the right-handed neutrino Yukawa coupling can be much smaller, and its effect on the scalar self coupling evolution is consequently reduced. The difference in the effect between figures 5.1 and 5.3 with a relatively small shift in the values of a suggests that, at least for $M_0 \leq 10^{10}$, the divergence from the Standard Model result could be less significant than that due to errors on the initial conditions, plotted in figure 4.9.

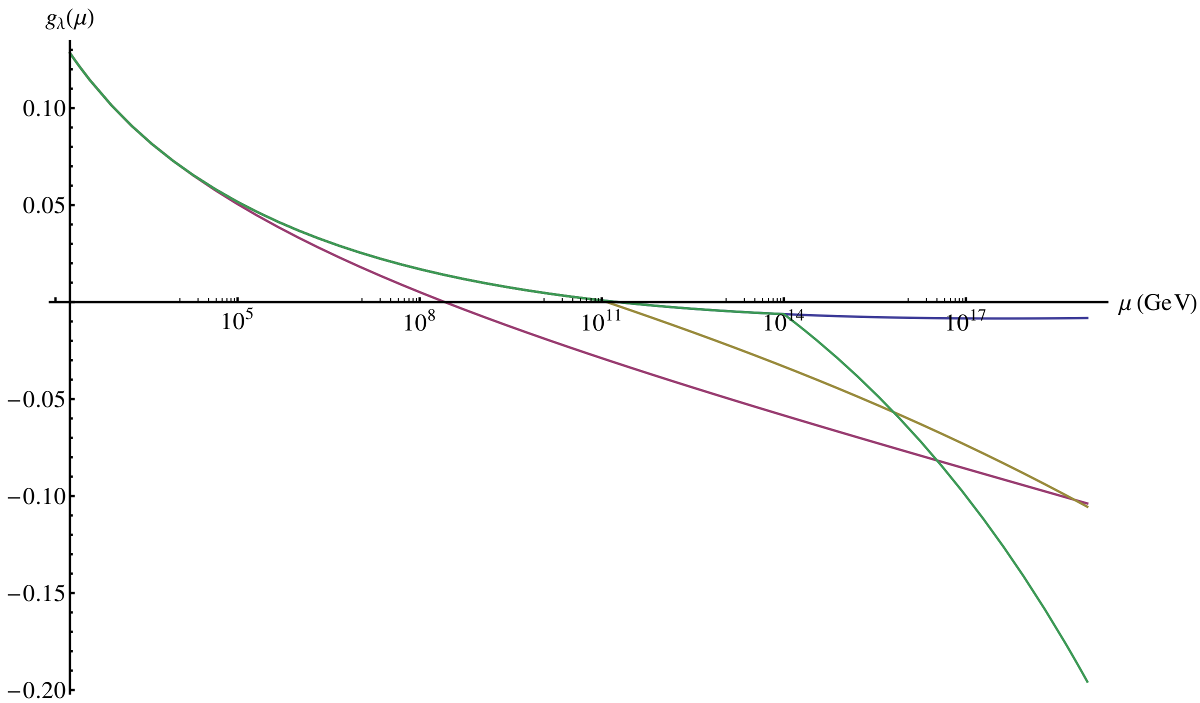


Figure 5.3.: The Higgs self coupling g_λ with the introduction of massive right handed neutrinos at $M_\nu = 10^4, 10^{11}, 10^{14}$ GeV and $a = 6.97$, 2.47 and 0.6 in magenta, yellow and green respectively. The Standard Model result is plotted in blue. The vertical axis is at $\mu = M_t$

Chapter 6.

Fourth Generation Fermions

One of the simplest extensions to the Standard Model is the inclusion of a fourth generation of fermions at a higher mass scale than the observed three generations. The number of generations in the Standard Model is not a prediction, but rather a consequence of the observation of only three generations of fermions. The most natural such extension would be to assume that the fourth generation consists an additional set of fermions, identical to the previous three generations except for their masses. Additionally, the close unitarity of the observed three generational CKM matrix requires that the mixings of the fourth generation quarks to the observed three be very small.

Constraints on these models come from direct detection, providing lower limits on the masses of fourth generation fermions, and from requiring that Yukawa couplings to the fourth generation remain perturbative, giving an upper limit. An additional problem is that any fourth-generation neutrino must have a mass greater than at least $\frac{M_Z}{2}$ given the strict limits on the number of neutrinos below this from Z decays into $\nu\overline{\nu}$ pairs.

6.1. Theory and Experimental Constraints

The addition to the Standard Model of a fourth generation of fermions (SM4) poses no real theoretical issues. That said, there are however very serious experimental issues with a fourth generation. Since this work was undertaken, new data from the LHC, and data from the Tevatron, reevaluated in the light of the now-known existence of a Higgs-like particle at ≈125 GeV have made a fourth generation, with no additional new particles, disfavoured [37]. The overabundance of $H \to \gamma\gamma$ events at the LHC is in direct opposition with the predictions of SM4, which would result in a significant reduction of this signal, either due to a fourth neutrino mass $\lesssim \frac{M_H}{2}$ reducing the branching ratio, or due to large cancellations between the fermion loop and W contributions at next-to-leading order [38]. Similarly, Higgs production by radiation from a Z boson is suppressed in SM4, again in conflict with Tevatron data, which shows a slight enhancement over the Standard Model [39]. Finally, the channel $H \to \tau\tau$ is predicted to have a significant enhancement in the SM4, which is not seen in the most recent LHC data.

6.2. Results

For our calculations, we have assumed that all of the fourth generation fermions have masses greater than that of the top quark, in order that each new fermion can be introduced sequentially at its corresponding mass — in this sense, the masses we are assigning to the fourth generation are pole masses. We also keep the order of the fermion masses the same as in previous Standard Model generations, with neutrino lightest, followed by the lepton, the down-type quark, and most massive, the up-type quark. We take the number of generations $n_G = 3$ below the scale where we introduce the new top quark.

Figure 6.1 demonstrates, instead of the Higgs self coupling directly, the energy at which the Higgs self coupling becomes zero. The parameters used were $M_H = 126$ GeV, $M_t = 172.9$ GeV and the masses of the other 3 fourth generation quarks being $M_B = 230$ GeV, $M_L = 200$ GeV and $M_N = 185$ GeV for the down-type quark, lepton and neutrino respectively.In general, from this and from similar graphs for the other masses involved, we can see that the presence of a fourth generation of fermions tends to push g_λ towards zero much faster than only three generations of quarks. The scale at which the scalar vacuum becomes unstable is around 350 GeV for the entire range of masses we considered, in comparison with energies of $\approx 10^{10}$ GeV for the Standard Model. Given the most significant term in the Standard Model scalar coupling beta function around the electroweak scale is that due to a single top loop, and is negative, this result is not surprising. The factor of n_G in the beta functions is treated naively to be 3 up to the scale where all four fourth generation fermions have been introduced, at which point we take $n_G = 4$. This graph in fact highlights the only interesting result in the mass range we considered — $\pm$ 30 GeV for M_T, 20 GeV for M_B and M_L and 10 GeV for M_N. All the other masses showed a monotonic decrease in the scale at which g_λ crossed zero with increasing fermion mass.

As a further exercise, we also changed the way we introduced the new physics, keeping the central mass values as in the previous case, bar M_N, which we take to be 100 GeV, but introducing all of the new physics simultaneously at M_T, with the results plotted in figure 6.2. In this way, changing from $n_G = 3$ to 4 is consistent. For this method, we find that increasing M_T also increases the scale at which g_λ crosses zero — again an expected result, as the running of the coupling becomes more negative with the addition of more fermions.

Regardless of the method by which the new fermionic content is introduced, the scale at which the Standard Model vacuum expectation becomes unstable is very close to the

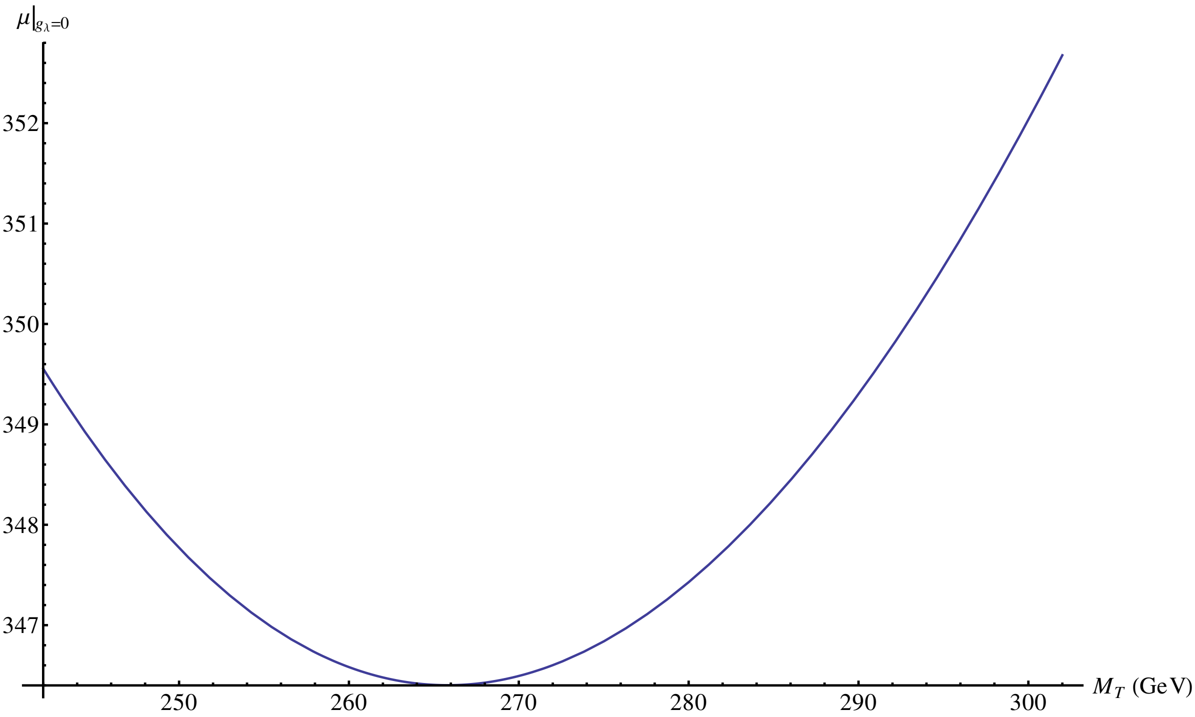


Figure 6.1.: Energy scale at which $g_\lambda = 0$ plotted against the mass of the fourth generation up-type quark, M_T, with fourth generation fermions introduced at their pole masses.

mass of the heaviest fermion, indicating that additional new physics would be needed at roughly the same scale as the new fermions in order to counteract the decrease in the beta function. The most obvious candidate would be to introduce additional scalars, as the one-loop terms $g_\lambda g_t^2$ and g_λ^2 are the largest positive contributions to the scalar self-coupling beta function at M_t. The model considered in the following chapter, the Left-Right Symmetric Model, has a greatly expanded scalar content, which will allow us to explore this direction. Another option we could consider along these lines, although we have not done so in this thesis, would be Supersymmetry.

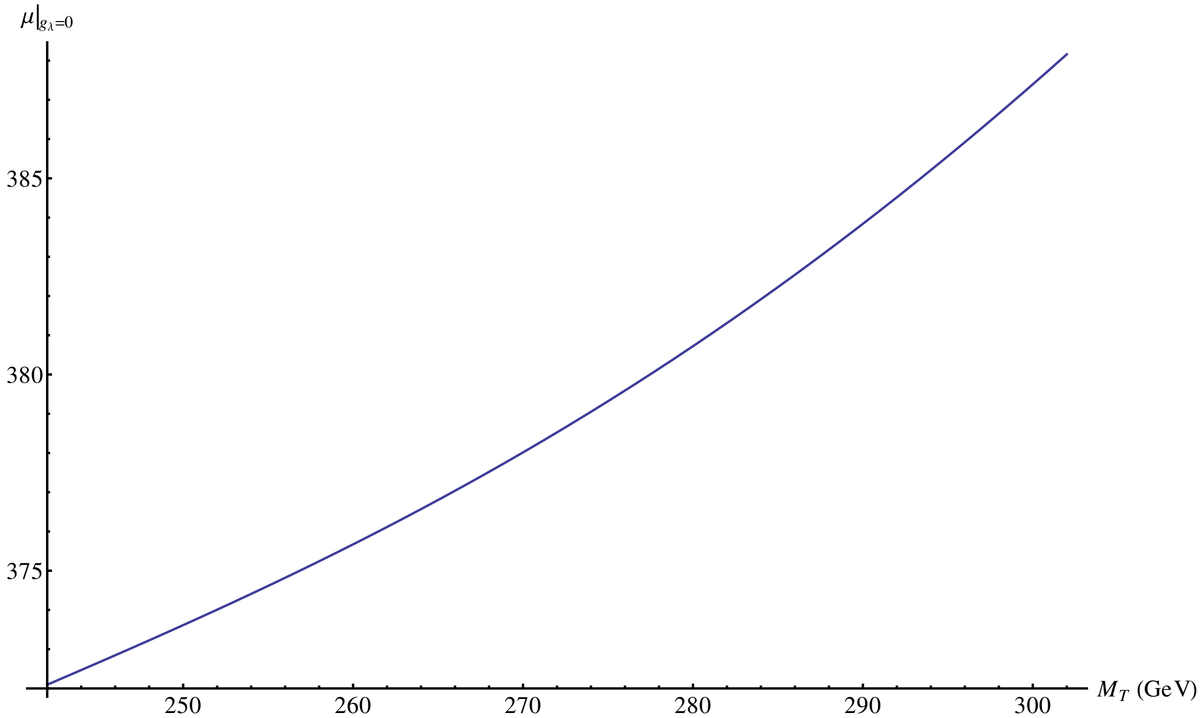


Figure 6.2.: Energy scale at which $g_\lambda = 0$ plotted against the mass of the fourth generation up-type quark, M_T, with all fourth generation content introduced at M_T.

Chapter 7.

Left-Right Symmetric Models

A less minimal, but more phenomenologically interesting extension to the Standard Model is the family of Left-Right Symmetric Models [40, 41, 42, 43]. These are motivated by the observation that the gauge group of the Standard Model, $\mathrm{SU}(3)_\mathrm{C} \otimes \mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_\mathrm{Y}$ contains an interaction that only acts on left-handed fermions. By extending the gauge group to $\mathrm{SU}(3)_\mathrm{C} \otimes \mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{SU}(2)_\mathrm{R} \otimes \mathrm{U}(1)_{B-L}$ we can describe a model which, at high energy, treats left- and right-handed fermions equally.

The absence of such an observed right-handed interaction leads to the conclusion that there must be a second spontaneous symmetry breaking scale above the electroweak scale of the Standard Model. Combined with a sufficiently weak coupling constant, the consequently more massive right-handed W and Z analogues are difficult to observe at energies below the second symmetry breaking scale. The pattern of symmetry breaking is thus

$$\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{SU}(2)_\mathrm{R} \otimes \mathrm{U}(1)_{B-L} \xrightarrow{\text{Left-Right}} \mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_\mathrm{Y} \xrightarrow{\text{Electroweak}} \mathrm{U}(1)_\mathrm{EM} \tag{7.1}$$

Although a spontaneous symmetry breaking $\mathrm{SU}(2)_{\mathrm{R}} \to \mathrm{U}(1)_{\mathrm{Y}}$ is not ruled out by gauge considerations, in order to recover the observed electromagnetic charges of the different fermions, we require an additional U(1) symmetry as part of the symmetry breaking. As a consequence of this, the conserved quantity of the U(1) gauge becomes $B-L$, or baryon number minus lepton number, as opposed to hypercharge in the Standard Model. This would raise the status of the observed conservation of baryon minus lepton number in the Standard Model to a fundamental result of the gauge structure, whilst simultaneously removing some of the arbitrary nature of the hypercharge values in the Standard Model.

7.1. Theory

Although there is no experimental difficulty with our observations of a universe that contains left-handed interactions with no right-handed counterpart, it is theoretically unsatisfying to have no explanation for this. The simplest explanation would be that at higher energy scales full left-right symmetry is restored, including a right-handed gauge interaction. As a result of this constraint, right-handed fermions would appear as $\mathrm{SU}(2)_{\mathrm{R}}$ doublets, $\mathrm{SU}(2)_{\mathrm{L}}$ singlets, and as such right-handed neutrinos must exist, which is already hinted at due to the observation of neutrino flavor-changing. The new gauge couplings we name g_1 for the new $\mathrm{U}(1)_{B-L}$ gauge group, g_2 for $\mathrm{SU}(2)_{\mathrm{L}}$ as in the Standard Model, g_3 for $\mathrm{SU}(2)_{\mathrm{R}}$ and g_4 for $\mathrm{SU}(3)_{\mathrm{color}}$. The simplest possible way of generating Yukawa terms, and thus mass terms for these fermions is by altering the Higgs sector of the Standard Model to a bidoublet scalar Φ, which transforms independently under both $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$ [42]. It is worth pointing out that as the scalar sector of the Standard Model is symmetric under SO(4), which is covered by SU(2)⊗SU(2), we are again making a local symmetry of an existing global symmetry.

$$\mathcal{L}_{\mathrm{Yukawa}} = Y\overline{\psi}_{\mathrm{L}}\Phi\psi_{\mathrm{R}} + \widetilde{Y}\overline{\psi}_{\mathrm{L}}\widetilde{\Phi}\psi_{\mathrm{R}} + h.c. \tag{7.2}$$

Here, Y and $\widetilde{Y}$ are the Yukawa couplings and the bidoublet scalars Φ and $\widetilde{\Phi}$ have the structure

$$\Phi = \begin{pmatrix} \phi_1^0 & \phi_1^+ \\ \phi_2^- & \phi_2^0 \end{pmatrix}, \qquad \widetilde{\Phi} = \tau_2 \Phi^* \tau_2 = \begin{pmatrix} \phi_2^{0*} & -\phi_2^+ \\ \phi_1^- & \phi_1^{0*} \end{pmatrix} \tag{7.3}$$

with both having the same gauge transformation

$$\Phi \to U_{\rm L} \Phi U_{\rm R}^\dagger, \qquad \widetilde{\Phi} \to U_{\rm L} \widetilde{\Phi} U_{\rm R}^\dagger. \tag{7.4}$$

We also have transformations for the left- and right-handed fermions

$$\psi_{\rm L} \to U_{\rm L} \psi_{\rm L}, \qquad \psi_{\rm R} \to U_{\rm R} \psi_{\rm R} \tag{7.5}$$

from which we can see that the Yukawa term in equation (7.2) is gauge invariant.

However, this basic setup leaves us short of two important components. Firstly, in order to recover the observed electromagnetic charges of the fermions, which differs between quark and lepton doublets, we need to promote the global $B-L$ symmetry of the Standard Model to a local U(1) gauge symmetry, which contributes a component to the electromagnetic charge $Q_{\rm EM}$ which we have, analogously with the Standard Model, as

$$Q_{\rm EM} = I_{\rm L}^3 + I_{\rm R}^3 + \frac{1}{2} Q_{B-L} \tag{7.6}$$

with $I_{\rm L,R}^3$ the third components of left and right isospin respectively. Particles that are charged under left and right isospin are, for example the ϕ_1^0, for which $I_{\rm L}^3(\phi_1^0) = \frac{1}{2}$ and $I_{\rm R}^3(\phi_1^0) = -\frac{1}{2}$. Solving for the fermion electric charges, we find that quarks must have a U(1) charge of $\frac{1}{3}$, and leptons a charge of -1, values which match the value of baryon number minus lepton number, $B-L$. This is a much more attractive scenario, theoretically, than the seemingly arbitrary hypercharge values found in the Standard Model, which have no motivation beyond breaking to the correct electromagnetic charge, and satisfying the conditions required in order to cancel gauge anomalies.

The second missing component is less obvious in the context of the Standard Model, but remembering that we now have right-handed neutrinos, which are not directly observed experimentally, we must have a significant mass difference between left- and right-handed neutrinos. The above Yukawa terms do not allow for different masses for the left and right doublets, so we must introduce new, Majorana Yukawa terms involving new specifically left- or right-handed scalars. A way to obtain tree-level Majorana mass terms is for these scalars to be triplets under their respective gauge groups [40] and to have $B-L$ charge of 2, written as

$$\Delta_{\mathrm{L,R}} = \begin{pmatrix} \frac{1}{\sqrt{2}\delta^{+}_{\mathrm{L,R}}} & \delta^{++}_{\mathrm{L,R}} \\ \delta^{0}_{\mathrm{L,R}} & \frac{1}{\sqrt{2}\delta^{+}_{\mathrm{L,R}}} \end{pmatrix}, \qquad \Delta_{\mathrm{L,R}} \to U_{\mathrm{L,R}}\Delta_{\mathrm{L,R}}U^{\dagger}_{\mathrm{L,R}}. \tag{7.7}$$

These new scalars allow us to write Majorana terms for the leptons

$$\mathcal{L}_{\mathrm{Majorana}} = M\left(\psi^{T}_{\mathrm{L}}C\tau_2\Delta_{\mathrm{L}}\psi_{\mathrm{L}} + \psi^{T}_{\mathrm{R}}C\tau_2\Delta_{\mathrm{R}}\psi_{\mathrm{R}} + h.c.\right) \tag{7.8}$$

Equivalent terms for the quarks are forbidden, as such an interaction would not conserve $B-L$. When the full left-right symmetry is broken to the observed Standard Model symmetry group, the left- and right-handed scalars can obtain different vacuum expectation values, and in doing so give differing masses to the left- and right-handed neutrinos. The vacuum expectation value for Δ_{R} appears on the uncharged component, and so only the neutrinos receive a Majorana mass term.

In order to discuss the relation of the left-right symmetric Yukawa couplings to the familiar Standard Model Yukawa couplings g_t, g_b and g_τ, we first need to discuss the structure of the vacuum expectation values in the three scalar representations. In terms of real variables, we can write [43]

$$\langle\Phi\rangle = \begin{pmatrix} k_1 e^{i\alpha_1} & 0 \\ 0 & k_2 e^{i\alpha_2} \end{pmatrix}, \qquad \langle\Delta_{\mathrm{L,R}}\rangle = \begin{pmatrix} 0 & 0 \\ v_{\mathrm{L,R}} e^{i\theta_{\mathrm{L,R}}} & 0 \end{pmatrix}. \tag{7.9}$$

This can be further restricted by using our freedom to make independent left and right gauge transformations to absorb two of the complex phases. The usual choice is to absorb α_2 and θ_{L}, whilst dropping the indices from the remaining phases to give us

$$\langle\Phi\rangle = \begin{pmatrix} k_1 e^{i\alpha} & 0 \\ 0 & k_2 \end{pmatrix}, \qquad \langle\Delta_{\mathrm{L}}\rangle = \begin{pmatrix} 0 & 0 \\ v_{\mathrm{L}} & 0 \end{pmatrix}, \qquad \langle\Delta_{\mathrm{R}}\rangle = \begin{pmatrix} 0 & 0 \\ v_{\mathrm{R}} e^{i\theta} & 0 \end{pmatrix}. \tag{7.10}$$

As our Lagrangian is explicitly CP invariant these two complex phases, generated spontaneously by symmetry breaking, are the only source of CP violation in the model [41]. Observationally, we also obtain some constraints on the values that the 4 parameters can take. In order to agree with the Standard Model electroweak symmetry breaking scale, we require $k_1^2 + k_2^2 = (246\ \mathrm{GeV})^2$, v_{L} must be much smaller than $k_{1,2}$ to preserve the left-handed W-Z mass ratio, $\frac{M^2_{W_{\mathrm{L}}}}{M^2_{Z_{\mathrm{L}}}} \sim \cos^2\theta_W$ and v_{R} must be much larger than $k_{1,2}$ in order to give sufficiently large masses to the right-handed gauge bosons and neutrinos.

We can now write down the Dirac mass terms for the quarks and leptons, and rearrange to find the values of the Yukawa couplings in terms of fermion masses and vacuum expectation values. As we are making the approximation that the only fermions with nonzero masses are the top, bottom and tau, we can neglect details relating to CKM-analogous matrices, and assume that there is no mixing of flavor. Given quark Yukawa couplings of Y_q and $\widetilde{Y}_q$, and lepton couplings Y_l and $\widetilde{Y}_l$ we can expand with the expectation values of the scalars to obtain Dirac mass terms

$$\bar{b}\left(\widetilde{Y}_q k_1 e^{i\alpha} + Y_q k_2\right) b + \bar{t}\left(Y_q k_1 e^{-i\alpha} + \widetilde{Y}_q k_2\right) t + \bar{\tau}\left(\widetilde{Y}_l k_1 e^{i\alpha} + Y_l k_2\right) \tau, \tag{7.11}$$

with the additional constraint from the negligible ν_τ mass that

$$Y_l k_1 e^{-i\alpha} + \widetilde{Y}_l k_2 = 0. \tag{7.12}$$

As long as we have $k_1^2 \neq k_2^2$ we can invert these equations to obtain [41]

$$Y_q = \frac{k_2 m_b - k_1 m_t e^{i\alpha}}{k_2^2 - k_1^2}, \tag{7.13}$$

$$\widetilde{Y}_q = \frac{k_2 m_t - k_1 m_b e^{-i\alpha}}{k_2^2 - k_1^2}, \tag{7.14}$$

$$Y_l = \frac{k_2 m_\tau}{k_2^2 - k_1^2}, \tag{7.15}$$

$$\widetilde{Y}_l = \frac{-k_1 m_\tau e^{-i\alpha}}{k_2^2 - k_1^2}. \tag{7.16}$$

Of further consideration in reproducing known Standard Model phenomenology is the existence of flavour-changing neutral currents in the Left-Right Symmetric Model. Non-observance of such currents indicates that any particles mediating such currents must exist at a mass much larger than the electroweak symmetry breaking scale, v_{EW}. As a consequence of this we can safely exclude models where the complex phase α is not very close to zero [44], which would allow some neutral scalars to have masses of the order of v_{EW}. On the other hand, the CP violating phase θ, which is only of concern for right-handed neutrinos, has consequently little effect on currently observable physics and any amount of CP violation in this sector is consistent with experimental results [41].

The expanded scalar bidoublet, along with the two new scalar triplets results in a much more complex scalar sector, and even with the requirement that the Lagrangian must be invariant under the interchange of left- and right-handed particles, the scalar sector contains 14 different coupling constants, of which g_{λ_1} is equivalent to the single scalar coupling in the Standard Model. The full, most generic scalar interaction Lagrangian invariant under left-right symmetry, up to dimension-4, is given by [41]

$$\begin{aligned}
\mathcal{L}_{\text{scalar}} = {} & g_{\lambda_1} \left[\operatorname{Tr}\left(\Phi^\dagger\Phi\right)\right]^2 \\
& + g_{\lambda_2} \left[\operatorname{Tr}\left(\widetilde{\Phi}\Phi^\dagger\right)^2 + \operatorname{Tr}\left(\widetilde{\Phi}^\dagger\Phi\right)^2\right] \\
& + g_{\lambda_3} \operatorname{Tr}\left(\widetilde{\Phi}\Phi^\dagger\right) \operatorname{Tr}\left(\widetilde{\Phi}^\dagger\Phi\right) \\
& + g_{\lambda_4} \operatorname{Tr}\left(\Phi\Phi^\dagger\right) \left[\operatorname{Tr}\left(\widetilde{\Phi}\Phi^\dagger\right) + \operatorname{Tr}\left(\widetilde{\Phi}^\dagger\Phi\right)\right] \\
& + g_{\rho_1} \left[\operatorname{Tr}\left(\Delta_{\mathrm{L}}\Delta_{\mathrm{L}}^\dagger\right)^2 + \operatorname{Tr}\left(\Delta_{\mathrm{R}}\Delta_{\mathrm{R}}^\dagger\right)^2\right] \\
& + g_{\rho_2} \left[\operatorname{Tr}(\Delta_{\mathrm{L}}\Delta_{\mathrm{L}}) \operatorname{Tr}\left(\Delta_{\mathrm{L}}^\dagger\Delta_{\mathrm{L}}^\dagger\right) + \operatorname{Tr}(\Delta_{\mathrm{R}}\Delta_{\mathrm{R}}) \operatorname{Tr}\left(\Delta_{\mathrm{R}}^\dagger\Delta_{\mathrm{R}}^\dagger\right)\right] \\
& + g_{\rho_3} \operatorname{Tr}\left(\Delta_{\mathrm{L}}\Delta_{\mathrm{L}}^\dagger\right) \operatorname{Tr}\left(\Delta_{\mathrm{R}}\Delta_{\mathrm{R}}^\dagger\right) \\
& + g_{\rho_4} \left[\operatorname{Tr}(\Delta_{\mathrm{L}}\Delta_{\mathrm{L}}) \operatorname{Tr}\left(\Delta_{\mathrm{R}}^\dagger\Delta_{\mathrm{R}}^\dagger\right) + \operatorname{Tr}(\Delta_{\mathrm{R}}\Delta_{\mathrm{R}}) \operatorname{Tr}\left(\Delta_{\mathrm{L}}^\dagger\Delta_{\mathrm{L}}^\dagger\right)\right] \\
& + g_{\alpha_1} \operatorname{Tr}\left(\Phi^\dagger\Phi\right) \left[\operatorname{Tr}\left(\Delta_{\mathrm{L}}^\dagger\Delta_{\mathrm{L}}\right) + \operatorname{Tr}\left(\Delta_{\mathrm{R}}^\dagger\Delta_{\mathrm{R}}\right)\right] \\
& + g_{\alpha_2} \left[\operatorname{Tr}\left(\widetilde{\Phi}^\dagger\Phi\right) + \operatorname{Tr}\left(\widetilde{\Phi}\Phi^\dagger\right)\right] \left[\operatorname{Tr}\left(\Delta_{\mathrm{L}}\Delta_{\mathrm{L}}^\dagger\right) + \operatorname{Tr}\left(\Delta_{\mathrm{R}}\Delta_{\mathrm{R}}^\dagger\right)\right] \\
& + g_{\alpha_3} \left[\operatorname{Tr}\left(\Phi\Phi^\dagger\Delta_{\mathrm{L}}\Delta_{\mathrm{L}}^\dagger\right) + \operatorname{Tr}\left(\Phi^\dagger\Phi\Delta_{\mathrm{R}}\Delta_{\mathrm{R}}^\dagger\right)\right] \\
& + g_{\beta_1} \left[\operatorname{Tr}\left(\Phi\Delta_{\mathrm{R}}\Phi^\dagger\Delta_{\mathrm{L}}^\dagger\right) + \operatorname{Tr}\left(\Phi^\dagger\Delta_{\mathrm{L}}\Phi\Delta_{\mathrm{R}}^\dagger\right)\right] \\
& + g_{\beta_2} \left[\operatorname{Tr}\left(\widetilde{\Phi}\Delta_{\mathrm{R}}\Phi^\dagger\Delta_{\mathrm{L}}^\dagger\right) + \operatorname{Tr}\left(\widetilde{\Phi}^\dagger\Delta_{\mathrm{L}}\Phi\Delta_{\mathrm{R}}^\dagger\right)\right] \\
& + g_{\beta_3} \left[\operatorname{Tr}\left(\Phi\Delta_{\mathrm{R}}\widetilde{\Phi}^\dagger\Delta_{\mathrm{L}}^\dagger\right) + \operatorname{Tr}\left(\Phi^\dagger\Delta_{\mathrm{L}}\widetilde{\Phi}\Delta_{\mathrm{R}}^\dagger\right)\right] .
\end{aligned} \tag{7.17}$$

It is not possible to create SU(2) invariant terms with an odd power of Φ and $\widetilde{\Phi}$, or U(1) invariant terms with odd powers of $\Delta_{\mathrm{L,R}}$. This therefore enumerates all possible scalar potential terms up to dimension-4. We also have three possible mass terms, which are relevant in terms of the minimisation conditions for the scalar potential,

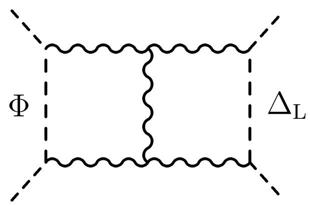
$$\mathcal{L}_{\text{scalar mass}} = -\mu_1^2 \operatorname{Tr}\left(\Phi^\dagger\Phi\right) - \mu_2^2 \left(\operatorname{Tr}\left(\Phi^\dagger\widetilde{\Phi}\right) + \operatorname{Tr}\left(\widetilde{\Phi}^\dagger\Phi\right)\right)$$

$$- \mu_3^2 \left(\mathrm{Tr}\left(\Delta_{\mathrm{L}}^\dagger \Delta_{\mathrm{L}} \right) + \mathrm{Tr}\left(\Delta_{\mathrm{R}}^\dagger \Delta_{\mathrm{R}} \right) \right) . \tag{7.18}$$

Although we expect the renormalisation group equations for any two elements of the scalar coupling matrix Λ_{abcd} of equal value to be equivalent, this is not to say that the equations need be identical. For example if we consider the form of the potential for the coupling g_{α_1},

$$\mathrm{Tr}\left(\Phi^\dagger \Phi\right) \left[\mathrm{Tr}\left(\Delta_{\mathrm{L}}^\dagger \Delta_{\mathrm{L}} \right) + \mathrm{Tr}\left(\Delta_{\mathrm{R}}^\dagger \Delta_{\mathrm{R}} \right) \right] \tag{7.19}$$

we see that this incorporates two terms from the most generic scalar potential, given the same coupling due to Left-Right symmetry. We can take the two-loop term in the RGE due to gauge self interactions, described by the term A^g_{abcd} (in this case connecting two Φ and two Δ_{L} Higgs fields).



(7.20)

Since Δ_{L} does not interact with $\mathrm{SU}(2)_{\mathrm{R}}$ gauge bosons, there will only be terms proportional to g_2^6, and as such, the corresponding element of the RGE tensor will not be symmetric with respect to interchange of Left- and Right-handed gauge couplings. However, if we consider instead the tensor element corresponding to changing the external fields, $\Delta_{\mathrm{L}} \leftrightarrow \Delta_{\mathrm{R}}$, $\Phi \leftrightarrow \Phi^\dagger$, we find that this term is indeed proportional to g_3^6. Hence, as long as strict Left-Right symmetry is maintained, these two RGEs will cause their corresponding tensor elements to evolve identically.

The parameter space of the Left-Right Symmetric Model is sufficiently large to warrant a discussion on the initial parameters chosen. For the gauge couplings, we require that, at energy scales where the Left-Right Symmetry is unbroken, the couplings g_2 and g_3 should have the same value. Since g_2 in the Left-Right Symmetric Model becomes g_2 of the Standard Model after symmetry breaking, their values must be equivalent at $\mu = v_{\rm R}$. Equally, the SU(3) coupling must also have the same value at the new physics threshold as in the Standard Model. The final gauge coupling, g_1 must combine with the coupling of the other broken gauge group, g_3, to give the correct $\mathrm{U}(1)_Y$ coupling from the Standard Model, and therefore the threshold values of the gauge couplings of the Left-Right Symmetric Model are totally determined by the Standard Model values at the threshold.

For the four Yukawa couplings, Y_q, $\widetilde{Y}_q$, Y_l and $\widetilde{Y}_l$, the situation is more complicated. All four couplings are potentially complex, and although they are dependent on the masses of the heaviest fermions, they also depend on the two Φ vacuum expectation values as well as the complex phase α seen in equation (7.10). An obvious constraint on the values these can take is that we want both of the quark couplings to have perturbative magnitudes — since the Yukawa couplings only appear in $Y^\dagger Y$ pairs, only the magnitude affects the evolution. As the lepton Yukawa couplings depend on the Dirac neutrino mass and the τ mass, which are both much smaller than the top mass, they remain perturbative for a much larger range of vacuum expectation values than the quark couplings. The figures 7.1 and 7.2 plot the magnitudes of the two quark Yukawa couplings at the new physics scale, against the value of k_1, for complex phase angles of 0, $\frac{\pi}{2}$ and π in blue, magenta and yellow respectively.

Evidently, α has little effect on the magnitude of the couplings, as the ratios of the top and bottom quark masses is so large, and values of k_1 too close to $\frac{246}{\sqrt{2}} \approx 174$ GeV are non-perturbative, as in this region the denominator of the Yukawa coupling $k_-^2 = k_2^2 - k_1^2$

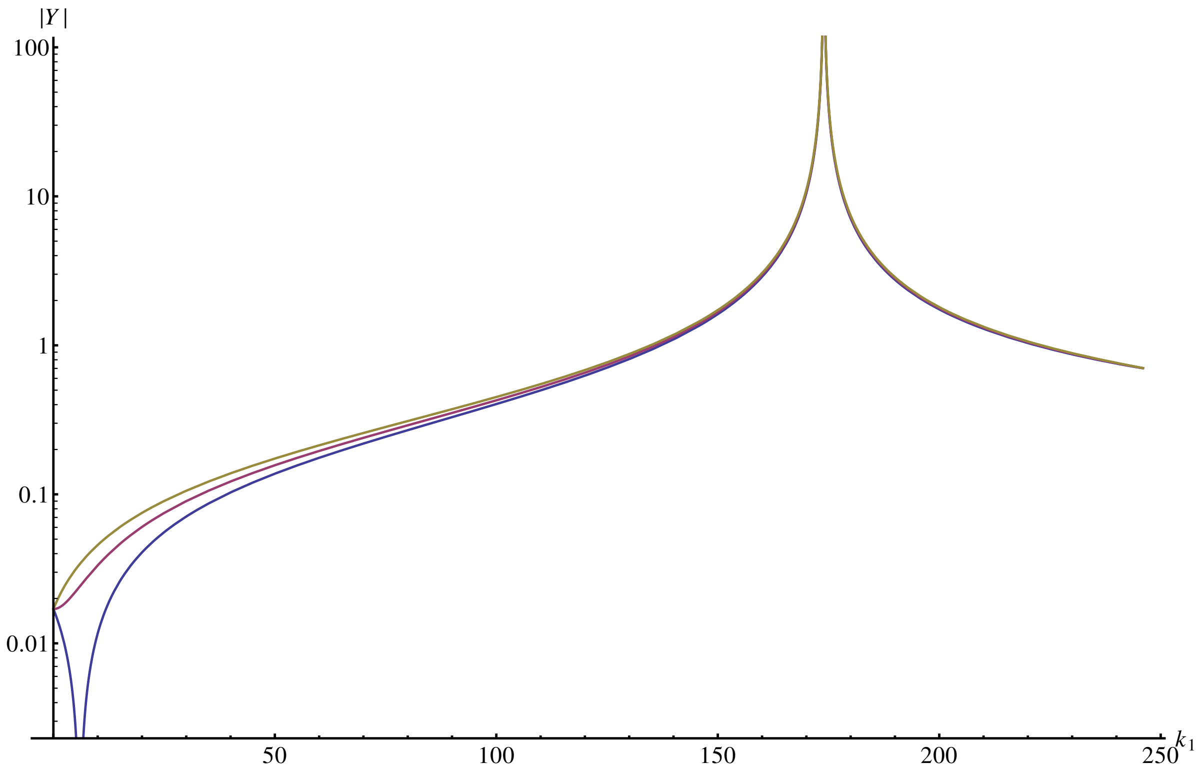


Figure 7.1.: Variation of Y_q with k_1 in GeV. α =0 (blue), $\frac{\pi}{2}$ (magenta) and π (yellow)

becomes small. Reasonable values for the vacuum expectation values, which are still of roughly the same order of magnitude are given by $k_1 \approx 80$ GeV, $k_2 \approx 230$ GeV. Taking $2k_1 = k_2 \approx 220$ GeV gives a magnitude of $\widetilde{Y}_q \approx 1.03$.

We also need to select a value for the new physics scale, v_R, above which the full Left-Right symmetry is manifest. A lower bound for this vacuum expectation value is calculated in [41], based on the constraints on the masses of light and heavy neutrinos, and is derived from the expression

$$v_\mathrm{R}^2 \approx k_+^2 \frac{m_N^2}{m_N m_\nu + m_e^2}$$

where m_ν and m_N are the masses of the light and heavy neutrino mass eigenstates respectively, and m_e is the electron mass. Substituting the lower bound for m_N with the upper bound on m_ν gives us a lower bound on v_R of 2.7x10^7 GeV. Although we have

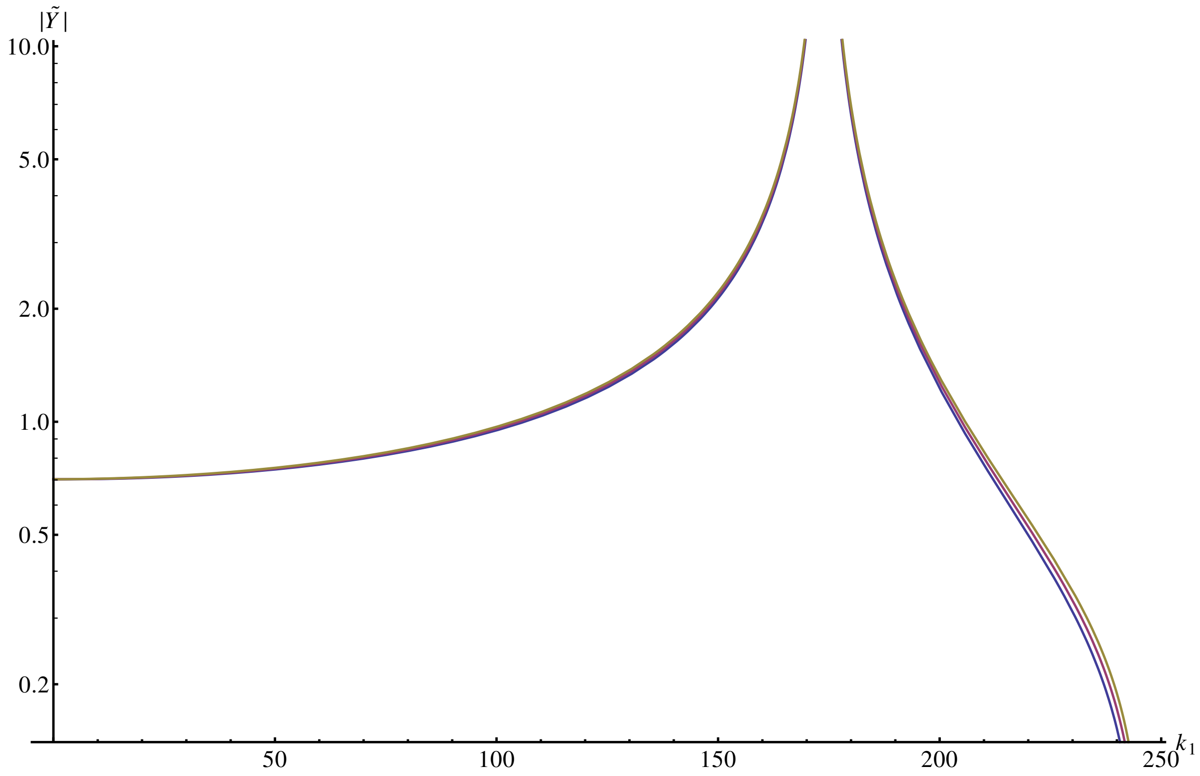


Figure 7.2.: Variation of $\widetilde{Y}_q$ with k_1 in GeV. α =0 (blue), $\frac{\pi}{2}$ (magenta) and π (yellow)

not included Majorana mass terms in our calculation, this is due to difficulties with the calculation, rather than a desire to exclude the effect of Majorana couplings from our model. As such, we feel that this limit is still valid for our model, and will attempt to estimate the effect that including Majorana terms in our model would have on our conclusions.

7.1.1. Beta Functions for the Reduced Left-Right Symmetric Model

There is only one other problem left to solve before we can evolve all the couplings in the Left-Right Symmetric Model, which is to establish how the model behaves in the energy range from the top mass M_t up to the new physics scale M_{NP}. As some of the scalar bosons, as well as 3 of the vector bosons coupling to right-handed particles, will acquire masses of order v_R, we want to decouple these and re-evaluate the beta functions in their absence. This can also reduce the number of effective scalar couplings, either by removing all the interactions associated with a coupling, or by making two or more couplings degenerate by eliminating those terms that previously differentiated them. In addition we need to consider making a change of basis, as the mass eigenstates are, in most cases, not the same as the gauge eigenstates. Finally, we need to recalculate the Yukawa and gauge beta functions, as the number of possible scalars substitutable into a scalar loop has been reduced.

To determine which scalar degrees of freedom we should decouple from the model, we used the results of [41], particularly the values in tables 7, 8 and 9. These results assume maximal CP violation in the lepton sector, ($\theta = \frac{\pi}{2}$) and none in the quark sector, ($\alpha = 0$). Although the amount of CP violation in the Standard Model is close to maximal, taking $\alpha = \frac{\pi}{2}$ results in unacceptably large flavor-changing neutral currents. A quark sector with no CP violation suppresses these currents as all the neutral scalar particles involved in flavor-changing interactions have masses of the order of v_R.

We rewrite the components of the scalar bidoublet in the form [41]

$$\phi^0_+ = \frac{1}{k_+}\left(-k_2\phi^0_1 + k_1 e^{i\alpha}\phi^{0*}_2\right), \tag{7.21}$$

$$\phi^0_- = \frac{1}{k_+}\left(k_1 e^{-i\alpha}\phi^0_1 + k_2\phi^{0*}_2\right), \tag{7.22}$$

$$\phi^+_+ = \frac{1}{k_+}\left(-k_2\phi^+_1 + k_1\phi^+_2\right), \tag{7.23}$$

$$\phi^+_- = \frac{1}{k_+}\left(k_1\phi^+_1 + k_2\phi^+_2\right). \tag{7.24}$$

In this basis, the Yukawa couplings are diagonal for the ϕ^0_- and hence all of the flavour changing interactions are due to ϕ^0_+. Therefore in order to render the effects of these flavor-changing currents unobservable, we merely need to give ϕ^0_+ a mass at the scale of v_R. For either minimal or maximal lepton CP violation, this condition is satisfied for $\theta = 0$. For our chosen set of conditions, we have that the additional scalar degrees of freedom with masses at the scale of v_{R} are the imaginary component of δ^0_{R}, ϕ^+_- and δ^{++}_{R}. Accounting for the two real neutral scalars and two complex, singly-charged scalars needed to give mass to the $W_{\mathrm{L,R}}$, Z and Z' vector bosons, and the Standard Model Higgs, we have two additional real neutral scalars, a complex, singly-charged scalar and a complex, doubly-charged scalar with masses at roughly the Electroweak scale.

The couplings we have renamed in the reduced model, in addition to having taken $g_{\alpha_3} = 0$ and setting $g_{\beta_2} = -2g_{\beta_1}$, are

$$g_\gamma = g_{\alpha_1} + \frac{4k_1k_2g_{\alpha_2}}{k_+^2}, \tag{7.25}$$

$$g_\delta = g_{\lambda_1} + \frac{4k_1k_2\left(2k_1k_2g_{\lambda_2} + k_1k_2g_{\lambda_3} + k_+^2g_{\lambda_4}\right)}{k_+^4}. \tag{7.26}$$

In order to eliminate the seven real, massive scalar degrees of freedom from the model, we simply perform a change of basis on the Lagrangian prior to extracting the coupling matrix, and then set those massive degrees of freedom to zero. We can then perform the same process as for the unreduced model to generate a rank-4, 13-dimensional tensor of scalar couplings. We can make a series of rotations on the W^3_{R} and B' to obtain the B boson gauging $\mathrm{U}(1)_Y$of the Standard Model. This rotation is similar to the symmetry-breaking rotation in the Standard Model that produces the photon and Z boson, being

$$Z' = \frac{g_3W^3_{\mathrm{R}} - g_1B'}{\sqrt{g_1^2 + g_3^2}}, \tag{7.27}$$

$$B = \frac{g_1W^3_{\mathrm{R}} + g_3B'}{\sqrt{g_1^2 + g_3^2}}. \tag{7.28}$$

An easy check that this procedure is correct is that the gauge-fermion coupling tensor t for the new U(1) group should reproduce the hypercharge values for the Standard Model as a result of this rotation. Once the heavy vector bosons have also been eliminated and the coupling tensors have been recalculated the only remaining refinement needed is to eliminate any now-redundant couplings in the scalar potential. This redefinition of couplings is done by hand, and is retrospectively applied to the scalar coupling tensor. Once this has been done, the remaining steps in the calculation of the beta function are again the automatic standard methods already outlined in Chapter 3.

7.2. Results

With the preliminary work of calculating the two sets of beta functions done, the only remaining step is to numerically integrate them. As for the Standard Model, we evaluate only the gauge coupling beta functions, neglecting the top Yukawa and scalar self-interactions below the top mass, before switching to the reduced Left-Right Symmetric Model between M_t and the left-right symmetry breaking scale, v_{R}. Finally we use the full Left-Right Symmetric Model beta functions from v_{R} up to M_{Planck}.

Initial conditions for the scalar sector are much more complicated than in the Standard Model — the scalar self-coupling g_λ is analogous to g_δ in the reduced Left-Right Symmetric Model, and the potential minimisation conditions allow us to determine three other couplings, but the other ten are free parameters. The minimisation conditions are obtained by taking the scalar interaction sector $\mathcal{L}_{\mathrm{scalar}}$ given above, substituting the vacuum expectation values of the scalar fields and taking derivatives with respect to the six parameters of the expectation values. Three of those, corresponding to the derivatives with respect to v_{R}, k_1 and k_2, are used to eliminate the three mass parameters, leaving us with three conditions [40]

$$v_{\mathrm{L}} v_{\mathrm{R}} \left(2 g_{\rho_1} - g_{\rho_3}\right) = g_{\beta_1} k_1 k_2 \cos(\theta - \alpha) + g_{\beta_2} k_1^2 \cos(\theta - 2\alpha) + g_{\beta_3} k_2^2 \cos\theta, \tag{7.29}$$

$$0 = g_{\beta_1} k_1 k_2 \sin(\theta - \alpha) + g_{\beta_2} k_1^2 \sin(\theta - 2\alpha) + g_{\beta_3} k_2^2 \sin\theta, \tag{7.30}$$

$$\begin{aligned} 0 &= v_\mathrm{L} v_\mathrm{R} \left[g_{\beta_1} \left(k_1^2 \sin(\theta - 2\alpha) + k_2^2 \sin\theta \right) + \left(g_{\beta_2} + g_{\beta_3} \right) 2 k_1 k_2 \sin(\theta - \alpha) \right] \\ &\quad + k_1 k_2 \sin\alpha \left[\left(8 g_{\lambda_2} - 4 g_{\lambda_3} \right) \left(k_1^2 - k_2^2 \right) - g_{\alpha_3} \left(v_\mathrm{L}^2 + v_\mathrm{R}^2 \right) \right] . \end{aligned} \tag{7.31}$$

The third relation contains a problem in that g_{α_3} is the only term containing a factor of v_R^2, all other factors of v_R being multiplied by a factor of v_L, which recalling the ordering of the vacuum expectation values $v_\mathrm{L} \ll k_1, k_2 \ll v_\mathrm{R}$, allows us to have the combined factor of the same magnitude as $k_{1,2}^2$. Taking a non-zero g_{α_3} requires either an extremely fine-tuned set of coupling values in order to keep the other couplings perturbative, or as argued in [40], that the CP violating angles θ and α are 0 — their argument eliminates the opposite pair of angles from the vacuum expectation values, but the results are equivalent.

Alternatively, we can introduce a new discrete symmetry $\Phi \to \widetilde{\Phi}$ with $\Delta_{\mathrm{L,R}}$ unchanged. In this case, we must either set $g_{\alpha_3} = 0$, or modify the g_{α_3} term to

$$\begin{aligned} \mathcal{L}_{\alpha_3} = g_{\alpha_3} \Big[&\mathrm{Tr}\Big(\Phi \Phi^\dagger \Delta_\mathrm{L} \Delta_\mathrm{L}^\dagger \Big) + \mathrm{Tr}\Big(\Phi^\dagger \Phi \Delta_\mathrm{R} \Delta_\mathrm{R}^\dagger \Big) \\ &+ \mathrm{Tr}\Big(\widetilde{\Phi} \widetilde{\Phi}^\dagger \Delta_\mathrm{L} \Delta_\mathrm{L}^\dagger \Big) + \mathrm{Tr}\Big(\widetilde{\Phi}^\dagger \widetilde{\Phi} \Delta_\mathrm{R} \Delta_\mathrm{R}^\dagger \Big) \Big] . \end{aligned} \tag{7.32}$$

In this case, the interactions described by the expanded term are identical to the interactions from the g_{α_1} term and can therefore be eliminated. In implementing this condition, we also need to set $g_{\beta_2} = g_{\beta_3}$. When we calculate the three potential minimisation conditions given above, we find that the dependence on g_{α_3} has disappeared while the remaining conditions are unaffected.

We can compare the gauge coupling results obtained with our new beta functions to those obtained from the Standard Model. As expected, since there are no new strongly coupled particles, we see that the strong coupling g_4 is barely affected by the change

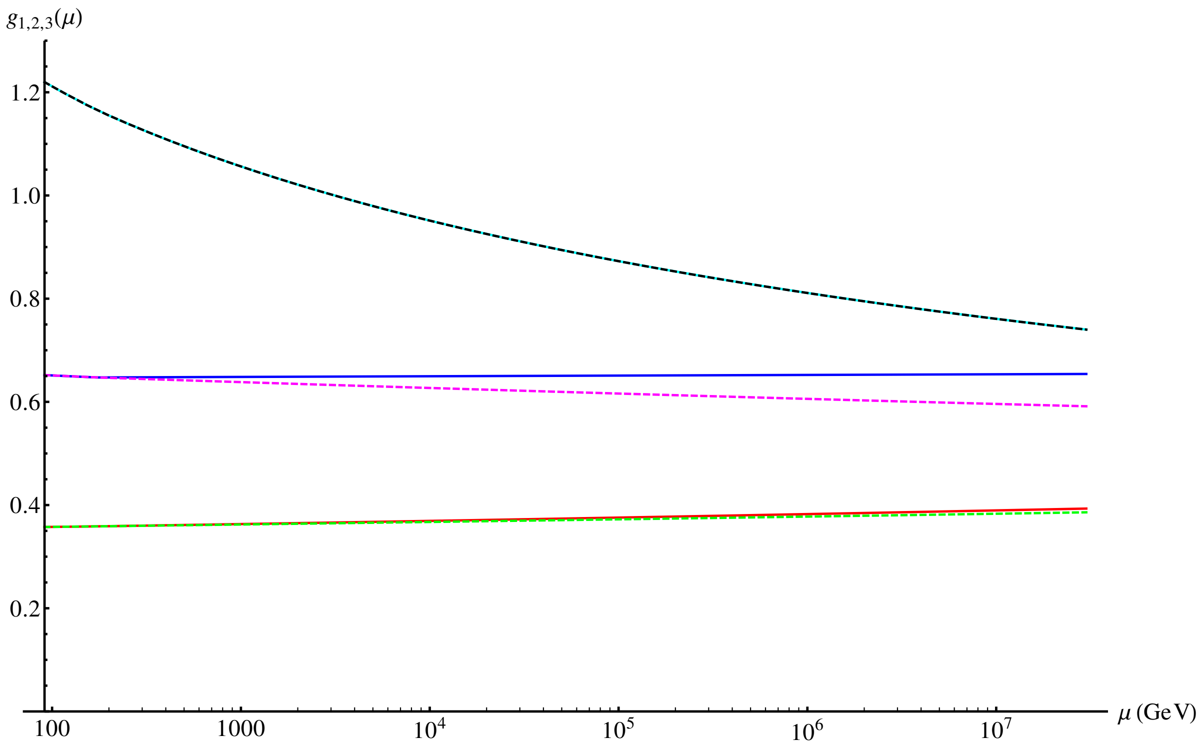


Figure 7.3.: Plain lines give g_1 in red, g_2 in blue and g_4 in black, for the LRSM. The corresponding dashed lines give the Standard Model result.

of model — at one loop the beta function is unchanged. On the other hand, there is a positive change for the weak and hypercharge couplings due to the presence of new scalar particles, with the most significant change being to g_2, largely due to the scalar triplet $\Delta_{\rm L}$, all of whose components have masses of order the Electroweak scale.

This change to the running of the left-handed weak coupling is more significant when we consider the running of the scalar coupling g_{ρ_1}. Plotted in figure 7.4, for a trial set of initial conditions for those scalar couplings not determined from Standard Model parameters, we can see that the coupling grows steadily from its initial value, to a value that is only just still perturbative (considering expansion in $\frac{g_{\rho_1}}{4\pi}$) by the time it reaches the new physics scale $v_{\rm R}$.

The reason for this strong positive growth can be explored if we borrow a method from [18], and expand the beta function in descending order of the magnitude of its

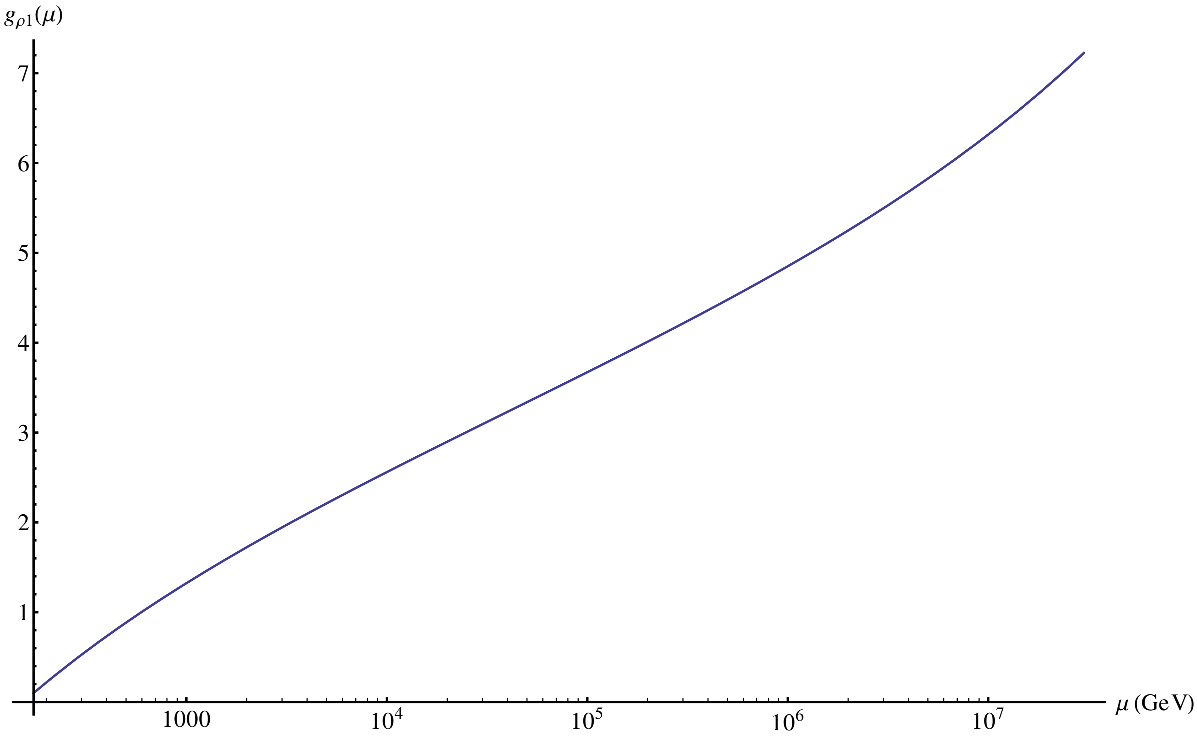


Figure 7.4.: g_{ρ_1} starting at $g_{\rho_1}(M_t) = 0.1$.

coefficients, after making the substitution

$$g_x \to G_x g_x(\mu_0)\,. \tag{7.33}$$

This allows us to identify and compare the most significant terms in the evolution at a given energy scale. For g_{ρ_1} at the top mass, the five largest terms are

$$\beta_{\rho 1|\mu=M_t} = \left(9.6G_2^2 - 1.8G_2^6\right)10^{-1} + \left(9.8G_1^2G_2^2 - 2.9G_1^2G_2^2 - 2.5G_2^2G_{\rho_1}\right)10^{-2} + \mathcal{O}\left(10^{-3}\right). \tag{7.34}$$

Clearly, the dominant term here is that due to the left-handed weak coupling, while the first term that depends on a scalar coupling is ≈ 40 times smaller. However that term is proportional to g_{ρ_1}, and is negative, so in the interests of understanding if it might be possible to keep g_{ρ_1} perturbative through a judicious choice of scalar couplings we also present the leading terms from the expansion of the full Left-Right Symmetric Model beta function at $\mu = v_{\mathrm{R}}$.

$$\beta_{\rho 1|\mu=v_{\mathrm{R}}} = \left(-1.9G_2^2 G_{\rho_1} + 1.5G_{\rho_1}^2 + 1.0G_2^4\right) + \left(2.4G_2^4 G_{\rho_1} + 2.1G_2^2 G_{\rho_1}^2 \right.$$
$$\left. -1.9G_2^6 - 1.6G_{\rho_1}^3 + 1.2G_1^2 G_2^2 + 1.2G_{\rho_1} G_{\rho_2}\right) 10^{-1} + \mathcal{O}\left(10^{-2}\right). \tag{7.35}$$

Here we can see that although the single largest term is now negative, the two subsequent terms more than compensate for this, with the $g_{\rho_1}^2$ term growing faster with g_{ρ_1} than $g_2^2 g_{\rho_1}$. We also note that the three leading terms are all from one-loop order, corroborating our assertion that the coupling is still perturbative at $\mu = v_{\mathrm{R}}$. The next two negative terms are proportional to g_2^6 and $g_{\rho_1}^3$, with the latter only comparing with the $g_{\rho_1}^2$ term when the coupling is distinctly non-perturbative.

As has been noted previously, we did not include the effect of massive Majorana neutrinos in our calculation, however we shall make an estimate of the effect that their inclusion would have. As seen in Chapter 5, and in the results of [28], the addition of massive right-handed neutrinos has, at least with regard to the scalar self-coupling of the Standard Model, a fairly strong negative effect. However, if we want to compare to a different scalar coupling we should look first at the relative size of similar Yukawa coupling effects. For the g_{ρ_1} coupling, to two loop order, there is only a single term in the beta function for the reduced Left-Right Symmetric Model containing a power of g_t, given, after substitution using (7.33), by

$$\beta_{\rho_1} = \cdots - 2.8G_t^2 G_\gamma^2 10^{-5} \ldots, \tag{7.36}$$

which is 4 orders of magnitude less significant than the leading order terms — and therefore, given a perturbative neutrino Majorana coupling, should still be much less significant than the g_2^4 term. Similarly, if we look at the beta function for the full LRSM at v_{R}, the relevant terms there are

$$\beta_{\rho_1} = \ldots \left(-5.4 G_t^2 G_{\alpha_2}^2 - 2.0 G_t^2 G_{\alpha_1} G_{\alpha_2}\right) 10^{-4} - 2.0 G_t^2 G_{\alpha_1}^2 10^{-5} \ldots \tag{7.37}$$

which are again of a similar scale with respect to the leading order terms.

From these results we can conclude that, barring exceptional values of the scalar couplings, we can expect the large number of scalar particles coupling to SU(2) gauge bosons to push the coupling for the term

$$g_{\rho_1} \left(\mathrm{Tr}\left(\Delta_{\mathrm{R}} \Delta_{\mathrm{R}}^{\dagger}\right)^2 + \mathrm{Tr}\left(\Delta_{\mathrm{L}} \Delta_{\mathrm{L}}^{\dagger}\right)^2 \right) \tag{7.38}$$

to non-perturbative values at an energy far short of the Planck scale. Indeed, it would seem difficult to construct a model that included a scalar triplet and yet forbade such a coupling generically — $\mathrm{Tr}\left(\phi\phi^{\dagger}\right)$ must be a gauge singlet.

Chapter 8.

Conclusions

The major result of this thesis is the creation of a framework within Mathematica for automatically calculating beta functions for all couplings in a general gauge theory model containing gauge, Yukawa, and quartic scalar couplings. We performed a survey of existing results for the two loop Standard Model beta functions, with which to compare our result. In addition to this, the framework was used to calculate the beta functions for a specific model, an example of a Left-Right Symmetric Model, and to model the running of the couplings in the model from the Z boson mass all the way up to the Planck scale, in comparison with two other BSM theories, the Standard Model plus right-handed neutrinos, and the Standard Model plus a fourth generation.

For the Standard Model, we found that, within the 1σ errors on the three of the initial conditions — the top pole mass, the Higgs pole mass and the strong coupling at the Z pole mass — the scalar self coupling could remain positive all the way to the Planck scale. However, with the addition of right-handed neutrinos, or especially a fourth generation of fermions, the g_λ can cross zero at scales far short of the Planck scale. For the right-handed neutrino case, the result is strongly dependent on the value of a, which, if close to zero, could allow for quite significant right-handed neutrino masses to have only a small effect on the evolution of the scalar self coupling.

By calculating the running of the beta functions for the Left-Right Symmetric Model, we found that it should be a generic expectation that the quartic coupling of a scalar triplet — in the case of the LRSM, the term proportional to g_{ρ_1} — should grow steadily, becoming non-perturbative. This is initially due to large contributions from the SU(2) couplings, and as such is unavoidable, without introducing new interactions whose contributions would cancel those due to the SU(2) coupling. As such, we believe that this could potentially be a problem with any model containing a scalar triplet.

More generically, we can see from the results of the four models considered that the balance of scalar coupling running in the Standard Model is difficult to replicate with additional content — having all scalar couplings remain perturbative, and for the right choice of top and Higgs masses, remaining ≥ 0 all the way up to the Planck scale. For both the four generations and right-handed neutrino cases, the additional particle content causes the scalar coupling to become negative at a scale much lower than the Planck scale, while in the Left-Right Symmetric Model, one of the scalar triplet couplings becomes non-perturbative at a scale just above the $\mathrm{SU(2)_L \otimes SU(2)_R}$ breaking scale.

That the scalar couplings are so susceptible to becoming negative is related to the form of their beta functions — the gauge coupling beta functions are all proportional to the relevant gauge coupling cubed, while the top Yukawa beta function is proportional to itself. These beta functions are thus protected against becoming negative, since as they approach zero their beta functions also tend to zero. For scalar couplings, there is no equivalent proportionality, and therefore a sufficiently negative beta function can be unmitigated as the coupling tends to zero.

Several extensions of the work presented should be considered; in addition to the coupling beta functions, reference [11] includes generalised expressions for the wave functions renormalisations for scalar, fermion and vector fields, and methods to calculate these could also be implemented. The example of the Left-Right Symmetric Model also highlights an additional omission from the framework, that of Majorana couplings. As we noted above, the gauge invariance condition for Majorana couplings differs from that of Yukawa couplings, and without that check we decided to omit the couplings. However, an equivalent gauge invariance condition exists, and with such a check we should be able to confidently include the effects of Majorana fermions.

Appendix A.

Mathematica Notebooks

All the notebooks used for the derivation of results in this thesis are available at www.ph.ed.ac.uk/˜s0458973

Chapter 3 `Test_vs_SM_Generalisation.nb`

Derives the beta functions for the Standard Model from the generic beta functions.

Chapter 4 `Integrator_3L_4G_Errors.nb`

Evolves the Standard Model couplings.

Chapter 5 `Integrator_BU_Neutrino.nb`

Evolves the SM+ν_{R} couplings.

Chapter 6 `Integrator_4Gen_Actual.nb`

Evolves the SM4 couplings.

Chapter 7 `LRSM_Generalisation.nb`, `Partial_LRSM.nb`, `Integrator_Generic.nb`

The first two notebooks derive the beta functions for the LRSM and reduced LRSm respectively. The third file takes the beta functions as inputs and evolves all of the LRSM couplings.

All of the notebooks should work as standalone files, except for `Integrator_Generic.nb`, which requires the input of the data files generated by `LRSM_Generalisation.nb` and `Partial_LRSM.nb` which contain the beta functions for their respective models.

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