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**Topics in affine and discrete
harmonic analysis**

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Jonathan Hickman*)

Publications

This thesis incorporates material adapted from published work of the author and also work which has been submitted for publication. In particular, results and proofs discussed in Chapter 2 and Chapter 3 appear in [Hic] and [Hic14], respectively. The final chapter details some unpublished joint work of the author and J. Wright.

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Lay Summary

The thesis investigates two related questions in harmonic analysis, both of which are essentially geometric in character.

- The first question concerns averaging: vaguely, if one “averages out” an object then it typically becomes more “regular” or “smooth” and a natural problem is to quantify this process. For instance, suppose the objects are functions on the (Euclidean) plane. If one begins with a function which has a tall and thin distribution, then, at least naïvely, one would expect an averaging process to flatten this out to a broad and fat distribution. Such “flattening” can be precisely measured using mathematical objects known as L^p -norms. Here averages over curves are considered and it is shown in certain cases that the degree of smoothing induced by the averaging process depends, in a very precise manner, on the geometry of the curves. In particular, the crucial consideration is the extent to which the curves bend and twist in space: as soon as one of the curves has a portion which is straight, no matter how tiny this portion may be, the corresponding averaging process fails to smooth out the functions in the desired manner.
- The second question can be explained by analogy with the basic idea of a filter in signal engineering. A filter is a process which attenuates all frequencies of a signal which lie outside some fixed range; a common example is a low pass filter which, as the name suggests, removes all high frequencies. Mathematically, a signal can be thought of as a function f where $f(t)$ is the amplitude at time t . Furthermore, applying a low pass filter can be described in terms of a mathematical operation on f known as a Fourier transform. It is natural to consider Fourier transforms not only of functions of a single variable but also functions on, say, the plane. Now one can consider the mathematical formulation of a filter in this context; the “frequencies” of the function f correspond to points on the plane and it is natural to consider what happens when one restricts these frequencies to some fixed set. In this thesis the restriction of the frequencies to curves and (in higher dimensions) surfaces is investigated. Again, the behaviour of interest is governed by geometric considerations. It is stressed that this work is not related to signal engineering in a literal sense: the analogy was made purely to aid the exposition. However, the study of “Fourier restriction phenomena” is an important part of modern harmonic analysis and is closely related to central questions in the study of differential equations and certain branches of geometry.

Abstract

In this thesis a number of problems in harmonic analysis of a geometric flavour are discussed and, in particular, the Lebesgue space mapping properties of certain averaging and Fourier restriction operators are studied. The first three chapters focus on the perspective afforded by affine-geometrical considerations whilst the remaining chapter considers some discrete variants of these problems.

In Chapter 1 there is an overview of the basic affine theory of the aforementioned operators and, in particular, the affine arc-length and surface measures are introduced.

Chapter 2 presents work of the author, submitted for publication, concerning an operator which takes averages of functions on Euclidean space over both translates and dilates of a fixed polynomial curve. Moreover, the averages are taken with respect to the affine arc-length; this allows one to prove Lebesgue space estimates with a substantial degree of uniformity in the constants. The sharp range of uniform estimates is obtained in all dimensions except for an endpoint.

Chapter 3 presents some work of the author, published in *Mathematika*, concerning a family of Fourier restriction operators closely related to the averaging operators discussed in Chapter 2. Specifically, a Fourier restriction estimate is obtained for a broad class of conic surfaces by introducing a certain measure which exhibits a special kind of affine invariance. Again, the sharp range of estimates is obtained, but the results are limited to the case of 2-dimensional cones.

Finally, Chapter 4 discusses some recent joint work of the author and Jim Wright considering the restriction problem over rings of integers modulo a prime power. The sharp range of estimates is obtained for Fourier restriction to the moment curve in finitely-generated free modules over such rings. This is achieved by lifting the problem to the p -adics and applying a classical argument of Drury in this setting. This work aims to demonstrate that rings of integers offer a simplified model for the Euclidean restriction problem.

Notation

The majority of the notation used herein is standard in the harmonic analysis literature. As a rough guide, it is noted that throughout this thesis:

- n will refer to the dimension of some space (typically \mathbb{R}^n);
- The Hölder conjugate of $1 \leq p \leq \infty$ is denoted p' ;
- The cardinality of a finite set B is denoted $\#B$;
- The (n -dimensional) Lebesgue measure of a Lebesgue measurable set $B \subseteq \mathbb{R}^n$ is denoted $|B|$;
- $\overline{\text{conv}}E$ denotes the closed convex hull of $E \subseteq \mathbb{R}^n$;
- $\text{supp}f$ denotes the support of a function f ;
- $C_c^k(\Omega)$ denotes the class of C^k -functions on Ω of compact support and $C_c(\Omega)$ is defined similarly;
- $\mathcal{S}(\mathbb{R}^n)$ denotes the class of Schwartz functions on \mathbb{R}^n .

In addition:

- The following ‘wiggles’ notation will be used extensively. If $X, Y \geq 0$ and L is a list of objects, then the notation $X \lesssim_L Y$ or $Y \gtrsim_L X$ signifies $X \leq C_L Y$ for some constant C_L which, unless otherwise stated,¹ depends only on n , any relevant Lebesgue exponents and on the objects featured in the list. This situation is also described by “ X is $O_L(Y)$ ”. As an example, in the estimate

$$\|Tf\|_{L^q(\Omega_2)} \lesssim \|f\|_{L^p(\Omega_1)} \tag{0.0.1}$$

the implied constant may depend on p and q but this will often be suppressed in the notation. In addition, $X \sim_L Y$ indicates $X \lesssim_L Y \lesssim_L X$.

Whilst this notation may at first sight appear ambiguous, it is useful to highlight the relevant dependencies.

- It will be often be tacitly assumed that a given L^p -inequality, such as one of the form (0.0.1), is *a priori* in the sense that it holds for a suitable dense class of functions belonging to the relevant L^p space(s). For instance, when considering functions on \mathbb{R}^n this dense class can often be taken to be either $C_c^\infty(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n)$.

¹A notable exception is in Chapter 2 where the dependence of constants on the degree of a certain polynomial will also be suppressed. This, however, is made clear at the beginning of the aforementioned chapter.

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Chapter 1

Introduction: affine harmonic analysis

1.1 The rôle of curvature

Primarily, this thesis investigates a number of problems in harmonic analysis involving operators whose definition depends on some submanifold (or family of submanifolds) of Euclidean space. In each case the differential geometry of the underlying submanifold (or family) determines the mapping properties of the associated operator and, in particular, various notions of curvature play a crucial rôle. To provide some simple and well-known examples, let Σ be a smooth hypersurface¹ in \mathbb{R}^n where $n \geq 2$ and λ a C_c^∞ -density on Σ ; that is, λ is the product of the surface measure σ on Σ and a smooth, non-negative function of compact (non-empty) support. With this set up, one may consider the following, rather general, questions.

Example 1 (Convolution/Averaging operator). For which Lebesgue exponents p, q does the convolution estimate

$$\|f * \lambda\|_{L^q(\mathbb{R}^n)} \lesssim \lambda \|f\|_{L^p(\mathbb{R}^n)} \quad (1.1.1)$$

hold? Notice (1.1.1) is trivially valid when $p = q$ for all $1 \leq p \leq \infty$ since λ is a finite measure. A classical (and very general) theorem of Hörmander [Hör60] prohibits any $L^p - L^q$ inequality for $p > q$ and simple dimensionality and duality considerations (observed by testing (1.1.1) and its dual formulation against the characteristic function of a small ball) then show that a necessary condition for (1.1.1) is that $(1/p, 1/q)$ lies in the triangle

$$\overline{\text{conv}}\left\{(0, 0), (1, 1), \left(\frac{n}{n+1}, \frac{1}{n+1}\right)\right\}. \quad (1.1.2)$$

Example 2 (Stein-Tomas restriction estimates). For which Lebesgue exponents p does the Fourier restriction estimate

$$\|\hat{f}|_\Sigma\|_{L^2(\lambda)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad (1.1.3)$$

hold? Notice (1.1.3) is trivially valid for $p = 1$. Testing the inequality against some simple

¹The term “smooth” is used rather loosely in this discussion and typically can be taken to mean C^k where $k \in \mathbb{N}_0 \cup \{\infty\}$ is large enough to ensure all subsequent definitions make sense. For instance, in order to define curvature functions one should assume at least a C^2 condition on a hypersurface and at least a C^n condition on a curve in \mathbb{R}^n .

examples (in particular, the so-called Knapp example²) shows that a necessary condition for the above restriction estimate is given by $1 \leq p \leq 2(n+1)/(n+3)$.

Both Example 1 and Example 2 admit a range of trivial estimates which are always valid for any choice of (Σ, λ) . In each case there is also a necessary condition on the exponents which describes the largest possible range of estimates one may hope to achieve. These observations lead to a more specific and precise formulation of the questions posed above.

Question 3. *Under what conditions on (Σ, λ) is the respective necessary condition also sufficient for (1.1.1) or (1.1.3)?*

If the respective necessary condition is sufficient, then the pair (Σ, λ) is said to be *non-degenerate* with respect to either the convolution or the restriction problem.

It transpires that for Example 1 and Example 2 non-degeneracy can be characterised in purely differential-geometric terms. In particular, in either case (Σ, λ) is non-degenerate if and only if the Gaussian curvature of Σ is everywhere non-vanishing in the support of λ . For the convolution problem, the estimate (1.1.1) was proved for the full non-degenerate range under the non-vanishing curvature hypothesis by Strichartz [Str70] and Littman [Lit73] (for further discussion of this result see [SS11, Chapter 8]). For the restriction problem, inequalities of the form (1.1.3) were first considered by Stein in unpublished work dating back to the late 1960s; sharp estimates were later established by Stein and Tomas³ [Tom75, Tom79, Ste86]. The necessity of the curvature condition, for both the convolution and restriction problems, is treated in [IL00]. It is remarked that it is not entirely surprising that the characterisation of non-degeneracy is the same in both cases, since estimates of the form (1.1.1) and (1.1.3) are closely related.⁴

A more ambitious version of Example 2 is to attempt to determine the precise $L^p - L^q$ mapping properties of the restriction operator.

Example 4 (Full restriction problem). For which Lebesgue exponents p, q does the Fourier restriction estimate

$$\|\hat{f}|_{\Sigma}\|_{L^q(\lambda)} \lesssim_{\lambda} \|f\|_{L^p(\mathbb{R}^n)} \tag{1.1.4}$$

hold? When $p = 1$, (1.1.4) is trivially valid for all $1 \leq q \leq \infty$. Furthermore, by testing the inequality and its dual formulation against some simple examples (one being the aforementioned Knapp example) one obtains the necessary conditions

$$1 \leq p < \frac{2n}{n+1}, \quad 1 \leq q \leq \frac{n-1}{n+1}p', \tag{1.1.5}$$

leading to a notion of non-degeneracy for the full restriction problem.

The problem of classifying the non-degenerate (Σ, λ) for the full restriction problem is well-understood when $n = 2$: it is known in this case (Σ, λ) is non-degenerate if and only

²For the Knapp example in other (related) contexts see Remark 8 below and Chapter 4.

³To be precise, Stein's work established the sharp result under the non-vanishing curvature hypothesis whilst Tomas' argument gave estimates only in the restricted range $1 \leq p < 2(n+1)/(n+3)$, missing the endpoint. It was observed by Carbery that Tomas' proof can be adapted to give a restricted-type inequality when $p = 2(n+1)/(n-3)$ and, more recently, Bak and Seeger [BS11] developed these methods to obtain the full range of strong-type estimates. Bak and Seeger thereby gave an alternative and, it transpires, more robust proof of Stein's theorem which can be applied to prove restriction inequalities in more general and abstract settings (see also [Moc00, Mit02]).

⁴For instance, the so called TT^* method of Tomas [Tom75] effectively reduces the study of (1.1.3) to proving $L^p - L^q$ estimates for the convolution operator $f \mapsto f * \tilde{\lambda}$ and both this and $f \mapsto f * \lambda$ can be analysed using similar techniques.

if the usual non-vanishing curvature condition is satisfied. This essentially follows from the characterisation of non-degeneracy for the Stein-Tomas problem together with the classical estimates of Fefferman and Zygmund [Fef70, Zyg74]. In contrast with Example 2, which has been resolved for all $n \geq 2$, in higher dimensions few sharp results are known for the full restriction problem. In particular, for $n \geq 3$ it is unknown whether (1.1.5) implies (1.1.4) for *any* (Σ, λ) satisfying the non-vanishing curvature condition.⁵ Establishing sharp Fourier restriction estimates for natural hypersurfaces such as the sphere and paraboloid is a major open problem in harmonic analysis with deep connections to many important questions in Fourier analysis, PDE, geometric measure theory and beyond. The reader is referred to [Tao04, Wol03] for further background to this fascinating topic.

There are various refinements of the characterisation of non-degeneracy given above. One example is the following theorem which, for simplicity, considers the $n = 2$ case only.

Theorem 5. *Suppose $n = 2$, (Σ, λ) are as above (so $\Sigma \subseteq \mathbb{R}^2$ is a plane curve) and that the curvature κ of Σ vanishes to at most finite order on $\text{supp } \lambda$. Let m be the maximum order to which κ vanishes on this set.*

i) The convolution estimate (1.1.1) is valid for $(1/p, 1/q)$ lying in trapezium

$$\overline{\text{conv}}\left\{(0, 0), (1, 1), \left(\frac{2}{m+3}, \frac{1}{m+3}\right), \left(\frac{m+2}{m+3}, \frac{m+1}{m+3}\right)\right\}. \quad (1.1.6)$$

ii) If $m \geq 1$, then the restriction estimate (1.1.4) is valid for exponents p, q satisfying

$$1 \leq p \leq \frac{m+4}{m+3}, \quad 1 \leq q \leq \frac{1}{m+3}p'. \quad (1.1.7)$$

Both results are sharp in the sense that (1.1.6) or (1.1.7) is a necessary condition for the corresponding estimate to hold.

Remark 6. As noted above, when $m = 0$ a slightly stronger condition on the exponents is required for the restriction estimate to hold.

The estimates in Theorem 5 for the convolution operators are proved, *inter alia*, in [BOS02] with near-sharp results appearing earlier in [See98]. The Fourier restriction inequality described in Theorem 5 is due to Sogge [Sog87], building upon work of Sjölin [Sjö74]. To clarify, the hypothesis of Theorem 5 is that $m := \max\{j(x) : x \in \text{supp } \lambda\} < \infty$ where $j = j(x)$ is defined for each $x \in \Sigma$ to be the least (non-negative) integer such that $\kappa^{(j)}(x) \neq 0$. In the literature this is referred to as a *type k* or *finite type* condition, with the convention $k := m + 2$, and is equivalent to a bound on the order of contact of the curve with any affine line in \mathbb{R}^2 .⁶ The definition of the type k condition readily extends to hypersurfaces in \mathbb{R}^n by considering either the vanishing of the Gaussian curvature or the order of contact of Σ with affine hyperplanes (for further discussion see [Ste93, Chapter VIII]).

Remark 7. The situation in higher dimensions is significantly more complicated and forms the subject of ongoing research. For instance, in the case of Examples 1 and 2, the sharp range

⁵Some sharp restriction results are known for hypersurfaces in \mathbb{R}^3 and \mathbb{R}^4 which do not satisfy the non-vanishing curvature condition, so that the sharp range of exponents is strictly smaller than that given by (1.1.5) for the corresponding value of n . The most important of these is the sharp restriction theorem of Wolff [Wol01] for the light cone in \mathbb{R}^4 .

⁶The integer k is the maximum order of contact of an affine line with Σ at a point in $\text{supp } \lambda$, where a non-tangential intersection is defined to have order 1. Note that $x \in \Sigma$ has order 2 contact with the tangent line to Σ at x and so $k \geq 2$, with equality if and only if the non-vanishing curvature condition is satisfied.

of estimates for finite type (Σ, λ) typically cannot be expressed solely in terms of the maximum order of contact of Σ with any affine hyperplane. Rather, in this case the key geometric information is given in terms of Newton polyhedra associated to Σ . As an example of work in this direction, a treatment of the Stein-Tomas restriction problem for finite type (Σ, λ) can be found in the recent papers [IM11, IMa, IMb].

Remark 8. In anticipation of what follows, it is useful to note that the necessary condition (1.1.6) for the type k convolution problem is obtained by considering a Knapp-type example; that is, one tests the convolution estimate against the characteristic function of a small rectangle, adapted to the geometry of the curve Σ . Indeed, for simplicity suppose Σ is parametrised by $t \mapsto (t, t^k + O(t^{k+1}))$ for some $2 \leq k =: m + 2$ in a neighbourhood of the origin and the intersection of the curve with this neighbourhood is contained in $\text{supp } \lambda$ (this is the archetypical example of a pair (Σ, λ) satisfying the hypotheses of Theorem 5). For $0 < \delta \ll 1$ let $R_\delta := [0, \delta] \times [0, \delta^k]$ and observe $\delta \chi_{\tilde{R}_\delta} \lesssim_\lambda \chi_{R_\delta} * \lambda$ where \tilde{R}_δ is some fixed dilate of R_δ . Hence, if (1.1.1) holds, then $\delta |R_\delta|^{1/q} \lesssim_\lambda |R_\delta|^{1/p}$ is valid for all sufficiently small $\delta > 0$ and this forces

$$\frac{1}{p} \leq \frac{1}{q} + \frac{1}{m+1}. \quad (1.1.8)$$

Intersecting the set of $(1/p, 1/q)$ for which (1.1.8) holds with (1.1.2), one obtains (1.1.6). For more general curves one can argue in a similar fashion, using Taylor's theorem and exploiting the symmetries of the problem to essentially reduce the situation to the above case.

Assuming a finite type condition, as in Theorem 5, is one way to obtain a precise and quantitative description of the relationship between curvature and the Lebesgue space mapping properties of operators of the kind introduced. This will not be pursued directly here; rather, the aim will be to study the mapping properties of certain operators using an approach based on affine-geometric considerations.

1.2 The affine approach

When considering a hypersurface Σ with vanishing Gaussian curvature κ , one may study the effect of introducing a mitigating power of $|\kappa|$ in either the definition of the convolution operator or the formulation of the restriction problem. That is, rather than formulate (1.1.1) or (1.1.3) with respect to a C_c^∞ -density λ , one can take the measure to be $d\mu(x) := |\kappa(x)|^{\text{power}} d\sigma(x)$, for some choice of positive power. The weight $|\kappa(x)|^{\text{power}}$ has the potential to ameliorate the effect of any flat points on the surface and, provided that the power is chosen correctly, one can hope to achieve $L^p - L^q$ boundedness for the full range of exponents corresponding to the non-degenerate case. This strategy follows the example of numerous authors (notably Sjölin [Sjö74], Drury [Dru90] and Drury and Marshall [DM85, DM87]) who, in considering convolution and Fourier restriction problems involving degenerate (curves or) surfaces, replaced the underlying surface measure with affine (arc-length or) surface measure. This measure has the desired effect of dampening any degeneracies of the (curve or) surface and also makes the problem affine invariant.

To make the above discussion precise, observe the surface measure σ on Σ is isometrically invariant in the sense that whenever $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry and σ_S is the surface measure

on $\Sigma_S := S(\Sigma)$, it follows that

$$\int_{\Sigma} f \circ S(x) \, d\sigma(x) = \int_{\Sigma_S} f(x) \, d\sigma_S(x) \quad (1.2.1)$$

for all $f \in C_c(\mathbb{R}^n)$. By weighting σ by a certain power of the Gaussian curvature one obtains a measure which is not only isometrically invariant, but invariant under the entire equi-affine group, viz.

Lemma 9. *Let $\Sigma \subset \mathbb{R}^n$ be a smooth hypersurface with Gaussian curvature κ and surface measure σ and $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an invertible affine transformation. Consider the hypersurface $\Sigma_T := T(\Sigma)$, let κ_T denote its curvature and σ_T its surface measure. Then*

$$|\det T|^{(n-1)/(n+1)} \int_{\Sigma} f \circ T(x) |\kappa(x)|^{1/(n+1)} \, d\sigma(x) = \int_{\Sigma_T} f(x) |\kappa_T(x)|^{1/(n+1)} \, d\sigma_T(x) \quad (1.2.2)$$

for all $f \in C_c(\mathbb{R}^n)$, where $\det T$ is defined below.

Any invertible affine transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by $Tx = Ax + a$ for all $x \in \mathbb{R}^n$ for some unique choice of $A \in \text{GL}(n, \mathbb{R})$ and $a \in \mathbb{R}^n$; in this case, $\det T := \det A$. If $|\det T| = 1$, so that T is volume-preserving, then T is said to be an *equi-affine* transformation.

The proof of Lemma 9 is a simple exercise in manipulating the various basic differential-geometric definitions; further details can be found in [Chrc, §11].

Definition 10. The measure $|\kappa(x)|^{1/(n+1)} \, d\sigma(x)$ introduced Lemma 9 is referred to as the *affine surface measure* on Σ .

Now consider the effect of replacing the density λ appearing in the earlier examples with the affine surface measure μ . Note that μ may not be finite, but a common feature of (1.1.1) and (1.1.3) is that both inequalities become *equi-affine invariant* under this choice of measure. In particular, for an invertible affine transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$, let μ_T be the affine surface measure on $\Sigma_T := T(\Sigma)$ and observe

$$\|f * \mu\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \iff \|f * \mu_T\|_{L^q(\mathbb{R}^n)} \leq C |\det T|^{e_c(n;p,q)} \|f\|_{L^p(\mathbb{R}^n)} \quad (1.2.3)$$

and

$$\|\hat{f}|_{\Sigma}\|_{L^q(\mu)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \iff \|\hat{f}|_{\Sigma_T}\|_{L^q(\mu_T)} \leq C |\det T|^{e_r(n;p,q)} \|f\|_{L^p(\mathbb{R}^n)} \quad (1.2.4)$$

where

$$e_c(n;p,q) := \frac{n-1}{n+1} + \frac{1}{q} - \frac{1}{p} \quad \text{and} \quad e_r(n;p,q) := \frac{n-1}{n+1} \frac{1}{q} - \frac{1}{p'}$$

Therefore, if T is an equi-affine transformation, one may transform (Σ, μ) under T and completely preserve the above convolution and restriction inequalities. Further, in the non-trivial endpoint range of (p, q) for the respective operators the exponents $e_c(n;p,q)$ and $e_r(n;p,q)$ vanish and so the estimates become invariant under the full group of invertible affine transformations.

The problem now becomes to prove (1.1.1) and (1.1.3), formulated in terms of the measure μ , for the non-degenerate range of exponents whenever Σ belongs to a large class of hypersurfaces which includes examples with vanishing Gaussian curvature. Moreover, the addition symmetry observed in (1.2.3) and (1.2.4) suggests it should be possible to establish **uniform** estimates

for endpoint exponents; that is, the constants appearing in the inequalities should be, to some degree, independent of the choice of hypersurface.

Here are two examples of results in this direction.

Theorem 11 (Gressman, $n = 2$ case [Gre13]). *Let Σ be a smooth convex plane curve with affine arc-length measure μ .*

i) For $p = 3/2$ and $q = 3$,

$$\|f * \mu\|_{L^q(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)} \tag{1.2.5}$$

where the implied constant is absolute and, in particular, is independent of the choice of Σ .

ii) If, in addition, the measure μ is finite, then (1.2.5) holds whenever $(1/p, 1/q)$ belongs to $\overline{\text{conv}}\{(0, 0), (1, 1), (2/3, 1/3)\}$ with a constant depending on p, q and the total variation of μ .

It is remarked that a slightly weaker version of Theorem 11 with a restricted strong-type endpoint was established earlier by D. Oberlin [Obe99, Obe00].

Theorem 12 (Sjölin [Sjö74] (see also [Obe01])). *Let Σ be a smooth convex plane curve with affine arc-length measure μ .*

i) For $1 \leq p < 4/3$ and $p' = 3q$,

$$\|\hat{f}|_{\Sigma}\|_{L^q(\Sigma, \mu)} \lesssim \|f\|_{L^p(\mathbb{R}^2)} \tag{1.2.6}$$

where the implied constant is independent of the choice of curve Σ .

ii) If, in addition, the measure μ is finite, then (1.2.6) holds for all $1 \leq p < 4/3$ and $p' \geq 3q$ with a constant depending on p, q and the total variation of μ .

Here a smooth convex plane curve Σ is a C^2 curve which admits a supporting line through each of its points. Hence, given $x \in \Sigma$ one can perform a rotation of \mathbb{R}^2 so that in a neighbourhood of x the transformed curve is parametrised by the graph of some C^2 convex function.

Remark 13. An example due to Sjölin [Sjö74] shows that the above restriction and convolution estimates are not possible for certain curves which are highly oscillatory,⁷ in the sense that the sign of the curvature changes from positive to negative (or *vice versa*) many times. Since any convex curve has a non-negative curvature function it is therefore natural to consider uniform estimates over the class of such curves.

Remark 14. It is reasonable to conjecture analogous results hold for hypersurfaces in \mathbb{R}^n . In [Gre13], Gressman established a substantial generalisation of Theorem 11 which is valid in any dimension $n \geq 2$ and concerns a much broader class of operators (see also the earlier work of D. Oberlin [Obe99, Obe00]). As one may expect, the Fourier restriction problem appears to be considerably more difficult. In fact, obtaining uniform estimates for the Stein-Tomas range (corresponding to Example 2) is an open problem, although there are a number of partial results in this direction [Obe04, AKS06, CKZ07, CKZ13, Obe12, Hic14]. The conjectured estimates for the restriction problem in higher dimensions are discussed in more detail in Section 3.5 below.

⁷In particular, Sjölin considers the curve parametrised by $(t, \sin(t^{-k})e^{-1/t})$ for $0 < t < 1$, where k is a large integer. Counterexamples are constructed for higher-dimensional versions of the restriction problem in [CZ02].

The proofs of both Theorem 11 and Theorem 12, which are elegant and pleasantly short, are discussed in the following subsections.

The construction of the affine surface measure and its introduction to these problems has arguably been rather *ad hoc*. However, one can show that μ is, in some sense, the optimal measure for which the convolution and Stein-Tomas restriction inequalities hold. This is made precise by the following theorem, which is valid in all dimensions $n \geq 2$.

Theorem 15 (D. Oberlin [Obe00] and Nicola [Nic08]). *Let $\Sigma \subseteq \mathbb{R}^n$ be a smooth hypersurface with surface measure σ and Gaussian curvature κ . Let λ be a C_c -density on Σ ; that is, $d\lambda(x) = \omega(x)d\sigma(x)$ where $\omega \in C_c(\Sigma)$ is non-negative. Suppose either of the following estimates hold:*

$$\|f * \lambda\|_{L^{n+1}(\mathbb{R}^{n+1})} \leq C \|f\|_{L^{(n+1)/n}(\mathbb{R}^n)} \tag{1.2.7}$$

or

$$\|\hat{f}|_\Sigma\|_{L^2(\lambda)} \leq \sqrt{C} \|f\|_{L^{2(n+1)/(n+3)}(\mathbb{R}^n)}. \tag{1.2.8}$$

Then the pointwise estimate

$$\omega(x) \lesssim C |\kappa(x)|^{1/(n+1)}$$

holds for all $x \in \Sigma$. Here the implied constant depends on the dimension n only.⁸

Hence the affine surface measure is, in some sense, the largest possible density on Σ with which one can hope to achieve the non-degenerate range of estimates for the convolution operator or the L^2 -based restriction inequalities. The convolution case is due to D. Oberlin [Obe00] whilst the result for Fourier restriction is due to Nicola [Nic08], building upon earlier work of Iosevich and Lu [IL00].

1.3 Gressman’s theorem

Here a more detailed discussion of Theorem 11 is provided. In particular, the statement of the result is extended to $n \geq 2$ and the theorem proved in this more general context.

In higher dimensions the hypotheses on the hypersurface Σ are slightly more involved. Let $D \subset \mathbb{R}^{n-1}$ for $n \geq 2$ be a connected open set and $\phi : D \rightarrow \mathbb{R}$ a C^2 mapping. Denote the Hessian determinant of ϕ by $\text{Hess } \phi(u)$; that is,

$$\text{Hess } \phi(u) := \det \left[\left(\frac{\partial^2 \phi}{\partial u_i \partial u_j} (u) \right)_{i,j=1,\dots,n-1} \right]$$

for all $u \in D$.

Let $\Gamma(u) := (u, \phi(u))$, consider the hypersurface

$$\Sigma := \{\Gamma(u) : u \in D\} \tag{1.3.1}$$

and let μ denote the affine surface measure on Σ . With the above choice of parametrisation, μ can be expressed by a particularly simple formula, viz.

$$\int_\Sigma f(x) d\mu(x) = \int_U f \circ \Gamma(u) |\text{Hess } \phi(u)|^{1/(n+1)} du \tag{1.3.2}$$

⁸In fact, if (1.2.8) holds, then the constant can be taken to be absolute.

for all $f \in C_c(\mathbb{R}^n)$, where $U := \{u \in D : |\text{Hess } \phi(u)| > 0\}$. After performing an isometric transformation, any hypersurface can be locally parametrised as a graph of the above form and so the hypothesis (1.3.1) is extremely mild.

Consider the maps $\Phi: U \rightarrow \mathbb{R}^{n-1}$ and $\Psi: \mathbb{R}^n \times U^n \rightarrow (\mathbb{R}^n)^n$ defined by

$$\Phi(u) := \nabla \phi(u) \quad \text{and} \quad \Psi(x; u_1, \dots, u_n) := (x - \Gamma(u_1), \dots, x - \Gamma(u_n)). \quad (1.3.3)$$

In general dimensions, the convexity hypothesis is replaced with the multiplicity condition that there exists some $m \in \mathbb{N}$ such that

$$\#\{u \in U : \Phi(u) = y\} \leq m \quad \text{for almost every } y \in \mathbb{R}^{n-1} \quad (1.3.4)$$

and

$$\#\{(x; \mathbf{u}) \in \mathbb{R}^n \times U^n : \Psi(x; \mathbf{u}) = \mathbf{y}\} \leq m \quad \text{for almost every } \mathbf{y} \in (\mathbb{R}^n)^n. \quad (1.3.5)$$

Having made these definitions the main theorem can be stated.

Theorem 16 (Gressman [Gre13]). *Let $\phi: D \rightarrow \mathbb{R}$ be as above and suppose the multiplicity conditions (1.3.4) and (1.3.5) hold. Then*

$$\|\mu * f\|_{L^{n+1}(\mathbb{R}^n)} \lesssim_m \|f\|_{L^{(n+1)/n}(\mathbb{R}^n)}. \quad (1.3.6)$$

Here the implied constant is uniform in the sense it depends on the multiplicity m and dimension n only and otherwise not on the choice of ϕ .

The assumptions of the theorem are weak and apply to natural classes of hypersurfaces.

Example 17. 1) When $n = 2$ the conditions (1.3.4) and (1.3.5) are easily seen to hold with $m = 2$ whenever $\phi: D \rightarrow \mathbb{R}$ is convex (here $D \subseteq \mathbb{R}$ is an open interval). In particular, Theorem 11 follows from a special case of Theorem 16.

2) For $n \geq 2$, if $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ is any polynomial mapping for which the Gaussian curvature does not vanish everywhere, then (1.3.4) and (1.3.5) hold with a constant m which depends on the degree of ϕ only. This is a consequence of a multiplicity lemma for polynomial mappings; the details are provided at the end of this section. In view of Sjölin's example, the dependence on the degree is natural, since the degree controls the number 'oscillations' of the hypersurface (see [CZ02] for a more detailed discussion of the necessity of multiplicity conditions in the related context of Fourier restriction estimates).

The key ingredient in the proof of Theorem 16 is a reverse-type inequality for certain multilinear functionals due to Gressman. Specifically, let $\text{vol}(u_1, \dots, u_n)$ denote the volume of the simplex in \mathbb{R}^{n-1} whose vertices are given by the vectors $u_1, \dots, u_n \in \mathbb{R}^{n-1}$ and consider the n -linear functional T_γ^n defined by

$$T_\gamma^n(F_1, \dots, F_n) := \int_{\mathbb{R}^{n-1}} \dots \int_{\mathbb{R}^{n-1}} \prod_{j=1}^n F_j(u_j) \text{vol}(u_1, \dots, u_n)^\gamma du_1 \dots du_n$$

for all n -tuples of non-negative measurable functions $F_j: \mathbb{R}^{n-1} \rightarrow [0, \infty)$. It is useful to note

that the volume element can be expressed by the formula

$$\text{vol}(u_1, \dots, u_n) = \frac{1}{(n-1)!} \left| \det \begin{bmatrix} u_{1,1} & u_{1,2} & \dots & u_{1,n-1} & 1 \\ u_{2,1} & u_{2,2} & \dots & u_{2,n-1} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{n,1} & u_{n,2} & \dots & u_{n,n-1} & 1 \end{bmatrix} \right| \quad (1.3.7)$$

where $u_j = (u_{j,1}, \dots, u_{j,n-1})$ for $j = 1, \dots, n$.

Lemma 18 (Gressman [Gre11]). *For $0 < \gamma < \infty$ and $1/p_j - 1 < \gamma$ for $1 \leq j \leq n$ which satisfy $\sum_{j=1}^n 1/p_j = n + \gamma$, the inequality*

$$\prod_{j=1}^n \|F_j\|_{L^{p_j}(\mathbb{R}^{n-1})} \lesssim_{\gamma} T_{\gamma}^n(F_1, \dots, F_n) \quad (1.3.8)$$

holds for all measurable functions $F_j : \mathbb{R}^{n-1} \rightarrow [0, \infty)$.

The reader is referred to [Gre11] for the full proof of this lemma. The proof of the $n = 2$ case follows from classical estimates for fractional integral operators and is included for interest. Indeed, note that for $n = 2$ the multilinear functional becomes

$$T_{-\gamma}(F_1, F_2) := \int_{\mathbb{R}^2} \prod_{j=1}^2 F_j(x_j) |x_1 - x_2|^{-\gamma} dx_1 dx_2,$$

which can be bounded by the classical Hardy-Littlewood-Sobolev inequality.

Lemma 19 (Hardy-Littlewood-Sobolev inequality). *For $0 < \gamma < 1$ and exponents $1 \leq p_1, p_2 \leq \infty$ satisfying $1 - 1/p_1 < \gamma$ for $j = 1, 2$ and $1/p_1 + 1/p_2 = 2 - \gamma$, the inequality*

$$T_{-\gamma}(F_1, F_2) \lesssim_{\gamma} \prod_{j=1}^2 \|F_j\|_{L^{p_j}(\mathbb{R})} \quad (1.3.9)$$

holds for all measurable functions $F_j : \mathbb{R} \rightarrow [0, \infty)$.

Thus, in the $n = 2$ case Lemma 18 is a reverse-type Hardy-Littlewood-Sobolev inequality and, furthermore, is a simple consequence of Lemma 19 and Hölder's inequality.

Proof (of Lemma 18, case $n = 2$). Fix γ, p_1 and p_2 as in the statement of the $n = 2$ case of the lemma, let F_1, F_2 be non-negative and measurable and choose $\mu \geq 2$ such that $0 < (\mu'\gamma)/\mu < 1$ and

$$q_1 := \frac{p_1}{(p_1 - 1/\mu)\mu'} \quad q_2 := \frac{p_2}{(p_2 - 1/\mu)\mu'}$$

satisfy $1 < q_1, q_2$. Write

$$\|F_1\|_{L^{p_1}(\mathbb{R})}^{p_1} \|F_2\|_{L^{p_2}(\mathbb{R})}^{p_2} = \int_{\mathbb{R}^2} \prod_{j=1}^2 F_j(x_j)^{p_j} |x_1 - x_2|^{\gamma/\mu} |x_1 - x_2|^{-\gamma/\mu} dx$$

and apply Hölder's inequality to deduce

$$\|F_1\|_{L^{p_1}(\mathbb{R})}^{p_1} \|F_2\|_{L^{p_2}(\mathbb{R})}^{p_2} \leq (T_{\gamma}(F_1, F_2))^{1/\mu} (T_{-(\mu'\gamma)/\mu}(F_1^{(p_1-1/\mu)\mu'}, F_2^{(p_2-1/\mu)\mu'}))^{1/\mu'}.$$

Since $1/q_1 + 1/q_2 = 2 - (\mu'\gamma)/\mu$ one may invoke (1.3.9) to conclude

$$\begin{aligned} \left(T_{-(\mu'\gamma)/\mu}(F_1^{(p_1-1/\mu)\mu'}, F_2^{(p_2-1/\mu)\mu'})\right)^{1/\mu'} &\lesssim \gamma \|F_1^{(p_1-1/\mu)\mu'}\|_{L^{q_1}(\mathbb{R})}^{1/\mu'} \|F_2^{(p_2-1/\mu)\mu'}\|_{L^{q_2}(\mathbb{R})}^{1/\mu'} \\ &= \|F_1\|_{L^{p_1}(\mathbb{R})}^{p_1-1/\mu} \|F_2\|_{L^{p_2}(\mathbb{R})}^{p_2-1/\mu}, \end{aligned}$$

from which the desired estimate follows. \square

Remark 20. For general $n \geq 2$, the estimate (1.3.8) is a direct consequence of a ‘forward’ inequality for the multilinear functional T_γ^n (which can be thought of as a n -linear generalisation of the bilinear Hardy-Littlewood-Sobolev inequality) together with Hölder’s inequality; the details can be found in [Gre11].

Given the above lemma, the proof of Theorem 16 is remarkably short.

Proof (of Theorem 16). It suffices to prove the inequality

$$\|f^{n/(n+1)} * \mu\|_{L^{n+1}(\mathbb{R}^n)}^{n+1} \lesssim_m \|f\|_{L^1(\mathbb{R}^n)}^n \quad (1.3.10)$$

for all continuous $f : \mathbb{R}^n \rightarrow [0, \infty)$. Fix such an f and $x \in \mathbb{R}^n$ and observe

$$f^{n/(n+1)} * \mu(x) = \int_U f(x - \Gamma(u))^{n/(n+1)} |\text{Hess } \phi(u)|^{1/(n+1)} du.$$

Introduce the change of variables $\Phi : U \rightarrow \mathbb{R}^n$ given by $\Phi(u) = \nabla \phi(u)$, noting that the associated Jacobian is given by $|\text{Hess } \phi(u)|$. Thus, taking $(n+1)$ -th powers,

$$\begin{aligned} f^{n/(n+1)} * \mu(x)^{n+1} &= \left(\int_{\Phi(U)} \sum_{u \in U: \Phi(u)=s} f(x - \Gamma(u))^{n/(n+1)} |\text{Hess } \phi(u)|^{-n/(n+1)} ds \right)^{n+1} \\ &\leq m \left(\int_{\Phi(U)} \left(\sum_{u \in U: \Phi(u)=s} f(x - \Gamma(u)) |\text{Hess } \phi(u)|^{-1} \right)^{n/(n+1)} ds \right)^{n+1}, \end{aligned}$$

where the final inequality is due to Hölder and the hypothesis (1.3.4). Applying inequality (1.3.8) to the right-hand side of the preceding equation, taking $\gamma = 1$, $p_j = n/(n+1)$ and

$$F_j(s) := \sum_{u \in U: \Phi(u)=s} f(x - \Gamma(u)) |\text{Hess } \phi(u)|^{-1}$$

for $1 \leq j \leq n$, yields

$$\begin{aligned} f^{n/(n+1)} * \mu(x)^{n+1} &\lesssim_m \int_{\Phi(U)^n} \prod_{j=1}^n \sum_{u_j \in U: \Phi(u_j)=s_j} f(x - \Gamma(u_j)) |\text{Hess } \phi(u_j)|^{-1} \text{vol}(\mathbf{s}) ds \\ &\sim \int_{U^n} \prod_{j=1}^n f(x - \Gamma(u_j)) V(\mathbf{u}) d\mathbf{u} \end{aligned}$$

where $\mathbf{s} = (s_1, \dots, s_n) \in (\mathbb{R}^{n-1})^n$ and $V(\mathbf{u}) := (n-1)! \times \text{vol}(\nabla \phi(u_1), \dots, \nabla \phi(u_n))$. Now integrate both sides of the above inequality with respect to $x \in \mathbb{R}^n$ to give

$$\|\mu * f^{n/(n+1)}\|_{L^{n+1}(\mathbb{R}^n)}^{n+1} \lesssim_m \int_{\mathbb{R}^n} \int_{U^n} \prod_{j=1}^n f(x - \Gamma(u_j)) V(\mathbf{u}) d\mathbf{u} dx. \quad (1.3.11)$$

To conclude the proof, consider the change of all $n(n-1) + n = n^2$ variables under the transformation $\Psi: \mathbb{R}^n \times U^n \rightarrow (\mathbb{R}^n)^n$ defined in (1.3.3). It is claimed that the absolute value of the Jacobian determinant of this change of variables is in fact $V(\mathbf{u})$; once this claim is established, by invoking (1.3.5) one may bound the integral appearing in the right-hand side of (1.3.11) by

$$\int_{\Psi(\mathbb{R}^n \times U^n)} \sum_{(x, \mathbf{u}): \Psi(x, \mathbf{u}) = \mathbf{y}} \prod_{j=1}^n f(x - \Gamma(u_j)) \, d\mathbf{y} \leq m \|f\|_{L^1(\mathbb{R}^n)}^n,$$

and this gives the desired inequality (1.3.10).

By rearranging the rows and columns of the Jacobian matrix one obtains (modulo a sign) the block matrix

$$\begin{bmatrix} I_{n-1} & 0_{n-1} & \cdots & 0_{n-1} & I_{n-1} & 0_{n-1,1} \\ 0_{n-1} & I_{n-1} & \cdots & 0_{n-1} & I_{n-1} & 0_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_{n-1} & 0_{n-1} & \cdots & I_{n-1} & I_{n-1} & 0_{n-1,1} \\ B_1(u_1) & B_2(u_2) & \cdots & B_n(u_n) & 0_{n,n-1} & 1_{n,1} \end{bmatrix}$$

where:

- I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix;
- For $A \in \{0, 1\}$ the block A_j denotes the $j \times j$ and $A_{i,j}$ the $i \times j$ matrix whose entries are all given by A ;
- $B_j(u)$ denotes the $n \times (n-1)$ matrix whose entries are all zero except for the j th row which is given by $\nabla\phi(u)$.

For $1 \leq j \leq n-1$ subtract column k from column $n^2 - n + j$ whenever $k \equiv j \pmod{n-1}$ to obtain the matrix

$$\begin{bmatrix} I_{n-1} & 0_{n-1} & \cdots & 0_{n-1} & 0_n \\ 0_{n-1} & I_{n-1} & \cdots & 0_{n-1} & 0_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{n-1} & 0_{n-1} & \cdots & I_{n-1} & 0_n \\ B_1(u_1) & B_2(u_2) & \cdots & B_n(u_n) & M(\mathbf{u}) \end{bmatrix} \quad (1.3.12)$$

where $M(\mathbf{u})$ is precisely the matrix appearing in the expression (1.3.7) for the volume element with each u_j replaced with $\nabla\phi(u_j)$. Consequently, the determinant of $M(\mathbf{u})$ is $V(\mathbf{u})$ and, moreover, since the matrix (1.3.12) is block-triangular, it also has determinant $V(\mathbf{u})$. As the Jacobian determinant of the change of variables is (modulo a sign) equal to the determinant of the matrix (1.3.12), the claim is proved. \square

To conclude this section the special case of polynomial hypersurfaces is discussed and the details of Example 17 are given. The key observation is the following multiplicity lemma, which will also be useful in later chapters.

Lemma 21 (Polynomial multiplicity lemma). *Let $Q: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a polynomial mapping; that is, $Q(t) = (Q_j(t))_{j=1}^d$ for all $t \in \mathbb{R}^d$ where each $Q_j: \mathbb{R}^d \rightarrow \mathbb{R}$ is a polynomial in d variables. Suppose the Jacobian determinant J_Q of Q does not vanish everywhere. Then for almost every*

$x \in \mathbb{R}^d$ the set $Q^{-1}(\{x\})$ is finite. Moreover, for almost every $x \in \mathbb{R}^d$ the inequality

$$\#Q^{-1}(\{x\}) \leq \prod_{j=1}^d \deg(Q_j) \quad (1.3.13)$$

holds, where $\deg(Q_j)$ denotes the degree of Q_j .

The simple proof of this lemma, which is based on Bézout's theorem, appears in [Chr98] but is included here for completeness.

Proof (of Lemma 21). Since the zero locus Z of J_Q is a proper algebraic subset of \mathbb{R}^d it has measure zero. Furthermore, as Q is a polynomial (and therefore locally Lipschitz) mapping the image $Q(Z)$ of Z under Q has measure zero (this is a special case of Sard's theorem). It is claimed that (1.3.13) holds for all $x \in \mathbb{R}^d \setminus Q(Z)$. Indeed, fixing such an x notice that $Q^{-1}(\{x\}) = \bigcap_{j=1}^d V_j^x$ where $\{V_j^x\}_{j=1}^d$ are algebraic sets given by

$$V_j^x := \{t \in \mathbb{R}^d : Q_j(t) - x_j = 0\}.$$

Bézout's theorem implies that the cardinality of this intersection is either uncountable or at most $\prod_{j=1}^d \deg(Q_j)$.⁹ It therefore suffices to show that $Q^{-1}(\{x\})$ is not uncountable; this is achieved by proving each point of the set is isolated. By the choice of x , whenever $t_0 \in Q^{-1}(\{x\})$ the vectors $\{\nabla Q_j(t_0)\}_{j=1}^d$ span \mathbb{R}^d . Thus the V_j^x are smooth hypersurfaces in a neighbourhood of t_0 which, of course, intersect at t_0 and are transversal at this point of intersection. It follows that t_0 must be an isolated point of $Q^{-1}(\{x\})$, as required. \square

If $\phi: D \rightarrow \mathbb{R}$ is a polynomial mapping, then so too are the associated change of variables $\Phi: U \rightarrow \mathbb{R}^{n-1}$ and $\Psi: \mathbb{R}^n \times U^n \rightarrow (\mathbb{R}^n)^n$, defined in (1.3.3), and the degrees of the components of Φ and Ψ depend only on $\deg \phi$. Recall from the proof of Theorem 16 that the Jacobian determinants of Φ and Ψ are given, up to a choice of sign, by

$$\text{Hess } \phi(u) \quad \text{and} \quad \text{vol}(\nabla \phi(u_1), \dots, \nabla \phi(u_n)), \quad (1.3.14)$$

respectively. Thus, if both these polynomials do not vanish everywhere, then Lemma 21 ensures the corresponding convolution estimate (1.3.6) holds with an implied constant depending only on $\deg \phi$. Clearly the second polynomial in (1.3.14) is non-zero if and only if there exist n linearly independent unit vectors which are each normal to the hypersurface at some point. This is valid if there exists a single point of non-zero Gaussian curvature, in which case $\text{Hess } \phi(u)$ does not vanish everywhere. Hence, either the Gaussian curvature is zero at every point on the hypersurface and the situation is completely degenerate, or there exists a point at which the curvature is non-zero in which case Theorem 16 applies.

⁹Recall, Bézout's theorem states that for any collection Q_1, \dots, Q_n of homogeneous polynomials on $\mathbb{C}\mathbb{P}^n$ the number of intersection points of the associated hypersurfaces $\{z \in \mathbb{C}\mathbb{P}^n : Q_j(z) = 0\}$ (counted with multiplicity) is either uncountable or precisely $\prod_{j=1}^n \deg(Q_j)$. The real version used here follows by homogenising the polynomials and taking the domain of the resulting functions to be $\mathbb{C}\mathbb{P}^n$. One then applies Bézout's theorem in complex projective space, de-homogenises and restricts to real-value intersection points. See, for example, [Has07] pp 223-224.

1.4 Sjölin's theorem

The proof of Sjölin's Fourier restriction theorem exploits phenomena particular to the $n = 2$ case and, consequently, the methods used are less effective when studying uniform restriction to hypersurfaces in \mathbb{R}^n .¹⁰ The present discussion, in contrast with the previous section, is therefore limited to the case of plane curves. As before, one can exploit the symmetries of the problem to reduce to the situation where the curve Σ is parametrised by a graph. Explicitly, let

$$\Sigma := \{(u, \phi(u)) : u \in D\}$$

where $D \subseteq \mathbb{R}$ is an open interval and $\phi: D \rightarrow \mathbb{R}$ is a C^2 convex function and let μ denote the associated affine arc-length measure. It is easy to see Sjölin's theorem is equivalent to the validity of the *extension estimate*

$$\|(gd\mu)^\vee\|_{L^{p'}(\mathbb{R}^2)} \lesssim \|g\|_{L^{q'}(\mu)} \quad (1.4.1)$$

for all pairs of exponents (p, q) satisfying the hypotheses of Theorem 12 part i). More precisely, here (1.4.1) is an *a priori* estimate in the sense that one is required to show that the inequality holds whenever $g \in C_c(\Sigma)$, say.

Proof (of Theorem 12). By definition,

$$(gd\mu)^\vee(x) = \int_U e^{2\pi i(x_1 t + x_2 \phi(t))} g(t, \phi(t)) \phi''(t)^{1/3} dt$$

where $U := \{t \in D : \phi''(t) > 0\}$. Define $f(t) := g(t, \phi(t)) \phi''(t)^{1/3}$ and $\Omega := \{(s, t) \in U \times U; s < t\}$ and observe

$$(gd\mu)^\vee(x)^2 = 2 \int_{\Omega} e^{2\pi i(x_1(t_1+t_2) + x_2(\phi(t_1)+\phi(t_2)))} \prod_{j=1}^2 f(t_j) dt.$$

Consider the change of variables $\Psi: \Omega \rightarrow \mathbb{R}^2$ given by

$$\Psi(t) := (t_1 + t_2, \phi(t_1) + \phi(t_2)),$$

noting that the convexity of ϕ ensures this function is injective with non-vanishing Jacobian. Thus

$$(gd\mu)^\vee(x)^2 = 2 \int_{\Psi(\Omega)} e^{2\pi i x \cdot u} \prod_{j=1}^2 f \circ t_j(u) |\phi' \circ t_1(u) - \phi' \circ t_2(u)|^{-1} du, \quad (1.4.2)$$

where the t_j are now functions of u . Taking the $L^{p'}(\mathbb{R}^2)$ -norm of both sides of (1.4.2) one deduces that

$$\|(gd\mu)^\vee\|_{L^{p'}(\mathbb{R}^2)}^2 \sim \left(\int_{\mathbb{R}^2} \left| \int_{\Psi(\Omega)} e^{2\pi i x \cdot u} \prod_{j=1}^2 f \circ t_j(u) |\phi' \circ t_1(u) - \phi' \circ t_2(u)|^{-1} du \right|^{p'/2} dx \right)^{2/p'}.$$

The right-hand side of this expression is in fact the $L^{p'/2}(\mathbb{R}^2)$ norm of the (inverse) Fourier

¹⁰However, one can effectively apply these techniques to study restriction to space curves (rather than hypersurfaces) in \mathbb{R}^n ; see, for instance, [Dru85] and the recent paper [Sto]. The topic of Fourier restriction to space curves will be considered in more detail later in this introduction and in Chapter 4.

transform of the function

$$F(u) := \chi_{U \times U}(u) \prod_{j=1}^2 |f \circ t_j(u)| |\phi' \circ t_1(u) - \phi' \circ t_2(u)|^{-1}.$$

Defining r by $2/p' + 1/r = 1$, and observing $1 \leq r < 2$, it follows from the Hausdorff-Young inequality that

$$\|(gd\mu)^\vee\|_{L^{p'}(\mathbb{R}^2)}^2 \lesssim \left(\int_{\Psi(\Omega)} \prod_{j=1}^2 |f \circ t_j(u)|^r |\phi' \circ t_1(u) - \phi' \circ t_2(u)|^{-r} du \right)^{1/r}$$

and changing back the variables yields

$$\|(gd\mu)^\vee\|_{L^{p'}(\mathbb{R}^2)}^2 \lesssim \left(\int_{\Omega} \prod_{j=1}^2 |f(t_j)|^r |\phi'(t_1) - \phi'(t_2)|^{1-r} dt \right)^{1/r}.$$

Consider a second change of variables $\Phi: U \times U \rightarrow \mathbb{R}^2$ given by $\Phi(t) := (\phi'(t_1), \phi'(t_2))$. Once again, the convexity of ϕ ensures this is valid and hence

$$\|(gd\mu)^\vee\|_{L^{p'}(\mathbb{R}^2)} \lesssim \left(\int_{\phi'(U) \times \phi'(U)} \prod_{j=1}^2 |f \circ t(\xi_j)|^r \phi'' \circ t(\xi_j)^{-1} |\xi_1 - \xi_2|^{-(r-1)} d\xi \right)^{1/2r}.$$

Apply the Hardy-Littlewood-Sobolev inequality (1.3.9) to the right-hand side of this inequality with functions

$$F_j(\xi) := |f \circ t(\xi)|^r \phi'' \circ t(\xi)^{-1}$$

and exponents $\gamma = r - 1, p_1 = p_2 = \tau$ where $2/\tau = 3 - r$ to deduce

$$\|(gd\mu)^\vee\|_{L^{p'}(\mathbb{R}^2)} \lesssim \left(\int_{\phi'(U)} |f \circ t(\xi)|^{\tau r} \phi'' \circ t(\xi)^{-\tau} d\xi \right)^{1/\tau r}.$$

Simple arithmetic shows $\tau r = q'$ and therefore, changing back the variables once more,

$$\begin{aligned} \|(gd\mu)^\vee\|_{L^{p'}(\mathbb{R}^2)} &\lesssim \left(\int_U |f(t)|^{q'} \phi''(t)^{1-\tau} dt \right)^{1/q'} \\ &= \left(\int_U |f(t)|^{q'} \phi''(t)^{-q'/p'} dt \right)^{1/q'} \\ &= \left(\int_D |g(t, \phi(t))|^{q'} \phi''(t)^{1/3} dt \right)^{1/q'} \end{aligned}$$

where the last equality is simply by recalling the definition of f and U and the choice of exponents $p' = 3q$. This concludes the proof. \square

1.5 A firmer foundation for the affine theory

The preceding subsections have developed the affine theory in the context of Example 1 and Example 2. It is stressed, however, that there is a burgeoning theory of affine harmonic analysis which applies to the study of a wide range of operators. To place the affine perspective on a firmer foundation it would be convenient to have an alternative definition of the affine surface

measure which is amenable to generalisation. Indeed, the current definition applies only to hypersurfaces and it would be useful, for instance, to have a notion of affine measure associated to k -surfaces in \mathbb{R}^n or even general Borel subsets. It transpires that the affine surface measure can be viewed as a specific instance of a more general notion of α -dimensional affine measure introduced by D. Oberlin [Obe03].

To motivate what follows, recall the Knapp example introduced in Remark 8. There it was observed that the influence of the curvature on the mapping properties of the convolution operator can be gauged by considering a covering of the curve (or k -surface) by rectangles or, more generally, parallelepipeds. In particular, if x is a flat point on the surface, then it is possible to cover a δ -neighbourhood of x using a very small parallelepiped; this leads to constraints on the range of viable $L^p - L^q$ estimates. Variants of this Knapp example are rather ubiquitous: they lead to necessary conditions for restriction inequalities and can be applied to study broad classes of operators such as generalised Radon transforms (in this case the parallelepipeds become Carnot-Carathéodory balls, see [CNSW99, TW03]).

The α -dimensional affine measure is defined to be the following variant of the classical α -dimensional Hausdorff measure \mathcal{H}^α .

Definition 22. Given a subset $E \subseteq \mathbb{R}^n$ and $\alpha, \delta > 0$ consider sums of the form $\sum_{R \in \mathfrak{R}} |R|^{\alpha/n}$ where \mathfrak{R} is a collection of parallelepipeds which satisfy the following two conditions:

- i) The collection covers E ; that is, $E \subseteq \bigcup_{R \in \mathfrak{R}} R$;
- ii) Each $R \in \mathfrak{R}$ has diameter bounded above by δ .

Define $\mathcal{A}_\delta^\alpha(E)$ to be the infimum of such sums and, noting for fixed α this quantity is non-decreasing in δ , let

$$\mathcal{A}^\alpha(E) := \lim_{\delta \rightarrow 0_+} \mathcal{A}_\delta^\alpha(E).$$

One may verify the set function \mathcal{A}^α defines a metric outer measure on \mathbb{R}^n and hence restricts to a true measure on the σ -algebra of Borel subsets.

Definition 23. The affine dimension of a Borel set $E \subseteq \mathbb{R}^n$ is defined by

$$\dim_a(E) := \inf\{\alpha > 0 : \mathcal{A}^\alpha(E) < \infty\}.$$

The measures \mathcal{A}^α are clearly equi-affine invariant; that is, $\mathcal{A}^\alpha(TE) = \mathcal{A}^\alpha(E)$ whenever $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an equi-affine transformation and $E \subseteq \mathbb{R}^n$ is a Borel subset. There is also the trivial inequality $\mathcal{H}^\alpha(E) \leq \mathcal{A}^\alpha(E)$ which, in general, is strict. Indeed, if E lies in a hyperplane, then $\dim_a(E) = 0$ whilst the Hausdorff dimension can, of course, be as large as $n - 1$. This is one indication that the \mathcal{A}^α are sensitive to curvature.

A key result is that, for a specific choice of α , the \mathcal{A}^α measure restricted to a hypersurface Σ agrees with the affine surface measure introduced earlier.

Proposition 24 (D. Oberlin). *Let $\Sigma \subseteq \mathbb{R}^n$ be a hypersurface with surface measure σ and Gaussian curvature κ . For any Borel set $E \subseteq \Sigma$ one has*

$$\mathcal{A}^{n(n-1)/(n+1)}(E) \sim \int_E |\kappa(x)|^{1/(n+1)} d\sigma(x).$$

Remark 25. As a corollary one observes that any smooth hypersurface Σ has affine dimension $\dim_a(\Sigma) \leq (n - 1)/(n + 1)$.

The proof of Proposition 24 for $n = 2$ appears in [Obe03], the general case is unpublished work of D. Oberlin (see [Obe00] for related results in all dimensions).

1.6 Affine geometry of space curves

Thus far the discussion has been limited to hypersurfaces; it is natural, however, to also consider similar questions posed for k -dimensional submanifolds of \mathbb{R}^n . The situation for $2 \leq k \leq n - 2$ is, in general, poorly understood and will not be addressed in this thesis, but when $k = 1$ (that is, the case of curves in \mathbb{R}^n) there exists a well-developed theory. Here the *affine arc-length measure* of a space curve will be introduced via an *ad hoc* construction; this measure will then be shown to arise from the \mathcal{A}^α introduced in the previous section.

The curvature of a smooth space curve γ in \mathbb{R}^n is measured by $n - 1$ curvature functions $\kappa_1, \dots, \kappa_{n-1}$ and, in analogy with the hypersurface case, one can weight the arc-length measure σ on γ by some product of powers of the $|\kappa_j|$'s to obtain a measure which is invariant under equi-affine transformation. It transpires that the correct choice of weight is

$$\omega_\gamma(x) := \left| \prod_{j=1}^{n-1} \kappa_j(x)^{n-j} \right|^{2/n(n+1)}, \quad (1.6.1)$$

as can be seen from the following analogue of Lemma 9.

Lemma 26. *Let γ be as above and $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ an invertible affine transformation. Consider the curve $\gamma_T := T(\gamma)$, let σ_T denote its arc-length measure and define ω_{γ_T} accordingly. Then*

$$|\det T|^{2/n(n+1)} \int_\gamma f \circ T(x) \omega_\gamma(x) d\sigma(x) = \int_{\gamma_T} f(x) \omega_{\gamma_T}(x) d\sigma_T(x) \quad (1.6.2)$$

for all $f \in C_c(\mathbb{R}^n)$.

Again, this identity is a consequence of the various basic differential-geometric definitions.

Definition 27. The measure $\omega_\gamma(x) d\sigma(x)$ introduced in Lemma 26 is referred to as the affine arc-length measure on γ .

The notions of affine surface and arc-length measures are unified by considering the \mathcal{A}^α introduced earlier. In particular, for some specific choice of dimension α , the affine arc-length measure on γ is essentially the restriction of \mathcal{A}^α to γ .

Proposition 28 (D. Oberlin). *Let γ be a smooth curve in \mathbb{R}^n with arc-length measure σ . For any Borel set $E \subseteq \gamma$ one has*

$$\mathcal{A}^{2/(n+1)}(E) \sim \int_E \omega_\gamma(x) d\sigma(x).$$

Remark 29. As a corollary one observes that any smooth curve γ has affine dimension $\dim_a(\gamma) \leq 2/(n + 1)$.

When $n = 2$ Proposition 28 coincides with Proposition 24 and is treated in [Obe03]. For $n \geq 3$ the result is unpublished work of D. Oberlin.

It is useful to note that if γ is a parametrised curve, then its affine arc-length measure admits a particularly simple expression. In particular, abusing notation and identifying γ with

its parametrisation, let $\gamma: I \rightarrow \mathbb{R}^n$ for some open interval $I \subseteq \mathbb{R}^n$ and define the *torsion* function associated to γ by

$$L_\gamma(t) := \det[\gamma^{(1)}(t) \dots \gamma^{(n)}(t)], \quad (1.6.3)$$

where $\gamma^{(j)}$ is the column vector given by taking the j th derivative of each of the components of γ . If $\lambda_\gamma := |L_\gamma|^{2/n(n+1)}$, then

$$\int_\gamma f(x) d\mu(x) = \int_I f \circ \gamma(t) \lambda_\gamma(t) dt \quad (1.6.4)$$

for all $f \in C_c(\mathbb{R}^n)$ (compare this formula with the formula (1.3.2) for affine surface measure on a graph parametrised hypersurface).

There is a well-developed affine theory for operators defined with respect to curves. Two typical results in this direction are quoted below, the first of which is fundamental to much of the subsequent discussion.

Theorem 30 (Stovall [Sto10]). *Let $P: \mathbb{R} \rightarrow \mathbb{R}^n$ be a polynomial mapping, μ the affine arc-length measure on the curve parametrised by P and define $p_n := (n+1)/2$ and $q_n := n(n+1)/2(n-1)$.*

i) For $(1/p, 1/q) \in \overline{\text{conv}}\{(1/p_n, 1/q_n), (1/q'_n, 1/p'_n)\}$,

$$\|f * \mu\|_{L^q(\mathbb{R}^n)} \lesssim_{\deg P} \|f\|_{L^p(\mathbb{R}^n)} \quad (1.6.5)$$

where the implied constant is uniform in the sense it does not depend on the coefficients of P .

ii) If the domain of P is restricted to a finite interval I , then (1.6.5) holds whenever $(1/p, 1/q)$ belongs to $\overline{\text{conv}}\{(0, 0), (1, 1), (1/p_n, 1/q_n), (1/q'_n, 1/p'_n)\}$ with a constant depending on the total variation of μ .

This is the analogue of Theorem 16 for space curves, albeit restricted to the polynomial case. The archetypical example is given by taking $P := h: \mathbb{R} \rightarrow \mathbb{R}^n$ to be the so-called *moment curve*

$$h(t) := \left(t, \frac{t^2}{2}, \dots, \frac{t^n}{n!}\right); \quad (1.6.6)$$

in this case the weight $\lambda_h \equiv 1$ is constant. The $L^p - L^q$ mapping properties of convolution with affine arc-length measure on the moment curve were investigated in low dimensions in a number of papers [Lit73, Obe87, Obe97, GSW99, Obe99] before being completely determined in all dimensions by Stovall [Sto09] using powerful new methods developed by Christ [Chrd, Chr98, Chra]. The proof of Theorem 30 is also based Christ's arguments, which are combinatorial in nature, and polynomial decomposition of Dendrinos and Wright [DW10] (see also [DLW09, DFGW10]). These techniques underpin much of the work presented in the following chapter. It is remarked that a more concise proof of Theorem 30, based on the same underlying arguments, can be found in [DS15].

Theorem 31 (Stovall [Sto]). *Let $P: \mathbb{R} \rightarrow \mathbb{R}^n$ be a polynomial mapping and μ the affine arc-length measure on the curve parametrised by P .*

i) For $1 \leq p < (n^2 + n + 2)/(n^2 + n)$ and $p' = n(n+1)q/2$,

$$\|\hat{f}\|_{L^q(\mu)} \lesssim_{\deg P} \|f\|_{L^p(\mathbb{R}^n)} \quad (1.6.7)$$

where the implied constant is uniform in the sense it does not depend on the coefficients of P .

ii) If the domain of P is restricted to a finite interval I , then (1.6.7) holds whenever $1 \leq p < (n^2 + n + 2)/(n^2 + n)$ and $p' \geq n(n+1)q/2$ with a constant depending on the total variation of μ .

This is a partial generalisation of Sjölin's theorem. The proof of Theorem 31 relies on arguments dating back to the early work of Christ [Chr85] and Drury [DM85] on restriction to curves (the latter established Theorem 31 in the special case of the moment curve) and also the aforementioned polynomial decomposition of Dendrinos and Wright [DW10], combined with additional square function estimates. Prior to Stovall's work estimates were known to hold for a restricted range of exponents [DW10, BOS13] and for restricted classes of polynomial curves [DM85, BOS08, BOS13].

Remark 32. Both Theorem 30 and Theorem 31 are sharp in the sense that no other uniform estimates are possible outside the range appearing in i) and, provided that μ is not the zero measure, no estimates are possible outside the range appearing in ii). This is easy to see in the convolution case, but for the restriction operator the proof requires rather involved oscillatory integral estimates [ACK87].

Remark 33. In light of Sjölin's counterexample it is natural to consider the class of polynomial curves and attempt to prove estimates which depend only on the degree of the polynomials. It is an interesting problem to consider uniform estimates over other classes of curves; this has been explored by a number of authors [BOS08, DM13, Sto14, DS15].

The topic of Fourier restriction to polynomial curves will be investigated in detail in the final chapter of this thesis. There, however, the problem is formulated in certain non-Euclidean settings and, in particular, the classical method of Drury [DM85] is used to study restriction operators defined over the finite rings $\mathbb{Z}/p^k\mathbb{Z}$ for p prime.

Chapter 2

Dilated averages over polynomial curves

2.1 Averages over dilates

Theorem 30 concerns a convolution operator which can be thought of as taking weighted averages over translates of a fixed polynomial curve. The study of this operation can be subsumed in the study of a related operator which takes averages not only over translates but also over dilates. This chapter, which presents the details of a paper (submitted for publication) of the author [Hic], investigates questions pertaining to the L^p -mapping properties of ‘dilated averages’. In particular, let $P : I \rightarrow \mathbb{R}^n$ denote a (parametrisation of a) polynomial curve in \mathbb{R}^n where $I \subseteq \mathbb{R}$ is an open interval. Consider the operator A defined, at least initially, on the space of for all test functions f on \mathbb{R}^n by

$$Af(x, r) := \int_I f(x - rP(t))\lambda_P(t) dt \quad \text{for all } (x, r) \in \mathbb{R}^n \times [1, 2]. \quad (2.1.1)$$

Here λ_P is the power of the (absolute value of the) torsion function (1.6.3) appearing in the formula (1.6.4) for the affine arc-length. A natural problem is to establish the range of (p, q, s) for which there is an a priori mixed-norm estimate either of the form¹

$$\|Af\|_{L_r^s L_x^q(\mathbb{R}^n \times [1, 2])} \lesssim_{\deg P} \|f\|_{L^p(\mathbb{R}^n)} \quad (2.1.2)$$

or

$$\|Af\|_{L_x^q L_r^s(\mathbb{R}^n \times [1, 2])} \lesssim_{\deg P} \|f\|_{L^p(\mathbb{R}^n)}, \quad (2.1.3)$$

where again one seeks uniform estimates in the sense that the implied constants are independent of the choice of coefficients of the polynomials. This question subsumes the study of the L^p mapping properties of both single averages $f * \mu$ (where μ is the affine arc-length measure on P) and certain maximal functions associated to space curves.

¹Given measure spaces (X_j, Σ_j, μ_j) for $j = 1, 2$ and F a measurable function on the product space $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \otimes \mu_2)$, the mixed $L_{x_1}^{p_1} L_{x_2}^{p_2}$ norm of F is defined by

$$\|F\|_{L_{x_1}^{p_1} L_{x_2}^{p_2}(X_1 \times X_2)} := \| \|F\|_{L^{p_2}(X_2)} \|_{L^{p_1}(X_1)}$$

for $1 \leq p_1, p_2 \leq \infty$. Here $\|F\|_{L^{p_2}(X_2)}$ is the measurable function on X_1 given by $x_1 \mapsto \|F(x_1, \cdot)\|_{L^{p_2}(X_2)}$. The subscript x_j 's appearing in the norm refer to the notation used for the variables.

- Taking $s = \infty$ in (2.1.2) reduces matters to determining the set of (p, q) for which the ‘single averages’

$$f * \mu_r(x) := \int_0^1 f(x - rP(t)) \lambda_P(t) dt \tag{2.1.4}$$

are strong-type (p, q) uniformly for $r \in [1, 2]$. This problem falls under the scope of Theorem 30.

- On the other hand, if one sets $s = \infty$ in (2.1.3) the situation is very different. For simplicity, suppose $P = h$ is the moment curve introduced in (1.6.6). One now wishes to understand the $L^p - L^q$ mapping properties of the maximal function \mathcal{M} associated to h , defined by

$$\mathcal{M}f(x) := \sup_{1 \leq r \leq 2} |f * \mu_r(x)|,$$

where μ_r is defined as in the previous bullet-point. A celebrated theorem of Bourgain [Bou86] established $L^p - L^p$ mapping properties for $d = 2$; this result was extended by Schlag [Sch97] who proved an almost-sharp range of $L^p - L^q$ estimates.² However, the problem of determining even the $L^p - L^p$ range remains open in all other dimensions. Some partial results in this direction are given in [PS07].

When considering maximal functions M defined with respect to more general curves it is no longer possible to develop an affine-invariant theory based on the affine arc-length measure μ_γ . For instance, it is shown in [Sch97] that the circular maximal function is unbounded from $L^p - L^q$ when the exponents satisfy the condition $1/q = 1/p - 1/3$. If M is defined with respect to μ_γ , then affine-invariant estimates can only hold when $1/q = 1/p - 1/3$. Simple examples show that such an M may also fail to be bounded on the full range of exponents corresponding to the (non-degenerate) case of circular averages [Mar95]. One could, however, consider replacing $|L_\gamma(t)|^{1/3}$ with some other power of the torsion $|L_\gamma(t)|^\sigma$ for suitably chosen σ and attempt to describe the range of estimates in terms of σ . Problems of this kind were considered in, for instance, the thesis of Marletta [Mar95] (see also [Mar99]).

In this chapter the special case of (2.1.2) and (2.1.3) where $s = q$ is considered. In particular, Theorem 34 below almost completely determines the set of (p, q) for which A is bounded from $L_x^p(\mathbb{R}^n)$ to $L_{x,r}^q(\mathbb{R}^n \times [1, 2])$. Testing the inequality on some simple examples (see Section 2.2) shows that, provided that the torsion does not vanish identically, such a bound is possible only if $(1/p, 1/q)$ lies in the trapezium

$$\mathcal{T}_n := \overline{\text{conv}}\{(0, 0), (1, 1), (1/p_1, 1/q_1), (1/p_2, 1/q_2)\}$$

where

$$\left(\frac{1}{p_1}, \frac{1}{q_1}\right) := \left(\frac{1}{n}, \frac{n-1}{n(n+1)}\right) \quad \text{and} \quad \left(\frac{1}{p_2}, \frac{1}{q_2}\right) := \left(\frac{n^2-n+2}{n(n+1)}, \frac{n-1}{n+1}\right).$$

This condition is shown to be sufficient for boundedness, at least up to an endpoint. In addition, the almost-sharp range of uniform estimates are obtained along the endpoint line $\overline{\text{conv}}\{(1/p_1, 1/q_1), (1/p_2, 1/q_2)\}$.

Theorem 34. *Let $n \geq 2$ and $P: \mathbb{R} \rightarrow \mathbb{R}^n$ be a polynomial curve.*

²More precisely, both Bourgain and Schlag studied the circular maximal function rather than the parabolic variant discussed here. However, in this context both objects can be understood via the same techniques.

i) For $(1/p, 1/q) \in \mathcal{T}_n \setminus \{(1/p_1, 1/q_1)\}$ satisfying $1/q = 1/p - 2/n(n+1)$,

$$\|Af\|_{L_{x,r}^q(\mathbb{R}^n \times [1,2])} \lesssim_{\deg P} \|f\|_{L_x^p(\mathbb{R}^n)} \tag{2.1.5}$$

where the implied constant is uniform in the sense that it does not depend on the coefficients of P .

ii) If the domain of P is restricted to a finite interval I , then (2.1.5) holds for all $(1/p, 1/q) \in \mathcal{T}_n \setminus \{(1/p_1, 1/q_1)\}$ with a constant depending on the total variation of the associated affine arc-length measure.

The proof of Theorem 34 proceeds by establishing a uniform restricted weak-type inequality at the endpoint (p_1, q_1) . Therefore, except for the question of whether this weak-type endpoint inequality can be strengthened the theorem completely determines the L^p mapping properties (and uniform mapping properties) of A .

The $n = 2$ case of Theorem 34 (when the curve is also a hypersurface) is already known to hold with a strong-type inequality at the (p_1, q_1) endpoint. In the case of the moment curve (which is now just a parabola), this result essentially appears, for example, in the work of Strichartz [Str77] (see also [SS97]). In addition, away from the endpoint the estimates for the parabola also follow from more recent work of Gressman [Gre06], utilising methods which are rather combinatorial in nature. The combinatorial techniques found in [Gre06] are akin to the arguments found in the present chapter; both are based on the earlier work of Christ [Chr98] mentioned in the introduction. For arbitrary polynomial curves, the $n = 2$ case follows from a very general theorem due to Gressman [Gre13]. Indeed, Gressman’s theorem, *inter alia*, establishes the hypersurface analogue of Theorem 34 in all dimensions, up to and including all the relevant endpoints. The proof of the hypersurface case is very similar to the argument used to prove Theorem 16 and, indeed, both results appear in [Gre13].

Remark 35. In [Str77, SS97] it is observed that the critical $L_x^2 - L_{x,r}^6$ inequality for dilated averages over circles is equivalent to a Stein-Tomas Fourier restriction theorem for a conic surface and connections between this theory and estimates for certain evolution equations are also discussed. This suggests there should be a connection between the $n = 2$ case of Theorem 34 and Fourier restriction to conical surfaces; this topic is investigated in the following chapter.

For $n \geq 3$ the results appear to be new and, indeed, no previous (non-trivial) partial results are known to the author, even in the special case of the moment curve. It is remarked that the connection between the theory of dilated averages, Fourier restriction and analysis of PDE appears to be confined to the hypersurface setting but nevertheless Theorem 34 is arguably of interest in its own right.

Remark 36. It is natural to ask whether the restricted weak-type (p_1, q_1) endpoint can be strengthened to a strong-type estimate; this is certainly the case in dimension $n = 2$ where the inequality is a consequence of the aforementioned theorem of Gressman [Gre13]. Furthermore, one may recover the strong-type bound for $n = 2$ by combining the analysis contained within the present chapter with an extrapolation method due to Christ [Chrd] (see also [Sto09]). It is possible that the argument can be adapted to the case where n belongs to a certain congruence class modulo 3 to (potentially) establish the strong-type bound in this situation. A more detailed discussion of the validity of the strong-type endpoint appears below in Remark 51.

Theorem 34 belongs to a growing body of works which have applied variants of the geometric and combinatorial arguments due to Christ [Chr98] to the study of operators collectively known as generalised Radon transforms, of which A is an example. Essentially these operators are defined for any point y belonging to Σ an n -dimensional manifold by integration over a k -dimensional manifold M_y , which depends on y , where $k < n$ is referred to here as the *dimension of the associated family*. The techniques of [Chr98] have fruitfully been applied and developed in, for instance, [Chra, Chrb, CE02, CE08, DLW09, Gre04, Gre09, Sto14, Sto09, Sto10, TW03] to study the Lebesgue mapping properties of one-dimensional generalised Radon transforms R which are, roughly, operators R for which R and its adjoint R^* are both generalised Radon transforms given by integration over some family of curves. The approach has been less successful when considering R which are *unbalanced* in the sense that R and R^* are both generalised Radon transforms but the dimensions of the associated families are not equal, although it has still produced results in some specific cases, for example [EO10, Gre06, Gre13]. The dilated averaging operator (2.1.1) fits into this framework by setting $\Sigma := \mathbb{R}^n \times (1, 2)$ and for each $(x, r) \in \Sigma$ defining $M_{(x,r)}$ to be the curve parametrised by $t \mapsto x - r\gamma(t)$. Observe that although A is defined by integration over curves, the adjoint of A is defined by integration over 2-surfaces and hence the operator is unbalanced.

The structure of this chapter is as follows. In the following section the necessary conditions on (p, q) for A to be restricted weak type (p, q) are discussed. In Section 2.3 standard methods together with estimates for single averages are combined to reduce the proof of Theorem 34 to proving a single restricted weak-type inequality. Christ’s method of refinements is also reviewed and used to establish the simple case of Theorem 34 when $n = 3$ and $P = h$ is the moment curve. The remaining sections develop this method to be applicable in the general situation.

Notation. A word of explanation concerning notation is in order: throughout this chapter C and c will be used to denote various positive constants whose value may change from line to line but will always depend only on the dimension n , relevant Lebesgue exponents **and** the degree $\deg P$ of some fixed polynomial. If $X, Y \geq 0$, then the notation $X \lesssim Y$ or $Y \gtrsim X$ signifies $X \leq CY$ and this situation is also described by “ X is $O(Y)$ ”. In addition, $X \sim Y$ indicates $X \lesssim Y \lesssim X$. In particular, in the present chapter the dependence on $\deg P$ will be suppressed in the ‘wiggles’ notation.

2.2 Necessary conditions

Let $P = h: [0, 1] \rightarrow \mathbb{R}^n$ be a compact piece of the moment curve and suppose the operator A from Theorem 34 satisfies a restricted weak-type (p, q) inequality for some $1 \leq p, q < \infty$. Here it is shown that the exponents p, q must satisfy four conditions,³ each corresponding to an edge of the trapezium \mathcal{T}_n . The first three conditions also appear in the study of the averaging operator $f \mapsto f * \mu$ and are deduced by the same reasoning (see Remark 8). The remaining condition does not appear in the theory of single averages and here the dilation parameter plays a non-trivial rôle, although the arguments are only marginally different from those used to examine $f * \mu$. Let $P = h: [0, 1] \rightarrow \mathbb{R}^n$ be a compact piece of the moment curve and suppose the operator A from Theorem 34 satisfies a restricted weak-type (p, q) inequality for some $1 \leq p, q < \infty$.

³The argument presented herein can be adapted to show the necessity of the conditions on the exponents for general P . This relies on applying various affine transformations to reduce to the case where P is of a particularly simple form, the details are omitted.

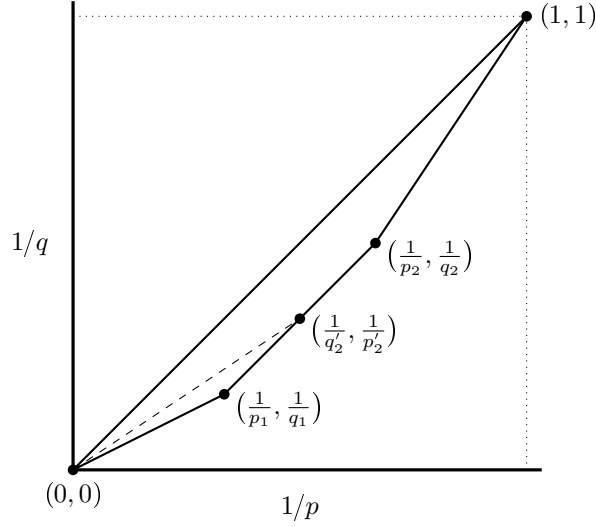


Figure 2.1: If the torsion of P does not vanish identically, then the operator A of Theorem 34 is restricted weak-type (p, q) if and only if $(1/p, 1/q)$ belongs to the (closed set bounded by the) bold trapezium. For comparison, the single average $f \mapsto f * \mu$ is restricted weak-type (p, q) if and only if $(1/p, 1/q)$ belongs to the smaller trapezium formed by introducing the dashed line.

Here it is shown that the exponents p, q must satisfy four conditions,⁴ each corresponding to an edge of the trapezium \mathcal{T}_n . The first three conditions also appear in the study of the averaging operator $f \mapsto f * \mu$ and are deduced by the same reasoning (see Remark 8). The remaining condition does not appear in the theory of single averages and here the dilation parameter plays a non-trivial rôle, although the arguments are only marginally different from those used to examine $f * \mu$.

To begin, a slight modification of a general theorem of Hörmander [Hör60] implies $p \leq q$.

For the second condition, let $R(\delta) := \prod_{j=1}^n [-\delta^j/j!, \delta^j/j!]$ and note that

$$A\chi_{R(\delta)}(x, r) = |\{t \in [0, 1] : x - rh(t) \in R(\delta)\}|.$$

If $x \in (1/2)R(\delta)$, then whenever $t \in [0, \delta/4]$ it follows that

$$|x_j - rt^j| \leq \frac{\delta^j}{2j!} + \frac{2}{j!} \frac{\delta^j}{4^j} \leq \delta^j \quad \text{for } j = 1, \dots, n$$

and therefore

$$A\chi_{R(\delta)}(x, r) \geq \frac{\delta}{4} \chi_{(1/2)R(\delta)}(x).$$

Consequently, applying the hypothesised restricted weak-type estimate,

$$|R(\delta)| \lesssim |\{(x, r) \in \mathbb{R}^n \times [1, 2] : A\chi_{R(\delta)}(x, r) > \delta/8\}| \lesssim \left(\frac{1}{\delta} |R(\delta)|^{1/p}\right)^q.$$

Observe $|R(\delta)| \sim \delta^{n(n+1)/2}$ and so the preceding inequality implies

$$\delta^{n(n+1)/(2q)} \lesssim \delta^{n(n+1)/(2p)-1} \quad \text{for all } 0 < \delta < 1.$$

⁴The argument presented herein can be adapted to show the necessity of the conditions on the exponents for general P . This relies on applying various affine transformations to reduce to the case where P is of a particularly simple form, the details are omitted.

The exponents (p, q) must therefore satisfy the relation

$$\frac{1}{q} \geq \frac{1}{p} - \frac{2}{n(n+1)}.$$

The third condition is established by testing A on $\chi_{B(\delta)}$, the characteristic function of a ball $B(\delta) \subset \mathbb{R}^n$ of radius $0 < \delta < 1$, centred at the origin. It is easy to see

$$A\chi_{B(\delta)}(x, r) \gtrsim \delta\chi_{\mathcal{N}_r(\delta)}(x) \quad (2.2.1)$$

where $\mathcal{N}_r(\delta)$ is a $\delta/3$ -neighbourhood of the r -dilate of the moment curve; that is, the set of all points $x \in \mathbb{R}^n$ for which $|x - rh(t_0)| < \delta/3$ for some $t_0 \in [0, 1]$. The hypothesised restricted weak-type estimate together with (2.2.1) imply

$$\begin{aligned} |\{(x, r) \in \mathbb{R}^n \times [1, 2] : x \in \mathcal{N}_r(\delta)\}| &\leq |\{(x, r) \in \mathbb{R}^n \times [1, 2] : A\chi_{B(\delta)}(x, r) > C\delta\}| \\ &\lesssim \left(\frac{1}{\delta} |B(\delta)|^{1/p}\right)^q. \end{aligned}$$

Observe $|B(\delta)| \sim \delta^n$ whilst $|\mathcal{N}_r(\delta)| \gtrsim \delta^{n-1}$ for all $r \in [1, 2]$ and so the preceding inequality implies

$$\delta^{(n-1)/q} \lesssim \delta^{n/p-1} \quad \text{for all } 0 < \delta < 1.$$

Thus the exponents must satisfy the relation

$$\frac{1}{q} \geq \frac{n}{n-1} \frac{1}{p} - \frac{1}{n-1}.$$

The final condition on $(1/p, 1/q)$ is deduced by considering the adjoint A^* of A . A simple computation yields

$$A^*g(x) = \int_1^2 \int_0^1 g(x + rh(t), r) dt dr$$

for suitable functions g defined on $\mathbb{R}^n \times [1, 2]$. The hypothesis on (p, q) is equivalent to the assumption that A^* is restricted weak-type (q', p') . For $B(\delta)$ as above, let $F(\delta)$ denote the set $B(\delta) \times [1, 1 + c\delta]$ for some small constant c . Observe

$$A^*\chi_{F(\delta)}(x) \gtrsim \delta^2\chi_{\mathcal{N}_1(\delta)}(-x)$$

where $\mathcal{N}_1(\delta)$ is as defined above. Therefore,

$$|\mathcal{N}_1(\delta)| \leq |\{x \in \mathbb{R}^n : A^*\chi_{F(\delta)}(x) \gtrsim \delta^2\}| \lesssim \left(\frac{1}{\delta^2} |F(\delta)|^{1/q'}\right)^{p'}.$$

Finally, $|F(\delta)| \sim \delta^{n+1}$ whilst $|\mathcal{N}_1(\delta)| \gtrsim \delta^{n-1}$ and so the preceding inequality implies

$$\delta^{(n-1)/p'} \lesssim \delta^{(n+1)/q'-2} \quad \text{for all } 0 < \delta < 1.$$

It follows that the exponents must satisfy the relation $(n+1)/q' - 2 \leq (n-1)/p'$ which can be rewritten as

$$\frac{1}{q} \geq \frac{n-1}{n+1} \frac{1}{p}.$$

2.3 An overview of the refinement method

It remains to show the conditions on (p, q) described in Theorem 34 are sufficient to ensure A satisfies a type (p, q) inequality with the desired uniformity. Real interpolation immediately reduces matters to establishing a uniform restricted weak-type (p_1, q_1) and strong type (p_2, q_2) estimate for A . The latter is easily dealt with by appealing to the existing literature. Indeed, Theorem 30 implies the estimate

$$\|Af(\cdot, r)\|_{L_x^{q_2}(\mathbb{R}^n)} \lesssim \|f\|_{L_x^{p_2}(\mathbb{R}^n)} \quad (2.3.1)$$

holds for all $r \in [1, 2]$. Taking $L_r^{q_2}([1, 2])$ -norms of both sides of (2.3.1) yields the uniform type (p_2, q_2) inequality for A and the proof of Theorem 34 is therefore reduced to establishing the following Proposition.

Proposition 37. *For $n \geq 2$ the inequality*

$$\langle A\chi_E, \chi_F \rangle \lesssim |E|^{1/n} |F|^{(n^2+1)/n(n+1)} \quad (2.3.2)$$

is valid for all pairs of Borel sets $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^n \times [1, 2]$ of finite Lebesgue measure.

The proof of Proposition 37 will utilise the geometric and combinatorial techniques introduced by Christ in [Chr98], which were briefly mentioned in the introduction and earlier in this chapter. Collectively these techniques are known as *the method of refinements*. In this section the rudiments of the method are reviewed. It is instructive to consider the proof of the analogue of Proposition 37 in three dimensions ($n = 3$) for $P = h: [0, 1] \rightarrow \mathbb{R}^3$ a compact piece of the moment curve. In this situation the arguments are extremely simple and only a crude version of the refinement procedure is required.

Let E and F denote fixed sets satisfying the hypotheses of Proposition 37 for $n = 3$. Assume, without loss of generality, that $\langle A\chi_E, \chi_F \rangle \neq 0$ where A is the operator from Theorem 34, defined with respect to the specific choice of curve given above. One wishes to establish the inequality

$$\langle A\chi_E, \chi_F \rangle \lesssim |E|^{1/3} |F|^{5/6}.$$

Defining constants α and β by the equation $\langle A\chi_E, \chi_F \rangle = \alpha|F| = \beta|E|$, one may rewrite the preceding inequality as a lower bound on the measure of E ; explicitly,

$$|E| \gtrsim \alpha^6(\beta/\alpha). \quad (2.3.3)$$

The basic idea behind Christ's method is to attempt to prove (2.3.3) by using iterates of A and A^* to construct a natural parameter set $\Omega \subset \mathbb{R}^3$ and parametrising function $\Phi: \Omega \rightarrow E$ with a number of special properties. First of all, Φ must have bounded multiplicity so, by applying the change of variables formula,

$$|E| \gtrsim \int_{\Omega} |J_{\Phi}(t)| dt$$

where J_{Φ} denotes the Jacobian of Φ . It then remains to bound this integral from below by some expression in terms of α and β , which is possible provided that the parametrisation has been carefully constructed.

Following [Chr98], define

$$\begin{aligned} F_1 &:= \{(x, r) \in F : A\chi_E(x, r) > \alpha/2\}, \\ E_1 &:= \{y \in E : A^*\chi_{F_1}(y) > \beta/4\}. \end{aligned}$$

It is not difficult to see the assumptions on E and F imply $\langle A\chi_{E_1}, \chi_{F_1} \rangle \neq 0$ and therefore E_1 is non-empty. Fix $y_0 \in E_1$ and define a map $\Phi_1: [1, 2] \times [0, 1] \rightarrow \mathbb{R}^3 \times [1, 2]$ by

$$\Phi_1(r_1, t_1) := \begin{pmatrix} y_0 + r_1 h(t_1) \\ r_1 \end{pmatrix}. \quad (2.3.4)$$

Note that the set

$$\Omega_1 := \{(r_1, t_1) \in [1, 2] \times [0, 1] : \Phi_1(r_1, t_1) \in F_1\}$$

satisfies $|\Omega_1| > \beta/4$. Similarly, define a map $\Phi_2: [1, 2] \times [0, 1]^2 \rightarrow \mathbb{R}^3$ by

$$\Phi_2(r_1, t_1, t_2) := y_0 + r_1 h(t_1) - r_1 h(t_2) \quad (2.3.5)$$

and observe for each $(r_1, t_1) \in \Omega_1$ the set

$$\Omega_2(r_1, t_1) := \{t_2 \in [0, 1] : \Phi_2(r_1, t_1, t_2) \in E\}$$

satisfies $|\Omega_2(r_1, t_1)| > \alpha/2$. Finally, define the structured set

$$\Omega_2 := \{(r_1, t_1, t_2) \in [1, 2] \times [0, 1]^2 : (r_1, t_1) \in \Omega_1 \text{ and } t_2 \in \Omega_2(r_1, t_1)\}.$$

Now, $\Omega := \Omega_2 \subset \mathbb{R}^3$ is the parameter set alluded to above and $\Phi := \Phi_2|_{\Omega} : \Omega \rightarrow E$ the parametrising function. Observe Φ is well-defined by the preceding observations and the polynomial nature of this map ensures it has almost everywhere bounded multiplicity.⁵ The absolute value of the Jacobian $J_{\Phi}(r_1, t_1, t_2)$ of Φ may be expressed as

$$r_1^2 \left| \det \begin{pmatrix} 1 & 1 & t_2 - t_1 \\ t_1 & t_2 & \frac{t_2^2}{2} - \frac{t_1^2}{2} \\ \frac{t_1^2}{2} & \frac{t_2^2}{2} & \frac{t_2^3}{3!} - \frac{t_1^3}{3!} \end{pmatrix} \right| = \frac{r_1^2}{2} \left| \int_{t_1}^{t_2} V(t_1, t_2, x) dx \right|$$

where $V(x_1, \dots, x_m) := \prod_{1 \leq i < j \leq m} (x_j - x_i)$ denotes, and will always denote, the m -variable Vandermonde polynomial. The sign of $V(t_1, t_2, x)$ does not change as x varies between t_1 and t_2 and so modulus signs can be placed inside the integral in the above expression, leading to the bound $|J_{\Phi}(r_1, t_1, t_2)| \gtrsim |t_1 - t_2|^4$. Consequently, by applying the change of variables formula,

$$|E| \gtrsim \int_{\Omega} |J_{\Phi}(r_1, t_1, t_2)| dt_2 dr_1 dt_1 \gtrsim \int_{\Omega_1} \int_{\Omega_2(r_1, t_1)} |t_1 - t_2|^4 dt_2 dr_1 dt_1 \gtrsim \alpha^5 \beta$$

and this concludes the proof of (2.3.3) and hence establishes Theorem 34 for this special case.

The remainder of the chapter will develop this elementary argument in order to prove Proposition 37 in any dimension n and for any polynomial curve P .

⁵That is, for almost every $x \in \mathbb{R}^n$ the cardinality of the pre-image $\Phi^{-1}(\{x\})$ is no greater than some fixed (finite) constant. This is, for instance, a consequence of Lemma 21 from the introduction.

2.4 The polynomial decomposition theorem of Dendrinos and Wright

The refinement method essentially reduces the problem of establishing the restricted weak-type inequality (2.3.2) from Proposition 37 to estimating a Jacobian determinant associated with a certain naturally arising change of variables. In the case of the moment curve this Jacobian takes a particularly simple form involving a Vandermonde polynomial $V(t)$. For a general polynomial curve $P: \mathbb{R} \rightarrow \mathbb{R}^n$ one is led to consider expressions of the form

$$J_P(t) := \det(P'(t_1) \dots P'(t_d)) \tag{2.4.1}$$

for $t = (t_1, \dots, t_n) \in \mathbb{R}^n$.⁶ The multivariate polynomial J_P can be effectively estimated by comparing it with the Vandermonde polynomial and a certain geometric quantity expressed in terms of the torsion function (whose definition is recalled below). This leads to what is referred to here (and in [DW10]) as a *geometric inequality* for J_P . It is often the case that such a comparison is not possible globally; however, an important theorem due to Dendrinos and Wright [DW10] demonstrates the existence of a decomposition of the real line into a bounded number of intervals, $\mathbb{R} = \bigcup_{m=1}^C \overline{I_m}$, such that such a geometric inequality holds on each constituent interval I_m . Furthermore, the torsion function has a particularly simple form when restricted to an I_m : it is comparable to a centred monomial. Restricting the analysis to an interval arising from the Dendrinos-Wright decomposition therefore significantly simplifies the situation and allows for an effective estimation of the Jacobian J_P .

In order to state the decomposition lemma, recall the torsion of the curve P is defined to be the polynomial function

$$L_P(t) := \det(P^{(1)}(t) \dots P^{(n)}(t))$$

where $P^{(i)}$ denotes the i th derivative of P .

Theorem 38 (Dendrinos and Wright [DW10]). *Let $P: \mathbb{R} \rightarrow \mathbb{R}^n$ be a polynomial curve of degree d such that $L_P \neq 0$. There exists an integer $C = C_{n,d}$ and a decomposition $\mathbb{R} = \bigcup_{m=1}^C \overline{I_m}$ where the I_m are pairwise disjoint open intervals with the following properties:*

1) *Whenever $\mathbf{t} = (t_1, \dots, t_n) \in I_m^n$ the geometric inequality*

$$|J_P(\mathbf{t})| \gtrsim \prod_{i=1}^n |L_P(t_i)|^{1/n} |V(\mathbf{t})|$$

holds.

2) *For every $1 \leq m \leq C$ there exists a positive constant D_m , a non-negative integer $K_m \lesssim 1$ and a real number $b_m \in \mathbb{R} \setminus I_m$ such that*

$$|L_P(t)| \sim D_m |t - b_m|^{K_m} \quad \text{for all } t \in I_m.$$

Theorem 38 originally appeared in [DW10] where it was used to prove a version of Theorem 31 with a restricted range of exponents; subsequently Stovall [Sto] augmented these methods

⁶For the moment curve $h(t) := (t, t^2/2!, \dots, t^n/n!)$, one immediately observes that $J_h(t) = cV(t)$.

with additional square function estimates to prove Theorem 31 in full. Dendrinos, Laghi and Wright [DLW09] applied the decomposition to establish uniform estimates for convolution with affine arc-length on polynomial curves in low dimensions; their results were then extended to all dimensions by Stovall [Sto10] to give Theorem 30. Many of the methods used in the remaining sections of this chapter are based on those found in [DLW09, Sto10].

Fixing a polynomial P for which $L_P \not\equiv 0$, to prove Proposition 37 it suffices to establish the analogous uniform restricted weak-type inequalities for the local operators

$$A(x, r) := \int_I f(x - rP(t)) \lambda_P(t) dt$$

where I is any bounded interval. Furthermore, one may assume I lies completely within one of the intervals I_m produced by the decomposition (indeed, A can always be expressed as a sum of a bounded number of operators of the same form for which this property holds). Observing the translation, reflection and scaling invariance of the problem, one may assume $D_m = 1$, $b_m = 0$ and $I \subset (0, \infty)$ with $|I| = 1$ without any loss of generality. Similar reductions were made in [Sto10] where further details can be found. Notice under these hypotheses, $|\lambda_P(t)| \sim t^K$ uniformly on I for some non-negative integer $K \lesssim 1$.

Henceforth A will denote the operator defined by

$$Af(x, r) := \int_I f(x - rP(t)) d\mu_P(t) \tag{2.4.2}$$

where μ_P is now the weighted measure $d\mu_P(t) := \lambda_P(t)dt$; λ_P is redefined as $\lambda_P(t) := t^{2K/n(n+1)}$ and the integer K and interval I satisfy the above properties. It remains to prove the analogue of the restricted weak-type inequality (2.3.2) from Proposition 37 for this operator.

To close this section it is remarked that Stovall [Sto10] established an upper bound for certain derivatives of J_P on the set I^n in terms of J_P itself. This estimate will be of use in the forthcoming analysis and is recorded presently for the reader's convenience.

Proposition 39 (Stovall [Sto10]). *Let $S \subseteq \{1, \dots, n\}$ be a non-empty set of indices. Whenever $t = (t_1, \dots, t_n) \in I^n$, one has the estimate*

$$\left| \prod_{j \in S} \frac{\partial}{\partial t_j} J_P(t) \right| \lesssim \sum_{T \subseteq S} \sum_{u, \epsilon} \left(\prod_{j \in S \setminus T} t_j^{-1} \right) \left(\prod_{j \in T} t_j^{-\epsilon(j)} |t_j - t_{u(j)}|^{\epsilon(j)-1} \right) |J_P(t)|$$

where the outer sum is over all subsets T of S and the inner sum is over all functions $u: T \rightarrow \{1, \dots, n\}$ with the property $u(j) \neq j$ for all $j \in T$ and all $\epsilon: T \rightarrow \{0, 1\}$.

2.5 Parameter towers

Having made the reductions of the previous section, fix Borel sets $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^n \times [1, 2]$ of finite Lebesgue measure such that $\langle A\chi_E, \chi_F \rangle \neq 0$ where A is of the special form described in (2.4.2). As in Section 2.3, the quantities

$$\alpha := \frac{1}{|F|} \langle A\chi_E, \chi_F \rangle \quad \text{and} \quad \beta := \frac{1}{|E|} \langle \chi_E, A^* \chi_F \rangle$$

play a dominant rôle in the analysis. Indeed, by some simple algebra the inequality (2.3.2) can be restated in terms of α and β as either

$$|E| \gtrsim \alpha^{n(n+1)/2} (\beta/\alpha)^{(n-1)/2} \quad (2.5.1)$$

or

$$|F| \gtrsim \alpha^{n(n+1)/2} (\beta/\alpha)^{(n+1)/2}. \quad (2.5.2)$$

The proof will proceed by attempting to establish either one of these estimates by applying a variant of the refinement procedure described earlier. In view of the $L_x^{p_2} - L_{x,r}^{q_2}$ estimate established in Section 2.3, henceforth it is assumed without loss of generality that $\alpha > \beta$. Indeed, the restricted weak-type (p_2, q_2) inequality implies

$$|E| \gtrsim \alpha^{n(n+1)/2} (\beta/\alpha)^n$$

from which (2.5.1) follows in the case $\alpha \leq \beta$.

As in Section 2.3, either (2.5.1) or (2.5.2) will be established by constructing suitable parameter domain Ω and parametrising function Φ where Ω is some structured set. In this section the basic structure of such a domain Ω is described.

Consider a collection $\{\Omega_j\}_{j=1}^D$ of Borel measurable sets either of the form⁷

$$\Omega_j \subseteq [1, 2]^{\lfloor j/2 \rfloor} \times I^j \quad \text{for } j = 1, \dots, D \quad (2.5.3)$$

or

$$\Omega_j \subseteq [1, 2]^{\lceil j/2 \rceil} \times I^j \quad \text{for } j = 1, \dots, D. \quad (2.5.4)$$

In order to be concise it is useful to let $\langle x \rangle$ ambiguously denote either $[x]$ or $\lceil x \rceil$ for any $x \in \mathbb{R}$, where it is understood the notation is consistent within any given equation. Thus (2.5.3) and (2.5.4) are considered simultaneously by writing

$$\Omega_j \subseteq [1, 2]^{\langle j/2 \rangle} \times I^j \quad \text{for } j = 1, \dots, D.$$

Assume each Ω_j has positive $(j + \langle j/2 \rangle)$ -dimensional measure. The following definitions, which borrow terminology from [Chra, Chrb], are fundamental in what follows.

Definition 40. i) A collection $\{\Omega_j\}_{j=1}^D$ of the above form is a (parameter) tower of height $D \in \mathbb{N}$ if for any $1 < j \leq D$ and $r_1, \dots, r_{\langle j/2 \rangle} \in [1, 2]$ and $t_1, \dots, t_j \in I$ the following holds:

$$(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j \Rightarrow (\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1}$$

where $\mathbf{r}_k := (r_1, \dots, r_{\langle k/2 \rangle})$ and $\mathbf{t}_k := (t_1, \dots, t_k)$ for $k = j-1, j$.

ii) If a tower is described as “type 1” (respectively, “type 2”) this indicates the constituent sets are of the form described in (2.5.3) (respectively, (2.5.4)). Thus, when considering type 1 (respectively, type 2) towers the symbol $\langle x \rangle$ is interpreted as $[x]$ (respectively, $\lceil x \rceil$) for any $x \in \mathbb{R}$. Notice in a type j tower the initial set Ω_1 is j -dimensional, for $j = 1, 2$.

iii) Given a type 1 (respectively, type 2) tower $\{\Omega_j\}_{j=1}^D$, fix $1 < j \leq D$. For each $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in$

⁷Here $\lceil x \rceil := \min\{m \in \mathbb{N} : m \geq x\}$ and $\lfloor x \rfloor := \max\{m \in \mathbb{N} : m \leq x\}$ for any $x \in \mathbb{R}$.

Ω_{j-1} define the associated fibre $\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})$ to be the set

$$\begin{cases} \{t_j \in I : (\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j\} & \text{if } j \text{ is odd (respectively even)} \\ \{(r_{\langle j/2 \rangle}, t_j) \in [1, 2] \times I : (\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j\} & \text{if } j \text{ is even (respectively odd)}. \end{cases}$$

For example, the collection $\{\Omega_j\}_{j=1}^2$ defined in Section 2.3 constitutes a type 2 tower. In what follows, type 1 towers will be of primary interest. The elements of the various levels of a type 1 tower are typically denoted using the following notation:

$$\mathbf{t}_1 = t_1 \in \Omega_1, \quad (\mathbf{r}_2, \mathbf{t}_2) = (r_1, t_1, t_2) \in \Omega_2, \quad (\mathbf{r}_3, \mathbf{t}_3) = (r_1, t_1, t_2, t_3) \in \Omega_3,$$

$$(\mathbf{r}_4, \mathbf{t}_4) = (r_1, r_2, t_1, t_2, t_3, t_4) \in \Omega_4, \quad (\mathbf{r}_5, \mathbf{t}_5) = (r_1, r_2, t_1, t_2, t_3, t_4, t_5) \in \Omega_5, \dots$$

Recall that certain the mappings Φ_1 and Φ_2 defined in (2.3.4) and (2.3.5) were associated to the tower constructed in Section 2.3. Presently the analogues of these mappings in the general situation are discussed. First of all one associates to every $(x_0, r_0) \in \mathbb{R}^n \times [1, 2]$ and $y_0 \in \mathbb{R}^n$ a family of functions.

i) Given $(x_0, r_0) \in \mathbb{R}^n \times [1, 2]$ define the functions $\Psi_j(x_0, r_0; \cdot) : [1, 2]^{[j/2]} \times I^j \rightarrow \mathbb{R}^n$ by

$$\Psi_j(x_0, r_0; \mathbf{r}_j, \mathbf{t}_j) = x_0 + \sum_{k=1}^j (-1)^k r_{[k/2]} P(t_k). \quad (2.5.5)$$

for all $\mathbf{r}_j = (r_1, \dots, r_{[j/2]}) \in [1, 2]^{[j/2]}$ and $\mathbf{t}_j = (t_1, \dots, t_j) \in I^j$.

ii) Given $y_0 \in \mathbb{R}^n$ define the functions $\Psi_j(y_0; \cdot) : [1, 2]^{[j/2]} \times I^j \rightarrow \mathbb{R}^n$ by

$$\Psi_j(y_0; \mathbf{r}_j, \mathbf{t}_j) = y_0 + \sum_{k=1}^j (-1)^{k+1} r_{[k/2]} P(t_k) \quad (2.5.6)$$

for all $\mathbf{r}_j = (r_1, \dots, r_{[j/2]}) \in [1, 2]^{[j/2]}$ and $\mathbf{t}_j = (t_1, \dots, t_j) \in I^j$.

To any tower one associates a family of mappings on the constituent sets, defined in terms of the Ψ_j functions.

Definition 41. Suppose $\{\Omega_j\}_{j=1}^D$ is a type 1 (respectively, type 2) tower and fix some $z_0 = (x_0, r_0) \in \mathbb{R}^n \times [1, 2]$ (respectively, $z_0 = y_0 \in \mathbb{R}^n$). The family of mappings $\{\Phi_j\}_{j=1}^D$ associated to these objects is defined as follows:

i) For $1 \leq j \leq D$ odd (respectively, even) let $\Phi_j : \Omega_j \rightarrow \mathbb{R}^n$ denote the map

$$\Phi_j(\mathbf{r}_j, \mathbf{t}_j) := \Psi_j(z_0; \mathbf{r}_j, \mathbf{t}_j).$$

ii) For $1 \leq j \leq D$ even (respectively, odd) let $\Phi_j : \Omega_j \rightarrow \mathbb{R}^n \times [1, 2]$ denote the map

$$\Phi_j(\mathbf{r}_j, \mathbf{t}_j) := \begin{pmatrix} \Psi_k(z_0, \mathbf{r}_j, \mathbf{t}_j) \\ r_{\langle j/2 \rangle} \end{pmatrix}.$$

Referring back to the simple case discussed earlier, (appropriate restrictions of) the functions

defined in (2.3.4) and (2.3.5) are easily seen to constitute the family associated to the point y_0 and tower $\{\Omega_j\}_{j=1}^2$ constructed in Section 2.3.

For notational convenience define

$$\kappa := \frac{n(n+1)}{2K+n(n+1)}. \quad (2.5.7)$$

Recalling the definition of μ_P from (2.4.2), it is also useful to let ν_P denote the measure given by the product of Lebesgue measure on $[1, 2]$ with μ_P . Hence, for any Borel set $R \subseteq [1, 2] \times I$,

$$\nu_P(R) = \int_1^2 \int_I \chi_R(r, t) \lambda_P(t) dt dr.$$

Initially the following lemma is used to construct a suitable parameter tower.

Lemma 42. *There exists a point $(x_0, r_0) \in F$ and a type 1 tower $\{\Omega_j\}_{j=1}^{n+1}$ with the following properties:*

1) *Whenever $(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j$ it follows that*

$$\alpha^\kappa = \max\{\alpha, \beta\}^\kappa \lesssim t_1 < t_2 < \cdots < t_j.$$

2) *For $1 \leq j \leq n+1$ odd:*

i) $\Phi_j(\Omega_j) \subseteq E$;

ii) $\mu_P(\Omega_1) \gtrsim \alpha$ and if $j > 1$, then $\mu_P(\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})) \gtrsim \alpha$ whenever $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1}$;

iii) *If $j > 1$ and $(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j$, then*

$$\int_{t_{j-1}}^{t_j} \lambda_P(t) dt \gtrsim \alpha.$$

3) *For $1 < j \leq n+1$ even:*

i) $\Phi_j(\Omega_j) \subseteq F$;

ii) $\nu_P(\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})) \gtrsim \beta$ whenever $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1}$;

iii) *If $(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j$, then*

$$\int_{t_{j-1}}^{t_j} \lambda_P(t) dt \gtrsim \beta.$$

Lemma 42 is a slight modification of a recent result due to Dendrinos and Stovall [DS15], based on a fundamental construction due to Christ [Chr98]. Rather than present a proof of Lemma 42, a stronger statement, Lemma 44, is established below.

To conclude this section it is noted that a tower admitting all the properties described in the previous lemma automatically satisfies a certain separation condition. This observation was also used in [DS15].

Corollary 43. *Let $\{\Omega_j\}_{j=1}^{n+1}$ be a tower with all the properties described in Lemma 42.*

i) *Suppose $1 < j \leq n+1$ is odd. Then, for all $(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j$ it follows that*

$$t_j - t_i \gtrsim \alpha t_i^{-2K/n(n+1)} \quad \text{for } 1 \leq i \leq j-1.$$

ii) Suppose $1 < j \leq n + 1$ is even. Then, for all $(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j$ it follows that

$$t_j - t_{j-1} \gtrsim \beta t_{j-1}^{-2K/n(n+1)}; \quad t_j - t_i \gtrsim \alpha t_i^{-2K/n(n+1)} \quad \text{for } 1 \leq i \leq j - 2.$$

Proof. Let $1 < j \leq n + 1$ be either odd or even and $1 \leq i \leq j - 1$. If j is even, then further suppose $i \leq j - 2$. For $(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j$, properties 1) and 2) iii) of the construction ensure there exists some $t_i < s_i < t_j$ for which

$$\int_{t_i}^{s_i} \lambda_P(t) dt \sim \alpha.$$

Consequently,

$$s_i^{1/\kappa} \sim \int_0^{s_i} \lambda_P(t) dt = \int_0^{t_i} \lambda_P(t) dt + \int_{t_i}^{s_i} \lambda_P(t) dt \sim t_i^{1/\kappa} + \alpha$$

and, since $\alpha \lesssim t_i^{1/\kappa}$ holds by property 1), one concludes that $s_i \sim t_i$. Thence,

$$|t_j - t_i| \geq |s_i - t_i| \gtrsim \left(\int_{t_i}^{s_i} \lambda_P(t) dt \right) t_i^{-2K/n(n+1)} \gtrsim \alpha t_i^{-2K/n(n+1)}.$$

The remaining case when j is even and $i = j - 1$ can be dealt with in a similar fashion, applying property 3) iii). \square

2.6 Improved parameter towers

The properties detailed in Lemma 42, though useful, are insufficient for the present purpose. Observe that although the even fibres of the tower constructed in Lemma 42 are two-dimensional sets, consisting of points $(r_{j/2}, t_j) \in [1, 2] \times I$, all the bounds are decidedly one-dimensional in the sense that they are in terms of the t_j variables and there is little reference to the dilation parameters. An additional refinement is necessary to take advantage the higher dimensionality of the even fibres.

Lemma 44. *Fix $0 < \delta \ll 1$ a small parameter. There exists a point $(x_0, r_0) \in F$ and a tower $\{\Omega_k\}_{k=1}^{n+1}$ satisfying all the properties of Lemma 42 with the additional property that for each even $1 < j \leq n + 1$ either*

$$|t_j - t_{j-1}| \geq \delta(\alpha\beta)^{1/2} t_{j-1}^{-2K/n(n+1)}$$

holds for all $(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j$, or

$$|t_j - t_{j-1}| < \delta(\alpha\beta)^{1/2} t_{j-1}^{-2K/n(n+1)} \quad \text{and} \quad |r_{j/2} - r_{j/2-1}| \gtrsim (\beta/\alpha)^{1/2}$$

both hold for all $(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j$. If the former case the index j is designated “red”, whilst in the latter j is designated “blue”. The odd vertices are achromatic; that is, they are not assigned a colour.

Remark 45. The partition of the indices into the sets of odd, red and blue indices plays a very similar rôle to the construction of the band structure in [Chr98] and in particular the “slicing method” of [Chra, Chr98] will be utilised.

The result is essentially established as follows. By applying the argument of Dendrinos and Stovall [DS15] one obtains an initial tower with the properties stated in Lemma 42. To ensure

the additional property described in Lemma 44 holds one further refines the tower, appealing to the following lemma (or, more precisely, the resulting corollary).

Lemma 46. *Let $\{\tilde{\Omega}_j\}_{j=1}^D$ be a type 1 tower of even height D and*

$$\{A(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) : (\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \in \tilde{\Omega}_{D-1}\}$$

a collection of measurable subsets of $[1, 2] \times I$. Then there exists a tower $\{\Omega_j\}_{j=1}^D$ satisfying:

a) $\Omega_1 \subseteq \tilde{\Omega}_1$ and $\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \subseteq \tilde{\Omega}_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})$ for all $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1}$ and all $1 < j \leq D$;

b) The following estimates hold:

$$i) \mu_P(\Omega_1) \geq \frac{1}{2}\mu_P(\tilde{\Omega}_1).$$

ii) Whenever $1 < j \leq D$ is odd,

$$\mu_P(\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})) \geq \frac{1}{2}\mu_P(\tilde{\Omega}_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}))$$

for all $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1}$.

iii) Whenever $1 < j \leq D$ is even,

$$\nu_P(\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})) \geq \frac{1}{2}\nu_P(\tilde{\Omega}_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}))$$

for all $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1}$.

c) Precisely one of the following holds:

i) $\Omega_D(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \subseteq A(\mathbf{r}_{D-1}, \mathbf{t}_{D-1})$ for all $(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \in \Omega_{D-1}$;

ii) $\Omega_D(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \cap A(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) = \emptyset$ for all $(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \in \Omega_{D-1}$.

Proof. Define a sequence of sets $\omega_{D-k} \subseteq \tilde{\Omega}_{D-k}$ for $1 \leq k \leq D-1$ recursively as follows. For all $(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \in \tilde{\Omega}_{D-1}$ let

$$\omega_D(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) := \tilde{\Omega}_D(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \cap A(\mathbf{r}_{D-1}, \mathbf{t}_{D-1})$$

and define ω_{D-1} to be the set

$$\left\{ (\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \in \tilde{\Omega}_{D-1} : \nu_P(\omega_D(\mathbf{r}_{D-1}, \mathbf{t}_{D-1})) \geq \frac{1}{2}\nu_P(\tilde{\Omega}_D(\mathbf{r}_{D-1}, \mathbf{t}_{D-1})) \right\}.$$

Hence, ω_{D-1} is the set of points $(\mathbf{r}, \mathbf{t}) \in \tilde{\Omega}_{D-1}$ with the property that most of the associated fibre $\tilde{\Omega}_D(\mathbf{r}, \mathbf{t})$ lies in $A(\mathbf{r}, \mathbf{t})$.

Now suppose ω_{D-k} has been defined for some $1 \leq k \leq D-2$ and let $(\mathbf{r}_{D-k-1}, \mathbf{t}_{D-k-1}) \in \tilde{\Omega}_{D-k-1}$. If k is odd, then $D-k$ is also odd and $\omega_{D-k}(\mathbf{r}_{D-k-1}, \mathbf{t}_{D-k-1})$ is defined to be

$$\{t_{D-k} \in \tilde{\Omega}_{D-k}(\mathbf{r}_{D-k-1}, \mathbf{t}_{D-k-1}) : (\mathbf{r}_{D-k}, \mathbf{t}_{D-k}) \in \omega_{D-k}\}$$

where $\mathbf{r}_{D-k} = \mathbf{r}_{D-k-1}$ and $\mathbf{t}_{D-k} = (\mathbf{t}_{D-k-1}, t_{D-k})$; throughout this chapter, similar notation will be used for elements belonging to levels of various parameter towers without further

comment. Let

$$\omega_{D-k-1} := \left\{ (\mathbf{r}, \mathbf{t}) \in \tilde{\Omega}_{D-k-1} : \mu_P(\omega_{D-k}(\mathbf{r}, \mathbf{t})) \geq \frac{1}{2} \mu_P(\tilde{\Omega}_{D-k}(\mathbf{r}, \mathbf{t})) \right\}$$

so that ω_{D-k-1} is the set of points $(\mathbf{r}, \mathbf{t}) \in \tilde{\Omega}_{D-k-1}$ with the property that most of the associated fibre $\tilde{\Omega}_{D-k}(\mathbf{r}, \mathbf{t})$ lies in $\omega_{D-k}(\mathbf{r}, \mathbf{t})$.

Similarly, if k is even, then $D - k$ is even and $\omega_{D-k}(\mathbf{r}_{D-k-1}, \mathbf{t}_{D-k-1})$ is defined to be

$$\left\{ (r_{(D-k)/2}, t_{D-k}) \in \tilde{\Omega}_{D-k}(\mathbf{r}_{D-k-1}, \mathbf{t}_{D-k-1}) : (\mathbf{r}_{D-k}, \mathbf{t}_{D-k}) \in \omega_{D-k} \right\}$$

and one completes the recursive definition by letting

$$\omega_{D-k-1} := \left\{ (\mathbf{r}, \mathbf{t}) \in \tilde{\Omega}_{D-k-1} : \nu_P(\omega_{D-k}(\mathbf{r}, \mathbf{t})) \geq \frac{1}{2} \nu_P(\tilde{\Omega}_{D-k}(\mathbf{r}, \mathbf{t})) \right\}.$$

Having constructed the sequence ω_{D-k} for $1 \leq k \leq D - 1$, suppose $\mu_P(\omega_1) \geq \frac{1}{2} \mu_P(\tilde{\Omega}_1)$. If one defines $\Omega_1 := \omega_1$ and $\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) := \omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})$ for $1 < j \leq D$, then one may construct a tower inductively by setting

$$\Omega_j := \{ (\mathbf{r}_j, \mathbf{t}_j) \in [1, 2]^{(j-1)/2} \times [0, 1]^j : (\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1} \text{ and } t_j \in \Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \} \quad (2.6.1)$$

for $j > 1$ odd and

$$\Omega_j := \{ (\mathbf{r}_j, \mathbf{t}_j) \in [1, 2]^{j/2} \times [0, 1]^j : (\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1} \text{ and } (r_{j/2}, t_j) \in \Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \} \quad (2.6.2)$$

for j even. It immediately follows from the definitions that the resulting tower $\{\Omega_j\}_{j=1}^D$ satisfies the properties a), b) and c) i) stated in the lemma.

On the other hand, if $\mu_P(\omega_1) < \frac{1}{2} \mu_P(\tilde{\Omega}_1)$, then define $\Omega_1 := \tilde{\Omega}_1 \setminus \omega_1$ and let

$$\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) := \tilde{\Omega}_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \setminus \omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})$$

for $1 < j \leq D$ so that properties a) and b) i) and c) ii) clearly hold for the resulting tower $\{\Omega_j\}_{j=1}^D$, again defined by (2.6.1) and (2.6.2). To prove b) ii), suppose $1 < j \leq D$ is odd and $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1}$. Thus, $(r_{(j-1)/2}, t_{j-1}) \in \Omega_{j-1}(\mathbf{r}_{j-2}, \mathbf{t}_{j-2})$ and, by the definition of $\omega_{j-1}(\mathbf{r}_{j-2}, \mathbf{t}_{j-2})$, it follows that $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \tilde{\Omega}_{j-1} \setminus \omega_{j-1}$. Finally, the definition of ω_{j-1} ensures

$$\mu_P(\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})) = \mu_P(\tilde{\Omega}_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})) - \mu_P(\omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})) \geq \frac{1}{2} \mu_P(\tilde{\Omega}_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})).$$

A similar argument shows b) iii) also holds, completing the proof. \square

By repeatedly applying the previous lemma one may deduce the following corollary.

Corollary 47. *Let $\{\tilde{\Omega}_j\}_{j=1}^D$ be a tower of height D and for each even $1 < k \leq D$ let*

$$\left\{ A_k(\mathbf{r}_{k-1}, \mathbf{t}_{k-1}) : (\mathbf{r}_{k-1}, \mathbf{t}_{k-1}) \in \tilde{\Omega}_{k-1} \right\}$$

a collection of measurable subsets of $[1, 2] \times I$. Then there exists a tower $\{\Omega_j\}_{j=1}^D$ satisfying:

a) $\Omega_1 \subseteq \tilde{\Omega}_1$ and $\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \subseteq \tilde{\Omega}_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})$ for all $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1}$ and all $1 < j \leq D$;

b) The following estimates hold for the fibres:

$$i) \mu_P(\Omega_1) \geq 2^{-\lfloor D/2 \rfloor} \mu_P(\tilde{\Omega}_1).$$

ii) Whenever $1 < j \leq D$ is odd,

$$\mu_P(\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})) \geq 2^{-\lfloor D/2 \rfloor} \mu_P(\tilde{\Omega}_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}))$$

holds for all $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1}$.

iii) Whenever $1 < j \leq D$ is even,

$$\nu_P(\Omega_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})) \geq 2^{-\lfloor D/2 \rfloor} \nu_P(\tilde{\Omega}_j(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}))$$

holds for all $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \Omega_{j-1}$.

c) For each even $1 < k \leq D$ precisely one of the following holds:

$$i) \Omega_k(\mathbf{r}_{k-1}, \mathbf{t}_{k-1}) \subseteq A_k(\mathbf{r}_{k-1}, \mathbf{t}_{k-1}) \text{ for all } (\mathbf{r}_{k-1}, \mathbf{t}_{k-1}) \in \Omega_{k-1};$$

$$ii) \Omega_k(\mathbf{r}_{k-1}, \mathbf{t}_{k-1}) \cap A_k(\mathbf{r}_{k-1}, \mathbf{t}_{k-1}) = \emptyset \text{ for all } (\mathbf{r}_{k-1}, \mathbf{t}_{k-1}) \in \Omega_{k-1}.$$

Proof. Proceed by induction on D , the case $D = 1$ being vacuous. Let $1 < D$ and fix a tower $\{\tilde{\Omega}_j\}_{j=1}^D$. Apply the induction hypothesis to $\{\tilde{\Omega}_j\}_{j=1}^{D-1}$ to obtain a tower $\{\omega_j\}_{j=1}^{D-1}$ satisfying the properties a), b) and c) of the corollary, with D replaced by $D-1$. For each $(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \in \omega_{D-1}$ define $\omega_D(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) := \tilde{\Omega}_D(\mathbf{r}_{D-1}, \mathbf{t}_{D-1})$. If D is odd define ω_D to be

$$\{(\mathbf{r}_D, \mathbf{t}_D) \in \tilde{\Omega}_D : t_D \in \omega_D(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \text{ and } (\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \in \omega_{D-1}\};$$

and similarly, if D is even, then define ω_D to be

$$\{(\mathbf{r}_D, \mathbf{t}_D) \in \tilde{\Omega}_D : (r_{D/2}, t_D) \in \omega_D(\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \text{ and } (\mathbf{r}_{D-1}, \mathbf{t}_{D-1}) \in \omega_{D-1}\}.$$

In the case D is odd, the proof is completed by letting $\Omega_j := \omega_j$ for $j = 1, \dots, D$. In the even case, apply Lemma 46 to the tower $\{\omega_j\}_{j=1}^D$ to obtain a refinement $\{\Omega_j\}_{j=1}^D$. This refinement is easily seen to have the desired properties. \square

Having stated these trivial refinement results one may proceed to prove Lemma 44.

Proof (of Lemma 44). Observe, defining $\Sigma := \mathbb{R}^n \times [1, 2] \times I$ it follows that

$$\langle A\chi_E, \chi_F \rangle = \int_{\Sigma} \chi_U(x, r, t) \lambda_P(t) dx dr dt$$

where $\lambda_P(t) := t^{2K/n(n+1)}$ and

$$U := \{(x, r, t) \in \Sigma : x - rP(t) \in E \text{ and } (x, r) \in F\}.$$

Writing $I := (a, b)$ where $a \geq 0$ and $b - a = 1$, a method due to Dendrinos and Stovall [DS15] may be applied to produce a sequence $\{U_k\}_{k=0}^{\infty}$ of decreasing subsets of U of pairwise comparable measure such that for all $k \geq 1$ either

$$\int_t^b \chi_{U_{k-1}}(x, r, \tau) \lambda_P(\tau) d\tau \geq 4^{-(k+1)} \alpha \tag{2.6.3}$$

or

$$\int_1^2 \int_t^b \chi_{U_{k-1}}(x - rP(t) + \rho P(\tau), \rho, \tau) \lambda_P(\tau) d\tau d\rho \geq 4^{-(k+1)} \beta \quad (2.6.4)$$

holds for all $(x, r, t) \in U_k$. Specifically, the $\{U_k\}_{k=0}^\infty$ can be chosen so that

$$\int_\Sigma \chi_{U_0}(x, r, t) \lambda_P(t) dx dr dt \geq \frac{1}{2} \langle A\chi_E, \chi_F \rangle$$

and for all $k \geq 1$:

i) $U_k \subseteq U_{k-1}$ and

$$\int_\Sigma \chi_{U_k}(x, r, t) \lambda_P(t) dx dr dt \geq \frac{1}{4} \int_\Sigma \chi_{U_{k-1}}(x, r, t) \lambda_P(t) dx dr dt;$$

ii) If $k \not\equiv n \pmod{2}$ (respectively, $k \equiv n \pmod{2}$), then (2.6.3) (respectively (2.6.4)) holds for all $(x, r, t) \in U_k$.

iii) Furthermore, for each k , if $t \in I$ is such that $(x, r, t) \in U_k$ for some $(x, r) \in \mathbb{R}^n \times [1, 2]$, then $t \geq (\alpha/2\kappa)^\kappa$ where κ is as in (2.5.7).

This construction is due to Dendrinos and Stovall [DS15], however the details are included in the final section of the chapter for completeness.

Fix $(x_0, r_0, t_0) \in U_{n+1}$. The next step is to use the sets $\{U_k\}_{k=0}^{n+1}$ to construct an initial tower $\{\tilde{\Omega}_j\}_{j=1}^{n+1}$ satisfying the properties of Lemma 42 and such that whenever $(\mathbf{r}_j, \mathbf{t}_j) \in \tilde{\Omega}_j$ for some $1 \leq j \leq n+1$, it follows that

$$\begin{cases} (\Psi_j(\mathbf{r}_j, \mathbf{t}_j), r_{j/2}, t_j) \in U_{n+1-j} & \text{if } 0 \leq j \leq n+1 \text{ is even} \\ (\Psi_{j-1}(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}), r_{\lfloor j/2 \rfloor}, t_j) \in U_{n+1-j} & \text{if } 1 \leq j \leq n+1 \text{ is odd} \end{cases}$$

where $\Psi_j(\mathbf{r}_j, \mathbf{t}_j) := \Psi_j(x_0, r_0; \mathbf{r}_j, \mathbf{t}_j)$ for $j \geq 1$ is as defined in (2.5.5) and $\Psi_0(\mathbf{r}_0, \mathbf{t}_0) := x_0$. It is convenient to consider $(\mathbf{r}_0, \mathbf{t}_0)$ as some arbitrary object (say, $(\mathbf{r}_0, \mathbf{t}_0) := (r_0, t_0)$) and Ψ_0 as a function on the singleton set $\tilde{\Omega}_0 := \{(\mathbf{r}_0, \mathbf{t}_0)\}$, taking the value x_0 .

The tower $\{\tilde{\Omega}_j\}_{j=1}^{n+1}$ is constructed recursively. To begin, define $\tilde{\Omega}_0$ as above; suppose $\tilde{\Omega}_j$ has been defined for some $0 \leq j \leq n$ and fix $(\mathbf{r}_j, \mathbf{t}_j) \in \tilde{\Omega}_j$. The argument splits into two cases, depending on the parity of j .

Case 1: j is even.

Since $(\Psi_j(\mathbf{r}_j, \mathbf{t}_j), r_{j/2}, t_j) \in U_{n+1-j}$ and $n+1-j \not\equiv n \pmod{2}$, one may apply (2.6.3) to deduce

$$\int_{t_j}^b \chi_{U_{n-j}}(\Psi_j(\mathbf{r}_j, \mathbf{t}_j), r_{j/2}, \tau) \lambda_P(\tau) d\tau \geq 4^{-(n+2-j)} \alpha. \quad (2.6.5)$$

It is therefore possible to choose $t_j < s_j < b$ with the property

$$\int_{t_j}^{s_j} \lambda_P(\tau) d\tau = 4^{-(n+5/2-j)} \alpha. \quad (2.6.6)$$

Define the set

$$\tilde{\Omega}_{j+1}(\mathbf{r}_j, \mathbf{t}_j) := \{t_{j+1} \in (s_j, b] : (\Psi_j(\mathbf{r}_j, \mathbf{t}_j), r_{j/2}, t_{j+1}) \in U_{n-j}\},$$

noting, by (2.6.5) and (2.6.6), that this has measure $\mu_P(\tilde{\Omega}_{j+1}(\mathbf{r}_j, \mathbf{t}_j)) \geq 4^{-(n+5/2-j)}\alpha$. To complete the recursive step in this case, if $j = 0$ let $\tilde{\Omega}_1 := \tilde{\Omega}_1(\mathbf{r}_0, \mathbf{t}_0)$ whilst for $j > 0$ define

$$\tilde{\Omega}_{j+1} := \{(\mathbf{r}_{j+1}, \mathbf{t}_{j+1}) : (\mathbf{r}_j, \mathbf{t}_j) \in \tilde{\Omega}_j \text{ and } t_{j+1} \in \tilde{\Omega}_{j+1}(\mathbf{r}_j, \mathbf{t}_j)\}.$$

Case 2: j is odd.

Since $(\Psi_{j-1}(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}), r_{\lfloor j/2 \rfloor}, t_j) \in U_{n+1-j}$ and $n+1-j \equiv n \pmod{2}$, one may apply (2.6.4) to deduce

$$\int_1^2 \int_{t_j}^b \chi_{U_{n-j}}(\Psi_j(\mathbf{r}_j, \mathbf{t}_j) + \rho P(\tau), \rho, \tau) \lambda_P(\tau) d\tau d\rho \geq 4^{-(n+2-j)}\beta. \quad (2.6.7)$$

Here the identity

$$\Psi_j(\mathbf{r}_j, \mathbf{t}_j) = \Psi_{j-1}(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) - r_{\lfloor j/2 \rfloor} P(t_j)$$

has been applied, which is a consequence of the definition (2.5.5). It is therefore possible to choose $t_j < s_j < b$ and $t_j < \tilde{s}_j$ with the properties

$$\int_{t_j}^{s_j} \lambda_P(\tau) d\tau = 4^{-(n+5/2-j)}\beta, \quad \int_{t_j}^{\tilde{s}_j} \lambda_P(\tau) d\tau = 2^{-(n+3-j)}(\alpha\beta)^{1/2}. \quad (2.6.8)$$

The factor $(\alpha\beta)^{1/2}$ is chosen in order to exploit the 2-dimensionality of the fibres in the present case; this will lead to the improved separation properties for the t_j variables associated to the red indices described in the statement of Lemma 44.

Define $\tilde{\Omega}_{j+1}(\mathbf{r}_j, \mathbf{t}_j)$ to be the set

$$\{(r_{\lfloor j/2 \rfloor}, t_{j+1}) \in D(\mathbf{r}_j, \mathbf{t}_j) : (\Psi_j(\mathbf{r}_j, \mathbf{t}_j) + r_{\lfloor j/2 \rfloor} P(t_{j+1}), r_{\lfloor j/2 \rfloor}, t_{j+1}) \in U_{n-j}\}$$

where $D(\mathbf{r}_j, \mathbf{t}_j) := ([1, 2] \times (s_j, b]) \setminus R(\mathbf{r}_j, \mathbf{t}_j)$ for the rectangle

$$R(\mathbf{r}_j, \mathbf{t}_j) := \{(r, t) \in \mathbb{R}^2 : |r - r_{\lfloor j/2 \rfloor}| \leq 2^{-(n+4-j)}(\beta/\alpha)^{1/2} \text{ and } t_j \leq t \leq \tilde{s}_j\}.$$

Observe, by (2.6.8) it follows that

$$\int_1^2 \int_I \chi_{R(\mathbf{r}_j, \mathbf{t}_j)}(r, t) \lambda_P(t) dt dr \leq 4^{-(n+3-j)}\beta.$$

Thus, by (2.6.7) and (2.6.8), the set $\tilde{\Omega}_{j+1}(\mathbf{r}_j, \mathbf{t}_j)$ has measure $\nu_P(\tilde{\Omega}_{j+1}(\mathbf{r}_j, \mathbf{t}_j)) \geq 4^{-(n+3-j)}\beta$.

Finally, to complete the recursive definition let

$$\tilde{\Omega}_{j+1} := \{(\mathbf{r}_{j+1}, \mathbf{t}_{j+1}) : (\mathbf{r}_j, \mathbf{t}_j) \in \tilde{\Omega}_j \text{ and } (r_{\lfloor j/2 \rfloor}, t_{j+1}) \in \tilde{\Omega}_{j+1}(\mathbf{r}_j, \mathbf{t}_j)\}.$$

It is easy to verify the collection $\{\tilde{\Omega}_j\}_{j=1}^{n+1}$ forms a tower satisfying all the properties of Lemma 42. Finally, Corollary 47 is applied to further refine this tower to ensure the additional property stated in Lemma 44 holds. For each $1 < j \leq n+1$ even, define

$$A(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) := \{(r, t) \in [1, 2] \times I : t - t_{j-1} > \delta(\alpha\beta)^{1/2} t_{j-1}^{-2K/n(n+1)}\}$$

for all $(\mathbf{r}_{j-1}, \mathbf{t}_{j-1}) \in \tilde{\Omega}_{j-1}$. Letting $\{\Omega_j\}_{j=1}^{n+1}$ denote the refined tower, the existence of which is guaranteed by Corollary 47, it is easy to see this has all the desired properties. In particular, for each $1 < j \leq n+1$ even, precisely one of the following holds:

- a) For all $(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j$ one has $|t_j - t_{j-1}| > \delta(\alpha\beta)^{1/2} t_{j-1}^{-2K/n(n+1)}$.
- b) For all $(\mathbf{r}_j, \mathbf{t}_j) \in \Omega_j$ one has $|t_j - t_{j-1}| \leq \delta(\alpha\beta)^{1/2} t_{j-1}^{-2K/n(n+1)}$. In this case, observe for $t_{j-1} \leq \tau \leq t_j$ it follows that

$$|\tau - t_{j-1}| \leq \delta(\alpha\beta)^{1/2} t_{j-1}^{-2K/n(n+1)} \lesssim \delta \alpha (\alpha^\kappa)^{-2K/n(n+1)} \lesssim \delta t_{j-1}$$

by condition iii) of the sets U_k described above. Thus $\tau \lesssim t_{j-1}$ and consequently

$$\int_{t_{j-1}}^{t_j} \lambda_P(\tau) d\tau \lesssim \delta(\alpha\beta)^{1/2}.$$

If δ is chosen from the outset to be sufficiently small depending only on n and K , then it follows that $t_{j-1} < t_j < \tilde{s}_{j-1}$ by the preceding inequality and the definition of \tilde{s}_{j-1} from (2.6.8). Since $(r_{j/2}, t_j) \notin R(\mathbf{r}_{j-1}, \mathbf{t}_{j-1})$, one concludes that $|r_{j/2} - r_{j/2-1}| > 2^{-(n+5-j)}(\beta/\alpha)^{1/2}$.

□

2.7 Definition of the parameter domain

Henceforth fix a tower $\{\Omega_j\}_{j=1}^{n+1}$ satisfying the properties stated in Lemma 44 with a suitably small choice of $0 < \delta \ll 1$ so as to satisfy all the forthcoming requirements of the proof. It is stressed that the subsequent argument will require δ to be chosen depending only on the admissible parameters n and $\deg P$; a careful examination of what follows shows such a choice of δ is always possible.

Observe that the set Ω_{n+1} is of dimension $\lfloor 3(n+1)/2 \rfloor$; one requires a domain of either dimension n or $n+1$ to effectively parametrise either the set E or F . Two methods can be applied to remedy this excess of variables. The first is to simply consider the tower defined only to a lower level; that is, only work with $\{\Omega_j\}_{j=1}^N$ for some $N \leq n+1$. The second method is to consider the whole tower $\{\Omega_j\}_{j=1}^{n+1}$ and freeze a number of the variables t_j . What is essentially meant by this is that for some set of indices $\mathcal{I} \subset \{1, \dots, n+1\}$ and choice of $(s_i)_{i \in \mathcal{I}} \subset \mathbb{R}^{\#\mathcal{I}}$ each set Ω_j is replaced with

$$\{(r, t) \in \Omega_j : t_i = s_i \text{ for all } i \in \mathcal{I} \cap \{1, \dots, j\}\}.$$

In order to optimise the subsequent Jacobian estimates, both methods are combined below and the variables to be frozen are chosen according to the ‘‘colour’’ of their indices. In particular only t_i for i a blue index will be frozen. In light of this discussion define the function $\zeta : \{1, \dots, n+1\} \rightarrow \{1, 2\}$ as follows:

$$\zeta(j) := \begin{cases} 2 & \text{if } j \text{ is red} \\ 1 & \text{otherwise} \end{cases} \quad \text{for } j = 1, \dots, n+1.$$

One can think of $\zeta(j)$ as the number of variables contributed by the fibres of the j th floor of the tower after the blue variables have been frozen. Note that there exists a least $1 < N \leq n+1$

such that

$$Z(N) := \sum_{j=1}^N \zeta(j) \in \{n, n+1\}; \tag{2.7.1}$$

in particular, the number of variables contributed by the first N floors corresponds to the dimension of either E or F . At this point the proof splits into a number of different cases depending on the parity of N and the value of $Z(N)$.

Definition 48. For $j \in \{n, n+1\}$, the notation $\text{Case}(1, j)$ (respectively, $\text{Case}(2, j)$) refers to the case where N is odd (respectively, even) and $Z(N) = j$.

Ostensibly there are four distinct cases; however, the minimality of N precludes $\text{Case}(1, n+1)$ and so it suffices to consider the remaining three cases.

The preceding observations lead to the definition of a suitable parameter domain Ω and mapping Φ which will form the focus of study for the remainder of the chapter. By the nature of the definitions of Ω and Φ , it will be convenient in some situations to introduce a relabelling of the relevant indices to replace the established “red, blue, odd” system. All these definitions depend on which $\text{Case}(i, j)$ happens to hold.

Case(1, n) and Case(2, n + 1):

The simplest situation corresponds to when either $\text{Case}(1, n)$ or $\text{Case}(2, n+1)$ holds. In both instances one defines $\Omega := \Omega_N$ and $\Phi := \Phi_N$. In the former case $\Phi : \Omega \rightarrow E$ whilst in the latter $\Phi : \Omega \rightarrow F$ by the properties 2. i) and 3. i) of Lemma 42, respectively. Here no relabelling is required: each even index $1 \leq j \leq N$ retains its original colour whilst the odd indices remain achromatic (that is, they have no colour assigned to them).

Case(2, n):

This situation is slightly more complicated. Fix $t_0 \in \Omega_1$ and consider the family of sets $\{\Omega_j^*\}_{j=1}^N$ defined by

$$\Omega_j^* := \{(t_1, \dots, t_j) : (t_0, t_1, \dots, t_j) \in \Omega_{j+1}\} \quad \text{for } j = 1, \dots, N.$$

It is easy to see $\{\Omega_j^*\}_{j=1}^N$ constitutes a type 2 tower. Let $y_0 := x_0 - r_0 P(t_0)$ and let $\{\Phi_j^*\}_{j=1}^N$ denote the family of mappings associated to $\{\Omega_j^*\}_{j=1}^N$ and the point y_0 , as defined in Definition 41. Define $\Omega := \Omega_N^*$ and $\Phi := \Phi_N^*$ and observe, by property 2. i) of Lemma 42, that $\Phi : \Omega \rightarrow E$. In this case the colouring system of the indices is redefined. In particular,

- i) The index 1 is designated red;
- ii) The odd indices $1 < j \leq N$ are designated red (respectively, blue) if $j+1$ was red (respectively, blue) in the previous scheme;
- iii) The even indices are re-designated achromatic.

2.8 Freezing variables and families of parametrisations

Rather than use a single function to parametrise E or F , here the slicing method of Christ [Chra, Chr98] is used to construct a family of maps G_σ . To begin some notation is introduced.

Let M denote the number of non-blue indices in $\{1, \dots, N\}$ and label these indices $l_1 < l_2 < \dots < l_M$. Similarly, let e (respectively, d) denote the number of red (respectively, blue) indices

so that $M = N - d$. For the reader's convenience the following table indicates the relationship between these parameters in the various cases.

	N	n
Case(1, n)	$2e + 2d + 1$	$3e + 2d + 1$
Case(2, $n + 1$)	$2e + 2d$	$3e + 2d - 1$
Case(2, n)	$2e + 2d$	$3e + 2d$

(2.8.1)

These computations follow immediately from the definition of N in (2.7.1).

Now let $1 \leq \mu_1 < \dots < \mu_e \leq M$ be such that $\{l_{\mu_i}\}_{i=1}^e$ is precisely the set of red indices. Rather than the blue indices themselves, it will be useful to enumerate those indices which lie directly before a blue index. Irrespective of which case happens to hold, any blue index is at least 2 and so there are precisely d indices lying directly before a blue index. In particular, let $1 \leq \nu_1 < \dots < \nu_d \leq M$ be such that $\{l_{\nu_j} + 1\}_{j=1}^d$ are precisely the blue indices.

Define functions τ and σ on Ω by

$$\begin{aligned} \tau &= (\tau_1, \dots, \tau_M) &:= (t_{l_1}, \dots, t_{l_M}) \\ s &= (s_{\nu_1}, \dots, s_{\nu_d}) &:= (t_{l_{\nu_j}+1} - t_{l_{\nu_j}})_{j=1}^d \\ \sigma &= (\sigma_{\nu_1}, \dots, \sigma_{\nu_d}) &:= (s_{\nu_j} \tau_{\nu_j}^{2K/n(n+1)})_{j=1}^d. \end{aligned}$$

Finally let $\rho = (\rho_{\mu_1}, \dots, \rho_{\mu_e}, \rho_{\nu_1}, \dots, \rho_{\nu_d})$ where ρ_{μ_i} (respectively, ρ_{ν_j}) is the dilation variable arising from the fibres of floor l_{μ_i} (respectively, $l_{\nu_j} + 1$) of the tower. More precisely, $\rho_{\mu_j} := r_{\lfloor l_{\mu_j}/2 \rfloor}$ for $1 \leq i \leq e$ whilst $\rho_{\nu_j} := r_{\lfloor (l_{\nu_j}+1)/2 \rfloor}$ for $1 \leq j \leq d$.

Observe that the map

$$\varphi : (r, t) \mapsto (\rho, \tau, \sigma) \tag{2.8.2}$$

is a valid change of variables with Jacobian determinant satisfying

$$\left| \det \frac{\partial \varphi}{\partial (r, t)} \right| = \prod_{j=1}^d \tau_{\nu_j}^{2K/n(n+1)}.$$

For $\sigma \in \mathbb{R}^d$ define the parameter set $\omega(\sigma) := \{(\rho, \tau) : \varphi^{-1}(\rho, \tau, \sigma) \in \Omega\}$ and let

$$W := \{\sigma \in \mathbb{R}^d : \omega(\sigma) \neq \emptyset\} \subseteq [0, \delta(\alpha\beta)^{1/2}]^d \tag{2.8.3}$$

where the inclusion follows from properties of the blue indices. Finally, consider the mapping G_σ on $\omega(\sigma)$ by

$$G_\sigma(\rho, \tau) := \Phi \circ \varphi^{-1}(\rho, \tau, \sigma).$$

By (2.8.1), it follows that in Case(1, n) and Case(2, n) the maps G_σ are functions of n variables and take values in E whilst in Case(2, $n + 1$) the G_σ are functions of $n + 1$ variables and take values in F . Hence in each case the maps G_σ have the desirable property that the domain and co-domain are of equal dimension.

Furthermore, the polynomial nature of maps G_σ imply each has bounded multiplicity. Indeed, as a consequence of Lemma 21, if J_σ denotes the Jacobian of G_σ , then in Case(1, n) and

Case(2, n) one concludes that the estimate

$$|E| \gtrsim \int_{\omega(\sigma)} |J_\sigma(\rho, \tau)| \, d\rho d\tau$$

holds for all $\sigma \in W$. In Case(2, $n + 1$) there is a similar estimate but with $|E|$ replaced with $|F|$ on the left-hand side of the above expression. Thus, in order to establish either (2.5.1) or (2.5.2) in the present cases it suffices to prove a suitable estimate for the Jacobian $|J_\sigma(\rho, \tau)|$ on the set $\omega(\sigma)$.

2.9 Reduction to Jacobian estimates

The key step in the proof is to estimate the Jacobian determinant J_σ of the map G_σ . It is convenient to introduce the notation

$$\eta := \begin{cases} 0 & \text{if either Case(1, } n) \text{ or Case(2, } n + 1) \text{ holds} \\ 1 & \text{if Case(2, } n) \text{ holds} \end{cases}.$$

Lemma 49. *Let $\sigma \in W$, where W is as defined in (2.8.3). Then*

$$|J_\sigma(\rho, \tau)| \gtrsim \alpha^{n(n+1)/2-M} (\beta/\alpha)^{(d+e-\eta)/2} \prod_{l=1}^M \tau_l^{2K/n(n+1)}$$

for all $(\rho, \tau) \in \omega(\sigma)$.

Theorem 34 is a direct consequence of Lemma 49.

Proof (of Theorem 34, assuming Lemma 49). To prove Theorem 34 it suffices to show the estimate (2.5.1) holds in both Case(1, n) and Case(2, n) and (2.5.2) holds in Case(2, $n + 1$). Indeed, recall both (2.5.1) and (2.5.2) are equivalent to the desired endpoint restricted weak-type (p_1, q_1) inequality (2.3.2) for A . For notational convenience, let

$$|X| := \begin{cases} |E| & \text{if either Case(1, } n) \text{ or Case(2, } n) \text{ holds} \\ |F| & \text{if Case(2, } n + 1) \text{ holds} \end{cases}.$$

Apply Lemma 49 to each $\sigma \in W$ together with the Multiplicity Lemma to deduce in all cases

$$\begin{aligned} |X| &\gtrsim \int_{\omega(\sigma)} |J_\sigma(\rho, \tau)| \, d\rho d\tau \\ &\gtrsim \alpha^{n(n+1)/2-M} (\beta/\alpha)^{(d+e-\eta)/2} \int_{\omega(\sigma)} \prod_{k=1}^M \tau_k^{2K/n(n+1)} \, d\rho d\tau. \end{aligned}$$

Integrating both sides of the preceding inequality over W , it follows that

$$(\alpha\beta)^{d/2} |X| \gtrsim \alpha^{n(n+1)/2-M} (\beta/\alpha)^{(d+e-\eta)/2} \int_{\varphi(\Omega)} \prod_{k=1}^M \tau_k^{2K/n(n+1)} \, d\rho d\tau d\sigma \quad (2.9.1)$$

where φ is the map defined in (2.8.2). By a change of variables, the integral on the right-hand

side of (2.9.1) can be written as

$$\begin{aligned} \int_{\varphi(\Omega)} \prod_{k=1}^M \tau_k^{2K/n(n+1)} d\rho d\tau d\sigma &= \int_{\Omega} \prod_{k=1}^M t_{l_k}^{2K/n(n+1)} \left| \det \frac{\partial \varphi}{\partial(r, t)}(r, t) \right| dr dt \\ &= \int_{\Omega} \prod_{k=1}^M t_{l_k}^{2K/n(n+1)} \prod_{j=1}^d t_{l_{\nu_j}}^{2K/n(n+1)} dr dt. \end{aligned}$$

Arguing as in the last step of the proof of Lemma 44, one may deduce $t_{l_{\nu_j}} \sim t_{l_{\nu_j+1}}$ for $1 \leq j \leq d$ provided that the parameter δ from Lemma 44 is chosen to be sufficiently small (depending only on $\deg P$ and n). The previous expression is therefore bounded below by a constant multiple of

$$\int_{\Omega} \prod_{k=1}^N t_k^{2K/n(n+1)} dr dt.$$

Applying Fubini's theorem and the estimates for the μ_P -measure of the fibres of the Ω_j , one may easily deduce the above integral is at least a constant multiple of $\alpha^N (\beta/\alpha)^{\lfloor N/2 \rfloor}$. Thence, combining these observations and multiplying both sides of (2.9.1) by $\alpha^{-d} (\beta/\alpha)^{-d/2}$, one arrives at the estimate

$$|X| \gtrsim \alpha^{n(n+1)/2 - M + N - d} (\beta/\alpha)^{(e-\eta)/2 + \lfloor N/2 \rfloor}.$$

Recalling $M = N - d$ and (2.8.1), this is easily seen to be the desired estimate. \square

To complete the proof of Theorem 37 it remains to prove Lemma 49.

2.10 The proof of the Jacobian estimates: Case(1, n)

Here the proof of Lemma 49 in Case(1, n) is discussed in detail. The same arguments can be adapted to treat the remaining cases, as demonstrated in the following section.

Proof (of Lemma 49, assuming Case(1, n) holds). The arguments here, which are based primarily on those of [Chr98, Sto10], are somewhat lengthy; it is convenient, therefore, to present the proof as a series of steps.

Compute the Jacobian matrix.

Recalling the definition of the mapping $\Phi := \Phi_N$, one may use the established index notation to express $\Phi \circ \varphi^{-1}$ as

$$\begin{aligned} \Phi \circ \varphi^{-1}(\rho, \tau, \sigma) &= x_0 - r_0 P(\tau_1) + \sum_{i=1}^e \rho_{\mu_i} (P(\tau_{\mu_i}) - P(\tau_{\mu_i+1})) \\ &\quad + \sum_{j=1}^d \rho_{\nu_j} (P(\tau_{\nu_j} + s_{\nu_j}(\tau, \sigma)) - P(\tau_{\nu_j+1})). \end{aligned}$$

One immediately deduces that

$$\frac{\partial G_{\sigma}}{\partial \rho_{\mu_i}}(\rho, \tau) = P(\tau_{\mu_i}) - P(\tau_{\mu_i+1}) \quad \text{for } i = 1, \dots, e$$

whilst

$$\frac{\partial G_\sigma}{\partial \rho_{\nu_j}}(\rho, \tau) = P(\tau_{\nu_j} + s_{\nu_j}(\tau, \sigma)) - P(\tau_{\nu_{j+1}}) \quad \text{for } j = 1, \dots, d$$

which identifies $d+e$ of the columns of the Jacobian matrix. The remaining columns correspond to differentiation with respect to the τ variables and are readily computed by expressing $\Phi \circ \varphi^{-1}$ as

$$\begin{aligned} \Phi \circ \varphi^{-1}(\rho, \tau, \sigma) &= x_0 + \sum_{i=1}^e (\rho_{\mu_i} P(\tau_{\mu_i}) - \rho_{\mu_{i-1}}^* P(\tau_{\mu_{i-1}})) \\ &\quad + \sum_{j=1}^d (\rho_{\nu_j} P(\tau_{\nu_j} + s_{\nu_j}(\tau, \sigma)) - \rho_{\nu_{j-1}}^* P(\tau_{\nu_j})) - \rho_M^* P(\tau_M) \end{aligned}$$

where the $\rho_{\mu_{i-1}}^*$, $\rho_{\nu_{j-1}}^*$ and ρ_M^* are defined in the obvious manner; for instance, if ρ_{ν_j} corresponds to the parameter r_k via the change of variables φ , then $\rho_{\nu_{j-1}}^*$ is understood to correspond to the parameter r_{k-1} . Thus for $i = 1, \dots, e$ one has

$$\frac{\partial G_\sigma}{\partial \tau_{\mu_i}}(\rho, \tau) = \rho_{\mu_i} P'(\tau_{\mu_i}) \quad \text{and} \quad \frac{\partial G_\sigma}{\partial \tau_{\mu_{i-1}}}(\rho, \tau) = -\rho_{\mu_{i-1}}^* P'(\tau_{\mu_{i-1}})$$

whilst

$$\frac{\partial G_\sigma}{\partial \tau_M}(\rho, \tau) = -\rho_M^* P'(\tau_M),$$

accounting for a further $2e + 1$ columns to the Jacobian matrix. To compute the remaining d columns differentiate G_σ with respect to the τ_{ν_j} to give

$$\begin{aligned} \frac{\partial G_\sigma}{\partial \tau_{\nu_j}}(\rho, \tau) &= \rho_{\nu_j} P'(\tau_{\nu_j} + s_{\nu_j}(\tau, \sigma)) - \rho_{\nu_{j-1}}^* P'(\tau_{\nu_j}) \\ &\quad - \frac{2K}{n(n+1)} \frac{s_{\nu_j}(\tau, \sigma)}{\tau_{\nu_j}} \rho_{\nu_j} P'(\tau_{\nu_j} + s_{\nu_j}(\tau, \sigma)) \end{aligned} \quad (2.10.1)$$

for $j = 1, \dots, d$.

Compare J_σ with J_P .

The estimation of the Jacobian J_σ will be achieved by comparing it to the more tractable expression J_P , introduced in (2.4.1). Once such a comparison is established, J_σ can then be bounded by means of the geometric inequality of Dendrinos and Wright (that is, Theorem 38). This inequality is guaranteed to hold in the appropriate setting due to the reductions made earlier in the chapter.

To begin, express the Jacobian determinant in the form of an integral

$$J_\sigma(\rho, \tau) = \pm \int_{R(\tau) \times B_\sigma(\tau)} \wp_\sigma(\rho, \tau, x) dx$$

where $\wp_\sigma(\rho, \tau, x)$ is a multi-variate polynomial and

$$R(\tau) := \prod_{i=1}^e (\tau_{\mu_i}, \tau_{\mu_{i+1}}) \quad \text{and} \quad B_\sigma(\tau) := \prod_{i=1}^d (\tau_{\nu_j} + s_{\nu_j}, \tau_{\nu_{j+1}})$$

are rectangles.⁸ The polynomial $\wp_\sigma(\rho, \tau, x)$ is the product of $C(\rho) = \rho_M^* \prod_{i=1}^e \rho_{\mu_i-1}^* \rho_{\mu_i}$ and the determinant of the matrix $A_\sigma(\rho, \tau, x)$ obtained from original Jacobian matrix by making the following changes:

- The column $P(\tau_{\mu_i}) - P(\tau_{\mu_i+1})$ is replaced with $P'(x_i)$ for $i = 1, \dots, e$.
- The column $P(\tau_{\nu_j} + s_{\nu_j}) - P(\tau_{\nu_j+1})$ is replaced with $P'(x_{m+j})$ for $j = 1, \dots, d$.
- The columns $\rho_{\mu_i} P'(\tau_{\mu_i})$ and $-\rho_{\mu_i-1}^* P'(\tau_{\mu_i-1})$ are replaced with $P'(\tau_{\mu_i})$ and $P'(\tau_{\mu_i-1})$, respectively, for all $i = 1, \dots, e$. In addition, $-\rho_M^* P'(\tau_M)$ is replaced with and $P'(\tau_M)$.
- The remaining columns d are unaltered; in other words, they agree with the corresponding columns of the Jacobian matrix.

Notice the unaltered columns are those corresponding to differentiation by τ_{ν_j} and are of the form given in (2.10.1). Each may be expressed as the sum of three terms

$$\frac{\partial G_\sigma}{\partial \tau_{\nu_j}}(\rho, \tau) = \sum_{i=1}^3 T_{\sigma,j}^i(\rho, \tau) \quad (2.10.2)$$

where, writing $c := -2K/n(n+1)$,

$$\begin{aligned} T_{\sigma,j}^1(\rho, \tau) &:= (\rho_{\nu_j} - \rho_{\nu_j-1}^*) P'(\tau_{\nu_j}), \\ T_{\sigma,j}^2(\rho, \tau) &:= c \frac{s_{\nu_j}(\tau, \sigma)}{\tau_{\nu_j}} \rho_{\nu_j} P'(\tau_{\nu_j} + s_{\nu_j}(\tau, \sigma)), \\ T_{\sigma,j}^3(\rho, \tau) &:= \rho_{\nu_j} (P'(\tau_{\nu_j} + s_{\nu_j}(\tau, \sigma)) - P'(\tau_{\nu_j})). \end{aligned}$$

The multi-linearity of the determinant and (2.10.2) are now applied to express $\det A_\sigma(\rho, \tau, x)$ as a sum of determinants of more elementary matrices. In order to present concisely the resulting expression it is useful to introduce some notation. In particular, for $S \subseteq \mathcal{N} := \{\nu_1, \dots, \nu_d\}$, let Δ_S denote the function of ρ given by

$$\Delta_S(\rho) := \prod_{\nu \in S} (\rho_\nu - \rho_{\nu-1}^*)$$

and $R_{\sigma,S}(\tau) \subset \mathbb{R}^{\#S}$ the rectangle

$$R_{\sigma,S}(\tau) := \prod_{\nu \in S} (\tau_\nu, \tau_\nu + s_\nu).$$

With this notation $\det A_\sigma(\rho, \tau, x)$ equals

$$\sum_S \Delta_{S_1}(\rho) \left(\prod_{\nu \in S_2} \frac{c \rho_\nu s_\nu}{\tau_\nu} \right) \left(\prod_{\nu \in S_3} \rho_\nu \right) \int_{R_{\sigma,S_3}(\tau)} \left(\prod_{\nu \in S_3} \frac{\partial}{\partial y_\nu} \right) J_P(\xi_S(y), x) dy, \quad (2.10.3)$$

at least up to a sign, where the sum ranges over all partitions $\mathcal{S} := (S_1, S_2, S_3)$ of \mathcal{N} and for any such partition $\xi_S(y) = (\xi_{S,l}(\tau, \sigma, y))_{l=1}^M$ is defined by

$$\xi_{S,l}(\tau, \sigma, y) := \begin{cases} \tau_l + s_l & \text{if } l \in S_2, \\ y_l & \text{if } l \in S_3, \\ \tau_l & \text{otherwise.} \end{cases}$$

⁸For notational convenience the dependence of s_ν on (τ, σ) has been suppressed.

If $S_3 = \emptyset$, then the integral appearing in (2.10.3) is interpreted as $J_P(\xi_S, x)$.

The term of the sum in (2.10.3) corresponding to the unique partition for which $S_1 = \mathcal{N}$ is simply $\Delta(\rho)J_P(\tau, x)$ where $\Delta(\rho) := \Delta_{\mathcal{N}}(\rho)$; the sum of the remaining terms is denoted by $E_\sigma(\rho, \tau, x)$. Thus, (2.10.3) equals

$$\Delta(\rho)J_P(\tau, x) + E_\sigma(\rho, \tau, x). \quad (2.10.4)$$

In conclusion, the Jacobian J_σ can be expressed in terms of (an integral of) the function J_P together with some error term.

Control the error.

It will be shown that provided that δ is chosen sufficiently small, depending only on n and $\deg P$, the right-hand summand of (2.10.4) is subordinate to the left-hand summand. Only a bounded number of terms of (2.10.3) are non-zero and the error is therefore a sum of $O(1)$ terms which will be estimated individually.

By the properties of the parameter tower, $s_\nu \lesssim \delta(\beta/\alpha)\tau_\nu$ for all $\nu \in \mathcal{N}$. Hence, for any $S_2 \subseteq \mathcal{N}$ one has

$$\prod_{\nu \in S_2} \frac{c\rho_\nu s_\nu}{\tau_\nu} \lesssim \delta^{\#S_2} (\beta/\alpha)^{\#S_2} \lesssim \delta^{\#S_2} \Delta_{S_2}(\rho) \quad (2.10.5)$$

where the final inequality is due to the definition of the blue indices. A suitable error bound would follow from a similar estimate for each of the integrals appearing in (2.10.3). In particular, fixing some partition $\mathcal{S} = (S_1, S_2, S_3)$ of \mathcal{N} , it suffices to prove

$$\int_{R_{\sigma, S_3}(\tau)} \left| \left(\prod_{\nu \in S_3} \frac{\partial}{\partial y_\nu} \right) J_P(\xi_S(y), x) \right| dy \lesssim \delta^{\#S_3} \Delta_{S_3}(\rho) |J_P(\tau, x)|. \quad (2.10.6)$$

Indeed, once (2.10.6) is established, the error bound

$$\begin{aligned} |E_\sigma(\rho, \tau, x)| &\lesssim \left(\sum_{\substack{\mathcal{S}=(S_1, S_2, S_3) \\ S_1 \neq \mathcal{N}}} \delta^{\#S_2 + \#S_3} \prod_{j=1}^3 \Delta_{S_j}(\rho) \right) |J_P(\tau, x)| \\ &\lesssim \delta \Delta(\rho) |J_P(\tau, x)| \end{aligned} \quad (2.10.7)$$

immediately follows, noting the factor $\prod_{\nu \in S_3} \rho_\nu$ from (2.10.3) is $O(1)$ whenever it appears in a non-zero term of the sum.

If $S_3 = \emptyset$, then (2.10.6) is trivial. Fix a partition \mathcal{S} as above with S_3 non-empty and some $y \in R_{\sigma, S_3}(\tau)$ and consider the ratio

$$\left| \frac{\left(\prod_{\nu \in S_3} \frac{\partial}{\partial y_\nu} \right) J_P(\xi_S(y), x)}{J_P(\xi_S(y), x)} \right|. \quad (2.10.8)$$

For notational convenience write $\xi = (\xi_1, \dots, \xi_d) := \xi_S(y)$. Using the derivative estimate from Proposition 39 one may bound (2.10.8) by a linear combination of $O(1)$ terms (with $O(1)$ coefficients) of the form

$$\left(\prod_{\nu \in T_1} y_\nu^{-1} \right) \left(\prod_{\nu \in T_2} y_\nu^{-\epsilon(\nu)} |y_\nu - \xi_{u(\nu)}|^{\epsilon(\nu)-1} \right) \left(\prod_{\nu \in T_3} y_\nu^{-\epsilon(\nu)} |y_\nu - x_{v(\nu)}|^{\epsilon(\nu)-1} \right) \quad (2.10.9)$$

where:

- (T_1, T_2, T_3) is a partition of S_3 ;
- $u: T_2 \rightarrow \{1, \dots, M\}$ is a function with the property $u(j) \neq j$ for all $j \in T_2$;
- $v: T_3 \rightarrow \{1, \dots, d + e\}$ (with no additional conditions) and
- $\epsilon: T_2 \cup T_3 \rightarrow \{0, 1\}$.

To prove (2.10.6) it therefore suffices to establish a suitable bound for the integral of the product of (2.10.9) and $|J_P(\xi, x)|$ over the set $R_{\sigma, S_3}(\tau)$. The first step is to estimate (2.10.9) by applying the following observations.

i) Given $y \in R_{\sigma, S_3}(\tau)$, by the definition of the parameter tower the estimates

$$\begin{aligned} y_\nu &\geq \tau_\nu = \tau_\nu^{1/\kappa} \tau_\nu^{-2K/n(n+1)} \gtrsim \alpha \tau_\nu^{-2K/n(n+1)}; \\ |y_\nu - \xi_{u(\nu)}| &\geq |\tau_\nu - \tau_{u(\nu)}| - \delta \alpha (\tau_\nu^{-2K/n(n+1)} + \tau_{u(\nu)}^{-2K/n(n+1)}); \\ |y_\nu - x_{v(\nu)}| &\geq |\tau_\nu - x_{v(\nu)}| - \delta \alpha \tau_\nu^{-2K/n(n+1)} \end{aligned}$$

hold for $\nu \in \mathcal{N}$.

ii) Since the indices l_ν for $\nu \in \mathcal{N}$ are those that directly precede a blue index (and so l_ν is odd), Corollary 43 ensures $\tau_\nu - \tau_u \gtrsim \alpha \tau_u^{-2K/n(n+1)}$ for all $1 \leq u < \nu$. Moreover, the ordering of the variables then guarantees

$$\tau_\nu - \tau_u \gtrsim \alpha \tau_\nu^{-2K/n(n+1)} \quad \text{whenever } 1 \leq u < \nu.$$

iii) On the other hand, since the labelling l_k omits the blue indices, for any $\nu \in \mathcal{N}$ and $\nu < u \leq M$ one must have $l - l_\nu \geq 2$ where l is the index such that $\tau_u = t_l$. Consequently, by applying Corollary 43 in this case one concludes that

$$\tau_u - \tau_\nu \gtrsim \alpha \tau_\nu^{-2K/n(n+1)} \quad \text{whenever } \nu < u \leq M.$$

Combining these observations one immediately deduces that

$$|y_\nu - \xi_{u(\nu)}| \gtrsim \alpha \tau_\nu^{-2K/n(n+1)}$$

for all $\nu \in \mathcal{N}$, provided that δ is chosen initially to be sufficiently small in the earlier application of Lemma 44.

It would be useful to have a similar bound for the terms $|y_\nu - x_l|$. At present such an estimate is not possible due to the potential lack of separation between the τ_ν and x_l variables. To remedy this, temporarily assume the addition separation hypothesis

$$|\tau_\nu - x_l| \gtrsim \alpha \tau_\nu^{-2K/n(n+1)} \tag{2.10.10}$$

for all $\nu \in \mathcal{N}$ and all $1 \leq l \leq d + e$. Presently it is shown that this separation hypothesis leads to desirable control over the error term $E_\sigma(\rho, \tau, x)$; the following step is then to modify the existing set-up so that (2.10.10) indeed holds without the need of additional assumptions.

The preceding discussion, together with the identity $|R_{\sigma,S}(\tau)| = \prod_{\nu \in S} s_\nu$, implies (2.10.9) is controlled by

$$\begin{aligned} \alpha^{-\#S_3} \prod_{\nu \in S_3} \tau_\nu^{2K/n(n+1)} &\lesssim \delta^{\#S_3} (\beta/\alpha)^{\#S_3/2} |R_{\sigma,S_3}(\tau)|^{-1} \\ &\lesssim \delta^{\#S_3} |\Delta_{S_3}(\rho)| |R_{\sigma,S_3}(\tau)|^{-1} \end{aligned}$$

provided that δ is chosen to be sufficiently small. Observe, both of the above inequalities are simple consequences of the definition of the blue indices. Consequently, the left-hand side of (2.10.6) may be bounded by

$$\delta^{\#S_3} \Delta_{S_3}(\rho) \frac{1}{|R_{\sigma,S_3}(\tau)|} \int_{R_{\sigma,S_3}(\tau)} |J_P(\xi_S, x)| \, dy$$

and so (2.10.6), and thence (2.10.7), would follow if

$$|J_P(\xi(y), x)| \sim |J_P(\tau, x)| \quad \text{for all } y \in R_{\sigma,S_3}(\tau).$$

This approximation is readily deduced by combining Proposition 39 with Grönwall's inequality (for a proof of Grönwall's inequality see, for instance, [Tao06a, Chapter 1]).

Hence, the estimate (2.10.7) is established under the assumption of the separation hypothesis (2.10.10).

Enforce separation.

In the previous section it was shown if (2.10.10) were to hold for each $\nu \in \mathcal{N}$ uniformly over all $x = (x_1, \dots, x_{d+e}) \in R_{\sigma,S_3}(\tau)$, then by choosing $0 < \delta \ll 1$ sufficiently small one may control the integrand by the easily-understood function $|\Delta(\rho)| |J_P(\tau, x)|$. Clearly for fixed i the estimate (2.10.10) cannot hold for at least one value of l , since as x varies over $R(\tau) \times B_\sigma(\tau)$ some x_l can stray close to τ_{ν_i} in the boundary regions. To remedy this problem one simply removes a suitable small portion of $R(\tau) \times B_\sigma(\tau)$ from the boundary, observing that this can be done without greatly diminishing the size of the integral to be estimated. Given $0 < \epsilon < 1/2$, $1 \leq i \leq e$ and $1 \leq j \leq d$, define the ϵ -truncate of $R_i(\tau) := (\tau_{\mu_i}, \tau_{\mu_i+1})$ and $B_{\sigma,j}(\tau) := (\tau_{\nu_j} + s_{\nu_j}, \tau_{\nu_j+1})$ by

$$R_i^\epsilon(\tau) := (\tau_{\mu_i} + \epsilon |R_i(\tau)|, \tau_{\mu_i+1} - \epsilon |R_i(\tau)|)$$

and

$$B_{\sigma,j}^\epsilon(\tau) := (\tau_{\nu_j} + s_{\nu_j} + \epsilon |B_{\sigma,j}(\tau)|, \tau_{\nu_j+1} - \epsilon |B_{\sigma,j}(\tau)|),$$

respectively. Moreover, define the ϵ -truncates of the associated rectangles to be $R^\epsilon(\tau) := \prod_{i=1}^e R_i^\epsilon(\tau)$ and $B_\sigma^\epsilon(\tau) := \prod_{j=1}^d B_{\sigma,j}^\epsilon(\tau)$. Lemma 50 below establishes the existence of some constant $0 < c_0 < 1/2$, depending only on n and $\deg P$, such that

$$|J_\sigma(\rho, \tau)| \geq \left| \int_{D(\tau)} \wp_\sigma(\rho, \tau, x) \, dx \right| - \frac{1}{2} \int_{D(\tau)} |\wp_\sigma(\rho, \tau, x)| \, dx. \quad (2.10.11)$$

where $D(\tau) := R^{c_0}(\tau) \times B_\sigma^{c_0}(\tau)$.

It is easy to show that for all $x \in D(\tau)$ the condition (2.10.10) holds with a uniform constant.

Observe

$$|B_{\sigma,j}(\tau)| = \tau_{\nu_j+1} - (\tau_{\nu_j} + s_{\nu_j}) = t_{l_{(\nu_j+1)}} - t_{l_{\nu_j+1}},$$

where the brackets in the subscript are included to aid the clarity of exposition. Since $l_{\nu_j} + 1$ is, by definition, a blue index it follows that $l_{(\nu_j+1)}$ is odd and, consequently,

$$|B_{\sigma,j}(\tau)| \gtrsim \alpha t_{l_{(\nu_j+1)}}^{-2K/n(n+1)} = \alpha (\tau_{\nu_j} + s_{\nu_j})^{-2K/n(n+1)}$$

by Corollary 43 part i). Furthermore, recalling $s_{\nu_j} \lesssim \delta(\beta/\alpha)\tau_{\nu_j} \lesssim \tau_{\nu_j}$, it follows that

$$|B_{\sigma,j}(\tau)| \gtrsim \alpha \tau_{\nu_j}^{-2K/n(n+1)}. \quad (2.10.12)$$

Now suppose $x_l \in B_{\sigma,j_0}^{co}(\tau)$ for some fixed $j_0 \in \{1, \dots, d\}$. It is clear from the definition of the parameter domain that if $j \neq j_0$, then (2.10.10) holds for $\nu = \nu_j$. Similarly, if $x_l \in R_{i_0}^{co}(\tau)$ for some fixed $i_0 \in \{1, \dots, e\}$, then (2.10.10) holds for all $\nu \in \mathcal{N}$. It remains to verify (2.10.10) when $x_l \in B_{\sigma,j_0}^{co}(\tau)$ and $j = j_0$, but this is immediate from the definition of the truncation and the bound (2.10.12).

Consequently, for $x \in D(\tau)$ and δ sufficiently small (2.10.7) holds and thus the estimate

$$|\wp_{\sigma}(\rho, \tau, x)| \gtrsim |\Delta(\rho)| |J_P(\tau, x)| \quad (2.10.13)$$

is valid on $D(\tau)$. Furthermore, it is claimed that as x varies over $D(\tau)$ the sign of $\wp_{\sigma}(\rho, \tau, x)$ is unchanged. Once this observation is established the right-hand side of (2.10.11) can be written as

$$\frac{1}{2} \int_{D(\tau)} |\wp_{\sigma}(\rho, \tau, x)| dx \gtrsim |\Delta(\rho)| \int_{D(\tau)} |J_P(\tau, x)| dx$$

To prove the claim, note that the ordering of the components of the $(r, t) \in \Omega$ implies the sign of $V(\tau, x)$ is fixed as x varies over $D(\tau)$; the geometric inequality guaranteed by Theorem 38 therefore ensures that the sign of $J_P(\tau, x)$ is also fixed (and is non-zero). The estimate (2.10.13) now implies the claim.

Bound J_P and apply the properties of Ω .

Combining the estimate guaranteed by Theorem 38 and the preceding observations one deduces that

$$|J_{\sigma}(\rho, \tau)| \gtrsim |\Delta(\rho)| \prod_{l=1}^M |L_P(\tau_l)|^{1/n} \int_{D(\tau)} \prod_{k=1}^{d+e} |L_P(x_k)|^{1/n} |V(\tau, x)| dx.$$

Over the domain of integration the estimate

$$|V(\tau, x)| \gtrsim \alpha^{n(n-1)/2 - M(M-1)/2} |V(\tau)| \prod_{l=1}^M \tau_l^{-K(n-M)/n(n+1)} \prod_{k=1}^{d+e} x_k^{-K(n-1)/n(n+1)}$$

is valid owing to both (2.10.10) and the additional separation enforced by truncating the set $R(\tau)$. Furthermore, the construction of the (type 1) parameter tower ensures

$$|V(\tau)| \gtrsim \alpha^{M(M-1)/2} (\beta/\alpha)^{e/2} \prod_{l=1}^M \tau_l^{-K(M-1)/n(n+1)}. \quad (2.10.14)$$

Since the properties of the blue intervals imply $|\Delta(\rho)| \gtrsim (\beta/\alpha)^{d/2}$, one may combine the preceding inequalities to deduce

$$|J_\sigma(\rho, \tau)| \gtrsim \alpha^{n(n-1)/2} (\beta/\alpha)^{(d+e)/2} \left(\int_{D(\tau)} \prod_{k=1}^{n-M} x_k^{2K/n(n+1)} dx \right) \prod_{l=1}^M \tau_l^{2K/n(n+1)}. \quad (2.10.15)$$

Here the approximation $L_P(t) \sim t^K$ has been applied, which was a consequence of the decomposition theorem.

Finally, the integral on the right-hand side of the above expression is easily seen to satisfy

$$\int_{D(\tau)} \prod_{k=1}^{n-M} x_k^{2K/n(n+1)} dx \gtrsim \alpha^{n-M},$$

concluding the proof. □

It remains to state and prove the lemma which justifies the estimate (2.10.11). In general, for $0 < \epsilon < 1/2$ the ϵ -truncation I^ϵ of a finite open interval $I = (a, b)$ is defined as $I^\epsilon := (a + \epsilon(b-a), b - \epsilon(b-a))$. If I_1, \dots, I_K is a family of finite open intervals, the ϵ -truncation R^ϵ of the associated rectangle $R := \prod_{j=1}^K I_j$ is defined simply by $R^\epsilon := \prod_{j=1}^K I_j^\epsilon$.

Lemma 50. *Given any $M, K \in \mathbb{N}$ there exists a constant $0 < c_{M,K} < 1/2$ with the following property. For all $0 < \epsilon < c_{M,K}$ there exists $C_{M,K}(\epsilon) > 0$ such that for any collection I_1, \dots, I_K of finite open intervals with associated rectangle R one has*

$$\int_{R \setminus R^\epsilon} |p(x)| dx \leq C_{M,K}(\epsilon) \int_{R^\epsilon} |p(x)| dx$$

whenever p is a polynomial of degree at most M in $x = (x_1, \dots, x_K)$. Moreover, $\lim_{\epsilon \rightarrow 0} C_{M,K}(\epsilon) = 0$ for any fixed M, K .

Once the lemma is established, taking d, e and M to be as defined in the previous proof and $K := d + e$, the inequality (2.10.11) (at least in Case(1, n)) follows by choosing c_0 sufficiently small so that $0 < C_{M,K}(c_0) < 1/2$.

Proof (of Lemma 50). By homogeneity it suffices to consider the case $I_1 = \dots = I_K = (0, 1)$ and a simple inductive procedure further reduces the problem to the case $K = 1$. Fixing M and letting $I = (0, 1)$, the proof is now a simple consequence of the equivalence of norms on finite-dimensional spaces: if $C_M < \infty$ is defined to be the supremum of the ratio $\|p\|_{L^\infty(I)}/\|p\|_{L^1(I)}$ over all polynomials of degree at most M , then

$$\int_{I \setminus I^\epsilon} |p(x)| dx \leq 2\epsilon C_M \left(\int_{I \setminus I^\epsilon} |p(x)| dx + \int_{I^\epsilon} |p(x)| dx \right).$$

Provided that $0 < \epsilon < C_M/2$ one may take $C_{M,1}(\epsilon) := 2\epsilon C_M/(1 - 2\epsilon C_M)$, completing the proof. □

2.11 The proof of the Jacobian estimates: Case(2, $n + 1$) and Case(2, n)

The argument used to prove Lemma 49 in Case(1, n) can easily be adapted to establish the result in the remaining cases. The necessary modifications are sketched below; the precise details are left to the patient reader.

Adapting the arguments to Case(2, $n + 1$).

To prove the inequality in Case(2, $n + 1$) only a minor modification of the preceding argument is needed. Notice by the minimality of the parameter N defined in (2.7.1) it follows that the index N is red and so $\mu_e = M$. Here $\Phi \circ \varphi^{-1}$ maps into $\mathbb{R}^n \times [1, 2]$ and is given by

$$\Phi \circ \varphi^{-1}(\rho, \tau, \sigma) = \begin{pmatrix} \Psi_N(x_0, r_0; \varphi^{-1}(\rho, \tau, \sigma)) \\ \rho_{\mu_m} \end{pmatrix}$$

where

$$\begin{aligned} \Psi_N(x_0, r_0; \varphi^{-1}(\rho, \tau, \sigma)) &= x_0 - r_0 P(\tau_1) + \sum_{i=1}^{e-1} \rho_{\mu_i} (P(\tau_{\mu_i}) - P(\tau_{\mu_i+1})) \\ &\quad + \sum_{j=1}^d \rho_{\nu_j} (P(\tau_{\nu_j} + s_{\nu_j}) - P(\tau_{\nu_j+1})) + \rho_{\mu_e} P(\tau_{\mu_e}). \end{aligned}$$

The Jacobian matrix is now an $(n+1) \times (n+1)$ matrix. The columns given by differentiating G_σ with respect to ρ_{μ_i} are

$$\begin{pmatrix} P(\tau_{\mu_i}) - P(\tau_{\mu_i+1}) \\ 0 \end{pmatrix} \text{ for } j = 1, \dots, e-1 \text{ and } \begin{pmatrix} P(\tau_{\mu_e}) \\ 1 \end{pmatrix}.$$

For remaining columns, the first n components are precisely the components of the corresponding columns in the previous case and the $n+1$ component is 0. Expanding the determinant across row $(n+1)$, the methods used earlier in the proof can be applied to deduce

$$J_\sigma(\rho, \tau) = \pm \int_{R(\tau) \times B_\sigma(\tau)} \wp_\sigma(\rho, \tau, x) dx$$

where $\wp_\sigma(\rho, \tau, x)$ is the determinant of a $n \times n$ matrix and

$$R(\tau) := \prod_{i=1}^{e-1} (\tau_{\mu_i}, \tau_{\mu_i+1}); \quad B_\sigma(\tau) := \prod_{j=1}^d (\tau_{\nu_j} + s_{\nu_j}, \tau_{\nu_j+1}).$$

The key difference is now the integral is over a rectangle of dimension $d + e - 1$ (rather than $d + e$). Define the truncated domain $D(\tau)$ in analogous manner to the previous case. Notice from (2.8.1) it follows that $n - M = d + e - 1$, which is precisely the dimension of the set $D(\tau)$ in the present situation. Arguing as before, the inequality (2.10.15) also holds in this setting and from this one obtains the required estimate.

Adapting the argument to Case(2, n).

Here the map $\Phi \circ \varphi^{-1}$ is given by

$$\begin{aligned} \Phi \circ \varphi^{-1}(\rho, \tau, \sigma) &= y_0 + \sum_{i=1}^e \rho_{\mu_i} (P(\tau_{\mu_i}) - P(\tau_{\mu_i+1})) \\ &\quad + \sum_{j=1}^d \rho_{\nu_j} (P(\tau_{\nu_j} + s_{\nu_j}(\tau, \sigma)) - P(\tau_{\nu_j+1})) \end{aligned}$$

and thus the columns of the Jacobian matrix essentially agree with those of Case(1, n), with the exception that now there is no column corresponding to $\partial G_\sigma / \partial \tau_{\mu_1-1}$.

The above arguments now carry through almost verbatim; the only substantial difference in this situation is that the Vandermonde estimate (2.10.14) becomes

$$|V(\tau)| \gtrsim \alpha^{M(M-1)/2} (\beta/\alpha)^{(e-1)/2}$$

due to the fact that the parameter tower in this situation is of type 2, as opposed to type 1 in both of the previous cases.

Remark 51. At the beginning of the chapter the possibility of strengthening the restricted weak-type (p_1, q_1) estimate from Proposition 37 to a strong-type estimate was discussed. It was remarked that the strong-type estimate in dimension $n = 2$ follows from a result of Gressman [Gre13], but can also be established by combining the analysis contained within the present chapter with an extrapolation method due to Christ [Chrd] (see also [Sto09]). Here some further details are sketched. The key ingredients in Christ’s extrapolation technique are certain ‘trilinear’ variants of the estimates (2.5.1) and (2.5.2). Recall, to prove the weak-type bound it sufficed to show *either* (2.5.1) *or* (2.5.2) holds since both these estimates are equivalent. This equivalence breaks down when one passes to the trilinear setting and to establish the strong-type inequality one must prove *both* the trilinear version of (2.5.1) *and* the trilinear version of (2.5.2) hold. This can be achieved in the $n = 2$ case by introducing an ‘inflation’ argument (see [Chra] and also [Gre06]). One may attempt to apply the same techniques in higher dimensions but now the Jacobian arising from the inflation is rather complicated. The question of whether or not this Jacobian can be effectively estimated remains unresolved. It is possible that the inflation argument is not required when n belongs to a certain congruence class modulo 3 and potentially the strong-type bound could be established more directly from existing arguments in this situation.

2.12 The method of Dendrinos and Stovall

To conclude the chapter the details the construction of the sequence of sets $\{U_k\}_{k=1}^\infty$ featured in Lemma 44 are presented. The argument here is due to Dendrinos and Stovall [DS15]. At this point some preliminary definitions and remarks are pertinent. Observe

$$\langle A\chi_E, \chi_F \rangle = \int_{\Sigma} \chi_F(\pi_1(x, r, t)) \chi_E(\pi_2(x, r, t)) \lambda_P(t) dx dr dt$$

where $\Sigma := \mathbb{R}^n \times [1, 2] \times I$ and $\pi_1: \Sigma \rightarrow \mathbb{R}^n \times [1, 2]$ and $\pi_2: \Sigma \rightarrow \mathbb{R}^n$ are the mappings

$$\pi_1(x, r, t) := (x, r), \quad \pi_2(x, r, t) := x - rP(t).$$

Define the π_j -fibres to be the sets $\pi_j^{-1} \circ \pi_j(x, r, t)$ for $(x, r, t) \in \Sigma$ and $j = 1, 2$. Thus, the π_1 -fibres form a partition of Σ into a continuum of curves (which are simply parallel lines) whilst the π_2 -fibres partition Σ into a continuum of 2-surfaces. Writing

$$U := \pi_1^{-1}(F) \cap \pi_2^{-1}(E) = \{(x, r, t) \in \Sigma : \pi_1(x, r, t) \in F \text{ and } \pi_2(x, r, t) \in E\}$$

it follows that

$$\langle A\chi_E, \chi_F \rangle = \int_{\Sigma} \chi_U(x, r, t) \lambda_P(t) dx dr dt.$$

The sets $\{U_k\}_{k=0}^{\infty}$ are defined recursively. To construct the initial set U_0 , let

$$B_0 := \{(x, r, t) \in U : 0 < t < (\alpha/2\kappa)^{\kappa}\}.$$

Then, recalling the definition of λ_P and applying Fubini's theorem, it follows that

$$\begin{aligned} \int_{\Sigma} \chi_{B_0}(x, r, t) \lambda_P(t) dx dr dt &= \int_F \int_0^{(\alpha/2\kappa)^{\kappa}} \chi_E(x - rP(t)) \lambda_P(t) dt dx dr \\ &\leq \frac{1}{2} \alpha |F| = \frac{1}{2} \int_{\Sigma} \chi_U(x, r, t) \lambda_P(t) dx dr dt. \end{aligned}$$

Define $U_0 := U \setminus B_0$ so that

$$\int_{\Sigma} \chi_{U_0}(x, r, t) \lambda_P(t) dx dr dt \geq \frac{1}{2} \int_{\Sigma} \chi_U(x, r, t) \lambda_P(t) dx dr dt.$$

Note that this definition will ensure property iii) holds for the sequence of refinements. Now suppose the set U_{k-1} has been defined for some $k \geq 1$ and satisfies the conditions stipulated in the proof of Lemma 44.

Case $k \equiv n \pmod{2}$.

In order to ensure the property (2.6.4) holds in this case, the following refinement procedure is applied. Let B_{k-1} denote the set

$$\left\{ (x, r, t) \in U_{k-1} : \int_1^2 \int_I \chi_{U_{k-1}}(x - rP(t) + \rho P(\tau), \rho, \tau) \lambda_P(\tau) d\tau d\rho \leq 4^{-(k+1/2)} \beta \right\}.$$

The map $(\rho, \tau) \mapsto (x - rP(t) + \rho P(\tau), \rho, \tau)$ parametrises the fibre $\pi_2^{-1}(\pi_2(x, r, t))$ and so B_{k-1} is precisely the set of all points belonging to π_2 -fibres which have a ‘‘small’’ intersection with U_{k-1} . Removing the parts of U_{k-1} lying in these fibres should not significantly diminish the

measure of the set and indeed, by Fubini's theorem and a simple change of variables,

$$\begin{aligned}
 \int_{\Sigma} \chi_{B_{k-1}}(x, r, t) \lambda_P(t) dt dx dr &= \int_{\mathbb{R}^n \times [1, 2]} \int_I \chi_{B_{k-1}}(x + rP(t), r, t) \lambda_P(t) dt dx dr \\
 &\leq \int_{\{x \in E : T_{k-1}(x) \leq 4^{-(k+1/2)} \beta\}} T_{k-1}(x) dx \\
 &\leq 4^{-(k+1/2)} \beta |E| \\
 &\leq \frac{1}{2} \int_{\Sigma} \chi_{U_{k-1}}(x, r, t) \lambda_P(t) dx dr dt
 \end{aligned} \tag{2.12.1}$$

where

$$T_{k-1}(x) := \int_1^2 \int_I \chi_{U_{k-1}}(x + \rho P(\tau), \rho, \tau) \lambda_P(\tau) d\tau d\rho.$$

Note that the inequality (2.12.1) is due to property i) of the sets U_j for $1 \leq j \leq k-1$, stated in Lemma 44. Thence, letting $U'_{k-1} := U_{k-1} \setminus B_{k-1}$ it follows that

$$\int_{\Sigma} \chi_{U'_{k-1}}(x, r, t) \lambda_P(t) dx dr dt \geq \frac{1}{2} \int_{\Sigma} \chi_{U_{k-1}}(x, r, t) \lambda_P(t) dx dr dt. \tag{2.12.2}$$

Now, recalling $I = (a, b)$, define B'_{k-1} to be the set

$$\left\{ (x, r, t) \in U'_{k-1} : \int_1^2 \int_t^b \chi_{U'_{k-1}}(x - rP(t) + \rho P(\tau), \rho, \tau) \lambda_P(\tau) d\tau d\rho \leq 4^{-(k+1)} \beta \right\}.$$

Given $x \in \pi_2(U'_{k-1})$, the fibre-wise nature of the definition of U'_{k-1} implies for $(x + rP(t), r, t) \in U'_{k-1}$ if and only if $(x + rP(t), r, t) \in U_{k-1}$ and consequently

$$\int_1^2 \int_I \chi_{U'_{k-1}}(x + rP(t), r, t) \lambda_P(t) dt dr \geq 4^{-(k+1/2)} \beta. \tag{2.12.3}$$

On the other hand,

$$\int_1^2 \int_I \chi_{B'_{k-1}}(x + rP(t), r, t) \lambda_P(t) dt dr = 4^{-(k+1)} \beta. \tag{2.12.4}$$

Indeed, the left-hand side can be expressed as

$$\nu_P(\{(r, t) \in K(x) : \nu_P(K(x) \cap ([1, 2] \times (t, b))) \leq 4^{-(k+1)} \beta\})$$

for $K(x) \subseteq [1, 2] \times I$ a measurable subset. The identity (2.12.4) is now a consequence of the fact that for any measure ν on \mathbb{R}^2 which is, say, absolutely continuous with respect to Lebesgue measure,

$$\nu(\{(r, t) \in K : \nu(K \cap (\mathbb{R} \times (t, \infty))) \leq u\}) = u$$

for all $0 < u < \nu(K)$ and all $K \subseteq \mathbb{R}^2$ measurable.

Thence, combining (2.12.3) and (2.12.4) it follows that

$$\int_1^2 \int_I \chi_{B_{k-1}}(x + rP(t), r, t) \lambda_P(t) dt dr \leq \frac{1}{2} \int_1^2 \int_I \chi_{U'_{k-1}}(x + rP(t), r, t) \lambda_P(t) dt dr$$

whenever $x \in \pi_2(U'_{k-1})$. Defining $U_k := U'_{k-1} \setminus B'_{k-1}$, one observes that

$$\begin{aligned} \int_{\Sigma} \chi_{U_k}(x, r, t) \lambda_P(t) dx dr dt &= \int_{\pi_2(U'_{k-1})} \int_1^2 \int_I \chi_{U_k}(x + rP(t), r, t) \lambda_P(t) dt dr dx \\ &\geq \frac{1}{2} \int_{\pi_2(U'_{k-1})} \int_1^2 \int_I \chi_{U'_{k-1}}(x + rP(t), r, t) \lambda_P(t) dt dr dx \\ &= \frac{1}{4} \int_{\Sigma} \chi_{U_{k-1}}(x, r, t) \lambda_P(t) dx dr dt. \end{aligned}$$

Moreover, the set U_k is easily seen to satisfy (2.6.4).

Case $k \not\equiv n \pmod{2}$.

It remains to define the set U_k under the assumption $k \not\equiv n \pmod{2}$, ensuring property (2.6.3) is satisfied. Here one is concerned with the fibres of the map π_1 . Define

$$B_{k-1} := \left\{ (x, r, t) \in U_{k-1} : \int_I \chi_{U_{k-1}}(x, r, \tau) \lambda_P(\tau) d\tau \leq 4^{-(k+1/2)} \alpha \right\}.$$

Notice that the map $\tau \mapsto (x, r, \tau)$ parametrises the fibre $\pi_1^{-1}(\pi_1(x, r, t))$ and so B_{k-1} is the collection of all points (x, r, t) in U_{k-1} which belong to π_1 -fibres which have a “small” intersection with U_{k-1} . Reasoning analogously to the previous case, if one defines $U'_{k-1} := U_{k-1} \setminus B_{k-1}$ it follows that (2.12.2) holds in this case. Finally, let

$$B'_{k-1} := \left\{ (x, r, t) \in U'_{k-1} : \int_t^b \chi_{U'_{k-1}}(x, r, \tau) \lambda_P(\tau) d\tau \leq 4^{-(k+1)} \alpha \right\}$$

and $U_k := U'_{k-1} \setminus B'_{k-1}$. Again arguing as in the previous case, it follows that

$$\int_{\Sigma} \chi_{U_k}(x, r, t) \lambda_P(t) dx dr dt \geq \frac{1}{4} \int_{\Sigma} \chi_{U_{k-1}}(x, r, t) \lambda_P(t) dx dr dt.$$

This recursive procedure defines a sequence of sets with all the desired properties.

Chapter 3

An affine Fourier restriction theorem for conical surfaces

3.1 Averages over circles and Fourier restriction to light cones

Let σ denote the arc-length measure on the unit circle S^1 (which agrees with the affine arc-length measure since S^1 has curvature everywhere equal to 1) and define the dilated averaging operator

$$Af(x, r) := \int_{S^1} f(x - ry) d\sigma(y).$$

For this example, the endpoint key inequality is

$$\|Af\|_{L^6(\mathbb{R}^2 \times [1, 2])} \lesssim \|f\|_{L^2(\mathbb{R}^2)}. \quad (3.1.1)$$

In contrast with averages over general space curves, averages over plane curves are (at least in the non-degenerate case) amenable to Fourier transform techniques. For instance, (3.1.1) can be established by comparing A with a certain Fourier integral operator and then applying the TT^* method combined with fractional integral estimates; this follows an argument used to prove (closely-related) classical Strichartz estimates for the wave equation, described in [Sog08] (see also [Tao06b]). Furthermore, by applying the Fourier transform it can be shown that (3.1.1) is equivalent to the following Stein-Tomas restriction theorem for the light cone.

Theorem 52 (Strichartz [Str77]). *The following Stein-Tomas restriction inequality holds:*

$$\left(\int_{\mathbb{R}^2} |\hat{F}(\xi, |\xi|)|^2 \frac{d\xi}{|\xi|} \right)^{1/2} \lesssim \|F\|_{L^{6/5}(\mathbb{R}^3)}. \quad (3.1.2)$$

Many of these observations were essentially present in the original paper of Strichartz [Str77], where the higher dimensional situation is also considered (see also [Ste93, Chapter VIII]). The weight $|\xi|^{-1}$ appearing on the left-hand side of (3.1.2) gives rise to the natural Lorentz-invariant measure on the light cone; with this choice of measure it is possible to obtain estimates for the full (non-compact) cone as above, see [Tao04] for further discussion. The connection between dilated averages over circles and Fourier restriction was briefly mentioned in Remark 35; in this chapter some of the consequences of this relationship are investigated.

The proof that (3.1.2) implies (3.1.1) is straight-forward (as is the proof of the converse). Note, provided that f is sufficiently regular,

$$Af(x, r) = \int_{\widehat{\mathbb{R}}^2} e^{2\pi i x \cdot \xi} \hat{f}(\xi) \hat{\sigma}(r\xi) d\xi$$

and recall that the Fourier transform of the measure σ can be expressed as

$$\hat{\sigma}(\xi) = \sum_{\pm} \frac{a_{\pm}(\xi) e^{\pm 2\pi i |\xi|}}{(1 + |\xi|)^{1/2}} \quad (3.1.3)$$

where $a_{\pm} \in C^{\infty}(\widehat{\mathbb{R}}^2)$ satisfy¹

$$\left| \left(\frac{\partial}{\partial \xi} \right)^{\alpha} a_{\pm}(\xi) \right| \lesssim_{\alpha} (1 + |\xi|)^{-|\alpha|_1} \quad \alpha \in \mathbb{N}_0^2.$$

The identity (3.1.3) follows from elementary stationary phase computations, see [Ste93, Chapter VIII] or [Sog93, Chapter 1] for details. Thus, in order to establish (3.1.1) it essentially suffices to show the same inequality but now with

$$Af(x, r) := \int_{\widehat{\mathbb{R}}^2} e^{2\pi i(x \cdot \xi + r|\xi|)} \frac{a(r\xi)}{(1 + r|\xi|)^{1/2}} \hat{f}(\xi) d\xi$$

for some $a \in C^{\infty}(\widehat{\mathbb{R}}^2)$ satisfying the same estimates as the a_{\pm} . Furthermore, defining

$$m(\xi, r) := \frac{a(r\xi)|\xi|^{1/2}}{(1 + r|\xi|)^{1/2}},$$

it follows that

$$\left| \left(\frac{\partial}{\partial \xi} \right)^{\alpha} m(\xi, r) \right| \lesssim_{\alpha} |\xi|^{-|\alpha|_1} \quad \alpha \in \mathbb{N}_0^2,$$

where the implied constant is independent of $r \in [1, 2]$. Now define

$$\mathcal{A}f(x, r) := \int_{\widehat{\mathbb{R}}^2} e^{2\pi i(x \cdot \xi + r|\xi|)} \hat{f}(\xi) \frac{d\xi}{|\xi|^{1/2}} \quad (3.1.4)$$

and observe, for A as (re)defined above, $Af(r, \cdot) = \mathfrak{M}_r \mathcal{A}f(r, \cdot)$ where \mathfrak{M}_r is the multiplier operator associated to the function $m(\cdot, r) : \widehat{\mathbb{R}}^n \rightarrow \mathbb{C}$. Thus, by the Hörmander-Mikhlin multiplier theorem (see [Ste70, Chapter IV]),

$$\|Af(\cdot, r)\|_{L_x^6(\mathbb{R}^n)} \lesssim \|\mathcal{A}f(\cdot, r)\|_{L_x^6(\mathbb{R}^n)}$$

and to prove (3.1.1) it suffices to show the same inequality holds but with \mathcal{A} in place of A . Take $\eta \in C_c^{\infty}(\mathbb{R})$ with $\text{supp } \eta \in [1, 2]$ and $0 \leq \eta \leq 1$ and note

$$\begin{aligned} \int_1^2 \int_{\mathbb{R}^n} \mathcal{A}f(x, r) \overline{g(x, r) \eta(r)} dx dr &= \int_{\mathbb{R}} \int_{\widehat{\mathbb{R}}^n} e^{2\pi i r |\xi|} \hat{f}(\xi) \overline{\hat{G}(\xi, r)} \frac{d\xi}{|\xi|^{1/2}} dr \\ &= \int_{\widehat{\mathbb{R}}^n} \hat{f}(\xi) \int_{\mathbb{R}} e^{-2\pi i r |\xi|} \hat{G}(\xi, r) dr \frac{d\xi}{|\xi|^{1/2}} \end{aligned}$$

where $G(x, r) := g(x, r)\eta(r)$. The inner integral is just the Fourier transform of $r \mapsto \hat{G}(\xi, r)$

¹Here $|\alpha|_1 := \alpha_1 + \alpha_2$ for $\alpha \in \mathbb{N}_0^2$.

evaluated at $|\xi|$. Thus,

$$\left| \int_1^2 \int_{\mathbb{R}^2} Af(x, r) \overline{g(x, r)} \eta(r) dx dr \right| \leq \|\hat{f}\|_{L^2(\mathbb{R}^2)} \left(\int_{\hat{\mathbb{R}}^2} |\hat{G}(\xi, |\xi|)|^2 \frac{d\xi}{|\xi|} \right)^{1/2}$$

and, since $\|G\|_{L^{6/5}(\mathbb{R}^3)} \leq \|g\|_{L^{6/5}(\mathbb{R}^3)}$, (3.1.1) now follows from (3.1.2) and Plancherel's theorem. The converse statement can be proven in a similar fashion.

3.2 Averages over convex curves and restriction to conical surfaces

Now consider the more general situation where μ is the affine arc-length measure on some smooth plane curve Σ and A is defined to be the dilated averaging operator

$$Af(x, r) := \int_{\Sigma} f(x - ry) d\mu(y). \quad (3.2.1)$$

It is natural to ask whether in general an $L_x^2 - L_{x,r}^6$ estimate for this operator corresponds to some weighted conical restriction inequality. In the previous chapter A was defined with respect to polynomial curves, but here it is more natural to let $\Sigma \subset \mathbb{R}^2$ be given by the boundary of some centred convex body. That is, Σ equals $\partial\Omega$ where Ω is a compact, convex set with smooth boundary for which $0 \in \Omega$ is an interior point. It transpires that estimates for A correspond to Fourier restriction estimates for the conical surface generated by the boundary of the *polar body* Ω^* . This correspondence is, in general, **not** a rigorous equivalence, but merely a heuristic. It does, however, suggest what the correct affine formulation of the conical restriction problem should be.

In order to proceed it is necessary to recall some fundamental facts concerning the relationship between a convex body and its polar body. Let \mathcal{K}_0 denote the class of centred convex bodies in \mathbb{R}^2 . The polar body Ω^* of $\Omega \in \mathcal{K}_0$ is defined by

$$\Omega^* := \{\xi \in \hat{\mathbb{R}}^2 : |\langle x, \xi \rangle| \leq 1 \text{ for all } x \in \Omega\};$$

later it will be useful to think of Ω^* as lying in the frequency space, but this has no particular significance at present.

Proofs of the following results are both easily deduced and widely available (see [Sch14], for instance).

- a) For $\Omega \in \mathcal{K}_0$, the polar body Ω^* is dual to Ω in the sense that $\Omega^* \in \mathcal{K}_0$ and, moreover, $\Omega^{**} = \Omega$.
- b) Suppose $\Omega \in \mathcal{K}_0$ is smooth; that is, $\Sigma := \partial\Omega$ is a smooth hypersurface. Then Ω together with its boundary Σ can be expressed as

$$\Omega = \{x \in \mathbb{R}^n : \phi(x) \leq 1\} \quad \text{and} \quad \Sigma = \{x \in \mathbb{R}^n : \phi(x) = 1\}$$

where $\phi : \mathbb{R}^n \rightarrow [0, \infty)$ is smooth away from 0, homogeneous of degree 1 and satisfies $\nabla\phi(x) \neq 0$ for $x \in \mathbb{R}^n \setminus \{0\}$.

- c) If $\Omega \in \mathcal{K}_0$ is smooth, then the polar body Ω^* is also smooth. Defining $\phi^* : \hat{\mathbb{R}}^2 \rightarrow [0, \infty)$ by

$\phi^*(\xi) := \sup_{x \in \Omega} \langle \xi, x \rangle$ and letting Σ^* denote the boundary of Ω^* , it follows that

$$\Omega^* = \{\xi \in \widehat{\mathbb{R}}^2 : \phi^*(\xi) \leq 1\} \quad \text{and} \quad \Sigma^* = \{\xi \in \widehat{\mathbb{R}}^2 : \phi^*(\xi) = 1\}.$$

Furthermore, ϕ^* is smooth away from the origin, homogeneous of degree 1 and satisfies $\nabla \phi^*(\xi) \neq 0$ for $\xi \in \widehat{\mathbb{R}}^2 \setminus \{0\}$.

d) Suppose Ω is strictly convex. Then given $x \in \Sigma$ there exists a unique point $x^* \in \Sigma^*$, known as the dual point to x , satisfying $\langle x, x^* \rangle = 1$.

The final fact is not so elementary; it is a special case of (2) from [Hug96].

e) Let $\Omega \in \mathcal{K}_0$ be smooth and strictly convex. The curvature $\kappa(x)$ of Σ at a point $x \in \Sigma$ is related to the curvature $\kappa^*(x^*)$ of Σ^* at the dual point $x^* \in \Sigma^*$ by

$$\kappa(x)^{1/3} \kappa^*(x^*)^{1/3} = \langle x, \nu(x) \rangle \langle x^*, \nu^*(x^*) \rangle \quad (3.2.2)$$

where ν and ν^* are the (outward) Gauss maps for Σ and Σ^* , respectively. Moreover, it is clear the right-hand side of this identity may be written as

$$\left\langle x, \frac{\nabla \phi(x)}{|\nabla \phi(x)|} \right\rangle \left\langle x^*, \frac{\nabla \phi^*(x^*)}{|\nabla \phi^*(x^*)|} \right\rangle = \frac{\phi(x) \phi^*(x^*)}{|\nabla \phi(x)| |\nabla \phi^*(x^*)|} = \frac{1}{|\nabla \phi(x)| |\nabla \phi^*(x^*)|}$$

where the first equality is due to Euler's homogeneous function theorem.

Returning to the averaging operator, fix Ω and Σ as above and let A be as defined in (3.2.1). As before, the key estimate is

$$\|Af\|_{L^6(\mathbb{R}^2 \times [1,2])} \lesssim \|f\|_{L^2(\mathbb{R}^2)}. \quad (3.2.3)$$

In the non-degenerate case (where the curvature of Σ is non-vanishing) this is equivalent to a conical restriction theorem. In particular, let Σ^* and $\phi^* : \widehat{\mathbb{R}}^2 \rightarrow [0, \infty)$ be as defined in c) and consider the conical surface \mathcal{C}^* given by

$$\mathcal{C}^* := \{(\xi, \phi^*(\xi)) : \xi \in \widehat{\mathbb{R}}^2\} \subset \widehat{\mathbb{R}}^3.$$

Define the weight function

$$w^*(\xi) := \langle M(\phi^*)(\xi) \nabla \phi^*(\xi), \nabla \phi^*(\xi) \rangle \phi^*(\xi)$$

where $M(\phi^*)$ is the matrix-valued function

$$M(\phi^*) := \begin{pmatrix} -\frac{\partial^2 \phi^*}{\partial \xi_2^2} & \frac{\partial^2 \phi^*}{\partial \xi_1 \partial \xi_2} \\ \frac{\partial^2 \phi^*}{\partial \xi_1 \partial \xi_2} & -\frac{\partial^2 \phi^*}{\partial \xi_1^2} \end{pmatrix}.$$

Notice $M(\phi^*)$ is the negative of the adjugate of the Hessian matrix of ϕ^* . One may easily verify w^* is smooth away from the origin and homogeneous of degree 0. With these definitions it transpires (3.2.3) corresponds to

$$\left(\int_{\widehat{\mathbb{R}}^2} |\widehat{F}(\xi, \phi(\xi))|^2 w^*(\xi)^{1/3} \frac{d\xi}{\phi^*(\xi)} \right)^{1/2} \lesssim \|F\|_{L^{6/5}(\mathbb{R}^3)}. \quad (3.2.4)$$

Of course, the prototypical example is given by taking $\Sigma = S^1$, in which case $\Sigma^* = S^1$ so that $\phi^*(\xi) = |\xi|$ and C^* is the light cone considered in the previous section and, furthermore, (3.2.4) reduces to (3.1.2).

In the non-degenerate case (3.2.3) can be shown to be equivalent to (3.2.4), provided that one allows constants which may depend on the choice of curve Σ . The details of this argument are given in the following section. For general convex curves the correspondence is merely a heuristic, which arises from ignoring certain (important) error terms in stationary phase estimates.

3.3 Equivalence in the non-degenerate case

Suppose Σ has non-vanishing curvature (and therefore Ω is strictly convex). Here the equivalence of (3.2.3) and (3.2.4) is discussed; this is intended to give a glimpse of the relationship between the weight w^* and the affine arc-length measure on Σ and is only for expository purposes. To proceed as in Section 3.1 one must obtain a suitable expression for the Fourier transform of affine arc-length measure on Σ , which is given by the formula

$$\hat{\mu}(\xi) = \int_{\Sigma} e^{-2\pi i x \cdot \xi} \kappa(x)^{1/3} d\sigma(x).$$

Since Σ is locally graph parametrised, by applying a partition of unity and various rotations it suffices to consider the integral

$$\int_{\mathbb{R}} e^{-2\pi i(t\xi_1 + \gamma(t)\xi_2)} \beta(t) \gamma''(t)^{1/3} dt \quad (3.3.1)$$

where $\beta \in C_c^\infty(\mathbb{R})$ is supported on an interval and satisfies $0 \leq \beta \leq 1$ and $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth convex function satisfying $\phi(t, \gamma(t)) = 1$ for $t \in \text{supp } \beta$. The non-vanishing curvature condition implies the Gauss map is a diffeomorphism from $\{(t, \gamma(t)) : \beta(t) \neq 0\}$ onto an open neighbourhood U of S^1 . For ξ belonging to the open cone Γ generated by U , let ξ' denote the unique point in Σ^* for which $\phi^*(\xi)\xi' = \xi$ and $t(\xi)$ the unique $t \in \mathbb{R}$ such that $\beta(t) \neq 0$ and ξ is normal to Σ at $(t, \gamma(t))$. The Gauss map is chosen to correspond to the outward normal vector field; in this case the set U and, consequently, Γ lie in the lower-half plane.

Write (3.3.1) as

$$e^{-2\pi i \langle (t(\xi), \gamma(t(\xi))), \xi \rangle} \int_{\mathbb{R}} e^{2\pi i \phi^*(\xi') \Phi(t, \xi')} \beta(t) \gamma''(t)^{1/3} dt \quad (3.3.2)$$

where

$$\Phi(t, \xi') := -\langle (t, \gamma(t)), \xi' \rangle + \langle (t(\xi), \gamma(t(\xi))), \xi' \rangle.$$

For fixed ξ the function $t \mapsto \Phi(t, \xi')$ has a critical point at $t(\xi)$; explicitly,

$$\Phi(t, \xi')|_{t=t(\xi)} = \Phi'_t(t, \xi')|_{t=t(\xi)} = 0.$$

Furthermore, the curvature condition implies this critical point is non-degenerate in the sense that

$$\Phi''_{tt}(t, \xi')|_{t=t(\xi)} = -\xi'_2 \gamma''(t(\xi)) > 0.$$

One is now in a position to apply standard asymptotic expansion formulae for oscillatory inte-

grals (see [Ste93, Chapter VIII]). In particular, it follows that (3.3.2) can be expressed as

$$a(\xi)\phi^*(\xi)^{-1/2}|\xi'_2|^{-1/2}\gamma''(t(\xi))^{-1/6}e^{2\pi i\langle x(\xi), \xi \rangle} \quad (3.3.3)$$

where $x(\xi) = (t(\xi), \gamma(t(\xi)))$ and $a(\xi) \in C^\infty(\widehat{\mathbb{R}}^2)$ satisfies

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha a(\xi) \right| \lesssim_\alpha (1 + |\xi|)^{-|\alpha|} \quad \alpha \in \mathbb{N}_0^2.$$

One would like to write (3.3.3) in a more palatable form. The map $x \mapsto \langle x, \xi \rangle$ defined on Σ attains its extrema at the point at which ξ is normal to the curve: indeed,

$$\frac{\partial^2}{\partial t^2} [t\xi_1 + \gamma(t)\xi_2] \Big|_{t=t(\xi)} = \gamma''(t(\xi))\xi_2 < 0.$$

Thence, $x(\xi)$ is a global maximum for the aforementioned map and therefore

$$\phi^*(\xi) = \sup_{x \in \Sigma} \langle x, \xi \rangle = \langle x(\xi), \xi \rangle. \quad (3.3.4)$$

This simplifies the expression appearing in the exponential. One would like to also write the factor $|\xi'_2|^{-1/2}\gamma''(t(\xi))^{-1/6}$ in terms of ϕ and ϕ^* . To do this, differentiate the equation $\phi(t, \gamma(t)) = 1$ with respect to t to obtain

$$\phi_1(t, \gamma(t)) + \gamma'(t)\phi_2(t, \gamma(t)) = 0 \quad (3.3.5)$$

where $\phi_j = \partial_{x_j}\phi$ for $j = 1, 2$. Since $\nabla\phi(t, \gamma(t)) \neq 0$, it follows that $\phi_2(t, \gamma(t)) \neq 0$ and differentiating (3.3.5) leads to the expression

$$\gamma''(t) = -\frac{\phi_{11}\phi_2^2 - 2\phi_{12}\phi_1\phi_2 + \phi_{22}\phi_1^2}{\phi_2^3}(t, \gamma(t)) \quad (3.3.6)$$

Note that the numerator is the weight

$$w(t, \gamma(t)) := \langle D^2\phi(x)\nabla\phi(x), \nabla\phi(x) \rangle \phi(x) \Big|_{x=(t, \gamma(t))}$$

which is similar to that appearing in (3.2.4), but with ϕ^* replaced with ϕ . For the denominator, observe both ξ and $\nabla\phi(x(\xi))$ are normal to the curve at $x(\xi)$ and hence are scalar multiples of one another. Applying (3.3.4) and appealing to Euler's homogeneous function theorem yields

$$\phi^*(\nabla\phi(x(\xi))) = \langle x(\xi), \nabla\phi(x(\xi)) \rangle = \phi(x(\xi)) = 1 \quad (3.3.7)$$

and it follows immediately that

$$\nabla\phi(x(\xi)) = \phi^*(\xi)^{-1}\xi = \xi' \quad (3.3.8)$$

so, in particular, $\phi_2(x(\xi)) = \xi'_2$. Substituting these identities into (3.3.3) one obtains

$$a(\xi) \frac{w(x(\xi))^{-1/6}}{\phi^*(\xi)^{1/2}} e^{2\pi i\phi^*(\xi)}.$$

The above expression can in turn be written as

$$a(\xi) \frac{w^*(\xi)^{1/6}}{\phi^*(\xi)^{1/2}} e^{2\pi i \phi^*(\xi)} \tag{3.3.9}$$

by invoking the fact

$$w(x(\xi))w^*(\xi) = w(x(\xi))w^*(x(\xi)^*) = 1, \tag{3.3.10}$$

which is established presently. Indeed, from (3.3.7) and (3.3.8) one concludes that $\xi' = x(\xi)^*$ and the first equality now follows by the homogeneity of w^* . For the second equality, note that (3.3.6) implies the formula

$$\kappa(x) = \frac{w(x)}{|\nabla\phi(x)|^3} \quad \text{for all } x \in \Sigma \tag{3.3.11}$$

and, by a similar argument,

$$\kappa^*(\xi) = \frac{w^*(\xi)}{|\nabla\phi^*(\xi)|^3} \quad \text{for all } \xi \in \Sigma^*$$

so that (3.3.10) is now an immediate consequence of the identity (3.2.2).

Thus far $\xi \in \Gamma$; if ξ lies outside Γ then the phase function appearing in (3.3.2) has no critical points. In this case the principle of non-stationary phase applies and as such (3.3.2) can be written as a function of the form (3.3.9) for all $\xi \in \widehat{\mathbb{R}}^2$. The Fourier transform of μ is then given as a sum of similar functions.

Thus, in the non-degenerate case, the averaging operator defined at the start of the section can be represented as a sum of Fourier integral operators of the form

$$\tilde{A}f(x, t) := \int_{\widehat{\mathbb{R}}^2} e^{2\pi i(x \cdot \xi + t\phi^*(\xi))} \hat{f}(\xi) a(r\xi) \frac{w^*(r\xi)^{1/6}}{\phi^*(r\xi)^{1/2}} d\xi.$$

By a simple multiplier argument, one concludes that the estimate (3.2.3) would follow from the same inequality but with

$$\mathcal{A}f(x, t) := \int_{\widehat{\mathbb{R}}^2} e^{2\pi i(x \cdot \xi + t\phi^*(\xi))} \hat{f}(\xi) \frac{w^*(\xi)^{1/6}}{\phi^*(\xi)^{1/2}} d\xi$$

in place of A . The desired estimate for \mathcal{A} is then easily seen to be a consequence of the restriction estimate (3.2.4) by duality. As before, the converse statement can be treated in a similar fashion.

The above argument is severely limited in that it applies only to non-degenerate Σ and it may introduce an undesirable dependence in the constants (which makes the appearance of the weight somewhat superfluous). Nevertheless, it does indicate a likely candidate for the correct measure for an affine conical restriction theorem. The following section considers the problem of establishing inequalities such as (3.2.4), but now without reference to estimates for dilated averages.

3.4 An affine restriction theorem for 2-dimensional cones

In [Nic09] Nicola gave an alternative proof of the sharp $L^p - L^q$ Fourier restriction theorem for the conical surface $\{(\xi, |\xi|) : \xi \in \widehat{\mathbb{R}}^2\}$ lying in the frequency space $\widehat{\mathbb{R}}^3$ with measure $d\xi/|\xi|$, originally due to Barceló [Bar85] (see [Bar86, DG93, Obe02, Bus] for related results). Explicitly, this states whenever $1 \leq p < 4/3$ and $q = p'/3$ one has

$$\left(\int_{\widehat{\mathbb{R}}^2} |\widehat{F}(\xi, |\xi|)|^q \frac{d\xi}{|\xi|} \right)^{1/q} \lesssim \|F\|_{L^p(\mathbb{R}^3)}. \tag{3.4.1}$$

In this section, which presents published work of the author from [Hic14], it is observed Nicola's arguments can easily be adapted to give results in an affine-invariant setting. In particular, it is shown that Sjölin's affine restriction estimate for convex plane curves (Theorem 12) implies a variant of the conical restriction theorem where one may replace the circular cone with any member of a broad class of conic surfaces given by dilating convex curves, provided that the measure $d\xi/|\xi|$ is substituted with the weighted measure introduced earlier in this chapter. To make this precise, let $\Omega \in \mathcal{K}_0$ be smooth and define Σ, ϕ with respect to Ω as in the previous sections. Consider the cone

$$\mathcal{C} := \{(\xi, \phi(\xi)) : \xi \in \widehat{\mathbb{R}}^2\} \subseteq \widehat{\mathbb{R}}^3,$$

which is thought of as lying in the frequency space, and define the weight function

$$w(\xi) := \langle M(\phi)(\xi) \nabla \phi(\xi), \nabla \phi(\xi) \rangle \phi(\xi)$$

where $M(\phi)$ is the negative of the adjugate of the Hessian matrix of ϕ . The desired restriction estimate for the whole cone is as follows:

Proposition 53. *For ϕ and w as above, if $1 \leq p < 4/3$ and $q = p'/3$, then*

$$\left(\int_{\widehat{\mathbb{R}}^2} |\widehat{F}(\xi, \phi(\xi))|^q w(\xi)^{1/3} \frac{d\xi}{\phi(\xi)} \right)^{1/q} \leq C \|F\|_{L^p(\mathbb{R}^3)}. \tag{3.4.2}$$

Here C is a universal constant in the sense that it depends on p only and, in particular, not the choice of conical surface.

Remark 54. a) In the prototypical case $\Sigma = S^1$, $w(\xi) = 1$ and the original restriction estimate (3.4.1) is recovered.

b) The inequality (3.4.2) exhibits certain kind of affine invariance. If (3.4.2) holds for a fixed ϕ , then it is easily seen to hold *with the same constant C* whenever ϕ replaced with any function of the form $\phi \circ X$ for $X \in \text{GL}(2, \mathbb{R})$.

As indicated above, the proof of the proposition is given simply by observing that the arguments of Nicola in [Nic09] may be adapted to work in this setting. The exposition will therefore be terse; the reader is directed to the aforementioned paper [Nic09] for further details.

Before giving the proof some preliminary remarks are in order. For each $t \geq 0$ let $\Sigma_t := t\Sigma$ denote the t -dilata of Σ so that the cone \mathcal{C} may be expressed as a disjoint union of a continuum of slices:

$$\mathcal{C} = \bigcup_{t \geq 0} \Sigma_t \times \{t\}.$$

Let $d\sigma_t$ denote the surface measure on Σ_t with $d\sigma := d\sigma_1$ and κ the curvature of Σ . It will be shown that the conic restriction estimate is related to Theorem 12 via the *co-area formula*

$$\int_{\widehat{\mathbb{R}}^2} g(\xi) d\xi = \int_0^\infty \int_{\Sigma_t} g(\xi) \frac{d\sigma_t(\xi)}{|\nabla\phi(\xi)|} dt, \quad (3.4.3)$$

valid for all non-negative continuous functions g on $\widehat{\mathbb{R}}^2$. For a proof of this identity see [Fed69].

Proof (of Proposition 53). Fixing exponents p, q satisfying the hypotheses of the proposition, it suffices to establish the dual *extension* estimate

$$\|(ud\mu_C)^\vee\|_{L^{p'}(\mathbb{R}^3)} \lesssim \|u\|_{L^{q'}(C, d\mu_C)} \quad (3.4.4)$$

where $d\mu_C$ denotes the weighted conic measure so that

$$(ud\mu_C)^\vee(x, t) = \int_{\widehat{\mathbb{R}}^2} e^{2\pi i(x \cdot \xi + t\phi(\xi))} u(\xi, \phi(\xi)) w(\xi)^{1/3} \frac{d\xi}{\phi(\xi)}.$$

By applying the co-area formula together with a change of variables one obtains

$$\begin{aligned} (ud\mu_C)^\vee(x, t) &= \int_0^\infty \int_{\Sigma_s} e^{2\pi i(x \cdot \xi + ts)} u(\xi, s) \frac{w(\xi)^{1/3}}{|\nabla\phi(\xi)|} d\sigma_s(\xi) \frac{ds}{s} \\ &= \int_0^\infty e^{2\pi ixs} \int_{\Sigma} e^{2\pi isx \cdot \xi'} u(s\xi', s) \kappa(\xi')^{1/3} d\sigma(\xi') ds, \end{aligned}$$

where here the formula (3.3.11) has been applied. Notice that the last integral is the value at t of the inverse Fourier transform of the function

$$\chi_{[0, \infty)}(s) \int_{\Sigma} e^{2\pi isx \cdot \xi'} u(s\xi', s) \kappa(\xi')^{1/3} d\sigma(\xi').$$

Apply the Lorentz space version of the Hausdorff-Young inequality to obtain

$$\begin{aligned} \|(ud\mu_C)^\vee\|_{L^{p', t}(\mathbb{R}^3)} &\lesssim \left\| \int_{\Sigma} e^{2\pi isx \cdot \xi'} u(s\xi', s) \kappa(\xi')^{1/3} d\sigma(\xi') \right\|_{L_x^{p'} L_s^{p, p'}(\mathbb{R}^2 \times \mathbb{R}_+)} \\ &\lesssim \left\| \int_{\Sigma} e^{2\pi isx \cdot \xi'} u(s\xi', s) \kappa(\xi')^{1/3} d\sigma(\xi') \right\|_{L_s^{p, p'} L_x^{p'}(\mathbb{R}^2 \times \mathbb{R}_+)} \end{aligned}$$

where the second inequality is due to the interchange lemma from [Nic09]. By a change of variables and an appeal to the dual formulation of Sjölin's theorem one deduces that

$$\begin{aligned} \left\| \int_{\Sigma} e^{2\pi isx \cdot \xi'} u(s\xi', s) \kappa(\xi')^{1/3} d\sigma(\xi') \right\|_{L_x^{p'}(\mathbb{R}^2)} &= s^{-2/p'} \|(u(s \cdot, s) \kappa(\cdot)^{1/3} d\sigma)^\vee\|_{L_x^{p'}(\mathbb{R}^2)} \\ &\lesssim s^{-2/p'} \|u(s \cdot, s)\|_{L_\xi^{q'}(\Sigma, \kappa^{1/3} d\sigma)}. \end{aligned}$$

Observe the hypotheses on the exponents imply $q' \leq p'$ and $1/p - 1/q' = 2/p'$. Thus, by the

nesting of Lorentz spaces and Lorentz version of Hölder's inequality,

$$\begin{aligned} \|(u d\mu_C)^\vee\|_{L_{x,t}^{p'}(\mathbb{R}^3)} &\lesssim \|s^{-2/p'} u(s \cdot, s)\|_{L_\xi^{q'}(\Sigma, \kappa^{1/3} d\sigma)} \|L_s^{p,p'}(\mathbb{R}_+)\| \\ &\lesssim \|s^{-2/p'} u(s \cdot, s)\|_{L_\xi^{q'}(\Sigma, \kappa^{1/3} d\sigma)} \|L_s^{p,q'}(\mathbb{R}_+)\| \\ &\lesssim \|s^{-2/p'}\|_{L_s^{p'/2, \infty}(\mathbb{R}_+)} \|u(s \cdot, s)\|_{L_\xi^{q'}(\Sigma, \kappa^{1/3} d\sigma)} \|L_s^{q'}(\mathbb{R}_+)\|. \end{aligned}$$

Finally recall

$$\|s^{-2/p'}\|_{L_s^{p'/2, \infty}(\mathbb{R}_+)} = \sup_{\alpha > 0} \alpha |\{s > 0 : s^{-2/p'} > \alpha\}|^{2/p'} = 1$$

whilst by an easy computation, essentially a reversal of the identities used at the start of the proof, one deduces that

$$\|u(s \cdot, s)\|_{L_\xi^{q'}(\Sigma, \kappa^{1/3} d\sigma)} \|L_s^{q'}(\mathbb{R}_+)\| = \|u\|_{L^{q'}(C, d\mu_C)}$$

and thence the required estimate. \square

By applying Hölder's inequality one obtains a sharp restriction theorem for a compact piece of the cone. In particular, consider restriction to the surface $S := \{(\xi, \phi(\xi)) : \xi \in \Delta\}$ where $\Delta := \{\xi \in \widehat{\mathbb{R}}^2 : 1 \leq \phi(\xi) \leq 2\}$.

Corollary 55. *For $1 \leq p < 4/3$ and $1 \leq q \leq p'/3$,*

$$\left(\int_{\Delta} |\widehat{F}(\xi, \phi(\xi))|^q w(\xi)^{1/3} d\xi \right)^{1/q} \lesssim_S \|F\|_{L^p(\mathbb{R}^3)}.$$

It is interesting to note that some applications of the preceding weighted restriction inequalities to the unweighted theory. Clearly, if the curvature of Σ is non-vanishing, then $w(\xi)$ is bounded below by some positive constant. Thus the weighted results imply both the sharp restriction theorem for the compact piece of the cone with surface measure and for the whole cone with the scale-invariant measure in the non-degenerate case.

The proposition can also be used to obtain the results when Σ is of finite type and gives the sharp range of exponents, except for an endpoint (for direct proofs of the finite type results see [Bar86, Bus, Hic14]). First note that for sub-critical exponents $1 \leq p < (k+1)/k$ and $(k+1)/p' < 1/q$ where $k \geq 3$ is the type of Σ , the estimate

$$\|\widehat{F}|_S\|_{L^q(S, d\sigma)} \lesssim_S \|F\|_{L^p(\mathbb{R}^3)}$$

is a simple consequence of the previous corollary and Hölder's inequality. One may also obtain results on the critical line $(k+1)/p' = 1/q$ by applying a simple interpolation argument, of the type described in [BS11, Remark 2.2].

Corollary 56. *Suppose Σ is of finite type, let S be as above and $d\sigma$ denote surface measure on S . For $1 \leq p < (k+2)/(k+1)$ and $q = p'/(k+1)$ where $k \geq 3$ is the type of Σ , the following estimate holds:*

$$\|\widehat{F}|_S\|_{L^q(S, d\sigma)} \lesssim_S \|F\|_{L^p(\mathbb{R}^3)}. \quad (3.4.5)$$

Remark 57. The sharp range for which (3.4.5) holds is given by $1 \leq p \leq (k+2)/(k+1)$ and therefore (56) is almost optimal (see [Hic14] for details). A more general version of Corollary

56, with additional Lorentz space estimates, was proved by Buschenhenke [Bus] using a very different method.

Proof (of Corollary 56). By interpolation with the trivial $(p, q) = (1, \infty)$ estimate, it suffices to show the restricted weak-type version of (3.4.5) holds for all $1 < p < (k+2)/(k+1)$ and $q = p'/(k+1)$. Fix a pair of exponents (p, q) satisfying these hypotheses and let

$$\rho := \frac{k-2-p(k-3)}{k-1-p(k-2)} \quad \tau := \frac{q(k+1)-(k-2)}{3}.$$

It is easy to verify $1 \leq \rho < 4/3$ and $\tau = \rho'/3$ and so the pair of exponents (ρ, τ) satisfies the conditions of Corollary 55. Now partition Δ into sets Δ_j defined as follows:

$$\Delta_j := \{\xi \in \Delta : 2^j \leq w(\xi) < 2^{j+1}\}.$$

Fix a measurable subset $E \subset \mathbb{R}^3$ of finite measure and $\alpha > 0$ and consider

$$\begin{aligned} |\{\xi \in \Delta_j : |\widehat{\chi}_E(\xi, \phi(\xi))| > \alpha\}| &\leq \frac{1}{\alpha^\tau} \int_{\Delta_j} |\widehat{\chi}_E(\xi, \phi(\xi))|^\tau d\xi \\ &\leq \frac{2^{-j/3}}{\alpha^\tau} \int_{\Delta_j} |\widehat{\chi}_E(\xi, \phi(\xi))|^\tau w(\xi)^{1/3} d\xi \\ &\lesssim_k \left(\frac{2^{-j/3\tau}}{\alpha} |E|^{1/\rho} \right)^\tau \end{aligned}$$

where the last inequality follows by applying Corollary 55. On the other hand, using the homogeneity of the weight one observes

$$\begin{aligned} |\{\xi \in \Delta_j : |\widehat{\chi}_E(\xi, \phi(\xi))| > \alpha\}| &\leq \frac{1}{\alpha} \int_{\Delta_j} |\widehat{\chi}_E(\xi, \phi(\xi))| d\xi \\ &\lesssim_\phi \frac{1}{\alpha} \sigma(\Sigma_j) |E| \end{aligned}$$

for $\Sigma_j = \{\xi' \in \Sigma : 2^j \leq w(\xi') < 2^{j+1}\}$. By applying the sub-level set version of van der Corput's lemma (see, for instance, [CCW99]) together with the curvature hypothesis, one may deduce the estimate $\sigma(\Sigma_j) \lesssim_\phi 2^{j/(k-2)}$. To conclude the proof note, for suitably chosen $J \in \mathbb{Z}$,

$$\begin{aligned} |\{\xi \in \Delta : |\widehat{\chi}_E(\xi, \phi(\xi))| > \alpha\}| &= \sum_{j=-\infty}^{\infty} |\{\xi \in \Delta_j : |\widehat{\chi}_E(\xi, \phi(\xi))| > \alpha\}| \\ &\lesssim_\phi \sum_{j=-\infty}^{\infty} \min \left\{ \left(\frac{2^{-j/3\tau}}{\alpha} |E|^{1/\rho} \right)^\tau, \frac{2^{j/(k-2)}}{\alpha} |E| \right\} \\ &\lesssim_k \left(\frac{2^{-J/3\tau}}{\alpha} |E|^{1/\rho} \right)^\tau + \frac{2^{J/(k-2)}}{\alpha} |E| \\ &\lesssim \frac{1}{\alpha^{3(\tau-1)/(k+1)+1}} |E|^{3(\tau/\rho-1)/(k+1)+1} \end{aligned}$$

where the last inequality is given by picking J to optimise the estimate. By the definition of the exponents ρ and τ it follows that

$$|\{\xi \in \Delta : |\widehat{\chi}_E(\xi, \phi(\xi))| > \alpha\}| \lesssim_\phi \left(\frac{1}{\alpha} |E|^{1/p} \right)^q,$$

as required. □

3.5 Conjectured results in higher dimensions

Now consider the analogous problem in higher dimensions. Let $\Sigma \subset \widehat{\mathbb{R}}^n$ be a smooth hypersurface, given by the boundary of some centred convex body. As before there exists $\phi : \widehat{\mathbb{R}}^n \rightarrow [0, \infty)$ smooth away from the origin, homogeneous of degree 1 and such that $\phi(\xi) = 1$ if and only if $\xi \in \Sigma$. Define the weight w by

$$w(\xi) := \langle M(\phi)(\xi) \nabla \phi(\xi), \nabla \phi(\xi) \rangle \phi(\xi)$$

where M is an $(n-1) \times (n-1)$ matrix-valued function given by the negative of the adjugate of the Hessian matrix of ϕ . It is not difficult to show if κ denotes the Gaussian curvature of Σ , then

$$\kappa(\xi) = \frac{w(\xi)}{|\nabla \phi(\xi)|^{n+1}} \quad \text{for all } \xi \in \Sigma. \tag{3.5.1}$$

By considering the conjectured $L^p - L^q$ bounds for the prototypical case of the light cone $\{(\xi, |\xi|) : \xi \in \widehat{\mathbb{R}}^n\}$ (as described in, for example, [Tao04]), the following conjecture is natural.

Conjecture 58. For $1 \leq p < 2n/(n+1)$ and $q = (n-1)p'/(n+1)$,

$$\left(\int_{\widehat{\mathbb{R}}^n} |\widehat{F}(\xi, \phi(\xi))|^q w(\xi)^{1/(n+1)} \frac{d\xi}{\phi(\xi)} \right)^{1/q} \lesssim \|F\|_{L^p(\mathbb{R}^{n+1})}.$$

Here the constant is uniform in the sense that it is independent of the choice of ϕ .

In order to proceed as before one would need an n -dimensional analogue of Sjölin's theorem. In light of the conjectured results for the restriction operator associated to the $(n-1)$ -dimensional sphere in $\widehat{\mathbb{R}}^n$ (see, for example, [Tao04]), the following conjecture is also natural.

Conjecture 59 (Affine restriction conjecture). For $\Sigma \subset \widehat{\mathbb{R}}^n$ as above and $1 \leq p < 2n/(n+1)$ and $q = (n-1)p'/(n+1)$,

$$\|\widehat{f}|_{\Sigma}\|_{L^q(\Sigma, \kappa^{1/(n+1)} d\sigma)} \lesssim \|f\|_{L^p(\widehat{\mathbb{R}}^n)},$$

where κ denotes the Gaussian curvature of Σ and $d\sigma$ surface measure. Here the constant is uniform in the sense that it is independent of the choice of Σ .

One can adapt the proof of Proposition 53 to show that if the affine restriction conjecture holds for some Σ and choice of exponents p, q satisfying the hypotheses of Conjecture 59, then the estimate for the corresponding cone holds for the same pair of exponents.

It is remarked that a number of partial results are known regarding the affine restriction conjecture. Many of these pertain to surfaces of revolution in \mathbb{R}^3 : for affine restriction in this special case and related results see [AKS06, CKZ07, Obe04, Sha07, Sha09]. Other classes of surfaces have been considered in [CZ02, Obe12, CKZ13]. Interesting connections between the affine restriction conjecture and the affine isoperimetric inequality have been observed and discussed in [CZ02, Sha07, Sha09].

As a final remark it is noted that arguments from [IL00, Nic08] can be adapted in order to study the weight function introduced above. In particular, the following proposition demonstrates that w is a natural choice of weight for the conic restriction problem.

Proposition 60 (Hickman [Hic14]). *Whenever $0 \leq \psi \in C(\widehat{\mathbb{R}}^n)$ is a weight for which the conic restriction estimate*

$$\left(\int_{\widehat{\mathbb{R}}^n} |\widehat{F}(\xi, \phi(\xi))|^2 \psi(\xi) \frac{d\xi}{\phi(\xi)} \right)^{1/2} \leq \sqrt{C} \|F\|_{L^{2(n+1)/(n+3)}(\mathbb{R}^{n+1})}$$

holds, it follows that

$$\psi(\xi) \lesssim C w(\xi)^{1/(n+1)}$$

holds for all $\xi \in \widehat{\mathbb{R}}^n \setminus \{0\}$. Here the implied constant depends on the dimension n only.

This is, of course, the analogue of the second part of Theorem 15 from the first chapter and can be proved using similar methods.

Chapter 4

Discrete analogues of the Fourier restriction problem

4.1 Introduction: Fourier restriction over finite fields.

This final chapter describes recent joint work of the author and J. Wright which considers a formulation of the Fourier restriction problem over rings of integers modulo a prime power.

Understanding (Euclidean) Fourier restriction phenomena in high dimensions appears to be a very difficult problem. In particular, for the paraboloid or sphere the sharp range of estimates for the associated Fourier restriction operators are unknown for $n \geq 3$. There have been a number of partial results which have, for instance, established restriction inequalities for a restricted range of Lebesgue exponents (one example being, of course, the Stein-Tomas estimate discussed in the introduction); details can be found in the survey article [Tao04], although there have been major advances in the theory since its publication, including [BCT06, BG11, BD]. In view of the apparent complexity of the restriction conjecture, it is prudent to attempt to pose a simplified version of the problem to act as a ‘toy model’. One way to approach this is to replace the underlying real field \mathbb{R} with a discrete object such as a finite field \mathbb{F}_q .¹ This follows the example of Wolff [Wol99] who famously posed a version of the Kakeya conjecture from geometric measure theory in the finite field setting; this led to an array of interesting and important developments in geometric combinatorics [Dvi09, Qui10, Gut10, CV13, GK15]. The Kakeya and restriction conjectures are closely related and, inspired by Wolff’s work, Tao and Mockenhaupt [MT04] succeeded in formulating a restriction problem over finite fields. Since \mathbb{F}_q has an essentially trivial topological structure, working in this setting dramatically simplifies issues related to scale which typically arise in the real-variable case. To motivate what follows the rudiments of this restriction theory over finite fields is presently discussed.

Since the vector space \mathbb{F}_q^n is a finite abelian group it admits a Fourier analysis and, furthermore, is self-dual (in the Pontryagin sense). Here the Haar measure on \mathbb{F}_q^n is taken to be counting measure whilst on $\widehat{\mathbb{F}_q^n}$ (which is identified with \mathbb{F}_q^n) it is taken to be normalised counting measure; with these choices Fourier inversion and Plancherel’s theorem hold. Fix $\psi \in \widehat{\mathbb{F}_q}$ a non-principal character and for any function $f \in \ell^1(\mathbb{F}_q^n)$ (that is, for *any* $f: \mathbb{F}_q^n \rightarrow \mathbb{C}$) define its

¹Here q is the prime power corresponding to the cardinality of the field.

Fourier transform $\hat{f} \in \ell^1(\mathbb{F}_q^n)$ by

$$\hat{f}(\xi) := \sum_{x \in \mathbb{F}_q^n} f(x) \psi(-x \cdot \xi) \quad \text{for all } \xi \in \mathbb{F}_q^n,$$

where $x \cdot \xi := x_1 \xi_1 + \dots + x_n \xi_n$.

Now define the d -dimensional ‘submanifold’ Σ in \mathbb{F}_q^n to be the image of some injective polynomial mapping $\gamma: \mathbb{F}_q^d \rightarrow \mathbb{F}_q^n$; this is endowed with normalised counting measure σ and one may consider restriction estimates of the form

$$\|\hat{f}|_\Sigma\|_{L^s(\sigma)} \lesssim_\gamma \|f\|_{L^r(\mathbb{F}_q^n)}, \quad (4.1.1)$$

where the left-hand side of this expression is given by

$$\left(\frac{1}{|\#\mathbb{F}_q^d|} \sum_{t \in \mathbb{F}_q^d} |\hat{f} \circ \gamma(t)|^s \right)^{1/s}$$

when $1 \leq s < \infty$ and (4.1.1) is interpreted in the obvious manner when $s = \infty$. Trivially,

$$\|\hat{f}|_\Sigma\|_{L^\infty(\sigma)} \leq \|f\|_{L^1(\mathbb{F}_q^n)}$$

and, moreover, by the equivalence of norms on finite-dimensional vector spaces it follows that (4.1.1) is valid for **all** $1 \leq r, s \leq \infty$ with an implied constant that depends on q . Thus, for a fixed finite field \mathbb{F}_q the question of determining the (r, s) for which the estimate (4.1.1) holds is completely trivial. Ostensibly, therefore, it is unclear how one may meaningfully pose a restriction conjecture in this finite setting. However, it was observed in [MT04] that an effective model of the Euclidean problem is obtained by simultaneously considering (4.1.1) over all \mathbb{F}_q and aiming to prove estimates which are uniform in q . With this requirement the situation is no longer entirely trivial and simple examples show (4.1.1) cannot hold with a uniform constant for every pair of Lebesgue exponents.

In the Euclidean formulation of the restriction problem necessary conditions are obtained by testing the estimate against the Knapp example, given by a small rectangle adapted to the geometry of the surface (see Remark 8 or Section 2.2 for a discussion of the Knapp example in related contexts). Crucially, in a finite field there is only one choice of scale and so the characteristic function of such a rectangle is simply modelled by a Dirac delta mass δ_x , concentrated at some $x \in \Sigma$. Defining $\hat{f} := \delta_x$, it follows that $\|f\|_{\ell^r(\mathbb{F}_q^n)} = q^{-n/r'}$ whilst $\|\hat{f}|_\Sigma\|_{\ell^s(\sigma)} = q^{-d/s}$ so that if (4.1.1) is to hold uniformly in q , then the exponents necessarily satisfy the condition $r' \geq sn/d$.

For concreteness, henceforth assume $\text{char } \mathbb{F}_q > n$, $d = 1$ and $\gamma := h: \mathbb{F}_q \rightarrow \mathbb{F}_q^n$ is the moment curve

$$h(t) := \left(t, \frac{t^2}{2}, \dots, \frac{t^n}{n!} \right), \quad (4.1.2)$$

the relevance of which was discussed in the Euclidean context in Section 1.6. Note that the assumption on the characteristic ensures the definition of h makes sense.

Now consider testing the dual formulation of the problem against the constant function 1. If (4.1.1) holds uniformly in q , it follows that

$$\|\check{\sigma}\|_{\ell^{r'}(\mathbb{F}_q^n)} \lesssim 1 \quad (4.1.3)$$

where

$$\check{\sigma}(x) := \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi \left(x_1 t + x_2 \frac{t^2}{2} + \cdots + x_n \frac{t^n}{n!} \right).$$

By Hölder's inequality,

$$\|\check{\sigma}\|_{\ell^2(\mathbb{F}_q^n)} \leq q^{n(1/2-1/r')} \|\check{\sigma}\|_{\ell^{r'}(\mathbb{F}_q^n)}$$

and, applying Plancherel's theorem,

$$\|\check{\sigma}\|_{\ell^2(\mathbb{F}_q^n)} = q^{n-1} \|\sigma\|_{\ell^2(\widehat{\mathbb{F}}_q^n)} = q^{(n-1)/2},$$

where here σ is identified with the function $\sigma(\xi) := q^{-1} \chi_{\{h(t): t \in \mathbb{F}_q\}}$ and the q^{n-1} factor arises from the choice of normalisation of the Haar measure. Consequently, if (4.1.3) holds uniformly in q , then $r' \geq 2n$.

Remark 61. Ostensibly it is unclear whether the above argument produces a sharp necessary condition. However, it is a well-known fact that

$$|\check{\sigma}(x)| \lesssim q^{-1/2} \quad \text{for } x \in \mathbb{F}_q^n \setminus \{0\} \quad (4.1.4)$$

and from this it follows (4.1.3) holds whenever $r' \geq 2n$. This gives moral support to the previous analysis. The estimate (4.1.4) follows from (deep) work of A. Weil [Wei48] on the Riemann hypothesis over curves in finite fields.

It transpires that the necessary conditions suggested by the above observations are also sufficient for this version of the finite field restriction problem.

Theorem 62 (Mockenhaupt and Tao [MT04]). *For $\text{char } \mathbb{F}_q > n$ and $\gamma := h$ given by the moment curve as above, the restriction estimate (4.1.1) holds uniformly in q if and only if*

$$1 \leq r \leq \frac{2n}{2n-1} \quad \text{and} \quad r' \geq sn. \quad (4.1.5)$$

The sufficiency of (4.1.5) can be proven by adapting a method of Christ [Chr85] to the finite field setting. It is interesting to note that Christ's argument does not produce the sharp range of exponents in the original context of restriction to the moment curve in \mathbb{R}^n , which is given by

$$1 \leq r < \frac{n^2 + n + 2}{n(n+1)} \quad \text{and} \quad r' \geq \frac{n(n+1)}{2} s. \quad (4.1.6)$$

Thus, there is some discrepancy between the behaviour of the finite field and real formulations of the restriction problem. This is partly due to the lack of available scales in \mathbb{F}_q^n , which leads to the weaker necessary condition $r' \geq sn$, and partly due to the improved decay rate (4.1.4) for the Fourier transform of the 'arc-length' measure σ .

Other interesting new phenomena of the kind described above was observed in [MT04] and in more recent work such as [IK10, LL12, Lew15].

4.2 Fourier restriction over rings of integers.

The finite field formulation of the restriction conjecture admits an interesting theory but, as observed in the previous section, it does not model all the relevant aspects of the Euclidean

problem. This is manifested in the weaker conditions on the Lebesgue exponents appearing in the statement of Theorem 62 compared with (4.1.6). There are, therefore, some inherent limitations in using Fourier analysis over finite fields to model Euclidean restriction phenomena. In addition, Wolff’s finite field Kakeya conjecture was completely resolved by Dvir [Dvi09], using a remarkably elementary argument; this has inspired some progress on the original real-variable problem (notably [Gut10]), but it appears Dvir’s methods cannot be translated into the setting of \mathbb{R}^n (at least in any reasonably direct manner).

One difference between \mathbb{F}_q^n and \mathbb{R}^n is that the natural topology on the former is discrete and therefore \mathbb{F}_q^n admits only one choice of scale. This contrast manifests itself in many different ways: for instance, when considering the \mathbb{F}_q^n analogue of the Knapp example one is led to the weaker necessary condition $r' \geq sn/d$ for the finite field restriction problem. In light of these observations and developments it is natural to ask whether one may formulate an alternative discrete version of the restriction conjecture which is endowed with a non-trivial notion of scale.

One possibility is to consider the restriction problem defined over rings of integers modulo a prime power. Fixing a prime p , consider the rings $G_\alpha := \mathbb{Z}/p^\alpha\mathbb{Z}$ for each $\alpha \in \mathbb{N}$ and define the Fourier transform $\hat{f} \in \ell^1(\hat{G}_\alpha^n)$ of a function $f \in \ell^1(G_\alpha^n)$ by

$$\hat{f}(\xi) := \sum_{x \in G_\alpha^n} f(x) e^{2\pi i x \cdot \xi / p^\alpha},$$

where the exponential is interpreted in the obvious manner. If $\gamma: G_\alpha^d \rightarrow G_\alpha^n$ is an injective polynomial mapping, one can consider Fourier restriction to the d -dimensional ‘submanifold’ Σ given by the image of γ . In particular, the set Σ is endowed with normalised counting measure and one now wishes to consider inequalities of the form

$$\|\hat{f}|_\Sigma\|_{L^s(\sigma)} \lesssim_\gamma \|f\|_{L^r(G_\alpha^n)}. \tag{4.2.1}$$

Since (4.2.1) involves norms over finite dimensional spaces, the estimate holds for all $1 \leq r, s \leq \infty$ with constants depending on p and α . Analogous to the finite field case, one can consider the problem of obtaining estimates which are uniform in p and α , or just α . The key difference between this and the finite field formulation is that here, for large α , there is a natural way to define a wide array of different scales. These scales arise by considering the highest power of p which divides a given $x \in \mathbb{Z}/p^\alpha\mathbb{Z}$; this is closely related to the definition of the p -adic valuation on \mathbb{Z} and, indeed, much of the subsequent analysis will be carried out in the p -adic setting (or, more precisely, in the more general setting of non-archimedean local fields). The restriction problem over rings of integers was first posed in unpublished work of J. Wright, who also recognised the key importance of the aforementioned notion of scaling and considered the analogous formulation of the Kakeya problem.²

As an example, let $p > n$, $k = 1$ and as before define $\gamma := h: G_\alpha \rightarrow G_\alpha^n$ to be the moment curve, given by the formula (4.1.2). In this situation, numbers of divisors can be used to construct Knapp-type examples which lead to a relatively strong necessary condition on the Lebesgue exponents for (4.2.1). Indeed, suppose α is large, let δ be some positive integer

²The Kakeya conjecture in the context of local fields is discussed in [EOT10], apparently without prior knowledge of the earlier work of J. Wright. Unpublished results of J. Wright on the local field Kakeya problem have been subsequently and independently rediscovered by Dummit and Hablicsek [DH13] and Fraser [Fra].

satisfying $n\delta < \alpha$ and, defining the ‘rectangle’

$$R_\delta := \{\xi \in \hat{G}_\alpha^n : p^{\delta j} |\xi_j| \text{ for } 1 \leq j \leq n\},$$

let $f_\delta: G_\alpha^n \rightarrow \mathbb{C}$ be given by $\hat{f}_\delta = \chi_{R_\delta}$. In this case the left-hand side of (4.2.1) equals $p^{-\delta/s}$ whilst, by a simple computation,

$$f(x) = \frac{1}{p^{\alpha n}} \sum_{\xi \in R_\delta} e^{2\pi i x \cdot \xi / p^\alpha} = p^{-\delta n(n+1)/2} \chi_{R_\delta^*}(x)$$

where R_δ^* is the dual rectangle

$$R_\delta^* := \{x \in G_\alpha^n : p^{\alpha - \delta j} |x_j| \text{ for } 1 \leq j \leq n\}.$$

Consequently,

$$\|f\|_{\ell^r(G_\alpha^n)} = p^{-\delta n(n+1)/2} [\#R_\delta^*]^{1/r} = p^{-\delta n(n+1)/2r'}.$$

Hence, if (4.2.1) holds uniformly in α , then

$$p^{-\delta/s} \lesssim p^{-\delta n(n+1)/2r'}$$

holds for all $\delta \in \mathbb{N}$ which forces

$$r' \geq n(n+1)s/2.$$

This is precisely one of the conditions (4.1.6) on the exponents for the Euclidean problem. Therefore, this initial analysis suggesting that working over $\mathbb{Z}/p^\alpha\mathbb{Z}$, and using the number of divisors as a notion of scale, may result in a model that very closely resembles the original Fourier restriction problem.

4.3 Basic properties of (non-archimedean) local fields

The following sections present some positive results on the restriction problem over $\mathbb{Z}/p^\alpha\mathbb{Z}$ obtained by the author and J. Wright. The methods used herein not only apply to the rings $\mathbb{Z}/p^\alpha\mathbb{Z}$, but to any sequence G_α of quotient rings associated to a (non-archimedean) local field K . Moreover, in order to study the problem over the G_α it will be useful to *lift* the analysis to K and consider Fourier restriction in the context of local fields. This lifting procedure will be described in a subsequent section. Presently the relevant definitions and basic results concerning local fields are briefly presented, in order to facilitate the subsequent analysis. For a more detailed exposition one may consult, for instance, [Sch06, §8 - §12] or [Cas86, Chapter 4].

Non-archimedean valuations

Let K be a field with a non-archimedean valuation $|\cdot|_K$. Hence, $|\cdot|_K: K \rightarrow [0, \infty)$ satisfies:

- i) $|x|_K = 0$ if and only if $x = 0$;
- ii) $|xy|_K = |x|_K |y|_K$ for all $x, y \in K$;
- iii) $|x + y|_K \leq \max\{|x|_K, |y|_K\}$ for all $x, y \in K$.

As a consequence of i) and ii), the image $|K^\times|_K$ of $K^\times = K \setminus \{0\}$ under $|\cdot|_K$ is a multiplicative subgroup of $(0, \infty)$; the valuation on K is said to be *discrete* if there exists some $\pi \in K$ such that

$$|\pi|_K \in (0, 1) \quad \text{and} \quad |K^\times|_K = \{|\pi|_K^n : n \in \mathbb{Z}\}. \quad (4.3.1)$$

Property iii) is referred to as the strong triangle inequality or non-archimedean property of the valuation. It is easy to see it implies $|x+y|_K = \max\{|x|_K, |y|_K\}$ whenever $|x|_K \neq |y|_K$ for some $x, y \in K$. It is remarked that $|\cdot|_K$ is said to be *archimedean* if it satisfies i), ii) and the usual triangle inequality, but iii) fails.

Two valuations $|\cdot|_K, |\cdot|'_K$ on K are said to be *equivalent* if they generate the same topology on K ; this is the case if and only if there exists some $c > 0$ such that $|\cdot|'_K = |\cdot|_K^c$. In particular, if $|\cdot|_K$ is discrete, then for any prescribed value $0 < r < 1$ there exists a valuation $|\cdot|'_K$ which is equivalent to $|\cdot|_K$ such that $|\pi|'_K = r$.

Rings of integers, uniformisers and series expansions

The set

$$\mathfrak{o} := \{x \in K : |x|_K \leq 1\}$$

is a local ring, referred to as *the ring of integers* of K . Its unique maximal ideal is given by

$$\mathfrak{p} := \{x \in \mathfrak{o} : |x|_K < 1\}$$

and the quotient ring $\mathfrak{o}/\mathfrak{p}$ (which is a field) is referred to as the *residue class field* of K .

If the valuation is discrete and $\pi \in K$ satisfies (4.3.1), then $\mathfrak{p} = \pi\mathfrak{o}$ and π is referred to as a *uniformiser*. Observe the choice of uniformiser is unique up to multiplication by an element of $\mathfrak{o}^\times = \{x \in \mathfrak{o} : |x|_K = 1\}$. Furthermore, each non-zero ideal in \mathfrak{o} is given by $\pi^\alpha\mathfrak{o}$ for a unique choice of $\alpha \in \mathbb{N}_0$.

Still assuming $|\cdot|_K$ is discrete, suppose $R \subset \mathfrak{o}$ is mapped bijectively to the residue class field under the canonical projection and, for simplicity, $0 \in R$. Then for each $a \in K^\times$ there exists a unique $N \in \mathbb{Z}$ and R -valued sequence $(a_j)_{j=N}^\infty$ such that

$$a = \sum_{j \geq N} a_j \pi^j = \pi^N (a_N + \pi a_{N+1} + \dots), \quad (4.3.2)$$

where $a_N \neq 0$. Conversely, if K is metrically complete, then the right-hand side of (4.3.2) defines an element of K for any choice of R -valued sequence $(a_j)_{j=N}^\infty$.

Local fields

Assume $|\cdot|_K$ is non-trivial in the sense that there exists a non-zero element $a \in K$ such that $|a|_K \neq 1$ and further assume K is metrically complete. In this situation one can show \mathfrak{o} is compact if and only if the valuation is discrete and the residue class field is finite. If these equivalent properties are satisfied, then K is said to be a (*non-archimedean*) *local field*.

Henceforth K denotes a local field with valuation $|\cdot|_K$ and uniformiser π . Thus, for some prime power $q = p^f$ the residue class field is isomorphic to \mathbb{F}_q . By possibly replacing the valuation with some equivalent version, one may assume without loss of generality $|\cdot|_K$ is normalised so that

$$|\pi|_K = q^{-1}.$$

This choice of normalisation will be tacitly assumed throughout the following sections.

Finally, for a local field K observe each of the quotient rings $G_\alpha := \mathfrak{o}/\pi^\alpha \mathfrak{o}$ is finite.

Key examples and the classification of local fields

There are two key examples of local fields.

Example 63 (p -adic numbers). The field of p -adic numbers \mathbb{Q}_p is the archetypical example of a local field. The ring of integers corresponds to the ring of p -adic integers \mathbb{Z}_p and the residue class field is isomorphic to \mathbb{F}_p . The uniformiser can be taken to be $p \in \mathbb{Q}_p$ so that (4.3.2) becomes the familiar series expansion of a p -adic number. Notice the usual p -adic valuation automatically has the desired normalisation.

The example of \mathbb{Q}_p is relevant to the problem of Fourier restriction over rings of integers modulo a prime power: observe in this case the sequence of quotient rings is given by $G_\alpha = \mathbb{Z}_p/p^\alpha \mathbb{Z}_p \cong \mathbb{Z}/p^\alpha \mathbb{Z}$ for $\alpha \in \mathbb{N}$.

Example 64 (Formal Laurent series in \mathbb{F}_q). Let k be a field and consider the set $k((X))$ of formal Laurent series over k . If $a_j(x)$ denotes the j th coefficient of $x \in k((X))$, then an addition and multiplication on $k((X))$ is defined by

$$a_j(x + y) := a_j(x) + a_j(y) \quad \text{and} \quad a_j(x.y) := \sum_{i \in \mathbb{Z}} a_i(x)a_{j-i}(y) \quad \text{for all } x, y \in k((X)).$$

With these operations $k((X))$ is a field. Now consider $K := \mathbb{F}_q((X))$ where \mathbb{F}_q is the field of cardinality $q = p^f$ for some prime p and, in analogy with the p -adics, define a valuation on K by

$$|x|_K := q^{-N} \quad \text{whenever } x = \sum_{j=N}^{\infty} a_j(x)X^j \text{ and } a_N(x) \neq 0$$

and $|0|_K := 0$. Then K is a local field with respect to this valuation. The ring of integers corresponds to the ring $\mathbb{F}_q[[X]]$ of formal power series over \mathbb{F}_q and the residue class field is isomorphic to \mathbb{F}_q . The uniformiser can be taken to be the linear polynomial X ; in this case the series expansion holds by definition. The valuation described above has the correct normalisation. Notice the key difference between this example and the p -adics is that here the arithmetic operations, defined with respect to the coefficients of the series expansion, no longer “carry” and, consequently, $\text{char } K = p$.

Remark 65. Any local field is a locally compact abelian group under addition and therefore admits a Fourier analysis (this will be developed in more detail in subsequent sections). Fourier analysis over the local field $\mathbb{F}_2((X))$ corresponds to the study of Walsh-Fourier series. This has played a prominent rôle as a discrete model for problems related to Carleson’s theorem and time-frequency analysis [DL12, DP14].

From Example 63 and Example 64 one may construct a plethora of non-isomorphic local fields by forming finite field extensions. Indeed, if K is a local field with valuation $|\cdot|_K$ and $L: K$ is a finite field extension, then there exists a unique valuation $|\cdot|_L$ on L which extends $|\cdot|_K$ and, furthermore, L is a local field with respect to $|\cdot|_L$.³ In fact, this exhausts the list of all possible local fields:

³This is perhaps surprising. Indeed, for any field extension $L: \mathbb{C}$ with $L \neq \mathbb{C}$ it is not possible to extend the usual (archimedean) valuation $|\cdot|$ on \mathbb{C} to L . This is one formulation of the Gelfand-Mazur theorem.

- If $\text{char } K = 0$, then K is isomorphic to a finite extension of \mathbb{Q}_p .
- If $\text{char } K = p$, then K is isomorphic to $\mathbb{F}_q((X))$ for some $q = p^f$.

It is remarked $\mathbb{F}_{p^f}((X))$ is a degree f extension of $\mathbb{F}_p((X))$. A proof of the above classification can be found in, for example, [Jac80, Chapter 9, §12].

4.4 Fourier analysis on local fields and their quotient rings

The analysis over $\mathbb{Z}/p^\alpha\mathbb{Z}$ described in Section 4.2 readily generalises to the setting of a sequence of quotient rings $G_\alpha := \mathfrak{o}/\pi^\alpha\mathfrak{o}$ of an arbitrary local field K . Here the rudiments of this theory are discussed.

Fourier analysis on quotient rings

Since G_α^n is a finite abelian group it is self-dual (in the Pontryagin sense). Moreover, if $\psi_\alpha \in \widehat{G}_\alpha$ is any non-principal character, then for any $\xi \in G_\alpha^n$ the map

$$\psi_\alpha^\xi(x) := \psi_\alpha(x \cdot \xi) \quad \text{for all } x \in G_\alpha^n \tag{4.4.1}$$

defines a character on G_α^n and the function $G_\alpha^n \rightarrow \widehat{G}_\alpha^n$ given by $\xi \mapsto \psi_\alpha^\xi$ is an isomorphism of topological groups. Although \widehat{G}_α^n and G_α^n will continue to be distinguished notationally, the identification $\widehat{G}_\alpha^n \cong G_\alpha^n$ will be heavily exploited and here \widehat{G}_α^n is essentially considered a ‘copy’ of G_α^n .

Definition 66. 1) Define the Fourier transform $\hat{f} \in \ell^1(\widehat{G}_\alpha^n)$ of $f \in \ell^1(G_\alpha^n)$ by

$$\hat{f}(\xi) := \sum_{x \in G_\alpha^n} f(x)\psi_\alpha(-x \cdot \xi) \quad \text{for all } \xi \in \widehat{G}_\alpha^n.$$

2) Similarly, define the inverse Fourier transform $\check{g} \in \ell^1(G_\alpha^n)$ of $g \in \ell^1(\widehat{G}_\alpha^n)$ by

$$\check{g}(x) := \frac{1}{\#\widehat{G}_\alpha^n} \sum_{\xi \in \widehat{G}_\alpha^n} g(\xi)\psi_\alpha(x \cdot \xi) \quad \text{for all } x \in G_\alpha^n.$$

Note that the Haar measure on G_α^n is taken to be counting measure, whilst on \widehat{G}_α^n it is taken to be counting measure normalised to have mass 1. With this choice of normalisation the inversion formula and Plancherel’s theorem hold (and are completely elementary).

A number of difficulties will arise when working in this discrete setting and, in order to circumvent them, it will be useful to lift the analysis to the local field K . The two theories - that is, Fourier analysis in the continuous setting of K and in the discrete setting of the G_α - are closely entwined via a correspondence principle described in a later subsection.

Fourier analysis on local fields

In this and the following subsection certain aspects of analysis over K are described in view of applications to the discrete Fourier restriction problem introduced in Section 4.2.

The vector space K^n is endowed with the norm

$$\|x\| := \max\{|x_j|_K : 1 \leq j \leq n\} \quad \text{for } x = (x_1, \dots, x_n) \in K^n. \tag{4.4.2}$$

This induces a system of balls

$$B_r(x) := \{y \in K^n : \|x - y\| \leq r\} \quad \text{for } x \in K^n \text{ and } r > 0$$

which are simultaneously open and closed; observe $B_{q^l}(0) = [\pi^l \mathfrak{o}]^n$ for each $l \in \mathbb{Z}$. Furthermore, with respect to the topology arising from these balls, K^n is a locally compact abelian (LCA) group under addition and therefore admits a Fourier analysis. Furthermore, if \hat{K}^n denotes the Pontryagin dual of K^n , then it transpires $\hat{K}^n \cong K^n$; in particular:

Proposition 67. *There exists some $\psi \in \hat{K}$ with the property that the restriction of ψ to \mathfrak{o} is a principal character whilst the restriction of ψ to $\pi^{-1}\mathfrak{o}$ is non-principal. Furthermore, given any $\xi \in K^n$ if one defines $\psi_\xi: K^n \rightarrow \mathbb{C}$ by $\psi_\xi(x) := \psi(x \cdot \xi)$ for $x \in K^n$ where $x \cdot \xi := \sum_{j=1}^n x_j \xi_j$, then $\xi \mapsto \psi_\xi$ is an isomorphism between K^n and \hat{K}^n .*

The reader is referred to [Tai75] for a proof of this proposition and a detailed exposition of many of the ideas discussed in the present section.

Henceforth, $\psi \in \hat{K}$ denotes a fixed character which admits the properties described in the preceding proposition and the character group \hat{K}^n is considered a ‘copy’ of K^n .

Both K^n and \hat{K}^n admit a Haar measure which is uniquely defined up to a choice of scaling factor. These measures are normalised so that the unit ball of each vector space (which is simply the n -fold product of the ring of integers of the underlying field) has mass 1.

Definition 68. 1) For $f \in L^1(K^n)$ define its Fourier transform $\hat{f}: \hat{K}^n \rightarrow \mathbb{C}$ by

$$\hat{f}(\xi) := \int_{K^n} f(x)\psi(-x \cdot \xi) dx \quad \text{for all } \xi \in \hat{K}^n. \quad (4.4.3)$$

2) Similarly, for $g \in L^1(\hat{K}^n)$ define its inverse Fourier transform $\check{g}: K^n \rightarrow \mathbb{C}$ by

$$\check{g}(x) := \int_{\hat{K}^n} g(\xi)\psi(x \cdot \xi) d\xi \quad \text{for all } x \in K^n.$$

The notation $dx, d\xi, dt \dots$ indicates here and in the subsequent text that the integration is with respect to the Haar measure on K^n or \hat{K}^n , for some dimension n (the value of n will always be clear from the context).

Remark 69. These definitions can be extended in the obvious manner in order to define the Fourier and inverse Fourier transforms of bounded, regular, complex-valued Borel measures on K^n and \hat{K}^n , respectively. The details are left to the reader.

The uncertainty principle and the Schwartz-Bruhat class

Many techniques in Euclidean Fourier analysis are guided by various heuristic principles which are collectively known as ‘the uncertainty principle’. One manifestation of this is that for $f \in L^1(\mathbb{R}^n)$ the following (non-rigorous) statement roughly holds:

$$f \text{ is spatially localised at scale } R \iff \hat{f} \text{ is essentially constant at scale } R^{-1}.$$

For instance, if f is supported on or concentrated in a ball of radius R , then one should, in principle, be able to treat the function \hat{f} as if it were constant on balls of radius R^{-1} . In this situation one says f has frequency uncertainty at scale R^{-1} .

It is possible to turn these heuristics into precise statements in a wide range of contexts but often there are significant limitations in how far they can be applied. For instance, one limiting factor is that it is not possible, in general, for a function on $\widehat{\mathbb{R}}^n$ to be simultaneously localised in both the spatial and frequency domain, viz.

Lemma 70. *If $f \in \mathcal{S}(\mathbb{R}^n)$ is such that both f and \hat{f} are compactly supported, then $f = 0$.*

Recall, $\mathcal{S}(\mathbb{R}^n)$ denotes the class of Schwartz functions on \mathbb{R}^n .

In the local field setting many of the technicalities which arise as a result of Lemma 70 are not present and the uncertainty principle is arguably more pronounced.

Definition 71. Let K be a local field.

- a) For $k, l \in \mathbb{N}$ denote by $\mathcal{S}(K^n; k, l)$ the vector subspace of $L^1(K^n)$ consisting of all $f: K^n \rightarrow \mathbb{C}$ which are supported on $B_{q^l}(0)$ and constant on cosets of $B_{q^{-k}}(0)$. A function $f \in \mathcal{S}(K^n; k, l)$ is said to have spatial uncertainty at scale q^{-k} and to be spatially localised to scale q^l .
- b) The union of the $\mathcal{S}(K^n; k, l)$ over all $k, l \in \mathbb{N}$ is denoted $\mathcal{S}(K^n)$. Note that $f \in \mathcal{S}(K^n)$ if and only if it is a finite linear combination of characteristic functions of balls.

The space $\mathcal{S}(K^n)$ is the local field analogue of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ from Euclidean analysis. It is remarked that both spaces can be viewed as particular instances of a more general construction, namely the *Schwartz-Bruhat* class on an arbitrary LCA group [Bru61, Osb75].⁴ At present only two properties of $\mathcal{S}(K^n)$ are noted; a more comprehensive discussion of these spaces can be found in, for instance, [Tai75]. First of all, it is a simple consequence of the Stone-Weierstrass theorem that $\mathcal{S}(K^n)$ is dense in $L^r(K^n)$ for $1 \leq r < \infty$. Secondly, and of direct relevance to the subsequent discussion, the following simple observation suggests Fourier analysis over local fields is a simpler affair compared with the Euclidean theory.

Lemma 72. *If $f \in \mathcal{S}(K^n; k, l)$, then $\hat{f} \in \mathcal{S}(\hat{K}^n; l, k)$.*

Notice, in stark contrast with Lemma 70, any $f \in \mathcal{S}(K^n)$ is simultaneously localised in both the spatial and frequency domains. Furthermore, each such f has spatial and frequency uncertainty at scales consistent with the uncertainty principle.⁵

A simple correspondence principle

By definition, any $f \in \mathcal{S}(K^n; k, l)$ can be considered a function of the cosets of $\pi^k \mathfrak{o}$ in $\pi^{-l} \mathfrak{o}$ and therefore, by rescaling, f can be identified with some $F \in \ell^1(G_{k+l}^n)$. This observation forms the basis of a correspondence principle which allows one to ‘lift’ certain problems over the rings G_α to problems over K .

Definition 73. For each $\alpha, n \in \mathbb{N}$ let $\iota_{\alpha, n}$ denote the canonical bijection $\iota_{\alpha, n}: G_\alpha^n \rightarrow \{x \in K^n : x = \sum_{j=0}^{\alpha-1} x_j \pi^j\}$.

Often it will be convenient to drop one or both of the subscripts and write either ι or ι_α for $\iota_{\alpha, n}$, if the meaning is apparent from the context. In general, ι_α is **not** an additive group homomorphism; however, the following identities hold:

$$\iota_\alpha(x + y) \equiv \iota_\alpha(x) + \iota_\alpha(y) \pmod{\pi^\alpha \mathfrak{o}^n} \quad \text{for all } x, y \in G_\alpha^n$$

⁴Strictly speaking, these spaces are, by definition, *topological* vector spaces; a description of their topology is not provided here as it plays no rôle in the forthcoming analysis.

⁵This ‘perfect localisation’ makes Fourier analysis on K particularly amenable to time-frequency techniques, cf. Remark 65.

and

$$\iota_\alpha(xy) \equiv \iota_\alpha(x)\iota_\alpha(y) \pmod{\pi^\alpha \mathfrak{o}} \quad \text{for all } x, y \in G_\alpha.$$

Using this notation, it is useful to refine Definition 66 so that the Fourier transform on G_α^n is compatible with the Fourier transform on K^n .

Definition 74. For ψ as above,

$$\psi_\alpha(x) := \psi(\pi^{-\alpha}\iota_\alpha(x)) \quad \text{for all } x \in G_\alpha^n \quad (4.4.4)$$

defines a non-principal character on G_α^n .

Henceforth it is assumed the character appearing in Definition 66 is given by (4.4.4).

With this choice of character one may define a natural linear bijection between $\mathcal{S}(K^n; k, l)$ and $\ell^1(G_{k+l}^n)$.

Definition 75. For $k, l \in \mathbb{N}_0$, $k + l \neq 0$ let $T_{k,l}: \mathcal{S}(K^n; k, l) \rightarrow \ell^1(G_{k+l}^n)$ be the map given by

$$T_{k,l}f(x) := q^{-kn}f(\pi^{-l}\iota_{k+l}(x)) \quad \text{for all } x \in G_{k+l}^n.$$

The choice of normalisation of the Haar measure on the dual group \widehat{G}_{k+l}^n suggests it is natural to consider rescaled versions of the maps $T_{k,l}$ in the dual setting.

Definition 76. For $k, l \in \mathbb{N}_0$, $k + l \neq 0$ let $\widehat{T}_{l,k}: \mathcal{S}(\widehat{K}^n; l, k) \rightarrow \ell^1(\widehat{G}_{k+l}^n)$ be the map given by

$$\widehat{T}_{l,k}f(x) := f(\pi^{-k}\iota_{k+l}(x)) \quad \text{for all } x \in \widehat{G}_{k+l}^n.$$

Clearly both $T_{k,l}$ and $\widehat{T}_{l,k}$ are linear mappings and are easily seen to be bijective.

Lemma 77. For any $f \in \mathcal{S}(K^n; k, l)$ the following hold:

- a) $\|T_{k,l}f\|_{\ell^r(G_{k+l}^n)} = q^{-kn/r'}\|f\|_{L^r(K^n)}$ for $1 \leq r \leq \infty$;
- b) $\|\widehat{T}_{l,k}\widehat{f}\|_{\ell^s(\widehat{G}_{k+l}^n)} = q^{ln/s}\|f\|_{L^s(\widehat{K}^n)}$ for $1 \leq s \leq \infty$.

Proof. Fix $f \in \mathcal{S}(K^n; k, l)$ and observe

$$\|T_{k,l}f\|_{\ell^r(G_{k+l}^n)}^r = q^{-knr} \sum_{z \in G_{k+l}^n} |f(\pi^{-l}\iota(z))|^r = q^{-knr} \sum_{y \in \mathfrak{Im}(\pi^{-l}\iota)} |f(y)|^r$$

where $\mathfrak{Im}(\pi^{-l}\iota) \subset K^n$ denotes the image of the injective mapping $z \mapsto \pi^{-l}\iota(z)$ on G_{k+l}^n . Since f is constant on cosets of $\pi^k\mathfrak{o}^n$, it follows that the above expression may be written as

$$q^{-kn(r-1)} \sum_{y \in \mathfrak{Im}(\pi^{-l}\iota)} \int_{B_{q^{-k}}(y)} |f(x)|^r dx.$$

One may easily observe $\mathfrak{Im}(\pi^{-l}\iota)$ forms a complete set of coset representatives of $\pi^k\mathfrak{o}^n$ in $\pi^{-l}\mathfrak{o}^n$ and hence

$$\|T_{k,l}f\|_{\ell^r(G_{k+l}^n)}^r = q^{-kn(r-1)} \int_{B_{q^l}(0)} |f(x)|^r dx.$$

Taking the $1/r$ -power of the preceding identity completes the proof of a) whilst b) follows from the same argument, *mutatis mutandis*. \square

A similar argument shows the mappings $T_{k,l}$ and $\widehat{T}_{l,k}$ behave well with respect to taking Fourier transforms.

Lemma 78. *The diagram*

$$\begin{array}{ccc} \mathcal{S}(K^n; k, l) & \xrightarrow{\mathcal{F}} & \mathcal{S}(\widehat{K}^n; l, k) \\ T_{k,l} \downarrow & & \downarrow \widehat{T}_{l,k} \\ \ell^1(G_{k+l}^n) & \xrightarrow{\mathcal{F}} & \ell^1(\widehat{G}_{k+l}^n) \end{array}$$

commutes, where each occurrence of \mathcal{F} denotes the appropriate Fourier transform.

Proof. Given $f \in \mathcal{S}(K^n; k, l)$ observe

$$\begin{aligned} (T_{k,l}f)^\wedge(\xi) &= q^{-kn} \sum_{z \in G_{k+l}^n} f(\pi^{-l}\iota(z))\psi(-\pi^{-(k+l)}\iota(z)\cdot\iota(\xi)) \\ &= q^{-kn} \sum_{y \in \mathfrak{Im}(\pi^{-l}\iota)} f(y)\psi(-\pi^{-k}y\cdot\iota(\xi)) \end{aligned}$$

for any $\xi \in \widehat{G}_{k+l}^n$. Arguing as in the proof of Lemma 77, one deduces that the right-hand side of the above expression is equal to

$$\sum_{y \in \mathfrak{Im}(\pi^{-l}\iota)} \int_{B_{q^{-k}}(y)} f(x)\psi(-\pi^{-k}x \cdot \iota(\xi)) dx = \int_{B_{q^l}(0)} f(x)\psi(-x \cdot \pi^{-k}\iota(\xi)) dx$$

and, recalling the definition of $\widehat{T}_{l,k}f^\wedge(\xi)$, this concludes the proof. \square

It will be convenient to have explicit formulae for the inverses of $T_{k,l}$ and $\widehat{T}_{l,k}$.

Definition 79. Note that $x \in K^n$ can be uniquely written as

$$x = \sum_{j=N}^{\infty} x_j \pi^j$$

where each $x_j \in R^n$, $x_N \neq 0$ and R is, as in (4.3.2), a set of representatives for the residue class field with $0 \in R$. Define the projection of x modulo $\pi^\alpha \mathfrak{o}^n$ to be the element $\{x\}_\alpha := \sum_{j=N}^{\alpha-1} x_j \pi^j \in K^n$ where the sum is understood to equal 0 if $N \geq \alpha$.

Having made this definition, one easily observes that

$$T_{k,l}^{-1}F(x) = \begin{cases} q^{kn} F \circ \iota_{k+l}^{-1}(\{\pi^l x\}_{k+l}) & \text{if } x \in B_{q^l}(0) \\ 0 & \text{otherwise} \end{cases}$$

for all $F \in \ell^1(G_{k+l}^n)$, whilst

$$\widehat{T}_{l,k}^{-1}\widehat{F}(\xi) = \begin{cases} \widehat{F} \circ \iota^{-1}(\{\pi^k \xi\}_{k+l}) & \text{if } \xi \in B_{q^k}(0) \\ 0 & \text{otherwise} \end{cases} \quad (4.4.5)$$

for all $\widehat{F} \in \ell^1(\widehat{G}_{k+l}^n)$.

4.5 The Fourier restriction problem over local fields and their quotient rings

In this section Fourier restriction is considered with respect to ‘ d -surfaces’ in both \hat{K}^n and the sequence of quotient rings \hat{G}_α^n and it is shown that the resulting theories are equivalent.

Fourier restriction over local fields

For $1 \leq d \leq n - 1$ let $\gamma: \mathfrak{o}^d \rightarrow \hat{K}^{n-d}$ be a polynomial mapping and consider the d -dimensional surface which is graph parametrised by γ , viz.

$$\Sigma := \{\Gamma(t) : t \in \mathfrak{o}^d\},$$

where $\Gamma: \mathfrak{o}^d \rightarrow \hat{K}^n$ is given by $\Gamma(t) := (t, \gamma(t))$. This is a compact subset of \hat{K}^n to which one may associate a natural measure σ given by

$$\int_{\Sigma} h \, d\sigma := \int_{\mathfrak{o}^d} h \circ \Gamma(t) \, dt$$

for all, say, continuous functions $h: \Sigma \rightarrow \mathbb{C}$.

Problem 80 (Restriction over local fields). *For which Lebesgue exponents r, s does the a priori inequality*

$$\|\hat{f}|_{\Sigma}\|_{L^s(\Sigma, \sigma)} \leq C_{\Sigma, r, s} \|f\|_{L^r(K^n)} \tag{4.5.1}$$

hold?

In this context, an estimate is *a priori* if it holds for all $f \in \mathcal{S}(K^n)$. Recall, for $1 \leq r < \infty$ the class $\mathcal{S}(K^n)$ is dense in $L^r(K^n)$ and for $f \in \mathcal{S}(K^n)$ the function $\mathfrak{R}f := \hat{f}|_{\Sigma}$ is explicitly (and therefore well-) defined by the formula (4.4.3). Consequently, once (4.5.1) is established for some $1 \leq r, s \leq \infty$ with $r < \infty$ one may extend the operator \mathfrak{R} uniquely to a bounded linear operator on the whole of $L^r(K^n)$ into $L^s(\Sigma)$.

Definition 81. For $1 \leq r, s \leq \infty$ let $\mathfrak{R}(\Sigma; r \rightarrow s) \in [0, \infty]$ denote the infimum over all values of $C_{\Sigma, r, s}$ for which (4.5.1) holds, with the convention that $\mathfrak{R}(\Sigma; r \rightarrow s) = \infty$ if the estimate fails.

The range of exponents for which $\mathfrak{R}(\Sigma; r \rightarrow s) < \infty$ will depend on the geometry of Σ .

Fourier restriction over quotient rings

The estimate (4.5.1) can be translated into the setting of the G_α via the correspondence principle outlined earlier, leading to an equivalent discrete formulation of Problem 80. Here Fourier restriction is introduced in the context of the quotient rings; the equivalence of the two formulations is detailed in the following subsection.

For each $\alpha \in \mathbb{N}$ consider the polynomial mapping $\gamma_\alpha: G_\alpha^d \rightarrow G_\alpha^{n-d}$ whose components are given by the corresponding components of γ modulo $\pi^\alpha \mathfrak{o}$. Explicitly,

$$\gamma_\alpha := \iota_\alpha^{-1} \circ \gamma \circ \iota_\alpha,$$

where the ι_α are as in Definition 73. Define

$$\Sigma_\alpha := \{\Gamma_\alpha(t) : t \in G_\alpha^d\}$$

where $\Gamma_\alpha: G_\alpha^d \rightarrow \widehat{G}_\alpha^n$ is given by $\Gamma_\alpha(t) := (t, \gamma_\alpha(t))$, noting $\Sigma_\alpha = \iota_\alpha^{-1}\{\Sigma\}_\alpha$. Endow each Σ_α with the normalised counting measure σ_α given by

$$\int_{\Sigma_\alpha} H \, d\sigma_\alpha := \frac{1}{\#G_\alpha^d} \sum_{t \in G_\alpha^d} H \circ \Gamma_\alpha(t)$$

for all $H: \Sigma_\alpha \rightarrow \mathbb{C}$.

Problem 82 (Restriction over quotient rings). *For which Lebesgue exponents r, s does the inequality*

$$\|\hat{F}|_{\Sigma_\alpha}\|_{\ell^s(\Sigma_\alpha, \sigma_\alpha)} \leq C_{\Sigma, r, s} \|F\|_{\ell^r(G_\alpha^n)} \quad (4.5.2)$$

hold for all $F \in \ell^1(G_\alpha^n)$ **uniformly** in $\alpha \in \mathbb{N}$?

Observe the function spaces appearing in Problem 82 are all finite-dimensional and hence, by equivalence of norms, it is clear that for any fixed α the inequality (4.5.2) holds with a constant depending on α for all $1 \leq r, s \leq \infty$. The non-trivial component of the problem is to obtain estimates which are independent of α .

Definition 83. For $1 \leq r, s \leq \infty$ let $\mathfrak{R}(\Sigma_\alpha; r \rightarrow s) \in [0, \infty]$ denote the smallest value of $C_{\Sigma, r, s}$ with which (4.5.2) holds uniformly in $\alpha \in \mathbb{N}$, with the convention $\mathfrak{R}(\Sigma_\alpha; r \rightarrow s) = \infty$ if the uniform estimate fails.

Such uniform estimates cannot hold for all choices of $1 \leq r, s \leq \infty$. Furthermore, as in the local field case, the range of exponents for which $\mathfrak{R}(\Sigma_\alpha; r \rightarrow s) < \infty$ will depend on geometric considerations.

Equivalence of restriction over local fields and quotient rings

The two formulations of the restriction problem outlined above are equivalent in a very precise sense.

Proposition 84. *For Σ as above and all $1 \leq r, s \leq \infty$,*

$$\mathfrak{R}(\Sigma, r \rightarrow s) = \mathfrak{R}(\Sigma_\alpha, r \rightarrow s).$$

This observation is based on considering *local restriction* estimates in the non-archimedean setting. In particular, for any $l \in \mathbb{N}$ the validity of (4.5.1) immediately implies the local version

$$\|\hat{f}|_\Sigma\|_{L^s(\Sigma, \sigma)} \leq C_{\Sigma, r, s} \|f\|_{L^r(B_{q^l}(0))} \quad (4.5.3)$$

whenever $f \in \mathcal{S}(K^n)$ is supported in $B_{q^l}(0)$. Furthermore, (4.5.1) is equivalent to (4.5.3) holding uniformly for all $l \in \mathbb{N}$. A striking aspect of the local field restriction theory is that (4.5.3) is equivalent to an estimate in discrete setting of G_l^n , as shown below. This observation leads directly to the proof of Proposition 84.

In the Euclidean context, the study of various analogues of (4.5.3) forms an invaluable part of restriction theory. The spatial localisation allows for application of (rigorous forms of) the uncertainty principle, leading to various useful reformulations of the problem. Much of the Euclidean theory can easily be adapted to the local field setting, often with simplified proofs. For example, one expects the spatial localisation to scale q^l in (4.5.3) to induce a frequency uncertainty at scale q^{-l} and one manifestation of this is the following lemma.

Lemma 85. *The estimate (4.5.3) is equivalent to*

$$\|\hat{f}\|_{L^s(\mathcal{N}_{q^{-l}}(\Sigma))} \leq C_{\Sigma,r,s} q^{-l(n-d)/s} \|f\|_{L^r(B_{q^l}(0))} \quad (4.5.4)$$

for all $f \in \mathcal{S}(K^n)$ supported in $B_{q^l}(0)$, where the constants appearing in both inequalities are identical.

Here,

$$\mathcal{N}_r(E) := \{y \in \hat{K}^n : \|x - y\| \leq r \text{ for some } x \in E\}$$

denotes the r -neighbourhood of a set $E \subseteq \hat{K}^n$.

This is the non-archimedean version of a well-known fact in the Euclidean case (see, for instance, [Tao04]). The simple proof is postponed until the end of the section. The connection between Problem 82 and local restriction theory now becomes apparent due to the simple observation

$$\mathcal{N}_{q^{-l}}(\Sigma) = \left\{ x \in K^n : \{\pi^k x\}_{k+l} \in \{\pi^k \Sigma\}_{k+l} \right\}.$$

Letting

$$(\pi^k \Sigma)_{k+l} := \iota_{k+l}^{-1}(\{\pi^k \Sigma\}_{k+l}) \subseteq \hat{G}_{k+l}^n$$

and recalling (4.4.5), it follows that

$$\chi_{\mathcal{N}_{q^{-l}}(\Sigma)} = \hat{T}_{l,k}^{-1} \chi_{(\pi^k \Sigma)_{k+l}}. \quad (4.5.5)$$

The neighbourhood $\mathcal{N}_{q^{-l}}(\Sigma) \subset \hat{K}^n$ can therefore be thought of as corresponding to a family of ‘surfaces’ $(\pi^k \Sigma)_{k+l}$ in the discrete groups \hat{G}_{k+l}^n .

Lemma 86. *For any $l \in \mathbb{N}_0$ the inequality (4.5.4) holds for all $f \in \mathcal{S}(K^n)$ supported in $B_{q^l}(0)$ if and only if*

$$\|\hat{F}|_{\Sigma_l}\|_{\ell^s(\Sigma_l, \sigma_l)} \leq C_{\Sigma,r,s} \|F\|_{\ell^r(G_l^n)} \quad (4.5.6)$$

holds for all $F \in \ell^1(G_l^n)$, where the constants appearing in both inequalities are identical.

Proposition 84 is now an immediate consequence of Lemma 85 and Lemma 86, which are proved presently.

Proof (of Lemma 86). Fix $f \in \mathcal{S}(K^n; k, l)$ for some $k \in \mathbb{N}_0$ and observe (4.5.5) together with Lemma 78 imply

$$\chi_{\mathcal{N}_{q^{-l}}(\Sigma)} \hat{f} = \hat{T}_{l,k}^{-1} (\chi_{(\pi^k \Sigma)_{k+l}} \hat{T}_{l,k} \hat{f}) = \hat{T}_{l,k}^{-1} (\chi_{(\pi^k \Sigma)_{k+l}} (T_{k,l} f)^\wedge).$$

From this and Lemma 77 b) one deduces that

$$\|\hat{f}\|_{L^s(\mathcal{N}_{q^{-l}}(\Sigma))} = q^{-l(n-d)/s} \left(\frac{1}{\#(\pi^k \Sigma)_{k+l}} \sum_{\xi \in (\pi^k \Sigma)_{k+l}} |(T_{k,l} f)^\wedge(\xi)|^s \right)^{1/s}.$$

Combining these observations together with Lemma 77 a), it follows that the local restriction estimate (4.5.4) holds for all $f \in \mathcal{S}(K^n)$ supported in $B_{q^l}(0)$ if and only if for every $k \in \mathbb{N}_0$ the estimate

$$\left(\frac{1}{\#(\pi^k \Sigma)_{k+l}} \sum_{\xi \in (\pi^k \Sigma)_{k+l}} |\hat{F}(\xi)|^s \right)^{1/s} \leq C_{\Sigma,r,s} q^{kn/r'} \|F\|_{\ell^r(G_{k+l}^n)} \quad (4.5.7)$$

holds for all $F \in \ell^1(G_{k+l}^n)$. Observe (4.5.6) is precisely the $k = 0$ case of the above inequality; it therefore suffices to show that (4.5.6) implies (4.5.7) holds for all $k \in \mathbb{N}_0$. For $F \in \ell^1(G_{k+l}^n)$ define the function $F_l \in \ell^1(G_l^n)$ by

$$F_l(z) := \sum_{x \equiv z \pmod{\pi^l}} F(x) \quad \text{for all } z \in G_l^n$$

where the notation $x \equiv z \pmod{\pi^l}$ indicates the sum is over the set of $x \in G_{k+l}^n$ such that⁶

$$x \equiv \iota_{k+l}^{-1} \circ \iota_l(z) \pmod{\iota_{k+l}^{-1}(\pi^l)G_{k+l}^n}.$$

For any $\xi \in G_{k+l}^n$ it follows that

$$\hat{F}(\iota_{k+l}^{-1}(\pi^k)\xi) = \sum_{z \in G_l^n} \sum_{x \equiv z \pmod{\pi^l}} F(x) \psi(-\pi^{-l} \iota_{k+l}(x) \cdot \iota_{k+l}(\xi)) = \hat{F}_l \circ \iota_l^{-1} \circ \iota_{k+l}(\xi).$$

Observing the identity

$$\Gamma_l = \iota_l^{-1} \circ \iota_{k+l} \circ \Gamma_{k+l} \circ \iota_{k+l}^{-1} \circ \iota_l,$$

it follows that

$$\sum_{\xi \in (\pi^k \Sigma)_{k+l}} |\hat{F}(\xi)|^s = \sum_{t \in G_l^d} |\hat{F}(\iota_{k+l}^{-1}(\pi^k)\Gamma_{k+l} \circ \iota_{k+l}^{-1} \circ \iota_l(t))|^s = \sum_{t \in G_l^d} |\hat{F}_l \circ \Gamma_l(t)|^s.$$

Applying (4.5.6) and noting $\#(\pi^k \Sigma)_{k+l} = \#\Sigma_l$, one concludes that

$$\begin{aligned} \left(\frac{1}{\#(\pi^k \Sigma)_{k+l}} \sum_{\xi \in (\pi^k \Sigma)_{k+l}} |\hat{F}(\xi)|^s \right)^{1/s} &\leq C_{\Sigma, r, s} \|F_l\|_{\ell^r(G_l^n)} \\ &\leq C_{\Sigma, r, s} q^{kn/r'} \|F\|_{\ell^r(G_{k+l}^n)}, \end{aligned}$$

where the final estimate is due to Hölder's inequality. \square

The proof of Proposition 84 has therefore been reduced to verifying Lemma 85. This is easily achieved by modifying Euclidean arguments.

Proof (of Lemma 85). Assume (4.5.3) holds and let $f \in \mathcal{S}(K^n)$ with $\text{supp } f \subseteq B_{q^l}(0)$. It follows from (4.5.3) and translation invariance that

$$\left(\int_{\Sigma} \int_{B_{q^{-l}}(0)} |\hat{f}(\xi + \eta)|^s d\eta d\sigma(\xi) \right)^{1/s} \leq C_{\Sigma, r, s} q^{-ln/s} \|f\|_{L^r(B_{q^l}(0))}.$$

The left-hand (double) integral may be decomposed as

$$\sum_{z \leq \pi^l} \int_{B_{q^{-l}}(z)} \int_{B_{q^{-l}}(\Gamma(t))} |\hat{f}(\eta)|^s d\eta dt,$$

where the sum is over all $z \in \mathfrak{o}^d$ whose series expansion is of the form $z = \sum_{j=0}^{l-1} z_j \pi^j$. If $t \in B_{q^{-l}}(z)$, then $\Gamma(t) \in B_{q^{-l}}(\Gamma(z))$ and so $B_{q^{-l}}(\Gamma(t)) = B_{q^{-l}}(\Gamma(z))$. From this identity one

⁶Equivalently, the sum is over the set $\iota_{k+l}^{-1}(\{B_{q^{-l}}(\iota_l(z))\}_{k+l})$.

deduces that the above expression is equal to

$$\sum_{z \leq \pi^l} |B_{q^{-l}}(z)| \int_{B_{q^{-l}}(\Gamma(z))} |\hat{f}(\eta)|^s d\eta = q^{-ld} \int_{\mathcal{N}_{q^{-l}}(\Sigma)} |\hat{f}(\eta)|^s d\eta,$$

where the final equality is due to the simple observation that the $B_{q^{-l}}(\Gamma(z))$ form a partition of $\mathcal{N}_{q^{-l}}(\Sigma)$. The desired estimate (4.5.4) now immediately follows.

Now suppose (4.5.4) holds and again let $f \in \mathcal{S}(K^n)$ with $\text{supp } f \subseteq B_{q^l}(0)$. Hence $f = f \chi_{B_{q^l}(0)}$, which leads to the reproducing formula $\hat{f} = \hat{f} * \hat{\chi}_{B_{q^l}(0)}$ and, in particular,

$$\|\hat{f}|_{\Sigma}\|_{L^s(\Sigma)} = \|\hat{f} * \hat{\chi}_{B_{q^l}(0)}|_{\Sigma}\|_{L^s(\Sigma)}.$$

To bound the latter quantity, notice

$$\hat{f} * \hat{\chi}_{B_{q^l}(0)} = \int_{\mathcal{N}_{q^{-l}}(\Sigma)} K(\xi, \eta) \hat{f}(\eta) d\eta$$

where $K(\xi, \eta) := \hat{\chi}_{B_{q^l}(0)}(\xi - \eta)$ for $(\xi, \eta) \in \Sigma \times \mathcal{N}_{q^{-l}}(\Sigma)$ is easily seen to satisfy

$$\sup_{\xi \in \Sigma} \int_{\mathcal{N}_{q^{-l}}(\Sigma)} |K(\xi, \eta)| d\eta \leq 1 \quad \text{and} \quad \sup_{\eta \in \mathcal{N}_{q^{-l}}(\Sigma)} \int_{\Sigma} |K(\xi, \eta)| d\sigma(\xi) \leq q^{l(n-d)}.$$

Hence, by the classical Schur test

$$\|\hat{f} * \hat{\chi}_{B_{q^l}(0)}\|_{L^s(\Sigma)} \leq q^{l(n-d)/s} \|\hat{f}\|_{L^s(\mathcal{N}_{q^{-l}}(\Sigma))}$$

and applying the hypothesised estimate yields (4.5.3). □

4.6 Fourier restriction to curves in $[\mathbb{Z}/p^\alpha\mathbb{Z}]^n$

Having developed a fairly general restriction theory, this section considers the more concrete problem of Fourier restriction to the moment curve. For simplicity, the main result is stated over rings of integers modulo a prime power. Fix a dimension $n \geq 2$ and let $p > n$ be a prime. For $\alpha \in \mathbb{N}$ let $\gamma_\alpha: \mathbb{Z}/p^\alpha\mathbb{Z} \rightarrow [\mathbb{Z}/p^\alpha\mathbb{Z}]^n$ be given by the familiar polynomial mapping

$$t \mapsto \left(t, \frac{t^2}{2}, \dots, \frac{t^n}{n!}\right), \tag{4.6.1}$$

noting this is well-defined by the hypothesis on p . The uniform mapping properties of the family of restriction operators associated to these curves are completely determined by the following theorem and subsequent remarks.

Theorem 87 (Hickman and Wright). *For p and γ_α as above, the restriction estimate*

$$\left(\frac{1}{p^\alpha} \sum_{t \in \mathbb{Z}/p^\alpha\mathbb{Z}} |\hat{F} \circ \gamma_\alpha(t)|^s\right)^{1/s} \lesssim_p \left(\sum_{x \in [\mathbb{Z}/p^\alpha\mathbb{Z}]^n} |F(x)|^r\right)^{1/r} \tag{4.6.2}$$

holds uniformly in α provided that

$$1 \leq r < \frac{n^2 + n + 2}{n(n+1)} \quad \text{and} \quad r' \geq \frac{n(n+1)}{2}s. \quad (4.6.3)$$

The condition $r' \geq n(n+1)s/2$ is necessary, as discussed in Section 4.2. On the other hand, it was shown by J. Wright that the condition $1 \leq r < (n^2 + n + 2)/(n^2 + n)$ is also necessary. As in the finite field case, this condition arises from determining the range of r' for which

$$\|\check{\sigma}_\alpha\|_{\ell^{r'}(\mathbb{Z}/p^\alpha\mathbb{Z})^n} \lesssim 1, \quad (4.6.4)$$

where the bound is understood to be uniform in α and σ_α is the normalised counting measure on γ_α . J. Wright employed number-theoretic results of G. I. Arkipov, V. N. Chubarikov and A. A. Karatsuba (see [ACK87]) to prove the following:

Proposition 88 (J. Wright, unpublished). *The inequality (4.6.4) holds if $r < (n^2 + n + 2)/(n^2 + n)$ and fails if $r \geq (n^2 + n + 2)/(n^2 + n)$.*

Thus, Theorem 87 is sharp.

Remark 89. When $n = 2$ the Fourier transform $\check{\sigma}(x)$ is given by a Gauss sum and the condition $r < 4/3$ can be shown to be necessary for (4.6.4) by applying elementary identities for such sums.

Remark 90. The condition (4.6.3) on the Lebesgue exponents corresponds precisely to the sharp range of Lebesgue space estimates for restriction to a compact piece of the moment curve in \mathbb{R}^n . Thus, the behaviour of the discrete model closely matches that of the original real-variable operator. This is in contrast with the finite field formulation: there the sharp range was given by the weaker conditions $1 \leq r \leq 2n/(2n-1)$ and $r' \geq ns$.

To prove Theorem 87 the correspondence principle will be exploited. In particular, the analogous result will be proven over local fields K satisfying either $\text{char}(K) > n$ or $\text{char}(K) = 0$. Fixing such a field, let $\gamma: K \rightarrow \hat{K}^n$ be the moment curve, defined by the same formula as that appearing in (4.6.1). The condition on $\text{char}(K)$ ensures this is well-defined. Furthermore, it is assumed $|j|_K = 1$ for all $1 \leq j \leq n$ so that the restriction of γ to the ring of integers \mathfrak{o} maps into \mathfrak{o}^n . By a slight abuse of notation, the image of γ will also be denoted by γ .

Theorem 91 (Hickman and Wright). *For K and γ as above, the restriction estimate*

$$\|\hat{f}|_\gamma\|_{L^s(\gamma)} \lesssim_q \|f\|_{L^r(K^n)}$$

holds whenever

$$1 \leq r < \frac{n^2 + n + 2}{n(n+1)} \quad \text{and} \quad r' = \frac{n(n+1)}{2}s. \quad (4.6.5)$$

Here q denotes the cardinality of the residue class field of K .

The more restrictive hypotheses on the Lebesgue exponents reflect the fact that here the curve is non-compact. Restricting the domain of the map γ to \mathfrak{o} , one may immediately deduce Theorem 87 as a consequence of the $K = \mathbb{Q}_p$ case of Theorem 91 and the correspondence principle, together with Hölder's inequality.

Analytic preliminaries: a change of variables formula

A rather routine but nevertheless important tool in the proof of Theorem 91 is a local field variant of the classical change of variables formula.

Proposition 92 (Change of variables formula). *Let $U \subseteq K^n$ be open, $\Phi: U \rightarrow K^n$ be a polynomial mapping and suppose J_Φ is non-vanishing on U . Then*

$$\int_U f(u)|J_\Phi(u)|_K du = \int_{\Phi(U)} \sum_{u \in U: \Phi(u)=x} f(u) dx$$

holds for all non-negative, measurable functions f on U .

Implicit in the statement of Proposition 92 is the fact that the function appearing in the right-hand integral is measurable.

Here J_Φ denotes the Jacobian determinant of Φ where, for the sake of simplicity, the derivatives of a polynomial over K can be understood in the formal sense. Variants of this result can be found in [Igu00, §7.4] and [Sch06, §A.7], but for the reader’s convenience a simple proof of Proposition 92 is presented at the end of this chapter.

Remark 93. Under the hypotheses of Proposition 92, if $f: \Phi(U) \rightarrow [0, \infty]$ is measurable, then

$$\int_U f \circ \Phi(u) du = \int_{\Phi(U)} f(x) \sum_{u \in U: \Phi(u)=x} |J_\Phi(u)|_K^{-1} dx. \tag{4.6.6}$$

The analogous result in the discrete setting is trivial. For $G_\alpha = \mathfrak{o}/\pi^\alpha \mathfrak{o}$, let $U_\alpha \subseteq G_\alpha^n$ and $\Phi_\alpha: U_\alpha \rightarrow G_\alpha^n$ be a polynomial mapping. If $F: \Phi_\alpha(U_\alpha) \rightarrow \mathbb{C}$ is any function, then⁷

$$\sum_{u \in U_\alpha} F \circ \Phi_\alpha(u) = \sum_{x \in \Phi_\alpha(U_\alpha)} F(x) \sum_{u \in U: \Phi(u)=x} 1. \tag{4.6.7}$$

Thus, both (4.6.6) and (4.6.7) suggest that to effectively utilise a change of variables one must have sufficient understanding of

$$\{u \in U : \Phi(u) = x\},$$

which is a solution set of a system of polynomial equations. In what follows, the key advantage of the local field formulation of the restriction problem (as opposed to working directly over the quotient rings) is that it is significantly easier to determine the solution sets of the polynomial equations of interest when working over K .

A convolution identity

By the usual duality arguments, Theorem 91 is equivalent to determining the Lebesgue space mapping properties of the extension operator associated to γ , defined by

$$\mathcal{E}_\gamma g(x) := \int_K g(t)\psi(x \cdot \gamma(t)) dt$$

⁷Of course, there is nothing special about G_α^n and Φ_α other than the fact the former is a finite set and the latter is a function. The restriction to this particular case is merely for expository purposes.

for $g \in L^1(K)$. By real interpolation it will also suffice to consider weak-type estimates for \mathcal{E} . Fixing $E \subseteq K$ a Borel set and letting τ be an arbitrary exponent,

$$\|\mathcal{E}_\gamma \chi_E\|_{L^{n\tau}(K^n)} = \|(\chi_E d\sigma)^\vee \cdots (\chi_E d\sigma)^\vee\|_{L^\tau(K^n)}^{1/n} = \|(\Delta_E)^\vee\|_{L^\tau(K^n)}^{1/n}$$

where Δ_E is the density function associated to the measure $\chi_E d\sigma * \cdots * \chi_E d\sigma$ and σ is the usual measure on γ . Now, for $s = (s_1, \dots, s_n) \in K^n$ define

$$\Phi(s) := \sum_{j=1}^n \gamma(s_j)$$

and observe

$$\int \phi(\xi) \chi_E d\sigma * \cdots * \chi_E d\sigma(\xi) = \int_{K^n} \phi \circ \Phi(s) \prod_{j=1}^n \chi_E(s_j) ds$$

for any non-negative, continuous $\phi: \hat{K}^n \rightarrow \mathbb{C}$. At this point it would be useful to apply the change of variables formula with respect to the transformation Φ . One may readily verify the absolute value of the Jacobian arising from this change of variables is the Vandermonde determinant

$$v(s) := \prod_{1 \leq i < j \leq n} |s_j - s_i|_K. \quad (4.6.8)$$

Furthermore, it will be shown in the following section that Φ is precisely $n!$ to 1 off a set of measure zero; in particular, $\Phi^{-1}\{\Phi(u)\}$ is the set of elements obtained by permuting the components of u and so

$$\#\{s \in K^n : \Phi(s) = \xi\} \leq n!. \quad (4.6.9)$$

Therefore, applying Proposition 92, the density of the measure can be expressed by

$$\Delta_E(\xi) = \sum_{s \in K^n : \Phi(s) = \xi} \prod_{j=1}^n \chi_E(s_j) v(s)^{-1} \quad (4.6.10)$$

for $\xi \in \hat{K}^n$. The multiplicity bound (4.6.9) implies the right-hand sum has at most $n!$ terms for every $\xi \in \hat{K}^n$.

Counting solutions to polynomial equations

The argument employed to prove (4.6.9) is robust and, informally, works whenever the polynomial system $\Phi(s) = \xi$ is defined over an integral domain. More precisely, for a ring⁸ R let $\mathbf{X} := (X_1, \dots, X_n)$ be indeterminates, let $p_k \in R[\mathbf{X}]$ denote the k th power sum

$$p_k(\mathbf{X}) := \sum_{j=1}^n X_j^k,$$

and fix $y = (y_1, \dots, y_n) \in R^n$. Consider the system of polynomial equations in \mathbf{X} given by

$$p_k(\mathbf{X}) = p_k(y) \quad \text{for } 1 \leq k \leq n \quad (4.6.11)$$

⁸Here all rings are assumed to be commutative and admit a multiplicative identity.

and let $\mathcal{N}(R; y)$ denote the cardinality of its solution set. For any permutation κ of $\{1, \dots, n\}$ one obtains a trivial solution $(y_{\kappa(1)}, \dots, y_{\kappa(n)})$ to (4.6.11). Moreover, if R is an integral domain, then it is easily shown the *only* solutions to (4.6.11) are given by permuting the components of y , leading to the following proposition.

Proposition 94. *Suppose R is an integral domain. Then $\mathcal{N}(R; y) \leq n!$ for all $y \in R^n$.*

The multiplicity bound (4.6.9) is a special case of this well-known result. The following is an elaboration of an argument appearing in [MT04] which gives a simple proof of Proposition 94. The key ingredients are the classical(!) Newton-Girard formulae.

Lemma 95 (Newton-Girard formulae). *On any ring R the identity*

$$ke_k(\mathbf{X}) = \sum_{i=1}^k (-1)^{i-1} e_{k-i}(\mathbf{X}) p_i(\mathbf{X}) \quad \text{for } 1 \leq k \leq n$$

holds, where $e_k \in R[\mathbf{X}]$ is the degree k elementary symmetric polynomial in \mathbf{X} .

Corollary 96. *Suppose R is a ring for which $1, \dots, n$ are not zero divisors. If $P \in R[\mathbf{X}]$ is any symmetric polynomial and x is a solution to (4.6.11), then $P(x) = P(y)$.*

Proof. By the fundamental theorem for symmetric polynomials there exists a unique polynomial $Q \in R[\mathbf{X}]$ such that

$$P(\mathbf{X}) = Q(e_1(\mathbf{X}), \dots, e_n(\mathbf{X})).$$

If x is a solution to (4.6.11), then the Newton-Girard formula together with a simple induction procedure show $e_k(x) = e_k(y)$ for each $1 \leq k \leq n$, from which the result follows. \square

Proof (of Proposition 94). For R an integral domain and $y \in R^n$, suppose x is a solution to (4.6.11) and consider the symmetric polynomial $P \in R[Z][\mathbf{X}]$ given by

$$P(\mathbf{X}) := \prod_{i=1}^n (Z - X_i).$$

Applying Corollary 96, it follows that

$$\prod_{1 \leq i \leq n} (Z - x_i) = \prod_{1 \leq i \leq n} (Z - y_i). \tag{4.6.12}$$

This is only possible if there exists a permutation κ on $\{1, \dots, n\}$ such that $x_i = y_{\kappa(i)}$ for $1 \leq i \leq n$. Indeed, (4.6.12) implies

$$\prod_{1 \leq i \leq n} (x_1 - y_i) = 0$$

and so, since R is an integral domain, there exists $\kappa(1) \in \{1, \dots, n\}$ such that $x_1 = y_{\kappa(1)}$. Now consider the polynomial

$$\prod_{2 \leq i \leq n} (Z - x_i) = \prod_{\substack{1 \leq i \leq n \\ i \neq \kappa(1)}} (Z - y_i)$$

and iterate the preceding argument to arrive at the desired conclusion. \square

Remark 97. This argument breaks down completely if R is not an integral domain. In particular, the cardinality of the solution set of the same system $\Phi(s) = \xi$ defined over the quotient

rings $\mathbb{Z}/p^\alpha\mathbb{Z}$ (so ξ is taken to be in $[\mathbb{Z}/p^\alpha\mathbb{Z}]^n$) is, in general, much more difficult to bound. This is the key advantage in working over the local field K , rather than the quotient rings.

The proof of the restriction theorem

The majority of the prerequisites for the proof of Theorem 91 have now been amassed. The argument follows the classical method of *off-spring curves* due to Drury [DM85]. This involves an inductive procedure. In particular, consider the (formal) inequality

$$\|\mathcal{E}_\gamma g\|_{L^{r'}(K^n)} \lesssim_q \|g\|_{L^{s'}(K)} \quad (4.6.13)$$

where \mathcal{E}_γ is the extension operator introduced earlier. Let $\mathfrak{R}^*(\gamma; s' \rightarrow r')$ denote the best constant in (4.6.13), with the usual convention $\mathfrak{R}^*(\gamma; s' \rightarrow r') = \infty$ if the estimate fails. By duality, Theorem 91 is equivalent to showing $\mathfrak{R}^*(\gamma; s' \rightarrow r') < \infty$ for all (r, s) satisfying (4.6.5). Trivially, $\mathfrak{R}^*(\gamma; 1 \rightarrow \infty) \leq 1$ which serves as a base case for the induction. The proof of Theorem 91 therefore reduces to verifying the following proposition.

Proposition 98. *If $\mathfrak{R}^*(\gamma; s'_0 \rightarrow n(n+1)s_0/2) \lesssim_q 1$ for some $1 \leq s'_0 < (n^2 + n + 2)/2$, then $\mathfrak{R}^*(\gamma; s' \rightarrow n(n+1)s/2) \lesssim_q 1$ for all $s' \geq 1$ satisfying*

$$\frac{n}{s'} > \frac{2}{n+2} + \frac{n-2}{n+2} \frac{1}{s'_0}. \quad (4.6.14)$$

Applying the proposition to the trivial estimate given by $s'_0 := 1$ shows Theorem 91 holds for r satisfying $1 \leq r < n(n+2)/(n^2 + 2n - 2)$; successive applications of the proposition can then be used to improve upon this result. Moreover, observing the sequence $(x_k)_{k \geq 0}$ defined by

$$\begin{cases} x_0 & := 1 \\ x_{k+1} & := \frac{2}{n(n+2)} + \frac{n-2}{n(n+2)} x_k \quad \text{for } k \geq 0 \end{cases}$$

is monotone decreasing and converges to $2/(n^2 + n + 2)$, it follows that the theorem can be proved for any value of r in the desired range by applying Proposition 98 a finite number of times.

The argument used to prove Proposition 98 makes substantial use of special symmetries of the moment curve, manifested in the identity (4.6.16) below. Given $\beta = (\beta_1, \dots, \beta_n) \in \hat{K}^n$ define the *off-spring curve* $\gamma_\beta: K \rightarrow \hat{K}^n$ by $\gamma_\beta(t) := B\gamma(t) + \beta$ where

$$B := \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ \beta_1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{n-2} & \beta_{n-3} & \dots & 1 & 0 \\ \beta_{n-1} & \beta_{n-2} & \dots & \beta_1 & 1 \end{pmatrix}.$$

Furthermore, for $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in K^{n-1}$ define $\gamma_\alpha := \gamma_{\beta(\alpha)}$ where $\beta(\alpha) := (0, \alpha)$. For each $\alpha \in K^{n-1}$ the curve γ_α is an affine image of γ and, moreover, by the change of variables formula one may easily show $\mathfrak{R}^*(\gamma; s' \rightarrow r') = \mathfrak{R}^*(\gamma_\alpha; s' \rightarrow r')$ for all α (here $\mathfrak{R}^*(\gamma_\alpha; s' \rightarrow r')$ is defined in the obvious manner, by replacing all occurrences of γ with γ_α in the definition of $\mathfrak{R}^*(\gamma; s' \rightarrow r')$).

The following observations are easily verified.

Lemma 99. *With the above notation,*

i) *The map $K^{n-1} \times K \rightarrow \hat{K}^n$ given by $(\alpha; t) \mapsto \gamma_\alpha(t)$ is bijective and the absolute value of the associated Jacobian determinant is everywhere equal to 1.*

ii) *Defining*

$$\Phi_t(s) := \frac{1}{n} \Phi(s_1 + t, \dots, s_n + t), \quad (4.6.15)$$

the identity

$$\Phi_t(s) = \gamma_{\Phi_0(s)}(t) \quad (4.6.16)$$

holds for all $s = (s_1, \dots, s_n) \in K^n$ and $t \in K$.

For any $f: \hat{K} \rightarrow \mathbb{C}$, define the auxiliary function

$$F(f): K^{n-1} \times K \cong \hat{K}^n \rightarrow \mathbb{C}, \quad F(f)(\alpha; t) := f \circ \gamma_\alpha(t).$$

Lemma 100. *With the above notation and assuming the hypothesis of Proposition 98, whenever $(1/\rho, 1/\mu)$ lies in the triangle $\text{conv}\{(1, 1), (1, 1/s'_0), (1/2, 1/2)\}$ and τ is defined by*

$$\frac{(n+2)(n-1)}{2} \frac{1}{\rho} + \frac{1}{\mu} + \frac{n(n+1)}{2} \frac{1}{\tau} = \frac{n(n+1)}{2} \quad (4.6.17)$$

it follows that

$$\|\check{f}\|_{L^\tau(K^n)} \lesssim_q \|F(f)\|_{L^\rho_\alpha L^\mu_t(\hat{K}^n)}.$$

Proof. By real interpolation (see [Tri78, §1.18.4]) it suffices to prove the lemma in the following special cases:

- a) $1 \leq \rho = \mu \leq 2$ and $\tau = \mu'$;
- b) $\rho = 1$, $\mu = s'_0$ and $\tau = n(n+1)s_0/2$.

The proof of a) follows from the Hausdorff-Young inequality together with the observation $\|f\|_{L^\tau(\hat{K}^n)} = \|F(f)\|_{L^\tau_\alpha L^\tau_t(\hat{K}^n)}$, which is itself a consequence of Lemma 99 i) and Proposition 92.

For b), once again applying Lemma 99 i) it follows that

$$\check{f}(x) = \int_{K^{n-1}} \int_K F(f)(\alpha; t) \psi(x \cdot \gamma_\alpha(t)) dt d\alpha$$

and so

$$\|\check{f}\|_{L^{n(n+1)s_0/2}(K^n)} \leq \int_{K^{n-1}} \|\mathcal{E}_{\gamma_\alpha} F(f)(\alpha; \cdot)\|_{L^{n(n+1)s_0/2}(K)} d\alpha.$$

Applying the hypothesis of Proposition 98 then yields the desired estimate. \square

Proof (of Proposition 98). Recalling the definition of Δ_E , let $\Delta_E^n(\xi) := \Delta_E(n\xi)$ so, by the hypothesis $|n|_K = 1$, it follows that $(\Delta_E)^\vee(n^{-1}x) = (\Delta_E^n)^\vee(x)$ and

$$\|\mathcal{E}_{\gamma\chi_E}\|_{L^{n\tau}(K^n)} = \|(\Delta_E)^\vee\|_{L^\tau(K^n)}^{1/n} = \|(\Delta_E^n)^\vee\|_{L^\tau(K^n)}^{1/n} \lesssim_q \|F(\Delta_E^n)\|_{L^\rho_\alpha L^\mu_t(\hat{K}^n)}^{1/n}$$

whenever $1 \leq \rho, \mu, \tau \leq \infty$ satisfy the hypothesis of Lemma 100 or, equivalently,

$$\left(\frac{2}{s'_0} - 1\right) \frac{1}{\rho} + \left(1 - \frac{1}{s'_0}\right) \leq \frac{1}{\mu} \leq \frac{1}{\rho} \quad (4.6.18)$$

and τ satisfies (4.6.17).

Recalling (4.6.10) and (4.6.15), it follows that

$$\Delta_E^n(\gamma_\alpha(t)) = \sum_{s \in K^n: \Phi_t(s) = \gamma_\alpha(t)} \prod_{j=1}^n \chi_E(s_j + t) v(s)^{-1}$$

and, applying (4.6.16) and Lemma 99 i), one deduces that

$$F(\Delta_E^n)(\alpha; t) = \sum_{s \in U: \Phi_0(s) = (0, \alpha)} \prod_{j=1}^n \chi_E(s_j + t) v(s)^{-1}.$$

Thus, by Minkowski's inequality,

$$\|F(\Delta_E^n)(\alpha; \cdot)\|_{L_t^\mu(K)} \leq \sum_{s: \Phi_0(s) = (0, \alpha)} w(s)^{1/\mu} v(s)^{-1}$$

where

$$w(s) := \int_K \prod_{j=1}^n \chi_E(s_j + t) dt.$$

Therefore, taking L_α^ρ -norms and applying the multiplicity bound (4.6.9) together with Hölder's inequality,

$$\|F(\Delta_E^n)\|_{L_\alpha^\rho L_t^\mu(\hat{K}^n)} \lesssim \left(\int_{K^{n-1}} \sum_{s: \Phi_0(s) = (0, \alpha)} w(s)^{\rho/\mu} v(s)^{-\rho} d\alpha \right)^{1/\rho}.$$

Changing variables once again, it immediately follows that

$$\|F(\Delta_E^n)\|_{L_\alpha^\rho L_t^\mu(\hat{K}^n)} \lesssim \left(\int_{K^{n-1}} w(0, \alpha)^{\rho/\mu} v(0, \alpha)^{-(\rho-1)} d\alpha \right)^{1/\rho}.$$

Suppose (ρ, μ) further satisfy

$$\frac{n}{n+2} < \frac{1}{\rho} < 1; \quad \frac{n+2}{2} \frac{1}{\rho} - \frac{n}{2} < \frac{1}{\mu}. \quad (4.6.19)$$

The former condition ensures $\eta' := 2/n(\rho-1)$ satisfies $1 < \eta' < \infty$ and so, applying the Lorentz space version of Hölder's inequality, one deduces that

$$\|\mathcal{E}_\gamma \chi_E\|_{L^{n\tau}(K^n)} \lesssim_q \|w(0, \cdot)^{\rho/\mu}\|_{L^{\eta', 1}(K^{n-1})}^{1/\rho n} \|v(0, \cdot)^{-(\rho-1)}\|_{L^{\eta', \infty}(K^{n-1})}^{1/\rho n}.$$

It will be proved below in Corollary 102 that the Vandermonde factor is $O_q(1)$. In addition,

$$\|w(0, \cdot)\|_{L^\infty(K^{n-1})} \leq \|\chi_E\|_{L^1(K)}; \quad \|w(0, \cdot)\|_{L^1(K^{n-1})} = \|\chi_E\|_{L^1(K)}^n$$

and therefore, since the second condition of (4.6.19) guarantees $1 < \rho\eta'/\mu < \infty$, one may readily deduce

$$\|w(0, \cdot)^{\rho/\mu}\|_{L^{\eta', 1}(K^{n-1})} \lesssim \|\chi_E\|_{L^1(K)}^{\rho/\mu + (n-1)/\eta'}.$$

Now let $\delta > 0$ and suppose s' satisfies the hypotheses of the proposition and (4.6.14) is within δ of equality, in the sense that the left-hand side of (4.6.14) differs by at most δ from the right-hand side. To conclude the proof it suffices to show that, provided that δ is chosen suitably

small, there exists (ρ, μ) satisfying (4.6.18) and (4.6.19) such that

$$\frac{n}{s'} = n - \left(\frac{n(n+1)}{2} - \frac{(n+2)(n-1)}{2} \frac{1}{\rho} - \frac{1}{\mu} \right). \quad (4.6.20)$$

Indeed, assuming (4.6.20) one immediately observes

$$\frac{1}{s'} = \frac{1}{\rho n} \left(\frac{\rho}{\mu} + \frac{n-1}{\eta} \right)$$

by the definition of η whilst, defining τ by (4.6.17),

$$\frac{1}{s'} = 1 - \frac{n+1}{2} \frac{1}{\tau}.$$

Substituting these values into the estimates established above,

$$\|\mathcal{E}_\gamma \chi_E\|_{L^{n(n+1)s/2}(K^n)} \lesssim_q \|\chi_E\|_{L^{s'}(K)}$$

and so, for all s' such that (4.6.14) is within δ of equality, \mathcal{E}_γ is restricted strong-type $(s', n(n+1)s/2)$. A simple interpolation argument now concludes the proof of Proposition 98.

It remains to establish the existence of a pair (ρ, μ) with the desired properties. Let μ be chosen (depending on ρ) so that equality holds in the first inequality of (4.6.18); that is,

$$\frac{1}{\mu} := \left(\frac{2}{s'_0} - 1 \right) \frac{1}{\rho} + \left(1 - \frac{1}{s'_0} \right). \quad (4.6.21)$$

Then the conditions of (4.6.18) and (4.6.19) are satisfied provided that

$$\frac{n}{n+2} < \frac{1}{\rho} < \frac{(n+2)s'_0 - 2}{(n+4)s'_0 - 4}.$$

Substituting (4.6.21) into (4.6.20), it suffices to show there exists some ρ satisfying the preceding inequalities such that

$$\frac{n}{s'} = \left(\frac{(n+2)(n-1)}{2} + \frac{2}{s'_0} - 1 \right) \frac{1}{\rho} - \frac{n(n-1)}{2} + 1 - \frac{1}{s'_0}.$$

The left-hand side of the above expression is a strictly increasing linear function of $1/\rho$ which equals

$$\frac{2}{n+2} + \frac{n-2}{n+2} \frac{1}{s'_0}$$

when evaluated at $1/\rho = n/(n+2)$. Thus, provided that $\delta > 0$ is sufficiently small depending on n and s'_0 , there exists a suitable choice of ρ . \square

Estimates for Vandermonde operators

It remains to establish the Lorentz space estimate for the Vandermonde determinant used in the above argument. The main ingredient is a multilinear inequality due to Christ.

Proposition 101 (Christ's multilinear inequality [Chr85]). *Let Ω be a σ -finite LCA group with Haar measure dx and for $1 \leq j < k \leq n$ let $g_{j,k}$ be a non-negative measurable function on Ω .*

Then

$$\int_{\Omega^n} \prod_{i=1}^n |f_i(x_i)| \prod_{1 \leq j < k \leq n} g_{j,k}(x_j - x_k) dx_1 \dots dx_n \lesssim \prod_{i=1}^n \|f_i\|_{L^s(\Omega)} \prod_{1 \leq j < k \leq n} \|g_{j,k}\|_{L^{t,\infty}(\Omega)} \quad (4.6.22)$$

holds whenever $s^{-1} + t^{-1}(n-1)/2 = 1$ and $1 \leq s < n$.

Although Proposition 101 is presented in the Euclidean setting in [Chr85], the proof is based on a fairly robust interpolation procedure which easily allows for the above generalisation.

The case of present interest is given by taking the LCA group to be the local field K and defining $g_{j,k}(t) := |t|_K^{-\alpha}$ for all j, k and some $0 \leq \alpha$, so that $\|g_{j,k}\|_{L^{1/\alpha,\infty}(K)} = 1$. With this choice, the kernel of the multilinear functional defined by the left-hand side of (4.6.22) is a power of the reciprocal of the Vandermonde determinant (4.6.8) and Proposition 101 can be applied to obtain the desired Lorentz space bound for $v(0, \cdot)^{-(\rho-1)}$. In particular, there is the following corollary, which is the local field version of [Dru85, Lemma 1].

Corollary 102. *For $n \geq 2$, $\eta' = 2/n(\rho-1)$ and $n/(n+2) < 1/\rho < 1$ it follows that*

$$\|v(0, \cdot)^{-(\rho-1)}\|_{L^{\eta',\infty}(K^{n-1})} \lesssim_q 1.$$

where q denotes the cardinality of the residue class field of K .

Proof. For $\lambda > 0$, by the discrete nature of the valuation,

$$|\{\alpha \in K^{n-1} : v(0, \alpha)^{-(\rho-1)} \geq \lambda\}| = |\{\alpha \in K^{n-1} : v(0, \alpha)^{-(\rho-1)} \geq q^{k_\lambda}\}|$$

where $k_\lambda := \max\{k \in \mathbb{Z} : \lambda \geq q^k\}$. Furthermore, by the homogeneity of the Vandermonde determinant, it follows that

$$\begin{aligned} |\{\alpha \in K^{n-1} : v(0, \alpha)^{-(\rho-1)} \geq \lambda\}| &= q^{-k_\lambda m(n-1)} \int_{K^{n-1}} \chi_{\{v(0, \alpha)^{-(\rho-1)} \geq q^{k_\lambda}\}} (\pi^{k_\lambda m} \alpha) d\alpha \\ &= q^{-k_\lambda m(n-1)} \int_{K^{n-1}} \chi_E(\alpha) d\alpha \end{aligned}$$

where $m := 2/(n(n-1)(\rho-1))$ and $E := \{\alpha \in K^{n-1} : v(0, \alpha) \leq 1\}$. Recalling the definition of η' , one deduces that

$$\lambda |\{\alpha \in K^{n-1} : v(0, \alpha)^{-(\rho-1)} \geq \lambda\}|^{1/\eta'} \leq q |E|^{1/\eta'}$$

and the problem is therefore reduced to showing E has finite measure. Furthermore, by symmetry it suffices to demonstrate

$$\tilde{E} := \{\alpha \in E : |\alpha_1|_K \leq |\alpha_2|_K \leq \dots \leq |\alpha_{n-1}|_K\}$$

has finite measure. When $n = 2$ this is immediate; henceforth assume $n \geq 3$. Consider the change of variables $t := \alpha_{n-1}$ and $s_j := \alpha_j t^{-1}$ for $1 \leq j \leq n-2$, which is injective off a null set, so that

$$v(0, \alpha) = t^{n(n-1)/2} \prod_{i=1}^{n-2} |s_j(1-s_j)|_K \prod_{1 \leq j < k \leq n-2} |s_k - s_j|_K =: t^{n(n-1)/2} u(s)$$

and

$$|\tilde{E}| = \int_{B_1(0)} \int_{B_{u(s)^{-2/n(n-1)}}(0)} |t|_K^{n-2} dt ds \lesssim \int_{B_1(0)} u(s)^{-2/n} ds,$$

where the final estimate is easily seen by recalling $|\{s \in K^{n-1} : |s|_K = q^k\}| = q^k(1 - 1/q)$. Now apply Proposition 101 in $n - 2$ dimensions with $s := n/3$, $t := n/2$, $g_{j,k}(s) := |s|_K^{-2/n}$ and $f_i(s) := \chi_{B_1(0)}(s)|s(1-s)|_K^{-2/n}$. A simple calculation shows

$$\begin{aligned} \int_{B_1(0)} |s(1-s)|_K^{-2/3} ds &= \int_{B_{q^{-1}}(0)} |s|_K^{-2/3} ds + \int_{\{|1-s|_K=1\}} |s|_K^{-2/3} ds \\ &\leq 2 \int_{B_1(0)} |s|_K^{-2/3} ds \lesssim 1, \end{aligned}$$

which concludes the proof. □

4.7 Elementary calculus on local fields

For completeness a discussion of elementary calculus over local fields is presented, focusing on the change of variables formula integral to the above argument. For the most part, the arguments are fairly direct translations of those used in the familiar real variable setting, however there are some notable differences between the two theories. All of the results presented in this section are well-known and a more thorough treatment of the theory can be found [Sch06, §27] or [Igu00, §7.4]. This presentation is based on that of [Sch06, §27], however [Sch06, §27] limits the discussion to the univariate case.

Uniformly differentiable functions and the inverse function theorem

In real analysis the inverse function theorem is often stated under a C^1 hypothesis on the function. Given an open set $U \subseteq K^n$, if one simply assumes a function $f: U \rightarrow K^n$ is (Fréchet) differentiable with continuous derivative, then f may fail to be injective in any neighbourhood of some regular point [Sch06, §26]. Hence, the naïve local field analogue of the inverse function theorem fails. To save the situation one must assume f satisfies a stronger hypothesis than a C^1 condition, namely *uniform differentiability*.

Definition 103. A function $f: U \rightarrow K^n$ on an open set $U \subseteq K^n$ is uniformly differentiable at a point $a \in K^n$ if there exists some $f'(a) \in \text{Mat}(n, K)$ such that

$$\lim_{(x,y) \rightarrow (a,a)} \frac{\|f(x) - f(y) - f'(a)(x - y)\|}{\|x - y\|} = 0 \tag{4.7.1}$$

where the limit is taken in $U \times U \setminus \{(x, x) : x \in K^n\}$. Furthermore, if $f'(a)$ is invertible, then a is said to be a regular point (for f).

This condition is stronger than Fréchet differentiability at a , which simply requires the existence of some $f'(a) \in \text{Mat}(n, K)$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(a+h) - f(a) - f'(a)h\|}{\|h\|} = 0; \tag{4.7.2}$$

indeed, whereas in (4.7.1) the limit may approach (a, a) from a wide array of directions, in (4.7.2) the approach is restricted to a single line.

Definition 104. A function $f: U \rightarrow K^n$ on an open set $U \subseteq K^n$ is uniformly differentiable on U if it is uniformly differentiable at all the points $a \in U$. The set of such functions is denoted $C_u^1(U \rightarrow K^n)$.

In general, $C_u^1(U \rightarrow K^n)$ forms a proper subset of the collection of functions $f: U \rightarrow K^n$ which are differentiable with continuous derivative.

Remark 105. One may easily formulate a notion of uniform differentiability in the real variable setting. However, the mean value theorem implies that real variable uniform differentiability is equivalent to the familiar C^1 condition.

Theorem 106 (Inverse function theorem). *Suppose $f \in C_u^1(U \rightarrow K^n)$ and $a \in U$ is a regular point.*

i) *There exists some $r_0 > 0$ such that*

$$\|f(x) - f(y)\| = \|f'(a)(x - y)\| \quad (4.7.3)$$

whenever $x, y \in B_{r_0}(a)$. Furthermore, for any $0 < r \leq r_0$ the function f restricts to a bijection from $B_r(a)$ to an open set $f(B_r(a))$.

ii) *If $g: f(B_r(a)) \rightarrow B_r(a)$ is the local inverse for f , then g is uniformly differentiable at $f(a)$ with $g'(f(a)) = f'(a)^{-1}$.*

Proof. i) Since $f'(a)$ is invertible, defining⁹ $2\|f'(a)^{-1}\|\lambda := 1$, it follows that

$$\lambda\|x\| \leq \|f'(a)x\| \quad (4.7.4)$$

for all $x \in K^n$. From the definition of uniform differentiability, there exists $r_0 > 0$ such that

$$\|f(x) - f(y) - f'(a)(x - y)\| < \lambda\|x - y\| \leq \|f'(a)(x - y)\|$$

whenever $x, y \in B_{r_0}(a)$. The identity (4.7.3) now follows from the non-archimedean property of the absolute value and immediately implies f is injective on $B_r(a)$ for all $0 < r \leq r_0$. It remains to show that the image of $B_r(a)$ under f is an open set. To this end, fix $y_0 \in f(B_r(a))$ so that $y_0 = f(x_0)$ for some $x_0 \in B_r(a)$. Let $y_1 \in B_{\lambda r}(y_0)$ and define

$$\varphi(u) := u - f'(a)^{-1}(f(u) - y_1)$$

for all $u \in U$. Note

$$\|\varphi(x) - \varphi(y)\| = \|f'(a)^{-1}(f(x) - f(y) - f'(a)(x - y))\| \leq \frac{1}{2}\|x - y\|$$

for $x, y \in B_{r_0}(a)$. Furthermore,

$$\|\varphi(x_0) - x_0\| = \|f'(a)^{-1}(y_0 - y_1)\| \leq \frac{r}{2}$$

and so whenever $u \in B_r(a)$ it follows that

$$\|\varphi(u) - a\| \leq \max\{\|\varphi(u) - \varphi(x_0)\|, \|\varphi(x_0) - x_0\|, \|x_0 - a\|\} \leq r,$$

⁹Here $\|f'(a)^{-1}\|$ denotes the operator norm of $f'(a)^{-1}$ as a linear operator $K^n \rightarrow K^n$.

from which one concludes that $\varphi: B_r(a) \rightarrow B_r(a)$ is a contraction mapping. Since $B_r(a)$ is closed it is therefore complete and, by the contraction mapping theorem, there exists a unique $x_1 \in B_r(a)$ such that $f(x_1) = y_1$, and so $B_{\lambda r}(y_0) \subseteq f(B_r(a))$, as required.

ii) By part i) it suffices to show for any $\epsilon > 0$ there exists $r > 0$ such that

$$\frac{\|g \circ f(x) - g \circ f(y) - f'(a)^{-1}(f(x) - f(y))\|}{\|f(x) - f(y)\|} < \epsilon$$

whenever $x, y \in B_r(a)$ with $x \neq y$. This, however, follows immediately from (4.7.4), (4.7.3) and the definition of $f'(a)$. □

The proof of the change of variables formula

Here a more general version of Proposition 92 is established, which is valid for $C_u^1(U \rightarrow K^n)$ change of variables (any polynomial mapping $U \rightarrow K^n$ is clearly $C_u^1(U \rightarrow K^n)$). The first step is to consider the simple case where the change of variables is linear.

Lemma 107 (Linear change of variables). *Let f be a non-negative measurable function on K^n and $A \in \text{GL}(n, K)$. Then*

$$|\det A|_K \int_{K^n} f(x) dx = \int_{K^n} f \circ A^{-1}(x) dx. \tag{4.7.5}$$

The proof relies on the following elementary matrix factorisation lemma, the proof of which is omitted (see, for instance, [Igu00, §7.4] for further details).

Lemma 108. *Any $A \in \text{GL}(n, K^n)$ can be factorised as $A = B'DB$ where $B, B' \in \text{GL}(n, \mathfrak{o})$ and D is a diagonal matrix whose non-zero entries are given by integral powers of π .*

Proof (of Lemma 107). By the monotone convergence theorem it suffices to prove the identity whenever f is the characteristic function of a Borel set. The integral on the right-hand side of (4.7.5) defines a Haar measure on K^n and therefore, by the uniqueness of such measures, it follows that for some constant $\rho(A) > 0$, the identity $|AE| = \rho(A)|E|$ holds for all Borel $E \subseteq K^n$. Thus, the problem is reduced to showing $\rho(A) = |\det A|_K$.

Note that $A \mapsto \rho(A)$ is a multiplicative group homomorphism and $\rho(A) = |A\mathfrak{o}^n|$. Let $D = \text{diag}[\pi^{k_1}, \dots, \pi^{k_n}]$ be the a diagonal matrix arising from the factorisation of A guaranteed by Lemma 108. Then $|\det A|_K = |\det D|_K$ and, from the above observations, $\rho(A) = \rho(D)$. Finally,

$$\rho(D) = |D\mathfrak{o}^n| = \prod_{j=1}^n |\pi^{k_j} \mathfrak{o}| = \prod_{j=1}^n |\pi^{k_j}|_K = |\det D|_K,$$

and the desired identity is now immediate. □

Proposition 109 (C_u^1 change of variables). *Let $U \subseteq K^n$ be open, $\Phi \in C_u^1(U \rightarrow K^n)$ and suppose every $u \in U$ is regular for Φ . Then*

$$\int_U f(u) |\det \Phi'(u)|_K du = \int_{\Phi(U)} \sum_{u \in U: \Phi(u)=x} f(u) dx$$

whenever f is a non-negative measurable function on U .

Proof (of the change of variables formula). It again suffices to prove the identity for $f = \chi_E$ where $E \subseteq U$ is a Borel set and, furthermore, one may assume E is non-empty, compact and open. For any $u \in E$, by the regularity hypothesis and the inverse function theorem, there exists some $r_0 > 0$ such that Φ bijects from $B_r(u)$ onto the open set $\Phi(B_r(u))$ for all $0 < r \leq r_0$. By compactness and the continuity of Φ' , given $\epsilon > 0$ one may partition of E into finitely many balls $\{B_j\}_{j=1}^N$ where $B_j = B_{r_j}(u_j)$ is mapped bijectively onto an open set by Φ and is such that

$$|\det \Phi'(u) - \det \Phi'(u_j)|_K < \epsilon |E|^{-1} \quad (4.7.6)$$

for all $u \in B_j$.

Recalling (4.7.3), one may further assume $\Phi \circ \Phi'(u_j)^{-1}$ is an isometry on $\Phi'(u_j)(B_j)$. Since the Haar measure is invariant under isometric transformation,¹⁰

$$|\Phi(B_j)| = |\Phi \circ \Phi'(u_j)^{-1}(\Phi'(u_j)(B_j))| = |\Phi'(u_j)(B_j)| = |\det \Phi'(u_j)|_K |B_j|$$

where the last inequality is due to Lemma 107.

On the other hand, since Φ is injective on the balls B_j , one may write

$$\chi_{\Phi(B_j)}(x) = \sum_{u \in B_j: \Phi(u)=x} \chi_E(u)$$

for all $x \in \Phi(U)$ and so

$$\begin{aligned} \int_{\Phi(U)} \sum_{u \in U: \Phi(u)=x} \chi_E(u) \, dx &= \sum_{j=1}^N \int_{\Phi(U)} \sum_{u \in B_j: \Phi(u)=x} \chi_E(u) \, dx \\ &= \sum_{j=1}^N \int_{B_j} |\det \Phi'(u_j)|_K \, du. \end{aligned}$$

From this and (4.7.6) one deduces that

$$\left| \int_{\Phi(U)} \sum_{u \in U: \Phi(u)=x} f(u) \, dx - \int_U f(u) |\det \Phi'(u)|_K \, dx \right| < \epsilon,$$

and, since $\epsilon > 0$ was arbitrary, the result follows. \square

¹⁰This is immediately obvious by, for instance, noticing the Haar measure must agree with the n -dimensional Hausdorff measure on K^n .

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