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Multilinear Geometric Inequalities

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Abstract

In this thesis we explore a multilinear duality principle of Carbery–Hänninen–Valdimarsson for multilinear geometric inequalities arising in Harmonic Analysis. In Chapter 2, we give a brief survey of how this theory manifests for a number of classical inequalities. The material in Chapter 2 is mostly expository except for two original results: Theorem 2.17, which gives an explicit factorisation of the classical Young convolution inequality on $\mathbb{R}$, and Proposition 2.10, establishing the equivalence of two natural formulations of an affine-invariant Finner inequality. The material in Section 2.3 is also new.

In Chapter 3, we discuss a certain structural duality between instances of the geometric Brascamp–Lieb inequality, and briefly explore how this manifests on the level of factorisations. Most of the material presented in this chapter is not original research. However, the connection between Lemma 3.2 and Proposition 3.7 in the setting of the geometric Brascamp–Lieb inequality is new. In particular, we give a new proof for a special case of Proposition 3.7, which only relies on the sharpness of the geometric Brascamp–Lieb inequality along with a generalised version of Lemma 3.2.

In Chapter 4 we study factorisations due to Guth for the endpoint multilinear Keulegan inequality, and factorisations due to Zhang for a corresponding generalisation to k_j -planes. In particular, we uncover interesting connections to classical notions of convex geometry. The main contributions of this chapter stem from the statements and proofs of Theorems 4.7 and 4.13. In particular, these two geometric observations allow us to translate Guth’s visibility theorem, stated below as Theorem 4.33, into statements about convex bodies, which lead us to the formulation of Theorems 4.8 and 4.14.

In Chapter 5 we study a sequence of generalised versions of Guth’s directed volume - see Definition 4.3. The main contribution of this chapter is Theorem 5.1, giving a sharp chain of inequalities for such geometric quantities. Furthermore, we conjecture restricted inequalities of the same type, which are optimal for boundaries of parallelotopes and related to a vector-valued generalisation of the classical Maclaurin inequality. In Chapter 6, we prove various special cases of this vector-valued variant of the Maclaurin inequality, and highlight connections to open problems in convex geometry.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Finlay McIntyre)

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Notation and terminology

We use fairly standard notation $X \lesssim Y$ to denote the estimate $X \leq CY$ for some absolute constant C . If we need the implied constant C to depend on additional parameters, we indicate this by subscripts, thus for instance $X \lesssim_d Y$ denotes the estimate $X \leq C_d Y$ for some quantity C_d depending only on d . Furthermore, we write $X \sim Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold. Similarly, we write $X \sim_d Y$ if $X \lesssim_d Y$ and $Y \lesssim_d X$. For any positive integer n , we denote by $[n]$ the set of integers $\{1, \dots, n\}$.

Given a subspace $V \subset \mathbb{R}^d$, we denote by π_V the orthogonal projection from $\mathbb{R}^d$ to V . For any vector $u \in \mathbb{R}^d$ we denote by π_u the orthogonal projection from $\mathbb{R}^d$ to the 1-dimensional subspace spanned by u . For any hypersurface $Z \subset \mathbb{R}^d$, which is simply an embedded manifold or algebraic variety of dimension $(d-1)$, we denote by σ_Z the surface measure on Z which coincides with the restricted $(d-1)$ -dimensional Hausdorff measure $\mathcal{H}^{d-1} \llcorner Z$. Given a hypersurface $Z \subset \mathbb{R}^d$, we will often impose the condition that Z has an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$, which means that at $\mathcal{H}^{d-1}$ -a.e point $p \in Z$ the tangent space of Z at p is well-defined and isomorphic to $\mathbb{R}^{d-1}$.

We denote by $\mathbb{B}_2^d$ the standard Euclidean unit ball in $\mathbb{R}^d$, and κ_d its volume. We denote by $\mathcal{K}^d$ the space of non-empty, compact convex subsets of $\mathbb{R}^d$, and we say that $K \in \mathcal{K}^d$ is a **convex body** if it has non-empty interior. This definition differs slightly from that used in [37], where the term convex body is used to describe any non-empty, compact, convex subset of $\mathbb{R}^d$. Since we refer to [37] for many of the results used in Chapters 4, 5 and 6, the reader should be aware of this discrepancy in convention. Given a convex set $K \subset \mathbb{R}^d$, we denote by $|K|$ the d -dimensional volume of K . For any k -dimensional subspace $V \subset \mathbb{R}^d$ we denote by $|\pi_V K|$ and $|K \cap V|$ the k -dimensional volumes of the convex sets $\pi_V K$ and $K \cap V$ respectively.

Chapter 1

Introduction

In this thesis we begin by introducing a certain multilinear duality principle developed by Carbery–Hänninen–Valdimarsson in [16]. Given positive linear operators $T_1, \dots, T_m$ defined from L^1 -spaces into measurable functions on some σ -finite measure space $(X, d\mu)$, we consider inequalities of the form

$$\left\| \prod_{j=1}^m (T_j f_j)^{\alpha_j} \right\|_{L^p(X)} \leq A \prod_{j=1}^m \|f_j\|_{L^1(Y_j)}^{\alpha_j}, \quad (1.1)$$

where $0 \leq \alpha_j \leq 1$, $\sum_{j=1}^m \alpha_j = 1$, $p \geq 1$ and $A > 0$ is some finite constant. One class of examples worth bearing in mind is that of the Brascamp–Lieb inequalities, which generalise a wide range of sharp functional inequalities including the multilinear Hölder, Loomis–Whitney and Young’s convolution inequality. These take the form

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j(L_j x)^{p_j} dx \leq \text{BL}(\mathbf{L}, \mathbf{p}) \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}, \quad (1.2)$$

where the mappings $L_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d_j}$ are linear surjections, the exponents $p_j \in [0, 1]$, and $\text{BL}(\mathbf{L}, \mathbf{p})$ denotes the smallest possible constant. It can happen that (1.2) does not hold for any finite constant, in which case we write $\text{BL}(\mathbf{L}, \mathbf{p}) = \infty$. The Brascamp–Lieb inequality is often referred to as a multilinear inequality, since it is equivalent to

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j(L_j x) dx \leq \text{BL}(\mathbf{L}, \mathbf{p}) \prod_{j=1}^m \|f_j\|_{L^{q_j}(\mathbb{R}^{d_j})}, \quad (1.3)$$

where we have simply replaced the $f_j^{p_j}$ with f_j and p_j with $\frac{1}{q_j}$. Although in general (1.1) will not be multilinear in the above sense, we will informally refer to such examples as multilinear norm inequalities.

The notion of multilinear duality presents an approach for proving such norm inequalities and can be best described by the following result:

Theorem 1.1 ([16]). *Suppose that each T_j saturates X in the sense that there exists a non-negative $h_j \in L^1(Y_j)$ such that $\|h_j\|_{L^1(Y_j)} \leq 1$ and $T_j h_j > 0$ a.e. on X . Then the following statements are equivalent:*

$$(i) \quad \left\| \prod_{j=1}^m (T_j f_j)^{\alpha_j} \right\|_{L^p(X)} \leq A \prod_{j=1}^m \|f_j\|_{L^1(Y_j)}^{\alpha_j} \text{ for all } f_j \in L^1(Y_j);$$

(ii) For every nonnegative $M \in L^{p'}(X)$ there exist nonnegative measurable functions $M_1, \dots, M_m$, such that

$$M(x) \leq \prod_{j=1}^m M_j(x)^{\alpha_j} \quad a.e. \quad (1.4)$$

and for $1 \leq j \leq m$ and every $f_j \in L^1(Y_j)$

$$\int_X M_j(T_j f_j) d\mu \leq A \|M\|_{L^{p'}(X)} \|f_j\|_{L^1(Y_j)}; \quad (1.5)$$

(iii) For every nonnegative $M \in L^{p'}(X)$ there exist nonnegative measurable functions $M_1, \dots, M_m$, such that

$$M(x) \leq \prod_{j=1}^m M_j(x)^{\alpha_j} \quad a.e. \quad (1.6)$$

and for every $f_j \in L^1(Y_j)$ where $1 \leq j \leq m$, we have

$$\prod_{j=1}^m \left(\int_X M_j(T_j f_j) d\mu \right)^{\alpha_j} \leq A \|M\|_{L^{p'}(X)} \prod_{j=1}^m \|f_j\|_{L^1(Y_j)}^{\alpha_j}. \quad (1.7)$$

Remark 1.2. It is not difficult to see that we can make each M_j slightly smaller to attain equality in either (1.4) or (1.6) while preserving conditions (1.5), (1.7) respectively. So the inequality signs in conditions (1.4) and (1.6) can be replaced with equality without changing the result.

It is clear that (ii) directly implies (iii). For the reverse implication suppose we have functions M_j satisfying (1.6) and (1.7). Now defining $L_j = \sup_{\|f_j\| \leq 1} \int_X M_j(T_j f_j) d\mu$ and $\lambda_j = L_j^{-1} \prod_{k=1}^m L_k^{\alpha_k}$, we see that functions $\tilde{M}_j := \lambda_j M_j$ will satisfy (1.4) and (1.5). Although (ii) and (iii) are completely equivalent, in some cases it will be more convenient to work with one or the other. Informally we refer to any such family of functions $M_1, \dots, M_m$ as a *factorisation* for the corresponding multilinear inequality, which derives from the almost-everywhere pointwise condition

$$M(x) \leq \prod_{j=1}^m M_j(x)^{\alpha_j}.$$

It is not difficult to verify that (ii) and (iii) both imply (i), this follows by an application of Hölder's inequality combined with a simple duality argument. Indeed, by linear duality of $L^p(X)$ we have

$$\left\| \prod_{j=1}^m (T_j f_j)^{\alpha_j} \right\|_{L^p(X)} = \sup_{\substack{M \in L^{p'}(X) \\ \|M\|_{p'} \leq 1}} \int_X \prod_{j=1}^m (T_j f_j)^{\alpha_j} M d\mu.$$

Now suppose we know that (ii) holds, then one can write

$$\begin{aligned}
\int_X \prod_{j=1}^m (T_j f_j)^{\alpha_j} M d\mu &\leq \int_X \prod_{j=1}^m (T_j f_j)^{\alpha_j} \prod_{j=1}^m M_j^{\alpha_j} d\mu = \int_X \prod_{j=1}^m (M_j T_j f_j)^{\alpha_j} d\mu \\
&\leq \prod_{j=1}^m \left(\int_X \prod_{j=1}^m M_j T_j f_j d\mu \right)^{\alpha_j} \\
&\leq \prod_{j=1}^m \left(A \|M\|_{L^{p'}(X)} \|f_j\|_{L^{p_j}(Y_j)} \right)^{\alpha_j} \\
&\leq A \|M\|_{L^{p'}(X)} \prod_{j=1}^m \|f_j\|_{L^{p_j}(Y_j)}^{\alpha_j} \\
&= A \prod_{j=1}^m \|f_j\|_{L^{p_j}(Y_j)}^{\alpha_j}.
\end{aligned}$$

So it is clear that in order to prove such a norm inequality, it is enough find a factorisation. However, Theorem 1.1 tells us formally that the task of finding factorisations is actually equivalent to proving such norm inequalities.

In Chapter 2 we explore how this theory manifests in the context of multilinear geometric inequalities by discussing some known constructions of factorisations appearing in [16] and [5]. Most, if not all the examples discussed below can be seen as generalisations of the following classical geometric inequality for functions:

Theorem 1.3 (Loomis–Whitney - [29]). *For $1 \leq j \leq d$ define $\pi_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ by $\pi_j(x) = (x_1, \dots, \hat{x}_j, \dots, x_d)$. Then following inequality holds*

$$\int_{\mathbb{R}^d} \prod_{j=1}^d F_j(\pi_j x) dx \leq \prod_{j=1}^d \|F_j\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

It is not immediate that this falls under the scope of Theorem 1.1; however, setting $f_j^{\frac{1}{d-1}} = F_j$ allows us to rewrite the above inequality as

$$\int_{\mathbb{R}^d} \prod_{j=1}^d f_j(\pi_j x)^{\frac{1}{d-1}} dx \leq \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} f_j \right)^{\frac{1}{d-1}}, \quad (1.8)$$

then taking both sides to the power $\frac{d-1}{d}$ yields

$$\left\| \prod_{j=1}^d (f_j \circ \pi_j)^{\frac{1}{d}} \right\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \prod_{j=1}^d \|f_j\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{d}}.$$

Note that $\pi_j = \pi_{e_j^\perp}$, where $e_1, \dots, e_d$ denote the standard orthonormal basis of $\mathbb{R}^d$. What happens if we choose a different basis to work with? This question motivates the formulation of an "affine-invariant" version of the Loomis–Whitney inequality.

Theorem 1.4 (Affine-invariant Loomis–Whitney [16]). *Let $\omega_1, \dots, \omega_d \in \mathbb{R}^d$ be unit vectors forming a basis of $\mathbb{R}^d$, then*

$$\int_{\mathbb{R}^d} \prod_{j=1}^d f_j(\pi_{\omega_j^\perp} x)^{\frac{1}{d-1}} dx \leq |\omega_1 \wedge \dots \wedge \omega_d|^{-\frac{1}{d-1}} \prod_{j=1}^d \left(\int_{\omega_j^\perp} f_j \right)^{\frac{1}{d-1}},$$

where we denote by $|\omega_1 \wedge \dots \wedge \omega_d|$ the volume of the parallelotope with edges $\omega_1, \dots, \omega_d$.

As with the classical Loomis–Whitney inequality, this version falls under the scope of Theorem 1.1 when written in the following way:

$$\left\| \prod_{j=1}^d \left(f_j \circ \pi_{\omega_j^\perp} \right)^{\frac{1}{d}} \right\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d}} \prod_{j=1}^m \|f_j\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{d}}.$$

In particular, we can deduce the existence of a factorisation. There are many other ways to generalise Theorem 1.3, such as the classical Finner inequality, which also has an affine-invariant version. In Chapter 2 we review these examples in more detail and discuss some known methods for constructing appropriate factorisations explicitly following ideas from [16] and [5].

There are still many examples of multilinear norm inequalities we do not know how to factorise, meaning we are not able to explicitly construct factorisations which are guaranteed to exist by Theorem 1.1. In some cases, we are only able to find factorisations which prove a given inequality with some sub-optimal constant. One of the simplest such example is a special case of Young’s convolution inequality, which can be stated as follows

Theorem 1.5 (Young’s convolution inequality on $\mathbb{R}$). *For any $F \in L^p(\mathbb{R})$, $G \in L^q(\mathbb{R})$ and $p, q, r \in [1, \infty]$ satisfying*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

we have the following bound

$$\|F * G\|_r \leq \|F\|_p \|G\|_q. \quad (1.9)$$

In order to write down a sharp form of the above inequality, we define for any $q \in [0, \infty]$

$$A_q := \frac{q^{\frac{1}{q}}}{q^{\frac{1}{q'}}}. \quad (1.10)$$

The following result is originally due to Beckner [8]:

Theorem 1.6 (Sharp Young convolution inequality). *For any $F \in L^p(\mathbb{R})$, $G \in L^q(\mathbb{R})$ and $p, q, r \in [1, \infty]$, the following inequality is sharp*

$$\|F * G\|_r \leq A_p A_q A_r \|F\|_p \|G\|_q.$$

As we will see in Chapter 2, one can rewrite (1.9) as a trilinear norm inequality which falls directly under the scope of Theorem 1.1 - see Theorem 2.16. If one is not too concerned about getting the sharp constant, one can build factorisations establishing (1.9) using a method of interpolation described in [16], Section 7. The explicit construction of such a factorisation is new and appears in the proof of Theorem 2.17.

It is still not clear how to factorise the sharp Young convolution inequality. In Chapter 3 we show how the sharp Young convolution inequality can be seen as a special case of the geometric Brascamp–Lieb inequalities, which are of the form

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j(\langle x, u_j \rangle)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}} g_j \right)^{p_j}, \quad (1.11)$$

for unit vectors $u_1, \dots, u_m \in \mathbb{R}^d$ and exponents $p_1, \dots, p_m > 0$. Unlike the affine-invariant Loomis–Whitney or Finner inequalities which holds for any basis of unit vectors, (1.11) only holds if the following condition is satisfied for all $x \in \mathbb{R}^d$:

$$\sum_{i=1}^m p_i \langle x, u_i \rangle u_i = x. \quad (1.12)$$

Another difference worth noting here is that the number of vectors is allowed to be different from the dimension d . Due to this last observation, it is difficult to use ideas from the factorisation of the affine-invariant Loomis–Whitney or Finner inequalities to help in constructing factorisations here. Let us introduce the following fairly standard notation to write the above condition in a more succinct way:

Definition 1.7. For any unit vector $u \in \mathbb{R}^d$ we define the linear map $u \otimes u: \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$u \otimes u(x) = \langle x, u \rangle u.$$

Immediately we see that (1.12) can be rewritten as

$$\sum_{i=1}^m p_i u_i \otimes u_i = I_d, \quad (1.13)$$

where I_d denotes the identity operator on $\mathbb{R}^d$. To be concise in referring to unit vectors and exponents satisfying this last identity we make the following definition:

Definition 1.8. Given exponents $p_1, \dots, p_m > 0$ and unit vectors $u_1, \dots, u_m \in \mathbb{R}^d$ satisfying (1.13), we refer to the tuple $(\mathbf{u}, \mathbf{p}) = ((u_1, \dots, u_m), (p_1, \dots, p_m))$ as **geometric Brascamp–Lieb datum**.

Using this notation, one can formulate the classical geometric Brascamp–Lieb inequality of Keith Ball as follows

Theorem 1.9 (Classical geometric Brascamp–Lieb inequality). *Given geometric Brascamp–Lieb datum $(\mathbf{u}, \mathbf{p})$, the following inequality holds*

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j(\langle x, u_j \rangle)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}} g_j \right)^{p_j}, \quad (1.14)$$

and for each $t > 0$, the standard Gaussian functions $g_j(y) = e^{-\pi t |y|^2}$ are extremals.

To see how this example falls under the scope of Theorem 1.1, set $p = \sum_{j=1}^m p_j$ and take both sides of (1.14) to the power $\frac{1}{p}$ to get

$$\left\| \prod_{j=1}^m (g_j \circ \pi_{u_j})^{\frac{p_j}{p}} \right\|_{L^p(\mathbb{R})} \leq \prod_{j=1}^m \|g_j\|_{L^1 \mathbb{R}}^{\frac{p_j}{p}}.$$

By equating traces of both sides in (1.13) we get

$$p := \sum_{j=1}^m p_j = d \geq 1.$$

Clearly $0 \leq \frac{p_j}{p} \leq 1$ for each j , so we can apply the multilinear duality principle and deduce the existence of factorisations for these inequalities too.

In Chapter 3 we discuss a certain structural duality between pairs of geometric Brascamp–Lieb data, which arises from the following elementary geometric observation:

Lemma 1.10. *Let $1 \leq d \leq m$. Suppose we have unit vectors $u_1, \dots, u_m \in \mathbb{R}^d$ and exponents $p_1, \dots, p_m \geq 0$ such that*

$$\sum_{i=1}^m p_i u_i \otimes u_i = I_d,$$

then there exists an orthonormal basis $v_1, \dots, v_m \in \mathbb{R}^m$ such that

$$\pi_{\mathbb{R}^d} v_i = p_i^{\frac{1}{2}} u_i,$$

where $\pi_{\mathbb{R}^d}$ denotes projection onto the first d coordinates of $\mathbb{R}^m$.

This elegant result was used in [7], and for completeness we give a short elementary proof in Section 3.1. Essentially any geometric Brascamp–Lieb datum tuple can be written as the "projection" of an orthonormal basis. The reverse implication is more obvious but also useful to note here, this is the fact that taking the orthogonal projection of any orthonormal basis onto a lower dimensional subspace will result in a tuple of geometric Brascamp–Lieb datum:

Lemma 1.11. *Fix $1 \leq k \leq m$. Suppose we are given an orthonormal basis $v_1, \dots, v_m \in \mathbb{R}^m$ and a k -dimensional subspace H of $\mathbb{R}^m$. Let π_H denote orthogonal projection onto H and for $i = 1, \dots, m$ define*

$$w_i = \begin{cases} \frac{\pi_H v_i}{\|\pi_H v_i\|} & \text{if } \|\pi_H v_i\| \neq 0 \\ \mathbf{0} & \text{if } \|\pi_H v_i\| = 0 \end{cases}$$

and

$$\tilde{p}_i = \|\pi_H v_i\|^{\frac{1}{2}}.$$

It follows that

$$\sum_{i=1}^m \tilde{p}_i w_i \otimes w_i = I_H,$$

where I_H denotes the identity operator on H .

At the beginning of Section 3.3 we give a short derivation of the above statement.

Suppose we apply Lemma 1.10 with geometric Brascamp–Lieb datum $(\mathbf{u}, \mathbf{p})$, then we get an orthonormal basis $v_1, \dots, v_m \in \mathbb{R}^m$ such that

$$\pi_{\mathbb{R}^d} v_i = p_i^{\frac{1}{2}} u_i.$$

Now we can apply this Lemma 1.11 to $v_1, \dots, v_m$, with $k = m - d$ and H being the subspace spanned by the last $(m - d)$ -coordinates of $\mathbb{R}^m$, to get a second tuple of geometric Brascamp–Lieb datum $(\mathbf{w}, \tilde{\mathbf{p}})$ such that

$$\pi_{\mathbb{R}^{m-d}} v_i = (\tilde{p}_i)^{\frac{1}{2}} w_i.$$

In other words, the above two lemmas combined suggest a natural "dual pairing" of geometric Brascamp–Lieb data. In particular, we are interested in how this reveals itself at the level of factorisations. As we will see in Chapter 3, this structural duality of geometric Brascamp–Lieb data turns out to be the manifestation of a more general Fourier duality of Brascamp–Lieb

inequalities used in [9]. It would be very useful if one could establish a connection between the factorisations of two geometric Brascamp–Lieb inequalities which are related via this structural duality. At the end of Section 3.2, we see how this could potentially help us construct factorisations to the sharp Young convolution inequality. There is no known procedure for computing factorisations for the geometric Brascamp–Lieb inequality with sharp constant 1. However, following the interpolation method laid out in [16], Section 7, one can construct factorisations which in each case yield some non-optimal constant. These more general constructions essentially build upon the arguments used in the proof of Theorem 2.17, but require a heavier dose of linear algebra. The details for explicitly calculating such factorisations do not provide much further insight so we will not cover them in this thesis.

In Chapter 4 we discuss a proof of the endpoint multilinear Keakeya inequality, which can be formulated as a far-reaching generalisation of Theorem 1.4:

Theorem 1.12 (Endpoint multilinear Keakeya inequality – functional form [24],[15]). *There exists a constant $C_d > 0$ such that for any linearly independent unit vectors $\omega_1, \dots, \omega_d \in \mathbb{R}^d$ and $f_1, \dots, f_d: \mathbb{S}^{d-1} \times \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ we have*

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} \left(\prod_{j=1}^d f_j(\omega_j, \pi_{\omega_j^\perp} x) \right) |\omega_1 \wedge \cdots \wedge \omega_d| d\sigma(\omega_1) \cdots d\sigma(\omega_d) \right)^{\frac{1}{d-1}} dx \leq C_d \prod_{j=1}^d \left(\int_{\mathbb{S}^{d-1} \times \mathbb{R}^{d-1}} f_j \right)^{\frac{1}{d-1}}, \quad (1.15)$$

where σ denotes the normalised surface measure on $\mathbb{S}^{d-1}$.

This example does not fall under the scope of multilinear duality, due to the non-constant factor of $|\omega_1 \wedge \cdots \wedge \omega_d|$ appearing inside the integral on the left hand side. However, the approach taken in [15] is to prove the existence of a certain factorisation using various geometric tools introduced by Guth in [24]. The study of this factorisation argument constitutes a large portion of Chapter 4. In particular, we highlight various underlying connections to results in classical convex geometry which have not previously been explored. The notion of directional surface area plays a key role in constructing the required factorisation - see Definition 4.3. The new perspective we give stems from the following geometric observation:

Theorem 1.13. *For every hypersurface $Z \subset \mathbb{R}^d$ of finite $(d-1)$ -dimensional measure with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$, there exists a unique compact convex set $L_Z \subset \mathbb{R}^d$, symmetric about the origin, such that for all $u \in \mathbb{S}^{d-1}$*

$$\int_Z |u \cdot n(x)| d\sigma_Z(u) = |\pi_u L_Z|.$$

This result follows from Lemma 4.25 combined with an elementary fact about the support function of a centrally symmetric convex set mentioned in Section 4.1.1.

Using this language we establish the following consequence of Guth’s visibility theorem, the precise statement of which is given in Theorem 4.5:

Theorem 1.14 (Visibility theorem in terms of convex bodies - tube version [15]). *Let $\mathcal{T}$ be some finite family of 1-tubes in $\mathbb{R}^d$. Given any finitely supported non-negative function M defined on the standard lattice of unit cubes in $\mathbb{R}^d$, denoted $\mathcal{Q}$, with $\|M\|_{\ell^m(\mathcal{Q})} = 1$, there exists*

a family of centrally symmetric convex bodies $\{K(Q)\}_{Q \in \text{supp}(M)}$ in $\mathbb{R}^d$ depending on the choice of M and $\mathcal{T}$ such that

$$\max_{T \in \mathcal{T}} \sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| \lesssim_{d,m} 1,$$

and for each $Q \in \text{supp}(M)$

$$|K(Q)|^{\frac{1}{d}} \gtrsim_{d,m} M(Q),$$

where the implicit constants depend only on the dimension d and the degree of multilinearity m .

Here the term "convex body" refers to any compact, convex subset of $\mathbb{R}^d$ with non-empty interior. This result does not appear explicitly in the literature but is proved implicitly in [24] and [15]. In the first section of Chapter 4, we see how this last result can be used directly to construct a factorisation for the endpoint multilinear Kakeya inequality. There is a natural generalisation to Theorem 1.15 where tubes are replaced by neighbourhoods of k_j -planes, which was first established by Zhang in [39]. As we will see, one can adapt factorisation from the setting of tubes to prove this generalisation. The notion of directional volume, implicitly used by Zhang, will play a central role in this argument - see Definition 4.12. In Chapter 4 we prove the following generalisation of Theorem 1.13:

Theorem 1.15. *For every hypersurface $Z \subset \mathbb{R}^d$ of finite $(d-1)$ -dimensional measure with an almost-everywhere defined unit normal, there exists a unique compact convex set $L_Z \subset \mathbb{R}^d$, symmetric about the origin, such that for $1 \leq k \leq d$ and any k -dimensional subspace $V \subset \mathbb{R}^d$*

$$\int_Z \cdots \int_Z |V^\perp \wedge n(x_1) \wedge \cdots \wedge n(x_k)| d\sigma_Z(x_1) \cdots d\sigma_Z(x_k) = k! |\pi_V L_Z|.$$

This result is essentially Proposition 4.44 combined with Lemma 4.25 and Definition 4.26.

In Chapter 5, we spend some time trying to understand the directional volume as a geometric object in its own right. Moreover, we study a sequence of natural generalisations and prove a chain of inequalities between them. A special case of these inequalities turns out to be closely related to the following classical refinement of the arithmetic-geometric mean inequality:

Theorem 1.16 (Maclaurin inequality). *For any sequence $x_1, \dots, x_n$ of positive real numbers and $1 < k \leq n$*

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [n]} x_{i_1} \cdots x_{i_k}}{\binom{n}{k}} \right)^{\frac{1}{k}} \leq \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [n]} x_{i_1} \cdots x_{i_{k-1}}}{\binom{n}{k-1}} \right)^{\frac{1}{k-1}},$$

with equality if and only if the numbers are all equal.

It should be noted that much of the material appearing in Chapter 5 is currently being prepared for publication. As we will see in Chapter 5, for any d -dimensional box R with side lengths $\ell_1, \dots, \ell_d$ the following expression will arise as an interesting geometric quantity

$$\left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{\ell_{i_1} \cdots \ell_{i_k}} \right)^{\frac{1}{k}}.$$

It is not difficult to see that we can apply Theorem 1.16 with $n = d$ and $x_i = \frac{1}{\ell_i}$ to get an inequality for each pair of consecutive expressions. More generally, for any d -dimensional parallelotope P with edges parallel to $v_1, \dots, v_d \in \mathbb{R}^d$ of length $\|v_1\|, \dots, \|v_d\|$ and outward facing normals $n_1, \dots, n_d \in \mathbb{S}^{d-1}$ we will see an appearance of the following expression

$$\left(\sum_{1 \leq i_1 < \dots < i_k \leq d} \left(\frac{\prod_{j=1}^k |v_1 \wedge \dots \wedge \widehat{v_{i_j}} \wedge \dots \wedge v_d|}{\|v_{i_1}\| \dots \|v_{i_k}\|} \right) \cdot |n_{i_1} \wedge \dots \wedge n_{i_k}| \right)^{\frac{1}{k}}.$$

Letting $\tilde{n}_j = \frac{|v_1 \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge v_d|}{\|v_j\|} n_j$ this last expression becomes

$$\left(\sum_{1 \leq i_1 < \dots < i_k \leq d} |\tilde{n}_{i_1} \wedge \dots \wedge \tilde{n}_{i_k}| \right)^{\frac{1}{k}}.$$

In light of this, it seems natural to ask if one can prove the following generalisation of the Maclaurin inequalities for vectors:

Conjecture 1.17 (Vector-valued Maclaurin inequalities). *For any m -tuple of vectors $v_1, \dots, v_m \in \mathbb{R}^d$ and $1 < k \leq d \leq m$, the following inequality holds*

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{m}{k}} \right)^{\frac{1}{k}} \leq \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|}{\binom{m}{k-1}} \right)^{\frac{1}{k-1}},$$

with equality if and only if $m = d$ and the v_i are orthogonal vectors of equal length.

Note that in the special case where $m = d$ and all the vectors are orthogonal to each other the wedge products all become products and we recover the original Maclaurin inequalities. Indeed, if the v_i form an orthogonal family then for any $\{i_1, \dots, i_r\} \subset [d]$ we have $|v_{i_1} \wedge \dots \wedge v_{i_r}| = \prod_{j=1}^r \|v_{i_j}\|$. The above statement in general is not known, but there are various special cases which we have been able to establish. In Chapter 6 we discuss in detail these partial results along with some interesting variants. As we will see, there are nice connections to be made between these seemingly elementary inequalities and bounds for the intrinsic volumes of a certain class of convex bodies, namely zonotopes and zonoids. Most of the material covered in Chapter 6 appears in a joint paper with Silouanos Brazitikos, which has been accepted for publication - see [38].

Chapter 2

Multilinear norm inequalities and factorisation

In [16], the authors introduce a duality theory for studying a wide class of multilinear norm inequalities. Moreover, they prove that each inequality among a general class of multilinear norm inequalities will hold if and only if there exists a factorisation of every function in a corresponding L^p -space. A recurrent theme for us is the search for factorisations of various multilinear inequalities from Harmonic Analysis. In the first section we use the multilinear duality principle mentioned above to reformulate various classical multilinear norm inequalities as factorisations statements. Furthermore, we describe how to explicitly construct known factorisations for the Loomis–Whitney and Finner inequalities along with their affine-invariant versions. Using an interpolation method given in [16], we construct explicit factorisations for non-sharp versions of the Young convolution inequality and describe how one could use a similar construction to build factorisations of the classical geometric Brascamp–Lieb inequality. In the second section we explore how factorisations arise in the setting of the endpoint multilinear Keakey inequality of Guth and similarly for a generalisation to k_j -planes originally due to Zhang. Interestingly, these examples do not fall within the scope of multilinear duality but the existence of a suitable "factorisation" is known. In fact, this is exactly the approach taken in [15], which has many points of contact with arguments given in [24]. In the third section we diverge somewhat and talk a little about an elegant example of a non-linear Brascamp–Lieb inequality. Let us begin with some examples to illustrate how these factorisations arise.

It should be noted that most of the material in this chapter is expository and not original work. However, the formulation and proof Proposition 2.10 is joint work with Tony Carbery and Silouanos Brazitikos - this result appears implicitly in [5]. The formulation of Theorem 2.17 is new - the construction is based on an algorithmic method of multilinear interpolation discussed in [16]. Lastly, the material appearing in Section 2.3 is also original work.

2.1 Classical multilinear norm inequalities

Here we consider some classical inequalities which fall under the scope of multilinear duality. For a handful of classical examples, one can actually write down explicit factorisations. In particular, this is true for the Loomis–Whitney and Finner inequalities as well as their affine-invariant generalisations. We include details for these constructions following ideas from [16] and [5]. These factorisations build somewhat on each other with the most general being for the affine invariant Finner inequality. However, there are still many inequalities of a similar form which are more difficult to factorise. Some of the simplest such inequalities are special cases

of the geometric Brascamp–Lieb inequalities. Although there is no known general construction for factorisations which yield the geometric Brascamp–Lieb inequality with sharp constant, we discuss how to factorise individual non-sharp versions using a certain multilinear interpolation used in [16]. Specialising this general procedure to the setting of Young’s convolution inequality, we are able to write down an explicit factorisation yielding the non-sharp constant 1.

Loomis–Whitney inequality As we mentioned earlier, most of the examples considered can be seen as generalisations of the classical Loomis–Whitney inequality for functions, which can be stated as follows:

Theorem 2.1 (Loomis–Whitney). *For $1 \leq j \leq d$ define $\pi_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ by $\pi_j(x) = (x_1, \dots, \hat{x}_j, \dots, x_d)$. Then following inequality holds*

$$\int_{\mathbb{R}^d} \prod_{j=1}^d f_j(\pi_j x)^{\frac{1}{d-1}} dx \leq \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} f_j \right)^{\frac{1}{d-1}}.$$

Let us begin by running through the easier half of Theorem 1.1, which tells us that given a certain factorisation we can establish the above result. Firstly take both sides to the power $\frac{d-1}{d}$ to get

$$\left\| \prod_{j=1}^d (f_j \circ \pi_j)^{\frac{1}{d}} \right\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \prod_{j=1}^d \|f_j\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{d}}.$$

By duality, this is equivalent to the statement that for any $g \in L^d(\mathbb{R}^d)$ with $\|g\|_d = 1$, we have

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d f_j(\pi_j x) \right)^{\frac{1}{d}} g(x) dx \leq \prod_{j=1}^d \|f_j\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{d}}. \quad (2.1)$$

Suppose that for $1 \leq j \leq d$, we can find functions $g_j: \mathbb{R}^d \rightarrow \mathbb{R}$ such that for every $x \in \mathbb{R}^d$

$$g(x) \leq \prod_{j=1}^d g_j(x)^{\frac{1}{d}},$$

and for $1 \leq j \leq d$ and almost every $(x_1, \dots, \hat{x}_j, \dots, x_d) \in \mathbb{R}^{d-1}$

$$\int_{\mathbb{R}} g_j(x) dx_j \leq 1.$$

Then by applying Hölder’s inequality and identifying $\mathbb{R}^d \cong \mathbb{R}^{d-1} \times \mathbb{R}$ we can deduce (2.1) as

follows:

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(\prod_{j=1}^d f_j(\pi_j x) \right)^{\frac{1}{d}} g(x) dx &\leq \int_{\mathbb{R}^d} \left(\prod_{j=1}^d f_j(\pi_j x) g_j(x) \right)^{\frac{1}{d}} dx \\
&\leq \prod_{j=1}^d \left(\int_{\mathbb{R}^d} f_j(\pi_j x) g_j(x) dx \right)^{\frac{1}{d}} \\
&= \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} f_j(x_1, \dots, \widehat{x}_j, \dots, x_d) \left(\int_{\mathbb{R}} g_j(x) dx_j \right) dx_1 \cdots \widehat{dx}_j \cdots dx_d \right)^{\frac{1}{d}} \\
&\leq \prod_{j=1}^d \left(\left(\sup_{(x_1, \dots, \widehat{x}_j, \dots, x_d) \in \mathbb{R}^{d-1}} \int_{\mathbb{R}} g_j(x) dx_j \right) \left(\int_{\mathbb{R}^{d-1}} f_j \right) \right)^{\frac{1}{d}} \\
&\leq \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} f_j \right)^{\frac{1}{d}} \\
&= \prod_{j=1}^d \|f_j\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{d}}.
\end{aligned}$$

As consequence of the more difficult implication from Theorem 1.1, we know that such functions $g_1, \dots, g_d$ must exist. In this case such a factorisation can be constructed explicitly:

Theorem 2.2 (Factorisation for the Loomis–Whitney inequality). *For every $g \in L^d(\mathbb{R}^d)$ with $\|g\|_d = 1$, we can construct functions $g_1, \dots, g_d$ on $\mathbb{R}^d$ such that for almost every $x \in \mathbb{R}^d$*

$$g(x) = \prod_{j=1}^d g_j(x)^{\frac{1}{d}}, \quad (2.2)$$

and for $1 \leq j \leq d$ and almost every $(x_1, \dots, \widehat{x}_j, \dots, x_d) \in \mathbb{R}^{d-1}$

$$\int_{\mathbb{R}} g_j(x) dx_j = 1. \quad (2.3)$$

Following ideas from [16], let us construct a factorisation explicitly satisfying conditions (2.2) and (2.3).

Proof of Theorem 2.2. Firstly let us take both sides of (2.2) to the power d to get

$$g(x)^d = \prod_{j=1}^d g_j(x),$$

Our approach will be to take factors g_j out from g one by one, ensuring that each factor satisfies condition (2.3). For every $x \in \mathbb{R}^d$, provided the integrals are finite, we can begin by writing

$$g(x)^d = \left(\frac{g(x)^d}{\int_{\mathbb{R}} g(x)^d dx_1} \right) \int_{\mathbb{R}} g(x)^d dx_1. \quad (2.4)$$

Let $g_1(x)$ denote the term inside the brackets on the right hand side, as a function of x . By construction, for every $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}} g_1(x) dx_1 = \int_{\mathbb{R}} \left(\frac{g(x)^d}{\int_{\mathbb{R}} g(x)^d dx_1} \right) dx_1 = \frac{\int_{\mathbb{R}} g(x)^d dx_1}{\int_{\mathbb{R}} g(x)^d dx_1} = 1.$$

Continuing in this way, again provided the integrals are finite, we can write the second factor in (2.4) as

$$\int_{\mathbb{R}} g(x)^d dx_1 = \left(\frac{\int_{\mathbb{R}} g(x)^d dx_1}{\int_{\mathbb{R}} \int_{\mathbb{R}} g(x)^d dx_1 dx_2} \right) \int_{\mathbb{R}} \int_{\mathbb{R}} g(x)^d dx_1 dx_2.$$

As we did for g_1 , set $g_2(x)$ equal to the term inside the brackets on the right hand side, as a function of x . By a similar argument, for every $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}} g_2(x) dx_2 = 1.$$

Furthermore, by construction we also have

$$g(x)^d = g_1(x) g_2(x) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} g(x)^d dx_1 dx_2 \right).$$

Following this pattern, let us define

$$g_j(x) = \frac{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(x)^d dx_1 \cdots dx_{j-1}}{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(x)^d dx_1 \cdots dx_j}.$$

It follows that

$$g(x)^d = g_1(x) \cdots g_d(x) \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(x)^d dx_1 \cdots dx_d \right) = g_1(x) \cdots g_d(x) \|g\|_d = g_1(x) \cdots g_d(x).$$

which deals with condition (2.2). By direct computation we see that for each j and every $x \in \mathbb{R}^d$

$$\int_{\mathbb{R}} g_j(x) dx_j = 1.$$

which deals with condition (2.3). □

Remark 2.3. Notice that for the above construction we were actually able to get equality in both (2.2) and (2.3). Theorem 1.1 only guarantees inequalities in both conditions, or at best equality for the pointwise factorisation - see Remark 1.2.

Finner's inequality The next example has more of a combinatorial flavour to it but can be seen as a direct generalisation of the Loomis–Whitney inequality.

Theorem 2.4 (Finner - [21]). *Suppose we have exponents $p_1, \dots, p_m > 0$ and subsets $S_1, \dots, S_m \subset [d]$. For $1 \leq j \leq m$ define a subspace as follows*

$$E_j := \text{span}\{e_i \mid i \in S_j\}.$$

If we have for every $1 \leq i \leq d$

$$\sum_{j=1}^m p_j \chi_{S_j}(i) = 1,$$

then the following inequality holds

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j (\pi_{E_j} x)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{E_j} f_j \right)^{p_j}.$$

Remark 2.5. Set $m = d$, all the p_j 's equal to $\frac{1}{d-1}$ and $S_j = \{i \in [d] \mid i \neq j\}$ for $1 \leq j \leq m$. Clearly for every $i \in \{1, \dots, d\}$ we have

$$\sum_{j=1}^m \frac{1}{d-1} \chi_{\{1, \dots, \hat{j}, \dots, d\}}(i) = \sum_{j=1}^m \frac{1}{d-1} (1 - \delta_{ij}) = 1,$$

so we immediately recover Theorem 2.1.

Although this example does not at first look like it falls under the scope of multilinear duality, we can remedy this by setting $p = \sum_{j=1}^m p_j$ and rewriting the inequality as

$$\left\| \prod_{j=1}^m (f_j \circ \pi_{E_j})^{\frac{p_j}{p}} \right\|_{L^p(\mathbb{R}^d)} \leq \prod_{j=1}^m \left(\int_{E_j} f_j \right)^{\frac{p_j}{p}}. \quad (2.5)$$

Since we have $p_j > 0$ for every j and $\sum_{j=1}^m p_j \chi_{S_j}(i) = 1$ for every i , it is clear that $p \geq 1$ and $0 \leq \frac{p_j}{p} \leq 1$ for every j . Applying a very similar duality argument as before, by Hölder's inequality it suffices to prove that for any function h in the unit sphere of $L^{p'}(\mathbb{R}^d)$ there exist functions $h_1, \dots, h_m: \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying two conditions analogous to (2.2) and (2.3). Firstly for almost every $x \in \mathbb{R}^d$

$$h(x) \leq \prod_{j=1}^m h_j(x)^{\frac{p_j}{p}}.$$

For any $S \subset \{1, \dots, d\}$ and $E = \text{span}\{e_i \mid i \in S\}$, let $(x_i)_{i \in S}$ be coordinates on E and $(x_i)_{i \notin S}$ be coordinates on $E^\perp$. We slightly abuse notation and write $x = (x_1, \dots, x_d) = (x_i)_{i \in S} + (x_i)_{i \notin S}$. The second condition is that for each j and almost every $(x_i)_{i \notin S_j} \in E_j$

$$\int_{E_j^\perp} h_j(x) \prod_{i \notin S_j} dx_i \leq 1,$$

where we are integrating on the left hand side just over coordinates $(x_i)_{i \notin S_j}$, thus we are left with a function of $(x_i)_{i \in S}$. Again, the existence of such functions $h_1, \dots, h_m$ follows directly from Theorem 1.1. As with the Loomis–Whitney inequality, here we can actually construct these functions explicitly:

Theorem 2.6 (Factorisation for Finner's inequality). *For every $h \in L^{p'}(\mathbb{R}^d)$ with $\|h\|_{p'} = 1$, we can construct functions $h_1, \dots, h_m$ on $\mathbb{R}^d$ such that for every almost every $x \in \mathbb{R}^d$*

$$h(x) = \prod_{j=1}^m h_j(x)^{\frac{p_j}{p}}. \quad (2.6)$$

and almost every $(x_i)_{i \notin S_j} \in E_j$

$$\int_{E_j^\perp} h_j(x) \prod_{i \notin S_j} dx_i = 1, \quad (2.7)$$

Given that Finner's inequality directly generalised the Loomis–Whitney inequality, it seems natural to try and adapt the factorisation given by Theorem 2.2 to satisfy conditions (2.6) and (2.7). Indeed, this is exactly the approach taken in [5], which we will now discuss.

Proof of Theorem 2.6. Recall that in order to factorise the Loomis–Whitney inequality, we started by essentially writing the test function g as a telescoping product. Similarly, we begin by writing $h(x)^{p'}$ as follows

$$h(x)^{p'} = \left(\frac{h(x)^{p'}}{\int_{\mathbb{R}} h(x)^{p'} dx_d} \right) \left(\frac{\int_{\mathbb{R}} h(x)^{p'} dx_d}{\int_{\mathbb{R}} \int_{\mathbb{R}} h(x)^{p'} dx_d dx_{d-1}} \right) \cdots \left(\frac{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} h(x)^{p'} dx_d \cdots dx_2}{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} h(x)^{p'} dx_d \cdots dx_1} \right) \cdot \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} h(x)^{p'} dx_d \cdots dx_1 \right),$$

where the last factor is just the $L^{p'}$ norm of h so is equal to 1. So we can write

$$h(x)^{p'} = \gamma_d(x_1, x_2, \dots, x_d) \gamma_{d-1}(x_1, x_2, \dots, x_{d-1}) \cdots \gamma_1(x_1),$$

where

$$\gamma_i(x_1, \dots, x_j) := \frac{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} h(x)^{p'} dx_d \cdots dx_{i+1}}{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} h(x)^{p'} dx_d \cdots dx_i}.$$

Furthermore, for each i , by construction

$$\int_{\mathbb{R}} \gamma_i(x_1, \dots, x_i) dx_i = 1 \tag{2.8}$$

uniformly in $x_1, x_2, \dots, x_{i-1}$. The hypothesis in Finner's inequality is that for all $i \in \{1, \dots, d\}$ we have $\sum_{j=1}^m p_j \chi_{S_j}(i) = 1$. Let $b_{ij} = 1 - \chi_{S_j}(i)$. Then for all i

$$\sum_{j=1}^m \left(\frac{p_j}{p} \right) b_{ij} = \frac{\sum_{j=1}^m p_j (1 - \chi_{S_j}(i))}{p} = \frac{p - \sum_{j=1}^m p_j \chi_{S_j}(i)}{p} = \frac{p - 1}{p} = \frac{1}{p'}.$$

Now, for $1 \leq j \leq m$, define

$$h_j(x) = \prod_{i=1}^d \gamma_i(x_1, \dots, x_i)^{b_{ij}}.$$

It follows that

$$\prod_{j=1}^m h_j(x)^{\frac{p_j}{p}} = \prod_{j=1}^m \prod_{i=1}^d \gamma_i(x_1, \dots, x_i)^{\left(\frac{p_j}{p}\right) b_{ij}} = \prod_{i=1}^d \gamma_i(x_1, \dots, x_i)^{\sum_{j=1}^m \left(\frac{p_j}{p}\right) b_{ij}} = \prod_{i=1}^d \gamma_i(x_1, \dots, x_i)^{\frac{1}{p'}} = h(x),$$

which is what we needed for condition (2.6). Now, it suffices to check that for $1 \leq j \leq m$ we have

$$\sup_{(x_i)_{i \in S_j} \in E_j} \left(\int_{E_j^\perp} h_j(x) \prod_{i \notin S_j} dx_i \right) \leq 1.$$

To see why this is true, first note that for fixed $(x_i)_{i \in S_j} \in E_j$

$$\int_{E_j^\perp} h_j(x) \prod_{i \notin S_j} dx_{i'} = \int_{E_j^\perp} \prod_{i'=1}^d \gamma_i(x_1, \dots, x_{i'})^{b_{ij}} \prod_{i \notin S_j} dx_{i'} = \int_{E_j^\perp} \prod_{i' \notin S_j} \gamma_i(x_1, \dots, x_{i'})^{b_{ij}} \prod_{i' \notin S_j} dx_{i'}.$$

Recall that each $\gamma_{i'}$ only depends on variables $x_1, \dots, x_{i'}$, so if $k = \max\{i' \in S_j^c\}$ the only $\gamma_{i'}$ in the above integral which depends on x_k will be γ_k . This will be the inner most integral and will be equal to 1 by (2.8). By a similar argument, then second inner most integral will be integration over $x_{k'}$, where k' denotes the second largest index in S_j^c . Furthermore, there will only be one $\gamma_{i'}$ left which depends on $x_{k'}$, namely $\gamma_{k'}$. Again by (2.8), this integral will be equal to 1. Performing the integrals one by one in this manner yields

$$\int_{E_j^\perp} \prod_{i' \notin S_j} \gamma_{i'}(x_1, \dots, x_{i'})^{b_{i'j}} \prod_{i' \notin S_j} dx_{i'} = 1.$$

Hence, we have shown that uniformly in $(x_i)_{i \in S_j}$

$$\int_{E_j^\perp} h_j(x) \prod_{i' \notin S_j} dx_{i'} = 1,$$

which concludes the argument. $\square$

In both the Loomis–Whitney and Finner inequalities, the standard basis vectors play a key role in formulating the inequality. In the Loomis–Whitney inequality, the functions f_j on the left hand side are composed with projections $\pi_j := \pi_{e_j^\perp}$ and in Finner's inequality the analogous maps are π_{E_j} for $E_j = \text{span}\{e_i \mid i \in S_j\}$. It is natural to ask whether one can prove similar inequalities where we replace the standard basis vectors by some other basis $\omega_1, \dots, \omega_d$. This is exactly how we arrive at affine-invariant versions of the above two inequalities, which we now discuss. As we will see, the factorisations we have given for the Loomis–Whitney and Finner inequalities can be adapted to this affine setting.

Affine invariant Loomis–Whitney inequality There seem to be two most natural ways to make the Loomis–Whitney inequality "affine-invariant". Let $\omega_1, \dots, \omega_d \in \mathbb{R}^d$ be unit vectors forming a basis of $\mathbb{R}^d$, which will replace the role of $e_1, \dots, e_d$ in the classical Loomis–Whitney inequality. Note that for each $j \in \{1, \dots, d\}$ we defined

$$\pi_j := \pi_{e_j^\perp} = \pi_{\text{span}\{e_i \mid i \neq j\}},$$

which suggests two possible replacements for π_j , namely $\pi_{\omega_j^\perp}$ or $\pi_{\text{span}\{\omega_i \mid i \neq j\}}$. In fact, we have the following two sharp inequalities which could both be called affine-invariant versions of the Loomis–Whitney inequality:

Theorem 2.7 ([16]). *Let $\omega_1, \dots, \omega_d \in \mathbb{R}^d$ be unit vectors forming a basis of $\mathbb{R}^d$, then we have the following sharp inequality*

$$\int_{\mathbb{R}^d} \prod_{j=1}^d f_j \left(\pi_{\omega_j^\perp} x \right)^{\frac{1}{d-1}} dx \leq |\omega_1 \wedge \dots \wedge \omega_d|^{-\frac{1}{d-1}} \prod_{j=1}^d \left(\int_{\omega_j^\perp} f_j \right)^{\frac{1}{d-1}}, \quad (2.9)$$

where we denote by $|\omega_1 \wedge \dots \wedge \omega_d|$ the volume of the parallelotope with edges $\omega_1, \dots, \omega_d$.

Alternatively, we have

Theorem 2.8 ([5]). *Let $u_1, \dots, u_d \in \mathbb{R}^d$ be unit vectors forming a basis of $\mathbb{R}^d$ and define $U_j := \text{span}\{u_i \mid i \neq j\}$, then we have the following sharp inequality*

$$\int_{\mathbb{R}^d} \prod_{j=1}^d f_j \left(\pi_{U_j} x \right)^{\frac{1}{d-1}} dx \leq \frac{\prod_{j=1}^d |\wedge_{i \neq j} u_i|}{|u_1 \wedge \dots \wedge u_d|} \prod_{j=1}^d \left(\int_{U_j} f_j \right)^{\frac{1}{d-1}}. \quad (2.10)$$

As it turns out, these two results are completely equivalent. To prove this, we establish a natural pairing of d -tuples of linearly independent unit vectors in $\mathbb{R}^d$, with the property that if $\omega_1, \dots, \omega_d$ and $u_1, \dots, u_d$ are such a pair, then for each j

$$\omega_j^\perp = \text{span}\{u_i \mid i \neq j\},$$

and

$$|\omega_1 \wedge \dots \wedge \omega_d|^{-\frac{1}{d-1}} \frac{\prod_{j=1}^d \left| \bigwedge_{i \neq j} u_i \right|^{\frac{1}{d-1}}}{|u_1 \wedge \dots \wedge u_d|}.$$

In this way, one can see that (2.9) and (2.10) are actually just two ways of writing the same inequality. This last observation combined with the fact that this pairing exhausts all d -tuples of linearly independent unit vectors is enough to establish the equivalence of Theorems 2.7 and 2.8.

Firstly we state the following simple lemma:

Lemma 2.9. *Let $\omega_1, \dots, \omega_d$ be unit vectors forming a basis of $\mathbb{R}^d$ and $v_1, \dots, v_d \in \mathbb{R}^d$ be uniquely determined by*

$$\langle \omega_i, v_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then for every $S \subset \{1, \dots, d\}$ we have

$$\left| \bigwedge_{i \in S} v_i \right| = \frac{\left| \bigwedge_{i \notin S} \omega_i \right|}{|\omega_1 \wedge \dots \wedge \omega_d|},$$

where we let the trivial wedge product of no vectors equal 1.

Before proving this last statement, let us establish the equivalence between these two forms of the affine-invariant Loomis–Whitney inequality:

Proposition 2.10. *Theorem 2.7 and Theorem 2.8 are the same up to a change of notation*

My contribution to the formulation and proof of this result is acknowledged in [5].

Proof of Proposition 2.10. Let us begin by making more precise this one-to-one pairing mentioned just before the statement of Lemma 2.9. Fix linearly independent unit vectors $\omega_1, \dots, \omega_d \in \mathbb{R}^d$ and denote by A the matrix with rows $\omega_1, \dots, \omega_d$. By assumption A has an inverse, so we can let $v_1, \dots, v_d \in \mathbb{R}^d$ denote the columns of A^{-1} , which will also be linearly independent but not necessarily all of unit length. We can of course now define a new family of linearly independent unit vectors by setting $u_j := \frac{v_j}{\|v_j\|}$ for $j = 1, \dots, d$. Since $AA^{-1} = I_d$, it is immediate that

$$\langle \omega_i, v_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \quad (2.11)$$

In particular, whenever $i \neq j$ we have $\omega_i \in v_j^\perp = u_j^\perp$. Starting with this new family of linearly independent unit vectors $u_1, \dots, u_d$, one can apply the same process in reverse to exactly recover

the original family $\omega_1, \dots, \omega_d$. To verify this, let B denote the invertible matrix with rows $u_1, \dots, u_d$ and set $\tilde{v}_1, \dots, \tilde{v}_d$ to be the columns of B^{-1} . As before, since $BB^{-1} = I_d$ we have

$$\langle \tilde{v}_i, u_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \quad (2.12)$$

It follows that whenever $i \neq j$ we have $\tilde{v}_i \in u_j^\perp$. Since the intersection $\bigcap_{i \neq j} u_j^\perp$ is a one dimensional subspace, for each i we have $\frac{\tilde{v}_i}{\|\tilde{v}_i\|} = \sigma \omega_i$ with $\sigma = \pm 1$. Combining what we already have, for each i one can write

$$1 = \langle \tilde{v}_i, u_i \rangle = \left\langle \sigma \|\tilde{v}_i\| \omega_i, \frac{v_i}{\|v_i\|} \right\rangle = \frac{\sigma \|\tilde{v}_i\|}{\|v_i\|} \langle \omega_i, v_i \rangle = \frac{\sigma \|\tilde{v}_i\|}{\|v_i\|}.$$

Since both $\|\tilde{v}_i\|$ and $\|v_i\|$ are positive we can see that σ must be equal to 1, thus $\frac{\tilde{v}_i}{\|\tilde{v}_i\|} = \omega_i$ for each i .

It should now be clear that families of d linearly independent unit vectors always come in pairs. As a result, it suffices to prove that for any such pair of families $\omega_1, \dots, \omega_d$ and $u_1, \dots, u_d$, related in the manner described above, (2.9) and (2.10) are two ways of writing the same inequality. Firstly, By condition (2.11) we have $\text{span}\{v_j \mid i \neq j\} \subset \omega_j^\perp$, but the v_j are linearly independent so in fact we can write

$$\omega_j^\perp = \text{span}\{v_j \mid i \neq j\} = \text{span}\{u_i \mid i \neq j\}.$$

With notation of (2.10), this can be written $\omega_j^\perp = U_j$ for $j = 1, \dots, d$. It remains to prove that

$$|\omega_1 \wedge \dots \wedge \omega_d|^{-\frac{1}{d-1}} = \frac{\prod_{j=1}^d \left| \bigwedge_{i \neq j} u_i \right|^{\frac{1}{d-1}}}{|u_1 \wedge \dots \wedge u_d|}.$$

This can be deduced using Lemma 2.9 as follows

$$\begin{aligned} |\omega_1 \wedge \dots \wedge \omega_d|^{-\frac{1}{d-1}} &= \frac{1}{|\omega_1 \wedge \dots \wedge \omega_d|^{\frac{d}{d-1}-1}} = \frac{\prod_{j=1}^d |\omega_1 \wedge \dots \wedge \omega_d|^{-\frac{1}{d-1}}}{|\omega_1 \wedge \dots \wedge \omega_d|^{-1}} \\ &= \frac{\prod_{j=1}^d \left(\frac{\|\omega_j\|}{|\omega_1 \wedge \dots \wedge \omega_d|} \right)^{\frac{1}{d-1}}}{|\omega_1 \wedge \dots \wedge \omega_d|^{-1}} \\ &= \frac{\prod_{j=1}^d \left| \bigwedge_{i \neq j} v_i \right|^{\frac{1}{d-1}}}{|v_1 \wedge \dots \wedge v_d|} \\ &= \frac{\prod_{j=1}^d \left(\prod_{i \neq j} \|v_i\| \right)^{\frac{1}{d-1}} \left| \bigwedge_{i \neq j} u_i \right|^{\frac{1}{d-1}}}{\left(\prod_{j=1}^d \|v_j\| \right) |u_1 \wedge \dots \wedge u_d|} \\ &= \frac{\prod_{j=1}^d \left| \bigwedge_{i \neq j} u_i \right|^{\frac{1}{d-1}}}{|u_1 \wedge \dots \wedge u_d|} \end{aligned}$$

□

It remains to prove Lemma 2.9, which we establish via the following slightly more general statement:

Lemma 2.11. *Let $\omega_1, \dots, \omega_d$ be unit vectors forming a basis of $\mathbb{R}^d$ and $v_1, \dots, v_d \in \mathbb{R}^d$ be uniquely determined by*

$$\langle \omega_i, v_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then for every subset $S \subset \{1, \dots, d\}$ of cardinality k and any choice of vectors $u_1, \dots, u_k \in \mathbb{R}^d$ we have

$$|u_1 \wedge \dots \wedge u_k| \left| \bigwedge_{i \in S} v_i \right| = \frac{|u_1 \wedge \dots \wedge u_k \wedge (\bigwedge_{i \notin S} \omega_i)|}{|\omega_1 \wedge \dots \wedge \omega_d|},$$

where we let the trivial wedge product of no vectors equal 1.

Of course, if we choose $u_1, \dots, u_k$ to be some orthonormal set spanning $\{\omega_i \mid i \notin S\}^\perp$ then we immediately recover Lemma 2.9. Using the language of multilinear algebra, Lemma 2.11 is equivalent to the statement that for any $S \subset \{1, \dots, d\}$ we have $\star(\bigwedge_{i \in S} v_i) = \pm |\omega_1 \wedge \dots \wedge \omega_d|^{-1} \bigwedge_{i \notin S} \omega_i$, where $\star$ denotes the Hodge star operator.

Proof of Lemma 2.11. By a simple reordering of indices, it suffice to prove that for every $1 \leq k \leq d$ and any choice of vectors $u_1, \dots, u_k \in \mathbb{R}^d$ we have

$$|u_1 \wedge \dots \wedge u_k| |v_1 \wedge \dots \wedge v_k| = \frac{|u_1 \wedge \dots \wedge u_k \wedge \omega_{k+1} \wedge \dots \wedge \omega_d|}{|\omega_1 \wedge \dots \wedge \omega_d|}. \quad (2.13)$$

Note that by assumption $\omega_1, \dots, \omega_d$ are linearly independent, so for each $1 \leq i \leq k$ we can write

$$u_i = \sum_{l=1}^d c_{i,l} \omega_l.$$

Using this, the numerator on right hand side of (2.13) can be written as

$$\left| \left(\sum_{l=1}^d c_{1,l} \omega_l \right) \wedge \dots \wedge \left(\sum_{l=1}^d c_{k,l} \omega_l \right) \wedge \omega_{k+1} \wedge \dots \wedge \omega_d \right| = \det \left(\{c_{i,j}\}_{ij} \right) |\omega_1 \wedge \dots \wedge \omega_d|.$$

Hence the right hand side is equal to

$$\det \left(\{c_{i,j}\}_{ij} \right).$$

On the other hand we can write the left hand side as follows

$$\begin{aligned} |u_1 \wedge \dots \wedge u_k| |v_1 \wedge \dots \wedge v_k| &= \det \left(\{ \langle u_i, v_j \rangle \}_{ij} \right) = \det \left(\left\{ \left\langle \sum_{l=1}^d c_{i,l} \omega_l, v_j \right\rangle \right\}_{ij} \right) \\ &= \det \left(\left\{ \sum_{l=1}^d c_{i,l} \langle \omega_l, v_j \rangle \right\}_{ij} \right) \\ &= \det \left(\{c_{i,j}\}_{ij} \right). \end{aligned}$$

□

For simplicity we will work with Theorem 2.7, but all the arguments we present below can easily be translated into the notation of Theorem 2.8. As with the previous two examples, one can reformulate the affine-invariant Loomis–Whitney inequality in terms of factorisations. Taking both sides of (2.9) to the power $\frac{d-1}{d}$ we get

$$\left\| \prod_{j=1}^d \left(f_j \circ \pi_{\omega_j^\perp} \right)^{\frac{1}{d}} \right\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d}} \prod_{j=1}^d \|f_j\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{d}},$$

which is now in the correct form to apply multilinear duality. By linear duality, this is equivalent to the statement that for any $g \in L^d(\mathbb{R}^d)$ with $\|g\|_d = 1$, we have

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d f_j \left(\pi_{\omega_j^\perp} x \right) \right)^{\frac{1}{d}} g(x) dx \leq |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d}} \prod_{j=1}^d \|f_j\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{d}}. \quad (2.14)$$

As we did with the classical Loomis–Whitney inequality, let us run through the easier half of Theorem 1.1 in order to understand the role of factorisations in this affine-invariant setting. Suppose that for $1 \leq j \leq d$, we can find functions $g_j: \mathbb{R}^d \rightarrow \mathbb{R}$ such that for every $x \in \mathbb{R}^d$

$$g(x) \leq \prod_{j=1}^d g_j(x)^{\frac{1}{d}},$$

and for $1 \leq j \leq d$ and almost every $u \in \omega_j^\perp$

$$\int_{\mathbb{R}} g_j(u + t_j \omega_j) dt_j \leq |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d}}.$$

Then by applying Hölder's inequality and identifying $\mathbb{R}^d \cong \omega_j^\perp \times \mathbb{R}$ we can deduce (2.14) as follows:

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\prod_{j=1}^d f_j \left(\pi_{\omega_j^\perp} x \right) \right)^{\frac{1}{d}} g(x) dx &\leq \int_{\mathbb{R}^d} \left(\prod_{j=1}^d f_j \left(\pi_{\omega_j^\perp} x \right) g_j(x) \right)^{\frac{1}{d}} dx \\ &\leq \prod_{j=1}^d \left(\int_{\mathbb{R}^d} f_j \left(\pi_{\omega_j^\perp} x \right) g_j(x) dx \right)^{\frac{1}{d}} \\ &= \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} f_j(u) \left(\int_{\mathbb{R}} g_j(u + t_j \omega_j) dt_j \right) du \right)^{\frac{1}{d}} \\ &\leq \prod_{j=1}^d \left(\left(\sup_{u \in \omega_j^\perp} \int_{\mathbb{R}} g_j(u + t_j \omega_j) dt_j \right) \left(\int_{\omega_j^\perp} f_j(u) du \right) \right)^{\frac{1}{d}} \\ &\leq \prod_{j=1}^d \left(\left(|\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d}} \right) \left(\int_{\omega_j^\perp} f_j(u) du \right) \right)^{\frac{1}{d}} \\ &\leq |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d}} \prod_{j=1}^d \left(\int_{\omega_j^\perp} f_j \right)^{\frac{1}{d}} \\ &= |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d}} \prod_{j=1}^d \|f_j\|_{L^1}^{\frac{1}{d}}. \end{aligned}$$

As consequence of Theorem 1.1, we know that such functions $g_1, \dots, g_d$ must exist. Similarly to the examples mentioned above, we can actually construct factorisations for the affine-invariant Loomis–Whitney inequality explicitly:

Theorem 2.12 (Factorisation for the affine-invariant Loomis–Whitney inequality [16]). *Fix linearly independent unit vectors $\omega_1, \dots, \omega_d \in \mathbb{R}^d$. Then for every $g \in L^d(\mathbb{R}^d)$ with $\|g\|_d = 1$ we can construct functions $g_1, \dots, g_d$ on $\mathbb{R}^d$ such that for almost every $x \in \mathbb{R}^d$*

$$g(x) = \prod_{j=1}^d g_j(x)^{\frac{1}{d}}, \quad (2.15)$$

and for $1 \leq j \leq d$ and almost every $u \in \omega_j^\perp$

$$\int_{\mathbb{R}} g_j(u + t_j \omega_j) dt_j = |\omega_1 \wedge \dots \wedge \omega_d|^{-\frac{1}{d}}. \quad (2.16)$$

Following ideas from [16], let us now adapt the factorisation for the classical Loomis–Whitney inequality to this affine-invariant setting.

Proof of Theorem 2.12. The construction here is very similar to the factorisation of the classical Loomis–Whitney inequality, the main difference being that vectors $\omega_1, \dots, \omega_d$ will replace the role played by standard basis vectors. In the factorisation of the Loomis–Whitney inequality we initially wrote the test function as a telescoping product; similarly, here we begin by writing

$$g(x)^d = \tilde{g}_1(x) \cdots \tilde{g}_d(x) \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(x + s_1 \omega_1 + \cdots + s_d \omega_d)^d ds_1 \cdots ds_d \right),$$

where

$$\tilde{g}_j(x) := \frac{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(x + s_1 \omega_1 + \cdots + s_{j-1} \omega_{j-1})^d ds_1 \cdots ds_{j-1}}{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(x + s_1 \omega_1 + \cdots + s_j \omega_j)^d ds_1 \cdots ds_j}.$$

It is not hard to see that for each j and every $y \in \mathbb{R}^d$ we have

$$\begin{aligned} \int_{\mathbb{R}} \tilde{g}_j(y + t_j \omega_j) dt_j &= \int_{\mathbb{R}} \frac{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(y + t_j \omega_j + s_1 \omega_1 + \cdots + s_{j-1} \omega_{j-1})^d ds_1 \cdots ds_{j-1}}{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(y + t_j \omega_j + s_1 \omega_1 + \cdots + s_j \omega_j)^d ds_1 \cdots ds_j} dt_j \\ &= \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(y + s_1 \omega_1 + \cdots + s_{j-1} \omega_{j-1} + t_j \omega_j)^d ds_1 \cdots ds_{j-1} dt_j}{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(y + s_1 \omega_1 + \cdots + (t_j + s_j) \omega_j)^d ds_1 \cdots ds_j} = \\ &= \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(y + s_1 \omega_1 + \cdots + s_{j-1} \omega_{j-1} + t_j \omega_j)^d ds_1 \cdots ds_{j-1} dt_j}{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(y + s_1 \omega_1 + \cdots + \tilde{s} \omega_j)^d ds_1 \cdots ds_{j-1} d\tilde{s}} \\ &= 1, \end{aligned}$$

where all we have done really is make a change of variables $\tilde{s} = (t_j + s_j)$. By another simple change of variables $y = x + s_1 \omega_1 + \cdots + s_d \omega_d$, we can calculate the last factor as follows

$$\begin{aligned} \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(x + s_1 \omega_1 + \cdots + s_d \omega_d)^d ds_1 \cdots ds_d \right) &= \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(y)^d |\omega_1 \wedge \cdots \wedge \omega_d|^{-1} dy \right) \\ &= |\omega_1 \wedge \cdots \wedge \omega_d|^{-1} \|g\|_{L^d(\mathbb{R}^d)} \\ &= |\omega_1 \wedge \cdots \wedge \omega_d|^{-1}. \end{aligned}$$

Bearing this in mind, we can distribute this constant factor evenly among the factors $\tilde{g}_j$. More precisely, define new functions

$$g_j(x) := |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d}} \tilde{g}_j(x).$$

It follows that

$$g(x)^d = g_1(x) \cdots g_d(x),$$

which deals with condition (2.15). In order to verify condition (2.16), for $y \in \mathbb{R}^d$ write $y = \pi_{\omega_j^\perp} y + \langle \omega_j, y \rangle \omega_j$, then by a change of variables

$$\int_{\mathbb{R}} g_j(y + t_j \omega_j) dt_j = \int_{\mathbb{R}} g_j(\pi_{\omega_j^\perp} y + \langle \omega_j, y \rangle \omega_j + t_j \omega_j) dt_j = \int_{\mathbb{R}} g_j(\pi_{\omega_j^\perp} y + \tilde{t}_j \omega_j) d\tilde{t}_j.$$

Since $y \in \mathbb{R}^d$ was arbitrary, it is clear that for every $u \in \omega_j^\perp$

$$\int_{\mathbb{R}} g_j(u + t_j \omega_j) dt_j = |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d}}.$$

□

Affine-invariant Finner inequality Generalising the previous example we now state an affine-invariant version of Finner's inequality. As with the affine-invariant Loomis–Whitney inequality there seem to be two natural ways of formulating such an inequality, but up to a change of notation they will be identical - here we just state one of the formulations which can be seen as a direct generalisation of Theorem 2.7.

Theorem 2.13 (Affine-invariant Finner inequality - [5]). *Suppose we have linearly independent unit vectors $\omega_1, \dots, \omega_d \in \mathbb{R}^d$, exponents $p_1, \dots, p_m > 0$ and subsets $S_1, \dots, S_m \subset [d]$. Set $p = \sum_{j=1}^m p_j$ and for $1 \leq j \leq m$ define the following subspace of $\mathbb{R}^d$*

$$H_j := \text{span}\{\omega_i \mid i \in S_j^c\}.$$

If we have for every $1 \leq i \leq d$

$$\sum_{j=1}^m p_j \chi_{S_j}(i) = 1,$$

then the following inequality holds

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j \left(\pi_{H_j^\perp} x \right)^{p_j} dx \leq \frac{\prod_{j=1}^m \left| \bigwedge_{i \in S_j} \omega_i \right|^{\frac{p_j}{p}}}{|\omega_1 \wedge \cdots \wedge \omega_d|^{p-1}} \prod_{j=1}^m \left(\int_{H_j^\perp} f_j \right)^{p_j}.$$

As with all the previous examples let us begin by rewriting the affine-invariant Finner inequality as follows

$$\left\| \prod_{j=1}^m \left(f_j \circ \pi_{H_j^\perp} \right)^{\frac{p_j}{p}} \right\|_{L^p(\mathbb{R}^d)} \leq \frac{\prod_{j=1}^m \left| \bigwedge_{i \in S_j} \omega_i \right|^{\frac{p_j}{p}}}{|\omega_1 \wedge \cdots \wedge \omega_d|^{\frac{p-1}{p}}} \prod_{j=1}^m \left(\int_{H_j^\perp} f_j \right)^{\frac{p_j}{p}}. \quad (2.17)$$

Now it is clear that this falls under the scope of multilinear duality we can reformulate the above result as a statement about factorisations:

Theorem 2.14 (Factorisation for the affine-invariant Finner inequality - [5]). *Fix linearly independent unit vectors $\omega_1, \dots, \omega_d \in \mathbb{R}^d$, exponents $p_1, \dots, p_m > 0$ and subsets $S_1, \dots, S_m \subset [d]$. Set $p = \sum_{j=1}^m p_j$ and for $1 \leq j \leq m$ define the following subspace of $\mathbb{R}^d$*

$$H_j := \text{span}\{\omega_i \mid i \in S_j^c\}.$$

Suppose that for $1 \leq i \leq d$ we have

$$\sum_{j=1}^m p_j \chi_{S_j}(i) = 1.$$

Then for every $g \in L^{p'}(\mathbb{R}^d)$ with $\|g\|_{p'} = 1$, we can construct functions $g_1, \dots, g_m$ on $\mathbb{R}^d$ such that for almost every $x \in \mathbb{R}^d$

$$g(x) = \prod_{j=1}^m g_j(x)^{\frac{p_j}{p}} \quad (2.18)$$

and for $1 \leq j \leq m$ and almost every $y_j \in H_j^\perp$

$$\int_{H_j} g_j(y_j + z_j) dz_j = \frac{\left| \bigwedge_{i \in S_j} \omega_i \right|}{\left| \omega_1 \wedge \dots \wedge \omega_d \right|^{\frac{p-1}{p}}}. \quad (2.19)$$

Using a very similar duality argument to the previous example, it is not difficult to see that this result directly implies Theorem 2.13. An explicit construction of factorisations for the affine-invariant Finner inequality can be found in [5]. Their argument is similar in spirit to the proof of Theorem 2.12 but more technical. Essentially, one can build a factorisation satisfying the hypotheses of Theorem 2.14 by taking appropriate products of the factorisation we constructed above for the affine-invariant Loomis–Whitney. This is analogous to the way in which factorisations for the orthogonal Finner inequality were built using the construction given for factorising the orthogonal Loomis–Whitney inequality.

Next we talk about a slightly different way to generalise the orthogonal examples mentioned earlier. In the Loomis–Whitney inequality, the order of multilinearity, usually denoted by m , is equal to the dimension of the ambient space, denoted by d . This is not the case for Finner’s inequality, where m is allowed to be strictly larger than d , provided that the exponents p_j satisfy a certain combinatorial condition. However, in all the previous examples there is a fixed basis of d vectors which plays a crucial role, namely to define the linear maps or projections appearing in the given inequality. What happens if we instead use a larger family of vectors in place of this fixed basis? In particular, if $m > d$, can one formulate an inequality similar to the above examples where the linear maps are all defined in terms of a some family $v_1, \dots, v_m$?

Young’s convolution inequality One of the simplest examples of such an inequality is that of the classical Young convolution inequality, which is typically stated as follows

Theorem 2.15 (Young’s convolution inequality on $\mathbb{R}$). *For any $F \in L^p(\mathbb{R})$, $G \in L^q(\mathbb{R})$ with $p, q, r \in [1, \infty]$ satisfying*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

we have the bound

$$\|F * G\|_r \leq \|F\|_p \|G\|_q.$$

As with all the other examples, it is not immediately obvious that this falls under the scope of multilinear duality. To remedy this, we begin by applying linear duality and rewriting this as a trilinear norm inequality. More precisely, using linear duality of L^p -spaces we know that Theorem 2.15 is equivalent to the following statement: suppose we have $p, q, r' \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r'} = 2.$$

Then, for any $F \in L^p(\mathbb{R})$, $G \in L^q(\mathbb{R})$ and $H \in L^{r'}(\mathbb{R})$, we have the bound

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x) G(x-y) dx \right) H(y) dy \leq \|F\|_p \|G\|_q \|H\|_{r'}.$$

Replacing F, G, H with $f^{\frac{1}{p}}, g^{\frac{1}{q}}, h^{\frac{1}{r'}}$ respectively allows us to rewrite the above as

$$\int_{\mathbb{R}^2} f(x)^{\frac{1}{p}} g(x-y)^{\frac{1}{q}} h(y)^{\frac{1}{r'}} dx dy \leq \left(\int_{\mathbb{R}} f \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} g \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} h \right)^{\frac{1}{r'}}.$$

Letting e_1, e_2 denote the standard basis vectors on $\mathbb{R}^2$ and using the fact that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r'} = 2$, we can rewrite the last inequality as

$$\left\| f(\langle \cdot, e_1 \rangle)^{\frac{1}{2p}} g(\langle \cdot, (e_1 - e_2) \rangle)^{\frac{1}{2q}} h(\langle \cdot, e_2 \rangle)^{\frac{1}{2r'}} \right\|_{L^2(\mathbb{R}^2)} \leq \|f\|_{L^1(\mathbb{R})}^{\frac{1}{2p}} \|g\|_{L^1(\mathbb{R})}^{\frac{1}{2q}} \|h\|_{L^1(\mathbb{R})}^{\frac{1}{2r'}}, \quad (2.20)$$

which is exactly in the form to apply Theorem 1.1. For clarity, let us now restate the Young convolution inequality as follows

Theorem 2.16 (Young's convolution inequality - trilinear form). *For any $f_1 \in L^{p_1}(\mathbb{R})$, $f_2 \in L^{p_2}(\mathbb{R})$, $f_3 \in L^{p_3}(\mathbb{R})$ with $p_1, p_2, p_3 \in [0, 1]$ satisfying*

$$p_1 + p_2 + p_3 = 2$$

we have the bound

$$\left\| f_1(\langle \cdot, e_1 \rangle)^{\frac{p_1}{2}} f_2(\langle \cdot, (e_1 - e_2) \rangle)^{\frac{p_2}{2}} f_3(\langle \cdot, e_2 \rangle)^{\frac{p_3}{2}} \right\|_{L^2(\mathbb{R}^2)} \leq \prod_{j=1}^3 \|f_j\|_{L^1(\mathbb{R})}^{\frac{p_j}{2}}.$$

Indeed, we can now apply Theorem 1.1 directly and reformulate Young's convolution inequality as the following factorisation statement:

Theorem 2.17 (Factorisation for Young's convolution inequality). *If we fix exponents $p_1, p_2, p_3 \in [0, 1]$ satisfying*

$$p_1 + p_2 + p_3 = 2,$$

then for every $g \in L^2(\mathbb{R}^2)$ with $\|g\|_2 = 1$, there exist functions g_1, g_2, g_3 on $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^2$

$$g(x) = g_1(x)^{\frac{p_1}{2}} g_2(x)^{\frac{p_2}{2}} g_3(x)^{\frac{p_3}{2}}, \quad (2.21)$$

and we have the following bound:

$$\left(\sup_{t \in \mathbb{R}} \int_{e_1^\perp} g_1(te_1 + v) dv \right)^{\frac{p_1}{2}} \left(\sup_{t \in \mathbb{R}} \int_{(e_1 - e_2)^\perp} g_2(t(e_1 - e_2) + \sqrt{2}v) dv \right)^{\frac{p_2}{2}} \left(\sup_{t \in \mathbb{R}} \int_{e_2^\perp} g_3(te_2 + v) dv \right)^{\frac{p_3}{2}} \leq 1. \quad (2.22)$$

Furthermore, we can construct g_1, g_2, g_3 explicitly, and for every $g \in L^2(\mathbb{R}^2)$, provided that $p_1, p_2, p_3 > 0$, conditions (2.21) and (2.22) are satisfied by the following functions:

$$\begin{aligned} g_1(x) &:= \left(\sqrt{2} \right)^{\frac{p_1 + p_2 - p_3}{4p_1}} \frac{g(x)^2}{\int_{\mathbb{R}} g(x + se_2)^2 ds}; \\ g_2(x) &:= (\sqrt{2})^{-\frac{1}{2}} \frac{g(x)^{\frac{p_2 + p_3 - p_1}{p_2}} \left(\int_{\mathbb{R}} g(x + se_2)^2 ds \right)^{\frac{p_1 + p_2 - p_3}{2p_2}}}{\left(\int_{\mathbb{R}} g(x + s(e_1 + e_2))^2 ds \right)^{\frac{p_2 + p_3 - p_1}{2p_2}}}; \\ g_3(x) &:= \left(\sqrt{2} \right)^{\frac{p_2 + p_3 - p_1}{4p_3}} \left(\int_{\mathbb{R}} g(x + se_2)^2 ds \right)^{\frac{p_1 + p_3 - p_2}{2p_3}} \left(\int_{\mathbb{R}} g(x + s(e_1 + e_2))^2 ds \right)^{\frac{p_2 + p_3 - p_1}{2p_3}}. \end{aligned}$$

If one of the p_j 's is equal to zero, then in order to satisfy conditions (2.21) and (2.22), we simply modify the corresponding g_j , to be any bounded function such that

$$\sup_{t \in \mathbb{R}} \int_{u_j^\perp} g_j(tu_j + v) dv < \infty,$$

where u_1, u_2, u_3 denote unit vectors in the directions $e_1, (e_1 - e_2), e_2$ respectively.

Remark 2.18. Unlike all the previous examples, noting the form of condition (2.22), we have chosen to use the equivalence of (i) and (iii) rather than (i) and (ii) in the statement of Theorem 1.1. This is simply for convenience and as we mentioned just after Remark 1.2, by scaling each of the g_j by some constant factor we could instead deduce a factorisation satisfying conditions analogous to those in (ii), which is what we chose to do for all the previous examples.

As a reality check, let us verify that this factorisation statement does in fact imply the Young convolution inequality as stated in Theorem 2.16.

Theorem 2.17 $\implies$ Theorem 2.16. Firstly, using linear duality it suffices to prove that for any $g \in L^2(\mathbb{R}^2)$ with $\|g\|_2 = 1$

$$\int_{\mathbb{R}^2} f_1(\langle x, e_1 \rangle)^{\frac{p_1}{2}} f_2(\langle x, (e_1 - e_2) \rangle)^{\frac{p_2}{2}} f_3(\langle x, e_2 \rangle)^{\frac{p_3}{2}} g(x) dx \leq \prod_{j=1}^3 \|f_j\|_{L^1(\mathbb{R})}^{\frac{p_j}{2}}.$$

Using (2.21) along with the multilinear Hölder inequality, we can deduce that

$$\begin{aligned} & \int_{\mathbb{R}^2} f_1(\langle x, e_1 \rangle)^{\frac{p_1}{2}} f_2(\langle x, (e_1 - e_2) \rangle)^{\frac{p_2}{2}} f_3(\langle x, e_2 \rangle)^{\frac{p_3}{2}} g(x) dx \\ & \leq \int_{\mathbb{R}^2} f_1(\langle x, e_1 \rangle)^{\frac{p_1}{2}} f_2(\langle x, (e_1 - e_2) \rangle)^{\frac{p_2}{2}} f_3(\langle x, e_2 \rangle)^{\frac{p_3}{2}} \prod_{j=1}^3 g_j(x)^{\frac{p_j}{2}} dx \\ & \leq \left(\int_{\mathbb{R}^2} f_1(\langle x, e_1 \rangle) g_1(x) dx \right)^{\frac{p_1}{2}} \left(\int_{\mathbb{R}^2} f_2(\langle x, (e_1 - e_2) \rangle) g_2(x) dx \right)^{\frac{p_2}{2}} \left(\int_{\mathbb{R}^2} f_3(\langle x, e_2 \rangle) g_3(x) dx \right)^{\frac{p_3}{2}}. \end{aligned}$$

We will now show that this last quantity is bounded above by $\prod_{j=1}^3 \|f_j\|_{L^1(\mathbb{R})}^{\frac{p_j}{2}}$. In order to use condition (2.22), we need to factor out $\|f_1\|_{L^1(\mathbb{R})}, \|f_2\|_{L^1(\mathbb{R})}, \|f_3\|_{L^1(\mathbb{R})}$ from the three respective integrals. Firstly, by identifying $\mathbb{R}^2 \cong e_1 \oplus e_1^\perp$, we can apply Fubini's theorem to get

$$\int_{\mathbb{R}^2} f_1(\langle x, e_1 \rangle) g_1(x) dx = \int_{\mathbb{R}} f_1(t) \left(\int_{e_1^\perp} g_1(te_1 + v) dv \right) dt \leq \|f_1\|_{L^1(\mathbb{R})} \sup_{t \in \mathbb{R}} \left(\int_{e_1^\perp} g_1(te_1 + v) dv \right).$$

Similarly, identifying $\mathbb{R}^2 \cong e_2 \oplus e_2^\perp$ we can write

$$\int_{\mathbb{R}^2} f_3(\langle x, e_2 \rangle) g_3(x) dx = \int_{\mathbb{R}} f_3(t) \left(\int_{e_2^\perp} g_3(te_2 + v) dv \right) dt \leq \|f_3\|_{L^1(\mathbb{R})} \sup_{t \in \mathbb{R}} \left(\int_{e_2^\perp} g_3(te_2 + v) dv \right).$$

Before applying Fubini's theorem in the remaining integral, let us make the following simple change of variables:

$$\int_{\mathbb{R}^2} f_2(\langle x, (e_1 - e_2) \rangle) g_2(x) dx = \int_{\mathbb{R}^2} f_2 \left(\left\langle y, \frac{(e_1 - e_2)}{\sqrt{2}} \right\rangle \right) g_2 \left(\frac{y}{\sqrt{2}} \right) \frac{1}{2} dy.$$

Identifying $\mathbb{R}^2 \cong \frac{e_1 - e_2}{\sqrt{2}} \oplus \left(\frac{e_1 - e_2}{\sqrt{2}} \right)^\perp$, we can then write

$$\int_{\mathbb{R}^2} f_2 \left(\left\langle y, \frac{(e_1 - e_2)}{\sqrt{2}} \right\rangle \right) g_2 \left(\frac{y}{\sqrt{2}} \right) \frac{1}{2} dy = \int_{\mathbb{R}} f_2(t) \left(\int_{\left(\frac{e_1 - e_2}{\sqrt{2}} \right)^\perp} g_2 \left(\frac{t(e_1 + e_2)}{2} + \frac{w}{\sqrt{2}} \right) \frac{1}{2} dw \right) dt.$$

Now by making a change of variables $v = \frac{w}{\sqrt{2}}$, we get

$$\begin{aligned} & \int_{\mathbb{R}} f_2(t) \left(\int_{\left(\frac{e_1 - e_2}{\sqrt{2}} \right)^\perp} g_2 \left(\frac{t(e_1 + e_2)}{2} + \frac{w}{\sqrt{2}} \right) \frac{1}{2} dw \right) dt \\ &= \int_{\mathbb{R}} f_2(t) \left(\int_{\left(\frac{e_1 - e_2}{\sqrt{2}} \right)^\perp} g_2 \left(\frac{t(e_1 + e_2)}{2} + \sqrt{2}v \right) dv \right) dt \\ &\leq \|f_2\|_{L^1(\mathbb{R})} \sup_{t \in \mathbb{R}} \left(\int_{\left(\frac{e_1 - e_2}{\sqrt{2}} \right)^\perp} g_2 \left(\frac{t(e_1 + e_2)}{2} + \sqrt{2}v \right) dv \right) \\ &\leq \|f_2\|_{L^1(\mathbb{R})} \sup_{\tilde{t} \in \mathbb{R}} \left(\int_{(e_1 - e_2)^\perp} g_2 \left(\tilde{t}(e_1 - e_2) + \sqrt{2}v \right) dv \right). \end{aligned}$$

Combining everything we have, it follows by (2.22) that

$$\begin{aligned} & \left(\int_{\mathbb{R}^2} f_1(x) g_1(x, y) dx dy \right)^{\frac{p_1}{2}} \left(\int_{\mathbb{R}^2} f_2(x - y) g_2(x, y) dx dy \right)^{\frac{p_2}{2}} \left(\int_{\mathbb{R}^2} f_3(y) g_3(x, y) dx dy \right)^{\frac{p_3}{2}} \\ &\leq \left(\prod_{j=1}^3 \|f_j\|_{L^1(\mathbb{R})}^{\frac{p_j}{2}} \right) \left(\sup_{t \in \mathbb{R}} \int_{e_1^\perp} g_1(te_1 + v) dv \right)^{\frac{p_1}{2}} \left(\sup_{t \in \mathbb{R}} \int_{(e_1 - e_2)^\perp} g_2 \left(t(e_1 - e_2) + \sqrt{2}v \right) dv \right)^{\frac{p_2}{2}} \\ &\quad \cdot \left(\sup_{t \in \mathbb{R}} \int_{e_2^\perp} g_3(te_2 + v) dv \right)^{\frac{p_3}{2}} \\ &\leq \left(\prod_{j=1}^3 \|f_j\|_{L^1(\mathbb{R})}^{\frac{p_j}{2}} \right), \end{aligned}$$

which concludes the proof. $\square$

Following ideas from [16], we will now discuss how to prove Theorem 2.17 directly. The main part of the proof can be seen as an interpolation at the level of factorisations. For $k = 0, 1$, fix exponents $p_{k,1}, p_{k,2}, p_{k,3} \in [0, 1]$ such that $p_{k,1} + p_{k,2} + p_{k,3} = 2$ and let $g_{k,1}, g_{k,2}, g_{k,3}$ be a factorisation proving the following case of the Young convolution inequality:

$$\left\| f_1(\langle \cdot, e_1 \rangle)^{\frac{p_{k,1}}{2}} f_2(\langle \cdot, (e_1 - e_2) \rangle)^{\frac{p_{k,2}}{2}} f_3(\langle \cdot, e_2 \rangle)^{\frac{p_{k,3}}{2}} \right\|_{L^2(\mathbb{R}^2)} \leq A_k \prod_{j=1}^3 \|f_j\|_{L^1(\mathbb{R})}^{\frac{p_{k,j}}{2}}.$$

To be precise, we require $g_{k,1}, g_{k,2}, g_{k,3}$ to satisfy conditions analogous to (ii) in the statement of Theorem 1.1. Given $\theta \in [0, 1]$, we will see that functions $g_{\theta,j} := g_{0,j}^\theta g_{1,j}^{(1-\theta)}$ define a factorisation for the Young convolution inequality with exponents $p_{\theta,j} := \theta p_{0,j} + (1-\theta)p_{1,j}$ written explicitly as

$$\left\| f_1(\langle \cdot, e_1 \rangle)^{\frac{p_{\theta,1}}{2}} f_2(\langle \cdot, (e_1 - e_2) \rangle)^{\frac{p_{\theta,2}}{2}} f_3(\langle \cdot, e_2 \rangle)^{\frac{p_{\theta,3}}{2}} \right\|_{L^2(\mathbb{R}^2)} \leq A_0^\theta A_1^{1-\theta} \prod_{j=1}^3 \|f_j\|_{L^1(\mathbb{R})}^{\frac{p_{\theta,j}}{2}}.$$

In fact, we will show that $g_{\theta,1}, g_{\theta,2}, g_{\theta,3}$ satisfy conditions analogous to (iii) in the statement of Theorem 1.1. To be concise, we will write the 3-tuple of exponents as

$$(p_{\theta,1}, p_{\theta,2}, p_{\theta,3}) = \theta(p_{0,1}, p_{0,2}, p_{0,3}) + (1-\theta)(p_{1,1}, p_{1,2}, p_{1,3}),$$

where the addition is component-wise. Note that in the special case where $A_0 = A_1 = 1$ we also have $A_0^\theta A_1^{1-\theta} = 1$ for every θ . To see why this interpolation between factorisations will be useful, consider the following special case of the Young convolution inequality:

$$\left\| f_1(\langle \cdot, e_1 \rangle)^{\frac{1}{4}} f_2(\langle \cdot, (e_1 - e_2) \rangle)^{\frac{1}{2}} f_3(\langle \cdot, e_2 \rangle)^{\frac{1}{4}} \right\|_{L^2(\mathbb{R}^2)} \leq A \|f_1\|_{L^1(\mathbb{R})}^{\frac{1}{4}} \|f_2\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \|f_3\|_{L^1(\mathbb{R})}^{\frac{1}{4}}. \quad (2.23)$$

Suppose we would like to construct a factorisation which proves the above inequality with $A = 1$. Setting $\theta = \frac{1}{2}$, then of course we have

$$\left(\frac{1}{2}, 1, \frac{1}{2}\right) = \theta(1, 1, 0) + (1-\theta)(0, 1, 1). \quad (2.24)$$

Thus, following the framework outlined above, we should interpolate between the following two inequalities:

$$\left\| f_1(\langle \cdot, e_1 \rangle)^{\frac{1}{2}} f_2(\langle \cdot, (e_1 - e_2) \rangle)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^2)} \leq A_0 \|f_1\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \|f_2\|_{L^1(\mathbb{R})}^{\frac{1}{2}};$$

$$\left\| f_2(\langle \cdot, (e_1 - e_2) \rangle)^{\frac{1}{2}} f_3(\langle \cdot, e_2 \rangle)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^2)} \leq A_1 \|f_2\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \|f_3\|_{L^1(\mathbb{R})}^{\frac{1}{2}}.$$

If we were able to construct factorisations proving the above two inequalities with constants A_0, A_1 respectively, then using the procedure described above we could construct a third factorisation proving (2.23) with constant $A = A_0^{\frac{1}{2}} A_1^{\frac{1}{2}}$. It is not difficult to verify that after applying a simple changes of variables in each of the left hand sides, these last two inequalities can be rewritten as follows

$$\left\| f_1(\langle \cdot, e_1 \rangle)^{\frac{1}{2}} f_2(\langle \cdot, e_2 \rangle)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^2)} \leq A_0 \|f_1\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \|f_2\|_{L^1(\mathbb{R})}^{\frac{1}{2}},$$

$$\left\| f_2(\langle \cdot, e_1 \rangle)^{\frac{1}{2}} f_3(\langle \cdot, e_2 \rangle)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^2)} \leq A_1 \|f_2\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \|f_3\|_{L^1(\mathbb{R})}^{\frac{1}{2}}.$$

Setting $A_0 = A_1 = 1$, we see that both these inequalities hold with equality and can be viewed as trivial cases of the orthogonal Loomis–Whitney, which we already know how to factorise explicitly. In a similar way to how we defined the functions $g_{\theta,j}$, one can use the factorisation for the orthogonal Loomis–Whitney inequality to define a new factorisation which proves (2.23) with constant $A = A_0^{\frac{1}{2}} A_1^{\frac{1}{2}} = 1$. Now that we have a rough idea of how this interpolation will be used, let us show how to construct such factorisations explicitly for the general case.

Proof of Theorem 2.17. For ease of notation, let u_1, u_2, u_3 denote the unit vectors $e_1, \frac{(e_1 - e_2)}{\sqrt{2}}, e_2$ respectively. Fix exponents $p_1, p_2, p_3 \in [0, 1]$ satisfying

$$p_1 + p_2 + p_3 = 2, \quad (2.25)$$

and $g \in L^2(\mathbb{R}^2)$ with $\|g\|_2 = 1$. It suffices to show that there exist functions g_1, g_2, g_3 on $\mathbb{R}^2$ such that

$$g(x) \leq g_1(x)^{\frac{p_1}{2}} g_2(x)^{\frac{p_2}{2}} g_3(x)^{\frac{p_3}{2}}, \quad (2.26)$$

and the following bound holds

$$\left(\sup_{t \in \mathbb{R}} \int_{u_1^\perp} g_1(tu_1 + v) dv \right)^{\frac{p_1}{2}} \left(\sup_{t \in \mathbb{R}} \int_{u_2^\perp} g_2(tu_2 + \sqrt{2}v) dv \right)^{\frac{p_2}{2}} \left(\sup_{t \in \mathbb{R}} \int_{u_3^\perp} g_3(tu_3 + v) dv \right)^{\frac{p_3}{2}} \leq 1.$$

Applying a simple change of variables $v \mapsto \sqrt{2}^{-1}v$ inside the second integral, this becomes

$$\left(\sup_{t \in \mathbb{R}} \int_{u_1^\perp} g_1(tu_1 + v) dv \right)^{\frac{p_1}{2}} \left(\sup_{t \in \mathbb{R}} \int_{u_2^\perp} g_2(tu_2 + v) dv \right)^{\frac{p_2}{2}} \left(\sup_{t \in \mathbb{R}} \int_{u_3^\perp} g_3(tu_3 + v) dv \right)^{\frac{p_3}{2}} \leq (\sqrt{2})^{\frac{p_2}{2}}.$$

Then, for $j = 1, 2, 3$ fix a unit vector $\omega_j \in u_j^\perp$, so that we can rewrite this last bound as

$$\left(\sup_{v \in \omega_1^\perp} \int_{\mathbb{R}} g_1(v + t_1 \omega_1) dt_1 \right)^{\frac{p_1}{2}} \left(\sup_{v \in \omega_2^\perp} \int_{\mathbb{R}} g_2(v + t_2 \omega_2) dt_2 \right)^{\frac{p_2}{2}} \left(\sup_{v \in \omega_3^\perp} \int_{\mathbb{R}} g_3(v + t_3 \omega_3) dt_3 \right)^{\frac{p_3}{2}} \leq (\sqrt{2})^{\frac{p_2}{2}}. \quad (2.27)$$

In the spirit of (2.24), let us begin by writing (p_1, p_2, p_3) as the convex combination of $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$. Firstly, define the following three numbers:

$$\theta_1 = \frac{p_2 + p_3 - p_1}{2};$$

$$\theta_2 = \frac{p_1 + p_3 - p_2}{2};$$

$$\theta_3 = \frac{p_1 + p_2 - p_3}{2}.$$

Note that for any permutation $\sigma \in S_3$ we have

$$\frac{p_{\sigma(2)} + p_{\sigma(3)} - p_{\sigma(1)}}{2} = \frac{2 - p_{\sigma(1)} - p_{\sigma(1)}}{2} = (1 - p_{\sigma(1)}),$$

where all we have used is condition (2.25). Since $p_1, p_2, p_3 \in [0, 1]$, it follows immediately that $\theta_1, \theta_2, \theta_3 \in [0, 1]$. Moreover, again using (2.25) we see that

$$\theta_1 + \theta_2 + \theta_3 = \frac{2(p_1 + p_2 + p_3) - (p_1 + p_2 + p_3)}{2} = 1.$$

Now we can write (p_1, p_2, p_3) as the following convex combination:

$$\begin{aligned} (p_1, p_2, p_3) &= (\theta_2 + \theta_3, \theta_1 + \theta_3, \theta_1 + \theta_2) \\ &= \theta_1(0, 1, 1) + \theta_2(1, 0, 1) + \theta_3(1, 1, 0). \end{aligned}$$

Following a similar route to the model example described above with $(p_1, p_2, p_3) = (\frac{1}{2}, 1, \frac{1}{2})$, we will build the desired factorisation by interpolating between factorisations for the following three trivial cases of the affine-invariant Loomis–Whitney inequality:

$$\left\| \left(f_2 \circ \pi_{\omega_2^\perp} \right)^{\frac{1}{2}} \left(f_3 \circ \pi_{\omega_3^\perp} \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^2)} = |\omega_2 \wedge \omega_3|^{-\frac{1}{2}} \|f_2\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \|f_3\|_{L^1(\mathbb{R})}^{\frac{1}{2}},$$

$$\left\| \left(f_1 \circ \pi_{\omega_1^\perp} \right)^{\frac{1}{2}} \left(f_3 \circ \pi_{\omega_3^\perp} \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^2)} = |\omega_1 \wedge \omega_3|^{-\frac{1}{2}} \|f_1\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \|f_3\|_{L^1(\mathbb{R})}^{\frac{1}{2}},$$

$$\left\| \left(f_1 \circ \pi_{\omega_1^\perp} \right)^{\frac{1}{2}} \left(f_2 \circ \pi_{\omega_2^\perp} \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^2)} = |\omega_1 \wedge \omega_2|^{-\frac{1}{2}} \|f_1\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \|f_2\|_{L^1(\mathbb{R})}^{\frac{1}{2}}.$$

Firstly, applying Theorem 2.12 with $d = 2$ for linearly independent unit vectors ω_2, ω_3 and the function $g \in L^2(\mathbb{R}^2)$ we fixed at the beginning of this proof, we get functions $\gamma_{1,\omega_2}, \gamma_{1,\omega_3}$ on $\mathbb{R}^2$ such that

$$g(x) = \gamma_{1,\omega_2}(x)^{\frac{1}{2}} \gamma_{1,\omega_3}(x)^{\frac{1}{2}} \quad (2.28)$$

and the following two inequalities hold:

$$\sup_{v \in \omega_2^\perp} \int_{\mathbb{R}} \gamma_{1,\omega_2}(v + t_2 \omega_2) dt_2 \leq |\omega_2 \wedge \omega_3|^{-\frac{1}{2}}, \quad (2.29)$$

$$\sup_{v \in \omega_3^\perp} \int_{\mathbb{R}} \gamma_{1,\omega_3}(v + t_3 \omega_3) dt_3 \leq |\omega_2 \wedge \omega_3|^{-\frac{1}{2}}. \quad (2.30)$$

In fact, using the general construction given in the proof of Theorem 2.12, we can write down such functions explicitly by setting

$$\gamma_{1,\omega_2}(x) := \left(\frac{g(x)^2}{\int_{\mathbb{R}} g(x + s_2 \omega_2)^2 ds_2} \right) |\omega_2 \wedge \omega_3|^{-\frac{1}{2}},$$

and

$$\gamma_{1,\omega_3}(x) := \left(\frac{\int_{\mathbb{R}} g(x + s_2\omega_2)^2 ds_2}{\int_{\mathbb{R}} \int_{\mathbb{R}} g(x + s_2\omega_2 + s_3\omega_3)^2 ds_2 ds_3} \right) |\omega_2 \wedge \omega_3|^{-\frac{1}{2}}.$$

In the proof of Theorem 2.12 it was shown that such functions satisfy conditions (2.28), (2.29) and (2.30). Similarly, let us define

$$\gamma_{2,\omega_1}(x) := \left(\frac{g(x)^2}{\int_{\mathbb{R}} g(x + s_1\omega_1)^2 ds_1} \right) |\omega_1 \wedge \omega_3|^{-\frac{1}{2}},$$

and

$$\gamma_{2,\omega_3}(x) := \left(\frac{\int_{\mathbb{R}} g(x + s_1\omega_1)^2 ds_1}{\int_{\mathbb{R}} \int_{\mathbb{R}} g(x + s_1\omega_1 + s_3\omega_3)^2 ds_1 ds_3} \right) |\omega_1 \wedge \omega_3|^{-\frac{1}{2}}.$$

Then the following three conditions are satisfied:

$$g(x) = \gamma_{2,\omega_1}(x)^{\frac{1}{2}} \gamma_{2,\omega_3}(x)^{\frac{1}{2}}; \quad (2.31)$$

$$\sup_{v \in \omega_1^\perp} \int_{\mathbb{R}} \gamma_{2,\omega_1}(v + t_1\omega_1) dt_1 \leq |\omega_1 \wedge \omega_3|^{-\frac{1}{2}}; \quad (2.32)$$

$$\sup_{v \in \omega_3^\perp} \int_{\mathbb{R}} \gamma_{2,\omega_3}(v + t_3\omega_3) dt_3 \leq |\omega_1 \wedge \omega_3|^{-\frac{1}{2}}. \quad (2.33)$$

Lastly, define

$$\gamma_{3,\omega_1}(x) := \left(\frac{g(x)^2}{\int_{\mathbb{R}} g(x + s_1\omega_1)^2 ds_1} \right) |\omega_1 \wedge \omega_2|^{-\frac{1}{2}},$$

and

$$\gamma_{3,\omega_2}(x) := \left(\frac{\int_{\mathbb{R}} g(x + s_1\omega_1)^2 ds_1}{\int_{\mathbb{R}} \int_{\mathbb{R}} g(x + s_1\omega_1 + s_2\omega_2)^2 ds_1 ds_2} \right) |\omega_1 \wedge \omega_2|^{-\frac{1}{2}},$$

which will satisfy three corresponding conditions:

$$g(x) = \gamma_{3,\omega_1}(x)^{\frac{1}{2}} \gamma_{3,\omega_2}(x)^{\frac{1}{2}}; \quad (2.34)$$

$$\sup_{v \in \omega_1^\perp} \int_{\mathbb{R}} \gamma_{3,\omega_1}(v + t_1\omega_1) dt_1 \leq |\omega_1 \wedge \omega_2|^{-\frac{1}{2}}; \quad (2.35)$$

$$\sup_{v \in \omega_2^\perp} \int_{\mathbb{R}} \gamma_{3,\omega_2}(v + t_2\omega_2) dt_2 \leq |\omega_1 \wedge \omega_2|^{-\frac{1}{2}}. \quad (2.36)$$

Combining conditions (2.28), (2.31), (2.34) we can write

$$g(x) = g(x)^{\theta_1} g(x)^{\theta_2} g(x)^{\theta_3} = \left(\gamma_{1,\omega_2}(x)^{\frac{1}{2}} \gamma_{1,\omega_3}(x)^{\frac{1}{2}} \right)^{\theta_1} \left(\gamma_{2,\omega_1}(x)^{\frac{1}{2}} \gamma_{2,\omega_3}(x)^{\frac{1}{2}} \right)^{\theta_2} \left(\gamma_{3,\omega_1}(x)^{\frac{1}{2}} \gamma_{3,\omega_2}(x)^{\frac{1}{2}} \right)^{\theta_3}.$$

Firstly, suppose that $p_1, p_2, p_3 > 0$, and define the following three functions:

$$\begin{aligned} g_1(x) &:= \gamma_{2,\omega_1}(x)^{\frac{\theta_2}{p_1}} \gamma_{3,\omega_1}(x)^{\frac{\theta_3}{p_1}}, \\ g_2(x) &:= \gamma_{1,\omega_2}(x)^{\frac{\theta_1}{p_2}} \gamma_{3,\omega_2}(x)^{\frac{\theta_3}{p_2}}, \\ g_3(x) &:= \gamma_{1,\omega_3}(x)^{\frac{\theta_1}{p_3}} \gamma_{2,\omega_3}(x)^{\frac{\theta_2}{p_3}}. \end{aligned}$$

It follows immediately that

$$g(x) = \prod_{j=1}^3 g_j(x)^{\frac{p_j}{2}},$$

which deals with (2.26). By construction we have the following three identities:

$$\begin{aligned} \frac{\theta_2}{p_1} + \frac{\theta_3}{p_1} &= 1; \\ \frac{\theta_1}{p_2} + \frac{\theta_3}{p_2} &= 1; \\ \frac{\theta_1}{p_3} + \frac{\theta_2}{p_3} &= 1. \end{aligned}$$

In particular, using the first of these identities we can apply Hölder's inequality along with conditions (2.29),(2.30) to deduce the following bound:

$$\begin{aligned} \sup_{v \in \omega_1^\perp} \int_{\mathbb{R}} g_1(v + t_1 \omega_1) dt_1 &= \sup_{v \in \omega_1^\perp} \int_{\mathbb{R}} \gamma_{2,\omega_1}(v + t_1 \omega_1)^{\frac{\theta_2}{p_1}} \gamma_{3,\omega_1}(v + t_1 \omega_1)^{\frac{\theta_3}{p_1}} dt_1 \\ &\leq \sup_{v \in \omega_1^\perp} \left(\int_{\mathbb{R}} \gamma_{2,\omega_1}(v + t_1 \omega_1) dt_1 \right)^{\frac{\theta_2}{p_1}} \left(\int_{\mathbb{R}} \gamma_{3,\omega_1}(v + \tilde{t}_1 \omega_1) d\tilde{t}_1 \right)^{\frac{\theta_3}{p_1}} \\ &\leq \left(\sup_{v \in \omega_1^\perp} \int_{\mathbb{R}} \gamma_{2,\omega_1}(v + t_1 \omega_1) dt_1 \right)^{\frac{\theta_2}{p_1}} \left(\sup_{\tilde{v} \in \omega_1^\perp} \int_{\mathbb{R}} \gamma_{3,\omega_1}(\tilde{v} + \tilde{t}_1 \omega_1) d\tilde{t}_1 \right)^{\frac{\theta_3}{p_1}} \\ &\leq \left(|\omega_1 \wedge \omega_3|^{-\frac{1}{2}} \right)^{\frac{\theta_2}{p_1}} \left(|\omega_1 \wedge \omega_2|^{-\frac{1}{2}} \right)^{\frac{\theta_3}{p_1}}. \end{aligned}$$

By similar arguments we can deduce the following two bounds for g_2 and g_3 :

$$\sup_{v \in \omega_2^\perp} \int_{\mathbb{R}} g_2(v + t_2 \omega_2) dt_2 \leq \left(|\omega_2 \wedge \omega_3|^{-\frac{1}{2}} \right)^{\frac{\theta_1}{p_2}} \left(|\omega_1 \wedge \omega_2|^{-\frac{1}{2}} \right)^{\frac{\theta_3}{p_2}};$$

$$\sup_{v \in \omega_3^\perp} \int_{\mathbb{R}} g_3(v + t_3 \omega_3) dt_3 \leq \left(|\omega_2 \wedge \omega_3|^{-\frac{1}{2}} \right)^{\frac{\theta_1}{p_3}} \left(|\omega_1 \wedge \omega_3|^{-\frac{1}{2}} \right)^{\frac{\theta_2}{p_3}}.$$

To establish (2.27), we simply observe that

$$\begin{aligned}
& \left(\sup_{v \in \omega_1^\perp} \int_{\mathbb{R}} g_1(v + t_1 \omega_1) dt_1 \right)^{\frac{p_1}{2}} \left(\sup_{v \in \omega_2^\perp} \int_{\mathbb{R}} g_2(v + t_2 \omega_2) dt_2 \right)^{\frac{p_2}{2}} \left(\sup_{v \in \omega_3^\perp} \int_{\mathbb{R}} g_3(v + t_3 \omega_3) dt_3 \right)^{\frac{p_3}{2}} \\
& \leq \left(|\omega_1 \wedge \omega_3|^{-\frac{\theta_2}{2p_1}} |\omega_1 \wedge \omega_2|^{-\frac{\theta_3}{2p_1}} \right)^{\frac{p_1}{2}} \left(|\omega_2 \wedge \omega_3|^{-\frac{\theta_1}{2p_2}} |\omega_1 \wedge \omega_2|^{-\frac{\theta_3}{2p_2}} \right)^{\frac{p_2}{2}} \\
& \quad \cdot \left(|\omega_2 \wedge \omega_3|^{-\frac{\theta_1}{2p_3}} |\omega_1 \wedge \omega_3|^{-\frac{\theta_2}{2p_3}} \right)^{\frac{p_3}{2}} \\
& \leq |\omega_1 \wedge \omega_2|^{-\frac{\theta_3}{2}} |\omega_1 \wedge \omega_3|^{-\frac{\theta_2}{2}} |\omega_2 \wedge \omega_3|^{-\frac{\theta_1}{2}} \\
& = \left(\sqrt{2} \right)^{\frac{\theta_3}{2}} 1^{\frac{\theta_2}{2}} \left(\sqrt{2} \right)^{\frac{\theta_1}{2}} \\
& = \left(\sqrt{2} \right)^{\frac{\theta_1 + \theta_3}{2}} \\
& = \left(\sqrt{2} \right)^{\frac{p_2}{2}},
\end{aligned}$$

where we use that $|\omega_1 \wedge \omega_2| = |\omega_2 \wedge \omega_3| = (\sqrt{2})^{-1}$ and $|\omega_1 \wedge \omega_3| = 1$. Let us now express the functions g_1, g_2, g_3 directly in terms of g , writing things out explicitly we get

$$\begin{aligned}
g_1(x) &:= \gamma_{2,\omega_1}(x)^{\frac{\theta_2}{p_1}} \gamma_{3,\omega_1}(x)^{\frac{\theta_3}{p_1}} = \left(\left(\frac{g(x)^2}{\int_{\mathbb{R}} g(x + s_1 \omega_1)^2 ds_1} \right) |\omega_1 \wedge \omega_3|^{-\frac{1}{2}} \right)^{\frac{\theta_2}{p_1}} \\
& \quad \cdot \left(\left(\frac{g(x)^2}{\int_{\mathbb{R}} g(x + s_1 \omega_1)^2 ds_1} \right) |\omega_1 \wedge \omega_2|^{-\frac{1}{2}} \right)^{\frac{\theta_3}{p_1}} \\
& = \left(\sqrt{2}^{\frac{1}{2}} \right)^{\frac{\theta_3}{p_1}} \left(\frac{g(x)^2}{\int_{\mathbb{R}} g(x + s_1 \omega_1)^2 ds_1} \right)^{\frac{\theta_2}{p_1} + \frac{\theta_3}{p_1}} \\
& = \left(\sqrt{2} \right)^{\frac{p_1 + p_2 - p_3}{4p_1}} \frac{g(x)^2}{\int_{\mathbb{R}} g(x + s_1 e_2)^2 ds_1},
\end{aligned}$$

$$\begin{aligned}
g_2(x) &:= \gamma_{1,\omega_2}(x)^{\frac{\theta_1}{p_2}} \gamma_{3,\omega_2}(x)^{\frac{\theta_3}{p_2}} = \left(\left(\frac{g(x)^2}{\int_{\mathbb{R}} g(x + s_2 \omega_2)^2 ds_2} \right) |\omega_2 \wedge \omega_3|^{-\frac{1}{2}} \right)^{\frac{\theta_1}{p_2}} \\
&\quad \cdot \left(\left(\frac{\int_{\mathbb{R}} g(x + s_1 \omega_1)^2 ds_1}{\int_{\mathbb{R}} \int_{\mathbb{R}} g(x + s_1 \omega_1 + s_2 \omega_2)^2 ds_1 ds_2} \right) |\omega_1 \wedge \omega_2|^{-\frac{1}{2}} \right)^{\frac{\theta_3}{p_2}} \\
&= \frac{g(x)^{\frac{\theta_1}{p_2}} \left(\int_{\mathbb{R}} g(x + s_1 \omega_1)^2 ds_1 \right)^{\frac{\theta_3}{p_2}}}{\left(\int_{\mathbb{R}} g(x + s_2 \omega_2)^2 ds_2 \right)^{\frac{\theta_1}{p_2}} |\omega_1 \wedge \omega_2|^{-\frac{\theta_1}{2p_2}} |\omega_1 \wedge \omega_2|^{-\frac{\theta_3}{2p_2}}} \\
&= \frac{g(x)^{\frac{\theta_1}{p_2}} \left(\int_{\mathbb{R}} g(x + s_1 \omega_1)^2 ds_1 \right)^{\frac{\theta_3}{p_2}}}{\left(\int_{\mathbb{R}} g(x + s_2 \omega_2)^2 ds_2 \right)^{\frac{\theta_1}{p_2}}} |\omega_2 \wedge \omega_3|^{-\frac{\theta_1}{2p_2}} |\omega_1 \wedge \omega_2|^{\frac{\theta_3}{2p_2}} \\
&= \frac{g(x)^{\frac{\theta_1}{p_2}} \left(\int_{\mathbb{R}} g(x + s_1 e_2)^2 ds_1 \right)^{\frac{\theta_3}{p_2}}}{\left(\int_{\mathbb{R}} g\left(x + s_2 \left(\frac{e_1 + e_2}{\sqrt{2}}\right)\right)^2 ds_2 \right)^{\frac{\theta_1}{p_2}}} \left(\sqrt{2}^{\frac{1}{2}} \right)^{\frac{\theta_1}{2p_2}} \left(\sqrt{2}^{\frac{1}{2}} \right)^{-\frac{\theta_3}{2p_2}} \\
&= \frac{g(x)^{\frac{\theta_1}{p_2}} \left(\int_{\mathbb{R}} g(x + s_1 e_2)^2 ds_1 \right)^{\frac{\theta_3}{p_2}}}{\left(\int_{\mathbb{R}} g(x + \tilde{s}_2(e_1 + e_2))^2 \sqrt{2} d\tilde{s}_2 \right)^{\frac{\theta_1}{p_2}}} \left(\sqrt{2} \right)^{\frac{\theta_1}{2p_2}} \left(\sqrt{2} \right)^{-\frac{\theta_3}{2p_2}} \\
&= \frac{g(x)^{\frac{\theta_1}{p_2}} \left(\int_{\mathbb{R}} g(x + s_1 e_2)^2 ds_1 \right)^{\frac{\theta_3}{p_2}}}{\left(\int_{\mathbb{R}} g(x + \tilde{s}_2(e_1 + e_2))^2 d\tilde{s}_2 \right)^{\frac{\theta_1}{p_2}}} \left(\sqrt{2} \right)^{-\frac{\theta_1}{2p_2}} \left(\sqrt{2} \right)^{-\frac{\theta_3}{2p_2}} \\
&= \frac{g(x)^{\frac{\theta_1}{p_2}} \left(\int_{\mathbb{R}} g(x + s_1 e_2)^2 ds_1 \right)^{\frac{\theta_3}{p_2}}}{\left(\int_{\mathbb{R}} g(x + \tilde{s}_2(e_1 + e_2))^2 d\tilde{s}_2 \right)^{\frac{\theta_1}{p_2}}} \left(\sqrt{2} \right)^{-\frac{1}{2}} \\
&= \frac{g(x)^{\frac{p_2 + p_3 - p_1}{p_2}} \left(\int_{\mathbb{R}} g(x + s_1 e_2)^2 ds_1 \right)^{\frac{p_1 + p_2 - p_3}{2p_2}}}{\left(\int_{\mathbb{R}} g(x + \tilde{s}_2(e_1 + e_2))^2 d\tilde{s}_2 \right)^{\frac{p_2 + p_3 - p_1}{2p_2}}} (\sqrt{2})^{-\frac{1}{2}},
\end{aligned}$$

and lastly

$$\begin{aligned}
g_3(x) &:= \gamma_{1,\omega_3}(x)^{\frac{\theta_1}{p_3}} \gamma_{2,\omega_3}(x)^{\frac{\theta_2}{p_3}} \\
&= \left(\left(\frac{\int_{\mathbb{R}} g(x + s_2 \omega_2)^2 ds_2}{\int_{\mathbb{R}} \int_{\mathbb{R}} g(x + s_2 \omega_2 + s_3 \omega_3)^2 ds_2 ds_3} \right) |\omega_2 \wedge \omega_3|^{-\frac{1}{2}} \right)^{\frac{\theta_1}{p_3}} \\
&\quad \cdot \left(\left(\frac{\int_{\mathbb{R}} g(x + s_1 \omega_1)^2 ds_1}{\int_{\mathbb{R}} \int_{\mathbb{R}} g(x + s_1 \omega_1 + s_3 \omega_3)^2 ds_1 ds_3} \right) |\omega_1 \wedge \omega_3|^{-\frac{1}{2}} \right)^{\frac{\theta_2}{p_3}} \\
&= \left(\int_{\mathbb{R}} g(x + s_2 \omega_2)^2 ds_2 \right)^{\frac{\theta_1}{p_3}} \left(\int_{\mathbb{R}} g(x + s_1 \omega_1)^2 ds_1 \right)^{\frac{\theta_2}{p_3}} |\omega_2 \wedge \omega_3|^{\frac{\theta_1}{2p_3}} |\omega_1 \wedge \omega_3|^{\frac{\theta_2}{2p_3}} \\
&= \left(\int_{\mathbb{R}} g \left(x + s_2 \left(\frac{e_1 + e_2}{\sqrt{2}} \right) \right)^2 ds_2 \right)^{\frac{\theta_1}{p_3}} \left(\int_{\mathbb{R}} g(x + s_1 e_2)^2 ds_1 \right)^{\frac{\theta_2}{p_3}} (\sqrt{2})^{-\frac{\theta_1}{2p_3}} \\
&= \left(\int_{\mathbb{R}} g(x + \tilde{s}_2 (e_1 + e_2))^2 \sqrt{2} d\tilde{s}_2 \right)^{\frac{\theta_1}{p_3}} \left(\int_{\mathbb{R}} g(x + s_1 e_2)^2 ds_1 \right)^{\frac{\theta_2}{p_3}} (\sqrt{2})^{-\frac{\theta_1}{2p_3}} \\
&= \left(\int_{\mathbb{R}} g(x + \tilde{s}_2 (e_1 + e_2))^2 d\tilde{s}_2 \right)^{\frac{\theta_1}{p_3}} \left(\int_{\mathbb{R}} g(x + s_1 e_2)^2 ds_1 \right)^{\frac{\theta_2}{p_3}} (\sqrt{2})^{\frac{\theta_1}{2p_3}} \\
&= \left(\int_{\mathbb{R}} g(x + s(e_1 + e_2))^2 ds \right)^{\frac{p_2 + p_3 - p_1}{2p_3}} \left(\int_{\mathbb{R}} g(x + s_1 e_2)^2 ds_1 \right)^{\frac{p_1 + p_3 - p_2}{2p_3}} (\sqrt{2})^{\frac{p_2 + p_3 - p_1}{4p_3}}.
\end{aligned}$$

It remains to consider the endpoint case where one of the p_j 's is equal to zero. Suppose that $p_i = 0$ for some fixed $i \in \{1, 2, 3\}$, then necessarily by (2.25) we have $p_j = 1$ for $j \neq i$. So we are in the case where

$$p_j = \begin{cases} 0 & j = i \\ 1 & j \neq i. \end{cases}$$

Moreover, by definition of $\theta_1, \theta_2, \theta_3$ it follows that

$$\theta_j = \begin{cases} 1 & j = i \\ 0 & j \neq i. \end{cases}$$

Now for $j \neq i$, similarly to the non-endpoint cases, we define

$$g_j(x) := \prod_{k \neq j} \gamma_{k,\omega_j}(x)^{\frac{\theta_k}{p_j}} = \gamma_{i,\omega_j}(x).$$

Then combining (2.29), (2.30), (2.32), (2.33), (2.35) and (2.36), we clearly have

$$\sup_{v \in \omega_j^\perp} \int_{\mathbb{R}} g_j(v + t_j \omega_j) dt_j = \sup_{v \in \omega_j^\perp} \int_{\mathbb{R}} \gamma_{i,\omega_j}(v + t_j \omega_j) dt_j \leq \left| \bigwedge_{k \neq i} \omega_k \right|^{-\frac{1}{2}}.$$

Now we just need $g_i: \mathbb{R}^d \rightarrow \mathbb{R}$ to be any bounded function such that

$$\sup_{v \in \omega_i^\perp} \int_{\mathbb{R}} g_i(v + t_i \omega_i) dt_i < \infty.$$

This can be easily achieved by fixing any bounded integrable function $\gamma_i: \mathbb{R} \rightarrow \mathbb{R}$ and defining

$$g_i(x) := \gamma_i(\pi_{\omega_i} x).$$

It should be clear that

$$g(x) = \prod_{j \neq i} \gamma_{i, \omega_j}(x)^{\frac{1}{2}} = \prod_{j \neq i} g_j(x)^{\frac{1}{2}} = \prod_{j=1}^3 g_j(x)^{\frac{p_j}{2}},$$

which establishes (2.26). Furthermore, by construction we have

$$\prod_{j=1}^3 \left(\sup_{v \in \omega_j^\perp} \int_{\mathbb{R}} g_j(v + t_j \omega_j) dt_j \right)^{\frac{p_j}{2}} = \prod_{j \neq i} \left(\sup_{v \in \omega_j^\perp} \int_{\mathbb{R}} \gamma_{i, \omega_j}(v + t_j \omega_j) dt_j \right)^{\frac{p_j}{2}} \leq \left(\left| \bigwedge_{k \neq i} \omega_k \right|^{-\frac{1}{4}} \right)^2 = \left| \bigwedge_{k \neq i} \omega_k \right|^{-\frac{1}{2}}.$$

It is not difficult to see that for every choice of i , we have $\left| \bigwedge_{k \neq i} \omega_k \right|^{-\frac{1}{2}} = (\sqrt{2})^{\frac{p_2}{2}}$, which verifies (2.27). This concludes the proof. $\square$

As it turns out, this method can be used to construct (suboptimal) factorisations for a much wider class of multilinear norm inequalities.

Geometric Brascamp–Lieb inequalities Given exponents $p_1, \dots, p_m > 0$ and unit vectors $u_1, \dots, u_m \in \mathbb{R}^d$, we refer to the tuple $(\mathbf{u}, \mathbf{p}) = ((u_1, \dots, u_m), (p_1, \dots, p_m))$ as **geometric Brascamp–Lieb datum** if the following identity is satisfied:

$$\sum_{i=1}^m p_i u_i \otimes u_i = I_d. \quad (2.37)$$

Using this language, we state the following result originally due to Keith Ball:

Theorem 2.19 (Classical geometric Brascamp–Lieb inequality - [6]). *Given geometric Brascamp–Lieb datum $(\mathbf{u}, \mathbf{p})$, the following inequality holds*

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j(\langle x, u_j \rangle)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}} g_j \right)^{p_j}, \quad (2.38)$$

and for each $t > 0$, the standard Gaussian functions $g_j(y) = e^{-\pi t |y|^2}$ are extremals.

To see how (2.38) falls under the scope of Theorem 1.1, set $p = \sum_{j=1}^m p_j$ and take both sides to the power $\frac{1}{p}$ to get

$$\left\| \prod_{j=1}^m (g_j \circ \pi_{u_j})^{\frac{p_j}{p}} \right\|_{L^p(\mathbb{R})} \leq \prod_{j=1}^m \|g_j\|_{L^1(\mathbb{R})}^{\frac{p_j}{p}}.$$

By equating the trace of both sides in (2.37), it follows that

$$p := \sum_{j=1}^m p_j = d \geq 1.$$

Clearly $0 \leq \frac{p_j}{p} \leq 1$ for each j , so applying the multilinear duality principle we can deduce existence of factorisations for these inequalities too:

Theorem 2.20 (Factorisation for classical geometric Brascamp–Lieb). *Suppose we are given geometric Brascamp–Lieb datum $(\mathbf{u}, \mathbf{p})$ and set $p = \sum_{j=1}^m p_j$. Then for every $g \in L^{p'}(\mathbb{R}^d)$ with $\|g\|_{p'} = 1$, there exist functions $g_1, \dots, g_m$ on $\mathbb{R}^d$ such that for almost every $x \in \mathbb{R}^d$*

$$g(x) \leq \prod_{j=1}^d g_j(x)^{\frac{p_j}{p}}, \quad (2.39)$$

and we have the following bound

$$\prod_{j=1}^m \left(\sup_{t \in \mathbb{R}} \int_{u_j^\perp} g_j(tu_j + v) dv \right)^{\frac{p_j}{p}} \leq 1. \quad (2.40)$$

As we will see in Section 3.2, the sharp Young convolution inequality on $\mathbb{R}$, due to Beckner [8], can be derived as a special case of the geometric Brascamp–Lieb inequality in $\mathbb{R}^2$. The interpolation method outlined in the discussion above for the non-sharp Young convolution inequality, can be adapted to the setting of geometric Brascamp–Lieb inequalities. Unfortunately, this technique only yields suboptimal factorisations, that is, ones which prove Theorem 2.19 with a non-sharp constant. A detailed algorithm for finding such factorisation, in the more general setting of Brascamp–Lieb inequalities, can be found in Section 7 of [16], from which one can deduce the following result:

Theorem 2.21 (Factorisation for classical geometric Brascamp–Lieb). *There exists some constant $A \geq 1$ such that, if we are given geometric Brascamp–Lieb datum $(\mathbf{u}, \mathbf{p})$ and set $p = \sum_{j=1}^m p_j$, then for every $g \in L^{p'}(\mathbb{R}^d)$ with $\|g\|_{p'} = 1$, one may construct functions $g_1, \dots, g_m$ on $\mathbb{R}^d$ such that for almost every $x \in \mathbb{R}^d$*

$$g(x) \leq \prod_{j=1}^d g_j(x)^{\frac{p_j}{p}}, \quad (2.41)$$

and we have the following bound

$$\prod_{j=1}^m \left(\sup_{t \in \mathbb{R}} \int_{u_j^\perp} g_j(tu_j + v) dv \right)^{\frac{p_j}{p}} \leq A. \quad (2.42)$$

2.2 The endpoint multilinear Kekeya inequalities

In this section we discuss the endpoint multilinear Kekeya inequality proved first by Guth in [24], along with a generalisation to k_j -planes due to Zhang [39]. Both arguments rely on a heavy dose of algebraic topology, along with extremely clever adaptations of the polynomial method. In [15], the authors manage to use elements of Guth’s argument to present a proof in the language of factorisations, which can also be generalised to prove Zhang’s extension to k_j -planes. The details for this generalisation can be found in [14]. In Chapter 4, following ideas from [15] and [14], we sketch proofs for the results of Guth and Zhang from a new perspective. In particular, we highlight useful connections to notions from classical convex geometry which are not explicitly mentioned elsewhere. Firstly, let us explore how factorisations arise in this setting.

Endpoint multilinear Keakeya inequality The endpoint multilinear Keakeya inequality can be seen as a far-reaching generalisation of the affine-invariant Loomis–Whitney inequality which we wrote earlier as

$$\int_{\mathbb{R}^d} \prod_{j=1}^d f_j \left(\pi_{\omega_j^\perp} x \right)^{\frac{1}{d-1}} dx \leq |\omega_1 \wedge \cdots \wedge \omega_d|^{-\frac{1}{d-1}} \prod_{j=1}^d \left(\int_{\omega_j^\perp} f_j \right)^{\frac{1}{d-1}}.$$

By putting the constant factor inside the integral on the left hand side this becomes

$$\int_{\mathbb{R}^d} \prod_{j=1}^d f_j \left(\pi_{\omega_j^\perp} x \right)^{\frac{1}{d-1}} |\omega_1 \wedge \cdots \wedge \omega_d|^{\frac{1}{d-1}} dx \leq \prod_{j=1}^d \left(\int_{\omega_j^\perp} f_j \right)^{\frac{1}{d-1}}.$$

It seems natural to ask whether a similar estimate holds if one averages over all choices of $\omega_1, \dots, \omega_d \in \mathbb{S}^{d-1}$. In other words, for $f_1, \dots, f_d: \mathbb{S}^{d-1} \times \mathbb{R}^{d-1}$ do we have

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} \left(\prod_{j=1}^d f_j \left(\omega_j, \pi_{\omega_j^\perp} x \right) \right) |\omega_1 \wedge \cdots \wedge \omega_d| d\sigma(\omega_1) \cdots d\sigma(\omega_d) \right)^{\frac{1}{d-1}} dx \\ \lesssim_d \prod_{j=1}^d \left(\int_{\mathbb{S}^{d-1} \times \mathbb{R}^{d-1}} f_j \right)^{\frac{1}{d-1}}, \end{aligned} \quad (2.43)$$

where σ denotes the normalised surface measure on $\mathbb{S}^{d-1}$? As it turns out, this is equivalent to the endpoint multilinear Keakeya inequality originally due to Guth, which is concerned with bounding the intersection of 1-tubes in $\mathbb{R}^d$:

Definition 2.22. Given a line $\ell \subset \mathbb{R}^d$, we refer to the 1-neighbourhood $T \subset \mathbb{R}^d$ of ℓ as a **1-tube**. Moreover, for any 1-tube $T \subset \mathbb{R}^d$, we denote by $E(T)$ the 1-dimensional subspace of which T forms the 1-neighbourhood.

Using this language, the endpoint multilinear Keakeya inequality can be stated as follows

Theorem 2.23 ([24],[12]). *There exists a finite constant $C_d > 0$ such that if $\mathcal{T}_1, \dots, \mathcal{T}_d$ are families of 1-tubes in $\mathbb{R}^d$, we have*

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\sum_{T_1 \in \mathcal{T}_1} \cdots \sum_{T_d \in \mathcal{T}_d} \left(\prod_{j=1}^d \alpha_{T_j} \chi_{T_j}(x) \right) |E(T_1) \wedge \cdots \wedge E(T_d)| \right)^{\frac{1}{d-1}} dx \\ \leq C_d \prod_{j=1}^d \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \right)^{\frac{1}{d-1}}. \end{aligned} \quad (2.44)$$

It is not difficult to show that the two forms of inequality (2.43) and (2.44) are actually equivalent. Firstly, consider the case where each f_j is essentially a sum of Dirac delta functions in the first argument, then for each fixed ω_j in its support, $f_j(\omega_j, \cdot)$ will be supported in and constant on finitely many unit balls. In this way, we exactly recover (2.44) as a special case of (2.43). To be completely rigorous, we would need to take sequences of functions which, when restricted to the first argument, approximate a sum of Dirac delta functions in the sense of distributions. In order to show the reverse implication, note that the normalised surface measure on $\mathbb{S}^{d-1}$ can be approximated by sums of Dirac delta functions, so it suffices to establish

(2.43) where we replace $d\sigma$ by a finite sum of Dirac delta functions. By density, it suffices to consider f_j which are constant in the second argument at some small scale, and using the scale invariance of (2.43), we can immediately reduce to the case where each f_j is constant in the second argument at unit scale. Thus, we have reduced to (2.44). One should think of these two formulations as a discrete and continuous version of the same result.

In [15], the authors prove a slight generalisation of (2.44), where the order of multilinearity is allowed to be less than the dimension of the ambient space:

Theorem 2.24 ([24],[12],[15]). *Let $2 \leq m \leq d$. There exists a finite constant $C_{d,m} > 0$ such that if $\mathcal{T}_1, \dots, \mathcal{T}_m$ are families of 1-tubes in $\mathbb{R}^d$, we have*

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\sum_{T_1 \in \mathcal{T}_1} \cdots \sum_{T_m \in \mathcal{T}_m} \left(\prod_{j=1}^m \alpha_{T_j} \chi_{T_j}(x) \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m-1}} dx \\ \leq C_{d,m} \prod_{j=1}^m \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \right)^{\frac{1}{m-1}}. \end{aligned} \quad (2.45)$$

Let us begin by attempting to reformulate this last result as we did with the previous examples, in the hope of using multilinear duality. Following ideas from [15], we partition $\mathbb{R}^d$ into unit cubes $\mathcal{Q}$ and note that for fixed 1-tubes $T_1, \dots, T_m$ and $x \in \mathbb{R}^d$ we have

$$\prod_{j=1}^m \alpha_{T_j} \chi_{T_j}(x) \sim_{d,m} \prod_{j=1}^m \left(\alpha_{T_j} \sum_{Q_j \in \mathcal{Q}: T_j \cap Q_j \neq \emptyset} \chi_{Q_j}(x) \right) = \sum_{Q_1 \in \mathcal{Q}: T_1 \cap Q_1 \neq \emptyset} \cdots \sum_{Q_m \in \mathcal{Q}: T_m \cap Q_m \neq \emptyset} \prod_{j=1}^m \alpha_{T_j} \chi_{Q_j}(x).$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\sum_{T_1 \in \mathcal{T}_1} \cdots \sum_{T_m \in \mathcal{T}_m} \left(\prod_{j=1}^m \alpha_{T_j} \chi_{T_j}(x) \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m-1}} dx \\ \sim_{d,m} \int_{\mathbb{R}^d} \left(\sum_{T_1 \in \mathcal{T}_1} \sum_{Q_1 \in \mathcal{Q}: T_1 \cap Q_1 \neq \emptyset} \cdots \sum_{T_m \in \mathcal{T}_m} \sum_{Q_m \in \mathcal{Q}: T_m \cap Q_m \neq \emptyset} \left(\prod_{j=1}^m \alpha_{T_j} \chi_{Q_j}(x) \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m-1}} dx \\ = \int_{\mathbb{R}^d} \left(\sum_{Q_1 \in \mathcal{Q}: T_1 \cap Q_1 \neq \emptyset} \sum_{Q_m \in \mathcal{Q}: T_m \cap Q_m \neq \emptyset} \cdots \sum_{j=1}^m \left(\prod_{j=1}^m \alpha_{T_j} \chi_{Q_j}(x) \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m-1}} dx \\ = \sum_{Q \in \mathcal{Q}} \left(\int_Q \left(\sum_{Q_1 \in \mathcal{Q}: T_1 \cap Q_1 \neq \emptyset} \sum_{Q_m \in \mathcal{Q}: T_m \cap Q_m \neq \emptyset} \cdots \sum_{j=1}^m \left(\prod_{j=1}^m \alpha_{T_j} \chi_{Q_j}(x) \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m-1}} dx \right) \\ = \sum_{Q \in \mathcal{Q}} \left(\left(\sum_{Q_1 \in \mathcal{Q}: T_1 \cap Q_1 \neq \emptyset} \sum_{Q_m \in \mathcal{Q}: T_m \cap Q_m \neq \emptyset} \cdots \sum_{j=1}^m \left(\prod_{j=1}^m \alpha_{T_j} \chi_{Q_j}(Q) \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m-1}} \right) \\ = \sum_{Q \in \mathcal{Q}} \left(\sum_{T_1 \cap Q \neq \emptyset} \cdots \sum_{T_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{T_j} \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m-1}} \end{aligned}$$

Using this we see that (2.45) can be reformulated as the following inequality:

$$\sum_{Q \in \mathcal{Q}} \left(\sum_{T_1 \in \mathcal{T}_1: T_1 \cap Q \neq \emptyset} \cdots \sum_{T_m \in \mathcal{T}_m: T_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{T_j} \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m-1}} \lesssim_{d,m} \prod_{j=1}^m \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \right)^{\frac{1}{m-1}}. \quad (2.46)$$

Note that this last inequality can be rewritten as

$$\left\| \left(\sum_{T_1 \in \mathcal{T}_1: T_1 \cap Q \neq \emptyset} \cdots \sum_{T_m \in \mathcal{T}_m: T_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{T_j} \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m}} \right\|_{\ell^{\frac{m}{m-1}}(\mathcal{Q})}^{\frac{m}{m-1}} \lesssim_{d,m} \prod_{j=1}^m \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \right)^{\frac{1}{m}}. \quad (2.47)$$

Due to the non-constant factor of $|E(T_1) \wedge \cdots \wedge E(T_d)|$, the expression inside the norm on the left hand side cannot be written as a geometric mean of linear operators. As a result, this example does not fall under the scope of multilinear duality. Nevertheless, we can still write down a reasonable candidate for a factorisation and essentially a variant of the easier half of Theorem 1.1 will hold. By linear duality, establishing (2.47) is equivalent to proving that for every $M \in \ell^m(\mathcal{Q})$ with $\|M\|_{\ell^m(\mathcal{Q})} = 1$

$$\sum_{Q \in \mathcal{Q}} \left(\sum_{T_1 \in \mathcal{T}_1: T_1 \cap Q \neq \emptyset} \cdots \sum_{T_m \in \mathcal{T}_m: T_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{T_j} \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m}} M(Q) \lesssim_{d,m} \prod_{j=1}^m \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \right)^{\frac{1}{m}}. \quad (2.48)$$

By density, we may further assume that M is supported on finitely many cubes. Suppose that for all such M there exist S_j defined on $\mathcal{Q} \times \mathcal{T}_j$ for $j = 1, \dots, m$ satisfying the following conditions, for all $Q \in \mathcal{Q}$ whenever $T_j \cap Q \neq \emptyset$ for all j

$$|E(T_1) \wedge \cdots \wedge E(T_m)| M(Q)^m \lesssim_{d,m} S_1(Q, T_1) \cdots S_m(Q, T_m), \quad (2.49)$$

and for all j and $T_j \in \mathcal{T}_j$

$$\sum_{Q \in \mathcal{Q}: Q \cap T_j \neq \emptyset} S_j(Q, T_j) \lesssim_{d,m} 1. \quad (2.50)$$

These two conditions are very similar to the factorisation conditions required in previous examples. The main difference here being that (2.49) is a pointwise factorisation of $|E(T_1) \wedge \cdots \wedge E(T_m)| M(Q)^m$, which is a function defined on $\mathcal{Q} \times \mathcal{T}_1 \times \cdots \times \mathcal{T}_m$, rather than the test function M itself. In spite of this, we will refer to functions $S_1, \dots, S_m$ as a factorisation of (2.48). The

above two estimates, along with the multilinear Hölder inequality yield

$$\begin{aligned}
& \sum_{Q \in \mathcal{Q}} \left(\sum_{T_1 \in \mathcal{T}_1: T_1 \cap Q \neq \emptyset} \cdots \sum_{T_m \in \mathcal{T}_m: T_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{T_j} \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m}} M(Q) \\
&= \sum_{Q \in \mathcal{Q}} \left(\sum_{T_1 \in \mathcal{T}_1: T_1 \cap Q \neq \emptyset} \cdots \sum_{T_m \in \mathcal{T}_m: T_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{T_j} \right) |E(T_1) \wedge \cdots \wedge E(T_m)| M(Q)^m \right)^{\frac{1}{m}} \\
&\lesssim_{d,m} \sum_{Q \in \mathcal{Q}} \left(\sum_{T_1 \in \mathcal{T}_1: T_1 \cap Q \neq \emptyset} \cdots \sum_{T_m \in \mathcal{T}_m: T_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{T_j} S_j(Q, T_j) \right) \right)^{\frac{1}{m}} \\
&= \sum_{Q \in \mathcal{Q}} \left(\prod_{j=1}^m \sum_{T_j \in \mathcal{T}_j: T_j \cap Q \neq \emptyset} \alpha_{T_j} S_j(Q, T_j) \right)^{\frac{1}{m}} \\
&\leq \prod_{j=1}^m \left(\sum_{Q \in \mathcal{Q}} \sum_{T_j \in \mathcal{T}_j: T_j \cap Q \neq \emptyset} \alpha_{T_j} S_j(Q, T_j) \right)^{\frac{1}{m}} \\
&= \prod_{j=1}^m \left(\sum_{T_j \in \mathcal{T}_j} \sum_{Q \in \mathcal{Q}: Q \cap T_j \neq \emptyset} \alpha_{T_j} S_j(Q, T_j) \right)^{\frac{1}{m}} \\
&= \prod_{j=1}^m \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \sum_{Q \in \mathcal{Q}: Q \cap T_j \neq \emptyset} S_j(Q, T_j) \right)^{\frac{1}{m}} \\
&\lesssim_{d,m} \prod_{j=1}^m \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \right)^{\frac{1}{m}}.
\end{aligned}$$

Unlike all the previous examples, the existence of a factorisation satisfying conditions (2.49) and (2.50) is not guaranteed by Theorem 1.1. However, one can actually prove the existence of such a factorisation directly:

Theorem 2.25 (Factorisation for endpoint multilinear Kakeya inequality - [24],[12],[15]). *Let $\mathcal{Q}$ denote the unit lattice of cubes on $\mathbb{R}^d$ and $2 \leq m \leq d$. Then, given any finitely supported nonnegative function $M \in \ell^m(\mathcal{Q})$ with $\|M\|_m = 1$, there exist S_j defined on $\mathcal{Q} \times \mathcal{T}_j$ for $j = 1, \dots, m$ such that for all $Q \in \mathcal{Q}$, whenever $T_j \cap Q \neq \emptyset$ for all j ,*

$$|E(T_1) \wedge \cdots \wedge E(T_m)| M(Q)^m \lesssim_{d,m} S_1(Q, T_1) \cdots S_m(Q, T_m),$$

and for all j and $T_j \in \mathcal{T}_j$

$$\sum_{Q \in \mathcal{Q}: Q \cap T_j \neq \emptyset} S_j(Q, T_j) \lesssim_{d,m} \left(\sum_Q M(Q)^m \right)^{\frac{1}{m}},$$

where the implicit constants both depend only on the dimension d and multilinearity m .

In Chapter 4 we describe a new approach for proving this result, which utilises underlying connections to the field of convex geometry. Essentially, we translate of the original proof given in [15] into the language of convex bodies. Nevertheless, we feel this new perspective provides some additional geometric intuition for an otherwise fairly technical argument.

Multilinear Keakeya inequality for k_j -planes In the endpoint multilinear Keakeya inequality, 1-tubes played a crucial role. Since any 1-tube is just the 1-neighbourhood of a line, one natural way to generalise this setup is by considering 1-neighbourhoods of higher dimensional planes:

Definition 2.26. Given a k -dimensional subspace $V \subset \mathbb{R}^d$, we refer to the 1-neighbourhood $P \subset \mathbb{R}^d$ of V as a **1-slab of dimension k** . Moreover, for any k -dimensional 1-slab $P \subset \mathbb{R}^d$, we denote by $E(P)$ the k -dimensional subspace of which P forms the 1-neighbourhood.

The main result in [39] implies a generalisation of Theorem 2.24 along these lines, which we will refer to as the endpoint multilinear Keakeya inequality for k_j -planes.

Theorem 2.27 ([39]). *Let $2 \leq m \leq d$. There exists a constant $C_{d,m}$ such that if $\mathcal{P}_1, \dots, \mathcal{P}_m$ are families of 1-slabs in $\mathbb{R}^d$ of dimensions $k_1, \dots, k_m$ respectively and $\sum_{j=1}^m k_j = d$, then we have*

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\sum_{P_1 \in \mathcal{P}_1} \cdots \sum_{P_m \in \mathcal{P}_m} \left(\prod_{j=1}^m \alpha_{P_j} \chi_{P_j}(x) \right) |E(P_1) \wedge \cdots \wedge E(P_m)| \right)^{\frac{1}{m-1}} dx \\ \leq C_{d,m} \prod_{j=1}^m \left(\sum_{P_j \in \mathcal{P}_j} \alpha_{P_j} \right)^{\frac{1}{m-1}}. \end{aligned}$$

By a similar reduction to the case of tubes it suffices to establish the following inequality for every finitely supported $M \in \ell^m(\mathcal{Q})$ with $\|M\|_{\ell^m(\mathcal{Q})} = 1$,

$$\begin{aligned} \sum_{Q \in \mathcal{Q}} \left(\sum_{P_1 \in \mathcal{P}_1: P_1 \cap Q \neq \emptyset} \cdots \sum_{P_m \in \mathcal{P}_m: P_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{P_j} \right) |E(P_1) \wedge \cdots \wedge E(P_m)| \right)^{\frac{1}{m}} M(Q) \\ \lesssim_{d,m} \prod_{j=1}^m \left(\sum_{P_j \in \mathcal{P}_j} \alpha_{P_j} \right)^{\frac{1}{m}}. \end{aligned}$$

Again, due to the non-constant factor of $|E(P_1) \wedge \cdots \wedge E(P_m)|$, this example does not quite fall under the scope of multilinear duality. Nevertheless, as with the endpoint multilinear Keakeya inequality for tubes, we can still write down a reasonable candidate for a factorisation. Suppose that for every such M there exist S_j defined on $\mathcal{Q} \times \mathcal{P}_j$ for $j = 1, \dots, m$ satisfying the following conditions, for all $Q \in \text{supp}(M)$

$$|E(P_1) \wedge \cdots \wedge E(P_m)| M(Q)^m \lesssim_{d,m} S_1(Q, P_1) \cdots S_m(Q, P_m), \quad (2.51)$$

and for all j and $P_j \in \mathcal{P}_j$

$$\sum_{Q: Q \cap P_j \neq \emptyset} S_j(Q, P_j) \lesssim_{d,m} 1. \quad (2.52)$$

Then we can apply the above two estimates along with the multilinear Hölder inequality to get

$$\begin{aligned}
& \sum_{Q \in \mathcal{Q}} \left(\sum_{P_1 \in \mathcal{P}_1: P_1 \cap Q \neq \emptyset} \cdots \sum_{P_m \in \mathcal{P}_m: P_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{P_j} \right) |E(P_1) \wedge \cdots \wedge E(P_m)| \right)^{\frac{1}{m}} M(Q) \\
&= \sum_{Q \in \mathcal{Q}} \left(\sum_{P_1 \in \mathcal{P}_1: P_1 \cap Q \neq \emptyset} \cdots \sum_{P_m \in \mathcal{P}_m: P_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{P_j} \right) |E(P_1) \wedge \cdots \wedge E(P_m)| M(Q)^m \right)^{\frac{1}{m}} \\
&\lesssim_{d,m} \sum_{Q \in \mathcal{Q}} \left(\sum_{P_1 \in \mathcal{P}_1: P_1 \cap Q \neq \emptyset} \cdots \sum_{P_m \in \mathcal{P}_m: P_m \cap Q \neq \emptyset} \left(\prod_{j=1}^m \alpha_{P_j} S_j(Q, P_j) \right) \right)^{\frac{1}{m}} \\
&= \sum_{Q \in \mathcal{Q}} \left(\prod_{j=1}^m \sum_{P_j \in \mathcal{P}_j: P_j \cap Q \neq \emptyset} \alpha_{P_j} S_j(Q, P_j) \right)^{\frac{1}{m}} \\
&\leq \prod_{j=1}^m \left(\sum_{Q \in \mathcal{Q}} \sum_{P_j \in \mathcal{P}_j: P_j \cap Q \neq \emptyset} \alpha_{P_j} S_j(Q, P_j) \right)^{\frac{1}{m}} \\
&= \prod_{j=1}^m \left(\sum_{P_j \in \mathcal{P}_j} \sum_{Q \in \mathcal{Q}: Q \cap P_j \neq \emptyset} \alpha_{P_j} S_j(Q, P_j) \right)^{\frac{1}{m}} \\
&= \prod_{j=1}^m \left(\sum_{P_j \in \mathcal{P}_j} \alpha_{P_j} \sum_{Q \in \mathcal{Q}: Q \cap P_j \neq \emptyset} S_j(Q, P_j) \right)^{\frac{1}{m}} \\
&\lesssim_{d,m} \prod_{j=1}^m \left(\sum_{P_j \in \mathcal{P}_j} \alpha_{P_j} \left(\sum_Q M(Q)^m \right)^{\frac{k_j}{d}} \right)^{\frac{1}{m}} \\
&= \left(\sum_Q M(Q)^m \right)^{\frac{\sum_{j=1}^m k_j}{dm}} \prod_{j=1}^m \left(\sum_{P_j \in \mathcal{P}_j} \alpha_{P_j} \right)^{\frac{1}{m}} \\
&= \left(\sum_Q M(Q)^m \right)^{\frac{1}{m}} \prod_{j=1}^m \left(\sum_{P_j \in \mathcal{P}_j} \alpha_{P_j} \right)^{\frac{1}{m}} \\
&= \prod_{j=1}^m \left(\sum_{P_j \in \mathcal{P}_j} \alpha_{P_j} \right)^{\frac{1}{m}}.
\end{aligned}$$

As with the previous example, the existence of a factorisation satisfying conditions (2.51) and (2.52) is not guaranteed by Theorem 1.1. In Chapter 4, we will discuss how to adapt the proof of Theorem 2.25 to work in this more general setting. As we will see, one can generalise the arguments in [15], following ideas from [39], to prove the existence of such a factorisation directly. The next result implies Theorem 2.27 via the duality argument laid out above.

Theorem 2.28 (Factorisation for k_j -planes endpoint multilinear Takeya inequality - [39]). *Let $\mathcal{Q}$ denote the unit lattice of cubes on $\mathbb{R}^d$ and $2 \leq m \leq d$. Then, given any finitely supported*

nonnegative function $M \in L^m(\mathcal{Q})$ with $\|M\|_m = 1$, there exist S_j defined on $\mathcal{Q} \times \mathcal{P}_j$ for $j = 1, \dots, m$ such that for all $Q \in \mathcal{Q}$, whenever $P_j \cap Q \neq \emptyset$ for all j ,

$$|E(P_1) \wedge \dots \wedge E(P_m)| M(Q)^m \lesssim_{d,m} S_1(Q, P_1) \dots S_m(Q, P_m),$$

and for all j and $P_j \in \mathcal{P}_j$

$$\sum_{Q \in \mathcal{Q}: Q \cap P_j \neq \emptyset} S_j(Q, P_j) \lesssim_{d,m} 1,$$

where the implicit constants both depend only on the dimension d and multilinearity m .

In Chapter 4, similarly to case of tubes, we describe a new convex geometric approach for proving of this result based on the arguments given in [39].

2.3 Examples of nonlinear Brascamp–Lieb inequalities with Loomis–Whitney exponents

As mentioned earlier, the Brascamp–Lieb inequality can be written

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j(L_j x)^{p_j} dx \leq \text{BL}(\mathbf{L}, \mathbf{p}) \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j} \quad (2.53)$$

where the mappings $L_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d_j}$ are linear surjections, the exponents $p_j \in [0, 1]$, and $\text{BL}(\mathbf{L}, \mathbf{p})$ denotes the smallest possible constant. It seems natural to ask whether we can replace the linear surjections L_j by more general nonlinear maps. The so-called nonlinear Brascamp–Lieb inequality, first appearing in [10] and later studied to great detail in [9], exactly addresses this question by replacing the L_j with maps which are smooth submersions in a neighbourhood of a point:

Theorem 2.29 ([9]). *Fix exponents $p_1, \dots, p_m \in [0, 1]$ and suppose that $B_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d_j}$ are C_2 submersions in a neighbourhood of a point x_0 with $dB(x_0) = L_j$, where $1 \leq j \leq m$. Then, for each $\varepsilon > 0$ there exists a neighbourhood U of x_0 such that*

$$\int_U \prod_{j=1}^m f_j(B_j x)^{p_j} dx \leq (1 + \varepsilon) \text{BL}(\mathbf{L}, \mathbf{p}) \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}. \quad (2.54)$$

In some special cases, one can interpret the value of $\text{BL}(\mathbf{L}, \mathbf{p})$ as a counterbalance for the lack of transversality between the maps $L_1, \dots, L_m$. This is particularly apparent in the affine-invariant Loomis–Whitney inequality, where the constant $|\omega_1 \wedge \dots \wedge \omega_d|^{-\frac{1}{d-1}}$ grows as the ω_j approach being linearly dependent and the projections $\pi_{\omega_1}, \dots, \pi_{\omega_d}$ become less ‘transversal’. With this in mind, it seems feasible to consider a global version of the nonlinear Brascamp–Lieb inequality provided we are willing to incorporate a nonconstant weight function in the integral on the left hand side. In other words, if we take general (potentially nonlinear) surjections $B_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d_j}$ and exponents $p_j \in [0, 1]$ for $1 \leq j \leq m$, under what circumstances can we expect to have

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j(B_j x)^{p_j} w(x) dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}, \quad (2.55)$$

for some weight function $w: \mathbb{R}^d \rightarrow \mathbb{R}$? This is trivially satisfied for w identically zero so it makes sense to seek some maximal weight function. Of course, in the special case where each of L_j is linear, this maximal weight function is constant and equal to $\text{BL}(\mathbf{L}, \mathbf{p})^{-1}$, which may be equal to zero in the case where $\text{BL}(\mathbf{L}, \mathbf{p})$ is infinite.

Provided that each B_j is nice enough to ensure that $f_j \circ B_j$ is measurable, and w is locally integrable, then inequalities of this form fall under the scope of multilinear duality. In particular, we have the following consequence of Theorem 1.1:

Theorem 2.30. *Suppose we are given a locally integrable weight function $w: \mathbb{R}^d \rightarrow \mathbb{R}$, and surjections $B_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d_j}$ for $1 \leq j \leq m$, such that $f_j \circ B_j$ is measurable for any measurable function $f_j: \mathbb{R}^{d_j} \rightarrow \mathbb{R}$. Fix exponents $p_1, \dots, p_m \in [0, 1]$, and set $p = \sum_{j=1}^m p_j$, then the following two statements are equivalent:*

(i) *For integrable $f_j: \mathbb{R}^{d_j} \rightarrow \mathbb{R}$, we have*

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j(B_j x)^{p_j} w(x) dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{p_j}.$$

(ii) *For every nonnegative function $M: \mathbb{R}^d \rightarrow \mathbb{R}$ such that $wM \in L^p(\mathbb{R}^d)$, there exist non-negative measurable functions $M_1, \dots, M_m$ such that for almost every $x \in \mathbb{R}^d$*

$$M(x) \leq \prod_{j=1}^m M_j(x)^{\frac{p_j}{p}},$$

and for $1 \leq j \leq m$ and every $f_j \in L^1(\mathbb{R}^{d_j})$

$$\int_{\mathbb{R}^d} f_j(B_j x) M_j(x) w(x) dx \leq \|f_j\|_{L^1(\mathbb{R}^{d_j})} \int_{\mathbb{R}^d} M(x) w(x) dx.$$

Although the above result guarantees the existence of factorisations for a wide class of nonlinear Brascamp–Lieb inequalities, we do not know how to construct any such factorisation explicitly.

In this section, we briefly survey a few examples of nonlinear Brascamp–Lieb inequalities which are global in the sense described above, and in each case we identify the weight function w as the Jacobian of a particular change of variables. To begin, consider the following such example, which was suggested to us by Jonathan Bennett:

$$\int_{\mathbb{R}^3} f_1(x_1, x_2) f_2\left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2}\right) f_3\left(x_3, -\frac{x_1 x_3}{x_2}\right) \frac{(x_1^2 + x_2^2)^{\frac{1}{2}} |x_3|^{\frac{1}{2}}}{|x_2|^{\frac{3}{2}}} dx \leq \|f_1\|_2 \|f_2\|_2 \|f_3\|_2. \quad (2.56)$$

Proposition 2.31. *Inequality (2.56) holds.*

Proof. Firstly, since $f_1(x_1, x_2)$ does not depend on x_3 , we can factor out the L^2 -norm of f_1 by

applying the Cauchy–Schwarz inequality to the integral in x_3 :

$$\begin{aligned}
& \int_{\mathbb{R}^3} f_1(x_1, x_2) f_2 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right) f_3 \left(x_3, -\frac{x_1 x_3}{x_2} \right) \frac{(x_1^2 + x_2^2)^{\frac{1}{2}} |x_3|^{\frac{1}{2}}}{|x_2|^{\frac{3}{2}}} dx \\
&= \int_{\mathbb{R}^2} f_1(x_1, x_2) \left(\int_{\mathbb{R}} f_2 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right) f_3 \left(x_3, -\frac{x_1 x_3}{x_2} \right) \frac{(x_1^2 + x_2^2)^{\frac{1}{2}} |x_3|^{\frac{1}{2}}}{|x_2|^{\frac{3}{2}}} dx_3 \right) dx_1 dx_2 \\
&\leq \|f_1\|_2 \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right) f_3 \left(x_3, -\frac{x_1 x_3}{x_2} \right) \frac{(x_1^2 + x_2^2)^{\frac{1}{2}} |x_3|^{\frac{1}{2}}}{|x_2|^{\frac{3}{2}}} dx_3 \right)^2 dx_1 dx_2 \right)^{\frac{1}{2}} \\
&= \|f_1\|_2 \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right) f_3 \left(x_3, -\frac{x_1 x_3}{x_2} \right) |x_3|^{\frac{1}{2}} dx_3 \right)^2 \frac{(x_1^2 + x_2^2)}{|x_2|^3} dx_1 dx_2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Now ignoring the factor of $\|f_1\|_2$, we can again apply the Cauchy–Schwarz inequality to the integral in x_3 to get

$$\begin{aligned}
& \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right) f_3 \left(x_3, -\frac{x_1 x_3}{x_2} \right) \frac{(x_1^2 + x_2^2)^{\frac{1}{2}} |x_3|^{\frac{1}{2}}}{|x_2|^{\frac{3}{2}}} dx_3 \right)^2 dx_1 dx_2 \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right)^2 dx_3 \right) \left(\int_{\mathbb{R}} f_3 \left(x_3, -\frac{x_1 x_3}{x_2} \right)^2 \frac{(x_1^2 + x_2^2) |x_3|}{|x_2|^3} dx_3 \right) dx_1 dx_2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The right hand side can then be rewritten as the following integral over $\mathbb{R}^4$:

$$\left(\int_{\mathbb{R}^4} f_2 \left(y_1 + y_3, y_2 - \frac{y_1 y_3}{y_2} \right)^2 f_3 \left(y_4, -\frac{y_1 y_4}{y_2} \right)^2 \frac{(y_1^2 + y_2^2) |y_4|}{|y_2|^3} dy \right)^{\frac{1}{2}}.$$

Let $G: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be defined by $G(y_1, y_2, y_3, y_4) = (y_1 + y_3, y_2 - \frac{y_1 y_3}{y_2}, y_4, -\frac{y_1 y_4}{y_2})$, and note that

$$JG(y) = \left| \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ -\frac{y_3}{y_2} & 1 + \frac{y_1 y_3}{y_2^2} & -\frac{y_1}{y_2} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{y_4}{y_2} & \frac{y_1 y_4}{y_2^2} & 0 & -\frac{y_1}{y_2} \end{pmatrix} \right| = \frac{(y_1^2 + y_2^2) |y_3|^{\frac{1}{2}} |y_4|^{\frac{1}{2}}}{|y_2|^3}$$

So by a change of variables $z = G(y)$ we get

$$\begin{aligned}
\left(\int_{\mathbb{R}^4} f_2 \left(y_1 + y_3, y_2 - \frac{y_1 y_3}{y_2} \right)^2 f_3 \left(y_4, -\frac{y_1 y_4}{y_2} \right)^2 \frac{(y_1^2 + y_2^2) |y_3|}{|y_2|^3} dy \right)^{\frac{1}{2}} &= \left(\int_{\mathbb{R}^4} f_2(z_1, z_2)^2 f_3(z_3, z_4)^2 dz \right)^{\frac{1}{2}} \\
&= \|f_2\|_2 \|f_3\|_2,
\end{aligned}$$

and we're done. $\square$

Note that, in terms of numerology, (2.56) has a similar form to that of the classical Loomis–Whitney inequality in $\mathbb{R}^2$, which states that for integrable $f_1, f_2, f_3: \mathbb{R}^2 \rightarrow \mathbb{R}$, we have the following sharp inequality:

$$\int_{\mathbb{R}^3} f_1(\pi_{e_1^\perp} x) f_2(\pi_{e_2^\perp} x) f_3(\pi_{e_3^\perp} x) dx \leq \|f_1\|_2 \|f_2\|_2 \|f_3\|_2. \quad (2.57)$$

Although (2.56) appears to be a nonlinear Brascamp–Lieb inequality in its current form, it is not clear whether the inequality is inherently nonlinear or simply a classical Brascamp–Lieb inequality in disguise. In particular, it seems natural to ask whether (2.56) can be reduced to (2.57), by simply applying a change of variables (i.e. diffeomorphism) in $\mathbb{R}^3$ on the left hand side and changes of variables in each copy of $\mathbb{R}^2$ on the right hand side. To illustrate what we mean, suppose we are given changes of variables $G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $C_1, C_2, C_3: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, then we can rewrite (2.57) as follows

$$\int_{\mathbb{R}^3} \prod_{j=1}^3 f_j \left(\pi_{e_j^\perp} \circ G(x) \right) JG(x) dx \leq \prod_{j=1}^3 \left(\int_{\mathbb{R}^2} f_j \left(C_j^{-1}(y_j) \right)^2 JC_j(y_j)^{-1} dy_j \right)^{\frac{1}{2}},$$

where $JG(x)$ denotes the Jacobian of G at x , and $JC_j(y_j)$ the Jacobian of C_j at y_j . Replacing f_j by $f_j \circ C_j$, which will also be an integrable function on $\mathbb{R}^2$, this last inequality becomes

$$\int_{\mathbb{R}^3} \prod_{j=1}^3 f_j \left(C_j \circ \pi_{e_j^\perp} \circ G(x) \right) |JG(x)| dx \leq \prod_{j=1}^3 \left(\int_{\mathbb{R}^2} f_j(y_j)^2 |JC_j(y_j)|^{-1} dy_j \right)^{\frac{1}{2}}.$$

In the special case where each C_j is linear, we can incorporate the corresponding Jacobians in the integral on the left hand side to get

$$\int_{\mathbb{R}^3} \prod_{j=1}^3 f_j \left(C_j \circ \pi_{e_j^\perp} \circ G(x) \right) \left(\prod_{j=1}^3 |JC_j| \right)^{\frac{1}{2}} |JG(x)| dx \leq \prod_{j=1}^3 \|f_j\|_2. \quad (2.58)$$

Written in this form, we see that it is very possible to have a seemingly more complicated (possibly non-linear) Brascamp–Lieb inequality in $\mathbb{R}^3$, which can be reduced to the classical Loomis–Whitney inequality, using only changes of variables.

A rather simple, yet illustrative example of this is the affine-invariant Loomis–Whitney inequality introduced earlier, which states that for integrable $f_1, f_2, f_3: \mathbb{R}^2 \rightarrow \mathbb{R}$, and any linearly independent unit vectors $\omega_1, \omega_2, \omega_3 \in \mathbb{R}^3$, we have the following sharp inequality

$$\int_{\mathbb{R}^3} f_1 \circ \pi_{\omega_1^\perp}(x) f_2 \circ \pi_{\omega_2^\perp}(x) f_3 \circ \pi_{\omega_3^\perp}(x) |\omega_1 \wedge \omega_2 \wedge \omega_3|^{\frac{1}{2}} dx \leq \|f_1\|_2 \|f_2\|_2 \|f_3\|_2, \quad (2.59)$$

where $|\omega_1 \wedge \omega_2 \wedge \omega_3|$ denotes the determinant of the matrix with columns $\omega_1, \omega_2, \omega_3$. To see how this can be written in the form of (2.58), let $\Phi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by $\Phi(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) = \lambda_1 \omega_1 + \lambda_2 \omega_2 + \lambda_3 \omega_3$, and set $G = \Phi^{-1}$. For each j , define $C_j := \pi_{\omega_j^\perp} \circ G^{-1} \circ \pi_{e_j^\perp}^*$, so that

$$\pi_{\omega_j^\perp} = C_j \circ \pi_{e_j^\perp} \circ G.$$

Observe that for every $x \in \mathbb{R}^3$ we have

$$JG(x) = J\Phi(x)^{-1} = |\omega_1 \wedge \omega_2 \wedge \omega_3|^{-1}.$$

Furthermore, $C_j(\sum_{i \neq j} \lambda_i e_i) = \sum_{i \neq j} \lambda_i \pi_{\omega_j^\perp} \omega_i$, so for every $y_j \in e_j^\perp$ we have

$$|JC_j(y_j)| = \left| \bigwedge_{i \neq j} \pi_{\omega_j^\perp} \omega_i \right| = |\omega_1 \wedge \omega_2 \wedge \omega_3|.$$

Using this, we can write the left hand side of (2.58) as follows

$$\begin{aligned}
& \int_{\mathbb{R}^3} \prod_{j=1}^3 f_j \left(C_j \circ \pi_{e_j^\perp} \circ G(x) \right) \left(\prod_{j=1}^3 |JC_j| \right)^{\frac{1}{2}} |JG(x)| dx \\
&= \int_{\mathbb{R}^3} \prod_{j=1}^3 f_j \left(\pi_{\omega_j^\perp} x \right) \left(\prod_{j=1}^3 |\omega_1 \wedge \omega_2 \wedge \omega_3| \right)^{\frac{1}{2}} |\omega_1 \wedge \omega_2 \wedge \omega_3|^{-1} dx \\
&= \int_{\mathbb{R}^3} \prod_{j=1}^3 f_j \left(\pi_{\omega_j^\perp} x \right) |\omega_1 \wedge \omega_2 \wedge \omega_3|^{\frac{1}{2}} dx,
\end{aligned}$$

which shows that the affine-invariant Loomis–Whitney inequality is indeed a special case of (2.58).

It might not be much of a surprise that one can derive the affine-invariant Loomis–Whitney inequality from the classical Loomis–Whitney inequality using changes of variables. However, this next example appears to have a similar form to (2.57), but can also be written in the form of (2.58):

$$\int_{\mathbb{R}^3} f_1(x_1, x_1 + x_3) f_2\left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2}\right) f_3\left(x_1, x_2 - \frac{x_1 x_3}{x_2}\right) \left| \frac{x_2^2 + x_1 x_3}{x_2^2} \right| dx \leq \|f_1\|_2 \|f_2\|_2 \|f_3\|_2. \quad (2.60)$$

Let us define $G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $G(x) = (x_2 - \frac{x_1 x_3}{x_2}, x_1, x_1 + x_3)$, then we have

$$JG(x) = \left| \det \begin{pmatrix} -\frac{x_3}{x_2} & 1 + \frac{x_1 x_3}{x_2^2} & -\frac{x_1}{x_2} \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right| = \left| 1 + \frac{x_1 x_3}{x_2^2} \right| = \left| \frac{x_2^2 + x_1 x_3}{x_2^2} \right|.$$

In this way, we can rewrite (2.60) as follows

$$\int_{\mathbb{R}^3} f_1 \left(\pi_{e_1^\perp} \circ G(x) \right) f_2 \left(\pi_{e_2^\perp} \circ G(x) \right) f_3 \left(\pi_{e_3^\perp} \circ G(x) \right) JG(x) dx \leq \|f_1\|_2 \|f_2\|_2 \|f_3\|_2,$$

which reduces immediately to the classical Loomis–Whitney inequality by a change of variables.

In contrast, we will now show that (2.56) cannot be reduced to the classical Loomis–Whitney inequality by changes of variables, which suggests there is something truly non-linear in the structure of this example:

Proposition 2.32. *There do not exist diffeomorphisms $G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $C_1, C_2, C_3: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ the following identities hold:*

$$\begin{aligned}
C_1 \circ \pi_{e_1^\perp} \circ G(x_1, x_2, x_3) &= (x_1, x_2); \\
C_2 \circ \pi_{e_2^\perp} \circ G(x_1, x_2, x_3) &= \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right); \\
C_3 \circ \pi_{e_3^\perp} \circ G(x_1, x_2, x_3) &= \left(x_3, -\frac{x_1 x_3}{x_2} \right).
\end{aligned}$$

Before proving this last statement, we state the following simple lemma:

Lemma 2.33. Fix $v \in \mathbb{R}$. Then for any real number $\alpha \leq \frac{v^2}{4}$, there exists $t \in \mathbb{R}$ such that

$$v = t + \frac{\alpha}{t}.$$

Proof. Fix $\alpha \leq \frac{v^2}{4}$, then multiplying through by t , we can rewrite the equation in the statement of the lemma as

$$vt = t^2 + \alpha.$$

This can certainly be solved by setting

$$t = \frac{v}{2} + \sqrt{\frac{v^2}{4} - \alpha},$$

which is real since $\alpha \leq \frac{v^2}{4}$. □

Proof of Proposition 2.32. Assume for a contradiction that such diffeomorphisms exists, that is $G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $C_1, C_2, C_3: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ the following identities hold:

$$\begin{aligned} C_1 \circ \pi_{e_1^\perp} \circ G(x) &= (x_1, x_2); \\ C_2 \circ \pi_{e_2^\perp} \circ G(x) &= \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right); \\ C_3 \circ \pi_{e_3^\perp} \circ G(x) &= \left(x_3, -\frac{x_1 x_3}{x_2} \right). \end{aligned}$$

For ease of notation, let us write $G(x) = (G_1(x), G_2(x), G_3(x))$. Taking inverses, which exist by assumption, we can rewrite the above three equalities in the following way:

$$\begin{aligned} C_1^{-1}(x_1, x_2) &= (G_2(x_1, x_2, x_3), G_3(x_1, x_2, x_3)); \\ C_2^{-1}\left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2}\right) &= (G_1(x_1, x_2, x_3), G_3(x_1, x_2, x_3)); \\ C_3^{-1}\left(x_3, -\frac{x_1 x_3}{x_2}\right) &= (G_1(x_1, x_2, x_3), G_2(x_1, x_2, x_3)). \end{aligned}$$

Since each C_i^{-1} is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, we can write

$$\begin{aligned} C_1^{-1}(u, v) &= (P_1(u, v), P_2(u, v)), \\ C_2^{-1}(u, v) &= (Q_1(u, v), Q_2(u, v)), \\ C_3^{-1}(u, v) &= (R_1(u, v), R_2(u, v)), \end{aligned}$$

where P_1, P_2, Q_1, Q_2, R_1 , and R_2 are all functions from $\mathbb{R}^2$ to $\mathbb{R}$. Now we have the following six identities:

- i) $P_1(x_1, x_2) = G_2(x_1, x_2, x_3);$
- ii) $P_2(x_1, x_2) = G_3(x_1, x_2, x_3);$
- iii) $Q_1\left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2}\right) = G_1(x_1, x_2, x_3);$

$$\text{iv)} \quad Q_2 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right) = G_3(x_1, x_2, x_3);$$

$$\text{v)} \quad R_1 \left(x_3, -\frac{x_1 x_3}{x_2} \right) = G_1(x_1, x_2, x_3);$$

$$\text{vi)} \quad R_2 \left(x_3, -\frac{x_1 x_3}{x_2} \right) = G_2(x_1, x_2, x_3).$$

Firstly, note that by (ii) we know that $G_3(x_1, x_2, x_3)$ does not depend on x_3 . Using (iv), we can immediately deduce that $Q_2 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right)$ is also independent of x_3 , and since this expression is symmetric in x_1 and x_3 it must also be independent of x_1 . Thus we can write

$$Q_2 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right) = g(x_2). \quad (2.61)$$

Combining (iii) and (v), we can write

$$R_1 \left(x_3, -\frac{x_1 x_3}{x_2} \right) = Q_1 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right). \quad (2.62)$$

Since the expression on the right hand side is symmetric in x_1 and x_3 , it follows that for any $y_1, y_2, y_3 \in \mathbb{R}$ with $y_2 \neq 0$, we have

$$R_1 \left(y_3, -\frac{y_1 y_3}{y_2} \right) = R_1 \left(y_1, -\frac{y_1 y_3}{y_2} \right).$$

This implies that $R_1(u, v)$ only depends on v . Indeed, fix $u, \tilde{u}, v \in \mathbb{R}$ with $v \neq 0$, and set $y_3 = u$, $y_1 = \tilde{u}$ and $y_2 = -\frac{u\tilde{u}}{v}$. It follows that

$$R_1(u, v) = R_1 \left(y_3, -\frac{y_1 y_3}{y_2} \right) = R_1 \left(y_1, -\frac{y_1 y_3}{y_2} \right) = R_1(\tilde{u}, v).$$

Furthermore, for any $u, \tilde{u} \in \mathbb{R}$ and $y_2 \neq 0$ we have

$$R_1(u, 0) = R_1 \left(u, -\frac{u \cdot 0}{y_2} \right) = R_1 \left(0, -\frac{u \cdot 0}{y_2} \right) = R_1 \left(0, -\frac{\tilde{u} \cdot 0}{y_2} \right) = R_1 \left(\tilde{u}, -\frac{\tilde{u} \cdot 0}{y_2} \right) = R_1(\tilde{u}, 0).$$

Hence, we can write

$$R_1(u, v) = h(v).$$

So from (2.62) we immediately deduce that

$$Q_1 \left(x_1 + x_3, x_2 - \frac{x_1 x_3}{x_2} \right) = h \left(-\frac{x_1 x_3}{x_2} \right). \quad (2.63)$$

We claim that by (2.61) and (2.63), one can write

$$Q_1(u, v) = a(v),$$

and

$$Q_2(u, v) = b(v),$$

which together contradict the assumption that C_2 is injective. To see why this is true, fix $v \in \mathbb{R}$ and some $\alpha \leq \frac{v^2}{4}$, and apply Lemma 2.33 to get $t \in \mathbb{R}$ such that

$$v = t + \frac{\alpha}{t}.$$

Fix $u, \tilde{u} \in \mathbb{R}$, then using (2.63) we can deduce that

$$Q_1(u, v) = Q_1\left(u, t + \frac{\alpha}{t}\right) = h\left(\frac{\alpha}{t}\right) = Q_1\left(\tilde{u}, t + \frac{\alpha}{t}\right) = Q_1(\tilde{u}, v).$$

Now using (2.61), it follows that

$$Q_2(u, v) = Q_2\left(u, t + \frac{\alpha}{t}\right) = g(t) = Q_2\left(\tilde{u}, t + \frac{\alpha}{t}\right) = Q_2(\tilde{u}, v).$$

We conclude that both $Q_1(u, v)$ and $Q_2(u, v)$ are independent of u , which as we mentioned above, contradicts the assumption that $(u, v) \mapsto (Q_1(u, v), Q_2(u, v))$ is a diffeomorphism. $\square$

2.3.1 More examples with Loomis–Whitney exponents

In an attempt to find a generalisation of (2.56), we survey some similar examples. In particular, we try to understand how the proof of Proposition 2.31 could be adapted to a more general setup. Suppose we have surjections $B_1, B_2, B_3: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, we are interested in finding a non-zero weight function $w = w_{B_1, B_2, B_3}: \mathbb{R}^3 \rightarrow \mathbb{R}$, such that the following inequality holds:

$$\int_{\mathbb{R}^3} f_1 \circ B_1(x) f_2 \circ B_2(x) f_3 \circ B_3(x) w(x) dx \leq \|f_1\|_2 \|f_2\|_2 \|f_3\|_2. \quad (2.64)$$

What conditions do we need to impose on the B_i in order to be able to apply a similar procedure as in the proof of Proposition 2.31? That is, when can we apply an initial change of variables to factor out $f_1 \circ B_1$ from one of the integrals, and then apply Hölder's inequality twice followed by a final change of variable in $\mathbb{R}^4$ to decouple the integration and be left with a product of L^2 -norms?

Note that, for each of the examples we present below, the proof of Proposition 2.32 can be adapted to prove that the given inequality does not reduce to a standard multilinear Brascamp–Lieb inequality. The first two examples are simply constructed by modifying the non-linear maps appearing in the left hand side of (2.56).

Example 1. The non-linearity in this example is much simpler than our original example but illustrates clearly how the procedure works:

$$\begin{aligned} & \int_{\mathbb{R}^3} f_1(x_1, x_2) f_2(x_1, x_3) f_3(x_2 + x_3, x_2 x_3) |x_2 - x_3|^{\frac{1}{2}} dx \\ & \leq \|f_1\|_2 \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2(x_1, x_3) f_3(x_2 + x_3, x_2 x_3) |x_2 - x_3|^{\frac{1}{2}} dx_3 \right)^2 dx_1 dx_2 \right)^{\frac{1}{2}} \\ & \leq \|f_1\|_2 \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2(x_1, x_3)^2 dx_3 \right) \left(\int_{\mathbb{R}} f_3(x_2 + x_4, x_2 x_4)^2 |x_2 - x_4| dx_4 \right) dx_1 dx_2 \right)^{\frac{1}{2}} \\ & = \|f_1\|_2 \left(\int_{\mathbb{R}^4} f_2(x_1, x_3)^2 f_3(x_2 + x_4, x_2 x_4)^2 |x_2 - x_4| dx \right)^{\frac{1}{2}} \\ & = \|f_1\|_2 \left(\int_{\mathbb{R}^4} f_2(y_1, y_3)^2 f_3(y_2, y_4)^2 dy \right)^{\frac{1}{2}} = \|f_1\|_2 \|f_2\|_2 \|f_3\|_2, \end{aligned}$$

where in the second to last step we made a change of variables $y = G(x)$, with $G(x_1, x_2, x_3, x_4) = (x_1, x_3, x_2 + x_4, x_2x_4)$ and $JG = |x_2 - x_4|$.

Example 2. This example is very similar (2.56) but we replace x_2 by x_2^2 in one of the expressions:

$$\begin{aligned}
& \int_{\mathbb{R}^3} f_1(x_1, x_2) f_2\left(x_1 + x_3, x_2^2 - \frac{x_1x_3}{x_2}\right) f_3\left(x_3, -\frac{x_1x_3}{x_2}\right) \frac{|2x_2^3 - x_1^2|^{\frac{1}{2}}|x_3|^{\frac{1}{2}}}{|x_2|^{\frac{3}{2}}} dx \\
& \leq \|f_1\|_2 \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2\left(x_1 + x_3, x_2^2 - \frac{x_1x_3}{x_2}\right) f_3\left(x_3, -\frac{x_1x_3}{x_2}\right) \frac{|2x_2^3 - x_1^2|^{\frac{1}{2}}|x_3|^{\frac{1}{2}}}{|x_2|^{\frac{3}{2}}} dx_3 \right)^2 dx_1 dx_2 \right)^{\frac{1}{2}} \\
& \leq \|f_1\|_2 \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2\left(x_1 + x_3, x_2^2 - \frac{x_1x_3}{x_2}\right)^2 dx_3 \right) \left(\int_{\mathbb{R}} f_3\left(x_3, -\frac{x_1x_3}{x_2}\right)^2 \frac{|2x_2^3 - x_1^2||x_4|}{|x_2|^3} dx_4 \right) dx_1 dx_2 \right)^{\frac{1}{2}} \\
& = \|f_1\|_2 \left(\int_{\mathbb{R}^4} f_2\left(x_1 + x_3, x_2^2 - \frac{x_1x_3}{x_2}\right)^2 f_3\left(x_4, -\frac{x_1x_4}{x_2}\right)^2 \frac{|2x_2^3 - x_1^2||x_4|}{|x_2|^3} dx \right)^{\frac{1}{2}} \\
& = \|f_1\|_2 \left(\int_{\mathbb{R}^4} f_2(y_1, y_2)^2 f_3(y_3, y_4)^2 dy \right)^{\frac{1}{2}} = \|f_1\|_2 \|f_2\|_2 \|f_3\|_2,
\end{aligned}$$

where in the second to last step we made a change of variables $y = G(x)$, with $G(x_1, x_2, x_3, x_4) = (x_1 + x_3, x_2^2 - \frac{x_1x_3}{x_2}, x_4, -\frac{x_1x_4}{x_2})$ and $JG = \frac{|2x_2^3 - x_1^2||x_4|}{|x_2|^3}$.

Note that (2.56) can be written equivalently in the following form:

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} f_1(x) f_2(x + y) f_3(y) \delta_0(x \cdot y) \|x\|^{\frac{1}{2}} \|y\|^{\frac{1}{2}} dx dy \leq \|f_1\|_2 \|f_2\|_2 \|f_3\|_2. \quad (2.65)$$

To generate the next few examples, we replace $x \cdot y = 0$ with other algebraic hypersurfaces in $\mathbb{R}^2 \times \mathbb{R}^2$.

Example 3. In this example we replace $\delta_0(x \cdot y)$ with $\delta_0(y_1y_2 + x_1x_2)$. For ease of notation we set $x_3 := y_1$. Define the weight function $w(x, y)|_{x_1x_2+y_1y_2=0} = \frac{|x_2|^{\frac{1}{2}}|x_1^2+x_3^2|^{\frac{1}{2}}}{|x_3|^{\frac{3}{2}}}$, then

$$\begin{aligned}
& \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_1(x) f_2(x + y) f_3(y) w(x, y) \delta_0(x_1x_2 + y_1y_2) dx dy \\
& = \int_{\mathbb{R}^3} f_1(x_1, x_2) f_2\left(x_1 + x_3, x_2 - \frac{x_1x_2}{x_3}\right) f_3\left(x_3, -\frac{x_1x_2}{x_3}\right) \frac{|x_2|^{\frac{1}{2}}|x_1^2+x_3^2|^{\frac{1}{2}}}{|x_3|^{\frac{3}{2}}} dx \\
& \leq \|f_1\|_2 \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2\left(x_1 + x_3, x_2 - \frac{x_1x_2}{x_3}\right) f_3\left(x_3, -\frac{x_1x_2}{x_3}\right) \frac{|x_2|^{\frac{1}{2}}|x_1^2+x_3^2|^{\frac{1}{2}}}{|x_3|^{\frac{3}{2}}} dx_3 \right)^2 dx_1 dx_2 \right)^{\frac{1}{2}} \\
& \leq \|f_1\|_2 \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2\left(x_1 + x_3, x_2 - \frac{x_1x_2}{x_3}\right)^2 \frac{|x_2||x_1^2+x_3^2|}{|x_3|^2} dx_3 \right) \left(\int_{\mathbb{R}} f_3\left(x_3, -\frac{x_1x_2}{x_3}\right)^2 \frac{1}{|x_4|} dx_4 \right) dx_1 dx_2 \right)^{\frac{1}{2}} \\
& \leq \|f_1\|_2 \left(\int_{\mathbb{R}^4} f_2\left(x_1 + x_3, x_2 - \frac{x_1x_2}{x_3}\right)^2 f_3\left(x_4, -\frac{x_1x_2}{x_4}\right)^2 \frac{|x_2||x_1^2+x_3^2|}{|x_4||x_3|^2} dx \right)^{\frac{1}{2}} \\
& = \|f_1\|_2 \left(\int_{\mathbb{R}^4} f_2(z_1, z_2)^2 f_3(z_3, z_4)^2 dz \right)^{\frac{1}{2}} = \|f_1\|_2 \|f_2\|_2 \|f_3\|_2,
\end{aligned}$$

where we have made a change of variables $z = G(x)$, with $G(x_1, x_2, x_3, x_4) = (x_1 + x_3, x_2 - \frac{x_1 x_2}{x_3}, x_4, -\frac{x_1 x_2}{x_4})$ so that $JG = \frac{|x_2||x_1^2 + x_3^2|}{|x_4||x_3|^2}$.

Example 4. In this example we replace $\delta_0(x \cdot y)$ with $\delta_0(y_2 - x_1 x_2 y_1)$. For ease of notation we set $x_3 := y_1$. Define the weight function $w(x, y)|_{y_2 - x_1 x_2 y_1 = 0} = |x_2|^{\frac{1}{2}} |x_3|^{\frac{1}{2}} |x_1^2 + 1|^{\frac{1}{2}}$, then

$$\begin{aligned}
& \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_1(x) f_2(x + y) f_3(y) w(x, y) \delta_0(y_2 - x_1 x_2 y_1) dx dy \\
&= \int_{\mathbb{R}^3} f_1(x_1, x_2) f_2(x_1 + x_3, x_2 + x_1 x_2 x_3) f_3(x_3, x_1 x_2 x_3) |x_2|^{\frac{1}{2}} |x_3|^{\frac{1}{2}} |x_1^2 + 1|^{\frac{1}{2}} dx \\
&\leq \|f_1\|_2 \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2(x_1 + x_3, x_2 + x_1 x_2 x_3) f_3(x_3, x_1 x_2 x_3) |x_2|^{\frac{1}{2}} |x_3|^{\frac{1}{2}} |x_1^2 + 1|^{\frac{1}{2}} dx_3 \right)^2 dx_1 dx_2 \right)^{\frac{1}{2}} \\
&\leq \|f_1\|_2 \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f_2(x_1 + x_3, x_2 + x_1 x_2 x_3)^2 dx_3 \right) \left(\int_{\mathbb{R}} f_3(x_3, x_1 x_2 x_3)^2 |x_2| |x_4| |x_1^2 + 1| dx_4 \right) dx_1 dx_2 \right)^{\frac{1}{2}} \\
&= \|f_1\|_2 \left(\int_{\mathbb{R}^4} f_2(x_1 + x_3, x_2 + x_1 x_2 x_3) f_3(x_4, x_1 x_2 x_4) |x_2| |x_4| |x_1^2 + 1| dx \right)^{\frac{1}{2}} \\
&= \|f_1\|_2 \left(\int_{\mathbb{R}^4} f_2(z_1, z_2)^2 f_3(z_3, z_4)^2 dz \right)^{\frac{1}{2}} = \|f_1\|_2 \|f_2\|_2 \|f_3\|_2,
\end{aligned}$$

where we have made a change of variables $z = G(y)$, with $G(x_1, x_2, x_3, x_4) = (x_1 + x_3, x_2 + x_1 x_2 x_3, x_4, x_1 x_2 x_4)$ so that $JG = |x_2| |x_4| |x_1^2 + 1|$.

Note that in all of the above examples, the final Jacobian satisfies the following factorisation property

$$JG(x_1, x_2, x_3, x_4) = a(x_1, x_2, x_3) b(x_1, x_2, x_4).$$

It is not clear how to interpret this condition, but perhaps there is some underlying structure connecting these examples.

Chapter 3

Geometric Brascamp–Lieb inequalities

In this chapter we discuss some ideas on attempting to factorise the geometric Brascamp–Lieb inequality of Keith Ball, which can be stated as follows

Theorem 3.1 (Classical geometric Brascamp–Lieb inequality - [6]). *Suppose we have exponents $p_1, \dots, p_m > 0$ and unit vectors $u_1, \dots, u_m \in \mathbb{R}^d$ such that*

$$\sum_{i=1}^m p_i u_i \otimes u_i = I_d. \quad (3.1)$$

Then the following inequality holds

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j(\langle x, u_j \rangle)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}} g_j \right)^{p_j}, \quad (3.2)$$

and for each $t > 0$, the standard Gaussian functions $g_j(y) = e^{-\pi t |y|^2}$ are extremals.

As we mentioned in Chapter 2, it is possible to "algorithmically" factorise special cases (3.2) if we are willing to lose the sharp constant 1 on the right hand side. However, this argument will involve multiple stages of interpolation and in general the final constant will be far from optimal. Unlike the Loomis–Whitney or Finner inequalities, the linear maps $\langle \cdot, u_j \rangle : \mathbb{R}^d \rightarrow \mathbb{R}$ are not defined in terms of some fixed basis or family of d vectors. Instead, we use a family of m vectors, where m in general will be greater than the dimension of the ambient space. It should be noted that most of the factorisations presented in Chapter 2 were constructed using some fixed underlying basis. In the Loomis–Whitney and Finner inequalities this role was played by the standard basis vectors $e_1, \dots, e_d$, whereas for both affine-invariant versions we began with an arbitrary basis of unit vectors $\omega_1, \dots, \omega_d$. It is not clear whether one can adapt the factorisations of these other examples to the setting of Theorem 3.1.

Although multilinear duality guarantees the existence of factorisations for the geometric Brascamp–Lieb inequality with sharp constant 1, we are interested in whether one can prove this fact directly. In an effort to do exactly this, we explore a certain structural duality of geometric Brascamp–Lieb inequalities. In some sense, such inequalities always arise in "dual pairs". This is illustrated through the following elementary result, which I first came across in [7]:

Lemma 3.2. *Let $1 \leq d \leq m$. Suppose we have unit vectors $u_1, \dots, u_m \in \mathbb{R}^d$ and exponents $p_1, \dots, p_m > 0$ such that*

$$\sum_{i=1}^m p_i u_i \otimes u_i = I_d,$$

then there exists an orthonormal basis $v_1, \dots, v_m \in \mathbb{R}^m$ such that

$$\pi_{\mathbb{R}^d} v_i = p_i^{\frac{1}{2}} u_i,$$

where $\pi_{\mathbb{R}^d}$ denotes projection onto the first d coordinates of $\mathbb{R}^m$. Moreover, letting $\pi_{\mathbb{R}^{m-d}}$ denote projection onto the final $(m-d)$ coordinates of $\mathbb{R}^m$, and setting

$$w_i = \begin{cases} \frac{\pi_{\mathbb{R}^{m-d}} v_i}{\|\pi_{\mathbb{R}^{m-d}} v_i\|} & \text{if } p_i \neq 1 \\ \mathbf{0} & \text{if } p_i = 1, \end{cases}$$

we have

$$\pi_{\mathbb{R}^{m-d}} v_i = (1 - p_i)^{\frac{1}{2}} w_i,$$

and

$$\sum_{i=1}^m (1 - p_i) w_i \otimes w_i = I_{m-d}.$$

For completeness, we give a proof of this result in Section 3.1. In light of Lemma 3.2, one might hope to establish a relationship between the corresponding "dual pair" of geometric Brascamp–Lieb inequalities, namely

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j (\langle x, u_j \rangle)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}} g_j \right)^{p_j} \quad (3.3)$$

and

$$\int_{\mathbb{R}^{m-d}} \prod_{j=1}^m h_j (\langle y, w_j \rangle)^{1-p_j} dy \leq \prod_{j=1}^m \left(\int_{\mathbb{R}} h_j \right)^{1-p_j}. \quad (3.4)$$

Let us introduce the following fairly standard notation to aid our discussion:

Definition 3.3. Fix m -tuples $\mathbf{p} = (p_1, \dots, p_m)$ of positive exponents and $\mathbf{v} = (v_1, \dots, v_m)$ of unit vectors in $\mathbb{R}^d$. The Brascamp–Lieb constant $\text{BL}(\mathbf{v}, \mathbf{p}) \in [0, \infty]$ is defined to be the smallest constant such that the following inequality holds

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j (\langle x, v_j \rangle)^{p_j} dx \leq \text{BL}(\mathbf{v}, \mathbf{p}) \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j \right)^{p_j},$$

for all integrable $f_j: \mathbb{R} \rightarrow \mathbb{R}$.

Using this language, essentially Theorem 3.1 tells us that if (3.1) is satisfied, then $\text{BL}(\mathbf{u}, \mathbf{p})$ is equal to 1.

Definition 3.4. The tuple $(\mathbf{v}, \mathbf{p})$ is referred to as **geometric Brascamp–Lieb datum** if and only the following identity holds

$$\sum_{i=1}^m p_i v_i \otimes v_i = I_d.$$

Using notation of Lemma 3.2 we have the following pair of geometric Brascamp–Lieb data

$$(\mathbf{u}, \mathbf{p}) = ((u_1, \dots, u_m), (p_1, \dots, p_m))$$

and

$$(\mathbf{w}, \mathbf{1-p}) = ((w_1, \dots, w_m), ((1-p_1), \dots, (1-p_m))).$$

To reiterate what we were alluding to above, it would be interesting to know if there was a deeper connection between the constants $\text{BL}(\mathbf{u}, \mathbf{p})$ and $\text{BL}(\mathbf{w}, \mathbf{1-p})$. For instance, can one establish a relationship between the two constants without reference to Theorem 3.1? In the final few sections of this chapter we discuss some rough ideas in response to this sort of question. Firstly in Section 3.3, we will discuss some initial attempts to prove directly that

$$\text{BL}(\mathbf{u}, \mathbf{p}) \cdot \text{BL}(\mathbf{w}, \mathbf{1-p}) \leq 1.$$

Subsequently, in the Section 3.4 we explore a possible connection between factorisations of (3.3) and (3.4) respectively. It would be very interesting if one could use a known factorisation of the former to construct a factorisation of the latter or vice versa. In particular, this would allow us to use the factorisation of the multilinear Hölder inequality given in [16] to construct a factorisation for the Young convolution inequality on $\mathbb{R}$ with sharp constant, which we are not currently able to do. The details of this implication are given at the end of Section 3.2, where we derive the sharp Young convolution inequality as a special case of Theorem 3.1. Although this last fact is well known to be true, the derivation we give provides a nice geometric perspective.

In [9], the authors observe an alternative more general notion of duality which, in the setting geometric Brascamp–Lieb inequalities is closely related the "dual pairing" suggested by Lemma 3.2. To formulate this precisely, we must first introduce a more general form of multilinear inequality:

Definition 3.5. Given $g_j: \mathbb{R}^{d_j} \rightarrow \mathbb{R}$ for $j = 1, \dots, m$, we define the tensor product function $g_1 \otimes \dots \otimes g_m: \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} \rightarrow \mathbb{R}$ by

$$g_1 \otimes \dots \otimes g_m(y) = g_1(y_1) \dots g_m(y_m).$$

The geometric Brascamp–Lieb inequality can be understood as a Lebesgue space bound on form

$$(g_1 \dots, g_m) = \int_{\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}} g_1 \otimes \dots \otimes g_m d\sigma_H,$$

where the integration is with respect to the Lebesgue measure on some subspace H of $\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}$. Indeed, as we will see in Section 3.1, there always exists a d -dimensional subspace $H \subset \mathbb{R} \times \dots \times \mathbb{R} \cong \mathbb{R}^m$ such that (3.2) can be rewritten as

$$\int_{\mathbb{R}^m} g_1 \otimes \dots \otimes g_m d\sigma_H \lesssim \prod_{j=1}^m \|g_j\|_{L^{q_j}(\mathbb{R})},$$

where now $d\sigma_H$ denotes the restriction of the d -dimensional Hausdorff measure to H . Using notation from [10], let us make the following definition

Definition 3.6. Fix a d -dimensional subspace H of $\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}$ and an m -tuple $\mathbf{q} = (q_1, \dots, q_m) \in [1, \infty]^m$. The Brascamp–Lieb constant $\text{BL}(H, \mathbf{q}) \in [0, \infty]$ is defined to be the smallest constant such that the following inequality holds

$$\int_{\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}} f_1 \otimes \dots \otimes f_m d\sigma_H \leq \text{BL}(H, \mathbf{q}) \prod_{j=1}^m \|f_j\|_{L^{q_j}(\mathbb{R}^{d_j})}, \quad (3.5)$$

for all integrable $f_j: \mathbb{R}^{d_j} \rightarrow \mathbb{R}$.

Although we will only be concerned with geometric Brascamp–Lieb inequalities, any Brascamp–Lieb inequality can be written in the form of (3.5), which is why we refer to $\text{BL}(H, \mathbf{q})$ as a Brascamp–Lieb constant. In general, there is a direct relationship between the two constants $\text{BL}(H, \mathbf{q})$ and $\text{BL}(H^\perp, \mathbf{q}')$, where $\mathbf{q}'$ denotes the tuple of dual exponents $(q'_1, \dots, q'_m)$. In order to state this formally, for any $r \in [1, \infty]$ we define the number A_r as follows

$$A_r := \left(\frac{r^{\frac{1}{r}}}{r'^{\frac{1}{r'}}} \right)^{\frac{1}{2}}.$$

Now for any m -tuple of exponents $\mathbf{q} = (q_1, \dots, q_m) \in [1, \infty]^m$ set

$$B_{\mathbf{q}} = \prod_{j=1}^m A_{q_j}.$$

Using this notation, we have the following duality principle for this more general form of Brascamp–Lieb inequality:

Proposition 3.7 ([9]). *For any subspace $H \subset \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}$ and exponents $q_1, \dots, q_m \in [1, \infty]$ we have*

$$\text{BL}(H, \mathbf{q}) = B_{\mathbf{q}} \text{BL}(H^\perp, \mathbf{q}').$$

The proof given in [9] relies on Lieb’s Theorem along with an application of a generalised Parseval identity. Below we discuss a special case of Proposition 3.7 related to the dual pairing of geometric Brascamp–Lieb data suggested by Lemma 3.2. Given geometric Brascamp–Lieb datum $(\mathbf{u}, \mathbf{p})$ with each $p_j \in (0, 1)$, the relationship between $\text{BL}(H, \mathbf{q})$ and $\text{BL}(H^\perp, \mathbf{q}')$ can be seen more explicitly in the case where $q_j := p_j^{-1}$ and

$$H := \left\{ \left(p_1^{\frac{1}{2}} \langle x, u_1 \rangle, \dots, p_m^{\frac{1}{2}} \langle x, u_m \rangle \right) \in \mathbb{R}^m \mid x \in \mathbb{R}^d \right\}.$$

In Section 3.1, we provide a new, more geometric proof of Proposition 3.7 for such H , using only the sharpness of Theorem 3.1 along with some tools from elementary geometry. Moreover, we discuss how the structural duality suggested by Lemma 3.2 is actually a manifestation of the more general duality given by Proposition 3.7.

3.1 Duality for general geometric Brascamp–Lieb inequalities

In this section we discuss the connection between the two forms of duality mentioned above, but for a more general version of the geometric Brascamp–Lieb inequality which can be stated as follows

Theorem 3.8 (General Geometric Brascamp–Lieb - [7]). *For $1 \leq j \leq m$, if we have exponents $p_j > 0$ and linear transformations $P_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d_j}$ such that*

$$\sum_{j=1}^m p_j P_j^* P_j = I_d, \quad (3.6)$$

and $P_j P_j^* = I_{d_j}$ for $1 \leq j \leq m$. Then the following inequality holds

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j(P_j x)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} g_j \right)^{p_j},$$

and for each $t > 0$, the standard Gaussian functions $g_j(y) = e^{-\pi t \|y\|^2}$ are extremals.

Firstly, we note that it suffices to consider the special case where each $P_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d_j}$ is actually an orthogonal projection. The following notation will be used throughout this chapter:

Definition 3.9. For any subspace $H \subset \mathbb{R}^N$, we define $\pi_H: \mathbb{R}^N \rightarrow H$ to be orthogonal projection onto H . Moreover, $\pi_H^*: H \rightarrow \mathbb{R}^N$ will always denote the adjoint.

Theorem 3.10 (Geometric Brascamp–Lieb for orthogonal projections - [7]). *For $1 \leq j \leq m$, suppose we have exponents $p_j > 0$ and d_j -dimensional subspaces $U_j \subset \mathbb{R}^d$ such that*

$$\sum_{j=1}^m p_j \pi_{U_j}^* \pi_{U_j} = I_d. \quad (3.7)$$

Then the following inequality holds

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j(\pi_{U_j} x)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} g_j \right)^{p_j},$$

and for each $t > 0$, the standard Gaussian functions $g_j(y) = e^{-\pi t \|y\|^2}$ are extremals.

At the end of this section we show that Theorems 3.8 and 3.10 are completely equivalent using a simple change of variables along with the singular value decomposition of each P_j .

Analogous to the setting of Theorem 3.1, we have the following generalisation of Lemma 3.2 which suggests a certain structural duality between pairs of geometric Brascamp–Lieb data in this more general setup too:

Lemma 3.11. *Suppose we have subspaces $U_1, \dots, U_m \subset \mathbb{R}^d$ of dimensions $d_1, \dots, d_m$ respectively, and exponents $p_1, \dots, p_m > 0$ such that*

$$\sum_{j=1}^m p_j \pi_{U_j}^* \pi_{U_j} = I_d,$$

If we define $N = \sum_{j=1}^m d_j$, then there exist orthogonal subspaces $V_1, \dots, V_m \subset \mathbb{R}^N$ of dimensions $d_1, \dots, d_m$ respectively such that for $1 \leq j \leq m$

$$\pi_{\mathbb{R}^d}(V_j) = U_j$$

and

$$(\pi_{V_j} \pi_{\mathbb{R}^d}^*)^* \pi_{V_j} \pi_{\mathbb{R}^d}^* = p_j \pi_{U_j}^* \pi_{U_j}.$$

If we set $W_j = \pi_{\mathbb{R}^{N-d}}(V_j)$, then for every j with $p_j < 1$, we have $\dim(W_j) = d_j$. Moreover, the following two identities hold

$$(\pi_{V_j} \pi_{\mathbb{R}^{N-d}}^*)^* \pi_{V_j} \pi_{\mathbb{R}^{N-d}}^* = (1 - p_j) \pi_{W_j}^* \pi_{W_j},$$

$$\sum_{j=1}^m (1 - p_j) \pi_{W_j}^* \pi_{W_j} = I_{N-d}.$$

In Section 3.1.2, we show how to deduce this last result as a consequence of Lemma 3.2, and for completeness we provide a full proof of the latter. In light of Lemma 3.11, one might hope to establish a relationship between the following two inequalities

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j (\pi_{U_j} x)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} g_j \right)^{p_j} \quad (3.8)$$

and

$$\int_{\mathbb{R}^{N-d}} \prod_{j=1}^m h_j (\pi_{W_j} y)^{1-p_j} dy \leq \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} h_j \right)^{1-p_j}. \quad (3.9)$$

Note on exponents $p_1, \dots, p_m$. Let us briefly discuss why the case where $0 < p_1, \dots, p_m < 1$ is already the most general setting for the geometric Brascamp–Lieb inequality. As a direct consequence of Lemma 3.11, we can write

$$(\pi_{V_j} \pi_{\mathbb{R}^d}^*)^* \pi_{V_j} \pi_{\mathbb{R}^d}^* = p_j \pi_{U_j}^* \pi_{U_j}.$$

Applying both sides of the above identity to some vector $u \in U_j$ and taking norms, we can immediately deduce that $0 < p_j \leq 1$ for $1 \leq j \leq m$.

Supposing $p_k = 1$ for some $1 \leq k \leq m$, we will show that (3.8) can easily be reduced to

$$\int_{\mathbb{R}^{d-d_k}} \prod_{j \neq k} g_j (\pi_{U_j} x)^{p_j} dx \leq \prod_{j \neq k} \left(\int_{\mathbb{R}^{d_j}} g_j \right)^{p_j}, \quad (3.10)$$

which will be a geometric Brascamp–Lieb inequality in $\mathbb{R}^{d-d_k}$.

Proof of reduction. Firstly assume for a contradiction that $V_k \cap \mathbb{R}^{N-d} \neq \{\mathbf{0}\}$. Defining $W_k = \pi_{\mathbb{R}^{N-d}}(V_k)$, by Lemma 3.11 we have

$$(\pi_{V_k} \pi_{\mathbb{R}^{N-d}}^*)^* \pi_{V_k} \pi_{\mathbb{R}^{N-d}}^* = (1 - p_k) \pi_{W_k}^* \pi_{W_k}.$$

It follows that for any $v \in V_k \cap \mathbb{R}^{N-d}$

$$v = (\pi_{V_k} \pi_{\mathbb{R}^{N-d}}^*)^* \pi_{V_k} \pi_{\mathbb{R}^{N-d}}^*(v) = (1 - p_k) \pi_{W_k}^* \pi_{W_k}(v) = \mathbf{0},$$

which gives a contradiction since $V_k \cap \mathbb{R}^{N-d} \neq \{\mathbf{0}\}$ by assumption. So we have shown that $V_k \subset \mathbb{R}^d$. Again using Lemma 3.11 we know that

$$U_k = \pi_{\mathbb{R}^d}(V_k),$$

so we can write

$$V_k = \pi_{\mathbb{R}^d}^*(U_k).$$

Since $V_j \perp V_k$ for $j \neq k$, it follows immediately that the same is true for the U_j . More precisely, for $j \neq k$ we have $U_j \perp U_k$. Using this we can apply Fubini's theorem as follows

$$\begin{aligned} \int_{\mathbb{R}^d} \prod_{j=1}^m g_j (\pi_{U_j} x)^{p_j} dx &= \int_{U_k} g_k(u) \int_{U_k^\perp} \prod_{j \neq k} g_j(\pi_{U_j}(u+v))^{p_j} dv du \\ &= \left(\int_{U_k} g_k \right) \int_{U_k^\perp} \prod_{j \neq k} g_j(\pi_{U_j}(v))^{p_j} dv \\ &= \left(\int_{\mathbb{R}^{d_k}} g_k \right) \left(\int_{\mathbb{R}^{d-d_k}} \prod_{j \neq k} g_j(\pi_{U_j}(y))^{p_j} dy \right). \end{aligned}$$

Furthermore, by assumption we have

$$\sum_{j=1}^m p_j \pi_{U_j}^* \pi_{U_j} = I_d.$$

Combining this with the fact that $U_j \subset U_k^\perp$ for $j \neq k$, we get

$$\sum_{j \neq k} p_j \pi_{U_j}^* \pi_{U_j} = I_d - \pi_{U_k}^* \pi_{U_k} = I_{U^\perp} = I_{d-d_k},$$

so indeed (3.10) is a geometric Brascamp–Lieb inequality in $\mathbb{R}^{d-d_k}$. $\square$

In light of the above, we may assume without loss of generality that $0 < p_1, \dots, p_m < 1$.

3.1.1 Tensor product formulation

As it turns out, the relationship between (3.8) and (3.9) can be understood as a special case of Proposition 3.7. To see why, let us begin by rewriting the above two inequalities in the form of (3.5).

Replacing $g_j^{p_j}$ with g_j and $h_j^{1-p_j}$ with h_j , and setting $q_j = \frac{1}{p_j}$ the two inequalities (3.8) and (3.9) become

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j (\pi_{U_j} x) dx \leq \prod_{j=1}^m \|g_j\|_{L^{q_j}(\mathbb{R}^{d_j})} \quad (3.11)$$

and

$$\int_{\mathbb{R}^{m-d}} \prod_{j=1}^m h_j (\pi_{W_j} y) dy \leq \prod_{j=1}^m \|h_j\|_{L^{q'_j}(\mathbb{R}^{d_j})}. \quad (3.12)$$

Note that for each j we can make simple changes of variables to get

$$\|g_j\|_{L^{q_j}(\mathbb{R}^{d_j})} = \left(q_j^{\frac{1}{2}}\right)^{\frac{d_j}{q_j}} \left\|g_j\left(q_j^{\frac{1}{2}} \cdot\right)\right\|_{L^{q_j}(\mathbb{R}^{d_j})}$$

and

$$\|h_j\|_{L^{q'_j}(\mathbb{R}^{d_j})} = \left(q'_j^{\frac{1}{2}}\right)^{\frac{d_j}{q'_j}} \|h_j\left(q'_j^{\frac{1}{2}}\cdot\right)\|_{L^{q'_j}(\mathbb{R}^{d_j})}.$$

So replacing each g_j with $g_j\left(q_j^{-\frac{1}{2}}\cdot\right)$ and h_j with $h_j\left(q'_j^{-\frac{1}{2}}\cdot\right)$, we can rewrite (3.11) and (3.12) as follows

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j\left(q_j^{-\frac{1}{2}}\pi_{U_j}x\right) dx \leq \prod_{j=1}^m \left(q_j^{\frac{1}{2}}\right)^{\frac{d_j}{q_j}} \prod_{j=1}^m \|g_j\|_{L^{q_j}(\mathbb{R}^{d_j})} \quad (3.13)$$

and

$$\int_{\mathbb{R}^{m-d}} \prod_{j=1}^m h_j\left(q'_j^{-\frac{1}{2}}\pi_{W_j}y\right) dy \leq \prod_{j=1}^m \left(q'_j^{\frac{1}{2}}\right)^{\frac{d_j}{q'_j}} \prod_{j=1}^m \|h_j\|_{L^{q'_j}(\mathbb{R}^{d_j})}. \quad (3.14)$$

For the last step we need the following small lemma:

Lemma 3.12. *Suppose we have an m -tuple of positive exponents $c_1, \dots, c_m$ and linear transformations $B_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d_j}$, such that*

$$\sum_{i=j}^m c_i B_j^* B_j = I_d, \quad (3.15)$$

and $B_j B_j^* = I_{d_j}$ for $1 \leq j \leq m$. If we define

$$H := \left\{ \left(c_1^{\frac{1}{2}} B_1 x, \dots, c_m^{\frac{1}{2}} B_m x \right) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} \mid x \in \mathbb{R}^d \right\},$$

then

$$\int_{\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}} f_1 \otimes \dots \otimes f_m d\mu_H = \int_{\mathbb{R}^d} \prod_{j=1}^m f_j\left(c_j^{\frac{1}{2}} B_j x\right) dx,$$

where $d\mu_H$ denotes the restriction of the d -dimensional Hausdorff measure to H .

Proof of Lemma 3.12. Firstly notice that for any $x, y \in \mathbb{R}^d$

$$\begin{aligned} \left\langle (c_1^{\frac{1}{2}} B_1(x), \dots, c_m^{\frac{1}{2}} B_m(x)), (c_1^{\frac{1}{2}} B_1(y), \dots, c_m^{\frac{1}{2}} B_m(y)) \right\rangle &= \sum_{j=1}^m \left\langle c_j^{\frac{1}{2}} B_j(x), c_j^{\frac{1}{2}} B_j(y) \right\rangle \\ &= \sum_{j=1}^m \langle c_j B_j^* B_j x, y \rangle \\ &= \left\langle \sum_{j=1}^m c_j B_j^* B_j x, y \right\rangle \\ &= \langle x, y \rangle, \end{aligned}$$

where the last equality is a direct application of (3.15). Let us define $v_i := (c_1^{\frac{1}{2}} B_1(e_i), \dots, c_m^{\frac{1}{2}} B_m(e_i))$ for $i = 1, \dots, d$, it follows that $\{v_1, \dots, v_d\} \subset \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}$ is an orthonormal set of vectors spanning H . Let us fix coordinates $y = (y_1, \dots, y_d)$ on H such that

$$\pi_H^*(y) = \sum_{j=1}^m y_j v_j.$$

Then we can write

$$\int_{\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}} \bigotimes_{j=1}^m f_j d\mu_H = \int_{\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}} \prod_{j=1}^m f_j (\pi_{\mathbb{R}^{d_j}}(x)) d\mu_H(x) = \int_{\mathbb{R}^d} \prod_{j=1}^m f_j (\pi_{\mathbb{R}^{d_j}} \circ \pi_H^*(y)) dy.$$

Observe that for any $y \in \mathbb{R}^d$

$$\pi_{\mathbb{R}^{d_j}} \circ \pi_H^*(y) = \pi_{\mathbb{R}^{d_j}} \left(\sum_{i=1}^d y_i v_i \right) = \sum_{i=1}^d y_i \pi_{\mathbb{R}^{d_j}}(v_i) = \sum_{i=1}^d y_i c_j^{\frac{1}{2}} B_j(e_i) = c_j^{\frac{1}{2}} B_j \left(\sum_{i=1}^d y_i e_i \right) = c_j^{\frac{1}{2}} B_j(y).$$

So in fact we can write

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j (\pi_{\mathbb{R}^{d_j}} \circ \pi_H^*(y)) dy = \int_{\mathbb{R}^d} \prod_{j=1}^m f_j (c_j^{\frac{1}{2}} B_j(y)) dy,$$

which completes the proof. $\square$

Let us define the following two particular subspaces of $\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}$:

$$U := \left\{ \left(q_1^{-\frac{1}{2}} \pi_{U_1} x, \dots, q_m^{-\frac{1}{2}} \pi_{U_m} x \right) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} \mid x \in \mathbb{R}^d \right\};$$

$$W := \left\{ \left(q'_1{}^{-\frac{1}{2}} \pi_{W_1} y, \dots, q'_m{}^{-\frac{1}{2}} \pi_{W_m} y \right) \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} \mid y \in \mathbb{R}^{N-d} \right\}.$$

Note that we are implicitly fixing coordinates on each of the subspaces U_j and W_j . The precise choice of bases are given below, and then used in the proof of Lemma 3.13. By direct applications of Lemma 3.12 we can now write (3.13) and (3.14) as

$$\int_{\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}} g_1 \otimes \dots \otimes g_m d\mu_U \leq \prod_{j=1}^m \left(q_j^{\frac{1}{2}} \right)^{\frac{d_j}{q_j}} \prod_{j=1}^m \|g_j\|_{L^{q_j}(\mathbb{R}^{d_j})}, \quad (3.16)$$

and

$$\int_{\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}} h_1 \otimes \dots \otimes h_m d\mu_W \leq \prod_{j=1}^m \left(q'_j{}^{\frac{1}{2}} \right)^{\frac{d_j}{q'_j}} \prod_{j=1}^m \|h_j\|_{L^{q'_j}(\mathbb{R}^{d_j})}, \quad (3.17)$$

with $d\mu_U$ and $d\mu_W$ denoting the restrictions of the d -dimensional Hausdorff measure to U and W respectively. At this point, we note that the constants $\prod_{j=1}^m \left(q_j^{\frac{1}{2}} \right)^{\frac{d_j}{q_j}}, \prod_{j=1}^m \left(q'_j{}^{\frac{1}{2}} \right)^{\frac{d_j}{q'_j}}$ appearing in (3.16), (3.17) respectively, are sharp by the reversibility of the above derivation combined with the fact that we began with sharp geometric Brascamp–Lieb inequalities (3.8) and (3.9). In other words, we have shown that

$$\text{BL}(U, \mathbf{q}) = \prod_{j=1}^m \left(q_j^{\frac{1}{2}} \right)^{\frac{d_j}{q_j}},$$

and

$$\text{BL}(W, \mathbf{q}') = \prod_{j=1}^m \left(q'_j{}^{\frac{1}{2}} \right)^{\frac{d_j}{q'_j}}.$$

With notation of Proposition 3.7, it follows immediately that

$$\text{BL}(U, \mathbf{q}) = B_{\mathbf{q}} \text{BL}(W, \mathbf{q}').$$

In fact, we can see this as a special case of Proposition 3.7 given the following result

Lemma 3.13. $W = U^\perp$.

Fixing coordinates of U_j and W_j Before proving this last result, we need to make a small note about fixing coordinates on each of the subspaces U_j and W_j , the choices of which slightly alter the definition of U and W . Essentially, we are free to use any basis for U_j , provided we choose the correct "corresponding" basis for W_j . In fact, this correspondence comes from the fact that any choice of basis $v_{j,1}, \dots, v_{j,d_j}$ for V_j induces natural choices of basis $u_{j,1}, \dots, u_{j,d_j}$ and $w_{j,1}, \dots, w_{j,d_j}$ on U_j and W_j respectively, which we define below. To begin, note that by definition of orthogonal projection, for all $X \in \mathbb{R}^N$ we have

$$\pi_{V_j} X = \sum_{i=1}^{d_j} \langle X, \pi_{V_j}^* v_{j,i} \rangle v_{j,i}. \quad (3.18)$$

Setting $u_{j,i} := p_j^{-\frac{1}{2}} \pi_{U_j} \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i}$, let us show that $u_{j,1}, \dots, u_{j,d_j}$ form a basis for U_j . Applying $\pi_{\mathbb{R}^d} \pi_{V_j}^*$ to both sides of (3.18) and setting $X = \pi_{\mathbb{R}^d}^* y$, we can deduce that for any $y \in \mathbb{R}^d$

$$(\pi_{V_j} \pi_{\mathbb{R}^d}^*)^* \pi_{V_j} \pi_{\mathbb{R}^d}^* (y) = \sum_{i=1}^{d_j} \langle y, \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i} \rangle \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i}.$$

Now by Lemma 3.11 we know that

$$(\pi_{V_j} \pi_{\mathbb{R}^d}^*)^* \pi_{V_j} \pi_{\mathbb{R}^d}^* = p_j \pi_{U_j} \pi_{U_j}^*.$$

Immediately, we can deduce that for any $y \in \mathbb{R}^d$

$$\pi_{U_j}^* \pi_{U_j} (y) = \sum_{i=1}^{d_j} \langle y, p_j^{-\frac{1}{2}} \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i} \rangle p_j^{-\frac{1}{2}} \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i}.$$

Applying π_{U_j} to both sides, this becomes

$$\pi_{U_j} (y) = \sum_{i=1}^{d_j} \langle y, p_j^{-\frac{1}{2}} \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i} \rangle p_j^{-\frac{1}{2}} \pi_{U_j} \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i},$$

and noting that $\pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i} \in \pi_{\mathbb{R}^d}(V_j) = U_j$, we can then write

$$\begin{aligned} \sum_{i=1}^{d_j} \langle y, p_j^{-\frac{1}{2}} \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i} \rangle p_j^{-\frac{1}{2}} \pi_{U_j} \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i} &= \sum_{i=1}^{d_j} \langle y, p_j^{-\frac{1}{2}} \pi_{U_j}^* \pi_{U_j} \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i} \rangle p_j^{-\frac{1}{2}} \pi_{U_j} \pi_{\mathbb{R}^d} \pi_{V_j}^* v_{j,i} \\ &= \sum_{i=1}^{d_j} \langle y, \pi_{U_j}^* u_{j,i} \rangle u_{j,i}. \end{aligned}$$

So for all $y \in \mathbb{R}^d$, we have shown that

$$\pi_{U_j} (y) = \sum_{i=1}^{d_j} \langle y, \pi_{U_j}^* u_{j,i} \rangle u_{j,i},$$

which implies that vectors $u_{j,1}, \dots, u_{j,d_j}$ form a basis for U_j . Similarly, assuming without loss of generality that $0 < p_1, \dots, p_m < 1$, one can show that for all $x \in \mathbb{R}^{N-d}$

$$\pi_{W_j} (x) = \sum_{i=1}^{d_j} \langle x, (1 - p_j)^{-\frac{1}{2}} \pi_{W_j}^* \pi_{W_j} \pi_{\mathbb{R}^{N-d}} \pi_{V_j}^* v_{j,i} \rangle (1 - p_j)^{-\frac{1}{2}} \pi_{W_j} \pi_{\mathbb{R}^{N-d}} \pi_{V_j}^* v_{j,i}.$$

Setting $w_{j,i} := (1 - p_j)^{-\frac{1}{2}} \pi_{W_j} \pi_{\mathbb{R}^{N-d}} \pi_{V_j}^* v_{j,i}$, it follows that for all $x \in \mathbb{R}^{N-d}$

$$\pi_{W_j}(x) = \sum_{i=1}^{d_j} \left\langle y, \pi_{W_j}^* w_{j,i} \right\rangle w_{j,i},$$

hence the vectors $w_{j,1}, \dots, w_{j,d_j}$ form a basis for W_j .

In order to prove Lemma 3.13, the choice of bases for each pair U_j, W_j must be induced from some choice of basis on V_j in the manner described above.

Proof of Lemma 3.13. Observe that for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^{N-d} \cong \mathbb{R}^N$ we have

$$\begin{aligned} & \left\langle \left(p_1^{\frac{1}{2}} \pi_{U_1} x, \dots, p_m^{\frac{1}{2}} \pi_{U_m} x \right), \left((1 - p_1)^{\frac{1}{2}} \pi_{W_1} y, \dots, (1 - p_m)^{\frac{1}{2}} \pi_{W_m} y \right) \right\rangle \\ &= \sum_{j=1}^m \left\langle p_j^{\frac{1}{2}} \pi_{U_j} x, (1 - p_j)^{\frac{1}{2}} \pi_{W_j} y \right\rangle \\ &= \sum_{j=1}^m \left\langle \pi_{V_j} \pi_{\mathbb{R}^d}^* x, \pi_{V_j} \pi_{\mathbb{R}^{N-d}}^* y \right\rangle \\ &= \sum_{j=1}^m \left\langle \sum_{i=1}^{d_j} \left\langle \pi_{\mathbb{R}^d}^* x, v_{j,i} \right\rangle v_{j,i}, \sum_{i=1}^{d_j} \left\langle \pi_{\mathbb{R}^{N-d}}^* y, v_{j,i} \right\rangle v_{j,i} \right\rangle \\ &= \sum_{j=1}^m \sum_{i=1}^{d_j} \left\langle \pi_{\mathbb{R}^d}^* x, v_{j,i} \right\rangle \left\langle \pi_{\mathbb{R}^{N-d}}^* y, v_{j,i} \right\rangle \\ &= \left\langle \pi_{\mathbb{R}^d}^* x, \pi_{\mathbb{R}^{N-d}}^* y \right\rangle \\ &= 0, \end{aligned}$$

where in the second to last step we used the fact that $v_{1,1}, \dots, v_{1,d_1}, \dots, v_{m,1}, \dots, v_{m,d_m}$ form an orthonormal basis of $\mathbb{R}^N$. Since we have $\dim(W) = N - d = N - \dim(U)$, It follows that $W = U^\perp$. $\square$

3.1.2 Proof of Lemma 3.11

As mentioned earlier, Lemma 3.11 can be proven using the simple characterisation of geometric Brascamp–Lieb datum given by Lemma 3.2. Before proving Lemma 3.2 directly, let us first show how to deduce Lemma 3.11.

Proof of Lemma 3.11. Let $u_{j,1}, \dots, u_{j,d_j} \in U_j$ be an orthonormal basis. Now for any $x \in \mathbb{R}^d$, we can write

$$\pi_{U_j}(x) = \sum_{i=1}^{d_j} \left\langle x, \pi_{U_j}^* u_{j,i} \right\rangle u_{j,i}.$$

It follows that

$$\pi_{U_j}^* \pi_{U_j}(x) = \sum_{i=1}^{d_j} \left\langle x, \pi_{U_j}^* u_{j,i} \right\rangle \pi_{U_j}^* u_{j,i} = \sum_{i=1}^{d_j} \left(\pi_{U_j}^* u_{j,i} \right) \otimes \left(\pi_{U_j}^* u_{j,i} \right) (x).$$

Thus,

$$\sum_{j=1}^m \sum_{i=1}^{d_j} p_j \left(\pi_{U_j}^* u_{j,i} \right) \otimes \left(\pi_{U_j}^* u_{j,i} \right) = \sum_{j=1}^m p_j \pi_{U_j}^* \pi_{U_j} = I_d,$$

so by Lemma 3.2 there exists an orthonormal basis $v_{1,1}, \dots, v_{1,d_1}, \dots, v_{m,1}, \dots, v_{m,d_m} \in \mathbb{R}^N$ such that

$$\pi_{\mathbb{R}^d} v_{j,i} = p_j^{\frac{1}{2}} \pi_{U_j}^* u_{j,i}.$$

For $1 \leq j \leq m$, define $V_j \subset \mathbb{R}^N$ to be the subspace spanned by $v_{j,1}, \dots, v_{j,d_j}$, so that clearly $V_j \perp V_k$ whenever $j \neq k$. Moreover, $U_j = \pi_{\mathbb{R}^d}(V_j)$ by definition. To verify that

$$\left(\pi_{V_j} \pi_{\mathbb{R}^d}^* \right)^* \pi_{V_j} \pi_{\mathbb{R}^d}^* = p_j \pi_{U_j}^* \pi_{U_j},$$

taking $x \in \mathbb{R}^d$ we simply write

$$\begin{aligned} \left(\pi_{V_j} \pi_{\mathbb{R}^d}^* \right)^* \pi_{V_j} \pi_{\mathbb{R}^d}^* (x) &= \pi_{\mathbb{R}^d} \left(\pi_{V_j}^* \pi_{V_j} \left(\pi_{\mathbb{R}^d}^* x \right) \right) = \pi_{\mathbb{R}^d} \left(\sum_{i=1}^{d_j} \langle \pi_{\mathbb{R}^d}^* x, v_{j,i} \rangle v_{j,i} \right) \\ &= \sum_{i=1}^{d_j} \langle x, \pi_{\mathbb{R}^d} v_{j,i} \rangle \pi_{\mathbb{R}^d} v_{j,i} \\ &= \sum_{i=1}^{d_j} \left\langle x, p_j^{\frac{1}{2}} \pi_{U_j}^* u_{j,i} \right\rangle p_j^{\frac{1}{2}} \pi_{U_j}^* u_{j,i} \\ &= p_j \pi_{U_j}^* \left(\sum_{i=1}^{d_j} \langle x, \pi_{U_j}^* u_{j,i} \rangle u_{j,i} \right) \\ &= p_j \pi_{U_j}^* \pi_{U_j} (x). \end{aligned}$$

Furthermore, setting

$$w_{j,i} = \begin{cases} \frac{\pi_{\mathbb{R}^{N-d}} v_{j,i}}{\|\pi_{\mathbb{R}^{N-d}} v_{j,i}\|} & \text{if } p_j \neq 1 \\ \mathbf{0} & \text{if } p_j = 1, \end{cases}$$

by Lemma 3.2 we have

$$\pi_{\mathbb{R}^{N-d}} v_{j,i} = (1 - p_j)^{\frac{1}{2}} w_{j,i},$$

and

$$\sum_{j=1}^m \sum_{i=1}^{d_j} (1 - p_j) w_{j,i} \otimes w_{j,i} = I_{N-d}.$$

Now defining W_j to be the subspace spanned by $w_{j,1}, \dots, w_{j,d_j}$, we clearly have $W_j = \pi_{\mathbb{R}^{N-d}}(V_j)$. Note that

$$\begin{aligned} &\langle v_{j,i}, v_{j,k} \rangle \\ &= \langle \pi_{\mathbb{R}^d}^* \pi_{\mathbb{R}^d} v_{j,i} + \pi_{\mathbb{R}^{N-d}}^* \pi_{\mathbb{R}^{N-d}} v_{j,i}, \pi_{\mathbb{R}^d}^* \pi_{\mathbb{R}^d} v_{j,k} + \pi_{\mathbb{R}^{N-d}}^* \pi_{\mathbb{R}^{N-d}} v_{j,k} \rangle \\ &= \left\langle \pi_{\mathbb{R}^d}^* \left(p_j^{\frac{1}{2}} \pi_{U_j}^* u_{j,i} \right) + \pi_{\mathbb{R}^{N-d}}^* \left((1 - p_j)^{\frac{1}{2}} w_{j,i} \right), \pi_{\mathbb{R}^d}^* \left(p_j^{\frac{1}{2}} \pi_{U_j}^* u_{j,k} \right) + \pi_{\mathbb{R}^{N-d}}^* \left((1 - p_j)^{\frac{1}{2}} w_{j,k} \right) \right\rangle \\ &= \left\langle \pi_{\mathbb{R}^d}^* \left(p_j^{\frac{1}{2}} \pi_{U_j}^* u_{j,i} \right), \pi_{\mathbb{R}^d}^* \left(p_j^{\frac{1}{2}} \pi_{U_j}^* u_{j,k} \right) \right\rangle + \left\langle \pi_{\mathbb{R}^{N-d}}^* \left((1 - p_j)^{\frac{1}{2}} w_{j,i} \right), \pi_{\mathbb{R}^{N-d}}^* \left((1 - p_j)^{\frac{1}{2}} w_{j,k} \right) \right\rangle \\ &= p_j \langle u_{j,i}, u_{j,k} \rangle + (1 - p_j) \langle w_{j,i}, w_{j,k} \rangle. \end{aligned}$$

So in fact whenever $p_j < 1$, the vectors $w_{j,1}, \dots, w_{j,d_j}$ form an orthonormal spanning set of W_j , thus $\dim(W_j) = d_j$ and we can write

$$\pi_{W_j}^* \pi_{W_j} = \sum_{i=1}^{d_j} w_{j,i} \otimes w_{j,i}.$$

It follows that

$$\sum_{j=1}^m (1 - p_j) \pi_{W_j}^* \pi_{W_j} = \sum_{j=1}^m \sum_{i=1}^{d_j} (1 - p_j) w_{j,i} \otimes w_{j,i} = I_{N-d}.$$

To conclude the proof we simply note that for any $y \in \mathbb{R}^{N-d}$ we can write

$$\begin{aligned} (\pi_{V_j} \pi_{\mathbb{R}^{N-d}}^*)^* \pi_{V_j} \pi_{\mathbb{R}^{N-d}}(y) &= \pi_{\mathbb{R}^{N-d}} \left(\pi_{V_j}^* \pi_{V_j} (\pi_{\mathbb{R}^{N-d}}^* y) \right) = \pi_{\mathbb{R}^{N-d}} \left(\sum_{i=1}^{d_j} \langle \pi_{\mathbb{R}^{N-d}}^* y, v_{j,i} \rangle v_{j,i} \right) \\ &= \sum_{i=1}^{d_j} \langle y, \pi_{\mathbb{R}^{N-d}} v_{j,i} \rangle \pi_{\mathbb{R}^{N-d}} v_{j,i} \\ &= \sum_{i=1}^{d_j} \left\langle y, (1 - p_j)^{\frac{1}{2}} w_{j,i} \right\rangle (1 - p_j)^{\frac{1}{2}} w_{j,i} \\ &= (1 - p_j) \sum_{i=1}^{d_j} \langle y, w_{j,i} \rangle w_{j,i} \\ &= (1 - p_j) \pi_{W_j}^* \pi_{W_j}(y). \end{aligned}$$

□

In order to prove Lemma 3.2 we use the following preliminary result

Lemma 3.14. *Suppose $m > n$ and we have row vectors $v_1, \dots, v_m \in \mathbb{R}^n$ such that*

$$\sum_{j=1}^m v_j^* v_j = I_n,$$

with $v_j^* v_j$ denoting the $n \times n$ matrix equal to the product of the column vector v_j^* and the row vector v_j . Then there exist $\tilde{v}_1, \dots, \tilde{v}_m \in \mathbb{R}^{n+1}$ with $\pi_{\mathbb{R}^n} \tilde{v}_j = v_j$ for $1 \leq j \leq m$ and

$$\sum_{j=1}^m \tilde{v}_j^* \tilde{v}_j = I_{n+1}$$

Proof. Suppose $u \in \mathbb{R}^n$ and $a \in \mathbb{R}$ so that $\tilde{u} = (u, a) \in \mathbb{R}^{n+1}$, then we have

$$u^* u = \begin{pmatrix} u_1 u_1 & \cdots & u_1 u_n \\ \vdots & & \vdots \\ u_n u_1 & \cdots & u_n u_n \end{pmatrix},$$

and

$$\tilde{u}^* \tilde{u} = \begin{pmatrix} u_1 u_1 & \cdots & u_1 u_n & u_1 a \\ \vdots & & \vdots & \vdots \\ u_n u_1 & \cdots & u_n u_n & u_n a \\ a u_1 & \cdots & a u_n & a^2 \end{pmatrix}.$$

Similarly for $a_1, \dots, a_m$ let $\tilde{v}_j = (v_j, a_j)$, It follows that

$$\sum_{j=1}^m \tilde{v}_j^* \tilde{v}_j = \sum_{j=1}^m \begin{pmatrix} v_{j,1}v_{j,1} & \cdots & v_{j,1}v_{j,n} & v_{j,1}a_j \\ \vdots & & \vdots & \vdots \\ v_{j,n}v_{j,1} & \cdots & v_{j,n}v_{j,n} & v_{j,n}a_j \\ a_jv_{j,1} & \cdots & a_jv_{j,n} & a_j^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & \sum_{j=1}^m v_{j,1}a_j \\ 0 & 1 & & & \vdots & \vdots \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & 1 & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \sum_{j=1}^m v_{j,n}a_j \\ \sum_{j=1}^m a_jv_{j,1} & \cdots & \cdots & \cdots & \sum_{j=1}^m a_jv_{j,n} & \sum_{j=1}^m a_j^2 \end{pmatrix}.$$

So it suffices to find $a_1, \dots, a_m \in \mathbb{R}$ such that for $1 \leq i \leq n$

$$\sum_{j=1}^n v_{j,i}a_j = 0$$

and

$$\sum_{j=1}^m a_j^2 = 1. \quad (3.19)$$

The first set of equations is a linear system in $a_1, \dots, a_m$ and since the v_j span $\mathbb{R}^n$ the space of solutions has dimension $m - n \geq 1$. This can be seen as a subspace $H \subset \mathbb{R}^m$ of dimension ≥ 1 . Note that (3.19) is simply the equation for the n -sphere in $\mathbb{R}^m$, which will necessarily intersect H at some point. This concludes the proof. $\square$

Now we are ready to prove Lemma 3.2.

Proof of Lemma 3.2. Apply Lemma 3.14 iteratively, starting with vectors $p_1^{\frac{1}{2}}u_1, \dots, p_m^{\frac{1}{2}}u_m \in \mathbb{R}^d$, to get $v_1, \dots, v_m \in \mathbb{R}^m$ such that

$$\sum_{j=1}^m v_j \otimes v_j = \sum_{j=1}^m v_j^* v_j = I_m. \quad (3.20)$$

We abuse notation slightly allowing $\sum_{j=1}^m v_j^* v_j$ to both represent an $m \times m$ matrix and the corresponding linear map from $\mathbb{R}^m$ to itself. By construction we have

$$\pi_{\mathbb{R}^d} v_j = \pi_{\mathbb{R}^d} \circ \cdots \circ \pi_{\mathbb{R}^{m-2}} \circ \pi_{\mathbb{R}^{m-1}} v_j = p_j^{\frac{1}{2}} u_j.$$

Note that (3.20) can only hold if $v_1, \dots, v_m$ form an orthonormal basis of $\mathbb{R}^m$. By Pythagoras theorem we have

$$\|\pi_{\mathbb{R}^{m-d}} v_j\|^2 = \|v_j\|^2 - \|\pi_{\mathbb{R}^d} v_j\|^2 = 1 - p_j.$$

For each j let us define

$$w_j = \begin{cases} \frac{\pi_{\mathbb{R}^{m-d}} v_j}{\|\pi_{\mathbb{R}^{m-d}} v_j\|} & \text{if } \|\pi_{\mathbb{R}^{m-d}} v_j\| \neq 0 \\ \mathbf{0} & \text{if } \|\pi_{\mathbb{R}^{m-d}} v_j\| = 0, \end{cases}$$

It follows immediately that $\pi_{\mathbb{R}^{m-d}} v_j = (1 - p_j)^{\frac{1}{2}} w_j$ for each j and

$$\begin{aligned}
\sum_{j=1}^m (1 - p_j) w_j \otimes w_j &= \sum_{j=1}^m \left((1 - p_j)^{\frac{1}{2}} w_j \right) \otimes \left((1 - p_j)^{\frac{1}{2}} w_j \right) \\
&= \sum_{j=1}^m (\pi_{\mathbb{R}^{m-d}} v_j) \otimes (\pi_{\mathbb{R}^{m-d}} v_j) \\
&= \pi_{\mathbb{R}^{m-d}} \circ \left(\sum_{j=1}^m v_j \otimes v_j \right) \circ \pi_{\mathbb{R}^{m-d}}^* \\
&= \pi_{\mathbb{R}^{m-d}} \circ I_m \circ \pi_{\mathbb{R}^{m-d}}^* \\
&= I_{m-d}.
\end{aligned}$$

□

3.1.3 Equivalence of Theorems 3.10 and 3.8

For completeness we now give some justification for our initial reduction to Theorem 3.10. Firstly, let us recall the following well known fact about real matrices

Theorem 3.15 (Singular Value Decomposition for real matrices). *Let M an $n \times m$ matrix with real entries, then we can write*

$$M = A \Sigma B^*$$

where

- A is an $n \times n$ orthogonal matrix,
- Σ is an $n \times m$ diagonal matrix with non-negative real numbers on the diagonal,
- B is an $m \times m$ orthogonal matrix, and B^* denotes the transpose.

Using this we can prove the following lemma

Lemma 3.16. *Theorem 3.8 and Theorem 3.10 are equivalent.*

Proof of Lemma 3.16. The forward direction of this equivalence is clear since any orthogonal projection $\pi_U: \mathbb{R}^d \rightarrow \mathbb{R}^k$ satisfies $\pi_U \pi_U^* = I_k$.

Suppose we are given exponents $p_j > 0$ and linear transformations $P_j: \mathbb{R}^d \rightarrow \mathbb{R}^{d_j}$ for $1 \leq j \leq m$ such that

$$\sum_{i=1}^m p_i P_i^* P_i = I_d, \quad (3.21)$$

and for $1 \leq j \leq m$

$$P_j P_j^* = I_{d_j}. \quad (3.22)$$

Apply Theorem 3.15 to the matrices associated to each linear transform P_j to get orthogonal $d \times d$ matrices $B_1, \dots, B_m$, and for $1 \leq j \leq m$ a diagonal $d_j \times d$ matrix Σ_j and an orthogonal $d_j \times d_j$ matrix A_j , such that

$$P_j = A_j \Sigma_j B_j^*.$$

Here we have made a slight abuse of notation; P_j denotes a linear transform while A_j , Σ_j and B_j denote matrices associated to linear transformations. Observe that by (3.22) we have

$$\begin{aligned} I_{d_j} &= P_j P_j^* = A_j \Sigma_j B_j^* B_j \Sigma_j^* A_j^* \\ &= A_j \Sigma_j \Sigma_j^* A_j^*. \end{aligned} \quad (3.23)$$

By definition we have $A_j^* A_j = I_{d_j}$, so using (3.23) we can write

$$I_{d_j} = A_j^* A_j = A_j^* I_{d_j} A_j = A_j^* A_j \Sigma_j \Sigma_j^* A_j^* A_j = \Sigma_j \Sigma_j^*.$$

Note that if Σ_j has diagonal entries $\lambda_{j,1}, \dots, \lambda_{j,d_j} \geq 0$ then $\Sigma_j \Sigma_j^*$ is a diagonal $d_j \times d_j$ matrix with diagonal entries $\lambda_{j,1}^2, \dots, \lambda_{j,d_j}^2$. It follows that $\lambda_{j,1} = \dots = \lambda_{j,d_j} = 1$, thus Σ_j is associated to the orthogonal projection from $\mathbb{R}^d$ onto the first d_j -coordinates. Supposing $\mathbb{R}^{d_j} \subset \mathbb{R}^d$ denotes the subspace spanned by the first d_j coordinate vectors, then defining $U_j = B_j(\mathbb{R}^{d_j}) \subset \mathbb{R}^d$ we have

$$\Sigma_j B_j^* = \pi_{\mathbb{R}^{d_j}} \circ B_j^* = \pi_{U_j}.$$

Here we have used B_j^* to denote both a matrix and the associated linear transformation. Also, using (3.21) directly we have

$$\begin{aligned} I_d &= \sum_{j=1}^m p_j P_j^* P_j = \sum_{j=1}^m p_j B_j \Sigma_j^* A_j^* A_j \Sigma_j B_j^* \\ &= \sum_{j=1}^m p_j B_j \Sigma_j^* I_{d_j} \Sigma_j B_j^* \\ &= \sum_{j=1}^m p_j (\Sigma_j B_j^*)^* \Sigma_j B_j^* \\ &= \sum_{j=1}^m p_j \pi_{U_j}^* \pi_{U_j}, \end{aligned}$$

Now we can apply Theorem 3.10 to deduce that we have the following sharp inequality

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j (\pi_{U_j} x)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} g_j \right)^{p_j}.$$

In each of the $\mathbb{R}^{d_j}$'s on the right hand side, let us make an orthogonal change of variables using linear transformations associated to the matrix A_j from above, so we can rewrite this last inequality as

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j (\pi_{U_j} x)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} g_j (A_j^{-1} z_j) dz_j \right)^{p_j}.$$

Replacing each $g_j(\cdot)$ with $g_j(A_j^{-1} \cdot)$, this becomes

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j (A_j \circ \pi_{U_j} x)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} g_j \right)^{p_j},$$

where A_j denotes both the matrix and the associated linear transformation. By construction, we already know that for each j

$$A_j \circ \pi_{U_j} = A_j \Sigma_j B_j^* = P_j,$$

hence we can rewrite this last inequality as

$$\int_{\mathbb{R}^d} \prod_{j=1}^m g_j(P_j x)^{p_j} dx \leq \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} g_j \right)^{p_j}.$$

This inequality is sharp due to the reversibility of these last few steps. $\square$

3.2 Young's convolution inequality

In this section we see how the structural duality discussed in the previous section appears explicitly for the Young convolution inequality on $\mathbb{R}$, which incidentally can be seen as a special case of the geometric Brascamp–Lieb inequality. For every $s \in [0, 1]$ let us define

$$C_s := \left(\frac{(1-s)^{(1-s)}}{s^s} \right)^{\frac{1}{2}}.$$

Using this notation, we can state Beckner's sharp version of the Young convolution inequality on $\mathbb{R}$ as follows

Theorem 3.17 (Sharp Young convolution inequality on $\mathbb{R}$ - [8]). *Suppose we have exponents $p_1, p_2, p_3 \in [0, 1]$ such that*

$$p_1 + p_2 + p_3 = 2.$$

Then the following inequality holds

$$\int_{\mathbb{R}^2} f_1(x)^{p_1} f_2(x-y)^{p_2} f_3(y)^{p_3} dx dy \leq C_{p_1} C_{p_2} C_{p_3} \left(\int_{\mathbb{R}} f_1 \right)^{p_1} \left(\int_{\mathbb{R}} f_2 \right)^{p_2} \left(\int_{\mathbb{R}} f_3 \right)^{p_3}, \quad (3.24)$$

and for each $t > 0$, the standard Gaussian functions $g_j(y) = e^{-\pi t |y|^2}$ are extremals.

Let us derive Theorem 3.17 as a special case of Theorem 3.1. Firstly, note that (3.24) can also be written

$$\int_{\mathbb{R}^2} f_1(\langle X, e_1 \rangle)^{p_1} f_2(\langle X, e_1 - e_2 \rangle)^{p_2} f_3(\langle X, e_2 \rangle)^{p_3} dX \leq C_{p_1} C_{p_2} C_{p_3} \left(\int_{\mathbb{R}} f_1 \right)^{p_1} \left(\int_{\mathbb{R}} f_2 \right)^{p_2} \left(\int_{\mathbb{R}} f_3 \right)^{p_3}, \quad (3.25)$$

where e_1, e_2 denote the standard unit basis vectors of $\mathbb{R}^2$. Suppose we make a linear change of variables $Y = (A^T)^{-1} X$ in the integral on the left hand side, where A is some 2×2 invertible matrix, then we can rewrite (3.25) as

$$\int_{\mathbb{R}^2} f_1(\langle Y, A e_1 \rangle)^{p_1} f_2(\langle Y, A(e_1 - e_2) \rangle)^{p_2} f_3(\langle Y, A e_2 \rangle)^{p_3} dY \leq \frac{\prod_{j=1}^3 C_{p_j}}{|\det(A^T)|} \prod_{j=1}^3 \left(\int_{\mathbb{R}} f_j \right)^{p_j}.$$

Furthermore, we can make linear changes of variables in each of the respective integrals on the right hand side, which will essentially be dilations by $b_1, b_2, b_3 \in \mathbb{R} \setminus \{0\}$, so that the above inequality becomes

$$\int_{\mathbb{R}^2} f_1(\langle Y, Ae_1 \rangle)^{p_1} f_2(\langle Y, A(e_1 - e_2) \rangle)^{p_2} f_3(\langle Y, Ae_2 \rangle)^{p_3} dY \leq \frac{\prod_{j=1}^3 C_{p_j}}{|\det(A^T)| \prod_{j=1}^3 b_j^{p_j}} \prod_{j=1}^3 \left(\int_{\mathbb{R}} f_j(b_j^{-1} t_j) dt_j \right)^{p_j}.$$

Setting $g_j(\cdot) = f_j(b_j^{-1} \cdot)$ this becomes

$$\int_{\mathbb{R}^2} g_1(\langle Y, b_1 Ae_1 \rangle)^{p_1} g_2(\langle Y, b_2 A(e_1 - e_2) \rangle)^{p_2} g_3(\langle Y, b_3 Ae_2 \rangle)^{p_3} dY \leq \frac{\prod_{j=1}^3 C_{p_j}}{|\det(A^T)| \prod_{j=1}^3 b_j^{p_j}} \prod_{j=1}^3 \left(\int_{\mathbb{R}} g_j \right)^{p_j}.$$

For convenience, let us define $u_1 := b_1 Ae_1$, $u_2 := b_2 A(e_1 - e_2)$ and $u_3 := b_3 Ae_2$. With this notation the above inequality can be written

$$\int_{\mathbb{R}^2} g_1(\langle Y, u_1 \rangle)^{p_1} g_2(\langle Y, u_2 \rangle)^{p_2} g_3(\langle Y, u_3 \rangle)^{p_3} dY \leq \frac{\prod_{j=1}^3 C_{p_j}}{|\det(A^T)| \prod_{j=1}^3 b_j^{p_j}} \left(\int_{\mathbb{R}} g_1 \right)^{p_1} \left(\int_{\mathbb{R}} g_2 \right)^{p_2} \left(\int_{\mathbb{R}} g_3 \right)^{p_3}, \quad (3.26)$$

In order to show the sharp Young convolution inequality as a special case of the geometric Brascamp–Lieb inequality, it suffices to choose A, b_1, b_2, b_3 such that each of the u_j has unit length and the following identity holds

$$\sum_{j=1}^3 p_j u_j \otimes u_j = I_2. \quad (3.27)$$

All we have done to arrive at (3.26) from (3.24) is change variables and alter notation. By the sharpness of Theorem 3.17, it follows that (3.26) must also be sharp. So assuming that (3.27) holds, using Theorem 3.1 we can deduce that $\frac{\prod_{j=1}^3 C_{p_j}}{|\det(A^T)| \prod_{j=1}^3 b_j^{p_j}} = 1$.

To begin, we state a small geometric lemma which essentially gives us good candidates for the above defined $u_1, u_2, u_3 \in \mathbb{R}^2$.

Lemma 3.18. *Suppose we have $p_1, p_2, p_3 \in [0, 1]$ such that*

$$p_1 + p_2 + p_3 = 2.$$

Then there exists an orthonormal basis $v_1, v_2, v_3 \in \mathbb{R}^3$ such that for each j

$$\|\pi_{\mathbb{R}^2} v_j\| = p_j^{\frac{1}{2}}.$$

Note that for any orthonormal basis $v_1, v_2, v_3 \in \mathbb{R}^3$, we have

$$\begin{aligned} I_2 &= \pi_{\mathbb{R}^2} \circ I_3 \circ \pi_{\mathbb{R}^2}^* = \pi_{\mathbb{R}^2} \circ \left(\sum_{j=1}^3 v_j \otimes v_j \right) \circ \pi_{\mathbb{R}^2}^* = \sum_{j=1}^3 \pi_{\mathbb{R}^2} \circ v_j \otimes v_j \circ \pi_{\mathbb{R}^2}^* \\ &= \sum_{j=1}^3 \pi_{\mathbb{R}^2} v_j \otimes \pi_{\mathbb{R}^2} v_j. \end{aligned}$$

Furthermore, if any of the p_j 's is equal to zero then necessarily the other two exponents must be equal to one. It is not difficult to see that Theorem 3.17 becomes trivial in this case, so for

our purposes we will always be working with $p_1, p_2, p_3 > 0$. Bearing this in mind, let us define $w_j := \frac{\pi_{\mathbb{R}^2} v_j}{\|\pi_{\mathbb{R}^2} v_j\|}$ for $j = 1, 2, 3$, then along with the p_1, p_2, p_3 from the above lemma, these vectors will satisfy

$$\sum_{j=1}^3 p_j w_j \otimes w_j = I_2. \quad (3.28)$$

Before proving Lemma 3.18, let us show how one can choose A, b_1, b_2, b_3 such that $u_j = w_j$ for each j . Firstly, note that it is impossible for all the w_j 's to be multiples of the same vector, thus we may assume that neither w_1 or w_2 is a multiple of w_3 . Without loss of generality we may also assume that $w_1 = e_1$, so we require that $b_1 A e_1 = e_1$ and $\|b_1 A e_1\| = 1$ which implies that

$$A = \begin{pmatrix} b_1^{-1} & c_1 \\ 0 & c_2 \end{pmatrix}.$$

Note that multiplying any of the v_j 's given by Lemma 3.18 by -1 will not affect the lengths or whether the three vectors form an orthonormal basis. In light of this, we may assume that w_3 lies in the upper half-space of $\mathbb{R}^2$, so for some $c \in [-1, 1]$ we can write

$$w_3 = \begin{pmatrix} c \\ \sqrt{1-c^2} \end{pmatrix}.$$

We require that $w_3 = u_3 = b_3 A e_2$, which implies that

$$A = \begin{pmatrix} b_1^{-1} & b_3^{-1} c \\ 0 & b_3^{-1} \sqrt{1-c^2} \end{pmatrix}.$$

So all we are left to fix are the numbers b_1, b_2, b_3 . The last requirement is that

$$w_2 = u_2 = b_2 A(e_1 + e_2) = \frac{b_2}{b_1} w_1 + \frac{b_2}{b_3} w_3.$$

By assumption w_1 and w_3 span $\mathbb{R}^2$, and that w_2 is not a multiple of w_3 , so for $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$ we can write

$$w_2 = \alpha w_1 + \beta w_3.$$

Fix $b_2 = 1$ and $b_1 = \frac{1}{\alpha}$. If $\beta \neq 0$ we are done by fixing $b_3 = \frac{1}{\beta}$. If $\beta = 0$, then $w_2 = w_1$ and the inequality becomes a trivial consequence of Fubini's theorem.

All that is left for us to prove is Lemma 3.18:

Proof of Lemma 3.18. Suppose we have $p_1, p_2, p_3 \in [0, 1]$ such that

$$p_1 + p_2 + p_3 = 2.$$

Fix some orthonormal basis $v_1, v_2, v_3 \in \mathbb{R}^3$, then it suffices to prove that there exists a two dimensional subspace $H \subset \mathbb{R}^2$ such that for each j

$$\|\pi_H v_j\| = p_j^{\frac{1}{2}}.$$

Let $n \in \mathbb{R}^2$ be some unit vector normal to H . It follows that for each j

$$\|\pi_H v_j\|^2 = 1 - \|\pi_n v_j\|^2.$$

Choose spherical coordinates (r, θ, φ) on $\mathbb{R}^3$ with $\theta \in [0, \pi]$ measuring the angle from the positive v_1 axis and $\varphi \in [0, 2\pi)$ measuring the angle in the plane spanned by v_2 and v_3 from the positive v_2 axis. In the case where θ equals 0 or π we make the convention that $\varphi = 0$. Note that by definition we immediately have

$$\|\pi_n v_j\| = \cos(\theta).$$

Since $p_1 \in [0, 1]$ we can of course choose $\theta \in [0, \pi]$ such that

$$\|\pi_H v_j\|^2 = 1 - \cos(\theta)^2 = p_1.$$

Now let us calculate $\|\pi_n v_2\|$ and $\|\pi_n v_3\|$. Firstly, note that by definition

$$\|\pi_{v_2} n\| = \sin(\theta) \cos(\varphi),$$

and

$$\|\pi_{v_3} n\| = \sin(\theta) \sin(\varphi).$$

For any unit vectors u, v it is not difficult to see that $\|\pi_u v\| = \|\pi_v u\|$, thus we have

$$\|\pi_H v_2\|^2 = 1 - \sin(\theta)^2 \cos(\varphi)^2,$$

and

$$\|\pi_H v_3\|^2 = 1 - \sin(\theta)^2 \sin(\varphi)^2.$$

Since we have $p_1 = 1 - \cos(\theta)^2 = \sin(\theta)^2$, it suffices to show that we can choose $\varphi \in [0, 2\pi]$ such that

$$p_2 = 1 - p_1 \cos(\varphi)^2,$$

and

$$p_3 = 1 - p_1 \sin(\varphi)^2.$$

Since $p_1 + p_2 + p_3 = 2$ and $p_3 \in [0, 1]$ we have

$$1 \geq p_2 = 2 - p_3 - p_1 \geq 1 - p_1.$$

It is now clear that we can choose $\varphi \in [0, 2\pi]$ satisfying the condition for p_2 and the condition on p_3 follows as a consequence since $\sin(\varphi)^2 + \cos(\varphi)^2 = 1$. $\square$

Duality for the Young convolution inequality Now we have shown that sharp Young convolution inequality on $\mathbb{R}$ can be formulated as

$$\int_{\mathbb{R}^2} g_1(\langle Y, u_1 \rangle)^{p_1} g_2(\langle Y, u_2 \rangle)^{p_2} g_3(\langle Y, u_3 \rangle)^{p_3} dY \leq \left(\int_{\mathbb{R}} g_1 \right)^{p_1} \left(\int_{\mathbb{R}} g_2 \right)^{p_2} \left(\int_{\mathbb{R}} g_3 \right)^{p_3},$$

with unit vectors $u_1, u_2, u_3 \in \mathbb{R}^2$ satisfying the following identity

$$\sum_{j=1}^3 p_j u_j \otimes u_j = I_2.$$

Applying Lemma 3.2 we get unit vectors $w_1, w_2, w_3 \in \mathbb{R}$, such that

$$\sum_{j=1}^3 (1 - p_j) w_j \otimes w_j = I_1.$$

If one of the p_j 's is equal to 1 then we are essentially left with two such unit vectors w_j . Without loss of generality we may assume that all the w_j are equal to 1 (as opposed to -1). It follows that the corresponding geometric Brascamp–Lieb inequality is simply the trilinear Hölder inequality on $\mathbb{R}$, which can be stated as

$$\int_{\mathbb{R}} g_1(t)^{p_1} g_2(t)^{p_2} g_3(t)^{p_3} dt \leq \left(\int_{\mathbb{R}} g_1 \right)^{p_1} \left(\int_{\mathbb{R}} g_2 \right)^{p_2} \left(\int_{\mathbb{R}} g_3 \right)^{p_3}.$$

Although we are not yet able to factorise the sharp Young convolution inequality, explicit factorisations of the multilinear Hölder inequality appear in [16]. In Section 3.4, we briefly discuss possible connections between factorisations for such a "dual pair" of geometric Brascamp–Lieb inequalities.

3.3 Attempting to prove sharp constant for a geometric Brascamp–Lieb inequality using the sharp constant for the dual inequality

In this section we attempt to establish more explicit connection between pairs of geometric Brascamp–Lieb inequalities which we consider "dual" in the sense of Lemma 3.2. For $1 \leq d \leq m$, let $\mathbb{R}^d$ denote the subspace spanned by the first d standard basis vectors and $\mathbb{R}^{m-d} \cong (\mathbb{R}^d)^\perp$ the subspace spanned by the last $(m - d)$ standard basis vectors. Given an orthonormal basis $v_1, \dots, v_m \in \mathbb{R}^m$, define the following two families of vectors

$$u_j = \begin{cases} \frac{\pi_{\mathbb{R}^d} v_j}{\|\pi_{\mathbb{R}^d} v_j\|} & \text{if } \|\pi_{\mathbb{R}^d} v_j\| \neq 0 \\ \mathbf{0} & \text{if } \|\pi_{\mathbb{R}^d} v_j\| = 0, \end{cases}$$

$$w_j = \begin{cases} \frac{\pi_{\mathbb{R}^{m-d}} v_j}{\|\pi_{\mathbb{R}^{m-d}} v_j\|} & \text{if } \|\pi_{\mathbb{R}^{m-d}} v_j\| \neq 0 \\ \mathbf{0} & \text{if } \|\pi_{\mathbb{R}^{m-d}} v_j\| = 0. \end{cases}$$

Furthermore, set $p_j = \|\pi_{\mathbb{R}^d} v_j\|^2$ so that for all j we have

$$\pi_{\mathbb{R}^d} v_j = p_j^{\frac{1}{2}} u_j,$$

and

$$\pi_{\mathbb{R}^{m-d}} v_j = (1 - p_j)^{\frac{1}{2}} w_j.$$

Since $v_1, \dots, v_m$ form an orthonormal basis, it follows that $(\mathbf{u}, \mathbf{p})$ and $(\mathbf{w}, \mathbf{1} - \mathbf{p})$ are both examples of geometric Brascamp–Lieb data, where $\mathbf{1} - \mathbf{p}$ denotes the m -tuple $(1-p_1, \dots, 1-p_m)$. To see why, simply observe that

$$\begin{aligned}
\sum_{j=1}^m p_j u_j \otimes u_j &= \sum_{j=1}^m \left(p_j^{\frac{1}{2}} u_j \right) \otimes \left(p_j^{\frac{1}{2}} u_j \right) \\
&= \sum_{j=1}^m (\pi_{\mathbb{R}^d} v_j) \otimes (\pi_{\mathbb{R}^d} v_j) \\
&= \pi_{\mathbb{R}^d} \circ \left(\sum_{j=1}^m v_j \otimes v_j \right) \circ \pi_{\mathbb{R}^d}^* \\
&= \pi_{\mathbb{R}^d} \circ I_m \circ \pi_{\mathbb{R}^d}^* \\
&= I_d,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^m (1-p_j) w_j \otimes w_j &= \sum_{j=1}^m \left((1-p_j)^{\frac{1}{2}} w_j \right) \otimes \left((1-p_j)^{\frac{1}{2}} w_j \right) \\
&= \sum_{j=1}^m (\pi_{\mathbb{R}^{m-d}} v_j) \otimes (\pi_{\mathbb{R}^{m-d}} v_j) \\
&= \pi_{\mathbb{R}^{m-d}} \circ \left(\sum_{j=1}^m v_j \otimes v_j \right) \circ \pi_{\mathbb{R}^{m-d}}^* \\
&= \pi_{\mathbb{R}^{m-d}} \circ I_m \circ \pi_{\mathbb{R}^{m-d}}^* \\
&= I_{m-d}.
\end{aligned}$$

Essentially, Lemma 3.2 tells us we can view any geometric Brascamp–Lieb datum tuple as part of such a "dual pair". With the notation defined above, we propose the following question: can one prove without reference to Theorem 3.1 that

$$BL(\mathbf{u}, \mathbf{p}) BL(\mathbf{w}, \mathbf{1} - \mathbf{p}) \leq BL(\mathbf{v}, \mathbf{1})? \quad (3.29)$$

Of course by Theorem 3.1 we immediately have

$$BL(\mathbf{u}, \mathbf{p}) BL(\mathbf{w}, \mathbf{1} - \mathbf{p}) = 1 \cdot 1 = 1 = BL(\mathbf{v}, \mathbf{1}).$$

Firstly note that (3.29) can be rewritten as

$$\begin{aligned}
\sup_{\|f_j\|_1=1} \left(\int_{\mathbb{R}^d} \prod_{j=1}^m f_j(\langle y, u_j \rangle)^{p_j} dy \right) \cdot \sup_{\|g_j\|_1=1} \left(\int_{\mathbb{R}^{m-d}} \prod_{j=1}^m g_j(\langle z, w_j \rangle)^{1-p_j} dz \right) & \leq 1
\end{aligned} \quad (3.30)$$

With this in mind, let us consider the following weaker inequality

$$\sup_{\|h_j\|_1} \left(\int_{\mathbb{R}^d} \prod_{j=1}^m h_j(\langle y, u_j \rangle)^{p_j} dy \right) \left(\int_{\mathbb{R}^{m-d}} \prod_{j=1}^m h_j(\langle z, w_j \rangle)^{1-p_j} dz \right) \leq 1. \quad (3.31)$$

If one could prove this last inequality, then (3.30) would reduce to the following question: for unit vectors $u_1, \dots, u_m, w_1, \dots, w_m$ and functions $f_1, \dots, f_m, g_1, \dots, g_m$, do there exist functions $h_1, \dots, h_m$ such that

$$\begin{aligned} \left(\int_{\mathbb{R}^d} \prod_{j=1}^m f_j (\langle x, u_j \rangle)^{p_j} dx \right) & \left(\int_{\mathbb{R}^{m-d}} \prod_{j=1}^m g_j (\langle x, w_j \rangle)^{1-p_j} dx \right) \\ & \leq \left(\int_{\mathbb{R}^d} \prod_{j=1}^m h_j (\langle x, u_j \rangle)^{p_j} dx \right) \left(\int_{\mathbb{R}^{m-d}} \prod_{j=1}^m h_j (\langle x, w_j \rangle)^{1-p_j} dx \right)? \end{aligned} \quad (3.32)$$

Trying to prove (3.31) for characteristic functions on intervals presents a nice elementary problem in convex geometry, and in the few simple cases I have checked can be solved directly. For $1 \leq j \leq m$ suppose $I_j \subset \mathbb{R}$ is an interval, then for functions of the form $h_j = \frac{1}{|I_j|} \chi_{I_j}$ we can rewrite (3.31) as

$$\left| \bigcap_{j=1}^m u_j^\perp \times I_j \right| \left| \bigcap_{j=1}^m w_j^\perp \times I_j \right| \leq \prod_{j=1}^m |I_j| \quad (3.33)$$

For sums of characteristic functions $h_j = \frac{\sum_{i=1}^{N_j} \chi_{I_{j,i}}}{\sum_{i=1}^{N_j} |I_{j,i}|}$ this becomes

$$\left(\sum_{i_1, \dots, i_m} \frac{\left| \bigcap_{j=1}^m u_j^\perp \times I_{j,i_j} \right|}{\prod_{j=1}^m |I_{j,i_j}|^{p_j}} \right) \left(\sum_{i_1, \dots, i_m} \frac{\left| \bigcap_{j=1}^m w_j^\perp \times I_{j,i_j} \right|}{\prod_{j=1}^m |I_{j,i_j}|^{1-p_j}} \right) \leq 1. \quad (3.34)$$

It also seems plausible that (3.32) should hold for characteristic functions and arbitrary (unit) vectors $u_1, \dots, u_m, w_1, \dots, w_m$. Both integrals seem to be maximised for centred intervals so perhaps one can take the largest centred interval between the f_j and g_j for each j and then define h_j in this way. To see why this might be true, let us consider the special case where all intervals are of the same size. Suppose we have unit intervals $I_1, \dots, I_m, J_1, \dots, J_m \subset \mathbb{R}$, and we define $f_k = \chi_{I_k}$ and $g_k = \chi_{J_k}$ for $1 \leq k \leq m$, then setting $Q = [-\frac{1}{2}, \frac{1}{2}]$ we always have

$$\left| \bigcap_{k=1}^m u_k^\perp \times I_k \right| \leq \left| \bigcap_{k=1}^m u_k^\perp \times Q \right|,$$

and

$$\left| \bigcap_{k=1}^m w_k^\perp \times J_k \right| \leq \left| \bigcap_{k=1}^m w_k^\perp \times Q \right|,$$

By defining $h_k = \chi_Q$ for $1 \leq k \leq m$ we immediately get (3.32).

3.4 Attempting to factorise a geometric Brascamp–Lieb inequality using a factorisation of the dual inequality

Let us now briefly explore a possible connection between factorisations of any "dual pair" of geometric Brascamp–Lieb inequalities. Assuming the same setup as in the previous section, by

Theorem 3.1, we have that the following two inequalities are sharp

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j (\langle y, u_j \rangle)^{p_j} dy \leq \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j \right)^{p_j} \quad (3.35)$$

and

$$\int_{\mathbb{R}^{m-d}} \prod_{j=1}^m f_j (\langle z, w_j \rangle)^{1-p_j} dz \leq \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j \right)^{1-p_j}. \quad (3.36)$$

Note that by Fubini's theorem, since $v_1, \dots, v_m \in \mathbb{R}^m$ form an orthonormal basis, we have the following identity

$$\int_{\mathbb{R}^m} \prod_{j=1}^m f_j (\langle x, v_j \rangle) dx = \prod_{j=1}^m \|f_j\|_{L^1(\mathbb{R})}. \quad (3.37)$$

Since we already know how to factorise Fubini's identity and some other special cases of the geometric Brascamp–Lieb inequalities, it would be interesting to use the notion of lifting or projecting (in the spirit of Lemma 3.2) to find factorisations for other geometric Brascamp–Lieb inequalities we do not currently know how to factorise, where perhaps we are able to do so for the corresponding "dual" inequality. The reverse implication of Lemma 3.2 is the fact that given an orthonormal basis $v_1, \dots, v_m$ of $\mathbb{R}^m$ we can always project to get geometric Brascamp–Lieb datum on $\mathbb{R}^d$ and $\mathbb{R}^{m-d}$ respectively. This is exactly what we observed at the beginning of the previous section. One might hope that a version of this is true in the setting of factorisations: given a factorisation of Fubini's identity, can we in some way project to factorisations of (3.35) and (3.36)? In general, applying Theorem 1.1 one can reformulate (3.37), (3.35) and (3.36) as the following respective factorisation statements.

1. For any $M \in L^{\frac{m}{m-1}}(\mathbb{R}^m)$ with $\|M\|_{\frac{m}{m-1}} = 1$ there exist functions $M_1, \dots, M_m: \mathbb{R}^m \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}^m$

$$M(x) \leq \prod_{j=1}^m M_j(x)^{\frac{1}{m}},$$

and

$$\prod_{j=1}^m \left(\sup_{t_j \in \mathbb{R}} \int_{v_j^\perp} M_j(t_j v_j + \tilde{v}_j) d\tilde{v}_j \right) \leq 1.$$

2. For any $N \in L^{(\sum_{j=1}^m \|\pi_{\mathbb{R}^d} v_j\|^2)'}(\mathbb{R}^d)$ with $\|N\|_{(\sum_{j=1}^m \|\pi_{\mathbb{R}^d} v_j\|^2)'} = 1$ there exist functions $N_1, \dots, N_m: \mathbb{R}^d \rightarrow \mathbb{R}$ such that for all $y \in \mathbb{R}^d$

$$N(y) \leq \prod_{j=1}^m N_j(y)^{\frac{\|\pi_{\mathbb{R}^d} v_j\|^2}{\sum_{i=1}^m \|\pi_{\mathbb{R}^d} v_i\|^2}},$$

and

$$\prod_{j=1}^m \left(\sup_{s_j \in \mathbb{R}} \int_{(\pi_{\mathbb{R}^d} v_j)^\perp} N_j(s_j \pi_{\mathbb{R}^d} v_j + \tilde{u}_j) d\tilde{u}_j \right)^{\frac{\|\pi_{\mathbb{R}^d} v_j\|^2}{\sum_{i=1}^m \|\pi_{\mathbb{R}^d} v_i\|^2}} \leq 1.$$

3. For any $L \in L^{(\sum_{j=1}^m \|\pi_{\mathbb{R}^{m-d} v_j}\|^2)'}(\mathbb{R})$ with $\|L\|_{(\sum_{j=1}^m \|\pi_{\mathbb{R}^{m-d} v_j}\|^2)'} = 1$ there exist functions $L_1, \dots, L_m: \mathbb{R}^{m-d} \rightarrow \mathbb{R}$ such that for all $z \in \mathbb{R}^{m-d}$

$$L(z) \leq \prod_{j=1}^m L_j(z)^{\frac{\|\pi_{\mathbb{R}^{m-d} v_j}\|^2}{\sum_{i=1}^m \|\pi_{\mathbb{R}^{m-d} v_j}\|^2}},$$

and

$$\prod_{j=1}^m \left(\sup_{r_j \in \mathbb{R}} \int_{(\pi_{\mathbb{R}^{m-d} v_j})^\perp} L_j(r_j \pi_{\mathbb{R}^{m-d} v_j} + \tilde{w}_j) d\tilde{w}_j \right)^{\frac{\|\pi_{\mathbb{R}^{m-d} v_j}\|^2}{\sum_{i=1}^m \|\pi_{\mathbb{R}^{m-d} v_j}\|^2}} \leq 1.$$

It may be more convenient to write $(\pi_{\mathbb{R}^d v_j})^\perp = v_j^\perp \cap \mathbb{R}^d$ and $(\pi_{\mathbb{R}^{m-d} v_j})^\perp = v_j^\perp \cap \mathbb{R}^{m-d}$. With this notation we consider the special case where $\|\pi_{\mathbb{R}^d v_i}\| = \|\pi_{\mathbb{R}^d v_j}\|$ for every $i \neq j$ (so necessarily $\|\pi_{\mathbb{R}^{m-d} v_i}\| = \|\pi_{\mathbb{R}^{m-d} v_j}\|$ for every $i \neq j$). Let us denote $c_n = \|\pi_{\mathbb{R}^d v_j}\|^2$ and $c_{m-d} = \|\pi_{\mathbb{R}^{m-d} v_j}\|^2$, then the factorisation statements for (3.37), (3.35) and (3.36) respectively are as follows

1. For any $M \in L^{\frac{m}{m-1}}(\mathbb{R}^m)$ with $\|M\|_{\frac{m}{m-1}} = 1$ there exist functions $M_1, \dots, M_m: \mathbb{R}^m \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}^m$

$$M(x) \leq \prod_{j=1}^m M_j(x)^{\frac{1}{m}},$$

and

$$\prod_{j=1}^m \left(\sup_{t_j \in \mathbb{R}} \int_{v_j^\perp} M_j(t_j v_j + \tilde{v}_j) d\tilde{v}_j \right) \leq 1.$$

2. For any $N \in L^{\frac{c_d m}{c_d m - 1}}(\mathbb{R}^d)$ with $\|N\|_{\frac{c_d m}{c_d m - 1}} = 1$ there exist functions $N_1, \dots, N_m: \mathbb{R}^d \rightarrow \mathbb{R}$ such that for all $y \in \mathbb{R}^d$

$$N(y) \leq \prod_{j=1}^m N_j(y)^{\frac{1}{m}},$$

and

$$\prod_{j=1}^m \left(\sup_{s_j \in \mathbb{R}} \int_{v_j^\perp \cap \mathbb{R}^d} N_j(t_j \pi_{\mathbb{R}^d v_j} + \tilde{u}_j) d\tilde{u}_j \right) \leq 1.$$

3. For any $L \in L^{\frac{c_{m-d} m}{c_{m-d} m - 1}}(\mathbb{R}^{m-d})$ with $\|L\|_{\frac{c_{m-d} m}{c_{m-d} m - 1}} = 1$ there exist functions $L_1, \dots, L_m: \mathbb{R}^{m-d} \rightarrow \mathbb{R}$ such that for all $z \in \mathbb{R}^{m-d}$

$$L(z) \leq \prod_{j=1}^m L_j(z)^{\frac{1}{m}},$$

and

$$\prod_{j=1}^m \left(\sup_{r_j \in \mathbb{R}} \int_{v_j^\perp \cap \mathbb{R}^{m-d}} L_j(t_j \pi_{\mathbb{R}^{m-d} v_j} + \tilde{w}_j) d\tilde{w}_j \right) \leq 1.$$

Chapter 4

Factorisation of the endpoint multilinear Keakeya inequalities and connections with convex geometry

Earlier we introduced the endpoint multilinear Keakeya inequality, originally due to Guth, as a far-reaching generalisation of the affine-invariant Loomis–Whitney inequality. In this chapter, we discuss new perspectives on some known polynomial method based proofs of the following result:

Theorem 4.1 (Endpoint multilinear Keakeya inequality - [24],[12], [15]). *Let $2 \leq m \leq d$. There exists a finite constant $C_{d,m} > 0$ such that if $\mathcal{T}_1, \dots, \mathcal{T}_m$ are families of 1-tubes in $\mathbb{R}^d$, we have*

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\sum_{T_1 \in \mathcal{T}_1} \cdots \sum_{T_m \in \mathcal{T}_m} \left(\prod_{j=1}^m \alpha_{T_j} \chi_{T_j}(x) \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m-1}} dx \\ \leq C_{d,m} \prod_{j=1}^m \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \right)^{\frac{1}{m-1}}. \end{aligned} \quad (4.1)$$

Although the multilinear duality principle explored in [16] does not apply here, one can prove this result via the following formally stronger statement:

Theorem 4.2 (Factorisation for the endpoint multilinear Keakeya inequality - [24],[15]). *Let $\mathcal{Q}$ denote the unit lattice of cubes on $\mathbb{R}^d$ and $2 \leq m \leq d$. Then, given any finitely supported nonnegative function $M \in \ell^m(\mathcal{Q})$ with $\|M\|_m = 1$, there exist functions S_j defined on $\mathcal{Q} \times \mathcal{T}_j$ for $j = 1, \dots, m$ such that for all $Q \in \mathcal{Q}$, whenever $T_j \cap Q \neq \emptyset$ for all j , we have*

$$|E(T_1) \wedge \cdots \wedge E(T_m)| M(Q)^m \lesssim_{d,m} S_1(Q, T_1) \cdots S_m(Q, T_m), \quad (4.2)$$

and for all j and $T_j \in \mathcal{T}_j$

$$\sum_{Q \in \mathcal{Q}: Q \cap T_j \neq \emptyset} S_j(Q, T_j) \lesssim_{d,m} \left(\sum_Q M(Q)^m \right)^{\frac{1}{m}}, \quad (4.3)$$

where the implicit constants both depend only on the dimension d and degree of multilinearity m .

Indeed, this was the approach taken in [15]. As we saw in Chapter 2, the existence of such a factorisation directly implies Theorem 4.1. Two main ingredients needed to establish Theorem 4.2 are that of *visibility* $\text{vis}(Z)$ and *directional surface area* $\text{surf}_e(Z)$ of a hypersurface $Z \subset \mathbb{R}^d$. To aid our discussion, let us define these notions precisely here:

Definition 4.3 (Directional surface area). Let $Z \subset \mathbb{R}^d$ be a hypersurface with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$. The **directional surface area** of Z in the direction $u \in \mathbb{R}^d$ is defined as

$$\text{surf}_u(Z) := \int_Z |u \cdot n(x)| d\sigma_Z(x).$$

In [24], [39] and [41], this same quantity is referred to as directed volume and denoted by $V_Z(u)$. The language and notation we adopt follows that used in [15], but our definition differs slightly in that we allow u to be any vector in $\mathbb{R}^d$, rather than restricting to the unit sphere. This slight modification is more for convenience than anything else. In order to define visibility, we associate a fundamental centrally-symmetric convex set $K(Z)$ to each piece of hypersurface $Z \subset \mathbb{R}^d$, where we set

$$K(Z) := \{u \in \mathbb{B}_2^d : \text{surf}_u(Z) \leq 1\}.$$

In [24] and [39], this same set is referred to as the fading zone. The notation $K(Z)$ we use here matches that in [15]. To see why $K(Z)$ is indeed a convex set, simply observe that for $u_0, u_1 \in K(Z)$ and $0 \leq t \leq 1$

$$\text{surf}_{(1-t)u_0+tu_1}(Z) \leq (1-t)\text{surf}_{u_0}(Z) + t\text{surf}_{u_1}(Z) \leq 1.$$

$K(Z)$ is clearly centrally-symmetric since $\text{surf}_u(Z) = \text{surf}_{-u}(Z)$. The volume of $K(Z)$ encodes some important information about the original hypersurface Z , which leads us to the definition of visibility:

Definition 4.4 (Visibility). Let $Z \subset \mathbb{R}^d$ be a hypersurface with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$. The **visibility** of Z is defined as follows

$$\text{vis}(Z) := |K(Z)|^{-\frac{1}{d}}.$$

This matches the definition used in [15] but differs from that of [24] and [39] by a power of d .

It should be noted that the notions of directional surface area and visibility are indispensable to arguments given in [24], [15] and [39]. Although these geometric quantities play such a crucial role in all known proofs of Theorem 4.1, there seems to be little intuition as to why this should be the case. In an attempt to shed light on some less explored connections to convex geometry, we will review the proof of Theorem 4.2 given in [15]. Their argument uses an extremely clever adaptation of the polynomial method, inspired by ideas from [24], and hinges on a deep topological theorem due to Guth:

Theorem 4.5 (Guth's Visibility Theorem - [24]). *Given a non-negative finitely supported function G defined on the unit lattice $\mathcal{Q}$ of cubes in $\mathbb{R}^d$ and sufficiently small $\varepsilon > 0$, there exists a non-zero polynomial p such that*

$$\deg p \leq C_d \left(\sum_{Q \in \mathcal{Q}} G(Q)^d \right)^{\frac{1}{d}}$$

and such that if we set $Z_p = \{x \in \mathbb{R}^d \mid p(x) = 0\}$, then for all $Q \in \mathcal{Q}$ we have

$$\text{vis}_\varepsilon(Z_p \cap Q) \geq C_d G(Q).$$

Remark 4.6. The notion of mollified visibility, denoted vis_ε , is used as a replacement for standard visibility to avoid certain topological complications. These modifications are discussed in Section 4.1.2

Even after having attempted to digest the notion of visibility, the implications of this result are quite mysterious. Furthermore, how this polynomial p is then used construct a factorisation seems rather miraculous.

In Section 4.1 we introduce various useful tools from convex geometry, which will allow us to reinterpret the notions of directional surface area and visibility as two closely related geometric quantities. In particular, given a convex set $K \subset \mathbb{R}^d$, we rely on the properties of the polar set K° - see Definition 4.21. Using the language of convex geometry, we give a new perspective on proving Theorem 4.2. Although we do not claim to provide an entirely new proof of this last result, our approach does highlight important connections to results from convex geometry which are not explicitly explored in [24] or [15]. The main contribution of this chapter stems from the following observation, which we prove in Section 4.1:

Theorem 4.7. *For every hypersurface $Z \subset \mathbb{R}^d$ of finite $(d - 1)$ -dimensional measure with an almost-everywhere defined unit normal, there exists a unique centrally-symmetric, compact, convex set $L_Z \subset \mathbb{R}^d$, such that for all $u \in \mathbb{R}^d$*

$$\text{surf}_u(Z) = |\pi_u L_Z|,$$

and

$$\text{vis}(Z) = |L_Z^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d}}.$$

As we will see at the end of Section 4.1.1, this unique convex set L_Z is related to the above-mentioned fundamental centrally-symmetric convex set $K(Z)$ via the following identity:

$$K(Z) = L_Z^\circ \cap \mathbb{B}_2^d.$$

In the special case where Z is the boundary of a convex set $K \subset \mathbb{R}^d$ with non-empty interior, this unique convex L_Z is coincides with a classical notion from convex geometry, the projection body of K , usually denoted ΠK - see Remark 4.27.

Let us now briefly describe how Theorem 4.7 gives us a nice way of interpreting the role played by Theorem 4.5 in the proofs of the endpoint multilinear Kakeya inequality given in [24] and [15]. Firstly, suppose we are given a finitely supported non-negative function M defined on $\mathcal{Q}$ with $\|M\|_{\ell^m(\mathcal{Q})} = 1$, and we apply Theorem 4.5 with $G(Q) = \lambda M(Q)^{\frac{d}{m}}$ for some $\lambda > 0$ to get a polynomial hypersurface Z_p . At the end of Section 4.3, we show that by choosing $\lambda > 0$ large enough, we can ensure that for each cube $Q \in \text{supp}(M)$, the unique convex set $L_{Z_p \cap Q}$ given by Theorem 4.7 is a convex body, and that the polar body $L_{Z_p \cap Q}^\circ$ is contained in the standard unit ball. This allows us to write the visibility as

$$\text{vis}(Z_p \cap Q) = |L_{Z_p \cap Q}^\circ|^{-\frac{1}{d}}. \quad (4.4)$$

A similar argument of applying Theorem 4.5 with $G(Q) = \lambda M(Q)^{\frac{d}{m}}$ appears in [24] and [15], where $\lambda > 0$ is chosen large enough to ensure that for each $Q \in \text{supp}(M)$ and every unit vector u we have $\text{surf}_u(Z_p \cap Q) \gtrsim 1$. Essentially, this last condition implies that $L_{Z_p \cap Q}^\circ$ is contained in a some large dilation of the standard unit ball. This observation is not mentioned explicitly in either paper, but we feel it provides further geometric insight for the λ -scaling argument we have

just described. In Section 4.3, we combine (4.4) with the well known fact from convex geometry that $|K||K^\circ| \sim_d 1$ for any convex body $K \subset \mathbb{R}^d$ - see discussion just after the statement of Lemma 4.24 for more details - to deduce that for each $Q \in \text{supp}(M)$ we can write

$$\text{vis}(Z_p \cap Q) \sim_d |L_{Z_p \cap Q}|^{\frac{1}{d}}.$$

In light of this, we state the following consequence of Theorem 4.5, which can be formulated purely in terms of convex bodies:

Theorem 4.8 (Visibility theorem in terms of convex bodies - tube version [24], [15]). *Let $\mathcal{T}$ be some finite family of 1-tubes in $\mathbb{R}^d$. Given any finitely supported non-negative function M defined on $\mathcal{Q}$ with $\|M\|_{\ell^m(\mathcal{Q})} = 1$, there exists a family of centrally symmetric convex bodies $\{K(Q)\}_{Q \in \text{supp}(M)}$ in $\mathbb{R}^d$ depending on the choice of M and $\mathcal{T}$ such that*

$$\max_{T \in \mathcal{T}} \sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| \lesssim_{d,m} 1, \quad (4.5)$$

and for each $Q \in \text{supp}(M)$

$$|K(Q)|^{\frac{1}{d}} \gtrsim_{d,m} M(Q), \quad (4.6)$$

where the implicit constants depend only on the dimension d and the degree of multilinearity m .

This result does not appear explicitly in [24] or [15], but essentially both papers provide a construction for such a family of convex bodies using the polynomial given by a clever application of Theorem 4.5, the proofs of which given in [24] and [15] are highly non-constructive.

In Section 4.3.1, we prove Theorem 4.8 using Theorem 4.5 as a black box. The argument we provide is similar in many ways to the one presented in [15]. However, our new perspective highlights the appearance of certain geometric quantities via Theorem 4.7, which are not mentioned in [15] or [24]. Once we have convex bodies satisfying the conditions of Theorem 4.8, cooking up a factorisation is rather straightforward: for $1 \leq j \leq m$ and any 1-tube $T \subset \mathbb{R}^d$, define

$$S_j(Q, T) := |\pi_{E(T)}(K(Q))|.$$

It is worth noting here that the right hand side in the above definition is essentially independent of j , and the only dependence on j comes when we restrict each S_j to be a function on $\mathcal{Q} \times \mathcal{T}_j$. This is in complete contrast with the factorisations discussed in Chapter 2. The functions $S_1, \dots, S_m$ satisfy (4.3) as a direct consequence of (4.5). To verify (4.2), we appeal to an elementary geometric inequality which can be seen as a special case of the affine-invariant Finner inequality:

Lemma 4.9. *For any measurable set $E \subset \mathbb{R}^d$ and vectors $\omega_1, \dots, \omega_m$*

$$|\omega_1 \wedge \dots \wedge \omega_m| |E| \lesssim_{d,m} |\pi_{\omega_1} E| \cdots |\pi_{\omega_m} E|.$$

In Section 4.3 we also give a direct proof of this using only changes of variables. Applying Lemma 4.9 with $E = K$, and using (4.6) we get

$$|E(T_1) \wedge \dots \wedge E(T_m)| M(Q)^m \lesssim_{d,m} |E(T_1) \wedge \dots \wedge E(T_m)| |K(Q)| \leq |\pi_{E(T_1)} K| \cdots |\pi_{E(T_m)} K|,$$

which deals with (4.2).

In the first three sections of this chapter, we present the ideas laid out above in more detail, taking Theorem 4.5 as a black box. In Section 4.1, we introduce tools from convex geometry needed to establish Theorem 4.7, which follows as a consequence of Lemma 4.25. Furthermore, we prove a variant of the same result for a mollification of the directional surface area, namely Lemma 4.31. In Section 4.2, we use the way of assigning a convex set to any hypersurface suggested by Theorem 4.7, and reinterpret Theorem 4.5 as a statement purely about convex sets - see Theorem 4.34. Section 4.3 is dedicated to proving that Theorem 4.5 implies Theorem 4.8. As mentioned, this argument highlights important connections between classical notions of convex geometry and the polynomial method based proofs given in [24] and [15]. In particular, our new perspective helps one better understand the role of Guth's visibility theorem in constructing a factorisation for the endpoint multilinear Kakeya inequality.

Theorem 4.1 deals with the intersection of tubes, which are simply neighbourhoods of lines. As we saw in Chapter 2, one natural way of generalising this set-up is to replace tubes by neighbourhoods of higher-dimensional planes. The main result in [39] implies a generalisation of Theorem 4.1, with families of 1-tubes replaced by families of k_j -dimensional 1-slabs:

Theorem 4.10 (Endpoint multilinear Kakeya inequality for k_j -planes - [39]). *Let $2 \leq m \leq d$. There exists a constant $C_{d,m} > 0$ such that if $\mathcal{P}_1, \dots, \mathcal{P}_m$ are families of 1-slabs in $\mathbb{R}^d$ of dimensions $k_1, \dots, k_m$ respectively and $\sum_{j=1}^m k_j = d$, then we have*

$$\int_{\mathbb{R}^d} \left(\sum_{P_1 \in \mathcal{P}_1} \cdots \sum_{P_m \in \mathcal{P}_m} \left(\prod_{j=1}^m \alpha_{P_j} \chi_{P_j}(x) \right) |E(P_1) \wedge \cdots \wedge E(P_m)| \right)^{\frac{1}{m-1}} dx \leq C_{d,m} \prod_{j=1}^m \left(\sum_{P_j \in \mathcal{P}_j} \alpha_{P_j} \right)^{\frac{1}{m-1}}. \quad (4.7)$$

Again, this does not fall under the scope of multilinear duality. However, following the arguments given in [39], one can adapt the framework developed in [15] to prove the existence of a factorisation for this generalisation to k_j -planes:

Theorem 4.11 (Factorisation for k_j -planes endpoint multilinear Kakeya inequality - [39]). *Let $\mathcal{Q}$ denote the unit lattice of cubes on $\mathbb{R}^d$ and $2 \leq m \leq d$. Given any finitely supported non-negative function $M \in L^m(\mathcal{Q})$ with $\|M\|_m = 1$, there exist functions S_j defined on $\mathcal{Q} \times \mathcal{P}_j$ for $j = 1, \dots, m$ such that for all $Q \in \mathcal{Q}$, whenever $P_j \cap Q \neq \emptyset$ for all j ,*

$$|E(P_1) \wedge \cdots \wedge E(P_m)| M(Q)^m \lesssim_{d,m} S_1(Q, P_1) \cdots S_m(Q, P_m), \quad (4.8)$$

and for all j and $P_j \in \mathcal{P}_j$

$$\sum_{Q \in \mathcal{Q}: Q \cap P_j \neq \emptyset} S_j(Q, P_j) \lesssim_{d,m} 1, \quad (4.9)$$

where the implicit constants both depend only on the dimension d and the degree of multilinearity m .

Similarly to the tube case, the proof of this result also relies heavily on Guth's visibility theorem. In addition, we need the following generalisation of the directional surface area, which was first introduced by Zhang [39]:

Definition 4.12 (Directional volume). Let $Z \subset \mathbb{R}^d$ be a hypersurface with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$. The **directional volume** of Z with respect to a k -dimensional subspace $V \subset \mathbb{R}^d$ is defined as

$$\text{surf}_V(Z) := \int_Z \cdots \int_Z |V^\perp \wedge n(x_1) \wedge \cdots \wedge n(x_k)| d\sigma_Z(x_1) \cdots d\sigma_Z(x_k).$$

In analogy with the case for tubes, we will prove the following generalisation of Theorem 4.7 in Section 4.4.1:

Theorem 4.13. *For every hypersurface $Z \subset \mathbb{R}^d$ of finite $(d-1)$ -dimensional measure with an almost-everywhere defined unit normal, there exists a unique compact convex set $L_Z \subset \mathbb{R}^d$, symmetric about the origin, such that for $1 \leq k \leq d$ and any k -dimensional subspace $V \subset \mathbb{R}^d$*

$$\text{surf}_V(Z) = k! |\pi_V L_Z|,$$

and

$$\text{vis}(Z) = |L_Z^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d}}.$$

Taking $k = 1$ we exactly recover Theorem 4.7. Similarly to the setting of tubes, it seems desirable to have each polar set L_Z° contained in the standard unit ball. Exactly as we described above, given a finitely supported non-negative function M defined on $\mathcal{Q}$ with $\|M\|_{\ell^m(\mathcal{Q})} = 1$, we can apply Theorem 4.5 with $G(Q) = \lambda M(Q)^{\frac{d}{m}}$ for some large enough $\lambda > 0$ to get a polynomial hypersurface Z_p , such that each polar set $L_{Z_p \cap Q}^\circ$ is contained in the standard unit ball. Details for this scaling argument are given at the beginning of Section 4.4.2.

In Section 4.4, we use this same λ -scaling argument to prove the following generalisation of Theorem 4.8 for k_j -planes:

Theorem 4.14 (Guth's Visibility Theorem for convex bodies - k_j plane version - [39]). *Let $2 \leq m \leq d$ and suppose we have families of 1-slabs $\mathcal{P}_1, \dots, \mathcal{P}_m$ in $\mathbb{R}^d$ of dimensions $k_1, \dots, k_m$ respectively with $\sum_{j=1}^m k_j = d$. Given a finitely supported non-negative function $M \in L^m(\mathcal{Q})$ with $\|M\|_m = 1$, there exists a family of centrally symmetric convex bodies $\{K(Q)\}_{Q \in \text{supp}(M)}$ in $\mathbb{R}^d$, depending on $\mathcal{P}_1, \dots, \mathcal{P}_m$ and M , such that for $1 \leq j \leq m$*

$$\max_{P_j \in \mathcal{P}_j} \sum_{Q: Q \cap P_j \neq \emptyset} |\pi_{E(P_j)} K(Q)| \lesssim_{d,m} 1, \quad (4.10)$$

and for each $Q \in \text{supp}(G)$

$$|K(Q)|^{\frac{1}{m}} \gtrsim_{d,m} M(Q), \quad (4.11)$$

where the implicit constants both depend only on the dimension d and the degree of multilinearity m .

This result does not appear explicitly in [39] or [41], which contain the only known proofs of Theorem 4.10, but essentially both papers provide a construction for such a family of convex bodies using the polynomial given by a clever application of Theorem 4.5. In both papers, the authors use a variant of the λ -scaling we described above, which involves multiplying the polynomial outputted from Theorem 4.5 by a particular choice of linear factors. As in the setting of tubes, this ensures that the corresponding polar convex sets $L_{Z_p \cap Q}^\circ$ given by Theorem 4.13 are all contained in a some large dilation of the standard unit ball. This last observation is noted explicitly in [41], Theorem 2.1.

In Section 4.4, roughly following course notes of Anthony Carbery [14], which borrow ideas from [39] and [15], we present a proof of Theorem 4.11 via Theorem 4.14. Again, we take Guth's visibility theorem as a black box. As with the case for tubes, given convex bodies satisfying the conditions of Theorem 4.8, it is not too difficult to cook up a factorisation: for $1 \leq j \leq m$ and any k_j -slab P , define

$$S_j(Q, P) := |\pi_{E(P)}(K(Q))|.$$

Again, it is worth noting here that the right hand side in the above definition is essentially independent of j . Similarly to the case for tubes, the functions $S_1, \dots, S_m$ satisfy (4.9) as a direct consequence of (4.10). To verify (4.8), we appeal to a slight generalisation of Lemma 4.9 which can also be seen as a special case of the affine-invariant Finner inequality:

Theorem 4.15. *Suppose we have $V_1, \dots, V_m \subset \mathbb{R}^d$ with each V_j denoting a k_j -dimensional plane, and $\sum_{j=1}^m k_j = d$. Then, given any measurable set $E \subset \mathbb{R}^d$, the following inequality holds*

$$|E| \lesssim_{d,m} \frac{|\pi_{V_1} E| \cdots |\pi_{V_m} E|}{|V_1 \wedge \cdots \wedge V_m|}.$$

At the beginning of Section 4.4, we provide a direct proof of this last result using change of variables. Applying Theorem 4.15 with $K = E$, and using (4.11), we can write

$$|E(P_1) \wedge \cdots \wedge E(P_m)| M(Q)^m \lesssim_{d,m} |E(P_1) \wedge \cdots \wedge E(P_m)| |K(Q)| \leq |\pi_{E(P_1)} K| \cdots |\pi_{E(P_m)} K|,$$

which deals with (4.8).

In the last section of this chapter, we discuss in detail the proof of Theorem 4.14, taking Theorem 4.5 as a black box. In Section 4.4.1, we introduce some tools from convex geometry required to prove Theorem 4.13, which follows by Proposition 4.44. Furthermore, as with the case for tubes, we also prove a variant of the same result for a mollification of the directional volume, namely Proposition 4.54. In Section 4.4.2, we show how to deduce Theorem 4.14 as a consequence of Theorem 4.5. It should be noted that this final part of the argument is remarkably similar to the proof of Theorem 4.8. Upon reading the arguments given in [39], it is not immediately obvious how these arise as natural generalisations of the ideas introduced in [24] and [15]. In [41], the author develops a very general framework using the language of exterior algebra and pushforward measures which more clearly generalises that of [15], but this seems a little complicated or abstract considering the rather concrete statements of Theorems 4.1 and 4.10. However, our perspective is very geometric in nature and seems to provide a more natural framework for studying multilinear Keakeya inequalities. It is our hope that using this new perspective, one might gain insight into more general forms of perturbed or Keakeya–Brascamp–Lieb inequalities.

4.1 Directional surface area and visibility

In order to utilise Guth’s visibility theorem, we must first spend some time understanding the notions of directional surface area and visibility. Along the way, we will draw attention to the appearance of these quantities in areas of convex geometry. These connections do not seem to have been explored previously in this context, and we feel our perspective sheds new light on the role of polynomial methods in proving endpoint multilinear Keakeya inequalities. In particular, we will see how one arrives at the statement of Theorem 4.8. Recall that for any hypersurface $Z \subset \mathbb{R}^d$ with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$, we define

$$\text{surf}_u(Z) := \int_Z |u \cdot n(x)| d\sigma_Z(x).$$

Suppose that $e \in \mathbb{S}^{d-1}$, it is not difficult to see that if Z is the graph of some function $\Gamma: \Omega \subset e^\perp \rightarrow \mathbb{R}$ then $\text{surf}_e(Z)$ is simply the $(d-1)$ -dimensional surface area of Ω . More generally, one

can think of $\text{surf}_e(Z)$ as the volume of $\pi_{e^\perp} Z$ counted with geometric multiplicity. In particular, we have the following alternative formula for the directional surface area

$$\text{surf}_e(Z) = \int_{e^\perp} |Z \cap \pi_{e^\perp}^{-1}(y)| dy, \quad (4.12)$$

where $|Z \cap \pi_{e^\perp}^{-1}(y)|$ denotes the cardinality of the set $Z \cap \pi_{e^\perp}^{-1}(y)$. A simple case of the above is when Z forms the boundary of a convex body. Indeed, if Z forms the boundary of some convex body $K \subset \mathbb{R}^d$, then any line will either intersect Z in exactly two points or be tangent to K . The latter will only happen for measure zero portion of lines pointing in a given direction. Let us formalise this statement with the following result:

Lemma 4.16. *Suppose that Z forms the boundary of some convex body $K \subset \mathbb{R}^d$ with an almost-everywhere defined outward facing normal $n: Z \rightarrow \mathbb{S}^{d-1}$. Then for every $e \in \mathbb{S}^{d-1}$, we have*

$$\text{surf}_e(Z) = 2 |\pi_{e^\perp} K|.$$

One special case of the above is when Z forms the boundary of an ellipsoid, which can also be written as the zero set of a degree two polynomial. In fact, using (4.12) one can see Lemma 4.16 as an example of a more general statement for polynomial hypersurfaces:

Lemma 4.17 (Cylinder estimate - [24], Lemma 2.1). *If $Z_p \subset \mathbb{R}^d$ is the zero set of a non-zero polynomial p and $T \subset \mathbb{R}^d$ is a 1-tube, then*

$$\text{surf}_{E(T)}(Z_p \cap T) \lesssim_d \deg p.$$

The cylinder estimate plays a crucial role in the polynomial method part of the argument. In some sense, the role of this estimate is analogous to that of Bézout's theorem in the polynomial method arguments used in incidence geometry. It should be noted however, that the proof of Lemma 4.17 relies on a special case of Bézout's theorem.

Proof of Lemma 4.17. The projection $\pi_{E(T)^\perp} T$ is a unit disc. It follows that the function $|Z_p \cap T \cap \pi_{E(T)^\perp}^{-1}(y)|$ is supported in the unit disc. However, Z_p is polynomial hypersurface of degree $\deg(p)$, so any line will either intersect Z_p in at most $\deg(p)$ points or be completely contained in Z_p . Since p is not the zero polynomial, only a measure zero set of lines $\pi_{E(T)^\perp}^{-1}(y)$ for y in the unit disc can be contained in Z_p . It follows that $|Z_p \cap \pi_{E(T)^\perp}^{-1}(y)| \lesssim_d \deg(p)$ for almost every y in the unit disc. Thus, by (4.12) we have

$$\text{surf}_{E(T)}(Z_p \cap T) = \int_{E(T)^\perp} |Z_p \cap T \cap \pi_{E(T)^\perp}^{-1}(y)| dy \lesssim_d \deg(p).$$

□

As mentioned at the beginning of this chapter, using the notion of directional surface area, one defines the visibility of Z , denoted $\text{vis}(Z)$, as the volume of the following convex set

$$K(Z) := \{u \in \mathbb{B}_2^d : \text{surf}_u(Z) \leq 1\}.$$

Here we diverge slightly in an attempt to better understand $K(Z)$ using language from classical convex geometry.

4.1.1 Polar sets and projection bodies

Let us overlook the condition that vectors in $K(Z)$ must lie within the unit ball and consider a slight variant:

$$\tilde{K}(Z) := \{u \in \mathbb{R}^d : \text{surf}_u(Z) \leq 1\}.$$

A fundamental concept in classical convex geometry is that of the support function: for every non-empty closed convex set $A \subset \mathbb{R}^d$, the support function of A , denoted $h_A: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$, is defined by

$$h_A(x) := \sup\{x \cdot a \mid a \in A\}.$$

Furthermore, every non-empty closed convex set is uniquely determined by its support function, which follows from the fact that we can always represent A as the following intersection of half-spaces

$$A = \bigcap_{x \in \mathbb{R}^d} \{y \in \mathbb{R}^d \mid x \cdot y \leq h_A(x)\}.$$

In the special case where A is also bounded we have the following result:

Lemma 4.18. *For any non-empty compact convex set $A \subset \mathbb{R}^d$, the support function h_A is a continuous function from $\mathbb{R}^d$ to $\mathbb{R}$.*

Proof. Firstly, since A is bounded, there exists some finite number $M > 0$ such that by Cauchy-Schwarz we can write

$$h_A(x) = \sup\{x \cdot a \mid a \in A\} \leq \sup\{\|x\| \cdot \|a\| \mid a \in A\} \leq M\|x\|,$$

which implies that $h_A: \mathbb{R}^d \rightarrow \mathbb{R}$. Fix $\epsilon > 0$ and let $x, y \in \mathbb{R}^d$ such that $\|x - y\| < \delta$ for some $\delta > 0$ we will fix later. By the Cauchy-Schwarz inequality, it follows that for every $a \in A$

$$(x - y) \cdot a \leq \|x - y\| \|a\| < \delta \|a\|,$$

and by rearranging we get

$$x \cdot a < \delta \|a\| + y \cdot a.$$

Taking the supremum over all $a \in A$ we get

$$h_A(x) < \delta M + h_A(y),$$

thus

$$h_A(x) - h_A(y) < \delta M.$$

Similarly we can deduce that

$$h_A(y) - h_A(x) < \delta M,$$

hence

$$|h_A(x) - h_A(y)| < \delta M.$$

Setting $\delta = \frac{\epsilon}{M}$, we have

$$\|x - y\| < \delta \implies |h_A(x) - h_A(y)| < \epsilon,$$

which exactly tells us that $h_A: \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuous function. □

Given a function $h: \mathbb{R}^d \rightarrow \mathbb{R}$, it is not immediately obvious whether there exists a convex set A such that $h = h_A$. It is not difficult to see using properties of the supremum that any support function of any convex set A is a sublinear, or in other words $h_A(x + y) \leq h_A(x) + h_A(y)$ for any $x, y \in \mathbb{R}^d$. As it turns out, the condition of being sublinear is also sufficient, provided the function in question is real valued:

Theorem 4.19 ([37], Theorem 1.7.1). *If $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is a sublinear function, then there exists a unique compact convex set $K \subset \mathbb{R}^d$ with support function f .*

Given a centrally-symmetric convex set $K \subset \mathbb{R}^d$, we have the following geometric representation of the support function $h_K(u)$:

Lemma 4.20. *Suppose that $K \subset \mathbb{R}^d$ is a non-empty closed convex set which is symmetric about the origin, then we can write the support function of K as follows*

$$h_K(u) = \frac{\|u\|}{2} |\pi_u K|. \quad (4.13)$$

Proof. Using the fact that $\pi_u K$ is a convex set, we can write

$$\begin{aligned} |\pi_u K| &= \left| \left\{ \left\langle x, \frac{u}{\|u\|} \right\rangle \in \mathbb{R} \mid x \in K \right\} \right| = \sup \left\{ \left\langle x, \frac{u}{\|u\|} \right\rangle \in \mathbb{R} \mid x \in K \right\} + \sup \left\{ \left\langle x, -\frac{u}{\|u\|} \right\rangle \in \mathbb{R} \mid x \in K \right\} \\ &= h_K \left(\frac{u}{\|u\|} \right) + h_K \left(-\frac{u}{\|u\|} \right) \\ &= 2h_K \left(\frac{u}{\|u\|} \right) \\ &= \frac{2}{\|u\|} h_K(u), \end{aligned}$$

where the last two equalities follow from the fact that K is symmetric about the origin and that the support function is positive homogenous respectively. $\square$

The support function can also be used to define a very useful operation defined on the class of convex sets:

Definition 4.21 (Polar set). For any convex set A with support function h_A , the polar set of A is defined as

$$A^\circ := \{y \in \mathbb{R}^d \mid h_A(y) \leq 1\}.$$

Of course, the polar set A° will also be convex by the sublinearity of h_A as a function on $\mathbb{R}^d$. More precisely, for any $x, y \in A^\circ$ and $t \in [0, 1]$ we can write

$$h_A(tx + (1-t)y) \leq th_A(x) + (1-t)h_A(y) \leq t + (1-t) = 1, \quad (4.14)$$

which implies that $tx + (1-t)y \in A^\circ$. Note that taking the polar of a convex set reverses any inclusion relations with other convex sets:

Lemma 4.22. *If $A \subset B$, then $A^\circ \supset B^\circ$.*

Proof. Unpacking the definition of the support function, we see that $A \subset B$ implies that $h_A(u) \leq h_B(u)$ for all $u \in \mathbb{R}^d$. If $y \in B^\circ$ then $h_B(y) \leq 1$, so $h_A(y) \leq h_B(y) \leq 1$, which implies that $y \in A^\circ$. It follows that $A^\circ \supset B^\circ$. $\square$

This allows us to deduce the following useful result:

Lemma 4.23. *If $A \subset \mathbb{R}^d$ is compact and convex, then A° will be a closed convex subset of $\mathbb{R}^d$ with non-empty interior*

Proof. Note that by Lemma 4.18 we know that h_A is a continuous function from $\mathbb{R}^d$ to $\mathbb{R}$. Furthermore, by definition, the polar set A° is equal to the preimage of the closed set $(-\infty, 1] \subset \mathbb{R}$ under the map h_A , so it follows that A° is also closed. Since A is bounded, there exists some large $R > 0$ such that

$$A \subset R\mathbb{B}_2^d.$$

The support function of $R\mathbb{B}_2^d$ is simply defined by $h_{R\mathbb{B}_2^d}(x) = R\|x\|$, so immediately we can deduce that $(R\mathbb{B}_2^d)^\circ = R^{-1}\mathbb{B}_2^d$. Now by Lemma 4.22 we can write

$$R^{-1}\mathbb{B}_2^d \subset A^\circ,$$

hence A° has non-empty interior. As shown by (4.14), convexity follows by sublinearity of h_A as a function on $\mathbb{R}^d$. $\square$

Another nice property is that taking the polar set twice yields the same convex set, provided the original convex set contains the origin:

Lemma 4.24 ([37], Theorem 1.6.1). *For any convex set A , the bipolar $A^{\circ\circ}$ is the closed convex hull of A and 0. In particular, if A is a centrally symmetric compact convex set (or more generally a compact convex set containing the origin) then $A^{\circ\circ} = A$*

A useful fact about the dual body of a centrally-symmetric compact convex set A with non-empty interior, is that the volume of A° is comparable to the reciprocal of the volume of A . More precisely, if $A \subset \mathbb{R}^d$ is a centrally-symmetric convex body we have

$$|A^\circ| |A| \sim_d 1. \quad (4.15)$$

The quantity $|A^\circ| |A|$ is known as the Mahler volume, and was first introduced by Kurt Mahler in [30]. In the same paper, Mahler conjectured the sharp upper and lower bounds for $|A^\circ| |A|$ to be attained respectively for euclidean balls and dual pairs of L^1 and L^∞ -balls. This upper bound is known to be sharp as a consequence of the Blaschke-Santaló's inequality [11] [36], but the sharp lower bound is yet to be established. To achieve crude upper and lower bounds, which will suffice for our purposes, one can directly verify the statement for ellipsoids and then apply the John ellipsoid theorem as stated in [27].

In addition to polar sets, let us introduce another useful operation defined on the class of convex bodies. For any compact convex set $K \subset \mathbb{R}^d$, the projection body of K , usually denoted by ΠK , is defined to be the unique compact convex set with support function

$$h_{\Pi K}(u) := \|u\| |\pi_{u^\perp} K|,$$

for $u \in \mathbb{R}^d$. Although it is not immediate that there exists a convex set with such support function, this follows as a consequence of Theorem 4.19 by observing that $\|u\| |\pi_{u^\perp} K|$ is a real-valued sublinear function on $\mathbb{R}^d$. To see why this last assertion is true, let us further assume that K has non-empty interior and the boundary ∂K has an almost-everywhere defined outward facing unit normal $n: \partial K \rightarrow \mathbb{S}^{d-1}$, then we can write

$$|\pi_{u^\perp} K| = \frac{1}{2} \int_{\partial K} \left| \left(\frac{u}{\|u\|} \right) \cdot n(x) \right| d\sigma_{\partial K}(x). \quad (4.16)$$

It follows immediately that for $u, v \in \mathbb{R}^d$

$$\begin{aligned} h_{\Pi K}(u + v) &= \frac{1}{2} \int_{\partial K} |(u + v) \cdot n(x)| d\sigma_{\partial K}(x) \leq \frac{1}{2} \int_{\partial K} |u \cdot n(x)| + |v \cdot n(x)| d\sigma_{\partial K}(x) \\ &= \left(\frac{1}{2} \int_{\partial K} |u \cdot n(x)| d\sigma_{\partial K}(x) \right) \\ &\quad + \left(\frac{1}{2} \int_{\partial K} |v \cdot n(x)| d\sigma_{\partial K}(x) \right) \\ &= h_{\Pi K}(u) + h_{\Pi K}(v), \end{aligned}$$

so clearly $h_{\Pi K}$ is sublinear and real-valued. In the general case, it is a well-know fact from classical convex geometry that for any compact convex set $K \subset \mathbb{R}^d$, there exists a finite Borel measure on $\mathbb{S}^{d-1}$, often referred to as the surface area measure and denoted $S_{d-1}(K, \cdot)$, such that for any $e \in \mathbb{S}^{d-1}$

$$|\pi_{e^\perp} K| = \int_{\mathbb{S}^{d-1}} |e \cdot v| dS_{d-1}(K, v).$$

So using the same argument as above, we can deduce that $h_{\Pi K}$ is sublinear and real-valued in this more general setting too. Details for the existence of such measures can be found in the first few chapters of [37].

As it turns out, we can use the notions of polar sets and projection bodies to describe the convex set $K(Z)$ introduced above. In the special case where the hypersurface Z is the boundary of a convex body $K \subset \mathbb{R}^d$, using (4.16) we can deduce that the support function of ΠK is defined by

$$h_{\Pi K}(u) = \frac{1}{2} \text{surf}_u(Z). \quad (4.17)$$

Hence, using the definition of the polar set we can write

$$\tilde{K}(Z) = (\Pi K)^\circ,$$

thus

$$K(Z) = (\Pi K)^\circ \cap \mathbb{B}_2^d.$$

It seems natural to ask whether we can do the same for a more general class of hypersurfaces in $\mathbb{R}^d$, which leads us to the statement of this next result:

Lemma 4.25. *Suppose $Z \subset \mathbb{R}^d$ is a hypersurface of finite $(d - 1)$ -dimensional measure with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$. Then there exists a unique compact convex set $K \subset \mathbb{R}^d$ with support function $h(u) = \frac{1}{2} \text{surf}_u(Z)$.*

Proof. Let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be defined by

$$f(u) = \frac{1}{2} \text{surf}_u(Z).$$

By finiteness of $\mathcal{H}^{d-1}(Z)$, we know that

$$\text{surf}_u(Z) = \int_Z |u \cdot n(x)| d\sigma_Z(x) \leq \|u\| \int_Z d\sigma_Z(x) < \infty,$$

thus f is real-valued. Furthermore, f is sublinear since

$$\begin{aligned}\text{surf}_{u+v}(Z) &= \int_Z |(u+v) \cdot n(x)| d\sigma_Z(x) \leq \int_Z |u \cdot n(x)| + |v \cdot n(x)| d\sigma_Z(x) \\ &= \int_Z |u \cdot n(x)| d\sigma_Z(x) + \int_Z |v \cdot n(x)| d\sigma_Z(x) \\ &= \text{surf}_u(Z) + \text{surf}_v(Z).\end{aligned}$$

So by Theorem 4.19 there exists a unique compact convex set $K \subset \mathbb{R}^d$ with support function f . $\square$

Using this we can extend the notion of a projection body for convex bodies to hypersurfaces:

Definition 4.26 (Projection body of a hypersurface). Let $Z \subset \mathbb{R}^d$ be a hypersurface of finite $(d-1)$ -dimensional measure with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$. The **projection body** of Z , denoted by ΠZ , is defined to be the unique compact convex set with support function $h_{\Pi Z}: \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$h_{\Pi Z}(u) = \frac{1}{2} \text{surf}_u(Z)$$

Remark 4.27. Note that if Z is the boundary of a convex body K , then $\Pi Z = \Pi K$, so our new definition of projection body for a hypersurface coincides with the classical definition of the projection body for a convex body.

It follows immediately that $\tilde{K}(Z) = (\Pi Z)^\circ$, thus $K(Z) = (\Pi Z)^\circ \cap \mathbb{B}_2^d$. Moreover,

$$\text{vis}(Z) = |(\Pi Z)^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d}}$$

In the case where $\Pi Z \subset \mathbb{B}_2^d$, by (4.15) we have

$$\text{vis}(Z) \sim_d |\Pi Z|^{\frac{1}{d}}$$

Using Lemma 4.20 we can write

$$\text{surf}_u(Z) = \|u\| |\pi_u(\Pi Z)|.$$

So, in the special case where $\Pi Z \subset \mathbb{B}_2^d$, these two quantities are closely related geometric objects. In the proof of Theorem 4.8, we will apply Guth's visibility theorem for a particular choice of G to get a polynomial hypersurface Z_p , which essentially has the property that $\Pi(Z_p \cap Q) \subset \mathbb{B}_2^d$ for every cube Q in the unit lattice. This should give some indication for how we combine Guth's result with the cylinder estimate above to establish Theorem 4.8.

The main reasons for introducing the directional surface area and visibility are to use Guth's visibility theorem, which guarantees the existence of a polynomial of bounded degree whose zero set has large visibility in each cube of the standard unit lattice. In order to state this result precisely, we cannot use the definition of visibility given above. This is due to the fact that, in some sense, the above formulation of visibility is not continuous on the space of polynomial hypersurfaces. In order to circumvent this issue, we redefine visibility for bounded pieces of polynomial hypersurfaces by averaging the original definition over a small region of neighbouring polynomial hypersurfaces. Below we introduce the mollified versions of directional surface area and visibility, and using some classical theory from convex geometry, we show how to rewrite these new mollified quantities in terms of convex bodies.

4.1.2 Mollified directed volumes and visibility

Although we have introduced the notion of visibility for any compact hypersurfaces, for our purposes we will always be working with subsets of polynomial hypersurfaces in $\mathbb{R}^d$, that is zero sets of polynomials in d real variables. One can identify the space of polynomials in d real variables of degree at most k with $\mathbb{R}^{\binom{d+k}{d}}$. Since the class of polynomial hypersurfaces is invariant under scalar multiplications of the defining polynomials, it will be convenient to work instead with unit sphere $\mathbb{S}^{\binom{d+k}{d}-1} \subset \mathbb{R}^{\binom{d+k}{d}}$. In what follows we make this identification and each polynomial p of degree $\leq k$ is considered as a point on the unit sphere $\mathbb{S}^{\binom{d+k}{d}-1}$. Furthermore, we denote by $B_\varepsilon(p)$ the spherical cap of radius ε around p . The mollification of directional surface area is defined by averaging over spherical caps of radius ε in $\mathbb{S}^{\binom{d+k}{d}}$ with respect to surface measure on the sphere:

Definition 4.28 (Directional surface area - mollified version). Fix $\varepsilon > 0$ and a bounded set $U \subset \mathbb{R}^d$. Let p be a non-zero polynomial in d real variables and define

$$Z_p := \{x \in \mathbb{R}^d \mid p(x) = 0\}.$$

The **mollified directional surface area** of $Z_p \cap U$ in the direction $u \in \mathbb{R}^d$ as

$$\text{surf}_{u,\varepsilon}(Z_p \cap U) := \int_{B_\varepsilon(p)} \left(\int_{Z_{p'} \cap U} |u \cdot n(x)| d\sigma_{Z_{p'}}(x) \right) dp'.$$

The cylinder estimate stated at the beginning of this section immediately implies the following mollified version:

Lemma 4.29 (Mollified cylinder estimate - [24], Lemma 5.1). *Let $\varepsilon > 0$, if $Z_p \subset \mathbb{R}^d$ is a polynomial hypersurface and $T \subset \mathbb{R}^d$ is a 1-tube, then*

$$\text{surf}_{E(T),\varepsilon}(Z_p \cap T) \lesssim_d \deg p,$$

In order to define the mollified version of visibility, as before we begin by associating a fundamental centrally-symmetric compact convex set to each bounded piece of polynomial hypersurface $Z_p \cap U$:

$$K_\varepsilon(Z_p \cap U) := \{u \in \mathbb{B}_2^d : \text{surf}_{u,\varepsilon}(Z_p \cap U) \leq 1\}.$$

As with $K(Z)$, it is not difficult to see that $K_\varepsilon(Z_p \cap U)$ will always be a centrally-symmetric compact convex set in $\mathbb{R}^d$. Furthermore, the d -dimensional volume of $K_\varepsilon(Z_p \cap U)$ will encode some information regarding the geometry of Z_p inside U :

Definition 4.30 (Visibility - mollified version). Fix $\varepsilon > 0$ and a bounded set $U \subset \mathbb{R}^d$. Let Z_p be a polynomial hypersurface, the **mollified visibility** of $Z_p \cap U$ is defined as

$$\text{vis}_\varepsilon(Z_p \cap U) := |K_\varepsilon(Z_p \cap U)|^{-\frac{1}{d}}.$$

Note that, just as we had with the unmollified directional surface area, $\text{surf}_{u,\varepsilon}(Z_p \cap U)$ will be real-valued and sublinear so must be the support function for some compact convex set:

Lemma 4.31. *Fix $\varepsilon > 0$. Let $Z_p \subset \mathbb{R}^d$ be a polynomial hypersurface and $U \subset \mathbb{R}^d$ some bounded set. Then there exists a unique compact convex set $K \subset \mathbb{R}^d$ with support function $h(u) = \frac{1}{2} \text{surf}_{u,\varepsilon}(Z_p \cap U)$.*

Proof. Let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be defined by

$$f(u) = \frac{1}{2} \text{surf}_{u,\varepsilon}(Z_p \cap U).$$

By finiteness of $\mathcal{H}^{d-1}(Z_p \cap U)$, we know that

$$\text{surf}_{u,\varepsilon}(Z_p \cap U) = \int_{B_\varepsilon(p)} \left(\int_{Z_{p'} \cap U} |u \cdot n(x)| d\sigma_{Z_{p'}}(x) \right) dp' < \infty,$$

thus f is real-valued. Furthermore, f is sublinear since

$$\begin{aligned} \text{surf}_{u+v,\varepsilon}(Z_p \cap U) &= \int_{B_\varepsilon(p)} \left(\int_{Z_{p'} \cap U} |(u+v) \cdot n(x)| d\sigma_{Z_{p'}}(x) \right) dp' \\ &\leq \int_{B_\varepsilon(p)} \left(\int_{Z_{p'} \cap U} |u \cdot n(x)| + |v \cdot n(x)| d\sigma_{Z_{p'}}(x) \right) dp' \\ &= \int_{B_\varepsilon(p)} \left(\int_{Z_{p'} \cap U} |u \cdot n(x)| d\sigma_{Z_{p'}}(x) \right) dp' + \int_{B_\varepsilon(p)} \left(\int_{Z_{p'} \cap U} |v \cdot n(x)| d\sigma_{Z_{p'}}(x) \right) dp' \\ &= \text{surf}_{u,\varepsilon}(Z_p \cap U) + \text{surf}_{v,\varepsilon}(Z_p \cap U). \end{aligned}$$

So by Theorem 4.19 there exists a unique compact convex set $K \subset \mathbb{R}^d$ with support function f . $\square$

Using this we can define a mollified version of projection body for hypersurfaces defined above:

Definition 4.32 (Mollified projection body of a hypersurface). Fix $\varepsilon > 0$. Let $Z_p \subset \mathbb{R}^d$ be a polynomial hypersurface and $U \subset \mathbb{R}^d$ some bounded set. The **mollified projection body** of $Z_p \cap U$, denoted by $\Pi_\varepsilon(Z_p \cap U)$, is defined to be the unique compact convex set with support function $h_{\Pi_\varepsilon(Z_p \cap U)}: \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$h_{\Pi_\varepsilon(Z_p \cap U)}(u) = \frac{1}{2} \text{surf}_{u,\varepsilon}(Z_p \cap U).$$

It follows that,

$$\text{vis}_\varepsilon(Z_p \cap U) = |(\Pi_\varepsilon(Z_p \cap U))^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d}}$$

As before, in the case where $\Pi_\varepsilon(Z_p \cap U) \subset \mathbb{B}_2^d$, by (4.15) we have

$$\text{vis}_\varepsilon(Z_p \cap U) \sim_d |\Pi_\varepsilon(Z_p \cap U)|^{\frac{1}{d}},$$

and using Lemma 4.20 we can write

$$\text{surf}_{u,\varepsilon}(Z_p \cap U) = \|u\| |\pi_u(\Pi_\varepsilon(Z_p \cap U))|.$$

As we will see, the polynomial hypersurface Z_p given by a clever application of the visibility theorem, will have the property that $\Pi_\varepsilon(Z_p \cap Q) \subset \mathbb{B}_2^d$ for every cube Q in the unit lattice. This will be exactly what allows us to formulate Theorem 4.8 as a consequence of Guth's visibility theorem.

4.2 Guth's visibility theorem and connections to convex geometry

Let us now restate the far-reaching result of Guth:

Theorem 4.33 (Guth's Visibility Theorem - [24]). *There exists a constant $C_d > 0$ such that, given a non-negative finitely supported function G defined on the unit lattice $\mathcal{Q}$ of cubes in $\mathbb{R}^d$ and sufficiently small $\varepsilon > 0$, there exists a non-zero polynomial p satisfying the following two conditions*

$$\deg p \leq C_d \left(\sum_{Q \in \mathcal{Q}} G(Q)^d \right)^{\frac{1}{d}},$$

and if we set $Z_p = \{x \in \mathbb{R}^d \mid p(x) = 0\}$, then for all $Q \in \mathcal{Q}$ we have

$$G(Q) \leq C_d \text{vis}_\varepsilon(Z_p \cap Q).$$

Using the language developed in the previous section we can use Guth's result to deduce the following statement purely in terms of convex sets:

Theorem 4.34 ([24]). *There exists a constant $c_d > 0$ such that, given a non-negative finitely supported function G defined on the unit lattice $\mathcal{Q}$ of cubes in $\mathbb{R}^d$, there exists a family of centrally-symmetric compact convex sets $\{K(Q)\}_{Q \in \mathcal{Q}}$ in $\mathbb{R}^d$, such that for any 1-tube $T \subset \mathbb{R}^d$*

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| \leq c_d \left(\sum_{Q \in \mathcal{Q}} G(Q)^d \right)^{\frac{1}{d}}$$

and for all $Q \in \mathcal{Q}$ we have

$$G(Q)^d \leq c_d |K(Q)^\circ \cap \mathbb{B}_2^d|^{-1}.$$

Proof. Let p be the polynomial given by Theorem 4.33. The mollified cylinder estimate, stated above as Lemma 4.29, tells us that for any tube $T \in \mathcal{T}$ we have

$$\text{surf}_{E(T), \varepsilon}(Z_p \cap T) \leq \deg p.$$

Set $K(Q) = \Pi_\varepsilon(Z_p \cap Q)$, which was defined above as the unique compact convex set in $\mathbb{R}^d$ with support function $h(u) = \text{surf}_{u, \varepsilon}(Z_p \cap Q)$. Since $\text{surf}_{u, \varepsilon}(Z_p \cap Q)$ is an even function with respect to u , it follows that $K(Q)$ is symmetric about the origin, which implies that

$$\text{surf}_{u, \varepsilon}(Z_p \cap Q) = |\pi_u K(Q)|.$$

Moreover, for any 1-tube $T \subset \mathbb{R}^d$

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| = \sum_{Q: Q \cap T \neq \emptyset} \text{surf}_{E(T), \varepsilon}(Z_p \cap Q) \sim_d \text{surf}_{E(T), \varepsilon}(Z_p \cap T).$$

Hence,

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| \lesssim_d \deg(p) \leq C_d \left(\sum_{Q \in \mathcal{Q}} G(Q)^d \right)^{\frac{1}{d}}.$$

By construction, we can write

$$\text{vis}_\varepsilon(Z_p \cap U) = |K(Q)^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d}},$$

which immediately gives

$$G(Q) \leq C_d |K(Q)^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d}}.$$

□

In the case where each $K(Q)^\circ$ is contained in the unit ball for every $Q \in \text{supp}(G)$, this last statement is a little easier to digest. Indeed, supposing this is the case, then clearly each $K(Q)^\circ$ is compact. Using a combination of Lemmas 4.23 and 4.24, we can then deduce that each $K(Q)$ has non-empty interior, so is in fact a convex body, which implies that $K(Q)^\circ$ is also a convex body. Crude bounds for the Mahler volume give us $|A||A^\circ| \sim_d 1$ for any convex body $A \subset \mathbb{R}^d$. In particular, we can write

$$|K(Q)^\circ \cap \mathbb{B}_2^d|^{-1} = |K(Q)^\circ|^{-1} \sim_d |K(Q)|.$$

Then the conclusion of Theorem 4.34 would be that for any 1-tube $T \subset \mathbb{R}^d$

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| \lesssim_d \left(\sum_{Q \in \mathcal{Q}} G(Q)^d \right)^{\frac{1}{d}}$$

and for all $Q \in \mathcal{Q}$ we have

$$G(Q)^d \lesssim_d |K(Q)|.$$

Intuitive example A simple case I like to keep in mind is when the function G is constant and supported in an N^d cube-shaped collection $\mathcal{C}$ of cubes $Q \in \mathcal{Q}$. Let the family of tubes $\mathcal{T}$ form a grid and such that each tube only intersects cubes from an axis parallel line of cubes in $\mathcal{C}$. If $d = 2$, this last condition means that each tube will only intersect cubes along a row or column of $\mathcal{C}$. Define each $K(Q)$ to be a centred cube of side length $G(Q) = a$, then clearly $|K(Q)| = G(Q)^d$ for each Q . The sum $\left(\sum_{Q \in \mathcal{Q}} G(Q)^d \right)^{\frac{1}{d}}$ will be equal to the side length of the large cube formed by taking the union of all cubes $Q \in \mathcal{C}$, which is clearly Na . There are N cubes along any row or column in $\mathcal{C}$, so we also have

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| = \sum_{Q: Q \cap T \neq \emptyset} G(Q) = \sum_{Q: Q \cap T \neq \emptyset} a = Na.$$

Hence,

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| = \left(\sum_{Q \in \mathcal{Q}} G(Q)^d \right)^{\frac{1}{d}}.$$

At the end of Sections 4.3 and 4.4, we applying Theorem 4.34 with $G(Q) = \lambda M(Q)^{\frac{d}{m}}$ for large enough $\lambda > 0$, to deduce Theorems 4.8 and 4.14 respectively. Essentially, we choose $\lambda > 0$ large enough with respect the maximum value of $M(Q)^{-1}$ so that each $K(Q)^\circ$ is contained in the unit ball, which in turn implies that each $K(Q)$ is a convex body. Let us now derive a

condition on the input function $G(Q)$ to ensure that each $K(Q)^\circ$ coming from Theorem 4.34 is contained in the unit ball, which will help us later to choose $\lambda > 0$ suitable large. This scaling argument also appears in [24], [15] and [39].

To begin, we state a geometric lemma which follows as a simple consequence of the John ellipsoid theorem as stated in [27]. This next result can be understood as a preliminary, which combined with the cylinder estimate would prove Lemma 3 or Lemma 6 in [15]. In particular, the proof we provide is very similar to the proof of Lemma 3 in [15].

Lemma 4.35. *Let $K \subset \mathbb{R}^d$ be a closed, centrally symmetric convex set with non-empty interior and suppose there exists a 1-dimensional subspace $V \subset \mathbb{R}^d$ such that*

$$|K \cap V| > 1.$$

Then there exists another 1-dimensional subspace $U \subset \mathbb{R}^d$ such that

$$|K \cap \mathbb{B}_2^d| > d^{-\frac{d}{2}} \kappa_d |K \cap U|^{d-1}.$$

Note that in the above statement $|K \cap \mathbb{B}_2^d|$ denotes the d -dimensional measure of $K \cap \mathbb{B}_2^d$, which has non-empty interior by assumption.

Proof. Since K is a closed, centrally symmetric, convex subset of $\mathbb{R}^d$ with non-empty interior, we have that $K \cap \mathbb{B}_2^d$ is a convex body. In particular, we can apply the John ellipsoid theorem to get a d -dimensional ellipsoid $E \subset \mathbb{R}^d$ such that

$$E \subset K \cap \mathbb{B}_2^d \subset d^{\frac{1}{2}} E.$$

Let $\ell_1 \leq \dots \leq \ell_d$ be the lengths of the principal axes of E . Combining the fact that $K \cap \mathbb{B}_2^d \subset d^{\frac{1}{2}} E$ with the existence of a 1-dimensional subspace $V \subset \mathbb{R}^d$ satisfying

$$|K \cap V| > 1,$$

we can immediately deduce that $d^{\frac{1}{2}} \ell_d > 1$. Furthermore, there must exist another 1-dimensional subspace $U \subset \mathbb{R}^d$ such that

$$|K \cap U| \leq d^{\frac{1}{2}} \ell_1.$$

It follows that $|K \cap U| \leq d^{\frac{1}{2}} \ell_j$ for every j . Using the fact that $E \subset K \cap \mathbb{B}_2^d$, we can write

$$\begin{aligned} |K^\circ \cap \mathbb{B}_2^d| &\geq |E| = \ell_1 \cdots \ell_d \kappa_d > \ell_1 \cdots \ell_{d-1} d^{-\frac{1}{2}} \kappa_d \geq \left(|K \cap U| d^{-\frac{1}{2}} \right)^{d-1} d^{-\frac{1}{2}} \kappa_d \\ &= |K \cap U|^{d-1} d^{-\frac{d}{2}} \kappa_d, \end{aligned}$$

which gives us the desired bound. □

For our purposes the contrapositive statement will be more convenient:

Corollary 4.36. *Let $K \subset \mathbb{R}^d$ be a closed, centrally symmetric, convex set with non-empty interior and suppose that for all 1-dimensional subspaces $U \subset \mathbb{R}^d$ we have*

$$|K \cap \mathbb{B}_2^d| \leq d^{-\frac{d}{2}} \kappa_d |K \cap U|^{d-1}.$$

Then for every 1-dimensional subspace $V \subset \mathbb{R}^d$ we also have

$$|K \cap V| \leq 1.$$

Clearly if the conclusion to this corollary holds, then $K \subset \mathbb{B}_2^d$. As we noted earlier, it seems desirable to have $K(Q)^\circ \subset \mathbb{B}_2^d$ for each convex set $K(Q)$ given by Theorem 4.34. In light of this, it would make sense to apply Corollary 4.36 for each of the $K(Q)^\circ$. Theorem 4.34 gives us a family of compact convex sets $\{K(Q)\}_Q$, such that for any 1-tube $T \subset \mathbb{R}^d$ we have

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| \leq c_d \left(\sum_{Q \in \mathcal{Q}} G(Q)^d \right)^{\frac{1}{d}},$$

and for all $Q \in \mathcal{Q}$ we have

$$G(Q)^d \leq c_d |K(Q)^\circ \cap \mathbb{B}_2^d|^{-1}.$$

In order to apply Corollary 4.36 with $K = K(Q)^\circ$, we need to first ensure that each $K(Q)^\circ$ is closed, centrally symmetric, convex and has non-empty interior. Moreover, we need that for each Q

$$|K(Q)^\circ \cap \mathbb{B}_2^d| \leq d^{-\frac{d}{2}} \kappa_d |K \cap U|^{d-1} \quad (4.18)$$

for all 1-dimensional subspaces $U \subset \mathbb{R}^d$. The first of four conditions follows directly from Lemma 4.23, since we know that each $K(Q)$ is compact and convex, and by definition of polar sets $K(Q)$ being centrally symmetric implies the same for $K(Q)^\circ$. For what kind of function G , would (4.18) hold for all 1-dimensional subspaces $U \subset \mathbb{R}^d$? This question motivates the following lemma:

Lemma 4.37. *Let $\mathcal{Q}$ denote the unit lattice of cubes in $\mathbb{R}^d$, and $c_d > 0$ the constant appearing in the statement of Theorem 4.34. Suppose we have some finitely supported non-negative function $G: \mathcal{Q} \rightarrow \mathbb{R}$ satisfying*

$$G(Q)^{\frac{d}{d-1}} \geq d^{\frac{d}{2(d-1)}} \kappa_d^{-\frac{1}{d-1}} c_d^{\frac{d}{d-1}} \|G\|_{\ell^d(\mathcal{Q})}, \quad (4.19)$$

and $\{K(Q)\}_{Q \in \mathcal{Q}}$ is family of centrally-symmetric compact convex sets in $\mathbb{R}^d$ such that for any 1-tube $T \subset \mathbb{R}^d$

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| \leq c_d \|G\|_{\ell^d(\mathcal{Q})}$$

and for all $Q \in \mathcal{Q}$ we have

$$G(Q)^d \leq c_d |K(Q)^\circ \cap \mathbb{B}_2^d|^{-1}.$$

Then it follows that $K(Q)^\circ \subset \mathbb{B}_2^d$ for all $Q \in \mathcal{Q}$.

In order to prove this let us first state and prove a small lemma which will allow us to compare the quantities $|\pi_{E(T)} K(Q)|$ and $|K(Q)^\circ \cap E(T)|$:

Lemma 4.38. *Suppose that K is a compact convex subset of $\mathbb{R}^d$ which is symmetric about the origin. It follows that for any 1-dimensional subspace $U \subset \mathbb{R}^d$*

$$|\pi_U K| = 4 |K^\circ \cap U|^{-1}.$$

Proof. Firstly, note that since K is compact, convex and symmetric about the origin, the same will be true for $\pi_U K$ as a subset of U . More precisely, letting $u \in \mathbb{R}^d$ be a unit vector spanning U , we can write

$$\pi_U K = \{tu \mid t \in [-a, a]\},$$

for some $a \in [0, \infty)$. So we have $|\pi_U K| = 2a$. Let h_K denote support function of K , then by Lemma 4.20 for any $x \in U$ we have

$$h_K(x) = \frac{\|x\|}{2} |\pi_U K|.$$

Now using this along with the definition of the polar set, we can write

$$\begin{aligned} K^\circ \cap U &= \{x \in U \mid h_K(x) \leq 1\} = \{x \in U \mid |\pi_U K| \leq 2\|x\|^{-1}\} \\ &= \{tu \mid t \in \mathbb{R}, |\pi_U K| \leq 2t^{-1}\} \\ &= \{tu \mid 2a \leq 2t^{-1}\} \\ &= \{tu \mid t \in [-a^{-1}, a^{-1}]\}. \end{aligned}$$

It follows that $|K^\circ \cap U| = 2a^{-1}$, thus we have

$$|\pi_U K| = 4 |K^\circ \cap U|^{-1},$$

which concludes the proof □

Now we are ready to prove Lemma 4.37:

Proof of Lemma 4.37. Using (4.19), we can deduce that

$$\begin{aligned} \sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| &\leq c_d \|G\|_{\ell^d(\mathcal{Q})} \leq d^{-\frac{d}{2(d-1)}} \kappa_d^{\frac{1}{d-1}} c_d^{-\frac{1}{d-1}} G(Q)^{\frac{d}{d-1}} \\ &\leq d^{-\frac{d}{2(d-1)}} \kappa_d^{\frac{1}{d-1}} |K(Q)^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d-1}} \\ &\leq 4 d^{-\frac{d}{2(d-1)}} \kappa_d^{\frac{1}{d-1}} |K(Q)^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d-1}}. \end{aligned}$$

In particular for any 1-tube $T \subset \mathbb{R}^d$ and Q such that $Q \cap T \neq \emptyset$

$$|\pi_{E(T)} K(Q)| \leq 4 d^{-\frac{d}{2(d-1)}} \kappa_d^{\frac{1}{d-1}} |K(Q)^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d-1}}.$$

Since each $K(Q)$ is compact, convex and symmetric about the origin, we can apply Lemma 4.38 to get

$$|\pi_{E(T)} K(Q)| = 4 |K(Q)^\circ \cap E(T)|^{-1}.$$

It follows that for any $T \subset \mathbb{R}^d$ and Q such that $Q \cap T \neq \emptyset$

$$|K(Q)^\circ \cap E(T)|^{-1} \leq d^{-\frac{d}{2(d-1)}} \kappa_d^{\frac{1}{d-1}} |K(Q)^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d-1}},$$

or equivalently

$$|K(Q)^\circ \cap \mathbb{B}_2^d| \leq d^{-\frac{d}{2}} \kappa_d |K(Q)^\circ \cap E(T)|^{d-1},$$

This is equivalent to the statement that for all 1-dimensional subspaces $U \subset \mathbb{R}^d$

$$|K(Q)^\circ \cap \mathbb{B}_2^d| \leq d^{-\frac{d}{2}} \kappa_d |K(Q)^\circ \cap U|^{d-1},$$

Since $K(Q)$ is centrally symmetric, the same is true for $K(Q)^\circ$, so by Corollary 4.36 we have that for every 1-dimensional subspace $V \subset \mathbb{R}^d$

$$|K(Q)^\circ \cap V| \leq 1,$$

which implies that $K(Q)^\circ \subset \mathbb{B}_2^d$. □

As we will see below, Theorems 4.8 and 4.14 follow from more or less direct applications of Theorem 4.34 for functions G satisfying (4.19).

4.3 Factorisation of the endpoint multilinear Keakeya inequality for tubes

We now discuss how to prove the endpoint multilinear Keakeya inequality for tubes, stated above as Theorem 4.1, following ideas from [15]. Given families of 1-tubes $\mathcal{T}_1, \dots, \mathcal{T}_m$ in $\mathbb{R}^d$, we are interested in proving the following estimate:

$$\int_{\mathbb{R}^d} \left(\sum_{T_1 \in \mathcal{T}_1} \cdots \sum_{T_m \in \mathcal{T}_m} \left(\prod_{j=1}^m \alpha_{T_j} \chi_{T_j}(x) \right) |E(T_1) \wedge \cdots \wedge E(T_m)| \right)^{\frac{1}{m-1}} dx \leq C_{d,m} \prod_{j=1}^m \left(\sum_{T_j \in \mathcal{T}_j} \alpha_{T_j} \right)^{\frac{1}{m-1}},$$

As discussed in the previous chapter, this can be reduced to proving the existence of a certain factorisation, which we formalised earlier as Theorem 4.2. With the set up from above, the precise statement is that for any finitely supported function $M: \mathcal{Q} \rightarrow \mathbb{R}_{\geq 0}$ with $\|M\|_{\ell^m(\mathcal{Q})} = 1$, there exist functions $S_j: \mathcal{Q} \times \mathcal{T}_j \rightarrow \mathbb{R}_{\geq 0}$, such that whenever $T_j \cap Q \neq \emptyset$ for all j

$$|E(T_1) \wedge \cdots \wedge E(T_m)| M(Q)^m \lesssim_{d,m} S_1(Q, T_1) \cdots S_d(Q, T_m), \quad (4.20)$$

and for every j

$$\max_{T_j \in \mathcal{T}_j} \sum_{Q \in \mathcal{Q}: Q \cap T_j \neq \emptyset} S_j(Q, T_j) \lesssim_{d,m} 1. \quad (4.21)$$

The aim of this section is to review a proof of this last statement using arguments from [15] and highlight certain connections to convex geometry which do not seem to have been explored in this context. These connections manifest themselves partly in the statement and proof of Theorem 4.8, which provides us with a family of centrally-symmetric convex bodies $\{K(Q)\}_{\text{supp}(M)}$ such that for any 1-tube $T \subset \mathbb{R}^d$

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)}(K(Q))| \lesssim_{d,m} 1, \quad (4.22)$$

and for each $Q \in \text{supp}(M)$

$$|K(Q)|^{\frac{1}{m}} \gtrsim_{d,m} M(Q). \quad (4.23)$$

Let us begin by using this family to construct a factorisation satisfying the conditions stated above. For $1 \leq j \leq m$ and $T_j \in \mathcal{T}$, we simply define

$$S_j(Q, T_j) := |\pi_{E(T_j)}(K(Q))|.$$

These functions satisfy (4.21) as a direct consequence of (4.22). To verify (4.20), as mentioned at the beginning of this chapter, we appeal to an elementary geometric inequality, which can be seen as a special case of the affine-invariant Finner inequality. Indeed, combining Lemma 4.9 for $E = K(Q)$ with (4.23) yields

$$|E(T_1) \wedge \cdots \wedge E(T_m)| M(Q)^m \lesssim_{d,m} |E(T_1) \wedge \cdots \wedge E(T_m)| |K(Q)| \leq |\pi_{\omega_1} K(Q)| \cdots |\pi_{\omega_m} K(Q)|,$$

For completeness we provide a direct proof of Lemma 4.9 here using change of variables.

Proof of Lemma 4.9. Firstly note that

$$|E| = \int_{\mathbb{R}^d} \chi_E(x) dx \leq \int_{\mathbb{R}^d} \prod_{i=1}^d \chi_{\pi_{\omega_i}^{-1}(\pi_{\omega_i} E)}(x) dx = \int_{\mathbb{R}^d} \prod_{i=1}^d \chi_{\pi_{\omega_i} E}(\pi_{\omega_i} x) dx.$$

Let A the matrix with rows $\omega_1, \dots, \omega_d$ and define $v_1, \dots, v_d$ to be the rows of A^{-1} , so in particular we have $\langle v_i, \omega_j \rangle$ equal to 1 if $i = j$ and zero otherwise. It follows that for each i and any $y \in \mathbb{R}^d$

$$\pi_{\omega_i}(A^{-1}y) = \pi_{\omega_i}(v_1 y_1 + \cdots + v_d y_d) = y_i \omega_i.$$

Making the change of variables $y = Ax$ yields

$$\begin{aligned} \int_{\mathbb{R}^d} \prod_{i=1}^d \chi_{\pi_{\omega_i} E}(\pi_{\omega_i} x) dx &= \int_{\mathbb{R}^d} \prod_{i=1}^d \chi_{\pi_{\omega_i} E}(\pi_{\omega_i}(A^{-1}y)) |\omega_1 \wedge \cdots \wedge \omega_m|^{-1} dy \\ &= \int_{\mathbb{R}^d} \prod_{i=1}^d \chi_{\pi_{\omega_i} E}(y_i \omega_i) |\omega_1 \wedge \cdots \wedge \omega_m|^{-1} dy \\ &= |\omega_1 \wedge \cdots \wedge \omega_m|^{-1} \prod_{i=1}^d \int_{\mathbb{R}} \chi_{\pi_{\omega_i} E}(y_i) dy_i \\ &= |\omega_1 \wedge \cdots \wedge \omega_m|^{-1} \prod_{i=1}^d |\pi_{\omega_i} E|. \end{aligned}$$

It follows that

$$|E| \leq |\omega_1 \wedge \cdots \wedge \omega_m|^{-1} \prod_{i=1}^d |\pi_{\omega_i} E|,$$

so dividing both sides by $|\omega_1 \wedge \cdots \wedge \omega_m|^{-1}$ we get the desired result. $\square$

Let us next discuss exactly how to prove the existence of centrally-symmetric convex bodies satisfying conditions (4.22) and (4.23).

4.3.1 Proof of Theorem 4.8

Suppose are given a finitely supported function $M: \mathcal{Q} \rightarrow \mathbb{R}_{\geq 0}$ with $\|M\|_{\ell^m(\mathcal{Q})} = 1$. The desired result follows from an application of Theorem 4.34 with $G(Q) = \lambda M(Q)^{\frac{m}{d}}$ for a suitable large $\lambda > 0$. To begin, choose λ large enough so that for all $Q \in \text{supp}(M)$

$$\lambda \geq d^{\frac{d}{2}} \kappa_d^{-1} c_d^d M(Q)^{-m}, \quad (4.24)$$

where c_d is the potentially very large constant appearing in the statement of Theorem 4.34. Rearranging this we get

$$\lambda^{\frac{d}{d-1}} M(Q)^{\frac{m}{d-1}} \geq d^{\frac{d}{2(d-1)}} \kappa_d^{-\frac{1}{d-1}} c_d^{\frac{d}{d-1}} \lambda.$$

By construction we have

$$\|G\|_{\ell^d(\mathcal{Q})} = \|\lambda M^{\frac{m}{d}}\|_{\ell^d(\mathcal{Q})} = \lambda \|M\|_{\ell^m(\mathcal{Q})}^{\frac{m}{d}} = \lambda,$$

so we can rewrite the previous inequality as

$$G(Q)^{\frac{d}{d-1}} \geq d^{\frac{d}{2(d-1)}} \kappa_d^{-\frac{1}{d-1}} c_d^{\frac{d}{d-1}} \|G\|_{\ell^d(\mathcal{Q})}. \quad (4.25)$$

Apply Theorem 4.34 with the above defined finitely supported non-negative function $G(Q)$, to get a family of centrally-symmetric compact convex sets $\{L(Q)\}_Q$ such that for any 1-tube $T \subset \mathbb{R}^d$

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} L(Q)| \leq c_d \lambda \quad (4.26)$$

and for all $Q \in \mathcal{Q}$ we have

$$\lambda^d M(Q)^m \leq c_d |L(Q)^\circ \cap \mathbb{B}_2^d|^{-1}. \quad (4.27)$$

By Lemma 4.37, we can deduce that each of the $L(Q)^\circ$ lies within the unit ball. Combining lemmas 4.23, 4.22 and 4.24 with the assumption that each $L(Q)$ is compact, we conclude that $L(Q)$ and $L(Q)^\circ$ are convex bodies for each Q in the support of M . Using crude bounds for the Mahler volume we can then write

$$|L(Q)^\circ \cap \mathbb{B}_2^d|^{-1} = |L(Q)^\circ|^{-1} \sim_d |L(Q)|.$$

For each Q in the support of M set $K(Q) = \lambda^{-1} L(Q)$, then conditions (4.26) and (4.27) become

$$\sum_{Q: Q \cap T \neq \emptyset} |\pi_{E(T)} K(Q)| \leq c_d,$$

for any 1-tube $T \subset \mathbb{R}^d$, and for all $Q \in \text{supp}(M)$

$$M(Q)^m \leq c_d |K(Q)|,$$

where we have simply used the fact

$$\lambda^{-1} |\pi_{E(T)} L(Q)| = |\pi_{E(T)} (\lambda^{-1} L(Q))|,$$

and

$$\lambda^{-d} |L(Q)| = |\lambda^{-1} L(Q)|.$$

This concludes the proof.

4.4 Factorisation of the endpoint multilinear Keakeya inequality for k_j -planes

Now we will adapt the methods presented above to prove the endpoint multilinear Keakeya inequality for k_j -planes, which we stated above as Theorem 4.10. Our arguments more or less follow the original proof of Zhang but will be formulated in terms of factorisations as was done in [15]. I first came across the details of this proof in a set of course notes - [14]. As with the previous example, the contribution we give here is to highlight certain connections to convex geometry which do not seem to have been explored in this setting. Given families of 1-slabs $\mathcal{P}_1, \dots, \mathcal{P}_m$ in $\mathbb{R}^d$ of dimensions $k_1, \dots, k_m$ respectively with $\sum_{j=1}^m k_j = d$, we are interested in proving the following estimate

$$\int_{\mathbb{R}^d} \left(\sum_{P_1 \in \mathcal{P}_1} \cdots \sum_{P_m \in \mathcal{P}_m} \left(\prod_{j=1}^m \alpha_{P_j} \chi_{P_j}(x) \right) |E(P_1) \wedge \cdots \wedge E(P_m)| \right)^{\frac{1}{m-1}} dx \lesssim_{d,m} \prod_{j=1}^m \left(\sum_{P_j \in \mathcal{P}_j} \alpha_{P_j} \right)^{\frac{1}{m-1}}.$$

As discussed in the previous chapter, this can also be reduced to proving the existence of a certain factorisation, which we formalised earlier as Theorem 4.11. With the set up from above, the precise statement is that for any finitely supported function $M: \mathcal{Q} \rightarrow \mathbb{R}_{\geq 0}$ with $\|M\|_{\ell^m(\mathcal{Q})} = 1$, there exist a functions $S_j: \mathcal{Q} \times \mathcal{P}_j \rightarrow \mathbb{R}_{\geq 0}$, such that whenever $P_j \cap Q \neq \emptyset$ for all j

$$|E(P_1) \wedge \cdots \wedge E(P_m)| M(Q)^m \lesssim_{d,m} S_1(Q, P_1) \cdots S_m(Q, P_m), \quad (4.28)$$

and for all j and $P_j \in \mathcal{P}_j$

$$\sum_{Q \in \mathcal{Q}: Q \cap P_j \neq \emptyset} S_j(Q, P_j) \lesssim_{d,m} \left(\sum_Q M(Q)^m \right)^{\frac{1}{m}}. \quad (4.29)$$

The aim of this section is to discuss a proof of this last statement via Theorem 4.14, which provides us with a family of centrally-symmetric, compact, convex subsets $\{K(Q)\}_{sup(M)}$ such that

$$\max_{P_j \in \mathcal{P}_j} \sum_{Q: Q \cap P_j \neq \emptyset} |\pi_{E(P_j)} K(Q)| \lesssim_{d,m} 1, \quad (4.30)$$

and for each $Q \in \text{supp}(G)$

$$|K(Q)|^{\frac{1}{m}} \gtrsim_{d,m} M(Q). \quad (4.31)$$

Let us begin by using this family to construct a factorisation satisfying the conditions stated above. For $1 \leq j \leq m$ and any k_j -dimensional 1-slab P_j , define

$$S_j(Q, P_j) := |\pi_{E(P_j)}(K(Q))|.$$

These functions satisfy (4.29) as a direct consequence of (4.30). To verify (4.28), we apply Lemma 4.15 with $E = K(Q)$ combined with (4.31) to get

$$|E(P_1) \wedge \cdots \wedge E(P_m)| M(Q)^m \lesssim_{d,m} |E(P_1) \wedge \cdots \wedge E(P_m)| |K(Q)| \leq |\pi_{E(P_1)} K(Q)| \cdots |\pi_{E(P_m)} K(Q)|.$$

For completeness we now provide a simple direct proof of Lemma 4.15.

Proof of Lemma 4.15. The proof is via a simple change of variables. Firstly note that

$$|E| = \int_{\mathbb{R}^d} \chi_E(x) dx \leq \int_{\mathbb{R}^d} \prod_{j=1}^m \chi_{\pi_{V_j}^{-1}(\pi_{V_j} E)}(x) dx = \int_{\mathbb{R}^d} \prod_{j=1}^m \chi_{(\pi_{V_j} E)}(\pi_{V_j} x) dx.$$

Begin by fixing an orthonormal basis $v_1, \dots, v_{k_1}$ for V_1 . Next, choose an orthonormal basis $v_{k_1+1}, \dots, v_{k_1+k_2}$ for V_2 . Continue in this way until we have vectors $v_1, \dots, v_d$ such that for $j = 2, \dots, m$ vectors $v_{k_{j-1}+1}, \dots, v_{k_{j-1}+k_j}$ forms an orthonormal basis for V_j . In particular setting $k_0 = 0$ we have

$$|V_1 \wedge \dots \wedge V_m| = \left| \bigwedge_{j=1}^m \bigwedge_{i=1}^{k_j} v_{k_{j-1}+i} \right|.$$

Let A the matrix with rows $v_1, \dots, v_d$ and define $u_1, \dots, u_d$ to be the rows of A^{-1} , so in particular we have $\langle u_i, v_j \rangle$ equal to 1 if $i = j$ and zero otherwise. It follows that any $y \in \mathbb{R}^d$

$$\pi_{V_1}(A^{-1}y) = \pi_{V_1}(u_1 y_1 + \dots + u_d y_d) = \sum_{i=1}^{k_1} y_i v_i,$$

and for each $j \geq 2$

$$\pi_{V_j}(A^{-1}y) = \pi_{V_j}(u_1 y_1 + \dots + u_d y_d) = \sum_{i=k_{j-1}+1}^{k_{j-1}+k_j} y_i v_i.$$

Making the change of variables $y = Ax$ yields

$$\begin{aligned} \int_{\mathbb{R}^d} \prod_{i=1}^m \chi_{\pi_{V_j} E}(\pi_{V_j} x) dx &= \int_{\mathbb{R}^d} \prod_{i=1}^m \chi_{\pi_{V_j} E}(\pi_{V_j} (A^{-1}y)) |V_1 \wedge \dots \wedge V_m|^{-1} dy \\ &= \int_{\mathbb{R}^d} \chi_{\pi_{V_1} E} \left(\sum_{i=1}^{k_1} y_i v_i \right) \prod_{i=2}^m \chi_{\pi_{V_j} E} \left(\sum_{i=k_{j-1}+1}^{k_{j-1}+k_j} y_i v_i \right) |V_1 \wedge \dots \wedge V_m|^{-1} dy \\ &= |V_1 \wedge \dots \wedge V_m|^{-1} \int_{\mathbb{R}^d} \chi_{\pi_{V_1} E} \left(\sum_{i=1}^{k_1} y_i v_i \right) \prod_{i=2}^m \chi_{\pi_{V_j} E} \left(\sum_{i=k_{j-1}+1}^{k_{j-1}+k_j} y_i v_i \right) dy. \end{aligned}$$

For each $j \geq 2$, since $v_{k_{j-1}+1}, \dots, v_{k_{j-1}+k_j}$ forms an orthonormal basis for V_j we have

$$\chi_{\pi_{V_j} E} \left(\sum_{i=k_{j-1}+1}^{k_{j-1}+k_j} y_i v_i \right) = \chi_{\pi_{V_j} E} (y_{k_{j-1}+1}, \dots, y_{k_{j-1}+k_j}),$$

moreover

$$\chi_{\pi_{V_1} E} \left(\sum_{i=1}^{k_1} y_i v_i \right) = \chi_{\pi_{V_1} E} (y_1, \dots, y_{k_1}).$$

It follows that

$$\begin{aligned}
& \int_{\mathbb{R}^d} \chi_{\pi_{V_1} E} \left(\sum_{i=1}^{k_1} y_i v_i \right) \prod_{i=2}^m \chi_{\pi_{V_j} E} \left(\sum_{i=k_{j-1}+1}^{k_{j-1}+k_j} y_i v_i \right) dy \\
&= \int_{\mathbb{R}^d} \chi_{\pi_{V_1} E} (y_1, \dots, y_{k_1}) \prod_{i=2}^m \chi_{\pi_{V_j} E} (y_{k_{j-1}+1}, \dots, y_{k_{j-1}+k_j}) dy \\
&= \left(\int_{\mathbb{R}^{k_1}} \chi_{\pi_{V_1} E} (y_1, \dots, y_{k_1}) dy_1, \dots, dy_{k_1} \right) \prod_{i=2}^m \left(\int_{\mathbb{R}^{k_j}} \chi_{\pi_{V_j} E} (y_{k_{j-1}+1}, \dots, y_{k_{j-1}+k_j}) dy_{k_{j-1}+1}, \dots, dy_{k_j} \right) \\
&= \prod_{j=1}^m |\pi_{V_j} E|.
\end{aligned}$$

Hence,

$$|E| \leq |V_1 \wedge \dots \wedge V_m|^{-1} \prod_{j=1}^m |\pi_{V_j} E|.$$

□

We now discuss how to prove Theorem 4.14 via a clever application of Guth's visibility theorem. The directional surface area played a key role in the previous section, we begin by introducing a more general quantity better adapted to the setting of k_j -planes.

4.4.1 Directional volume

Recall that for any hypersurface $Z \subset \mathbb{R}^d$ with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$, we define the directional surface area via the following integral formula

$$\text{surf}_u(Z) := \int_Z |u \cdot n(x)| d\sigma_Z(x).$$

Observe that we can rewrite this as

$$\text{surf}_u(Z) := \int_Z |u^\perp \wedge n(x)| d\sigma_Z(x),$$

where $|u^\perp \wedge n(x)|$ is defined to be equal to $|v_1 \wedge \dots \wedge v_{d-1} \wedge n(x)|$ for any orthonormal basis $\{v_1, \dots, v_{d-1}\}$ of $u^\perp$. As mentioned above - see Definition 4.12 - we use the following generalisation of directional surface area introduced implicitly by Zhang in [39]:

$$\text{surf}_V(Z) := \int_Z \dots \int_Z |V^\perp \wedge n(x_1) \wedge \dots \wedge n(x_k)| d\sigma_Z(x_1) \dots d\sigma_Z(x_k).$$

By Lemma 4.25, for any $(d-1)$ -dimensional hypersurface $Z \subset \mathbb{R}^d$ of finite measure with an almost everywhere defined unit normal, there exists a unique centrally symmetric compact convex set $\Pi Z \subset \mathbb{R}^d$, which we refer to as the projection body of Z , such that

$$\text{surf}_u(Z) = |\pi_u \Pi Z|.$$

In what follows, we show that for $1 \leq k \leq d$ and any k -dimensional subspace $V \subset \mathbb{R}^d$ one can also write

$$\text{surf}_V(Z) = k! |\pi_V(\Pi Z)|.$$

Zonoids and volume formulas In convex geometry we say that any polytope P is a zonotope if it is a linear combination of line segments, in the sense of Minkowski summation. That is, one can write

$$P = \sum_{i=1}^m \lambda_i L_i,$$

for line segments $L_1, \dots, L_m$ and positive numbers $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. For $\delta \ll 1$, let $L_1, \dots, L_N$ be unit line segments centred at the origin with directions forming a δ -separated subset of $\mathbb{S}^{d-1}$ with $N \sim_d \delta^{-(d-1)}$, and define the following polytope

$$P_\delta = \sum_{i=1}^N \lambda_i L_i.$$

It is not difficult to see that in some sense P_δ gets closer and closer to the sphere $\mathbb{S}^{d-1}$ as $\delta \downarrow 0$. This convergence can be describes precisely using the notion of Hausdorff distance:

Definition 4.39 (Hausdorff distance). For any two non-empty subsets $A, B \subset \mathbb{R}^d$, the **Hausdorff distance** between A and B , denoted by $d_H(A, B)$ is defined by

$$d_H(A, B) = \inf \{ \varepsilon \geq 0 \mid A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon \},$$

where

$$X_\varepsilon := \bigcup_{x \in X} \{ y \in \mathbb{R}^d \mid \|x - y\|_2 \leq \varepsilon \}.$$

Taking limits of zonotopes with respect to Hausdorff distance yields a much larger class of convex sets with various nice properties:

Definition 4.40 (Zonoids). We say that a non-empty compact convex set $K \subset \mathbb{R}^d$ is a **zonoid** if it can be written as the limit of a sequence of zonotopes with respect to Hausdorff distance.

A canonical example of such a convex set is the sphere, but more generally the class of ellipsoids is contained in the class of zonoids. The following result gives us a useful characterisation for a the class of zonoids:

Theorem 4.41 ([37], Theorem 3.5.3). *A non-empty compact convex set $K \subset \mathbb{R}^d$ centred at the origin is a zonoid if and only if one can write*

$$h_K(u) = \int_{\mathbb{S}^{d-1}} |u \cdot v| d\rho(v), \tag{4.32}$$

for some finite even measure ρ on $\mathbb{S}^{d-1}$.

For simplicity, we introduce the following standard terminology:

Definition 4.42 (Generating measure). If $K \subset \mathbb{R}^d$ is a zonoid centred at the origin with support function given by (4.32), for some even measure ρ , then we say that ρ is the **generating measure** of K .

We are justified here in saying "the" generating measure because even measures satisfying (4.32) are always unique - see [37], Theorem 3.5.4. Earlier we encountered an example of a zonoid, namely the projection body ΠK of a non-empty compact convex set $K \subset \mathbb{R}^d$. Recall that using Lemma 4.25, we were able to extend the classical notion of projection body to hypersurfaces $Z \subset \mathbb{R}^d$ of finite $(d-1)$ -dimensional measure with almost-everywhere defined unit normals. As it turns out, ΠZ will be a zonoid for any such $Z \subset \mathbb{R}^d$:

Lemma 4.43. *For any hypersurface $Z \subset \mathbb{R}^d$ of finite $(d-1)$ -dimensional measure with an almost-everywhere defined unit normal, the unique compact convex set ΠZ with support function $h_{\Pi Z}(u) = \text{surf}_u(Z)$ is a zonoid.*

Proof. By a simple change of variables we see that

$$h_{\Pi Z}(u) = \frac{1}{2} \int_Z |u \cdot n(x)| d\sigma_Z(x) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} |u \cdot v| dn_*[\sigma_Z](v),$$

where $n_*[\sigma_Z]$ denotes the pushforward of σ_Z under n , defined for any measurable $A \subset \mathbb{S}^{d-1}$ as

$$n_*[\sigma_Z](A) = \sigma_Z(n^{-1}(A)).$$

If $n_*[\sigma_Z]$ is even, this tells us that ΠZ is a zonoid with generating measure $\frac{n_*[\sigma_Z]}{2}$. Although $n_*[\sigma_Z]$ is not in general an even measure, note that $n_*[\sigma_Z] + (-n)_*[\sigma_Z]$ is always even, where $-n$ denotes the inward facing normal. Moreover,

$$\int_{\mathbb{S}^{d-1}} |u \cdot v| d(-n)_*[\sigma_Z] = \int_Z |u \cdot -n(x)| d\sigma_Z(x) = \int_Z |u \cdot n(x)| d\sigma_Z(x) = \int_{\mathbb{S}^{d-1}} |u \cdot v| dn_*[\sigma_Z],$$

thus

$$h_{\Pi Z}(u) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} |u \cdot v| dn_*[\sigma_Z](v) = \int_{\mathbb{S}^{d-1}} |u \cdot v| d\left(\frac{n_*[\sigma_Z] + (-n)_*[\sigma_Z]}{4}\right)(v),$$

which implies that ΠZ is a zonoid. $\square$

Using this last observation we will exploit various nice properties of zonoids to establish the following result which generalises the fact that

$$\text{surf}_u(Z) = h_{\Pi Z}(u) = |\pi_u(\Pi Z)|.$$

Proposition 4.44. *Let $Z \subset \mathbb{R}^d$ be a nice enough compact piece of hypersurface and $V \subset \mathbb{R}^d$ be a k -dimensional subspace, then we have the following representation for the directional volume*

$$\text{surf}_V(Z) = k! |\pi_V(\Pi Z)|,$$

where the right-hand side denotes k -dimensional measure of $\pi_V(\Pi Z)$.

In order to prove this we first need to introduce a fundamental quantity from classical convex geometry:

Definition 4.45. The **mixed volume** $V(K_1, \dots, K_d)$ of compact convex subsets $K_1, \dots, K_d \subset \mathbb{R}^d$ is defined by

$$V(K_1, \dots, K_d) := \frac{1}{d!} \sum_{k=1}^d (-1)^{d+k} \sum_{\{i_1, \dots, i_d\} \subset [d]} |K_{i_1} + \dots + K_{i_k}|.$$

A comprehensive discussion of various interesting properties and historical origins of the mixed volume can be found in Chapter 5 of [37]. In particular, let us state the following result:

Theorem 4.46 ([37], Theorem 5.1.7). *For any $m \in \mathbb{N}$, the mixed volume satisfies the following identity*

$$|\lambda_1 K_1 + \cdots + \lambda_m K_m| = \sum_{i_1, \dots, i_d=1}^m \lambda_{i_1} \cdots \lambda_{i_d} V(K_{i_1}, \dots, K_{i_d}) \quad (4.33)$$

for arbitrary compact convex subsets $K_1, \dots, K_m \subset \mathbb{R}^d$ and $\lambda_1, \dots, \lambda_m \geq 0$. Furthermore, there is a symmetric map S from $(\mathcal{K}^d)^{d-1}$ into the space of finite Borel measures on $\mathbb{S}^{d-1}$, the **mixed area measure**, such that for arbitrary compact convex subsets $K_1, \dots, K_d \subset \mathbb{R}^d$ we have the following identity

$$V(K_1, \dots, K_d) = \frac{1}{d} \int_{\mathbb{S}^{d-1}} h_{K_1}(u) dS(K_2, \dots, K_d)(u), \quad (4.34)$$

where h_{K_1} denotes the support function of K_1 .

Using this we can immediately deduce the following useful properties for the mixed volume:

Lemma 4.47 (Properties of the mixed volume). *The mixed volume satisfies the following three properties:*

1. **Volume:** For any compact convex set $K \subset \mathbb{R}^d$, $V(\underbrace{K, \dots, K}_d)$ is equal to the d -dimensional Lebesgue measure of K
2. **Symmetry:** V is symmetric in its arguments
3. **Multilinearity:** V is linear in each argument with respect to Minkowski addition and multiplication by non-negative reals

Proof. The first property follows directly from (4.33) by setting $K_1 = K, \lambda_1 = 1$ and $\lambda_2 = \cdots = \lambda_d = 0$. The second property follows directly from the definition of the mixed volume. For the third property, let $\lambda, \mu > 0$ and note that by definition of the support function we have

$$\begin{aligned} h_{\lambda K + \mu \tilde{K}}(u) &= \sup\{u \cdot x \mid x \in \lambda K + \mu \tilde{K}\} = \sup\{u \cdot (y + \tilde{y}) \mid y \in \lambda K, \tilde{y} \in \mu \tilde{K}\} \\ &= \sup\{u \cdot y + u \cdot \tilde{y} \mid y \in \lambda K, \tilde{y} \in \mu \tilde{K}\} \\ &= \sup\{u \cdot y \mid y \in \lambda K\} + \sup\{u \cdot \tilde{y} \mid \tilde{y} \in \mu \tilde{K}\} \\ &= \sup\{\lambda(u \cdot z) \mid z \in K\} + \sup\{\mu(u \cdot \tilde{z}) \mid \tilde{z} \in \tilde{K}\} \\ &= \lambda \sup\{u \cdot z \mid z \in K\} + \mu \sup\{u \cdot \tilde{z} \mid \tilde{z} \in \tilde{K}\} \\ &= \lambda h_K(u) + \mu h_{\tilde{K}}(u). \end{aligned}$$

So by (4.34) we can write

$$\begin{aligned} &V(\lambda K + \mu \tilde{K}, K_2, \dots, K_d) \\ &= \frac{1}{d} \int_{\mathbb{S}^{d-1}} h_{\lambda K + \mu \tilde{K}}(u) dS(K_2, \dots, K_d)(u) \\ &= \frac{1}{d} \int_{\mathbb{S}^{d-1}} (\lambda h_K(u) + \mu h_{\tilde{K}}(u)) dS(K_2, \dots, K_d)(u) \\ &= \lambda \left(\frac{1}{d} \int_{\mathbb{S}^{d-1}} h_K(u) dS(K_2, \dots, K_d)(u) \right) + \mu \left(\frac{1}{d} \int_{\mathbb{S}^{d-1}} h_{\tilde{K}}(u) dS(K_2, \dots, K_d)(u) \right) \\ &= \lambda V(K, K_2, \dots, K_d) + \mu V(\tilde{K}, K_2, \dots, K_d). \end{aligned}$$

This linearity in the first argument can be extended to every argument by using the fact we have already proven that the mixed volume is symmetric in its arguments. $\square$

Next we state a classical result from [37] which gives us a nice formula for the mixed volume of zonoids:

Theorem 4.48 ([37], Theorem 5.3.2). *Let $Z_1, \dots, Z_j \subset \mathbb{R}^d$ be zonoids with generating measures $\rho_1, \dots, \rho_j$. Given non-empty, compact, convex subsets $K_1, \dots, K_{d-j} \subset \mathbb{R}^d$ and $1 \leq j \leq d$, we have the following formula for the mixed volume:*

$$\begin{aligned} V(Z_1, \dots, Z_j, K_1, \dots, K_{d-j}) \\ = \frac{2^j(d-j)!}{d!} \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} |u_1 \wedge \cdots \wedge u_j| V(\pi_{\text{span}\{u_1, \dots, u_j\}^\perp} K_1, \dots, \pi_{\text{span}\{u_1, \dots, u_j\}^\perp} K_{d-j}) d\rho_1(u_1) \cdots d\rho_j(u_j). \end{aligned}$$

A special case of the above gives us a nice way to represent the mixed volume of a projection body of a hypersurface:

Corollary 4.49. *Given hypersurfaces $Z_1, \dots, Z_d \subset \mathbb{R}^d$ of finite $(d-1)$ -dimensional measure and almost-everywhere defined unit normals $n_1, \dots, n_d$ respectively, we have the following formula for the mixed volume of $\Pi Z_1, \dots, \Pi Z_d$*

$$V(\Pi Z_1, \dots, \Pi Z_d) = \frac{1}{d!} \int_{Z_1} \cdots \int_{Z_d} |n_1(y_1) \wedge \cdots \wedge n_d(y_d)| d\sigma_{Z_1}(y_1) \cdots d\sigma_{Z_d}(y_d).$$

Proof. In the proof of Lemma 4.43, we saw that the generating measure of ΠZ_j is given by

$$\rho_j = \frac{(n_j)_*[\sigma_{Z_j}] + (-n_j)_*[\sigma_{Z_j}]}{4},$$

where $n_j: Z_j \rightarrow \mathbb{S}^{d-1}$ is some choice of "upward-facing" unit normal to Z_j , and $-n_j$ the corresponding "downward-facing" unit normal. Applying Theorem 4.48 yields

$$\begin{aligned} V(\Pi Z_1, \dots, \Pi Z_d) \\ = \frac{2^j}{d!} \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} |u_1 \wedge \cdots \wedge u_d| d \left(\frac{(n_1)_*[\sigma_{Z_1}] + (-n_1)_*[\sigma_{Z_1}]}{4} \right) (u_1) \\ \quad \cdots d \left(\frac{(n_d)_*[\sigma_{Z_d}] + (-n_d)_*[\sigma_{Z_d}]}{4} \right) (u_d) \\ = \frac{1}{d!} \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} |u_1 \wedge \cdots \wedge u_d| d(n_1)_*[\sigma_{Z_1}](u_1) \cdots d(n_d)_*[\sigma_{Z_d}](u_d). \end{aligned}$$

By a simple change of variables we have

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} |u_1 \wedge \cdots \wedge u_d| d(n_1)_*[\sigma_{Z_1}](u_1) \cdots d(n_d)_*[\sigma_{Z_d}](u_d) \\ = \int_{Z_1} \cdots \int_{Z_d} |n_1(y_1) \wedge \cdots \wedge n_d(y_d)| d\sigma_{Z_1}(y_1) \cdots d\sigma_{Z_d}(y_d), \end{aligned}$$

so we conclude that

$$V(\Pi Z_1, \dots, \Pi Z_d) = \frac{1}{d!} \int_{Z_1} \cdots \int_{Z_d} |n_1(y_1) \wedge \cdots \wedge n_d(y_d)| d\sigma_{Z_1}(y_1) \cdots d\sigma_{Z_d}(y_d).$$

$\square$

The next result is originally due to Fedotov and can be found in [13]:

Theorem 4.50 ([37], Theorem 5.3.1). *Let $E \in G_{d,k}$ and $L_1, \dots, L_{d-k}$ be compact convex subsets of $E^\perp$. If $K_1, \dots, K_k \in \mathcal{K}_d$, then*

$$\binom{d}{k} V(K_1, \dots, K_k, L_1, \dots, L_{d-k}) = V_E(\pi_E K_1, \dots, \pi_E K_k) V_{E^\perp}(L_1, \dots, L_{d-k}),$$

where $V_E, V_{E^\perp}$ denote the mixed volumes on $E, E^\perp$ respectively.

Now we are ready to establish the nice formula mentioned above for the directional volume.

Proof of Proposition 4.44. Fix $u_1, \dots, u_{d-k} \in \mathbb{R}^d$ to be an orthonormal family of vectors spanning $V^\perp$. Let $H_1, \dots, H_{d-k}$ be unit measure $(d-1)$ -dimensional discs with unit normals $u_1, \dots, u_{d-k}$ respectively. Then by definition we have

$$\begin{aligned} \text{surf}_V(Z) &= \int_Z \cdots \int_Z |V^\perp \wedge n(x_1) \wedge \cdots \wedge n(x_k)| d\sigma_Z(x_1) \cdots d\sigma_Z(x_k) \\ &= \int_{H_{d-k}} \cdots \int_{H_1} \int_Z \cdots \int_Z |u_1 \wedge \cdots \wedge u_{d-k} \wedge n(x_1) \wedge \cdots \wedge n(x_k)| d\sigma_Z(x_1) \\ &\quad \cdots d\sigma_Z(x_k) d\sigma_{H_1}(y_1) \cdots d\sigma_{H_{d-k}}(y_{d-k}). \end{aligned}$$

Now applying Corollary 4.49 to this last expression gives us

$$\begin{aligned} &\int_{H_{d-k}} \cdots \int_{H_1} \int_Z \cdots \int_Z |u_1 \wedge \cdots \wedge u_{d-k} \wedge n(x_1) \wedge \cdots \wedge n(x_k)| d\sigma_Z(x_1) \\ &\quad \cdots d\sigma_Z(x_k) d\sigma_{H_1}(y_1) \cdots d\sigma_{H_{d-k}}(y_{d-k}) \\ &= d! V(\Pi Z, \dots, \Pi Z, \Pi H_1, \dots, \Pi H_{d-k}). \end{aligned}$$

Observe that for $1 \leq j \leq d-k$ we have $\Pi H_j = L_j$, where L_j is a unit line segment centred at the origin with direction u_j . By $(d-k)$ consecutive applications of Theorem 4.50 we then get

$$\begin{aligned} d! V(\Pi Z, \dots, \Pi Z, \Pi H_1, \dots, \Pi H_{d-k}) &= \frac{d!}{d(d-1) \cdots (d-k)} |\pi_V(\Pi Z)| \\ &= k! |\pi_V(\Pi Z)|. \end{aligned}$$

This concludes the proof. □

Mollified directional volume Just as with the directional surface area, it will be convenient to work with a mollified versions of the directional volume defined for subsets of polynomial hypersurfaces. In what follows we make the same identification described at the beginning of Section 4.1.2 between non-zero polynomials in d real variables of degree less than k and the unit sphere $\mathbb{S}^{\binom{d+k}{d}-1}$.

Definition 4.51 (Directional volume - mollified version). Fix $\varepsilon > 0$ and a bounded set $U \subset \mathbb{R}^d$. Let $p_1, \dots, p_k$ be non-zero polynomials in d real variables and for each $1 \leq j \leq d$ define

$$Z_{p_j} = \{x \in \mathbb{R}^d \mid p_j(x) = 0\}.$$

For any k -dimensional subspace $V \subset \mathbb{R}^d$, we define the **mollified directional volume** of the k -tuple $(Z_{p_1} \cap U, \dots, Z_{p_k} \cap U)$ with respect to V as

$$\begin{aligned} & \text{surf}_{V,\varepsilon}(Z_{p_1} \cap U, \dots, Z_{p_k} \cap U) \\ &:= \int_{B_\varepsilon(p_k)} \cdots \int_{B_\varepsilon(p_1)} \left(\int_{Z_{p'_k} \cap U} \cdots \int_{Z_{p'_1} \cap U} |V^\perp \wedge n(x_1) \wedge \cdots \wedge n(x_k)| d\sigma_{Z_{p'_1}}(x_1) \cdots d\sigma_{Z_{p'_k}}(x_k) \right) dp'_1 \cdots dp'_k. \end{aligned}$$

In particular if $Z_{p_1} = \cdots = Z_{p_k} = Z_p$ we write

$$\text{surf}_{V,\varepsilon}(Z_p \cap U) = \text{surf}_{V,\varepsilon}(Z_p \cap U, \dots, Z_p \cap U).$$

For this mollified directional volume we also have a version of the cylinder estimate:

Lemma 4.52 (Mollified cylinder estimate for k -planes - [39], Theorem 5.2). *Fix $\varepsilon > 0$. If $Z_p \subset \mathbb{R}^d$ is a polynomial hypersurface and $P \subset \mathbb{R}^d$ is a k -dimensional 1-slab, we have*

$$\sum_{Q \in \mathcal{Q}: Q \cap P \neq \emptyset} \text{surf}_{E(P),\varepsilon}(Z_p \cap Q) \lesssim_d (\deg p)^k,$$

As with the unmollified directional volume, it turns out that for a certain zonoid, which we will denote by $\Pi_\varepsilon(Z_p \cap Q)$, one can represent the directional volume more geometrically as follows

$$\text{surf}_{V,\varepsilon}(Z_p \cap U) = k! |\pi_V(\Pi_\varepsilon(Z_p \cap U))|.$$

In order to see why this is true, we begin by showing that the non-empty compact convex set with support function $h_{\Pi_\varepsilon(Z_p \cap Q)}(u) = \text{surf}_{u,\varepsilon}(Z_p \cap U)$ is in fact a zonoid. This fact combined with Corollary 4.49 will help us unpack the definition of $\text{surf}_{V,\varepsilon}(Z_p \cap U)$ in terms of $\Pi_\varepsilon(Z_p \cap Q)$.

Proposition 4.53. *Fix $\varepsilon > 0$. For every polynomial hypersurface Z_p and bounded set $U \subset \mathbb{R}^d$, the non-empty compact convex set $\Pi_\varepsilon(Z_p \cap U)$ with support function $h_{\Pi_\varepsilon(Z_p \cap U)}(u) = \text{surf}_{u,\varepsilon}(Z_p \cap U)$ is a zonoid.*

Proof. By Theorem 4.41, it suffices to show that there exists an even measure $\mu_{p,Q,\varepsilon}$ on $\mathbb{S}^{d-1}$ such that for every $u \in \mathbb{R}^d$

$$\text{surf}_{u,\varepsilon}(Z_p \cap Q) = \int_{\mathbb{S}^{d-1}} |u \cdot v| d\mu_{p,Q,\varepsilon}(v).$$

Denote by $\mu_{p,U}$ the even measure described in the proof of Lemma 4.43 such that

$$\text{surf}_u(Z_p \cap U) = \int_{\mathbb{S}^{d-1}} |u \cdot v| d\mu_{p,U}(v).$$

Note that we have the following positive linear functional on $C(\mathbb{S}^{d-1})$ defined by

$$\psi(f) := \int_{B_\varepsilon(p)} \left(\int_{\mathbb{S}^{d-1}} f(v) d\mu_{p',U}(v) \right) dp'.$$

So by Riesz representation theorem for measures, we know there exists a unique regular Borel measure $\mu_{p,U,\varepsilon}$ on $\mathbb{S}^{d-1}$ such that

$$\psi(f) = \int_{\mathbb{S}^{d-1}} f(v) d\mu_{p,U,\varepsilon}(v),$$

for every $f \in C(\mathbb{S}^{d-1})$, in particular

$$\oint_{B_\varepsilon(p)} \left(\int_{\mathbb{S}^{d-1}} |u \cdot v| d\mu_{p',U}(v) \right) dp' = \int_{\mathbb{S}^{d-1}} |u \cdot v| d\mu_{p,U,\varepsilon}(v).$$

It remains to show that $\mu_{p,U,\varepsilon}$ is an even measure, that is $\mu_{p,U,\varepsilon}(-A) = \mu_{p,U,\varepsilon}(A)$ for every $\mu_{p,U,\varepsilon}$ -measurable set $A \subset \mathbb{S}^{d-1}$. Let $\phi_{A,n}: \mathbb{S}^{d-1} \rightarrow \mathbb{R}$ be an increasing sequence of bump functions converging pointwise to χ_A , also define $\phi_{-A,n}(u) = \phi_{A,n}(-u)$. By the dominated convergence theorem and the fact that each $\mu_{p,U}$ is even, we can write

$$\begin{aligned} \mu_{p,U,\varepsilon}(A) &= \lim_{n \rightarrow \infty} \int_{\mathbb{S}^{d-1}} \phi_{A,n}(u) d\mu_{p,U,\varepsilon}(u) \\ &= \lim_{n \rightarrow \infty} \oint_{B_\varepsilon(p)} \left(\int_{\mathbb{S}^{d-1}} \phi_{A,n}(u) d\mu_{p',U}(u) \right) dp' \\ &= \lim_{n \rightarrow \infty} \oint_{B_\varepsilon(p)} \left(\int_{\mathbb{S}^{d-1}} \phi_{-A,n}(u) d\mu_{p',U}(u) \right) dp' \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{S}^{d-1}} \phi_{-A,n}(u) d\mu_{p,U,\varepsilon}(u) \\ &= \mu_{p,U,\varepsilon}(-A). \end{aligned}$$

□

Now we are ready to state and prove a version of Proposition 4.44 for the mollified directional volume:

Proposition 4.54. *Fix $\varepsilon > 0$. Let $Z_p \subset \mathbb{R}^d$ be a polynomial hypersurface and $U \subset \mathbb{R}^d$ a bounded set. For any k -dimensional subspace $V \subset \mathbb{R}^d$ we have the following representation for the mollified directional volume*

$$\text{surf}_{V,\varepsilon}(Z_p \cap U) = k! |\pi_V(\Pi_\varepsilon(Z_p \cap U))|.$$

Proof. Recall from the proof of Proposition 4.53 that we have

$$\oint_{B_\varepsilon(p)} \left(\int_{\mathbb{S}^{d-1}} f(v) d\mu_{p',U}(v) \right) dp' = \int_{\mathbb{S}^{d-1}} f(v) d\mu_{p,U,\varepsilon}(v)$$

for every $f \in C(\mathbb{S}^{d-1})$. In particular, for $1 \leq j \leq k$ any fixed $\omega_1, \dots, \omega_{j-1}, \omega_{j+1}, \dots, \omega_k \in \mathbb{S}^{d-1}$ we have

$$\begin{aligned} &\oint_{B_\varepsilon(p)} \left(\int_{\mathbb{S}^{d-1}} |V^\perp \wedge \omega_1 \wedge \dots \wedge \omega_{j-1} \wedge \omega \wedge \omega_{j+1} \wedge \dots \wedge \omega_k| d\mu_{p',U}(\omega) \right) dp' \\ &= \int_{\mathbb{S}^{d-1}} |V^\perp \wedge \omega_1 \wedge \dots \wedge \omega_{j-1} \wedge \omega \wedge \omega_{j+1} \wedge \dots \wedge \omega_k| d\mu_{p,U,\varepsilon}(\omega). \end{aligned}$$

Using this for each $1 \leq j \leq k$ along with numerous applications of Fubini's theorem, it follows that

$$\begin{aligned} &\text{surf}_{V,\varepsilon}(Z_p \cap U) \\ &= \oint_{B_\varepsilon(p)} \dots \oint_{B_\varepsilon(p)} \left(\int_{Z_{p'_1} \cap U} \dots \int_{Z_{p'_k} \cap U} |V^\perp \wedge n(x_1) \wedge \dots \wedge n(x_k)| d\sigma_{Z_{p'_1}}(x_1) \dots d\sigma_{Z_{p'_k}}(x_k) \right) dp'_1 \dots dp'_k \\ &= \oint_{B_\varepsilon(p)} \dots \oint_{B_\varepsilon(p)} \left(\int_{\mathbb{S}^{d-1}} \dots \int_{\mathbb{S}^{d-1}} |V^\perp \wedge \omega_1 \wedge \dots \wedge \omega_k| d\mu_{p'_1,U}(\omega_1) \dots d\mu_{p'_k,U}(\omega_k) \right) dp'_1 \dots dp'_k \\ &= \int_{\mathbb{S}^{d-1}} \dots \int_{\mathbb{S}^{d-1}} |V^\perp \wedge \omega_1 \wedge \dots \wedge \omega_k| d\mu_{p,U,\varepsilon}(\omega_1) \dots d\mu_{p,U,\varepsilon}(\omega_k). \end{aligned}$$

Fix $u_1, \dots, u_{d-k} \in \mathbb{R}^d$ to be an orthonormal family of vectors spanning $V^\perp$. Let $H_1, \dots, H_{d-k}$ be unit measure $(d-1)$ -dimensional discs with unit normals $u_1, \dots, u_{d-k}$ respectively, and for $1 \leq j \leq d-k$ define $\mu_{H_j} := \frac{1}{2} (\delta_{u_j} + \delta_{-u_j})$ which is in fact just the generating measure for H_j . Now we can write

$$\begin{aligned} & \text{surf}_{V,\varepsilon}(Z_p \cap U) \\ &= \int_{H_{d-k}} \cdots \int_{H_1} \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} |u_1 \wedge \cdots \wedge u_{d-k} \wedge \omega_1 \wedge \cdots \wedge \omega_k| d\mu_{p,U,\varepsilon}(\omega_1) \\ & \quad \cdots d\mu_{p,U,\varepsilon}(\omega_k) d\mu_{H_1}(y_1) \cdots d\mu_{H_{d-k}}(y_{d-k}). \end{aligned}$$

Then by applying Theorem 4.48, we get

$$\begin{aligned} & \int_{H_{d-k}} \cdots \int_{H_1} \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} |u_1 \wedge \cdots \wedge u_{d-k} \wedge \omega_1 \wedge \cdots \wedge \omega_k| d\mu_{p,U,\varepsilon}(\omega_1) \\ & \quad \cdots d\mu_{p,U,\varepsilon}(\omega_k) d\mu_{H_1}(y_1) \cdots d\mu_{H_{d-k}}(y_{d-k}) \\ &= d! V(\Pi_\varepsilon(Z_p \cap U), \dots, \Pi_\varepsilon(Z_p \cap U), \Pi H_1, \dots, \Pi H_{d-k}). \end{aligned}$$

Observe that for $1 \leq j \leq d-k$ we have $\Pi H_j = L_j$, where L_j is a unit line segment centred at the origin with direction u_j . By $(d-k)$ consecutive applications of Theorem 4.50 we then get

$$\begin{aligned} d! V(\Pi_\varepsilon(Z_p \cap U), \dots, \Pi_\varepsilon(Z_p \cap U), \Pi H_1, \dots, \Pi H_{d-k}) &= \frac{d!}{d(d-1) \cdots (d-k)} |\pi_V(\Pi_\varepsilon(Z_p \cap U))| \\ &= k! |\pi_V(\Pi_\varepsilon(Z_p \cap U))|. \end{aligned}$$

This concludes the proof. $\square$

Using the language developed above, we will be able to use Guth's visibility theorem to deduce a statement purely in terms of convex bodies.

4.4.2 Proof of Theorem 4.14

Suppose are given a finitely supported function $M: \mathcal{Q} \rightarrow \mathbb{R}_{\geq 0}$ with $\|M\|_{\ell^m(\mathcal{Q})} = 1$. Just as we saw in the tube setting, the desired result follows from an application of Guth's visibility theorem with $G(Q) = \lambda M(Q)^{\frac{m}{d}}$ for a suitable $\lambda > 0$. Again, we will choose $\lambda > 0$ to ensure the compact, centrally symmetric, convex sets $\Pi_\varepsilon(Z_p \cap Q)$ defined in Section 4.1.2 each satisfy the condition that the polar set $(\Pi_\varepsilon(Z_p \cap Q))^\circ$ lies within the unit ball. As a consequence of Lemma 4.37, similarly to the tube setting, we know this can be achieved by requiring that for all $Q \in \text{supp}(M)$

$$\lambda \gtrsim d^{\frac{d}{2}} \kappa_d^{-1} c_d^d M(Q)^{-m}, \quad (4.35)$$

where c_d is the potentially very large constant appearing in Theorem 4.34. Applying Guth's visibility theorem, with $G(Q) = \lambda M(Q)^{\frac{m}{d}}$ and λ satisfying (4.35), to get a polynomial p satisfying the following two conditions

$$\deg p \lesssim_d \lambda, \quad (4.36)$$

and if we set $Z_p = \{x \in \mathbb{R}^d \mid p(x) = 0\}$, then for all $Q \in \mathcal{Q}$ we have

$$\lambda M(Q)^{\frac{m}{d}} \lesssim_d \text{vis}_\varepsilon(Z_p \cap Q). \quad (4.37)$$

By definition of mollified visibility we can write

$$\text{vis}_\varepsilon(Z_p \cap Q) = |(\Pi_\varepsilon(Z_p \cap Q))^\circ \cap \mathbb{B}_2^d|^{-\frac{1}{d}}.$$

As noted above, $(\Pi_\varepsilon(Z_p \cap Q))^\circ$ will actually lie in the unit ball so combining lemmas 4.22, 4.23 and 4.24 with the fact that each $\Pi_\varepsilon(Z_p \cap Q)$ is compact, we can deduce that $\Pi_\varepsilon(Z_p \cap Q)$ and $(\Pi_\varepsilon(Z_p \cap Q))^\circ$ are convex bodies for each Q in the support of M . In particular, using crude bounds for the Mahler volume we can write

$$\text{vis}_\varepsilon(Z_p \cap Q) = |(\Pi_\varepsilon(Z_p \cap Q))^\circ|^{-\frac{1}{d}} \sim_d |\Pi_\varepsilon(Z_p \cap Q)|^{\frac{1}{d}}.$$

Hence (4.37) becomes

$$|\Pi_\varepsilon(Z_p \cap Q)| \gtrsim_d \lambda^d M(Q)^m. \quad (4.38)$$

The k -plane version of the cylinder estimate, stated above as Lemma 4.52, tells us that for any k -dimensional 1-slab $P \subset \mathbb{R}^d$

$$\sum_{Q: Q \cap P \neq \emptyset} \text{surf}_{E(P), \varepsilon}(Z_p \cap Q) \lesssim_d \text{surf}_{E(P), \varepsilon}(Z_p \cap P) \lesssim_d (\deg p)^k.$$

Furthermore, by Proposition 4.44 we know that

$$\text{surf}_{V, \varepsilon}(Z_p \cap Q) = k! |\pi_V(\Pi_\varepsilon(Z_p \cap Q))|,$$

so from (4.36) we can deduce that

$$\sum_{Q: Q \cap P \neq \emptyset} |\pi_{E(P)}(\Pi_\varepsilon(Z_p \cap Q))| \lesssim_d \lambda^k. \quad (4.39)$$

Letting $K(Q) = \lambda^{-1} \Pi_\varepsilon(Z_p \cap Q)$ we see that (4.38) becomes

$$|K(Q)|^{\frac{1}{m}} \gtrsim_d M(Q).$$

Moreover, for any k -plane $V \subset \mathbb{R}^d$

$$|\pi_V K(Q)| = |\pi_V(\lambda^{-1} \Pi_\varepsilon(Z_p \cap Q))| = \lambda^{-k} |\pi_V(\Pi_\varepsilon(Z_p \cap Q))|$$

Hence, given families $\mathcal{P}_1, \dots, \mathcal{P}_m$, with each $\mathcal{P}_j$ denoting a family of k_j -dimensional 1-slabs, we can deduce from (4.39) that for $1 \leq j \leq m$

$$\max_{P_j \in \mathcal{P}_j} \sum_{Q: Q \cap P_j \neq \emptyset} |\pi_{E(P_j)} K(Q)| \lesssim_{d, m} 1,$$

which concludes the proof.

Now we go on a little detour in an attempt to study the directional volumes and their generalisations from a more geometric perspective. One might hope that a better understanding of these quantities in their own right would shed more light on why they are useful in the setting of multilinear Kakeya inequalities.

Chapter 5

Directional volumes and their generalisations

The directional volume, denoted by $\text{surf}_V(Z_1, \dots, Z_k)$, played a crucial role in constructing a factorisation for the k_j -plane endpoint multilinear Kakeya inequality. Recall that for any k -dimensional subspace $V \subset \mathbb{R}^d$ and family of hypersurfaces $Z_1, \dots, Z_k \subset \mathbb{R}^d$, each with an almost-everywhere defined unit normal $n_j: Z_j \rightarrow \mathbb{S}^{d-1}$, we define

$$\text{surf}_V(Z_1, \dots, Z_k) = \int_{Z_k} \cdots \int_{Z_1} |V^\perp \wedge n(x_1) \wedge \cdots \wedge n(x_k)| d\sigma_{Z_1}(x_1) \cdots d\sigma_{Z_k}(x_k). \quad (5.1)$$

In this chapter, we attempt to understand these quantities and their natural generalisation from a purely geometric point of view. It should be noted that much of the material appearing in this chapter is currently being prepared for publication.

Firstly, notice that we can always choose pieces of hyperplane section $H_1, \dots, H_{d-k}$ such that

$$\text{surf}_V(Z_1, \dots, Z_k) = \text{surf}_{\mathbb{R}^d}(Z_1, \dots, Z_k, H_1, \dots, H_{d-k}).$$

It follows that the case where $k = d$ is already the most general. Now consider the following sequence of quantities which generalise (5.1): for hypersurfaces $Z_1, \dots, Z_k \subset \mathbb{R}^d$, each with an almost-everywhere defined unit normal $n_j: Z_j \rightarrow \mathbb{S}^{d-1}$, we define

$$\mathcal{H}_k^{d-1}(Z_1, \dots, Z_k) := \left(\int_{Z_k} \cdots \int_{Z_1} |n_1(x_1) \wedge \cdots \wedge n_k(x_k)| d\sigma_{Z_1}(x_1) \cdots d\sigma_{Z_k}(x_k) \right)^{\frac{1}{k}}. \quad (5.2)$$

In particular, for a single hypersurface $Z \subset \mathbb{R}^d$, with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$, and $1 \leq k \leq d$ we define

$$\mathcal{H}_k^{d-1}(Z) := \left(\int_Z \cdots \int_Z |n(x_1) \wedge \cdots \wedge n(x_k)| d\sigma_Z(x_1) \cdots d\sigma_Z(x_k) \right)^{\frac{1}{k}}. \quad (5.3)$$

The aim of this chapter is to study the relationship between quantities $\mathcal{H}_1^{d-1}(Z), \dots, \mathcal{H}_d^{d-1}(Z)$.

To begin we note that for any $1 \leq j, k \leq d$ with $1 \leq j + k \leq d$, as a simple consequence of the Hadamard-Fischer inequality, we have

$$|n(x_1) \wedge \cdots \wedge n(x_j) \wedge n(x_{j+1}) \wedge \cdots \wedge n(x_{j+k})| \leq |n(x_1) \wedge \cdots \wedge n(x_j)| |n(x_{j+1}) \wedge \cdots \wedge n(x_{j+k})|.$$

It follows immediately that

$$\begin{aligned}
\mathcal{H}_{j+k}^{d-1}(Z)^{j+k} &= \int_Z \cdots \int_Z |n(x_1) \wedge \cdots \wedge n(x_j) \wedge n(x_{j+1}) \wedge \cdots \wedge n(x_{j+k})| d\mathcal{H}^{d-1}(x_1) \dots \mathcal{H}^{d-1}(x_{j+k}) \\
&\leq \int_Z \cdots \int_Z |n(x_1) \wedge \cdots \wedge n(x_j)| |n(x_{j+1}) \wedge \cdots \wedge n(x_{j+k})| d\mathcal{H}^{d-1}(x_1) \dots \mathcal{H}^{d-1}(x_{j+k}) \\
&= \left(\int_Z \cdots \int_Z |n(x_1) \wedge \cdots \wedge n(x_j)| d\mathcal{H}^{d-1}(x_1) \dots \mathcal{H}^{d-1}(x_j) \right) \\
&\quad \cdot \left(\int_Z \cdots \int_Z |n(x_{j+1}) \wedge \cdots \wedge n(x_{j+k})| d\mathcal{H}^{d-1}(x_{j+1}) \dots \mathcal{H}^{d-1}(x_{j+k}) \right) \\
&= \mathcal{H}_j^{d-1}(Z)^j \mathcal{H}_k^{d-1}(Z)^k.
\end{aligned}$$

Taking both sides to the power $\frac{1}{j+k}$, this becomes

$$\mathcal{H}_{j+k}^{d-1}(Z) \leq \mathcal{H}_j^{d-1}(Z)^{\frac{j}{j+k}} \mathcal{H}_k^{d-1}(Z)^{\frac{k}{j+k}}. \quad (5.4)$$

One can view this as a quasi-decreasing condition on the sequence $\mathcal{H}_1^{d-1}(Z), \dots, \mathcal{H}_d^{d-1}(Z)$. Indeed, whenever $1 \leq k \leq m \leq d$ and $m = nk$, it follows that

$$\mathcal{H}_m^{d-1}(Z) = \mathcal{H}_{nk}^{d-1}(Z) \leq \left(\mathcal{H}_k^{d-1}(Z)^{\frac{k}{nk}} \right)^n = \mathcal{H}_k^{d-1}(Z).$$

Furthermore, if we already knew that $(\mathcal{H}_j^{d-1}(Z))_j$ was a decreasing sequence, then we could immediately deduce (5.4) as follows

$$\mathcal{H}_{j+k}^{d-1}(Z) = \mathcal{H}_{j+k}^{d-1}(Z)^{\frac{j}{j+k}} \mathcal{H}_{j+k}^{d-1}(Z)^{\frac{k}{j+k}} \leq \mathcal{H}_j^{d-1}(Z)^{\frac{j}{j+k}} \mathcal{H}_k^{d-1}(Z)^{\frac{k}{j+k}}.$$

With this in mind, one might naively hope to prove the following stronger chain of inequalities

$$\mathcal{H}_d^{d-1}(Z) \leq \mathcal{H}_{d-1}^{d-1}(Z) \leq \cdots \leq \mathcal{H}_1^{d-1}(Z) \quad (5.5)$$

Let us now state the main result of this chapter:

Theorem 5.1. *For any hypersurface $Z \subset \mathbb{R}^d$ of finite $(d-1)$ -dimensional measure with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$, we have the following chain of inequalities*

$$C_{d,d} \mathcal{H}_d^{d-1}(Z) \leq C_{d-1,d} \mathcal{H}_{d-1}^{d-1}(Z) \leq \cdots \leq C_{1,d} \mathcal{H}_1^{d-1}(Z), \quad (5.6)$$

where we define

$$C_{j,d} := \left[\frac{\kappa_{d-j}}{j! \binom{d}{j} \kappa_d} \right]^{\frac{1}{j}},$$

with κ_{d-j} denoting the $(d-j)$ -dimensional volume of the unit ball in $\mathbb{R}^{d-j}$. Furthermore, if Z is the boundary of a ball in $\mathbb{R}^d$ then all the inequalities become equalities.

Note that we do not claim to have characterised all equality cases. In order to prove the above result, we will need some tools from convex geometry which were developed in the previous chapter. Recall that for any hypersurface $Z \subset \mathbb{R}^d$ of finite $(d-1)$ -dimensional measure with

an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$, we defined the projection body of Z , denoted by ΠZ , to be the unique compact convex set with support function

$$h_{\Pi Z}(u) = \int_Z |u \cdot n(x)| d\sigma_Z(x).$$

By Proposition 4.44, we know that the directional volume $\text{surf}_V(Z)$ is equal to the k -dimensional volume of $\pi_V(\Pi Z)$, where V denotes some k -dimensional subspace of $\mathbb{R}^d$. The proof of this required the notion of mixed volume, which we write $V: (\mathcal{K}^d)^d \rightarrow \mathbb{R}_{\geq 0}$ - see Definition 4.45. Let us recall three useful properties which we proved earlier in Lemma 4.47:

1. **Volume:** For any non-empty compact convex set $K \subset \mathbb{R}^d$, $V(\underbrace{K, \dots, K}_d)$ is equal to the d -dimensional Lebesgue measure of K
2. **Symmetry:** V is symmetric in its arguments
3. **Multilinearity:** V is linear in each argument with respect to Minkowski addition and multiplication by non-negative reals

Using a similar argument to the proof of Proposition 4.44, in Section 5.3 we show that

$$\mathcal{H}_k^{d-1}(Z) = \left(\frac{2^j \kappa_{d-k}}{k! \binom{d}{k}} V(\Pi Z[k], \mathbb{B}_2^d[d-k]) \right)^{\frac{1}{k}},$$

where we use the concise notation

$$V(\Pi Z[k], \mathbb{B}_2^d[d-k]) := V(\underbrace{\Pi Z, \dots, \Pi Z}_k, \underbrace{\mathbb{B}_2^d, \dots, \mathbb{B}_2^d}_{d-k}).$$

Following this, at the end of Section 5.3, we deduce Theorem 5.1 as a consequence of the following classical inequality proved independently by Fenchel [19, 20] and Aleksandrov [1, 2, 3, 4]:

Theorem 5.2 (Aleksandrov–Fenchel inequality). *For non-empty, compact, convex subsets $K_1, \dots, K_d \subset \mathbb{R}^d$ the following inequality holds:*

$$V(K_1, K_2, \dots, K_d)^2 \geq V(K_1, K_1, K_3, \dots, K_d) V(K_2, K_2, K_3, \dots, K_d).$$

In the special case where $Z \subset \mathbb{R}^d$ is the boundary of a box, we are able to prove a version of Theorem 5.1 which is extremised for cubes instead of balls. In Section 5.2 we give a proof of the following result using an elementary refinement of the arithmetic-geometric mean inequality called the Maclaurin inequality:

Theorem 5.3. *Suppose $Z = \partial R$ where $R = [0, \ell_1] \times \dots \times [0, \ell_d] + \eta$ for some $\ell_1, \dots, \ell_d \in \mathbb{R}_{\geq 0}$ and $\eta \in \mathbb{R}^d$. Then we have the following chain of inequalities*

$$B_{d,d} \mathcal{H}_d^{d-1}(Z) \leq B_{d-1,d} \mathcal{H}_{d-1}^{d-1}(Z) \leq \dots \leq B_{1,d} \mathcal{H}_1^{d-1}(Z),$$

where

$$B_{j,d} := \left[\frac{1}{j! \binom{d}{j}} \right]^{\frac{1}{j}}.$$

Furthermore, equality for each inequality is attained if and only if $Z = \partial Q$ for some d -dimensional cube $Q = [0, \ell]^d + \eta \subset \mathbb{R}^d$ with $\ell \in \mathbb{R}_{\geq 0}$ and $\eta \in \mathbb{R}^d$.

Note that for $1 < j \leq d$ we can write

$$\frac{B_{j-1,d}}{B_{j,d}} = \left[\frac{\left(j! \binom{d}{j}\right)^{j-1}}{\left((j-1)! \binom{d}{j-1}\right)^j} \right]^{\frac{1}{j(j-1)}} = \left[\frac{(d-j+1)!(d-j+1)^{j-1}}{d!} \right]^{\frac{1}{j(j-1)}} < 1.$$

So $B_{j-1,d} < B_{j,d}$, which tells us that Theorem 5.3 is an improvement on our initial guess of (5.5).

At the end of Section 5.2, we propose a generalisation of the Maclaurin inequality, namely the vector-valued Maclaurin inequality, which allows one to extend Theorem 5.3 to hold whenever Z is the boundary of a parallelotope. In light of the partial results presented in Chapter 6 for proving the so-called vector-valued Maclaurin inequality, we state the following conjecture:

Conjecture 5.4. *Suppose $Z = \partial P$ for some parallelotope $P \subset \mathbb{R}^d$. Then we have the following chain of inequalities*

$$B_{d,d} \mathcal{H}_d^{d-1}(Z) \leq B_{d-1,d} \mathcal{H}_{d-1}^{d-1}(Z) \leq \cdots \leq B_{1,d} \mathcal{H}_1^{d-1}(Z),$$

where

$$B_{j,d} := \left[\frac{1}{j! \binom{d}{j}} \right]^{\frac{1}{j}}.$$

Furthermore, equality for each inequality is attained if and only if $Z = \partial Q$ for some d -dimensional cube $Q = [0, \ell]^d + \eta \subset \mathbb{R}^d$ with $\ell \in \mathbb{R}_{\geq 0}$ and $\eta \in \mathbb{R}^d$.

This conjecture follows directly from the above-mentioned generalisation of the Maclaurin inequality, which we propose in Chapter 6 as Conjecture 6.9. The discussion at the end of Section 5.2 illustrates how one can deduce Conjecture 5.4 directly from Conjecture 6.9. One can easily show that the Loomis-Whitney inequality implies the isoperimetric inequality with non-sharp constant, which could be thought of as an isoperimetric inequality for parallelotopes where cubes are the extremisers rather than balls. In some sense we are trying to prove a refinement of the isoperimetric inequality, so it seems natural to explore variants which would be extremised by cubes. The vector-valued Maclaurin inequality can be seen as the most general cube-extremised version of Theorem 5.1.

In Section 5.4, we spend some time trying to understand $\mathcal{H}_2^1(C)$ for curves $C \subset \mathbb{R}^2$. In particular, using elementary methods we show that for any symmetric convex set K with boundary ∂K

$$\mathcal{H}_2^1(C) = 2\sqrt{2} \text{Area}(K)^{\frac{1}{2}}.$$

5.1 Affine invariance

Let us begin by trying to understand how these quantities act under affine transformations of $\mathbb{R}^d$. Firstly, notice that for any hypersurface $Z \subset \mathbb{R}^d$, the quantities $\mathcal{H}_1^{d-1}(Z), \dots, \mathcal{H}_d^{d-1}(Z)$ are all unchanged by euclidean transformations of Z . It is not clear however, how these quantities change if we apply some linear change of variables to the ambient space. As it turns out, for any linear map $A: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and hypersurface surface $Z \subset \mathbb{R}^d$ with an almost-everywhere defined unit normal, we have

$$\mathcal{H}_d^{d-1}(A(Z)) = \det(A)^{\frac{d-1}{d}} \mathcal{H}_d^{d-1}(Z).$$

On the other hand, for $k = 1$ and any $d \in \mathbb{N}$, we have

$$\mathcal{H}_1^{d-1}(Z) = \left(\int_Z |n| d\mathcal{H}^{d-1} \right)^{\frac{1}{d-1}} = \mathcal{H}^{d-1}(Z),$$

which is certainly not invariant under volume preserving linear transformations. Let us state the affine invariance for $k = d$ as the following lemma:

Lemma 5.5. *Suppose $A: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is some invertible linear transformation and $Z \subset \mathbb{R}^d$ is a $(d-1)$ -dimensional surface with an almost everywhere defined unit normal, then we have*

$$\mathcal{H}_d^{d-1}(A(Z)) = |\det(A)|^{\frac{d-1}{d}} \mathcal{H}_d^{d-1}(Z).$$

Proof. Let $V \subset \mathbb{R}^d$ be some a $(d-1)$ -dimensional plane through the origin with unit normal $n_V \in \mathbb{S}^{d-1}$, then we have

$$\mathcal{H}^{d-1}(A(V \cap B(x, r))) = \frac{|\det(A)|}{\left\langle A n, \frac{(A^\top)^{-1} n_V}{\|(A^\top)^{-1} n_V\|} \right\rangle} \mathcal{H}^{d-1}(V \cap B(x, r)).$$

Notice that

$$\left\langle A n, \frac{(A^\top)^{-1} n_V}{\|(A^\top)^{-1} n_V\|} \right\rangle = \frac{\langle n, A^\top \circ (A^\top)^{-1} n_V \rangle}{\|(A^\top)^{-1} n_V\|} = \frac{\|n_V\|}{\|(A^\top)^{-1} n_V\|} = \frac{1}{\|(A^\top)^{-1} n_V\|},$$

so in fact we can write

$$\mathcal{H}^{d-1}(A(V \cap B(x, r))) = \det(A) \|(A^\top)^{-1} n_V\| \mathcal{H}^{d-1}(V \cap B(x, r)).$$

Using a special case of the generalised area formula - see [18], Theorem 3.2.22 - this idea can be extended to surface measures: for any $\mathcal{H}^{d-1} \llcorner Z$ -integrable function $g: Z \rightarrow \mathbb{R}$, one can show that

$$\int_Z g(x) \det(A) \|(A^\top)^{-1} \circ n(x)\| d\mathcal{H}^{d-1}(x) = \int_{A(Z)} g \circ A^{-1}(y) d\mathcal{H}^{d-1}(y). \quad (5.7)$$

Consider the function $f: (Z)^d \rightarrow \mathbb{R}$ defined by

$$f(x_1, \dots, x_d) = \left| \frac{(A^\top)^{-1} \circ n(x_1)}{\|(A^\top)^{-1} \circ n(x_1)\|} \wedge \dots \wedge \frac{(A^\top)^{-1} \circ n(x_d)}{\|(A^\top)^{-1} \circ n(x_d)\|} \right|.$$

Using properties of the wedge product, we can write

$$\begin{aligned} & \left| \frac{(A^\top)^{-1} \circ n(x_1)}{\|(A^\top)^{-1} \circ n(x_1)\|} \wedge \dots \wedge \frac{(A^\top)^{-1} \circ n(x_d)}{\|(A^\top)^{-1} \circ n(x_d)\|} \right| \\ &= |\det((A^\top)^{-1})| \left| \frac{n(x_1)}{\|(A^\top)^{-1} \circ n(x_1)\|} \wedge \dots \wedge \frac{n(x_d)}{\|(A^\top)^{-1} \circ n(x_d)\|} \right| \\ &= \frac{1}{|\det(A)|} \left| \frac{n(x_1)}{\|(A^\top)^{-1} \circ n(x_1)\|} \wedge \dots \wedge \frac{n(x_d)}{\|(A^\top)^{-1} \circ n(x_d)\|} \right| \\ &= \frac{|n(x_1) \wedge \dots \wedge n(x_d)|}{|\det(A)| \prod_{j=1}^d \|(A^\top)^{-1} \circ n(x_j)\|} \end{aligned}$$

It follows immediately that

$$f(x_1, \dots, x_d) = \frac{|n(x_1) \wedge n(x_2) \wedge \dots \wedge n(x_d)|}{|\det(A)| \prod_{j=1}^d \|(A^\top)^{-1} \circ n(x_j)\|}.$$

Denote by $\tilde{n}(y)$ the unit normal to $A(Z)$ at y , we can write

$$\tilde{n}(A(x)) = \frac{(A^\top)^{-1} \circ n(x)}{\|(A^\top)^{-1} \circ n(x)\|}.$$

So by definition, for any $y_1, \dots, y_d \in A(Z)$

$$f(A^{-1}(y_1), \dots, A^{-1}(y_d)) = |\tilde{n}(y_1) \wedge \dots \wedge \tilde{n}(y_d)|.$$

Applying (5.7) d -times, each time fixing $(x_1, \dots, \hat{x}_j, \dots, x_d) \in (Z)^{d-1}$ and considering $g(x_j) = f(x_1, \dots, x_j, \dots, x_d)$, yields

$$\begin{aligned} & \int_{A(Z)} \dots \int_{A(Z)} |\tilde{n}(y_1) \wedge \dots \wedge \tilde{n}(y_d)| d\mathcal{H}^{d-1}(y_1) \dots \mathcal{H}^{d-1}(y_k) \\ &= |\det(A)|^{d-1} \int_Z \dots \int_Z |n(x_1) \wedge \dots \wedge n(x_k)| d\mathcal{H}^{d-1}(x_1) \dots \mathcal{H}^{d-1}(x_k), \end{aligned}$$

which implies that

$$\mathcal{H}_d^{d-1}(A(Z)) = |\det(A)|^{\frac{d-1}{d}} \mathcal{H}_d^{d-1}(Z).$$

□

5.2 Special case where Z is the boundary of a box

One of the simplest examples in $\mathbb{R}^d$ for which we can calculate the quantities $\mathcal{H}_1^{d-1}(Z), \dots, \mathcal{H}_d^{d-1}(Z)$ explicitly is where Z forms the boundary of box. At the end of this section, we use the direct computation given by Lemma 5.7 to deduce Theorem 5.3 as an immediate consequence of the classical Maclaurin inequality, stated below as Theorem 5.9.

In what follows, we will always be considering the outwards facing normal to the boundary of any box (or more generally any convex body). To begin, let us calculate each $\mathcal{H}_k^{d-1}(\partial Q)$ where Q is a d -dimensional cube in $\mathbb{R}^d$. This more or less reduces to counting k -tuples of $(d-1)$ -dimensional faces:

Lemma 5.6. *Let $Q = [0, \ell]^d + \eta \subset \mathbb{R}^d$ with $\ell \in \mathbb{R}_{\geq 0}$ and $\eta \in \mathbb{R}^d$. Then for $1 \leq k \leq d$*

$$\mathcal{H}_k^{d-1}(\partial Q) = 2\ell^{d-1} \left(\frac{d!}{(d-k)!} \right)^{\frac{1}{k}}$$

Proof. Note that the quantities $\mathcal{H}_k^{d-1}(Z)$ are invariant under rigid transformations of Z , so without loss of generality we may assume that $Q = [0, \ell]^d$. Let $\mathcal{F}_{d-1}(Q) = \{F_1^+, F_1^-, \dots, F_d^+, F_d^-\}$ denote the set of $(d-1)$ -dimensional faces of Q such that F_j^+ has outward facing normal e_j

and F_j^- has outward facing normal $-e_j$. Furthermore define $\mathcal{F}^+(Q) := \{F_1^+, \dots, F_d^+\}$ and $\mathcal{F}^-(Q) := \{F_1^-, \dots, F_d^-\}$. It follows that

$$\begin{aligned}
\mathcal{H}_k^{d-1}(\partial Q) &:= \left(\int_{\partial Q} \cdots \int_{\partial Q} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= \left(\int_{\bigcup_{F \in \mathcal{F}_{d-1}^-(Q)} F} \cdots \int_{\bigcup_{F \in \mathcal{F}_{d-1}^-(Q)} F} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= \left(2^k \int_{\bigcup_{F \in \mathcal{F}_{d-1}^+(Q)} F} \cdots \int_{\bigcup_{F \in \mathcal{F}_{d-1}^+(Q)} F} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= \left(2^k \sum_{\{i_1, \dots, i_k\} \subset [d]} \ell^{(d-1)k} |e_{i_1} \wedge \cdots \wedge e_{i_k}| \right)^{\frac{1}{k}} = 2\ell^{d-1} \left(\frac{d!}{(d-k)!} \right)^{\frac{1}{k}},
\end{aligned}$$

□

From this calculation we see that for $2 \leq k \leq d$

$$\left(\frac{1}{k! \binom{d}{k}} \right)^{\frac{1}{k}} \mathcal{H}_k^{d-1}(\partial Q) = \left(\frac{1}{(k-1)! \binom{d}{k-1}} \right)^{\frac{1}{k-1}} \mathcal{H}_{k-1}^{d-1}(\partial Q).$$

This is exactly the equality case for Theorem 5.3. Notice that $\left(\frac{d!}{(d-k)!} \right)^{\frac{1}{k}}$ is simply the twice geometric mean of the numbers $\{d, (d-1), \dots, (d-(k-1))\}$. The geometric mean of the k number less than or equal to d decreases as we increase k , which immediately gives the following weaker chain of inequalities

$$\mathcal{H}_d^{d-1}(\partial Q) \leq \mathcal{H}_{d-1}^{d-1}(\partial Q) \leq \cdots \leq \mathcal{H}_1^{d-1}(\partial Q).$$

Now let us consider the case where Z is the boundary of a d -dimensional box with side lengths which are not necessarily all equal.

Lemma 5.7. *Let $R \subset \mathbb{R}^d$ be a box with side lengths $l_1, \dots, l_d$, then for $1 \leq k \leq d$ we have*

$$\mathcal{H}_k^{d-1}(\partial R) = 2(k!)^{\frac{1}{k}} (l_1 \cdots l_d) \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_k}} \right)^{\frac{1}{k}}$$

Proof. Again, since the quantities $\mathcal{H}_k^{d-1}(Z)$ are invariant under rigid transformations of Z , without loss of generality we may assume that $R = [0, l_1] \times [0, l_2] \times \cdots \times [0, l_d] \subset \mathbb{R}^d$. The set of edges with direction parallel to e_j all have side length l_j . Just as we did above, let $\mathcal{F}_{d-1}(R) = \{F_1^+, F_1^-, \dots, F_d^+, F_d^-\}$ denote the set of $(d-1)$ -dimensional faces of R such that F_j^+ has outward facing normal e_j and F_j^- has outward facing normal $-e_j$. Furthermore define

$\mathcal{F}^+(R) := \{F_1^+, \dots, F_d^+\}$ and $\mathcal{F}^-(R) := \{F_1^-, \dots, F_d^-\}$. It follows that

$$\begin{aligned}
\mathcal{H}_k^{d-1}(\partial R) &:= \left(\int_{\partial R} \cdots \int_{\partial R} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= \left(\int_{\bigcup_{\mathcal{F}_{d-1}^-(R)} F} \cdots \int_{\bigcup_{\mathcal{F}_{d-1}^-(R)} F} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= \left(2^k \int_{\bigcup_{\mathcal{F}_{d-1}^+(R)} F} \cdots \int_{\bigcup_{\mathcal{F}_{d-1}^+(R)} F} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= \left(2^k k! \sum_{\{i_1, \dots, i_k\} \subset [d]} \int_{F_{i_1}^+} \cdots \int_{F_{i_k}^+} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= 2(k!)^{\frac{1}{k}} \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \mathcal{H}_{d-1}(F_{i_1}^+) \cdots \mathcal{H}_{d-1}(F_{i_k}^+) \cdot |e_{i_1} \wedge \cdots \wedge e_{i_k}| \right)^{\frac{1}{k}} \\
&= 2(k!)^{\frac{1}{k}} \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \prod_{j=1}^k (l_1 \cdots \widehat{l_{i_j}} \cdots l_d) \right)^{\frac{1}{k}} \\
&= 2(k!)^{\frac{1}{k}} \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{(l_1 \cdots l_d)^k}{l_{i_1} \cdots l_{i_k}} \right)^{\frac{1}{k}} \\
&= 2(k!)^{\frac{1}{k}} (l_1 \cdots l_d) \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_k}} \right)^{\frac{1}{k}}
\end{aligned}$$

□

Remark 5.8. In addition let $\mathcal{F}_k(R)$ denote the set of k -dimensional faces of R . Then we could

have written the above quantity alternatively as

$$\begin{aligned}
&= 2(k!)^{\frac{1}{k}} \mathcal{H}^d(R)^{\frac{k-1}{k}} \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{l_{i_1} \cdots l_{i_k}}{l_{i_1} \cdots l_{i_k}} \right)^{\frac{1}{k}} \\
&= 2(k!)^{\frac{1}{k}} \mathcal{H}^d(R)^{\frac{k-1}{k}} \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} l_{i_1} \cdots l_{i_{d-k}} \right)^{\frac{1}{k}} \\
&= 2(k!)^{\frac{1}{k}} \mathcal{H}^d(R)^{\frac{k-1}{k}} \left(\frac{1}{2^k} \sum_{F \in \mathcal{F}_{d-k}(R)} \mathcal{H}^{d-k}(F) \right)^{\frac{1}{k}} \\
&= (k!)^{\frac{1}{k}} \mathcal{H}^d(R)^{\frac{k-1}{k}} \left(\sum_{F \in \mathcal{F}_{d-k}(R)} \mathcal{H}^{d-k}(F) \right)^{\frac{1}{k}},
\end{aligned}$$

where the second to last equality follows from the fact that every $(d-k)$ -dimensional face of R has exactly 2^k translates of itself which are also faces of R . Note that

$$\mathcal{H}_1^{d-1}(\partial R) = (1!)^1 \mathcal{H}^d(R)^0 \left(\sum_{F \in \mathcal{F}_{d-1}(R)} \mathcal{H}^{d-1}(F) \right)^1 = \mathcal{H}^{d-1}(\partial R)$$

and

$$\mathcal{H}_d^{d-1}(\partial R) = (d!)^{\frac{1}{d}} \mathcal{H}^d(R)^{\frac{d-1}{d}} \left(\sum_{F \in \mathcal{F}_0(R)} \mathcal{H}^0(F) \right)^{\frac{1}{d}} = (d!)^{\frac{2}{d}} \mathcal{H}^d(R)^{\frac{d-1}{d}} (2^d)^{\frac{1}{d}} \sim d \cdot \mathcal{H}^d(R)^{\frac{d-1}{d}}.$$

The isoperimetric inequality states that for any subset $E \subset \mathbb{R}^d$

$$\mathcal{H}^{d-1}(\partial E) \geq d \cdot \mathcal{H}^d(E)^{\frac{d-1}{d}}.$$

In particular, this gives

$$\mathcal{H}_1^{d-1}(\partial R) \geq \mathcal{H}_d^{d-1}(R).$$

In order to establish Theorem 5.3, we first need to state a classical refinement of the arithmetic-geometric mean inequality:

Theorem 5.9 (The Maclaurin inequality - [34]). *For $0 \leq k \leq n$, define the k th symmetric average of an n -tuple of positive numbers $X = (x_1, \dots, x_n)$ as*

$$S_k(X) := \frac{\sum_{\{i_1, \dots, i_k\} \subset [n]} x_{i_1} \cdots x_{i_k}}{\binom{n}{k}}.$$

Then following chain of inequalities holds:

$$\text{AM} = S_1(X) \geq (S_2(X))^{\frac{1}{2}} \geq (S_3(X))^{\frac{1}{3}} \geq \cdots \geq (S_n(X))^{\frac{1}{n}} = \text{GM}$$

A proof of this result using elementary methods can be found in [25] - see proof of Theorem 52. Now we are ready to prove the corresponding chain of inequalities for boundaries of d -dimensional boxes.

Proof of Theorem 5.3. By Lemma 5.7 for $k = 1, \dots, d$ we can write

$$\mathcal{H}_k^{d-1}(\partial R) = 2(k!)^{\frac{1}{k}} (l_1 \cdots l_d) \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_k}} \right)^{\frac{1}{k}}.$$

For $1 \leq k < d$, the k -th Maclaurin inequality applied to the sequence of positive numbers $\frac{1}{l_1}, \dots, \frac{1}{l_d}$ gives

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_k}}}{\binom{d}{k}} \right)^{\frac{1}{k}} \geq \left(\frac{\sum_{\{i_1, \dots, i_{k+1}\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_{k+1}}}}{\binom{d}{k+1}} \right)^{\frac{1}{k+1}},$$

thus

$$\begin{aligned} B_{k,d} \mathcal{H}_k^{d-1}(\partial R) &= \left[\frac{1}{k! \binom{d}{k}} \right]^{\frac{1}{k}} 2(k!)^{\frac{1}{k}} (l_1 \cdots l_d) \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_k}} \right)^{\frac{1}{k}} \\ &= 2(l_1 \cdots l_d) \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_k}}}{\binom{d}{k}} \right)^{\frac{1}{k}} \\ &\leq 2(l_1 \cdots l_d) \left(\frac{\sum_{\{i_1, \dots, i_{k+1}\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_{k+1}}}}{\binom{d}{k+1}} \right)^{\frac{1}{k+1}} \\ &= \left[\frac{1}{(k+1)! \binom{d}{k+1}} \right]^{\frac{1}{k+1}} 2((k+1)!)^{\frac{1}{k+1}} (l_1 \cdots l_d) \left(\sum_{\{i_1, \dots, i_{k+1}\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_{k+1}}} \right)^{\frac{1}{k+1}} \\ &= B_{k+1,d} \mathcal{H}_{k+1}^{d-1}(\partial R). \end{aligned}$$

□

Special case where Z is the boundary of a parallelotope Let us attempt to follow a similar procedure for boundaries of d -dimensional parallelotopes:

Lemma 5.10. *Let $P \subset \mathbb{R}^d$ be a parallelotope with side lengths $l_1, \dots, l_d$ and linearly independent directions $v_1, \dots, v_d$, then for $1 \leq k \leq d$ we have*

$$\mathcal{H}_k^{d-1}(\partial P) = 2(k!)^{\frac{1}{k}} l_1 \cdots l_d \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_k}} \left(\prod_{j=1}^k |v_1 \wedge \cdots \wedge \widehat{v_{i_j}} \wedge \cdots \wedge v_d| \right) \cdot |n_{i_1} \wedge \cdots \wedge n_{i_k}| \right)^{\frac{1}{k}}.$$

Proof. As noted earlier, the quantities $\mathcal{H}_k^{d-1}(Z)$ are invariant under rigid transformations of Z , so without loss of generality we may assume that one of the vertices is the origin and one of the faces lies in an axis plane and that the whole parallelotope lies in the upper half plane with respect to that axis plane. Let P denote a d -dimensional parallelotope with edges of lengths $l_1, \dots, l_d$ and directions $v_1, \dots, v_d$, each edge has 2^{d-1} translates which are also edges of P . As before, let $\mathcal{F}_{d-1}(P) = \{F_1^+, F_1^-, \dots, F_d^+, F_d^-\}$ denote the set of $(d-1)$ -dimensional faces of P such that F_j^+ has outward facing normal n_j and F_j^- has outward facing normal $-n_j$. Furthermore define $\mathcal{F}^+(R) := \{F_1^+, \dots, F_d^+\}$ and $\mathcal{F}^-(R) := \{F_1^-, \dots, F_d^-\}$, where F_j^+ is defined to be the face whose defining plane contains the origin. Observe that

$$\begin{aligned}
\mathcal{H}_k^{d-1}(\partial P) &:= \left(\int_{\partial P} \cdots \int_{\partial P} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= \left(\int_{\bigcup_{\mathcal{F}_{d-1}(P)} F} \cdots \int_{\bigcup_{\mathcal{F}_{d-1}(P)} F} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= \left(2^k \int_{\bigcup_{\mathcal{F}_{d-1}^+(P)} F} \cdots \int_{\bigcup_{\mathcal{F}_{d-1}^+(P)} F} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= \left(2^k (k!) \sum_{\{i_1, \dots, i_k\} \subset [d]} \int_{F_{i_1}^+} \cdots \int_{F_{i_k}^+} |n(z_1) \wedge \cdots \wedge n(z_k)| d\mathcal{H}^{d-1}(z_1) \dots d\mathcal{H}^{d-1}(z_k) \right)^{\frac{1}{k}} \\
&= 2(k!)^{\frac{1}{k}} \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \mathcal{H}_{d-1}(F_{i_1}^+) \cdots \mathcal{H}_{d-1}(F_{i_k}^+) \cdot |n_{i_1} \wedge \cdots \wedge n_{i_k}| \right)^{\frac{1}{k}} \\
&= 2(k!)^{\frac{1}{k}} \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \left(\prod_{j=1}^k l_1 \cdots \widehat{l_{i_j}} \cdots l_d |v_1 \wedge \cdots \wedge \widehat{v_{i_j}} \wedge \cdots \wedge v_d| \right) \cdot |n_{i_1} \wedge \cdots \wedge n_{i_k}| \right)^{\frac{1}{k}} \\
&= 2(k!)^{\frac{1}{k}} l_1 \cdots l_d \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_k}} \left(\prod_{j=1}^k |v_1 \wedge \cdots \wedge \widehat{v_{i_j}} \wedge \cdots \wedge v_d| \right) \cdot |n_{i_1} \wedge \cdots \wedge n_{i_k}| \right)^{\frac{1}{k}}.
\end{aligned}$$

□

Using Lemma 5.10, we see that Conjecture 5.4 reduces to the following estimate

$$\begin{aligned}
&\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_k}} \left(\prod_{j=1}^k |v_1 \wedge \cdots \wedge \widehat{v_{i_j}} \wedge \cdots \wedge v_d| \right) \cdot |n_{i_1} \wedge \cdots \wedge n_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}} \\
&\leq \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} \frac{1}{l_{i_1} \cdots l_{i_{k-1}}} \left(\prod_{j=1}^{k-1} |v_1 \wedge \cdots \wedge \widehat{v_{i_j}} \wedge \cdots \wedge v_d| \right) \cdot |n_{i_1} \wedge \cdots \wedge n_{i_{k-1}}|}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}.
\end{aligned} \tag{5.8}$$

As we saw above, if the vectors $v_1, \dots, v_d$ are orthogonal so that P is a box, then (5.8) follows directly from the classical Maclaurin inequality. Setting $u_i = l_i^{-1} |v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_d| n_i$, we see that (5.8) becomes

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |u_{i_1} \wedge \dots \wedge u_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}} \leq \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |u_{i_1} \wedge \dots \wedge u_{i_{k-1}}|}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}. \quad (5.9)$$

If we are able to establish this last inequality for any linearly independent vectors $u_1, \dots, u_d \in \mathbb{R}^d$ and $1 < k \leq d$, then Conjecture 5.4 follows as a result. In Chapter 6, we establish various special cases of (5.9) and conjecture a more general form of inequality - see Conjecture 6.9.

5.3 Proof of Theorem 5.1

Let us now prove the main result of this chapter using tools from convex geometry. In particular, for any hypersurface $Z \subset \mathbb{R}^d$ with an almost-everywhere defined unit normal $n: Z \rightarrow \mathbb{S}^{d-1}$, we will prove the following chain of inequalities:

$$C_{d,d} \mathcal{H}_d^{d-1}(Z) \leq C_{d-1,d} \mathcal{H}_{d-1}^{d-1}(Z) \leq \dots \leq C_{1,d} \mathcal{H}_1^{d-1}(Z), \quad (5.10)$$

where we define

$$C_{j,d} := \left[\frac{\kappa_{d-j}}{j! \binom{d}{j}} \right]^{\frac{1}{j}}.$$

Furthermore, we will show that equality for each inequality is attained when Z is the boundary of some ball in $\mathbb{R}^d$.

In Chapter 4, we encountered the projection body of a surface Z with an almost-everywhere defined normal $n: Z \rightarrow \mathbb{S}^{d-1}$, denoted ΠZ , which was defined to be the unique convex set with support function

$$h_{\Pi Z}(u) = \int_Z |u \cdot n(x)| d\sigma_Z.$$

Furthermore, ΠZ is a zonoid, which means one can write the support function as

$$h_{\Pi Z}(u) = \int_{\mathbb{S}^{d-1}} |u \cdot v| d\rho(u),$$

for some finite even measure ρ on $\mathbb{S}^{d-1}$, which is referred to as the generating measure of ΠZ . This can be verified by setting

$$\rho = \frac{n_*[\sigma_Z] + (-n)_*[\sigma_Z]}{4},$$

where $f_*\mu$ denotes the pushforward of a measure μ by a function f defined by

$$f_*\mu(A) = \mu(f^{-1}(A)).$$

Theorem 4.48 gave us a convenient formula for calculating mixed volumes of zonoids. Let us now introduce a special case of the mixed volume which will simplify notation:

Definition 5.11 (Intrinsic Volumes). For any non-empty compact convex set $K \subset \mathbb{R}^d$ we define the j -th intrinsic volume

$$V_j(K) := \binom{d}{j} \frac{1}{\kappa_{d-j}} V(K[j], \mathbb{B}_2^d[d-j]).$$

By Theorem 4.48 we immediately get the following formula for the intrinsic volume of a zonoid

Theorem 5.12. Suppose $K \subset \mathbb{R}^d$ is a zonoid with generating measure ρ and $1 \leq j \leq d$, then we have

$$V_j(K) = \frac{2^j}{j!} \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} |u_1 \wedge \cdots \wedge u_j| d\rho(u_1) \cdots d\rho(u_j).$$

Applying this with $K = \Pi Z$, we immediately get the following connection between the directional volumes and intrinsic volumes:

Lemma 5.13. For any compact surface $Z \subset \mathbb{R}^d$ with an almost-everywhere defined normal $n: Z \rightarrow \mathbb{S}^{d-1}$, and $1 \leq j \leq d$, we have

$$V_j(\Pi Z) = \frac{2^j}{j!} \mathcal{H}_j^{d-1}(Z)^j \quad (5.11)$$

Let us now restate the well-known Aleksandrov–Fenchel inequalities for mixed volumes.

Theorem 5.14 (Aleksandrov–Fenchel). For non-empty, compact, convex subsets $K_1, \dots, K_d \subset \mathbb{R}^d$ the following inequality holds

$$V(K_1, K_2, \dots, K_d)^2 \geq V(K_1, K_1, K_3, \dots, K_d) V(K_2, K_2, K_3, \dots, K_d).$$

As a direct consequence we get the following result:

Theorem 5.15 (Log-concavity of intrinsic volumes). For any non-empty, compact, convex subset $K \subset \mathbb{R}^d$ and $1 \leq j < d$ we have

$$\left[\frac{\kappa_{d-j}}{\binom{d}{j}} V_j(K) \right]^2 \geq \left[\frac{\kappa_{d-(j+1)}}{\binom{d}{j+1}} V_{j+1}(K) \right] \left[\frac{\kappa_{d-(j-1)}}{\binom{d}{j-1}} V_{j-1}(K) \right].$$

Furthermore, equality holds if K is a ball centred at the origin.

Proving Theorem 5.14 $\implies$ Theorem 5.15. By the Aleksandrov–Fenchel inequality we have

$$\begin{aligned} \left[\frac{\kappa_{d-j}}{\binom{d}{j}} V_j(K) \right]^2 &= V(K[j], \mathbb{B}_2^d[d-j])^2 \geq V(K[j+1], \mathbb{B}_2^d[d-(j+1)]) V(K[j-1], \mathbb{B}_2^d[d-(j-1)]) \\ &= \left[\frac{\kappa_{d-j-1}}{\binom{d}{j+1}} V_{j+1}(K) \right] \left[\frac{\kappa_{d-j+1}}{\binom{d}{j-1}} V_{j-1}(K) \right]. \end{aligned}$$

If $K = r\mathbb{B}_2^d$ for some $r > 0$, then by properties of the mixed volume

$$\begin{aligned} V(K[j], \mathbb{B}_2^d[d-j])^2 &= r^{2j} \kappa_d^2 = [r^{j+1} \kappa_d] [r^{j-1} \kappa_d] \\ &= V(K[j+1], \mathbb{B}_2^d[d-(j+1)]) V(K[j-1], \mathbb{B}_2^d[d-(j-1)]), \end{aligned}$$

which concludes the proof. $\square$

We now state a small lemma which translates log-concavity statement into one about monotonicity.

Lemma 5.16. *For any sequence $a_0, a_1, \dots, a_m \in \mathbb{R}_{>0}$ with $a_0 = 1$. If for $1 \leq j < m$ we have*

$$a_j^2 \geq a_{j+1}a_{j-1}, \quad (5.12)$$

then it follows that

$$(a_m)^{\frac{1}{m}} \leq (a_{m-1})^{\frac{1}{m-1}} \leq \dots \leq (a_2)^{\frac{1}{2}} \leq a_1. \quad (5.13)$$

If we have equality in each case of (5.12), then we get all equalities in (5.13).

Proof. Fix $1 \leq k < m$, and multiply together cases of (5.12) for $j \leq k$ to get

$$(a_0a_2)(a_1a_3)^2 \dots (a_{k-1}a_{k+1})^k \leq (a_1)^2(a_2)^4 \dots (a_k)^{2k},$$

which reduces to

$$a_{k+1}^k \leq a_k^{k+1}.$$

Taking both sides to the power $\frac{1}{k(k+1)}$ yields

$$(a_{k+1})^{\frac{1}{k+1}} \leq (a_k)^{\frac{1}{k}}.$$

□

We are now ready to prove the main chain of inequalities.

Proof of Theorem 5.1. For $0 \leq j \leq d$ let us begin by defining

$$b_j = \frac{\kappa_{d-j}}{\binom{d}{j}} V_j(\Pi Z).$$

As an immediate consequence of Theorem 5.15, for $1 \leq j \leq d-1$ we get

$$b_j^2 \geq b_{j-1}b_{j+1}. \quad (5.14)$$

Notice that for any choice of Z , we have $b_0 = \kappa_d \neq 0$, so we can construct new sequence $a_0, a_1, \dots, a_d$ by setting

$$a_j = \frac{b_j}{b_0}.$$

Clearly $a_0 = 1$, and from (5.14) we can immediately deduce that for $1 \leq j \leq d-1$

$$a_j^2 \geq a_{j-1}a_{j+1}. \quad (5.15)$$

Directly applying Lemma 5.16, we then get

$$(a_d)^{\frac{1}{d}} \leq (a_{d-1})^{\frac{1}{d-1}} \leq \dots \leq (a_2)^{\frac{1}{2}} \leq a_1. \quad (5.16)$$

Using Lemma 5.13, this last chain of inequalities can be rewritten as

$$C_{d,d}\mathcal{H}_d^{d-1}(Z) \leq C_{d-1,d}\mathcal{H}_{d-1}^{d-1}(Z) \leq \dots \leq C_{2,d}\mathcal{H}_2^{d-1}(Z) \leq C_{1,d}\mathcal{H}_1^{d-1}(Z).$$

If Z is the boundary of a ball in $\mathbb{R}^d$, then immediately from the definition we know that the projection body ΠZ is a ball centred at the origin. By Theorem 5.15, each case of (5.15) then becomes an equality, which by Lemma 5.16 implies that each inequality in (5.16) becomes equality too. □

5.4 Understanding generalisations of the directional volumes from a geometric perspective

In this section we spend some time trying to understand what these quantities are from a more geometric perspective. In particular, we calculate $\mathcal{H}_2^1(C)$ for a large class of curves $C \subset \mathbb{R}^2$.

Symmetric curves The first non-trivial case we will study is where C forms a closed convex loop in $\mathbb{R}^2$.

Lemma 5.17. *For any convex body $K \subset \mathbb{R}^2$ which is symmetric about the origin we have*

$$\mathcal{H}_2^1(\partial K) = 2\sqrt{2}|K|^{\frac{1}{2}}.$$

Remark 5.18. Notice that $\mathcal{H}_2^1(Z)$ is invariant under orthogonal transformations, so the lemma above holds for any convex body which is symmetric about any point $x \in \mathbb{R}^2$.

Sketch of proof for Lemma 5.17. Firstly we will show that for any convex graph $C \in \mathbb{R}^2$ we have

$$\mathcal{H}_2^1(C)^2 = 4|\text{Conv}(C)|.$$

Let C be a convex graph in $\mathbb{R}^2$, so that for some function $\gamma : [0, 1] \rightarrow \mathbb{R}$, we have

$$C = \{(x, \gamma(x)) \mid x \in [0, 1]\}.$$

Now define $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ by $\Gamma(x) = (x, \gamma(x))$, and let $n(z)$ denote the unit normal to $z \in C$. Without loss of generality we may assume that $\gamma(0) = 0$. Since $J\Gamma(x) = \sqrt{1 + \gamma'(x)^2}$, applying the area formula - see [17], Section 3.3 for statement and proof - gives

$$\begin{aligned} \mathcal{H}_2^1(C)^2 &:= \int_C \int_C |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_2) d\mathcal{H}^1(z_1) \\ &= \int_0^1 \int_0^1 \left| \det \begin{pmatrix} \frac{1}{\sqrt{1+\gamma'(x)^2}} & \frac{1}{\sqrt{1+\gamma'(y)^2}} \\ \frac{-\gamma'(x)}{\sqrt{1+\gamma'(x)^2}} & \frac{-\gamma'(y)}{\sqrt{1+\gamma'(y)^2}} \end{pmatrix} \right| \sqrt{1 + \gamma'(x)^2} \sqrt{1 + \gamma'(y)^2} d\mathcal{L}^{n-1}(x) d\mathcal{L}^{n-1}(y) \\ &= \int_0^1 \int_0^1 |\gamma'(x) - \gamma'(y)| dx dy \\ &= 2 \int_0^1 \int_0^y \gamma'(y) - \gamma'(x) dx dy \\ &= 2 \int_0^1 y\gamma'(y) - \gamma(y) + \gamma(0) dy \\ &= 2 \int_0^1 [y\gamma(y)]' - 2\gamma(y) dy \\ &= 2 \left(\gamma(1) - 2 \int_0^1 \gamma(y) dy \right) = 4|\text{Conv}(C)|. \end{aligned}$$

Claim 5.19. For every convex body $K \subset \mathbb{R}^2$, there exists a line L which splits ∂K into two convex graphs C_1 and C_2 . Moreover, if K is symmetric about the origin we can ensure that $C_2 = -C_1$.

Sketch of proof of claim. For convex polygons the statement is easy to verify. Consider the boundary ∂P of a convex polygon $P \subset \mathbb{R}^2$, let $N_1, \dots, N_k$ be the unit normals to the edges ordered by the angle they make with the positive x -axis. Let $v_1, \dots, v_k$ be the vertices of ∂P such that N_i is normal to the edge connecting vertices v_i and v_{i+1} . Observe that N_1 lies in the first quadrant. Set $j_1 \in \{1, \dots, k\}$ to be the smallest index such that

$$N_1 \cdot N_{j_1} < 0,$$

and set $j_2 \in \{1, \dots, k\}$ to be the smallest index larger than j_1 such that

$$N_1 \cdot N_{j_2} > 0.$$

Then for $j_1 \leq i \leq j_2 - 1$ we have

$$N_1 \cdot N_i < 0.$$

It follows that the line L passing through v_{j_1} and v_{j_2} splits ∂P as required.

Suppose that P is symmetric about the origin. Orthogonal transformations, in particular rotations, will not affect the problem so without loss of generality we may assume that v_1 lies on the positive x -axis. It follows that the x -axis splits ∂P into two graphs C_1, C_2 and by symmetry we have $C_2 = -C_1$.

We can approximate any convex body K using a sequence of polygons and it seems clear that the corresponding lines would converge to a line which splits ∂K into two graphs. $\square$

Let $K \subset \mathbb{R}^2$ be a symmetric convex body with $n(z)$ denoting the outward facing normal at $z \in \partial K$. Apply the previous claim to get a line L passing through the origin which splits ∂K into two convex graphs C_1, C_2 such that $C_2 = -C_1$. Without loss of generality we may assume that C_1 and C_2 are graphs over the x -axis, then for some $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$ we have

$$C_1 = \{(x, f(x)) \mid x \in A\}.$$

Let us define $F : A \rightarrow \mathbb{R}^2$ by $F(x) = (x, f(x))$, so that we have $C_1 = F(A)$. It follows that

$$C_2 = \{(-x, -f(x)) \mid x \in A\}.$$

Let us define $G : A \rightarrow \mathbb{R}^2$ by $G(x) = (-x, -f(x))$. Observe that $JG = JF$, so by the area formula we have

$$\begin{aligned} \int_{F(A)} \int_{G(A)} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_2) d\mathcal{H}^1(z_1) &= \int_{F(A)} \int_A |n(z_1) \wedge n((-x, -f(x)))| JG dx d\mathcal{H}^1(z_1) \\ &= \int_{F(A)} \int_A |n(z_1) \wedge n((x, f(x)))| JF dx d\mathcal{H}^1(z_1) \\ &= \int_{F(A)} \int_{F(A)} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_1) d\mathcal{H}^1(z_2) \\ &= \mathcal{H}_2^1(F(A))^2. \end{aligned}$$

Similarly, we have

$$\int_{G(A)} \int_{F(A)} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_2) d\mathcal{H}^1(z_1) = \mathcal{H}_2^1(G(A))^2.$$

Hence

$$\begin{aligned}
\mathcal{H}_2^1(\partial K)^2 &= \int_{\partial K} \int_{\partial K} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_1) d\mathcal{H}^1(z_2) \\
&= \int_{C_1 \cup C_2} \int_{C_1 \cup C_2} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_1) d\mathcal{H}^1(z_2) \\
&= \mathcal{H}_2^1(C_1) + \mathcal{H}_2^1(C_2) + \int_{C_2} \int_{C_1} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_1) d\mathcal{H}^1(z_2) \\
&\quad + \int_{C_1} \int_{C_2} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_2) d\mathcal{H}^1(z_1) \\
&= \mathcal{H}_2^1(C_1) + \mathcal{H}_2^1(C_2) + \mathcal{H}_2^1(C_2) + \mathcal{H}_2^1(C_1) \\
&= 2(4|\text{Conv}(C_1)| + 4|\text{Conv}(C_2)|) \\
&= 8|K|.
\end{aligned}$$

□

Corollary 5.20. *For any convex body $K \subset \mathbb{R}^2$ which is symmetric about the origin we have*

$$\mathcal{H}_2^1(\partial K) = \sqrt{2} (\mathcal{H}^2(K - K))^{\frac{1}{2}},$$

where $K - K$ denotes the difference body of K defined as

$$K - K := \{k_1 - k_2 | k_1, k_2 \in K\}.$$

Proof. Notice that for symmetric K we have $K - K = 2K$.

□

Non-symmetric curves Let Γ_1 and Γ_2 be two convex graphs whose union form the boundary of a convex set K . From the proof of Lemma 5.17, recall that

$$\mathcal{H}_2^1(\Gamma_1 \cup \Gamma_2) = \mathcal{H}_2^1(\Gamma_1) + \mathcal{H}_2^1(\Gamma_2) + 2 \int_{\Gamma_2} \int_{\Gamma_1} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_1) d\mathcal{H}^1(z_2).$$

Let us study the following quantity which we will refer to as the cross term:

$$\mathcal{H}_2^1(\Gamma_1, \Gamma_2) = \int_{\Gamma_2} \int_{\Gamma_1} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_1) d\mathcal{H}^1(z_2).$$

Recall that we have already proven that $\mathcal{H}_2^1(\Gamma) = 4\mathcal{H}^2(\text{Conv}(\Gamma))$ for any convex graph Γ , furthermore we have

$$\mathcal{H}_2^1(\Gamma_1 \cup \Gamma_2) = \mathcal{H}_2^1(\Gamma_1) + \mathcal{H}_2^1(\Gamma_2) + 2\mathcal{H}_2^1(\Gamma_1, \Gamma_2).$$

Let us restrict to the case where Γ_1 and Γ_2 are graphs of functions $\gamma_1 : [0, 1] \rightarrow \mathbb{R}$ and $\gamma_2 : [0, 1] \rightarrow \mathbb{R}$ such that

$$\gamma_1(0) = \gamma_1(1) = \gamma_2(1) = \gamma_2(0) = 0.$$

Let Γ_1 be a graph lying below the x -axis and we will consider a series of piecewise linear curves $(\Gamma_2^j)_j$ approximating Γ_2 , such that each $\Gamma_1 \cup \Gamma_2^j$ form the boundary of some convex set. The simplest case we can consider is an approximation of Γ_2 by the single line segment $\Gamma_2^0 = L^0$ joining $\gamma_1(0)$ and $\gamma_1(1)$, namely the interval $[0, 1]$ of the x -axis. Applying the mean value

theorem and using the strict convexity of Γ_1 , define $c \in [0, 1]$ to be the unique point such that $\gamma'_1(c) = 0$. Now we see that

$$\begin{aligned}\mathcal{H}_1^2(\Gamma_1, \Gamma_2^0) &= \int_{\Gamma_2^0} \int_{\Gamma_1} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_1) d\mathcal{H}^1(z_2) \\ &= \int_0^1 |\gamma'_1(t)| dt = \int_0^c -\gamma'_1(t) dt + \int_c^1 \gamma'_1(t) dt \\ &= 2(c\gamma_1(1) - \gamma_1(c)) = 2 \det \begin{pmatrix} -c & 1-c \\ -\gamma_1(c) & \gamma_1(1) - \gamma_1(c) \end{pmatrix} \\ &= 4\mathcal{H}^2(T),\end{aligned}$$

where T is the triangle with vertices $(0, 0), (c, \gamma_1(c))$ and $(1, 0)$. Note that $4\mathcal{H}^2(T) = 2\mathcal{H}^2(Q)$ where Q is the rectangle containing Γ_1 , I mention this because I have feeling that this cross term could be related to the area of the smallest symmetric convex body containing Γ_1 , which in this case would be Q . Thus we have

$$\mathcal{H}_1^2(\Gamma_1 \cup \Gamma_2^0) = 8\mathcal{H}^2(\text{Conv}(\Gamma_1 \cup \Gamma_2^0)) + 4\mathcal{H}^2(Q \setminus \text{Conv}(\Gamma_1 \cup \Gamma_2^0))$$

Next consider the union of two line segments $\Gamma_2^1 = L_0^1 \cup L_1^1$ approximating Γ_2 . Set L_0^1 is the line segment joining the points $(0, 0)$ and $(\frac{1}{2}, \gamma_2(\frac{1}{2}))$, then set L_1^1 to be the line segment joining the points $(\frac{1}{2}, \gamma_2(\frac{1}{2}))$ and $(1, 0)$. Similar to before we wish to define unique points $c_0, c_1 \in [0, 1]$, at which the tangent vector $(1, \gamma'_1(t))$ is parallel to the directions of the lines L_0^1 and L_1^1 respectively (note that $c_1 \leq c_0$). Now we have

$$\begin{aligned}\mathcal{H}_2^1(\Gamma_1, \Gamma_2^1) &= \int_{\Gamma_2^1} \int_{\Gamma_1} |n(z_1) \wedge n(z_2)| d\mathcal{H}^1(z_1) d\mathcal{H}^1(z_2) \\ &= \int_0^{\frac{1}{2}} \int_0^1 \left| \gamma'_1(t) - 2\gamma_2\left(\frac{1}{2}\right) \right| dt ds + \int_{\frac{1}{2}}^1 \int_0^1 \left| \gamma'_1(t) + 2\gamma_2\left(\frac{1}{2}\right) \right| dt ds \\ &= \frac{1}{2} \left[\int_0^{c_0} 2\gamma_2\left(\frac{1}{2}\right) - \gamma'_1(t) dt + \int_{c_0}^1 \gamma'_1(t) - 2\gamma_2\left(\frac{1}{2}\right) dt \right. \\ &\quad \left. + \int_0^{c_1} -2\gamma_2\left(\frac{1}{2}\right) - \gamma'_1(t) dt + \int_{c_1}^1 \gamma'_1(t) + 2\gamma_2\left(\frac{1}{2}\right) dt \right] \\ &= 2 \left[\gamma_2\left(\frac{1}{2}\right) (c_0 - c_1) - \frac{1}{2}(\gamma_1(c_0) + \gamma_1(c_1)) \right].\end{aligned}$$

Observe that if $|\gamma'_1(0)|, |\gamma'_1(1)| < \infty$ we can choose $\gamma_2(\frac{1}{2})$ large enough so that $c_0 = 0$ and $c_1 = 1$, then we would have $\gamma_1(c_0) = \gamma_1(c_1) = 0$ hence

$$\mathcal{H}_2^1(\Gamma_1, \Gamma_2^1) = 4|\text{Conv}(\Gamma_2^1)|.$$

It follows that

$$\begin{aligned}\mathcal{H}_2^1(\Gamma_1 \cup (L_0^1 \cup L_1^1)) &= 8[\mathcal{H}^2(\text{Conv}(L_0^1 \cup L_1^1)) + \mathcal{H}^2(\text{Conv}(\Gamma_1))] \\ &\quad + 4[\mathcal{H}^2(\text{Conv}(L_0^1 \cup L_1^1)) - \mathcal{H}^2(\text{Conv}(\Gamma_1))].\end{aligned}$$

In general, if $\Gamma_2^j := \bigcup_{i=1}^n L_i^{n-1}$ represents the union of n line segments approximating Γ_2 , we get

$$\mathcal{H}_1^2(\Gamma_1, \Gamma_2^j) = 2 \left[\sum_{i=1}^{n-1} \gamma_2\left(\frac{i}{n}\right) (c_{i-1} - c_i) - \frac{1}{n} \left(\sum_{i=0}^{n-1} \gamma_1(c_i) \right) \right].$$

For very simple examples of convex bodies with piecewise linear boundaries, this quantity seems related to the area of some smallest symmetric convex body which contains the original convex body. Note the following examples. Let T be a triangle, then we have

$$\mathcal{H}_2^1(\partial T) = 8\mathcal{H}^2(T) + 4\mathcal{H}^2(Q \setminus T)$$

where Q_T the smallest parallelogram containing T (this also is true if we set Q_T to be the smallest symmetric hexagon containing T). Next let R be any quadrilateral, then we have

$$\mathcal{H}_2^1(\partial R) = 8\mathcal{H}^2(R) + 4\mathcal{H}^2(Q \setminus R),$$

where Q seems to be the smallest symmetric hexagon containing R . Consider a regular pentagon P , then it turns out that

$$\mathcal{H}_2^1(\partial P) < 8\mathcal{H}^2(P) + 4\mathcal{H}^2(P \setminus D)$$

where D is the regular decagon made by sticking a rotated version of P onto itself and taking the convex hull. This seemed the obvious candidate for the smallest symmetric convex body containing P .

Remark 5.21. For the three above examples we actually have

$$\mathcal{H}_2^1(K) = \sqrt{2} (\mathcal{H}^2(K - K))^{\frac{1}{2}}$$

Sarron's formula Let $X = [x_1 x_2 \dots, x_n] \subset \mathbb{R}^2$ be a simple polyhedron and for $1 \leq j \leq n$ let α_i be the angles measured from the vertical line anticlockwise to the edge $[x_i x_{i+1}]$ (where the last edge considered is $[x_n x_1]$). Let ℓ_i denote the length of the edge $[x_i x_{i+1}]$. If we further assume that X is convex so that $\alpha_j - \alpha_i \in [0, \pi]$ for $i < j$, then by Lemma 5.17

$$\begin{aligned} |X| &= \frac{1}{2 \cdot 2^2} \sum_{i=1}^n \sum_{j=1}^n \ell_i \ell_j \sin(\alpha_j - \alpha_i) = \frac{1}{2} \sum_{i < j} \ell_i \ell_j |\sin(\alpha_j - \alpha_i)| \\ &= \frac{1}{2} \sum_{i < j} \ell_i \ell_j \sin(\alpha_j - \alpha_i). \end{aligned}$$

It was rather miraculous when Silouanos Brazitikos brought to our attention the appearance of the following neat formula originally due to Sarron, which is mentioned as an exercise in [35], page 372:

Lemma 5.22 (Sarron's formula). *For any simple polyhedron $X = [x_1 x_2 \dots, x_n] \subset \mathbb{R}^2$ we have*

$$|X| = \frac{1}{2} \sum_{i < j} \ell_i \ell_j \sin(\alpha_j - \alpha_i).$$

Although Lemma 5.17 only applies to convex sets, Sarron's formula also holds if X is non-convex where

$$\frac{1}{2} \sum_{i < j} \ell_i \ell_j |\sin(\alpha_j - \alpha_i)| \neq \frac{1}{2} \sum_{i < j} \ell_i \ell_j \sin(\alpha_j - \alpha_i),$$

which rules out the possibility of Lemma 5.17 holding for non-convex sets.

Chapter 6

Vector-valued Maclaurin inequalities

Most of the material covered in this chapter appears in a joint paper with Silouanos Brazitikos, which has been accepted for publication - see [38]. Recall that in Section 5.2, where we calculated the quantities $\mathcal{H}_1^{d-1}(\partial R), \dots, \mathcal{H}_d^{d-1}(\partial R)$ for any d -dimensional box $R \subset \mathbb{R}^d$, we encountered a certain chain of inequalities for the k -means of any sequence of positive numbers:

Theorem 6.1 (The Maclaurin inequality - [34]). *For any sequence $x_1, \dots, x_m$ of positive real numbers and $1 < k \leq m$*

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [m]} x_{i_1} \cdots x_{i_k}}{\binom{m}{k}} \right)^{\frac{1}{k}} \leq \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [m]} x_{i_1} \cdots x_{i_{k-1}}}{\binom{m}{k-1}} \right)^{\frac{1}{k-1}}.$$

It should be noted that Theorem 6.1 gives a refinement of the arithmetic-geometric inequality since the geometric mean and arithmetic mean appear as the smallest and largest quantities respectively in the chain of inequalities given by this result.

Suppose now we want to prove something similar for vectors. Let $v_1, \dots, v_m \in \mathbb{R}^d$ with $d \leq m$, and for any $\{i_1, \dots, i_k\} \subset [m]$ with $k \leq d$ define $|v_{i_1} \wedge \cdots \wedge v_{i_k}|$ to be the k -dimensional volume of the parallelotope spanned by $v_{i_1}, \dots, v_{i_k}$. If we fix $m = d$, then by the arithmetic-geometric mean inequality along with basic facts about the volume of a parallelotope, we can write

$$|v_1 \wedge \cdots \wedge v_d|^{\frac{1}{d}} \leq \prod_{j=1}^d \|v_j\|^{\frac{1}{d}} \leq \frac{\sum_{j=1}^d \|v_j\|}{d}. \quad (6.1)$$

In light of the Maclaurin inequality, one might hope to prove a similar refinement of (6.1):

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}} \leq \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}, \quad (6.2)$$

for $1 < k \leq d$. Note that if $v_1, \dots, v_d$ are orthogonal, then any k -fold wedge product $|v_{i_1} \wedge \cdots \wedge v_{i_k}|$ will just be equal to a k -fold product of numbers, namely $\|v_{i_1}\| \cdots \|v_{i_k}\|$, so from (6.2) we exactly recover the classical Maclaurin inequality. At the end of Section 5.2, we saw that if one could establish (6.2), it would follow that for any parallelotope $P \subset \mathbb{R}^d$

$$\left[\frac{1}{k! \binom{d}{k}} \right]^{\frac{1}{k}} \mathcal{H}_k^{d-1}(\partial P) \leq \left[\frac{1}{(k-1)! \binom{d}{k-1}} \right]^{\frac{1}{k-1}} \mathcal{H}_{k-1}^{d-1}(\partial P),$$

with equality if $P = [0, \ell]^d$ for any $\ell > 0$.

Although it is not known whether (6.2) holds in general, using elementary techniques we have been able to establish various special cases.

Theorem 6.2. (6.2) holds for linearly independent vectors $v_1, \dots, v_d \in \mathbb{R}^d$ and $k = 2, 3, d-1, d$ in all dimensions d

Given the difficulty in proving special cases of (6.2), let us consider the following weaker inequality between non-consecutive k -means:

$$\left(\frac{\sum_{\{i_1, \dots, i_l\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_l}|}{\binom{d}{l}} \right)^{\frac{1}{l}} \leq \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}}, \quad (6.3)$$

where $1 < k < l \leq d$. Using techniques developed in the proof of Theorem 6.2, we are able to establish various cases of this non-consecutive variant

Theorem 6.3. (6.3) holds for linearly independent vectors $v_1, \dots, v_d \in \mathbb{R}^d$ in the following cases:

1. $k = 1$ and every $l \in \{2, \dots, d\}$.
2. $k = 2$ and $l = d - 1$.

Remark 6.4. Of course using Theorem 6.2 we also get the result for any combination among $k = 1, 2$ and $l = d - 1, d$.

As it turns out, we are also able to prove an ℓ^2 -version of (6.2) using some neat facts about determinants:

Theorem 6.5 (Vector-valued Maclaurin inequality with squares). *For linearly independent vectors $v_1, \dots, v_d \in \mathbb{R}^d$ and $1 < k \leq d$, the following inequality holds*

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|^2}{\binom{d}{k}} \right)^{\frac{1}{k}} \leq \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|^2}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}.$$

In light of this, it seems natural to ask whether we can replace the squares with other powers. That is, for which range of p could we hope to prove inequalities of the form

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|^p}{\binom{d}{k}} \right)^{\frac{1}{kp}} \leq \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|^p}{\binom{d}{k-1}} \right)^{\frac{1}{(k-1)p}} ? \quad (6.4)$$

This question is not so interesting in the setting of classical Maclaurin inequality. Indeed, for $-\infty < p < \infty$ we can replace the positive numbers $x_1, \dots, x_m$ with their p -th powers which will also be a sequence of positive numbers. In this way, Theorem 6.1 is equivalent to the statement that for any positive numbers $x_1, \dots, x_m$ and $-\infty < p < \infty$ we have

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [m]} x_{i_1}^p \dots x_{i_k}^p}{\binom{m}{k}} \right)^{\frac{1}{k}} \leq \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [m]} x_{i_1}^p \dots x_{i_{k-1}}^p}{\binom{m}{k-1}} \right)^{\frac{1}{(k-1)}}.$$

Furthermore, taking $p \rightarrow \infty$ this last inequality becomes

$$\left(\max_{\{i_1, \dots, i_k\} \subset [m]} x_{i_1} \cdots x_{i_k} \right)^{\frac{1}{k}} \leq \left(\max_{\{i_1, \dots, i_{k-1}\} \subset [m]} x_{i_1} \cdots x_{i_{k-1}} \right)^{\frac{1}{k-1}},$$

which holds trivially. Similarly, taking $p \rightarrow -\infty$ yields

$$\left(\max_{\{i_1, \dots, i_k\} \subset [m]} x_{i_1}^{-1} \cdots x_{i_k}^{-1} \right)^{\frac{1}{k}} \leq \left(\max_{\{i_1, \dots, i_{k-1}\} \subset [m]} x_{i_1}^{-1} \cdots x_{i_{k-1}}^{-1} \right)^{\frac{1}{k-1}},$$

which is exactly the previous inequality with positive numbers $x_1^{-1}, \dots, x_m^{-1}$. In the setting of vectors it is not immediately obvious how altering the value of p changes the nature of the inequality. Note that the p -th power mean of the positive numbers $|v_{i_1} \wedge \cdots \wedge v_{i_k}|$ is exactly

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|^p}{\binom{m}{k}} \right)^{\frac{1}{p}},$$

which is defined for all $p \in [-\infty, \infty]$. For $p = 0$ we recover the geometric mean by observing that

$$\lim_{p \rightarrow 0} \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|^p}{\binom{m}{k}} \right)^{\frac{1}{p}} = \left(\prod_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}| \right)^{\frac{1}{\binom{m}{k}}}.$$

Moreover, taking $p = \infty$ reveals the maximum since

$$\lim_{p \rightarrow \infty} \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|^p}{\binom{m}{k}} \right)^{\frac{1}{p}} = \max_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|.$$

Using an inequality of Szasz's for subdeterminants, we can establish a generalisation with $m \geq d$ vectors for the endpoint cases of (6.4) with $p = 0$ and $p = \infty$.

Theorem 6.6. *For any m -tuple of vectors $v_1, \dots, v_m \in \mathbb{R}^d$ and $1 < k \leq d \leq m$, the following inequality holds*

$$\left(\prod_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}| \right)^{\frac{1}{\binom{m}{k}}} \leq \left(\prod_{\{i_1, \dots, i_{k-1}\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}| \right)^{\frac{1}{\binom{m}{k-1}}}$$

Theorem 6.7. *Fix vectors $v_1, \dots, v_m \in \mathbb{R}^d$ with $1 \leq d \leq m$. Then for any $1 < k \leq d$, the following inequality holds*

$$\left(\max_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}| \right)^{\frac{1}{k}} \leq \left(\max_{\{i_1, \dots, i_{k-1}\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}| \right)^{\frac{1}{k-1}}. \quad (6.5)$$

Remark 6.8. The inequality corresponding to $p = -\infty$ can be written

$$\left(\min_{\{i_1, \dots, i_{k-1}\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}| \right)^{\frac{1}{k-1}} \leq \left(\min_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}| \right)^{\frac{1}{k}},$$

which does not hold in general.

In light of results stated above, we make the following conjecture in the restricted range $p \in [0, \infty]$:

Conjecture 6.9 (The vector-valued Maclaurin inequality). *Fix vectors $v_1, \dots, v_m \in \mathbb{R}^d$ with $1 \leq d \leq m$. Then for all $p \in [0, \infty]$ and $2 \leq k \leq d$, the following inequality holds*

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_k}|^p}{\binom{m}{k}} \right)^{\frac{1}{kp}} \leq \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|^p}{\binom{m}{k-1}} \right)^{\frac{1}{(k-1)p}},$$

with equality if and only if $m = d$ and the v_i form an orthonormal basis.

In Section 6.3, we discuss some connections between these seemingly elementary inequalities and open questions in convex geometry. In the case where $p = 1$, which was our original motivation for studying these inequalities, the vector-valued Maclaurin inequality can be rewritten in terms of intrinsic volumes. In this geometric form, these seemingly elementary inequalities are closely related to the far-reaching Aleksandrov–Fenchel inequality from convex geometry. This provides us with a direct connection between the Aleksandrov–Fenchel inequality and Newton’s inequality:

Theorem 6.10 (Newton’s inequality - [34]). *Let $x_1, \dots, x_m$ be positive real number. For any $1 < k < m$ the following inequality holds*

$$\left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [m]} x_{i_1} \dots x_{i_{k-1}}}{\binom{m}{k-1}} \right) \left(\frac{\sum_{\{i_1, \dots, i_{k+1}\} \subset [m]} x_{i_1} \dots x_{i_{k+1}}}{\binom{m}{k+1}} \right) \leq \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [m]} x_{i_1} \dots x_{i_k}}{\binom{m}{k}} \right)^2.$$

Indeed, by a simple application of Lemma 5.16, one can deduce the classical Maclaurin inequality as a consequence of Theorem 6.10. Similarly one would be able to deduce the vector-valued Maclaurin inequality for $p = 1$ from a corresponding vector-valued version of Newton’s inequality of the form

$$\left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|}{\binom{m}{k-1}} \right) \left(\frac{\sum_{\{i_1, \dots, i_{k+1}\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_{k+1}}|}{\binom{m}{k+1}} \right) \leq \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{m}{k}} \right)^2, \quad (6.6)$$

where $2 \leq k \leq d - 1$. It is worth noting that up until this point, in the literature, the Aleksandrov–Fenchel inequality has been referred to as a Newton-type inequality, simply because it is of the same form (square greater than a product).

To illustrate the connection between the vector-valued Maclaurin inequality and the Aleksandrov–Fenchel inequality, let us denote by P the polytope defined as the following Minkowski sum of line segments

$$P = \sum_{j=1}^m \frac{1}{2} [-v_j, v_j].$$

Using this notation, Conjecture 6.9 for $p = 1$ exactly says that for $1 < k \leq d$ we have

$$\left(\frac{V_k(P)}{\binom{m}{k}} \right)^{\frac{1}{k}} \leq \left(\frac{V_{k-1}(P)}{\binom{m}{k-1}} \right)^{\frac{1}{k-1}}. \quad (6.7)$$

This form of inequality is closely related to the Aleksandrov-Fenchel inequality. In particular, Conjecture 6.9 for $p = 1$ would recover the dimension free estimate of McMullen [31] restricted to the class of zonoids. Furthermore, (6.7) provides the equality case for a generalisation of certain reverse isoperimetric-type inequalities proved in [26] for the intrinsic volumes. In Section 6.3, we discuss these connections in more detail and suggest an elegant method for establishing the $p = 1$ case of Conjecture 6.9 using the Aleksandrov-Fenchel inequality directly, which requires the existence of a convex body satisfying certain projection conditions. In $\mathbb{R}^2$, we know how to construct such a convex body explicitly but we do not know how to establish the existence for $d \geq 3$. Finding such a convex body seems to be an interesting problem.

Using a more crude application of the Aleksandrov-Fenchel inequality, one can establish the vector-valued Maclaurin inequality for $p = 1$ and $m = d$ with a constant greater than 1, which we presume is not sharp:

Theorem 6.11. *For any d -tuple of vectors $v_1, \dots, v_d \in \mathbb{R}^d$ and $2 < k \leq d$, the following inequality holds*

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}} \leq \frac{2(d-k+1)}{(d-k+2)} \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}$$

Remark 6.12. The $k = 2$ case is missing here, but from Theorem 6.2 we already know this holds in all dimensions with constant 1.

6.1 Principal minors and Szasz's inequality

Let us begin by reviewing some classical results of linear algebra in an attempt to write these inequalities involving k -fold wedge product in terms of determinants. In fact, once we have done this it is not difficult to prove Theorems 6.5, 6.6 and 6.7. Let us recall the following definition of a principal minor

Definition 6.13 (Principal minors). Let M be an $n \times n$ matrix. For $S \subset [n]$, define M_S to be submatrix constructed by removing rows and edges with indexes not in S . The set of principal submatrices of M is defined as $\{M_S \mid S \subset [n]\}$. Furthermore, the set of principal minors of M is defined as $\{\det(M_S) \mid S \subset [n]\}$. For $1 \leq k \leq n$, we define the principal k -submatrices and principal k -minors of M by adding the condition that $|S| = k$.

Firstly we will rewrite the square of each k -dimensional volume $|v_{i_1} \wedge \dots \wedge v_{i_k}|^2$ as the determinant of a $k \times k$ submatrix of some fixed $m \times m$ matrix. Let A denote the $d \times m$ matrix with columns $v_1, \dots, v_m$ and B the $d \times k$ matrix with columns $v_1, \dots, v_k$ for some $1 \leq k \leq d$. By definition, we have

$$|v_1 \wedge \dots \wedge v_k|^2 = \det(B^T B).$$

Furthermore, $B^T B$ is the principal k -submatrix of $A^T A$ constructed by removing the last $(d-k)$ rows and columns. In this way we see that the sum of terms $|v_{i_1} \wedge \dots \wedge v_{i_k}|^2$ over all $\{i_1, \dots, i_k\} \subset [d]$ is equal to the sum of all principal k -minors of $A^T A$. The next lemma then allows us to work with the sum of principal k -minors of $A^T A$.

Lemma 6.14 (Sum of principal minors). *Let M be a $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct). For $1 \leq k \leq n$, It follow that*

$$\sum_{|S|=k} \det(M_S^T M_S) = \sum_{|S|=k} \prod_{i \in S} \lambda_i^2.$$

A proof of this lemma can be found in [32] - see (7.1.6). Now we are ready to establish the vector-valued Maclaurin inequality with $p = 2$:

Proof of Theorem 6.5 Here we let A denote the square matrix with columns $v_1, \dots, v_d$ and eigenvalues $\lambda_1, \dots, \lambda_d$ (not necessarily distinct.) Then for $\{i_1, \dots, i_k\} \subset [d]$, define $A_{i_1, \dots, i_k}$ to be the $d \times k$ matrix with columns $v_{i_1}, \dots, v_{i_k}$. Now we use Lemma 6.14 along with Theorem 6.1, which is precisely the classical Maclaurin inequality, to write

$$\begin{aligned} \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|^2}{\binom{d}{k}} \right)^{\frac{1}{k}} &= \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} \det(A_{i_1, \dots, i_k}^T A_{i_1, \dots, i_k})}{\binom{d}{k}} \right)^{\frac{1}{k}} \\ &= \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} \lambda_{i_1}^2 \dots \lambda_{i_k}^2}{\binom{d}{k}} \right)^{\frac{1}{k}} \\ &\leq \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d]} \lambda_{i_1}^2 \dots \lambda_{i_{k-1}}^2}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}} \\ &= \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d]} \det(A_{i_1, \dots, i_{k-1}}^T A_{i_1, \dots, i_{k-1}})}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}} \\ &= \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|^2}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}. \end{aligned}$$

Question It would be interesting to investigate whether one can find a $d \times d$ matrix M such that for every $A \subset [d]$ with $|A| = k$

$$\det(M_A) = \left| \bigwedge_{i \in A} v_i \right|,$$

and for every $B \subset [d]$ with $|B| = k - 1$

$$\det(M_B) = \left| \bigwedge_{j \in B} v_j \right|.$$

If there exists such a matrix M , it would allow us to establish the vector-valued Maclaurin inequality with $p = 1$ using a very similar argument to the one given in the proof of Theorem 6.5.

The following inequality for determinants of sub-matrices is originally due to Szasz:

Lemma 6.15 (Szasz). *Let M be some $n \times n$ matrix. For $1 < k \leq n$ the following inequality holds*

$$\left(\prod_{|A|=k} \det(M_A) \right)^{\frac{1}{\binom{n-1}{k-1}}} \leq \left(\prod_{|B|=k-1} \det(M_B) \right)^{\frac{1}{\binom{n-1}{k-2}}}.$$

A proof of this using elementary methods can be found in [23]. Simple applications of Lemma 6.15 allow us to deduce the vector-valued Maclaurin inequality with $p = 0$ and $p = \infty$.

Proof of Theorem 6.6. Let A denote the $d \times m$ matrix with columns $v_1, \dots, v_m$ and $A_{i_1, \dots, i_k}$ the $d \times k$ matrix with columns $v_{i_1}, \dots, v_{i_k}$ for some $1 < k \leq d$, then we have

$$|v_{i_1} \wedge \dots \wedge v_{i_k}|^2 = \det(A_{i_1, \dots, i_k}^T A_{i_1, \dots, i_k}).$$

As we mentioned earlier, $A_{i_1, \dots, i_k}^T A_{i_1, \dots, i_k}$ can also be seen as a principal submatrix of $A^T A$, which is an $m \times m$ matrix. Hence, by Szasz's lemma we have

$$\left(\prod_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_k}|^2 \right)^{\frac{1}{\binom{m-1}{k-1}}} \leq \left(\prod_{\{i_1, \dots, i_{k-1}\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|^2 \right)^{\frac{1}{\binom{m-1}{k-2}}}.$$

Taking square root, and noting that $\binom{m-1}{k-1} = \binom{m}{k} \cdot \frac{k}{m}$ and $\binom{m-1}{k-2} = \binom{m}{k-1} \cdot \frac{k-1}{m}$, the above inequality can be written as

$$\left(\left(\prod_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_k}| \right)^{\frac{1}{\binom{m}{k}}} \right)^{\frac{1}{k}} \leq \left(\left(\prod_{\{i_1, \dots, i_{k-1}\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}| \right)^{\frac{1}{\binom{m}{k-1}}} \right)^{\frac{1}{k-1}}, \quad (6.8)$$

□

Remark 6.16. In (6.8) we see that we have geometric means appearing inside the first set of parenthesis on both sides. Intriguingly, simply replacing these geometric means by arithmetic means reveals the vector-valued Maclaurin inequality with $p = 1$. So if we think of the vector-valued Maclaurin inequality with $p = 1$ as a chain of inequalities for a sequence of arithmetic means, then (6.8) can be thought of as an analogous chain of inequalities for the corresponding geometric means.

Next we deal with the case $p = \infty$.

Proof of Theorem 6.7. Let $\{i_1, \dots, i_k\} \subset [m]$ be the indices where the left hand side is maximised and B be the matrix with columns $v_{i_1}, \dots, v_{i_k}$. Now, using Szasz's inequality for $M = B^T B$, $n = k$ and taking square root of both sides we get

$$\begin{aligned} |v_{i_1} \wedge \dots \wedge v_{i_k}| &\leq \left(\prod_{j=1}^k |v_{i_1} \wedge \dots \wedge \widehat{v_{i_j}} \wedge \dots \wedge v_{i_k}| \right)^{\frac{1}{k-1}} \leq \left(\prod_{j=1}^k \left(\max_{\{i_1, \dots, i_{k-1}\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}| \right) \right)^{\frac{1}{k-1}} \\ &= \left(\max_{\{i_1, \dots, i_{k-1}\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}| \right)^{\frac{k}{k-1}}, \end{aligned}$$

which concludes the proof. □

6.2 Partial results for $p = 1$ and monotonicity argument

In this section we prove the special cases of the vector-valued Maclaurin inequality with $p = 1$ and $m = d$ listed in Theorem 6.2 along with the non-consecutive cases listed in Theorem 6.3. Our method is somewhat inspired by a monotonicity argument given in [25] to prove the classical Maclaurin inequality. Given vectors $v_1, \dots, v_d \in \mathbb{R}^d$ and $1 \leq k < l \leq d$, we attempt to construct a second family of orthogonal vectors $\tilde{v}_1, \dots, \tilde{v}_d \in \mathbb{R}^d$, depending on the choice of k and l , such that for some $C \geq 1$ we have

$$\sum_{\{i_1, \dots, i_l\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_l}| \leq C^{\frac{l}{2}} \sum_{\{i_1, \dots, i_l\} \subset [d]} |\tilde{v}_{i_1} \wedge \dots \wedge \tilde{v}_{i_l}|,$$

and

$$\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}| \geq C^{-\frac{k}{2}} \sum_{\{i_1, \dots, i_k\} \subset [d]} |\tilde{v}_{i_1} \wedge \dots \wedge \tilde{v}_{i_k}|.$$

If such an orthogonal family exists, then combining the above two estimates with Theorem 6.1 applied to positive numbers $\|v_1\|, \dots, \|v_d\|$, we can write

$$\begin{aligned} \left(\frac{\sum_{\{i_1, \dots, i_l\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_l}|}{\binom{d}{l}} \right)^{\frac{1}{l}} &\leq C^{\frac{1}{2}} \left(\frac{\sum_{\{i_1, \dots, i_l\} \subset [d]} |\tilde{v}_{i_1} \wedge \dots \wedge \tilde{v}_{i_l}|}{\binom{d}{l}} \right)^{\frac{1}{l}} \\ &= C^{\frac{1}{2}} \left(\frac{\sum_{\{i_1, \dots, i_l\} \subset [d]} \|\tilde{v}_{i_1}\| \dots \|\tilde{v}_{i_l}\|}{\binom{d}{l}} \right)^{\frac{1}{l}} \\ &\leq C^{\frac{1}{2}} \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} \|\tilde{v}_{i_1}\| \dots \|\tilde{v}_{i_k}\|}{\binom{d}{k}} \right)^{\frac{1}{k}} \\ &= C^{\frac{1}{2}} \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |\tilde{v}_{i_1} \wedge \dots \wedge \tilde{v}_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}} \\ &\leq C \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}}. \end{aligned}$$

In order to streamline the argument slightly, we introduce the following convenient notation:

$$S_r(v_1, \dots, v_d) := \sum_{\{i_1, \dots, i_r\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_r}|.$$

Notice that by symmetry, instead of constructing a whole family of orthogonal vectors, it suffices to construct $\tilde{v}_1 \in \{v_2, \dots, v_d\}^\perp$ such that

$$S_l(v_1, \dots, v_d) \leq C^{\frac{1}{2}} S_l(\tilde{v}_1, v_2, \dots, v_d),$$

and

$$S_k(v_1, \dots, v_d) \geq C^{-\frac{1}{2}} S_k(\tilde{v}_1, v_2, \dots, v_d).$$

Furthermore, we only need to worry about the terms involving v_1 , so these last two conditions can be reduced to the following two estimates

$$\sum_{\{i_1, \dots, i_{l-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{l-1}}| \leq C^{\frac{1}{2}} \sum_{\{i_1, \dots, i_{l-1}\} \subset [d] \setminus 1} |\tilde{v}_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{l-1}}|, \quad (6.9)$$

and

$$\sum_{\{i_1, \dots, i_{k-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-1}}| \geq C^{-\frac{1}{2}} \sum_{\{i_1, \dots, i_{k-1}\} \subset [d] \setminus 1} |\tilde{v}_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|. \quad (6.10)$$

Indeed, supposing that (6.9) and (6.10) both hold, then using the fact that $C \geq 1$, we can write

$$\begin{aligned} S_l(v_1, \dots, v_d) &= \sum_{\{i_1, \dots, i_{l-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{l-1}}| + \sum_{\{j_1, \dots, j_l\} \neq 1} |v_{j_1} \wedge \dots \wedge v_{j_l}| \\ &\leq C^{\frac{1}{2}} \sum_{\{i_1, \dots, i_{l-1}\} \subset [d] \setminus 1} |\tilde{v}_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{l-1}}| + C^{\frac{1}{2}} \sum_{\{j_1, \dots, j_l\} \neq 1} |v_{j_1} \wedge \dots \wedge v_{j_l}| \\ &= C^{\frac{1}{2}} S_l(\tilde{v}_1, v_2, \dots, v_d), \end{aligned}$$

and

$$\begin{aligned} S_k(v_1, \dots, v_d) &= \sum_{\{i_1, \dots, i_{k-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-1}}| + \sum_{\{j_1, \dots, j_k\} \neq 1} |v_{j_1} \wedge \dots \wedge v_{j_k}| \\ &\geq C^{-\frac{1}{2}} \sum_{\{i_1, \dots, i_{k-1}\} \subset [d] \setminus 1} |\tilde{v}_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-1}}| + C^{-\frac{1}{2}} \sum_{\{j_1, \dots, j_k\} \neq 1} |v_{j_1} \wedge \dots \wedge v_{j_k}| \\ &= C^{-\frac{1}{2}} S_k(\tilde{v}_1, v_2, \dots, v_d), \end{aligned}$$

Note that $\tilde{v}_1$ is orthogonal to the vectors $v_2, \dots, v_d$, so for any $\{v_{i_1}, \dots, v_{i_r}\} \subset \{v_2, \dots, v_d\}$

$$|\tilde{v}_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_r}| = \|\tilde{v}_1\| |v_{i_1} \wedge \dots \wedge v_{i_r}|.$$

Hence we can combine (6.9) and (6.10) as follows

$$C^{-\frac{1}{2}} \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|}{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|} \right) \leq \|\tilde{v}_1\| \leq C^{\frac{1}{2}} \left(\frac{\sum_{\{i_1, \dots, i_{k-2}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-2}}|}{\sum_{\{i_1, \dots, i_{k-2}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{k-2}}|} \right).$$

So the question is can we choose a length $\|\tilde{v}_1\|$ satisfying the above? Let us summarise what we have just derived with the following result:

Theorem 6.17. Fix $1 \leq k < l \leq d$ and $C \geq 1$. Suppose that for any $v_1, \dots, v_d \in \mathbb{R}^d$

$$\frac{\sum_{\{i_1, \dots, i_{l-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{l-1}}|}{\sum_{\{i_1, \dots, i_{l-1}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{l-1}}|} \leq C \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|}{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|} \right), \quad (6.11)$$

where we interpret the $k = 1$ case as

$$\frac{\sum_{\{i_1, \dots, i_{l-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{l-1}}|}{\sum_{\{i_1, \dots, i_{l-1}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{l-1}}|} \leq C \|v_1\|.$$

Then, for any $\omega_1, \dots, \omega_d \in \mathbb{R}^d$, we have

$$\left(\frac{S_l(\omega_1, \dots, \omega_d)}{\binom{d}{l}} \right)^{\frac{1}{l}} \leq C \left(\frac{S_k(\omega_1, \dots, \omega_d)}{\binom{d}{k}} \right)^{\frac{1}{k}}.$$

Using a certain duality between families of vectors given by Lemma 2.9, we can prove the following useful fact:

Lemma 6.18. *If the vectors $v_1, \dots, v_d \in \mathbb{R}^d$ are linearly independent, then (6.11) holds for $k = p$ and $l = q$ if and only if it also holds $k = d - q + 1$ and $l = d - p + 1$.*

Proof. Given linearly independent vectors $v_1, \dots, v_d \in \mathbb{R}^d$, let $u_1, \dots, u_d \in \mathbb{R}^d$ be a second family uniquely determined by

$$\langle v_i, u_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

By Lemma 2.9, we know that for all $A \subset [d]$

$$\left| \bigwedge_{i \in A} v_i \right| = \frac{\left| \bigwedge_{i \notin A} u_i \right|}{|u_1 \wedge \dots \wedge u_d|}.$$

It follows that for any $1 \leq r \leq d$, we have

$$\begin{aligned} \frac{\sum_{\{i_1, \dots, i_{r-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{r-1}}|}{\sum_{\{i_1, \dots, i_{r-1}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{r-1}}|} &= \frac{\sum_{\{i_1, \dots, i_{r-1}\} \subset [d] \setminus 1} \frac{|\bigwedge_{j \notin \{1, i_1, \dots, i_{r-1}\}} u_j|}{|u_1 \wedge \dots \wedge u_d|}}{\sum_{\{i_1, \dots, i_{r-1}\} \subset [d] \setminus 1} \frac{|\bigwedge_{j \notin \{i_1, \dots, i_{r-1}\}} u_j|}{|u_1 \wedge \dots \wedge u_d|}} \\ &= \frac{\sum_{\{i_1, \dots, i_{r-1}\} \subset [d] \setminus 1} \left| \bigwedge_{j \notin \{1, i_1, \dots, i_{r-1}\}} u_j \right|}{\sum_{\{i_1, \dots, i_{r-1}\} \subset [d] \setminus 1} \left| \bigwedge_{j \notin \{i_1, \dots, i_{r-1}\}} u_j \right|}} \\ &= \frac{\sum_{\{j_1, \dots, j_{d-r}\} \subset [d] \setminus 1} |u_{j_1} \wedge \dots \wedge u_{j_{d-r}}|}{\sum_{\{j_1, \dots, j_{d-r}\} \subset [d] \setminus 1} |u_1 \wedge u_{j_1} \wedge \dots \wedge u_{j_{d-r}}|}} \\ &= \frac{\sum_{\{j_1, \dots, j_{(d-r+1)-1}\} \subset [d] \setminus 1} |u_{j_1} \wedge \dots \wedge u_{j_{(d-r+1)-1}}|}{\sum_{\{j_1, \dots, j_{(d-r+1)-1}\} \subset [d] \setminus 1} |u_1 \wedge u_{j_1} \wedge \dots \wedge u_{j_{(d-r+1)-1}}|}} \end{aligned}$$

Thus the statement that for any $v_1, \dots, v_d \in \mathbb{R}^d$

$$\frac{\sum_{\{i_1, \dots, i_{q-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{q-1}}|}{\sum_{\{i_1, \dots, i_{q-1}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{q-1}}|} \leq \frac{\sum_{\{i_1, \dots, i_{p-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{p-1}}|}{\sum_{\{i_1, \dots, i_{p-1}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{p-1}}|},$$

is equivalent to the statement that for any $u_1, \dots, u_d \in \mathbb{R}^d$

$$\frac{\sum_{\{i_1, \dots, i_{(d-p+1)-1}\} \subset [d] \setminus 1} |u_1 \wedge u_{i_1} \wedge \dots \wedge u_{i_{(d-p+1)-1}}|}{\sum_{\{i_1, \dots, i_{(d-p+1)-1}\} \subset [d] \setminus 1} |u_{i_1} \wedge \dots \wedge u_{i_{(d-p+1)-1}}|} \leq \frac{\sum_{\{i_1, \dots, i_{(d-q+1)-1}\} \subset [d] \setminus 1} |u_1 \wedge u_{i_1} \wedge \dots \wedge u_{i_{(d-q+1)-1}}|}{\sum_{\{i_1, \dots, i_{(d-q+1)-1}\} \subset [d] \setminus 1} |u_{i_1} \wedge \dots \wedge u_{i_{(d-q+1)-1}}|}.$$

□

For most applications we will only be concerned with taking $C = 1$ and $k = l - 1$, so we state the following special case of Theorem 6.17 as a separate result:

Theorem 6.19. Fix $1 < k \leq d$. Suppose that for any $v_1, \dots, v_d \in \mathbb{R}^d$

$$\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|}{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|} \leq \frac{\sum_{\{i_1, \dots, i_{k-2}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-2}}|}{\sum_{\{i_1, \dots, i_{k-2}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{k-2}}|}, \quad (6.12)$$

where we interpret the $k = 2$ case as

$$\frac{\sum_{i \in [d] \setminus 1} |v_1 \wedge v_i|}{\sum_{i \in [d] \setminus 1} \|v_i\|} \leq \|v_1\|.$$

Then, for any $\omega_1, \dots, \omega_d \in \mathbb{R}^d$, we have

$$\left(\frac{S_k(\omega_1, \dots, \omega_d)}{\binom{d}{k}} \right)^{\frac{1}{k}} \leq \left(\frac{S_{k-1}(\omega_1, \dots, \omega_d)}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}.$$

Similarly, for convenience we state the following special case of Lemma 6.18:

Lemma 6.20. If the vectors $v_1, \dots, v_d \in \mathbb{R}^d$ are linearly independent, then (6.12) holds for $k = r$ if and only if it also holds $k = d - r + 2$.

6.2.1 Proof of Theorem 6.2

Now we will prove various special cases of (6.12) which then imply the corresponding cases listed in Theorem 6.2:

Lemma 6.21. If $k = 2$ or $k = d$, then (6.12) holds for arbitrary $v_1, \dots, v_d \in \mathbb{R}^d$.

Proof. To begin, let us deal with the case $k = d$. Observe that for $2 \leq i \leq d$ we have

$$\frac{|v_1 \wedge \dots \wedge v_d|}{|v_2 \wedge \dots \wedge v_d|} = \|\pi_{\{v_2, \dots, v_d\}^\perp} v_1\| \leq \|\pi_{\{v_2, \dots, \widehat{v}_i, \dots, v_d\}^\perp} v_1\| = \frac{|v_1 \wedge \dots \wedge \widehat{v}_i \wedge \dots \wedge v_d|}{|v_2 \wedge \dots \wedge \widehat{v}_i \wedge \dots \wedge v_d|}.$$

Rearranging this gives

$$|v_1 \wedge \cdots \wedge v_d| |v_2 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d| \leq |v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d| |v_2 \wedge \cdots \wedge v_d|.$$

Summing over i yields

$$\sum_{i=2}^d |v_1 \wedge \cdots \wedge v_d| |v_2 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d| \leq \sum_{i=2}^d |v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d| |v_2 \wedge \cdots \wedge v_d|$$

which implies that

$$\frac{|v_1 \wedge \cdots \wedge v_d|}{|v_2 \wedge \cdots \wedge v_d|} \leq \frac{\sum_{i=2}^d |v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d|}{\sum_{i=2}^d |v_2 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d|},$$

which is exactly (6.12) for $k = d$. Now for the case where $k = 2$. Simply note that

$$\sum_{i=2}^d |v_1 \wedge v_i| \leq \|v_1\| \sum_{i=2}^d \|v_i\|,$$

which immediately gives

$$\frac{\sum_{i=2}^d |v_1 \wedge v_i|}{\sum_{i=2}^d \|v_i\|} \leq \|v_1\|,$$

which concludes the proof. □

Next we deal with the cases $k = 3$ and $k = d - 1$, which require a little more work:

Lemma 6.22. *If $k = 3$ or $k = d - 1$, then (6.12) holds in all dimensions d .*

In order to prove Lemma 6.22, we first prove the following variant of (6.12):

Lemma 6.23. *For linearly independent vectors $v_1, \dots, v_d \in \mathbb{R}^d$, we have*

$$\frac{\sum_{\{i_1, \dots, i_{d-2}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|}{\sum_{\{i_1, \dots, i_{d-2}\} \subset [d] \setminus 1} |v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|} \leq \frac{\sum_{1 < i \leq d} |v_1 \wedge v_i|}{\sum_{1 < i \leq d} \|v_i\|}. \quad (6.13)$$

To prove Lemma 6.23 we need to use an elementary fact regarding barycentric coordinates with respect to a simplex. The next result is originally due to Möbius [33]:

Proposition 6.24 (Barycentric coordinates with respect to a simplex). *Let $v_1, \dots, v_d \in \mathbb{R}^{d-1}$ be the vertices of a $(d - 1)$ -simplex Δ . Given a vector $u \in \Delta$, there exists a unique d -tuple $(\beta_1, \dots, \beta_d) \in \mathbb{R}^d$ with $\sum_{j=1}^d \beta_j = 1$, satisfying the following identity*

$$\sum_{j=1}^d \beta_j v_j = u.$$

Furthermore, we can write such numbers $\beta_1, \dots, \beta_d$ explicitly using the following formula

$$\beta_j = \frac{|(v_1 - u) \wedge \cdots \wedge \widehat{(v_j - u)} \wedge \cdots \wedge (v_d - u)|}{2\text{Vol}(\Delta)}.$$

Proof of Lemma 6.23. By rearranging it suffices to prove

$$\left(\sum_{\{i_1, \dots, i_{d-2}\} \subset [d] \setminus 1} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{d-2}}| \right) \left(\sum_{1 < i \leq d} \|v_i\| \right) \leq \left(\sum_{1 < i \leq d} |v_1 \wedge v_i| \right) \cdot \left(\sum_{\{i_1, \dots, i_{d-2}\} \subset [d] \setminus 1} |v_{i_1} \wedge \dots \wedge v_{i_{d-2}}| \right). \quad (6.14)$$

For $j = 1, \dots, d-2$, it is always true that

$$\frac{|v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{d-2}}|}{|v_{i_1} \wedge \dots \wedge v_{i_{d-2}}|} = \left\| \pi_{\{v_{i_1}, \dots, v_{i_{d-2}}\}^\perp} v_1 \right\| \leq \left\| \pi_{v_{i_j}^\perp} v_1 \right\| = \frac{|v_1 \wedge v_{i_j}|}{\|v_{i_j}\|}.$$

Rearranging this we get

$$|v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{d-2}}| \|v_{i_j}\| \leq |v_1 \wedge v_{i_j}| |v_{i_1} \wedge \dots \wedge v_{i_{d-2}}|. \quad (6.15)$$

Summing over $\{i_1, \dots, i_{d-2}\} \subset [d]$ and $j \in \{1, \dots, d-2\}$, we get

$$\sum_{\{i_1, \dots, i_{d-2}\} \subset [d] \setminus 1} \sum_{j=1}^{d-2} |v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{d-2}}| \|v_{i_j}\| \leq \sum_{\{i_1, \dots, i_{d-2}\} \subset [d] \setminus 1} \sum_{j=1}^{d-2} |v_1 \wedge v_{i_j}| |v_{i_1} \wedge \dots \wedge v_{i_{d-2}}|. \quad (6.16)$$

Multiplying out the brackets in (6.14), we see that after using (6.16), it suffices to prove the following estimate involving terms $|v_1 \wedge v_{i_1} \wedge \dots \wedge v_{i_{d-2}}| \|v_i\|$ with $i \notin \{i_1, \dots, i_{d-2}\}$:

$$\sum_{j=2}^d |v_1 \wedge v_2 \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge v_d| \|v_j\| \leq \sum_{j=2}^d |v_1 \wedge v_j| |v_2 \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge v_d|.$$

Without loss of generality we may assume that

$$\frac{|v_1 \wedge v_2|}{\|v_2\|} \leq \dots \leq \frac{|v_1 \wedge v_d|}{\|v_d\|}, \quad (6.17)$$

then, bearing this in mind, we can apply particular cases of (6.15) to each term on the left hand side to get

$$\begin{aligned} \sum_{j=1}^d |v_1 \wedge v_2 \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge v_d| \|v_j\| &\leq \frac{|v_1 \wedge v_2|}{\|v_2\|} \left(\sum_{j=3}^d |v_2 \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge v_d| \|v_j\| \right) \\ &\quad + \frac{|v_1 \wedge v_3|}{\|v_3\|} |v_3 \wedge \dots \wedge v_d| \|v_2\|. \end{aligned}$$

Supposing we know that

$$\sum_{j=3}^d |v_2 \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge v_d| \|v_j\| \geq |v_3 \wedge \dots \wedge v_d| \|v_2\|, \quad (6.18)$$

then by a simple application of the rearrangement inequality for numbers, using (6.18) and (6.17), we see that

$$\begin{aligned}
& \frac{|v_1 \wedge v_2|}{\|v_2\|} \left(\sum_{j=3}^d |v_2 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_d| \|v_j\| \right) + \frac{|v_1 \wedge v_3|}{\|v_3\|} |v_3 \wedge \cdots \wedge v_d| \|v_2\| \\
& \leq \frac{|v_1 \wedge v_2|}{\|v_2\|} |v_3 \wedge \cdots \wedge v_d| \|v_2\| + \frac{|v_1 \wedge v_3|}{\|v_3\|} \left(\sum_{j=3}^d |v_2 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_d| \|v_j\| \right) \\
& \leq \sum_{j=2}^d |v_1 \wedge v_j| |v_2 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_d|,
\end{aligned}$$

which is what we want. It suffices to prove the following claim:

Claim 6.25. For linearly independent vectors $u_1, \dots, u_d \in \mathbb{R}^d$ we have

$$\sum_{j=2}^d |u_1 \wedge \cdots \wedge \widehat{u}_j \wedge \cdots \wedge u_d| \|u_j\| \geq |u_2 \wedge \cdots \wedge u_d| \|u_1\|.$$

Proof of Claim 6.25. Let $S = \text{Conv}(u_1, \dots, u_d)$. Since $u_1, \dots, u_d \in \mathbb{R}^d$ are linearly independent, we know that S is a $(d-1)$ -dimensional simplex. Moreover, this linear independence guarantees that S does not lie in a $(d-1)$ -dimensional subspace of $\mathbb{R}^d$. Denote by b the barycentre of S . It follows that $\pi_{b^\perp}(S) = \text{Conv}(\pi_{b^\perp}u_1, \dots, \pi_{b^\perp}u_d)$ is a $(d-1)$ -dimensional simplex containing the origin.

Define $w_j := \pi_{b^\perp}u_j \neq \mathbf{0}$, then by a simple reordering, we may assume that for $j \geq 2$

$$\frac{\|u_1\| |u_2 \wedge \cdots \wedge u_d|}{\|w_1\| |w_2 \wedge \cdots \wedge w_d|} \leq \frac{\|u_j\| |u_1 \wedge u_2 \wedge \cdots \wedge \widehat{u}_j \wedge \cdots \wedge u_d|}{\|w_j\| |w_1 \wedge w_2 \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_d|} \quad (6.19)$$

If we set

$$\tilde{w}_1 := \frac{\|u_1\| |u_2 \wedge \cdots \wedge u_d|}{\|w_1\| |w_2 \wedge \cdots \wedge w_d|} w_1,$$

then clearly $\tilde{S} = \text{Conv}(\tilde{w}_1, w_2, \dots, w_d)$ also forms a $(d-1)$ -dimensional simplex containing the origin. Furthermore, we have

$$\|\tilde{w}_1\| |w_2 \wedge \cdots \wedge w_d| = \frac{\|u_1\| |u_2 \wedge \cdots \wedge u_d|}{\|w_1\| |w_2 \wedge \cdots \wedge w_d|} \|w_1\| |w_2 \wedge \cdots \wedge w_d| = \|u_1\| |u_2 \wedge \cdots \wedge u_d|, \quad (6.20)$$

and for $j \geq 2$, we can use (6.19) to get

$$\begin{aligned}
& \|w_j\| |\tilde{w}_1 \wedge w_2 \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_d| \\
& = \|w_j\| \frac{\|u_1\| |u_2 \wedge \cdots \wedge u_d|}{\|w_1\| |w_2 \wedge \cdots \wedge w_d|} |w_1 \wedge w_2 \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_d| \\
& \leq \|w_j\| \frac{\|u_j\| |u_1 \wedge u_2 \wedge \cdots \wedge \widehat{u}_j \wedge \cdots \wedge u_d|}{\|w_j\| |w_1 \wedge w_2 \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_d|} |w_1 \wedge w_2 \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_d| \\
& = \|u_j\| |u_1 \wedge u_2 \wedge \cdots \wedge \widehat{u}_j \wedge \cdots \wedge u_d|.
\end{aligned} \quad (6.21)$$

For ease of notation, let us redefine $w_1 := \tilde{w}_1$. By Proposition 6.24, noting that $\mathbf{0} \in \tilde{S}$ and setting

$$\beta_j := \frac{|w_1 \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_d|}{2\text{Vol}_{d-1}(\tilde{S})},$$

we have

$$\sum_{j=1}^d \beta_j w_j = \mathbf{0},$$

which implies that

$$\sum_{j=1}^d |w_1 \wedge w_2 \wedge \cdots \wedge \tilde{w}_j \wedge \cdots \wedge w_d| w_j = \mathbf{0}.$$

By the reverse triangle inequality, we can deduce that

$$\begin{aligned} 0 &= \left\| \sum_{j=1}^d |w_1 \wedge w_2 \wedge \cdots \wedge \tilde{w}_j \wedge \cdots \wedge w_d| w_j \right\| \\ &\geq |w_2 \wedge \cdots \wedge w_d| \|w_1\| - \left\| \sum_{j=2}^d |w_1 \wedge w_2 \wedge \cdots \wedge \tilde{w}_j \wedge \cdots \wedge w_d| w_j \right\|. \end{aligned}$$

So by the standard triangle inequality we have

$$\begin{aligned} |w_2 \wedge \cdots \wedge w_d| \|w_1\| &\leq \left\| \sum_{j=2}^d |w_1 \wedge w_2 \wedge \cdots \wedge \tilde{w}_j \wedge \cdots \wedge w_d| w_j \right\| \\ &\leq \sum_{j=2}^d |w_1 \wedge w_2 \wedge \cdots \wedge \tilde{w}_j \wedge \cdots \wedge w_d| \|w_j\|. \end{aligned}$$

Combining this last inequality with (6.20) and (6.21), we can write

$$\begin{aligned} \|u_1\| |u_2 \wedge \cdots \wedge u_d| &= \|w_1\| |w_2 \wedge \cdots \wedge w_d| \leq \sum_{j=2}^d |w_1 \wedge w_2 \wedge \cdots \wedge \tilde{w}_j \wedge \cdots \wedge w_d| \|w_j\| \\ &\leq \sum_{j=2}^d |u_1 \wedge u_2 \wedge \cdots \wedge \tilde{u}_j \wedge \cdots \wedge u_d| \|u_j\| \end{aligned}$$

□

This concludes the proof of Lemma 6.23. □

By a simple argument we can now establish (6.12) for $k = 3$ and $k = d - 1$ in all dimensions:

Proof of Lemma 6.22. Directly applying Lemma 6.23 with $d = 4$, for any $v_1, \dots, v_4 \in \mathbb{R}^4$ we have

$$\frac{\sum_{\{i,j\} \subset \{2,3,4\}} |v_1 \wedge v_i \wedge v_j|}{\sum_{\{i,j\} \subset \{2,3,4\}} |v_i \wedge v_j|} \leq \frac{\sum_{k \in \{2,3,4\}} |v_1 \wedge v_k|}{\sum_{k \in \{2,3,4\}} \|v_k\|}.$$

Multiplying out the denominators and expanding the brackets, this last inequality can be rewritten as

$$\sum_{\{i,j\} \subset \{2,3,4\}} \sum_{k \in \{2,3,4\}} |v_1 \wedge v_i \wedge v_j| \|v_k\| \leq \sum_{\{i,j\} \subset \{2,3,4\}} \sum_{k \in \{2,3,4\}} |v_i \wedge v_j| |v_1 \wedge v_k|. \quad (6.22)$$

For higher dimensions simply note that using (6.22), we can deduce that

$$\begin{aligned}
\sum_{\{i,j\} \subset [d] \setminus 1} \sum_{k \in [d] \setminus 1} |v_1 \wedge v_i \wedge v_j| \|v_k\| &= \sum_{\{a,b,c\} \subset [d] \setminus 1} \sum_{\{i,j\} \subset \{a,b,c\}} \sum_{k \in \{a,b,c\}} |v_1 \wedge v_i \wedge v_j| \|v_k\| \\
&\leq \sum_{\{a,b,c\} \subset [d] \setminus 1} \sum_{\{i,j\} \subset \{a,b,c\}} \sum_{k \in \{a,b,c\}} |v_i \wedge v_j| |v_1 \wedge v_k| \\
&= \sum_{\{i,j\} \subset [d] \setminus 1} \sum_{k \in [d] \setminus 1} |v_i \wedge v_j| |v_1 \wedge v_k|,
\end{aligned}$$

which directly implies that

$$\frac{\sum_{\{i,j\} \subset [d] \setminus 1} |v_1 \wedge v_i \wedge v_j|}{\sum_{\{i,j\} \subset [d] \setminus 1} |v_i \wedge v_j|} \leq \frac{\sum_{k \in [d] \setminus 1} |v_1 \wedge v_k|}{\sum_{k \in [d] \setminus 1} \|v_k\|}.$$

By Lemma 6.20, we immediately get (6.12) for $k = d - 3 + 2 = d - 1$ in all dimensions. $\square$

Proof of Theorem 6.3 To deal with the non-consecutive listed in Theorem 6.3, we apply a similar argument to above. Firstly note that the case for $k = 1$ follows from a direct application of the original Maclaurin inequality:

$$\left(\frac{\sum_{\{i_1, \dots, i_l\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_l}|}{\binom{d}{l}} \right)^{\frac{1}{l}} \leq \left(\frac{\sum_{\{i_1, \dots, i_l\} \subset [d]} \prod_{j=1}^l \|v_{i_j}\|}{\binom{d}{l}} \right)^{\frac{1}{l}} \leq \frac{\sum_{j=1}^d \|v_j\|}{d}.$$

The other cases follow from Lemma 6.23 by an application of Theorem 6.17 with $l = d - 1$ and $k = 2$.

6.3 Connection to intrinsic volumes

As mentioned at the beginning of this chapter, the vector-valued Maclaurin inequality with $p = 1$ can be rewritten as a sequence of inequalities between intrinsic volumes of certain polytopes, which are all in some sense extremised by the cube. Firstly, let us restate a useful formula for calculating the mixed volume of a zonoid:

Theorem 6.26 (Theorem 5.3.2). *For $1 \leq j \leq d$ and $1 \leq i \leq j$, let $Z_i \subset \mathbb{R}^d$ be a zonoid with generating measure ρ_i and let $K_1, \dots, K_{d-j} \subset \mathbb{R}^d$ be convex bodies. Then*

$$\begin{aligned}
&V(Z_1, \dots, Z_j, K_1, \dots, K_{d-j}) \\
&= \frac{2^j (d-j)!}{d!} \int_{\mathbb{S}^{d-1}} \dots \int_{\mathbb{S}^{d-1}} |u_1 \wedge \dots \wedge u_j| v^{(d-j)}(\pi_{\{u_1, \dots, u_j\}^\perp} K_1, \dots, \pi_{\{u_1, \dots, u_j\}^\perp} K_{d-j}) d\rho_1(u_1) \dots d\rho_j(u_j),
\end{aligned}$$

where $v^{(d-j)}$ denotes the j -dimensional mixed volume.

In Chapter 4 this result appeared as Theorem 4.48 and was used to reinterpret the directional volume, then later in Chapter 5 we used a special case, namely Theorem 5.12, to establish a chain of inequalities between quantities $\mathcal{H}_1^{d-1}(Z), \mathcal{H}_2^{d-1}(Z)^{\frac{1}{2}}, \dots, \mathcal{H}_d^{d-1}(Z)^{\frac{1}{d}}$ for any nice enough hypersurface $Z \subset \mathbb{R}^d$.

Note that any zonotope $Z = \sum_{i=1}^m \alpha_i [-v_i, v_i]$ has a support function defined by

$$h_Z(u) = \sum_{i=1}^m \alpha_i |\langle u, v_i \rangle| = \int_{\mathbb{S}^{d-1}} |\langle u, v \rangle| d\rho(v),$$

where ρ is concentrated at $\pm v_i$ and assigns mass $\frac{\alpha_i}{2}$ to each of these points. So by Theorem 6.26, given $Z_1 = \sum_{k_1} \alpha_{k_1} [-v_{k_1}, v_{k_1}], \dots, Z_j = \sum_{k_j} \alpha_{k_j} [-v_{k_j}, v_{k_j}]$ and a fixed convex body $K \subset \mathbb{R}^d$, we have

$$V(Z_1, \dots, Z_j, K[d-j]) = \frac{2^j(d-j)!}{d!} \sum_{k_1, \dots, k_j} \alpha_{k_1} \cdots \alpha_{k_j} |v_{k_1} \wedge \cdots \wedge v_{k_j}| \left| \pi_{\{v_{k_1}, \dots, v_{k_j}\}^\perp} K \right| \quad (6.23)$$

Note that the k -th intrinsic volume of a convex body $L \subset \mathbb{R}^d$ for $k = 1, \dots, d$, can be defined as

$$V_k(L) = \binom{d}{k} \frac{V(L, k; \mathbb{B}_2^d, d-k)}{\kappa_{d-k}}.$$

Furthermore, the support function of any centred parallelotope P with edges of lengths $\|v_1\|, \dots, \|v_d\|$ and directions $\frac{v_1}{\|v_1\|}, \dots, \frac{v_d}{\|v_d\|}$, can be written as follows

$$h_P(u) = \sum_{i=1}^d \frac{1}{2} |\langle u, v_i \rangle|.$$

So using (6.23), we can write

$$\begin{aligned} V_k(P) &= \binom{d}{k} \frac{V(P, k; \mathbb{B}_2^d, d-k)}{\kappa_{d-k}} = \frac{\frac{2^k(d-k)!}{d!} \sum_{\{i_1, \dots, i_k\} \subset [d]} \left(\frac{1}{2}\right)^k |v_{i_1} \wedge \cdots \wedge v_{i_k}| \left| \pi_{\{v_{i_1}, \dots, v_{i_k}\}^\perp} \mathbb{B}_2^d \right|}{\kappa_{d-k}} \\ &= \binom{d}{k} \frac{k!(d-k)!}{d!} \sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \cdots \wedge v_{i_k}| \\ &= \sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|. \end{aligned}$$

In this language, Conjecture 6.9 for $p = 1$ implies that for all parallelotopes $P \subset \mathbb{R}^d$ and $1 < j \leq d$, the following inequality holds

$$\left(\frac{V_j(P)}{\binom{d}{j}} \right)^{\frac{1}{j}} \leq \left(\frac{V_{j-1}(P)}{\binom{d}{j-1}} \right)^{\frac{1}{j-1}}, \quad (6.24)$$

with equality if and only if P is a cube. Suppose instead we consider the zonotope $Z = \sum_{i=1}^m \frac{1}{2} [-v_i, v_i] \subset \mathbb{R}^d$, then by the same argument we have

$$\begin{aligned} V_k(Z) &= \binom{d}{k} \frac{V(Z, k; \mathbb{B}_2^d, d-k)}{\kappa_{d-k}} = \frac{\frac{2^k(d-k)!}{d!} \sum_{\{i_1, \dots, i_k\} \subset [m]} \left(\frac{1}{2}\right)^k |v_{i_1} \wedge \cdots \wedge v_{i_k}| \left| \pi_{\{v_{i_1}, \dots, v_{i_k}\}^\perp} \mathbb{B}_2^d \right|}{\kappa_{d-k}} \\ &= \binom{d}{k} \frac{k!(d-k)!}{d!} \sum_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}| \\ &= \sum_{\{i_1, \dots, i_k\} \subset [m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|. \end{aligned}$$

So for $k \leq d$ and $p = 1$, Conjecture 6.9 states that

$$\left(\frac{V_j(Z)}{\binom{m}{j}} \right)^{\frac{1}{j}} \leq \left(\frac{V_{j-1}(Z)}{\binom{m}{j-1}} \right)^{\frac{1}{j-1}}. \quad (6.25)$$

Rearranging this gives

$$\frac{V_{j-1}(Z)^j}{V_j(Z)^{j-1}} \geq \frac{\binom{m}{j-1}^j}{\binom{m}{j}^{j-1}}. \quad (6.26)$$

It follows that

$$\frac{V_{j-1}(Z)^j}{V_j(Z)^{j-1}} \frac{\binom{m}{j}^{j-1}}{\binom{m}{j-1}^j} \geq 1 \geq \frac{V_{j+1}(Z)^j}{V_j(Z)^{j+1}} \frac{\binom{m}{j}^{j+1}}{\binom{m}{j+1}^j},$$

which implies the following log-concavity:

$$V_j(Z)^2 \geq \frac{\binom{m}{j}^2}{\binom{m}{j-1} \binom{m}{j+1}} V_{j+1}(Z) V_{j-1}(Z).$$

This last inequality can be simplified to

$$V_j(Z)^2 \geq \frac{(j+1)(m-j+1)}{j(m-j)} V_{j+1}(Z) V_{j-1}(Z). \quad (6.27)$$

Note that (6.26) is an Aleksandrov-type inequality. Indeed, the Aleksandrov inequalities for quermassintegrals $W_1(K), \dots, W_{d-1}(K)$ are given by:

$$\left(\frac{W_i(K)}{\kappa_d} \right)^{\frac{1}{n-i}} \geq \left(\frac{W_j(K)}{\kappa_d} \right)^{\frac{1}{n-j}} \quad d > i > j \geq 0.$$

Taking into account that $V_j(K) = \binom{d}{j} \frac{W_{d-j}(K)}{\kappa_{d-j}}$ we can rewrite the last one as

$$\frac{V_{j-1}(K)^j}{V_j(K)^{j-1}} \geq \frac{V_{j-1}(\mathbb{B}_2^d)^j}{V_j(\mathbb{B}_2^d)^{j-1}}. \quad (6.28)$$

Therefore (6.26) is a restriction of the general (6.28), in the class of zonotopes and it is also sensitive to the number of segments which make up the given zonotope, while (6.28) is not.

Moreover, (6.27) can be compared with an inequality of McMullen, who proved a related dimension-free bound for the intrinsic volumes, namely

$$V_j(K)^2 \geq \frac{j+1}{j} V_{j+1}(K) V_{j-1}(K).$$

The factor that appears in the right-hand side of (6.27) is always at least $\frac{j+1}{j}$, which implies McMullen's inequality. Moreover, we attain this bound as $m \rightarrow \infty$. This should not be surprising as one can find a sequence of zonotopes that converge to $\mathbb{B}_2^d$ as $m \rightarrow \infty$.

Note also that in [26] the following inequalities were proved: for zonoids Z

$$\sup_{\Lambda \in GL(d)} \frac{V_j(\Lambda Z)}{V_1(\Lambda Z)^j} \geq \frac{1}{d^j} \binom{d}{j} \quad (6.29)$$

if $\dim(Z) = d$ and $j \geq 2$, with equality if and only if Z is a parallelotope. These are reverse, in a sense, to a consequence of (6.28), namely for any convex body K

$$\frac{V_j(K)}{V_1(K)^j} \leq \frac{V_j(\mathbb{B}_2^d)}{V_1(\mathbb{B}_2^d)^j}. \quad (6.30)$$

The equality case in (6.29) exactly says that for any parallelotope $P \subset \mathbb{R}^d$, the following inequality holds:

$$\left(\frac{V_j(P)}{\binom{d}{j}} \right)^{\frac{1}{j}} \leq \frac{V_1(P)}{\binom{d}{1}}. \quad (6.31)$$

This is of course a special case of (6.24). If this more general equality case can be established, one might hope to prove the corresponding generalisation to (6.29).

6.3.1 Argument for Theorem 6.11

Let us see how one can write the reduction stated above to the vector-valued Maclaurin inequality, in terms of intrinsic volumes. That is, can we write (6.12) purely in terms of intrinsic volumes? Fix $u \in \mathbb{R}^{m-1} \times \mathbb{R} \cong \mathbb{R}^m$ and $v_1, \dots, v_{m-1} \in \mathbb{R}^{m-1} \subset \mathbb{R}^m$, and let $Z = \sum_{i=1}^{m-1} \frac{1}{2}[-v_i, v_i] \subset \mathbb{R}^{m-1}$, then we have

$$\begin{aligned} V_k(\pi_{u^\perp} Z) &= \binom{m-1}{k} \frac{V(\pi_{u^\perp} Z, k; \mathbb{B}_2^{m-1}, m-k-1)}{\kappa_{m-k-1}} \\ &= \binom{m}{m-1} \binom{m-1}{k} \frac{V(Z, k; \mathbb{B}_2^m, m-k-1; \frac{1}{2}[-u, u])}{\kappa_{m-k-1}} \\ &= \frac{m!}{(m-k-1)!k!} \frac{\left(\frac{2^{k+1}(m-k-1)!}{(m)!} \right) \sum_{\{i_1, \dots, i_k\} \subset [m-1]} \left(\frac{1}{2} \right)^{k+1} |v_{i_1} \wedge \dots \wedge v_{i_k} \wedge u| \left| \pi_{\{v_{i_1}, \dots, v_{i_k}, u\}^\perp} \mathbb{B}_2^m \right|}{\kappa_{m-k-1}} \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq m-1} |v_{i_1} \wedge \dots \wedge v_{i_k} \wedge u|. \end{aligned}$$

In particular, suppose we let $Z = \sum_{i=2}^m \frac{1}{2}[-v_i, v_i]$ then we can rewrite (6.12) as

$$\frac{V_{k-1}(\pi_{u^\perp} Z)}{V_{k-1}(Z)} \leq \frac{V_{k-2}(\pi_{u^\perp} Z)}{V_{k-2}(Z)} \quad (6.32)$$

for $1 < k \leq m-1$. Theorem 1.2 in [22] implies the following result: let $K \subset \mathbb{R}^d$ be a convex body, then for any $u \in \mathbb{R}^d$ we have

$$\frac{V_{k-1}(\pi_{u^\perp} K)}{V_{k-1}(K)} \leq \frac{2(d-k+1)}{d-k+2} \frac{V_{k-2}(\pi_{u^\perp} K)}{V_{k-2}(K)}, \quad (6.33)$$

for all $3 \leq k \leq d$. In the special case where $m = d+1$ and u lies in the span of $v_1, \dots, v_d$, one can view (6.32) as the sharp version of (6.33) when we restrict to zonotopes. By Theorem 6.17, we can deduce the following result as a consequence of (6.33):

Theorem 6.27. *For any d -tuple of vectors $v_1, \dots, v_d \in \mathbb{R}^d$ and any $2 < k \leq d$, the following inequality holds:*

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}} \leq \frac{2(d-k+1)}{(d-k+2)} \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}$$

Note that the constant appearing on the right-hand side is greater than 1, but smaller than 2. As a consequence of Theorem 6.2, we know that the sharp constant is equal to 1 in dimensions $d = 3, 4$. Thus, it seems likely that the constant given in Theorem 6.27 is suboptimal.

Some brief remarks on known volume bounds for zonotopes and projections of the unit cube Although not directly related to our problem, we make some remarks about upper bounds for the intrinsic volumes of zonotopes and projections of the unit cube.

In [28], the authors prove a combinatorial upper bound for the intrinsic volumes of a zonotope which is the sum of unit line segments. Firstly, they show that k -th intrinsic volume of a zonotope can be written as the sum of k -fold wedge products. Then one can use some elementary norm inequalities to compare with the sum of squares, which as we have already seen is easy to work with using facts about the principle minors of a matrix.

It turns out that one can write any k -dimensional zonotope which is the sum of n -line segments, with $k \leq n$, as the orthogonal projection of the unit cube in $\mathbb{R}^n$ onto $\mathbb{R}^k$ up to some affine transformation, for details see Chapter 2 of [40]. In the same book, proofs of some known upper bounds for the k -dimensional volume of any orthogonal projection of the unit n -cube are given which are due to Chakerian and Filliman. The methods again rely on comparing the sum of k -fold wedge products with the sum of squares.

6.4 Attempting to prove Conjecture 6.9 for $p = 1$ using the Aleksandrov-Fenchel inequality

Earlier we saw that for a centred parallelotope P with edges of lengths $\|v_1\|, \dots, \|v_d\|$ and directions $\frac{v_1}{\|v_1\|}, \dots, \frac{v_d}{\|v_d\|}$, we have

$$h_P(u) = \sum_{i=1}^d \frac{1}{2} |\langle u, v_i \rangle|,$$

so by Theorem 6.26 for any convex body $L \subset \mathbb{R}^d$ we can write

$$V(P, k; L, d - k) = \frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}| \left| \pi_{\{v_{i_1}, \dots, v_{i_k}\}^\perp} L \right|}{\binom{d}{k}}.$$

An application of the Aleksandrov-Fenchel gives us

$$V(P, k; L, d - k)^2 \geq V(P, k - 1; L, d - k + 1) V(P, k + 1; L, d - k - 1).$$

In order to use this to prove the vector-valued Newton inequality, mentioned at the beginning of this chapter as (6.6), one might hope to choose a convex body $L \subset \mathbb{R}^d$ such that the quantity $|\pi_{\text{span}\{v_{i_1}, \dots, v_{i_k}\}^\perp} L|$ was constant with respect to the choice of k -tuple $v_{i_1}, \dots, v_{i_k}$, that is can we find L such that

$$\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}| |\pi_{\text{span}\{v_{i_1}, \dots, v_{i_k}\}^\perp} L| = C_k \sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|,$$

for $1 \leq k \leq d$. Once we have fixed $v_1, \dots, v_d$ we think of $C_k = |\pi_{\text{span}\{v_{i_1}, \dots, v_{i_k}\}^\perp} L|$ as a constant with respect to the choice of $v_{i_1}, \dots, v_{i_k} \in \{v_1, \dots, v_d\}$. The C_k may well depend on the choice

of $v_1, \dots, v_d \in \mathbb{R}^d$. Given such $L \subset \mathbb{R}^d$, we have

$$\begin{aligned} \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{d}{k}} \right) &= \frac{1}{C_k} \left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}| |\pi_{\text{span}\{v_{i_1}, \dots, v_{i_k}\}^\perp} L|}{\binom{d}{k}} \right) \\ &= \frac{1}{C_k} V(P, k; L, d - k). \end{aligned}$$

Then, provided that $C_k^2 \leq C_{k-1} C_{k+1}$ for $2 \leq k \leq d - 1$, by the Aleksandrov-Fenchel inequality we have

$$\begin{aligned} &\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{d}{k}} \right)^2 \\ &= \frac{1}{C_k^2} V(P, k; L, d - k)^2 \\ &\geq \frac{1}{C_k^2} V(P, k; L, d - k + 1) V(P, k; L, d - k - 1) \\ &\geq \left(\frac{1}{C_{k-1}} V(P, k; L, d - k + 1) \right) \left(\frac{1}{C_{k+1}} V(P, k; L, d - k - 1) \right) \\ &= \left(\frac{\sum_{\{i_1, \dots, i_{k-1}\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|}{\binom{d}{k-1}} \right) \left(\frac{\sum_{\{i_1, \dots, i_{k+1}\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_{k+1}}|}{\binom{d}{k+1}} \right). \end{aligned}$$

Note that for $L = \mathbb{B}_2^d$ and any $u_1, \dots, u_k \in \mathbb{R}^d$ we have $|\pi_{\text{span}\{u_1, \dots, u_k\}^\perp} L| = \kappa_{d-k}$. In particular

$$\left(\frac{\sum_{\{i_1, \dots, i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{d}{k}} \right) = \frac{1}{2^{2k} \kappa_{d-k}} V(\Pi P[k]; \mathbb{B}_2^d[d - k]).$$

Unfortunately $\kappa_{d-k}^2 \not\leq \kappa_{d-k+1} \kappa_{d-k-1}$, so the above argument will not work if we set $L = \mathbb{B}_2^d$. Consider the $\mathbb{R}^3$ case:

$$\kappa_2^2 = \pi^2 \not\leq 2 \cdot \frac{4\pi}{3} = \kappa_1 \kappa_3.$$

Note that $L \subset \mathbb{R}^d$ is allowed depend on the choice of $v_1, \dots, v_d \in \mathbb{R}^d$ provided that $|\pi_{\text{span}\{v_{i_1}, \dots, v_{i_k}\}^\perp} L|$ does not depend on the choice of $\{v_{i_1}, \dots, v_{i_k}\} \subset \{v_1, \dots, v_d\}$. Suppose instead that we would like $|\pi_{\text{span}\{u_1, \dots, u_k\}^\perp} L|$ constant for any choice of $u_1, \dots, u_k \in \mathbb{S}^{d-1}$, then there don't seem to be many options other than $L = \mathbb{B}_2^d$ (maybe this is more true if we impose that L should also be convex). My hope is that since we are only wanting $|\pi_{\text{span}\{v_{i_1}, \dots, v_{i_k}\}^\perp} L|$ to be constant for the finitely many choices of $v_{i_1}, \dots, v_{i_k} \in \{v_1, \dots, v_d\}$, perhaps we can find $L \subset \mathbb{R}^d$ such that $C_k = |\pi_{\text{span}\{v_{i_1}, \dots, v_{i_k}\}^\perp} L|$ satisfies

$$C_k^2 \leq C_{k-1} C_{k+1}.$$

Does there exist a convex body with projections all equal to 1? If we could in fact choose $L \subset \mathbb{R}^d$ such that $|\pi_{\text{span}\{v_{i_1}, \dots, v_{i_k}\}^\perp} L| = 1$ for all $1 \leq k \leq d$ and $v_{i_1}, \dots, v_{i_k} \in \{v_1, \dots, v_d\}$, then we would be done. Perhaps this is too much to ask for, although in $\mathbb{R}^2$ there are many possibilities for L and in $\mathbb{R}^3$ it does not seem immediately impossible but certainly not easy to find such L . In $\mathbb{R}^2$ we answer the following question: given $u_1, u_2 \in \mathbb{R}^2$ can we find $L \subset \mathbb{R}^2$ such that

$$|\pi_{u_1^\perp} L| = |\pi_{u_2^\perp} L| = |L| = 1.$$

Note that, if we impose the condition that L is convex, the conditions $|\pi_{u_1^\perp} L|, |\pi_{u_2^\perp} L| \leq 1$ simply states L must live in the intersection of two unit width strips $T_1, T_2 \subset \mathbb{R}^2$ in directions u_1 and u_2 respectively. To make these equalities we should just look for L which spans the width of each strip. It is not difficult to see that $|T_1 \cap T_2| = \frac{1}{|u_1 \wedge u_2|} \geq 1$ with equality if and only if $u_1 \perp u_2$. So we can certainly choose $L \subset T_1 \cap T_2$ with unit area which spans the width of each strip.

In $\mathbb{R}^3$ the problem is roughly as follows: given $u_1, u_2, u_3 \in \mathbb{R}^3$, can we find $L \subset \mathbb{R}^3$ whose projections $\pi_{u_1^\perp} L, \pi_{u_2^\perp} L, \pi_{u_3^\perp} L$ solve the respective 2-dimensional problems. For example, we would like that $L_1 = \pi_{u_1^\perp} L \subset u_1^\perp$ to satisfy

$$|L_1| = |\pi_{\tilde{u}_2^\perp} L_1| = |\pi_{\tilde{u}_3^\perp} L_1| = 1,$$

where $\tilde{u}_2 = \pi_{u_1^\perp} u_2$ and $\tilde{u}_3 = \pi_{u_1^\perp} u_3$ are considered as vectors living in $u_1^\perp$. If we have solutions L_1, L_2, L_3 to the 2-dimensional problems in $u_1^\perp, u_2^\perp, u_3^\perp$ respectively, then any solution $L \subset \mathbb{R}^3$ to the 3-dimensional problem will have a translate contained in the intersection of three tubes with cross sections L_1, L_2, L_3 respectively, that is for some $v \in \mathbb{R}^3$

$$L + v \subset \left(\pi_{u_1^\perp}^{-1} L_1 \right) \cap \left(\pi_{u_2^\perp}^{-1} L_2 \right) \cap \left(\pi_{u_3^\perp}^{-1} L_3 \right).$$

In particular, for such a convex set L with volume 1 to exist, it is necessary to have

$$\left| \left(\pi_{u_1^\perp}^{-1} L_1 \right) \cap \left(\pi_{u_2^\perp}^{-1} L_2 \right) \cap \left(\pi_{u_3^\perp}^{-1} L_3 \right) \right| \geq 1.$$

This is somewhat reminiscent of a special case of the affine-invariant Loomis–Whitney inequality: for any $K \subset \mathbb{R}^d$ and $\omega_1, \dots, \omega_d \in \mathbb{S}^{d-1}$

$$\left| \bigcap_{j=1}^d \pi_{\omega_j^\perp}^{-1} \left(\pi_{\omega_j^\perp} K \right) \right| \leq |\omega_1 \wedge \dots \wedge \omega_d|^{-\frac{1}{d-1}} \prod_{j=1}^d |\pi_{\omega_j^\perp} K|^{\frac{1}{d-1}}.$$

Suppose equality is attained for some $K_0 \subset \mathbb{R}^d$ with $|\pi_{\omega_j^\perp} K| = 1$ for $j = 1, \dots, d$, then we would have

$$\left| \bigcap_{j=1}^d \pi_{\omega_j^\perp}^{-1} \left(\pi_{\omega_j^\perp} K \right) \right| = |\omega_1 \wedge \dots \wedge \omega_d|^{-\frac{1}{d-1}} \geq 1,$$

with the final inequality being an equality if and only if the ω_i are all orthogonal. This suggests that a solutions to the problem I have described in $\mathbb{R}^3$ are related to "close to equality" cases for the above case of affine-invariant Loomis–Whitney.

Equivalent problem by taking dual bodies Recall the following two facts about convex bodies and their duals:

Lemma 6.28. *Given a convex body $K \subset \mathbb{R}^d$ and a subspace $V \subset \mathbb{R}^d$*

$$\pi_V(K^*) = (K \cap V)^*.$$

Lemma 6.29. *Let Q_d, O_d denote the unit cube and unit octahedron in $\mathbb{R}^d$ respectively. For any invertible linear transformation A we have*

$$|AQ_d| |(AQ_d)^*| = |AO_d| |(AO_d)^*| = \frac{4^d}{(d+1)!}.$$

If we restrict our search to L being either a parallelotope or some affine image of an octahedron then we can restate the question in $\mathbb{R}^3$ as follows: given planes V_1, V_2, V_3 , can we find L among parallelotopes or affine images of octahedrons such that the following holds:

1. $L \cap V_i$ and $L \cap V_i \cap V_j$ are parallelotopes or affine images of octahedrons for every choice of i, j ,
2. $|L| = \frac{4^3}{3!}$,
3. $|P \cap V_i| = \frac{4^2}{2!}$ for every choice of i ,
4. $|P \cap V_i \cap V_j| = \frac{4^1}{1!}$ for every choice of i, j .

6.5 Attempting to use Muirhead inequalities for $p = 1$

In this last section we briefly discuss an unsuccessful attempt to prove the vector-valued Maclaurin inequality for $p = 1$ using bunching or Muirhead inequalities.

Consider the following approach to proving Conjecture 6.9 with $p = 1$: take both sides to the power $k(k+1)$ and then multiply both sides by $\binom{d}{k} \binom{d}{k+1}$, so we want to show that

$$\binom{d}{k-1}^k \left(\sum_{\{i_1, \dots, i_k\} \subset [d]} |u_{i_1} \wedge \dots \wedge u_{i_k}| \right)^{k-1} \leq \binom{d}{k}^{k-1} \left(\sum_{\{i_1, \dots, i_{k-1}\} \subset [d]} |u_{i_1} \wedge \dots \wedge u_{i_{k-1}}| \right)^k. \quad (6.34)$$

Note that for $j = 2, \dots, d$ we have

$$\frac{|u_1 \wedge \dots \wedge u_d|}{\|u_1\| |u_2 \wedge \dots \wedge u_d|} = \|\pi_{\{u_2, \dots, u_d\}^\perp} u_1\| \leq \|\pi_{\{u_2, \dots, \hat{u}_j, \dots, u_d\}^\perp} u_1\| = \frac{|u_1 \wedge u_2 \wedge \dots \wedge \hat{u}_j \wedge \dots \wedge u_d|}{\|u_1\| |u_2 \wedge \dots \wedge \hat{u}_j \wedge \dots \wedge u_d|}. \quad (6.35)$$

Expanding out the brackets in (6.34), one might be tempted to try and use (6.35) in conjunction with Muirhead inequalities. Given n -tuples of real numbers $(\alpha) = (\alpha_1, \dots, \alpha_n)$ and $(\beta) = (\beta_1, \dots, \beta_n)$, Muirhead inequalities (sometimes referred to as bunching inequalities) tell us when we can say that for all real positive n -tuples $x = (x_1, \dots, x_n)$ the following inequality holds,

$$\sum_{\sigma \in S_n} x_{\sigma_1}^{\alpha_1} \dots x_{\sigma_n}^{\alpha_n} \leq \sum_{\sigma \in S_n} x_{\sigma_1}^{\beta_1} \dots x_{\sigma_n}^{\beta_n}.$$

For simplicity we define

$$[\alpha] := \sum_{\sigma \in S_n} x_{\sigma_1}^{\alpha_1} \cdots x_{\sigma_n}^{\alpha_n}.$$

This differs by a factor of $\frac{1}{n!}$ to the convention used in [25]. Since $[\alpha]$ is unchanged by permutations of (α) , in this context we regard to n -tuples the same if they differ only in arrangement. For two n -tuples $(\alpha), (\beta)$, we say that (α) is *majorized* by (β) and write $(\alpha) < (\beta)$ if the two n -tuples can be arranged to satisfy the following three conditions:

1. $\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n$;
2. $\alpha_1 \geq \cdots \geq \alpha_n, \beta_1 \geq \cdots \geq \beta_n$;
3. $\alpha_1 + \cdots + \alpha_r = \beta_1 + \cdots + \beta_r \quad (1 \leq r < n)$.

The following result due to Muirhead.

Theorem 6.30 (Muirhead inequalities). *For all positive real n -tuples $(x_1, \dots, x_n)$, $[\alpha] \leq [\beta]$ if and only if $(\alpha) < (\beta)$. Equality holds only if (α) and (β) are the same **or** $x_1 = \cdots = x_n$.*

The first non-trivial case of (6.34) is $k = 3$ and $d = 4$,

$$\binom{4}{2} \left(\sum_{\{i,j,k\} \subset [4]} |u_i \wedge u_j \wedge u_k| \right)^2 \leq \binom{4}{3} \left(\sum_{\{i,j\} \subset [4]} |u_i \wedge u_j| \right)^3. \quad (6.36)$$

To see if the using Muirhead inequalities has any chance of proving this we can firstly consider the case where the u_i are orthogonal, more precisely for positive $a_1, \dots, a_d \in \mathbb{R}$

$$\binom{4}{2}^3 \left(\sum_{\{i,j,k\} \subset [4]} a_i a_j a_k \right)^2 \leq \binom{4}{3}^2 \left(\sum_{\{i,j\} \subset [4]} a_i a_j \right)^3. \quad (6.37)$$

In this setting (6.35) becomes trivial, so we are interested in whether this particular case of the vector-valued Maclaurin inequality can be established via Muirhead inequalities. Each term on the left hand side of (6.37) will be one of the following two forms,

- $a_i^2 a_j^2 a_k^2$,
- $a_i^2 a_j^2 a_k a_l$.

Each term on the right hand side of (6.37) will be one of the following three forms,

- $a_i^3 a_j^3$,
- $a_i^3 a_j^2 a_k$,
- $a_i^3 a_j a_k a_l$,
- $a_i^2 a_j^2 a_k^2$,
- $a_i^2 a_j^2 a_k a_l$.

It follows that the only Muirhead inequalities we could possibly apply are the following:

1. $[(2, 2, 2, 0)] \leq [(3, 3, 0, 0)]$
2. $[(2, 2, 2, 0)] \leq [(3, 2, 1, 0)]$
3. $[(2, 2, 2, 0)] \leq [(2, 2, 1, 1)]$
4. $[(2, 2, 1, 1)] \leq [(3, 3, 0, 0)]$
5. $[(2, 2, 1, 1)] \leq [(3, 2, 1, 0)]$
6. $[(2, 2, 1, 1)] \leq [(3, 1, 1, 1)]$

Expanding out of the brackets in (6.36) we get

$$\begin{aligned}
& 6^3 ([(2, 2, 2, 0)] + 2[(2, 2, 1, 1)]) \\
& \leq 4^2 ([(3, 3, 0, 0)] + 3[(3, 2, 1, 0)] + 3![(3, 1, 1, 1)] + 3![(2, 2, 2, 0)] + (3 + 2 \cdot 3!)[(2, 2, 1, 1)]) .
\end{aligned}$$

Which reduces to

$$120[(2, 2, 2, 0)] + 192[(2, 2, 1, 1)] \leq 4^2[(3, 3, 0, 0)] + 4^2 \cdot 3[(3, 2, 1, 0)] + 4^2 \cdot 3![(3, 1, 1, 1)].$$

One can see that applications of Muirhead inequalities 1-6 alone will not be enough to establish this inequality.

Further questions

To conclude, we gather together a few further questions and suggest potential avenues for future research.

In Chapter 2, we discussed a variety of known factorisations for multilinear geometric inequalities, most of which generalised in some way or another the factorisation for the Loomis–Whitney inequality given in [16]. There are many inequalities we are still not able to factorise explicitly, the simplest of which is the sharp Young convolution inequality, and constructing new factorisations or developing more general methods for doing so is an open area of research. In Chapter 3, we discussed a certain duality of geometric Brascamp–Lieb data, which turned out to be the manifestation of a more general Fourier duality of Brascamp–Lieb inequalities used in [9]. Whether or not exploiting this structure can be used as a tool for constructing factorisations is not clear.

Another interesting question is whether the multilinear duality of Carbery–Hänninen–Valdimarsson can be extended to the setting of non-linear Brascamp–Lieb inequalities. At the end of Chapter 2, we wrote down what looks like a reasonable candidate for a factorisation of such an inequality, but there are no known examples where we are able to construct such functions explicitly.

In Chapter 4, we drew connections between the directed volume and visibility introduced by Guth, and the notion of projection body from classical convex geometry. This allows one to write an almost entirely convex geometric proof of endpoint multilinear Kakeya inequality. If one could prove Theorem 4.34 directly, we would have an alternative route for proving the endpoint multilinear Kakeya inequality which completely avoids the use of polynomials. This could be useful in studying generalisations of the multilinear Kakeya inequality, where tubes are replaced by neighbourhoods of Lipschitz curves. It would also be interesting to explore possible variants of Theorem 4.34, for example, we could impose the condition that every $K(Q)$ must be the same up to orthogonal transformations and dilations.

In Chapter 5 we study quantities $\mathcal{H}_1^{d-1}(Z), \dots, \mathcal{H}_d^{d-1}(Z)$, which generalise Guth’s directed volume for a hypersurface $Z \subset \mathbb{R}^d$. In particular, we establish a chain of inequalities, all of which become equalities when $Z = \mathbb{S}^{d-1}$. Trivially, we always have $\mathcal{H}_1^{d-1}(Z) = \mathcal{H}^{d-1}(Z)$, so the equality case gives us a way to calculate $\mathcal{H}_2^{d-1}(\mathbb{S}^{d-1}), \dots, \mathcal{H}_d^{d-1}(\mathbb{S}^{d-1})$. There are very few examples where we are able to calculate $\mathcal{H}_k^{d-1}(Z)$ for $k \geq 2$, and even a direct method for calculating $\mathcal{H}_k^{d-1}(\mathbb{S}^{d-1})$ would be of interest.

As we saw in Chapter 6, there are still many cases of the vector-valued Maclaurin inequality which have not been established. Although we have some results in the special cases of $p = 0, 1, 2, \infty$, it is not clear whether such an inequality is reasonable for general $p \in [0, \infty]$. It is not known whether one can use an interpolation-type argument to deal with intermediate values of p . If we take $p = 1, 2$, it should be noted that there are no known examples where $d < m$. The arguments given in Sections 6.1 and 6.2 only work if $m = d$, and it is not clear how one could adapt these methods to work for $m > d$.

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