

TWO PROBLEMS WITH DYNAMICAL SYMMETRY;
COULOMB AND ISOTROPIC OSCILLATOR POTENTIALS
ON A SPHERE

Thesis

submitted by

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DECLARATION

This thesis has been written by me
and the work contained within it is
my own except where otherwise
indicated.

ABSTRACT

This thesis examines some methods of solution of systems with dynamical symmetry by considering the Coulomb and isotropic oscillator potentials on a sphere. It is shown that the classical solutions provide closed orbits which is a criterion for dynamical symmetry. The extra constants of the motion, which are the generators of the symmetry groups, are found and it is shown that the groups are $SO(4)$ and $SU(3)$ respectively. However, the form of the commutation relations of the quantum-mechanical operators prevents the direct use of group representation theory.

An indirect technique, which Pauli used to solve the usual Coulomb problem, is employed to derive the energy eigenvalues and eigenfunctions of both systems. This technique makes use of the matrix elements of the operators in a basis of energy and angular-momentum eigenstates. This is shown to be equivalent to a method of Schrödinger for solving a special class of differential equations. The systems above are generalised to N dimensions and solved by this method.

For systems with dynamical symmetry the Schrödinger equation is separable in more than one set of coordinates. This is equivalent to choosing different bases of eigenstates. The sets of coordinates are found and the equations are separated in them but neither they nor the corresponding algebra of matrix elements has been solved.

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CHAPTER 1

INTRODUCTION

1. Symmetry in Physics

The concept of symmetry plays an important role in the solution of many problems in Physics. Noether's theorem tells us that for each symmetry of a physical system there exists a constant of the motion (e.g. Hill, 1951). Conversely, in Classical Mechanics, we know that "the constants of the motion are the generating functions of those infinitesimal canonical transformations which leave the Hamiltonian invariant" (Goldstein, 1950, p. 261). In Quantum Mechanics also, the constants of the motion will generate symmetry transformations, and wherever there is a symmetry the techniques of Group Theory are available to solve or at least to reduce the difficulty of the problem.

A well-known example is that of a particle in a central potential where the constant angular momentum is a consequence of the rotational invariance of the potential. The representation theory of $SO(3)$ enables us to separate the wave function into a radial part and a spherical harmonic function so that the problem is reduced to that of solving for the radial wave function.

There are however potentials for which there exist constants of the motion not associated with any such geometrical symmetry (e.g. Makarov et al., 1967). The symmetries associated with these constants are termed dynamical and, classically, are canonical transformations which mix the p

q variables. They give rise to the so-called "accidental" degeneracies of some quantum-mechanical systems. The occurrence of dynamical symmetries has provided alternative methods of solution for the well-known isotropic oscillator and Coulomb potentials.

In this thesis the more difficult problems of the oscillator and Coulomb potentials on a sphere will be solved by exploiting their dynamical symmetries. The Lagrangians of the two systems will be introduced in this chapter after a more detailed discussion of the potentials in flat space. The second chapter will be devoted to the oscillator potential and the third to the Coulomb potential. The classical systems are studied in detail and the quantised systems are solved completely for the energy levels and wave functions.

The generalisation of these systems to N dimensions is considered in the fourth chapter. A technique of Schrödinger (1940) for solving a particular class of differential equations is shown to be applicable because of the dynamical symmetry and to be equivalent to that used in the preceding chapters. The systems are once again solved completely.

In the fifth chapter another property of systems with dynamical symmetry is considered, namely the separability of the Schrödinger equation in different sets of coordinates. For each system two such sets of coordinates are found and the equations are separated in them.

2. Dynamical Symmetries for the Coulomb and Oscillator Potentials

In the classical Coulomb problem it was discovered long ago that the Runge-Lenz vector was a constant and could be used to integrate the equations of motion (Pauli, 1926; Lenz, 1924). For negative energy states the orbit obtained is an ellipse. Just as the constancy of the angular momentum vector means that the orbit lies in a fixed plane, so the constancy of the Runge-Lenz vector, which is parallel to the major axis of the ellipse and whose magnitude is proportional to the eccentricity, means that there is no precession of the orbit.

The Poisson Bracket algebra of the two vectors provides the algebra of $SO(4)$ as does the commutator algebra of the quantum-mechanical operators. The representation theory of $SO(4)$ can be used directly to obtain the energy levels of the quantised system and their degeneracy (e.g. Bander and Itzykson, 1966a; Englefield, 1972; Fonda and Ghirardi, 1970), as well as the wave functions.

The method of Pauli (1926) differs from this in that it does not use representation theory directly, but uses relations between the matrix elements of the Runge-Lenz vector in an energy-angular-momentum basis to obtain the same results. This method will be discussed in greater detail later as it is the fundamental technique used in this paper.

The $SO(4)$ symmetry was demonstrated explicitly in momentum space by Fock (1936) whose projection of the

momentum plane onto a sphere in four dimensions provided yet another solution of the Coulomb problem. (See also Bander and Itzykson, 1966a; Englefield, 1972).

The positive energy states give rise to the algebra $SO(3,1)$ and the representation theory of the Lorentz group has been used to analyse the scattering states of the Coulomb potential (e.g. Bander and Itzykson, 1966b). In this case the Fock projection of the momentum space is onto an hyperboloid which is the group-space of the Lorentz group.

For the classical isotropic oscillator potential the extra constant of the motion is a symmetric tensor of rank 2 under the rotation group, whose trace is proportional to the energy. Its eigenvectors are the minor and major axes of the elliptical orbit and the eigenvalues are proportional to the squares of the lengths of these axes (Fradkin, 1965). Once again this implies that there is no precession of the orbit.

In Quantum Mechanics the commutator algebra of the angular momentum and the constant tensor is that of $SU(3)$. The association of this symmetry group with the problem was first described by Jauch and Hill (1940). The representations of $SU(3)$ can be used to provide the energy levels and their degeneracy, although it is usual to reduce the commutator algebra to that of the well-known raising and lowering operators (e.g. Fonda and Ghirardi, 1970, pp. 228). That the technique Pauli used for the Coulomb problem can also be applied to this one will be demonstrated in the next chapter.

3. A Particle on a Sphere

Consider the classical problem of a particle of unit mass moving freely on the surface of a sphere in four dimensions. Let the sphere have radius $\lambda^{-\frac{1}{2}}$. The free Lagrangian

$$L = \frac{1}{2} \delta_{ij} \dot{q}^i \dot{q}^j + \frac{1}{2} \dot{q}^4 \dot{q}^4 \quad [i, j = 1, 2, 3] \quad (1.3.1)$$

is subject to the constraint

$$\lambda^{-1} = \delta_{ij} q^i q^j + q^4 q^4 \quad (1.3.2)$$

$$= q^2 + (q^4)^2 \quad (1.3.3)$$

and becomes

$$L = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j \quad (1.3.4)$$

where

$$g_{ij}(q) = \delta_{ij} + \frac{\lambda q_i q_j}{1 - \lambda q^2} \quad (1.3.5)$$

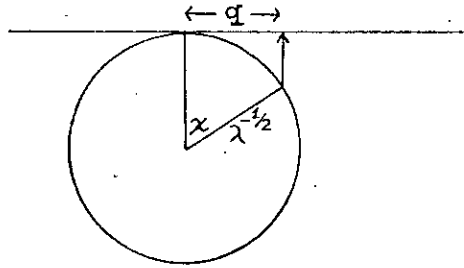
and

$$q_i = q^i \quad (1.3.6)$$

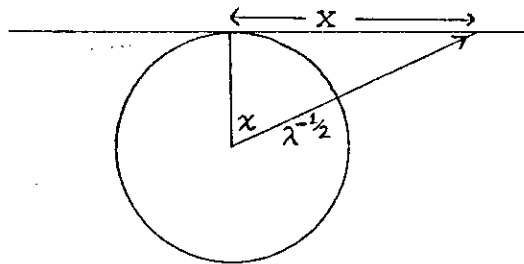
Charap (1973a) has derived this Lagrangian from the chiral-invariant theories of meson-meson interaction. The Lagrangian is clearly invariant under $SO(4)$ and the chiral-invariance arises from the equivalence of $SO(4)$ and $SU(2) \times SU(2)$.

The coordinates q^i are not always the most useful ones to use. Some useful choices have been listed by

Charap (1973b), all of which are projections from the surface of the sphere onto a tangential plane. In this paper two sets of coordinates will be used. The first set which will be consistently designated q^i is that used in (1.3.4) and is obtained from the sphere by the following projection:



The second set, which will be designated x^i is given by the tangential parametrisation:



(The projection angle α is a useful variable in this problem as will be shown in Chapter 3.)

The two sets of coordinates are connected by the equations

$$x^i = q^i (1 - \lambda q^2)^{-\frac{1}{2}} \tag{1.3.7}$$

and

$$q^i = x^i (1 + \lambda x^2)^{-\frac{1}{2}} . \tag{1.3.8}$$

In the x coordinates the Lagrangian becomes

$$L = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j \quad (1.3.9)$$

where

$$g_{ij}(x) = \frac{1}{(1+\lambda x^2)} \left(\delta_{ij} - \frac{\lambda x_i x_j}{1+\lambda x^2} \right) \quad (1.3.10)$$

and

$$x_i = x^i \quad (1.3.11)$$

The Hamiltonian derived from (1.3.4) is

$$H = \frac{1}{2} p_i \hat{g}^{ij} p_j \quad (1.3.12)$$

where

$$p_i = g_{ij} \dot{q}^j \quad (1.3.13)$$

and

$$\hat{g}^{ij} g_{jk} = \delta^i_k \quad (1.3.14)$$

When quantising this Hamiltonian there is an ambiguity in the ordering of the terms. A suitable form can be chosen by the following argument. The quantised Hamiltonian of a free particle in a flat space is proportional to the Laplacian operator and the generalisation of this to a curved space is the Laplace-Beltrami operator

$$\nabla = g^{-\frac{1}{2}} \partial_i g^{\frac{1}{2}} \hat{g}^{ij} \partial_j \quad (1.3.15)$$

where

$$g = \det \|g_{ij}\| \quad (1.3.16)$$

This is a suitable candidate for the quantised Hamiltonian if the derivatives are replaced by an Hermitian momentum

operator. The scalar product of two wave functions is constructed using the invariant integration on a curved space so that the Hermitian momentum is

$$p_i = -i\hbar g^{-\frac{1}{4}} \partial_i g^{\frac{1}{4}} \quad (1.3.17)$$

and the scalar product is

$$\langle \phi | \psi \rangle = \int d^3q g^{\frac{1}{2}}(q) \phi^*(q) \psi(q) \quad (1.3.18)$$

Using (1.3.15) and (1.3.17) the quantised Hamiltonian is

$$H = \frac{1}{2} g^{-\frac{1}{4}} p_i g^{\frac{1}{2}} \hat{g}^{ij} p_j g^{-\frac{1}{4}} \quad (1.3.19)$$

This is not the only possible choice, but De Witt (1957) showed that the various Hamiltonians differ only by terms involving the scalar curvature which in this problem is a constant. Therefore (1.3.19) is an unambiguous choice as far as the physics of the problem is concerned.

Charap (1973b) has solved both the classical and quantised systems in the x coordinates, the latter by using the Schrödinger equation, and obtains discrete energy levels. This is exactly what we would expect for a particle confined to a compact manifold, and applies even when we add in a potential such as the Coulomb potential which normally has continuum states in flat space.

4. The Lagrangians to be Solved

In this thesis I shall be considering the two systems obtained from (1.3.9) by including the potentials $\frac{1}{2}\omega^2 x^2$ and $-\mu/x$ in the Lagrangian. In the former case it is easier to work in the q coordinates so that the two classical Lagrangians are

$$L = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j - \frac{1}{2}\omega^2 q^2 / (1 - \lambda q^2) \quad (1.4.1)$$

and

$$L = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j + \mu/x . \quad (1.4.2)$$

The Hamiltonians, both quantised and classical, are obtained by adding the potentials to the Hamiltonian discussed in the last section.

Why should we expect these systems to exhibit dynamical symmetry? It is not merely because they are analogous to the flat space systems. Makarov et al. (1967) state that "In our opinion, the existence of dynamical symmetries in classical mechanics is connected with the existence of closed paths." (See also Bacry et al., 1966; Stöhle and Han, 1967). Do these systems provide closed paths? It can be shown that the orbit equation for a central potential $V(x)$ is

$$\left(\frac{dx}{d\theta}\right)^2 = \frac{x^4}{L^2} \left[(2E - \lambda L^2) - 2V(x) - \frac{L^2}{x^2} \right]. \quad (1.4.3)$$

(The derivation is the same as that given in Sections 3.1 and 3.2.) This equation is the same as that obtained for flat space but with a different constant. The only potentials which provide closed paths as solutions to this are precisely

those given above (Bertrand, 1873).

When the coordinates for each system have been fixed as in (1.4.1) and (1.4.2), it is only the vector character of the indices under the rotation group that is of interest and the distinction between covariant and contravariant indices can be disregarded. In the work that follows all the indices will be lowered.

The quantised form of the Lagrangian (1.4.1) has been solved completely by Lakshmanan and Eswaran (1975) who use a Schrödinger equation. The energy eigenvalues of the other system have been obtained by Schrödinger (1940). It will be shown in the following chapters that Pauli's method can be used to solve both problems exactly for the energy eigenvalues and the wave functions.

CHAPTER 2

THE OSCILLATOR POTENTIAL

1. The Classical Problem

The Lagrangian of the system is

$$L = \frac{1}{2}g_{ij}(q)\dot{q}_i\dot{q}_j - \frac{1}{2}\omega^2q^2/(1 - \lambda q^2), \quad (1.4.1)$$

where

$$g_{ij}(q) = \delta_{ij} + \lambda q_i q_j / (1 - \lambda q^2). \quad (1.3.5)$$

The momentum conjugate to q_i is

$$p_i = g_{ij} \dot{q}_j \quad (1.3.13)$$

giving the Hamiltonian

$$H = \frac{1}{2}p_i \hat{g}_{ij} p_j + \frac{1}{2}\omega^2q^2/(1 - \lambda q^2) \quad (2.1.1)$$

where

$$\hat{g}_{ij} = \delta_{ij} - \lambda q_i q_j. \quad (2.1.2)$$

The Hamiltonian is rotationally invariant so that

$$L_{ij} = q_i p_j - q_j p_i \quad (2.1.3)$$

is a constant of the motion. From this can be formed the usual angular momentum vector

$$L_i = \frac{1}{2}\epsilon_{ijk} L_{jk} \quad (2.1.4)$$

and its magnitude

$$L^2 = \frac{1}{2}L_{ij} L_{ij}. \quad (2.1.5)$$

The equation of motion obtained from the Lagrangian is

$$\ddot{q}_i + [\lambda g_{k\ell} \dot{q}_k \dot{q}_\ell + \omega^2 / (1 - \lambda q^2)] q_i = 0 \quad (2.1.6)$$

which can be rewritten as

$$\ddot{q}_i + [\bar{\omega}^2 + 2\lambda H] q_i = 0, \quad (2.1.7)$$

while the Hamiltonian can be rearranged to give

$$H + \frac{1}{2} \lambda L^2 = \frac{1}{2} [\dot{q} \cdot \dot{q} + (\omega^2 + 2\lambda H) q^2]. \quad (2.1.8)$$

This is the first integral of the equation of motion.

The constancy of the angular momentum implies in the usual way that the motion is planar and (2.1.7) provides two orthogonal harmonic oscillations so that the orbit must be an ellipse (or a circle or a straight line in the degenerate cases). In fact equations (2.1.7) and (2.1.8) are precisely those obtained for the usual harmonic oscillator if the replacements

$$\bar{\omega}^2 = \omega^2 + 2\lambda H \quad (2.1.9)$$

and

$$\bar{H} = H + \frac{1}{2} \lambda L^2 \quad (2.1.10)$$

are made.

From the equation of motion, it follows directly that

$$S_{ij} = \bar{\omega} q_i q_j + (1/\bar{\omega}) \dot{q}_i \dot{q}_j \quad (2.1.11)$$

is a constant of the motion and that

$$S_{ii} = 2\bar{H}/\bar{\omega}. \quad (2.1.12)$$

This constant is the analogue of that found by Fradkin (1965).

2. The Orbit

By a suitable choice of axes the solution to equation (2.1.7) can be written as

$$\underline{q} = (A \cos \bar{\omega}t, B \sin \bar{\omega}t, 0). \quad (2.2.1)$$

This is an ellipse whose major and minor axes coincide with the coordinate axes. A and B are found by satisfying equations (2.1.5) and (2.1.8). The solutions are

$$A^2 \bar{\omega}^2 = \bar{H} + (\bar{H}^2 - L^2 \bar{\omega}^2)^{\frac{1}{2}} \quad (2.2.2)$$

and

$$B^2 \bar{\omega}^2 = \bar{H} - (\bar{H}^2 - L^2 \bar{\omega}^2)^{\frac{1}{2}} \quad (2.2.3)$$

The two degenerate motions are the straight line with

$$L = 0 \quad (2.2.4)$$

and the circle with

$$\bar{H} = L\bar{\omega}. \quad (2.2.5)$$

(It is simple to show from (2.1.8) that $\bar{H} \geq L\bar{\omega}$.)

The choice of axes in (2.2.1) is very convenient because S_{ij} is diagonalised with

$$S_{11} = A^2 \bar{\omega} \quad , \quad (2.2.6)$$

$$S_{22} = B^2 \bar{\omega} \quad (2.2.7)$$

and

$$S_{33} = 0. \quad (2.2.8)$$

Thus the eigenvectors and eigenvalues of S_{ij} have the same interpretation as for the usual harmonic oscillator solution (Fradkin, 1965). This suggests that the same SU(3) invariance occurs for this problem as for the other.

3. SU(3) Symmetry in the Classical Problem

If a traceless tensor is formed from S_{ij} by

$$N_{ij} = S_{ij} - \delta_{ij} (2\bar{H}/3\bar{\omega}) \quad (2.3.1)$$

then we can look at the Poisson Bracket relations involving L_{ij} and N_{ij} . The three relations have the following forms:

$$\begin{aligned} [L_{ij}, L_{kl}]_{PB} &= \\ & -\delta_{jk} L_{il} + \delta_{jl} L_{ik} + \delta_{ik} L_{jl} - \delta_{il} L_{jk} ; \end{aligned} \quad (2.3.2)$$

$$\begin{aligned} [L_{ij}, N_{kl}]_{PB} &= \\ & -\delta_{jk} N_{il} - \delta_{jl} N_{ik} + \delta_{ik} N_{jl} + \delta_{il} N_{jk} ; \end{aligned} \quad (2.3.3)$$

$$\begin{aligned} [N_{ij}, N_{kl}]_{PB} &= \\ & \{ \delta_{jk} L_{il} + \delta_{jl} L_{ik} + \delta_{ik} L_{jl} + \delta_{il} L_{jk} \} \times \\ & \quad \{ 1 - 2\lambda\bar{H}/3\bar{\omega}^2 \} \\ & - (\lambda/\bar{\omega}) \{ L_{jk} N_{il} + L_{jl} N_{ik} + L_{ik} N_{jl} + L_{il} N_{jk} \} \\ & + (2\lambda/3\bar{\omega}) \delta_{kl} \{ L_{ir} N_{rj} + L_{jr} N_{ri} \} \\ & - (2\lambda/3\bar{\omega}) \delta_{ij} \{ L_{kr} N_{rl} + L_{lr} N_{rk} \} . \end{aligned} \quad (2.3.4)$$

The last equation has the required SU(3) form only when $\lambda = 0$. We can ask whether it is possible to find another constant of the motion, formed from these, that will satisfy the correct equation when $\lambda \neq 0$. Such a choice could be

$$\bar{S}_{ij} = f(H, L^2) S_{ij} + g(H, L^2) L_{is} L_{sj} . \quad (2.3.5)$$

\bar{S}_{ij} is required to be symmetric and this is the most general symmetric tensor possible since all symmetric combinations will reduce to these, for example

$$S_{is} S_{sj} = L_{is} L_{sj} + S_{ss} S_{ij} \quad (2.3.6)$$

If we now define

$$\bar{N}_{ij} = \bar{S}_{ij} - \frac{1}{3} \delta_{ij} \bar{S}_{kk} \quad (2.3.7)$$

and substitute this into (2.3.4) we are led to the following equations:

$$f^2 - S_{kk} fg + g^2 L^2 = 1 \quad ; \quad (2.3.8)$$

$$(\lambda/\bar{\omega}) f + S_{kk} \partial f / \partial (L^2) - 2g - 2L^2 \partial g / \partial (L^2) = 0 \quad ; \quad (2.3.9)$$

$$2f \partial f / \partial (L^2) - S_{kk} f \partial g / \partial (L^2) - g^2 = 0 \quad . \quad (2.3.10)$$

These are the necessary and sufficient conditions that \bar{N}_{ij} satisfies the correct SU(3) equations. (The three equations are not independent since

$$\partial / \partial (L^2) \times (2.3.8) + g \times (2.3.9) = (2.3.10) .)$$

The second equation integrates to give

$$f S_{kk} - 2g L^2 = (2/\bar{\omega}) \alpha(H) \quad (2.3.11)$$

$$= \bar{S}_{kk} \quad (2.3.12)$$

where α is an arbitrary function of H . Together, (2.3.11) and (2.3.8) give

$$f = \frac{(\alpha^2 - L^2 \bar{\omega}^2)^{\frac{1}{2}}}{(\bar{H}^2 - L^2 \bar{\omega}^2)^{\frac{1}{2}}} \quad (2.3.13)$$

and

$$g = \frac{1}{\bar{\omega}L^2} \left\{ \frac{\bar{H}(\alpha^2 - L^2 \bar{\omega}^2)^{\frac{1}{2}}}{(\bar{H}^2 - L^2 \bar{\omega}^2)^{\frac{1}{2}}} - \alpha \right\} \quad (2.3.14)$$

In the limit as $\lambda \rightarrow 0$ we must have $f = 1$ and $g = 0$, therefore

$$\alpha(H) \rightarrow H \quad \text{as} \quad \lambda \rightarrow 0. \quad (2.3.15)$$

Both f and g have a singularity at $\bar{H} = L\bar{\omega}$ which is the condition for a circular orbit. Obviously α must be chosen to cancel out this singularity so as not to exclude these orbits. The choice which does this is

$$\alpha(H) = (\bar{\omega} - \omega)\bar{\omega}/\lambda. \quad (2.3.16)$$

Here I have explicitly demonstrated the $SU(3)$ invariance which was suggested in the previous section. However, it is apparent that this technique is not terribly useful because of the complexity of the calculation. When the equations are quantised the problem becomes even more difficult.

4. The Quantised Hamiltonian

The quantised form of (2.1.1), obtained from (1.3.19) together with (1.4.1), is

$$H = \frac{1}{2}g^{-\frac{1}{4}} p_i g^{\frac{1}{2}} \hat{g}_{ij} p_j g^{-\frac{1}{4}} + \frac{1}{2}\omega^2 q^2 / (1 - \lambda q^2) \quad (2.4.1)$$

where

$$\hat{g}_{ij} = \delta_{ij} - \lambda q_i q_j \quad (2.1.2)$$

and

$$g = (1 - \lambda q^2)^{-1}. \quad (2.4.2)$$

On rearranging the terms in the Hamiltonian we obtain the form

$$H = \frac{1}{2} \{ p_k \hat{g}_{k\ell} p_\ell + kq^2 / (1 - \lambda q^2) + \frac{3}{2} \lambda \hbar^2 \} \quad (2.4.3)$$

where

$$k = \omega^2 + \frac{1}{2} \lambda^2 \hbar^2 . \quad (2.4.4)$$

This differs from the Hamiltonian of Lakshmanan and Eswaran (1975) only in the constant term.

The Hamiltonian is rotationally invariant so L_{ij} is still a constant of the motion but we have to find a suitably quantised form for S_{ij} . The properties we are looking for are that it be symmetric in its indices and Hermitian. As well, it needs to reduce to the appropriate quantities in the limits $\hbar \rightarrow 0$ and $\lambda \rightarrow 0$. Also, it would be useful if it satisfied an equation analogous to (2.1.12).

First we need an Hermitian replacement for \dot{q}_k . Inverting (1.3.13) and symmetrising, we get

$$\pi_k = \frac{1}{2} (\hat{g}_{k\ell} p_\ell + p_\ell \hat{g}_{\ell k}) . \quad (2.4.5)$$

Now if we define

$$\hat{\omega}^2 = k + 2\lambda H , \quad (2.4.6)$$

where k is given by (2.4.4), it can be shown that

$$S_{ij} = \frac{1}{2} \{ q_i \hat{\omega}^2 q_j + q_j \hat{\omega}^2 q_i + \lambda^2 \hbar^2 q_i q_j + \pi_i \pi_j + \pi_j \pi_i - 3\lambda \hbar^2 \delta_{ij} \} (1/\hat{\omega}) \quad (2.4.7)$$

is a constant of the motion. (This result and the following ones are proven using the commutators

$$[Q_i, \pi_j] = i\hbar \hat{g}_{ij} \quad (2.4.8)$$

and

$$[\pi_i, \pi_j] = -i\hbar\lambda L_{ij} .) \quad (2.4.9)$$

It satisfies all the criteria stated above and

$$S_{kk} = 2\bar{H}/\hat{\omega}. \quad (2.4.10)$$

As in (2.3.1) define

$$N_{ij} = S_{ij} - \delta_{ij} (2\bar{H}/3\hat{\omega}) . \quad (2.4.11)$$

The commutation relations analogous to the Poisson Bracket relations are

$$\begin{aligned} & [L_{ij}, L_{kl}] \\ & = i\hbar\{-\delta_{jk} L_{il} + \delta_{jl} L_{ik} + \delta_{ik} L_{jl} - \delta_{il} L_{jk}\}, \end{aligned} \quad (2.4.12)$$

$$\begin{aligned} & [L_{ij}, N_{kl}] \\ & = i\hbar\{-\delta_{jk} N_{il} - \delta_{jl} N_{ik} + \delta_{ik} N_{jl} + \delta_{il} N_{jk}\} \end{aligned} \quad (2.4.13)$$

and

$$\begin{aligned} & [N_{ij}, N_{kl}] \\ & = i\hbar\{\delta_{jk} L_{il} + \delta_{jl} L_{ik} + \delta_{ik} L_{jl} + \delta_{il} L_{jk}\} \times \\ & \quad \{1 - (\lambda^2 \hbar^2 / 4\hat{\omega}^2) - (2\lambda\bar{H}/3\hat{\omega}^2)\} \\ & \quad - i(\lambda\hbar/2\hat{\omega})\{L_{jk} N_{il} + L_{jl} N_{ik} + L_{ik} N_{jl} + L_{il} N_{jk} + \\ & \quad N_{jk} L_{il} + N_{jl} L_{ik} + N_{ik} L_{jl} + N_{il} L_{jk}\} \\ & \quad + i(\lambda\hbar/3\hat{\omega})\delta_{kl}\{L_{ir} N_{rj} + L_{jr} N_{ri} - N_{ir} L_{rj} - N_{jr} L_{ri}\} \end{aligned}$$

$$- i(\lambda\hbar/3\omega) \delta_{ij} \{L_{kr}N_{rl} + L_{lr}N_{rk} - N_{kr}L_{rl} - N_{lr}L_{rk}\}. \quad (2.4.14)$$

This result demonstrates how difficult it would be to carry out the same calculation as in Section 3, because we need to be concerned about the ordering of terms.

There is a further condition connecting L_{ij} and S_{ij} , namely that

$$S_{ij} L_j = 0, \quad (2.4.15)$$

where L_j is defined by (2.1.4), or

$$N_{ij} L_j = - (2\bar{H}/3\omega) L_i. \quad (2.4.16)$$

This result is the analogue of the classical result that the orbit of the particle lies in a fixed plane to which the angular momentum is normal.

5. Method of Solution

In the familiar isotropic oscillator problem where $\lambda = 0$ it is usual to introduce the raising and lowering operators and write the constants of the motion in terms of them (e.g. Fonda and Ghirardi, 1970, pp. 228). This reduces the commutation relations to very simple ones involving these operators.

As an alternative, one could use directly the representation theory of SU(3). The Casimir operators, formed from the Hermitian traceless matrix

$$T_{ij} = N_{ij} + i L_{ij} , \quad (2.5.1)$$

are

$$C_1 = \frac{1}{8} \text{trace} (T^2) \quad (2.5.2)$$

$$= H^2/3\omega^2 - 3/4 \quad (2.5.3)$$

and

$$C_2 = \frac{1}{16} \text{trace} (T^3 - 3T^2) \quad (2.5.4)$$

$$= (H/3\omega) (H^2/3\omega^2 - \frac{3}{4}) . \quad (2.5.5)$$

In the (a,b) representation of SU(3) the Casimir operators take the following values:

$$C_1 = \frac{1}{12}(a - b)^2 + \frac{1}{4}(a + b)(a + b + 4) \quad (2.5.6)$$

and

$$C_2 = \frac{1}{18}(a - b)(2a + b + 3)(2b + a + 3) \quad (2.5.7)$$

(Fonda and Ghirardi, 1970, p. 202). The first of these specifies the value of H while the second, used in conjunction with (2.5.5), allows only the solutions $a = 0$ or $b = 0$, but $a = 0$ gives a negative energy, so that we are restricted to the (a,0) representations of SU(3) and the energy is

$$H = \omega(a + \frac{3}{2}) . \quad (2.5.8)$$

The multiplicity of this representation is equal to the degeneracy of the energy level and is $\frac{1}{2}(a+1)(a+2)$.

Having explained this technique it must now be pointed out that it cannot be used when $\lambda \neq 0$ because we cannot obtain the SU(3) commutation relations that we need, as discussed in the previous section. It should also be apparent that an attempt to find raising and lowering operators will

be bedevilled by exactly the same problems.

It is in overcoming these problems that Pauli's method shows its usefulness. To my knowledge it has never been used in the context of the harmonic oscillator.

I make use of the fact that L_i and N_{ij} are respectively tensors of rank 1 and rank 2 under the rotation group. They can therefore be written in their spherical components as

$$L_0 = L_3, \quad (2.5.9)$$

$$L_{\pm 1} = \mp \frac{1}{\sqrt{2}} (L_1 \pm i L_2) \quad (2.5.10)$$

and

$$N_0 = \frac{\sqrt{3}}{2} N_{33}, \quad (2.5.11)$$

$$N_{\pm 1} = \mp \frac{1}{\sqrt{2}} (N_{13} \pm i N_{23}), \quad (2.5.12)$$

$$N_{\pm 2} = \frac{1}{2\sqrt{2}} (N_{11} - N_{22} \pm 2i N_{12}). \quad (2.5.13)$$

(In the remainder of this chapter $\hbar = 1$.)

Now I introduce the eigenstates $|\alpha, \ell, m\rangle$ where

$$L_0 |\alpha, \ell, m\rangle = m |\alpha, \ell, m\rangle, \quad (2.5.14)$$

$$L^2 |\alpha, \ell, m\rangle = \ell(\ell+1) |\alpha, \ell, m\rangle \quad (2.5.15)$$

and

$$H |\alpha, \ell, m\rangle = E_\alpha |\alpha, \ell, m\rangle. \quad (2.5.16)$$

When we perform calculations using the matrix elements of the operators between the states $\langle \alpha', \ell', m' |$ and $|\alpha, \ell, m\rangle$ we are removing one of the major difficulties

in solving the problem. We are no longer using operators but numbers which are not subject to the problems of ordering. Furthermore, the matrix elements can be simplified by using the Wigner-Eckart theorem (e.g. Edmonds, 1957, p. 75):

$$\begin{aligned} & \langle \alpha', \ell', m' | L_{\sigma} | \alpha, \ell, m \rangle \\ &= (-1)^{\ell' - m'} \begin{pmatrix} \ell' & 1 & \ell \\ -m' & \sigma & m \end{pmatrix} \langle \alpha', \ell' || L || \alpha, \ell \rangle, \end{aligned} \quad (2.5.17)$$

$$\begin{aligned} & \langle \alpha', \ell', m' | N_{\tau} | \alpha, \ell, m \rangle \\ &= (-1)^{\ell' - m'} \begin{pmatrix} \ell' & 2 & \ell \\ -m' & \tau & m \end{pmatrix} \langle \alpha', \ell' || N || \alpha, \ell \rangle. \end{aligned} \quad (2.5.18)$$

The commutation relations (2.4.12), (2.4.13) and (2.4.14) can be rewritten in terms of the spherical components. In practice only the following will be necessary:

$$[L_{+1}, L_{-1}] = -L_0, \quad (2.5.19)$$

$$[L_{+1}, N_{-1}] = -\sqrt{3} N_0 \quad (2.5.20)$$

and

$$\begin{aligned} [N_{+1}, N_{-1}] &= -L_0 \{ 1 - (\lambda^2/4\hat{\omega}^2) - (2\lambda\bar{H}/3\hat{\omega}^2) \\ &+ (\lambda/2\hat{\omega}) \{ L_{+1} N_{-1} + L_{-1} N_{+1} + \frac{2}{\sqrt{3}} L_0 N_0 \\ &+ N_{+1} L_{-1} + N_{-1} L_{+1} + \frac{2}{\sqrt{3}} N_0 L_0 \} \end{aligned} \quad (2.5.21)$$

Equation (2.4.16) can also be written in spherical components as

$$N_{+1} L_{-1} + N_{-1} L_{+1} - \frac{2}{\sqrt{3}} N_0 L_0 = (2\bar{H}/3\hat{\omega}) L_0. \quad (2.5.22)$$

These last four equations are sufficient to provide us with the solution of the problem. We take matrix elements of the equations but first we must have

$$\langle \alpha', \ell', m' | N_\sigma | \alpha, \ell, m \rangle = 0 \text{ unless } E_\alpha = E_{\alpha'}, \quad (2.5.23)$$

since N_σ is a constant of the motion.

Equation (2.5.22) provides us with two results:

$$\langle \alpha, \ell' | |N| | \alpha, \ell \rangle = 0 \text{ unless } \ell' - \ell = 0, \pm 2 \quad (2.5.24)$$

and

$$\begin{aligned} & \langle \alpha, \ell | |N| | \alpha, \ell \rangle \\ &= -\frac{1}{\sqrt{3}} \left[\frac{(2\ell+2)(2\ell+1)(2\ell)}{(2\ell+3)(2\ell-1)} \right]^{\frac{1}{2}} \left[E_\alpha + \frac{1}{2} \lambda \ell(\ell+1) \frac{1}{\omega_\alpha} \right] \end{aligned} \quad (2.5.25)$$

where

$$\hat{\omega}_\alpha^2 = k + 2\lambda E_\alpha. \quad (2.5.26)$$

Equation (2.5.19) gives the familiar result

$$L_{\pm 1} | \alpha, \ell, m \rangle = \bar{\mp} \{ \frac{1}{2} (\bar{\ell} \mp m) (\bar{\ell} \pm m + 1) \}^{\frac{1}{2}} | \alpha, \ell, m \pm 1 \rangle \quad (2.5.27)$$

while (2.5.20) just gives an identity.

The last and most important result is obtained from equation (2.5.21):

$$\begin{aligned} & \frac{8 \langle \alpha, \ell | |N| | \alpha, \ell+2 \rangle \langle \alpha, \ell+2 | |N| | \alpha, \ell \rangle}{(2\ell+4)(2\ell+3)(2\ell+2)} \\ & - 12 \frac{(2\ell-1)}{(2\ell+3)} \frac{\langle \alpha, \ell | |N| | \alpha, \ell \rangle^2}{(2\ell+2)(2\ell+1)(2\ell)} \end{aligned}$$

$$\begin{aligned}
 &= - \left\{ 1 - \frac{\lambda^2}{4\hat{\omega}_\alpha^2} - \frac{2\lambda}{3\hat{\omega}_\alpha^2} \left[E_\alpha + \frac{1}{2}\lambda\ell(\ell+1) \right] \right\} \\
 &+ \frac{2}{\sqrt{3}} \frac{(2\ell-1)(2\ell-3) \langle \alpha, \ell || N || \alpha, \ell \rangle \lambda}{\{(2\ell+3)(2\ell+2)(2\ell+1)(2\ell)(2\ell-1)\}^{\frac{1}{2}} \hat{\omega}_\alpha} \quad (2.5.28)
 \end{aligned}$$

On substituting (2.5.25) into this equation and rearranging the terms we get

$$\begin{aligned}
 &\frac{8 |\langle \alpha, \ell || N || \alpha, \ell+2 \rangle|^2}{(2\ell+4)(2\ell+3)(2\ell+2)} \\
 &= \frac{\left[E_\alpha - \frac{1}{2}\lambda\left(\ell^2+3\ell+\frac{3}{2}\right) \right]^2 - k\left(\ell+\frac{3}{2}\right)^2}{\hat{\omega}_\alpha^2 \left(\ell + \frac{3}{2}\right)^2} \quad (2.5.29)
 \end{aligned}$$

making use of

$$\langle \alpha, \ell+2 || N || \alpha, \ell \rangle^\dagger = \langle \alpha, \ell || N || \alpha, \ell+2 \rangle \quad (2.5.30)$$

The right-hand side of (2.5.29) can be rearranged to give

$$\frac{\lambda^2}{\hat{\omega}_\alpha^2 (2\ell+3)^2} \left\{ \left[\left(\ell+\frac{3}{2}\right)^2 - \left(\frac{3}{4} + \frac{2E_\alpha}{\lambda}\right) \right]^2 - \frac{4k}{\lambda^2} \left(\ell + \frac{3}{2}\right)^2 \right\} \quad (2.5.31)$$

The expression within the bracket is a quadratic in $\left(\ell + \frac{3}{2}\right)^2$ which has two distinct zeros on the positive side of the origin. As ℓ becomes large this expression must become negative at some point, while the left-hand side of (2.5.29) must always be non-negative. From any state $|\alpha, \ell, m\rangle$ we can always generate a state of higher ℓ by means of

$$\begin{aligned}
 N_{\tau} |\alpha, \ell, m\rangle &= \langle \alpha, \ell+2, m+\tau | N_{\tau} | \alpha, \ell, m\rangle | \alpha, \ell+2, m+\tau\rangle \\
 &+ \langle \alpha, \ell, m+\tau | N_{\tau} | \alpha, \ell, m\rangle | \alpha, \ell, m+\tau\rangle \\
 &+ \langle \alpha, \ell-2, m+\tau | N_{\tau} | \alpha, \ell, m\rangle | \alpha, \ell-2, m+\tau\rangle. \quad (2.5.32)
 \end{aligned}$$

unless there exists some integer n for which

$$\langle \alpha, n+2, m+\tau | N_{\tau} | \alpha, n, m\rangle = 0, \quad (2.5.33)$$

or equivalently

$$\langle \alpha, n+2 || N || \alpha, n\rangle = 0. \quad (2.5.34)$$

If we put this into equation (2.5.29) we obtain

$$E_{\alpha} = \sqrt{k} \left(n + \frac{3}{2} \right) + \frac{1}{2} \lambda \left(n^2 + 3n + \frac{3}{2} \right). \quad (2.5.35)$$

This enables us to identify α with n in all the previous results.

Equation (2.5.35) is the result obtained by Lakshmanan and Eswaran (1975) apart from the extra $\frac{3}{2}\lambda$ mentioned in Section 4.

Now the first part of the problem is solved. I will show in the next section that wave functions can be found explicitly using the results obtained in this section.

6. Finding the Wavefunctions

The method to be used here is an analogue of a common method of deriving the angular momentum eigenfunctions $Y_{\ell, m}$ (e.g. Edmonds, 1957, pp. 19). It makes use of recurrence relations obtained from (2.5.32) which generate all the wave functions from just one which can be found easily.

For this we will need to know the reduced matrix elements which can be obtained by substituting (2.5.35) into (2.5.29). I will define for convenience

$$c_{n, \ell} = \frac{2\sqrt{2} \langle n, \ell || N || n, \ell+2 \rangle \hat{\omega}_n}{\{(2\ell+4)(2\ell+3)(2\ell+2)\}^{\frac{1}{2}}} \quad (2.6.1)$$

so that

$$\begin{aligned} |c_{n, \ell}|^2 &= (n-\ell)(n+\ell+3) \left[\sqrt{k} + \frac{1}{2}\lambda(n-\ell) \right] \left[\sqrt{k} + \frac{1}{2}\lambda(n+\ell+3) \right] / \left(\ell + \frac{3}{2} \right)^2 \\ & \quad (2.6.2) \end{aligned}$$

Using (1.3.17) and (2.4.2) we can write

$$p_i = -i \left[\partial_i + \frac{1}{2}\lambda q_i (1-\lambda q^2)^{-1} \right]. \quad (2.6.3)$$

With this, the Hamiltonian (2.4.1) can be put into differential form and on introducing the usual spherical polar coordinates it can be written as

$$\begin{aligned} H &= \frac{1}{2} \left\{ - (1-\lambda q^2) \partial_q^2 - \frac{1}{q} (2 - 3\lambda q^2) \partial_q \right. \\ & \quad \left. + \frac{1}{q^2} L^2 + \omega^2 q^2 / (1 - \lambda q^2) \right\}. \quad (2.6.4) \end{aligned}$$

In applying equation (2.5.32) it is sufficient to find a differential form for N_0 . Making use of all the definitions and introducing a new variable

$$z = \lambda q^2 \quad (2.6.5)$$

we obtain the result that

$$\begin{aligned} N_0 = & \frac{\sqrt{3}}{2} \left\{ \cos^2 \theta \left[2H + \lambda L^2 - \frac{\lambda}{z} L^2 + 6\lambda(1-z) \partial_z \right] \right. \\ & + 2\sin\theta \cos\theta \left[2\lambda(1-z) \partial_z - \frac{\lambda}{z} \right] \partial_\theta \\ & \left. + \frac{\lambda}{z} \sin^2 \theta \partial_\theta^2 - 2\lambda(1-z) \partial_z - \frac{2}{3}(H + \frac{1}{2}\lambda L^2) \right\} \frac{1}{\omega} \end{aligned} \quad (2.6.6)$$

As we are dealing with angular momentum eigenstates, the wave functions associated with them can be written as

$$\psi_{n,\ell,m}(z,\theta,\phi) = X_{n,\ell}(z) Y_{\ell,m}(\theta,\phi) \quad (2.6.7)$$

When substituted in (2.5.32) this provides

$$\begin{aligned} & N_0 X_{n,\ell}(z) Y_{\ell,m}(\theta,\phi) \\ & = \langle n,\ell+2,m | N_0 | n,\ell,m \rangle X_{n,\ell+2}(z) Y_{\ell+2,m}(\theta,\phi) \\ & + \langle n,\ell,m | N_0 | n,\ell,m \rangle X_{n,\ell}(z) Y_{\ell,m}(\theta,\phi) \\ & + \langle n,\ell-2,m | N_0 | n,\ell,m \rangle X_{n,\ell-2}(z) Y_{\ell-2,m}(\theta,\phi) \end{aligned} \quad (2.6.8)$$

Using the orthogonality properties of the $Y_{\ell,m}$ and some associated relations (Edmonds, 1957, pp. 23) we can select in turn each of the three terms on the right-hand side

of (2.6.8). The $Y_{\ell, m}$ term provides an identity in $X_{n, \ell}$ while the other terms provide the recurrence relations that are needed:

$$\begin{aligned} & (2\ell-1)c_{n, \ell-2} X_{n, \ell-2}(z) \\ &= \{ (2n+3)(\sqrt{k+\lambda n} + \frac{1}{2}\lambda) - (2\ell-1)2\lambda(1-z)\partial_z - (2\ell-1)(\ell+1)\lambda/z \\ & \quad + \lambda\ell(\ell+1) - \lambda n(n+1) \} X_{n, \ell}(z) \end{aligned} \quad (2.6.9)$$

and

$$\begin{aligned} & (2\ell+3)c_{n, \ell}^* X_{n, \ell+2}(z) \\ &= \{ (2n+3)(\sqrt{k+\lambda n} + \frac{1}{2}\lambda) + (2\ell+3)2\lambda(1-z)\partial_z - (2\ell+3)\ell\lambda/z \\ & \quad + \lambda\ell(\ell+1) - \lambda n(n+1) \} X_{n, \ell}(z) \end{aligned} \quad (2.6.10)$$

If $\ell = n$ is put into this last equation, the left-hand side vanishes because of (2.6.2) and the result is the simple differential equation

$$\{ 2\lambda(1-z)\partial_z - \lambda n/z + \sqrt{k} + \lambda n + \frac{1}{2}\lambda \} X_{n, n}(z) = 0 \quad (2.6.11)$$

which has the solution

$$\begin{aligned} X_{n, n}(z) &= A_{n, n} z^{n/2} (1-z)^{\sqrt{k}/2\lambda + \frac{1}{4}} & z \leq 1 \\ &= 0 & z > 1 \end{aligned} \quad (2.6.12)$$

if finite boundary conditions at $z = \infty$ are imposed. All the $X_{n, \ell}(z)$ can be generated from this by means of equation

(2.6.9).

The wave functions are to be normalised according to (1.3.18) which leads to the result

$$|A_{n,n}|^2 = \frac{2\lambda^{3/2} \Gamma(\sqrt{k}/\lambda + n + 5/2)}{\Gamma(n + 3/2) \Gamma(\sqrt{k}/\lambda + 1)} \quad (2.6.13)$$

The recurrence relation (2.6.9) guarantees that the $X_{n,\ell}$ constructed from the properly normalised $X_{n,n}$ will themselves be properly normalised and, after some manipulation, I obtain the following result for the wave function:

$$\begin{aligned} & \psi_{n,\ell,m}(z, \theta, \phi) \\ &= \left[\frac{2\lambda^{3/2} (\sqrt{k}/\lambda + n + 3/2) \Gamma(\sqrt{k}/\lambda + \frac{1}{2}(n+\ell+3))}{[\frac{1}{2}(n-\ell)]! \Gamma(\frac{1}{2}(n+\ell+3)) \Gamma(\sqrt{k}/\lambda + \frac{1}{2}(n-\ell) + 1)} \right]^{\frac{1}{2}} \\ & \times z^{-\frac{1}{2}(\ell+1)} (1-z)^{-\sqrt{k}/2\lambda + \frac{1}{4}} \\ & \times \left(\frac{d}{dz} \right)^{\frac{1}{2}(n-\ell)} \left[z^{\frac{1}{2}(n+\ell+1)} (1-z)^{\sqrt{k}/\lambda + \frac{1}{2}(n-\ell)} \right] \\ & \times Y_{\ell,m}(\theta, \phi) \quad \text{for} \quad z \leq 1 \\ &= 0 \quad \text{for} \quad z > 1 \quad (2.6.14) \end{aligned}$$

The phase of the wave function is undetermined because the phases of the matrix elements were undetermined.

In the limit as $\lambda \rightarrow 0$ the wave function becomes

$$\begin{aligned} \psi_{n,\ell,m}(q,\theta,\phi) &= \left[\frac{2\omega^{\ell+3/2}}{[\frac{1}{2}(n-\ell)]! \Gamma(\frac{1}{2}(n+\ell+3))} \right]^{\frac{1}{2}} \\ &\times q^{-\ell-1} e^{\frac{1}{2}\omega q^2} \left(\frac{d}{d(q^2)} \right)^{\frac{1}{2}(n-\ell)} \left[q^{n+\ell+1} e^{-\omega q^2} \right] \\ &\times Y_{\ell,m}(\theta,\phi) \end{aligned} \tag{2.6.15}$$

which is precisely the solution of the usual three-dimensional isotropic oscillator. This function can also be written in terms of the Laguerre polynomials (Erdélyi et al., 1953b, p. 188).

The wave function of (2.6.15) can be written in terms of hypergeometric functions by means of the relations of Erdélyi et al. (1953a, pp. 101) giving

$$\begin{aligned} \psi_{n,\ell,m}(z,\theta,\phi) &= A_{n,\ell} \frac{\Gamma(\frac{1}{2}(n+\ell+3))}{\Gamma(\ell+3/2)} z^{\frac{1}{2}\ell} (1-z)^{\sqrt{k}/2\ell+\frac{1}{4}} \\ &\times F(\sqrt{k}/\ell + \frac{1}{2}(n+\ell+3), -\frac{1}{2}(n-\ell); \ell + 3/2; z) \\ &\times Y_{\ell,m}(\theta,\phi) \end{aligned} \tag{2.6.16}$$

This result is the same as that obtained by Lakshmanan and Eswaran (1975) apart from the factor of $(1-z)^{\frac{1}{4}}$ which is absent in their solution because of their choice of a flat space normalisation. Also their method of solution is not able to provide the correct normalisation constant.

CHAPTER 3

THE COULOMB POTENTIAL

1. The Classical Problem

In this chapter I shall be considering the Hamiltonian

$$H = \frac{1}{2}(1 + \lambda x^2)(p^2 + \lambda(\underline{x} \cdot \underline{p})^2) - \mu/x \quad (3.1.1)$$

and its quantised version. These coordinates have been chosen because they are most appropriate to the solution of the problem although initially the equations are more complicated.

This complication comes entirely from the form of the metric g_{ij} in equation (1.3.10) which has an overall multiplying factor. In fact all choices of coordinates, except the q -coordinates which give (1.3.5), will have some multiplying factor. The q -coordinates, however, are not appropriate to this problem because the potential takes the form $-\mu(1 - \lambda q^2)^{1/2}/q$, from equation (1.3.7), and the unwanted factor $(1 - \lambda q^2)^{1/2}$ keeps appearing in the equations.

Hamilton's equations of motion obtained from (3.1.1) are

$$\dot{x}_k = (1 + \lambda x^2) [p_k + \lambda x_k (\underline{x} \cdot \underline{p})] \quad (3.1.2)$$

and

$$\dot{p}_k = -\lambda [x_k (p^2 + \lambda (\underline{x} \cdot \underline{p})^2) + (1 + \lambda x^2) (\underline{x} \cdot \underline{p}) p_k] - \mu x_k / x^3 \quad (3.1.3)$$

We deduce, either from these equations or directly from the rotational invariance of the Hamiltonian, that the angular momentum is a constant. In this case we use the vector

$$L_i = \epsilon_{ijk} x_j \dot{p}_k \quad (3.1.4)$$

or, if we invert (3.1.2) and substitute it into this equation,

$$L_i = \frac{1}{(1+\lambda x^2)} \epsilon_{ijk} x_j \dot{x}_k \quad (3.1.5)$$

From this we obtain the result that

$$L^2 = \frac{1}{(1+\lambda x^2)^2} [x^2 \dot{x}^2 - (\underline{x} \cdot \dot{\underline{x}})^2] \quad (3.1.6)$$

The second equation of motion (3.1.3) can be cast into a more accessible form if we define

$$\Pi_k = p_k + \lambda x_k (\underline{x} \cdot \underline{p}) \quad (3.1.7)$$

for then

$$\dot{\Pi}_k = -\mu(1 + \lambda x^2) x_k / x^3 \quad (3.1.8)$$

With this result available it is easy to check that

$$R_i = \epsilon_{ijk} L_j \Pi_k + \mu x_i / x \quad (3.1.9)$$

is a constant of the motion. This vector is the analogue of the Runge-Lenz vector discussed in Section 1.2. Its magnitude is given by

$$R^2 = L^2(2E - \lambda L^2) + \mu^2 \quad (3.1.10)$$

where E is the energy numerically equivalent to the Hamiltonian.

What is the physical significance of replacing p_k by Π_k in this problem? In flat space with no potential, p_k is the generator of translations. In this curved space with no potential, Π_k is a constant, using (3.1.8), and therefore must generate a symmetry. We can guess this symmetry if

We calculate the Poisson Bracket relations

$$[L_{ij}, \Pi_k]_{P.B.} = \delta_{ik} \Pi_j - \delta_{jk} \Pi_i \quad (3.1.11)$$

and

$$[\Pi_i, \Pi_j]_{P.B.} = \lambda L_{ij} \quad (3.1.12)$$

This suggests making the identification

$$\Pi_k = \sqrt{\lambda} L_{4k} \quad (3.1.13)$$

$$= -\sqrt{\lambda} L_{4k} \quad (3.1.14)$$

to complete the set of SO(4) generators. This guess can be verified if both sides of (3.1.13) are evaluated using polar coordinates in four dimensions. Therefore the generators of translations in flat space have been replaced by generators of rotations on the sphere.

The reverse process, going from the sphere to the flat space by letting $\lambda = 0$, is an example of a contraction of a Lie group (e.g. Gilmore, 1974, pp. 450) in which the compact group SO(4) gives rise to the non-compact Euclidean group in three dimensions ISO(3) or E(3).

2. The Orbit

Using equation (3.1.2) we can write the energy in the following form

$$2E = \frac{1}{(1+\lambda x^2)} \left[\dot{\underline{x}}^2 - \frac{\lambda (\underline{x} \cdot \dot{\underline{x}})^2}{1+\lambda x^2} \right] - 2\mu/x \quad (3.2.1)$$

and further, using equation (3.1.6) we get

$$\frac{\dot{x}^2}{(1+\lambda x^2)^2} = (2E - \lambda L^2) + \frac{2\mu}{x} - \frac{L^2}{x^2} . \quad (3.2.2)$$

Now introduce the polar coordinates

$$\underline{x} = (x \cos \theta, x \sin \theta, 0) \quad (3.2.3)$$

into equation (3.1.6) to obtain

$$L^2 = \frac{x^4 \dot{\theta}^2}{(1+\lambda x^2)^2} \quad (3.2.4)$$

and into (3.2.2) together with this last result to obtain finally

$$\left(\frac{dx}{d\theta}\right)^2 = x^4 \left(\frac{(2E - \lambda L^2)}{L^2} + \frac{2\mu}{L^2 x} - \frac{1}{x^2} \right) . \quad (3.2.5)$$

This is a standard equation for the Coulomb problem (e.g. Goldstein, 1950, pp. 76) and its solution is

$$\frac{1}{x} = \frac{\mu}{L^2} (1 - \epsilon \cos \theta) \quad (3.2.6)$$

where

$$\epsilon^2 = \frac{1}{\mu^2} [\mu^2 + L^2 (2E - \lambda L^2)] \quad (3.2.7)$$

$$= R^2 / \mu^2 . \quad (3.2.8)$$

The Runge-Lenz vector (3.1.9) now takes the form

$$\underline{R} = (\mu\epsilon, 0, 0) . \quad (3.2.9)$$

If $\epsilon < 1$, equation (3.2.6) describes an ellipse, one of whose foci lies at the origin, and whose major axis lies along $\theta = 0$. The vector \underline{R} also points in this direction and the statement that it is a constant is equivalent to the

statement that there is no precession and the orbit is closed.

The condition that $\epsilon < 1$ is given by the physical condition that

$$E < \frac{1}{2}\lambda L^2 . \quad (3.2.10)$$

If $\epsilon > 1$, the orbit is an hyperbola; if $\epsilon = 1$, a parabola; if $\epsilon = 0$, a circle. We shall see in the next section what these mean in terms of the dynamical symmetries of the problem.

3. The Symmetries of the Classical Problem

As in the last chapter we use the classical Poisson Bracket relations between the constants of the motion to study the symmetry of the problem. These relations are

$$[L_i, L_j]_{P.B.} = \epsilon_{ijk} L_k , \quad (3.3.1)$$

$$[L_i, R_j]_{P.B.} = \epsilon_{ijk} R_k \quad (3.3.2)$$

and

$$[R_i, R_j]_{P.B.} = \epsilon_{ijk} (-2E + 2\lambda L^2) L_k . \quad (3.3.3)$$

This last equation is not in the right form for SO(4) symmetry. If instead we had the equations

$$[L_i, \hat{R}_j]_{P.B.} = \epsilon_{ijk} \hat{R}_k \quad (3.3.4)$$

$$\text{and } [\hat{R}_i, \hat{R}_j]_{P.B.} = \sigma \epsilon_{ijk} L_k \quad (3.3.5)$$

then for $\sigma = 1$ we would have so(4) symmetry and for $\sigma = -1$, SO(3,1) symmetry. In the familiar problem when $\lambda = 0$ this

is the distinction between bound states and scattering states (e.g. Bander and Itzykson, 1966a, 1966b).

Now let

$$\hat{R}_i = f(E, L^2) R_i \quad (3.3.6)$$

Equation (3.3.4) holds immediately while substitution of (3.3.6) into (3.3.5) leads to the condition

$$\frac{\partial}{\partial (L^2)} \{f^2 (\lambda L^4 - 2EL^2 - \mu^2)\} = \sigma \quad (3.3.7)$$

The general solution of this equation is

$$f^2 = \frac{\sigma L^2 + \alpha(E, \lambda)}{\lambda L^4 - 2EL^2 - \mu^2} \quad (3.3.8)$$

where α is an arbitrary function of E . This solution would be quite sufficient except that the denominator is proportional to the square of the eccentricity and is zero for a circular orbit. By a judicious choice of α we can cancel out this singularity.

When $\lambda = 0$ the choice of

$$\alpha(E, 0) = \sigma \mu^2 / 2E \quad (3.3.9)$$

$$\text{provides } f^2 = \sigma / (-2E) \quad (3.3.10)$$

so, for negative energy states $\sigma = 1$ and for positive energy states $\sigma = -1$ as required.

When $\lambda \neq 0$ we can choose

$$\alpha(E, \lambda) = -\frac{\sigma}{\lambda} \{E + \sqrt{E^2 + \lambda \mu^2}\} \quad (3.3.11)$$

which leaves us, after cancellation, with

$$f^2 = \sigma / \{ \lambda L^2 - E + \sqrt{E^2 + \lambda \mu^2} \} \quad (3.3.12)$$

The denominator is always positive so $\sigma = 1$. Therefore the system always has $SO(4)$ symmetry.

We should note here that for positive energy states the limit as $\lambda \rightarrow 0$ of (3.3.12) is infinite. Therefore we cannot obtain $SO(3, 1)$ symmetry in the limit. The reason for this is as follows: when $\lambda \neq 0$ all states are bound states and the limit can only give us bound states, which are those with negative energy. This will be seen also in the quantised solution as the limit of discrete energy levels can only lead to discrete levels. However it will be possible to obtain the scattering states if this limit is combined with a limit of large quantum numbers.

4. The Quantised Hamiltonian

As in the previous chapter the quantised form of (3.1.1) is obtained from (1.3.19) and (1.3.10) and is

$$H = \frac{1}{2} g^{-\frac{1}{4}} p_i g^{\frac{1}{2}} \hat{g}_{ij} p_j g^{-\frac{1}{4}} - \mu/x \quad (3.4.1)$$

$$\text{where } \hat{g}_{ij} = (1 + \lambda x^2) (\delta_{ij} + \lambda x_i x_j) \quad (3.4.2)$$

$$\text{and } g = (1 + \lambda x^2)^{-4} \quad (3.4.3)$$

Rearrangement of the terms in the Hamiltonian provides the form

$$H = \frac{1}{2} p_i \hat{g}_{ij} p_j - \mu/x - 3\lambda\hbar^2(1+\lambda x^2) . \quad (3.4.4)$$

The angular momentum L_i is unchanged from (3.1.4) but we need an Hermitian analogue to the \underline{R} of (3.1.9). It is sufficient to ensure that all the terms are properly symmetrised. First define

$$\Pi_k = p_k + \frac{1}{2}\lambda\{x_k(\underline{x}\cdot\underline{p}) + (\underline{p}\cdot\underline{x})x_k\} \quad (3.4.5)$$

then the symmetrised form of (3.1.9) is

$$R_i = \frac{1}{2}\epsilon_{ijk} \{L_j \Pi_k - \Pi_j L_k\} + \mu x_i/x . \quad (3.4.6)$$

We need to prove that

$$[H, R_i] = 0 , \quad (3.4.7)$$

which can be done by noting that the Hamiltonian can be written as

$$H = \frac{1}{2} \Pi_k \Pi_k + \frac{1}{2}\lambda L^2 - \mu/x \quad (3.4.8)$$

and using the commutators

$$[x_i, \Pi_j] = i\hbar(\delta_{ij} + \lambda x_i x_j) \quad (3.4.9)$$

$$\text{and } [\Pi_i, \Pi_j] = i\hbar\lambda \epsilon_{ijk} L_k . \quad (3.4.10)$$

The magnitude of \underline{R} is given by

$$R^2 = (L^2 + \hbar^2) 2H - \lambda L^4 + \mu^2 - 2\lambda\hbar^2 L^2 \quad (3.4.11)$$

while the condition that the orbit lie in a plane has its analogue in

$$\underline{L}\cdot\underline{R} = 0 . \quad (3.4.12)$$

The commutation relations between the constants are as follows:

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k, \quad (3.4.13)$$

$$[L_i, R_j] = i\hbar \epsilon_{ijk} R_k \quad (3.4.14)$$

and
$$[R_i, R_j] = i\hbar \epsilon_{ijk} (-2H + 2\lambda L^2) L_k. \quad (3.4.15)$$

As discussed in the previous chapter, it would be difficult to cast these equations into the form suitable for the usual group-theoretical solution, so Pauli's method is once again the most suitable one to use.

5. Finding the Energy Eigenvalues

The application of Pauli's technique to this problem is much simpler than that of the previous chapter. The spherical components of \underline{R} have the same form as those of \underline{L} given in (2.5.9) and (2.5.10) and the choice of eigenstates is the same as in (2.5.14), (2.5.15) and (2.5.16). The reduced matrix elements are obtained from (2.5.17) and once again $\hbar = 1$.

The commutation relations (3.4.13), (3.4.14) and (3.4.15) can be written in terms of the spherical components as can equation (3.4.12). Only the last two provide any new information and the two equations necessary are

$$[R_{-1}, R_{+1}] = (-2H + 2\lambda L^2) L_0 \quad (3.5.1)$$

and
$$-L_{+1} R_{-1} - L_{-1} R_{+1} + L_0 R_0 = 0. \quad (3.5.2)$$

We can also make use of the result that

$$R^2 = -R_{+1}R_{-1} - R_{-1}R_{+1} + R_0R_0 \quad (3.5.3)$$

As R is a constant we conclude immediately that

$$\langle \alpha', \ell', m' | R_0 | \alpha, \ell, m \rangle = 0 \text{ unless } E_{\alpha'} = E_{\alpha} \quad (3.5.4)$$

Equation (3.5.2) leads to the result that

$$\langle \alpha, \ell' || R || \alpha, \ell \rangle = 0 \text{ unless } \ell' = \ell \pm 1 \quad (3.5.5)$$

If we also make use of the property of the reduced matrix elements that

$$\langle \alpha, \ell || R || \alpha, \ell+1 \rangle^* = -\langle \alpha, \ell+1 || R || \alpha, \ell \rangle \quad (3.5.6)$$

then we can use equations (3.5.1) and (3.5.3) to obtain the following two results:

$$\begin{aligned} \frac{|\langle \alpha, \ell || R || \alpha, \ell+1 \rangle|^2}{(\ell+1)} - \frac{|\langle \alpha, \ell-1 || R || \alpha, \ell \rangle|^2}{\ell} \\ = (2\ell+1) \{ 2E_{\alpha} - 2\lambda\ell(\ell+1) \} \end{aligned} \quad (3.5.7)$$

$$\begin{aligned} \text{and } |\langle \alpha, \ell || R || \alpha, \ell+1 \rangle|^2 + |\langle \alpha, \ell-1 || R || \alpha, \ell \rangle|^2 \\ = (2\ell+1) \{ (\ell^2 + \ell + 1) 2E_{\alpha} + \mu^2 - \lambda\ell^2(\ell+1)^2 \\ - 2\lambda\ell(\ell+1) \} \end{aligned} \quad (3.5.8)$$

These two equations provide the solution

$$F_{\alpha, \ell} = \frac{|\langle \alpha, \ell || R || \alpha, \ell+1 \rangle|^2}{(\ell+1)} \quad (3.5.9)$$

$$= \mu^2 + 2E_{\alpha}(\ell+1)^2 - \lambda\ell(\ell+2)(\ell+1)^2 \quad (3.5.10)$$

As ℓ becomes large this last expression will become negative unless there exists some integer n for which $F_{\alpha,n} = 0$. When this happens

$$E_{\alpha} = -\frac{1}{2} \frac{\mu^2}{(n+1)^2} + \frac{1}{2} \lambda n(n+2) \quad (3.5.11)$$

and when we identify α with n we have completely solved the energy eigenvalue problem. Substitution of this result into (3.5.10) enables us to write

$$F_{n,\ell} = (n-\ell)(n+\ell+2) \left\{ \frac{\mu^2}{(n+1)^2} + \lambda(\ell+1)^2 \right\} \quad (3.5.12)$$

We can now see that all states are bound states, even those of positive energy, but as $\lambda \rightarrow 0$ we can only retrieve the bound states of the flat-space Coulomb potential as was discussed for the classical problem in Section 3.

6. Calculation of the Wave Functions

The wave functions will be calculated from the equation

$$\begin{aligned} R_0 |n,\ell,m\rangle &= \langle n,\ell+1,m | R_0 |n,\ell,m\rangle |n,\ell+1,m\rangle \\ &+ \langle n,\ell-1,m | R_0 |n,\ell,m\rangle |n,\ell-1,m\rangle \quad (3.6.1) \end{aligned}$$

We define for convenience the quantities

$$f_{n,\ell} = \langle n,\ell+1 || R || n,\ell \rangle / \sqrt{(\ell+1)} \quad (3.6.2)$$

$$\text{so that } F_{n,\ell} = |f_{n,\ell}|^2 \quad (3.6.3)$$

Using (1.3.17) and (3.4.3) we can write

$$p_i = -i \{ \partial_i - 2\lambda x_i (1 + \lambda x^2)^{-1} \} \quad (3.6.4)$$

and from (3.4.5) we get

$$\Pi_k = -i \{ \partial_i + \lambda x_i x_k \partial_k \} \quad (3.6.5)$$

If we now introduce the usual polar coordinates, we can write the spherical components of Π as

$$i\Pi_{\pm 1} = \mp \frac{1}{\sqrt{2}} e^{\pm i\phi} \left\{ (1 + \lambda x^2) \sin\theta \partial_x + \frac{1}{x} \cos\theta \partial_\theta \right. \\ \left. \pm \frac{i}{x} \csc\theta \partial_\phi \right\} \quad (3.6.6)$$

and

$$i\Pi_0 = (1 + \lambda x^2) \cos\theta \partial_x - \frac{1}{x} \sin\theta \partial_\theta \quad (3.6.7)$$

Now we can make the usual separation of the wave function as

$$\psi_{n,\ell,m}(x,\theta,\phi) = \bar{X}_{n,\ell}(x) Y_{\ell,m}(\theta,\phi) \quad (3.6.8)$$

When R_0 is written in terms of spherical components and applied to this wave function as in (3.6.1) we get the following form

$$R_0 \bar{X}_{n,\ell} Y_{\ell,m} = (i\Pi_{+1} \bar{X}_{n,\ell})(L_{-1} Y_{\ell,m}) \\ - (i\Pi_{-1} \bar{X}_{n,\ell})(L_{+1} Y_{\ell,m}) \\ + (i\Pi_0 \bar{X}_{n,\ell}) Y_{\ell,m} \\ + (\mu^{x_0}/x) \bar{X}_{n,\ell} Y_{\ell,m} \quad (3.6.9)$$

By use of the orthogonality properties of the $Y_{\ell,m}$ (e.g. Edmonds, 1957, pp. 23) equation (3.6.1) yields, after a long calculation, two recurrence relations, just as in the previous

chapter, which are

$$\begin{aligned}
 f_{n,\ell} \bar{X}_{n,\ell+1}(x) \\
 = \{(\ell+1)(1+\lambda x^2) \partial_x - \ell(\ell+1)/x + \mu\} \bar{X}_{n,\ell}(x)
 \end{aligned} \tag{3.6.10}$$

$$\begin{aligned}
 f_{n,\ell-1}^* \bar{X}_{n,\ell-1}(x) \\
 = \{-\ell(1+\lambda x^2) \partial_x - \ell(\ell+1)/x + \mu\} \bar{X}_{n,\ell}(x)
 \end{aligned} \tag{3.6.11}$$

These are not the easiest coordinates to work with, so I will change variables by means of

$$\sqrt{\lambda} x = \tan \chi, \tag{3.6.12}$$

where χ is the projection angle introduced in Section 1.3.

$$\text{With } \mu = \sqrt{\lambda} \hat{\alpha} \tag{3.6.13}$$

the recurrence relations become

$$\begin{aligned}
 \bar{X}_{n,\ell+1}(\chi) \\
 = \frac{\sqrt{\lambda}}{f_{n,\ell}} \{(\ell+1) \partial_\chi - \ell(\ell+1) \cot \chi + \hat{\alpha}\} \bar{X}_{n,\ell}(\chi)
 \end{aligned} \tag{3.6.14}$$

and

$$\begin{aligned}
 \bar{X}_{n,\ell-1}(\chi) \\
 = \frac{\sqrt{\lambda}}{f_{n,\ell-1}^*} \{-\ell \partial_\chi - \ell(\ell+1) \cot \chi + \hat{\alpha}\} \bar{X}_{n,\ell}(\chi)
 \end{aligned} \tag{3.6.15}$$

As $f_{n,n} = 0$, from (3.5.12) and (3.6.3) equation (3.6.14) will become a simple differential equation for $\bar{X}_{n,n}$, the solution of which is

$$\bar{X}_{n,n}(\chi) = A_{n,n} (\sin \chi)^n \exp\left(-\frac{\hat{\alpha} \chi}{(n+1)}\right) \tag{3.6.16}$$

I can simplify the recurrence relation (3.6.15) if I make the following definition:

$$\bar{X}_{n,\ell}(\chi) = A_{n,\ell}(\sin\chi)^{-\ell-1} e^{-\frac{1}{2}\hat{\beta}\chi} W_{n,\ell}(\chi) \quad (3.6.17)$$

where $A_{n,\ell-1} = -\frac{1}{2} \frac{\sqrt{\lambda}}{f_{n,\ell-1}^*} A_{n,\ell}$ (3.6.18)

and $\hat{\beta} = 2\hat{\alpha}/(n+1)$. (3.6.19)

From (3.6.16) comes

$$W_{n,n}(\chi) = (\sin\chi)^{2n+1} \quad (3.6.20)$$

and from (3.6.15) comes

$$W_{n,\ell-1}(\chi) = (\csc\chi) \{2\ell\partial_\chi - (n+\ell+1)\hat{\beta}\} W_{n,\ell}(\chi) \quad (3.6.21)$$

These two equations are what is needed to find the complete wave functions of this system. I shall devote the next section to proving that

$$W_{n,\ell}(\chi) = \frac{(2n+1)!}{(n+\ell+1)!} (\sin\chi)^{-(n-\ell+1)} e^{\hat{\beta}\chi} \times \left[(\sin\chi)^2 \frac{d}{d\chi} \right]^{(n-\ell)} \{ (\sin\chi)^{2\ell+2} e^{-\hat{\beta}\chi} \} \quad (3.6.22)$$

The wave functions are to be normalised in accord with (1.3.18), so if we normalise $\psi_{n,n,m}$ we obtain the result that

$$1 = |A_{n,n}|^2 \lambda^{-3/2} \int_0^{\pi/2} d\chi (\sin\chi)^{2n+2} e^{-\hat{\beta}\chi} \quad (3.6.23)$$

This integral is rather complicated to evaluate except in particular limits which will be examined later. However,

in principle, all the $A_{n,\ell}$ can be evaluated from it using (3.6.18).

7. Proof of the Form of the Wave Functions

In this section I wish to prove the following theorem which stated in full is:

Theorem

$$W_{n,\ell}(\chi) = \frac{(2n+1)!}{(n+\ell+1)!} (\sin\chi)^{-(n-\ell+1)} e^{\hat{\beta}\chi} \times \left[(\sin\chi)^2 \frac{d}{d\chi} \right]^{(n-\ell)} \{ (\sin\chi)^{2\ell+2} e^{-\hat{\beta}\chi} \} \quad (3.6.22)$$

satisfies the equations

$$W_{n,\ell-1}(\chi) = (\csc\chi) \{ 2\ell \partial_\chi - (n+\ell+1)\hat{\beta} \} W_{n,\ell}(\chi) \quad (3.6.21)$$

$$\text{and } W_{n,n}(\chi) = (\sin\chi)^{2n+1} \quad (3.6.20)$$

Proof: The last equation is clearly satisfied. In proving the first part of the theorem I shall derive the fully-expanded form of (3.6.22), prove some relations between the coefficients in this form, and show that these relations are those obtained by using the recurrence relation (3.6.21) on equation (3.6.22).

a) The expansion of $\left[(\sin\chi)^2 \frac{d}{d\chi} \right]^{(n-\ell)} \{ (\sin\chi)^{2\ell+2} e^{-\hat{\beta}\chi} \}$

will involve different forms for $(n-\ell)$ even and odd. This is expressed in the following lemma:

Lemma

$$\begin{aligned}
 & \left[(\sin \chi)^2 \frac{d}{d\chi} \right]^{2D} \{ (\sin \chi)^{2n-4D+2} e^{-\hat{\beta}\chi} \} \\
 = & \sum_{s=0}^D \hat{\beta}^{2s} \sum_{r=0}^{D-s} (-1)^r \hat{G}_{r,s}^{D,n} (\sin \chi)^{2n-2D+2r+2s+2} (\cos \chi)^{2D-2r-2s} e^{-\hat{\beta}\chi} \\
 - & \sum_{s=0}^D \hat{\beta}^{2s+1} \sum_{r=0}^{D-s} (-1)^r \hat{H}_{r,s}^{D,n} (\sin \chi)^{2n-2D+2r+2s+3} (\cos \chi)^{2D-2r-2s-1} \\
 & \times e^{-\hat{\beta}\chi} \quad (3.7.1)
 \end{aligned}$$

and

$$\begin{aligned}
 & \left[(\sin \chi)^2 \frac{d}{d\chi} \right]^{2D+1} \{ (\sin \chi)^{2n-4D} e^{-\hat{\beta}\chi} \} \\
 = & \sum_{s=0}^D \hat{\beta}^{2s} \sum_{r=0}^{D-s} (-1)^r \hat{E}_{r,s}^{D,n} (\sin \chi)^{2n-2D+2r+2s+1} (\cos \chi)^{2D-2r-2s+1} \\
 & \times e^{-\hat{\beta}\chi} \\
 - & \sum_{s=0}^D \hat{\beta}^{2s+1} \sum_{r=0}^{D-s} (-1)^r \hat{F}_{r,s}^{D,n} (\sin \chi)^{2n-2D+2r+2s+2} (\cos \chi)^{2D-2r-2s} \\
 & \times e^{-\hat{\beta}\chi} \quad (3.7.2)
 \end{aligned}$$

where

$$\hat{G}_{r,s}^{D,n} = \frac{1}{(r+s)!} \frac{(2D)!}{(2D-2r-2s)!} g_{r,s}^{D,n} \frac{(2n-2D+1)!}{(2n-4D+2r+2s+1)!}, \quad (3.7.3)$$

$$\hat{H}_{r,s}^{D,n} = \frac{1}{(r+s)!} \frac{(2D)!}{(2D-2r-2s-1)!} h_{r,s}^{D,n} \frac{(2n-2D+1)!}{(2n-4D+2r+2s+2)!}, \quad (3.7.4)$$

$$\hat{E}_{r,s}^{D,n} = \frac{1}{(r+s)!} \frac{(2D+1)!}{(2D-2r-2s+1)!} g_{r,s}^{D,n-1} \frac{(2n-2D)!}{(2n-4D+2r+2s-1)!}, \quad (3.7.5)$$

$$\hat{F}_{r,s}^{D,n} = \frac{1}{(r+s)!} \frac{(2D+1)!}{(2D-2r-2s)!} h_{r,s}^{D,n-1} \frac{(2n-2D)!}{(2n-4D+2r+2s)!}, \quad (3.7.6)$$

and where

$$g_{r,0}^{D,n} = \frac{(n-2D+r)!}{(n-2D)!} \quad , \quad (3.7.7)$$

$$g_{r,s}^{D,n} = \frac{1}{2} \sum_{t=0}^r \frac{(n-2D+r+s)!}{(n-2D+t+s)!} h_{t,s-1}^{D,n} \quad s \neq 0 \quad (3.7.8)$$

$$\text{and } h_{r,s}^{D,n} = \sum_{t=0}^r \frac{(r+s)!}{(t+s)!} \frac{(r+s)!}{(t+s)!} \frac{(2t+2s)!}{(2r+2s+1)!} \\ \times \frac{(2n-4D+2r+2s+1)!}{(2n-4D+2t+2s+1)!} \frac{(n-2D+t+s)!}{(n-2D+r+s)!} g_{t,s}^{D,n} \quad (3.7.9)$$

Proof I will need the following subsidiary results:

$$g_{r,s}^{D+1,n+2} = g_{r,s}^{D,n} \quad (3.7.10)$$

and

$$h_{r,s}^{D+1,n+2} = h_{r,s}^{D,n} \quad (3.7.11)$$

which hold because equations (3.7.7), (3.7.8) and (3.7.9) involve n and D only in the combination $n-2D$. The following results also hold:

$$g_{r,s}^{D,n} = (n-2D+r+s) g_{r-1,s}^{D,n} + \frac{1}{2} h_{r,s-1}^{D,n} \quad (3.7.12)$$

$$\text{and } (2r+2s+1) h_{r,s}^{D,n} = (r+s) (2n-4D+2r+2s+1) h_{r-1,s}^{D,n} + g_{r,s}^{D,n} \quad (3.7.13)$$

[Note that it is implicit in these equations and all that follow that any coefficients with negative suffices are defined to be zero.]

Using equations (3.7.1) and (3.7.2)

$$\left[(\sin\chi)^2 \frac{d}{d\chi} \right]^{2D+1} \{ (\sin\chi)^{2n-4D+2} e^{-\hat{\beta}\chi} \} \quad \text{and}$$

$$\left[(\sin\chi)^2 \frac{d}{d\chi} \right]^{2D+2} \{ (\sin\chi)^{2n-4D+2} e^{\hat{\beta}\chi} \} \quad \text{can each be}$$

written in two different ways. These will provide relations connecting the various coefficients as follows:

$$\begin{aligned} \hat{E}_{r,s}^{D,n+1} &= (2n-2D+2r+2s+2) \hat{G}_{r,s}^{D,n} \\ &+ (2D-2r-2s+2) \hat{G}_{r-1,s}^{D,n} + \hat{H}_{r,s-1}^{D,n} \end{aligned} \quad (3.7.14)$$

$$\begin{aligned} \hat{F}_{r,s}^{D,n+1} &= (2n-2D+2r+2s+3) \hat{H}_{r,s}^{D,n} \\ &+ (2D-2r-2s+1) \hat{H}_{r-1,s}^{D,n} + \hat{G}_{r,s}^{D,n} \end{aligned} \quad (3.7.15)$$

$$\begin{aligned} \hat{G}_{r,s}^{D+1,n+2} &= (2n-2D+2r+2s+3) \hat{E}_{r,s}^{D,n+1} \\ &+ (2D-2r-2s+3) \hat{E}_{r-1,s}^{D,n+1} + \hat{F}_{r,s-1}^{D,n+1} \end{aligned} \quad (3.7.16)$$

and

$$\begin{aligned} \hat{H}_{r,s}^{D+1,n+2} &= (2n-2D+2r+2s+4) \hat{F}_{r,s}^{D,n+1} \\ &+ (2D-2r-2s+2) \hat{F}_{r-1,s}^{D,n+1} + \hat{E}_{r,s}^{D,n+1} \end{aligned} \quad (3.7.17)$$

These equations are necessary and sufficient conditions that a set of coefficients satisfies (3.7.1) and (3.7.2). The coefficients given in equations (3.7.3) to (3.7.6) satisfy the above equations as identities. Therefore the lemma is proven.

b) I shall now derive some relations between the coefficients in these expressions but first I will need the following result:

Lemma:

$$g_{r,s}^{D,n-1} = g_{r,s}^{D,n} - (r+s)g_{r-1,s}^{D,n} \quad (3.7.24)$$

$$\text{and } h_{r,s}^{D,n-1} = h_{r,s}^{D,n} - (r+s)h_{r-1,s}^{D,n} \quad (3.7.25)$$

Proof: The proof is inductive on s . I require the result that

$$\begin{aligned} \frac{(n-2D+r)!}{(n-2D)!} &= \sum_{t=0}^r \frac{r!}{t!} \frac{r!}{t!} \frac{(2t)!}{(2r+1)!} \frac{(2n-4D+2r+1)!}{(2n-4D+2t+1)!} \\ &\times \frac{(n-2D+t)!}{(n-2D+r)!} \frac{(n-2D+t-1)!}{(n-2D-1)!} \end{aligned} \quad (3.7.26)$$

which is itself proven by induction on r . The details of it are straightforward.

Equation (3.7.24) is true for $s=0$ using (3.7.7), as is (3.7.25) using (3.7.9). Now, if (3.7.24) and (3.7.25) are true for $s-1$ then they can be shown to be true for s by using (3.7.8) and (3.7.9). Therefore they must be true for all values of s and the proof of the lemma is complete.

Using this result the following relations between the coefficients defined in (3.7.18) - (3.7.21) can be proven:

$$\begin{aligned} E_{r,s}^{D,n} &= (2n-4D)(2n-4D+2r+2s+1) G_{r,s}^{D,n} \\ &+ (2n-4D)(2D-2r-2s+2) G_{r-1,s}^{D,n} + (2n-2D+1) H_{r,s-1}^{D,n} \end{aligned} \quad (3.7.27)$$

$$\begin{aligned}
 F_{r,s}^{D,n} &= (2n-4D)(2n-4D+2r+2s+2) H_{r,s}^{D,n} \\
 &+ (2n-4D)(2D-2r-2s+1) H_{r-1,s}^{D,n} + (2n-2D+1) G_{r,s}^{D,n},
 \end{aligned}
 \tag{3.7.28}$$

$$\begin{aligned}
 G_{r,s}^{D+1,n} &= (2n-4D-2)(2n-4D+2r+2s-1) E_{r,s}^{D,n} \\
 &+ (2n-4D-2)(2D-2r-2s+3) E_{r-1,s}^{D,n} + (2n-2D) F_{r,s-1}^{D,n}
 \end{aligned}
 \tag{3.7.29}$$

$$\begin{aligned}
 \text{and } H_{r,s}^{D+1,n} &= (2n-4D-2)(2n-4D+2r+2s) F_{r,s}^{D,n} \\
 &+ (2n-4D-2)(2D-2r-2s+2) F_{r-1,s}^{D,n} + (2n-2D) E_{r,s}^{D,n}.
 \end{aligned}
 \tag{3.7.30}$$

c) All that remains to the proof of the theorem is to show that these last equations are the necessary and sufficient conditions that the forms defined in (3.7.22) and (3.7.23) satisfy the recurrence relation (3.6.21). This can be done by direct substitution in the relation. Thus the theorem is proven.

8. Some Properties of the Wave Functions

Two properties that will be examined here are the forms the wave functions take in the limits as $\lambda \rightarrow 0$ and as $\mu \rightarrow 0$. The first should be the usual solution to the bound state Coulomb problem while the second should correspond to the solution found by Charap (1973b). The latter should also correspond in part to the limiting solution of the oscillator problem as $\omega \rightarrow 0$.



a) In finding the limit as $\lambda \rightarrow 0$ of the radial wave functions it is useful to cast them into the following slightly more compact forms using (3.7.22), (3.7.23) and (3.6.17):

$$\begin{aligned} \bar{X}_{n,n-2D}(\chi) &= A_{n,n-2D} e^{-\frac{1}{2}\hat{\beta}\chi} (\tan\chi)^{n-2D} (\cos\chi)^n \\ &\times \left\{ \sum_{s=0}^D \hat{\beta}^{2s} \sum_{r=0}^{D-s} (-1)^r G_{r,s}^{D,n} (\tan\chi)^{2r+2s} \right. \\ &\left. - \sum_{s=0}^D \hat{\beta}^{2s+1} \sum_{r=0}^{D-s} (-1)^r H_{r,s}^{D,n} (\tan\chi)^{2r+2s+1} \right\} \end{aligned} \quad (3.8.1)$$

and

$$\begin{aligned} \bar{X}_{n,n-2D-1}(\chi) &= A_{n,n-2D-1} e^{-\frac{1}{2}\hat{\beta}\chi} (\tan\chi)^{n-2D-1} (\cos\chi)^n \\ &\times \left\{ \sum_{s=0}^D \hat{\beta}^{2s} \sum_{r=0}^{D-s} (-1)^r E_{r,s}^{D,n} (\tan\chi)^{2r+2s} \right. \\ &\left. - \sum_{s=0}^D \hat{\beta}^{2s+1} \sum_{r=0}^{D-s} (-1)^r F_{r,s}^{D,n} (\tan\chi)^{2r+2s+1} \right\}. \end{aligned} \quad (3.8.2)$$

When χ is replaced by x using (3.6.12), each term in the sums contains a factor λ^r so that in the limit only the $r = 0$ terms contribute. The coefficients can be evaluated by using the results that

$$g_{0,s}^{D,n} = s!/(2s)! \quad (3.8.3)$$

and

$$h_{0,s}^{D,n} = s!/(2s+1)! \quad (3.8.4)$$

which are obtained from (3.7.8) and (3.7.9). When the limit

of the normalisation constant has also been calculated, the results can be compared with the formulae for Laguerre polynomials given by Erdélyi et al. (1953b, p. 188). The solution obtained by this is

$$\begin{aligned} \psi_{n,\ell,m}(x,\theta,\phi) &= \\ &= \frac{2}{(n+1)^2} \left\{ \frac{(n-\ell)!}{(n+\ell+1)!} \right\}^{\frac{1}{2}} \\ &\times e^{-\frac{\mu x}{n+1}} \left(\frac{2\mu x}{n+1} \right)^\ell L_{n-\ell}^{2\ell+1} \left(\frac{2\mu x}{n+1} \right) Y_{\ell,m}(\theta,\phi) \end{aligned} \quad (3.8.5)$$

which is the familiar solution to the bound state Coulomb problem.

b) When $\mu = 0$ only the $\hat{\beta}^0$ terms in (3.8.1) and (3.8.2) remain and the radial wave functions can be rewritten as

$$\begin{aligned} \bar{\chi}_{n,n-2D}(\chi) &= A_{n,n-2D}(\mu=0) (\sin\chi)^{n-2D} \\ &\times \sum_{r=0}^D (-1)^r G_{r,0}^{D,n}(\sin\chi)^{2r} (\cos\chi)^{2D-2r} \end{aligned} \quad (3.8.6)$$

$$\begin{aligned} &= A_{n,n-2D}(\mu=0) (\sin\chi)^{n-2D} \\ &\times \sum_{t=0}^D (-1)^t \left\{ \sum_{s=t}^D \binom{s}{s-t} G_{s,0}^{D,n} \right\} (\cos\chi)^{2D-2t} \end{aligned} \quad (3.8.7)$$

and

$$\begin{aligned} \bar{X}_{n,n-2D-1}(\chi) &= A_{n,n-2D-1}(\mu=0) (\sin\chi)^{n-2D-1} \\ &\quad \sum_{r=0}^D (-1)^r E_{r,s}^{D,n} (\sin\chi)^{2r} (\cos\chi)^{2D+1-2r} \end{aligned} \quad (3.8.8)$$

$$\begin{aligned} &= A_{n,n-2D-1}(\mu=0) (\sin\chi)^{n-2D-1} \\ &\quad \sum_{t=0}^D (-1)^t \left\{ \sum_{s=t}^D \binom{s}{s-t} E_{s,0}^{D,n} \right\} (\cos\chi)^{2D+1-2t} \end{aligned} \quad (3.8.9)$$

The terms in the brackets can be evaluated by using the following result:

Lemma:

$$\sum_{r=0}^D \frac{(2D)! (n-2D+r)!}{r! (2D-2r)! (2n-4D+2r+1)!} = 2^{2D} \frac{n!}{(2n-2D+1)!} \quad (3.8.10)$$

and

$$\sum_{r=0}^D \frac{(2D+1)! (n-2D-1+r)!}{r! (2D+1-2r)! (2n-4D+2r-1)!} = 2^{2D+1} \frac{n!}{(2n-2D)!} \quad (3.8.11)$$

Proof: This can be done by induction on D . The results must hold for $D = 0$ and if both hold for D then they hold for $D+1$.

Now substitute $n-t$ for n , $D-t$ for D and $s-t$ for r in the above expressions. The resultant ones can be used to prove the following results, using (3.7.3), (3.7.5), (3.7.18) and (3.7.20):

$$\sum_{s=t}^D \binom{s}{s-t} G_{s,0}^{D,n} = \frac{2^{2D-2t} (2n+1)! (n-t)! (2D)!}{t! (2n-2D+1)! (n-2D)! (2D-2t)!} \quad (3.8.12)$$

and

$$\sum_{s=t}^D \binom{s}{s-t} E_{s,0}^{D,n} = \frac{2^{2D+1-2t} (2n+1)! (n-t)! (2D+1)!}{t! (2n-2D)! (n-2D-1)! (2D+1-2t)!} \quad (3.8.13)$$

Once again, using the expressions of Erdélyi et al. (1953b, p. 175) for Gegenbauer polynomials, equations (3.8.7) and (3.8.9) can be jointly written as

$$\bar{X}_{n,\ell}(\chi) = 2^{\ell+1} \ell! \left[\frac{(n+1)(n-\ell)! \lambda^{3/2}}{\pi (n+\ell+1)!} \right]^{\frac{1}{2}} \times (\sin \chi)^\ell C_{n-\ell}^{\ell+1}(\cos \chi) \quad (3.8.14)$$

The energy eigenvalue associated with this wave function is

$$E_n = \frac{1}{2} \lambda n(n+2) \quad (3.8.15)$$

This solution is the one obtained by Charap (1973b).

There must be some connection between these wave functions and those obtained for the oscillator problem in (2.6.14) with $\omega = 0$. However, there are important differences. In the Coulomb potential, the ℓ states within any energy level decrease by steps of 1 unit, while in the oscillator potential, the ℓ states decrease by steps of 2. Also, if we take the energy eigenvalues for $\omega = 0$ from (2.5.35) and (2.4.4) we obtain

$$E_{n'} = \frac{1}{2} \lambda (n'+1)(n'+3) \quad (3.8.16)$$

which provides the same levels as equation (3.8.15) except that the zero level is missing. The wave function of this

level is a constant from (3.8.14). Such a level cannot exist under the conditions of the oscillator problem because we require the wave function (2.6.14) to be zero at the equator of the sphere, i.e. $q = \lambda^{-\frac{1}{2}}$ or $\chi = \pi/2$.

(The oscillator wave functions must be zero at the equator because the potential is infinite here and provides a barrier separating the two hemispheres.) In fact, none of the wave functions of (3.8.14) with $n-l = 2D$ are zero at the equator so they cannot correspond to wave functions obtained from (2.6.14). If we choose the remaining wave functions where $n-l = 2D+1$ and identify n with $n'+1$ we can convert the Gegenbauer polynomials of (3.8.14) into a hypergeometric function in $z = \sin^2\chi$, using a formula of Erdélyi et al. (1953b, p. 176) and obtain the solutions to (2.6.16) with $\omega = 0$.

The oscillator wave functions retain their SU(3) symmetry when $\omega = 0$. Therefore the procedure just outlined selects wave functions with SU(3) symmetry from wave functions with SO(4) symmetry. Ravenhall et al. (1967) have shown that a similar phenomenon occurs in flat space when an infinite barrier is introduced into a Coulomb potential. It appears therefore that it is a relation between the representations of SU(3) and of SO(4) rather than a specific property of the potentials.

CHAPTER 4

GENERALISATION TO N DIMENSIONS

1. The Basic Hamiltonian

The classical Lagrangian (1.3.4) can be generalised to N dimensions merely by letting the indices range between 1 and N. The particle is now moving freely on a sphere in an N+1 dimensional space and the system is SO(N+1) invariant. Similarly, the quantisation procedure of Section 1.3 can be carried through unchanged, (except that in the x coordinates

$$g(x) = (1 + \lambda x^2)^{-N-1}, \quad (4.1.1)$$

and the quantised Hamiltonian is given by (1.3.19). The constant L_{ij} is defined in the usual way.

Working in the x coordinates, Π_k can be defined as in (3.4.5) and the Hamiltonian can be written as

$$H = \frac{1}{2} \Pi_k \Pi_k + \frac{1}{4} \lambda L_{jk} L_{jk}. \quad (4.1.2)$$

This form has been found already in equation (3.4.8) and its significance is clear when related to the classical results of Section 3.1. The basic commutation relations are

$$[x_i, \Pi_j] = i(\delta_{ij} + \lambda x_i x_j) \quad (4.1.3)$$

and

$$[\Pi_i, \Pi_j] = i\lambda L_{ij}. \quad (4.1.4)$$

With these it is straightforward to prove that

$$[H, \Pi_i] = 0 \quad (4.1.5)$$

and

$$[L_{ij}, \Pi_k] = i(\delta_{ik} \Pi_j - \delta_{jk} \Pi_i). \quad (4.1.6)$$

The L_{ij} satisfy the usual commutation relation (2.4.12) and we can conclude therefore that the Π_k are the extra generators of $SO(N+1)$ and can define

$$\Pi_k = -\sqrt{\lambda} L_{k, N+1} \quad (4.1.7)$$

$$= \sqrt{\lambda} L_{N+1, k} \quad (4.1.8)$$

If we write the larger set of generators as $L_{\mu\nu}$ then the Hamiltonian (4.1.2) becomes

$$H = \frac{1}{4}\lambda L_{\mu\nu} L_{\mu\nu} \quad (4.1.9)$$

$$= \frac{1}{2}\lambda L_{N+1}^2 \quad (4.1.10)$$

The energy eigenvalues will be given by the eigenvalues of this operator in representations of $SO(N+1)$ and are

$$E_n = \frac{1}{2}\lambda n(n+N-1) \quad (4.1.11)$$

(e.g. Stone, 1965).

2. The Coulomb and Oscillator Potentials

The form of the potentials is the same as in the earlier chapters. In this section I will list the constants of the motion and their commutation relations. The results can all be found in the same manner as before.

a) Coulomb potential

The generalised Runge-Lenz vector is a vector under the action of the SO(N) subgroup generated by the L_{ij} . It has the form

$$R_i = \frac{1}{2}(L_{ki}\Pi_k - \Pi_k L_{ik}) + \mu x_i/x \quad (4.2.1)$$

which is the same as (3.4.6). Its commutation relations are

$$[L_{ij}, R_k] = i(\delta_{ik}R_j - \delta_{jk}R_i) \quad (4.2.2)$$

and

$$[R_i, R_j] = i L_{ij} (-2H + 2\lambda L_N^2 + \frac{1}{2}\lambda(N-3)^2). \quad (4.2.3)$$

b) Oscillator potential

The symmetric tensor under SO(N) has the form

$$\begin{aligned} N_{ij} = & \frac{1}{2}\{q_i \hat{\omega}^2 q_j + q_j \hat{\omega}^2 q_i + \frac{1}{2}(N^2 - 2N - 1)\lambda^2 q_i q_j \\ & + \pi_i \pi_j + \pi_j \pi_i - N\lambda \delta_{ij}\} (1/\hat{\omega}) \\ & - \delta_{ij} (2H + \lambda L_N^2) / (N\hat{\omega}) \end{aligned} \quad (4.2.4)$$

where all the quantities are the same as in (2.4.7). Its commutation relations are

$$[L_{ij}, N_{k\ell}] = i\{-\delta_{jk}N_{i\ell} - \delta_{j\ell}N_{ik} + \delta_{ik}N_{j\ell} + \delta_{i\ell}N_{jk}\} \quad (4.2.5)$$

and

$$\begin{aligned}
 [\bar{N}_{ij}, \bar{N}_{kl}] &= i\{\delta_{jk}L_{il} + \delta_{jl}L_{ik} + \delta_{ik}L_{jl} + \delta_{il}L_{jk}\} \\
 &\quad \times \{1 + \frac{1}{4}(N^2 - 4N + 2)\lambda^2/\hat{\omega}^2 \\
 &\quad - (2H + \lambda L_N^2)(\lambda/N\hat{\omega}^2)\} \\
 &\quad - i(\lambda/2\hat{\omega})\{L_{jk}N_{il} + L_{jl}N_{ik} + L_{ik}N_{jl} + L_{il}N_{jk} \\
 &\quad + N_{jk}L_{il} + N_{jl}L_{ik} + N_{ik}L_{jl} + N_{il}L_{jk}\} \\
 &\quad + i(\lambda/N\hat{\omega})\delta_{kl}\{L_{ir}N_{rj} + L_{jr}N_{ri} - N_{ir}L_{rj} - N_{jr}L_{ri}\} \\
 &\quad - i(\lambda/N\hat{\omega})\delta_{ij}\{L_{kr}N_{rl} + L_{lr}N_{rk} - N_{kr}L_{rl} - N_{lr}L_{rk}\} \quad (4.2.6)
 \end{aligned}$$

In principle the same techniques could be applied here as in the earlier chapters. These constants are operators which connect different representations of the SO(N) subgroup and give rise to the raising and lowering operators from which we can generate the energy eigenvalues and eigenfunctions. However, the algebra would be extremely complicated. We can find these operators more directly by using a technique of Schrödinger (1940a, 1940b) in which the symmetry of the problem is bypassed. However, the technique works because of the symmetry.

3. Construction of the Operators

The Schrödinger equation for the basic Hamiltonian can be constructed directly from the Laplace-Beltrami operator (1.3.15) which can be written in the form

$$\begin{aligned} \nabla^2 = & \frac{(1+\lambda x^2)^2}{x^2} (x_i \partial_i)^2 + \frac{(N-2)(1+\lambda x^2)}{x^2} (x_i \partial_i) \\ & + \lambda(1+\lambda x^2) (x_i \partial_i) - \frac{(1+\lambda x^2)}{x^2} L_N^2. \end{aligned} \quad (4.3.1)$$

If we introduce some type of hyperspherical polar coordinates (e.g. Erdélyi et al., 1953b, p. 233) then the angular part is all contained in L_N^2 . Therefore the wave function can be separated into a radial part and a hyperspherical harmonic, that is, an eigenfunction of L_N^2 . We do not need to know the form of this function as we are only interested in the radial wave equation which is

$$\begin{aligned} & \{ -(1+\lambda x^2) \partial_x (1+\lambda x^2) \partial_x - (N-1) \frac{(1+\lambda x^2)}{x} \partial_x \\ & + \frac{(1+\lambda x^2)}{x^2} \ell(\ell+N-2) + 2V(x) - 2E_n \} X_{n,\ell}(x) = 0. \end{aligned} \quad (4.3.2)$$

Now introducing the angle χ as in (3.6.12) we can construct the equations for the two potentials:

$$\begin{aligned} & \{ \partial_\chi^2 + (N-1) \cot \chi \partial_\chi - \ell(\ell+N-2) \csc^2 \chi \\ & + 2\hat{\alpha} \cot \chi + 2E_n/\lambda \} \bar{X}_{n,\ell}(\chi) = 0 \end{aligned} \quad (4.3.3)$$

and

$$\begin{aligned} & \{ \partial_\chi^2 + (N-1) \cot \chi \partial_\chi - \ell(\ell+N-2) \csc^2 \chi \\ & - (\omega^2/\lambda^2) \tan^2 \chi + 2E_n/\lambda \} X_{n,\ell}(\chi) = 0. \end{aligned} \quad (4.3.4)$$

a) Coulomb potential

Extrapolating from equations (3.6.14) and (3.6.15) I define the following differential operators:

$$O_+ = \partial_\chi - \ell \cot \chi + \hat{\alpha}/(\ell + \frac{1}{2}(N-1)) \quad (4.3.5)$$

and

$$O_- = -\partial_\chi - (\ell + N - 1) \cot \chi + \hat{\alpha}/(\ell + \frac{1}{2}(N-1)). \quad (4.3.6)$$

The radial wave equation (4.3.3) can be written in two ways using these operators:

$$O_+ O_- \bar{X}_{n, \ell+1} - \{\hat{\alpha}^2 / (\ell + \frac{1}{2}(N-1))^2 - \ell(\ell + N - 1) + 2E_n / \lambda\} \bar{X}_{n, \ell+1} = 0 \quad (4.3.7)$$

and

$$O_- O_+ \bar{X}_{n, \ell} - \{\hat{\alpha}^2 / (\ell + \frac{1}{2}(N-1))^2 - \ell(\ell + N - 1) + 2E_n / \lambda\} \bar{X}_{n, \ell} = 0 \quad (4.3.8)$$

It is clear that $O_- \bar{X}_{n, \ell+1}$ is a solution of the second equation while $O_+ \bar{X}_{n, \ell}$ is a solution of the first so that we have the raising and lowering operators. These are not the same as those used by Schrödinger (1940a) but are the most suitable ones here as they agree with (3.6.14) and (3.6.15).

The wave functions are normalised according to (1.3.18) and, assuming that the hyperspherical harmonics are normalised to unity, the normalisation of the radial functions is

$$(\bar{X}_{n', \ell'}, \bar{X}_{n, \ell}) = \lambda^{-N/2} \int_0^{\pi/2} d\chi (\sin \chi)^{N-1} \bar{X}_{n', \ell'}^* \bar{X}_{n, \ell} \quad (4.3.9)$$

$$= \delta_{n', n} \delta_{\ell', \ell}. \quad (4.3.10)$$

With respect to this normalisation the above operators are adjoint. Therefore the constant term in (4.3.7) and (4.3.8) must be non-negative. By the usual arguments there must be an integer n such that this term is zero when $\ell = n$.

Therefore

$$E_n = -\frac{1}{2} \frac{\mu^2}{(n+\frac{1}{2}(N-1))^2} + \frac{1}{2}\lambda n(n+N-1) \quad (4.3.11)$$

and

$$O_- O_+ \bar{X}_{n,\ell} = \frac{(n-\ell)(n+\ell+N-1) \{ \hat{\alpha}^2 / (n+\frac{1}{2}(N-1))^2 + (\ell+\frac{1}{2}(N-1))^2 \} \bar{X}_{n,\ell}}{(\ell+\frac{1}{2}(N-1))^2} \quad (4.3.12)$$

$$= \frac{|f_{n,\ell}|^2}{\lambda (\ell+\frac{1}{2}(N-1))^2} \bar{X}_{n,\ell} \quad (4.3.13)$$

Making the assumption that the $\bar{X}_{n,\ell}$ are real the following equations hold:

$$O_+ \bar{X}_{n,\ell} = - |f_{n,\ell}| / [\sqrt{\lambda} (\ell+\frac{1}{2}(N-1))] \bar{X}_{n,\ell+1} \quad (4.3.14)$$

and

$$O_- \bar{X}_{n,\ell+1} = - |f_{n,\ell}| / [\sqrt{\lambda} (\ell+\frac{1}{2}(N-1))] \bar{X}_{n,\ell} \quad (4.3.15)$$

b) Oscillator potential

The operators which generalise (2.6.9) and (2.6.10) are

$$\hat{O}_+ = (2\ell+N) \cot \chi \partial \chi - (2\ell+N) \ell \cot^2 \chi - \ell(\ell+N-1) + 2E_n/\lambda \quad (4.3.16)$$

and

$$\begin{aligned} \hat{O}_- = & -(2\ell+N)\cot\chi\partial\chi - (2\ell+N)(\ell+N)\cot^2\chi \\ & - (\ell+1)(\ell+N) + 2E_n/\lambda \end{aligned} \quad (4.3.17)$$

with which the radial wave equation can be written as

$$\begin{aligned} \hat{O}_+\hat{O}_- X_{n,\ell+2} - \{ [2E_n/\lambda - (\ell+1)(\ell+N)] [2E_n/\lambda - \ell(\ell+N-1)] \\ - (2\ell+N)^2\omega^2/\lambda^2 \} X_{n,\ell+2} = 0 \end{aligned} \quad (4.3.18)$$

or

$$\begin{aligned} \hat{O}_-\hat{O}_+ X_{n,\ell} - \{ [2E_n/\lambda - (\ell+1)(\ell+N)] [2E_n/\lambda - \ell(\ell+N-1)] \\ - (2\ell+N)^2\omega^2/\lambda^2 \} X_{n,\ell} = 0 \end{aligned} \quad (4.3.19)$$

That these operators are adjoint is not apparent from their form. However, for consistency for $N = 3$ I assume that they are. The constant term, which must be non-negative, can be rewritten as

$$[2E_n/\lambda - (\ell^2 + N\ell + \frac{1}{2}N)]^2 - (2\ell+N)^2 k/\lambda^2 \quad (4.3.20)$$

$$= [(\ell + \frac{1}{2}N)^2 - (\frac{1}{4}N^2 - \frac{1}{2}N + 2E_n/\lambda)]^2 - (\ell + \frac{1}{2}N)^2 4k/\lambda^2. \quad (4.3.21)$$

For the same reasons as in Section 2.5 there must exist an n such that this term is zero when $\ell = n$. Therefore

$$E_n = (n + \frac{1}{2}N)\sqrt{k} + \frac{1}{2}\lambda(n^2 + Nn + \frac{1}{2}N) \quad (4.3.22)$$

and

$$\hat{O}_-\hat{O}_+ X_{n,\ell} = 4(n-\ell)(n+\ell+N)\{\sqrt{k}/\lambda + \frac{1}{2}(n-\ell)\}\{\sqrt{k}/\lambda + \frac{1}{2}(n+\ell+N)\}X_n, \quad (4.3.23)$$

$$= [(2\ell+N)^2 |c_{n,\ell}|^2/\lambda^2] X_{n,\ell} \quad (4.3.24)$$

As before the equations for the raising and lowering operators can be written as

$$\hat{O}_+ X_{n,\ell} = - (2\ell+N) |c_{n,\ell}|/\lambda X_{n,\ell+2} \quad (4.3.25)$$

and

$$\hat{O}_- X_{n,\ell+2} = - (2\ell+N) |c_{n,\ell}|/\lambda X_{n,\ell} \quad (4.3.26)$$

4. Calculation of the Wave Functions

a) The Oscillator Potential

The procedure is the same as in Section 2.6 except that initially I shall work with the variable χ . $X_{n,n}$ is found from (4.3.25) and has the form

$$X_{n,n}(\chi) = A_{n,n} (\sin\chi)^n (\cos\chi)^{\sqrt{k}/\lambda + \frac{1}{2}} \quad (4.4.1)$$

The other wave functions can be found by repeated application of (4.3.26), however the calculation can be simplified by the definition

$$X_{n,\ell} = A_{n,\ell} (\sin\chi)^{-(\ell+N-2)} (\cos\chi)^{\sqrt{k}/\lambda + \frac{1}{2}} Z_{n,\ell} \quad (4.4.2)$$

with which the recurrence relation becomes

$$\begin{aligned} Z_{n,\ell-2}(\chi) &= \csc^2\chi \{ (2\ell+N-4) \cot\chi \partial\chi \\ &\quad - (n+\ell+N-2) (2\sqrt{k}/\lambda + n - \ell + 2) \} Z_{n,\ell}(\chi) \end{aligned} \quad (4.4.3)$$

and

$$A_{n,\ell-2} = \frac{\lambda A_{n,\ell}}{(2\ell+N-4) |c_{n,\ell-2}|} \quad (4.4.4)$$

It is not too difficult to show that

$$z_{n,n-2D} = \sum_{r=0}^D (-1)^r B_{D,r}^{n,N} (\sin\chi)^{2n+N-4D+2r-2} (\cos\chi)^{2D-2r} \quad (4.4.5)$$

where

$$B_{D,r}^{n,N} = \frac{2^{2r-2D} D! \Gamma(\sqrt{k}/\lambda + D + 1)}{(D-r)! r! \Gamma(\sqrt{k}/\lambda + D + 1 - r)} \times \frac{\Gamma(2n+N-1) \Gamma(\frac{1}{2}(2n+N-4D+2r-1))}{\Gamma(\frac{1}{2}(2n+N-1)) \Gamma(2n+N-4D+2r-1)} \quad (4.4.6)$$

At this point we can change to the variable

$$z = \sin^2 \chi \quad (4.4.7)$$

and prove the following result

$$\begin{aligned} & \left(\frac{d}{dz}\right)^D \left[z^{\frac{1}{2}(2n+N-2D-2)} (1-z)^{\sqrt{k}/\lambda + D} \right] \\ &= \frac{\Gamma(2n+N-2D-1) \Gamma(\frac{1}{2}(2n+N-1))}{\Gamma(\frac{1}{2}(2n+N-2D-1)) \Gamma(2n+N-1)} (1-z)^{\sqrt{k}/\lambda} z_{n,n-2D}(z). \quad (4.4.8) \end{aligned}$$

When the normalisation constant is evaluated the result for the radial wave function is

$$\begin{aligned} X_{n,\ell}(z) &= \left[\frac{2\lambda^{N/2} (\sqrt{k}/\lambda + n + \frac{1}{2}N) \Gamma(\sqrt{k}/\lambda + \frac{1}{2}(n+\ell+N))}{[\frac{1}{2}(n-\ell)]! \Gamma(\frac{1}{2}(n+\ell+N)) \Gamma(\sqrt{k}/\lambda + \frac{1}{2}(n-\ell) + 1)} \right]^{\frac{1}{2}} \\ &\times z^{-\frac{1}{2}(\ell+N-2)} (1-z)^{-\sqrt{k}/2\lambda + \frac{1}{4}} \\ &\times \left[\frac{d}{dz} \right]^{\frac{1}{2}(n-\ell)} \left[z^{\frac{1}{2}(n+\ell+N-2)} (1-z)^{\sqrt{k}/\lambda + \frac{1}{2}(n-\ell)} \right] \quad (4.4.9) \end{aligned}$$

which can in turn be written as

$$\text{const.} \times \frac{\Gamma(\frac{1}{2}(n+\ell+N))}{\Gamma(\frac{1}{2}(2\ell+N))} z^{\frac{1}{2}\ell} (1-z)^{\sqrt{k}/2\lambda+\frac{1}{4}}$$

$$\times F(\sqrt{k}/\lambda+\frac{1}{2}(n+\ell+N), -\frac{1}{2}(n-\ell); \ell+\frac{1}{2}N; z) . \quad (4.4.10)$$

b) The Coulomb Potential

The procedure given in Sections 3.6 and 3.7 can be applied to equations (4.3.14) and (4.3.15). I will state the results without repeating the details.

$$\bar{X}_{n,\ell}(\chi) = A_{n,\ell} \frac{(2n+N-2)!}{(n+\ell+N-2)!} (\sin\chi)^{-(n+N-1)} e^{\frac{1}{2}\hat{\beta}\chi}$$

$$\times \left[(\sin\chi)^2 \frac{d}{dx} \right]^{(n-\ell)} \left[(\sin\chi)^{2\ell+N-1} e^{-\hat{\beta}\chi} \right] \quad (4.4.11)$$

where

$$A_{n,\ell-1} = \frac{\sqrt{\lambda}}{2|f_{n,\ell-1}|} A_{n,\ell} \quad (4.4.12)$$

and

$$\hat{\alpha} = \frac{1}{2}(n+\frac{1}{2}(N-1))\hat{\beta} . \quad (4.4.13)$$

The expanded forms of (4.4.11) are precisely those of (3.8.1) and (3.8.2) except that the coefficients are different. This is only to be expected from the form of the recurrence relations (4.3.5) and (4.3.6). The coefficients are

$$G_{r,s}^{D,n} = \frac{1}{(r+s)!} \frac{(2D)!}{(2D-2r-2s)!} g_{r,s}^{D,n} \frac{(2n+N-2)!}{(2n-4D+N+2r+2s-2)!} , \quad (4.4.14)$$

$$H_{r,s}^{D,n} = \frac{1}{(r+s)!} \frac{(2D)!}{(2D-2r-2s-1)!} h_{r,s}^{D,n} \frac{(2n+N-2)!}{(2n-4D+N+2r+2s-1)!} , \quad (4.4.15)$$

$$E_{r,s}^{D,n} = \frac{1}{(r+s)!} \frac{(2D+1)!}{(2D-2r-2s+1)!} \cdot g_{r,s}^{D,n-1} \frac{(2n+N-2)!}{(2n-4D+N+2r+2s-4)!} \quad (4.4.16)$$

and

$$F_{r,s}^{D,n} = \frac{1}{(r+s)!} \frac{(2D+1)!}{(2D-2r-2s)!} \cdot h_{r,s}^{D,n-1} \frac{(2n+N-2)!}{(2n-4D+N+2r+2s-3)!} \quad (4.4.17)$$

where

$$g_{r,0}^{D,n} = \frac{\Gamma(\frac{1}{2}(2n-4D+N+2r-1))}{\Gamma(\frac{1}{2}(2n-4D+N-1))} \quad (4.4.18)$$

$$g_{r,s}^{D,n} = \frac{1}{2} \sum_{t=0}^r \frac{\Gamma(\frac{1}{2}(2n-4D+N+2r+2s-1))}{\Gamma(\frac{1}{2}(2n-4D+N+2t+2s-1))} h_{t,s-1}^{D,n} \quad , \quad s \neq 0 \quad (4.4.19)$$

and

$$h_{r,s}^{D,n} = \sum_{t=0}^r \frac{(r+s)!}{(t+s)!} \frac{(r+s)!}{(t+s)!} \frac{(2t+2s)!}{(2r+2s+1)!} \\ \times \frac{\Gamma(2n-4D+N+2r+2s-1)}{\Gamma(2n-4D+N+2t+2s-1)} \frac{\Gamma(\frac{1}{2}(2n-4D+N+2t+2s-1))}{\Gamma(\frac{1}{2}(2n-4D+N+2r+2s-1))} g_{t,s}^{D,n} \quad (4.4.20)$$

CHAPTER 5

SEPARABILITY OF THE SCHRÖDINGER EQUATION

IN DIFFERENT SETS OF COORDINATES

1. Separability and Dynamical Symmetry

There is another interesting property associated with systems having dynamical symmetry, namely the separability of the Schrödinger equation in different sets of coordinates. (The classical equivalent of this is separability of the Hamilton-Jacobi equation, which separates in the same sets of coordinates. Therefore, the following analysis could be done equally well in a classical context.) Separability is not a property exclusive to these systems. Eisenhart (1948) has listed eleven sets of coordinates and the associated potentials for which the Schrödinger equation in three dimensions is separable. However, Makarov et al. (1967) have shown that if there exists a pair of commuting constants of the motion, quadratic in the momenta, then the Schrödinger equation is separable in one of these sets of coordinates and vice versa. If a system has a dynamical symmetry then in general it will be possible to choose the pair of commuting operators in more than one way. This is certainly true of the Coulomb and oscillator potentials and thus their Schrödinger equations are separable in more than one set of coordinates. In the oscillator problem the pairs of commuting operators $\{N_{11}, N_{22}\}$, $\{L^2, L_0^2\}$ and $\{L^2, N_0\}$ correspond to separability in cartesian, spherical polar and circular cylindrical coordinates. In the Coulomb problem the pairs $\{L^2, L_0^2\}$ and $\{L_0^2, R_0\}$ correspond to spherical polar and parabolic rotational coordinates.

Each system has its uses. For example, the last set of coordinates is particularly well suited to the solution of the Coulomb scattering problem (Landau and Lifshitz, 1958, pp. 516). In this type of problem, if we choose a basis of eigenfunctions of the commuting operators and the Hamiltonian, then we have a natural set of coordinates to go with it.

The interesting question now arises of whether it can be done for the Coulomb and oscillator potentials on the sphere. First, the analysis of Makarov et al. (1967) is not valid here. However, we have seen that the set $\{L^2, L_O^2\}$ does correspond to separability in spherical polar coordinates for both systems. We might guess therefore that their results are still applicable. Second, the set $\{N_{11}, N_{22}\}$ is no longer a commuting pair, as can be seen from (2.4.14). Therefore we want sets of coordinates which will correspond to the pairs $\{L_O^2, R_O\}$ and $\{L_O^2, N_O\}$. I am indebted to Dr. P.W. Higgs for the following analysis by which these coordinates are found.

2. Separable Coordinates on the Sphere

The metric form of the sphere is

$$ds^2 = g_{ij}(x) dx_i dx_j \quad (5.2.1)$$

where $g_{ij}(x)$ is given by (1.3.10). This quantity is invariant under all coordinate transformations. For example

$$\underline{x} = \lambda^{-\frac{1}{2}} (\tan\chi \sin\theta \cos\phi, \tan\chi \sin\theta \sin\phi, \tan\chi \cos\theta) \quad (5.2.2)$$

gives

$$ds^2 = \lambda^{-1} (d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2) \quad (5.2.3)$$

This form provides a lot of information. We can construct a free particle Schrödinger equation in these coordinates directly using (1.3.15) and, knowing the transformation (5.2.2), can add in the appropriate potential.

Exactly the same can be done in flat space. Circular cylindrical coordinates

$$\underline{x} = (\rho \cos \phi, \rho \sin \phi, z) \quad (5.2.4)$$

give

$$ds^2 = d\rho^2 + dz^2 + \rho^2 d\phi^2 \quad (5.2.5)$$

and parabolic rotational coordinates

$$\underline{x} = (\sqrt{\xi\eta} \cos \phi, \sqrt{\xi\eta} \sin \phi, \frac{1}{2}(\xi-\eta)) \quad (5.2.6)$$

give

$$ds^2 = \frac{1}{4}(\xi+\eta) \left[\frac{d\xi^2}{\xi} + \frac{d\eta^2}{\eta} \right] + \xi\eta d\phi^2 . \quad (5.2.7)$$

We want to obtain the generalisations of the last four equations to our curved space. It is reasonable to assume that the coordinates will have the form

$$\underline{x} = (F(\xi,\eta) \cos \phi, F(\xi,\eta) \sin \phi, G(\xi,\eta)). \quad (5.2.8).$$

This generates a metric form

$$ds^2 = A(\xi,\eta)d\xi^2 + B(\xi,\eta)d\eta^2 + K(\xi,\eta)d\xi d\eta + D(\xi,\eta)d\phi^2 , \quad (5.2.9)$$

where A, B and D are all positive.

A necessary condition for the Schrödinger equation to be separable is that

$$K(\xi, \eta) = 0 . \quad (5.2.10)$$

With this the equation becomes

$$\left[\frac{1}{\sqrt{ABD}} \frac{\partial}{\partial \xi} \sqrt{\frac{BD}{A}} \frac{\partial}{\partial \xi} + \frac{1}{\sqrt{ABD}} \frac{\partial}{\partial \eta} \sqrt{\frac{AD}{B}} \frac{\partial}{\partial \eta} + \frac{1}{D} \frac{\partial^2}{\partial \phi^2} + 2E \right] \psi(\xi, \eta, \phi) = 0 . \quad (5.2.11)$$

The terms lying between the derivative operators must be products of functions of ξ and η and can be written quite generally as

$$BD/A = [a(\xi)]^2 f(\xi)/g(\eta) \quad (5.2.12)$$

and

$$AD/B = [b(\eta)]^2 g(\eta)/f(\xi) . \quad (5.2.13)$$

The factor \sqrt{ABD} must be able to absorb the extraneous factors from between the derivatives and so must have the form

$$\sqrt{ABD} = \frac{h(\xi, \eta)}{\sqrt{f(\xi) g(\eta)}} . \quad (5.2.14)$$

The final condition for (5.2.11) to be separable is that h must have the form

$$h(\xi, \eta) = \sigma a(\xi) + \tau b(\eta) . \quad (5.2.15)$$

Using all these the metric form becomes

$$ds^2 = [\sigma a(\xi) + \tau b(\eta)] \left[\frac{d\xi^2}{a(\xi)f(\xi)} + \frac{d\eta^2}{b(\eta)g(\eta)} \right] + a(\xi)b(\eta)d\phi^2 \quad (5.2.16)$$

and the Schrödinger equation becomes

$$\{\sqrt{f} \partial_\xi a \sqrt{f} \partial_\xi + \sqrt{g} \partial_\eta b \sqrt{g} \partial_\eta + (\tau/a + \sigma/b) \partial_\phi^2 + 2E(\sigma a + \tau b)\} \psi(\xi, \eta, \phi) = 0 \quad (5.2.17)$$

which is clearly separable.

There is no information about the geometry of the system contained in (5.2.16). To impose more stringent conditions on it we make use of the fact that in a space of constant curvature λ the Riemann-Christoffel tensor satisfies

$$R_{ijkl} = \lambda(g_{ik}g_{jl} - g_{il}g_{jk}) \quad (5.2.18)$$

(e.g. Eisenhart, 1926, p. 83). This leads to the following equations:

$$4\lambda(\sigma a + \tau b)^3 = -(\sigma a + \tau b) \{ \tau [gb'^2 + g'bb' + 2gbb''] + \sigma [fa'^2 + f'aa' + 2faa''] \} + 2\{\tau^2 gbb'^2 + \sigma^2 faa'^2\} \quad (5.2.19)$$

$$4\lambda(\sigma a + \tau b)^2 = -(\sigma a + \tau b) \{ g'b' + 2gb'' \} + \tau gb'^2 - \sigma fa'^2 \quad (5.2.20)$$

and

$$4\lambda(\sigma a + \tau b)^2 = -(\sigma a + \tau b) \{ f'a' + 2fa'' \} + \sigma fa'^2 - \tau gb'^2 \quad (5.2.21)$$

In solving these equations we can make use of the freedom to rescale the coordinates. The particular choice which is most useful is that which makes $a(\xi) = \xi$ and $b(\eta) = \eta$. If we try polynomial solutions for $f(\xi)$ and $g(\eta)$ then we find that

$$f = \frac{4}{\sigma} \{p + f_1 \sigma \xi - \lambda \sigma^2 \xi^2\} \quad (5.2.22)$$

and

$$g = \frac{4}{\tau} \{p - f_1 \tau \eta - \lambda \tau^2 \eta^2\} \quad (5.2.23)$$

unless $\tau = 0$ for which

$$f = 4\xi(f_1 - \sigma\lambda\xi) \quad (5.2.24)$$

and

$$g = 4(g_0 - f_1\eta) \quad (5.2.25)$$

This provides us with a general framework for all systems of coordinates for which the free particle Schrödinger equation is separable. All the systems are in principle obtainable from one another. However, we require the systems which provide a separable equation when a specific potential is added. This potential must have the form

$$V(\xi, \eta) = \frac{p(\xi) + q(\eta)}{\sigma\xi + \tau\eta} \quad (5.2.26)$$

and be obtained from the oscillator or Coulomb potential by the change of coordinates (5.2.8). These coordinates are given in the next section. Once again I am grateful to Dr. P.W. Higgs for finding them.

3. Separable Coordinates for the Coulomb and Oscillator Potentials

a) $\sigma = 1, \tau = 0, g_0 = 1, f_1 = \sqrt{\lambda}$. Redefinition of the coordinates by $\xi = \lambda^{-\frac{1}{2}} \sin^2 \chi$ and $\eta = \lambda^{-\frac{1}{2}} \sin^2 \theta$ leads to (5.2.3) for which the potentials are $\frac{1}{2}\omega^2 \lambda^{-1} \tan^2 \chi$ and $-\hat{\alpha} \lambda \cot \chi$.

b) $\sigma = 1, \tau = 1, p = 1, f_1 = 0$.

This provides a metric form

$$ds^2 = \frac{1}{4}(\xi + \eta) \left[\frac{d\xi^2}{\xi(1-\lambda\xi^2)} + \frac{d\eta^2}{\eta(1-\lambda\eta^2)} \right] + \xi\eta d\phi^2 \quad (5.3.1)$$

which can be seen to reduce to (5.2.7) when $\lambda = 0$. To see whether this is the proper generalisation of (5.2.7) it is useful to redefine the variables by $\xi = \lambda^{-\frac{1}{2}} \sin \alpha$ and $\eta = \lambda^{-\frac{1}{2}} \sin \beta$ by which

$$ds^2 = \frac{1}{4\lambda} (\sin \alpha + \sin \beta) \left[\frac{d\alpha^2}{\sin \alpha} + \frac{d\beta^2}{\sin \beta} \right] + \frac{1}{\lambda} \sin \alpha \sin \beta d\phi^2. \quad (5.3.2)$$

This can be obtained from (5.2.3) by the transformation

$$\chi = \frac{1}{2}\alpha + \frac{1}{2}\beta \quad (5.3.3)$$

$$\cos \theta = \sin \frac{1}{2}(\alpha - \beta) \csc \frac{1}{2}(\alpha + \beta). \quad (5.3.4)$$

Making use of the result that

$$\tan \chi = \frac{\sin \alpha + \sin \beta}{\cos \alpha + \cos \beta} \quad (5.3.5)$$

we can show that the Coulomb potential is

$$-\frac{\mu}{x} = -\lambda \hat{\alpha} \frac{(\cos\alpha + \cos\beta)}{(\sin\alpha + \sin\beta)} \quad (5.3.6)$$

which agrees with the criterion of separability (5.2.26). As a final check we can show that in the limit as $\lambda \rightarrow 0$ (5.3.3) and (5.3.4) lead to the correct flat space transformation (5.2.6).

With all these results we can construct the Schrödinger equation and separate it using the wave function

$$\psi(\alpha, \beta, \phi) = F(\alpha) G(\beta) e^{im\phi} \quad (5.3.7)$$

to obtain the following equations:

$$\{\partial_{\alpha} \sin\alpha \partial_{\alpha} - \frac{1}{2}m^2 \csc\alpha + \frac{1}{2}\hat{\alpha} \cos\alpha + (E/2\lambda) \sin\alpha + \frac{1}{2}\delta\} F(\alpha) = 0 \quad (5.3.8)$$

$$\{\partial_{\beta} \sin\beta \partial_{\beta} - \frac{1}{2}m^2 \csc\beta + \frac{1}{2}\hat{\alpha} \cos\beta + (E/2\lambda) \sin\beta - \frac{1}{2}\delta\} G(\beta) = 0. \quad (5.3.9)$$

If we construct a differential form for R_0 in the above coordinates

$$R_0 = \sqrt{\lambda} \left[\frac{-2 \sin \beta}{(\sin\alpha + \sin\beta)} (\partial_{\alpha} \sin\alpha \partial_{\alpha}) + \frac{2 \sin\alpha}{(\sin\alpha + \sin\beta)} (\partial_{\beta} \sin\beta \partial_{\beta}) + \frac{1}{2} \frac{(\sin\alpha - \sin\beta)}{\sin\alpha \sin\beta} \partial_{\phi}^2 + \hat{\alpha} \frac{(\sin\alpha \cos\beta - \cos\alpha \sin\beta)}{\sin\alpha + \sin\beta} \right] \quad (5.3.10)$$

and apply it to (5.3.7) using the last two equations then we find that

$$R_0 \psi = \sqrt{\lambda} \delta \psi \quad (5.3.11)$$

thereby demonstrating that these coordinates are naturally associated with eigenstates of L_0 and R_0 .

The attempts to solve the above equations are still in a preliminary stage.

$$c) \quad \sigma = -1, \quad \tau = 1, \quad p_1 = -1, \quad f_1 = -\frac{3}{2}\sqrt{2\lambda}.$$

With the choice of variables $\xi = \sqrt{\frac{\lambda}{2}}\rho^2$ and $\eta = \sqrt{\frac{1}{2\lambda}}(2-\lambda z^2)$ the metric form becomes

$$ds^2 = [1 - \frac{1}{2}\lambda(\rho^2+z^2)] \left[\frac{d\rho^2}{(1-\lambda\rho^2)(1-\frac{1}{2}\lambda\rho^2)} + \frac{dz^2}{(1-\lambda z^2)(1-\frac{1}{2}\lambda z^2)} \right] + \rho^2 [1-\frac{1}{2}\lambda z^2] d\phi^2. \quad (5.3.12)$$

This certainly reduces to (5.2.5) when $\lambda = 0$. To check that these coordinates are the correct ones, once again it is convenient to change to angular variables by means of $\rho = \lambda^{-\frac{1}{2}} \sin\gamma$ and $z = \lambda^{-\frac{1}{2}} \sin\epsilon$ and the metric form becomes

$$ds^2 = \frac{1}{\lambda} [\cos^2\gamma + \cos^2\epsilon] \left[\frac{d\gamma^2}{(1+\cos^2\gamma)} + \frac{d\epsilon^2}{(1+\cos^2\epsilon)} \right] + \frac{1}{2\lambda} \sin^2\gamma (1 + \cos^2\epsilon) d\phi^2. \quad (5.3.13)$$

which can be obtained from (5.2.3) by the transformation

$$\cos \chi = \cos \gamma \cos \epsilon \quad (5.3.14)$$

$$\tan \theta = \frac{\sin \gamma (1 + \cos^2 \epsilon)^{\frac{1}{2}}}{\sin \epsilon (1 + \cos^2 \gamma)^{\frac{1}{2}}} \quad (5.3.15)$$

The oscillator potential can be written as

$$\frac{1}{2} \omega^2 x^2 = \frac{1}{2} \frac{\omega^2}{\lambda} \frac{\{\sec^2 \gamma - \cos^2 \gamma + \sec^2 \epsilon - \cos^2 \epsilon\}}{\cos^2 \gamma + \cos^2 \epsilon} \quad (5.3.16)$$

therefore the Schrödinger equation is separable and in the limit as $\lambda \rightarrow 0$ equations (5.3.14) and (5.3.15) reduce to the coordinate transformation given in (5.2.4).

Constructing the Schrödinger equation and separating it using

$$\psi(\alpha, \beta, \phi) = \bar{F}(\gamma) \bar{G}(\epsilon) e^{im\phi} \quad (5.3.17)$$

we obtain the following equations:

$$\begin{aligned} & \{ \sqrt{(1 + \cos^2 \gamma)} \csc \gamma \partial_{\gamma} \sqrt{(1 + \cos^2 \gamma)} \sin \gamma \partial_{\gamma} - 2m^2 \csc^2 \gamma \\ & - (\omega^2 / \lambda^2) (\sec^2 \gamma - \cos^2 \gamma) + (2E / \lambda) \cos^2 \gamma + \kappa \} \bar{F}(\gamma) = 0 \end{aligned} \quad (5.3.18)$$

and

$$\begin{aligned} & \{ \partial_{\epsilon} (1 + \cos^2 \epsilon) \partial_{\epsilon} + 2m^2 / (1 + \cos^2 \epsilon) - (\omega^2 / \lambda^2) (\sec^2 \epsilon - \cos^2 \epsilon) \\ & + (2E / \lambda) \cos^2 \epsilon - \kappa \} \bar{G}(\epsilon) = 0 \end{aligned} \quad (5.3.19)$$

Unfortunately finding a differential form for N_0 is very difficult so that proving that the wave function (5.3.17) is an eigenfunction of it cannot be done here.

Now let us consider some of the problems that confront us when trying to solve the separated equations (5.3.8),

(5.3.9), (5.3.18) and (5.3.19). It is not apparent that they could be converted into any standard form. However, if we take (5.3.8) and (5.3.9) and compare them with (4.3.3) then we can see that they have a very similar form. The difference is an extra term involving the separation constant. Even if we knew the value of this constant it would still be unclear how we could apply Schrödinger's technique.

Finding the separation constant is the first obstacle. We might hope that we could apply an algebraic technique as we know the commutation relations between all the constants of the motion and we have a set of eigenfunctions. When we try to construct the matrix elements and hence derive the eigenvalues, we see that the commutation relations (2.4.14) and (3.4.15) each involve terms in L^2 on their right-hand sides. As we are not dealing with eigenstates of L^2 the equations involving the matrix elements become very complex indeed. Thus it has not been possible to clear even the first obstacle.

CONCLUSION

In this thesis we have seen how two quite complicated systems can be solved by exploiting their dynamical symmetries. Ideally, we would like to be able to use Group Representation Theory on the commutator algebra of the constants of the motion which would avoid the introduction of particular bases or coordinate systems. We are unable to do this but we can fall back on an alternative and very powerful technique.

This technique involves the choice of a basis of eigenstates of a set of commuting operators. For each system there are two such bases and a set of coordinates to go with them. However, in both systems the angular-momentum basis is more suitable than the other basis owing to the form of the commutation relations. (This is not true of the systems in flat space.) With either choice of basis we can attempt to solve the problems algebraically by using the matrix elements of the operators or we can attempt to solve the Schrödinger equation by separating it in the appropriate coordinates. It is not surprising therefore that, when using the less suitable basis, the difficulties of one way should be mirrored in the other.

Having settled on the more suitable basis we can solve by either way for the energy eigenvalues and find that the flat space eigenvalues have been increased by a term dependent on the curvature. The energy eigenfunctions provide a distorted representation of the symmetry group.

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