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The Tilting Theory of Contraction Algebras

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Jennifer Louise August)

Abstract

This thesis focuses on a class of finite dimensional symmetric algebras arising in geometry, known as *contraction algebras*. The main results presented here combine to give a complete description of the derived equivalence class of such an algebra, providing the first concrete evidence towards a key conjecture in the Homological Minimal Model Programme.

More precisely, to each *minimal model* $f: X \rightarrow \text{Spec } R$ of a complete local isolated cDV singularity $\text{Spec } R$, Donovan–Wemyss associate a contraction algebra A . In this way, the collection of all minimal models of $\text{Spec } R$ gives a collection of contraction algebras. We provide a new proof that these algebras are all derived equivalent, thus showing that the corresponding derived category is an invariant of the singularity $\text{Spec } R$. Donovan–Wemyss conjecture that this invariant actually provides a classification of such singularities.

Given a contraction algebra A of a minimal model as above, we show that the *two-term tilting complexes* of A control the entire derived equivalence class of A . For the members of this class, we prove that the only basic algebras derived equivalent to A are the endomorphism algebras of these complexes. We further prove that these algebras precisely coincide with the collection of contraction algebras of $\text{Spec } R$, giving strong evidence to support the conjecture of Donovan–Wemyss.

To understand the structure of maps between the members of this class, namely *standard derived equivalences*, we use the wall and chamber structure given by the two-term tilting theory of A . We prove that this wall and chamber structure coincides with a hyperplane arrangement arising from the geometry and that the chambers of this arrangement are naturally labelled by the collection of contraction algebras. Using our new proof that the contraction algebras of $\text{Spec } R$ are derived equivalent, we establish that the combinatorics of the arrangement completely controls the structure of all the standard derived equivalences. This gives further evidence towards the Donovan–Wemyss conjecture by demonstrating we can recover the group structure of certain derived symmetries arising from the geometry, known as *flops*, just from the derived category of the contraction algebras.

Lay Summary

There are well-established strong links between the study of geometry and algebra. For example, we learn in school that you can describe a circle with radius 1 as all the points (x, y) satisfying the equation

$$x^2 + y^2 = 1.$$

Here, the fact there are two variables, namely x and y , tells us the shape lives in two dimensions; that is, we can draw it on a piece of paper. If we add in an extra variable z and consider the equation

$$x^2 + y^2 + z^2 = 1,$$

then the points (x, y, z) that satisfy this equation form a sphere in three dimensions. It is natural to then consider the set of points (x, y, z, w) satisfying

$$x^2 + y^2 + z^2 + w^2 = 1$$

as an analogue of the sphere living in four dimensions. Although we can no longer ‘see’ this shape living in higher dimensions, we can still use the properties of the associated equation to study it, and this idea is very broadly what happens in the mathematical field of algebraic geometry.

One particular question to ask is, when do two different equations define the same shape? One tool mathematicians can use to answer such a question is called an *invariant*. An invariant assigns a number to each shape such that, if two shapes are assigned a different number, then they can not be the same. For example, we can view the two circles defined by $x^2 + y^2 = 1$ and $x^2 + y^2 = 2^2$ as different because one has radius 1 and the other has radius 2. Thus, we may call the radius an invariant of circles. Other examples include both the length and breadth of a rectangle; if two rectangles have different length, then they can not be the same.

Although invariants can be very useful in determining when two shapes are different, they often do not carry enough information to tell when two shapes are the same. For example, if you only know the length of a rectangle, there are many different rectangles you could draw with that given length. However, if you know the radius of a circle then you actually know everything about the circle; we say that the radius *classifies* circles. Such a tool is incredibly useful in mathematics as it means we can study everything about the circle just by knowing the radius.

For more complicated shapes in higher dimensions, there is too much information to be captured by a single number and so to look for an invariant that will classify our chosen objects, we need something with more structure - this leads straight into the mathematical field of abstract algebra.

In this thesis, we study an algebraic invariant, known as the *derived category of a contraction algebra*, of geometric shapes called *complete local isolated compound Du Val singularities* which live in four dimensions. This invariant is conjectured to completely classify such shapes, and the key results in this thesis provide strong evidence to support this conjecture. In the process, we uncover some unexpectedly rich structure, and demonstrate that the algebraic invariant itself is as worthy of study as its geometric origin.

Publications

The material contained in this thesis is currently available in two preprints by the author, [Au] and [Au2]. The results of Chapter 5 are drawn from the first preprint [Au], although there are significant changes in the presentation to reflect the focus of this thesis on contraction algebras. The results of Chapters 4, 6 and 7 are all taken from [Au2].

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Chapter 1

Introduction

This thesis studies a class of symmetric, finite dimensional algebras, known as *contraction algebras*, which were introduced by Donovan–Wemyss as a tool in birational geometry.

Traditionally, algebraic geometry has relied on the correspondence between geometric spaces and commutative noetherian rings; indeed, given any such ring R , there is an associated space $\text{Spec } R$ defined using the prime ideals of R . The entire philosophy of algebraic geometry is that studying geometric properties of $\text{Spec } R$ is equivalent to studying algebraic properties of the ring R .

In general, noncommutative rings do not have enough prime ideals to make $\text{Spec } R$ interesting and thus comparatively, noncommutative rings have played very little role in enhancing our understanding of geometry. However, in recent years this has begun to change. In particular, starting with the McKay Correspondence for Kleinian singularities in 1980, a programme has been developed to understand resolutions of singularities, or more generally minimal models of singular spaces, via noncommutative rings. Contraction algebras are a part of this programme and so this introduction gives some background to the noncommutative approach, describing the role contraction algebras play and the key conjecture concerning them.

1.1 Kleinian Singularities and the McKay Correspondence

One basic idea in algebraic geometry is that smooth spaces are ‘nice’ and singular spaces are ‘bad’. Thus, given a singular variety X , the goal is to *resolve* X by finding a smooth space Y with a proper birational map $f: Y \rightarrow X$. In 1964, Hironaka earned his Fields medal by proving that such a map f , known as a *resolution of singularities*, always exists when the base field has characteristic zero [Hi].

Once existence is established, it is natural to ask for the ‘best’ possible resolution; one where Y can be considered as close as possible to the original space X . One way to define this is as a resolution $f: Y \rightarrow X$ such that every other resolution of X factors through f , known as a *minimal resolution*. For surfaces, these always exist and, moreover, are unique. For a special class of surfaces, known as *Kleinian singularities* (or sometimes *Du Val singularities*), the McKay correspondence [M] describes their minimal resolutions using representation theory.

Given a finite subgroup $G \leq \text{SL}(2, \mathbb{C})$, there is a natural linear action of G on the power series ring $\mathbb{C}[[x, y]]$. This action defines an invariant ring,

$$\mathbb{C}[[x, y]]^G := \{f \in \mathbb{C}[[x, y]] \mid g \cdot f = f \ \forall g \in G\},$$

and the associated surface $\text{Spec } \mathbb{C}[[x, y]]^G$ is called a Kleinian singularity.

Example 1.1.1. Take G to be the cyclic group of order n , generated by the matrix $g := \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, where ε is a primitive n^{th} root of unity. Then,

$$g \cdot x = \varepsilon x \quad \text{and} \quad g \cdot y = \varepsilon^{-1} y,$$

and one can show that the invariant ring is $R = \mathbb{C}[[x^n, y^n, xy]] \cong \mathbb{C}[[a, b, c]]/(ab - c^n)$. This is called the Kleinian singularity of type A_{n-1} .

Kleinian singularities each contain a unique singular point at the origin, and the minimal resolution $f: Y \rightarrow \text{Spec } R$ is an isomorphism away from this singular point, where the preimage is a finite chain of curves (see e.g. [Du]). The way these curves are arranged is encoded in the *dual graph* of the minimal resolution, which contains a vertex for each curve, and an edge between two vertices if the corresponding curves intersect. For example, the curve configuration below on the left corresponds to the dual graph on the right.



It is well known that the dual graph is always an ADE Dynkin diagram and further, any ADE Dynkin diagram appears as the dual graph of the minimal resolution of some Kleinian singularity [DV]. In this way, the ADE Dynkin diagrams classify the Kleinian singularities and a summary of this classification may be found in [LW, Chapter 6].

For example, the A_{n-1} singularity given in Example 1.1.1 will have a minimal resolution with $n - 1$ curves arranged in a single, non-branching chain as shown below.



Whilst the minimal resolutions of these singularities were understood by Du Val in the 1930s, McKay's observation in the 1980s was that we can describe this chain of curves using the representation theory of the group G .

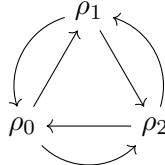

Definition 1.1.2. Given a finite subgroup $G \leq \text{SL}(2, \mathbb{C})$, let ρ_0, \dots, ρ_n be the irreducible representations of G , where ρ_0 denotes the trivial representation. Then the McKay quiver of G is defined to have vertices $\{0, \dots, n\}$ and m_{ij} arrows from $i \rightarrow j$, where

$$\rho \otimes \rho_i = \sum_j m_{ij} \rho_j,$$

and where ρ denotes the natural representation of G .

Using a case by case analysis, McKay showed that the dual graph of the minimal resolution of $\text{Spec } \mathbb{C}[[x, y]]^G$ is recovered from the McKay quiver of G by removing the vertex corresponding to the trivial representation ρ_0 and merging all double edges [M].

Example 1.1.3. Suppose G is the cyclic group of order three, generated by the matrix $g := \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, where ε is a primitive 3^{rd} root of unity. Then G has three irreducible representations, $\rho_i: G \rightarrow \text{GL}(\mathbb{C})$ where $\rho_i(g) = \varepsilon^i$ for $i = 0, 1, 2$, and the natural representation is given as $\rho = \rho_1 \oplus \rho_2$. Using the construction above, the McKay quiver for G and the corresponding dual graph are as follows.

$$\begin{array}{l}
\rho_0 \otimes \rho = \rho_1 \oplus \rho_2 \\
\rho_1 \otimes \rho = \rho_0 \oplus \rho_2 \\
\rho_2 \otimes \rho = \rho_0 \oplus \rho_1
\end{array}$$



After McKay's observation, there were several attempts to give a conceptual understanding as to why these two graphs should be linked. Both [GSV] and [AV] describe explicit bijections explaining why the curves in the minimal resolution and the non-trivial irreducible representations of G match up, but Auslander was the first to explain why the edges also coincide [Aus].

Theorem 1.1.4 (Auslander–McKay Correspondence). *Let $f: X \rightarrow \text{Spec } R$ be the minimal resolution of a Kleinian singularity $R := \mathbb{C}[[x, y]]^G$. Then there is a one-to-one correspondence*

$$\{\text{curves in the exceptional locus of } f\} \longleftrightarrow \{\text{non-free indecomposable objects in } \text{CM } R\}.$$

Moreover, the Auslander–Reiten (AR) quiver of $\text{CM } R$ is precisely the McKay quiver of G and so the dual graph of f may be recovered from the representation theory of R .

Here, $\text{CM } R$ is the category of *maximal Cohen–Macaulay R -modules*, and the AR quiver is designed to be a picture of this category, with a vertex for each indecomposable object and an arrow for each irreducible map. Auslander's work further showed that the quiver of the endomorphism algebra of the direct sum of all the indecomposable objects in $\text{CM } R$, known as the *Auslander algebra* of $\text{CM } R$, is precisely the AR quiver. In other words, he showed that there is an associated noncommutative algebra whose quiver can recover the dual graph of the minimal resolution.

The final step in this story for surfaces comes from noticing that the connection between the minimal resolution and the Auslander algebra goes much deeper than having corresponding underlying graphs. The following result has now been proved in many different ways by many different people, but the statement on derived categories appeared first in [KV], and both statements follow from [BKR].

Theorem 1.1.5. *Let $f: X \rightarrow \text{Spec } R$ be the minimal resolution of a Kleinian singularity, and suppose R, M_1, \dots, M_n are the indecomposable objects in the category $\text{CM } R$. Then there is a derived equivalence*

$$D^b\left(\text{mod End}_R\left(R \oplus \bigoplus_{i=1}^n M_i\right)\right) \simeq D^b(\text{coh } X).$$

Moreover, the space X can be recovered from the algebra $\text{End}_R\left(R \oplus \bigoplus_{i=1}^n M_i\right)$ using a process called quiver GIT.

As the derived category is an invariant which stores all the homological data of an object, the first statement shows that the homological information of the space X is precisely the same as the homological information of the algebra $\text{End}_R\left(R \oplus \bigoplus_{i=1}^n M_i\right)$. The second statement says that the algebra has enough information to completely recover the geometry and thus we can study everything about the geometry using the algebra, just as in commutative algebraic geometry.

1.2 Noncommutative Crepant Resolutions

In higher dimensions, things become more complicated as minimal resolutions may not exist. Instead, Reid introduced a new notion of what it means for the space Y in a resolution $f: Y \rightarrow X$ to be ‘close’ to the original space X . This technical notion, known as *crepancy*, requires the pullback along f of the canonical divisor of X to be the canonical divisor of Y .

For Kleinian singularities, the notions of crepant and minimal resolutions coincide and hence there is a unique crepant resolution. Unfortunately, even in dimension three this is no longer true and thus it is natural to ask how any two crepant resolutions are connected.

In this thesis, we will restrict to certain threefolds, known as *isolated compound Du Val (cDV) singularities* as these already have a rich theory and in fact, many more general threefolds can be reduced to this setting. We can think of cDV singularities as a threefold analogue of Kleinian surface singularities. Indeed, they are threefolds whose generic hyperplane cut to a surface is a Kleinian singularity (see Definition 3.1.2).

Example 1.2.1. One family of cDV singularities are the *cA* singularities, consisting of those that cut to Kleinian singularities of type A . These are of the form $\text{Spec } R$ for some ring $R \cong \mathbb{C}[[u, v, x, y]]/(uv - f(x, y))$. For example, when $f(x, y) = xy(x + y^2)$, $\text{Spec } R$ cuts to the A_2 surface singularity and has six different crepant resolutions. See §3.3 later.

As with the minimal resolutions of Kleinian singularities, the preimage of the singular point of an isolated cDV singularity under a crepant resolution is a finite chain of curves, and thus the dual graph can be similarly defined. Moreover, as any crepant resolution $f: X \rightarrow \text{Spec } R$ is an isomorphism away from the singular point, the entire resolution is determined by how these curves sit in the space X . Thus, one naive idea to create a new crepant resolution from a given one is to cut one of these curves out and replace it with a new curve. This is the basic idea behind the technical notion of a *flop*, and Kollár showed that any two crepant resolutions of an isolated cDV singularity are connected by a finite sequence of such flops [Ko]. Bridgeland then showed that a flop between two crepant resolutions $f: X \rightarrow \text{Spec } R$ and $g: Y \rightarrow \text{Spec } R$ induces a derived equivalence

$$D^b(\text{coh } X) \xrightarrow{\simeq} D^b(\text{coh } Y)$$

known as a *flop-functor*. Combining this with Kollár’s result leads to the following.

Theorem 1.2.2. [B, 1.1] *Let $f: X \rightarrow \text{Spec } R$ and $g: Y \rightarrow \text{Spec } R$ be any two crepant resolutions of an isolated cDV singularity. Then $D^b(\text{coh } X) \simeq D^b(\text{coh } Y)$.*

The idea of Van den Bergh was to recover this result by constructing a noncommutative algebra which is derived equivalent to both spaces. Inspired by the properties of the Auslander algebra in the Auslander–McKay Correspondence, this led to the introduction of noncommutative crepant resolutions.

Definition 1.2.3. *Suppose $\text{Spec } R$ is either a cDV singularity or a Kleinian singularity. A noncommutative crepant resolution (NCCR) of $\text{Spec } R$ is an algebra $A = \text{End}_R(M)$, where M is a reflexive R -module, and A has finite global dimension and is maximal Cohen–Macaulay as an R -module.*

Given a crepant resolution $f: X \rightarrow \text{Spec } R$, Van den Bergh's work followed the ideas in [AV] to further show how such an algebra A could be computed, leading to the following theorem, which can be thought of as a threefold analogue of Theorem 1.1.5.

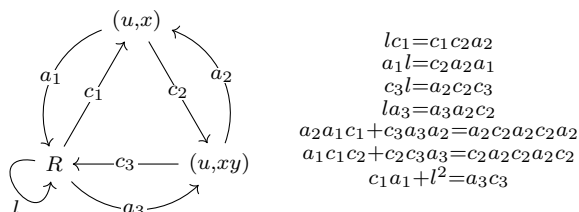
Theorem 1.2.4. [VdB2, 6.3.1] *Let $f: X \rightarrow \text{Spec } R$ be a crepant resolution of an isolated cDV singularity $\text{Spec } R$. Then there exists $M \in \text{CM } R$ such that $A := \text{End}_R(M)$ is an NCCR of R and further, there is a derived equivalence*

$$D^b(\text{coh } X) \xrightarrow{\cong} D^b(\text{mod } A).$$

As it was later shown that X can actually be recovered from the algebra A via quiver GIT [Ka, 5.2.4], this shows that, to study a crepant resolution of a cDV singularity, one can equivalently study the corresponding NCCR. In the case of surfaces, the algebra containing all the geometric information can be obtained by taking the endomorphism algebra of the direct sum of all indecomposable modules in $\text{CM } R$. For threefolds, each NCCR now only corresponds to a subset of the indecomposable modules in $\text{CM } R$. However, the subsets appearing can be characterised algebraically as giving the *cluster-tilting objects* in the stable category $\underline{\text{CM}} R$ [BIKR], and thus, we need no knowledge of the geometry to begin studying the crepant resolutions.

Example 1.2.5. Given a Kleinian singularity $\text{Spec } R$, the unique NCCR is given by the algebra from Theorem 1.1.5, which is the Auslander algebra of the category $\text{CM } R$. Moreover, it is known that this algebra can be described as the preprojective algebra of the corresponding extended Dynkin diagram [RVdB].

Example 1.2.6. For the cA_2 singularity given by $f(x, y) = xy(x + y^2)$, one NCCR is $\text{End}_R(R \oplus (u, x) \oplus (u, xy))$ and this has quiver and relations



We can see, even in this simple case, that the quivers and relations get more complicated in dimension three as loops can now appear.

1.3 Minimal Models

We have already noted that crepant resolutions may not be unique for cDV singularities, but perhaps more critically, they may not exist at all. For example, the cDV singularity of type A given by a polynomial $uv - f(x, y)$ will have a crepant resolution if and only if every irreducible factor of $f(x, y)$ has a linear term [BIKR, 5.7]. This issue inspired Mori and Reid to introduce the notion of a *minimal model*, where the basic idea is that the crepancy property (i.e. staying close to your original space) is more important than ending up with something smooth. We delay the precise definition to §3.1, but the basic idea is to ask for a crepant morphism $f: X \rightarrow \text{Spec } R$ such that the singularities of X are ‘not too bad’.

As X may not be smooth, these are not necessarily resolutions, but many of the ideas from the previous section carry over; the minimal models of a cDV singularity are all

connected by flops [K], a generalisation of Bridgeland’s result by Chen shows that they are all derived equivalent [C], and Van den Bergh’s construction of a noncommutative algebra of the form $\text{End}_R(M)$ can still be carried out. In fact, there are only two key differences:

1. $\text{End}_R(M)$ satisfies all the properties of an NCCR, except that of finite global dimension - this should come as no surprise if we recall the fact from algebraic geometry that $\text{Spec } R$ is smooth if and only if R has finite global dimension.
2. $\text{CM } R$ has no cluster-tilting objects [BIKR, 1.6], and thus the algebraic description of the modules giving these algebras breaks down.

To remedy this, Iyama–Wemyss introduced the new notion of *maximal modifying (MM) modules* (see Definition 3.2.2) which are intended to generalise cluster-tilting objects, and they show that they share many of the same properties [IW1]. With this notion, Wemyss obtains the following result, generalising and improving Theorem 1.2.4 for NCCRs.

Theorem 1.3.1. [We, 4.10, 6.2] *Let $\text{Spec } R$ be a complete local isolated cDV singularity. Then there is a bijection*

$$\{\text{minimal models of } \text{Spec } R\} \longleftrightarrow \{\text{basic MM modules in } \text{CM } R\},$$

where a module is basic if its indecomposable summands are pairwise non-isomorphic. Under this bijection, there is a one-to-one correspondence between the curves in the exceptional locus of a minimal model and the non-free indecomposable summands of the corresponding module. Moreover, given a minimal model $f: X \rightarrow \text{Spec } R$ and the corresponding MM module $M \in \text{CM } R$, the following hold.

1. There is a derived equivalence $D^b(\text{coh } X) \rightarrow D^b(\text{End}_R(M))$;
2. The dual graph of f can be recovered from the quiver of the algebra $\text{End}_R(M)$;
3. All the minimal models of $\text{Spec } R$ can be recovered from $\text{End}_R(M)$ using quiver GIT.

The endomorphism algebra of an MM module is known as a *maximal modification algebra* (MMA) and in the case when $\text{Spec } R$ has a crepant resolution (i.e. a smooth minimal model), these coincide with the NCCRs. However, this is a strictly more general result since crepant resolutions often do not exist, and it shows that no geometric information is lost by passing to the algebra. For example, if we wish to study flop functors between minimal models, then there are isomorphic derived equivalences between the MMAs, known as *mutation functors*, that we can study instead (see [We, 4.2] for details).

1.4 Contraction Algebras

The story so far shows how both the geometric and algebraic viewpoint have changed over time. On the geometric side, the focus has shifted from resolutions to minimal models, and the algebraic side has developed alongside this so that we may now study minimal models of isolated cDV singularities using solely noncommutative algebra.

Contraction algebras originally arose from a slightly different attempt to introduce noncommutative algebras into the study of geometry, using deformation theory. Traditionally, the input of a deformation functor is a commutative ring, and the functor

then returns the set of equivalence classes of deformations of some object over that ring. For example, given a *flopping contraction* $f: X \rightarrow \text{Spec } R$ (such as a minimal model) which contracts a single curve, we can look at the deformations of this curve. Donovan–Wemyss [DW1] show that noncommutative deformations of this curve control an autoequivalence of $D^b(\text{coh } X)$ called the flop-flop functor, and moreover, show that the commutative deformations alone would be insufficient. This provides more evidence that noncommutative algebra is a powerful tool in algebraic geometry.

In this setting, the contraction algebra can be defined as the representing object of the functor of noncommutative deformations of the curve and, as well as controlling the flop-flop functor, recovers all known numerical invariants of such flops [DW1, HT].

For more general threefold flopping contractions, the contraction algebra can be defined similarly as the representing object of a functor of pointed noncommutative deformations. While this deformation-theoretic approach was originally used to define contraction algebras, the role contraction algebras play in our story continues on from the study of MMAs, and thus we choose to use the following equivalent definition [We, 3.5].

Definition 1.4.1. *Given a minimal model $f: X \rightarrow \text{Spec } R$ of a complete local isolated cDV singularity, let $\text{End}_R(M)$ be the corresponding MMA. Then the contraction algebra of f is the stable endomorphism algebra*

$$\underline{\text{End}}_R(M) := \text{End}_R(M)/[R]$$

where $[R]$ denotes the two-sided ideal consisting of all morphisms which factor through $\text{add}(R)$.

Notice that as R is a summand of any MM module, the quiver of the contraction algebra is obtained from the corresponding MMA simply by deleting the vertex corresponding to R .

Example 1.4.2. As in Example 1.2.6, consider the NCCR $\text{End}_R(R \oplus (u, x) \oplus (u, xy))$ and its associated crepant resolution $f: X \rightarrow \text{Spec } R$. Then the contraction algebra of f is the algebra given by the following quiver and relations.

$$\underline{\text{End}}_R(R \oplus (u, x) \oplus (u, xy)) = \begin{array}{ccc} & a_2 & \\ & \curvearrowright & \\ 1 & & 2 \\ & \curvearrowleft & \\ & c_2 & \end{array} \quad \begin{array}{l} a_2 c_2 a_2 c_2 a_2 = 0 \\ c_2 a_2 c_2 a_2 c_2 = 0 \end{array}$$

In the setting of isolated cDV singularities, the contraction algebra of a minimal model $f: X \rightarrow \text{Spec } R$ is symmetric and finite dimensional, and the results of Theorem 1.3.1 show that, as for MMAs, the dual graph of f can be recovered from its quiver. Thus, the contraction algebra still retains some geometric information and so a natural question to ask is whether we can still recover all the geometric information. This question led to the following conjecture of Donovan–Wemyss.

Conjecture 1.4.3. *Suppose $f: X \rightarrow \text{Spec } R$ and $g: Y \rightarrow \text{Spec } S$ are smooth minimal models of complete local isolated cDV singularities. Then $R \cong S$ if and only if their contraction algebras are derived equivalent.*

This says that the contraction algebras of a cDV singularity are all derived equivalent and that we can recover R from this derived equivalence class. In other words,

if this conjecture were true, it means that we can study these singularities using only finite dimensional algebras, rather than the infinite dimensional MMAs.

The results of this thesis provide evidence towards this conjecture by describing the derived equivalence class of a contraction algebra, along the way showing that the derived category of a contraction algebra can recover:

1. the number of minimal models of $\text{Spec } R$;
2. how the minimal models are connected by flops;
3. all the contraction algebras of $\text{Spec } R$ and hence the geometric information they retain about the associated minimal models;
4. the hyperplane arrangement which controls all the relations between the flop functors.

Whilst this does not yet show that the derived category of the contraction algebra can recover $\text{Spec } R$, it does reduce the conjecture of Donovan–Wemyss to an isomorphism problem, instead of one involving derived equivalences (see §7.3).

We finish this introduction by remarking that, although contraction algebras are motivated by the geometry, they are worth studying purely for their algebraic properties. This thesis will show that there are finitely many basic algebras in the derived equivalence class of such an algebra, and furthermore, their derived equivalences are all controlled by the simplest possible derived equivalences (those induced by two-term tilting complexes). With such strong results, it might be expected that the class of contraction algebras is very small, with only a few examples. However, this is not the case. Even in the restricted setting of crepant resolutions, the quivers of possible contraction algebras form the seven infinite families shown in Figure 1.1 [Mo, 5.5].

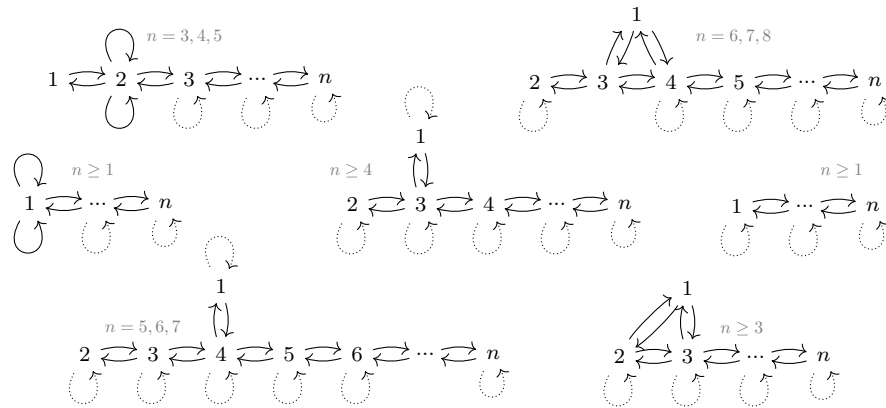


Figure 1.1: The possible quivers for contraction algebras of crepant resolutions of cDV singularities. Here a dotted loop represents that this loop may or may not be present.

It is further believed that *any* choice of potential for any such quiver gives a contraction algebra and hence, even in this restricted setting, the class of contraction algebras is huge. In particular, this class should overlap with many other classes of algebras, which suggests that the techniques developed for contraction algebras in this thesis may be adapted to study other classes as well.

1.5 Organisation of the Thesis

Chapters 2 and 3 will provide the necessary preliminaries, the first dealing with the algebraic notions and the second dealing with the geometric background, in particular defining contraction algebras. In Chapter 4, we construct standard derived equivalences between the contraction algebras of two minimal models that are connected by a simple flop. Chapter 5 will determine the members of the derived equivalence class of a contraction algebra and Chapter 6 will control the compositions of the standard derived equivalences constructed in Chapter 4. Finally, in Chapter 7, we will combine all these results to show that we have a complete picture of the derived equivalence class of a contraction algebra.

1.6 Notation and Conventions

Throughout, k will denote an algebraically closed field of characteristic zero. All rings considered will be noetherian, but we will distinguish between commutative and possibly noncommutative rings as follows:

- R, S, \dots for commutative rings (usually defining a space $\text{Spec } R$);
- Λ, Γ, \dots for possibly noncommutative rings.

For a ring Λ , we write:

- $\text{Mod } \Lambda$ for the category of right Λ -modules;
- $\text{mod } \Lambda$ for the category of finitely generated right Λ -modules;
- $\text{proj } \Lambda$ for the full subcategory of $\text{mod } \Lambda$ consisting of finitely generated projective Λ -modules;
- $\text{add}(M)$ for the full subcategory of $\text{mod } \Lambda$ consisting of summands of finite direct sums of copies of M , where $M \in \text{mod } \Lambda$;
- $(-)^* := \text{Hom}_\Lambda(-, \Lambda): \text{mod } \Lambda \rightarrow \text{mod } \Lambda^{\text{op}}$;
- $\text{K}^b(\text{proj } \Lambda)$ for the homotopy category of bounded complexes in $\text{proj } \Lambda$;
- $\text{D}(\Lambda) := \text{D}(\text{Mod } \Lambda)$ for the derived category of $\text{Mod } \Lambda$;
- $\text{D}^b(\Lambda) := \text{D}^b(\text{mod } \Lambda)$ for the bounded derived category of $\text{mod } \Lambda$.

If Λ is a finite dimensional k -algebra, then we will write:

- $D := \text{Hom}_k(-, k): \text{mod } \Lambda \rightarrow \text{mod } \Lambda^{\text{op}}$ for the standard k -duality;
- $\text{tilt } \Lambda$, $\text{silt } \Lambda$, $2\text{-tilt } \Lambda$, $2\text{-silt } \Lambda$, $2\text{-pretilt } \Lambda$, $2\text{-presilt } \Lambda$ for the sets of isomorphism classes of basic tilting, silting, 2-term tilting, 2-term silting, 2-term pretilting, 2-term presilting complexes of Λ -modules.

If \mathcal{C} denotes a k -linear, Hom-finite, Krull-Schmidt, 2-Calabi-Yau triangulated category (as in cluster-tilting theory), we will write:

- $\text{rig } \mathcal{D}$ for the collection of rigid objects in a full subcategory \mathcal{D} of \mathcal{C} ;
- $\text{mrig } \mathcal{C}$ for the collection of maximal rigid objects in \mathcal{C} ;

- if $N \in \text{rig } \mathcal{C}$, then $\text{mut}(N)$ denotes the collection of all rigid objects in \mathcal{C} which can be obtained from N by finite iterated mutation.

In the geometry, $\text{Spec } R$ will nearly always denote a complete local isolated cDV singularity and we write:

- $\text{CM } R$ for the full subcategory of $\text{mod } R$ consisting of maximal Cohen–Macaulay R -modules;
- $\underline{\text{Hom}}_R(M, N)$ for the quotient of $\text{Hom}_R(M, N)$ by the set of all morphisms factoring through $\text{add}(R)$;
- $\underline{\text{CM}} R$ for the stable category of $\text{CM } R$, where the objects are the same but the morphisms are defined as $\underline{\text{Hom}}_R(M, N)$.

instead. This section provides a short introduction to these categories, as well as giving some examples which will appear later in the thesis.

2.1.1 Definition

For a thorough background to triangulated categories, see [N] or [H].

Definition 2.1.1. A triangulated category $(\mathcal{C}, \Sigma, \mathcal{E})$ is an additive category \mathcal{C} with an autoequivalence $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$, which we will call the shift functor, and a class of triangles \mathcal{E} , each of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \quad (2.2)$$

which satisfy:

1. The class \mathcal{E} is closed under isomorphism i.e. if there is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ a \downarrow & & b \downarrow & & c \downarrow & & \downarrow \Sigma a \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

where the top row is in \mathcal{E} , and each vertical map is an isomorphism, then the bottom row must also be in \mathcal{E} .

2. For each object $X \in \mathcal{C}$, the sequence

$$X \xrightarrow{\text{id}} X \rightarrow 0 \rightarrow \Sigma X$$

is a triangle in \mathcal{E} .

3. Any morphism $f: X \rightarrow Y$ in \mathcal{C} can be completed to a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

4. A sequence (2.2) is in \mathcal{E} if and only if the sequence

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is in \mathcal{E} ; i.e. triangles are closed under rotation.

5. For each commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ a \downarrow & & b \downarrow & & \exists c \downarrow & & \downarrow \Sigma a \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

where the rows are triangles, there exists a map $c: Z \rightarrow Z'$ making the diagram commute.

6. The triangles in \mathcal{E} must satisfy the ‘Octahedral Axiom’ which will not be stated here, see e.g. [N, 1.4.6, 1.4.7].

There are many useful results which are a direct consequence of these axioms, for example, for any triangle as in (2.2) we can determine that the morphisms $g \circ f$, $h \circ g$ and $\Sigma(f) \circ h$ all vanish. Further, the triangle associated to a morphism $f: X \rightarrow Y$ is unique up to non-unique isomorphism, and if any two of a, b and c in 2.1.1(5) are isomorphisms, then so is the third. One of the key results we will use is the following, which shows that triangles mirror the behaviour of short exact sequences.

Proposition 2.1.2. [N, 1.1.10, 1.1.11] *Suppose $(\mathcal{C}, \Sigma, \mathcal{E})$ is a triangulated category and*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a triangle. Then for any object $U \in \mathcal{C}$, the sequences

$$\mathrm{Hom}_{\mathcal{C}}(U, X) \xrightarrow{f \circ -} \mathrm{Hom}_{\mathcal{C}}(U, Y) \xrightarrow{g \circ -} \mathrm{Hom}_{\mathcal{C}}(U, Z)$$

and

$$\mathrm{Hom}_{\mathcal{C}}(Z, U) \xrightarrow{- \circ g} \mathrm{Hom}_{\mathcal{C}}(Y, U) \xrightarrow{- \circ f} \mathrm{Hom}_{\mathcal{C}}(X, U)$$

are exact.

Since we can rotate triangles using the axiom 2.1.1(4), this shows that a triangle actually induces a long exact sequence

$$\begin{array}{ccccccc} \dots & \xrightarrow{\Sigma^{-1}f \circ -} & \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^{-1}Y) & \xrightarrow{\Sigma^{-1}g \circ -} & \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^{-1}Z) & \longrightarrow & \dots \\ & \searrow & \xrightarrow{f \circ -} & \mathrm{Hom}_{\mathcal{C}}(U, Y) & \xrightarrow{g \circ -} & \mathrm{Hom}_{\mathcal{C}}(U, Z) & \longrightarrow \dots \\ & & \searrow & \xrightarrow{\Sigma f \circ -} & \mathrm{Hom}_{\mathcal{C}}(U, \Sigma Y) & \xrightarrow{\Sigma g \circ -} & \mathrm{Hom}_{\mathcal{C}}(U, \Sigma Z) & \longrightarrow \dots \\ & & & \searrow & \xrightarrow{\Sigma^2 f \circ -} & \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^2 Y) & \xrightarrow{\Sigma^2 g \circ -} & \dots \end{array}$$

just as was the case with short exact sequences. This result will be used freely throughout this thesis.

Before introducing some examples of triangulated categories, we explain what it means for two triangulated categories to be considered the same.

Definition 2.1.3. *Suppose $(\mathcal{C}, \Sigma, \mathcal{E})$ and $(\mathcal{C}', \Sigma', \mathcal{E}')$ are triangulated categories.*

1. A pair (F, ϕ) , where $F: \mathcal{C} \rightarrow \mathcal{C}'$ is an additive functor and $\phi: F \circ \Sigma \rightarrow \Sigma' \circ F$ is a natural isomorphism, is called a triangle functor if for every triangle in \mathcal{E} such as (2.2), the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\phi_X \circ F(h)} \Sigma' F(X)$$

is a triangle in \mathcal{E}' .

2. We say $(\mathcal{C}, \Sigma, \mathcal{E})$ and $(\mathcal{C}', \Sigma', \mathcal{E}')$ are triangle equivalent if there exists a triangle functor (F, ϕ) between them where $F: \mathcal{C} \rightarrow \mathcal{C}'$ is an equivalence of categories.

Remark 2.1.4. In part (2) of the above definition, it is equivalent to require that there are triangle functors $(F, \phi): (\mathcal{C}, \Sigma, \mathcal{E}) \rightarrow (\mathcal{C}', \Sigma', \mathcal{E}')$ and $(G, \varphi): (\mathcal{C}', \Sigma', \mathcal{E}') \rightarrow (\mathcal{C}, \Sigma, \mathcal{E})$ such that F and G are mutually inverse equivalences of categories.

2.1.2 Key Examples

This subsection contains the examples of triangulated categories that will appear throughout this thesis.

Homotopy category of chain complexes

Throughout this subsection let \mathcal{A} denote an additive category, e.g. the module category of a ring or even the full subcategory consisting of finitely generated projective modules.

Definition 2.1.5. 1. A chain complex (M, d) for \mathcal{A} is a sequence

$$\dots \xrightarrow{d_{i-2}} M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \xrightarrow{d_{i+1}} \dots$$

consisting of objects $M_i \in \mathcal{A}$ and morphisms $d_i: M_i \rightarrow M_{i+1}$ in \mathcal{A} , such that $d_i \circ d_{i-1} = 0$ for all $i \in \mathbb{Z}$.

2. A chain map $f: (M, d) \rightarrow (N, \delta)$ is a sequence of morphisms $f_i: M_i \rightarrow N_i$ in \mathcal{A} making the following diagram commute:

$$\begin{array}{ccccccc} \dots & \xrightarrow{d_{i-2}} & M_{i-1} & \xrightarrow{d_{i-1}} & M_i & \xrightarrow{d_i} & M_{i+1} & \xrightarrow{d_{i+1}} & \dots \\ & & \downarrow f_{i-1} & & \downarrow f_i & & \downarrow f_{i+1} & & \\ \dots & \xrightarrow{\delta_{i-2}} & N_{i-1} & \xrightarrow{\delta_{i-1}} & N_i & \xrightarrow{\delta_i} & N_{i+1} & \xrightarrow{\delta_{i+1}} & \dots \end{array}$$

3. The category of chain complexes for \mathcal{A} , denoted $\mathcal{C}(\mathcal{A})$, has as objects all chain complexes in \mathcal{A} and morphisms consisting of all possible chain maps.

We will often abuse notation and refer to a complex (M, d) simply as M and thus write the set of all chain maps from (M, d) to (N, δ) as $\text{Hom}_{\mathcal{C}(\mathcal{A})}(M, N)$. We may also restrict to the full subcategories of $\mathcal{C}(\mathcal{A})$ consisting of *bounded complexes* and *right bounded complexes*; those in which only finitely many M_i are nonzero in the first case and those for which there exists an $n \in \mathbb{Z}$ such that $M_i = 0$ for all $i \geq n$ in the second case. We will denote these subcategories by $\mathcal{C}^b(\mathcal{A})$ and $\mathcal{C}^-(\mathcal{A})$ respectively.

Note that if \mathcal{A} is abelian, then $\mathcal{C}(\mathcal{A})$, $\mathcal{C}^-(\mathcal{A})$ and $\mathcal{C}^b(\mathcal{A})$ are all abelian. For example, the kernel of a chain map $f: (M, d) \rightarrow (N, \delta)$ is simply the chain map

$$\begin{array}{ccccccc} \dots & \xrightarrow{d_{i-2}} & \ker f_{i-1} & \xrightarrow{d_{i-1}} & \ker f_i & \xrightarrow{d_i} & \ker f_{i+1} & \xrightarrow{d_{i+1}} & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \xrightarrow{d_{i-2}} & M_{i-1} & \xrightarrow{d_{i-1}} & M_i & \xrightarrow{d_i} & M_{i+1} & \xrightarrow{d_{i+1}} & \dots \end{array}$$

where the maps in the top complex are well defined because f is a chain map and hence $f_{i+1} \circ d_i = \delta_i \circ f_i$. However, motivated by topology, it is common to identify some of the chain maps in $C(\mathcal{A})$ to end up with the homotopy category of chain complexes.

- Definition 2.1.6.** 1. Given two chain maps $f, g: (M, d) \rightarrow (N, \delta)$, define a chain homotopy between f and g as a sequence of morphisms $h_i: M_i \rightarrow N_{i-1}$ in \mathcal{A} such that $f_i - g_i = h_{i+1} \circ d_i + \delta_i \circ h_i$. If such a chain homotopy exists, we say f and g are chain homotopic, which defines an equivalence relation \sim on chain maps $(M, d) \rightarrow (N, \delta)$.
2. We define the homotopy category of chain complexes in \mathcal{A} , denoted $K(\mathcal{A})$, to have the same objects as $C(\mathcal{A})$ but where

$$\mathrm{Hom}_{K(\mathcal{A})}(M, N) := \mathrm{Hom}_{C(\mathcal{A})}(M, N) / \sim .$$

As before, we also consider the full subcategories of bounded complexes and right bounded complexes, denoted $K^b(\mathcal{A})$ and $K^-(\mathcal{A})$ respectively.

Notice that a chain map $f: (M, d) \rightarrow (N, \delta)$ will be the zero map in $K(\mathcal{A})$ if there exists morphisms $h_i: M_i \rightarrow N_{i-1}$ in \mathcal{A} such that $f_i = h_{i+1} \circ d_i + \delta_i \circ h_i$. With these extra maps that are now zero, even if \mathcal{A} is abelian, a chain map may not have a kernel in $K(\mathcal{A})$ and hence this category is not abelian in general. However, it is well known (see e.g. [W, 10.2]) that it has the structure of a triangulated category in the following way.

The shift functor is defined to be $[1]: K(\mathcal{A}) \rightarrow K(\mathcal{A})$, where $M[1]$ is the complex with $(M[1])_i = M_{i+1}$ and differentials $d[1]_i = -d_{i+1}$. Further, for any morphism $f: (M, d) \rightarrow (N, \delta)$, the *mapping cone* of f is defined to be the complex

$$C(f) := (M[1] \oplus N, \partial) \quad \text{where} \quad \partial_i := \begin{pmatrix} -d_{i+1} & 0 \\ f_{i+1} & \delta_i \end{pmatrix} .$$

Note that there are natural maps $N \rightarrow C(f) \rightarrow M[1]$ which are used to define a triangle

$$M \xrightarrow{f} N \rightarrow C(f) \rightarrow M[1]. \quad (2.3)$$

The collection of all triangles then consists of any sequence

$$X \rightarrow Y \rightarrow Z \rightarrow X[1]$$

which is isomorphic to a triangle as in (2.3).

Note that for the identity map $\mathrm{id}: M \rightarrow M$, the mapping cone $C(\mathrm{id})$ is not zero in $C(\mathcal{A})$ but it is easily checked that it is isomorphic to zero, or *null homotopic* in $K(\mathcal{A})$, and thus there are triangles

$$M \xrightarrow{\mathrm{id}} M \rightarrow 0 \rightarrow M[1].$$

Derived Categories

The main focus of this thesis is derived categories. This is an invariant associated to any ring which stores all the homological information about the ring. Recall our convention that all rings considered will be noetherian.

Definition 2.1.7. Let Λ be a ring and suppose (M, d) is a chain complex for $\text{mod } \Lambda$. The homology of the complex at M_i is

$$H^i(M) := \text{Ker } d_i / \text{Im } d_{i-1}.$$

The complex M is called exact at M_i if $H^i(M) = 0$, and we call M exact if $H^i(M) = 0$ for all $i \in \mathbb{Z}$.

A chain map $f: (M, d) \rightarrow (N, \delta)$ induces maps

$$H^i(f): H^i(M) \rightarrow H^i(N)$$

and if $H^i(f)$ is an isomorphism for all $i \in \mathbb{Z}$, f is called a *quasi-isomorphism*. Note that quasi-isomorphisms can be detected using the mapping cone construction above; f is a quasi-isomorphism if and only if $C(f)$ is exact.

In the derived category, two complexes which are quasi-isomorphic are isomorphic. Formally, this is achieved as follows.

Definition 2.1.8. If Λ is a ring, define the derived category $D(\text{mod } \Lambda)$ to be the Verdier localisation of $K(\text{mod } \Lambda)$ at the class of quasi-isomorphisms. Define $D^b(\text{mod } \Lambda)$ and $D^-(\text{mod } \Lambda)$ to be the full subcategories of $D(\text{mod } \Lambda)$ consisting of bounded complexes and right bounded complexes respectively.

Full technical details may be found in [N, 2.1] but the idea is that a formal inverse is added for each quasi-isomorphism. This can make morphisms in the derived category difficult to work with. For example, even when there is no direct quasi-isomorphism between two complexes M and N , they may still be isomorphic in $D(\text{mod } \Lambda)$ via a chain of quasi-isomorphisms such as

$$M \rightarrow X \leftarrow Y \rightarrow N.$$

However, any quasi-isomorphism from a right bounded complex of projectives is in fact an isomorphism in $K^-(\text{mod } \Lambda)$ (see e.g. [W, 10.4.7]) and hence

$$\text{Hom}_{K^-(\text{mod } \Lambda)}(P, X) \cong \text{Hom}_{D^-(\text{mod } \Lambda)}(P, X)$$

for any $P \in K^-(\text{proj } \Lambda)$, $X \in D^-(\text{mod } \Lambda)$. Thus, to understand $\text{Hom}_{D^b(\text{mod } \Lambda)}(X, Y)$, it is common to replace X with a quasi-isomorphic right bounded complex P of projective modules and then study $\text{Hom}_{K^-(\text{mod } \Lambda)}(P, Y)$ instead.

Due to standard constructions (see e.g. [N, 2.1]) and the formal definition above, we conclude that $D(\text{mod } \Lambda)$, $D^b(\text{mod } \Lambda)$ and $D^-(\text{mod } \Lambda)$ are all triangulated categories, where the triangles are sequences isomorphic to the images of triangles in the corresponding homotopy category.

In this thesis, our main focus will be the bounded derived categories of k -algebras and in particular, describing when two such algebras have equivalent derived categories.

Definition 2.1.9. Two finite dimensional algebras Λ and Γ are called *derived equivalent* if their derived categories $D^b(\text{mod } \Lambda)$ and $D^b(\text{mod } \Gamma)$ are triangle equivalent.

Note that two general rings Λ and Γ are defined to be derived equivalent if $D^b(\text{Mod } \Lambda)$ and $D^b(\text{Mod } \Gamma)$ are triangle equivalent. However, Rickard provides several equivalent characterisations, and in particular, for finite dimensional algebras, he shows Definition 2.1.9 is also equivalent [Ri1, 8.3]. As we will mostly work with $\text{mod } \Lambda$ in this thesis, we will ease notation by writing $D^b(\Lambda) := D^b(\text{mod } \Lambda)$ throughout.

where P is \mathcal{E} -projective. The object B is called a syzygy of A , and although there may be many choices for such a sequence, one can easily show that B is unique up to projective summands, and hence well-defined in $\underline{\mathcal{C}}$. One can further show that this construction induces a functor $\Omega: \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$ which is an equivalence. The shift functor Σ is then defined to be Ω^{-1} .

Given a morphism $f: X \rightarrow Y$ in $\underline{\mathcal{C}}$ consider the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_X} & P & \xrightarrow{p_X} & \Omega^{-1}X \\ f \downarrow & & \downarrow & & \parallel \\ Y & \xrightarrow{g} & C_f & \xrightarrow{h} & \Omega^{-1}X \end{array}$$

where the top row is the exact sequence defining $\Omega^{-1}X$, C_f is the pushout of i_X and f , and the map $h: C_f \rightarrow \Omega^{-1}X$ exists by universal property of pushouts. Then the triangle associated to f is defined to be

$$X \xrightarrow{f} Y \xrightarrow{g} C_f \xrightarrow{h} \Omega^{-1}X.$$

Full details that this gives $\underline{\mathcal{C}}$ the structure of a triangulated category can be found in [H, I.2]. In practice, however, we will not use the above construction to provide examples of triangles in $\underline{\mathcal{C}}$. Instead, we will use the following proposition, which shows that the exact sequences in $(\mathcal{C}, \mathcal{E})$ induce triangles in $\underline{\mathcal{C}}$.

Proposition 2.1.12. [H, I.2.7] *Suppose that $(\mathcal{C}, \mathcal{E})$ is a Frobenius category and*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is an exact sequence in \mathcal{E} . Then in $\underline{\mathcal{C}}$, there is a canonical triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \rightarrow \Omega^{-1}A.$$

2.2 Tilting Theory

In representation theory, two noetherian rings Λ and Γ are called *Morita equivalent* if there exists an equivalence $\text{mod } \Lambda \simeq \text{mod } \Gamma$. This was shown to be the case if and only if there exists a certain Λ -module P , satisfying some technical conditions, such that $\text{End}_\Lambda(P) \cong \Gamma$. In [H], Happel showed that if you relax the conditions on P , to that of a *tilting module*, then the resulting ring $\text{End}_\Lambda(P)$ will not necessarily be Morita equivalent to Λ , but it will be derived equivalent. However, the converse was not true; Λ and Γ could be derived equivalent without there being any tilting Λ -module P with $\text{End}_\Lambda(P) \cong \Gamma$. To determine precisely which rings are derived equivalent to Λ , Rickard introduced the notion of tilting complexes [Ri1], which we now recall.

Definition 2.2.1. *A complex $P \in \text{K}^b(\text{proj } \Lambda)$ is called:*

1. *presilting (respectively pretilting) if $\text{Hom}_{\text{K}^b(\text{proj } \Lambda)}(P, P[n]) = 0$ for all $n > 0$ (respectively for all $n \neq 0$).*
2. *silting (respectively tilting) if P is presilting (respectively pretilting) and P generates $\text{K}^b(\text{proj } \Lambda)$ as a triangulated category.*

Rickard showed that Λ and Γ are derived equivalent if and only if there exists a tilting complex $T \in \mathbf{K}^b(\text{proj } \Lambda)$ satisfying $\text{End}_{\mathbf{K}^b(\text{proj } \Lambda)}(T) \cong \Gamma$ [Ri1]. In other words, to determine all rings derived equivalent to some ring Λ , we must determine all the tilting complexes of Λ and their endomorphism rings.

2.2.1 Standard Derived Equivalences

The equivalence written down by Rickard in [Ri1] to prove the derived equivalence between Λ and $\text{End}_{\mathbf{K}^b(\text{proj } \Lambda)}(T)$ is difficult to work with and it was immediately recognised [Ri2, Ke2] that it is preferable and more natural, to ask for extra structure on the tilting complex T ; namely that of a complex of bimodules.

Given two noetherian k -algebras, Λ and Γ , and a Γ - Λ bimodule X , it is well known that there is an induced pair of adjoint functors

$$\text{Hom}_{\Lambda}(X, -): \text{mod } \Lambda \rightarrow \text{mod } \Gamma \quad \text{and} \quad - \otimes_{\Gamma} X: \text{mod } \Gamma \rightarrow \text{mod } \Lambda;$$

meaning that there are isomorphisms

$$\text{Hom}_{\Lambda}(Y \otimes_{\Gamma} X, Z) \cong \text{Hom}_{\Gamma}(Y, \text{Hom}_{\Lambda}(X, Z))$$

for all $Y \in \text{mod } \Gamma$ and $Z \in \text{mod } \Lambda$.

Similarly, a complex X of Γ - Λ -bimodules induces an adjoint pair of triangle functors

$$\mathbf{R}\text{Hom}_{\Lambda}(X, -): \mathbf{D}(\text{Mod } \Lambda) \rightarrow \mathbf{D}(\text{Mod } \Gamma) \quad \text{and} \quad - \otimes_{\Gamma}^{\mathbf{L}} X: \mathbf{D}(\text{Mod } \Gamma) \rightarrow \mathbf{D}(\text{Mod } \Lambda),$$

known as the derived functors (for details see [W, §10.5]). Since they are adjoint functors, one is an equivalence if and only if the other is and, in this case, they are mutually inverse.

Definition 2.2.2. *An equivalence $F: \mathbf{D}^b(\Lambda) \rightarrow \mathbf{D}^b(\Gamma)$ is called a standard derived equivalence if it is isomorphic to*

$$\mathbf{R}\text{Hom}_{\Lambda}(X, -)$$

for some complex X of Γ - Λ -bimodules. We call X a two-sided tilting complex.

It is not known whether all derived equivalences are standard but it is known that the composition of standard equivalences is again standard and further, the inverse of a standard equivalence is also standard [Ri2, 4.1]. To determine when a complex of bimodules is a two-sided tilting complex, we use the following.

Theorem 2.2.3. *Suppose that Λ and Γ are two finite dimensional k -algebras, and X is a complex of Γ - Λ -bimodules. The following are equivalent:*

1. $- \otimes_{\Gamma}^{\mathbf{L}} X: \mathbf{D}(\text{Mod } \Gamma) \rightarrow \mathbf{D}(\text{Mod } \Lambda)$ is an equivalence;
2. $- \otimes_{\Gamma}^{\mathbf{L}} X$ induces an equivalence $\mathbf{K}^b(\text{proj } \Gamma) \rightarrow \mathbf{K}^b(\text{proj } \Lambda)$;
3. $- \otimes_{\Gamma}^{\mathbf{L}} X$ induces an equivalence $\mathbf{D}^b(\text{mod } \Gamma) \rightarrow \mathbf{D}^b(\text{mod } \Lambda)$;
4. Viewing X as a complex of right Λ -modules;

- (a) *The map $\Gamma \rightarrow \text{Hom}_{\mathbf{D}(\text{Mod } \Lambda)}(X, X)$ induced by $- \otimes_{\Gamma}^{\mathbf{L}} X$ is an isomorphism and further $\text{Hom}_{\mathbf{D}(\text{Mod } \Lambda)}(X, X[n]) = 0$ for $n \neq 0$.*

- (b) X is quasi-isomorphic to some complex T in $\mathbf{K}^b(\text{proj } \Lambda)$.
- (c) The smallest triangulated full subcategory of $\mathbf{D}(\text{Mod } \Lambda)$ containing T and closed under isomorphism and direct summands is $\mathbf{K}^b(\text{proj } \Lambda)$.

Proof. The equivalence of (1), (2) and (4) is [Ke3, 8.1.4]. Then the equivalence of (3) with (2) is given by [Ri1, 8.3]. \square

Note that the conditions in part (4) ensure that any two-sided tilting complex X is quasi-isomorphic to a tilting complex for Λ . Moreover, as Λ is a finite dimensional k -algebra, [Ri2] shows that for any tilting complex $T \in \mathbf{D}^b(\Lambda)$, there exists a two-sided tilting complex \mathcal{T} such that $T \cong \mathcal{T}$ in $\mathbf{D}^b(\Lambda)$ (this also holds for more general rings using [Ke3]). As \mathcal{T} is two-sided, it induces the standard equivalence

$$\mathbf{RHom}_\Lambda(\mathcal{T}, -): \mathbf{D}^b(\Lambda) \rightarrow \mathbf{D}^b(\text{End}_\Lambda(T)),$$

and as $\mathcal{T} \cong T$, this maps $T \mapsto \text{End}_\Lambda(T)$. It was further shown in [Ke4, 2.1] that such a \mathcal{T} is unique in a suitable sense. For our purposes, the following suffices.

Proposition 2.2.4. [RZ, 2.3] *Suppose that Λ and Γ are finite dimensional k -algebras and that T and T' are two-sided tilting complexes for Λ with $\Gamma := \text{End}_\Lambda(T) \cong \text{End}_\Lambda(T')$. Then $T_\Lambda \cong T'_\Lambda$ if and only if there exists an automorphism $\alpha: \Gamma \rightarrow \Gamma$ such that*

$$T' \cong {}_\alpha\Gamma \otimes_\Gamma T$$

in the derived category of Γ - Λ bimodules.

In this way, if Λ is a finite dimensional k -algebra, we can say that any tilting complex $T \in \mathbf{K}^b(\text{proj } \Lambda)$ induces a unique (up to algebra automorphism) standard equivalence.

2.2.2 Mutation

Rickard's result shows that determining the derived equivalence class of a ring Λ is equivalent to finding all the tilting complexes in $\mathbf{K}^b(\text{proj } \Lambda)$. The idea of mutation is to produce new tilting complexes from old by removing an indecomposable summand and looking for a replacement which gives a new tilting complex. Unfortunately, with the restrictive conditions on tilting complexes, this is not always possible, but if we relax those conditions to that of silting complexes, this becomes very well behaved. This section provides a summary of this mutation procedure.

We will now restrict to the setting of finite dimensional k -algebras. For such a Λ , the bounded homotopy category $\mathbf{K}^b(\text{proj } \Lambda)$ is a k -linear Hom-finite, Krull-Schmidt category and we will write $\text{silt } \Lambda$ (respectively $\text{tilt } \Lambda$) for the class of isomorphism classes of basic silting (respectively tilting) complexes in $\mathbf{K}^b(\text{proj } \Lambda)$. Here, basic means that the indecomposable summands are pairwise non-isomorphic. Note that $\Lambda \in \text{tilt } \Lambda$ and so we always have an example of at least one tilting complex. In this setting, all basic silting complexes for Λ have the same number of indecomposable summands [AI, 2.28].

To define mutation requires the following. Suppose that \mathcal{D} is an additive category and \mathcal{S} is a class of objects in \mathcal{D} .

1. A morphism $f: X \rightarrow Y$ is called a *right \mathcal{S} -approximation* of Y if $X \in \mathcal{S}$ and the induced morphism $\text{Hom}(Z, X) \rightarrow \text{Hom}(Z, Y)$ is surjective for any $Z \in \mathcal{S}$.
2. A morphism $f: X \rightarrow Y$ is said to be *right minimal* if for any $g: X \rightarrow X$ such that $f \circ g = f$, then g must be an isomorphism.

3. A morphism $f: X \rightarrow Y$ is a *minimal right \mathcal{S} -approximation* if f is both right minimal and a right \mathcal{S} -approximation of Y .

There is also the dual notion of a minimal left \mathcal{S} -approximation.

Definition 2.2.5. [AI, 2.31] *Let $P \in \mathbf{K}^b(\text{proj } \Lambda)$ be a basic siltng complex for Λ , and write $P := \bigoplus_{i=1}^n P_i$ where each P_i is indecomposable. Consider a triangle*

$$P_i \xrightarrow{f} P' \rightarrow Q_i \rightarrow P_i[1]$$

where f is a minimal left $\text{add}(P/P_i)$ -approximation of P_i . Then $\mu_i(P) := P/P_i \oplus Q_i$ is also a siltng complex, known as the left mutation of P with respect to P_i .

Right mutation is defined dually and is an inverse operation to left mutation [AI, 2.33] so we denote it by μ_i^{-1} .

As noted above, the mutation of a tilting complex may not be a tilting complex. However, it is well known (see e.g. [AI, 2.8]) that if Λ is a symmetric algebra, i.e. $\Lambda \cong \text{Hom}_k(\Lambda, k)$ as Λ - Λ bimodules, then any siltng complex is a tilting complex and hence any mutation of a tilting complex is again a tilting complex. This will be the case in our setting later in the thesis.

To help control mutations, Aihara–Iyama introduced a partial order on $\text{silt } \Lambda$ [AI].

Definition 2.2.6. *Let P and Q be siltng complexes for Λ . If $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(P, Q[i]) = 0$ for all $i > 0$, then we say $P \geq Q$. Further, we write $P > Q$ if $P \geq Q$ and $P \not\cong Q$.*

This order can determine whether two siltng complexes are related by mutation.

Theorem 2.2.7. [AI, 2.35] *If P and Q are basic siltng complexes for Λ , then the following are equivalent:*

1. $Q = \mu_i(P)$ for some summand P_i of P .
2. $P > Q$ and there is no siltng complex T such that $P > T > Q$.

Theorem 2.2.8. [A, 3.5] *Suppose that $T \geq U$ are two basic siltng complexes for Λ . If there are finitely many siltng complexes P such that $T \geq P \geq U$, then U is obtained by iterated left mutation from T or equivalently, T is obtained by iterated right mutation from U .*

The final result of this section, which is implicit in the literature, shows how siltng complexes behave under derived equivalences.

Lemma 2.2.9. *Let Λ and Γ be finite dimensional k -algebras and let $F: \mathbf{D}^b(\Lambda) \rightarrow \mathbf{D}^b(\Gamma)$ be a triangle equivalence. Then the following statements hold.*

1. F maps siltng complexes to siltng complexes;
2. F preserves the siltng order;
3. If P is a siltng complex for Λ , then

$$F(\mu_i(P)) \cong \mu_i(F(P)) \quad \text{and} \quad F(\mu_i^{-1}(P)) \cong \mu_i^{-1}(F(P)).$$

Proof. For lack of a suitable reference, we give a proof.

1. A standard result of Rickard [Ri1, 6.2] states that F restricts to an equivalence

$$\mathrm{K}^b(\mathrm{proj} \Lambda) \rightarrow \mathrm{K}^b(\mathrm{proj} \Gamma).$$

Then, for any silting complex $P \in \mathrm{K}^b(\mathrm{proj} \Lambda)$, and any $i > 0$,

$$\begin{aligned} \mathrm{Hom}_{\mathrm{K}^b(\mathrm{proj} \Gamma)}(F(P), F(P)[i]) &\cong \mathrm{Hom}_{\mathrm{K}^b(\mathrm{proj} \Gamma)}(F(P), F(P[i])) \\ &\cong \mathrm{Hom}_{\mathrm{K}^b(\mathrm{proj} \Lambda)}(P, P[i]) = 0, \end{aligned}$$

and hence $F(P)$ is presilting. Further, since equivalences must send generators to generators, $F(P)$ must generate $\mathrm{K}^b(\mathrm{proj} \Gamma)$ as a triangulated category and hence $F(P)$ is a silting complex.

2. Suppose that $P \geq Q$ are silting complexes for Λ . Then for all $i > 0$,

$$\begin{aligned} \mathrm{Hom}_{\mathrm{K}^b(\mathrm{proj} \Gamma)}(F(P), F(Q)[i]) &\cong \mathrm{Hom}_{\mathrm{K}^b(\mathrm{proj} \Gamma)}(F(P), F(Q[i])) \\ &\cong \mathrm{Hom}_{\mathrm{K}^b(\mathrm{proj} \Lambda)}(P, Q[i]) = 0, \end{aligned}$$

and hence $F(P) \geq F(Q)$. If $P > Q$ then $P \not\cong Q$ and hence $F(P) \not\cong F(Q)$ as F is an equivalence. Thus $F(P) > F(Q)$ and so F preserves the silting order.

3. Suppose that P is a silting complex and consider $\mu_i(P)$. By Theorem 2.2.7, $P > \mu_i(P)$ and hence by part (2), $F(P) > F(\mu_i(P))$. Suppose that there exists a silting complex T of Γ such that $F(P) > T > F(\mu_i(P))$. Then, by part (2) applied to F^{-1} ,

$$P > F^{-1}(T) > \mu_i(P).$$

By part (1), $F^{-1}(T)$ is a silting complex for Λ and so this provides a contradiction using Theorem 2.2.7. Thus, again by Theorem 2.2.7, $F(\mu_i(P)) \cong \mu_j(F(P))$ for some j . However, by comparing the summands of $F(\mu_i(P))$ and $F(P)$, it is clear that $j = i$.

The second statement follows since

$$F(\mu_i^{-1}P) \cong \mu_i^{-1}\mu_i F(\mu_i^{-1}P) \cong \mu_i^{-1}F(\mu_i\mu_i^{-1}P) \cong \mu_i^{-1}F(P). \quad \square$$

2.2.3 Tilting-Discreteness

For symmetric algebras, we saw that we can always mutate at any summand of a tilting complex to get a new tilting complex. Thus, starting with the algebra itself, we can produce many tilting complexes. However, for most algebras, it is unknown whether all tilting complexes can be recovered in this way. This subsection recalls a class of algebras, known as *tilting-discrete algebras*, for which this question can be answered positively. These were first introduced in [A].

Definition 2.2.10. *A finite dimensional algebra Λ is said to be tilting-discrete if for any $P \in \mathrm{tilt} \Lambda$ and any $\ell > 1$, the set*

$$\ell\text{-tilt}_P \Lambda := \{ T \in \mathrm{tilt} \Lambda \mid P \geq T \geq P[\ell - 1] \}$$

is finite. Further, Λ is called 2-tilting-finite if $2\text{-tilt}_P \Lambda$ is a finite set for any $P \in \mathrm{tilt} \Lambda$.

Proposition 2.2.11. *If Λ is a tilting-discrete finite dimensional algebra then any tilting complex $T \in \mathrm{K}^b(\mathrm{proj} \Lambda)$ can be obtained from Λ by finite iterated mutation.*

Proof. Choose a tilting complex T . By [A, 2.9], there exists integers $m > n$ such that $\Lambda[n] \geq T \geq \Lambda[m]$. We now split the proof into two cases: when $n \geq 0$, and when $n < 0$.

If $n \geq 0$, the set $\{Q \in \text{tilt } \Lambda \mid \Lambda \geq Q \geq T\} \subseteq (m+1)\text{-tilt}_\Lambda \Lambda$ and hence is finite by the tilting-discrete assumption. Thus, by Theorem 2.2.8, T is obtained from Λ by iterated left mutation.

If $n < 0$, the set $(1-n)\text{-tilt}_{\Lambda[n]} \Lambda := \{Q \in \text{tilt } \Lambda \mid \Lambda[n] \geq Q \geq \Lambda\}$ is finite as Λ is tilting-discrete. Using Theorem 2.2.8, this shows $\Lambda[n]$ can be obtained by iterated right mutation from Λ . Further, $\{Q \in \text{tilt } \Lambda \mid \Lambda[n] \geq Q \geq T\} \subseteq (m-n+1)\text{-tilt}_{\Lambda[n]} \Lambda$ is also finite, showing T can be obtained from $\Lambda[n]$ by iterated left mutation, again using Theorem 2.2.8. Combining these mutation sequences proves the result. \square

In this thesis, we will show that our algebras of interest are tilting-discrete. To do this requires the following result, which establishes equivalent conditions for an algebra to be tilting-discrete.

Theorem 2.2.12. [AM, 2.11] *Let Λ be a finite dimensional symmetric algebra. Then the following are equivalent:*

1. Λ is tilting-discrete.
2. Λ is 2-tilting-finite.
3. $2\text{-tilt}_P \Lambda$ is a finite set for any tilting complex P which is given by iterated left mutation from Λ .

2.2.4 Two-Term Silting Complexes

Our final section on silting complexes focuses on so called two-term silting complexes of a finite dimensional algebra Λ .

Definition 2.2.13. *A presilting complex $P \in \text{K}^b(\text{proj } \Lambda)$ is called two-term if the terms are zero in every degree other than 0 and -1 , or equivalently by [A, 2.9], if $\Lambda \geq P \geq \Lambda[1]$.*

We denote the set of isomorphism classes of basic two-term silting (respectively presilting, tilting) complexes as $2\text{-silt } \Lambda$ (respectively $2\text{-presilt } \Lambda$, $2\text{-tilt } \Lambda$). The following shows that mutation of two-term silting complexes is particularly well behaved.

Proposition 2.2.14. [AIR, 3.8] *Suppose P and Q are basic two-term silting complexes for Λ . Then P and Q are related by single mutation if and only if they differ by exactly one indecomposable summand.*

This proposition arises from a link with cluster-tilting theory, introduced next.

2.3 Cluster-Tilting Theory

While the introduction of cluster-tilting theory came much later than tilting complexes, both can be viewed as generalisations of tilting modules. For tilting complexes, the motivation was to determine derived equivalences, whereas for cluster-tilting theory, the motivation was to find the right setting in which we can always mutate tilting modules, giving a categorification of the cluster algebras introduced by Fomin and Zelevinsky. As with tilting complexes, the mutation of tilting modules involves taking out one indecomposable summand and replacing it to get a new tilting module. In this

case, there is at most one choice of new summand but sometimes no such summand exists [HU].

To remedy this for hereditary algebras, [BMRRT] construct a larger category than the module category, known as the *cluster category* of the algebra. This is an orbit category of the derived category and so not only do all tilting modules for Λ lie in this category, but also all tilting modules for any hereditary algebra derived equivalent to Λ . The set of all such objects was shown to have ‘perfect’ mutation: for each summand you take out, there is always exactly one summand to replace it with to get another such object! This thesis uses the abstraction of this theory provided in [AIR].

Throughout this subsection, \mathcal{C} (the abstraction of the cluster category) will denote a k -linear Hom-finite, Krull-Schmidt, 2-Calabi-Yau triangulated category with shift functor Σ . The property 2-Calabi-Yau (2-CY) means there are bifunctorial isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(M, N[2]) \cong D \mathrm{Hom}_{\mathcal{C}}(N, M)$$

for all $M, N \in \mathcal{C}$ where $D := \mathrm{Hom}_k(-, k)$.

Definition 2.3.1. *Let $M \in \mathcal{C}$.*

1. M is called rigid if $\mathrm{Hom}_{\mathcal{C}}(M, \Sigma M) = 0$.
2. M is called maximal rigid if M is rigid and

$$\mathrm{add}(M) = \{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(M \oplus X, \Sigma(M \oplus X)) = 0\}.$$

Write $\mathrm{rig} \mathcal{C}$ for the set of isomorphism classes of basic rigid objects in \mathcal{C} and write $\mathrm{mrig} \mathcal{C}$ for the set of isomorphism basic maximal rigid objects.

Mutation is defined similarly to the mutation of silting complexes in §2.2.2.

Definition 2.3.2. *Suppose that $M := \bigoplus_{i=1}^n M_i$ is a basic rigid object in \mathcal{C} with each M_i indecomposable. Consider a triangle*

$$M_i \xrightarrow{f} V \xrightarrow{g} N_i \rightarrow \Sigma M_i \tag{2.4}$$

where f is a minimal left $\mathrm{add}(M/M_i)$ -approximation of M_i . Then $\nu_i(M) := M/M_i \oplus N_i$ is also a rigid object, known as the left mutation of M with respect to M_i . We call the triangle an exchange triangle.

Right mutation is defined dually and we denote it by ν_i^{-1} . As with silting complexes, right and left mutation are inverse operations.

Lemma 2.3.3. *For any rigid object $M \in \mathcal{C}$ and any i , then $\nu_i \nu_i^{-1} M \cong M \cong \nu_i^{-1} \nu_i M$.*

Proof. For lack of a reference, we sketch the proof. To show this, it is enough to show that, if f is a minimal left $\mathrm{add}(M/M_i)$ -approximation of M_i in the exchange triangle (2.4), then g is a minimal right $\mathrm{add}(M/M_i)$ -approximation of N_i and vice versa.

This is very similar to [GLS, 5.7, 5.8]: begin by assuming f is a minimal left $\mathrm{add}(M/M_i)$ -approximation of M_i . Applying $\mathrm{Hom}_{\mathcal{C}}(M/M_i, -)$ to the exchange triangle (2.4) and using that M is rigid, gives an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(M/M_i, V) \xrightarrow{g^{\circ-}} \mathrm{Hom}_{\mathcal{C}}(M/M_i, N_i) \rightarrow 0.$$

This shows that g is a right $\text{add}(M/M_i)$ -approximation. If g is not minimal, there is an isomorphism

$$\left(V \xrightarrow{g} N_i \right) \cong \left(W \oplus Z \xrightarrow{(g',0)} N_i \right)$$

where g' is minimal and $W, Z \in \text{add}(M/M_i)$. Completing these maps to triangles gives

$$\left(M_i \xrightarrow{f} V \xrightarrow{g} N_i \rightarrow \Sigma M_i \right) \cong \left(\left(W' \xrightarrow{f'} W \xrightarrow{g'} N_i \rightarrow \Sigma M_i \right) \oplus \left(Z \xrightarrow{\text{id}} Z \xrightarrow{0} 0 \rightarrow \Sigma Z \right) \right)$$

and hence by the uniqueness of cocones, $M_i \cong W' \oplus Z$. Since M_i is indecomposable, either W' or Z must be zero. If W' is zero, $M_i \cong Z$ and so $M_i \in \text{add}(Z) \subseteq \text{add}(M/M_i)$ which is a contradiction. Hence, $Z \cong 0$ and so g is minimal, as required. The other direction is a dual argument. \square

For maximal rigid objects M and N , it is shown in [IY, 5.3], that M is a mutation of N if and only if M and N differ by exactly one indecomposable summand. An easy consequence of this is that left and right mutation must coincide in this case, but this is not true in general for rigid objects.

For a rigid object $M \in \mathcal{C}$, we will write $\text{mut}(M)$ for the collection of isomorphism classes of basic rigid objects which can be obtained from M by iterated left or right mutation. If \mathcal{C} has finitely many maximal rigid objects, it follows from [AIR, 4.9] that, for any maximal rigid object M , $\text{mut}(M) = \text{mrig } \mathcal{C}$. In other words, starting with one maximal rigid object, we can obtain all others via mutation.

The main algebras studied in this thesis can be viewed as the endomorphism algebras of rigid objects in some suitable category. For this reason, we are interested in how the properties of such algebras, particularly their tilting theory, depend on the underlying category.

Note that for any object M in a Krull-Schmidt category \mathcal{D} , [Kr, 2.3] shows there is an equivalence

$$\text{Hom}_{\mathcal{D}}(M, -): \text{add}(M) \rightarrow \text{proj End}_{\mathcal{D}}(M). \quad (2.5)$$

So if $M := \bigoplus_{i=1}^n M_i$, where each M_i is indecomposable, then the indecomposable projective modules of $\text{End}_{\mathcal{D}}(M)$ are $\text{Hom}_{\mathcal{D}}(M, M_1), \dots, \text{Hom}_{\mathcal{D}}(M, M_n)$. In particular, the quiver for $\text{End}_{\mathcal{D}}(M)$ will have n vertices.

Moreover, if M is a rigid object in \mathcal{C} , this equivalence also allows us to relate the two-term silting theory of $\text{End}_{\mathcal{C}}(M)$ to rigid objects in the following subcategory of \mathcal{C} .

Definition 2.3.4. *Given $M \in \mathcal{C}$, define $M * \Sigma M$ to be the full subcategory of \mathcal{C} consisting of the objects N such that there exists a triangle*

$$M_1 \xrightarrow{f} M_0 \xrightarrow{g} N \rightarrow \Sigma M_1 \quad (2.6)$$

where $M_1, M_0 \in \text{add}(M)$.

It is easy to show that if M is rigid, then for any triangle such as (2.6), g is a right $\text{add}(M)$ approximation and further, by possibly changing M_0 and M_1 , we can choose g to be minimal. Since minimal approximations are unique up to isomorphism, this means that for each object $N \in M * \Sigma M$, there is a unique triangle

$$M_1 \xrightarrow{f} M_0 \xrightarrow{g} N \rightarrow \Sigma M_1$$

such that g is a right minimal $\text{add}(M)$ -approximation. By the equivalence (2.5), applying $\text{Hom}_{\mathcal{C}}(M, -)$ to f gives a two-term complex in $\text{K}^{\text{b}}(\text{proj End}_{\mathcal{C}}(M))$.

The following theorem is a slight generalisation of [AIR, 4.7], similar to that of [CZZ, 3.2]. By replacing the whole category \mathcal{C} in the proof of [AIR, 4.7] by $M * \Sigma M$, the proof in this case is identical, and so is not repeated here.

Theorem 2.3.5. [AIR, 4.7] *Let \mathcal{C} be a k -linear Hom-finite, Krull-Schmidt, 2-CY triangulated category and M be a rigid object of \mathcal{C} . If $\Lambda := \text{End}_{\mathcal{C}}(M)$, the process described above restricts to a bijection*

$$\text{rig}(M * \Sigma M) \longleftrightarrow \text{2-presilt } \Lambda$$

which preserves the number of summands.

Remark 2.3.6. 1. Since the bijection preserves the number of summands, rigid objects in $M * \Sigma M$ with the same number of summands as M must correspond to silting complexes.

2. If M is maximal rigid, it is shown in [ZZ, 2.5] that all rigid objects of \mathcal{C} lie in $M * \Sigma M$ and so this bijection restricts to

$$\text{mrig } \mathcal{C} \longleftrightarrow \text{2-silt } \Lambda.$$

Further, this bijection respects mutation since mutation on both sides corresponds to differing by exactly one indecomposable summand.

Finally, we note that Theorem 2.3.5 can also be used to show the following.

Lemma 2.3.7. *Let \mathcal{C} be a k -linear Hom-finite, Krull-Schmidt, 2-CY triangulated category. If there exists a maximal rigid object in \mathcal{C} , then any basic rigid object $M \in \mathcal{C}$ is the direct summand of some basic maximal rigid object.*

Proof. Let $N \in \mathcal{C}$ be a basic maximal rigid object and write $\Lambda := \text{End}_{\mathcal{C}}(N)$. Let ϕ denote the corresponding bijection from Theorem 2.3.5. Now take any basic rigid object $M \in \mathcal{C}$. By Remark 2.3.6(2), M is contained in $\text{rig}(N * \Sigma N)$ and thus $\phi(M) \in \text{2-presilt } \Lambda$. Hence, by [BZ, 3.1], there exists $P \in \text{2-presilt } \Lambda$ such that $\phi(M) \oplus P$ is a two-term silting complex for Λ . Mapping back across the bijection, and again using Remark 2.3.6(2), shows that $M \oplus \phi^{-1}(P)$ is maximal rigid object. \square

In particular, this shows that if there are finitely many maximal rigid objects in \mathcal{C} , there must also be finitely many rigid objects.

Chapter 3

Homological Minimal Model Programme Preliminaries

While the main theorems of this thesis are purely algebraic results, and use only algebraic techniques, the motivation for these results lies in geometry; namely in the study of the Minimal Model Programme (MMP). Very broadly, the MMP aims to find ‘best’ approximations of singular spaces, which are known as the *minimal models*.

This chapter provides a brief introduction to minimal models of 3-folds, before going on to discuss the homological approach to this situation which has been developed by Wemyss [We], along with coauthors such as Donovan [DW1, DW2, DW3, DW4], and Iyama [IW1, IW2]. This will involve introducing the following:

- minimal models, flopping contractions and flops;
- the algebras used to study this geometry, known as modification algebras and contraction algebras;
- modifying modules and their relation to cluster-tilting theory;
- a summary of the main results about modifying modules and algebras;
- the main conjecture about contraction algebras.

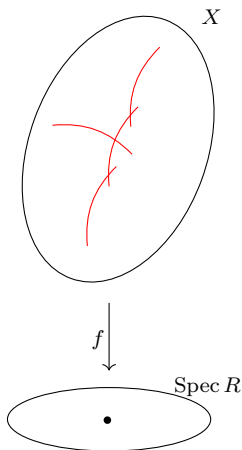
While all the results that are needed later in the thesis are precisely given, many of the technical geometric definitions are not presented here, though we do point to where definitions may be found. Our focus will be on conveying the idea behind a definition, and the consequences of the various technical assumptions which are suitable for our algebraic needs.

3.1 Minimal Models and Flopping Contractions

We begin by stating our general technical setup.

Setup 3.1.1. Take $f: X \rightarrow \operatorname{Spec} R$ to be a 3-fold flopping contraction where R is a complete local ring and X has at worst Gorenstein terminal singularities.

This setup ensures that $\operatorname{Spec} R$ has a unique singular point \mathfrak{m} and the preimage $C := f^{-1}(\mathfrak{m})$ of this point, known as the *exceptional locus*, consists of a finite chain of curves. In particular, giving C the reduced scheme structure, we have $C^{\text{red}} = \cup_{i=1}^n C_i$ where $C_i \cong \mathbb{P}^1$ [VdB, 3.4.1]. Outside of this point, f is an isomorphism.



The restriction to Gorenstein terminal singularities on X in Setup 3.1.1 can be thought of as a ‘niceness’ condition on X ; for example, it is satisfied if X is smooth but it also allows for some mild singularities, such as in the minimal models of Example 3.1.6. But the condition on X also forces conditions on $\text{Spec } R$. Namely, R must be a complete local isolated cDV singularity [R].

Definition 3.1.2. A three dimensional complete local \mathbb{C} -algebra R is a compound Du Val (cDV) singularity if R is isomorphic to

$$\mathbb{C}[[u, v, x, y]]/(f(u, v, x) + yg(u, v, x, y))$$

where $\mathbb{C}[[u, v, x]]/(f)$ is a Du Val (or equivalently Kleinian) surface singularity and g is arbitrary.

In other words, a cDV singularity is a 3-fold such that any generic slice to a surface is in fact a Kleinian singularity. Such surface singularities are very well understood (see e.g. [LW, Chapter 6]) and are classified by the simply laced Dynkin diagrams.

Minimal models are a special case of Setup 3.1.1 where X satisfies a further restriction on the singularities.

Definition 3.1.3. A 3-fold flopping contraction $f: X \rightarrow \text{Spec } R$ where R is complete local and X has at worst \mathbb{Q} -factorial terminal singularities (see [We, §2] for a definition) is called a minimal model of $\text{Spec } R$.

A special case is when X is smooth, in which case $f: X \rightarrow \text{Spec } R$ is called a *crepant resolution* of $\text{Spec } R$. The idea is that X can be considered a ‘best’ approximation of $\text{Spec } R$. It is well known that for a given 3-fold $\text{Spec } R$, there are multiple minimal models and thus it is natural to ask how many there are and how they are related. Both questions have answers; there are only ever finitely many minimal models [KM], and they are all connected by sequences of codimension two surgery operations, known as flops [K].

To explain the latter in more detail, recall the exceptional locus of f is $C^{\text{red}} = \cup_{i=1}^n C_i$. Choosing any such C_i , since R is complete local, it is possible to factorise f as

$$X \xrightarrow{g} X_{\text{con}} \xrightarrow{h} \text{Spec } R$$

where $g(C_j)$ is a single point if and only if $j = i$; that is, g contracts the curve C_i to a point and leaves everything else fixed (see e.g. [Ko, p25] or [Sc, §2]). For any such

factorisation, it is possible to find a birational map $g^+ : X^+ \rightarrow X_{\text{con}}$, satisfying some technical conditions detailed in [We, 2.6], which fits into a commutative diagram

$$\begin{array}{ccc}
 X & \overset{\phi}{\dashrightarrow} & X^+ \\
 \searrow g & & \swarrow g^+ \\
 & X_{\text{con}} & \\
 \swarrow f & & \searrow f^+ \\
 & \text{Spec } R & \\
 & \uparrow h & \\
 & \text{Spec } R &
 \end{array}$$

where ϕ is a birational equivalence [Ko, 2.4]. Such a map g^+ is unique and induces the map $f^+ : X^+ \rightarrow \text{Spec } R$ which we call the *simple flop* of f at the curve C_i . Note that f^+ still satisfies all the conditions in Setup 3.1.1 and moreover, f is a minimal model if and only if f^+ is [Ko, 4.11].

It is well known that the number of curves in the exceptional locus of f^+ matches that of f and there is a natural correspondence between them. In this way, we may think of flopping as choosing a curve in the exceptional locus, cutting it out and trying to replace it with a new curve in such a way that we still satisfy the setup in 3.1.1. If we fix an ordering C_1, \dots, C_n on the curves in f , this will also fix an ordering on the curves in f^+ and we will often abuse notation by using C_1, \dots, C_n to also refer to the curves in f^+ .

We further note that flopping is an involution; if $f^+ : X^+ \rightarrow \text{Spec } R$ is the simple flop of f at the curve C_i then $f : X \rightarrow \text{Spec } R$ is the simple flop of f^+ at the curve C_i . However, by choosing different curves to flop at each time, we can iterate this procedure to produce many new flopping contractions from a given one.

Definition 3.1.4. *With f as in the setup of 3.1.1, we define the simple flops graph of f as follows: the vertices consist of all the flopping contractions which can be obtained from f by a finite number of simple flops and there is an edge between two vertices if the corresponding flopping contractions are connected by a simple flop.*

In the setup of 3.1.1, all curves are individually floppable and hence the simple flops graph of any such flopping contraction will be an n regular graph, where n is the number of curves in the exceptional locus. Kawamata's result about minimal models says that the simple flops graph of any minimal model of $\text{Spec } R$ will contain precisely all the minimal models of $\text{Spec } R$.

Remark 3.1.5. The Minimal Model Programme (and the homological approach) work in more general settings where it may not be possible to flop each curve individually; the issue that arises is that by changing one curve, you are also forced to change others at the same time. However, as $\text{Spec } R$ in Setup 3.1.1 is complete local and isolated, this ensures that the simple flop at every curve does exist.

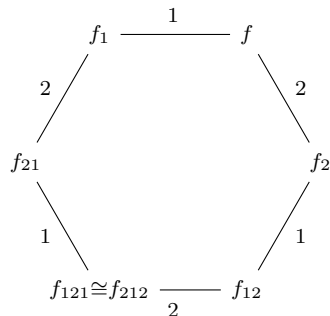
Example 3.1.6 (Atiyah Flop). Consider the singularity $R := \mathbb{C}[[u, v, x, y]]/(uv - xy)$ which has a unique singular point at the origin. Blowing up at the maximal ideal (u, v, x, y) gives a resolution $f : X \rightarrow \text{Spec } R$ where f is an isomorphism away from the origin and the exceptional locus is $\mathbb{P}^1 \times \mathbb{P}^1$. Although X is smooth, it is considered 'too far away' from the original space $\text{Spec } R$ to be a minimal model (the technical condition that f no longer satisfies is crepancy). However, we can obtain the minimal

models of $\text{Spec } R$ from this resolution; obtain X_1 by contracting the first \mathbb{P}^1 and X_2 by contracting the second. In both cases, the map f factors as

$$X \rightarrow X_i \xrightarrow{f_i} \text{Spec } R,$$

where we see the exceptional locus of f_i is a single \mathbb{P}^1 . Now, f_1 and f_2 are the minimal models of $\text{Spec } R$ and they are the simple flop of each other. Since all minimal models must be connected by a sequence of flops, this shows $\text{Spec } R$ has just two minimal models.

Example 3.1.7. Given the cA_2 singularity $R := \mathbb{C}[[u, v, x, y]]/(uv - xy(x + y))$ and any minimal model of $f: X \rightarrow \text{Spec } R$, the simple flops graph of f is



where $f_{i_m \dots i_1}$ denotes the flopping contraction defined iteratively by:

1. f_{i_1} is the simple flop of the f at curve C_{i_1} ;
2. $f_{i_j \dots i_1}$ is the flop of $f_{i_{j-1} \dots i_1}$ at the curve C_{i_j} for $1 < j \leq m$.

In summary, we can start from one minimal model and obtain all others via simple flops, and this information is stored in the simple flops graph. This should be reminiscent of the previous chapter, where one can start with one maximal rigid object and obtain all the others via mutation. However, unlike maximal rigid objects, which come with a procedure for calculating the mutation at a certain summand, calculating flops can be extremely difficult. One of the key successes of the Homological Minimal Model Programme is to find a connection between the two, and use the ease of computation on the algebraic side to determine geometric results.

3.1.1 Key Constructions

The idea of the Homological Minimal Model Programme is to use noncommutative algebra to study minimal models and more generally, 3-fold flopping contractions. This section, following [DW2, 3.5], details how to construct the algebras we will study.

Recall that in the setup of 3.1.1, the exceptional locus of f consists of a finite chain of curves C_1, \dots, C_n . For each i , let \mathcal{L}_i be the unique line bundle on X such that $\mathcal{L}_i \cdot C_j = \delta_{ij}$ [VdB, §3.4]. If the multiplicity of C_i is equal to 1, set $\mathcal{M}_i := \mathcal{L}_i$. Otherwise, define \mathcal{M}_i to be given by the maximal extension

$$0 \rightarrow \mathcal{O}_X^{\oplus(r-1)} \rightarrow \mathcal{M}_i \rightarrow \mathcal{L}_i \rightarrow 0$$

associated to a minimal set of $r - 1$ generators of $H^1(X, \mathcal{L}_i^*)$ as an R -module [VdB, 3.5.4]. Then, by [VdB, 3.5.5],

$$\mathcal{O}_X \oplus \bigoplus_{i=1}^n \mathcal{M}_i^*$$

is a tilting bundle on X . Associated to this is the algebra $A := \text{End}_X(\mathcal{O}_X \oplus \bigoplus_{i=1}^n \mathcal{M}_i^*)$, and a derived equivalence

$$\mathbf{R}\text{Hom}_X \left(\mathcal{O}_X \oplus \bigoplus_{i=1}^n \mathcal{M}_i^*, - \right) : \text{D}^b(\text{coh } X) \xrightarrow{\sim} \text{D}^b(\text{mod } A). \quad (3.1)$$

Pushing forward via f gives $f_*(\mathcal{O}_X) \cong R$ and, for each i , $f_*(\mathcal{M}_i^*) \cong N_i$ for some R -module N_i (see Theorem 3.2.4 for a characterisation of these modules). Since f is a flopping contraction, there is an isomorphism [VdB, 3.2.10]

$$A \cong \text{End}_R \left(R \oplus \bigoplus_{i=1}^n N_i \right).$$

In other words, for each curve in the exceptional locus, this construction yields an indecomposable R -module and by taking certain endomorphism algebras of these modules, we get the algebras in which we are interested.

Definition 3.1.8. *We call A the modification algebra associated to f and define the contraction algebra associated to f to be the stable endomorphism algebra*

$$A_{\text{con}} := \underline{\text{End}}_R \left(R \oplus \bigoplus_{i=1}^n N_i \right) := \text{End}_R \left(R \oplus \bigoplus_{i=1}^n N_i \right) / [R]$$

where $[R]$ denotes the ideal of all morphisms which factor through $\text{add } R$.

If further f is a minimal model of $\text{Spec } R$, we call A a maximal modification algebra (MMA) and we refer to the set of contraction algebras of all minimal models of $\text{Spec } R$ as the contraction algebras of $\text{Spec } R$.

Due to the equivalence (3.1), one might expect that the homological properties of the modification algebra A correspond to properties of X . This is indeed the case and, in fact, much more is also true. As the modification algebras are not the primary study of this thesis we only provide a brief selection of highlights from the many results about them but direct the reader to [We] for details. Note that for A as above, the quiver of such an algebra will have one vertex for each summand R, N_1, \dots, N_n .

1. The dual graph of f (which records how the curves C_1, \dots, C_n intersect with each other) can be obtained from the quiver of A by removing the vertex corresponding to R , merging double edges and removing loops [We, 4.10].
2. If X is smooth, the number of loops at a vertex in the quiver of A determines the normal bundle of the corresponding curve [We, 2.15].
3. If f is a minimal model, *all* minimal models of $\text{Spec } R$ can be recovered from A using a process called quiver GIT [We, 6.2].

The first two results show that modification algebras carry some geometric information but the final result shows that a single MMA in fact knows everything about all the minimal models of $\text{Spec } R$. In other words, we do not lose any geometric information by passing to the algebra.

As the quiver of the contraction algebra A_{con} can be obtained from that of A simply by deleting the vertex corresponding to R , we see that the first two results also hold for contraction algebras. A natural question to ask is whether the contraction algebra A_{con} , which in this setting is always finite dimensional, also completely controls the geometry. These ideas led to the following conjecture of Donovan–Wemyss.

Conjecture 3.1.9 (Donovan–Wemyss). *Suppose that $f: X \rightarrow \text{Spec } R$ and $g: Y \rightarrow \text{Spec } S$ are smooth minimal models of complete local isolated cDV singularities with associated contraction algebras A_{con} and B_{con} . Then $R \cong S$ if and only if A_{con} and B_{con} are derived equivalent.*

This says that R , and hence all the geometry, can be recovered from the derived category of a contraction algebra and if true, shows that the geometry can be studied using just finite dimensional algebras. This is the conjecture our later results will provide evidence towards.

Remark 3.1.10. The contraction algebra A_{con} can also be defined as the representing object of a certain deformation functor of the curves in C , but we will not need this fact in this thesis. See [DW2] for more details.

3.2 MM Modules

In the previous section, we constructed the modification algebra associated to a flopping contraction $f: X \rightarrow \text{Spec } R$. The construction started with a tilting bundle on X before pushing it forward to obtain an R module. This pushforward process made no difference to the resulting endomorphism algebra, but it has the huge advantage that the collection of minimal models of $\text{Spec } R$ now gives a collection of R -modules. Since these all belong to the same category, we can compare them and ask how they are related. We begin by giving an algebraic description of the R -modules that occur.

Definition 3.2.1. *Let (R, \mathfrak{m}) be a commutative noetherian local ring and choose $M \in \text{mod } R$. Then define the depth of M to be*

$$\text{depth}_R(M) = \min\{i \geq 0 \mid \text{Ext}_R^i(R/\mathfrak{m}, M) \neq 0\}.$$

We say M is maximal Cohen–Macaulay (CM) if $\text{depth}_R(M) = \dim(R)$ and write $\text{CM } R$ for the full subcategory of $\text{mod } R$ consisting of maximal Cohen–Macaulay modules.

The following definition of modifying modules was first given in [IW1, 4.1].

Definition 3.2.2. *Let R be a complete local isolated cDV singularity.*

1. *We denote by $(-)^*$ the functor $\text{Hom}_R(-, R): \text{mod } R \rightarrow \text{mod } R^{\text{op}}$ and call an R -module M reflexive if the natural map $M \rightarrow M^{**}$ is an isomorphism. The full subcategory of $\text{mod } R$ consisting of reflexive modules is denoted by $\text{refl } R$.*
2. *We call $M \in \text{refl } R$ a modifying module if $\text{End}_R(M) \in \text{CM } R$.*
3. *A modifying module M is called maximal modifying (MM) if for any $X \in \text{refl } R$ such that $X \oplus M$ is modifying, it must be that $X \in \text{add}(M)$.*

Remark 3.2.3. Modifying modules are also defined for more general rings R , allowing the results of the Homological MMP to work in a more general setting than 3.1.1, but we will not need this in this thesis.

Using this definition, the following gives a complete description of the R -modules appearing in the construction of modification and contraction algebras.

Theorem 3.2.4. [We, 4.10, 4.13] *If $\text{Spec } R$ is a complete local isolated cDV singularity, then the construction in §3.1.1 yields a bijection*

$$\{\text{flopping contractions } f: X \rightarrow \text{Spec } R\} \longleftrightarrow \{\text{basic modifying modules in CM } R\}$$

$$f \longmapsto R \oplus \bigoplus_{i=1}^n N_i$$

which restricts to a bijection

$$\{\text{minimal models of } \text{Spec } R\} \longleftrightarrow \{\text{basic MM modules in CM } R\}.$$

Note that the inverse of this map is constructed using a process called quiver GIT, which builds a space from the corresponding modification algebra. Thus, if we understand all the MM modules in $\text{CM } R$ for a complete local isolated cDV singularity $\text{Spec } R$, this will allow us to understand all the minimal models. Just as we pass between minimal models by flopping, we pass between modifying modules using mutation.

3.2.1 Mutation

To define mutation of modifying modules recall the definition of approximations from §2.2.2. Now suppose $M = \bigoplus_{i=0}^n M_i \in \text{CM } R$ is a basic modifying module with $M_0 \cong R$. To mutate at M_i , where $i \neq 0$, take a minimal right $\text{add}((M/M_i)^*)$ -approximation

$$V_i^* \xrightarrow{a_i} M_i^*,$$

and let $J_i := \text{Ker } a_i$ so that there is an exact sequence

$$0 \rightarrow J_i \xrightarrow{c_i} V_i^* \xrightarrow{a_i} M_i^*. \quad (3.2)$$

Define $K_i := J_i^*$ and $\mathfrak{v}_i M := M/M_i \oplus K_i$. The module $\mathfrak{v}_i M$ is again CM and is a modifying module [IW1, 6.10]. Further, if M is an MM module then $\mathfrak{v}_i M$ is also an MM module [IW1, 6.10].

Lemma 3.2.5. *The sequence (3.2) induces an exact sequence*

$$0 \rightarrow M_i \xrightarrow{b_i} V_i \xrightarrow{d_i} K_i \rightarrow 0 \quad (3.3)$$

such that d_i is a minimal right $\text{add}(M/M_i)$ -approximation of K_i and b_i is a minimal left $\text{add}(M/M_i)$ -approximation of M_i .

Proof. As $R \in \text{add}(M)$ and M is basic, it must be that $R \in \text{add}((M/M_i)^*)$ and hence applying $\text{Hom}_R(R, -)$ to (3.2) shows that

$$0 \rightarrow J_i \xrightarrow{c_i} V_i^* \xrightarrow{a_i} M_i^* \rightarrow 0 \quad (3.4)$$

is exact. Further, by [We, A.1(2)], c_i is a minimal left $\text{add}((M/M_i)^*)$ -approximation. Thus, applying $\text{Hom}_R(-, R)$ to (3.4) gives an exact sequence

$$0 \rightarrow M_i^{**} \xrightarrow{a_i^*} V_i^{**} \xrightarrow{c_i^*} K_i \rightarrow 0.$$

However, as M_i and V_i are both reflexive, this gives an exact sequence

$$0 \rightarrow M_i \xrightarrow{a_i^*} V_i \xrightarrow{c_i^*} K_i \rightarrow 0.$$

Finally, as M and $\nu_i M$ are modifying modules, [IW1, 4.3] shows that $\text{Ext}_R^1(M, M)$ and $\text{Ext}_R^1(\nu_i M, \nu_i M)$ are both zero. Thus, both $\text{Ext}_R^1(M/M_i, M_i)$ and $\text{Ext}_R^1(K_i, M/M_i)$ are zero, showing that $d_i := c_i^*$ is a right $\text{add}(M/M_i)$ -approximation and $b_i := a_i^*$ is a left $\text{add}(M/M_i)$ -approximation respectively. Minimality of both follows from minimality of a_i and c_i . \square

We refer to the sequence (3.3) as the *exchange sequence* defining $\nu_i M$ and we call $\nu_i M$ the left mutation of M at the summand M_i .

Similarly, right mutation uses an exchange sequence

$$0 \rightarrow J_i \xrightarrow{a_i} U_i \xrightarrow{c_i} M_i \rightarrow 0 \tag{3.5}$$

where c_i is a minimal right $\text{add}(M/M_i)$ -approximation of M_i and a_i is a minimal left $\text{add}(M/M_i)$ -approximation of J_i . It is clear from this that left and right mutation are inverse operations, but in fact, in our special setting of cDV singularities, they coincide.

Proposition 3.2.6. [We, 2.25] *With the setup of 3.1.1 and notation as above, there is an isomorphism $J_i \cong K_i$. Consequently, $\nu_i \nu_i M \cong M$.*

Thus, we do not need to distinguish between left and right mutation and will always refer to this simply as mutation.

Now, we can start with a modifying module M and mutate it at a summand M_i or alternatively, we can take the flopping contraction f corresponding to M via the bijection in Theorem 3.2.4, flop it at the curve corresponding to the summand M_i and map it back across the bijection to obtain another modifying module. The following states that these processes coincide.

Theorem 3.2.7. [We, 4.10] *Under the bijection in Theorem 3.2.4, a simple flop corresponds to a mutation at some summand M_i with $i \neq 0$.*

Remark 3.2.8. One direct consequence of this result is as follows. Using the notation from Example 3.1.7 for iterated flops, this shows that if M is the MM module associated to a flopping contraction $f: X \rightarrow \text{Spec } R$, then

$$\underline{\text{End}}_R(\nu_{i_m} \dots \nu_{i_1} M)$$

is the contraction algebra of $f_{i_m \dots i_1}$.

This result allows us to do mutations on the algebraic side, where they are much easier to compute, and then pull them back across to obtain geometric results. For example, this result is crucial towards the result that you can recover all minimal models from a given MMA.

3.2.2 Link with Cluster-Tilting Theory

It is clear to see that the mutation procedure defined above for modifying modules is very similar to the mutation procedure for rigid objects in cluster-tilting theory. In this section, we show that, in the setup of 3.1.1, they are actually equivalent.

Recall that we can define the stable category $\underline{\text{CM}} R$ of $\text{CM} R$ to have the same objects as $\text{CM} R$ but where

$$\underline{\text{Hom}}_R(M, N) := \text{Hom}_R(M, N)/[R]$$

where $[R]$ denotes any morphism which factors through $\text{add}(R)$. The following summary theorem asserts that the category $\text{CM} R$ and its stable category $\underline{\text{CM}} R$ have the appropriate properties to consider cluster-tilting theory. For details, and full references, see e.g. [BIKR, §1].

Proposition 3.2.9. *If R is a complete local isolated cDV singularity, $\text{CM} R$ is a Frobenius category with $\text{proj}(\text{CM} R) = \text{add}(R)$. Moreover, the stable category $\underline{\text{CM}} R$ is a Krull-Schmidt, Hom-finite, 2-Calabi-Yau triangulated category with shift functor given by the inverse syzygy functor Ω^{-1} satisfying $\Omega^2 \cong \text{id}$.*

This allows us to define rigid objects in the category $\underline{\text{CM}} R$ and the following shows that they coincide with the modifying modules.

Proposition 3.2.10. [IW1, 5.12] *Suppose that R is a complete local isolated cDV singularity. Then $M \in \text{CM} R$ is a modifying module if and only if M is rigid in $\underline{\text{CM}} R$ and $R \oplus M \in \text{CM} R$ is an MM module if and only if M is maximal rigid in $\underline{\text{CM}} R$.*

In particular, each contraction algebra in this thesis can be viewed as the endomorphism algebra of a rigid object in some k -linear, Hom-finite, Krull-Schmidt, 2-CY triangulated category. This viewpoint makes it clear that the contraction algebras are finite dimensional as they are endomorphism algebras in a Hom-finite category.

Remark 3.2.11. If we only consider complete local cDV singularities (not necessarily isolated), the category $\underline{\text{CM}} R$ loses the Hom-finite condition and, so although contraction algebras can still be defined, they are not necessarily finite dimensional.

Note that the property that the shift functor Ω^{-1} satisfies $\Omega^2 \cong \text{id}$ is not typical for a category considered in cluster-tilting theory. This leads to the following special property for contraction algebras.

Proposition 3.2.12. [BIKR, 7.1] *Let $f: X \rightarrow \text{Spec} R$ be as in Setup 3.1.1. Then the contraction algebra Λ_{con} associated to f is symmetric i.e. $\Lambda_{\text{con}} \cong \text{Hom}_k(\Lambda_{\text{con}}, k)$ as $\Lambda_{\text{con}}\text{-}\Lambda_{\text{con}}$ bimodules.*

Note that this also gives some duality on modifying modules. Given any modifying module M , the module ΩM will also be modifying and further, will have isomorphic corresponding contraction algebra. Combining $\Omega^2 \cong \text{id}$ with the 2-CY property also gives isomorphisms

$$\text{Ext}_R^1(M, \Omega N) \cong \underline{\text{Hom}}_R(M, N) \cong \text{Ext}_R^1(\Omega M, N)$$

for any $M, N \in \text{CM} R$, which we will use freely throughout.

Now that we have two ways to view modifying modules, we also have two ways to mutate them; we can mutate using the exchange sequence (3.3) in $\text{CM} R$ or we can descend to $\underline{\text{CM}} R$ and mutate M as a rigid object there (as in §2.3). The following well-known lemma shows that these are the same.

Lemma 3.2.13. *The sequence (3.3) induces an exchange triangle in $\underline{\text{CM}} R$. In particular, if M is a rigid object in $\underline{\text{CM}} R$, mutation of $R \oplus M$ as a modifying module and mutation of M as a rigid object coincide.*

Proof. Since (3.3) is an exact sequence in $\text{CM } R$, Proposition 2.1.12 shows it induces a triangle

$$M_i \xrightarrow{b_i} V_i \xrightarrow{d_i} K_i \rightarrow \Omega^{-1}M_i.$$

Applying $\underline{\text{Hom}}_R(-, M/M_i)$ to this triangle and using rigidity of $\nu_i M$ from Proposition 3.2.10 gives an exact sequence

$$\underline{\text{Hom}}_R(V_i, M/M_i) \xrightarrow{-\circ b_i} \underline{\text{Hom}}_R(M_i, M/M_i) \rightarrow 0$$

which shows b_i is a left $\text{add}(M/M_i)$ -approximation. The minimality of b_i in $\underline{\text{CM}} R$ follows from minimality in $\text{CM } R$. \square

Note that combining this result with Proposition 3.2.6 shows that mutation of rigid objects in $\underline{\text{CM}} R$ satisfies another property atypical for rigid objects in general cluster-tilting theory; namely, left mutation will always equal right mutation.

3.2.3 Equivalent Setups

The discussion so far in this chapter can be summarised as giving several equivalent setups to our original geometric setup in 3.1.1:

1. Let $f: X \rightarrow \text{Spec } R$ be as in Setup 3.1.1, and let M be the associated R module constructed as in §3.1.1.
2. Let $\text{Spec } R$ be a complete local isolated cDV singularity and choose a basic modifying module $M := \bigoplus_{i=0}^n M_i \in \text{CM } R$ such that $R \cong M_0$.
3. Let $\text{Spec } R$ be a complete local isolated cDV singularity and choose a basic rigid object $\bigoplus_{i=1}^n M_i \in \underline{\text{CM}} R$. Let $M := R \oplus \bigoplus_{i=1}^n M_i$.

In general, since the results in this thesis are purely algebraic, we will typically avoid the first setup, reserving it for the geometric corollaries we can obtain from our results. We will swap between the other two setups, usually using the second when the modification algebra (and hence the summand R) plays an important role. In either setup however, we will freely use the results available from the other point of view.

3.3 cA_n Singularities

As with the Kleinian surface singularities they generalise, cDV singularities can be grouped into types A, D and E , corresponding to the Dynkin diagrams of the same names. In this section, we introduce the first type, complete local cA_{t-1} singularities, as well as what is already known about their minimal models. These singularities can be written in the form

$$R := \mathbb{C}\llbracket u, x, x, v \rrbracket / (uv - f(x, y))$$

where t is the order of the polynomial $f(x, y)$ considered as a power series. We will only consider isolated singularities, which can be characterised using the irreducible

factors of $f(x, y)$; namely, if f factors into n irreducible polynomials f_1, \dots, f_n , then R is isolated precisely when $(f_i) \neq (f_j)$ for all $i \neq j$. In this case, the modifying modules in $\text{CM } R$ (or equivalently the rigid objects in $\underline{\text{CM}} R$) have been completely determined.

Definition 3.3.1. *Suppose $\mathbb{C}[[u, x, x, v]]/(uv - f_1 \dots f_n)$ is a cA_t singularity. Given $\sigma \in S_n$, where S_n is the symmetric group on n objects, define*

$$M_\sigma := (u, f_{\sigma(1)}) \oplus (u, f_{\sigma(1)}f_{\sigma(2)}) \oplus \dots \oplus (u, f_{\sigma(1)} \dots f_{\sigma(n-1)}).$$

Theorem 3.3.2. [IW2, 5.1] *For an isolated cA_t singularity $R := \mathbb{C}[[u, x, x, v]]/(uv - f_1 \dots f_n)$, the maximal rigid objects in $\underline{\text{CM}} R$ are precisely the objects $\{M_\sigma \mid \sigma \in S_n\}$. In particular, there are $n!$ maximal rigid objects, and each has $n - 1$ summands.*

If $\text{Spec } R$ has smooth minimal models, the maximal rigid objects are in fact cluster-tilting objects and in this setting Theorem 3.3.2 was first proved in [BIKR, 4.2].

When $n = 2$, there are only two rigid objects, which are both maximal. Thus, by Theorems 3.2.4 and 3.2.7, any cA_t singularity of this form has two minimal models, each with one curve in the exceptional locus, and they are each the flop of each other. Note that the Atiyah flop in Example 3.1.6 is an example of this and in that case, the contraction algebras are both simply \mathbb{C} .

If we choose $n = 3$, there are six maximal rigid objects in $\underline{\text{CM}} R$, each with two summands and the mutation graph, containing a vertex for each maximal rigid object and an edge connecting two vertices if they differ by exactly one indecomposable summand, is shown in Figure 3.1. We see that this matches the simple flops graph from Example 3.1.7.

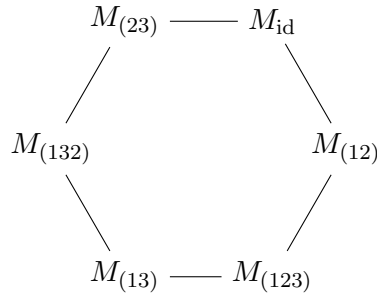


Figure 3.1: The graph shows the mutation graph of maximal rigid objects for a cA_t singularity given by $f(x, y) = f_1 f_2 f_3$.

Indeed, by Theorem 3.2.4, this shows there are six minimal models of $\text{Spec } R$, each with two curves in the exceptional locus. Choosing $f_1 = x$, $f_2 = y$ and $f_3 = x^2 + y^3$ the contraction algebras associated to these minimal models are given in Figure 3.2.

Remark 3.3.3. More generally, if Λ_{con} is the contraction algebra of a minimal model of a complete local cA_t singularity given by $uv - f_1 \dots f_n$, then the quiver of Λ_{con} will be the double of the A_{n-1} Dynkin quiver, possibly with up to two loops at each vertex [IW2, 5.29].

3.4 Derived Equivalences

Recall that if $f: X \rightarrow \text{Spec } R$ is a flopping contraction as in Setup 3.1.1, then it is possible to choose any curve in the exceptional locus of f , and flop at that curve to

$$\begin{aligned}
A_{\text{con}} &:= \underline{\text{End}}_R(M_{\text{id}}) \cong \underline{\text{End}}_R(M_{(13)}) \cong \begin{array}{c} \bullet \xleftarrow{a} \bullet \\ \bullet \xrightarrow{c} \bullet \\ \bullet \xrightarrow{l} \bullet \end{array} \quad \begin{array}{l} l^2 + acacac = 0 \\ cl = 0 \\ la = 0 \end{array} \\
B_{\text{con}} &:= \underline{\text{End}}_R(M_{(23)}) \cong \underline{\text{End}}_R(M_{(123)}) \cong \begin{array}{c} \bullet \xleftarrow{a} \bullet \\ \bullet \xrightarrow{c} \bullet \\ \bullet \xrightarrow{m} \bullet \end{array} \quad \begin{array}{l} l^2 = ac \quad m^3 = ca \\ la = 0 \quad am = 0 \\ cl = 0 \quad mc = 0 \end{array} \\
C_{\text{con}} &:= \underline{\text{End}}_R(M_{(132)}) \cong \underline{\text{End}}_R(M_{(12)}) \cong \begin{array}{c} \bullet \xleftarrow{a} \bullet \\ \bullet \xrightarrow{c} \bullet \\ \bullet \xrightarrow{m} \bullet \end{array} \quad \begin{array}{l} m^3 + caca = 0 \\ mc = 0 \\ am = 0 \end{array}
\end{aligned}$$

Figure 3.2: The quivers and relations of the contraction algebras of the minimal models of the cA_t singularity with $f_1 = x, f_2 = y$ and $f_3 = x^2 + y^3$.

obtain a new flopping contraction $f^+ : X^+ \rightarrow \text{Spec } R$. This construction can be shown to induce two derived equivalences

$$D^b(\text{coh } X) \rightarrow D^b(\text{coh } X^+) \quad \text{and} \quad D^b(\text{coh } X^+) \rightarrow D^b(\text{coh } X), \quad (3.6)$$

known as *Bridgeland-Chen flop functors* [B, C]. As these are not inverse to each other, a nontrivial autoequivalence of $D^b(\text{coh } X)$ can be obtained by considering their composition. More generally, we can think of any autoequivalence obtained as the composition of flop functors as a derived symmetry arising from birational geometry. To study these symmetries, we can use the associated algebras.

From the construction in §3.1.1, it can be seen that the modification algebra A associated to f satisfies

$$D^b(A) \simeq D^b(\text{coh } X).$$

Thus, if we wish to study the derived equivalences between flopping contractions, or even the autoequivalences of a single one, it is equivalent to study the derived equivalences between the modification algebras. Just as flops induce derived equivalences on the geometric side, the following shows mutation induces derived equivalences between the modification algebras.

Theorem 3.4.1. [IW1, 4.17, 6.14],[We, 2.20] *Let $\text{Spec } R$ be a complete local isolated cDV singularity and suppose that $M := \bigoplus_{i=0}^n M_i \in \text{CM } R$ is a basic modifying module with $M_0 \cong R$. Writing $\Lambda := \text{End}_R(M)$, the following statements hold.*

1. *For any $i \neq 0$, the bimodule $\text{Hom}_R(M, \nu_i M)$ is tilting of projective dimension one on both sides, which gives rise to a standard equivalence*

$$G_i := \mathbf{R}\text{Hom}_\Lambda(\text{Hom}_R(M, \nu_i M), -) : D^b(\Lambda) \xrightarrow{\sim} D^b(\text{End}_R(\nu_i M)).$$

2. *If further M is an MM module then, for any other MM module N , the bimodule $\text{Hom}_R(M, N)$ is tilting of projective dimension one on both sides, which gives rise*

to a standard derived equivalence

$$\mathbf{R}\mathrm{Hom}_\Lambda(\mathrm{Hom}_R(M, N), -): \mathrm{D}^b(\Lambda) \xrightarrow{\sim} \mathrm{D}^b(\mathrm{End}_R(N)).$$

It is shown in [We, 4.2] that, if M corresponds to f , then $\nu_i M$ corresponds to f^+ and G_i is functorially isomorphic to the inverse of the corresponding flop functor. Thus, to study the flop functors, it is equivalent to study the equivalences G_i between the modifying modules.

In the case of minimal models, this approach has yielded great success; it is shown in [HW] that the group structure of the flop functors is completely determined by the topology of an associated hyperplane arrangement. We leave the description of this hyperplane arrangement, and the precise results to Chapter 6, where we will make use of them to obtain similar results about contraction algebras.

Recall that the results of this thesis are intended to provide evidence towards Conjecture 3.1.9; that for two smooth minimal models $f: X \rightarrow \mathrm{Spec} R$, $g: Y \rightarrow \mathrm{Spec} S$ of complete local isolated cDV singularities, $R \cong S$ if and only if their contraction algebras are derived equivalent.

For this reason, this thesis focuses on understanding the derived equivalences between contraction algebras. We finish this section by recalling what was already known about such equivalences.

Theorem 3.4.2. [D, 5.5] *Let $\mathrm{Spec} R$ be a complete local isolated cDV singularity and suppose that $M := \bigoplus_{i=0}^n M_i \in \mathrm{CM} R$ is a basic MM module with $M_0 \cong R$. Then for $i \neq 0$, $\underline{\mathrm{End}}_R(M)$ and $\underline{\mathrm{End}}_R(\nu_i M)$ are derived equivalent.*

This result was proved using the cluster-tilting viewpoint. As we know all minimal models are connected by flops, Theorem 3.2.7 shows that all MM modules of R are connected by mutation and combining this with the above theorem then shows that all the contraction algebras of $\mathrm{Spec} R$ are derived equivalent. In other words, the ‘only if’ direction of Conjecture 3.1.9 holds. We will recover Theorem 3.4.2 using our new methods in Corollary 4.4.2.

3.5 Global Setting

In this final subsection, we remove the restrictions in Setup 3.1.1 that the base of the flopping contraction needs to be complete local, or even affine.

Setup 3.5.1. Take $f: X \rightarrow X_{\mathrm{con}}$ to be a 3-fold flopping contraction between quasi-projective varieties where X has at worst Gorenstein terminal singularities.

In this more general setup, there may now be multiple singular points, p_1, \dots, p_n , on the base X_{con} , but each one must still be isolated. Again, f is an isomorphism away from the singular points, and the preimage of each p_i is a finite collection of curves. The key properties of this setup are characterised in Figure 3.3.

In this more general setup, Donovan–Wemyss introduce a more general invariant given by a sheaf of algebras [DW4]. As with the construction of the contraction algebra, the construction involves a vector bundle $\mathcal{V} := \mathcal{O}_X \oplus \mathcal{V}_0$ on X satisfying

$$f_* \mathcal{E}nd_X(\mathcal{V}) \cong \mathcal{E}nd_{X_{\mathrm{con}}}(f_* \mathcal{V}).$$

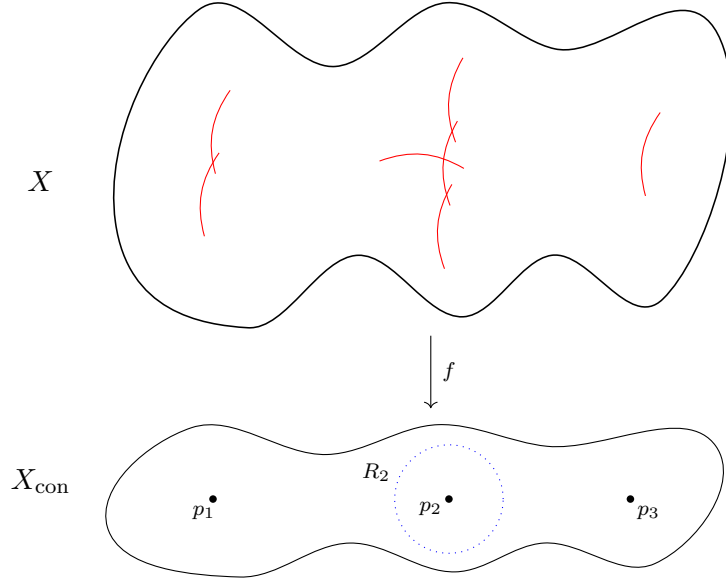


Figure 3.3: A diagram showing the key properties of Setup 3.5.1. The map f is an isomorphism away from the singular points p_1, p_2, p_3 of X_{con} , while the preimage of each p_i is a finite chain of curves. Around each p_i we may choose an affine neighbourhood $\text{Spec } R_i$ containing no other singular points.

Although this bundle may not be tilting (as it is in the complete local case) there is a technical condition on \mathcal{V} , detailed in [DW4, 2.3], which ensures that for any choice of affine open $\text{Spec } R$ in X_{con} , the bundle $\mathcal{V}|_{f^{-1}(\text{Spec } R)}$ is a tilting bundle.

With this bundle \mathcal{V} , they define the *sheaf of contraction algebras* to be

$$\mathcal{D} := f_* \mathcal{E}nd_X(\mathcal{V})/\mathcal{I}$$

where \mathcal{I} is the ideal sheaf of local sections that at each stalk factor through a finitely generated projective $\mathcal{O}_{X_{\text{con}},v}$ -module (see [DW4, 2.8] for details).

Writing Z for the locus of points on X_{con} at which f is not an isomorphism, [DW4, 2.16] showed that the support of the sheaf \mathcal{D} is precisely Z . In particular, in the setup of 3.5.1, the condition on X ensures that $Z = \{p_1, \dots, p_n\}$ where each p_i is an isolated singularity and thus

$$\mathcal{D} \cong \bigoplus_{i=1}^n \mathcal{D}_{p_i}$$

where \mathcal{D}_{p_i} is the $\mathcal{O}_{X_{\text{con}},p_i}$ -algebra given by the stalk of \mathcal{D} at p_i . Specifically, in the setup of 3.5.1, \mathcal{D} is a finite dimensional algebra which splits into a direct sum of algebras, one for each point p_i . As \mathcal{D}_{p_i} is a finite length module over $\mathcal{O}_{X_{\text{con}},p_i}$, there is an isomorphism $\widehat{\mathcal{D}}_{p_i} \cong \mathcal{D}_{p_i}$ of $\mathcal{O}_{X_{\text{con}},p_i}$ -algebras, where $\widehat{\mathcal{D}}_{p_i}$ denotes the completion of \mathcal{D}_{p_i} .

Alternatively, for each p_i , it is possible to choose an affine neighbourhood R_i of p_i which contains no other p_j . Localising if necessary, we can assume p_i is the unique closed point of R_i and setting $U_i := f^{-1}(\text{Spec } R_i)$, we can consider the map $f_i := f|_{U_i}$. Further, we can complete this map to obtain a map

$$\widehat{f}_i: \widehat{U}_i \rightarrow \text{Spec } \widehat{R}_i.$$

This map now satisfies the conditions of the complete local setup in 3.1.1 and thus we get an associated contraction algebra $A_i := \underline{\text{End}}_{\widehat{R}_i}(N_i)$ where N_i is a rigid object in $\underline{\text{CM}}\widehat{R}_i$.

Theorem 3.5.2. [DW4, 2.24] *The algebra \mathcal{D}_{p_i} is morita equivalent to A_i .*

In other words, the invariant in this more general setup is just an algebra which breaks into blocks, where each block is morita equivalent to the contraction algebra of a flopping contraction as in the setting of 3.1.1. Our results in the complete local setting later thus automatically globalise, and so for the remainder of the thesis we restrict to the local case.

Chapter 4

Construction of Derived Equivalences

In the 1980s, Rickard reduced the problem of determining if two algebras are derived equivalent to the search for a certain tilting complex (see Definition 2.2.1). In particular, two algebras A and B are derived equivalent if and only if there exists a tilting complex T for A such that $\text{End}_{\text{D}^b(A)}(T) \cong B$.

Since then, almost all derived equivalences in the literature have been proved using this result. In particular, it is used in [D] to show that two contraction algebras connected by a single mutation are derived equivalent (see Theorem 3.4.2). As noted in the preliminaries, combining this with the fact all contraction algebras are connected by simple mutations then shows all the contraction algebras of a complete local isolated cDV singularity are derived equivalent. In other words, one direction of Conjecture 3.1.9 about contraction algebras is already known.

However, the downside of using Rickard's result is that the explicit functor he writes down to prove the equivalence is very difficult to work with. If only the existence of such an equivalence is required, this is not an issue. However if, as in the situation of the contraction algebras, we wish to study the group structure of derived autoequivalences, we would ideally like so called *standard* derived equivalences (see Definition 2.2.2). This requires the construction of *two-sided tilting complexes*.

This chapter constructs such complexes for the contraction algebras which are further compatible with the known derived equivalences between the corresponding modification algebras. In this way, we can think of the functors we construct as analogues of the flop functors in geometry and this will be crucial in later chapters of this thesis. However, we also note that, as our two-sided construction is independent of the results in [D], this chapter recovers their one-sided results as a special case.

4.1 Summary of Results

Suppose that $\text{Spec } R$ is a complete local isolated cDV singularity and that $M := \bigoplus_{i=0}^n M_i \in \text{CM } R$ is a basic modifying module with $M_0 \cong R$. Recall from §3.2.1 that we can mutate M at the summand M_i , where $i \neq 0$, via an exchange sequence

$$0 \rightarrow M_i \xrightarrow{b_i} V_i \xrightarrow{d_i} K_i \rightarrow 0, \quad (4.1)$$

to get $\nu_i M := M/M_i \oplus K_i$. Consider the following algebras:

$$\begin{aligned} \Lambda &:= \text{End}_R(M) & \Gamma &:= \text{End}_R(\nu_i M) \\ \Lambda_{\text{con}} &:= \underline{\text{End}}_R(M) & \Gamma_{\text{con}} &:= \underline{\text{End}}_R(\nu_i M). \end{aligned}$$

For the modification algebras Λ and Γ , Theorem 3.4.1 gives an explicit derived equivalence induced by a tilting bimodule. However, as contraction algebras are symmetric by Proposition 3.2.12, it is well known that they have no nontrivial tilting modules and thus we have to look for tilting complexes of higher length. The bijection in Theorem 2.3.5 provides a method of construction for those of length two.

Proposition 4.1.1. *With notation as above, the complex*

$$\mathcal{P}_i := \left(\underline{\text{Hom}}_R(M, M_i) \xrightarrow{b_i \circ -} \underline{\text{Hom}}_R(M, V_i) \right) \oplus \left(0 \rightarrow \bigoplus_{j \neq i} \underline{\text{Hom}}_R(M, M_j) \right)$$

is a tilting complex for Λ_{con} .

Proof. Recall that as M is modifying, it is a rigid object in the stable category $\underline{\text{CM}} R$ and by Proposition 2.1.12, the exchange sequence (4.1) descends to a triangle

$$M_i \xrightarrow{b_i} V_i \xrightarrow{d_i} K_i \rightarrow \Omega^{-1} M_i$$

in $\underline{\text{CM}} R$ where d_i is a minimal right $\text{add}(M/M_i)$ -approximation by Lemma 3.2.5. An easy check using that M is rigid shows further that d_i is a minimal right $\text{add}(M)$ -approximation and hence in the triangle

$$M_i \xrightarrow{\begin{pmatrix} b_i \\ 0 \end{pmatrix}} V_i \oplus M/M_i \xrightarrow{\begin{pmatrix} d_i & 0 \\ 0 & \text{id} \end{pmatrix}} K_i \oplus M/M_i \rightarrow \Omega^{-1} M_i,$$

the map $\begin{pmatrix} d_i & 0 \\ 0 & \text{id} \end{pmatrix}$ is a minimal right $\text{add}(M)$ -approximation of $\nu_i M := K_i \oplus M/M_i$. Thus, by the bijection of Theorem 2.3.5, the complex \mathcal{P}_i is a silting complex. As [AI, 2.8] shows any silting complex for a symmetric algebra is in fact tilting, and Λ_{con} is symmetric by Proposition 3.2.12, this completes the proof. \square

Remark 4.1.2. This complex is the same tilting complex as considered in [D], although there it is shown directly from the definition that it is tilting.

Although the complex \mathcal{P}_i is tilting, it has no two-sided structure and hence to obtain a standard equivalence, we must lift \mathcal{P}_i to a two-sided tilting complex. We do this via the construction of the complex \mathcal{T}_i in the second part of the following, which is the main new definition in this chapter.

Setup 4.1.3. With notation as above, set:

1. $\mathcal{T}_i := \text{Hom}_R(M, \nu_i M)$ which, by Theorem 3.4.1, is a Γ - Λ bimodule giving an equivalence

$$- \otimes_{\Gamma}^{\mathbf{L}} \mathcal{T}_i : \text{D}^{\text{b}}(\Gamma) \rightarrow \text{D}^{\text{b}}(\Lambda); \tag{4.2}$$

2. $\mathcal{J}_i := \tau_{\geq -1}(\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} \mathcal{T}_i \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}})$ which is a complex of Γ_{con} - Λ_{con} -bimodules.

Here, $\tau_{\geq -1}$ is the truncation functor taking a complex

$$X := \cdots \rightarrow X_{i-1} \xrightarrow{d_{i-1}} X_i \xrightarrow{d_i} X_{i+1} \rightarrow \cdots$$

to the complex

$$\cdots 0 \rightarrow 0 \rightarrow X_{-1}/\text{Im}(d_{-2}) \xrightarrow{d_{-1}} X_0 \xrightarrow{d_0} X_1 \rightarrow \cdots$$

Note that, if X has zero homology in degrees -2 and lower, there is a quasi-isomorphism $X \rightarrow \tau_{\geq -1}X$.

Ultimately, we wish to show that the complex \mathcal{T}_i is a two-sided tilting complex lifting \mathcal{P}_i . This will require several steps, the first of which is to track the \mathcal{P}_i through the derived equivalence (4.2) between the modification algebras. We establish the following in §4.2.

Theorem 4.1.4. *With the set up of 4.1.3, there is an isomorphism $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \mathcal{P}_i$ in $D^b(\Lambda)$.*

This will then be used in Proposition 4.3.2 to show that \mathcal{T}_i is indeed a lift of \mathcal{P}_i to a complex of $\Gamma_{\text{con}}\text{-}\Lambda_{\text{con}}$ bimodules; that is, there is an isomorphism $\mathcal{T}_i \cong \mathcal{P}_i$ in $D^b(\Lambda_{\text{con}})$. Since \mathcal{T}_i is a complex of $\Gamma_{\text{con}}\text{-}\Lambda_{\text{con}}$ bimodules, it naturally induces a functor between the derived categories of Γ_{con} and Λ_{con} . In §4.3, we show that this functor is compatible with the known derived equivalence between the modification algebras Γ and Λ by establishing the following.

Theorem 4.1.5. *With the setup of 4.1.3, there is an isomorphism*

$$\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \mathcal{T}_i \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}}\Lambda$$

in the derived category of $\Gamma_{\text{con}}\text{-}\Lambda$ -bimodules. Consequently, there is a commutative diagram

$$\begin{array}{ccc} D(\Lambda_{\text{con}}) & \xrightarrow{-\otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}}} & D(\Lambda) \\ \uparrow -\otimes_{\Gamma_{\text{con}}}^{\mathbf{L}} \mathcal{T}_i & & \uparrow -\otimes_{\Gamma}^{\mathbf{L}} T_i \\ D(\Gamma_{\text{con}}) & \xrightarrow{-\otimes_{\Gamma_{\text{con}}} \Gamma_{\text{con}}} & D(\Gamma) \end{array} \quad (4.3)$$

where the functor on the right hand side is an equivalence.

Finally, in §4.4 we will use the commutative diagram to prove the following, which shows that \mathcal{T}_i is in fact a two-sided tilting complex, inducing a standard derived equivalence between the contraction algebras.

Corollary 4.1.6. *With the set up of 4.1.3, the map $\Gamma_{\text{con}} \rightarrow \text{End}_{D^b(\Lambda_{\text{con}})}(\mathcal{T}_i)$ induced by the functor $-\otimes_{\Gamma_{\text{con}}}^{\mathbf{L}} \mathcal{T}_i: D^b(\Gamma_{\text{con}}) \rightarrow D^b(\Lambda_{\text{con}})$ is an algebra isomorphism. Consequently, the functor $F_i := \mathbf{R}\text{Hom}_{\Lambda_{\text{con}}}(\mathcal{T}_i, -): D^b(\Lambda_{\text{con}}) \rightarrow D^b(\Gamma_{\text{con}})$ is a standard equivalence.*

In summary, for any simple mutation of modifying modules, this chapter provides a standard derived equivalence F_i between the corresponding contraction algebras. Moreover, as the functor $-\otimes_{\Gamma}^{\mathbf{L}} T_i$ is known to be isomorphic to the flop functor between minimal models, and F_i^{-1} fits into a commutative diagram with this functor, we think of the equivalence F_i^{-1} as the contraction algebra analogue of the flop functor.

4.2 Tracking Through Derived Equivalences

The goal of this section is to prove Theorem 4.1.4, namely that there is an isomorphism $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \mathcal{P}_i$ in $D^b(\Lambda)$. To do this, we introduce the following notation.

- Notation 4.2.1.**
1. The projective Λ -modules are $P_i := \text{Hom}_R(M, M_i)$ for $i = 0, \dots, n$.
 2. The projective Λ_{con} -modules are $A_i := \underline{\text{Hom}}_R(M, M_i)$ for $i = 1, \dots, n$.
 3. The projective Γ -modules are $Q_j := \text{Hom}_R(\mathbf{v}_i M, M_j)$ for $j = 0, \dots, n$ and $j \neq i$, and $Q_i := \text{Hom}_R(\mathbf{v}_i M, K_i)$.
 4. The projective Γ_{con} -modules are $B_j := \underline{\text{Hom}}_R(\mathbf{v}_i M, M_j)$ for $j = 1, \dots, n$ and $j \neq i$, and $B_i := \underline{\text{Hom}}_R(\mathbf{v}_i M, K_i)$.

For a general complex $X \in D^b(\Gamma)$, the derived tensor product $X \otimes_{\Gamma}^{\mathbf{L}} T_i$ is computed by first finding a complex $Y \in K(\text{proj } \Gamma)$ which is quasi-isomorphic to X and then computing the tensor product $Y \otimes_{\Gamma} T_i$. This second step only involves being able to calculate $Q \otimes_{\Gamma} T_i$ for any projective Γ -module Q , which our first lemma does.

Lemma 4.2.2. *The equivalence (4.2) restricts to an equivalence $\text{proj } \Gamma \rightarrow \text{add}(T_i)$, and thus*

$$Q_j \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \begin{cases} P_j & \text{if } j \neq i \\ \text{Hom}_R(M, K_i) & \text{if } j = i \end{cases}.$$

Moreover, if $f: N \rightarrow N'$ is any morphism in $\text{add}(\mathbf{v}_i M)$, then $\text{Hom}_R(\mathbf{v}_i M, f)$ maps to $\text{Hom}_R(M, f)$ under the equivalence.

Proof. As Γ is a projective Γ -module, it is clear $\Gamma \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \Gamma \otimes_{\Gamma} T_i \cong T_i$ and thus, as $-\otimes_{\Gamma}^{\mathbf{L}} T_i$ is an additive equivalence, it must restrict to an equivalence $\text{add}(\Gamma) \rightarrow \text{add}(T_i)$. Recalling that $\text{proj } \Gamma = \text{add}(\Gamma)$ completes the proof of the first statement.

For an indecomposable module $N \in \text{add}(\mathbf{v}_i M)$, the isomorphism can be described explicitly as

$$\begin{aligned} \phi_N: \text{Hom}_R(\mathbf{v}_i M, N) \otimes_{\Gamma} \text{Hom}_R(M, \mathbf{v}_i M) &\rightarrow \text{Hom}_R(M, N) \\ g \otimes g' &\mapsto g \circ g' \end{aligned}$$

with inverse given by $h \mapsto \text{pr} \otimes i \circ h$ where $\text{pr}: \mathbf{v}_i M \rightarrow N$ and $i: N \rightarrow \mathbf{v}_i M$ are the natural projection and inclusion maps. Thus, if $f: N \rightarrow N'$ is any map between indecomposables in $\text{add}(\mathbf{v}_i M)$, the equivalence (4.2) maps $\text{Hom}_R(\mathbf{v}_i M, f)$ to

$$\phi_{N'} \circ (\text{Hom}_R(\mathbf{v}_i M, f) \otimes \text{id}) \circ \phi_N^{-1}: \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, N')$$

satisfying

$$\begin{aligned} \phi_{N'} \circ (\text{Hom}_R(\mathbf{v}_i M, f) \otimes \text{id}) \circ \phi_N^{-1}(s) &= \phi_{N'} \circ (\text{Hom}_R(\mathbf{v}_i M, f) \otimes \text{id})(\text{pr} \otimes i \circ s) \\ &= \phi_{N'}(f \circ \text{pr} \otimes i \circ s) \\ &= f \circ \text{pr} \circ i \circ s \\ &= f \circ s \\ &= \text{Hom}_R(M, f)(s) \end{aligned}$$

for all $s: M \rightarrow N$. Recalling that $-\otimes_{\Gamma}^{\mathbf{L}} T_i$ is an additive functor then completes the proof. \square

Using Lemma 4.2.2, Γ_{con} -modules can now be tracked through the equivalence (4.2) if their projective resolution as a Γ -module can be computed. The following lemma helps with this. First, recall from §3.2.2 that for $M, N \in \text{CM } R$, there are isomorphisms

$$\text{Ext}_R^1(M, \Omega N) \cong \underline{\text{Hom}}_R(M, N) \cong \text{Ext}_R^1(\Omega M, N)$$

and these will be used freely throughout.

Lemma 4.2.3. *Suppose that N is a basic modifying module in $\text{CM } R$ with $R \in \text{add}(N)$ and that $N_j \not\cong R$ is an indecomposable summand of N . Setting $k = \text{rk}(N_j) + \text{rk}(\Omega N_j)$, then there is a projective resolution of $\underline{\text{Hom}}_R(N, N_j)$ as an $\text{End}_R(N)$ -module of the form*

$$0 \rightarrow \text{Hom}_R(N, N_j) \rightarrow \text{Hom}_R(N, R^k) \rightarrow \text{Hom}_R(N, R^k) \rightarrow \text{Hom}_R(N, N_j) \rightarrow 0.$$

Proof. Given N_j , Proposition 3.2.9 shows that $\Omega^2 \cong \text{id}$ on $\underline{\text{CM}} R$ and hence there are exact sequences

$$0 \rightarrow \Omega N_j \rightarrow R^k \rightarrow N_j \rightarrow 0 \quad (4.4)$$

$$0 \rightarrow N_j \rightarrow R^k \rightarrow \Omega N_j \rightarrow 0 \quad (4.5)$$

which come from taking the syzygy of N_j and ΩN_j respectively. Using that R is injective in $\text{CM } R$ to get $\text{Ext}_R^1(N, R) = 0$, applying $\text{Hom}_R(N, -)$ to (4.4) gives the exact sequence

$$0 \rightarrow \text{Hom}_R(N, \Omega N_j) \rightarrow \text{Hom}_R(N, R^k) \rightarrow \text{Hom}_R(N, N_j) \rightarrow \underline{\text{Hom}}_R(N, N_j) \rightarrow 0.$$

Similarly, since N is modifying and hence rigid in $\underline{\text{CM}} R$, applying $\text{Hom}_R(N, -)$ to (4.5) gives the exact sequence

$$0 \rightarrow \text{Hom}_R(N, N_j) \rightarrow \text{Hom}_R(N, R^k) \rightarrow \text{Hom}_R(N, \Omega N_j) \rightarrow 0.$$

Splicing these two together gives the required result. \square

Since $\Gamma_{\text{con}} := \bigoplus_{j=1}^n B_j$, to prove Theorem 4.1.4, it will be enough to track each B_j through the equivalence (4.2). We start with the case when $j \neq i$.

Lemma 4.2.4. *Under the setup of 4.1.3, when $j \neq i$, $B_j \otimes_{\Gamma}^{\mathbf{L}} T_i \cong A_j$ in $\text{D}^b(\Lambda)$.*

Proof. By Lemma 4.2.3 there is a projective resolution of B_j as a Γ -module of the form

$$0 \rightarrow Q_j \rightarrow Q_0^{n_j} \rightarrow Q_0^{n_j} \rightarrow Q_j \rightarrow 0.$$

Applying $- \otimes_{\Gamma} T_i$ termwise to this complex gives $B_j \otimes_{\Gamma}^{\mathbf{L}} T_i$ but, by Lemma 4.2.2, this is

$$0 \rightarrow P_j \rightarrow P_0^{n_j} \rightarrow P_0^{n_j} \rightarrow P_j \rightarrow 0,$$

where the maps are precisely those giving a projective resolution of A_j as a Λ -module, by Lemma 4.2.3. Thus, $B_j \otimes_{\Gamma}^{\mathbf{L}} T_i$ is quasi-isomorphic to A_j and hence isomorphic in $\text{D}^b(\Lambda)$. \square

To deal with the $j = i$ case, we consider the exact sequences

$$0 \rightarrow \Omega K_i \xrightarrow{f} R^k \xrightarrow{g} K_i \rightarrow 0 \quad (4.6)$$

$$0 \rightarrow K_i \xrightarrow{f'} R^k \xrightarrow{g'} \Omega K_i \rightarrow 0 \quad (4.7)$$

coming from taking syzygies of K_i and ΩK_i respectively.

Lemma 4.2.5. *Under the setup of 4.1.3, the complex $B_i \otimes_{\Gamma}^{\mathbf{L}} T_i$ is isomorphic in $D^b(\Lambda)$ to*

$$0 \rightarrow \mathrm{Hom}_R(M, K_i) \xrightarrow{f' \circ -} \mathrm{Hom}_R(M, R^k) \xrightarrow{f \circ g' \circ -} \mathrm{Hom}_R(M, R^k) \xrightarrow{g \circ -} \mathrm{Hom}_R(M, K_i) \rightarrow 0 \quad (4.8)$$

and hence has homology;

1. $\underline{\mathrm{Hom}}_R(M, K_i)$ in degree 0;
2. $\underline{\mathrm{Hom}}_R(M, \Omega K_i)$ in degree -1 ;
3. 0 elsewhere.

Proof. By Lemma 4.2.3 the sequence

$$0 \rightarrow Q_i \xrightarrow{f' \circ -} Q_0^n \xrightarrow{f \circ g' \circ -} Q_0^n \xrightarrow{g \circ -} Q_i \rightarrow 0$$

is a Γ -projective resolution of B_i and thus to get $B_i \otimes_{\Gamma}^{\mathbf{L}} T_i$, we can apply $- \otimes_{\Gamma} T_i$ termwise to this complex. By Lemma 4.2.2 this is exactly (4.8).

To compute the homology, apply $\mathrm{Hom}_R(M, -)$ to (4.6) to obtain the exact sequence

$$0 \rightarrow \mathrm{Hom}_R(M, \Omega K_i) \xrightarrow{f \circ -} \mathrm{Hom}_R(M, R^k) \xrightarrow{g \circ -} \mathrm{Hom}_R(M, K_i) \rightarrow \mathrm{Ext}_R^1(M, \Omega K_i) \rightarrow 0$$

which shows

$$\mathrm{Hom}_R(M, K_i) / \mathrm{Im}(g \circ -) \cong \mathrm{Ext}_R^1(M, \Omega K_i) \cong \underline{\mathrm{Hom}}_R(M, K_i).$$

This shows the homology in degree 0. Since the sequence is exact

$$\mathrm{Ker}(g \circ -) = \mathrm{Im}(f \circ -) \cong \mathrm{Hom}_R(M, \Omega K_i)$$

and further, $\mathrm{Im}(f \circ g' \circ -) \cong \mathrm{Im}(g' \circ -)$ as f is injective. Thus,

$$\mathrm{Ker}(g \circ -) / \mathrm{Im}(f \circ g' \circ -) \cong \mathrm{Hom}_R(M, \Omega K_i) / \mathrm{Im}(g' \circ -) \cong \underline{\mathrm{Hom}}_R(M, \Omega K_i)$$

where the last isomorphism comes from the exact sequence

$$0 \rightarrow \mathrm{Hom}_R(M, K_i) \xrightarrow{f' \circ -} \mathrm{Hom}_R(M, R^k) \xrightarrow{g' \circ -} \mathrm{Hom}_R(M, \Omega K_i) \rightarrow \mathrm{Ext}_R^1(M, K_i) \rightarrow 0$$

obtained by applying $\mathrm{Hom}_R(M, -)$ to the exact sequence (4.7). This sequence also shows that the complex (4.8) is exact elsewhere. \square

Since $B_i \otimes_{\Gamma}^{\mathbf{L}} T_i$ has zero homology outside degrees -1 and 0 , the complex can be truncated appropriately to give a quasi-isomorphic complex.

Corollary 4.2.6. *The complex $B_i \otimes_{\Gamma}^{\mathbf{L}} T_i$ is quasi-isomorphic to the truncated complex*

$$0 \rightarrow \mathrm{Hom}_R(M, R^k) / \mathrm{Im}(f \circ g' \circ -) \xrightarrow{g \circ -} \mathrm{Hom}_R(M, K_i) \rightarrow 0. \quad (4.9)$$

In particular, to prove Theorem 4.1.4, we now only need to show that the complexes of Λ -modules, (4.9) and

$$\underline{\mathrm{Hom}}_R(M, M_i) \xrightarrow{b_i \circ -} \underline{\mathrm{Hom}}_R(M, V_i),$$

are quasi-isomorphic. To do this, we construct a complex of projective Λ -modules which is quasi-isomorphic to both. This involves first finding projective resolutions of $\mathrm{Hom}_R(M, K_i)$ and $\mathrm{Hom}_R(M, R^k)/\mathrm{Im}(f \circ g' \circ -)$.

Lemma 4.2.7. 1. *The sequence*

$$0 \rightarrow \mathrm{Hom}_R(M, M_i) \xrightarrow{b_i \circ -} \mathrm{Hom}_R(M, V_i) \xrightarrow{d_i \circ -} \mathrm{Hom}_R(M, K_i) \rightarrow 0 \quad (4.10)$$

is exact and so is a projective resolution of $\mathrm{Hom}_R(M, K_i)$ as a Λ -module.

2. *The sequence*

$$0 \rightarrow \mathrm{Hom}_R(M, M_i) \xrightarrow{b_i \circ -} \mathrm{Hom}_R(M, V_i) \xrightarrow{f' \circ d_i \circ -} \mathrm{Hom}_R(M, R^k) \xrightarrow{f \circ g' \circ -} \mathrm{Hom}_R(M, R^k) \rightarrow \mathrm{Hom}_R(M, R^k)/\mathrm{Im}(f \circ g' \circ -) \rightarrow 0$$

is a projective resolution of $\mathrm{Hom}_R(M, R^k)/\mathrm{Im}(f \circ g' \circ -)$ as a Λ -module.

Proof. 1. As M is rigid, $\mathrm{Ext}_R^1(M, M_i) = 0$ and thus applying $\mathrm{Hom}_R(M, -)$ to the exchange sequence (4.1) shows (4.10) is exact.

2. The sequence

$$0 \rightarrow \mathrm{Hom}_R(M, K_i) \xrightarrow{f' \circ -} \mathrm{Hom}_R(M, R^k) \xrightarrow{f \circ g' \circ -} \mathrm{Hom}_R(M, R^k)$$

is exact using the proof of Lemma 4.2.5. Taking the cokernel and splicing this sequence with (4.10) gives the result. \square

To construct a projective complex quasi-isomorphic to the complex (4.9), we need maps between the projective resolutions constructed in Lemma 4.2.7. For this, the following lemma is useful.

Lemma 4.2.8. *There exist maps $s: R^k \rightarrow V_i$ and $x: \Omega K_i \rightarrow M_i$ such that the following diagram commutes.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_i & \xrightarrow{b_i} & V_i & \xrightarrow{d_i} & K_i & \longrightarrow & 0 \\ & & \uparrow x & & \uparrow s & & \uparrow \mathrm{id} & & \\ 0 & \longrightarrow & \Omega K_i & \xrightarrow{f} & R^k & \xrightarrow{g} & K_i & \longrightarrow & 0 \end{array}$$

Further, these maps give an exact sequence

$$0 \rightarrow \Omega K_i \xrightarrow{\begin{pmatrix} -f \\ x \end{pmatrix}} R^k \oplus M_i \xrightarrow{\begin{pmatrix} s & b_i \end{pmatrix}} V_i \rightarrow 0. \quad (4.11)$$

Proof. The map s exists because R^k is projective and d_i is a surjective map. Then the map x exists using the universal property of kernels.

Viewing the rows of the commutative diagram as complexes, this construction gives a chain map between two exact complexes which therefore must be a quasi-isomorphism. Thus, the mapping cone,

$$0 \rightarrow \Omega K_i \xrightarrow{\begin{pmatrix} -f \\ x \end{pmatrix}} R^k \oplus M_i \xrightarrow{\begin{pmatrix} -g & 0 \\ s & b_i \end{pmatrix}} K_i \oplus V_i \xrightarrow{(\text{id} \quad d_i)} K_i \rightarrow 0.$$

is necessarily exact. In the commutative diagram below,

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega K_i & \xrightarrow{\begin{pmatrix} -f \\ x \end{pmatrix}} & R^k \oplus M_i & \xrightarrow{(s \quad b_i)} & V_i & \xrightarrow{0} & 0 & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \begin{pmatrix} -d_i \\ \text{id} \end{pmatrix} & & \downarrow \begin{pmatrix} \text{id} & d_i \end{pmatrix} & & \\ 0 & \longrightarrow & \Omega K_i & \xrightarrow{\begin{pmatrix} -f \\ x \end{pmatrix}} & R^k \oplus M_i & \xrightarrow{\begin{pmatrix} -g & 0 \\ s & b_i \end{pmatrix}} & K_i \oplus V_i & \xrightarrow{(\text{id} \quad d_i)} & K_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow (\text{id} \quad d_i) & & \downarrow \text{id} & & \\ 0 & \longrightarrow & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & K_i & \xrightarrow{\text{id}} & K_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & 0 & & \end{array}$$

the first, second and fourth columns are obviously exact while the third column is exact as it is the mapping cone of the map:

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_i & \xrightarrow{-\text{id}} & V_i & \longrightarrow & 0 \\ & & \downarrow -d_i & & \downarrow d_i & & \\ 0 & \longrightarrow & K_i & \xrightarrow{\text{id}} & K_i & \longrightarrow & 0. \end{array}$$

Thus all the columns are exact and so we have a short exact sequence of complexes. Considering the long exact sequence of homology associated to this short exact sequence shows that the first row is exact since the second and third are. \square

This gives us everything we need to construct a complex of projective Λ -modules quasi-isomorphic to $B_i \otimes_{\Gamma}^{\mathbf{L}} T_i$.

Lemma 4.2.9. *With notation as above, and writing $R(X, Y) = \text{Hom}_R(X, Y)$, the chain map*

$$\begin{array}{ccccccc} R(M, M_i) & \xrightarrow{b_i \circ -} & R(M, V_i) & \xrightarrow{f'_i \circ d_i \circ -} & R(M, R^k) & \xrightarrow{\begin{pmatrix} f \circ g' \circ - \\ -x \circ g' \circ - \end{pmatrix}} & R(M, R^k) \oplus_{R(M, M_i)} & \xrightarrow{(s \circ -, b_i \circ -)} & R(M, V_i) \\ \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow (0 \quad \text{pr}) & & \downarrow d_i \circ - \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & R(M, R^k) / \text{Im}(f \circ g' \circ -) & \xrightarrow{g \circ -} & R(M, K_i) \end{array}$$

is a quasi-isomorphism where $\text{pr}: R(M, R^k) \rightarrow R(M, R^k) / \text{Im}(f \circ g' \circ -)$ denotes the natural surjection. In particular, $B_i \otimes_{\Gamma}^{\mathbf{L}} T$ is isomorphic in $D^b(\Lambda)$ to the complex in the top row.

Proof. First note that the diagram is a chain map because $d_i \circ b_i = 0$ and $d_i \circ s = g$ by construction. Also, by Lemma 4.2.7 and Lemma 4.2.8, there is a double complex

$$\begin{array}{ccc}
\mathrm{Hom}_R(M, M_i) & \xrightarrow{0} & 0 \\
\downarrow b_i \circ - & & \downarrow 0 \\
\mathrm{Hom}_R(M, V_i) & \xrightarrow{0} & 0 \\
\downarrow f' \circ d_i \circ - & & \downarrow 0 \\
\mathrm{Hom}_R(M, R^k) & \xrightarrow{x \circ g' \circ -} & \mathrm{Hom}_R(M, M_i) \\
\downarrow f \circ g' \circ - & & \downarrow b_i \circ - \\
\mathrm{Hom}_R(M, R^k) & \xrightarrow{-(s \circ -)} & \mathrm{Hom}_R(M, V_i) \\
\downarrow \mathrm{pr} & & \downarrow d_i \circ - \\
\mathrm{Hom}_R(M, R^k) / \mathrm{Im}(f \circ g' \circ -) & \xrightarrow{g \circ -} & \mathrm{Hom}_R(M, K_i)
\end{array}$$

where the columns are acyclic. A standard result from homological algebra says the total complex of a bounded double complex with acyclic columns is acyclic [W, 2.7.3]. However, the total complex of this double complex is precisely (up to \pm signs on the maps) the mapping cone of the given chain map and so the chain map must be a quasi-isomorphism. The final statement then follows by combining this quasi-isomorphism with that of Corollary 4.2.6. \square

Finally, it needs to be shown that the complex of projectives constructed in Lemma 4.2.9 is also quasi-isomorphic to

$$\underline{\mathrm{Hom}}_R(M, M_i) \xrightarrow{b_i \circ -} \underline{\mathrm{Hom}}_R(M, V_i).$$

Lemma 4.2.10. *Writing ${}_R(X, Y) := \mathrm{Hom}_R(X, Y)$, the following chain map is a quasi-isomorphism:*

$$\begin{array}{ccccccc}
{}_R(M, M_i) & \xrightarrow{b_i \circ -} & {}_R(M, V_i) & \xrightarrow{f' \circ d_i \circ -} & {}_R(M, R^k) & \xrightarrow{\begin{pmatrix} f \circ g' \circ - \\ -x \circ g' \circ - \end{pmatrix}} & {}_R(M, R^k) \oplus {}_R(M, M_i) & \xrightarrow{\begin{pmatrix} s \circ - \\ b_i \circ - \end{pmatrix}} & {}_R(M, V_i) \\
\downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow \begin{pmatrix} 0 \\ \mathrm{pr} \end{pmatrix} & & \downarrow \mathrm{pr} \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \underline{\mathrm{Hom}}_R(M, M_i) & \xrightarrow{b_i \circ -} & \underline{\mathrm{Hom}}_R(M, V_i).
\end{array}$$

Proof. Combining Lemma 4.2.9, Corollary 4.2.6 and Lemma 4.2.5, the top complex has homology:

1. $\underline{\mathrm{Hom}}_R(M, K_i)$ in degree 0;
2. $\underline{\mathrm{Hom}}_R(M, \Omega K_i)$ in degree -1 ;
3. 0 elsewhere.

Using Proposition 2.1.12 the exchange sequence (4.1) induces a triangle

$$M_i \xrightarrow{b_i} V_i \xrightarrow{d_i} K_i \rightarrow \Omega M_i$$

in $\underline{\text{CM}}R$. Applying $\underline{\text{Hom}}_R(M, -)$ and using rigidity of M in $\underline{\text{CM}}R$ gives an exact sequence

$$0 \rightarrow \underline{\text{Hom}}_R(M, \Omega K_i) \rightarrow \underline{\text{Hom}}_R(M, M_i) \xrightarrow{b_i \circ} \underline{\text{Hom}}_R(M, V_i) \xrightarrow{d_i \circ} \underline{\text{Hom}}_R(M, K_i) \rightarrow 0.$$

This shows the homology of the two complexes in the statement of the lemma are the same in each degree, are finite dimensional and are zero outside degrees 0 and -1 . Thus, to prove the claim it is enough to show the maps on homology induced by the given chain map are surjective in degrees 0 and -1 .

In degree 0, the map

$$\text{Hom}_R(M, V_i) \xrightarrow{\text{pr}} \underline{\text{Hom}}_R(M, V_i) \rightarrow \underline{\text{Hom}}_R(M, V_i) / \text{Im}(b_i \circ -)$$

is the composition of two surjective maps and hence is surjective. Also, any map in the image of the map $(s \circ -, b_i \circ -)$ is clearly in the kernel of this map and so the map on homology must be surjective.

In degree -1 , take $\alpha: M \rightarrow M_i$ such that $b_i \circ \alpha$ factors through $\text{add}(R)$; that is, $\text{pr}(\alpha) \in \text{Ker}(\underline{\text{Hom}}_R(M, M_i) \xrightarrow{b_i \circ} \underline{\text{Hom}}_R(M, V_i))$. In particular, $b_i \circ \alpha = \delta_2 \circ \delta_1$ for some $\delta_1: M \rightarrow R^m$ and $\delta_2: R^m \rightarrow V_i$, where $m \in \mathbb{N}$. To show the map on homology is surjective we need to show there exists $\phi: M \rightarrow R^k$ and $\phi': M \rightarrow M_i$ such that (ϕ, ϕ') is in the kernel of $(s \circ -, b_i \circ -)$ and $\text{pr}(\phi') = \text{pr}(\alpha)$.

Since R^m is projective, $\text{Hom}_R(R^m, -)$ is exact and hence applying this to the exact sequence (4.11) shows that

$$\text{Hom}_R(R^m, R^k) \oplus \text{Hom}_R(R^m, M_i) \xrightarrow{(s \circ -, b_i \circ -)} \text{Hom}_R(R^m, V_i)$$

is surjective. Thus there exists $\beta: R^m \rightarrow R^k$ and $\gamma: R^m \rightarrow M_i$ such that

$$\delta_2 = s \circ \beta + b_i \circ \gamma.$$

This gives

$$b_i \circ \alpha = s \circ \beta \circ \delta_1 + b_i \circ \gamma \circ \delta_1$$

and so $(-\beta \circ \delta_1, \alpha - \gamma \circ \delta_1)$ belongs to the kernel of

$$\text{Hom}_R(M, R^k) \oplus \text{Hom}_R(M, M_i) \xrightarrow{(s \circ -, b_i \circ -)} \text{Hom}_R(M, V_i).$$

Moreover, since $\text{pr}: \text{Hom}_R(M, M_i) \rightarrow \underline{\text{Hom}}_R(M, M_i)$ is the natural surjection, applying $(0, \text{pr})$ to $(-\beta \circ \delta_1, \alpha - \gamma \circ \delta_1)$ gives

$$\text{pr}(\alpha - \gamma \circ \delta_1) = \text{pr}(\alpha)$$

as $\gamma \circ \delta_1$ factors through $\text{add}(R)$. This shows the map on homology is surjective and hence is an isomorphism. \square

We now have all the results required to prove Theorem 4.1.4.

Proof of Theorem 4.1.4. Recall that we wish to show, with the setup of 4.1.3, there is

an isomorphism

$$\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \left(0 \rightarrow \bigoplus_{j \neq i} \underline{\text{Hom}}_R(M, M_j)\right) \oplus \left(\underline{\text{Hom}}_R(M, M_i) \xrightarrow{b_i \circ -} \underline{\text{Hom}}_R(M, V_i)\right).$$

in $\text{D}^b(\Lambda)$.

Since $\Gamma_{\text{con}} \cong \bigoplus_{j=1}^n B_j$, it is enough to show

$$B_j \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \underline{\text{Hom}}_R(M, M_j)$$

when $j \neq i$, which holds by Lemma 4.2.4, and further

$$B_i \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \underline{\text{Hom}}_R(M, M_i) \xrightarrow{b_i \circ -} \underline{\text{Hom}}_R(M, V_i),$$

which holds by combining Corollary 4.2.6 and Lemmas 4.2.9 and 4.2.10. \square

Remark 4.2.11. Note that an immediate corollary of Theorem 4.1.4 is that there is an isomorphism

$$\Gamma_{\text{con}} \cong \text{End}_{\text{D}^b(\Lambda)}(\mathcal{P}_i).$$

Thus we can recover Dugas's result (our Theorem 3.4.2) if we can prove

$$\text{End}_{\text{D}^b(\Lambda)}(\mathcal{P}_i) \cong \text{End}_{\text{D}^b(\Lambda_{\text{con}})}(\mathcal{P}_i).$$

To show there is an isomorphism of vector spaces is straightforward and indeed, this method will be used in the proof of Lemma 4.4.1. We can further prove the resulting isomorphism is a ring isomorphism by chasing elements through the isomorphism, but instead we choose to leave this result until after we have obtained the commutative diagram of Theorem 4.1.5, as this gives us a ring homomorphism for free.

4.3 The Commutative Diagram

The goal of this section is to prove that the diagram (4.3) commutes, or equivalently, there is an isomorphism $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \mathcal{I}_i \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}\Lambda}$ in the derived category of $\Gamma_{\text{con}}\text{-}\Lambda$ -bimodules. We first show that the isomorphism holds on one side, but to do this, we use the complex of projective Λ -modules constructed in Lemmas 4.2.9 and 4.2.10 which is quasi-isomorphic to both \mathcal{P}_i and $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i$. As we will make further use of this complex in the rest of this chapter, we begin by introducing some notation.

Notation 4.3.1. Under the setup of 4.1.3, for $j \neq i$ set \mathbb{P}_j to be the projective resolution

$$0 \rightarrow \text{Hom}_R(M, M_j) \rightarrow \text{Hom}_R(M, R^{n_j}) \rightarrow \text{Hom}_R(M, R^{n_j}) \rightarrow \text{Hom}_R(M, M_j) \rightarrow 0$$

of $\underline{\text{Hom}}_R(M, M_j)$ as a Λ -module, as in Lemma 4.2.3. Further, set \mathbb{P}_i to be the complex of projective Λ -modules

$$0 \rightarrow \text{Hom}_R(M, M_i) \rightarrow \text{Hom}_R(M, V_i) \rightarrow \text{Hom}_R(M, R^k) \rightarrow \text{Hom}(M, R^k \oplus M_i) \rightarrow \text{Hom}_R(M, V_i) \rightarrow 0,$$

constructed in Lemma 4.2.9. Then write $\mathbb{P} := \bigoplus_{j=1}^n \mathbb{P}_j$.

The following gives a summary of the results we will need about the complex \mathbb{P} .

Proposition 4.3.2. *Under the setup of 4.1.3, there are the following isomorphisms:*

1. $\mathbb{P} \cong \mathcal{P}_i \cong \Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i$ in $D^b(\Lambda)$;
2. $\mathbb{P} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}} \cong \mathcal{P}_i \oplus \mathcal{P}_i[3]$ in $D^b(\Lambda_{\text{con}})$;
3. $\mathcal{T}_i \cong \mathcal{P}_i$ in $D^b(\Lambda_{\text{con}})$;
4. $\mathcal{T}_i \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}} \cong \mathcal{P}_i \oplus \mathcal{P}_i[3]$ in $D^b(\Lambda_{\text{con}})$;
5. $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \mathcal{T}_i \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}\Lambda}$ in $D^b(\Lambda)$.

Proof. 1. By Lemma 4.2.10, \mathbb{P}_i is quasi-isomorphic to the complex

$$\underline{\text{Hom}}_R(M, M_i) \xrightarrow{b_i \circ -} \underline{\text{Hom}}_R(M, V_i).$$

and hence, the first isomorphism follows by construction, and the second by Theorem 4.1.4.

2. For any summand M_j of M , it is easily checked that there is an isomorphism

$$\underline{\text{Hom}}_R(M, M_j) \cong \text{Hom}_R(M, M_j) \otimes_{\Lambda} \Lambda_{\text{con}}$$

and thus it is clear, using the explicit form of \mathbb{P} , that, in $D^b(\Lambda_{\text{con}})$,

$$\mathbb{P} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}} \cong \mathbb{P} \otimes_{\Lambda} \Lambda_{\text{con}} \cong \mathcal{P}_i \oplus \mathcal{P}_i[3].$$

3. From (1), there is an isomorphism $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \cong \mathbb{P}$ in $D^b(\Lambda)$. Applying $-\otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}$ and using (2) leads to isomorphisms

$$\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}} \cong \mathbb{P} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}} \cong \mathcal{P}_i \oplus \mathcal{P}_i[3]$$

in $D^b(\Lambda_{\text{con}})$. Applying $\tau_{\geq -1}$ to both sides, and using that \mathcal{P}_i has non-zero terms only in degrees 0 and -1 , gives the result.

4. Viewing \mathcal{T}_i and \mathcal{P}_i as complexes of Λ -modules via the restriction of scalars functors, (3) shows that $\mathcal{T}_i \cong \mathcal{P}_i$ in $D^b(\Lambda)$. Thus, by (1), there is an isomorphism $\mathcal{T}_i \cong \mathbb{P}$ in $D^b(\Lambda)$. Now, applying $-\otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}$ and using (2) gives the result.

5. As above, viewing \mathcal{T}_i and \mathcal{P}_i as complexes of Λ -modules via the restriction of scalars functors, (3) shows that $\mathcal{T}_i \cong \mathcal{P}_i$ in $D^b(\Lambda)$. Now applying (1) completes the proof. \square

The final part of Proposition 4.3.2 shows that $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i$ and $\mathcal{T}_i \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}\Lambda}$ have the same homology in each degree (which is zero outside of degrees 0 and -1) and by Lemma 4.2.5, we know this homology is always finite dimensional. Therefore, to prove Theorem 4.1.5, it is enough to show there is a map in the derived category of Γ_{con} - Λ -bimodules between these two complexes which is injective on homology.

Suppose that

$$Q := \cdots \rightarrow Q_{-3} \xrightarrow{d_{-3}} Q_{-2} \xrightarrow{d_{-2}} Q_{-1} \xrightarrow{d_{-1}} Q_0 \rightarrow 0 \quad (4.12)$$

is a complex of Γ_{con} - Λ bimodules, projective as Λ -modules, which is quasi-isomorphic to $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i$. For example, taking the Cartan–Eilenberg resolution of $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i$ in the derived category of Γ_{con} - Λ bimodules would suffice.

Since the Q_i are projective as Λ -modules, the complex $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}$ is simply

$$\cdots \rightarrow Q_{-3} \otimes_{\Lambda} \Lambda_{\text{con}} \xrightarrow{d_{-3} \otimes \text{id}} Q_{-2} \otimes_{\Lambda} \Lambda_{\text{con}} \xrightarrow{d_{-2} \otimes \text{id}} Q_{-1} \otimes_{\Lambda} \Lambda_{\text{con}} \xrightarrow{d_{-1} \otimes \text{id}} Q_0 \otimes_{\Lambda} \Lambda_{\text{con}} \rightarrow 0.$$

There are natural maps $\partial_i: Q_i \rightarrow Q_i \otimes_{\Lambda} \Lambda_{\text{con}}$ given by $q \mapsto q \otimes 1$ and these induce a map of complexes $Q \rightarrow Q \otimes_{\Lambda} \Lambda_{\text{con}}$. Composing this map with the natural map from $Q \otimes_{\Lambda} \Lambda_{\text{con}}$ to the truncation $\tau_{\geq -1}(Q \otimes_{\Lambda} \Lambda_{\text{con}})$ gives the following map of complexes:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & Q_{-3} & \xrightarrow{d_{-3}} & Q_{-2} & \xrightarrow{d_{-2}} & Q_{-1} & \xrightarrow{d_{-1}} & Q_0 & \longrightarrow & 0 \\
& & \downarrow \partial_{-3} & & \downarrow \partial_{-2} & & \downarrow \partial_{-1} & & \downarrow \partial_0 & & \\
\dots & \longrightarrow & Q_{-3} \otimes_{\Lambda} \Lambda_{\text{con}} & \xrightarrow{d_{-3} \otimes \text{id}} & Q_{-2} \otimes_{\Lambda} \Lambda_{\text{con}} & \xrightarrow{d_{-2} \otimes \text{id}} & Q_{-1} \otimes_{\Lambda} \Lambda_{\text{con}} & \xrightarrow{d_{-1} \otimes \text{id}} & Q_0 \otimes_{\Lambda} \Lambda_{\text{con}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \frac{Q_{-1} \otimes_{\Lambda} \Lambda_{\text{con}}}{\text{Im}(d_{-2} \otimes \text{id})} & \xrightarrow{d_{-1} \otimes \text{id}} & Q_0 \otimes_{\Lambda} \Lambda_{\text{con}} & \longrightarrow & 0.
\end{array}$$

By construction, this is a map $\delta: \Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \rightarrow \mathcal{T}_i \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}\Lambda}$ in the derived category of Γ_{con} - Λ bimodules. Thus, to prove Theorem 4.1.5, we show the induced maps on homology, $H^i(\partial_i)$, are injective for $i = 0, -1$. To do this we will make the assumption that for $i = 0, -1$,

$$Q_i I \cap \text{Ker}(d_i) = (\text{Ker}(d_i))I \quad (4.13)$$

where I is the two-sided ideal of Λ such that $\Lambda_{\text{con}} = \Lambda/I$ (namely, $I = [R]$ consists of the endomorphisms of M factoring through $\text{add } R$). This holds trivially when $i = 0$ and is proved below in Lemmas 4.3.3 and 4.3.4 for the case $i = -1$.

With assumption (4.13), take $q + \text{Im}(d_{i-1}) \in H^i(Q)$ to be in the kernel of $H^i(\delta_i)$ (or equivalently such that $q \otimes 1 \in \text{Im}(d_{i-1} \otimes \text{id})$). Then, for some $p_j \in Q_{i-1}$ and $f_j \in \Lambda$,

$$\begin{aligned}
q \otimes 1 &= (d_{i-1} \otimes \text{id}) \left(\sum_j (p_j \otimes f_j) \right) \\
&= \sum_j (d_{i-1}(p_j) \otimes f_j) \\
&= d_{i-1} \left(\sum_j p_j f_j \right) \otimes 1.
\end{aligned}$$

Thus, $q - d_{i-1}(\sum_j p_j f_j) \in Q_i I \cap \text{Ker}(d_i)$ and so, by assumption (4.13),

$$q - d_{i-1}(\sum_j p_j f_j) = \sum_k q_k \lambda_k \quad (4.14)$$

for some $q_k \in \text{Ker}(d_i)$, $\lambda_k \in I$. Using Lemma 4.2.5, there are isomorphisms of Λ -modules

$$\phi_i: H^i(Q) \rightarrow \begin{cases} \underline{\text{Hom}}_R(M, \mathfrak{v}_i M) & \text{if } i = 0 \\ \underline{\text{Hom}}_R(M, \Omega K_i) & \text{if } i = -1, \end{cases}$$

and since the ϕ_i are Λ -module homomorphisms,

$$\phi_i \left(\sum_k q_k \lambda_k + \text{Im}(d_{i-1}) \right) = \sum_k \phi_i(q_k + \text{Im}(d_{i-1})) \lambda_k.$$

It is clear that $\underline{\text{Hom}}_R(M, \mathfrak{v}_i M)$ and $\underline{\text{Hom}}_R(M, \Omega K_i)$ are both annihilated on the right by I and hence each of the terms $\phi_i(q_k + \text{Im}(d_{i-1})) \lambda_k$ must be zero since $\lambda_k \in I$. Thus,

$\phi_i(\sum_k q_k \lambda_k + \text{Im}(d_{i-1})) = 0$ and hence, as ϕ_i is an isomorphism,

$$\sum_k q_k \lambda_k \in \text{Im}(d_{i-1}).$$

Using (4.14), this in turn shows that $q \in \text{Im}(d_{i-1})$ and so the map on homology is injective.

Thus all that remains to prove Theorem 4.1.5 is to verify assumption (4.13) in the case $i = -1$.

Lemma 4.3.3. *Suppose $\cdots \rightarrow P_{i-2} \xrightarrow{\delta_{i-2}} P_{i-1} \xrightarrow{\delta_{i-1}} P_i \xrightarrow{\delta_i} P_{i+1} \rightarrow \cdots$ is a complex of projective right Λ -modules and that I is the two-sided ideal of Λ such that $\Lambda_{\text{con}} = \Lambda/I$. If $\text{Tor}_2^\Lambda(P_{i+1}/\text{Im}(\delta_i), \Lambda_{\text{con}}) = 0$, then*

$$(\text{Ker}(\delta_i))I = P_i I \cap \text{Ker}(\delta_i).$$

Proof. First of all, note that the inclusion $(\text{Ker}(\delta_i))I \subseteq P_i I \cap \text{Ker}(\delta_i)$ is clear and so it is enough to show

$$P_i I \cap \text{Ker}(\delta_i) \subseteq (\text{Ker}(\delta_i))I.$$

There is an exact sequence

$$0 \rightarrow \text{Im}(\delta_i) \hookrightarrow P_{i+1} \rightarrow P_{i+1}/\text{Im}(\delta_i) \rightarrow 0.$$

Applying $-\otimes_\Lambda \Lambda_{\text{con}}$ and using that P_{i+1} is a projective Λ -module produces an exact sequence

$$0 \rightarrow \text{Tor}_2^\Lambda(P_{i+1}/\text{Im}(\delta_i), \Lambda_{\text{con}}) \rightarrow \text{Tor}_1^\Lambda(\text{Im}(\delta_i), \Lambda_{\text{con}}) \rightarrow 0$$

which, combined with the assumption in the statement, implies that $\text{Tor}_1^\Lambda(\text{Im}(\delta_i), \Lambda_{\text{con}}) = 0$. Thus, applying $-\otimes_\Lambda \Lambda_{\text{con}}$ to the exact sequence

$$0 \rightarrow \text{Ker}(\delta_i) \hookrightarrow P_i \xrightarrow{\delta_i} \text{Im}(\delta_i) \rightarrow 0$$

produces an exact sequence

$$0 \rightarrow \text{Ker}(\delta_i) \otimes_\Lambda \Lambda_{\text{con}} \hookrightarrow P_i \otimes_\Lambda \Lambda_{\text{con}} \xrightarrow{\delta_i \otimes 1} \text{Im}(\delta_i) \otimes_\Lambda \Lambda_{\text{con}} \rightarrow 0.$$

Now choose $p \in P_i I \cap \text{Ker}(\delta_i)$. Then $p \otimes 1$ belongs to $\text{Ker}(\delta_i) \otimes_\Lambda \Lambda_{\text{con}}$ and maps to zero in $P_i \otimes_\Lambda \Lambda_{\text{con}}$. Since the map is injective, this implies $p \otimes 1$ is zero in $\text{Ker}(\delta_i) \otimes_\Lambda \Lambda_{\text{con}}$ and hence $p \in (\text{Ker}(\delta_i))I$, completing the proof. \square

We next apply Lemma 4.3.3 to Q in (4.12) where, by Lemma 4.2.5, there are isomorphisms of Λ -modules,

$$Q_0/\text{Im}(d_{-1}) \cong H^0(Q) \cong \underline{\text{Hom}}_R(M, \nu_i M).$$

With this in mind, the following completes the proof of Theorem 4.1.5.

Lemma 4.3.4. *Under setup 4.1.3, $\text{Tor}_2^\Lambda(\underline{\text{Hom}}_R(M, \nu_i M), \Lambda_{\text{con}}) = 0$.*

Proof. We begin by constructing a projective resolution of Λ_{con} as a left Λ -module which is very similar to Lemma 4.2.3. Proposition 3.2.9 shows $\Omega^2 \cong \text{id}$ on $\underline{\text{CM}} R$ and hence there are exact sequences,

$$\begin{aligned} 0 \rightarrow M \rightarrow R^k \rightarrow \Omega M \rightarrow 0 \\ 0 \rightarrow \Omega M \rightarrow R^k \rightarrow M \rightarrow 0 \end{aligned}$$

for some $k \in \mathbb{N}$, arising from taking the syzygy of ΩM and M respectively. Applying $\text{Hom}_R(-, M)$ to these sequences, using that M is rigid and splicing yields the sequence

$$0 \rightarrow \text{Hom}_R(M, M) \rightarrow \text{Hom}_R(R^k, M) \rightarrow \text{Hom}_R(R^k, M) \rightarrow \text{Hom}_R(M, M) \rightarrow 0.$$

This is exact everywhere except for the last term where the homology is $\text{Ext}_R^1(\Omega M, M) \cong \underline{\text{Hom}}_R(M, M)$. Since all the terms are projective left Λ -modules this gives a projective resolution of $\underline{\text{Hom}}_R(M, M)$ which we call \mathcal{Q} .

Now $\text{Tor}_2^\Lambda(\underline{\text{Hom}}_R(M, \nu_i M), \Lambda_{\text{con}})$ is defined to be the homology in degree -2 of $\underline{\text{Hom}}_R(M, \nu_i M) \otimes_\Lambda \mathcal{Q}$ and thus, to show that it is zero, it is enough to show that

$$(\underline{\text{Hom}}_R(M, \nu_i M) \otimes_\Lambda \mathcal{Q})_{-2} \cong \underline{\text{Hom}}_R(M, \nu_i M) \otimes_\Lambda \text{Hom}_R(R^k, M) = 0$$

or equivalently, that $\underline{\text{Hom}}_R(M, \nu_i M) \otimes_\Lambda \text{Hom}_R(R, M) = 0$.

Take $f \otimes g \in \underline{\text{Hom}}_R(M, \nu_i M) \otimes_\Lambda \text{Hom}_R(R, M)$. Then, since R is a summand of M , there are maps $i: R \rightarrow M$ and $p: M \rightarrow R$ given by inclusion and projection such that $p \circ i = \text{id}$. Therefore,

$$f \otimes g = f \otimes g \circ (p \circ i) = f \otimes (g \circ p) \circ i = f \circ g \circ p \otimes i = 0$$

as $f \circ g \circ p$ factors through R . Thus $\underline{\text{Hom}}_R(M, \nu_i M) \otimes_\Lambda \text{Hom}_R(R, M) = 0$ as required. \square

Proof of Theorem 4.1.5. Consider the bimodule map $\delta: \Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \rightarrow \mathcal{T}_i \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}} \Lambda$ constructed on p54. Recall that, by Proposition 4.3.2, $\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i$ and $\mathcal{T}_i \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}} \Lambda$ have the same homology in each degree (which is zero outside of degrees 0 and -1) and by Lemma 4.2.5, this homology is always finite dimensional. Using Lemmas 4.3.4 and 4.3.3, the assumption (4.13) is satisfied and hence δ is injective on homology in degrees -1 and 0 and hence is an isomorphism on homology in all degrees. Thus, we conclude the map δ is an isomorphism in the derived category of $\Gamma_{\text{con}}\text{-}\Lambda$ bimodules. \square

4.4 The Equivalence

In this subsection we will prove Corollary 4.1.6, namely that the functor

$$- \otimes_{\Gamma_{\text{con}}}^{\mathbf{L}} \mathcal{T}_i: \text{D}^b(\Gamma_{\text{con}}) \rightarrow \text{D}^b(\Lambda_{\text{con}})$$

is an equivalence. By Theorem 2.2.3, and writing $\text{D}(\Gamma_{\text{con}}) := \text{D}(\text{Mod } \Gamma_{\text{con}})$ and $\text{D}(\Lambda_{\text{con}}) := \text{D}(\text{Mod } \Lambda_{\text{con}})$, it is enough to show that:

- (a) The map $\Gamma_{\text{con}} \rightarrow \text{Hom}_{\text{D}(\Lambda_{\text{con}})}(\mathcal{T}_i, \mathcal{T}_i)$ induced by $- \otimes_{\Gamma_{\text{con}}}^{\mathbf{L}} \mathcal{T}_i$ is an isomorphism and further $\text{Hom}_{\text{D}(\Lambda_{\text{con}})}(\mathcal{T}_i, \mathcal{T}_i[n]) = 0$ for $n \neq 0$.
- (b) \mathcal{T}_i is quasi-isomorphic to a complex $T \in \text{K}^b(\text{proj } \Lambda_{\text{con}})$.
- (c) The smallest triangulated full subcategory of $\text{D}(\Lambda_{\text{con}})$ containing T and closed under isomorphism and direct summands is $\text{K}^b(\text{proj } \Lambda_{\text{con}})$.

By Proposition 4.3.2(3), $\mathcal{T}_i \cong \mathcal{P}_i$ in $D^b(\Lambda_{\text{con}})$ and since \mathcal{P}_i is a tilting complex for Λ_{con} , conditions (b), (c) and the latter part of (a) are satisfied. The following lemma uses the commutative diagram (4.3) to show the first part of condition (a) also holds.

Lemma 4.4.1. *In the set up of 4.1.3, the map $\Gamma_{\text{con}} \rightarrow \text{Hom}_{D(\Lambda_{\text{con}})}(\mathcal{T}_i, \mathcal{T}_i)$ induced by $-\otimes_{\Gamma_{\text{con}}}^{\mathbf{L}} \mathcal{T}_i$ is an isomorphism.*

Proof. The commutative diagram (4.3) of Theorem 4.1.5 induces a commutative diagram of algebra morphisms

$$\begin{array}{ccc} \text{End}_{D(\Lambda_{\text{con}})}(\mathcal{T}_i) & \xrightarrow{\beta} & \text{End}_{D(\Lambda)}(\mathcal{T}_{i\Lambda}) \\ \alpha \uparrow & & \uparrow \delta \\ \text{End}_{D(\Gamma_{\text{con}})}(\Gamma_{\text{con}}) & \xrightarrow{\gamma} & \text{End}_{D(\Gamma)}(\Gamma_{\text{con}}\Gamma) \end{array}$$

where α is the map in the statement and δ is an isomorphism since $-\otimes_{\Gamma}^{\mathbf{L}} \mathcal{T}_i$ is an equivalence. Moreover, γ is an isomorphism since Γ_{con} is a Γ_{con} -module. Thus, $\delta \circ \gamma$ is an isomorphism and so β must be surjective.

Now, the restriction and extension of scalars adjunction gives an isomorphism of vector spaces,

$$\text{End}_{D(\Lambda)}(\mathcal{T}_{i\Lambda}) \cong \text{Hom}_{D(\Lambda_{\text{con}})}(\mathcal{T}_i \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}, \mathcal{T}_i). \quad (4.15)$$

Proposition 4.3.2(3) shows $\mathcal{T}_i \cong \mathcal{P}_i$ in $D^b(\Lambda_{\text{con}})$ and Proposition 4.3.2(4) shows $\mathcal{T}_i \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}} \cong \mathcal{P}_i \oplus \mathcal{P}_i[3]$ in $D^b(\Lambda_{\text{con}})$. This gives vector space isomorphisms,

$$\begin{aligned} \text{End}_{D(\Lambda)}(\mathcal{T}_i) &\cong \text{Hom}_{D(\Lambda_{\text{con}})}(\mathcal{P}_i \oplus \mathcal{P}_i[3], \mathcal{P}_i) && \text{(using (4.15))} \\ &\cong \text{Hom}_{D(\Lambda_{\text{con}})}(\mathcal{P}_i, \mathcal{P}_i) && \text{(using that } \mathcal{P}_i \text{ is tilting)} \\ &\cong \text{End}_{D(\Lambda_{\text{con}})}(\mathcal{T}_i). && \text{(using } \mathcal{P}_i \cong \mathcal{T}_i \text{ in } D^b(\Lambda_{\text{con}})) \end{aligned}$$

Since β is a surjective morphism of algebras between isomorphic finite dimensional vector spaces, it must therefore be an isomorphism. Thus, as β, γ and δ are isomorphisms, α must also be. \square

Proof of Corollary 4.1.6. Lemma 4.4.1 and the discussion above show that \mathcal{T}_i satisfies the conditions in Theorem 2.2.3, and hence we get the desired equivalence. \square

Note that, as $\mathcal{T}_i \cong \mathcal{P}_i$ in $D^b(\Lambda_{\text{con}})$ by Proposition 4.3.2, the above lemma also gives the following direct corollary.

Corollary 4.4.2. *Under the setup of 4.1.3, with \mathcal{P}_i as in Proposition 4.1.1, there is an isomorphism of k -algebras*

$$\underline{\text{End}}_R(\mathbf{v}_i M) \cong \text{End}_{K^b(\text{proj } \Lambda_{\text{con}})}(\mathcal{P}_i).$$

Remark 4.4.3. In particular, the results of this section show that for any basic rigid object $M \in \underline{\text{CM}}R$, the algebras $\underline{\text{End}}_R(M)$ and $\underline{\text{End}}_R(\mathbf{v}_i M)$ are derived equivalent, recovering Dugas's result, Theorem 3.4.2. Moreover, as we provide the two-sided tilting complex \mathcal{T}_i , and hence a standard equivalence between the algebras, we can think of our results as a two-sided version of both Theorem 3.4.2 and Theorem 4.1.1.

Chapter 5

The Members of the Derived Equivalence Class

This chapter focuses on the following problem: given a contraction algebra A , find all the basic algebras B such that A and B have equivalent derived categories.

By a well-known result of Rickard [Ri1], this problem is equivalent to first finding all the *tilting complexes* (see Definition 2.2.1) over A , and then computing their endomorphism rings. One approach to the first of these problems is to use *mutation*; an iterative procedure described in §2.2.2 which produces new tilting complexes from old. The naive hope is that starting from a given tilting complex, all others can be reached using mutation. However, for a general algebra, tilting complexes do not behave well enough for this to work. There are two key problems:

1. The mutation procedure does not always produce a tilting complex.
2. It is often possible to find two tilting complexes which are not connected by any sequence of mutations.

Both problems have motivated results in the literature; the first prompting the introduction of the weaker notion of *silting complexes* [KeV, AI] and the second resulting in restricting to a class of algebras known as *tilting-discrete* algebras [A, AM]. For symmetric algebras, such as the contraction algebras we are interested in, it is well known that the concepts of silting and tilting coincide or, in other words, the first problem does not occur. The goal of this chapter is to show that contraction algebras are also tilting-discrete, and thus, we exclude the possibility that the second problem can occur. This leads to the main result of the chapter which shows that the only algebras derived equivalent to a contraction algebra of a complete local isolated cDV singularity $\text{Spec } R$ are the other contraction algebras of $\text{Spec } R$. This result will not only provide evidence towards Conjecture 3.1.9, but it also allows us to create a ‘picture’ of the derived equivalence class that contains both all the members of the class and all standard derived equivalences between them, up to algebra automorphism.

5.1 Tilting-Discreteness

In this section, we prove that contraction algebras are tilting-discrete. This can be thought of as a 3-fold analogue of the results in [AM], where they show preprojective algebras of Dynkin type (those associated to Kleinian singularities) are tilting-discrete.

Setup 5.1.1. Let $\text{Spec } R$ be a complete local isolated cDV singularity, choose a basic rigid object $M := \bigoplus_{i=1}^n M_i \in \underline{\text{CM}} R$ and set $\Lambda_{\text{con}} := \underline{\text{End}}_R(M)$.

Choosing a summand M_i of M , and hence the summand $\underline{\text{Hom}}_R(M, M_i)$ of Λ_{con} , there are two ways we could consider mutating the algebra Λ_{con} :

1. Mutate M via an exchange sequence

$$0 \rightarrow M_i \xrightarrow{f_i} V_i \xrightarrow{g_i} K_i \rightarrow 0$$

to obtain $\nu_i M := M/M_i \oplus K_i$ (see §3.2.1 for details). Then consider $\underline{\text{End}}_R(\nu_i M)$ as the mutation of Λ_{con} .

2. View Λ_{con} as a tilting complex over itself and mutate to obtain the tilting complex $\mu_i \Lambda_{\text{con}}$ (see §2.2.2 for details). Then consider $\text{End}_{\text{K}^b(\text{proj } \Lambda_{\text{con}})}(\mu_i \Lambda_{\text{con}})$ as the mutation of Λ_{con} .

The following not only determines the tilting complex $\mu_i \Lambda_{\text{con}}$, but also shows that the two different mutation procedures give the same result.

Proposition 5.1.2. *With the setup above, there is an isomorphism*

$$\mu_i \Lambda_{\text{con}} \cong \left(0 \rightarrow \bigoplus_{j \neq i} \underline{\text{Hom}}_R(M, M_j) \right) \oplus \left(\underline{\text{Hom}}_R(M, M_i) \xrightarrow{f_i \circ -} \underline{\text{Hom}}_R(M, V_i) \right).$$

and hence there is a ring isomorphism

$$\underline{\text{End}}_R(\nu_i M) \cong \text{End}_{\text{K}^b(\text{proj } \Lambda_{\text{con}})}(\mu_i \Lambda_{\text{con}}).$$

Proof. Recall that the two-term complex on the left hand side is the tilting complex \mathcal{P}_i from Theorem 4.1.1 which has endomorphism ring $\underline{\text{End}}_R(\nu_i M)$ by Corollary 4.4.2. Now, Proposition 2.2.14 shows the complex must be $\mu_i \Lambda_{\text{con}}$ as it is a two-term silted complex differing from Λ by precisely the summand $\underline{\text{Hom}}_R(M, M_i)$. \square

In particular, this shows that for any basic rigid object $N := \bigoplus_{i=1}^n N_i$ in $\underline{\text{CM}} R$, and any i , that there exists a derived equivalence

$$\begin{aligned} \Psi: \text{D}^b(\underline{\text{End}}_R(N)) &\rightarrow \text{D}^b(\underline{\text{End}}_R(\nu_i N)), \\ \mu_i(\underline{\text{End}}_R(N)) &\mapsto \underline{\text{End}}_R(\nu_i N). \end{aligned}$$

To understand the derived equivalence class of Λ_{con} , we need to obtain the endomorphism algebra of any tilting complex of Λ_{con} ; the above proposition does this for any tilting complex of the form $\mu_i \Lambda_{\text{con}}$ and our next result extends this to any tilting complex reachable by iterated mutation. To state this result, recall that any such tilting complex may be written as

$$\mu_{i_m}^{\varepsilon_m} \dots \mu_{i_1}^{\varepsilon_1} \Lambda_{\text{con}}$$

for some $i_1, \dots, i_m \in \{1, \dots, n\}$ and $\varepsilon_j \in \{-1, 1\}$.

Proposition 5.1.3. *Under the setup of 5.1.1, consider any tilting complex $T := \mu_{i_m}^{\varepsilon_m} \dots \mu_{i_1}^{\varepsilon_1} \Lambda_{\text{con}}$. Then there is an isomorphism of k -algebras*

$$\text{End}_{\text{K}^b(\text{proj } \Lambda_{\text{con}})}(T) \cong \underline{\text{End}}_R(\nu_{i_m} \dots \nu_{i_1} M).$$

Proof. Begin by choosing a sequence of derived equivalences

$$\begin{aligned} \Psi_{i_j} : D^b(\underline{\text{End}}_R(\nu_{i_{j-1}} \dots \nu_{i_1} M)) &\rightarrow D^b(\underline{\text{End}}_R(\nu_{i_j} \dots \nu_{i_1} M)) \\ \mu_{i_j}^{\varepsilon_j}(\underline{\text{End}}_R(\nu_{i_{j-1}} \dots \nu_{i_1} M)) &\mapsto \underline{\text{End}}_R(\nu_{i_j} \dots \nu_{i_1} M). \end{aligned}$$

If $\varepsilon_j = 1$, then such an equivalence Ψ_{i_j} exists by the comments after Proposition 5.1.2. Moreover, applying those same comments to the mutation

$$\nu_{i_j} \dots \nu_{i_1} M \rightarrow \nu_{i_j} \nu_{i_j} \nu_{i_{j-1}} \dots \nu_{i_1} M \cong \nu_{i_{j-1}} \dots \nu_{i_1} M$$

gives the existence of a derived equivalence

$$\begin{aligned} \Theta_{i_j} : D^b(\underline{\text{End}}_R(\nu_{i_j} \dots \nu_{i_1} M)) &\rightarrow D^b(\underline{\text{End}}_R(\nu_{i_{j-1}} \dots \nu_{i_1} M)) \\ \mu_{i_j}(\underline{\text{End}}_R(\nu_{i_j} \dots \nu_{i_1} M)) &\mapsto \underline{\text{End}}_R(\nu_{i_{j-1}} \dots \nu_{i_1} M). \end{aligned}$$

and using Lemma 2.2.9, we see $\Theta_{i_j}^{-1}(\mu_{i_j}^{-1}(\underline{\text{End}}_R(\nu_{i_{j-1}} \dots \nu_{i_1} M))) \cong \underline{\text{End}}_R(\nu_{i_j} \dots \nu_{i_1} M)$. Taking $\Psi_{i_j} = \Theta_{i_j}^{-1}$ therefore gives the required functor.

We then prove by induction on m , the length of the mutation sequence, that $\Psi_{i_m} \circ \dots \circ \Psi_{i_1}(T) \cong \underline{\text{End}}_R(\nu_{i_m} \dots \nu_{i_1} M)$.

Base Case $m = 1$: If $m = 1$, then either $T := \mu_i \Lambda_{\text{con}}$ or $T := \mu_i^{-1} \Lambda_{\text{con}}$ for some $i \in \{1, \dots, n\}$ and the result follows directly from our choice of equivalence.

Inductive Step: Let $S := \mu_{i_{m-1}}^{\varepsilon_{m-1}} \dots \mu_{i_1}^{\varepsilon_1} \Lambda_{\text{con}}$ so by the inductive hypothesis,

$$\Psi_{i_{m-1}} \circ \dots \circ \Psi_{i_1}(S) \cong \underline{\text{End}}_R(\nu_{i_{m-1}} \dots \nu_{i_1} M).$$

As $T = \mu_{i_m}^{\varepsilon_m} S$, applying Lemma 2.2.9 to the equivalence $\Psi_{i_{m-1}} \circ \dots \circ \Psi_{i_1}$, and then using the inductive hypothesis, gives

$$\begin{aligned} \Psi_{i_m} \circ \dots \circ \Psi_{i_1}(T) &\cong \Psi_{i_m}(\mu_{i_m}^{\varepsilon_m}(\Psi_{i_{m-1}} \circ \dots \circ \Psi_{i_1}(S))) \\ &\cong \Psi_{i_m}(\mu_{i_m}^{\varepsilon_m}(\underline{\text{End}}_R(\nu_{i_{m-1}} \dots \nu_{i_1} M))) \\ &\cong \underline{\text{End}}_R(\nu_{i_m} \dots \nu_{i_1} M) \end{aligned}$$

where the last isomorphism follows by the choice of Ψ_{i_m} .

The statement then follows directly from the fact that equivalences are fully faithful. \square

This has the following direct corollary.

Corollary 5.1.4. *Under the setup of 5.1.1, any tilting complex obtained from Λ_{con} by finite iterated mutation (either left or right at each stage) has endomorphism algebra isomorphic to one of the set*

$$\{\underline{\text{End}}_R(N) \mid N \in \text{mut}(M)\}.$$

Proof. If T is a tilting complex for Λ_{con} obtained by iterated mutation then

$$T \cong \mu_{i_m}^{\varepsilon_m} \dots \mu_{i_1}^{\varepsilon_1} \Lambda_{\text{con}}$$

for some $i_1, \dots, i_m \in \{1, \dots, n\}$ and $\varepsilon_j \in \{-1, 1\}$. This mutation sequence defines a rigid object $\nu_{i_m} \dots \nu_{i_1} M$ which, by definition, belongs to $\text{mut}(M)$. By Proposition

5.1.3, $\text{End}_{K^{\text{b}}(\text{proj } \Lambda_{\text{con}})}(T) \cong \underline{\text{End}}_R(\nu_{i_m} \dots \nu_{i_1} M)$. \square

Using this result, to completely determine the basic algebras in the derived equivalence class of Λ_{con} , we just need to show that every basic tilting complex of Λ_{con} can be obtained by iterated right or left mutation from Λ_{con} . By Lemma 2.2.11, this will follow from showing that Λ_{con} is tilting-discrete.

Theorem 5.1.5. *Under the setup of 5.1.1, the contraction algebra Λ_{con} is a tilting-discrete algebra.*

Proof. Recall that Theorem 2.2.12 gives three equivalent conditions for a symmetric algebra to be tilting-discrete and we will check the third, namely that

$$2\text{-tilt}_T \Lambda_{\text{con}} := \{P \in \text{tilt } \Lambda_{\text{con}} \mid T \geq P \geq T[1]\}$$

is a finite set for any tilting complex T which is given by iterated left tilting mutation from Λ_{con} .

We first claim that $2\text{-tilt}_{\Lambda_{\text{con}}} \Lambda_{\text{con}} = 2\text{-tilt } \Lambda_{\text{con}}$ is finite. Recall from Theorem 2.3.5 that two-term silting (and hence in this case tilting by Proposition 3.2.12) complexes for Λ_{con} are in bijection with certain rigid objects in the full subcategory $M * \Sigma M$ of $\underline{\text{CM}} R$. However, as there are only ever finitely many minimal models of a complete local isolated cDV singularity, Theorem 3.2.4 shows that there are finitely many maximal rigid objects in $\underline{\text{CM}} R$. Finally, as Lemma 2.3.7 shows that any rigid object is a direct summand of a maximal rigid object, this shows there are only finitely many rigid objects in $\underline{\text{CM}} R$, and hence also in $M * \Sigma M$.

Now suppose T is a tilting complex obtained by iterated left mutation of Λ_{con} so,

$$T \cong \mu_{i_m} \dots \mu_{i_1} \Lambda_{\text{con}}$$

for some $i_1, \dots, i_m \in \{1, \dots, n\}$. If $N := \nu_{i_m} \dots \nu_{i_1} M$, then writing $\Gamma_{\text{con}} := \underline{\text{End}}_R(N)$, Proposition 5.1.3 shows there is a derived equivalence

$$F: \text{D}^{\text{b}}(\Lambda_{\text{con}}) \rightarrow \text{D}^{\text{b}}(\Gamma_{\text{con}})$$

mapping T to Γ_{con} . Thus, applying Lemma 2.2.9 to F , there is a bijection

$$\{P \in \text{tilt } \Lambda_{\text{con}} \mid T \geq P \geq T[1]\} \leftrightarrow \{Q \in \text{tilt } \Gamma_{\text{con}} \mid \Gamma_{\text{con}} \geq Q \geq \Gamma_{\text{con}}[1]\}.$$

By definition, the left hand side is $2\text{-tilt}_T \Lambda_{\text{con}}$, while the right hand side is $2\text{-tilt } \Gamma_{\text{con}}$. But since Γ_{con} is also the endomorphism algebra of a rigid object in $\underline{\text{CM}} R$, the first argument shows $2\text{-tilt } \Gamma_{\text{con}}$ must be finite and hence so is $2\text{-tilt}_T \Lambda_{\text{con}}$. \square

As Proposition 5.1.3 allows us to control the endomorphism algebras of tilting complexes reachable by mutation, combining this with the fact that contraction algebras are tilting-discrete leads to the main result of this section.

Corollary 5.1.6. *In the setup of 5.1.1, the following statements hold.*

1. *Any tilting complex of Λ_{con} can be obtained from Λ_{con} by finite iterated mutation.*
2. *The basic algebras derived equivalent to Λ_{con} are precisely the algebras*

$$\{\underline{\text{End}}_R(N) \mid N \in \text{mut}(M)\},$$

of which there are finitely many.

Proof. 1. Since Λ is tilting-discrete by Theorem 5.1.5 this follows directly from Proposition 2.2.11.

2. Suppose Γ is a basic algebra derived equivalent to Λ_{con} . Then $\Gamma \cong \text{End}_{\text{K}^{\text{b}}(\text{proj } \Lambda_{\text{con}})}(T)$ for some basic tilting complex $T \in \text{tilt } \Lambda_{\text{con}}$. However, by part (1),

$$T \cong \mu_{i_m}^{\varepsilon_m} \cdots \mu_{i_1}^{\varepsilon_1} \Lambda_{\text{con}}$$

for some $i_1, \dots, i_m \in \{1, \dots, n\}$ and $\varepsilon_j \in \{-1, 1\}$ and by Proposition 5.1.3

$$\Gamma \cong \text{End}_{\text{K}^{\text{b}}(\text{proj } \Lambda_{\text{con}})}(T) \cong \underline{\text{End}}_R(\nu_{i_m} \cdots \nu_{i_1} M)$$

as required. Finally, the set is finite as there are finitely many maximal rigid (and hence rigid by Lemma 2.3.7) objects in $\underline{\text{CM}} R$. \square

Remark 5.1.7. The results in this section hold more generally. The only properties required are the standard setup of cluster-tilting theory (see §2.3) along with two additional assumptions:

1. The shift functor Ω^{-1} on $\underline{\text{CM}} R$ satisfies $\Omega^2 \cong \text{id}$. This is required to show that the endomorphism algebras of rigid objects are symmetric, as in Proposition 3.2.12;
2. There are finitely many maximal rigid objects in $\underline{\text{CM}} R$, which is required in the proof of tilting-discreteness.

See [Au] for precise details.

5.2 Geometric Corollaries

In this section, we describe the geometric consequences of Corollary 5.1.6 and in particular, the evidence it provides towards Conjecture 3.1.9.

Let $f: X \rightarrow \text{Spec } R$ be a 3-fold flopping contraction as in Setup 3.1.1. Recall from §3.1 that we can flop f at any curve in the exceptional locus to obtain a new flopping contraction. Iterating this procedure allows us to create many flopping contractions from a given one.

Notation 5.2.1. Let f be a flopping contraction as in Setup 3.1.1 and suppose there are curves C_1, \dots, C_n in the exceptional locus of f . Given a sequence (i_1, \dots, i_m) where $i_j \in \{1, \dots, n\}$ we obtain a flopping contraction $f_{i_m \dots i_1}$ defined iteratively by:

1. f_{i_1} is the simple flop of the f at curve C_{i_1} ;
2. $f_{i_j \dots i_1}$ is the flop of $f_{i_{j-1} \dots i_1}$ at the curve C_{i_j} for $1 < j \leq m$.

We call the sequence (i_1, \dots, i_m) a *mutation sequence*.

For such a flopping contraction f , §3.1.1 shows how to construct a basic rigid object $M := \bigoplus_{i=1}^n M_i$ in $\underline{\text{CM}} R$ such that the corresponding contraction algebra is $\underline{\text{End}}_R(M)$. Moreover, the summands M_i of M naturally correspond to the curves C_i and Remark 3.2.8 shows that

$$\underline{\text{End}}_R(\nu_{i_m} \cdots \nu_{i_1} M)$$

is the contraction algebra of $f_{i_m \dots i_1}$. Combining these results with the results from the previous section gives the following.

Theorem 5.2.2. *Let $f: X \rightarrow \text{Spec } R$ be as in Setup 3.1.1 with associated contraction algebra $\Lambda_{\text{con}} := \underline{\text{End}}_R(N)$.*

1. *The endomorphism algebra of the tilting complex $T := \mu_{i_m}^{\varepsilon_m} \dots \mu_{i_1}^{\varepsilon_1} \Lambda_{\text{con}}$ is isomorphic to the contraction algebra of $f_{i_m \dots i_1}$.*
2. *The basic algebras derived equivalent to Λ_{con} are precisely the contraction algebras of flopping contractions $g: Y \rightarrow \text{Spec } R$, obtained by a sequence of iterated flops from f . In particular, there are finitely many such algebras.*

Proof. 1. By Proposition 5.1.3, there is an isomorphism

$$\text{End}_{\mathbf{K}^b(\text{proj } \Lambda_{\text{con}})}(T) \cong \underline{\text{End}}_R(\nu_{i_m} \dots \nu_{i_1} N).$$

Combining this with Remark 3.2.8 gives the result.

2. This follows by applying Remark 3.2.8 to part (2) of Corollary 5.1.6. □

In the special case of minimal models, it is well known that any two minimal models are connected by a sequence of simple flops. Thus, part (2) of Theorem 5.2.2 reduces to the following.

Corollary 5.2.3. *Let $f: X \rightarrow \text{Spec } R$ be a minimal model of a complete local isolated cDV singularity. Writing Λ_{con} for the associated contraction algebra, the basic algebras derived equivalent to Λ_{con} are precisely the contraction algebras of minimal models of $\text{Spec } R$.*

Recall from Conjecture 3.1.9 that it is expected that the derived category of the contraction algebras of minimal models of $\text{Spec } R$ can recover all the geometry of $\text{Spec } R$. The following shows that, to some extent, this is true.

Theorem 5.2.4. *Suppose that $f: X \rightarrow \text{Spec } R$ and $g: Y \rightarrow \text{Spec } S$ are minimal models of complete local isolated cDV singularities with associated contraction algebras Λ_{con} and Γ_{con} . If Λ_{con} and Γ_{con} are derived equivalent then there is a bijection*

$$\{\text{minimal models of } \text{Spec } R\} \longleftrightarrow \{\text{minimal models of } \text{Spec } S\}.$$

Further, the bijection preserves both mutation and contraction algebras.

Proof. Let $M \in \underline{\text{CM}} R$ be the maximal rigid object associated to f and let $N \in \underline{\text{CM}} S$ be the maximal rigid object associated to g so that

$$\Lambda_{\text{con}} := \underline{\text{End}}_R(M) \quad \text{and} \quad \Gamma_{\text{con}} := \underline{\text{End}}_S(N).$$

Since Γ_{con} is basic and derived equivalent to Λ_{con} , Corollary 5.2.3 shows that Γ_{con} must be isomorphic to a contraction algebra for some minimal model of $\text{Spec } R$. In particular, there exists a maximal rigid object $M' \in \underline{\text{CM}} R$ such that

$$\Gamma_{\text{con}} \cong \underline{\text{End}}_R(M').$$

Applying Theorem 2.3.5 and the remark afterwards, there are mutation preserving bijections

$$\begin{array}{ccccc} \text{mrig } \underline{\text{CM}} R & \longleftrightarrow & 2\text{-silt } \Gamma_{\text{con}} & \longleftrightarrow & \text{mrig } \underline{\text{CM}} S \\ M' & \leftarrow & \Gamma_{\text{con}} & \rightarrow & N \end{array}$$

Further, Theorem 3.2.4 shows that the minimal models of $\text{Spec } R$ are in bijection with $\text{mrig } \underline{\text{CM}} R$ and the minimal models of $\text{Spec } S$ are in bijection with $\text{mrig } \underline{\text{CM}} S$. Since all maps respect mutation, combining these bijections with Theorem 5.2.2 completes the proof. \square

In summary, this theorem shows that if $\text{Spec } R$ and $\text{Spec } S$ are complete local isolated cDV singularities with derived equivalent contraction algebras of minimal models then we can conclude that $\text{Spec } R$ and $\text{Spec } S$ have isomorphic simple flops graphs so the numbers of minimal models are the same. Moreover, the sets of contraction algebras of minimal models are the same in both cases. We shall refine these results further in Chapter 7

5.3 Standard Equivalences

This section translates our results about tilting complexes into results about standard derived equivalences. Continuing with the setup of 5.1.1, recall that we have fixed a basic rigid object $M \in \underline{\text{CM}} R$ and let $\Lambda_{\text{con}} := \underline{\text{End}}_R(M)$.

Definition 5.3.1. *Define the oriented mutation graph of M , denoted X_M , to have $\text{mut}(M)$ for the set of vertices and an arrow from $N \rightarrow N'$ if N' is the left mutation of N at some indecomposable summand.*

As in [HW], fixing a decomposition $M := \bigoplus_{i=1}^n M_i$, also fixes the ordering on decompositions of all the other vertices via mutation. This allows us to abuse notation by denoting any arrow $N \rightarrow \nu_i N$ by s_i . If M and N are related by mutation, then X_M and X_N will be isomorphic graphs. In particular, the mutation graphs of any two maximal rigid objects will be isomorphic.

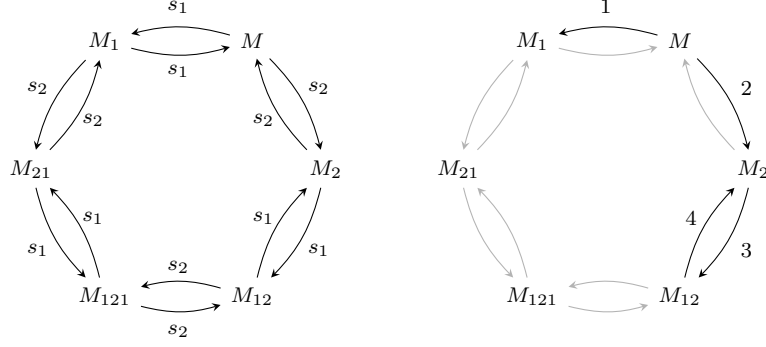
Remark 5.3.2. Note that if f is the flopping contraction associated to M , then by Theorem 3.2.7, X_M can also be defined as the double of the simple flops graph of f .

The idea of this section is that combinatorial paths in X_M control not only mutation of rigid objects but also tilting complexes of Λ_{con} .

Definition 5.3.3. *A path in X_M is a symbol $s_{i_m}^{\varepsilon_m} \dots s_{i_1}^{\varepsilon_1}$ with $i_1, \dots, i_m \in \{1, \dots, n\}$ and $\varepsilon_j \in \{-1, 1\}$, along with a specified starting vertex $N \in \text{mut}(M)$. The path s_i starting at N should be thought of as the path travelling along arrow s_i from vertex N and the path s_i^{-1} should be thought of travelling backwards along the arrow s_i incident to N . Longer paths are composed right to left as with function composition. A path is called positive if all the ε_i equal 1.*

Example 5.3.4. Consider an isolated cA_t singularity $R = \mathbb{C}[[u, v, x, y]]/(uv - f_1 f_2 f_3)$ and choose a maximal rigid object $M := M_1 \oplus M_2 \in \underline{\text{CM}} R$. Writing $M_{i_n \dots i_1} := \nu_{i_n} \dots \nu_{i_1} M$ to ease notation, the oriented mutation graph X_M is given below, which we can see is the double of the simple flops graph in Example 3.1.7. The path $\alpha := s_1 s_1 s_2 s_1^{-1}$ starting at vertex M_1 is shown in the diagram on the right, where we travel

along the first edge in the opposite orientation.



For each arrow $s_i: N \rightarrow \nu_i N$ in X_M , the main result of the previous chapter, Corollary 4.1.6, defines a standard derived equivalence

$$F_i: D^b(\underline{\text{End}}_R(N)) \rightarrow D^b(\underline{\text{End}}_R(\nu_i N))$$

induced by a certain two-term tilting complex \mathcal{T}_i . Combining this with Proposition 5.1.2 shows F_i maps the tilting complex $\mu_i(\underline{\text{End}}_R(N))$ to the algebra $\underline{\text{End}}_R(\nu_i N)$. If we label each s_i with the corresponding derived equivalence we obtain a picture where paths will correspond to the composition of functors. Travelling along an arrow backwards corresponds to using the inverse of the equivalence labelling that arrow. For this, we introduce the following notation.

Notation 5.3.5. Consider a path $\alpha := s_{i_m}^{\varepsilon_m} \dots s_{i_1}^{\varepsilon_1}$ in X_M starting at a basic rigid object $N \in \text{mut}(M)$. Then, writing $\Gamma_{\text{con}} := \underline{\text{End}}_R(N)$ set:

1. $\nu_\alpha N := \nu_{i_m} \dots \nu_{i_1} N$;
2. $\mu_\alpha \Gamma_{\text{con}} := \mu_{i_m}^{\varepsilon_m} \dots \mu_{i_1}^{\varepsilon_1} \Gamma_{\text{con}}$;
3. $F_\alpha := F_{i_m}^{\varepsilon_m} \circ \dots \circ F_{i_1}^{\varepsilon_1}: D^b(\Gamma_{\text{con}}) \rightarrow D^b(\underline{\text{End}}_R(\nu_\alpha N))$.

Example 5.3.6. Consider the path $\alpha := s_1 s_1 s_2 s_1^{-1}$ from Example 5.3.4. As right and left mutation are equal, and hence $\nu_i \nu_i M \cong M$, the object at the end of the path is

$$\begin{aligned} \nu_\alpha M_1 &:= \nu_1 \nu_1 \nu_2 \nu_1 M_1 \\ &\cong \nu_1 \nu_1 \nu_2 \nu_1 \nu_1 M \\ &\cong \nu_2 M \\ &\cong M_2 \end{aligned}$$

as in the diagram. Similarly, but without cancellation now as left and right mutation are different,

$$\mu_\alpha \underline{\text{End}}_R(M_1) := \mu_1 \mu_1 \mu_2 \mu_1^{-1}(\underline{\text{End}}_R(M_1))$$

and F_α is the composition

$$D^b(\underline{\text{End}}_R(M_1)) \xrightarrow{F_1^{-1}} D^b(\underline{\text{End}}_R(M)) \xrightarrow{F_2} D^b(\underline{\text{End}}_R(M_2)) \xrightarrow{F_1} D^b(\underline{\text{End}}_R(M_{12})) \xrightarrow{F_1} D^b(\underline{\text{End}}_R(M_2)).$$

Note that the tilting complex $\mu_\alpha \underline{\text{End}}_R(M_1)$ maps to $\underline{\text{End}}_R(M_2)$ under the equivalence F_α , by repeated use of Lemma 2.2.9.

Our first result of this section shows that this final observation holds more generally.

Proposition 5.3.7. *Under the setup of 5.1.1, and with notation as in 5.3.5,*

$$F_\alpha(\mu_\alpha \Gamma_{\text{con}}) \cong \underline{\text{End}}_R(\nu_\alpha N).$$

Proof. For each $j = 1, \dots, m$, the equivalence $F_{i_j}^{\varepsilon_j}$ constructed in Corollary 4.1.6 satisfies the conditions of the Ψ_{i_j} chosen in the proof of Proposition 5.1.3. Thus, this result follows in exactly the same way. \square

In other words, the derived equivalence F_α is the unique (up to algebra automorphism) standard derived equivalence induced by the tilting complex $\mu_\alpha \Gamma_{\text{con}}$.

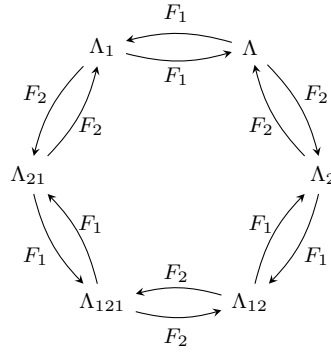
Corollary 5.3.8. *Let $\text{Spec } R$ be a complete local isolated cDV singularity and let $M \in \underline{\text{CM}} R$ be a basic rigid object. If X_M denotes the oriented mutation graph of M , and $N \in \text{mut}(N)$, then with notation as in 5.3.5, the following statements hold.*

1. *Up to algebra automorphism, any standard equivalence from $D^b(\Gamma_{\text{con}})$ is obtained as F_α for some (not necessarily positive) path α in X_M starting at N .*
2. *Up to algebra automorphism, any standard equivalence from $D^b(\Gamma_{\text{con}})$ is obtained by composition of the F_i and their inverses.*

Proof. 1. Take some standard equivalence $F: D^b(\Gamma_{\text{con}}) \rightarrow D^b(\Delta)$. Then $F^{-1}(\Delta)$ is a tilting complex for Γ_{con} and so as Γ_{con} is tilting-discrete by Proposition 5.1.5, Proposition 2.2.11 shows $F^{-1}(\Delta) \cong \mu_\alpha \Gamma_{\text{con}}$ for some path α in X_M starting at N . By Proposition 5.3.7, F_α is also a standard equivalence induced by $\mu_\alpha \Gamma_{\text{con}}$ and hence F and F_α must be the same up to an algebra automorphism, using Proposition 2.2.4.

2. This is a direct consequence of part (1). \square

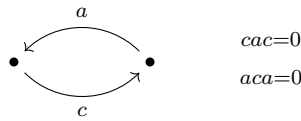
Example 5.3.9. Returning to Example 5.3.4, set $\Lambda := \underline{\text{End}}_R(M)$ and $\Lambda_{i_m \dots i_1} := \underline{\text{End}}_R(M_{i_m \dots i_1})$. Corollaries 5.1.6 and 5.3.8 show the diagram



can be thought of as a ‘picture’ of the derived equivalence class of Λ . The set of vertices consists of precisely the basic algebras in the derived equivalence class and the (not necessarily positive) paths determine all of the standard derived equivalences between these algebras. In other words, this picture not only contains all the members of the derived equivalence class, but also all the maps (up to some technicalities) between them. Notice that at this stage we can not say anything about when two paths give the same derived equivalence. This will be the focus of the next chapter.

Example 5.3.10. For any isolated cA_n singularity given by $f(x, y) = f_1 f_2$, there are two minimal models with isomorphic contraction algebras. For the Atiyah flop given by $f(x, y) = uv - xy$ they are both \mathbb{C} , whereas for the Pagoda flops, given by $f(x, y) = uv - (x - y^m)(x + y^m)$, they are both $\mathbb{C}[x]/(x^m)$. As these algebras are local, it is well known that they are the only basic members of a derived equivalence class. However, Corollary 5.2.3 recovers this result. In this case, the derived equivalence F from Corollary 4.1.6 is induced by the tilting complex $\Lambda_{\text{con}}[1]$ and thus, up to algebra automorphism, F is the shift functor. Hence, Corollary 5.3.8 further recovers the well-known fact that any standard derived autoequivalence of these algebras is in fact a shift functor up to algebra automorphism.

For the cA_2 singularity given by $f(x, y) = uv - xy(x + y)$, all six contraction algebras of the minimal models are given by the quiver with relations



and thus Corollary 5.2.3 shows it is the only basic member of its derived equivalence class (as in the case above). However, there are now extra tilting complexes and hence there are standard derived autoequivalences which are not simply shift functors. We will understand the structure of these equivalences in the next chapter.

Finally, for the cA_3 singularity given by $f(x, y) = uv - xy(x^2 + y^3)$ considered in §3.3, the quivers of the contraction algebras are shown in Figure 5.1. Corollary 5.2.3 shows they are the only basic members of a derived equivalence class.

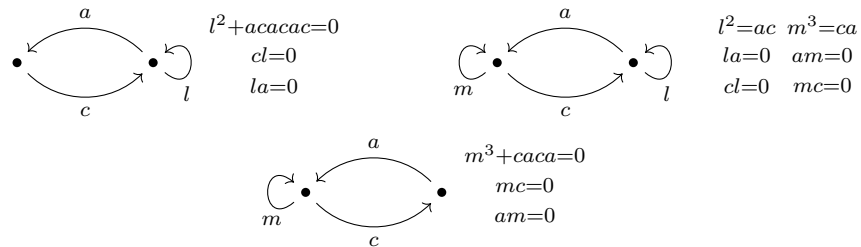


Figure 5.1: The quivers and relations of the contraction algebras of the minimal models of the cA_3 singularity with $f_1 = x, f_2 = y$ and $f_3 = x^2 + y^3$.

Chapter 6

The Structure of Derived Equivalences

In the previous chapters, we have shown that the contraction algebras of a complete local isolated cDV singularity $\text{Spec } R$ are all derived equivalent and further, there are no other algebras in the derived equivalence class. In addition, we have constructed standard derived equivalences between the contraction algebras which can be thought of as analogues of the flop functors between the minimal models. These functors can be used to label the arrows in the double of the simple flops graph of minimal models, or equivalently the oriented mutation graph of MM modules in $\text{CM } R$, in such a way that every standard derived equivalence, up to algebra automorphism, can be viewed as a path in this graph.

In this chapter, we consider which paths give rise to the same functors, or in other words, we determine the relations between our chosen equivalences. For the flop functors between the minimal models, this question was considered in [HW], where they associated a simplicial hyperplane arrangement $\mathcal{H} \subset \mathbb{R}^n$, and its complexification $\mathcal{H}_{\mathbb{C}} \subset \mathbb{C}^n$, to each minimal model, and used them to prove the following.

Theorem 6.0.1. [HW, 6.7] *Suppose that $f: X \rightarrow \text{Spec } R$ is a minimal model of a complete local isolated cDV singularity with associated hyperplane arrangement \mathcal{H} . Then the subgroup of $\text{Auteq}(\text{D}^b(\text{coh } X))$ consisting of flop functors, their compositions and inverses is isomorphic to $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}})$.*

The first result of this chapter shows that the hyperplane arrangement associated to f can be constructed from the two-term tilting theory of the corresponding contraction algebra. As a direct corollary, we obtain that the group of flop functors of a minimal model can be recovered from any one of the contraction algebras. This provides further evidence that the contraction algebras recover all the geometric information and hence further evidence towards Conjecture 3.1.9.

In the rest of the chapter, we use this hyperplane arrangement to control the composition of the derived equivalences between contraction algebras constructed in Chapter 4. This serves two purposes. Firstly, it demonstrates further that we can think of these equivalences as the analogue of the flop functors in the geometry as they satisfy precisely the same relations. Secondly, it gives the results needed so that in the next chapter, we can combine all the results of this thesis to show that the hyperplane arrangement gives a complete picture of the derived equivalence class of a contraction algebra.

6.1 Hyperplane Arrangements in the Homological Minimal Model Program

In this section, we describe the hyperplane arrangement associated to any MM module in $\text{CM } R$. This can be described in many different ways; as the quiver GIT chamber decomposition of the MMA (as in [We]), as the intersection of a Dynkin root system associated to the minimal model (as in [Pi]) and finally, algebraically via K-theory (as in [HW]). It is this final description we present here.

Setup 6.1.1. Let $\text{Spec } R$ be a complete local isolated cDV singularity and choose a basic MM module $M := \bigoplus_{i=0}^n M_i \in \text{CM } R$ where $M_0 \cong R$.

Under this setup, let $N := \bigoplus_{i=0}^n N_i \in \text{CM } R$ be any other MM module with $N_0 \cong R$. Then [IW1, 4.12] shows that for each $1 \leq i \leq n$, there exists an exact sequence

$$0 \rightarrow \bigoplus_{j=0}^n M_j^{b_{ij}} \xrightarrow{f_i} \bigoplus_{j=0}^n M_j^{a_{ij}} \xrightarrow{g_i} N_i \rightarrow 0 \quad (6.1)$$

where g_i is a minimal right $\text{add}(M)$ -approximation. As in [HW, §3.2], this collection of sequences defines a chamber

$$C_N := \left\{ \sum_{i=1}^n \vartheta_i \left(\sum_{j=1}^n (a_{ij} - b_{ij}) \mathbf{e}_j \right) \mid \vartheta_i > 0 \forall 1 \leq i \leq n \right\} \subset \mathbb{R}^n.$$

Theorem 6.1.2 ([DW3, 3.8] [HW, 4.6(2)]). *With the setup as in 6.1.1 and notation as above, the collection of chambers C_N , where N ranges across all MM modules in $\text{CM } R$, give all the chambers of a simplicial hyperplane arrangement \mathcal{H}_M in \mathbb{R}^n .*

Given a real simplicial hyperplane arrangement \mathcal{H} , the *oriented skeleton graph* of \mathcal{H} , denoted $X_{\mathcal{H}}$, has a vertex for each chamber and an arrow $v \rightarrow w$ if the corresponding chambers are separated by a codimension one wall.

Theorem 6.1.3 ([We, 6.9(5)]). *Under the setup of 6.1.1, the oriented skeleton graph $X_{\mathcal{H}_M}$ is isomorphic to the oriented mutation graph X_M of MM modules in $\text{CM } R$ (or equivalently, the maximal rigid objects in $\underline{\text{CM}} R$).*

In other words, crossing a wall in the hyperplane arrangement corresponds to mutation of MM modules. As a consequence of this, given any two MM modules $M, N \in \text{CM } R$, there is an isomorphism between $X_{\mathcal{H}_M}$ and $X_{\mathcal{H}_N}$ which fixes the modules and further, using Theorem 3.2.7, this graph is the double of the simple flops graph of minimal models of $\text{Spec } R$.

The first result of this section shows that the hyperplane arrangement \mathcal{H}_M , and hence the group structure of the flop functors via Theorem 6.0.1, can be recovered from a contraction algebra $\Lambda_{\text{con}} := \underline{\text{End}}_R(M)$. To do this, recall that if A is a finite dimensional k -algebra with indecomposable projective modules P_1, \dots, P_n , then the g -vector of a two-term complex

$$Q := \bigoplus_{i=1}^n P_i^{b_i} \rightarrow \bigoplus_{i=1}^n P_i^{a_i}$$

in degrees -1 and 0 is defined as $g(Q) := \sum_{i=1}^n (a_i - b_i) \mathbf{e}_i \in \mathbb{R}^n$.

Theorem 6.1.4. *Under the setup of 6.1.1, the g -vectors of the two-term tilting complexes of the contraction algebra $\Lambda_{\text{con}} := \underline{\text{End}}_R(M)$ determine the simplicial hyperplane arrangement \mathcal{H}_M .*

Proof. For each two-term tilting complex $Q := \bigoplus_{i=1}^n Q_i \in \mathbf{K}^b(\text{proj } \Lambda_{\text{con}})$ set

$$C_Q := \left\{ \sum_{i=1}^n \vartheta_i g(Q_i) \mid \vartheta_i > 0 \forall 1 \leq i \leq n \right\} \subset \mathbb{R}^n.$$

By the bijection in Theorem 2.3.5 and Remark 2.3.6(2) afterwards, Q corresponds to a basic maximal rigid object $N := \bigoplus_{i=1}^n N_i$ in $\underline{\text{CM}} R$ and hence to an MM module $\bigoplus_{i=0}^n N_i$ in $\text{CM} R$ with $N_0 \cong R$. For each summand, Proposition 2.1.12 shows the exact sequence (6.1) descends to a triangle

$$\bigoplus_{j=1}^n M_j^{b_{ij}} \xrightarrow{f_i} \bigoplus_{j=1}^n M_j^{a_{ij}} \xrightarrow{g_i} N_i$$

in $\underline{\text{CM}} R$ where g_i is still a minimal right $\text{add}(M)$ -approximation. This shows that, under the bijection of Theorem 2.3.5, N_i corresponds to the two-term complex

$$\bigoplus_{j=1}^n \underline{\text{Hom}}_R(M, M_j)^{b_{ij}} \rightarrow \bigoplus_{j=1}^n \underline{\text{Hom}}_R(M, M_j)^{a_{ij}}.$$

However, by assumption N_i corresponds to Q_i and thus, $g(Q_i) = \sum_{j=1}^n (a_{ij} - b_{ij}) \mathbf{e}_j$. Therefore, $C_Q = C_N$ and so, as the C_N sweep out the chambers of simplicial hyperplane arrangement \mathcal{H}_M by Theorem 6.1.2, the result follows. \square

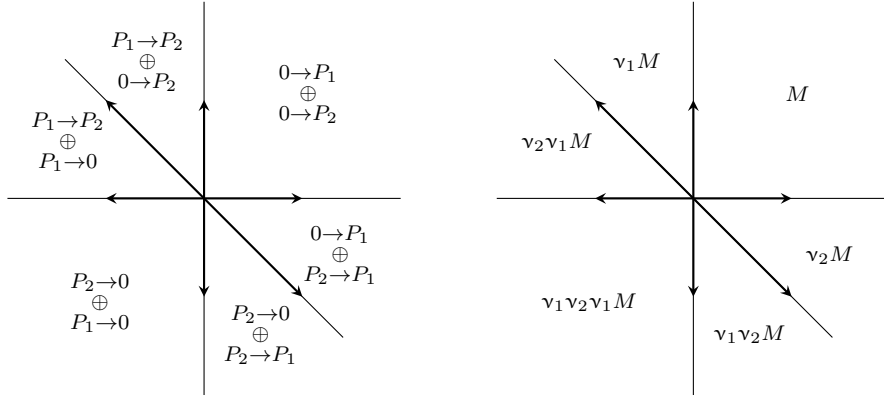
Combining with Theorem 6.0.1, the following is an immediate corollary.

Corollary 6.1.5. *Suppose that Λ_{con} is a contraction algebra of a minimal model $f: X \rightarrow \text{Spec } R$ for some complete local isolated cDV singularity $\text{Spec } R$. The subgroup of $\text{Auteq}(\mathbf{D}^b(\text{coh } X))$ formed by flop functors, their compositions and inverses can be recovered from Λ_{con} .*

Example 6.1.6. Consider the cA_2 singularity given by $uv - xy(x + y^2)$ and choose $M := R \oplus (u, x) \oplus (u, xy)$, which, by Theorem 3.3.2 is one of the MM modules in $\text{CM} R$. Then $\Lambda_{\text{con}} := \underline{\text{End}}_R(M)$ is given by the quiver and relations

$$\begin{array}{ccc} & a & \\ & \curvearrowright & \\ 1 & & 2 \\ & \curvearrowleft & \\ & c & \end{array} \quad \begin{array}{l} cacac=0 \\ acaca=0 \end{array}$$

Thus, Λ_{con} has two indecomposable projective modules, labelled P_1 and P_2 , and the wall and chamber structure constructed from the g -vectors of the two-term tilting theory is shown below on the left, where each chamber is labelled by a two-term tilting complex.



By Theorem 6.1.4, this is the same as the hyperplane arrangement on the right, where the chambers are labelled by the MM modules in $\text{CM } R$. Moreover, for each chamber, the two-term tilting complex and MM module labelling it correspond to each other via the bijection Theorem 2.3.5.

Remark 6.1.7. Note that the results in this section are stated only for MM modules and not for modifying modules (equivalently only for maximal rigid and not rigid more generally). In the latter case, the quiver GIT construction still gives an associated hyperplane arrangement whose chambers are labelled by all those modifying modules N that can be reached by mutation, namely $\text{mut}(M)$. Further, the oriented skeleton graph will still match the mutation graph. However, in this case, the current literature cannot guarantee the existence of the exact sequence (6.1) for each summand N_i and hence the algebraic description given in Theorem 6.1.2 does not necessarily exist. If this sequence always exists, Theorem 6.1.2 also holds in this more general setting, and hence so do the key results in [HW] and the rest of this thesis.

6.2 Deligne Groupoid Preliminaries

In the rest of this chapter, we use the hyperplane arrangement associated to a MM module to control the compositions of the functors F_i constructed in Chapter 4. The extra structure gained from the hyperplane arrangement comes in the form of an associated groupoid, called the Deligne Groupoid, which we now briefly review.

For the description of the Deligne Groupoid we follow [HW]. Let \mathcal{H} be a simplicial hyperplane arrangement and let $X_{\mathcal{H}}$ be its oriented skeleton graph.

A *positive path* of length n is then a formal symbol

$$p = a_n a_{n-1} \dots a_1$$

such that there exist vertices v_0, v_1, \dots, v_n and arrows $a_i: v_{i-1} \rightarrow v_i$. The *source* and *target* of such a p are defined to be $s(p) := v_0$ and $t(p) := v_n$ respectively. A positive path p is *minimal* if there is no positive path with the same endpoints that has shorter length. Positive minimal paths are called *atoms*.

There is an equivalence relation \sim on the set of positive paths in $X_{\mathcal{H}}$ given as the smallest equivalence relation satisfying:

1. If $p \sim q$ then $s(p) = s(q)$ and $t(p) = t(q)$.
2. If p and q are two atoms starting and ending at the same point then $p \sim q$.

3. If $p \sim q$, then $upr \sim uqr$ for all positive paths u and r with $t(r) = s(p) = s(q)$ and $s(u) = t(p) = t(q)$.

Write $[p]$ for the equivalence class of a positive path p .

Definition 6.2.1. For two chambers v, w in \mathcal{H} , let

$$\text{Path}_{\mathcal{H}}(v, w) := \{[p] \mid s(p) = v, t(p) = w\}.$$

The category $\mathcal{G}_{\mathcal{H}}^+$ is defined to have the vertices of $X_{\mathcal{H}}$ as objects and the $\text{Path}_{\mathcal{H}}(v, w)$ as morphisms. The Deligne Groupoid of \mathcal{H} , denoted $\mathcal{G}_{\mathcal{H}}$, is then defined to be the groupoid completion of $\mathcal{G}_{\mathcal{H}}^+$; that is, the objects are the same as for $\mathcal{G}_{\mathcal{H}}^+$ but a formal inverse is added for each morphism.

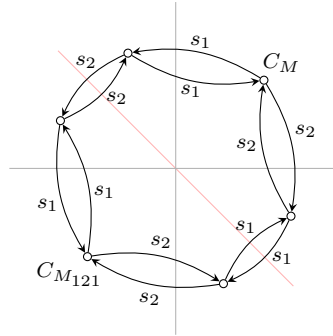
The following key theorem shows that the vertex groups of $\mathcal{G}_{\mathcal{H}}$ are all isomorphic and depend only on the structure of \mathcal{H} . To state it, recall that for a hyperplane H in \mathbb{R}^n , the complexification of H is simply $H \otimes_{\mathbb{R}} \mathbb{C}$ inside \mathbb{C}^n , and the complexification of a hyperplane arrangement \mathcal{H} is the collection of all the complexified hyperplanes in \mathcal{H} .

Theorem 6.2.2. [P1],[S] (see [P2, 2.1]) *If \mathcal{H} is a simplicial hyperplane arrangement, then for each chamber v of \mathcal{H} , there is an isomorphism $\text{Hom}_{\mathcal{G}_{\mathcal{H}}}(v, v) \cong \pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}})$ where $\mathcal{H}_{\mathbb{C}}$ is the complexification of \mathcal{H} .*

6.3 Strategy and Result for MMAs

Under the setup of 6.1.1, let \mathcal{H}_M be the associated hyperplane arrangement and X_M its oriented skeleton graph. As \mathcal{H}_M is simplicial, each chamber C_N of \mathcal{H}_M has n codimension one walls and, by Theorem 6.1.3, crossing a wall from C_N is equivalent to mutating N at an indecomposable summand. As in §5.3, we abuse notation and denote an arrow $C_N \rightarrow C_L$ in X_M by s_i if $L \cong v_i N$.

Example 6.3.1. Continuing Example 6.1.6, the hyperplane arrangement and oriented skeleton graph are given below.



Notice that the two paths $s_1 s_2 s_1$ and $s_2 s_1 s_2$ starting in C_M and traversing clockwise and anti-clockwise respectively to $C_{M_{121}}$ are both atoms, and hence are identified in the Deligne Groupoid.

Since \mathcal{H}_M is a simplicial hyperplane arrangement, there is an associated Deligne Groupoid $\mathcal{G}_{\mathcal{H}_M}$ whose vertex groups are all isomorphic to $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}})$ by Theorem

6.2.2. We will also associate to $\text{Spec } R$ the groupoid \mathbb{F} , whose vertices are the MM modules in $\text{CM } R$ and whose morphisms are

$$\text{Hom}_{\mathbb{F}}(M, N) := \{\text{standard derived equivalences } \underline{\text{End}}_R(M) \rightarrow \underline{\text{End}}_R(N)\},$$

taken up to isomorphism. As in [HW], our strategy to control the composition of derived equivalences between contraction algebras is to first construct a functor

$$\Phi: \mathcal{G}_{\mathcal{H}_M} \rightarrow \mathbb{F}.$$

We will then further show that this functor is faithful, from which it will immediately follow that there is an injective group homomorphism

$$\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}) \rightarrow \text{Auteq}(\text{D}^b(\underline{\text{End}}_R(N)))$$

for any MM module N in $\text{CM } R$.

To define such a functor Φ , it is natural to set $\Phi(C_N) := N$. For each arrow $s_i: N \rightarrow \nu_i N$ we need to choose an equivalence between the corresponding derived categories. These are precisely the equivalences F_i that we constructed in Chapter 4.

Notation 6.3.2. Suppose that $\text{Spec } R$ is a complete local isolated cDV singularity and that $N := \bigoplus_{i=0}^n N_i \in \text{CM } R$ is an MM module in $\text{CM } R$ with $N_0 \cong R$. If $i \neq 0$, consider the following algebras:

$$\Lambda := \text{End}_R(N), \quad \Lambda_{\text{con}} := \underline{\text{End}}_R(N), \quad \Gamma := \text{End}_R(\nu_i N), \quad \text{and} \quad \Gamma_{\text{con}} := \underline{\text{End}}_R(\nu_i N).$$

Additionally, define

1. $T_i := \text{Hom}_R(N, \nu_i N)$ and;
2. $\mathcal{T}_i := \tau_{\geq -1}(\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_i \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}})$.

By Theorem 3.4.1 and Corollary 4.1.6, these induce the following vertical equivalences

$$\begin{array}{ccc} \text{D}^b(\Lambda_{\text{con}}) & \xrightarrow{-\otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}}} & \text{D}^b(\Lambda) \\ F_i := \mathbf{R}\text{Hom}_{\Lambda_{\text{con}}}(\mathcal{T}_i, -) \downarrow & & \downarrow G_i := \mathbf{R}\text{Hom}_{\Lambda}(T_i, -) \\ \text{D}^b(\Gamma_{\text{con}}) & \xrightarrow{-\otimes_{\Gamma_{\text{con}}} \Gamma_{\text{con}}} & \text{D}^b(\Gamma) \end{array} \quad (6.2)$$

with inverses $F_i^{-1} := - \otimes_{\Gamma_{\text{con}}}^{\mathbf{L}} \mathcal{T}_i$ and $G_i^{-1} := - \otimes_{\Gamma}^{\mathbf{L}} T_i$. This diagram commutes by Theorem 4.1.5.

Remark 6.3.3. Note that we will abuse notation by using F_i to refer to any equivalence

$$\text{D}^b(\underline{\text{End}}_R(N)) \rightarrow \text{D}^b(\underline{\text{End}}_R(\nu_i N))$$

constructed as above, regardless of the choice of MM module N . Similarly, G_i will refer to any standard equivalence between MMAs induced by a tilting bimodule of the form $\text{Hom}_R(N, \nu_i N)$.

With this notation, we define $\Phi: \mathcal{G}_{\mathcal{H}} \rightarrow \mathbb{F}$ by mapping the arrow $s_i: N \rightarrow \nu_i N$ to the corresponding standard equivalence F_i . This construction will yield a functor

between the groupoids if and only if equivalent paths in $X_{\mathcal{H}_M}$ give isomorphic functors. In particular, the equivalences F_i must satisfy the relations on paths in the Deligne Groupoid. To check this, we need to be able to understand compositions of these functors. For the G_i , this is already known.

Theorem 6.3.4. [HW, 4.6] *Under the setup of 6.1.1, write $\Lambda := \text{End}_R(M)$ and iteratively mutate M to obtain*

$$N := \nu_{i_m} \nu_{i_{m-1}} \dots \nu_{i_1} M.$$

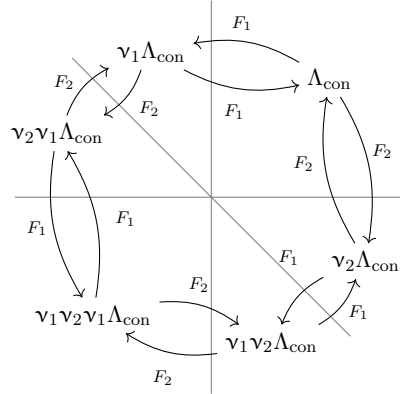
This defines a positive path $s_{i_m} \dots s_{i_1} : C_M \rightarrow C_N$ in $X_{\mathcal{H}_M}$. If this path is an atom then

$$G_{i_m} \circ \dots \circ G_{i_1} \simeq \mathbf{R}\text{Hom}_\Lambda(\text{Hom}_R(M, N), -).$$

Remark 6.3.5. Although [HW] only prove the above result for NCCRs (or equivalently, when the corresponding minimal model is smooth) the proof works with no modifications for MMAs.

In the next subsection, we will prove that a similar result holds for the F_i . To do this directly would be extremely difficult as the F_i are given by tilting complexes rather than tilting modules but our key idea is to use the commutative diagram (6.2), together with Theorem 6.3.4.

Example 6.3.6. Continuing Example 6.3.1, we label each chamber by its associated contraction algebra by writing $\nu_{i_m} \dots \nu_{i_1} \Lambda_{\text{con}} := \underline{\text{End}}_R(\nu_{i_m} \dots \nu_{i_1} M)$. Further labelling each wall crossing by the associated derived equivalence yields the following picture.



As in §5.3, the idea is that paths correspond to compositions of functors and as the paths $s_1 s_2 s_1$ and $s_2 s_1 s_2$ are equivalent in the Deligne Groupoid, we need to deduce $F_1 \circ F_2 \circ F_1 \cong F_2 \circ F_1 \circ F_2$.

6.4 Result for Contraction Algebras

The main result of this section will be to prove the analogue of Theorem 6.3.4 for contraction algebras. For this, the following two technical lemmas are required.

Lemma 6.4.1. *In the setup of 6.1.1, write $\Lambda := \text{End}_R(M)$ and $\Lambda_{\text{con}} := \underline{\text{End}}_R(M)$. Then the homology of $\Lambda_{\text{con}} \otimes_\Lambda^\mathbf{L} \Lambda_{\text{con}}$ is zero outside degrees -3 and 0 . In degree 0 , it is isomorphic to Λ_{con} as a $\Lambda_{\text{con}}\text{-}\Lambda_{\text{con}}$ bimodule.*

Proof. By Lemma 4.2.3, Λ_{con} has a deleted projective resolution as a right Λ -module of the form

$$0 \rightarrow \Lambda \rightarrow \text{Hom}_R(M, R^n) \rightarrow \text{Hom}_R(M, R^n) \rightarrow \Lambda \rightarrow 0.$$

Applying $-\otimes_{\Lambda} \Lambda_{\text{con}}$ gives

$$0 \rightarrow \Lambda_{\text{con}} \rightarrow 0 \rightarrow 0 \rightarrow \Lambda_{\text{con}} \rightarrow 0$$

and this shows $H^i(\Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}) = 0$ if $i \neq 0, -3$. To obtain the homology as bimodules in degree 0, note that it is isomorphic as $\Lambda_{\text{con}}\text{-}\Lambda_{\text{con}}$ -bimodules to $\text{Tor}_0^{\Lambda}(\Lambda_{\text{con}}, \Lambda_{\text{con}}) \cong \Lambda_{\text{con}} \otimes_{\Lambda} \Lambda_{\text{con}}$. Moreover, it is easily checked that the isomorphism $\Lambda_{\text{con}} \otimes_{\Lambda} \Lambda_{\text{con}} \rightarrow \Lambda_{\text{con}}$, defined via $\lambda_1 \otimes \lambda_2 \mapsto \lambda_1 \lambda_2$ is in fact an isomorphism of $\Lambda_{\text{con}}\text{-}\Lambda_{\text{con}}$ bimodules as required. \square

Lemma 6.4.2. *Suppose that Δ is a ring and Λ_{con} is the contraction algebra of some minimal model of a complete local isolated cDV singularity. Let X be a complex of $\Delta\text{-}\Lambda_{\text{con}}$ -bimodules whose homology vanishes in degrees other than $-1, 0$ and 1 . Then,*

$$\tau_{\geq -1}(X \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}) \cong X$$

in the derived category of $\Delta\text{-}\Lambda_{\text{con}}$ -bimodules.

Proof. First note that since the homology vanishes above degree 1, $\tau_{\leq 1}(X)$ is quasi-isomorphic to X and so we can assume that $X_i = 0$ for all $i > 1$ by instead considering the truncation. Now, for any complex Y of $\Lambda_{\text{con}}\text{-}\Lambda_{\text{con}}$ -bimodules, there is a triangle

$$\tau_{< -1}(Y) \rightarrow Y \rightarrow \tau_{\geq -1}(Y) \rightarrow \tau_{< -1}(Y)[1].$$

Taking $Y := \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}$, Lemma 6.4.1 identifies $\tau_{< -1}(Y)$ and $\tau_{\geq -1}(Y)$ as complexes in a single degree, and so gives a triangle

$$M[3] \rightarrow \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}} \rightarrow \Lambda_{\text{con}} \rightarrow M[4]$$

where M is some $\Lambda_{\text{con}}\text{-}\Lambda_{\text{con}}$ bimodule. Applying the functor $X \otimes_{\Lambda_{\text{con}}}^{\mathbf{L}} -$ then results in a triangle

$$X \otimes_{\Lambda}^{\mathbf{L}} M[3] \rightarrow X \otimes_{\Lambda_{\text{con}}}^{\mathbf{L}} \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}} \xrightarrow{\phi} X \rightarrow X \otimes_{\Lambda}^{\mathbf{L}} M[4] \quad (6.3)$$

in the derived category of $\Delta\text{-}\Lambda_{\text{con}}$ -bimodules. As truncation is a functor, the map ϕ induces a map

$$\phi^*: \tau_{\geq -1}(X \otimes_{\Lambda_{\text{con}}}^{\mathbf{L}} \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}) \rightarrow \tau_{\geq -1}(X)$$

and further, it is easy to check that ϕ and ϕ^* fit into the following commutative diagram

$$\begin{array}{ccccccc} X \otimes_{\Lambda}^{\mathbf{L}} M[3] & \longrightarrow & X \otimes_{\Lambda_{\text{con}}}^{\mathbf{L}} \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}} & \xrightarrow{\phi} & X & \longrightarrow & X \otimes_{\Lambda}^{\mathbf{L}} M[4] \\ & & \alpha \downarrow & & \downarrow \beta & & \\ & & \tau_{\geq -1}(X \otimes_{\Lambda_{\text{con}}}^{\mathbf{L}} \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}) & \xrightarrow{\phi^*} & \tau_{\geq -1}(X) & & \end{array}$$

where α and β are the natural maps to the truncations.

As X has vanishing homology in degrees other than -1 , 0 and 1 , it is clear $\tau_{\geq -1}(X) \cong X$, and hence to prove the result, it is enough to show ϕ^* is a quasi-isomorphism.

As homology is functorial, taking homology of the commutative diagram yields another commutative diagram

$$\begin{array}{ccccccc} H^i(X \otimes_{\Lambda}^{\mathbf{L}} M[3]) & \longrightarrow & H^i(X \otimes_{\Lambda_{\text{con}}}^{\mathbf{L}} \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}) & \xrightarrow{H^i(\phi)} & H^i(X) & \longrightarrow & H^i(X \otimes_{\Lambda}^{\mathbf{L}} M[4]) \\ & & \downarrow H^i(\alpha) & & \cong \downarrow H^i(\beta) & & \\ & & H^i(\tau_{\geq -1}(X \otimes_{\Lambda_{\text{con}}}^{\mathbf{L}} \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}})) & \xrightarrow{H^i(\phi^*)} & H^i(\tau_{\geq -1}(X)) & & \end{array}$$

where the top row is exact since it is part of the long exact sequence of homology of the triangle (6.3).

For $i < -1$, the truncation is constructed so that

$$H^i(\tau_{\geq -1}(X \otimes_{\Lambda_{\text{con}}}^{\mathbf{L}} \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}})) = 0 = H^i(\tau_{\geq -1}(X))$$

and hence $H^i(\phi^*)$ is an isomorphism. For $i \geq -1$, it is clear that $H^i(\alpha)$ and $H^i(\beta)$ are isomorphisms. Further, as $X_i = 0$ for all $i \geq 2$ and M is a module, $(X \otimes_{\Lambda}^{\mathbf{L}} M)_i = 0$ for all $i \geq 2$ as well. Thus, for $i \geq -1$,

$$H^i(X \otimes_{\Lambda}^{\mathbf{L}} M[3]) = 0 = H^i(X \otimes_{\Lambda}^{\mathbf{L}} M[4])$$

and hence $H^i(\phi)$ must be an isomorphism using exactness of the top row. This shows that for $i \geq -1$, the top and vertical maps in the commutative square are isomorphisms and hence $H^i(\phi^*)$ must also be. Thus $H^i(\phi^*)$ is an isomorphism for all i , and so ϕ^* is a quasi-isomorphism, as required. \square

The following is the main technical result of this section.

Theorem 6.4.3. *Under the setup of 6.1.1, write $\Lambda := \text{End}_R(M)$ and $\Lambda_{\text{con}} := \underline{\text{End}}_R(M)$. Iteratively mutate M to obtain*

$$N := \nu_{i_m} \nu_{i_{m-1}} \dots \nu_{i_1} M.$$

This defines a positive path $\alpha := s_{i_m} \dots s_{i_1} : C_M \rightarrow C_N$ in $X_{\mathcal{H}_M}$. If this path is an atom then, writing $\Gamma := \text{End}_R(N)$ and $\Gamma_{\text{con}} := \underline{\text{End}}_R(N)$,

$$F_{i_m} \circ \dots \circ F_{i_1} \simeq \mathbf{R}\text{Hom}_{\Lambda_{\text{con}}}(\tau_{\geq 1}(\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} \text{Hom}_R(M, N) \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}), -).$$

Proof. We begin by setting notation. For $1 \leq j \leq m$, define $\Gamma^j := \text{End}_R(\nu_{i_j} \dots \nu_{i_1} M)$ and $\Gamma_{\text{con}}^j := \underline{\text{End}}_R(\nu_{i_j} \dots \nu_{i_1} M)$. Further, with T_{i_j} and \mathcal{T}_{i_j} as in Notation 6.3.2, set

$$\mathcal{T}_{\alpha} := \mathcal{T}_{i_m} \otimes_{\Gamma_{\text{con}}^m}^{\mathbf{L}} \mathcal{T}_{i_{m-1}} \otimes_{\Gamma_{\text{con}}^{m-1}}^{\mathbf{L}} \dots \otimes_{\Gamma_{\text{con}}^2}^{\mathbf{L}} \mathcal{T}_{i_2} \otimes_{\Gamma_{\text{con}}^1}^{\mathbf{L}} \mathcal{T}_{i_1} \quad (6.4)$$

and

$$T_{\alpha} := T_{i_m} \otimes_{\Gamma^m}^{\mathbf{L}} T_{i_{m-1}} \otimes_{\Gamma^{m-1}}^{\mathbf{L}} \dots \otimes_{\Gamma^2}^{\mathbf{L}} T_{i_2} \otimes_{\Gamma^1}^{\mathbf{L}} T_{i_1} \quad (6.5)$$

so that $F_{i_1}^{-1} \circ \dots \circ F_{i_m}^{-1} \cong - \otimes_{\Gamma_{\text{con}}}^{\mathbf{L}} \mathcal{T}_{\alpha}$ and $G_{i_1}^{-1} \circ \dots \circ G_{i_m}^{-1} \cong - \otimes_{\Gamma}^{\mathbf{L}} T_{\alpha}$. Then, for each

F_i , there is a commutative diagram (4.3), and combining them shows that

$$G_{i_1}^{-1} \circ \cdots \circ G_{i_m}^{-1} \circ (- \otimes_{\Gamma_{\text{con}}} \Gamma_{\text{con}\Gamma}) \cong (- \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}\Lambda}) \circ F_{i_1}^{-1} \circ \cdots \circ F_{i_m}^{-1}.$$

Or in other words, there is an isomorphism

$$\mathcal{T}_\alpha \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}\Lambda} \cong \Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} T_\alpha \quad (6.6)$$

in the derived category of $\Gamma_{\text{con}}\text{-}\Lambda$ -bimodules.

By Theorem 6.3.4, $T_\alpha \cong \text{Hom}_R(M, N)$ as Γ - Λ -bimodules and hence, by Theorem 3.4.1, T_α is a tilting bimodule which has projective dimension one as a left Γ -module. Thus the right hand side of (6.6) has nonzero homology in at most degrees -1 and 0 . This shows \mathcal{T}_α also has nonzero homology in at most degrees -1 and 0 , as the tensor with Λ_{con} does not change the homology.

Applying $- \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}$ to both sides of (6.6) gives an isomorphism

$$\mathcal{T}_\alpha \otimes_{\Lambda_{\text{con}}} \Lambda_{\text{con}} \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}} \cong \Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} \text{Hom}_R(M, N) \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}$$

in the derived category of $\Gamma_{\text{con}}\text{-}\Lambda_{\text{con}}$ -bimodules. Thus, applying the truncation $\tau_{\geq -1}$ and Lemma 6.4.2 gives

$$\mathcal{T}_\alpha \cong \tau_{\geq -1}(\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} \text{Hom}_R(M, N) \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}})$$

in the derived category of $\Gamma_{\text{con}}\text{-}\Lambda_{\text{con}}$ -bimodules, completing the proof. \square

Corollary 6.4.4. *Suppose that $\text{Spec } R$ is a complete local isolated cDV singularity with MM module $M \in \text{CM } R$ and associated hyperplane arrangement \mathcal{H} . Then there is a well defined functor*

$$\Phi: \mathcal{G}_{\mathcal{H}} \rightarrow \mathbb{F}$$

which sends a chamber C_N to N and an arrow $s_i: N \rightarrow \mathfrak{v}_i N$ to the standard equivalence F_i (as in notation 6.3.2). In particular, for any contraction algebra Λ_{con} , there is a group homomorphism

$$\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}) \rightarrow \text{Auteq}(\text{D}^b(\Lambda_{\text{con}}))$$

where $\mathcal{H}_{\mathbb{C}}$ is the complexification of \mathcal{H} .

Proof. It is enough to show that for any two atoms $\alpha, \beta: C_L \rightarrow C_N$, we have that $\Phi(\alpha) \cong \Phi(\beta)$. Viewing α and β as paths in $X_{\mathcal{H}_L}$ via the isomorphism in the remark after Theorem 6.1.3, applying Theorem 6.4.3 in that setting gives the desired result. \square

In the next section, we will determine that this group homomorphism is injective and in the next chapter, we will combine this result with results from previous chapters to discuss how far this map is from being an isomorphism (see Remark 7.2.2). For now, note that a striking corollary of Corollary 6.4.4 is that the F_i satisfy higher length braid relations.

Corollary 6.4.5. *With notation as above, there is a functorial isomorphism*

$$\underbrace{\cdots F_j \circ F_i \circ F_j}_{m} \cong \underbrace{\cdots F_i \circ F_j \circ F_i}_{m} \quad (6.7)$$

for some m with $2 \leq m \leq 8$.

Proof. By Theorem 6.4.3, the two atoms corresponding to traversing either way round a codimension two wall in \mathcal{H} must be functorially isomorphic, and this establishes (6.7) for some m . The bounds $2 \leq m \leq 8$ follow from the bounds for flops in [DW3, §1.2] or [IW3]. \square

For example, it is known that the hyperplane arrangements appearing for a given cDV singularity are all intersection arrangements of the root system of the underlying Kleinian singularity [Pi],[We, 5.24, 5.25]. In particular, since all intersection arrangements of a type A arrangement are again type A [We, 6.5], this shows that for any cA_n singularity, $m = 3$ in Corollary 6.4.5.

Remark 6.4.6. Since this work was completed, an alternative strategy for producing Corollary 6.4.4 has appeared in [HK]. For any two MM modules M, N in $\text{CM } R$, write $\Lambda := \text{End}_R(M)$, $\Lambda_{\text{con}} := \underline{\text{End}}_R(M)$, $\Gamma := \text{End}_R(N)$ and $\Gamma_{\text{con}} := \underline{\text{End}}_R(N)$. Using DG categories, [HK, 4.23] shows there is a complex of $\Gamma_{\text{con}}\text{-}\Lambda_{\text{con}}$ -bimodules X , unique up to isomorphism, such that

$$\Gamma \Gamma_{\text{con}} \otimes_{\Gamma_{\text{con}}}^{\mathbf{L}} X \cong \text{Hom}_R(M, N) \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}$$

in the derived category of $\Gamma\text{-}\Lambda_{\text{con}}$ bimodules, and hence further making the square

$$\begin{array}{ccc} \text{D}^b(\Lambda_{\text{con}}) & \xleftarrow{-\otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}} & \text{D}^b(\Lambda) \\ -\otimes_{\Gamma_{\text{con}}}^{\mathbf{L}} X \uparrow & & \uparrow -\otimes_{\Gamma}^{\mathbf{L}} \text{Hom}_R(M, N) \\ \text{D}^b(\Gamma_{\text{con}}) & \xleftarrow{-\otimes_{\Gamma}^{\mathbf{L}} \Gamma_{\text{con}}} & \text{D}^b(\Gamma) \end{array}$$

commute. This result shows there exists a functor from the groupoid of flop functors between the minimal models of $\text{Spec } R$ to the groupoid \mathbb{F} and composing this with the functor in [DW3, 3.22] recovers Corollary 6.4.4.

Another corollary of Theorem 6.4.3 is the following analogue of Theorem 3.4.1(2) in the case of contraction algebras; namely, it provides a direct standard equivalence between any two contraction algebras of minimal models of $\text{Spec } R$.

Corollary 6.4.7. *Let $\text{Spec } R$ be a complete local isolated cDV singularity and suppose that $M, N \in \text{CM } R$ are two MM modules. Writing $\Lambda := \text{End}_R(M)$, $\Lambda_{\text{con}} := \underline{\text{End}}_R(M)$, $\Gamma := \text{End}_R(N)$ and $\Gamma_{\text{con}} := \underline{\text{End}}_R(N)$ there is a standard derived equivalence*

$$\mathbf{R}\text{Hom}_{\Lambda_{\text{con}}}(\tau_{\geq 1}(\Gamma_{\text{con}} \otimes_{\Gamma}^{\mathbf{L}} \text{Hom}_R(M, N) \otimes_{\Lambda}^{\mathbf{L}} \Lambda_{\text{con}}), -): \text{D}^b(\Lambda_{\text{con}}) \rightarrow \text{D}^b(\Gamma_{\text{con}}).$$

Proof. Given two MM modules $M, N \in \text{CM } R$, there must be an atom between their corresponding chambers in the hyperplane arrangement \mathcal{H}_M . Applying Theorem 6.4.3 to this atom gives the required functor. \square

6.5 Faithfulness

In the previous section, we showed that the derived equivalences F_i between contraction algebras satisfy the relations of the Deligne Groupoid of the associated hyperplane

arrangement. In this section, we show that these are the only relations they satisfy, or in other words, the functor from Corollary 6.4.4 is faithful.

This will be achieved by adapting the strategy used in [HW, §6]. First, as in [HW, 2.11], this problem can be immediately reduced to checking the functor is faithful on positive paths. Namely, if α, β are positive paths with $\Phi(\alpha) \cong \Phi(\beta)$, then we need to show $\alpha \sim \beta$. For this, we need an effective way of telling when two positive paths are equivalent, for which we will use the Deligne normal form.

6.5.1 Deligne Normal Form

As with the Deligne groupoid, our description of the Deligne normal form will follow [HW]. Take a hyperplane arrangement \mathcal{H} and its oriented skeleton graph $X_{\mathcal{H}}$. For positive paths p, q in $X_{\mathcal{H}}$ with $s(p) = s(q)$, we say p begins with q if there exists a positive path r such that $s(r) = t(q)$, $t(r) = t(p)$ and $p \sim rq$. For a positive path p let $\text{Begin}(p)$ denote the set of all atoms with which p begins.

Lemma 6.5.1. [P2, 2.2] *For each positive path p in $X_{\mathcal{H}}$, there exists a unique (up to the relations) atom α such that $\text{Begin}(p) = \text{Begin}(\alpha)$.*

Definition 6.5.2. *Take p to be any positive path in $X_{\mathcal{H}}$ and let α_1 be the unique atom such that $\text{Begin}(p) = \text{Begin}(\alpha_1)$. Then p begins with α_1 and so there exists a positive β such that*

$$p \sim \beta\alpha_1.$$

Continuing this process with β , we decompose p as

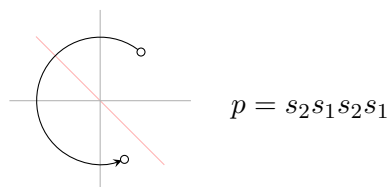
$$p \sim \alpha_n \dots \alpha_1.$$

which we refer to as the Deligne normal form of p .

If p and q are positive paths with Deligne normal forms $\alpha_n \dots \alpha_1$ and $\beta_m \dots \beta_1$ respectively, it is clear that $p \sim q$ if and only if $n = m$ and $\alpha_i \sim \beta_i$ for each i . The following well-known lemma is useful for determining the Deligne normal form of a given path.

Lemma 6.5.3. [HW, 5.1] *If $p \sim \alpha_n \dots \alpha_1$ is in Deligne normal form then for each $k \in \{2, \dots, n\}$, α_k must start (up to relations) by crossing the wall that α_{k-1} crosses through last.*

Example 6.5.4. Continuing Example 6.3.1, the positive path



starting in chamber C_M has Deligne normal form $(s_2)(s_1s_2s_1)$. Although $s_1s_2s_1$ does not appear to end with s_2 , under the relations it is equivalent to $s_2s_1s_2$ which clearly does.

6.5.2 Tracking Simple

The key idea of this section is that it is possible to compute the Deligne normal form of a positive path α by tracking where the functor $\Phi(\alpha)$ sends simple modules.

Notation 6.5.5. Under the setup of 6.1.1, we will abuse notation by referring to the simple modules of any contraction algebra $\Lambda_{\text{con}} := \underline{\text{End}}_R(N)$ as S_1, \dots, S_n , where the projective cover of S_i is $\underline{\text{Hom}}_R(N, N_i)$. Further, let $\mathcal{S} := \bigoplus_{i=1}^n S_i$. Note that each S_i is also a simple module when considered as a module over $\Lambda := \text{End}_R(N)$ and if we wish to view S_i in that way, we will write it as $(S_i)_\Lambda$.

The following technical lemma will be used repeatedly.

Lemma 6.5.6. *Under setup 6.1.1, let $N \in \text{CM } R$ be a basic MM module and write $\Lambda := \text{End}_R(N)$ and $\Lambda_{\text{con}} := \underline{\text{End}}_R(N)$. Suppose that X is a Λ -module, and $Y \in \text{D}^b(\Lambda_{\text{con}})$ is such that*

$$X[n] \cong Y \otimes_{\Lambda_{\text{con}}} (\Lambda_{\text{con}})_\Lambda$$

in $\text{D}^b(\Lambda)$. Then, X is a Λ_{con} -module and $X \cong Y[-n]$ in $\text{D}^b(\Lambda_{\text{con}})$.

Proof. Since $-\otimes_{\Lambda_{\text{con}}} (\Lambda_{\text{con}})_\Lambda$ is an exact functor, it preserves homology and hence there is an isomorphism of Λ -modules

$$\phi_i: H^i(X[n]) \rightarrow H^i(Y)$$

for all $i \in \mathbb{Z}$. When $i = -n$ this gives an isomorphism

$$\phi: X \rightarrow H^{-n}(Y).$$

But as $H^{-n}(Y)$ is a Λ_{con} -module, it is annihilated by I , where $\Lambda_{\text{con}} \cong \Lambda/I$. Hence, if $f \in I$, and $x \in X$, then

$$\phi(x \cdot f) = \phi(x) \cdot f = 0$$

and hence, as ϕ is an isomorphism, $x \cdot f = 0$. Thus, X is annihilated by I , and so X is a Λ_{con} -module. If $i \neq -n$, then the above shows that the homology of Y in degree i is zero and hence

$$Y[-n] \cong H^{-n}(Y) \cong X$$

in $\text{D}^b(\Lambda_{\text{con}})$ as required. \square

We now introduce some notation for the functors associated to a path α in the hyperplane arrangement, similar to §5.3.

Notation 6.5.7. Recall that associated to any arrow $s_i: N \rightarrow v_i N$ in $X_{\mathcal{H}_M}$, there are equivalences F_i and G_i as in notation 6.3.2. For a positive path $\alpha := s_{i_m} \dots s_{i_1}$, let

1. $F_\alpha := \Phi(\alpha) = \mathbf{R}\text{Hom}_{\Lambda_{\text{con}}}(\mathcal{T}_\alpha, -)$ where \mathcal{T}_α is defined as in (6.4);
2. G_α be the composition of the corresponding functors G_i , and T_α will be the tilting bimodule inducing G_α , defined as in (6.5).

The following known result tracks simple modules through the G_α .

Lemma 6.5.8. [HW, §5] *Under the setup of 6.1.1, let $\alpha: C_L \rightarrow C_N$ be an atom in $X_{\mathcal{H}_M}$. Writing $\Lambda := \text{End}_R(L)$ and $\Gamma := \text{End}_R(N)$, the following statements hold.*

1. $G_\alpha^{-1}((S_i)_\Gamma) \cong \begin{cases} \text{Tor}_1^\Gamma(S_i, T_\alpha)[1] & \text{if } \alpha \text{ ends (up to relations) with } s_i; \\ S_i \otimes_\Gamma T_\alpha & \text{otherwise.} \end{cases}$
2. *If α ends (up to relations) with s_i , then there exists $j \in \{1, \dots, n\}$ such that $S_j \hookrightarrow \text{Tor}_1^\Gamma(S_i, T_\alpha)$ and α starts (up to relations) with s_j .*

The commutative diagram (6.2) allows us to prove the corresponding result for contraction algebras.

Corollary 6.5.9. *Under the setup of 6.1.1, let $\alpha: C_L \rightarrow C_N$ be an atom in $X_{\mathcal{H}_M}$. Writing $\Lambda := \text{End}_R(L)$ and $\Gamma := \text{End}_R(N)$, then*

$$F_\alpha^{-1}(S_i) \cong \begin{cases} \text{Tor}_1^\Gamma(S_i, T_\alpha)[1] & \text{if } \alpha \text{ ends (up to relations) with } s_i; \\ S_i \otimes_\Gamma T_\alpha & \text{otherwise.} \end{cases}$$

Proof. The commutative diagram (6.2) shows that

$$F_\alpha^{-1}(S_i) \otimes_{\Lambda_{\text{con}}} (\Lambda_{\text{con}})_\Lambda \cong G_\alpha^{-1}((S_i)_\Gamma).$$

By Lemma 6.5.8, $G_\alpha^{-1}((S_i)_\Gamma)$ is a complex concentrated in either degree -1 or degree 0 . Applying Lemma 6.5.6 gives the result. \square

Note that this result shows that both $\text{Tor}_1^\Gamma(S_i, T_\alpha)$ and $S_i \otimes_\Gamma T_\alpha$ are Λ_{con} -modules which is not immediately obvious. The following is a technical lemma required for the main results of this chapter. To state it, recall that for a contraction algebra Λ_{con} , we denote by \mathcal{S} the direct sum of all the simple Λ_{con} -modules.

Lemma 6.5.10. *Under the setup of 6.1.1, let $N \in \text{CM } R$ be a basic MM module and let $\Lambda_{\text{con}} := \underline{\text{End}}_R(N)$. Suppose that $X \in \text{mod } \Lambda_{\text{con}}$ is nonzero.*

1. $\text{Ext}_{\Lambda_{\text{con}}}^{\geq p}(\mathcal{S}, X) = 0$ if and only if $\text{Ext}_{\Lambda_{\text{con}}}^{\geq p}(-, X) = 0$.
2. $\text{Ext}_{\Lambda_{\text{con}}}^i(-, \Lambda_{\text{con}}) = 0$ if $i \geq 1$.
3. $\text{Hom}_{\Lambda_{\text{con}}}(X, \Lambda_{\text{con}}) \neq 0$.

Proof. 1. Choose $Y \in \text{mod } \Lambda_{\text{con}}$. We need to show that $\text{Ext}_{\Lambda_{\text{con}}}^{\geq p}(Y, X) = 0$. Filtering Y by simple modules, an easy induction on the length of Y establishes the result.

2. Λ_{con} is a symmetric algebra by Proposition 3.2.12, and is therefore self-injective.

3. Since Λ_{con} is a symmetric algebra there is an isomorphism

$$\text{Hom}_{\Lambda_{\text{con}}}(X, \Lambda_{\text{con}}) \cong \text{Hom}_k(X, k)$$

for all $X \in \text{mod } \Lambda_{\text{con}}$ (see e.g. [Br, 2.7]). As $\text{Hom}_k(-, k): \text{mod } \Lambda_{\text{con}} \rightarrow \text{mod } \Lambda_{\text{con}}^{\text{op}}$ is a duality, the statement follows. \square

The following is the main technical result of this section and it mirrors [BT, 3.1] and [HW, 6.3].

Proposition 6.5.11. *Under the setup of 6.1.1, let $\alpha: C_L \rightarrow C_N$ be a positive path in $X_{\mathcal{H}_M}$ with Deligne normal form $\alpha = \alpha_k \dots \alpha_1$. Writing $\Lambda_{\text{con}} := \underline{\text{End}}_R(L)$ and $\Gamma_{\text{con}} := \underline{\text{End}}_R(N)$, then the following statements hold.*

1. $\text{Ext}_{\Gamma_{\text{con}}}^{\geq k+1}(-, F_{\alpha}(\Lambda_{\text{con}})) = 0$.
2. $\text{Ext}_{\Gamma_{\text{con}}}^k(S_i, F_{\alpha}(\Lambda_{\text{con}})) \neq 0$ if and only if α_k ends (up to relations) with s_i .
3. $\max\{p \mid \text{Ext}_{\Gamma_{\text{con}}}^p(S, F_{\alpha}(\Lambda_{\text{con}})) \neq 0\} = k$.

Proof. Part (3) is clearly a consequence of the first two parts. We prove parts (1) and (2) together using induction on k .

Base case: $k = 1$. There are two cases to consider, namely if α ends (up to relations) with s_i or not. From now on, for ease of reading, we will omit the statement ‘up to relations’.

1. If α does not end with s_i , then as $k = 1$, α is an atom and so by Corollary 6.5.9,

$$F_{\alpha}^{-1}(S_i) \cong X$$

for some Λ_{con} -module X . Then,

$$\begin{aligned} \text{Ext}_{\Gamma_{\text{con}}}^{\geq 1}(S_i, F_{\alpha}(\Lambda_{\text{con}})) &\cong \text{Ext}_{\Lambda_{\text{con}}}^{\geq 1}(F_{\alpha}^{-1}(S_i), \Lambda_{\text{con}}) \\ &\cong \text{Ext}_{\Lambda_{\text{con}}}^{\geq 1}(X, \Lambda_{\text{con}}) = 0 \end{aligned}$$

where the last equality holds by Lemma 6.5.10(2).

2. If α ends with s_i , then by Corollary 6.5.9,

$$F_{\alpha}^{-1}(S_i) \cong Y[1]$$

for some Λ_{con} -module Y . Then,

$$\begin{aligned} \text{Ext}_{\Gamma_{\text{con}}}^1(S_i, F_{\alpha}(\Lambda_{\text{con}})) &\cong \text{Ext}_{\Lambda_{\text{con}}}^1(F_{\alpha}^{-1}(S_i), \Lambda_{\text{con}}) \\ &\cong \text{Ext}_{\Lambda_{\text{con}}}^1(Y[1], \Lambda_{\text{con}}) \\ &\cong \text{Hom}_{\Lambda_{\text{con}}}(Y, \Lambda_{\text{con}}) \neq 0 \end{aligned}$$

where the last part is Lemma 6.5.10(3). A similar calculation gives

$$\text{Ext}_{\Gamma_{\text{con}}}^{\geq 2}(S_i, F_{\alpha}(\Lambda_{\text{con}})) \cong \text{Ext}_{\Lambda_{\text{con}}}^{\geq 1}(Y, \Lambda_{\text{con}}) = 0.$$

Combining the two cases establishes part (2) of the result when $k = 1$. Furthermore, we have that $\text{Ext}_{\Gamma_{\text{con}}}^{\geq 2}(S, F_{\alpha}(\Lambda_{\text{con}})) = 0$. Applying Lemma 6.5.10(1) then gives $\text{Ext}_{\Gamma_{\text{con}}}^{\geq 2}(-, F_{\alpha}(\Lambda_{\text{con}})) = 0$, also establishing part (1) for $k = 1$.

Inductive Step. We now assume that the result is true for all paths with fewer than k Deligne factors. Write $\alpha = \alpha_k \beta$ where $\beta := \alpha_{k-1} \dots \alpha_1$. Further, write Δ_{con} for the contraction algebra associated to the chamber at the end of β . By the inductive hypothesis,

$$\text{Ext}_{\Delta_{\text{con}}}^{\geq k}(-, F_{\beta}(\Lambda_{\text{con}})) = 0 \tag{6.8}$$

and further, $\text{Ext}_{\Delta_{\text{con}}}^{k-1}(S_i, F_{\beta}(\Lambda_{\text{con}})) \neq 0$ if and only if β ends with s_i . Again, we consider two cases:

1. If α_k does not end with s_i , then by Corollary 6.5.9,

$$F_{\alpha_k}^{-1}(S_i) \cong X$$

for some Δ_{con} -module X . Then,

$$\begin{aligned} \text{Ext}_{\Gamma_{\text{con}}}^{\geq k}(S_i, F_\alpha(\Lambda_{\text{con}})) &\cong \text{Ext}_{\Delta_{\text{con}}}^{\geq k}(F_{\alpha_k}^{-1}(S_i), F_\beta(\Lambda_{\text{con}})) \\ &\cong \text{Ext}_{\Delta_{\text{con}}}^{\geq k}(X, F_\beta(\Lambda_{\text{con}})) \\ &= 0 \end{aligned}$$

where the last equality holds by the inductive hypothesis (6.8).

2. If α_k ends with s_i , then by Corollary 6.5.9,

$$F_{\alpha_k}^{-1}(S_i) \cong Y[1]$$

where $Y := \text{Tor}_1^\Gamma(S_i, T_\alpha)$ is a Δ_{con} -module. Then

$$\text{Ext}_{\Gamma_{\text{con}}}^{\geq k+1}(S_i, F_\alpha(\Lambda_{\text{con}})) \cong \text{Ext}_{\Delta_{\text{con}}}^{\geq k}(Y, F_\beta(\Lambda_{\text{con}})) = 0,$$

again using the inductive hypothesis (6.8). Further,

$$\text{Ext}_{\Gamma_{\text{con}}}^k(S_i, F_\alpha(\Lambda_{\text{con}})) \cong \text{Ext}_{\Delta_{\text{con}}}^{k-1}(Y, F_\beta(\Lambda_{\text{con}}))$$

and hence it suffices to show that $\text{Ext}_{\Delta_{\text{con}}}^{k-1}(Y, F_\beta(\Lambda_{\text{con}})) \neq 0$. As $Y := \text{Tor}_1^\Gamma(S_i, T_\alpha)$, Lemma 6.5.8(2) shows there exists a simple module S_j of Δ_{con} such that $S_j \hookrightarrow Y$ and α_k starts with s_j . Applying $\text{Hom}_{\Delta_{\text{con}}}(-, F_\beta(\Lambda_{\text{con}}))$ to the short exact sequence

$$0 \rightarrow S_j \rightarrow Y \rightarrow Y/S_j \rightarrow 0,$$

gives a long exact sequence

$$\cdots \rightarrow \text{Ext}_{\Delta_{\text{con}}}^{k-1}(Y, F_\beta(\Lambda_{\text{con}})) \rightarrow \text{Ext}_{\Delta_{\text{con}}}^{k-1}(S_j, F_\beta(\Lambda_{\text{con}})) \rightarrow \text{Ext}_{\Delta_{\text{con}}}^k(Y/S_j, F_\beta(\Lambda_{\text{con}})) \rightarrow \cdots$$

where the last term is zero by the inductive hypothesis (6.8). Since α_k starts with s_j , Lemma 6.5.3 shows β must also end with s_j , otherwise α would not be in Deligne normal form. Thus, by the inductive hypothesis,

$$\text{Ext}_{\Delta_{\text{con}}}^{k-1}(S_j, F_\beta(\Lambda_{\text{con}})) \neq 0$$

and hence by the exact sequence, $\text{Ext}_{\Delta_{\text{con}}}^{k-1}(Y, F_\beta(\Lambda_{\text{con}})) \neq 0$, completing the proof. \square

Corollary 6.5.12. *The functor Φ from Corollary 6.4.4 is a faithful functor.*

Proof. Using Proposition 6.5.11, this now follows exactly as in [HW, 6.5] or [BT, 3.1]. Indeed, by [HW, 2.11], the problem is reduced to checking the functor is faithful on positive paths; namely, if α and β are positive paths starting and ending in the same chambers in $X_{\mathcal{H}_M}$ such that $F_\alpha \cong F_\beta$ then we need to show $\alpha \sim \beta$.

If $F_\alpha \cong F_\beta$ then Proposition 6.5.11 shows that the number of atoms in the Deligne normal form of both α and β must be the same. So we write the Deligne normal forms of α and β as

$$\alpha = \alpha_k \dots \alpha_1 \quad \text{and} \quad \beta = \beta_k \dots \beta_1.$$

By induction it will be enough to prove $\alpha_k \sim \beta_k$ and $F_{\alpha_{k-1} \dots \alpha_1} \cong F_{\beta_{k-1} \dots \beta_1}$. By symmetry, we can assume the length l of α_k is less than or equal to the length of β_k . Now, Proposition 6.5.11 shows that α and β must end with the same wall crossing.

We will denote this by s_{i_1} so that $\alpha_k \sim s_{i_1} \tilde{\alpha}_k$ and $\beta_k \sim s_{i_1} \tilde{\beta}_k$. Applying F_i^{-1} to $F_\alpha \cong F_\beta$ shows $F_{\tilde{\alpha}_k \alpha_{k-1} \dots \alpha_1} \cong F_{\tilde{\beta}_k \beta_{k-1} \dots \beta_1}$. Repeating the above argument shows we can write $\alpha_k \sim s_{i_1} \dots s_{i_l}$ and $\beta_k \sim s_{i_1} \dots s_{i_l} \gamma$ for some positive path γ . This also gives $F_{\alpha_{k-1} \dots \alpha_1} \cong F_{\gamma \beta_{k-1} \dots \beta_1}$ so all that remains to show is that γ is a length zero path. Note that $\gamma \beta_{k-1} \dots \beta_1$ is in Deligne normal form (as β was) and so if γ is not length zero, $\gamma \beta_{k-1} \dots \beta_1$ has k factors in the Deligne normal form, contradicting Proposition 6.5.11 applied to $F_{\alpha_{k-1} \dots \alpha_1} \cong F_{\gamma \beta_{k-1} \dots \beta_1}$. \square

The following is an immediate consequence of Corollary 6.5.12, and is the main result of this section.

Corollary 6.5.13. *Suppose that $f: X \rightarrow \text{Spec } R$ is a minimal model of a complete local isolated cDV singularity with associated hyperplane arrangement \mathcal{H} in \mathbb{R}^n and contraction algebra Λ_{con} . Then there is an injective group homomorphism*

$$\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}) \rightarrow \text{Auteq}(\text{D}^b(\Lambda_{\text{con}}))$$

where $\mathcal{H}_{\mathbb{C}}$ is the complexification of \mathcal{H} .

Chapter 7

The Complete Picture

This final short chapter combines the results from the previous chapters to show that for a contraction algebra, the two-term tilting theory controls the entire derived equivalence class and further, the associated hyperplane arrangement can be thought of as a complete picture of this derived equivalence class.

7.1 Two-Term Tilting Complexes

As in the previous chapter, our main setting will be Setup 6.1.1: let $\text{Spec } R$ be a complete local isolated cDV singularity and choose a basic MM module $M := \bigoplus_{i=0}^n M_i \in \text{CM } R$ where $M_0 \cong R$.

Recall from §6.1 that associated to such an M , there is a hyperplane arrangement \mathcal{H}_M with oriented skeleton graph X_M . The chambers of \mathcal{H}_M are labelled by the basic MM R -modules in such a way that crossing a wall corresponds to mutation and, as in §5.3, we use the following notation.

Notation 7.1.1. Under the setup of 6.1.1, choose a positive path $\alpha := s_{i_m} \dots s_{i_1}$ in X_M starting in chamber C_N . Let $\nu_\alpha N := \nu_{i_m} \dots \nu_{i_1} N$ and write $\Lambda_{\text{con}} := \underline{\text{End}}_R(N)$ and $\Gamma_{\text{con}} := \underline{\text{End}}_R(\nu_\alpha N)$. We write

1. $F_\alpha := F_{i_m} \circ \dots \circ F_{i_1} = \mathbf{R}\text{Hom}_{\Lambda_{\text{con}}}(\mathcal{T}_\alpha, -): \text{D}^b(\Lambda_{\text{con}}) \rightarrow \text{D}^b(\Gamma_{\text{con}})$, where \mathcal{T}_α is defined as in (6.4).
2. $\mu_\alpha \Lambda_{\text{con}} := \mu_{i_m} \dots \mu_{i_1} \Lambda_{\text{con}}$ which by Proposition 5.3.7 satisfies $\mathcal{T}_\alpha \cong \mu_\alpha \Lambda_{\text{con}}$.

In other words, associated to each positive path α starting at a chamber C_N , there is a derived equivalence F_α which is the unique (up to algebra automorphism) standard derived equivalence induced by the tilting complex $\mu_\alpha \underline{\text{End}}_R(N)$. This section determines precisely which paths are induced by two-term tilting complexes. Our main tool is the partial order on tilting complexes described in §2.2.2 and, in particular, Theorem 2.2.7, which shows that the partial order controls mutation. Moreover, recall from §2.2.4 that two-term tilting complexes for a finite dimensional algebra A can be characterised as precisely those tilting complexes T which satisfy

$$A \geq T \geq A[1].$$

Lemma 7.1.2. *Under the setup of 6.1.1, choose a chamber C_N and let $\Lambda_{\text{con}} := \underline{\text{End}}_R(N)$. If $T := \mu_{i_m} \dots \mu_{i_1} \Lambda_{\text{con}}$ is a two-term tilting complex for Λ_{con} , then the path $\alpha := s_{i_m} \dots s_{i_1}$ starting in chamber C_N must be an atom.*

Proof. We prove this by induction on m . When $m = 1$ this is clear as any path of length one must be an atom.

Now assume $m \geq 2$ and let $\beta := s_{i_{m-1}} \dots s_{i_1}$. By Theorem 2.2.7, we have

$$\Lambda_{\text{con}} > \mu_{\beta} \Lambda_{\text{con}} > T$$

and hence, as T is two-term,

$$\Lambda_{\text{con}} > \mu_{\beta} \Lambda_{\text{con}} > T \geq \Lambda_{\text{con}}[1]$$

so that $\mu_{\beta} \Lambda_{\text{con}}$ is also two-term. Thus, by the inductive hypothesis, β is an atom. Let us suppose that α is not. By [HW, 5.1], β must end (up to relations) with s_{i_m} , and hence there exists a positive path γ such that $\beta \sim s_{i_m} \gamma$. By the relations on paths in the Deligne groupoid, this implies that $\alpha \sim s_{i_m} s_{i_m} \gamma$. Now, as the assignment $\alpha \mapsto F_{\alpha}$ was shown in Corollary 6.4.4 to give a functor $\mathcal{G}_{\mathcal{H}} \rightarrow \mathbb{F}$, we must have $F_{\alpha} \cong F_{s_{i_m} s_{i_m} \gamma}$ and hence

$$T = \mu_{\alpha} \Lambda_{\text{con}} \cong \mathcal{T}_{\alpha} \cong \mathcal{T}_{s_{i_m} s_{i_m} \gamma} \cong \mu_{s_{i_m} s_{i_m} \gamma} \Lambda_{\text{con}}.$$

Using Theorem 2.2.7 and the fact that T is two-term,

$$\Lambda_{\text{con}} > \mu_{\gamma} \Lambda_{\text{con}} > \mu_{s_{i_m} s_{i_m} \gamma} \Lambda_{\text{con}} > T \geq \Lambda_{\text{con}}[1] \tag{7.1}$$

and hence $\mu_{\gamma} \Lambda_{\text{con}}$ is a two-term complex. But $\mu_{\gamma} \Lambda_{\text{con}}$ and T differ by at most one summand as T is obtained by mutating at the same summand twice and thus, by Proposition 2.2.14, T and $\mu_{\gamma} \Lambda_{\text{con}}$ are either isomorphic or related by a single mutation. However, combining (7.1) with Theorem 2.2.7 gives a contradiction in both cases, and thus α must be an atom. \square

Theorem 7.1.3. *Under the setup of 6.1.1, choose a basic maximal rigid object $N \in \text{CM } R$, and let $\Lambda_{\text{con}} := \underline{\text{End}}_R(N)$. Then there is a bijection*

$$\{\text{atoms starting in } C_N\} \longrightarrow \text{2-tilt } \Lambda_{\text{con}},$$

sending an atom α to the tilting complex $\mu_{\alpha} \Lambda_{\text{con}}$.

Proof. We first need to check this map is well defined, namely:

1. If $\alpha \sim \beta$, then $\mu_{\alpha} \Lambda_{\text{con}} \cong \mu_{\beta} \Lambda_{\text{con}}$;
2. For any atom α , $\mu_{\alpha} \Lambda_{\text{con}}$ is a two-term tilting complex.

The first statement follows easily from Corollary 6.4.4, where the assignment $\alpha \rightarrow F_{\alpha}$ is shown to yield a functor from the Deligne Groupoid to the groupoid \mathbb{F} . Indeed, if $\alpha \sim \beta$, then $F_{\alpha} \cong F_{\beta}$ and hence,

$$\mu_{\alpha} \Lambda_{\text{con}} \cong \mathcal{T}_{\alpha} \cong \mathcal{T}_{\beta} \cong \mu_{\beta} \Lambda_{\text{con}}.$$

For the second part, since α is a positive path, $\Lambda_{\text{con}} \geq \mu_{\alpha} \Lambda_{\text{con}}$ follows from Theorem 2.2.7. Moreover, as α is an atom, Theorem 6.4.3 shows \mathcal{T}_{α} is the truncation $\tau_{\geq -1}$ of a non-positive complex and thus is zero outside degrees 0 and -1 . In particular, \mathcal{T}_{α} has zero homology outside these degrees. Using [A, 2.7] and that Λ_{con} is symmetric, there is an isomorphism

$$\text{Hom}_{\text{K}^b(\text{proj } \Lambda_{\text{con}})}(\mu_{\alpha} \Lambda_{\text{con}}, \Lambda_{\text{con}}[i+1]) \cong D \text{Hom}_{\text{K}^b(\text{proj } \Lambda_{\text{con}})}(\Lambda_{\text{con}}[i+1], \mu_{\alpha} \Lambda_{\text{con}})$$

which, combining with $\mu_\alpha \Lambda_{\text{con}} \cong \mathcal{T}_\alpha$, leads to isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbb{K}^{\text{b}}(\text{proj } \Lambda_{\text{con}})}(\mu_\alpha \Lambda_{\text{con}}, \Lambda_{\text{con}}[i+1]) &\cong D \text{Hom}_{\mathbb{D}^{\text{b}}(\Lambda_{\text{con}})}(\Lambda_{\text{con}}[i+1], \mathcal{T}_\alpha) \\ &= DH^{-(i+1)}(\mathcal{T}_\alpha) = 0 \text{ if } i \neq 0, -1. \end{aligned}$$

This shows that $\mu_\alpha \Lambda_{\text{con}} \geq \Lambda_{\text{con}}[1]$. Thus we have $\Lambda_{\text{con}} \geq \mu_\alpha \Lambda_{\text{con}} \geq \Lambda_{\text{con}}[1]$, and hence $\mu_\alpha \Lambda_{\text{con}}$ is a two-term tilting complex, as required.

Next we show the map is bijective. By Theorem 6.1.4, the chambers of the hyperplane arrangement are labelled by the two-term tilting complexes of Λ_{con} . Further it is clear that atoms starting in a given chamber are also in bijection with the chambers. Hence,

$$\#\{\text{atoms starting in } C_N\} = \# \text{2-tilt } \Lambda_{\text{con}}$$

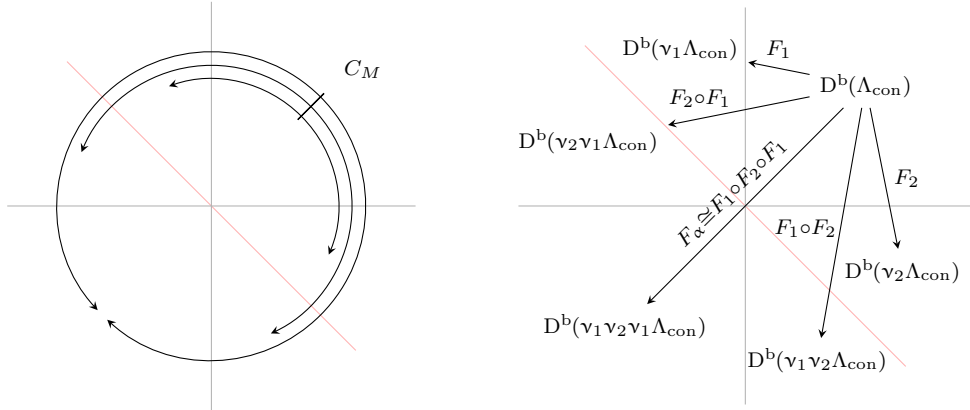
and so it is enough to show the given map is surjective.

As there are finitely many two-term complexes for Λ_{con} , Theorem 2.2.8 shows that any two-term complex can be obtained from Λ_{con} by iterated left mutation. So given any $P \in \text{2-tilt } \Lambda_{\text{con}}$, $P \cong \mu_\alpha \Lambda_{\text{con}}$ for some positive path α starting in C_N . Lemma 7.1.2 shows that α must be an atom. In particular, α maps to P under the given map, and so it is surjective. \square

Corollary 7.1.4. *In the set up of 6.1.1, choose a basic maximal rigid object N , and let $\Lambda_{\text{con}} := \underline{\text{End}}_R(N)$. Then standard equivalences from $\mathbb{D}^{\text{b}}(\Lambda_{\text{con}})$ induced by two-term tilting complexes of Λ_{con} are precisely (up to algebra automorphism) the F_α given by atoms α in $X_{\mathcal{H}_M}$ starting in chamber C_N .*

Proof. The bijection in Theorem 7.1.3 shows that the atoms starting in chamber C_N correspond precisely to the two-term tilting complexes for Λ_{con} . Given such an atom α , combining Proposition 5.1.3 and Proposition 2.2.4 then shows that F_α is the unique (up to algebra automorphism) standard equivalence induced by $\mu_\alpha \Lambda_{\text{con}}$. \square

Example 7.1.5. Consider the cA_2 singularity given by the polynomial $uv - xy(x+y^2)$. Given an MM module $M \in \text{CM } R$, we know the associated hyperplane arrangement from Example 6.1.6 and the left hand diagram below shows all the atoms starting in chamber C_M . Note that the two atoms going to the opposite chamber are identified in $\mathcal{G}_{\mathcal{H}}$. Writing $\nu_{i_m} \dots \nu_{i_1} \Lambda_{\text{con}} := \underline{\text{End}}_R(\nu_{i_m} \dots \nu_{i_1} M)$, Corollary 7.1.4 shows that the functors on the right hand side are all induced by two-term tilting complexes and further, they are the only standard equivalences from $\mathbb{D}^{\text{b}}(\Lambda_{\text{con}})$ (up to algebra automorphism) which are induced by two-term tilting complexes.



This result is as expected; the two-term tilting complexes can certainly be thought of as the simplest tilting complexes for Λ_{con} and Corollary 7.1.4 shows that they correspond precisely to the simplest paths, namely the atoms. However, the following shows that we can view the results of this section as a two-sided improvement of the bijection of Theorem 2.3.5 between maximal rigid objects in $\underline{\text{CM}} R$ and two-term tilting complexes of a contraction algebra.

Corollary 7.1.6. *In the set up of 6.1.1, choose a basic maximal rigid object N , and let $\Lambda_{\text{con}} := \underline{\text{End}}_R(N)$. Given an atom $\alpha: N \rightarrow \nu_\alpha N$, the bimodule complex \mathcal{T}_α is isomorphic in $\text{D}^b(\Lambda_{\text{con}})$ to the two-term tilting complex P of Λ_{con} associated to $\nu_\alpha N$ via Theorem 2.3.5. Then,*

$$\text{End}_{\Lambda_{\text{con}}}(P) \cong \underline{\text{End}}_R(\nu_\alpha N)$$

so that, in this case, the bijection of Theorem 2.3.5 preserves endomorphism rings.

Proof. By Proposition 5.1.3, \mathcal{T}_α is isomorphic in $\text{D}^b(\Lambda_{\text{con}})$ to $\mu_\alpha \Lambda_{\text{con}}$, which is two-term by Theorem 7.1.3. As the bijection of Theorem 2.3.5 preserves mutation, this shows that $\mu_\alpha \Lambda_{\text{con}}$ is precisely the two-term tilting complex associated to $\nu_\alpha N$, completing the proof of the first statement. Now, by Proposition 5.1.3, $F_\alpha(\mu_\alpha \Lambda_{\text{con}}) \cong \underline{\text{End}}_R(\nu_\alpha N)$ and so using that F_α is an equivalence gives the second statement. \square

7.2 Visualising the Derived Equivalence Class

Given a contraction algebra Λ_{con} of a complete local isolated cDV singularity $\text{Spec } R$, let \mathcal{H} be the hyperplane arrangement determined by two-term tilting complexes. By Theorem 6.1.4, this matches the hyperplane arrangement \mathcal{H}_M for some MM module $M \in \text{CM } R$ such that $\Lambda_{\text{con}} \cong \underline{\text{End}}_R(M)$. In particular, the chambers of \mathcal{H} naturally correspond to the MM modules in $\text{CM } R$ and paths in the oriented skeleton graph correspond to the derived equivalences F_α (compositions of the F_i constructed in Chapter 4). We now combine the results of this thesis into the following summary theorem. For clarity, note that if an arrow s_i is assigned the functor F_i , the path corresponding to travelling along this arrow backwards, namely s_i^{-1} , is assigned the functor F_i^{-1} .

Theorem 7.2.1. *Given a contraction algebra Λ_{con} of a complete local isolated cDV singularity $\text{Spec } R$, let \mathcal{H} be the hyperplane arrangement determined by two-term tilting complexes. Choosing an MM module $M \in \text{CM } R$ such that $\Lambda_{\text{con}} \cong \underline{\text{End}}_R(M)$, then the following hold.*

1. *The only basic algebras in the derived equivalence class of Λ_{con} are the contraction algebras of $\text{Spec } R$; or equivalently, the endomorphism algebras of two-term tilting complexes of Λ_{con} . In particular, there are finitely many such algebras.*
2. *Any standard derived equivalence, up to algebra automorphism, from Λ_{con} is obtained as F_α for a not-necessarily-positive path α in $X_{\mathcal{H}}$ starting at C_M . In particular, all standard derived equivalences are the composition of two-term tilts and their inverses.*
3. *The standard autoequivalences of $\text{D}^b(\Lambda_{\text{con}})$ are determined precisely, up to algebra automorphism, by paths $C_M \rightarrow C_N$ in the mutation graph, where N satisfies $\underline{\text{End}}_R(N) \cong \Lambda_{\text{con}}$.*

4. If $\alpha, \beta: C_M \rightarrow C_N$ are two paths, then $F_\alpha \cong F_\beta$ if and only if $\alpha \sim \beta$ in the Deligne Groupoid. In other words, the F_i satisfy precisely the Deligne Groupoid relations.
5. There is an injective group homomorphism $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}) \rightarrow \text{Auteq}(\text{D}^b(\Lambda_{\text{con}}))$.
6. The atoms starting at C_M determine precisely, up to algebra automorphism, the standard equivalences induced by two-term tilting complexes of Λ_{con} .
7. For any atom $\alpha: C_M \rightarrow C_N$, the two-sided tilting complex \mathcal{T}_α defining F_α is given in Corollary 6.4.7.

Proof. The first statement of part (1) is Corollary 5.2.3 while the second follows as the bijection between maximal rigid objects and two-term tilting complexes preserves endomorphism rings by Corollary 7.1.6. Part (2) is Corollary 5.3.8. Part (3) is a special case of (2), while parts (4) and (5) follow from Corollaries 6.5.12 and 6.5.13 respectively. Part (6) is Corollary 7.1.4 and part (7) is Corollary 6.4.7. \square

In this way, we can say that the two-term tilting complexes of Λ_{con} control the entire derived equivalence class and the hyperplane arrangement they define can be viewed as a ‘picture’ of this class; the contraction algebras (the basic members of the equivalence class) sit in the chambers and paths determine all standard derived equivalences. Using the Deligne Groupoid, which is also completely determined by the two-term tilting complexes of Λ_{con} , we are further able to control the composition of these equivalences and show the simplest tilting complexes correspond to the simplest paths. Thus, we obtain a complete understanding of the members of the derived equivalence class and of the standard equivalences between them from the combinatorics of the hyperplane arrangement. In particular, part (3) of Theorem 7.2.1 allows us to determine all the autoequivalences of a contraction algebra, and the relations between them.

Remark 7.2.2. Using part (2) of Theorem 7.2.1, we would like to say the injective group homomorphism

$$\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}) \rightarrow \text{Auteq}(\text{D}^b(\Lambda_{\text{con}}))$$

from Corollary 6.5.13 is almost surjective in the sense that it hits every standard equivalence, up to algebra automorphism. This might allow us to obtain a similar result to that of [Mi, 4.4]; determining the group of standard equivalences as a semi-direct product of a braid group (in our case this would be $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}})$) and certain algebra automorphisms of Λ_{con} . However, for a maximal rigid object M , the object ΩM is another maximal rigid object with isomorphic stable endomorphism algebra. In particular, each contraction algebra appears at least twice in our ‘picture’ of the derived equivalence and thus some autoequivalences are obtained as paths between their corresponding vertices, rather than as a loop at a single vertex. This shows the need for the groupoid picture when visualising the derived equivalence class.

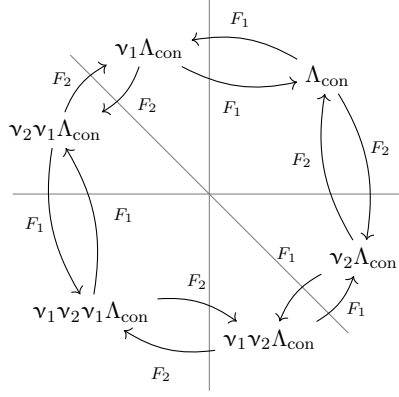
Example 7.2.3. Consider again the cA_2 singularity given by $uv - xy(x + y^2)$, and let $M := R \oplus (u, x) \oplus (u, xy)$ be one of the MM modules in CM R by Theorem 3.3.2. With $\Lambda_{\text{con}} := \underline{\text{End}}_R(M)$, there are isomorphisms $\Lambda_{\text{con}} \cong \nu_1 \nu_2 \nu_1 \Lambda_{\text{con}}$ and

$$\nu_2 \Lambda_{\text{con}} \cong \nu_2 \nu_1 \Lambda_{\text{con}} \cong \nu_1 \Lambda_{\text{con}} \cong \nu_1 \nu_2 \Lambda_{\text{con}}$$

and the quivers and relations are given as follows.

$$\Lambda_{\text{con}} \cong \begin{array}{c} \bullet \xleftarrow{a} \bullet \\ \bullet \xrightarrow{c} \bullet \end{array} \quad \begin{array}{l} cacac=0 \\ acaca=0 \end{array} \quad \nu_1 \Lambda_{\text{con}} \cong \begin{array}{c} \bullet \xleftarrow{a} \bullet \\ \bullet \xrightarrow{c} \bullet \end{array} \begin{array}{l} l^2=ac \\ la=0 \\ cl=0 \end{array}$$

As in Examples 6.1.6 and 6.3.6, the hyperplane arrangement associated to M is



and Theorem 7.2.1 shows this picture contains all the information about the derived equivalence class. In this case, there are two basic members of the class, and the braid relation $F_1 \circ F_2 \circ F_1 \cong F_2 \circ F_1 \circ F_2$ controls all compositions of functors. Moreover, the functor $F_1 \circ F_2 \circ F_1: D^b(\Lambda_{\text{con}}) \rightarrow D^b(\nu_1 \nu_2 \nu_1 \Lambda_{\text{con}})$ is in fact the shift functor followed by an algebra automorphism, as it is induced by the titing complex $\Lambda_{\text{con}}[1]$. However, we can not obtain this functor as any loop at Λ_{con} demonstrating that $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}})$ is not the whole derived autoequivalence group and so we need the entire picture.

7.3 Concluding Remarks

The main results of this thesis are purely algebraic and they demonstrate that contraction algebras have quite remarkable properties. For most finite dimensional algebras, we do not expect the derived equivalence class to even be finite, let alone to be able to describe all the derived equivalences and the relations between them, and yet in this case, we can describe everything with just finitely many two-term complexes.

However, contraction algebras were introduced as a tool in the geometry and so we finish by summarising the evidence these results give towards Conjecture 3.1.9. For convenience, we restate the conjecture here:

Conjecture (Donovan–Wemyss). *Suppose that $f: X \rightarrow \text{Spec } R$ and $g: Y \rightarrow \text{Spec } S$ are smooth minimal models of complete local isolated cDV singularities with associated contraction algebras Λ_{con} and Γ_{con} . Then $R \cong S$ if and only if Λ_{con} and Γ_{con} are derived equivalent.*

Recall that the 'only if' direction is known to be true either by iteratively using Dugas's result, Theorem 4.1.1, or using our new result Corollary 6.4.7. For the other direction, with the notation as in the conjecture, suppose that Λ_{con} and Γ_{con} are derived equivalent. Then the results in this thesis show the following:

1. $\text{Spec } R$ and $\text{Spec } S$ must have the same number of minimal models (Theorem 5.2.4);
2. $\text{Spec } R$ and $\text{Spec } S$ have the same set of contraction algebras, as both are simply the basic members of the derived equivalence class of Λ_{con} (Theorem 7.2.1(1));

3. The curves in the minimal models of $\text{Spec } R$ and $\text{Spec } S$ must have similar properties, as the contraction algebras encode the intersection theory;
4. The simple flops graphs of the minimal models are the same in both cases (Theorem 5.2.4);
5. The relations satisfied by the flop functors between the minimal models are the same (Theorem 7.2.1(4)).

Although this is not enough to prove the conjecture, it certainly provides significant evidence. Furthermore, it reduces the conjecture to the following statement:

Conjecture. *Suppose that $\text{Spec } R$ and $\text{Spec } S$ are complete local isolated cDV singularities with smooth minimal models. If there exists minimal models of $\text{Spec } R$ and $\text{Spec } S$ with isomorphic contraction algebras, then $R \cong S$.*

This restatement, removing derived categories and replacing them with isomorphism problems, is a significant reduction in complexity.

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