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**Double Hilbert transforms along
surfaces in the Heisenberg group**

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Marco Vitturi)

Abstract

We provide an L^2 theory for the local double Hilbert transform along an analytic surface $(s, t, \varphi(s, t))$ in the Heisenberg group \mathbb{H}^1 , that is operator

$$f \mapsto H_\varphi f(x) := \text{p.v.} \int_{|s|, |t| \leq 1} f(x \cdot (s, t, \varphi(s, t))^{-1}) \frac{ds}{s} \frac{dt}{t},$$

where \cdot denotes the group operation in \mathbb{H}^1 . This operator combines several features: it is a multi-parameter singular integral, its kernel is supported along a submanifold, and convolution is with respect to a homogeneous group structure. We reprove H_φ is always $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded (a result first obtained in [Str12]) to illustrate the method and then refine it to characterize the largest class of polynomials P of degree less than d such that the operator H_P is uniformly bounded when P ranges in the class. Finally, we provide examples of surfaces that can be treated by our method but not by the theory of [Str12].

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Nomenclature

$C^k(\Omega)$	The space of k -times differentiable functions supported on Ω
$C^\infty(\Omega)$	The space of infinitely differentiable functions supported on Ω
$C_c^\infty(\Omega)$	The space of infinitely differentiable, compactly supported functions supported on Ω
$C^\omega(\Omega)$	The space of real-analytic functions on Ω
$L^p(\Omega)$	The space of measurable functions f such that $\int_\Omega f(x) ^p dx < \infty$
$L^p(\Omega, d\mu)$	The space of μ -measurable functions f such that $\int_\Omega f(x) ^p d\mu(x) < \infty$
$\mathcal{S}(\Omega)$	The Schwartz space of smooth rapidly decreasing functions on Ω
$\mathcal{S}'(\Omega)$	The space of tempered distributions on Ω , dual of $\mathcal{S}(\Omega)$
$\mathcal{E}(\Omega)$	$C^\infty(\Omega)$ endowed with the topology given by uniform convergence on compacts
$\mathcal{E}'(\Omega)$	The space of compactly supported distributions on Ω , dual of $\mathcal{E}(\Omega)$
$A \lesssim_\nu B$	There exists a constant C_ν depending on the parameters ν such that $A \leq C_\nu B$
$A \sim B$	It holds that $A \lesssim B$ and $B \lesssim A$
$g = O_\nu(f)$	The same as $ g(x) \lesssim_\nu f(x) $
\mathbb{H}^1	The first Heisenberg group
$H^* f$	The maximal Hilbert transform of f
Mf	The Hardy-Littlewood maximal function of f
$a \wedge b$	The minimum between a and b
$a \vee b$	The maximum between a and b
p.v. \int	Principal value integral
$\mathbb{R}[X_1, \dots, X_n]$	The vector space of polynomials in n variables with coefficients in \mathbb{R}
$\mathcal{U}(\mathcal{H})$	The space of linear unitary operators on the Hilbert space \mathcal{H}
$\ T\ _{HS}$	The Hilbert-Schmidt norm of operator T , given by $\sqrt{\text{tr}(T^* T)}$

Chapter 1

Motivation

Since Calderón and Zygmund opened the way to real analytic techniques in the study of singular integral operators in their groundbreaking paper “*On the existence of certain singular integrals*” [CZ52], this has been one of the most active areas of research in harmonic analysis. A lot of effort has been poured in the extension of the now classical Calderón-Zygmund theory of singular integral operators, in multiple directions. A comprehensive survey would be beyond the scope of this thesis¹, but we mention here a subset of such directions the research has taken over the years that provides context for the work developed in here. Namely, the following areas (or ‘themes’) have been the object of much attention:

- i) *Singular integral operators with singular kernels supported along submanifolds*; also known as singular Radon transforms. The prototypical example of such operators is the Hilbert transform along a parabola, that is the operator

$$f \mapsto \text{p.v.} \int_{-\infty}^{\infty} f(x-t, y-t^2) \frac{dt}{t}.$$

The operator naturally arises when one applies the method of rotations to singular integral operators associated to parabolic differential operators (that is, singular integrals with kernels that are homogeneous with respect to parabolic dilations). It has been widely studied, and is known to be $L^p \rightarrow L^p$ bounded for any $1 < p < \infty$ (see for example [SW78], and [DRdF86]), but it is not completely understood yet: indeed, it is an open problem to determine whether the operator is bounded from L^1 to $L^{1,\infty}$. The best results in this direction have been achieved in [Chr88] by Christ, in which he proved $H_{\text{par}}^1 \rightarrow L^{1,\infty}$ boundedness, and in [STW04] by Seeger, Tao and Wright, in which they prove $L \log \log L \rightarrow L^{1,\infty}$ boundedness (both papers rely on suitable modifications of the Calderón-Zygmund decomposition).

More in general, one can replace the parabola with a generic curve γ (embedded in a generic \mathbb{R}^n) and ask which are the minimal regularity properties required for the operator to be $L^2 \rightarrow L^2$ bounded or to be $L^p \rightarrow L^p$ bounded in the full range $1 < p < \infty$. It was realized that when the curve has non-trivial curvature² then one has boundedness in the full range $1 < p < \infty$ (see the already mentioned

¹See [Ste98] for a survey of the historical development though.

²There are various notions of curvature that could be used to quantify this statement; a quite

[SW78]), and subsequently the case of flat curves was intensively studied. Cases of non convolution kernels and higher dimensional submanifolds have received much consideration too - see for example the very influential work of Christ, Nagel, Stein and Wainger [CNSW99].

- ii) *Singular integral operators in the setting of homogeneous groups.* A homogeneous group is a (connected, simply connected) nilpotent Lie Group N , identified with some \mathbb{R}^n through the exponential map³, which admits a group of automorphic one parameter dilations. The prototypical singular integral operator then takes the form

$$f \mapsto \int_N f(x \cdot y^{-1}) \mathcal{K}(y) d\mu(y),$$

where μ is the Haar measure on N and \mathcal{K} is a singular kernel which is homogeneous of critical degree with respect to the automorphic one parameter dilations of N . Notice the convolution is now with respect to the group operation. These operators arise naturally in the study of harmonic functions on symmetric spaces: if G is a connected semi-simple Lie group and $G = KAN$ is its Iwasawa decomposition (with K maximal compact subgroup, A abelian, N nilpotent), then harmonic functions on the symmetric space G/K can be expressed as Poisson-like integrals of their boundary values; the boundary in turn can be identified with N , and the dilations of N with a subgroup of A , thus leading to operators of the form above analogous to the Hilbert transform.

The simplest non-commutative example of a homogeneous group is the *Heisenberg group* \mathbb{H}^1 , that is the group given by endowing the space \mathbb{R}^3 with the group operation

$$(x, y, z) \cdot (x', y', z') := (x + x', y + y', z + z' + (xy' - x'y)/2);$$

the definition can be readily adapted to give the higher dimensional Heisenberg groups \mathbb{H}^n , which are identified with the spaces $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and group law

$$(\mathbf{x}, \mathbf{y}, z) \cdot (\mathbf{x}', \mathbf{y}', z') := (\mathbf{x} + \mathbf{x}', \mathbf{y} + \mathbf{y}', z + z' + (\mathbf{x} \cdot \mathbf{y}' - \mathbf{x}' \cdot \mathbf{y})/2),$$

where the \cdot on the right hand side denotes the inner product of \mathbb{R}^n . The Heisenberg group is endowed with the group of (non-isotropic) automorphic dilations given by

$$\delta_\lambda(x, y, z) := (\lambda x, \lambda y, \lambda^2 z),$$

that is the parabolic dilations.

Extensive research has focused early on on extending the Calderón-Zygmund theory of singular integral operators to operators defined in such homogeneous spaces, that is, with kernels homogeneous with respect to the non-isotropic dilations of a nilpotent Lie group. See for example Korányi and Vági [KV71].

- iii) *Multi-parameter singular integral operators.* The most trivial example of such

general example would be that of well-curvedness - that is, when the collection of vectors $(d/dt)^k \gamma(0)$ for $k \in \mathbb{N}$ spans the subspace of \mathbb{R}^n that contains the curve.

³Technically, one identifies the nilpotent Lie Group N with its Lie algebra \mathfrak{n} , which is then identified with \mathbb{R}^n as a vector field.

operators is the double Hilbert transform, that is the operator given by

$$f \mapsto \text{p.v.} \int_{-\infty}^{\infty} f(x-s, y-t) \frac{ds}{s} \frac{dt}{t},$$

which is obviously $L^p \rightarrow L^p$ bounded for all $1 < p < \infty$ because it factorizes in the composition of two directional Hilbert transforms. More in general, for the two parameter case, one can consider the space $\mathbb{R}^m \times \mathbb{R}^n$ together with the two parameter family of dilations $\delta: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n$ given by

$$\delta_{\lambda, \mu}(x, y) := (\lambda x, \mu y),$$

and consider generic kernels $K(x, y)$ that satisfy the two parameter homogeneity condition

$$K(\delta_{\lambda, \mu}(x, y)) = \lambda^{-m} \mu^{-n} K(x, y).$$

These kernels do not necessarily factorize (for example, they can take the form $\Omega(x, y)|x|^{-m}|y|^{-n}$ with $\Omega(\delta_{\lambda, \mu}(x, y)) = \Omega(x, y)$) and therefore the operators they give rise to have to be treated differently. Here the issues arise from the fact that the singularity of the kernel is no more a point as in the one parameter case, but in general it is a submanifold of \mathbb{R}^n , where n is the ambient dimension (for the double Hilbert transform it is indeed the union of two orthogonal lines). The theory for such operators has been developed by R. Fefferman in [Fef81] for the two parameter case, where it is proven that, under certain cancellation and smoothness hypotheses on the kernel K that mimic those of the double Hilbert transform kernel $\text{p.v.} 1/st$, the operator of convolution with K is $L^p \rightarrow L^p$ bounded for $1 < p < \infty$, and moreover it is $L \log L \rightarrow L^{1, \infty}$ bounded. Further major developments were given in [FS82], [Jou85]; too many works followed to survey them here. We content ourselves with mentioning one particular result that will be quite relevant for the following discussion, namely the work of Ricci and Stein [RS92]. In there they develop the L^p theory for singular integral operators (and associated maximal functions) with k -parameter convolution kernels in \mathbb{R}^n , with $1 \leq k \leq n$, that are roughly homogeneous with respect to a specified group of dilations. More precisely, they are of the form

$$K(x) = \sum_{I \in \mathbb{Z}^k} \mu_I^{(I)}(x); \tag{1.0.1}$$

here $\{\mu^{(I)}\}_{I \in \mathbb{Z}^k}$ is a family of distributions with support in $[-1, 1]^n$ that satisfy some uniform cancellation and smoothness conditions (both phrased in terms of their Fourier transforms), and the subscript I denotes k -parameter dilation

$$f_I(x) := \det(2^{-\Lambda I}) f(2^{-\Lambda I} x),$$

with $\Lambda = (\lambda_{ij})_{ij}$ an $n \times k$ matrix of non zero exponents and, for $I = (i_1, \dots, i_k)$, $x = (x_1, \dots, x_n)$,

$$2^{-\Lambda I} x := (2^{-\lambda_{11}i_1 - \dots - \lambda_{1k}i_k} x_1, \dots, 2^{-\lambda_{n1}i_1 - \dots - \lambda_{nk}i_k} x_n).$$

Notice that the dilation exponents specified in the matrix Λ can be negative. A unifying modern perspective on multi-parameter singular integrals is given by Street in [Str14].

Each one far from being impermeable, the areas mentioned above have intersected multiple times and each time produced new insights in the theory of singular integral operators. We give some examples.

Consider the case i) + ii), that is singular integral operators in homogeneous groups with kernels supported along submanifolds. One of the main results in this area was obtained by Ricci and Stein in [RS88], in which they prove $L^p \rightarrow L^p$ boundedness for all $1 < p < \infty$ for singular integral operators of the form

$$f \mapsto \text{p.v.} \int_{\mathcal{V}} f(x \cdot y^{-1}) K(y) \, d\sigma(y),$$

where \mathcal{V} is a connected analytic homogeneous submanifold of an ambient nilpotent group N (with automorphic dilations $\{\delta_r\}_r$) that generates the entire group, σ is the surface measure on \mathcal{V} , $K \in C^\infty(\mathcal{V})$ is such that⁴

$$\forall a, b > 0, \quad \int_{a < \|x\| < b} K(x) \, d\sigma(x) = 0$$

and $K(x) \, d\sigma(x)$ is homogeneous of critical degree⁵. They are also able to treat the case of singular integral operators in homogeneous groups whose kernels are homogeneous with respect to dilations that are not automorphisms of the group: they accomplish this feat by lifting the operators to operators in a larger ‘freer’ nilpotent Lie group where the dilations extend to automorphic ones instead, to which the result mentioned above applies; finally, they use a transference method to push the boundedness result onto the original operators.

Another result that belongs to the same area (the intersection of themes i) and ii)) and is directly relevant to the work developed here has been obtained by Kim in [Kim00], in which an L^2 theory for the Hilbert transform in the Heisenberg group \mathbb{H}^1 along curves of the form $\Gamma_\alpha(t) := (t, \gamma(t), \alpha t \gamma(t))$ is developed. Here $\alpha \in \mathbb{R}$ and $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be Lipschitz, and the resulting operator can be written as

$$f \mapsto H_{\Gamma_\alpha} f(x) := \text{p.v.} \int_{-\infty}^{\infty} f(x \cdot \Gamma_\alpha(t)^{-1}) \frac{dt}{t}.$$

In [Kim00] it is shown that if γ is even and convex on $[0, \infty)$, then for any α the operator H_{Γ_α} is $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded if and only if γ' has bounded doubling time, that is there exists $C > 1$ such that for all $t > 0$

$$\gamma'(Ct) \geq 2\gamma'(t);$$

⁴Here $\|\cdot\|$ denotes a homogeneous gauge on N , so in particular it satisfies $\|\delta_\lambda(x)\| = |\lambda|\|x\|$ for all $\lambda > 0, x \in N$.

⁵ K and σ need not be separately homogeneous.

the importance of the doubling time condition had already been appreciated before, since it was known to play a prominent role in the Euclidean case as well (see [NVWW83]; see also [CWW95] for related earlier work on flat curves in \mathbb{H}^1). The way the result is proven is by the use of the group Fourier transform on \mathbb{H}^1 , which - as we will see in Chapter 2, §2.3 - turns the question of the boundedness of H_{Γ_α} into the uniform boundedness (in λ) of a certain oscillatory singular integral operator on \mathbb{R} , namely

$$\phi \mapsto \text{p.v.} \int e^{i\lambda\gamma(x-t)((x+t)-\alpha(x-t))} \frac{\phi(t)}{x-t} dt;$$

this in turn is proven by an almost orthogonality argument. We will exploit the group Fourier transform analogously in our work.

Consider instead the case ii)+iii), that is to say multi-parameter singular integral operators on homogeneous groups. An important result in this area was obtained by Nagel, Ricci, Stein and Wainger in [NRSW12], in which they consider singular integral operators with flag kernels on graded nilpotent groups endowed with automorphic dilations. These operators arise naturally in the study of the Kohn laplacian on CR manifolds.

Flag kernels are kernels of product type that in terms of severity of the singularity sit in-between the case of one parameter singular kernels and that of the most general product kernels. More precisely, in the case of two parameters and Euclidean convolution to keep things easy, a flag kernel associated to flag $\{(0,0)\} \subset \{0\} \times \mathbb{R}^n \subset \mathbb{R}^m \times \mathbb{R}^n$ is a distribution K that agrees with a C^∞ function $K(x,y)$ away from the subspace $\{0\} \times \mathbb{R}^n$ and satisfies smoothness conditions of the form

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \lesssim |x|^{-m-|\alpha|} (|x|+|y|)^{-n-|\beta|} \quad \forall \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n,$$

and cancellation conditions that say that for every nice test function ψ and every $r > 0$, with $\psi_r := \psi \circ \delta_r$, kernels $K_{\psi_r}^{(1)}, K_{\psi_r}^{(2)}$ defined by

$$\begin{aligned} \langle K_{\psi_r}^{(1)}, \phi \rangle &:= \langle K, \psi_r \otimes \phi \rangle, \quad \forall \phi \text{ test function,} \\ \langle K_{\psi_r}^{(2)}, \phi \rangle &:= \langle K, \phi \otimes \psi_r \rangle, \quad \forall \phi \text{ test function,} \end{aligned}$$

are kernels of one parameter singular integrals on $\mathbb{R}^n, \mathbb{R}^m$ respectively. The definition for (graded) homogeneous groups is similar but more complicated to state, as it involves an entire sequence of nested subspaces on which the singularities lie; it will not concern us here.

In [NRSW12] the authors prove such operators as described above form a closed algebra under composition, and moreover they are $L^p \rightarrow L^p$ bounded for all $1 < p < \infty$.

A further crossover happened between the areas listed above that is most relevant for this thesis. It sits in the intersection of areas i) and iii): that is, the case of multi-parameter singular integral operators with kernels supported along a submanifold. The simplest non trivial example of this would be the operator given by the (local)

double Hilbert transform along a surface of the form $s, t \mapsto (s, t, s^m t^n)$, that is

$$f \mapsto T_{m,n} f(x, y, z) := \text{p.v.} \int \int_{|s|, |t| \leq 1} f(x-s, y-t, z-s^m t^n) \frac{ds}{s} \frac{dt}{t}.$$

Since the term $s^m t^n$ is a pure monomial, this operator is naturally associated to the group of two parameter dilations defined by

$$\delta_{\lambda, \mu}(x, y, z) = (\lambda x, \mu y, \lambda^m \mu^n z);$$

moreover, one can realize the kernel of such an operator in the form (1.0.1) and thus this operator, if certain smoothness and cancellation hypotheses are satisfied, falls under the scope of the theory of multi-parameter singular integrals developed in [RS92] described above in iii). But these cancellation conditions (phrased in terms of the multiplier vanishing along certain subspaces determined by the dilations) are quite non trivial, and thus need to be verified carefully. As it happens, they fail precisely when both m and n are odd; when at least one amongst m, n is even, the theory instead applies and proves such operators to be $L^p \rightarrow L^p$ bounded for all $1 < p < \infty$.

We can see directly that when m, n are odd then boundedness fails: indeed, the resulting operator is not even bounded on $L^2(\mathbb{R}^3)$. This can be seen by looking at the multiplier of $T_{m,n}$, which is given by

$$M_{m,n}(\xi, \eta, \lambda) := \text{p.v.} \int \int_{|s|, |t| \leq 1} e^{i(\xi s + \eta t + \lambda s^m t^n)} \frac{ds}{s} \frac{dt}{t};$$

consider then $M_{mn}(0, 0, \lambda)$, and reduce by a change of variable to $M_{11}(0, 0, \lambda)$, which is given by the expression

$$\text{p.v.} \int \int_{|s|, |t| \leq 1} e^{i\lambda st} \frac{ds}{s} \frac{dt}{t}.$$

We claim that this quantity is bounded from below by $\log \lambda$ for $\lambda > 0$ large, and is thus not bounded in L^∞ . The calculation is simple but very instructive, as it highlights very well some of the differences between the one parameter case and the multi-parameter case.

First of all, we have by integration by parts

$$\begin{aligned} \text{p.v.} \int_{|s| \leq 1} e^{i\lambda st} \frac{ds}{s} &= \text{p.v.} \int_{-\infty}^{\infty} e^{i\lambda st} \frac{ds}{s} - \text{p.v.} \int_{|s| \geq 1} e^{i\lambda st} \frac{ds}{s} \\ &= c_0 \text{sgn}(\lambda t) + \text{p.v.} \int_{|s| \geq 1} \frac{\frac{d}{ds}(e^{i\lambda st})}{i\lambda t} \frac{ds}{s} \\ &= c_0 \text{sgn}(\lambda t) + \left[\frac{e^{i\lambda st}}{i\lambda st} \right]_{-1}^1 \\ &\quad + \int_{|s| \geq 1} \frac{e^{i\lambda st}}{i\lambda t s^2} \frac{ds}{s} \\ &= c_0 \text{sgn}(\lambda t) + O((\lambda|t|)^{-1}). \end{aligned} \tag{1.0.2}$$

Moreover, since the kernel is odd,

$$\left| \text{p.v.} \int_{|s| \leq 1} e^{i\lambda st} \frac{ds}{s} \right| = \left| \text{p.v.} \int_{|s| \leq 1} (e^{i\lambda st} - 1) \frac{ds}{s} \right| \leq \int_{|s| \leq 1} \lambda |s| |t| \frac{ds}{|s|} = O(\lambda |t|). \quad (1.0.3)$$

Therefore, by splitting as follows,

$$\begin{aligned} \text{p.v.} \int \int_{|s|, |t| \leq 1} e^{i\lambda st} \frac{ds}{s} \frac{dt}{t} &= \text{p.v.} \int_{|t| \leq \lambda^{-1}} \left(\text{p.v.} \int_{|s| \leq 1} e^{i\lambda st} \frac{ds}{s} \right) \frac{dt}{t} \\ &\quad + \int_{\lambda^{-1} \leq |t| \leq 1} \left(\text{p.v.} \int_{|s| \leq 1} e^{i\lambda st} \frac{ds}{s} \right) \frac{dt}{t} \\ &=: I + II \end{aligned}$$

we have by (1.0.3) that

$$|I| \leq \int_{|t| \leq \lambda^{-1}} O(\lambda |t|) \frac{dt}{|t|} = O(1),$$

and by (1.0.2)

$$\left| II - \int_{\lambda^{-1} \leq |t| \leq 1} c_0 \text{sgn}(t) \frac{dt}{t} \right| \leq \int_{\lambda^{-1} \leq |t| \leq 1} O((\lambda |t|)^{-1}) \frac{dt}{|t|} = O(1),$$

and since

$$\int_{\lambda^{-1} \leq |t| \leq 1} c_0 \text{sgn}(t) \frac{dt}{t} = 2c_0 \log \lambda$$

the claim is proved.

It is evident from the calculations that the term that causes the logarithmic divergence in λ is precisely $\text{sgn}(t)/t = 1/|t|$; boundedness is destroyed by ‘a conspiracy of signs’, so to speak.

Remark 1.1. We digress here briefly to comment on the above calculation. As simple as it is, one can already derive an important consequence from it. C. Fefferman [Fef71] indeed used the fact to construct a continuous periodic function of two variables such that the rectangular partial sums of its double Fourier series are everywhere divergent; that is, there exists $f \in C(\mathbb{T}^2)$ such that

$$\lim_{\substack{M \rightarrow \infty, \\ N \rightarrow \infty}} \sum_{\substack{|m| \leq M, \\ |n| \leq N}} \hat{f}(m, n) e^{2\pi i(mx + ny)} = \infty \quad \forall (x, y) \in \mathbb{T}^2.$$

This is in sharp contrast with the one dimensional case, where Carleson’s theorem says that the partial sums of every $L^2(\mathbb{T})$ (and thus in particular $C(\mathbb{T})$) function converge to the function almost everywhere; a further indication of the stark difference between the one-parameter world and the multi-parameter one.

A more general type of operator can be obtained by replacing the single monomial $s^m t^n$ with a full polynomial $P(s, t) \in \mathbb{R}[s, t]$, to get the (local) double Hilbert transform along a polynomial surface

$$f \mapsto T_P f(x, y, z) := \text{p.v.} \int \int_{|s|, |t| \leq 1} f(x - s, y - t, z - P(s, t)) \frac{ds}{s} \frac{dt}{t}.$$

Now the monomials are tangled together and there is no (unique) dilation group associated to it. These operators have been considered by Carbery, Wainger and Wright in [CWW00], in which they found a necessary and sufficient condition on the polynomial P for T_P to be $L^p(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3)$ bounded for all $1 < p < \infty$. This condition, which obviously must include the condition for the monomial case discussed above, is stated in terms of the Newton diagram of P .

For $(p, q) \in \mathbb{N} \times \mathbb{N}$, let Q_{pq} be the quadrant of \mathbb{R}^2 given by

$$Q_{pq} := [p, \infty) \times [q, \infty);$$

the *Newton diagram* of $P(s, t) = \sum_{m,n} c_{mn} s^m t^n$ is the subset of the plane given by

$$\mathcal{N}(P) := \overline{\text{co}} \left(\bigcup_{m,n: c_{mn} \neq 0} Q_{mn} \right),$$

that is, it is the closed convex hull of the union of all the quadrants associated to each monomial with non-zero coefficient in P . With this definition, we can state their theorem as follows:

Theorem 1.1 ([CWW00]). *Let $P \in \mathbb{R}[s, t]$ and let $\mathcal{N}(P)$ be its Newton diagram. If for every (m, n) that is a corner of $\mathcal{N}(P)$ at least one of m, n is even, then the operator T_P is $L^p(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3)$ bounded for every $1 < p < \infty$. Else, the operator is unbounded on any L^p space.*

One can see that the condition for monomials is subsumed in the condition for the Newton diagram of P , since the diagram for a monomial is a single quadrant. The strategy of the proof is to reduce things to the monomial case, which is completely characterized by [RS92] as pointed out before. The way the authors achieve this, is by dividing up the integration region $|s|, |t| \leq 1$ into subregions in each of which one can replace $P(s, t)$ with one of its monomials, at the price of an L^p bounded error. Observe that for $p = 2$ it would suffice to estimate the multiplier of T_P , which is given by

$$M_P(\xi, \eta, \lambda) := \text{p.v.} \int \int_{|s|, |t| \leq 1} e^{i(\xi s + \eta t + \lambda P(s, t))} \frac{ds}{s} \frac{dt}{t}.$$

The relevance of the Newton diagram for the boundedness and decay of such oscillatory integrals has been known at least since [Var76]. We will encounter similar oscillatory integrals multiple times.

Remark 1.2. It is very interesting to notice that the naive extension of the statement of Theorem 1.1 to the three parameter case in \mathbb{R}^4 and operator

$$f \mapsto \text{p.v.} \int \int \int_{|s|, |t|, |u| \leq 1} f(x - s, y - t, z - u, w - P(s, t, u)) \frac{ds}{s} \frac{dt}{t} \frac{du}{u}$$

is false. Indeed, there exists a polynomial P such that its Newton polyhedron (defined in the obvious way) has all corners with all coordinates even, yet the operator is unbounded, even on $L^2(\mathbb{R}^4)$. See [CWW00] for details. Thus the situation for higher numbers of parameters is significantly more complicated. See [CWW09] by the same authors for further investigation of the triple Hilbert transform case.

Other results centred around similar operators have appeared: Pramanik and Yang have characterized in [PY08] the boundedness of the double Hilbert transform in \mathbb{R}^{d+2} along the surface given by $(s, t, \varphi_1(s, t), \dots, \varphi_d(s, t))$, where φ_j are real analytic functions; in another direction, Patel in [Pat08] has characterized the boundedness for the global double Hilbert transform along a polynomial surface, which now ends up depending on the convex hull of all the points (m, n) such that $c_{mn} \neq 0$.

So far we have discussed examples of results for any combination of two of the themes i), ii), iii) introduced above. Now, this thesis could be described as our modest contribution to the understanding of the case given by the combination of all *three* themes, i)+ii)+iii), that is of singular integral operators having all the features discussed: multi-parameter singular integral operators in a (non-commutative) homogeneous group, whose kernels are supported along submanifolds. Here we consider the simplest such operator, that is the *(local) double Hilbert transform along surfaces in \mathbb{H}^1* . This is the operator

$$f \mapsto H_\varphi f(x, y, z) := \text{p.v.} \int \int_{|s|, |t| \leq 1} f((x, y, z) \cdot (s, t, \varphi(s, t))^{-1}) \frac{ds}{s} \frac{dt}{t},$$

where φ will be taken to be a real analytic function in Chapter 3 and will further be restricted to being a polynomial in $\mathbb{R}[s, t]$ in Chapter 4 to produce uniformity results. Notice this is the simplest non-commutative analogue of the operator T_P studied in [CWW00], where by non-commutative we refer to the fact that the convolution is taken in a non-commutative homogeneous group.

What we will do in the next chapters is exploit the Fourier transform theory of \mathbb{H}^1 to develop an L^2 theory for such operators, which will allow us to prove the following results⁶:

Theorem. *For every φ real analytic in a neighbourhood of $(0, 0)$, the operator H_φ is $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded.*

Theorem. *Let $P(s, t) = \sum_{m, n} c_{mn} s^m t^n$ be a polynomial in $\mathbb{R}[s, t]$ of degree at most d , and suppose P is such that if $c_{mn} \neq 0$ then at least one amongst m, n is even. Then $\|H_P\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)}$ is uniformly bounded by a constant $C = C(d)$ depending only on d .*

Notice that the first theorem says the operator H_P is bounded for *all* polynomials P unconditionally, unlike in the Euclidean convolution case of operator T_P . That the situation in the Heisenberg group is going to be different is already evident from an example: if one takes $P(s, t) = st/2$ then, noticing that

$$(s, 0, 0) \cdot (0, t, 0) = (s, t, st/2)$$

one can factorize the operator $H_{st/2}$ as

$$\begin{aligned} H_{st/2} f(x, y, z) &= \text{p.v.} \int_{|t| \leq 1} \text{p.v.} \int_{|s| \leq 1} f((x, y, z) \cdot (0, t, 0)^{-1} \cdot (s, 0, 0)^{-1}) \frac{ds}{s} \frac{dt}{t} \\ &= \text{p.v.} \int_{|t| \leq 1} \mathcal{H}_1 f((x, y, z) \cdot (0, t, 0)^{-1}) \frac{dt}{t} \end{aligned}$$

⁶See Chapters 3 and 4 for the full statements.

$$= \mathcal{H}_2 \mathcal{H}_1 f(x, y, z),$$

where \mathcal{H}_i is the Hilbert transform in \mathbb{H}^1 in direction x_i , $i = 1, 2$. Since these operators are $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded, the operator $H_{st/2}$ is bounded as well. On the other hand, $T_{st/2}$ is certainly not.

Before moving to the next chapters where we address the problem outlined above, we want to give further context in order for the results presented here to be better understood, and in order to justify the work carried out in Chapter 5. Our work is certainly not the first one to address operators described by the combination i) + ii) + iii). Most notably, in recent years Stein and Street have introduced an L^p theory with broad scope for a large class of multi-parameter operators, better described as multi-parameter singular Radon transforms. They take the form

$$f \mapsto Tf(x) := \psi(x) \int_{\mathbb{R}^N} f(\gamma(\mathbf{t}; x)) K(\mathbf{t}) \, d\mathbf{t},$$

where K is a multi-parameter singular kernel, ψ a cutoff function and $\gamma(\mathbf{t}; x)$ is a smooth map such that $\gamma(\mathbf{0}; x) = x$.

The L^2 theory for such a general class of operators has been originally developed by Street in [Str12], and later has been extended to the full L^p theory by Stein and Street jointly in [SS13]; in the final of a series of three papers, [SS12], they rediscuss the case of real analytic submanifolds and show it admits significant simplifications with respect to the fully general case. It is important to notice at this stage that their results fundamentally rely on certain assumptions on the map $\gamma(\mathbf{t}; x)$, which will be explained below.

Their theory is very relevant to us, for several reasons. First of all, as by §17.8 of [Str12], their theory already covers the L^2 (and L^p) boundedness of H_φ when φ is real analytic. Here one takes

$$\gamma(s, t; x, y, z) := (x, y, z) \cdot (s, t, \varphi(s, t))^{-1} \quad (1.0.4)$$

and $K(s, t) := \text{p.v.} 1/st$. In Chapter 3 we thus reprove the result of [Str12] by different, more hands-on means; this will serve as an illustration of our method which will then be refined in Chapter 4 to prove the uniformity result stated above. Secondly, we point out that the uniformity result is new, and cannot possibly be achieved by means of the Stein-Street theory of multi-parameter singular Radon transforms.

Finally, it is not a priori clear how strong the above mentioned assumptions on $\gamma(\mathbf{t}; x)$ are, or in other words it is not clear how close they are to being necessary. The results presented in Chapter 5 show L^2 boundedness for certain other surfaces that *do not* satisfy the assumptions of Stein-Street, and therefore point to the fact that such assumptions are not close to being necessary. Here it is best to state what these assumptions are in order to be clear.

Apart from some technical assumptions of uniform regularity that are not easily stated and that we will therefore gloss over, the core of the conditions can be explained as follows. It was realized in [CNSW99] that smooth maps $\gamma(\mathbf{t}; x)$ such that $\gamma(\mathbf{0}; x) = x$ can be realized as the exponential of Taylor series whose coefficients are vector fields. More precisely, given $\gamma : \mathbb{R}^N \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ as above, one can find a

unique collection of smooth vector fields $\{X_\alpha, \alpha \in \mathbb{N}^N\}$ such that for any $M > 0$ and \mathbf{t} sufficiently small it is

$$\gamma(\mathbf{t}; x) = \exp\left(\sum_{\substack{\alpha \in \mathbb{N}^N, \\ |\alpha| < M}} \mathbf{t}^\alpha X_\alpha / \alpha!\right)(x) + O(|\mathbf{t}|^M)$$

for every x in a neighbourhood of the origin. Here there is an ambiguity in the definition of \exp when $N > 1$: for fixed \mathbf{t}, x we define the above exponential mapping to be

$$\exp\left(s\left(\sum_{|\alpha| < M} \mathbf{t}^\alpha X_\alpha / \alpha!\right)\right)\Big|_{s=1}(x);$$

thus here $\sum_{|\alpha| < M} \mathbf{t}^\alpha X_\alpha / \alpha!$ is a fixed vector field.

Now, if the kernel K is a product kernel on $\mathbb{R}^N = \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_k}$, we write analogously the multi-indices $\alpha \in \mathbb{N}^N$ as $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^{N_1} \times \dots \times \mathbb{N}^{N_k}$. We call *pure powers* those multi-indices α such that $\alpha_j = 0$ for all except one $j \in \{1, \dots, k\}$. Then the condition of Stein-Street can be stated as follows: for every $\delta \in (0, 1]^k$, the formal sum

$$\sum_{\alpha \in \mathbb{N}^N} \delta^\alpha \mathbf{t}^\alpha X_\alpha$$

must belong to the involutive distribution⁷ generated by the vector fields with formal degrees⁸

$$\delta^\beta \mathbf{t}^\beta X_\beta$$

where β ranges over all pure powers, and the distribution must be finitely generated as a C^∞ -module. This condition is usually summarized by saying that *the pure powers control the mixed powers*. Here we are being a bit sloppy for the sake of clarity: it is very important that such conditions hold “uniformly in δ ”, in a sense that has to be made precise. Here however we ignore these details.

For the case of the Heisenberg group \mathbb{H}^1 and γ as in (1.0.4), the Taylor expansion is given by⁹

$$\gamma(\mathbf{t}; x) = \exp\left(sX + tY + \varphi(s, t)Z\right)(x);$$

if $\sum_{mn} a_{mn} s^m t^n$ is the power series expansion of φ in a neighbourhood of the origin, the pure powers are vector fields

$$\delta_1 sX, \quad \delta_2 tY, \quad \delta_1^m s^m a_{m0} Z, \quad \delta_2^n t^n a_{0n} Z,$$

and since $[X, Y] = Z$ and all the other commutators vanish, the involutive distribution they generate is the span of

$$\delta_1 sX, \quad \delta_2 tY, \quad \delta_1^m \delta_2^n s^m t^n Z,$$

to which $sX + tY + \varphi(s, t)Z$ clearly belongs (in the sense that every truncate does). Hence the assumptions are verified and Stein-Street theory applies, as explained.

⁷Recall the involutive distribution of the collection of vector spaces \mathcal{X} is the smallest C^∞ module that contains \mathcal{X} and is closed with respect to the commutator operation on vector fields.

⁸That is where \mathbf{t} is kept as a variable, heuristically speaking.

⁹Here X, Y, Z are the left-invariant vector fields on \mathbb{H}^1 , given by $X = \partial_x - \frac{y}{2}\partial_z, Y = \partial_y + \frac{x}{2}\partial_z, Z = \partial_z$.

The surfaces we will consider in Chapter 5 are of a type for which the above assumptions are not verified, that is the pure powers cannot control the mixed powers and Stein-Street theory does not apply. One such surface is given for example by (s^3, t, st^2) , for which the pure powers are clearly

$$\delta_1^3 s^3 X, \quad \delta_2 t Y,$$

but $[\delta_1^3 s^3 X, \delta_2 t Y] = \delta_1^3 \delta_2 s^3 t Z$, and thus $\delta_1 \delta_2^2 s t^2 Z$ does not belong to the involutive distribution generated by the pure powers. However, our results in Chapter 5 show that the double Hilbert transform in \mathbb{H}^1 along such a surface is indeed $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded.

We conclude here our introduction, with the hope to have provided sufficient motivation to justify interest in the results presented in this work.

Chapter 2

Preliminaries

In this brief chapter we review certain facts that will play an important role in our later study. These are well known facts but deserve rigorous separate treatment, for the sake of clarity and with the intention to make this body of work as self contained as possible.

In the first section of this chapter we will state and prove the well known Van der Corput lemma and its variants, both in the one variable setting and in the many variables one. In the second section we review a result of Ricci and Stein regarding oscillatory singular integrals with polynomial phases, that is with kernels of the form $e^{iP(x,y)}K(x,y)$. In the third and final section we review the group Fourier transform theory on the Heisenberg group \mathbb{H}^1 , building it from its representation theory, and present a condition that characterizes the $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ boundedness of convolution operators on \mathbb{H}^1 .

References for the material in this chapter are [Ste93], [Fol89] and [RS87].

2.1 Van der Corput lemma

2.1.1

Oscillatory integrals are ubiquitous in harmonic analysis for obvious reasons and understanding them entails understanding the often subtle cancellation involved in the particular problem at hand - be it the boundedness of the multiplier of a convolution operator or the summability of the Bochner-Riesz means of multiple Fourier series.

The most fundamental result regarding certain general oscillatory integrals (so called of first kind) is the well known Van der Corput's lemma, which we state and prove.

Lemma 2.1 (Van der Corput). *Let ϕ be in $C^k((a,b))$ and suppose that for all $t \in (a,b)$ we have $|\phi^{(k)}(t)| \geq \mu$. Then*

i) if $k = 1$ and ϕ' is monotonic on (a,b) , then

$$\left| \int_a^b e^{i\lambda\phi(t)} dt \right| \lesssim (\lambda\mu)^{-1};$$

ii) if $k \geq 2$ then

$$\left| \int_a^b e^{i\lambda\phi(t)} dt \right| \lesssim_k (\lambda\mu)^{-1/k}.$$

Observe that the implicit constants do not depend on a, b and neither on ϕ ; they only depend on k , and this is very important.

Remark 2.1. It is easy to see by scaling that the exponent $1/k$ is the only possible one for an estimate like the above to hold. Indeed, suppose the estimate holds with $1/k$ replaced by some exponent $\alpha > 0$; by changing variables by replacing t with βt we have

$$\left| \int_a^b e^{i\lambda\phi(t)} dt \right| = \beta \left| \int_a^b e^{i\lambda\phi(\beta t)} dt \right| \lesssim \beta (\lambda\mu\beta^k)^{-\alpha},$$

and therefore it must be $\alpha = 1/k$ for the inequality to hold for all $\beta > 0$.

Proof. Let $k = 1$ and ϕ' monotonic first. Then we have by integration by parts that

$$\int_a^b e^{i\lambda\phi(t)} dt = \int_a^b \frac{\frac{d}{dt} e^{i\lambda\phi(t)}}{i\lambda\phi'(t)} dt = \left[\frac{e^{i\lambda\phi(t)}}{i\lambda\phi'(t)} \right]_a^b - \int_a^b e^{i\lambda\phi(t)} \frac{d}{dt} \left(\frac{1}{i\lambda\phi'(t)} \right) dt,$$

and therefore

$$\begin{aligned} \left| \int_a^b e^{i\lambda\phi(t)} dt \right| &\leq \left| \frac{e^{i\lambda\phi(a)}}{i\lambda\phi'(a)} \right| + \left| \frac{e^{i\lambda\phi(b)}}{i\lambda\phi'(b)} \right| \\ &\quad + \left| \int_a^b e^{i\lambda\phi(t)} \frac{d}{dt} \left(\frac{1}{i\lambda\phi'(t)} \right) dt \right| \\ &\leq \frac{2}{\lambda\mu} + \int_a^b \left| \frac{d}{dt} \left(\frac{1}{i\lambda\phi'(t)} \right) \right| dt \\ &= \frac{2}{\lambda\mu} + \left| \int_a^b \frac{d}{dt} \left(\frac{1}{i\lambda\phi'(t)} \right) dt \right| \\ &= \frac{2}{\lambda\mu} + \frac{1}{\lambda} \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \leq \frac{3}{\lambda\mu}. \end{aligned}$$

Notice we have used the monotonicity of ϕ' in passing from the second to the third line.

If $k \geq 2$ then we argue by induction. Suppose the theorem has been proved for $k-1$, and let $\delta > 0$, whose precise value is to be specified later. Since $\phi^{(k)}$ is single signed on (a, b) by assumption, $\phi^{(k-1)}$ can have at most one zero in (a, b) (and is monotonic). Suppose it does have a zero in t_0 , and split the integral as $\int_a^b = \int_a^{t_0-\delta} + \int_{t_0-\delta}^{t_0+\delta} + \int_{t_0+\delta}^b$. By the hypothesis on $\phi^{(k)}$ we have that outside $(t_0 - \delta, t_0 + \delta)$ it is $|\phi^{(k-1)}(t)| \geq \mu\delta$, and in there we apply the inductive hypothesis: thus

$$\left| \int_a^{t_0-\delta} e^{i\lambda\phi(t)} dt \right| \lesssim_{k-1} (\lambda\mu\delta)^{-1/(k-1)},$$

and similarly for the integral over $(t_0 + \delta, b)$. For the remaining term, since $|e^{i\lambda\phi(t)}| = 1$, we simply estimate

$$\left| \int_{t_0-\delta}^{t_0+\delta} e^{i\lambda\phi(t)} dt \right| \leq \delta.$$

We optimize in δ by choosing $\delta \sim (\lambda\mu)^{-1/k}$, which proves the result.

If instead $\phi^{(k-1)}$ does not have zeroes in (a, b) , observe that the minimum of $|\phi^{(k-1)}|$ must occur at either a or b ; suppose without loss of generality that it occurs in a and that $\phi^{(k-1)}$ is positive in (a, b) , then we split the integral as $\int_a^b = \int_a^{a+\delta} + \int_{a+\delta}^b$. The first integral is bounded by δ as before, and in the second one it must be $\phi^{(k-1)}(t) \geq \phi^{(k-1)}(a) + \mu\delta > \mu\delta$, and thus by inductive hypothesis the second interval is bounded again by $O_k((\lambda\mu\delta)^{-1/(k-1)})$, and one concludes exactly as before. \square

The statement in Van der Corput's lemma is interesting but is not so general, in that one rarely gets oscillatory integrals precisely of that form. It is therefore important to observe that the statement generalizes easily by integration by parts to the following corollary

Corollary 2.2. *Let $\phi \in C^k((a, b))$ and ψ be such that $\psi' \in L^1((a, b))$, and suppose that for all $t \in (a, b)$ it is $|\phi^{(k)}(t)| \geq \mu$. Then*

i) *if $k = 1$ and ϕ' is monotonic on (a, b) , then*

$$\left| \int_a^b e^{i\lambda\phi(t)} \psi(t) dt \right| \lesssim (\lambda\mu)^{-1} \left[|\psi(b)| + \int_a^b |\psi'(s)| ds \right];$$

ii) *if $k \geq 2$ then*

$$\left| \int_a^b e^{i\lambda\phi(t)} \psi(t) dt \right| \lesssim_k (\lambda\mu)^{-1/k} \left[|\psi(b)| + \int_a^b |\psi'(s)| ds \right].$$

Notice that, again, the implicit constants do not depend on a, b or ϕ , and they do not depend on ψ either (the dependence of the estimate on ψ is made explicit in the term in square brackets).

Proof. As anticipated, the proof is just an application of integration by parts. Define

$$F(u) := \int_a^u e^{i\lambda\phi(t)} dt,$$

so that

$$\begin{aligned} \int_a^b e^{i\lambda\phi(t)} \psi(t) dt &= \int_a^b F'(t) \psi(t) dt \\ &= \left[F(t) \psi(t) \right] - \int_a^b F(t) \psi'(t) dt, \end{aligned}$$

and therefore by Van der Corput's lemma the integral is bounded by the sum of

$$|F(b)\psi(b) - F(a)\psi(a)| = |F(b)\psi(b)| \lesssim_k |\psi(b)| (\lambda\mu)^{-1/k},$$

and

$$\left| \int_a^b F(t) \psi'(t) dt \right| \lesssim (\lambda\mu)^{-1/k} \int_a^b |\psi'(t)| dt,$$

since the bound on F is uniform in the endpoints of the integral. Thus the corollary is proved. \square

2.1.2

The estimate given by Corollary 2.2 is already quite powerful. Indeed, it is powerful enough to prove, for example, decay estimates for the Fourier transform of the surface measure $d\sigma$ of the sphere \mathbb{S}^{n-1} . These in turn allow one to prove Fourier restriction estimates for the sphere (the Stein-Tomas theorem), that is a-priori estimates of the form

$$\|\widehat{f}|_{\mathbb{S}^{n-1}}\|_{L^2(\mathbb{S}^{n-1}, d\sigma)} \lesssim_{p,n} \|f\|_{L^p(\mathbb{R}^n)}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, with $p \geq 1$ and sufficiently smaller than 2.

Nevertheless, we shall later need one further consequence of the Van der Corput's lemma, namely a multidimensional estimate. This estimate takes an analogous form to the one dimensional one, but the constant now depends on the phase as well.

Proposition 2.3. *Let $\psi \in L^\infty(\mathbb{R}^n)$ be compactly supported in the unit ball of \mathbb{R}^n and such that $\nabla\psi \in L^1(\mathbb{R}^n)$; let $\phi \in C^{k+1}(\mathbb{R}^n)$ and suppose that for a multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ we have*

$$|\partial_x^\alpha \phi(x)| \geq \mu > 0, \quad \forall x \in \text{Supp}(\psi).$$

Then

$$\left| \int_{\mathbb{R}^n} e^{i\lambda\phi(x)} \psi(x) dx \right| \lesssim_{n,|\alpha|, \|\phi\|_{\dot{C}^{|\alpha|+1}}} (\lambda\mu)^{-1/|\alpha|} [\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^1}].$$

The implicit constant stays bounded if the homogeneous $C^{|\alpha|+1}$ norm $\|\phi\|_{\dot{C}^{|\alpha|+1}}$ stays bounded.

Proof. Before we start with the actual proof, we claim that the vectors $(\omega \cdot \nabla)^k$ for $\omega \in \mathbb{S}^{n-1}$ span the vector space of k -th order derivatives, that is the span of ∂_x^α for all α s.t. $|\alpha| = k$. Indeed, introduce the bilinear form

$$\langle P(\nabla), Q(\nabla) \rangle := [Q(\nabla)](P),$$

where P, Q are homogeneous polynomials of degree k and $P(\nabla)$ is interpreted as $\sum_{|\beta|=k} c_\beta \partial_x^\beta$ (hence every k -th derivatives can be expressed as $P(\nabla)$ for some polynomial P). It is easy to see that this is actually a positive definite inner product on the real vector space of k -th order constant coefficients differential operators of \mathbb{R}^n : for example, for positive definiteness observe

$$\langle P(\nabla), P(\nabla) \rangle = \sum_{|\beta|=k} c_\beta \partial_x^\beta \left(\sum_{|\gamma|=k} c_\gamma x^\gamma \right) := \sum_{|\beta|=|\gamma|=k} \delta_{\beta\gamma} c_\beta c_\gamma \geq 0.$$

Thus it suffices to show that if $\langle P(\nabla), (\omega \cdot \nabla)^k \rangle = 0$ for all $\omega \in \mathbb{S}^{n-1}$ then P must identically

which in turn is equivalent to $\left(\frac{d}{dt}\right)^k P(t\omega) = 0$ for all ω ; but since P is homogeneous of degree k , this is impossible unless $P \equiv 0$.

Now to the actual proof. We normalize by replacing λ with $\lambda\mu$ and ϕ by ϕ/μ , so that our assumption is now that $|\partial_x^\alpha \phi| \geq 1$. By what was just proved for k -th order derivatives, for every x there exists $\omega(x) \in \mathbb{S}^{n-1}$ such that

$$|(\omega(x) \cdot \nabla)^k \phi(x)| \gtrsim_{n,k} 1.$$

Now, if we keep x fixed we have

$$|(\omega(x) \cdot \nabla)^k \phi(x) - (\omega(x) \cdot \nabla)^k \phi(x')| \leq \|\phi\|_{\dot{C}^{k+1}} |x - x'|,$$

and therefore if we choose an $\varepsilon = \varepsilon(n, k)$ sufficiently small we can find a countable covering of \mathbb{R}^n by balls $B_j := B(x_j, \varepsilon / \|\phi\|_{\dot{C}^{k+1}})$ of fixed radius such that any point belongs to at most $O(1)$ balls (that is the covering has bounded multiplicity) and such that for all $z \in B_j$

$$|(\omega(x_j) \cdot \nabla)^k \phi(z)| \gtrsim_{n,k} 1;$$

in particular, we have

$$\left| \left(\frac{d}{dt} \right)^k \Big|_{t=0} \phi(t\omega(x_j) + z) \right| \gtrsim_{n,k} 1.$$

Let η_j be a smooth partition of unity subordinated to the balls B_j , that is $\text{Supp}(\eta_j) \subset B_j$, $0 \leq \eta_j \leq 1$, $\sum_j \eta_j(x) \equiv 1$ for every x and $\sum_j |\nabla \eta_j(x)| \lesssim 1$ for all x . Pick one ball B_j and suppose without loss of generality that $\omega(x_j) = (1, 0, \dots, 0)$, then

$$\int e^{i\lambda\phi(x)} \psi(x) \eta_j(x) dx = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} e^{i\lambda\phi(t,y)} \psi(t,y) \eta_j(t,y) dt dy,$$

and the inner integral is bounded by Corollary 2.2 by

$$\begin{aligned} &\lesssim_{n,k} \lambda^{-1/k} \left[\sup_t |\psi(t,y) \eta_j(t,y)| + \int |\partial_{x_1}(\psi \eta_j)(t,y)| dt \right] \\ &\leq \lambda^{-1/k} \left[\|\psi\|_{L^\infty(\mathbb{R}^n)} \chi_{\pi_j(B_j)}(y) + \int |\eta_j(t,y) \partial_{x_1} \psi(t,y)| + |\psi(t,y) \partial_{x_1} \eta_j(t,y)| dt \right], \end{aligned}$$

where π_j is the projection in direction $\omega(x_j)$. Integration in y then gives

$$\begin{aligned} &\left| \int e^{i\lambda\phi(x)} \psi(x) \eta_j(x) dx \right| \\ &\lesssim_{n,k} \lambda^{-1/k} [\|\psi\|_{L^\infty(\mathbb{R}^n)} |\pi_j(B_j)| + \|\psi\|_{L^\infty(\mathbb{R}^n)} \|\nabla \eta_j\|_{L^1} + \|\nabla \psi\|_{L^1(B_j)}], \end{aligned}$$

and notice that $|\pi_j(B_j)| \sim_{n,k, \|\phi\|_{\dot{C}^{k+1}}} 1$; since $\text{Supp}(\psi) \subset B(0, 1)$, the number of balls B_j that contribute is finite and depends only on n, k and $\|\phi\|_{\dot{C}^{k+1}}$, and thus by summing in j we conclude the desired estimate, given that

$$\int e^{i\lambda\phi(x)} \psi(x) dx = \sum_j \int e^{i\lambda\phi(x)} \psi(x) \eta_j(x) dx.$$

□

Remark 2.2. The estimate given by the Proposition will be enough for our purposes, but is not optimal in several ways. The decay rate of $1/k$ is not necessarily sharp - the scaling argument does not work here because the variables are more than one. Indeed, for example, when the Hessian of the phase is non-singular one can prove¹ by the stationary phase method that the actual rate of decay is $\lambda^{-n/2}$, whereas the lemma

¹See for example [Ste93], Chapter VIII, §2.3.

above gives only decay $\lambda^{-1/2}$. Moreover, the dependence of the constant on the phase ϕ can be removed (at the price of worsening the exponent), as has been shown in [CCW99], where estimates uniform in the phase are given for multidimensional oscillatory integrals. It is currently an open problem to determine the minimal amount by which one has to worsen the exponent in order for such uniformity to hold.

There would be a lot more to say regarding Van der Corput type estimates, but what we have included so far will suffice for our purposes, and therefore we move on to the next section.

2.2 Oscillatory singular integrals

2.2.1

In the study of singular integrals in the more general setting of homogeneous groups, it is natural to address their L^2 boundedness by using the Fourier transform on such groups. As we will see in the next section of this chapter, this is operator-valued when the group is non-commutative, and it is often important to estimate norms of these values. These operators typically end up being oscillatory singular integrals; examples of their kernels (arising from one parameter singular integrals) have the form

$$e^{iP(x,y)} K(x, y),$$

where P is a polynomial in two variables (each in \mathbb{R}^n) and K is a kernel in $C^1(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) \text{ s.t. } x = y\})$ satisfying the so-called standard estimates, that is

$$\begin{aligned} |K(x, y)| &\lesssim |x - y|^{-n}, \\ |K(x, y) - K(x, y')| &\lesssim \frac{|y - y'|^\delta}{|x - y|^{n+\delta}} \quad \text{when } |y - y'| < \frac{1}{2}|x - y|, \\ |K(x, y) - K(x', y)| &\lesssim \frac{|x - x'|^\delta}{|x - y|^{n+\delta}} \quad \text{when } |x - x'| < \frac{1}{2}|x - y|, \end{aligned}$$

for some $0 < \delta \leq 1$.

This rather general example has been studied by Ricci and Stein in [RS87]: it is the first of a series of three papers ([RS87], [RS88] that was already discussed in Chapter 1, and [RS89]) on harmonic analysis on the nilpotent Lie groups - and the connection between them is the one hinted at above, that is, the fact that such operators arise as values of the multipliers of singular integrals in nilpotent groups.

The main result of Ricci and Stein is

Theorem 2.4 ([RS87]). *Let P and K be as defined above. Then the operator*

$$C_c^\infty(\mathbb{R}^n) \ni f \mapsto Tf(x) := \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x, y) f(y) dy$$

extends to an $L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ bounded operator for any $1 < p < \infty$, and more precisely

$$\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p,\deg P} \|f\|_{L^p(\mathbb{R}^n)}.$$

Notice how the constant depends on the degree of the polynomial P but does not otherwise depend on the coefficients of P . This is therefore a uniformity result akin to the one we will be presenting in Chapter 4. Indeed, take the one parameter singular integral in \mathbb{H}^1 with kernel supported along the curve $\gamma(t) := (t, t^\alpha, t^\beta)$ given by

$$f \mapsto \text{p.v.} \int_{-\infty}^{\infty} f(x \cdot \gamma(t)) \frac{dt}{t};$$

as the methods of next section can show, the $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ boundedness of this operator is equivalent to the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ boundedness uniformly in $\lambda > 0$ of the operator

$$\phi \mapsto \text{p.v.} \int_{-\infty}^{\infty} e^{i\lambda((x-y)^\beta + 4x(x-y)^\alpha - 2(x-y)^{\alpha+1})} \phi(y) \frac{dy}{x-y},$$

which is precisely of the form described above and therefore the theorem applies to it, giving boundedness uniformly in the coefficients - that is, uniformly in λ . This last operator can be considered a one parameter version of the ones we will have to deal with later, in which the scalar oscillatory factor will be replaced by a scalar oscillatory integral instead (also, compare the above operator with the ones considered in [Kim00]).

The proof of Theorem 2.4 is too long to be meaningfully illustrated in this brief chapter, and therefore we have chosen to omit it. We include a remark instead.

Remark 2.3. Theorem 2.4 does not say what happens at the endpoint $p = 1$ of the range of exponents. The proof cannot say anything because it works by interpolation: the operator is decomposed into a classical part with barely any oscillation T_0 and another part where oscillation occurs, T_∞ ; while the classical part is easily bounded, the oscillating part needs to be decomposed into dyadic annuli, $T_\infty = \sum_j T_j$, and the operators $T_j^* T_j$ are shown to have $L^p \rightarrow L^p$ norm bounded trivially by $O(1)$ and $L^2 \rightarrow L^2$ norm decaying geometrically in $|j|$, and real interpolation gives the summability of the norms of such pieces in any L^p , $1 < p < \infty$. Oscillatory integral estimates of Van der Corput type are used to prove these rapidly decaying bounds on $\|T_j^* T_j\|_{L^2 \rightarrow L^2}$.

Nevertheless, the operators turn out to be $L^1 \rightarrow L^{1,\infty}$ bounded as well - as was proven in [CC87]. In there, a rough Calderón-Zygmund decomposition is used to reduce the problem to an application of the $T^* T$ method to operators of the form $T_j^* T_k$, where the T_k are certain pieces of the original operator, different from the ones in [RS87]; then one needs good pointwise estimates on the kernels of such operators in $T^* T$ form, which are oscillatory integrals, and thus one is led to use once again Van der Corput estimates. As the reader might have guessed from the appearance of the Calderón-Zygmund decomposition, the argument needs L^2 bounds for the operators, which are assumed from [RS87].

2.3 The Fourier transform on the Heisenberg group

2.3.1 Representations of the Heisenberg group

We have previously introduced the Heisenberg group \mathbb{H}^1 as a purely algebraic object, that is the space \mathbb{R}^3 with group operation

$$(x, y, t) \cdot (x', y', t') := (x + x', y + y', t + t' + (xy' - x'y)/2);$$

but there is a way in which it arises naturally (technically speaking, a representation of it arises naturally) when considering certain symmetries of the Fourier transform. Indeed, consider functions in $L^2(\mathbb{R})$. The space $L^2(\mathbb{R})$ is invariant with respect to translation and modulation, that is to the families \mathcal{T}, \mathcal{M} of operators

$$\tau_y : f \mapsto f(\cdot - y)$$

and

$$M_\xi : f \mapsto e^{i\xi \cdot} f(\cdot),$$

where y, ξ range over \mathbb{R} . It is clear that each family itself is also a group of unitary transformations acting on $L^2(\mathbb{R})$, namely

$$\forall x, y \in \mathbb{R}, \quad \tau_x \circ \tau_y = \tau_{x+y},$$

and similarly

$$\forall \xi, \eta \in \mathbb{R}, \quad M_\xi \circ M_\eta = M_{\xi+\eta};$$

this in particular shows that they are commutative groups. The operators are linked to each other through the Fourier transform on $L^2(\mathbb{R})$: as is well known,

$$\widehat{\tau_y f}(\zeta) = e^{iy\zeta} \hat{f}(\zeta) = M_y \hat{f}(\zeta),$$

and vice versa a similar relationship holds with τ and M swapped.

It is natural to consider what is the group generated by the union of the two families of translations and modulations and what is its action on $L^2(\mathbb{R})$. One then considers the commutator of a translation and a modulation, and it can be easily seen that for any $f \in L^2(\mathbb{R})$ it is

$$[\tau_y, M_\xi] f(x) = e^{-i\xi y} f(x), \tag{2.3.1}$$

that is $[\tau_y, M_\xi]$ is the operator of scalar multiplication by the constant $e^{-i\xi y}$ with modulus 1 - so in particular we have that $[\mathcal{T}, \mathcal{M}]$ is isomorphic to $U(1) = \mathcal{U}(\mathbb{C})$, the unitary group of degree 1. As this commutes with both translations and modulations, one can see that the group generated by translations and modulations of $L^2(\mathbb{R})$ is equivalently the group G obtained by compositions of a modulation, a translation and a scalar multiplication (equivalently called phase shift) in this fixed order (although any order would do). The group operation \odot of this group is easily described: by the commutator relation above one has

$$e^{i\theta} \tau_y M_\xi f = e^{i(\theta - \xi y)} M_\xi \tau_y f,$$

and therefore

$$\begin{aligned}
 e^{i\theta} \tau_y M_\xi e^{i\theta'} \tau_{y'} M_{\xi'} &= e^{i(\theta+\theta')} \tau_y M_\xi \tau_{y'} M_{\xi'} \\
 &= e^{i(\theta+\theta'-\xi y)} M_\xi \tau_y \tau_{y'} M_{\xi'} \\
 &= e^{i(\theta+\theta'-\xi y)} M_\xi \tau_y \tau_{y'} M_{\xi'} \\
 &= e^{i(\theta+\theta'-\xi y+\xi(y+y'))} \tau_{y+y'} M_\xi M_{\xi'} \\
 &= e^{i(\theta+\theta'+\xi y')} \tau_{y+y'} M_{\xi+\xi'};
 \end{aligned}$$

in coordinates,

$$(\xi, y, \theta) \circ (\xi', y', \theta') = (\xi + \xi', y + y', \theta + \theta' + \xi y').$$

It is not hard to see that this group is isomorphic to the Heisenberg group as introduced before: let $\phi: G \rightarrow \mathbb{H}^1$ be given by

$$\phi(\xi, y, \theta) := (\xi, y, \theta - \xi y/2),$$

then

$$\begin{aligned}
 \phi(\xi, y, \theta) \cdot \phi(\xi', y', \theta') &= (\xi + \xi', y + y', \theta + \theta' - \frac{\xi y}{2} - \frac{\xi' y'}{2} + (\frac{\xi y'}{2} - \frac{\xi' y}{2})) \\
 &= (\xi + \xi', y + y', \theta + \theta' + \xi y' - \frac{1}{2}(\xi + \xi')(y + y')) \\
 &= \phi((\xi, y, \theta) \circ (\xi', y', \theta'));
 \end{aligned}$$

moreover, ϕ is clearly bijective.

The group G of translations, modulations and phase shifts is a subgroup of the group of unitary transformations of $L^2(\mathbb{R})$, denoted $\mathcal{U}(L^2(\mathbb{R}))$; more precisely, since it is also continuous, ϕ^{-1} is a (faithful) unitary representation of the Heisenberg group. It is moreover an irreducible unitary representation: indeed, suppose there is a unitary transformation $U \in \mathcal{U}(L^2(\mathbb{R}))$ that commutes with all elements of G ; then in particular it commutes with all translations, and therefore it is given by a multiplier, that is there exists m such that $\widehat{Uf}(\xi) = m(\xi) \hat{f}(\xi)$; but since U commutes with modulations, m must be constant, and therefore U is a scalar multiplication; by Schur's lemma then G is an irreducible representation.

It should be noted that the representation given by G is just one in a family of representations, because given any $\lambda \neq 0$ we can define

$$R^\lambda(x, y, t) = \phi^{-1}(\lambda x, y, \lambda t),$$

that is $R^\lambda(x, y, t)$ is the unitary operator

$$f \mapsto e^{i\lambda(t+xy/2)} \tau_y M_{\lambda x} f.$$

It is immediate to verify this is again an irreducible representation, and that R^1 is the representation above; moreover, any two such representations for distinct λ 's are not

unitarily equivalent. Indeed, suppose there is $U \in \mathcal{U}(L^2(\mathbb{R}))$ such that

$$U \circ R^\lambda = R^\mu \circ U,$$

then for every $t \in \mathbb{R}$ it must be

$$U(R^\lambda(0,0,t)f) = U(e^{i\lambda t}f) = e^{i\lambda t}Uf,$$

$$R^\mu(0,0,t)Uf = e^{i\mu t}Uf$$

and therefore it must be $\lambda = \mu$.

The celebrated Stone-Von Neumann theorem tells us that these are the only faithful irreducible representations of \mathbb{H}^1 ; namely

Theorem 2.5 (Stone-von Neumann). *Let $\pi : \mathbb{H}^1 \rightarrow \mathcal{U}(H)$ be a unitary representation of \mathbb{H}^1 such that $\pi(0,0,t)f = e^{i\lambda t}f$ for some $\lambda \neq 0$. Then there exists an orthogonal decomposition of the Hilbert space H ,*

$$H = \bigoplus_{\alpha \in A} H_\alpha,$$

such that $\pi|_{H_\alpha}$ is irreducible for every α and unitarily equivalent to R^λ .

We do not prove this theorem. A proof akin to the original one of Stone and Von Neumann can be found in [Fol89].

Notice that since $[\mathbb{H}^1, \mathbb{H}^1] = \{(0,0,t) \text{ s.t. } t \in \mathbb{R}\}$, every representation π of \mathbb{H}^1 must satisfy $\pi(0,0,t)f = e^{i\lambda t}f$ for some λ , possibly equal to 0, in which case the representation is non-faithful. These representations can be classified as well without too much effort, and the irreducible ones turn out to be of the form

$$r^{a,b}(x,y,t) := e^{-2\pi i(ax+by)}$$

for $a, b \in \mathbb{R}$, so in particular they are one dimensional.

Remark 2.4. The representations studied above hint at the role of the Heisenberg group in quantum physics. Indeed, we have that the *momentum* operator $D := -i \frac{d}{dx}$ can be thought of as the generator of translations, in the sense that under an appropriate interpretation of operator exponentiation and for $f \in \mathcal{S}(\mathbb{R})$ one has

$$e^{iyD}f = \tau_y f.$$

Similarly, the *position* operator X defined by $Xf(x) = xf(x)$ can be thought of as the generator of modulations, since

$$e^{i\xi X}f(x) = e^{i\xi x}f(x).$$

The commutator relation (2.3.1) has its correspondent in the fact that

$$[D, X] = -i\mathbb{1},$$

which expresses the well known canonical commutator relationship of quantum mechanics - from which Heisenberg's uncertainty principle follows. Indeed, suppose

that $f \in \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, then by (formal) self-adjointness of D, X it is

$$\begin{aligned} \|f\|_{L^2(\mathbb{R})}^2 &= \langle f, f \rangle \\ &= i\langle [D, X]f, f \rangle = i\langle Df, Xf \rangle - i\langle Xf, Df \rangle \\ &\leq 2\|Df\|_{L^2(\mathbb{R})}\|Xf\|_{L^2(\mathbb{R})}. \end{aligned}$$

2.3.2 Fourier transform on \mathbb{H}^1

Once one has a satisfactory representation theory for a group, it is possible to construct the Fourier transform on that group. We briefly review how this can be done in generality. Let G be a topological group and μ be a fixed multiple of the Haar measure on G (we assume the group is unimodular, for simplicity, that is the left and right Haar measures coincide). Recall that a unitary representation π of G is a continuous homomorphism $G \rightarrow \mathcal{U}(\mathcal{H}_\pi)$, where \mathcal{H}_π is a separable Hilbert space and $\mathcal{U}(\mathcal{H}_\pi)$ denotes the group of unitary linear transformations on \mathcal{H}_π . The representation is said to be irreducible if there exists no subspace V of \mathcal{H}_π that is invariant with respect to π - that is, there doesn't exist V such that $\pi(g)(V) \subset V$ for all $g \in G$. Two representations π, ρ are said to be unitarily equivalent if there exists a unitary map $U : \mathcal{H}_\pi \rightarrow \mathcal{H}_\rho$ such that

$$U \circ \pi(g) = \rho(g) \circ U \quad \forall g \in G.$$

Let then \widehat{G} be the collection of the unitarily equivalent classes of the irreducible representations of G ; then the Fourier transform of $f \in L^1(G, d\mu)$ can be defined formally as the function

$$\widehat{G} \ni \pi \mapsto \widehat{f}(\pi) := \int_G f(g)\pi(g^{-1}) d\mu(g), \quad (2.3.2)$$

which is to be interpreted in a distributional sense, that is for every $\phi \in \mathcal{H}_\pi$, $\widehat{f}(\pi)$ is given by

$$\widehat{f}(\pi)\phi = \int_G f(g)(\pi(g^{-1})\phi) d\mu(g),$$

where this integral is to be further interpreted as a vector-valued integral in the usual way, that is for every $\psi \in \mathcal{H}_\pi$ it is

$$\langle \widehat{f}(\pi)\phi, \psi \rangle = \int_G f(g)\langle \pi(g^{-1})\phi, \psi \rangle d\mu(g);$$

notice this quantity is finite, since $f \in L^1(G, d\mu)$ and by unitarity $|\langle \pi(g^{-1})\phi, \psi \rangle| \leq \|\phi\|_{\mathcal{H}_\pi}\|\psi\|_{\mathcal{H}_\pi}$.

Thus if the group is non-commutative then the Fourier transform is operator-valued. For sufficiently well-behaved groups and sufficiently well-behaved functions, one can have an inversion formula for the Fourier transform, that is there exists a measure ν on \widehat{G} such that

$$f(g) = \int_{\widehat{G}} \text{tr}(\widehat{f}(\pi)\pi(g)) d\nu(\pi).$$

We do not pursue the abstract approach here but rather see what this definition generates once applied to \mathbb{H}^1 . Given a function $f \in L^1(\mathbb{H}^1)$ (observe the Haar measure coincides with the Lebesgue measure) we define its Fourier transform to be

$$\lambda \mapsto \widehat{f}_{\mathbb{H}^1}(\lambda) := \int_{\mathbb{H}^1} f(x, y, t) R^\lambda(-x, -y, -t) dx dy dt,$$

where therefore $\widehat{f}_{\mathbb{H}^1}(\lambda)$ is understood to be the operator that acts on $\phi \in L^2(\mathbb{R})$ as

$$(\widehat{f}_{\mathbb{H}^1}(\lambda)\phi)(z) = \int_{\mathbb{H}^1} f(x, y, t) (R^\lambda(-x, -y, -t)\phi)(z) dx dy dt.$$

This operator is $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ bounded for every λ , since

$$\begin{aligned} |\langle \widehat{f}_{\mathbb{H}^1}(\lambda)\phi, \psi \rangle| &= \left| \int_{\mathbb{H}^1} f(x, y, t) \langle R^\lambda(-x, -y, -t)\phi, \psi \rangle dx dy dt \right| \\ &\leq \int_{\mathbb{H}^1} |f(x, y, t)| |\langle R^\lambda(-x, -y, -t)\phi, \psi \rangle| dx dy dt \\ &\leq \|f\|_{L^1(\mathbb{H}^1)} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})}, \end{aligned} \quad (2.3.3)$$

and notice that this fact is the analogue of the trivial Hausdorff-Young inequality for the classical (Euclidean) Fourier transform.

It should be noticed that in defining $\widehat{f}_{\mathbb{H}^1}$ we have not taken into account the representations $r^{a,b}$; indeed, it turns out they are irrelevant. Observe that by (2.3.2) we should have

$$\widehat{f}(r^{a,b})\phi = \int_{\mathbb{H}^1} e^{i(ax+by)} f(x, y, t)\phi dx dy dt = \widehat{f}(a, b, 0)\phi.$$

The reason why such representations turn out to be irrelevant is that there is a natural measure on $\widehat{\mathbb{H}^1}$, which is the measure that makes Plancherel's formula true and corresponds to the ν measure introduced above, and it turns out it gives zero measure to the set $\{r^{a,b}\}_{a,b \in \mathbb{R}}$. We build such measure explicitly below.

We explore the definition of Fourier transform on \mathbb{H}^1 in more detail. Observe that

$$\begin{aligned} (\widehat{f}_{\mathbb{H}^1}(\lambda)\phi)(z) &= \int f(x, y, t) (e^{-i\lambda(t-xy/2)} \tau_{-y} M_{-\lambda x} \phi)(z) dx dy dt \\ &= \int f(x, y, t) e^{-i\lambda(t-xy/2)} e^{-i\lambda x(z+y)} \phi(z+y) dx dy dt \\ &= \int f(x, y, t) e^{-i\lambda(t+xy/2+xz)} \phi(z+y) dx dy dt \\ &= \int f(x, u-z, t) e^{-i\lambda(t+x(u+z)/2)} \phi(u) dx du dt, \end{aligned}$$

and therefore $(\widehat{f}_{\mathbb{H}^1}(\lambda)\phi)$ is given by integrating ϕ against the kernel

$$\begin{aligned} K_f^\lambda(u, z) &:= \int f(x, u-z, t) e^{-i\lambda(t+x(u+z)/2)} dx dt \\ &= (\mathcal{F}_1 \mathcal{F}_3 f)(\lambda(u+z)/2, u-z, \lambda), \end{aligned} \quad (2.3.4)$$

where \mathcal{F}_j denotes the Euclidean Fourier transform in the j -th variable,

$$\mathcal{F}_j g(x_1, \dots, \xi, \dots, x_n) := \int e^{-i\xi x_j} g(x_1, \dots, x_j, \dots, x_n) dx_j.$$

In particular, we can see that $\widehat{f}_{\mathbb{H}^1}(\lambda)$ has finite Hilbert-Schmidt norm:

$$\begin{aligned} \|\widehat{f}_{\mathbb{H}^1}(\lambda)\|_{\text{HS}}^2 &= \int \int |K_f^\lambda(u, z)|^2 du dz \\ &= \int \int |(\mathcal{F}_1 \mathcal{F}_3 f)(\lambda(u+z)/2, u-z, \lambda)|^2 du dz \\ &= |\lambda|^{-1} \int \int |(\mathcal{F}_1 \mathcal{F}_3 f)(v, w, \lambda)|^2 dv dw; \end{aligned}$$

if we assume that $f \in L^1(\mathbb{H}^1) \cap L^2(\mathbb{H}^1)$, by Plancherel's theorem for the Euclidean Fourier transform it follows that $|\lambda| \|\widehat{f}_{\mathbb{H}^1}(\lambda)\|_{\text{HS}}^2$ is finite for a.e. λ . Actually, notice we have proven that for some constant c_0 it is

$$\int |f(x, y, t)|^2 dx dy dt = c_0 \int_{\mathbb{R}} \|\widehat{f}_{\mathbb{H}^1}(\lambda)\|_{\text{HS}}^2 |\lambda| d\lambda, \quad (2.3.5)$$

which is Plancherel's formula for the group Fourier transform. Thus the measure ν on \widehat{G} mentioned above is $|\lambda| d\lambda$, and consequently the set $\{r^{a,b}\}_{a,b \in \mathbb{R}}$ has measure zero as stated above.

By a limiting argument akin to the usual one for the Euclidean Fourier transform, one establishes

Proposition 2.6. *The Fourier transform on \mathbb{H}^1 , defined for $f \in L^1(\mathbb{H}^1) \cap L^2(\mathbb{H}^1)$ by*

$$f \mapsto \widehat{f}_{\mathbb{H}^1},$$

extends to a unitary mapping from $L^2(\mathbb{H}^1)$ to the Hilbert space of Hilbert-Schmidt valued functions², with norm $\|\cdot\|$ given by

$$\|T\|^2 := \int_{\mathbb{R}} \|T(\lambda)\|_{\text{HS}}^2 |\lambda| d\lambda.$$

One can verify that the Fourier transform on \mathbb{H}^1 satisfies

$$\widehat{(f \star g)}_{\mathbb{H}^1}(\lambda) = \widehat{f}_{\mathbb{H}^1}(\lambda) \circ \widehat{g}_{\mathbb{H}^1}(\lambda) \quad (2.3.6)$$

for $f, g \in L^1(\mathbb{H}^1)$ (here \star denotes convolution with respect to the group operation, that is $f \star g(x) = \int_{\mathbb{H}^1} f(y)g(x \cdot y^{-1}) dy$). Indeed, it is enough to verify it using the abstract definition given at the beginning of the subsection: by Fubini's theorem and a change of variable we have

$$\widehat{(f \star g)}_{\mathbb{H}^1}(\lambda) = \int_{\mathbb{H}^1} (f \star g)(x) R^\lambda(x^{-1}) dx$$

²We notice here that the operator valued function is $|\lambda| d\lambda$ -measurable and only defined up to sets of measure zero, analogously to what happens for the Euclidean Fourier transform.

$$\begin{aligned}
&= \int_{\mathbb{H}^1} \int_{\mathbb{H}^1} f(y) g(x \cdot y^{-1}) R^\lambda(x^{-1}) \, dy \, dx \\
&= \int_{\mathbb{H}^1} \int_{\mathbb{H}^1} f(y) R^\lambda(y^{-1}) g(x \cdot y^{-1}) R^\lambda(y) R^\lambda(x^{-1}) \, dx \, dy \\
&= \int_{\mathbb{H}^1} \int_{\mathbb{H}^1} f(y) R^\lambda(y^{-1}) g(z) R^\lambda(z^{-1}) \, dz \, dy \\
&= \left(\int_{\mathbb{H}^1} f(y) R^\lambda(y^{-1}) \, dy \right) \left(\int_{\mathbb{H}^1} g(z) R^\lambda(z^{-1}) \, dz \right) = \widehat{f}_{\mathbb{H}^1}(\lambda) \circ \widehat{g}_{\mathbb{H}^1}(\lambda).
\end{aligned}$$

2.3.3 Fourier transform of distributions in \mathbb{H}^1

In our work we will have to consider the Fourier transform of certain distributions, which needs therefore to be properly defined. Since our distributions will be compactly supported, the definition is the same as the one given in (2.3.2). Indeed, let $\mathcal{E}(\mathbb{R}^3)$ denote the space of $C^\infty(\mathbb{R}^3)$ functions endowed with the topology of uniform convergence on compacts; then it is known³ that $\mathcal{E}'(\mathbb{R}^3)$ can be identified with the subspace of $\mathcal{S}'(\mathbb{R}^3)$ consisting of the distributions with compact support. Recall that $\kappa \in \mathcal{S}'(\mathbb{R}^3)$ is said to have compact support if there exists $K \subset \mathbb{R}^3$ compact such that for all $\phi \in \mathcal{S}(\mathbb{R}^3)$ it holds that

$$\text{Supp}(\phi) \cap K = \emptyset \quad \Rightarrow \quad (\kappa, \phi) = 0.$$

Then the Fourier transform on \mathbb{H}^1 of $\kappa \in \mathcal{E}'(\mathbb{R}^3)$ can be given formally by

$$\widehat{\kappa}_{\mathbb{H}^1}(\lambda) := \int_{\mathbb{H}^1} \kappa(x) R^\lambda(-x) \, dx,$$

which is to be interpreted as the operator such that for every $\phi, \psi \in L^2(\mathbb{R})$ it holds that

$$\langle \widehat{\kappa}_{\mathbb{H}^1}(\lambda) \phi, \psi \rangle = (\kappa, \langle R^\lambda(-x) \phi, \psi \rangle).$$

Notice the pairing on the right hand side of the last equation makes sense, since R^λ is a continuous representation and \mathbb{H}^1 is a Lie group, and therefore $x \mapsto \langle R^\lambda(-x) \phi, \psi \rangle$ is a bounded C^∞ function on \mathbb{R}^3 . This can also be easily verified directly.

The equality (2.3.6) relating convolution and the group Fourier transform also holds for compactly supported distributions.

2.3.4 L^2 multipliers on \mathbb{H}^1

Once one has a well-behaved Fourier transform, the associated L^2 theory can be used to characterize the L^2 boundedness of convolution operators on \mathbb{H}^1 , where the convolution is taken with respect to the group operation. That is, given the operator

$$f \mapsto K \star f,$$

we want to find conditions on K that are equivalent to the operator being $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded. Here we consider only $K \in L^1(\mathbb{H}^1)$ or $K \in \mathcal{E}'(\mathbb{R}^3)$ for simplicity. Such

³See [Rud91], Chapter 6 for details.

a condition turns out to be quite simple to state:

Lemma 2.7. *Let $K \in L^1(\mathbb{H}^1)$ or $K \in \mathcal{E}'(\mathbb{R}^3)$. The operator*

$$f \mapsto Tf := K \star f$$

is $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded if and only if⁴

$$\sup_{\lambda} \|\widehat{K}_{\mathbb{H}^1}(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} < \infty.$$

Moreover,

$$\sup_{\lambda} \|\widehat{K}_{\mathbb{H}^1}(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} = \|T\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)}.$$

For a more general statement that holds for all left-invariant bounded linear operators on $L^2(\mathbb{H}^1)$, see [Dix77], Chapter 18.

Proof. First of all notice that given two integral operators $Tf(x) := \int K_1(x, y)f(y) dy$, $Sf(x) := \int K_2(x, y)f(y) dy$, we have

$$\begin{aligned} \|T \circ S\|_{\text{HS}}^2 &= \int \int \left| \int K_1(x, y)K_2(y, z) dy \right|^2 dx dz \\ &= \int \int |T(K_2(\cdot, z))(x)|^2 dx dz \\ &\leq \|T\|_{L^2 \rightarrow L^2}^2 \int \int |K_2(x, z)|^2 dx dz \\ &= \|T\|_{L^2 \rightarrow L^2}^2 \|S\|_{\text{HS}}^2. \end{aligned}$$

By Plancherel's formula (2.3.5) and the identity (2.3.6), it follows from the simple inequality above that

$$\begin{aligned} \|K \star f\|_{L^2(\mathbb{H}^1)} &= c_0 \int \|\widehat{K}_{\mathbb{H}^1}(\lambda) \circ \widehat{f}_{\mathbb{H}^1}(\lambda)\|_{\text{HS}}^2 |\lambda| d\lambda \\ &\leq c_0 \int \|\widehat{K}_{\mathbb{H}^1}(\lambda)\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)}^2 \|\widehat{f}_{\mathbb{H}^1}(\lambda)\|_{\text{HS}}^2 |\lambda| d\lambda \\ &\leq c_0 \sup_{\lambda} \|\widehat{K}_{\mathbb{H}^1}(\lambda)\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)}^2 \int \|\widehat{f}_{\mathbb{H}^1}(\lambda)\|_{\text{HS}}^2 |\lambda| d\lambda \\ &= \|f\|_{L^2(\mathbb{H}^1)} \sup_{\lambda} \|\widehat{K}_{\mathbb{H}^1}(\lambda)\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)}^2. \end{aligned}$$

Thus we have proven one direction of the stated equivalence, namely that

$$\|K \star \cdot\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)} \leq \sup_{\lambda} \|\widehat{K}_{\mathbb{H}^1}(\lambda)\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)}^2.$$

Now, we prove the converse. We begin with $K \in L^1(\mathbb{H}^1)$, and the idea will be to use approximations of the identity. Suppose that

$$\|K \star f\|_{L^2(\mathbb{H}^1)} \leq C \|f\|_{L^2(\mathbb{H}^1)}$$

⁴The supremum is to be interpreted as the essential supremum with respect to measure $|\lambda| d\lambda$ - or equivalently, Lebesgue measure in λ .

and let ψ be a bump function in \mathbb{R} , positive and supported in $B(0, 1)$, such that $\int |\psi(x)|^2 dx = 1$, and let

$$\psi_\delta(x) := \frac{1}{\delta^{1/2}} \psi\left(\frac{x}{\delta}\right)$$

be the L^2 -normalized dilate of ψ by δ . Then for any $\phi \in L^2(\mathbb{R})$ and for $\lambda_0 \neq 0$ define $\theta_{\delta, \lambda_0}$ by the identity

$$\mathcal{F}_1 \mathcal{F}_3 \theta_{\delta, \lambda_0}(\lambda(y+z)/2, y-z, \lambda) = \frac{1}{|\lambda|^{1/2}} \psi_\delta(z) \psi_\delta(\lambda - \lambda_0) \phi(y)$$

(observe the definition is well-posed). Then one has again by (2.3.5)

$$\begin{aligned} \|K \star \theta_{\delta, \lambda_0}\|_{L^2(\mathbb{H}^1)}^2 &= \int \|\widehat{K}_{\mathbb{H}^1}(\lambda) \circ \widehat{\theta}_{\delta, \lambda_0}(\lambda)\|_{\text{HS}}^2 |\lambda| d\lambda \\ &= \int \int \int \left| \int K_K^\lambda(u, y) K_{\theta_{\delta, \lambda_0}}^\lambda(y, z) dy \right|^2 du dz |\lambda| d\lambda \\ &= \int \int \int \left| \int K_K^\lambda(u, y) (\mathcal{F}_1 \mathcal{F}_3 \theta_{\delta, \lambda_0}(\lambda(y+z)/2, y-z, \lambda)) dy \right|^2 du dz |\lambda| d\lambda \\ &= \int \int \int |(\widehat{K}_{\mathbb{H}^1}(\lambda) \phi)(u)|^2 |\psi_\delta(z)|^2 |\psi_\delta(\lambda - \lambda_0)|^2 du d\lambda dz \\ &= \int \int |(\widehat{K}_{\mathbb{H}^1}(\lambda) \phi)(u)|^2 |\psi_\delta(\lambda - \lambda_0)|^2 du d\lambda. \end{aligned}$$

Now observe that $\lambda \mapsto \int |(\widehat{K}_{\mathbb{H}^1}(\lambda) \phi)(u)|^2 du$ is in L^∞ by (2.3.3), and thus it is in $L^1_{\text{loc}}(\mathbb{R})$; therefore by Lebesgue's differentiation theorem one has for $\delta \rightarrow 0$ that for a.e. λ_0

$$\int \int |(\widehat{K}_{\mathbb{H}^1}(\lambda) \phi)(u)|^2 |\psi_\delta(\lambda - \lambda_0)|^2 du d\lambda \rightarrow \int |(\widehat{K}_{\mathbb{H}^1}(\lambda_0) \phi)(u)|^2 du.$$

It follows that for sufficiently small δ one has

$$\begin{aligned} \int |(\widehat{K}_{\mathbb{H}^1}(\lambda_0) \phi)(u)|^2 du &\leq \|K \star \theta_{\delta, \lambda_0}\|_{L^2(\mathbb{H}^1)}^2 \\ &\leq C^2 \|\theta_{\delta, \lambda_0}\|_{L^2(\mathbb{H}^1)}^2 = C^2 \|\phi\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

and since ϕ is arbitrary it follows that

$$\|\widehat{K}_{\mathbb{H}^1}(\lambda_0)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C,$$

and thus the lemma is proved, since λ_0 is arbitrary as well.

It remains to extend the second part of the proof to $K \in \mathcal{E}'(\mathbb{R}^3)$ as well. The proof is a standard approximation argument; we sketch it briefly only because one has to be careful when dealing with non-commutative convolution such as the one in \mathbb{H}^1 . Let $\varphi \in C_c^\infty(\mathbb{R}^3)$ be such that $\int_{\mathbb{R}^3} \varphi(x) dx = 1$ and $\varphi(x) = \varphi(-x)$; thus if we apply an automorphic dilation and define

$$\varphi^\delta(x, y, z) := \frac{1}{\delta^4} \varphi\left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{z}{\delta^2}\right),$$

the family $\{\varphi^\delta\}_{\delta > 0}$ is an approximation to the identity in \mathbb{H}^1 (this can be verified

easily; see also [Wei40], Theorem 20.18, for Young's inequality for non-commutative convolutions). Consider then K^δ defined by

$$K^\delta := K \star \varphi^\delta,$$

that is K^δ is the distribution defined by

$$(K^\delta, \psi) = (K, \psi \star \tilde{\varphi}^\delta), \quad \forall \psi \in \mathcal{S}(\mathbb{R}^3),$$

where $\tilde{f}(x) = f(x^{-1})$ (the order of the elements in the convolution is important). One can see that K^δ is in $\mathcal{E}'(\mathbb{R}^3)$, and it can also be identified with function

$$x \mapsto (K, \mathcal{T}_x \tilde{\varphi}^\delta),$$

where $\mathcal{T}_x f(z) = f(z \cdot x^{-1})$ is the translation operator in \mathbb{H}^1 ; this is a $C^\infty(\mathbb{R}^3)$ -function, and therefore K^δ is in $C_c^\infty(\mathbb{R}^3)$ too, and thus in $L^1(\mathbb{H}^1)$. It follows from the previous part of the proof that for any $\delta > 0$ the operator $f \mapsto K^\delta \star f$ is bounded on $L^2(\mathbb{H}^1)$ if and only if

$$\sup_{\lambda \neq 0} \|\widehat{K^\delta}_{\mathbb{H}^1}(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} < \infty.$$

Now observe that since φ^δ is an approximation of the identity and

$$\langle K^\delta \star f, \psi \rangle = (K, \overline{\psi} \star \tilde{f} \star \tilde{\varphi}^\delta),$$

it follows that

$$\lim_{\delta \rightarrow 0} \|K^\delta \star \cdot\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)} \geq \|K \star \cdot\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)}$$

(simply approximate the operator norm on the right by an inner product $\langle K \star f, g \rangle$ for suitably chosen $f, g \in \mathcal{S}(\mathbb{R}^3)$). By Young's convolution inequality (for non-commutative groups) the reverse inequality holds even without the limit, since $\|\varphi^\delta\|_{L^1(\mathbb{H}^1)} = 1$, and therefore we have proven

$$\lim_{\delta \rightarrow 0} \|K^\delta \star \cdot\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)} = \|K \star \cdot\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)}. \quad (2.3.7)$$

On the other hand, we have by Hausdorff-Young's inequality for \mathbb{H}^1 (equation (2.3.3)), that

$$\|\widehat{K^\delta}(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq \|\widehat{K}(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \underbrace{\|\varphi^\delta\|_{L^1(\mathbb{H}^1)}}_{=1},$$

so that

$$\sup_{\lambda \neq 0} \|\widehat{K^\delta}(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq \sup_{\lambda \neq 0} \|\widehat{K}(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})}.$$

Finally, let $\varepsilon > 0$ be fixed and choose $\lambda_0 \neq 0$ such that

$$\sup_{\lambda \neq 0} \|\widehat{K}(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} - \|\widehat{K}(\lambda_0)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} < \varepsilon.$$

Observe that $\widehat{K}^\delta(\lambda_0) = \widehat{K}(\lambda_0) \circ \widehat{\varphi}^\delta(\lambda_0)$ and that by (2.3.4)

$$\widehat{\varphi}^\delta(\lambda_0)\phi(z) = \int \frac{1}{\delta} \varphi\left(\frac{\delta\lambda_0}{2}(u+z), \frac{u-z}{\delta}, \delta^2\lambda_0\right)\phi(u) du;$$

it is not hard to see that this implies $\widehat{\varphi}^\delta(\lambda_0)$ acts as an approximation of the identity (although not of convolution type) provided φ is chosen appropriately, and therefore we can prove analogously as before that

$$\lim_{\delta \rightarrow 0} \|\widehat{K}^\delta(\lambda_0)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \geq \sup_{\lambda \neq 0} \|\widehat{K}(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} - \varepsilon,$$

and since ε is arbitrary it follows that

$$\limsup_{\delta \rightarrow 0} \sup_{\lambda \neq 0} \|\widehat{K}^\delta(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} = \sup_{\lambda \neq 0} \|\widehat{K}(\lambda)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})}. \quad (2.3.8)$$

By combining (2.3.7) and (2.3.8) with the previous part for L^1 convolution kernels, we conclude the proof. \square

This concludes the preliminary chapter.

Chapter 3

Real analytic surfaces

In this chapter we begin to address the subject of this thesis by proving the $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ boundedness of the operator given by the (local) double Hilbert transform along the surface given by $(s, t, \varphi(s, t))$. This will provide a clear illustration of the method, which will then be refined in the next chapter to yield more precise results (in terms of the uniformity of the operator norm with respect to a suitable subspace of the surfaces that are graphs of polynomials).

We remark that this result already follows from the Stein-Street¹ theory in [Str12], but the method given here has little in common with it, and thus constitutes an entirely different proof of the fact. In Chapter 5 we will show the scope of the method goes beyond that of the Stein-Street theory in regard to a certain class of operators, at least as far as L^2 boundedness is concerned.

The chapter consists of two sections. In the first one we will state the theorem rigorously and we will prelude a series of reductions, and in the second section we will provide the proof of the theorem, given the reductions. We have included several remarks throughout to make the proof more transparent.

3.1 Statement and preliminary reductions

3.1.1 Statement and reduction to the multiplier

Let $\varphi \in C^\omega(U)$, that is a real analytic function of two variables in some neighbourhood of the origin U , such that $\varphi(0, 0) = 0$ and let ε be sufficiently small that for all $(s, t) \in [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \subset U$ one has

$$\varphi(s, t) = \sum_{k, \ell \in \mathbb{N}} c_{k\ell} s^k t^\ell,$$

in the sense that the rectangular sums of the power series are absolutely convergent and they converge to the values of the function, namely for all $(s, t) \in [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$

$$\varphi(s, t) = \lim_{K, L \rightarrow \infty} \sum_{\substack{k \leq K, \\ \ell \leq L}} c_{k\ell} s^k t^\ell.$$

¹Actually, the first paper in the series we are referring to is authored by Street only.

This will not be the only condition we impose on ε ; indeed, ε will be sufficiently small to justify a series of bounds (only involving the coefficients of φ) which will be pointed out as they arise.

We are interested in the $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ boundedness of the (local) double Hilbert transform along the surface given by $(s, t, \varphi(s, t))$, that is the operator defined formally as

$$f \mapsto H_\varphi f(x) := \text{p.v.} \int \int_{|s|, |t| \leq \varepsilon} f(x \cdot (s, t, \varphi(s, t))^{-1}) \frac{ds}{s} \frac{dt}{t}, \quad (3.1.1)$$

where $\varepsilon = \varepsilon(\varphi)$ is sufficiently small (it will become clear how small it needs to be). The definition requires some clarification of the principal value nature of operator H_φ : we show that it is a well defined operator at least on $C_c^\infty(\mathbb{R}^3)$ (or $\mathcal{S}(\mathbb{R}^3)$, and the proof is entirely similar). Indeed, by the notation of (3.1.1) we mean $H_\varphi f(x)$ denotes the quantity

$$\lim_{\eta, \theta \rightarrow 0} \int \int_{\substack{\eta \leq |s| \leq \varepsilon, \\ \theta \leq |t| \leq \varepsilon}} f(x \cdot (s, t, \varphi(s, t))^{-1}) \frac{ds}{s} \frac{dt}{t} =: \lim_{\eta, \theta \rightarrow 0} H_\varphi^{\eta, \theta} f(x),$$

if such a limit exists in some sense. We claim that if $f \in C_c^\infty(\mathbb{R}^3)$ (or $f \in \mathcal{S}(\mathbb{R}^3)$) the limit exists pointwise. This can be seen as follows: let

$$F_x(s, t) := f(x \cdot (s, t, \varphi(s, t))^{-1}),$$

then notice that by cancellation of the kernel we have

$$\begin{aligned} & \int \int_{\substack{\eta \leq |s| \leq \varepsilon, \\ \theta \leq |t| \leq \varepsilon}} F_x(s, t) \frac{ds}{s} \frac{dt}{t} \\ &= \int \int_{\substack{\eta \leq |s| \leq \varepsilon, \\ \theta \leq |t| \leq \varepsilon}} F_x(s, t) - F_x(s, 0) \frac{ds}{s} \frac{dt}{t} \\ &= \int \int_{\substack{\eta \leq |s| \leq \varepsilon, \\ \theta \leq |t| \leq \varepsilon}} \int_0^1 t \partial_2 F_x(s, tr) dr \frac{ds}{s} \frac{dt}{t} \\ &= \int \int_{\substack{\eta \leq |s| \leq \varepsilon, \\ \theta \leq |t| \leq \varepsilon}} \int_0^1 t \partial_2 F_x(s, tr) - t \partial_2 F_x(0, tr) dr \frac{ds}{s} \frac{dt}{t} \\ &= \int \int_{\substack{\eta \leq |s| \leq \varepsilon, \\ \theta \leq |t| \leq \varepsilon}} \int_0^1 \int_0^1 st \partial_1 \partial_2 F_x(su, tr) du dr \frac{ds}{s} \frac{dt}{t} \\ &= \int \int_{\substack{\eta \leq |s| \leq \varepsilon, \\ \theta \leq |t| \leq \varepsilon}} \int_0^1 \int_0^1 \partial_1 \partial_2 F_x(su, tr) du dr ds dt. \end{aligned}$$

Since we have assumed $f \in C_c^\infty(\mathbb{R}^3)$, it follows from dominated convergence that the limit in $\eta, \theta \rightarrow 0$ of the expression exists pointwise. A simple modification of the above argument shows that if $f \in C_c^\infty(\mathbb{R}^3)$ then $H_\varphi f$ belongs to $L^2(\mathbb{R}^3)$ (it suffices to use Riesz's Representation theorem). Therefore it makes sense to talk of the $L^2(\mathbb{H}^1) = L^2(\mathbb{R}^3)$ norm of $H_\varphi f$, and by Fatou's lemma we have

$$\int_{\mathbb{H}^1} |H_\varphi f(x)|^2 dx \leq \liminf_{\eta, \theta \rightarrow 0} \int_{\mathbb{H}^1} |H_\varphi^{\eta, \theta} f(x)|^2 dx;$$

thus, by a standard approximation argument using the density of $C_c^\infty(\mathbb{R}^3)$ in $L^2(\mathbb{R}^3)$, to prove Theorem 3.1 it suffices to prove the operators $H_\varphi^{\eta,\theta}$ are $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded independently of η, θ .

We will prove that this is indeed the case, so that we can state

Theorem 3.1. *Let $\varphi \in C^\omega([-\varepsilon, \varepsilon]^2)$ and let H_φ be defined as before. Then for every φ , the operator H_φ extends to an $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded operator.*

Recall that the corresponding operator in the Euclidean convolution case is not always bounded, not even on $L^2(\mathbb{R}^3)$ ([CWW00]).

Remark 3.1. The proof that we will give of Theorem 3.1 will not use in any substantial way the specific homogeneity properties of the Hilbert product kernel p.v. $1/st$. As such, the proof can be adapted to yield the same result for the slightly more general case where the kernel p.v. $1/st$ is replaced by a general tensor product kernel $K_1(s)K_2(t)$ where K_i is a Calderón-Zygmund one dimensional kernel, $i = 1, 2$, satisfying the usual cancellation and smoothness conditions.

In our proof of the theorem we will exploit the Fourier theory of \mathbb{H}^1 by studying the multiplier of H_φ . Indeed, we claim that by Lemma 2.7 of Chapter 2, §2.3, we have the following equivalence.

Lemma 3.2. *The operator H_φ is $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded if and only if the operators T_φ^λ on $L^2(\mathbb{R})$ given by*

$$T_\varphi^\lambda \phi(y) := \text{p.v.} \int_{|y-t| \leq \varepsilon} \left(\text{p.v.} \int_{|s| \leq \varepsilon} e^{i\lambda((y+t)s + \varphi(s, y-t))} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt \quad (3.1.2)$$

are $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ bounded uniformly in $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. It suffices by Lemma 2.7 of Chapter 2 to verify that the above expression for T_φ^λ is indeed the Fourier transform of the compactly supported distribution defined by

$$(\mathcal{K}, \psi) := \text{p.v.} \int \int_{\substack{|s| \leq \varepsilon, \\ |t| \leq \varepsilon}} \psi(s, t, \varphi(s, t)) \frac{ds}{s} \frac{dt}{t},$$

since $H_\varphi f = \mathcal{K} \star f$. As discussed above, we really are working with the truncations of the above distribution, but we avoid repeating so to ease the notation a bit, since the arguments don't depend on such truncations in any way. We have by (2.3.4) and formal manipulations that

$$\begin{aligned} (\widehat{\mathcal{K}}(\lambda)\phi)(y) &= \int \int_{\substack{|s| \leq \varepsilon, \\ |t| \leq \varepsilon}} e^{-i\lambda(\varphi(s, t) + st/2 - sy)} \phi(y+t) \frac{ds}{s} \frac{dt}{t} \\ &= \int \int_{\substack{|s| \leq \varepsilon, \\ |y-t| \leq \varepsilon}} e^{-i\lambda(\varphi(s, t-y) + s(t+y)/2)} \phi(t) \frac{ds}{s} \frac{dt}{t-y}. \end{aligned}$$

This completes the proof. □

We observe that since $T_\varphi^{-\lambda} \phi(y) = T_{-\varphi}^\lambda \tilde{\phi}(-y)$, with $\tilde{\phi}(t) := \phi(-t)$, it will suffice to consider the case $\lambda > 0$. We will therefore assume from now on that $\lambda > 0$.

3.1.2 Further reductions

Now we intend to operate further reductions on the operator T_φ^λ .

We decompose dyadically in the quantities y and $y - t$ to better localize the operator.

Thus, let $\chi := \chi_{[-1, -1/2] \cup [1/2, 1]}$; we write

$$\begin{aligned} T_\varphi^\lambda \phi(y) &= \sum_{n \in \mathbb{Z}} \chi(2^n y) \sum_{j \geq C_\varepsilon} \text{p.v.} \int_{2^{-j-1} < |y-t| \leq 2^{-j}} m(\lambda; y, t) \phi(t) dt, \\ &=: \sum_{n \in \mathbb{Z}} \sum_{j \geq C_\varepsilon} T_\varphi^{\lambda, n, j} \phi(y), \end{aligned}$$

where $C_\varepsilon = \log_2 |\varepsilon|^{-1}$ and $m(\lambda; y, t)$ is the function given by the oscillatory integral

$$m(\lambda; y, t) := \text{p.v.} \int_{|s| \leq \varepsilon} e^{i\lambda((y+t)s + \varphi(s, y-t))} \frac{ds}{s}.$$

We split the sum into two parts as

$$\begin{aligned} T_\varphi^\lambda \phi(y) &= \sum_{n \in \mathbb{Z}} \sum_{C_\varepsilon \leq j < n + C_0} T_\varphi^{\lambda, n, j} \phi(y) + \sum_{n \in \mathbb{Z}} \sum_{\substack{j \geq n + C_0, \\ j \geq C_\varepsilon}} T_\varphi^{\lambda, n, j} \phi(y) \\ &=: S_\varphi^\lambda \phi(y) + \sum_{n \in \mathbb{Z}} \sum_{\substack{j \geq n + C_0, \\ j \geq C_\varepsilon}} T_\varphi^{\lambda, n, j} \phi(y), \end{aligned}$$

where $C_0 > 0$ is some integer constant we are free to choose.

3.1.2.1

We claim that S_φ^λ is bounded, uniformly in λ ; but in order to proceed we need the following lemma.

Lemma 3.3. *For any $\lambda, y \in \mathbb{R}$ and t such that $|y - t| \leq \varepsilon$ we have that*

$$|m(\lambda; y, t)| = O_\varphi(1).$$

Proof. For every $k \in \mathbb{N}$, let $\ell(k)$ be the smallest $\ell \in \mathbb{N}$ such that $c_{k\ell} \neq 0$. Then let $K > 1$ be an integer such that $\ell(K)$ exists and for every $k > K$ it is $\ell(k) \geq \ell(K) =: L$; in other words, the coefficients $c_{K+p, \ell}$ for $p > 0$ are all zero if ℓ is below the threshold L . If such a K does not exist, the conclusion is essentially trivial (it follows from Corollary 2.2 of Chapter 2). Let

$$P(s, t) := c_{KL} s^K t^L + \sum_{\substack{k < K, \\ \ell \in \mathbb{N}}} c_{k\ell} s^k t^\ell,$$

$$\psi(s, t) := \varphi(s, t) - P(s, t);$$

notice that P is a polynomial and ψ is analytic. We want to show that

$$\left| m(\lambda; y, t) - \int_{|s| \leq \varepsilon} e^{i\lambda((y+t)s + P(s, y-t))} \right| = O_\varphi(1).$$

To do so, we bound the quantity on the left hand side by triangle inequality by

$$\begin{aligned} & \left| \int_{|s|\leq\theta} e^{i\lambda((y+t)s+\varphi(s,y-t))} - e^{i\lambda((y+t)s+P(s,y-t))} \frac{ds}{s} \right| \\ & + \left| \int_{\theta<|s|\leq\varepsilon} e^{i\lambda((y+t)s+\varphi(s,y-t))} \frac{ds}{s} \right| + \left| \int_{\theta<|s|\leq\varepsilon} e^{i\lambda((y+t)s+P(s,y-t))} \frac{ds}{s} \right| \\ & =: I + II + III, \end{aligned}$$

where θ is a parameter to be chosen. We can estimate by simple domination that

$$\begin{aligned} I & \lesssim \lambda \int_{|s|\leq\theta} |\psi(s,y-t)| \frac{ds}{|s|} \leq \lambda \sum_{\substack{k\geq K, \ell\geq L, \\ (k,\ell)\neq(K,L)}} |c_{k\ell}| \theta^k |y-t|^\ell \\ & \leq \lambda \theta^K |y-t|^L \sum_{\substack{k\geq K, \ell\geq L, \\ (k,\ell)\neq(K,L)}} |c_{k\ell}| \varepsilon^{k-K} \varepsilon^{\ell-L}; \end{aligned}$$

we suppose that ε is sufficiently small that

$$\sum_{\substack{k\geq K, \ell\geq L, \\ (k,\ell)\neq(K,L)}} |c_{k\ell}| \varepsilon^{k-K} \varepsilon^{\ell-L} \leq \frac{1}{10} |c_{KL}|,$$

which we can always achieve², so that we have the bound

$$I \lesssim \lambda |c_{KL}| \theta^K |y-t|^L. \quad (3.1.3)$$

Next we observe that the K -th derivative of the phase is given by

$$\left(\frac{d}{ds} \right)^K ((y+t)s + \varphi(s,y-t)) = c_{KL} K! (y-t)^L + \partial_s^K \psi(s,y-t),$$

and

$$\begin{aligned} |\partial_s^K \psi(s,y-t)| & \leq K! \sum_{\substack{k\geq K, \ell\geq L, \\ (k,\ell)\neq(K,L)}} |c_{k\ell}| \binom{k}{K} s^{k-K} (y-t)^\ell \\ & \leq K! |y-t|^L \sum_{\substack{k\geq K, \ell\geq L, \\ (k,\ell)\neq(K,L)}} |c_{k\ell}| \binom{k}{K} s^{k-K} (y-t)^{\ell-L}, \end{aligned}$$

and we can suppose that ε is sufficiently small that the sum is bounded by $\frac{1}{10} |c_{KL}|$ (this assumption implies the previous one, obviously). Therefore we have for all $|s| \leq \varepsilon$ that

$$\left| \left(\frac{d}{ds} \right)^K ((y+t)s + \varphi(s,y-t)) \right| \gtrsim_K |c_{KL}| |y-t|^L,$$

and by Corollary 2.2 of Chapter 2 (that is, by Van der Corput's lemma and integration

²By the well known fact that there exist $C, \alpha > 0$ such that $|c_{k\ell}| \leq C \alpha^{k+\ell}$.

by parts) we can bound II by

$$II \lesssim_K (\lambda |c_{KL}| |y-t|^L)^{-1/K} \frac{1}{\theta}; \quad (3.1.4)$$

moreover, the same estimate holds for III , since $\partial_s^K P(s, y-t) = c_{KL} K! (y-t)^L$. Therefore, if we choose $\theta := (\lambda |c_{KL}| |y-t|^L)^{-1/K} \wedge \varepsilon$ we see that by (3.1.3) and (3.1.4) we have

$$I + II + III \lesssim_K 1,$$

a bound which depends on φ only. We have thus reduced the problem to that of proving that

$$\int_{|s| \leq \varepsilon} e^{i\lambda((y+t)s + P(s, y-t))} \frac{ds}{s},$$

is $O_\varphi(1)$. We claim this is true with the bound depending only on K , since the phase is a polynomial in s (of degree exactly K). We postpone the proof of this fact; we will derive it in Corollary 4.4 of Chapter 4, §4.2 as a consequence of the oscillatory integrals estimates of Proposition 4.2. \square

With this lemma at hand we can show the claim we made before that S_φ^λ is uniformly bounded on $L^2(\mathbb{R})$ independently of λ . Indeed, notice we have (with $\gamma := 2^{-C_0}$)

$$S_\varphi^\lambda \phi(y) = \text{p.v.} \int_{\gamma|y| \leq |y-t| \leq \varepsilon} m(\lambda; y, t) \frac{\phi(t)}{y-t} dt,$$

and thus if we let

$$S\phi(y) := \text{p.v.} \int_{\gamma|y| \leq |y-t|} \frac{|\phi(t)|}{|y-t|} dt, \quad (3.1.5)$$

by Lemma 3.3 we can bound

$$|S_\varphi^\lambda \phi(y)| \lesssim_\varphi S\phi(y).$$

Let $K(y, t)$ be the kernel given by $\chi_{[\gamma|y|, +\infty)}(|y-t|) |y-t|^{-1}$, so that

$$S\phi(y) = \int K(y, t) |\phi(t)| dt,$$

and observe that this kernel is homogeneous of degree 1, that is

$$K(\mu y, \mu t) = \mu^{-1} K(y, t).$$

To show S is bounded on $L^2(\mathbb{R})$, it suffices by Schur's test³ to show that for some $0 < \beta < 1$ it is

$$\int K(y, t) \frac{1}{|y|^\beta} dy \lesssim \frac{1}{|t|^\beta}, \quad (3.1.6)$$

$$\int K(y, t) \frac{1}{|t|^\beta} dt \lesssim \frac{1}{|y|^\beta}. \quad (3.1.7)$$

³See for example Theorem 0.3.1 in [Sog93] for a statement.

This is a simple calculation: by homogeneity one has

$$\int K(y, t) \frac{1}{|t|^\beta} dt = |y|^\beta \int K(1, t) \frac{1}{|t|^\beta} dt = |y|^{-\beta} \int_{\gamma < |1-t|} \frac{1}{|1-t||t|^\beta} dt;$$

if $Q \gg 1$ we have

$$\int_{\gamma < |1-t| < Q} \frac{1}{|1-t||t|^\beta} dt \leq \frac{1}{\gamma} \int_{\gamma < |1-t| < Q} \frac{1}{|t|^\beta} dt < \infty,$$

and

$$\int_{Q < |1-t|} \frac{1}{|1-t||t|^\beta} dt \lesssim \int_{|t| \geq Q/2} \frac{1}{|t|^{1+\beta}} dt < \infty,$$

and therefore the bound (3.1.6) is proven. Bound (3.1.7) amounts to a similar calculation. Thus Schur's test proves that S and hence S_φ^λ is $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ bounded, with constant depending only on φ and not on λ .

3.1.2.2

Next we reduce the boundedness of the operator T_φ^λ uniformly in λ to establishing the boundedness of the operator $\sum_{\substack{j \geq n+C_0 \\ j \geq C_\varepsilon}} T_\varphi^{\lambda, n, j} =: T_\varphi^{\lambda, n}$ uniformly in λ and n . Namely,

we claim that

$$\left\| \sum_{n \in \mathbb{Z}} T_\varphi^{\lambda, n} \right\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \sim \sup_{n \in \mathbb{Z}} \left\| T_\varphi^{\lambda, n} \right\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})}. \quad (3.1.8)$$

Indeed, we have already shown that we can ignore S_φ^λ , and observe that since we are assuming $j \geq n + C_0$ we have $|y - t| \ll |y|$, and therefore $|t| \sim |y| \sim 2^{-n}$. The definition of $T_\varphi^{\lambda, n, j}$ then implies that for such j 's it is

$$T_\varphi^{\lambda, n, j} \phi(y) = T_\varphi^{\lambda, n, j} (\tilde{\chi}(2^n \cdot) \phi)(y),$$

where $\tilde{\chi}$ is the characteristic function of $[-2, -1/4] \cup [1/4, 2]$; therefore we have by orthogonality

$$\begin{aligned} \left\| \sum_{n \in \mathbb{Z}} T_\varphi^{\lambda, n} \phi \right\|_{L^2(\mathbb{R})}^2 &= \sum_{n \in \mathbb{Z}} \left\| T_\varphi^{\lambda, n} \phi \right\|_{L^2(\mathbb{R})}^2 \\ &= \sum_{n \in \mathbb{Z}} \left\| T_\varphi^{\lambda, n} (\tilde{\chi}(2^n \cdot) \phi) \right\|_{L^2(\mathbb{R})}^2 \\ &\leq \left(\sup_{n \in \mathbb{Z}} \left\| T_\varphi^{\lambda, n} \right\|_{L^2 \rightarrow L^2} \right)^2 \sum_{n \in \mathbb{Z}} \left\| \tilde{\chi}(2^n \cdot) \phi \right\|_{L^2(\mathbb{R})}^2 \\ &\sim \left(\sup_{n \in \mathbb{Z}} \left\| T_\varphi^{\lambda, n} \right\|_{L^2 \rightarrow L^2} \right)^2 \|\phi\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

thus proving one half of (3.1.8). The other half is trivial and follows from the first equality in the sequence above.

3.1.2.3

Finally, we proceed to some reductions regarding the phase

$$\Phi(s) := (y + t)s + \varphi(s, y - t)$$

of the integrand of $m(\lambda; y, t)$.

We write

$$\varphi(s, y - t) = \sum_{k \geq 0} s^k \psi_k(y - t), \quad (3.1.9)$$

where the ψ_k are in $C^\omega([-\varepsilon, \varepsilon])$ and are given by

$$\psi_k(y - t) := c_{k,0} + \sum_{\ell \geq \ell_k} c_{k,\ell} (y - t)^\ell =: c_{k,0} + \tilde{\psi}_k(y - t), \quad (3.1.10)$$

where ℓ_k is the smallest $\ell > 0$ such that $c_{k,\ell} \neq 0$ (if such an ℓ exists); equivalently, it is the smallest $\ell > 0$ such that $\psi_k^{(\ell)}(0) \neq 0$. To avoid confusion, notice ℓ_k does not necessarily coincide with $\ell(k)$ (as previously introduced in the proof of Lemma 3.3), the difference being that ℓ_k is required to be positive.

By writing $(y + t)s = 2ys - (y - t)s$ we can rewrite the phase as

$$\Phi(s) = 2ys + \varphi_1(s, y - t),$$

where

$$\varphi_1(s, y - t) := \varphi(s, y - t) - (y - t)s;$$

another thing we can do is to collect aside from the power expansion (3.1.9) the terms that depend only on $y - t$, that is $\psi_0(y - t)$, so that we can write

$$\Phi(s) = 2ys + \varphi_2(y - t, s) + \psi_0(y - t),$$

where

$$\varphi_2(y - t, s) := \varphi_1(s, y - t) - \psi_0(y - t).$$

Therefore if we let

$$\Phi_1(s) := 2ys + \varphi_2(y - t, s)$$

and

$$m_1(\lambda; y, t) := \text{p.v.} \int_{|s| \leq \varepsilon} e^{i\lambda\Phi_1(s)} \frac{ds}{s},$$

we can rewrite

$$T_\varphi^{\lambda,n} \phi(y) = \sum_{\substack{j \geq n + C_0, \\ j \geq C_\varepsilon}} \chi(2^j y) \int_{2^{-j-1} \leq |y-t| \leq 2^{-j}} m_1(\lambda; y, t) \phi(t) \frac{e^{i\lambda\psi_0(y-t)}}{y-t} dt.$$

Having collected the terms of the power expansion of φ that depend purely on $y - t$ it is also natural to collect the terms that only depend on s . These amount to

$$\bar{\varphi}(s) := \sum_{k \geq 1} c_{k,0} s^k;$$

with $\tilde{\psi}_k$ as defined in (3.1.10) above, we can write

$$\Phi_1(s) = 2ys + \bar{\varphi}(s) + \sum_{k \geq 1} \tilde{\psi}_k(y-t)s^k.$$

We can further assume that $\bar{\varphi}'(0) = 0$; indeed, if not, let $\bar{\varphi}'(0) = 2c$ and notice $2ys + 2cs = 2(y+c)s$, and therefore if we denote by τ_c the translation operator defined by $\tau_c f(x) := f(x-c)$ we have

$$\tau_c T_\varphi^\lambda(\tau_{-c}\phi)(y) = \int_{|y-t| \leq \varepsilon} m_1(\lambda; y-c, t-c) \phi(t) \frac{e^{i\lambda\psi_0(y-t)}}{y-t} dt,$$

and observe that the phase of $m_1(\lambda; y-c, t-c)$ is given by

$$2ys + (\bar{\varphi}(s) - 2cs) + \sum_{k \geq 1} \tilde{\psi}_k(y-t)s^k,$$

which is of the desired form. It is clear that the boundedness of T_φ^λ is equivalent to that of $\tau_c T_\varphi^\lambda \tau_{-c}$.

We conclude the series of reductions with a simple observation. We notice that there exists k_* such that for all $k > k_*$ and $\ell < \ell_{k_*}$ it is $\tilde{\psi}_k^{(\ell)}(0) = 0$, i.e. $\ell_k \geq \ell_{k_*}$ (notice that $\tilde{\psi}_{k_*}^{(\ell_{k_*})}(0) \neq 0$ by definition).

3.2 Proof of Theorem 3.1

3.2.1 Main Lemma

By the reductions in §3.1.2.1, §3.1.2.2 it will suffice to prove the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ boundedness of $T_\varphi^{\lambda, n}$ uniformly in λ and in n . We summarize the reductions made in §3.1.2.3 in the following lemma (notice ψ below corresponds to $\tilde{\psi}$ in §3.1.2.3 above).

Lemma 3.4. *Let $\Phi(s) = \Phi(y, t; s)$ be the phase given by*

$$\Phi(s) := 2ys + \bar{\varphi}(s) + \sum_{k \in N} \psi_k(y-t)s^k + \sum_{k > k_*} \psi_k(y-t)s^k,$$

where

1. $\Phi, \bar{\varphi}, \psi_k$ are all in $C^\omega([-\varepsilon, \varepsilon])$,
2. $\bar{\varphi}'(0) = 0$,
3. for every k there exists $\ell_k > 0$ such that $\psi_k^{(\ell)}(0) = 0$ for all $0 \leq \ell < \ell_k$ and $\psi_k^{(\ell_k)}(0) \neq 0$,
4. for $k > k_*$ and $\ell < \ell_{k_*}$, one has $\psi_k^{(\ell)}(0) = 0$ (or equivalently, $\ell_k \geq \ell_{k_*}$ for $k > k_*$),
5. N is the set of indices given by $N := \{k \text{ s.t. } 1 \leq k \leq k_*, \psi_k \neq 0\}$.

Define moreover

$$\mathcal{A}_k(j) := \frac{\lambda 2^{-j\ell_k}}{(\lambda|y|)^k}.$$

Let $m(\lambda; y, t)$ be given by

$$m(\lambda; y, t) := \text{p.v.} \int_{|s|<\varepsilon} e^{i\lambda\Phi(s)} \frac{ds}{s},$$

and $\Theta(\lambda; y)$ be given by

$$\Theta(\lambda; y) := \text{p.v.} \int_{|s|<\varepsilon} e^{i\lambda(2ys + \bar{\varphi}(s))} \frac{ds}{s}.$$

Then there exists ω monotone increasing function (that depends only on the phase Φ) such that

$$\int_0^1 \omega(r) \frac{dr}{r} < \infty$$

and such that for every $j \geq \max\{n + C_0, C_\varepsilon\}$ and for $2^{-j-1} \leq |y - t| \leq 2^{-j}$ we have

$$|m(\lambda; y, t) - \Theta(\lambda; y)| \leq \sum_{k \in \mathbb{N}} \omega\left(\mathcal{A}_k(j) \wedge \mathcal{A}_k(j)^{-1}\right), \quad (3.2.1)$$

Remark 3.2. It will suffice to choose $\omega(r) = Cr^\sigma$ for a sufficiently small $\sigma > 0$ depending on φ . We therefore assume from now on that ω is of this form, and therefore we will allow ourselves to say for any $0 < r < 1$ and $\sigma' = \sigma'(\varphi)$ arising in the proof, that $r^{\sigma'} \lesssim \omega(r)$.

Remark 3.3. The reason we have introduced N is that we will have to take specific derivatives of the phase to obtain suitable oscillatory integral estimates in the following, and it will be useful to have ruled out those that cannot help us.

Remark 3.4. The quantities $\mathcal{A}_k(j)$ arise by normalizing the phase in such a way that the linear contribution $2\lambda ys$ becomes just s' : indeed, with such a substitution, one has heuristically

$$|\lambda \psi_k(y - t) s^k| \sim 2^{-j\ell_k} \frac{\lambda |s'|^k}{\lambda^k |y|^k} = \mathcal{A}_k(j) |s'|^k,$$

where the implicit constant depends on φ .

We observe a couple of facts about the quantities \mathcal{A}_k : firstly that

$$\mathcal{A}_1(j) = 2^{-j\ell_1} / |y| \leq 2^{-j+n} \leq 2^{-C_0} < 1;$$

and secondly, notice that

$$2\mathcal{A}_k(j+1) \leq 2^{\ell_k} \mathcal{A}_k(j+1) = \mathcal{A}_k(j) = \mathcal{A}_k(j-1) 2^{-\ell_k} \leq \frac{1}{2} \mathcal{A}_k(j-1). \quad (3.2.2)$$

This in particular implies that quantities of the form

$$\sum_j \mathcal{A}_k(j)^\sigma \wedge \mathcal{A}_k(j)^{-\sigma}$$

for $\sigma > 0$ are uniformly bounded by $O_\sigma(1)$; that is, the bound is independent of all the other quantities involved.

Using Lemma 3.4 one can prove the desired uniform (in λ, n) $L^2 \rightarrow L^2$ boundedness of the operator $T_\varphi^{\lambda, n}$, and therefore prove Theorem 3.1. In particular, we claim that for $|y| \sim 2^{-n}$ the inequality

$$|T_\varphi^{\lambda, n} \phi(y)| \lesssim_\varphi S\phi(y) + |T_{\varphi_0} \phi(y)| + M\phi(y) \quad (3.2.3)$$

holds pointwise, where T_{φ_0} is the operator given by

$$T_{\varphi_0} \phi(y) := \text{p.v.} \int_{-\infty}^{\infty} e^{i\lambda\varphi_0(y-t)} \phi(t) \frac{dt}{y-t},$$

S is the operator with positive kernel defined in (3.1.5) and M is the Hardy-Littlewood maximal function.

Indeed, by adding and subtracting $\Theta(\lambda; y)$ from $m(\lambda; y, t)$ we can bound

$$\begin{aligned} |T_\varphi^{\lambda, n} \phi(y)| &\leq \left| \Theta(\lambda; y) \int_{|y-t| \leq 2^{-n-C_0}} \phi(t) \frac{e^{i\lambda\varphi_0(y-t)}}{y-t} dt \right| \\ &\quad + \left| \int_{|y-t| \leq 2^{-n-C_0}} (m(\lambda; y, t) - \Theta(\lambda; y)) \frac{\phi(t)}{y-t} dt \right|, \end{aligned}$$

and by triangle inequality and Lemma 3.4 applied to the second term we have then

$$|T_\varphi^{\lambda, n} \phi(y)| \lesssim |\Theta(\lambda; y)| \left| \int_{|y-t| \leq 2^{-n-C_0}} \phi(t) \frac{e^{i\lambda\varphi_0(y-t)}}{y-t} dt \right| \quad (3.2.4)$$

$$+ \sum_{\substack{j \geq n+C_0, \\ j \geq C_\varepsilon}} \omega_j \int_{|y-t| \sim 2^{-j}} \frac{|\phi(t)|}{|y-t|} dt, \quad (3.2.5)$$

where ω_j is shorthand for $\sum_{k \in \mathbb{N}} \omega(\mathcal{A}_k(j) \wedge \mathcal{A}_k(j)^{-1})$.

Since by Lemma 3.3 we have $|\Theta(\lambda; y)| = O_\varphi(1)$, term (3.2.4) is bounded by

$$\lesssim_\varphi \left| \int_{|y-t| \leq \gamma|y|} \phi(t) \frac{e^{i\lambda\varphi_0(y-t)}}{y-t} dt \right| \leq S\phi(y) + \left| \text{p.v.} \int_{-\infty}^{\infty} \phi(t) \frac{e^{i\lambda\varphi_0(y-t)}}{y-t} dt \right|$$

and thus is $L^2 \rightarrow L^2$ bounded uniformly in λ (and n) by §3.1.2.1 of this chapter and Theorem 2.4 of Chapter 2 (from [RS87]). As for (3.2.5), on the one hand we have for every j

$$\int_{|y-t| \sim 2^{-j}} \frac{|\phi(t)|}{|y-t|} dt \sim \frac{1}{2^{-j}} \int_{|y-t| \sim 2^{-j}} |\phi(t)| dt \leq M\phi(y);$$

on the other hand, by the hypothesis on the function ω and as noted above we have for any positive real $A > 0$ that

$$\sum_{j \in \mathbb{Z}} \omega((A2^{-j}) \wedge (A2^{-j})^{-1}) = O(1)$$

and the bound is independent of A , so by (3.2.2) one then has

$$\sum_{\substack{j \geq n+C_0, \\ j \geq C_\varepsilon}} \omega_j = \sum_{\substack{j \geq n+C_0, \\ j \geq C_\varepsilon}} \sum_{k \in N} \omega(\mathcal{A}_k(j) \wedge \mathcal{A}_k(j)^{-1}) = O_\varphi(1).$$

We therefore have for (3.2.5)

$$\sum_{\substack{j \geq n+C_0, \\ j \geq C_\varepsilon}} \omega_j \int_{|y-t| \sim 2^{-j}} \frac{|\phi(t)|}{|y-t|} dt \lesssim \sum_{\substack{j \geq n+C_0, \\ j \geq C_\varepsilon}} \omega_j M\phi(y) \lesssim M\phi(y),$$

which as is well known is $L^2 \rightarrow L^2$ bounded as well; thus (3.2.3) is proved, and with it the uniform boundedness in λ and n of $T_\varphi^{\lambda, n}$ is proved too.

It remains therefore to prove Lemma 3.4, which we do in the next subsection.

3.2.2 Proof of Lemma 3.4

Depending on the various parameters, it might be that $\mathcal{A}_k(j)$ is bigger or smaller than 1, and we introduce notation to treat these cases all at once. Given a bipartition⁴ $N = S \sqcup L$ such that $1 \in N \Rightarrow 1 \in S$ (because $\mathcal{A}_1(j) < 1$, as observed above), we set

$$J = J_{S,L} := \{j \text{ s.t. } j \geq \max\{n+C_0, C_\varepsilon\}, \mathcal{A}_k(j) \leq 1 \text{ if } k \in S, \mathcal{A}_k(j) > 1 \text{ if } k \in L\}.$$

Thus, for any given j , S is the set indexing the small coefficients $\mathcal{A}_k(j)$ and L is the one indexing the large ones, intuitively speaking. It is clear that as S, L range through all the allowed bipartitions of N , we obtain a disjoint partition of $[n+C, \infty) \subset \mathbb{N}$. Notice, because of (3.2.2), that there is a unique infinite component, which is $J_{N, \emptyset}$. We will proceed by removing terms from the phase of $m(\lambda; y, t)$; the errors that arise will be controlled by the quantities $\mathcal{A}_k(j)$.

3.2.2.1

Consider now S, L fixed and assume $j \in J$. The first thing we observe is that we can dispose of the large terms immediately; that is, the terms indexed by L . Indeed, suppose $L \neq \emptyset$ and write

$$\Phi_S(s) := 2ys + \bar{\varphi}(s) + \sum_{k \in S} \psi_k(y-t) s^k + \Psi(s, y-t),$$

where we let Ψ denote the tail of the phase, that is

$$\Psi(s, y-t) := \sum_{k > k_*} \psi_k(y-t) s^k;$$

⁴That is, $S \cap L = \emptyset$ and $S \cup L = N$.

then since $k > 1$ when $k \in L$

$$\begin{aligned}
 & \left| \text{p.v.} \int_{|s| < \varepsilon} e^{i\lambda\Phi(s)} \frac{ds}{s} - \text{p.v.} \int_{|s| < \varepsilon} e^{i\lambda\Phi_S(s)} \frac{ds}{s} \right| \\
 & \leq \int_{|s| \leq \varepsilon} \lambda \left| \sum_{k \in L} \psi_k(y-t) s^k \right| \frac{ds}{|s|} \\
 & \leq \lambda \sum_{k \in L} \left(\sum_{\ell \geq \ell_k} |c_{k\ell}| |y-t|^\ell \right) \int_{|s| \leq \varepsilon} |s|^{k-1} ds \\
 & \lesssim \lambda \sum_{k \in L} 2^{-j\ell_k} \left(\sum_{\ell \geq \ell_k} |c_{k\ell}| \varepsilon^{\ell-\ell_k} \frac{\varepsilon^k}{k} \right),
 \end{aligned}$$

and we can assume that ε is sufficiently small that the (finitely many) sums in brackets are finite and all uniformly in $O_\varphi(1)$. We claim that

$$\lambda 2^{-j\ell_k} \leq \mathcal{A}_k(j)^{-1/(k-1)} < 1,$$

so that in particular $\lambda 2^{-j\ell_k} \lesssim \omega(\mathcal{A}_k(j)^{-1})$. Indeed

$$\begin{aligned}
 \lambda 2^{-j\ell_k} \mathcal{A}_k(j)^{1/(k-1)} &= \lambda 2^{-j\ell_k} (\lambda^{-(k-1)} |y|^{-k} 2^{-j\ell_k})^{1/(k-1)} \\
 &= |y|^{-k/(k-1)} 2^{-j\ell_k} 2^{-j\ell_k/(k-1)} \leq (2^n 2^{-j\ell_k})^{k/(k-1)} < 1,
 \end{aligned}$$

since $j \geq \max\{n + C_0, C_\varepsilon\}$ and $\ell_k \geq 1$.

3.2.2.2

It suffices then to prove the lemma for phase Φ_S . At this point we will have to split the integral into several parts, all of which except one will be bounded in terms of some $\mathcal{A}_k(j)$.

We first split

$$\begin{aligned}
 \text{p.v.} \int_{|s| < \varepsilon} e^{i\lambda\Phi_S(s)} \frac{ds}{s} &= \text{p.v.} \int_{|s| < \theta} e^{i\lambda\Phi_S(s)} \frac{ds}{s} + \text{p.v.} \int_{\theta \leq |s| \leq \varepsilon} e^{i\lambda\Phi_S(s)} \frac{ds}{s} \quad (3.2.6) \\
 &=: I + II,
 \end{aligned}$$

where

$$\theta := \frac{1}{\lambda|y|} \min_{k \in S} \mathcal{A}_k(j)^{-\sigma_k}$$

and $\sigma_k > 0$ are parameters to be considered fixed, but which will be chosen later.

We deal with I first. We claim that

$$I - \Theta(\lambda; y)$$

is bounded by the right hand side of (3.2.1). Indeed, let

$$I - \Theta(\lambda; y) = \left(\int_{|s| < \theta} e^{i\lambda\Phi_S(s)} - e^{i\lambda(2ys + \bar{\varphi}(s))} \frac{ds}{s} \right) - \int_{\theta \leq |s| \leq \varepsilon} e^{i\lambda(2ys + \bar{\varphi}(s))} \frac{ds}{s}$$

$$=: I_1 - I_2.$$

We have

$$\begin{aligned} |I_1| &\leq \int_{|s|<\theta} \lambda \left| \sum_{k \in S} \psi_k(y-t) + \sum_{k > k_*} \psi_k(y-t) \right| |s|^k \frac{ds}{|s|} \\ &\lesssim_{\varphi} \sum_{k \in S} 2^{-j\ell_k} \lambda \int_{|s|<\theta} |s|^{k-1} ds + \lambda \sum_{k > k_*} \int_{|s|<\theta} |\psi_k(y-t)| |s|^{k-1} ds \\ &\lesssim_{\varphi} \sum_{k \in S} 2^{-j\ell_k} \lambda \int_{|s|<\theta} |s|^{k-1} ds + \lambda \sum_{k > k_*} |\psi_k(y-t)| \int_{|s|<\theta} |s|^{k_*-1} \varepsilon^{k-k_*} ds \\ &\lesssim_{\varphi} \sum_{k \in S} \lambda 2^{-j\ell_k} \int_{|s|<\theta} |s|^{k-1} ds + \lambda 2^{-j\ell_{k_*}} \int_{|s|<\theta} |s|^{k_*-1} ds \\ &\lesssim \sum_{k \in S} \lambda 2^{-j\ell_k} \theta^k + \lambda 2^{-j\ell_{k_*}} \theta^{k_*}. \end{aligned}$$

Again here we have assumed that ε is sufficiently small (depending only on the coefficients $c_{k\ell}$, thus on φ) to justify the bound

$$\sum_{k > k_*} |\psi_k(y-t)| \varepsilon^{k-k_*} \lesssim \lambda 2^{-j\ell_{k_*}};$$

one can easily check that this is possible to achieve. Similar assumptions will appear over and over throughout our argument, and since they are all similar and easily achieved we shall not comment them in detail anymore to avoid becoming overly pedantic (although we will point out where they occur).

Now we observe that for each $k \in S$

$$\lambda 2^{-j\ell_k} \theta^k = \frac{\lambda 2^{-j\ell_k}}{\lambda^k |y|^k} \min_{p \in S} \mathcal{A}_p(j)^{-k\sigma_p} = \mathcal{A}_k(j) \min_{p \in S} \mathcal{A}_p(j)^{-k\sigma_p} \leq \mathcal{A}_k(j)^{1-k\sigma_k},$$

and therefore if we choose $0 < \sigma_k < 1/k$ then $|I_1|$ will be bounded by $\sum_{k \in S} \omega(\mathcal{A}_k(j)) + \omega(\mathcal{A}_{k_*}(j) \wedge \mathcal{A}_{k_*}(j)^{-1})$.

Next we deal with I_2 . We have assumed that

$$\bar{\varphi}(s) = \sum_{k \geq k_0} c_{k,0} s^k$$

with $k_0 \geq 2$, because we have assumed in the hypotheses of Lemma 3.4 that $\bar{\varphi}'(0) = 0$. We then split further

$$\begin{aligned} I_2 &= \int_{\substack{\theta \leq |s| < \varepsilon, \\ c_0 |y|^{1/(k_0-1)} \leq |s|}} e^{i\lambda(2ys + \bar{\varphi}(s))} \frac{ds}{s} + \int_{\substack{\theta \leq |s| < \varepsilon, \\ |s| \leq c_0 |y|^{1/(k_0-1)}}} e^{i\lambda(2ys + \bar{\varphi}(s))} \frac{ds}{s} \\ &=: I_2' + I_2'', \end{aligned}$$

where c_0 is a sufficiently small constant depending on φ .

Remark 3.5. The motivation for choosing $c_0 |y|^{1/(k_0-1)}$ as cutoff is the following: the

k_0 -th order derivative of the phase is

$$\overline{\varphi}^{(k_0)}(s) = k_0! c_{k_0,0} + \sum_{k>k_0} \frac{k!}{(k-k_0)!} c_{k,0} s^{k-k_0}$$

and thus $\left| \left(\frac{d}{ds} \right)^{k_0} (2ys + \overline{\varphi}(s)) \right| \gtrsim_{\varphi} 1$, which allows us to use Van der Corput's lemma. This gives us

$$\begin{aligned} \left| \int_{\theta \leq |s| \leq \varepsilon} e^{i\lambda(2ys + \overline{\varphi}(s))} \frac{ds}{s} \right| &\lesssim \frac{1}{\lambda^{1/k_0}} \theta^{-1} = \frac{\lambda y}{\lambda^{1/k_0}} \max_{p \in S} \mathcal{A}_p(j)^{\sigma_p} \\ &= (\lambda^{k_0-1} |y|^{k_0})^{1/k_0} \max_{p \in S} \mathcal{A}_p(j)^{\sigma_p} \end{aligned}$$

and we would like this to be summable in j . This can be achieved if we can enforce, for some $0 < \beta < 1$,

$$(\lambda^{k_0-1} |y|^{k_0})^{1/k_0} \lesssim \mathcal{A}_p(j)^{-\sigma_p \beta}$$

for all $p \in S$. This is equivalent to

$$(\lambda^{k_0-1} |y|^{k_0})^{1/k_0} \lambda^{-\beta} |y|^{-\beta} \lesssim \theta^{\beta},$$

and we make λ disappear by choosing $\beta = (k_0 - 1)/k_0$, which then gives $|y| \lesssim \theta^{k_0-1}$ as condition, and hence the above splitting.

Now for the rigorous argument. We analyse I'_2 first. We have, as noted above, that $|\overline{\varphi}^{(k_0)}(s)| \gtrsim_{\varphi} 1$ for ε sufficiently small (depending only on φ), and therefore by Corollary 2.2 of Chapter 2

$$|I'_2| = \left| \int_{\substack{\theta \leq |s| < \varepsilon, \\ c_0 |y|^{1/(k_0-1)} \leq |s|}} e^{i\lambda(2ys + \overline{\varphi}(s))} \frac{ds}{s} \right| \lesssim \frac{1}{\lambda^{1/k_0}} \min\{\theta^{-1}, |y|^{-1/(k_0-1)}\};$$

by dominating the minimum by the geometric average $(\frac{1}{\theta})^{1/k_0} \left(\frac{1}{|y|^{1/(k_0-1)}} \right)^{1-1/k_0} = (\theta|y|)^{-1/k_0}$, we have

$$|I'_2| \lesssim \frac{1}{(\lambda|y|\theta)^{1/k_0}} = \max_{p \in S} \mathcal{A}_p(j)^{\sigma_p/k_0},$$

which is bounded by the right hand side of (3.2.1).

Next we deal with I''_2 . Although we still have the lower bound $|\overline{\varphi}^{(k_0)}(s)| \gtrsim_{\varphi} 1$, now the region of integration is not convenient anymore. Thus we observe instead that the first derivative of the phase gives

$$\left| \frac{d}{ds} (2ys + \overline{\varphi}(s)) \right| = |2y + \overline{\varphi}'(s)| \gtrsim 2|y| - k_0 c_{k_0,0} |s|^{k_0-1},$$

and since now $|s| \leq c_0 |y|^{1/(k_0-1)}$, for sufficiently small c_0 (depending only on φ) we have

$$\left| \frac{d}{ds} (2ys + \overline{\varphi}(s)) \right| \gtrsim_{\varphi} |y|. \quad (3.2.7)$$

We cannot apply Van der Corput's lemma because the phase will not in general be monotone, thus we compensate with some further derivative estimates. We observe

that, again by the specific region of integration, for ε sufficiently small it is

$$\left| \left(\frac{d}{ds} \right)^2 (2ys + \bar{\varphi}(s)) \right| \lesssim_{\varphi} k_0(k_0 - 1) |c_{k_0,0}| |s|^{k_0-2} \lesssim_{\varphi} \frac{|y|}{|s|}; \quad (3.2.8)$$

we therefore integrate by parts, letting $\Phi_0(s) := 2ys + \bar{\varphi}(s)$, to obtain

$$\begin{aligned} I_2'' &= \int_{\substack{\theta \leq |s| < \varepsilon, \\ |s| \leq c_0 |y|^{1/(k_0-1)}}} \frac{d}{ds} \left[e^{i\lambda\Phi_0(s)} \right] \frac{1}{\lambda\Phi_0'(s)} \frac{ds}{s} \\ &= \left[\frac{e^{i\lambda\Phi_0(s)}}{\lambda\Phi_0'(s)s} \right]_{\theta < |s| < c_0 |y|^{1/(k_0-1)}} \\ &\quad - \int_{\substack{\theta \leq |s| < \varepsilon, \\ |s| \leq c_0 |y|^{1/(k_0-1)}}} e^{i\lambda\Phi_0(s)} \frac{\Phi_0''(s)s}{\lambda(\Phi_0'(s)s)^2} ds \\ &\quad - \int_{\substack{\theta \leq |s| < \varepsilon, \\ |s| \leq c_0 |y|^{1/(k_0-1)}}} e^{i\lambda\Phi_0(s)} \frac{\Phi_0'(s)}{\lambda(\Phi_0'(s)s)^2} ds. \end{aligned}$$

Since we are assuming $|s| > \theta$ we have by (3.2.7) that

$$\left| \frac{e^{i\lambda\Phi_0(s)}}{\lambda\Phi_0'(s)s} \right| \lesssim_{\varphi} \frac{1}{\lambda|y|\theta} = \max_{p \in S} \mathcal{A}_p(j)^{\sigma_p},$$

which is dominated by the right hand side of (3.2.1). Next, by (3.2.7) and (3.2.8)

$$\begin{aligned} \left| \int_{\substack{\theta \leq |s| < \varepsilon, \\ |s| \leq c_0 |y|^{1/(k_0-1)}}} e^{i\lambda\Phi_0(s)} \frac{\Phi_0''(s)s}{\lambda(\Phi_0'(s)s)^2} ds \right| &\lesssim_{\varphi} \int_{\substack{\theta \leq |s| < \varepsilon, \\ |s| \leq c_0 |y|^{1/(k_0-1)}}} \frac{(|y|/|s|)|s|}{\lambda(|y|s)^2} ds \\ &\leq \frac{1}{\lambda|y|} \int_{\theta \leq |s| < \varepsilon} \frac{ds}{|s|^2} \\ &\lesssim \frac{1}{\lambda|y|\theta} = \max_{p \in S} \mathcal{A}_p(j)^{\sigma_p}, \end{aligned}$$

which is bounded by the right hand side of (3.2.1) again; and finally similarly

$$\left| \int_{\substack{\theta \leq |s| < \varepsilon, \\ |s| \leq c_0 |y|^{1/(k_0-1)}}} e^{i\lambda\Phi_0(s)} \frac{\Phi_0'(s)}{\lambda(\Phi_0'(s)s)^2} ds \right| \lesssim \int_{\substack{\theta \leq |s| < \varepsilon, \\ |s| \leq c_0 |y|^{1/(k_0-1)}}} \frac{1}{\lambda|y||s|^2} ds \lesssim \max_{p \in S} \mathcal{A}_p(j)^{\sigma_p}.$$

We have therefore dealt with contribution I from (3.2.6).

It remains to show that II from (3.2.6) is bounded by the right hand side of (3.2.1). We will have to analyse very carefully the derivatives of the phase this time, and to this aim we introduce some notation. Let k_{**} be the smallest $k \geq k_*$ such that $\bar{\varphi}^{(k)}(0) \neq 0$ (thus if $c_{k_*,0} \neq 0$, then $k_{**} = k_*$). Then define

$$T := \{k < k_{**} \text{ s.t. } \bar{\varphi}^{(k)}(0) \neq 0\}.$$

Finally, for $k \in (S \cup T) \setminus \{1\}$, define the (possibly empty) integration regions

$$R(k) := \{s : \forall r \in S \cup T, r > k \Rightarrow \xi_k \leq |s| < \xi_r\},$$

where

$$\xi_k := \begin{cases} (\delta_k |y|)^{1/(k-1)} & \text{if } k \in T \setminus S, \\ (\delta_k |y| 2^{j\ell_k})^{1/(k-1)} & \text{if } k \in S \setminus \{1\}, \end{cases}$$

for constants $\delta_k > 0$ to be fixed in the following (they will depend only on φ).

Remark 3.6. The reason for choosing the regions of integration as above is heuristically as follows. If $k \in S \setminus T$ we see that

$$\Phi_S^{(k)}(s) \approx \frac{k!}{(k'-k)!} c_{k',0} s^{k'-k} + k! \psi_k(y-t) + \text{other terms}$$

for k' the smallest $k \in T$ s.t. $k' > k$. Since $|\psi_k(y-t)| \sim 2^{-j\ell_k}$, the best lower bound we can hope for this derivative is $2^{-j\ell_k}$ itself. Suppose we had indeed this lower bound, and suppose we restrict ourselves to a region $B < |s|$; we need to choose B for Van der Corput's lemma to be most effective. Indeed, Van der Corput's lemma gives

$$\left| \int_{\substack{\theta \leq |s| \leq \varepsilon, \\ B < |s|}} e^{i\lambda \Phi_S(s)} \frac{ds}{s} \right| \lesssim \frac{1}{(\lambda 2^{-j\ell_k})^{1/k}} \frac{1}{B} = \frac{1}{\lambda |y| \mathcal{A}_k(j)^{1/k}} \frac{1}{B};$$

for this to be summable we would like $\lambda |y| B \gtrsim \mathcal{A}_k(j)^{-\alpha}$ for some $\alpha > 1/k$. Thus we need

$$B \gtrsim \frac{(\lambda^{k-1} |y|^{k 2^{j\ell_k}})^{\alpha}}{\lambda |y|},$$

and if we pick $\alpha = 1/(k-1)$ the λ term disappears from the expression and we see we can choose

$$B \gtrsim (|y| 2^{j\ell_k})^{1/(k-1)};$$

hence the choice of lower bound for $|s|$ in $R(k)$.

Next observe that if $k \in T \setminus S$ then the derivative is, for some $k' \in S$,

$$\Phi_S^{(k)}(s) \approx k! c_{k,0} + \frac{k!}{(k'-k)!} \psi_{k'}(y-t) s^{k'-k} + \text{other terms};$$

we can therefore have a lower bound of the form $|\Phi_S^{(k)}(s)| \gtrsim 1$ if we can ensure

$$|\psi_{k'}(y-t) s^{k'-k}| \ll 1 \Leftrightarrow |s|^{k'-k} \ll 2^{j\ell_{k'}}.$$

Now, if $\xi_k < |s| < \xi_{k'}$, then $|s|^{k'-k} < \xi_{k'}^{k'-1} / \xi_k^{k-1} \sim |y| 2^{j\ell_{k'}} / \xi_k^{k-1}$, so the condition we seek is automatically enforced if we choose $\xi_k^{k-1} \sim |y|$.

Now for the rigorous argument. We split according to the integration regions above (notice they are disjoint). We write

$$II = \int_{\substack{\theta \leq |s| \leq \varepsilon, \\ s \notin \bigcup_{k \in S \cup T} R(k)}} e^{i\lambda \Phi_S(s)} \frac{ds}{s} + \sum_{k \in S \cup T} \int_{\substack{\theta \leq |s| \leq \varepsilon, \\ s \in R(k)}} e^{i\lambda \Phi_S(s)} \frac{ds}{s}$$

$$=: II' + \sum_{k \in S \cup T} II_k.$$

We deal with the II_k first. As the remark above suggested, we have

Lemma 3.5. *There is an ε sufficiently small depending on φ only such that, for $|y-t| \sim 2^{-j}$, if $k \in T$ then*

$$|\Phi_S^{(k)}(s)| \gtrsim_{\varphi} 1 \quad \forall s \in R(k),$$

and if $k \in S \setminus T$ we have

$$|\Phi_S^{(k)}(s)| \gtrsim_{\varphi} 2^{-j\ell_k} \quad \forall s \in R(k).$$

We will prove the lemma in §3.2.3.

Now, assuming the lemma holds, if $k \in T$ (notice $k \geq k_0 \geq 2$) we have therefore by Corollary 2.2 of Chapter 2 that

$$\left| \int_{\substack{\theta \leq |s| \leq \varepsilon, \\ s \in R(k)}} e^{i\lambda\Phi_S(s)} \frac{ds}{s} \right| \lesssim_{\varphi} \frac{1}{\lambda^{1/k}} \min\{\theta^{-1}, \xi_k^{-1}\};$$

since $\xi_k \sim |y|^{1/(k-1)}$, by the same calculations done in the analysis of I'_2 we get that this is bounded by $\max_{p \in S} \mathcal{A}_p(j)^{\sigma_p/k}$, which is therefore bounded by the right hand side of (3.2.1), as desired.

Then assume otherwise that $k \in S \setminus T$; we see by the lemma above and by Corollary 2.2 of Chapter 2 that

$$\begin{aligned} \left| \int_{\substack{\theta \leq |s| \leq \varepsilon, \\ s \in R(k)}} e^{i\lambda\Phi_S(s)} \frac{ds}{s} \right| &\lesssim_{\varphi} \frac{1}{(\lambda 2^{-j\ell_k})^{1/k} \xi_k^{-1}} \\ &= \frac{2^{-j\ell_k(\frac{1}{k-1} - \frac{1}{k})}}{\lambda^{1/k} |y|^{1/(k-1)}} = \mathcal{A}_k(j)^{1/k(k-1)}, \end{aligned}$$

which is bounded by the right hand side of (3.2.1).

We are therefore left with estimating II' . Observe that $s \notin \bigcup_{k \in S \cup T} R(k)$ means that $s < \xi_k$ for all $k \in S \cup T$. In this region we have no good lower bounds for the higher order derivatives, and therefore we have to resort once again to the 1st derivative. We have

$$\Phi'_S(s) = 2y + \bar{\varphi}'(s) + \sum_{k \in S} k\psi_k(y-t)s^{k-1} + \frac{\partial}{\partial s} \Psi(s, y-t);$$

now, since $|s| < \xi_{k_0} \sim_{\varphi} |y|^{1/(k_0-1)}$ (because $k_0 \in T$),

$$|\bar{\varphi}'(s)| \lesssim_{\varphi} |s|^{k_0-1} \ll |y|,$$

and similarly for all $k \in S$ and ε sufficiently small (depending on φ only)

$$|k\psi_k(y-t)s^{k-1}| \lesssim_{\varphi} 2^{-j\ell_k} \xi_k^{k-1} \ll |y|$$

for sufficiently small constants δ_k ; finally, since $k_* \in S$,

$$\begin{aligned} \left| \frac{\partial}{\partial s} \Psi(s, y-t) \right| &= \left| \sum_{k > k_*} k \psi_k(y-t) s^{k-1} \right| \\ &\lesssim_{\varphi} 2^{-j\ell_{k_*}} \sum_{k > k_*} |s|^{k-1} \sim 2^{-j\ell_{k_*}} |s|^{k_*-1} \\ &\leq 2^{-j\ell_{k_*}} \xi_{k_*}^{k_*-1} \ll |y| \end{aligned}$$

as well. Thus we have shown that

$$|\Phi'_S(s)| \gtrsim_{\varphi} |y|$$

when $|s| < \min_k \xi_k$. We need now an upperbound on $\Phi''_S(s)$, which is

$$\Phi''_S(s) = \bar{\varphi}''(s) + \sum_{k \in S} k(k-1) \psi_k(y-t) s^{k-2} + \frac{\partial^2}{\partial s^2} \Psi(s, y-t);$$

from the calculations done for Φ'_S it follows that every term is bounded by $|y|/|s|$, so that

$$|\Phi''_S(s)| \lesssim_{\varphi} \frac{|y|}{|s|}$$

when $|s| < \min_k \xi_k$ (and again, for ε sufficiently small). Then from the same calculations done for I''_2 (just replace Φ_0 with Φ_S) it follows that II' is bounded by positive powers of $\mathcal{A}_k(j)$ for $k \in S$, which are therefore bounded by the right hand side of (3.2.1), as desired. This concludes the proof of Theorem 3.1, modulo the proof of Lemma 3.5, which is presented below. \square

3.2.3 Proof of Lemma 3.5

Suppose $k \in T$ as defined above. The k -th derivative of Φ_S is given by

$$\begin{aligned} \Phi_S^{(k)}(s) &= k! c_{k,0} + \sum_{k' > k} \frac{k'!}{(k'-k)!} c_{k',0} s^{k'-k} \\ &\quad + \sum_{\bar{k} \in S, \bar{k} \geq k} \frac{\bar{k}!}{(\bar{k}-k)!} \psi_{\bar{k}}(y-t) s^{\bar{k}-k} \\ &\quad + \sum_{\bar{k} > k_*} \frac{\bar{k}!}{(\bar{k}-k)!} \psi_{\bar{k}}(y-t) s^{\bar{k}-k}. \end{aligned}$$

If $k' \leq k_{**}$ is such that $c_{k',0} \neq 0$ then $k' \in T$ and therefore since $s \in R(k)$ means $\xi_k \leq |s| < \xi_{k'}$,

$$\left| \frac{k'!}{(k'-k)!} c_{k',0} s^{k'-k} \right| \lesssim_{\varphi} \frac{\xi_{k'}^{k'-1}}{\xi_k^{k-1}} = \frac{\delta_{k'}}{\delta_k} \ll 1,$$

provided we choose the constants δ_k rapidly decreasing in k . Next, if $k' > k_{**}$ we can bound

$$\left| \sum_{k' > k_{**}} \frac{k'!}{(k' - k)!} c_{k',0} s^{k' - k} \right| \lesssim_{\varphi} \sum_{k' > k_{**}} |s|^{k' - k} \sim_{\varphi} |s|^{k_{**} - k} \leq \frac{\xi_{k_{**}}^{k_{**} - 1}}{\xi_k^{k - 1}} = \frac{\delta_{k_{**}}}{\delta_k} \ll 1$$

as well. Observe moreover that, always for ε sufficiently small,

$$\left| \frac{\tilde{k}!}{(\tilde{k} - k)!} \psi_{\tilde{k}}(y - t) s^{\tilde{k} - k} \right| \lesssim_{\varphi} 2^{-j\ell_{\tilde{k}}} \frac{\xi_{\tilde{k}}^{\tilde{k} - 1}}{\xi_k^{k - 1}} = 2^{-j\ell_{\tilde{k}}} \frac{\xi_{\tilde{k}}^{\tilde{k} - 1}}{\delta_k |y|},$$

and this can be either $\delta_{\tilde{k}}/\delta_k$ if $k \notin T$ or $2^{-j\ell_{\tilde{k}}}\delta_{\tilde{k}}/\delta_k$ if $k \in T$; in both cases, the term is $\ll 1$, as desired. Finally, we have for ε sufficiently small

$$\left| \sum_{\bar{k} > k_*} \frac{\bar{k}!}{(\bar{k} - k)!} \psi_{\bar{k}}(y - t) s^{\bar{k} - k} \right| \lesssim_{\varphi} 2^{-j\ell_{k_*}} \frac{\xi_{k_*}^{k_* - 1}}{\xi_k^{k - 1}} \ll 1$$

too, by the same calculations above. Therefore we have proven that

$$\left| \Phi_S^{(k)}(s) \right| \gtrsim_{\varphi} 1$$

for $s \in R(k)$, $k \in T$.

Now suppose instead that $k \in S \setminus T$. Then the k -th derivative now takes the form

$$\begin{aligned} \Phi_S^{(k)}(s) &= \sum_{k' > k} \frac{k'!}{(k' - k)!} c_{k',0} s^{k' - k} \\ &\quad + k! \psi_k(y - t) + \sum_{\tilde{k} \in S, \tilde{k} > k} \frac{\tilde{k}!}{(\tilde{k} - k)!} \psi_{\tilde{k}}(y - t) s^{\tilde{k} - k} \\ &\quad + \sum_{\bar{k} > k_*} \frac{\bar{k}!}{(\bar{k} - k)!} \psi_{\bar{k}}(y - t) s^{\bar{k} - k}. \end{aligned}$$

We have, if $k' \leq k_{**}$, that

$$\left| \frac{k'!}{(k' - k)!} c_{k',0} s^{k' - k} \right| \lesssim_{\varphi} \frac{\xi_{k'}^{k' - 1}}{\xi_k^{k - 1}} = \frac{\delta_{k'} |y|}{\delta_k |y| 2^{j\ell_k}} = \frac{\delta_{k'}}{\delta_k} 2^{-j\ell_k} \ll 2^{-j\ell_k}.$$

Next we have for ε sufficiently small

$$|k! \psi_k(y - t)| \sim_{\varphi} 2^{-j\ell_k};$$

and for $S \ni \tilde{k} > k$ we have

$$\left| \frac{\tilde{k}!}{(\tilde{k} - k)!} \psi_{\tilde{k}}(y - t) s^{\tilde{k} - k} \right| \lesssim_{\varphi} 2^{-j\ell_{\tilde{k}}} \frac{\xi_{\tilde{k}}^{\tilde{k} - 1}}{\xi_k^{k - 1}} = \frac{\delta_{\tilde{k}} |y|}{\delta_k |y| 2^{j\ell_k}} \ll 2^{-j\ell_k}.$$

Finally, by usual calculations,

$$\left| \sum_{\bar{k} > k_*} \frac{\bar{k}!}{(\bar{k} - k)!} \psi_{\bar{k}}(y - t) s^{\bar{k} - k} \right| \lesssim_{\varphi} 2^{-j\ell_{k_*}} |s|^{k_* - k} \leq 2^{-j\ell_{k_*}} \frac{\xi_{k_*}^{k_* - 1}}{\xi_k^{k - 1}} = \frac{\delta_{k_*}}{\delta_k} 2^{-j\ell_k}.$$

Therefore we have proven

$$\left| \Phi_S^{(k)}(s) \right| \gtrsim_{\varphi} 2^{-j\ell_k}$$

for $s \in R(k)$, $k \in S \setminus T$, and thus we are done. □

Chapter 4

Uniformity results

In this chapter we address the L^2 boundedness of the operators under study from the point of view of uniformity. It makes sense indeed to restrict our analytic function φ to be a real polynomial in two variables, and for any fixed degree d we can look for the largest subspace of polynomials of degree at most d such that the $L^2 \rightarrow L^2$ norms of the resulting operators are uniformly bounded in the coefficients of the polynomial. We will see that this is the subspace of polynomials whose monomials have at least one even exponent; thus one recovers a situation analogous to that of the Euclidean translation invariant case.

In the first two sections we will introduce two known results that we will exploit in the proof. In the third section we state our main result. The remaining sections are devoted to the proofs of the statements in the third section.

4.1 A decomposition lemma

The key to the uniformity result mentioned in the introduction is a lemma that, given a polynomial p in one variable, allows one to partition \mathbb{R} into a finite (bounded in the degree of p) disjoint union of intervals, such that every interval is either of dyadic type, or it is such that p behaves essentially like one of its monomials on it. More precisely, define a *symmetric double interval* to be a set of the form $[-b, -a] \cup [a, b]$, for $0 \leq a < b$. Then we can state

Lemma 4.1 ([CRW98]). *Let $d \in \mathbb{N}$. There exists a constant $A = A(d) > 1$ such that the following holds. Let $p \in \mathbb{R}[t]$ be a polynomial of degree d , $p(t) = \sum_{j=0}^d c_j t^j$, then there exists a decomposition*

$$\mathbb{R} = \bigsqcup_{i \in \mathcal{I}} J_i,$$

where each J_i is a symmetric double interval, that satisfies the following properties:

- i) the cardinality of \mathcal{I} is bounded by $d + 1$;
- ii) each J_i is of (exactly) one of two types:
 - dyadic type: J_i is a symmetric double interval of the form $[-A^2\beta, -\beta] \cup [\beta, A^2\beta]$, for some $\beta > 0$;

- *gap type: there exists $k \in \{0, \dots, d\}$ such that for all $t \in J_i$ it holds*

$$|p(t)| \sim |c_k| |t|^k;$$

moreover, we have for the first derivative of p

$$|p'(t)| \sim \frac{|p(t)|}{|t|}.$$

We provide a proof of this lemma below, but first we digress a little to provide some context and to illustrate the usefulness of the above decomposition.

The lemma was introduced in [CRW98] in order to prove uniform boundedness results for one dimensional singular and maximal integral operators of the form

$$\begin{aligned} \mathcal{H}_p f(x) &:= \text{p.v.} \int_{-\infty}^{\infty} f(x-p(s)) \frac{ds}{s}, \\ \mathcal{M}_p f(x) &:= \sup_{t>0} \frac{1}{2t} \int_{-t}^t |f(x-p(s))| ds, \end{aligned}$$

with p a polynomial. Indeed, using the lemma above, one can decompose the integrals above as the sum of the integrals p.v. \int_J (with boundedly many terms in d) and estimate that if J is of gap type then $p(s)$ is essentially of the form s^k , and thus (assume k is odd for simplicity) after a change of variable the contribution to evaluate is essentially bounded by

$$\left| \text{p.v.} \int_{J'} f(x-s) \frac{ds}{s} \right|,$$

which in itself is bounded by $H^* f(x)$, where H^* denotes the maximal Hilbert transform; if J is of dyadic type, then the matter reduces to the maximal operator \mathcal{M}_p , since then

$$\left| \int_{\beta \leq |s| \leq A^2 \beta} f(x-p(s)) \frac{ds}{s} \right| \leq \frac{1}{\beta} \int_{-A^2 \beta}^{A^2 \beta} |f(x-p(s))| ds \leq 2A^2 \mathcal{M}_p f(x).$$

It is not hard to prove then that $\mathcal{M}_p f(x)$ is bounded pointwise by a multiple of $Mf(x)$, with M the Hardy-Littlewood maximal function, and therefore that $\mathcal{H}_p f(x)$ is pointwise bounded by $\lesssim_d H^* f(x) + Mf(x)$, where the constant depends on d but is otherwise uniform in the coefficients of p . Uniformity of the L^p operator norms for $1 < p < \infty$ follows immediately.

After this the lemma was used successfully in a number of other cases, all concerned with the uniformity of the bounds in terms of the coefficients. As examples of the applications, we mention here uniformity results for oscillatory singular integrals with a polynomial phase [FGW12] (in particular with kernels of the form p.v. $e^{ip(s)}/s$) and uniform restriction estimates for curves with polynomial coordinates [DW10]. In this second example, the authors reduce proving the restriction estimates to proving an inequality with some geometrical flavour for certain Jacobians, thanks to a well known argument of Christ from [Chr85]. More precisely: the restriction estimates for a curve $\Gamma : [0, 1] \rightarrow \mathbb{R}^n$, whose coordinates $\Gamma_j(t)$ are polynomials of degree at most d ,

are stated in terms of the *affine arclength measure* $dv = dv_\Gamma$, which is given on test functions in $C([0, 1])$ by

$$v(\phi) := \int_0^1 \phi(\Gamma(t)) |\det(\Gamma'(t) \ \cdots \ \Gamma^{(n)}(t))|^{2/n(n+1)} dt;$$

let Φ_Γ be the map

$$\Phi_\Gamma(t_1, \dots, t_n) := \Gamma(t_1) + \dots + \Gamma(t_n),$$

then for $1 \leq p < \frac{n^2+2n}{n^2+2n-2}$ the inequality

$$\|\widehat{f}\Big|_\Gamma\|_{L^{q(p)}(\Gamma, dv)} \leq C_{p,n,d} \|f\|_{L^p(\mathbb{R}^n)},$$

with $q(p) := \frac{2p'}{n(n+1)}$, is implied by two facts:

a) Φ_Γ is injective;

b) $|d\Phi_\Gamma(t_1, \dots, t_n)| \geq C_{p,n,d} \prod_{j=1}^n \left(|\det(\Gamma'(t_j) \ \cdots \ \Gamma^{(n)}(t_j))|^{1/n} \prod_{k>j} |t_k - t_j| \right)$.

These do not hold in general, but Dendrinos and Wright are able to decompose $\mathbb{R} = \bigsqcup_{J \in \mathcal{J}} J$ in a finite number of intervals (whose number is bounded in terms of d and n only) in such a way that both properties hold in J^n - the key result being that *b*) holds uniformly in the coefficients of Γ . To achieve this they use in a fundamental way the lemma above - although this is not enough on its own, and they need a second, more elementary decomposition to combine with the one provided by Lemma 4.1; the two are then used in tandem, exploiting the affine invariance of the problem. This ends up being quite technical in that case; for the problem considered in this thesis fortunately it will not be so.

Proof of Lemma 4.1. Let $A > 1$ be a constant that will be fixed later (and will depend on the degree d of p). We decompose the polynomial into the product of its (complex) irreducible factors, thus obtaining

$$p(t) = c_d (t - \alpha_1) \cdot \dots \cdot (t - \alpha_d), \tag{4.1.1}$$

where we have ordered the (complex) roots α_j in such a way that $|\alpha_j| \leq |\alpha_{j+1}|$. We assume that it actually is $|\alpha_j| < |\alpha_{j+1}|$ for any $j = 1, \dots, d-1$ - the other cases following with minor modifications. Then consider the sets (each a symmetric double interval except for G_0) defined as follows: for $j \in \{1, \dots, d\}$, let

$$D_j := \{t \in \mathbb{R} \text{ s.t. } A^{-1}|\alpha_j| \leq |t| < A|\alpha_j|\},$$

and

$$G_j := \{t \in \mathbb{R} \text{ s.t. } A|\alpha_j| \leq |t| < A^{-1}|\alpha_{j+1}|\},$$

$$G_d := \{t \in \mathbb{R} \text{ s.t. } A|\alpha_d| \leq |t|\},$$

with the caveat that for $j < d$ they could possibly be empty (if it happens that $A|\alpha_j| > A^{-1}|\alpha_{j+1}|$, that is, the two consecutive roots are comparable in size); moreover, define

$$G_0 := \{t \in \mathbb{R} \text{ s.t. } |t| < A^{-1}|\alpha_1|\}.$$

Then it is immediate that D_j is of dyadic type for all $j \in \{1, \dots, d\}$, and they can easily be made all disjoint from each other (notice it can be $D_j \cap D_{j+1} \neq \emptyset$ only if $G_j = \emptyset$); so it suffices to look at the G_j sets.

For $j \in \{1, \dots, d-1\}$ such that $G_j \neq \emptyset$ and $t \in G_j$ we have by the definition that for $k \leq j$

$$|t|(1 - A^{-1}) \leq ||t| - |\alpha_k|| \leq |t - \alpha_k| \leq |t| + |\alpha_k| \leq |t|(1 + A^{-1});$$

on the other hand, for $k > j$ the inequalities are reversed, namely

$$|\alpha_k|(1 - A^{-1}) \leq ||\alpha_k| - |t|| \leq |t - \alpha_k| \leq |t| + |\alpha_k| \leq |\alpha_k|(1 + A^{-1}).$$

By combining the two, since

$$|p(t)| = |c_d| \left(\prod_{k \leq j} |t - \alpha_k| \right) \left(\prod_{k > j} |t - \alpha_k| \right)$$

we have that

$$|c_d| |t|^j \left(\prod_{k > j} |\alpha_k| \right) (1 + A^{-1})^d \geq |p(t)| \geq |c_d| |t|^j \left(\prod_{k > j} |\alpha_k| \right) (1 - A^{-1})^d. \quad (4.1.2)$$

Now, it is well known that

$$c_j := (-1)^{d-j} c_d \sum_{\substack{S \subset \{1, \dots, d\}, \\ \#S = d-j}} \prod_{i \in S} \alpha_i,$$

and since $G_j \neq \emptyset$ it is $|\alpha_k| > A^2 |\alpha_{k'}|$ for all $mk > j \geq k'$, and therefore by triangle inequality

$$\begin{aligned} |c_j| &\geq |c_d| \left(\prod_{k > j} |\alpha_k| - \sum_{\substack{S \subset \{1, \dots, d\}, \\ \#S = d-j, \\ S \neq \{j+1, \dots, d\}}} \prod_{i \in S} |\alpha_i| \right) \\ &\geq |c_d| \left(\prod_{k > j} |\alpha_k| - \binom{d}{j} A^{-2} \prod_{k > j} |\alpha_k| \right) \\ &\geq \frac{1}{2} |c_d| \prod_{k > j} |\alpha_k|, \end{aligned}$$

provided A is sufficiently large depending on d . Similarly,

$$|c_j| \leq \frac{3}{2} |c_d| \prod_{k > j} |\alpha_k|,$$

so that (4.1.2) actually implies

$$(1 + A^{-1})^d |c_j| |t|^j \geq |p(t)| \geq (1 - A^{-1})^d |c_j| |t|^j$$

when $t \in G_j$. If $j = d$ instead, then the same analysis shows that, correctly, if $t \in G_d$

then

$$|p(t)| \sim |c_d| |t|^d;$$

if $j = 0$ instead, then similarly for $t \in G_0$

$$|p(t)| \sim |c_d| \prod_{j=1}^d |\alpha_j| = |c_0|.$$

It remains to prove the bounds on the derivative of p . In order to do this, we differentiate the rhs of (4.1.1) and see that we can write

$$p'(t) = \sum_{k=1}^d \frac{p(t)}{(t - \alpha_k)};$$

therefore for $t \in G_j$, $j \in \{1, \dots, d\}$, by the inequalities above,

$$\begin{aligned} |p'(t)| &\geq |p(t)| \left| \sum_{k \leq j} \frac{1}{(t - \alpha_k)} \right| - \sum_{k > j} \frac{|p(t)|}{|t - \alpha_k|} \\ &\geq |p(t)| \left| \sum_{k \leq j} \Re \frac{1}{(t - \alpha_k)} \right| - \sum_{k > j} \frac{|p(t)|}{(1 - A^{-1})|\alpha_k|} \\ &\geq |p(t)| \left| \sum_{k \leq j} \frac{t - \Re \alpha_k}{t^2 - |\alpha_k|^2} \right| - \sum_{k > j} \frac{|p(t)|}{(1 - A^{-1})A|t|} \\ &\geq |p(t)| \frac{j}{(1 - A^{-1})|t|} - |p(t)| \frac{(d - j)}{(A - 1)|t|} \geq \frac{1}{2} \frac{|p(t)|}{|t|}, \end{aligned}$$

for A sufficiently large depending on d only. Similarly,

$$\begin{aligned} |p'(t)| &\leq |p(t)| \sum_{k \leq j} \frac{1}{|t - \alpha_k|} + \sum_{k > j} \frac{|p(t)|}{|t - \alpha_k|} \\ &\leq |p(t)| \left(\frac{j}{(1 - A^{-1})|t|} + \frac{d - j}{(1 - A^{-1})A|t|} \right) \leq 2 \frac{|p(t)|}{|t|}, \end{aligned}$$

for sufficiently large A depending only on d . This concludes the proof. \square

Remark 4.1. It is evident from the proof that in a gap type symmetric double interval (or simply ‘gap’) one can obtain estimates for all derivatives of p as well, not just the first derivative. However, we will not need any such estimates on the derivatives in this chapter; we have included them only to show the precision of the decomposition. Notice that the implicit constants in the statement of the lemma ($|p(t)| \sim |c_j| |t|^j$) are absolute, that is they do not depend on any parameter of the problem (provided one chooses A sufficiently large, depending on d , as the proof shows).

4.2 Oscillatory integral estimates

We will need in our analysis to provide uniform bounds for quantities of the form

$$\left| \int_{\theta < |s| \leq 1} e^{i\lambda(2ys + P(s, y-t))} \frac{ds}{s} \right|,$$

as encountered in the previous chapter, where the polynomial P was replaced by a generic analytic function φ . There we estimated them more or less ad hoc, and the key fact we used was Lemma 3.5 that provided lower bounds on the derivatives of the phase $2ys + \varphi(s, y-t)$, but these lower bounds depended on the precise function φ in some implicit way. Now we need more precise estimates in regards to the dependence on the coefficients, and what we will come to is the following

Proposition 4.2 ([NW77]). *Let d be a positive integer; then there exists $\beta = \beta(d) > 0$ such that, if we let $p(s) = \sum_{j=1}^d c_j s^j$, then for all $j \in \{1, \dots, d\}$, $\lambda > 0$ and $\theta \in (0, 1)$ such that $\lambda |c_j| \theta^j \geq 1$ we have*

$$\left| \int_{\theta}^1 e^{i\lambda p(s)} \frac{ds}{s} \right| \lesssim_d (\lambda |c_j| \theta^j)^{-\beta}.$$

Remark 4.2. We were not aware that this proposition already explicitly existed in the literature, specifically in the work of Nagel and Wainger referenced above, and thus have initially provided a proof of our own. However, since their presentation is more elegant, we have chosen to adapt their proof instead. At the core, the proofs are essentially the same though.

Remark 4.3. In order to provide some context and to illustrate the potential of the proposition above for applications, we briefly describe what was the work that originally motivated the proposition in [NW77]. There, the authors are concerned with the L^2 boundedness of singular integrals whose kernels are invariant with respect to prescribed multi-parameter groups of dilations; that is, if $\mathcal{D} < \text{GL}(\mathbb{R}^n)$ is such a group of dilations in any number of parameters, realized as a group of diagonal matrices with positive diagonal entries, then the kernel K must satisfy

$$\forall D \in \mathcal{D}, \forall x \neq 0, \quad K(Dx) = \frac{1}{\det D} K(x).$$

The way they proceed is to use an appropriate method of rotations to reduce the $L^2(\mathbb{R}^n)$ boundedness of $f \mapsto K * f$ to the boundedness of a multi-parameter Hilbert transform along the surfaces given by the orbits of a point in $\mathbb{R}^n \setminus \{0\}$ under the action of \mathcal{D} , that is operators of the form

$$\text{p.v.} \int \cdots \int f(x - \sigma(t_1, \dots, t_k)) \frac{dt_1}{t_1} \cdots \frac{dt_k}{t_k},$$

with $\sigma(t_1, \dots, t_k) := (\prod_{j=1}^k t_j^{\alpha_{1j}}, \dots, \prod_{j=1}^k t_j^{\alpha_{nj}})$, for $(\alpha_{ij})_{\substack{i=1, \dots, n \\ j=1, \dots, k}}$ a real matrix of rank k .

Then the multiplier of such an operator is given by a multi-parameter oscillatory integral analogue of our $m(\lambda; y, t)$ (the phase being a polynomial in several variables),

and the estimate provided by Proposition 4.2 is used to bound such multiplier. The estimate is applied to just one of the parameters and then integration in the others is performed, which explains their need for precise estimates that have minimal dependence on the coefficients.

Remark 4.4. Observe that when $p(s) = As^k$ for $k > 1$ odd, a simple integration by parts argument shows that

$$\left| \int_{\theta}^1 e^{i\lambda As^k} \frac{ds}{s} \right| \lesssim (\lambda A \theta^k)^{-1}.$$

The exponent β that the proof provided below gives in this case is instead $-1/k^2$, which is much worse; but we are not interested in the optimal rate of decay here, as any positive exponent will do for us, as long as it is uniformly bounded from below in terms of d only.

The proof of Proposition 4.2 is based on Van der Corput's lemma and the following lemma that provides a lower bound on the derivatives of certain reparametrizations of p .

Lemma 4.3. *Let d be a positive integer. There exists $\delta = \delta(d) > 0$ such that the following holds: let $p(s) = \sum_{j=1}^d c_j s^j$ be any polynomial and define for $k \in \{1, \dots, d\}$ the functions q_k by $q_k(t) := p(t^{1/k})$. Then for all k and for all $|t| \in (0, 1)$*

$$\max_{1 \leq j \leq d} |q_k^{(j)}(t)| \geq \delta |c_k|.$$

Remark 4.5. The proposition holds more in general for functions of the form $\phi(s) = \sum_{j=1}^d c_j s^{\alpha_j}$, with $\alpha_j \in \mathbb{R}$, $\alpha_1 < \alpha_2 < \dots < \alpha_d$ and provided $\alpha_j > 0$, modifying the definition of the q_k 's as $q_k(t) := \phi(s^{1/\alpha_k})$. The proof needs no modifications worthy of remark.

It should be noted that the definition of q_k is heuristically motivated by the fact that the kernel ds/s behaves well with respect to changes of variable of the form $s \rightarrow s^\alpha$.

Proof of Lemma 4.3. Let

$$\mathbf{q}(t) := \begin{pmatrix} q_k'(t) \\ q_k''(t) \\ \vdots \\ q_k^{(d)}(t) \end{pmatrix}, \quad \mathbf{c} := \begin{pmatrix} c_k \\ c_1 \\ \vdots \\ c_d \end{pmatrix} =: \begin{pmatrix} c_k \\ \tilde{\mathbf{c}} \end{pmatrix};$$

it suffices to prove that for all $t \in (0, 1)$ we have that

$$\|\mathbf{q}(t)\| \geq \delta |c_k|,$$

where $\|\cdot\|$ is a fixed norm on \mathbb{R}^d . We observe that

$$q'_k(t) = c_k + \sum_{i \neq k} c_i \frac{i}{k} t^{i/k-1},$$

$$q_k^{(j)}(t) = \sum_{i \neq k} c_i A_{ij}^k t^{i/k-j},$$

where $A_{ij}^k := \binom{i}{k} \binom{i}{k} - 1 \cdots \binom{i}{k} - j + 1$. Then by defining the matrix

$$\mathbf{T}(t) := \begin{pmatrix} 1 & A_{11}^k t^{1/k-1} & A_{21}^k t^{2/k-1} & \cdots & A_{d1}^k t^{d/k-1} \\ 0 & A_{12}^k t^{1/k-2} & A_{22}^k t^{2/k-2} & \cdots & A_{d2}^k t^{d/k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & A_{1d}^k t^{1/k-d} & A_{2d}^k t^{2/k-d} & \cdots & A_{dd}^k t^{d/k-d} \end{pmatrix}$$

$$=: \begin{pmatrix} 1 & \mathbf{w}(t) \\ 0 & \tilde{\mathbf{T}}(t) \\ \vdots & \\ 0 & \end{pmatrix}$$

we can rewrite the expressions for the derivatives of q_k as

$$\mathbf{q}(t) = \mathbf{T}(t)\mathbf{c}.$$

We therefore have

$$\|\mathbf{q}(t)\| \sim |c_k + \mathbf{w}(t) \cdot \tilde{\mathbf{c}}| + \|\tilde{\mathbf{T}}(t)\tilde{\mathbf{c}}\|;$$

if $|\mathbf{w}(t) \cdot \tilde{\mathbf{c}}| < |c_k|/2$ or $|\mathbf{w}(t) \cdot \tilde{\mathbf{c}}| > 2|c_k|$ then clearly $\|\mathbf{q}(t)\| \gtrsim |c_k|$. Suppose then that $|\mathbf{w}(t) \cdot \tilde{\mathbf{c}}| \sim |c_k|$. Notice that the m -th row of $\tilde{\mathbf{T}}(t)$ is given by

$$\frac{1}{t^m} \left(\prod_{\ell=1}^m (D - \ell I) \right) \mathbf{w}(t),$$

where D is the diagonal matrix given by

$$\begin{pmatrix} \frac{1}{k} & & \\ & \ddots & \\ & & \frac{d}{k} \end{pmatrix}.$$

It then suffices to show that for some m it must be that

$$\left| \mathbf{w}(t) \cdot \left(\prod_{\ell=1}^m (D - \ell I) \right) \tilde{\mathbf{c}} \right| \gtrsim |\mathbf{w}(t) \cdot \tilde{\mathbf{c}}|,$$

since $t^{-m} \geq 1$. Suppose by contradiction that this is not the case, that is for any $\varepsilon > 0$ we can find $\tilde{\mathbf{c}}$ such that for all m

$$\left| \mathbf{w}(t) \cdot \left(\prod_{\ell=1}^m (D - \ell I) \right) \tilde{\mathbf{c}} \right| < \varepsilon |\mathbf{w}(t) \cdot \tilde{\mathbf{c}}|.$$

If $\{\mathbf{e}_i\}_{i \neq k}$ is a basis for the vectors $\tilde{\mathbf{c}}$, then observe

$$\left(\prod_{\ell=1}^m (D - \ell I) \right) \mathbf{e}_i = \frac{k}{i} A_{i,m+1}^k \mathbf{e}_i;$$

it follows that if the matrix $\Lambda := \left(\frac{k}{i} A_{i,m+1}^k \right)_{\substack{i \neq k, \\ m=1, \dots, d-1}}$ is invertible, then we can write

$$\tilde{\mathbf{c}} = \sum_m u_m \left(\prod_{\ell=1}^m (D - \ell I) \right) \tilde{\mathbf{c}},$$

where u_m are the coordinates of \mathbf{u} that satisfies

$$\Lambda \mathbf{u} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix};$$

since Λ depends only on d , it is clear that $\|\mathbf{u}\| \lesssim_d 1$, and therefore we would have

$$\begin{aligned} |\mathbf{w}(t) \cdot \tilde{\mathbf{c}}| &\leq \sum_m |u_m| \left| \mathbf{w}(t) \cdot \left(\prod_{\ell=1}^m (D - \ell I) \right) \tilde{\mathbf{c}} \right| \\ &\lesssim \varepsilon \|\mathbf{u}\| |\mathbf{w}(t) \cdot \tilde{\mathbf{c}}| \lesssim_d \varepsilon |\mathbf{w}(t) \cdot \tilde{\mathbf{c}}|, \end{aligned}$$

which is a contradiction if ε is too small.

Thus it remains to show that Λ is invertible. To do so, suppose there exists \mathbf{u} such that $\Lambda \mathbf{u} = \mathbf{0}$. But, by definition of the $A_{i,m+1}^k$'s, this means that the function given by

$$\phi(t) := \sum_{j \neq k} u_j t^{j/k-1}$$

is such that $\phi'(1) = \dots = \phi^{(d)}(1) = 0$, and it is immediate to see that this implies $\phi \equiv 0$. \square

We are now ready to prove the proposition.

Proof of Proposition 4.2. Let k be such that $c_k \neq 0$. Consider first the integral function

$$F(u) := \int_0^u e^{i\lambda p(s)} ds,$$

of which we want to estimate the absolute value in the case where $\lambda |c_k| u^k \geq 1$. By the substitution $s = us'$ we have

$$F(u) := u \int_0^1 e^{i\lambda p(us')} ds',$$

and we let $p^u(s) := p(us)$. Now we substitute $s' = t^{1/k}$ and we get

$$F(u) = \frac{u}{k} \int_0^1 e^{i\lambda q_k^u(t)} t^{1/k-1} dt.$$

We claim that we can partition $(0, 1)$ into at most $N(d)$ intervals J , so that on each J either

- i) $(q_k^u)'$ is monotone and $|(q_k^u)'(t)| \geq \delta u^k |c_k|$ for all $t \in J$,
- ii) or there is $j = j(J) > 1$ s.t. $|(q_k^u)^{(j)}(t)| \geq \delta u^k |c_k|$ for all $t \in J$.

Indeed, Lemma 4.3 shows that for every t at least one of the two lower bounds must hold, and this gives a partition of $(0, 1)$ into intervals by continuity. That the cardinality of the partition is bounded by a quantity depending only on d follows from the fact that q_k^u behaves essentially like a polynomial in this regard, and in particular it is easy to see (for example by induction in the degree d) first that $(q_k^u)''$ has at most d zeroes, and second that the sets $\{t \in (0, 1) \text{ s.t. } |(q_k^u)^{(j)}(t)| > |(q_k^u)^{(\ell)}(t)|\}$ can consist of at most $2d + 1$ intervals. This suffices to prove the claim.

Therefore it suffices, in order to estimate $|F(u)|$, to estimate one such interval $J = (a, b)$. Let $j(J) = j$, then we have by Van der Corput's lemma and the claim that

$$\begin{aligned} \frac{u}{k} \left| \int_a^b e^{i\lambda q_k^u(t)} t^{1/k-1} dt \right| &\lesssim_d \frac{u}{k} (\lambda |c_k| u^k)^{-1/j} \left(b^{1/k-1} + \int_a^b \left| \left(\frac{1}{k} - 1\right) t^{1/k-2} \right| dt \right) \\ &\lesssim_d u (\lambda |c_k| u^k)^{-1/j} a^{1/k-1}. \end{aligned}$$

If $a < \frac{1}{100}$ then we split the integral as $\int_a^b = \int_a^\varepsilon + \int_\varepsilon^b$ for ε to be chosen, and then we have

$$\frac{u}{k} \left| \int_a^\varepsilon e^{i\lambda q_k^u(t)} t^{1/k-1} dt \right| \leq \frac{u}{k} \int_a^\varepsilon t^{1/k-1} dt \leq u \varepsilon^{1/k},$$

and as seen above

$$\frac{u}{k} \left| \int_\varepsilon^b e^{i\lambda q_k^u(t)} t^{1/k-1} dt \right| \lesssim_d u (\lambda |c_k| u^k)^{-1/j} \varepsilon^{1/k-1};$$

now choose $\varepsilon \sim (\lambda |c_k| u^k)^{-1/j}$ and thus if $a < \frac{1}{100}$

$$\frac{u}{k} \left| \int_\varepsilon^b e^{i\lambda q_k^u(t)} t^{1/k-1} dt \right| \lesssim_d u (\lambda |c_k| u^k)^{-1/jk}.$$

Finally, if $a > \frac{1}{100}$, we simply have

$$\frac{u}{k} \left| \int_a^b e^{i\lambda q_k^u(t)} t^{1/k-1} dt \right| \lesssim_d u (\lambda |c_k| u^k)^{-1/j} \leq u (\lambda |c_k| u^k)^{-1/jk}.$$

Therefore, summing all the contributions from all the intervals J , we have shown

$$|F(u)| \lesssim_d N(d) \max_{1 \leq j \leq d} u (\lambda |c_k| u^k)^{-1/jk} \lesssim_d u (\lambda |c_k| u^k)^{-1/d^2}. \quad (4.2.1)$$

Now we can address the original oscillatory integral by means of integration by parts, thus writing

$$\int_\theta^1 e^{i\lambda p(s)} \frac{ds}{s} = \int_1^{\theta^{-1}} e^{i\lambda p^\theta(s)} \frac{ds}{s} = \int_1^{\theta^{-1}} \frac{F'(s)}{s} ds = \left[\frac{F(s)}{s} \right]_1^{\theta^{-1}} + \int_1^{\theta^{-1}} \frac{F(s)}{s^2} ds$$

(notice now F is defined in terms of p^θ rather than p , the polynomial whose k -th coefficient is $c_k\theta^k$). Then by (4.2.1) (and $0 < \theta < 1$)

$$\begin{aligned} |F(1)| &\lesssim_d (\lambda|c_k|\theta^k)^{-1/d^2}, \\ \frac{|F(\theta^{-1})|}{\theta^{-1}} &\lesssim_d (\lambda|c_k|\theta^k\theta^{-k})^{-1/d^2} \leq (\lambda|c_k|\theta^k)^{-1/d^2}, \\ \int_1^{\theta^{-1}} \frac{|F(s)|}{s^2} ds &\lesssim_d \int_1^{\theta^{-1}} \frac{(\lambda|c_k|\theta^k s^k)^{-1/d^2}}{s} ds \\ &\lesssim_d (\lambda|c_k|\theta^k)^{-1/d^2} \int_1^{\theta^{-1}} s^{-k/d^2-1} ds \lesssim_d (\lambda|c_k|\theta^k)^{-1/d^2}, \end{aligned}$$

and the estimate of Proposition 4.2 is thus proven with $\beta = 1/d^2$. \square

Before moving on to the next subsection, we show a simple corollary of Proposition 4.2 that we anticipated in the proof of Lemma 3.3 and that will be useful again to us. Historically, this fact was first proved in [SW70], which is antecedent to Proposition 4.2; the original motivation was once again the study of certain multipliers.

Corollary 4.4 (of Proposition 4.2; [SW70]). *Let $P(s, t) \in \mathbb{R}[s, t]$ and let*

$$m(\lambda; y, t) := \int_{|s| \leq 1} e^{i\lambda(2ys + P(s, y-t))} \frac{ds}{s};$$

then for all λ, y, t ,

$$|m(\lambda; y, t)| \lesssim_d 1.$$

Proof. Let $\Phi(s) := 2ys + P(s, y-t)$; we can bound

$$\begin{aligned} |m(\lambda; y, t)| &\leq \left| \int_{|s| \leq \theta_1} e^{i\lambda\Phi(s)} - e^{i\lambda P(s, y-t)} \frac{ds}{s} \right| \\ &\quad + \left| \int_{\theta_1 < |s| \leq 1} e^{i\lambda\Phi(s)} \frac{ds}{s} \right| + \left| \int_{|s| \leq \theta_1} e^{i\lambda P(s, y-t)} \frac{ds}{s} \right| \\ &=: I_1 + II_1 + III_1. \end{aligned}$$

We have as usual

$$I_1 \lesssim \lambda|y|\theta_1,$$

and by Proposition 4.2

$$II_1 \lesssim_d (\lambda|y|\theta_1)^{-\beta};$$

therefore, by choosing $\theta_1 = \min\{(\lambda|y|)^{-1}, 1\}$ we have $I_1 + II_1 \lesssim_d 1$. As for III_1 , we pick any one non null monomial $s^k Q_k(y-t)$ in $P(s, y-t)$ and repeat the same steps: we bound

$$\begin{aligned} III_1 &\leq \left| \int_{|s| \leq \theta_2} e^{i\lambda P(s, y-t)} - e^{i\lambda(P(s, y-t) - s^k Q_k(y-t))} \frac{ds}{s} \right| \\ &\quad + \left| \int_{\theta_2 < |s| \leq \theta_1} e^{i\lambda P(s, y-t)} \frac{ds}{s} \right| + \left| \int_{|s| \leq \theta_2} e^{i\lambda(P(s, y-t) - s^k Q_k(y-t))} \frac{ds}{s} \right| \\ &=: I_2 + II_2 + III_2. \end{aligned}$$

Then we have

$$I_2 \lesssim \lambda |Q_k(y-t)| \theta_2^k$$

and by Proposition 4.2, provided we choose $\theta_2 < \theta_1$,

$$II_2 \lesssim_d (\lambda |Q_k(y-t)| \theta_2^k)^{-\beta}$$

(notice one needs to rescale with the change of variable $s = \theta_2 s'$ to apply the proposition directly, but this changes the polynomial in the phase by multiplying every coefficient of power s^k by a factor θ_2^k , and the two effects cancel each other out). Now choose $\theta_2 = \min\{(\lambda |Q_k(y-t)|)^{-1/k}, \theta_1\}$, so that¹ $I_2 + II_2 \lesssim_d 1$, and we are left with estimating III_2 . But this is of the same form as III_1 , except that the phase has one less monomial; therefore we can apply the procedure iteratively collecting at most $O_d(1)$ at each step, until we arrive after some n steps (where obviously $n \leq d$) at a term

$$III_n = \left| \int_{|s| \leq \theta_n} e^{i\lambda s^k Q_k(y-t)} \frac{ds}{s} \right|$$

(this k is different than the one above), which we have to bound. Notice that necessarily $Q_k(y-t) \neq 0$. If k is even then the integrand is odd and therefore the integral is identically zero and there is nothing to bound; assume therefore that k is odd. Then we can use the change of variable $s = u^{1/k} / (\lambda |Q_k(y-t)|)$ to show that

$$III_n = \frac{1}{k} \left| \int_{|u| \leq \lambda |Q_k(y-t)| \theta_n^k} e^{iu \operatorname{sgn}(Q_k(y-t))} \frac{du}{u} \right|,$$

and thus we can have two estimates:

$$III_n = \frac{1}{k} \left| \int_{|u| \leq \lambda |Q_k(y-t)| \theta_n^k} e^{iu \operatorname{sgn}(Q_k(y-t))} - 1 \frac{du}{u} \right| \lesssim \lambda |Q_k(y-t)| \theta_n^k,$$

and by integration by parts

$$\begin{aligned} III_n &\leq \frac{\pi}{k} + \frac{1}{k} \left| \int_{|u| > \lambda |Q_k(y-t)| \theta_n^k} e^{iu \operatorname{sgn}(Q_k(y-t))} \frac{du}{u} \right| \\ &\lesssim 1 + \frac{1}{\lambda |Q_k(y-t)| \theta_n^k} + \left| \int_{|u| > \lambda |Q_k(y-t)| \theta_n^k} \frac{e^{iu \operatorname{sgn}(Q_k(y-t))}}{u^2} du \right| \\ &\lesssim 1 + (\lambda |Q_k(y-t)| \theta_n^k)^{-1}. \end{aligned}$$

By these two estimates, it follows immediately that $III_n \lesssim 1$, thus concluding the proof of the corollary. \square

¹Notice that if $\theta_2 = \theta_1$ then $II_2 \equiv 0$.

4.3 Main result

Let $P(s, t)$ be a polynomial in two variables of degree d in s , and let Q_k be the polynomials defined by the identity

$$P(s, t) =: \sum_{k=0}^d s^k Q_k(t).$$

As in the previous chapter, §3.1.1, we reduce the problem of establishing the $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ boundedness of the singular integral

$$H_P f(x, y, z) := \text{p.v.} \int \int_{|s|, |t| \leq 1} f((x, y, z) \cdot (s, t, P(s, t))^{-1}) \frac{ds}{s} \frac{dt}{t}$$

to establishing the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ boundedness of the singular integral

$$T_P^\lambda \phi(y) := \text{p.v.} \int_{|y-t| \leq 1} m(\lambda; y, t) \frac{\phi(t)}{y-t} dt$$

uniformly in $\lambda > 0$, where m is given by the oscillatory integral

$$m(\lambda; y, t) := \text{p.v.} \int_{|s| \leq 1} e^{i\lambda((y+t)s + P(s, t-y))} \frac{ds}{s};$$

this time though, for a suitable class of polynomials, we will want the estimates on $\|T_P^\lambda\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})}$ to be uniform in the coefficients of P too, as P ranges inside this class and his degree stays bounded. To be more explicit, let

$$\mathcal{E}_d := \{P(s, t) = \sum_{k, \ell \text{ s.t. } k+\ell \leq d} c_{k\ell} s^k t^\ell \in \mathbb{R}[s, t] \text{ s.t. if } c_{k\ell} \neq 0 \text{ then } k \text{ or } \ell \text{ is even}\}.$$

We will prove

Theorem 4.5. *Let d be a positive integer. Then*

$$\sup_{P \in \mathcal{E}_d} \sup_{\lambda > 0} \|T_P^\lambda\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} < \infty.$$

Equivalently,

$$\sup_{P \in \mathcal{E}_d} \|H_P\|_{L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)} < \infty.$$

We will also show that the given class is in a sense the largest possible, as Example 4.1 below will suggest. We have indeed

Proposition 4.6. *Let V be a subspace of $\mathbb{R}_d[s, t]$, the space of polynomials of degree at most d , such that $\mathbb{R}_d[s, t] \supseteq V \supset \mathcal{E}_d$, and such that the second inclusion is strict. Then*

$$\sup_{P \in V} \sup_{\lambda > 0} \|T_P^\lambda\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} = \infty.$$

As a warm up to the proof and as just anticipated above, we consider the following example that highlights the importance of the condition defining \mathcal{E}_d .

Example 4.1. Consider the polynomial $P(s, t) = -st + c_0 s^k t^\ell$ with both k and ℓ odd. We claim that the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ norm of T_P^λ grows like $\log|c_0|$ for $|c_0| \rightarrow \infty$. The phase of the integrand of m is given by

$$\Phi(s) = 2ys + c_0 s^k (y - t)^\ell;$$

define then the quantity

$$\mathcal{A}(j) = \frac{\lambda |c_0| 2^{-j\ell}}{(\lambda |y|)^k}$$

for $j \gg n$, where $|y| \sim 2^{-n}$, $|y - t| \sim 2^{-j}$. The only critical point of the phase is given by

$$\eta := \left(-\frac{2y}{kc_0(y-t)^\ell} \right)^{1/(k-1)},$$

when defined; notice in particular $|\eta| \sim (|y|/2^{-j\ell})^{1/(k-1)}$. Let J_0 be the set of indices j such that the critical point is outside the region of integration, that is

$$J_0 := \left\{ j \gg n \vee 0, \text{ s.t. } \frac{|y|}{|c_0| 2^{-j\ell}} \gtrsim 1 \right\};$$

thus if $j \in J_0$ one has $|\Phi'(s)| \gtrsim |y|$ for all $s \in [-1, 1]$. Then we can show that if $j \in J_0$, $|y - t| \sim 2^{-j}$, we have for some $\sigma > 0$ depending only on k

$$\left| m(\lambda; y, t) - \int_{|s| \leq 1} e^{i\lambda 2ys} \frac{ds}{s} \right| \lesssim \mathcal{A}(j)^\sigma \wedge \mathcal{A}(j)^{-\sigma}.$$

Indeed, if $\mathcal{A}(j) > 1$, simply bound the difference on the left by $O(\lambda |c_0| 2^{-j\ell})$, and notice that

$$\lambda |c_0| 2^{-j\ell} \mathcal{A}(j)^{1/(k-1)} = \mathcal{A}(j)^{k/(k-1)} (\lambda |y|)^k = \left(\frac{|c_0| 2^{-j\ell}}{|y|} \right)^{k/(k-1)},$$

and the last quantity is $\ll 1$ since $j \in J_0$, so that $O(\lambda |c_0| 2^{-j\ell}) = O(\mathcal{A}(j)^{-1/(k-1)})$. As for when $\mathcal{A}(j) \leq 1$, we split the contributions as

$$\begin{aligned} \left| m(\lambda; y, t) - \int_{|s| \leq 1} e^{i\lambda 2ys} \frac{ds}{s} \right| &\leq \left| \int_{|s| \leq \theta} e^{i\lambda \Phi(s)} \frac{ds}{s} - \int_{|s| \leq \theta} e^{i\lambda 2ys} \frac{ds}{s} \right| \\ &\quad + \left| \int_{\theta < |s| \leq 1} e^{i\lambda \Phi(s)} \frac{ds}{s} \right| + \left| \int_{\theta < |s| \leq 1} e^{i\lambda 2ys} \frac{ds}{s} \right| \\ &=: I + II + III; \end{aligned} \tag{4.3.1}$$

then we have by simple domination that $I = O(\lambda |c_0| 2^{-j\ell} \theta^k)$, and by Van der Corput's lemma and integration by parts the estimate for Φ' (notice it is monotone too) gives for $j \in J_0$ that $II + III = O((\lambda |y| \theta)^{-1})$. Optimization in θ (which leads to choose $\theta \sim \frac{1}{\lambda |y|} \mathcal{A}(j)^{-1/(k+1)}$) gives then

$$I + II + III \lesssim \mathcal{A}(j)^{1/(k+1)},$$

as claimed.

We then study what happens when $j \notin J_0$. In the case $\mathcal{A}(j) < 1$ we claim we can show for some $\sigma' > 0$ that

$$\left| m(\lambda; y, t) - \int_{|s| \leq 1} e^{i\lambda 2ys} \frac{ds}{s} \right| \lesssim \mathcal{A}(j)^{\sigma'},$$

or in other words that the estimate above still holds. The presence of a critical point in the interval $[-1, 1]$ makes the application of Van der Corput's lemma less trivial in this case. We choose two numbers μ, ν such that $0 < \nu < 1/k < \mu < 1/(k-1)$ and define

$$\theta_1 := \frac{1}{\lambda|y|} \mathcal{A}(j)^{-\nu}, \theta_2 := \frac{1}{\lambda|y|} \mathcal{A}(j)^{-\mu};$$

then $\theta_1 < \theta_2$ and since $|\eta| \sim \frac{1}{\lambda|y|} \mathcal{A}(j)^{-1/(k-1)}$ we have that the interval $(-\theta_2, \theta_2)$ does not contain critical points for Φ . We split accordingly

$$\begin{aligned} \left| m(\lambda; y, t) - \int_{|s| \leq 1} e^{i\lambda 2ys} \frac{ds}{s} \right| &\leq \left| \int_{|s| \leq \theta_1} e^{i\lambda \Phi(s)} \frac{ds}{s} - \int_{|s| \leq \theta_1} e^{i\lambda 2ys} \frac{ds}{s} \right| \\ &\quad + \left| \int_{\theta_1 < |s| \leq \theta_2} e^{i\lambda \Phi(s)} \frac{ds}{s} \right| + \left| \int_{\theta_1 < |s| \leq 1} e^{i\lambda 2ys} \frac{ds}{s} \right| \\ &\quad + \left| \int_{\theta_2 < |s| \leq 10|\eta|} e^{i\lambda \Phi(s)} \frac{ds}{s} \right| + \left| \int_{10|\eta| < |s| \leq 1} e^{i\lambda \Phi(s)} \frac{ds}{s} \right| \\ &=: I + II + III + IV + V. \end{aligned}$$

We can bound by simple domination $I \lesssim \lambda|c_0|2^{-j\ell}\theta_1^k = O(\mathcal{A}(j)^{1-\nu k})$; in the interval $(\theta_1, \theta_2]$ we can bound $|\Phi'(s)| \gtrsim |y|$ and therefore by Corollary 2.2 we have $II \lesssim (\lambda|y|\theta_1)^{-1} = \mathcal{A}(j)^\nu$, and similarly for III ; moreover, in $(10|\eta|, 1]$ we have $|\Phi'(s)| \gtrsim |y|$ too, therefore by the same argument we have $V \lesssim (\lambda|y||\eta|)^{-1} = \mathcal{A}(j)^{1/(k-1)}$. These contributions are all fine, and thus it suffices to tackle IV , where there is a critical point. In this case the best one can say is that $|\Phi^{(k)}(s)| = k!|c_0||y-t|^\ell \sim |c_0|2^{-j\ell}$ (for all s), and by Corollary 2.2 then $IV \lesssim (\lambda|c_0|2^{-j\ell}\theta_2^k)^{-1/k} = \mathcal{A}(j)^{\mu-1/k}$, which is fine too.

It remains to see what happens when $j \notin J_0$ and $\mathcal{A}(j) > 1$. In this case we claim that we have to drop the $2ys$ term instead from the phase, and more precisely that for some $\sigma'' > 0$

$$\left| m(\lambda; y, t) - \int_{|s| \leq 1} e^{i\lambda c_0 s^k (y-t)^\ell} \frac{ds}{s} \right| \lesssim \mathcal{A}(j)^{-\sigma''}.$$

Indeed, split into three parts as in (4.3.1). Then by simple domination we bound $I \lesssim \lambda|y|\theta$, and by Corollary 2.2 using the k -th derivative lower bound we have $II + III \lesssim (\lambda|c_0|2^{-j\ell}\theta^k)^{-1/k}$. We optimize in θ by choosing $\theta \sim \frac{1}{\lambda|y|} \mathcal{A}(j)^{-1/(2k)}$, which gives $I + II + III \lesssim \mathcal{A}(j)^{-1/(2k)}$ as claimed.

Now we wrap up: we can bound²

$$\begin{aligned} & \left| T_P^\lambda \phi(y) - \left(\int_{|s| \leq 1} e^{i\lambda 2ys} \frac{ds}{s} \right) \sum_{\substack{j \in J_0 \\ \text{or } \mathcal{A}(j) < 1}} \int_{|y-t| \sim 2^{-j}} \frac{\phi(t)}{y-t} dt \right. \\ & \quad \left. - \sum_{\substack{j \notin J_0 \\ \text{and } \mathcal{A}(j) > 1}} \int_{|y-t| \sim 2^{-j}} \left(\int_{|s| \leq 1} e^{i\lambda c_0 s^k (y-t)^\ell} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt \right| \\ & \lesssim \left(\sum_j \mathcal{A}(j)^\sigma \wedge \mathcal{A}(j)^{-\sigma} \right) M\phi(y) \lesssim M\phi(y), \end{aligned}$$

and since $\int_{|s| \leq 1} e^{i\lambda 2ys} \frac{ds}{s} = O(1)$ this means that we can bound pointwise

$$\left| T_P^\lambda \phi(y) - \sum_{\substack{j \notin J_0 \\ \text{and } \mathcal{A}(j) > 1}} \int_{|y-t| \sim 2^{-j}} \left(\int_{|s| \leq 1} e^{i\lambda 2ys} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt \right| \lesssim M\phi + H^* \phi,$$

where the right hand side has $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ norm uniformly bounded in c_0 . On the other hand, since k is odd, we have seen in Chapter 1 that for some constant C we have that

$$\int_{|s| \leq 1} e^{i\lambda c_0 s^k (y-t)^\ell} \frac{ds}{s} = \begin{cases} O(\lambda |c_0| 2^{-j\ell}) \\ C \operatorname{sgn}(y-t) + O((\lambda |c_0| 2^{-j\ell})^{-1}), \end{cases}$$

and therefore (since the errors above are summable in j) showing the lack of uniform boundedness of T_P^λ is reduced to showing that the operator given by

$$\begin{aligned} & \sum_{\substack{j \notin J_0, \\ \mathcal{A}(j) > 1, \lambda |c_0| 2^{-j\ell} > 1}} \int_{|y-t| \sim 2^{-j}} \operatorname{sgn}(y-t) \frac{\phi(t)}{y-t} dt \\ & = \sum_{\substack{j \notin J_0, \\ \mathcal{A}(j) > 1, \lambda |c_0| 2^{-j\ell} > 1}} \int_{|y-t| \sim 2^{-j}} \frac{\phi(t)}{|y-t|} dt \end{aligned}$$

is not uniformly bounded in c_0 . The conditions on the j 's boil down to

$$\lambda |c_0| 2^{-j\ell} > \max\{1, (\lambda |y|)^k\},$$

so if $\lambda |y| < 1$ we get the integration interval

$$(\lambda |c_0|)^{-1/\ell} < |y-t| \ll |y| \wedge 1 < 1/\lambda,$$

and if $\lambda |y| \geq 1$ we get instead

$$\frac{(\lambda |y|)^{k/\ell}}{\lambda |c_0|} < |y-t| \leq \min\{2^{-C_0} |y|, 1\}.$$

²See §4.4 for a more detailed explanation of the pointwise bounds.

Since the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ norm of the operator

$$\int_{A \leq |y-t| \leq B} \frac{|\phi(t)|}{|y-t|} dt$$

is proportional to $\log(A/B)$, we see that for λ, n fixed the norm of T_λ^P is bounded from below by $\log|c_0|$ for large c_0 .

Remark 4.6. Observe that the estimation above works out just fine, with minor modifications, for the case $P(s, t) = c_0 s^k t^\ell$ too (that is, without the term $-st$). Indeed, it suffices to introduce the quantity

$$\mathcal{B}(j) := \frac{2^{-j}}{|y|}$$

and show that replacing $(y+t)s$ by $2ys$ only gives rise to an error that is summable in j uniformly in d , and therefore is acceptable. See §4.4.2 below for details, where we perform exactly this replacement for the general form of the operator.

The next sections are devoted to the proofs of Theorem 4.5 and Proposition 4.6.

4.4 Proof of Theorem 4.5

We are now ready to prove the main theorem in this chapter.

We write $P(s, t) =: \sum_{k=1}^d s^k Q_k(t)$, where $P \in \mathcal{E}_d$; let also

$$Q_k(t) =: \sum_{\ell \leq d-k} c_{k\ell} t^\ell.$$

4.4.1

We apply Lemma 4.1 to every polynomial Q_k (whose degrees are uniformly bounded by d), which gives us for each $k \in \{1, \dots, d\}$ two families of symmetric double intervals $\{D_i^k\}_{i \in \mathcal{I}_k}, \{G_j^k\}_{j \in \mathcal{F}_k}$ (respectively, the dyadic type intervals and the gap type intervals) with the properties

- i) the cardinality of each family is bounded in terms of d only: $|\mathcal{I}_k| + |\mathcal{F}_k| \leq d - k + 1 \leq d + 1$;
- ii) together they form a disjoint partition of the real line: $\mathbb{R} = \bigsqcup_{i \in \mathcal{I}_k} D_i^k \sqcup \bigsqcup_{j \in \mathcal{F}_k} G_j^k$;
- iii) every symmetric double interval D_i^k is of the form $[a, A^2 a) \cup [-A^2 a, a)$ for some $a > 0$ (where $A = A(d)$ is the constant given by the lemma);
- iv) on every interval G_j^k the polynomial Q_k behaves like one of its monomials: $\forall j \in \mathcal{F}_k, \exists \ell \leq d - k$ s.t. $\forall t \in G_j^k, |Q_k(t)| \sim |c_{k\ell}| |t|^\ell$.

Now we build from these the family of symmetric double intervals of the form

$$I^1 \cap I^2 \cap \dots \cap I^d \cap [-1, 1],$$

where $I^k \in \{D_i^k\}_{i \in \mathcal{I}_k} \cup \{G_i^k\}_{i \in \mathcal{F}_k}$. These intervals form a disjoint partition of $[-1, 1]$ (by property *ii*) and their number is bounded by $(d+1)^d$ (by property *i*). Therefore, it will suffice to prove bounds uniform in the coefficients (for d fixed) for the operators \tilde{T}_λ^P , defined exactly like T_P^λ except for the fact that the integration in t is restricted to J , where J is of the form above.

Now, if in $J := I^1 \cap I^2 \cap \dots \cap I^d \cap [-1, 1]$ at least one of the I^k is of dyadic type (denote it by $I^{k_0} = [a, A^2a) \cup [-A^2a, -a)$, we can bound by Corollary 4.4

$$\begin{aligned} |\tilde{T}_P^\lambda \phi(y)| &= \left| \text{p.v.} \int_{y-t \in J} m(\lambda; y, t) \frac{\phi(t)}{y-t} dt \right| \leq \int_{y-t \in J} |m(\lambda; y, t)| \frac{|\phi(t)|}{|y-t|} dt \\ &\lesssim_d \int_{y-t \in I^{k_0}} \frac{|\phi(t)|}{|y-t|} dt \leq \frac{1}{a} \int_{-A^2a}^{A^2a} |\phi(y+t)| dt \lesssim_d M\phi(y). \end{aligned}$$

Therefore, the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ norm of \tilde{T}_P^λ for J as above is uniformly bounded in the coefficients and in λ , when d is fixed.

4.4.2

It therefore remains to prove the theorem for $J = I^1 \cap I^2 \cap \dots \cap I^d \cap [-1, 1]$ where for all k the symmetric double interval I^k is of gap type. This implies that for every $k \in \{1, \dots, d\}$ there exists $\ell_k \geq 0$ such that $|Q_k(y-t)| \sim |c_k \ell_k| |t|^{\ell_k}$; notice that, by the proof of Lemma 4.1, $s^k t^{\ell_k}$ must be a monomial appearing in P , and given that $P \in \mathcal{E}_d$ we have that at least one amongst k and ℓ_k will always be even.

Now we localize the singular integral as done before (see Chapter 3, §3.1.2.1): it suffices then to consider the operator given by

$$\chi(2^n y) \sum_{j \geq \max\{n+C_0, 0\}} \text{p.v.} \int_{\substack{|y-t| \sim 2^{-j} \\ y-t \in J}} m(\lambda; y, t) \frac{\phi(t)}{y-t} dt,$$

for some C_0 independent of P but possibly depending on d . Let $J_j := ([-2^{-j+1}, -2^{-j}) \cup [2^{-j}, 2^{-j+1})) \cap J$ (that is, we split J dyadically); assuming n fixed throughout the rest of the argument, we denote by \mathcal{G} the set of indices

$$\mathcal{G} := \{j \geq \max\{n+C_0, 0\} \text{ s.t. } J_j \neq \emptyset\},$$

so that we can rewrite the inner sum above as

$$\sum_{j \in \mathcal{G}} \text{p.v.} \int_{y-t \in J_j} m(\lambda; y, t) \frac{\phi(t)}{y-t} dt.$$

As by Chapter 3, §3.1.2.2, it suffices to bound this operator uniformly in λ and independently on n .

To study the boundedness of this operator we introduce again certain quantities that proved useful before - this time though, they will have to depend on the coefficients of P somehow. First of all, we take apart the exceptional set of those indices k such that the corresponding polynomial Q_k is essentially constant on J : define

$$E := \{k \in \{1, \dots, d\} \text{ s.t. } \ell_k = 0\}.$$

We define then, for $k \in \{1, \dots, d\} \setminus E$ and $j \in \mathcal{G}$,

$$\mathcal{A}_k(j) := \frac{\lambda |Q_k(2^{-j})|}{(\lambda |y|)^k};$$

notice that if $y - t \in J_j$ then $|Q_k(y - t)| \sim |c_{k, \ell_k}| |t|^{\ell_k} \sim |c_{k, \ell_k}| 2^{-j \ell_k} \sim |Q_k(2^{-j})|$. Recall that the phase of the integrand of $m(\lambda; y, t)$ is (upon replacing $P(s, t - y)$ with $\tilde{P}(s, y - t)$, where $\tilde{P}(s, t) := P(s, -t)$)

$$\Phi(s) := (y + t)s + P(s, y - t) = 2ys - (y - t)s + P(s, y - t);$$

before proceeding any further, we will dispense of the harmless term $-(y - t)s$. Indeed, let $\theta \in [0, 1]$ to be chosen later. We can bound

$$\begin{aligned} \left| m(\lambda; y, t) - \text{p.v.} \int_{|s| \leq 1} e^{i\lambda(2ys + P(s, t - y))} \frac{ds}{s} \right| &\leq \left| \text{p.v.} \int_{|s| \leq \theta} e^{i\lambda\Phi(s)} - e^{i\lambda(\Phi(s) + (y - t)s)} \frac{ds}{s} \right| \\ &+ \left| \text{p.v.} \int_{\theta < |s| \leq 1} e^{i\lambda\Phi(s)} \frac{ds}{s} \right| \\ &+ \left| \text{p.v.} \int_{\theta < |s| \leq 1} e^{i\lambda(\Phi(s) + (y - t)s)} \frac{ds}{s} \right| \\ &=: I + II + III. \end{aligned}$$

Then we can bound

$$I \lesssim \lambda |y - t| \theta \sim \lambda 2^{-j} \theta.$$

On the other hand, the coefficient of s in the phase Φ is $(y + t) + Q_1(y - t)$, and recall that $|y + t| \sim |y|$ and $|Q_1(y - t)| \sim |Q_1(2^{-j})|$ for $y - t \in J$. Thus if $|y| \gg |Q_1(2^{-j})|$ or $|y| \ll |Q_1(2^{-j})|$ the coefficient of s in the phase is bounded from below by some absolute multiple of $|y|$, except for at most $O(1)$ values of j . But observe that we can discard such values of j , since by the same argument used above in the case where J was contained in a dyadic type interval, we can bound by Corollary 4.4

$$\left| \text{p.v.} \int_{|y - t| \sim 2^{-j}} m(\lambda; y, t) \frac{\phi(t)}{y - t} dt \right| \lesssim_d M\phi(y),$$

and thus these terms contribute at most $O_d(1)$ to the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ norm of the operator T_P^λ , which is acceptable.

Thus we can assume that the coefficient of s in the phase Φ is bounded from below by $|y|$, and by Proposition 4.2 we have

$$II + III \lesssim_d (\lambda |y| \theta)^{-1}.$$

At this point it suffices to choose

$$\theta := \frac{1}{\lambda |y|} \left(\frac{|y|}{|y - t|} \right)^\delta$$

for $0 < \delta < 1$ to show that

$$I + II + III \lesssim_d \min\left\{\left(\frac{|y-t|}{|y|}\right)^{1-\delta}, \left(\frac{|y-t|}{|y|}\right)^\delta\right\},$$

which is a quantity that is summable in j . This implies that the error in omitting the term $-(y-t)s$ from the phase is bounded pointwise by a multiple of $M\phi$, and therefore contributes at most $O(1)$ to the L^2 norm of T_p^λ ; the argument has already been given in Example 4.1, but it will also be repeated below in §4.4.2.1.

We therefore can assume that the phase is of the form

$$\Phi(s) = 2ys + P(s, y-t),$$

and proceed, as now usual, in estimating $m(\lambda; y, t)$ by removing from such phase Φ certain monomials. In particular, let S, L form a bipartition³ of $\{1, \dots, d\} \setminus E$ and let

$$\mathcal{G}_{(S,L)} := \{j \in \mathcal{G} \text{ s.t. } \forall k \in S, \mathcal{A}_k(j) \leq 1 \text{ and } \forall k \in L, \mathcal{A}_k(j) > 1\}.$$

It is clear that as (S, L) ranges through the bipartitions, the sets $\mathcal{G}_{(S,L)}$ form a disjoint partition of \mathcal{G} ; the cardinality of this partition is at most 2^d , and therefore it suffices to bound uniformly in the coefficients the operator

$$T_{(S,L)}\phi(y) := \sum_{j \in \mathcal{G}_{(S,L)}} \text{p.v.} \int_{y-t \in J_j} m(\lambda; y, t) \frac{\phi(t)}{y-t} dt.$$

We claim that the following facts hold. Firstly

Lemma 4.7. *Let (S, L) be a fixed bipartition of $\{1, \dots, d\} \setminus E$ such that $L \neq \emptyset$. Then for all $j \in \mathcal{G}_{(S,L)}$ (as defined above) except at most $O(1)$ of them⁴, it is*

$$\left| m(\lambda; y, t) - \int_{|s| \leq 1} e^{i\lambda \sum_{k \in L \cup E} s^k Q_k(y-t)} \frac{ds}{s} \right| \lesssim_d \sum_{k \in S \cup L} \omega(\mathcal{A}_k(j) \wedge \mathcal{A}_k(j)^{-1}),$$

where ω is a monotone increasing function such that $\int_0^1 \omega(r) \frac{dr}{r} < \infty$.

The proof of this lemma also yields

Scholium 4.8. *Let (S, L) be a fixed bipartition of $\{1, \dots, d\} \setminus E$. Then for all $j \in \mathcal{G}_{(S,L)}$ (as defined above) except at most $O(1)$ of them it is*

$$\left| m(\lambda; y, t) - \int_{|s| \leq 1} e^{i\lambda(2ys + \sum_{k \in L \cup E} s^k Q_k(y-t))} \frac{ds}{s} \right| \lesssim_d \sum_{k \in S} \omega(\mathcal{A}_k(j)).$$

where ω is as above.

Finally

³That is $S \cap L = \emptyset$ and $S \cup L = \{1, \dots, d\}$.

⁴Which specific ones depends on y and P , but their number is bounded by an absolute constant. See the proof of the lemma for details.

Lemma 4.9. *Let $d \in \mathbb{N}$, $F \subset \mathbb{N}$, $\xi, \lambda \in \mathbb{R}$ and let $R \in \mathcal{E}_d$. Then*

$$\left| \sum_{j \in F} \sum_{i \geq 0} \text{p.v.} \int \int_{\substack{|s| \sim 1, \\ |t| \sim 1}} e^{i(\xi 2^{-j} t + \lambda R(2^{-i} s, 2^{-j} t))} \frac{ds}{s} \frac{dt}{t} \right| \lesssim_d 1;$$

in particular, the bound does not depend on F , on the coefficients of R or on the parameters ξ, λ .

Remark 4.7. Observe how Lemma 4.9 is essentially a two-parameter version of Corollary 4.4; as the example $R(s, t) = c_0 s t$ shows though, in more than one parameter it is essential that $R \in \mathcal{E}_d$ for the conclusion to hold.

We will prove the lemmas in §4.4.3; first we show how they imply the desired uniform L^2 boundedness of $T_{(S,L)}$.

4.4.2.1

We have to split into two cases.

We first address the case when $L \neq \emptyset$. Let $\omega_j := \sum_{k \in S \cup L} \omega(\mathcal{A}_k(j) \wedge \mathcal{A}_k(j)^{-1})$ and

$$m_L(\lambda; y, t) := \int_{|s| \leq 1} e^{i\lambda \sum_{k \in L \cup E} s^k Q_k(y-t)} \frac{ds}{s};$$

we have by Lemma 4.7 that

$$\begin{aligned} & \left| T_{(S,L)} \phi(y) - \sum_{j \in \mathcal{G}_{(S,L)}} \text{p.v.} \int_{y-t \in J_j} m_L(\lambda; y, t) \frac{\phi(t)}{y-t} dt \right| \\ & \lesssim_d \sum_{j \in \mathcal{G}_{(S,L)} \setminus \mathcal{G}_{\text{bad}}} \omega_j \int_{y-t \in J_j} \frac{|\phi(t)|}{|y-t|} dt \\ & \quad + \sum_{j \in \mathcal{G}_{\text{bad}}} \left(\left| \int_{y-t \in J_j} m(\lambda; y, t) \frac{\phi(t)}{y-t} dt \right| + \left| \int_{y-t \in J_j} m_L(\lambda; y, t) \frac{\phi(t)}{y-t} dt \right| \right), \end{aligned} \quad (4.4.1)$$

where \mathcal{G}_{bad} is the set of j 's we have to exclude, as per statement of Lemma 4.7. Since $|\mathcal{G}_{\text{bad}}| \lesssim 1$, we have by Corollary 4.4

$$\begin{aligned} & \sum_{j \in \mathcal{G}_{\text{bad}}} \left| \int_{y-t \in J_j} m(\lambda; y, t) \frac{\phi(t)}{y-t} dt \right| \\ & \leq \sum_{j \in \mathcal{G}_{\text{bad}}} \int_{y-t \in J_j} |m(\lambda; y, t)| \frac{|\phi(t)|}{|y-t|} dt \\ & \stackrel{(\text{Cor. 4.4})}{\lesssim_d} \sum_{j \in \mathcal{G}_{\text{bad}}} \int_{y-t \in J_j} \frac{|\phi(t)|}{|y-t|} dt \\ & \leq \sum_{j \in \mathcal{G}_{\text{bad}}} 2^j \int_{|y-t| \sim 2^{-j}} |\phi(t)| dt \\ & \leq \sum_{j \in \mathcal{G}_{\text{bad}}} M\phi(y) \lesssim M\phi(y); \end{aligned}$$

and similarly for the term containing m_L .

As for the remaining sum, we have similarly

$$\begin{aligned} \sum_{j \in \mathcal{G}_{(S,L)} \setminus \mathcal{G}_{\text{bad}}} \omega_j \int_{y-t \in J_j} \frac{|\phi(t)|}{|y-t|} dt &\leq \sum_{j \in \mathcal{G}_{(S,L)} \setminus \mathcal{G}_{\text{bad}}} \omega_j 2^j \int_{|y-t| \sim 2^{-j}} |\phi(t)| dt \\ &\leq \left(\sum_{j \in \mathcal{G}_{(S,L)} \setminus \mathcal{G}_{\text{bad}}} \omega_j \right) M\phi(y) \lesssim_d M\phi(y), \end{aligned}$$

since the sum in $j \in \mathcal{G}_{(S,L)} \setminus \mathcal{G}_{\text{bad}}$ is uniformly bounded with respect to the coefficients of P , as the quantities $\mathcal{A}_k(j)$ are geometrically distributed with ratio at least 2. Thus we have shown that the difference on the left hand side of (4.4.1) is pointwise bounded by a constant multiple of $M\phi$, where the constant depends only on d ; the L^2 norm of the difference is therefore bounded by $C_d \|\phi\|_{L^2(\mathbb{R})}$, which is acceptable. It suffices therefore to show that the operator

$$\sum_{j \in \mathcal{G}_{(S,L)}} \text{p.v.} \int_{y-t \in J_j} m_L(\lambda; y, t) \frac{\phi(t)}{y-t} dt$$

is bounded uniformly in the coefficients. But if we expand the definition of m_L we obtain

$$\sum_{j \in \mathcal{G}_{(S,L)}} \text{p.v.} \int_{y-t \in J_j} \left(\int_{|s| \leq 1} e^{i\lambda \sum_{k \in L \cup E} s^k Q_k(y-t)} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt,$$

which is now a *translation invariant* operator. Therefore, its $L^2(\mathbb{R})$ boundedness is equivalent to the L^∞ boundedness of its multiplier, which is evidently given by

$$M_\lambda(\xi) := \sum_{j \in \mathcal{G}_{S,L}} \text{p.v.} \int_{t \in J_j} \int_{|s| \leq 1} e^{i(\xi t + \lambda \sum_{k \in L \cup E} s^k Q_k(t))} \frac{ds}{s} \frac{dt}{t};$$

this is almost of the form we need in order to apply Lemma 4.9. We can indeed reduce exactly to that form by the following consideration: all symmetric double intervals J_j are of the form $[-2^{-j+1}, -2^{-j}] \cup [2^{-j}, 2^{-j+1})$ except for at most two, those that contain the endpoints of J ; but for any interval J_j we have by Corollary 4.4 that for any polynomial $R(s, t)$ of degree at most d

$$\begin{aligned} \left| \text{p.v.} \int_{t \in J_j} \int_{|s| \leq 1} e^{i(\xi t + \lambda R(s,t))} \frac{ds}{s} \frac{dt}{t} \right| &\leq \int_{t \in J_j} \left| \int_{|s| \leq 1} e^{i\lambda R(s,t)} \frac{ds}{s} \right| \frac{dt}{|t|} \\ &\lesssim_d \int_{t \in J_j} 1 \frac{dt}{|t|} \lesssim 1, \end{aligned}$$

because J_j is contained in a single dyadic (symmetric double) interval. Therefore those J_j that contain the endpoints of J can contribute at most $O_d(1)$ to the L^∞ norm of the multiplier $M_\lambda(\xi)$, which is acceptable. Thus we can assume that $\forall j \in \mathcal{G}_{(S,L)}$ it actually is $J_j = [-2^{-j+1}, -2^{-j}] \cup [2^{-j}, 2^{-j+1})$. Let for convenience $R(s, t) := \sum_{k \in L \cup E} s^k Q_k(t)$; R clearly belongs to \mathcal{E}_d . By splitting dyadically in s and rescaling

both s and t , we see that the multiplier $M_\lambda(\xi)$ is

$$M_\lambda(\xi) = \sum_{j \in \mathcal{G}(S,L)} \sum_{i \geq 0} \text{p.v.} \int \int_{\substack{|s| \sim 1, \\ |t| \sim 1}} e^{i(\xi 2^{-j} t + \lambda R(2^{-i} s, 2^{-j} t))} \frac{ds}{s} \frac{dt}{t};$$

now we can apply Lemma 4.9 straight away, which shows that $\|M_\lambda\|_{L^\infty(\mathbb{R})} \lesssim_d 1$, and thus we are done with the case $L \neq \emptyset$.

Suppose now that $L = \emptyset$ instead (so $S = \{1, \dots, d\} \setminus E$). We write $E = \{k_1, \dots, k_{|E|}\}$ for notational ease. By Scholium 4.8 we can remove from the phase the monomials $s^k Q_k(y-t)$ for which $k \in S$; the difference between $T_{(S, \emptyset)}$ and the resulting operator with kernel

$$\left(\text{p.v.} \int_{|s| \leq 1} e^{i\lambda(2ys + \sum_{k \in E} s^k Q_k(y-t))} \frac{ds}{s} \right) \frac{dt}{y-t}$$

is bounded pointwise by the Hardy-Littlewood maximal function as seen above. Notice that for all $k \in E$ it is $|Q_k(y-t)| \sim |c_{k0}|$, thus heuristically the phase is now independent of $y-t$. We show that this is indeed the case. To do so, introduce the polynomials

$$\tilde{Q}_k(t) := Q_k(t) - c_{k0} = \sum_{0 < \ell \leq d-k} c_{k\ell} t^\ell,$$

and

$$p(s) := \sum_{k \in E} c_{k0} s^k.$$

We essentially repeat the analysis done at the beginning. By applying Lemma 4.1 to each \tilde{Q}_k , we obtain $|E|$ disjoint partitions of \mathbb{R} into dyadic and gap type symmetric double intervals, with the properties we are now acquainted with. Denote $\{\tilde{D}_j^k\}_{j \in \tilde{\mathcal{J}}_k}$ the dyadic type intervals associated to $k \in E$ and by $\{\tilde{G}_j^k\}_{j \in \tilde{\mathcal{F}}_k}$ the gap type intervals, where $k \in E = \{k_1, \dots, k_{|E|}\}$. Then consider the symmetric double intervals of the form

$$\tilde{J} = \tilde{I}^{k_1} \cap \dots \cap \tilde{I}^{k_{|E|}} \cap J,$$

where \tilde{I}^{k_r} in $\{\tilde{D}_j^{k_r}\}_{j \in \tilde{\mathcal{J}}_{k_r}} \cup \{\tilde{G}_j^{k_r}\}_{j \in \tilde{\mathcal{F}}_{k_r}}$. They partition J and their cardinality is bounded by a quantity depending only on d , thus it suffices to bound the operator where $y-t$ is restricted to one of these intervals as before. If at least one of the \tilde{I}^{k_r} is of dyadic type, then one can proceed in the same way as before to show that the operator

$$\sum_{j \in \mathcal{G}(S,L)} \text{p.v.} \int_{\substack{y-t \in \tilde{J}, \\ |y-t| \sim 2^{-j}}} \left(\text{p.v.} \int_{|s| \leq 1} e^{i\lambda(2ys + \sum_{k \in E} s^k Q_k(y-t))} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt$$

is pointwise bounded by $C_d M\phi(y)$ for some constant $C_d > 0$ depending only on d . Thus it suffices to consider the case where all the \tilde{I}^{k_r} are of gap type, and therefore for all $k \in E$ there is ℓ'_k such that for all $t \in \tilde{J}$ it is $|\tilde{Q}_k(t)| \sim |c_{k\ell'_k}| |t|^{\ell'_k}$. But notice that now $\ell'_k > 0$ necessarily, as the polynomials \tilde{Q}_k do not have terms of degree zero. Thus we can introduce for all $k \in E$ the quantities

$$\mathcal{B}_k(j) := \frac{\lambda |\tilde{Q}_k(2^{-j})|}{(\lambda |y|)^k},$$

and for every bipartition (\tilde{S}, \tilde{L}) of E we define

$$\tilde{\mathcal{G}}_{(\tilde{S}, \tilde{L})} := \{j \in \mathcal{G}_{(S, L)} \text{ s.t. } \forall k \in \tilde{S}, \mathcal{B}_k(j) \leq 1 \text{ and } \forall k \in \tilde{L}, \mathcal{B}_k(j) > 1\},$$

which partition $\mathcal{G}_{(S, L)}$. Then it suffices to bound the operator

$$\sum_{j \in \tilde{\mathcal{G}}_{(\tilde{S}, \tilde{L})}} \text{p.v.} \int_{\substack{y-t \in \tilde{J}, \\ |y-t| \sim 2^{-j}}} \left(\text{p.v.} \int_{|s| \leq 1} e^{i\lambda(2ys + p(s) + \sum_{k \in E} s^k \tilde{Q}_k(y-t))} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt;$$

by appealing to Scholium 4.8 we remove from the phase the monomials $s^k \tilde{Q}_k(y-t)$ for $k \in \tilde{S}$ at the usual price of an additional term bounded pointwise by $C_d M\phi(y)$ (the reader may check that the proof of the Scholium allows the presence of an additional phase $p(s)$ in this particular case, the essential fact being that one only needs lower bounds on the linear phase in order to apply Proposition 4.2 in the case $\mathcal{B}_k(j) \leq 1$; see Remark 4.8 in §4.4.3).

At this point, we have to split into two further cases, depending on whether \tilde{L} is empty or not.

If $\tilde{L} \neq \emptyset$ we can apply Lemma 4.7 to reduce to the operator

$$\sum_{j \in \tilde{\mathcal{G}}_{(\tilde{S}, \tilde{L})}} \text{p.v.} \int_{\substack{y-t \in \tilde{J}, \\ |y-t| \sim 2^{-j}}} \left(\text{p.v.} \int_{|s| \leq 1} e^{i\lambda(p(s) + \sum_{k \in \tilde{L}} s^k \tilde{Q}_k(y-t))} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt;$$

indeed, the reader may check that the proof of Lemma 4.7 allows the presence of the additional term $p(s)$ in this case, for the reason that in $\tilde{J} \subset J$ it is $|c_{k0}| \gg |\tilde{Q}_k(t)|$ for all $t \in \tilde{J}$ (see again Remark 4.8 in §4.4.3 for details). But now the operator is translation invariant, and therefore we can prove it is $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ bounded uniformly in the coefficients of the polynomials involved by estimating its multiplier. This is clearly

$$\sum_{j \in \tilde{\mathcal{G}}_{(\tilde{S}, \tilde{L})}} \sum_{i \geq 0} \text{p.v.} \int_{\substack{|s| \sim 1, \\ |t| \sim 1}} e^{i(\xi 2^{-j} t + \lambda(p(2^{-i}s) + \sum_{k \in \tilde{L}} 2^{-ik} s^k \tilde{Q}_k(2^{-j}t))} \frac{ds}{s} \frac{dt}{t},$$

and another application of Lemma 4.9 shows this is $O_d(1)$, since the polynomial $p(s) + \sum_{k \in \tilde{L}} s^k \tilde{Q}_k(t)$ is clearly in \mathcal{E}_d .

If $\tilde{L} = \emptyset$ instead, there is no need to appeal to Lemma 4.7, since the operator is of the form

$$\sum_{j \in \tilde{\mathcal{G}}_{(\tilde{S}, \tilde{L})}} \text{p.v.} \int_{\substack{y-t \in \tilde{J}, \\ |y-t| \sim 2^{-j}}} \left(\text{p.v.} \int_{|s| \leq 1} e^{i\lambda(2ys + p(s))} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt,$$

and the oscillatory integral in brackets is therefore independent of $y-t$. Thus we can simply bound this operator pointwise by

$$\left| \text{p.v.} \int_{|s| \leq 1} e^{i\lambda(2ys + p(s))} \frac{ds}{s} \right| \sum_{j \in \tilde{\mathcal{G}}_{(\tilde{S}, \tilde{L})}} \text{p.v.} \int_{\substack{y-t \in \tilde{J}, \\ |y-t| \sim 2^{-j}}} \frac{\phi(t)}{y-t} dt,$$

and by Corollary 4.4 - and the simple observation that the j 's in $\tilde{\mathcal{G}}_{(\tilde{S}, \tilde{L})}$ are contiguous - this is bounded by a constant depending only on d times

$$\left| \sum_{j \in \tilde{\mathcal{G}}_{(\tilde{S}, \tilde{L})}} \text{p.v.} \int_{\substack{y-t \in \tilde{J}, \\ |y-t| \sim 2^{-j}}} \frac{\phi(t)}{y-t} dt \right| \lesssim H^* \phi(y),$$

which is $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ bounded.

It therefore only remains to prove the lemmas - which we do in the next subsection.

4.4.3 Proofs of Lemma 4.7 and Lemma 4.9

4.4.3.1 Proof of Lemma 4.7

We first show we can remove from the phase of m the terms $s^k Q_k(y-t)$ for those $k \in S$. This will prove Scholium 4.8.

Indeed, let $k \in S$ and as before denote by Φ the phase $\Phi(s) := 2ys + P(s, y-t)$; then we estimate for any $\theta \in (0, 1)$

$$\begin{aligned} \left| m(\lambda; y, t) - \int_{|s| \leq 1} e^{i\lambda(\Phi(s) - s^k Q_k(y-t))} \frac{ds}{s} \right| &\leq \left| \int_{|s| \leq \theta} e^{i\lambda\Phi(s)} - e^{i\lambda(\Phi(s) - s^k Q_k(y-t))} \frac{ds}{s} \right| \\ &\quad + \left| \int_{\theta < |s| \leq 1} e^{i\lambda\Phi(s)} \frac{ds}{s} \right| \\ &\quad + \left| \int_{\theta < |s| \leq 1} e^{i\lambda(\Phi(s) - s^k Q_k(y-t))} \frac{ds}{s} \right| \\ &=: I + II + III. \end{aligned}$$

Then easily

$$I \lesssim \lambda \theta^k |Q_k(y-t)|;$$

for the other two terms, we will apply instead Proposition 4.2, since the phase is a polynomial in s . In particular, we want to estimate in terms of the coefficient of s in Φ , but we need to be careful in doing so. Recall that we have arranged things so that $\ell_1 > 0$; the coefficient of the linear term s is given by $2y + Q_1(y-t)$, where $|Q_1(y-t)| \sim |c_1|2^{-j\ell_1}$. Now if $|y| \ll |c_1|2^{-j\ell_1}$ or $|y| \gg |c_1|2^{-j\ell_1}$, we have that $|2y + Q_1(y-t)| \gtrsim |y|$. If $|y| \sim |c_1|2^{-j\ell_1}$ we cannot give any such lower bound, but notice that this can only happen for $O(1)$ indices j for fixed y and P , given that $\ell_1 \geq 1$ (the coefficient c_1 is one of the coefficients of P , as per Lemma 4.1. We therefore ignore these j 's, since their number is controlled by an absolute constant, and assume $|2y + Q_1(y-t)| \gtrsim |y|$, where the implicit constant does not depend on any parameter. Then by Proposition 4.2 there exists a $\beta = \beta_d > 0$ s.t.

$$\begin{aligned} II &\lesssim_d (\lambda|y|\theta)^{-\beta}, \\ III &\lesssim_d (\lambda|y|\theta)^{-\beta}, \end{aligned}$$

since they have the same linear phase. Now let

$$\theta := \frac{1}{\lambda|y|} \frac{1}{\mathcal{A}_k(j)^\sigma} \wedge 1$$

for $\sigma > 0$ to be chosen; this gives

$$I \lesssim_d \frac{\lambda|Q_k(y-t)|}{(\lambda|y|)^k} \frac{1}{\mathcal{A}_k(j)^{\sigma k}} = \mathcal{A}_k(j)^{1-\sigma k},$$

and thus we will choose $\sigma < 1/k$ in order for this to be summable (since $k \in S$, $\mathcal{A}_k(j) < 1$). As for the other contributions, our choice of θ gives

$$II + III \lesssim_d \mathcal{A}_k(j)^{\beta\sigma},$$

which is certainly summable.

By repeating the above procedure for every $k \in S$ we show that for some $\sigma > 0$ sufficiently small (depending only on d) it is

$$\left| m(\lambda; y, t) - \int_{|s| \leq 1} e^{i\lambda(2ys + \sum_{k \in L \cup E} s^k Q_k(y-t))} \frac{ds}{s} \right| \lesssim_d \sum_{k \in S} \mathcal{A}_k(j)^\sigma;$$

it remains to remove the linear term $2ys$ to finish the proof.

Now let $R(s, y-t) := \sum_{k \in L \cup E} s^k Q_k(y-t)$; we can estimate for any $\theta \in (0, 1)$

$$\begin{aligned} & \left| \int_{|s| \leq 1} e^{i\lambda(2ys + R(s, y-t))} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda R(s, y-t)} \frac{ds}{s} \right| \\ & \leq \left| \int_{|s| \leq \theta} e^{i\lambda(2ys + R(s, y-t))} - e^{i\lambda R(s, y-t)} \frac{ds}{s} \right| \\ & \quad + \left| \int_{\theta < |s| \leq 1} e^{i\lambda(2ys + R(s, y-t))} \frac{ds}{s} \right| \\ & \quad + \left| \int_{\theta < |s| \leq 1} e^{i\lambda R(s, y-t)} \frac{ds}{s} \right| \\ & =: I' + II' + III'. \end{aligned}$$

It is immediate that

$$I' \lesssim \lambda|y|\theta.$$

Since $L \neq \emptyset$, by Proposition 4.2 we have for some $\beta > 0$ and any $k \in L$ that⁵

$$II' + III' \lesssim_d (\lambda|Q_k(y-t)|\theta^k)^{-\beta};$$

by choosing θ as before (this time though $k \in L$, so $\mathcal{A}_k(j) > 1$) we have

$$\begin{aligned} I' & \lesssim \mathcal{A}_k(j)^{-\sigma}, \\ II' + III' & \lesssim_d \mathcal{A}_k(j)^{-\beta(1-\sigma k)}, \end{aligned}$$

⁵Notice we can allow $k = 1$ and the argument still works.

which are summable in j if we choose $\sigma > 0$ smaller than $1/k$, thus finishing the proof. \square

Remark 4.8. As pointed out before, the above proof also applies with some modifications in the case where $L = \emptyset$. Indeed, write the phase as

$$\Phi(s) = 2ys + p(s) + \sum_{k \in E} s^k \tilde{Q}_k(y-t),$$

where p and \tilde{Q}_k are as defined in subsection 4.4, and take a bipartition (\tilde{S}, \tilde{L}) of E , and assume $j \in \tilde{\mathcal{G}}_{(\tilde{S}, \tilde{L})}$ as defined before. Then in order to remove the monomial $s^k \tilde{Q}_k(y-t)$ for $k \in \tilde{S}$ we split the integral as above and bound

$$I \lesssim \lambda |\tilde{Q}_k(y-t)| \theta^k;$$

to bound the other terms we only need a lower bound on the linear phase in order to apply Proposition 4.2, but $p(s)$ only contains monomials s^k for $k \geq 2$ and if $1 \in E$ one has $|2y + \tilde{Q}_k(y-t)| \gtrsim |y|$ for all j except at most $O(1)$; thus the proof goes through in the same way, showing

$$\left| \int_{|s| \leq 1} e^{i\lambda\Phi(s)} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda(\Phi(s) - s^k \tilde{Q}_k(y-t))} \frac{ds}{s} \right| \lesssim_d \omega(\mathcal{B}_k(j)),$$

and then all monomials $s^k \tilde{Q}_k(y-t)$ for $k \in \tilde{S}$ can be removed one at a time.

Next, suppose you have removed all monomials for $k \in \tilde{S}$ and that $\tilde{L} \neq \emptyset$. In this case we want to remove the term $2ys$ from the phase, and by splitting the integral as above we bound

$$I \lesssim \lambda |y| \theta,$$

and we need a lower bound on some coefficient of the polynomial phase to apply Proposition 4.2 again. If $k = 1$ the coefficient is

$$2y + \tilde{Q}_1(y-t),$$

and if $\mathcal{B}_1(j) > 1$ (which it has to if $1 \in \tilde{L}$) we see that (except for $O(1)$ indices of j , which is still acceptable) it is $|2y + \tilde{Q}_1(y-t)| \gtrsim |\tilde{Q}_1(2^{-j})|$ and therefore

$$I + II + III \lesssim_d \lambda |y| \theta + (\lambda |\tilde{Q}_1(2^{-j})| \theta)^{-\beta},$$

which is bounded by $\mathcal{B}_1(j)^{-\sigma'} \leq \omega(\mathcal{B}_1(j)^{-1})$ for some $\sigma' > 0$, by our usual choice of $\theta := (\lambda |y|)^{-1} \mathcal{B}_1(j)^{-\sigma}$ for $\sigma > 0$ sufficiently small (depending on d only). If instead there is a $k \in \tilde{L}$ with $k \neq 1$ the coefficient of the k -th power of s in the phase is (since $k \in E$)

$$|c_{k0} + \tilde{Q}_k(y-t)| = |Q_k(y-t)| \sim |c_{k0}| \gg |\tilde{Q}_k(2^{-j})|,$$

and therefore we have

$$I + II + III \lesssim_d \lambda |y| \theta + (\lambda |\tilde{Q}_k(2^{-j})| \theta^k)^{-\beta},$$

which is bounded by $\mathcal{B}_k(j)^{-\sigma''} \leq \omega(\mathcal{B}_k(j)^{-1})$ for some $\sigma'' > 0$ by the now obvious choice of θ , as claimed.

4.4.3.2 Proof of Lemma 4.9

For any pair of exponents $(m_0, n_0) \in \mathbb{N}^2$ such that $m_0 + n_0 \leq d$, define

$$\Lambda_{m_0 n_0} := \{(i, j) \in \mathbb{N} \times F \text{ s.t. } \forall (m, n) \in \mathbb{N}^2, |c_{mn}|2^{-im-jn} \leq |c_{m_0 n_0}|2^{-im_0-jn_0}\},$$

where the c_{mn} are the coefficients of $R(s, t)$. The number of possible pairs (m_0, n_0) is of course bounded in terms of d only, and therefore it suffices to prove

$$\left| \sum_{(i,j) \in \Lambda_{m_0 n_0}} \text{p.v.} \int \int_{\substack{|s| \sim 1 \\ |t| \sim 1}} e^{i(\xi 2^{-j} t + \lambda R(2^{-i} s, 2^{-j} t))} \frac{ds}{s} \frac{dt}{t} \right| \lesssim_d 1$$

for every pair.

Let therefore (m_0, n_0) be such a pair; we can always assume that $(m_0, n_0) \neq (0, 1)$. Indeed, if $c_{01} \neq 0$ we can simply extract factor $e^{i\lambda c_{01}(y-t)} = e^{i\lambda c_{01}y} e^{-i\lambda c_{01}t}$ from $m(\lambda; y, t)$ and we can make this oscillating factor in $c_{01}t$ disappear by modulating the function ϕ and the operator T_P^λ , analogously to what we have done for translation in Chapter 3, §3.1.2.3.

Define the *ray* through (m_0, n_0) to be the set

$$\Gamma_{m_0, n_0} := \{(m, n) \in \mathbb{N}^2 \text{ s.t. } \exists \gamma \in \mathbb{Q}, (m, n) = \gamma(m_0, n_0)\}.$$

If $(m, n) \in \Lambda_{m_0 n_0} \setminus \Gamma_{m_0, n_0}$, we will show that for

$$I_{ij} := \text{p.v.} \int \int_{\substack{|s| \sim 1 \\ |t| \sim 1}} (e^{i\Phi(2^{-i} s, 2^{-j} t)} - e^{i(\Phi(2^{-i} s, 2^{-j} t) - c_{mn} 2^{-im-jn} s^m t^n)}) \frac{ds}{s} \frac{dt}{t},$$

where $\Phi(s, t) := \xi t + \lambda R(s, t)$, it holds that

$$\sum_{(i,j) \in \Lambda_{m_0 n_0}} |I_{ij}| = O_d(1).$$

To begin with, we have by simple domination that

$$|I_{ij}| \lesssim \lambda |c_{mn}| 2^{-im-jn}.$$

On the other hand, we have that

$$\partial_s^{m_0} \partial_t^{n_0} \Phi(s, t) = \lambda m_0! n_0! c_{m_0 n_0} 2^{-im_0-jn_0} \left(1 + \sum_{\substack{m > m_0, \\ n > n_0}} b_{mn} \binom{m}{m_0} \binom{n}{n_0} s^{m-m_0} t^{n-n_0} \right),$$

where

$$b_{mn} := \frac{c_{mn} 2^{-im-jn}}{c_{m_0 n_0} 2^{-im_0-jn_0}};$$

notice $|b_{mn}| \leq 1$ by assumption. By pigeonholing, it is easy to see that there exist $p, q \in \mathbb{N}$ such that

$$|\partial_s^{m_0+p} \partial_t^{n_0+q} \Phi(s, t)| \gtrsim_d \lambda |c_{m_0 n_0}| 2^{-im_0-jn_0},$$

and therefore by an application of Van der Corput's lemma in two variables (Proposition 2.3 of Chapter 2) we have for some $\delta = \delta(d) > 0$

$$|I_{ij}| \lesssim_d (\lambda|c_{m_0 n_0}| 2^{-im_0 - jn_0})^{-\delta};$$

notice that we can assert that the implicit constant is independent of the polynomial R because it stays bounded if the homogeneous \dot{C}^N norm of the phase stays bounded, where $N = m_0 + n_0 + p + q + 1$, and this is indeed the case for $\sum_{m,n} b_{mn} s^m t^n$ in $|s|, |t| \sim 1$, since $|b_{mn}| \leq 1$. It is crucial here that $(m_0, n_0) \neq (0, 1)$. Thus we have

$$\sum_{(i,j) \in \Lambda_{m_0 n_0}} |I_{ij}| \lesssim_d \sum_{(i,j) \in \Lambda_{m_0 n_0}} (\lambda|c_{mn}| 2^{-im-jn}) \wedge (\lambda|c_{m_0 n_0}| 2^{-im_0 - jn_0})^{-\delta};$$

we claim this is finite and $O_d(1)$, and to show this it will be instrumental that (m, n) is not in the ray through (m_0, n_0) (if it were not so, the sum could only be bounded logarithmically in $\lambda|c_{m_0 n_0}|$). Indeed, let

$$L_k := \{(i, j) \in \Lambda_{m_0 n_0} \text{ s.t. } (i, j) \cdot (m, n) = k\}$$

so that we can write the sum as

$$\sum_{k \geq 0} \sum_{(i,j) \in L_k} (\lambda|c_{mn}| 2^{-k}) \wedge (\lambda|c_{m_0 n_0}| 2^{-im_0 - jn_0})^{-\delta}.$$

We have

$$\begin{aligned} \lambda|c_{mn}| 2^{-k} &\leq (\lambda|c_{m_0 n_0}| 2^{-im_0 - jn_0})^{-\delta} \\ \Leftrightarrow 2^{-im_0 - jn_0} &\leq \frac{(\lambda|c_{mn}| 2^{-k})^{-1/\delta}}{\lambda|c_{m_0 n_0}|}. \end{aligned}$$

Let $r \in \mathbb{Z}$ be such that $(\lambda|c_{mn}| 2^{-k})^{-1/\delta} (\lambda|c_{m_0 n_0}|)^{-1} \sim 2^{-r}$. Then the condition above is equivalent to $(i, j) \cdot (m_0, n_0) \gtrsim r$. Therefore we have a contribution to the sum of

$$\sum_{k \geq 0} \sum_{\substack{(i,j) \in L_k, \\ (i,j) \cdot (m_0, n_0) \gtrsim r}} \lambda|c_{mn}| 2^{-k}.$$

Since $(i, j) \in \Lambda_{m_0 n_0}$, for any $\theta \in [0, 1]$ we can bound the above contribution by

$$\sum_{k \geq 0} (\lambda|c_{mn}| 2^{-k})^\theta \sum_{\substack{(i,j) \in L_k, \\ (i,j) \cdot (m_0, n_0) \gtrsim r}} (\lambda|c_{m_0 n_0}| 2^{-im_0 - jn_0})^{1-\theta};$$

since (m_0, n_0) and (m, n) have distinct directions, as (i, j) ranges in L_k the values of $2^{-im_0 - jn_0}$ are all distinct, and therefore separated by a factor of at least 2 in ratio. This implies that the inner sum in the last expression is bounded by $O((\lambda|c_{m_0 n_0}| 2^{-r})^{1-\theta})$, and by definition of r this is actually $O((\lambda|c_{mn}| 2^{-k})^{-(1-\theta)/\delta})$. The sum we are considering is therefore bounded by

$$\sum_{k \geq 0} (\lambda|c_{mn}| 2^{-k})^{(\theta(1+\delta)-1)/\delta},$$

but we can choose a different θ for any k , so for those k s.t. $\lambda|c_{mn}|2^{-k} > 1$ choose $\theta > 0$ but sufficiently small so that $(\theta(1 + \delta) - 1)/\delta < 0$, and for the others choose $\theta < 1$ but sufficiently close to 1 so that $(\theta(1 + \delta) - 1)/\delta > 0$; the resulting sum is bounded by $O(1)$.

It remains to bound the complementary contribution, namely

$$\sum_{k \geq 0} \sum_{\substack{(i,j) \in L_k, \\ (i,j) \cdot (m_0, n_0) \lesssim r}} (\lambda|c_{m_0 n_0}|2^{-im_0 - jn_0})^{-\delta};$$

but the summand can be bounded for any $\theta \in [0, 1]$ by

$$(\lambda|c_{m_0 n_0}|2^{-im_0 - jn_0})^{-\delta\theta} (\lambda|c_{mn}|2^{-k})^{1-\theta},$$

and thus the contribution is bounded by

$$\sum_{k \geq 0} (\lambda|c_{mn}|2^{-k})^{1-(1+\delta)\theta},$$

and suitable choices of θ give that this sum is $O(1)$ too, thus proving the claim.

By iterating the above argument for every $(m, n) \in \Lambda_{m_0 n_0} \setminus \Gamma_{m_0 n_0}$ (whose cardinality is bounded in d), we are left with estimating

$$\sum_{(i,j) \in \Lambda_{m_0 n_0}} I'_{ij} := \sum_{(i,j) \in \Lambda_{m_0 n_0}} \text{p.v.} \int \int_{\substack{|s| \sim 1, \\ |t| \sim 1}} e^{i(\xi 2^{-j} t + \lambda \sum_{(m,n) \in \Gamma_{m_0 n_0}} c_{mn} 2^{-im-jn} s^m t^n)} \frac{ds}{s} \frac{dt}{t}.$$

Let $m'_0 := \min\{m \text{ s.t. } (m, n) \in \Gamma_{m_0, n_0}\}$ and n'_0 be such that $(m'_0, n'_0) \in \Gamma_{m_0 n_0}$. Notice that $c_{m'_0 n'_0}$ might well be zero, since the ray is only defined with respect to the position of (m_0, n_0) in the \mathbb{N}^2 lattice. We distinguish two cases.

If m'_0 is even, then all m in $\Gamma_{m_0 n_0}$ are even. Indeed, for any $(m, n) \in \Gamma_{m_0, n_0}$ there is $\gamma \in \mathbb{Q}$ is such that $(m, n) = \gamma(m'_0, n'_0)$, but by minimality it must actually be $\gamma \in \mathbb{N}$, and therefore m'_0 is a factor of all m appearing in $\Gamma_{m_0 n_0}$. In this case the phase is an even function in s , and since the kernel $\frac{ds}{s}$ is odd it follows that

$$\text{p.v.} \int \int_{\substack{|s| \sim 1, \\ |t| \sim 1}} e^{i(\xi 2^{-j} t + \lambda \sum_{(m,n) \in \Gamma_{m_0 n_0}} c_{mn} 2^{-im-jn} s^m t^n)} \frac{ds}{s} \frac{dt}{t} \equiv 0$$

for all (i, j) , and there is nothing to prove.

We assume therefore that m'_0 is odd. If n'_0 is odd as well, then all $(m, n) \in \Gamma_{m_0 n_0}$ s.t. $c_{mn} \neq 0$ must have both m and n even, but in particular all the m 's are even, and therefore by the same argument as before there is nothing to prove, as the phase is an even function. Thus we assume that m'_0 is odd and n'_0 is even - which implies that for all $(m, n) \in \Gamma_{m_0 n_0}$ s.t. $c_{mn} \neq 0$, n is even. Let

$$R_{m_0 n_0}(2^{-i} s, 2^{-j} t) := \sum_{(m,n) \in \Gamma_{m_0 n_0}} c_{mn} 2^{-im-jn} s^m t^n;$$

this polynomial as a function is even in t , and therefore we have

$$I'_{ij} = \text{p.v.} \int \int_{\substack{|s| \sim 1, \\ |t| \sim 1}} e^{i(\xi 2^{-j} t + \lambda R_{m_0 n_0}(2^{-i} s, 2^{-j} t))} - e^{i\lambda R_{m_0 n_0}(2^{-i} s, 2^{-j} t)} \frac{ds}{s} \frac{dt}{t},$$

and thus we have the estimate

$$|I'_{ij}| \lesssim |\xi| 2^{-j}. \quad (4.4.2)$$

Moreover, the function $e^{i\xi 2^{-j} t}$ is even in s , and therefore

$$I'_{ij} = \text{p.v.} \int \int_{\substack{|s| \sim 1, \\ |t| \sim 1}} e^{i\Psi(2^{-i} s, 2^{-j} t)} - e^{i\xi 2^{-j} t} \frac{ds}{s} \frac{dt}{t},$$

where $\Psi(s, t) = \xi t + \lambda R_{m_0 n_0}(s, t)$ is the unrescaled phase, which gives

$$|I'_{ij}| \lesssim \lambda \sum_{(m,n) \in \Gamma_{m_0 n_0}} |c_{mn}| 2^{-im-jn} \lesssim_d \lambda |c_{m_0 n_0}| 2^{-im_0-jn_0}. \quad (4.4.3)$$

As implicitly understood above, these two estimates alone do not suffice to give an $O_d(1)$ bound, and therefore we look for further estimates. We notice that we can estimate the mixed derivatives as done before for the full phase: there are $p, q \in \mathbb{N}$ such that

$$|\partial_s^{m_0+p} \partial_t^{n_0+p} \Psi(2^{-i} s, 2^{-j} t)| \gtrsim_d |c_{m_0 n_0}| 2^{-im_0-jn_0},$$

so that again by Proposition 2.3 we have for some $\delta > 0$

$$|I'_{ij}| \lesssim (\lambda |c_{m_0 n_0}| 2^{-im_0-jn_0})^{-\delta}. \quad (4.4.4)$$

Finally, observe that

$$\partial_t \Psi(2^{-i} s, 2^{-j} t) = \xi 2^{-j} + \lambda \sum_{(m,n) \in \Gamma_{m_0 n_0}} c_{mn} n 2^{-im-jn} s^m t^{n-1},$$

where again we can estimate

$$\left| \sum_{(m,n) \in \Gamma_{m_0 n_0}} c_{mn} n 2^{-im-jn} s^m t^{n-1} \right| \lesssim_d |c_{m_0 n_0}| 2^{-im_0-jn_0};$$

thus if $|\xi| 2^{-j} \gg \lambda |c_{m_0 n_0}| 2^{-im_0-jn_0}$, we have for all s, t in the integration domain that

$$|\partial_t \Psi(2^{-i} s, 2^{-j} t)| \gtrsim |\xi| 2^{-j}.$$

On the other hand,

$$\partial_t^2 \Psi(2^{-i} s, 2^{-j} t) = \lambda \sum_{(m,n) \in \Gamma_{m_0 n_0}} c_{mn} n(n-1) 2^{-im-jn} s^m t^{n-2},$$

so we easily have $|\partial_t^2 \Psi(2^{-i} s, 2^{-j} t)| \lesssim \lambda |c_{m_0 n_0}| 2^{-im_0-jn_0}$. Thus we can estimate by integration by parts

$$\begin{aligned}
\left| \int_{|t|\sim 1} e^{i\Psi(2^{-i}s, 2^{-j}t)} \frac{dt}{t} \right| &\leq 2 \sup_{|t|\sim 1} \left| \frac{e^{i\Psi(2^{-i}s, 2^{-j}t)}}{it\partial_t(\Psi(2^{-i}s, 2^{-j}t))} \right| \\
&\quad + \left| \int_{|t|\sim 1} e^{i\Psi(2^{-i}s, 2^{-j}t)} \frac{t\partial_t^2\Psi + \partial_t\Psi}{i(t\partial_t\Psi)^2} dt \right| \\
&\lesssim (|\xi|2^{-j})^{-1} + \int_{|t|\sim 1} \frac{|\partial_t^2\Psi|}{|t||\partial_t\Psi|^2} + \frac{1}{|t|^2|\partial_t\Psi|} dt \\
&\lesssim (|\xi|2^{-j})^{-1},
\end{aligned}$$

where we have written Ψ in place of $\Psi(2^{-i}s, 2^{-j}t)$ for brevity. It follows that

$$|I'_{ij}| \leq \int_{|s|\sim 1} \left| \int_{|t|\sim 1} e^{i\Psi(2^{-i}s, 2^{-j}t)} \frac{dt}{t} \right| \frac{ds}{|s|} \lesssim (|\xi|2^{-j})^{-1}. \quad (4.4.5)$$

We can now use estimates (4.4.2), (4.4.3), (4.4.4), (4.4.5) to conclude that $\sum_{i,j} |I'_{ij}| = O_d(1)$. Let L_k be as before and consider the contribution from those terms for which $\lambda|c_{m_0 n_0}|2^{-k} < 1$: it can be bounded for any $\theta_1, \theta_2, \theta_3 \geq 0$ s.t. $\theta_1 + \theta_2 + \theta_3 = 1$ (eventually different for every k) by

$$\sum_{\substack{k \geq 0: \\ \lambda|c_{m_0 n_0}|2^{-k} < 1}} (\lambda|c_{m_0 n_0}|2^{-k})^{\theta_1} \sum_{(i,j) \in L_k} (|\xi|2^{-j})^{\theta_2 - \theta_3};$$

by choosing $\theta_2 - \theta_3 > 0$ when $|\xi|2^{-j} < 1$ and viceversa, the inner sum is $O(1)$, and therefore if we also choose $\theta_1 > 0$ the entire sum is $O(1)$. Finally, consider the contribution from the terms for which $\lambda|c_{m_0 n_0}|2^{-k} \geq 1$; in this case one bounds the sum by

$$\sum_{\substack{k \geq 0: \\ \lambda|c_{m_0 n_0}|2^{-k} \geq 1}} (\lambda|c_{m_0 n_0}|2^{-k})^{-\delta\theta_1} \sum_{(i,j) \in L_k} (|\xi|2^{-j})^{\theta_2 - \theta_3},$$

and a similar reasoning shows this is $O(1)$ as well. This concludes the proof. \square

4.5 Proof of Proposition 4.6

We want to prove that if $V \supsetneq \mathcal{E}_d$ is a subspace of polynomials of degree at most d that strictly contains \mathcal{E}_d then the operator norm of H_P is not uniformly bounded as P ranges over V . It suffices to assume that there exists $P(s, t) = \sum_{m+n \leq d} a_{mn} s^m t^n \in \mathbb{R}_d[s, t]$ such that if $a_{mn} \neq 0$ then both m and n are odd, and such that the operator $H_{\mu P}$ is $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded uniformly in μ ; we will then obtain a contradiction from this.

Since we have already shown in Example 4.1 that the above assumption cannot hold in the special case $P(s, t) = s^m t^n$ with both m, n odd, the proof essentially reduces to the one given in [CW00] to show that the Euclidean double Hilbert transform along $(s, t, P(s, t))$ is not bounded if a corner of $\mathcal{N}(P)$ has both coordinates odd. We reproduce it here for the reader's benefit.

The idea is to show that suitable dilates of the kernel of $H_{\mu P}$ can be made to converge

to the kernel of $H_{\mu s^m t^n}$, for certain m, n odd. But this would imply that $H_{\mu s^m t^n}$ is $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded uniformly in μ too, and we know by Example 4.1 and Remark 4.6 that this is false.

Define then, for P as assumed above, the distribution \mathcal{K}_μ by

$$(\mathcal{K}_\mu, f) := \text{p.v.} \int \int_{|s|, |t| \leq 1} f(s, t, \mu P(s, t)) \frac{ds}{s} \frac{dt}{t} \quad \forall f \in \mathcal{S}(\mathbb{R}^3).$$

Let (m_0, n_0) be a corner of $\mathcal{N}(P)$ (thus m_0, n_0 are both odd), assume without loss of generality that $a_{m_0 n_0} = 1$, and notice there exist by convexity of $\mathcal{N}(P)$ two numbers $\alpha, \beta > 0$ such that $\alpha m_0 + \beta n_0 < \alpha m + \beta n$ for all $(m, n) \in \mathcal{N}(P)$ that are not equal to (m_0, n_0) . Then define for $\delta > 0$ the distribution \mathcal{K}_μ^δ by

$$(\mathcal{K}_\mu^\delta, f) := \text{p.v.} \int \int_{\substack{|s| \leq \delta^{-\alpha} \\ |t| \leq \delta^{-\beta}}} f(s, t, \mu \delta^{-\alpha m_0 - \beta n_0} P(\delta^\alpha s, \delta^\beta t)) \frac{ds}{s} \frac{dt}{t} \quad \forall f \in \mathcal{S}(\mathbb{R}^3);$$

notice that

$$\delta^{-\alpha m_0 - \beta n_0} P(\delta^\alpha s, \delta^\beta t) = s^{m_0} t^{n_0} + \sum_{(m, n) \neq (m_0, n_0)} \delta^{a(m-m_0) + b(n-n_0)} a_{mn} s^m t^n,$$

and the exponents of δ are all positive.

If $\Delta_{\lambda_1, \lambda_2}$ denotes the automorphic dilation operator

$$\Delta_{\lambda_1, \lambda_2} f(x, y, z) := f(\lambda_1 x, \lambda_2 y, \lambda_1 \lambda_2 z),$$

we have, with $\mu' := \mu \delta^{\alpha(m_0-1) + \beta(n_0-1)}$,

$$\mathcal{K}_\mu \star f = \Delta_{\delta^{-\alpha}, \delta^{-\beta}} (\mathcal{K}_{\mu'}^\delta \star \Delta_{\delta^\alpha, \delta^\beta} f),$$

and therefore the operators $f \mapsto \mathcal{K}_\mu \star f$ and $f \mapsto \mathcal{K}_{\mu'}^\delta \star f$ have the same $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ norm; in particular, the operator norm of $f \mapsto \mathcal{K}_\mu^\delta \star f$ is uniformly bounded in both δ and μ , by assumption. This allows one to make the two parameters μ, δ in \mathcal{K}_μ^δ independent, which is fundamental for the proof below to work (it allows us to bypass the fact that there are no three parameter automorphic dilations in \mathbb{H}^1).

Let \mathcal{L}_μ be the distribution defined by

$$(\mathcal{L}_\mu, f) := \text{p.v.} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s, t, \mu s^{m_0} t^{n_0}) \frac{ds}{s} \frac{dt}{t} \quad \forall f \in \mathcal{S}(\mathbb{R}^3);$$

we claim that $\mathcal{K}_\mu^\delta \rightarrow \mathcal{L}_\mu$ for $\delta \rightarrow 0$ as distributions. This will give the desired contradiction, since we know by Example 4.1 that the operator norm of $L^2(\mathbb{H}^1) \ni f \mapsto \mathcal{L}_\mu \star f \in L^2(\mathbb{H}^1)$ is not uniformly bounded in μ . Here we must point out that \mathcal{L}_μ involves integration over \mathbb{R} and not just $[-1, 1]$, thus $f \mapsto \mathcal{L}_\mu \star f$ is not exactly the same operator as in Example 4.1; however, it is not hard to modify the proof in the example to show that the result in there extends to \mathcal{L}_μ as well.

Define for shortness

$$P_\delta(s, t) := \delta^{-\alpha m_0 - \beta n_0} P(\delta^\alpha s, \delta^\beta t),$$

and take some numbers a, b such that $0 < a < \alpha$, $0 < b < \beta$ to be fixed later. We want to prove that for every μ and for every $f \in \mathcal{S}(\mathbb{R}^3)$ we have

$$(\mathcal{K}_\mu^\delta, f) \rightarrow (\mathcal{L}_\mu, f) \quad \text{as } \delta \rightarrow 0,$$

so we split the quantity on the left as

$$\begin{aligned} & \int \int_{\substack{|s| \leq \delta^{-a} \\ |t| \leq \delta^{-b}}} [f(s, t, \mu P_\delta(s, t)) - f(s, t, \mu s^{m_0} t^{n_0})] \frac{ds}{s} \frac{dt}{t} \\ & + \int \int_{\substack{\delta^{-a} < |s| \leq \delta^{-\alpha} \\ |t| \leq \delta^{-\beta}}} f(s, t, \mu P_\delta(s, t)) \frac{ds}{s} \frac{dt}{t} \\ & + \int \int_{\substack{|s| \leq \delta^{-a} \\ \delta^{-b} < |t| \leq \delta^{-\beta}}} f(s, t, \mu P_\delta(s, t)) \frac{ds}{s} \frac{dt}{t} \\ & + \int \int_{\delta^{-a} < |s|} f(s, t, \mu s^{m_0} t^{n_0}) \frac{ds}{s} \frac{dt}{t} \\ & + \int \int_{\substack{|s| \leq \delta^{-a} \\ \delta^{-b} < |t|}} f(s, t, \mu s^{m_0} t^{n_0}) \frac{ds}{s} \frac{dt}{t} \\ & =: I(\delta) + II(\delta) + III(\delta) + IV(\delta) + V(\delta). \end{aligned}$$

We notice right away that since the integrand is independent of δ ,

$$IV(\delta), V(\delta) \rightarrow 0 \quad \text{for } \delta \rightarrow 0.$$

Moreover, we can dispense of quantities $II(\delta), III(\delta)$ too. Indeed, by oddness of the kernel one has

$$\int \int_{\substack{\delta^{-a} < |s| \leq \delta^{-\alpha} \\ |t| \leq \delta^{-\beta}}} f(s, 0, \mu P_\delta(s, 0)) \frac{ds}{s} \frac{dt}{t} = 0,$$

and therefore can write

$$II(\delta) = \int \int_{\substack{\delta^{-a} < |s| \leq \delta^{-\alpha} \\ |t| \leq \delta^{-\beta}}} \int_0^1 \frac{\partial}{\partial \eta} \left(f(s, \eta t, \mu P_\delta(s, \eta t)) \right) d\eta \frac{ds}{s} \frac{dt}{t}.$$

But we have for any N (since $f \in \mathcal{S}(\mathbb{R}^3)$)

$$\begin{aligned} \left| \frac{\partial}{\partial \eta} \left(f(s, \eta t, \mu P_\delta(s, \eta t)) \right) \right| &= |\nabla f(s, \eta t, \mu P_\delta(s, \eta t)) \cdot (0, t, \mu t (\partial_2 P_\delta)(s, \eta t))| \\ &\lesssim \frac{C_N}{1 + |s|^N} (|t| + \mu |t| |\partial_2 P_\delta(s, \eta t)|); \end{aligned}$$

for some large B and some m it is then

$$\frac{\left| \frac{\partial}{\partial \eta} \left(f(s, \eta t, \mu P_\delta(s, \eta t)) \right) \right|}{|s||t|} \lesssim_N \frac{1 + \delta^{-B} |s|^m}{1 + |s|^N},$$

and therefore upon integration, for sufficiently large N it is

$$|II(\delta)| \lesssim_N \delta^{N^*},$$

for some $N^* > 0$, and therefore $II(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. A similar argument shows the same holds of $III(\delta)$.

It remains therefore to establish that $I(\delta) \rightarrow 0$. To do so, we split it further. Let $c > 0$ be a number to be fixed (it will depend on a, b), and split $I(\delta)$ as

$$\begin{aligned} & \int \int_{\substack{\delta^c < |s| \leq \delta^{-a} \\ \delta^c < |t| \leq \delta^{-b}}} [f(s, t, \mu P_\delta(s, t)) - f(s, t, \mu s^{m_0} t^{n_0})] \frac{ds}{s} \frac{dt}{t} \\ & + \int \int_{\substack{|s| \leq \delta^c \\ \delta^c < |t| \leq \delta^{-b}}} [f(s, t, \mu P_\delta(s, t)) - f(s, t, \mu s^{m_0} t^{n_0})] \frac{ds}{s} \frac{dt}{t} \\ & + \int \int_{\substack{\delta^c < |s| \leq \delta^{-a} \\ |t| \leq \delta^c}} [f(s, t, \mu P_\delta(s, t)) - f(s, t, \mu s^{m_0} t^{n_0})] \frac{ds}{s} \frac{dt}{t} \\ & + \int \int_{\substack{|s| \leq \delta^c \\ |t| \leq \delta^c}} [f(s, t, \mu P_\delta(s, t)) - f(s, t, \mu s^{m_0} t^{n_0})] \frac{ds}{s} \frac{dt}{t} \\ & =: I_1(\delta) + I_2(\delta) + I_3(\delta) + I_4(\delta). \end{aligned}$$

For $I_1(\delta)$, observe that in the range of integration it is

$$\begin{aligned} |f(s, t, \mu P_\delta(s, t)) - f(s, t, \mu s^{m_0} t^{n_0})| & \lesssim \|f\|_{C^1} |\mu| |P_\delta(s, t) - s^{m_0} t^{n_0}| \\ & \lesssim \sum_{\substack{m+n \leq \deg P \\ (m,n) \neq (m_0, n_0)}} \delta^{\alpha(m-m_0) + \beta(n-n_0)} \delta^{-am-bn} \lesssim \delta^\varepsilon \end{aligned}$$

for some $\varepsilon > 0$ and a, b sufficiently small. Therefore

$$|I_1(\delta)| \lesssim \int \int_{\substack{\delta^c < |s| \leq \delta^{-a} \\ \delta^c < |t| \leq \delta^{-b}}} \delta^\varepsilon \frac{ds}{|s|} \frac{dt}{|t|} \sim \delta^\varepsilon \log \delta,$$

and thus $I_1(\delta) \rightarrow 0$.

Now consider the contribution

$$\int \int_{\substack{|s| \leq \delta^c \\ |t| \leq \delta^c}} f(s, t, \mu P_\delta(s, t)) \frac{ds}{s} \frac{dt}{t}$$

to $I_4(\delta)$. Write $\mu P_\delta(s, t) = P_\delta^0(s, t) + P_\delta^1(s) + P_\delta^2(t)$, where $st \mid P_\delta^0(s, t)$; then we can remove $P_\delta^0(s, t)$ from the expression, since in this range of s, t one has $P_\delta^0(s, t) \lesssim |s||t|$ and therefore the error that arises is bounded by

$$\int \int_{\substack{|s| \leq \delta^c \\ |t| \leq \delta^c}} \|f\|_{C^1} |s||t| \frac{ds}{|s|} \frac{dt}{|t|} \lesssim \delta^{2c} \rightarrow 0.$$

We use a similar trick as above (although we exploit the removal of P_δ^0 by multiplying

the polynomials by the extra parameters, instead of their arguments; see below), namely we notice

$$\begin{aligned}
& f(s, t, P_\delta^1(s) + P_\delta^2(t)) - f(s, 0, P_\delta^1(s)) \\
&= \int_0^1 \frac{\partial}{\partial \eta} \left(\phi(s, \eta t, P_\delta^1(s) + \eta P_\delta^2(t)) \right) d\eta \\
&= \int_0^1 \int_0^1 \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi} \left(\phi(\xi s, \eta t, \xi P_\delta^1(s) + \eta P_\delta^2(t)) \right) d\xi d\eta \\
&\quad + \int_0^1 \frac{\partial}{\partial \eta} \left(\phi(0, \eta t, \eta P_\delta^2(t)) \right) d\eta.
\end{aligned}$$

Thus, if we let

$$E_\delta(s, t, \eta, \xi) := \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi} \left(\phi(\xi s, \eta t, \xi P_\delta^1(s) + \eta P_\delta^2(t)) \right),$$

we see that

$$\int \int_{\substack{|s| \leq \delta^c \\ |t| \leq \delta^c}} f(s, t, P_\delta^1(s) + P_\delta^2(t)) \frac{ds}{s} \frac{dt}{t} = \int_0^1 \int_0^1 \int \int_{\substack{|s| \leq \delta^c \\ |t| \leq \delta^c}} E_\delta(s, t, \eta, \xi) \frac{ds}{s} \frac{dt}{t} d\xi d\eta,$$

and since one can easily see that $|E_\delta(s, t, \eta, \xi)| \lesssim |s||t|$ it follows that this contribution is bounded by $O(\delta^{2c})$ as well, and thus vanishes in the limit $\delta \rightarrow 0$. The same calculations work (and are even easier) for the contribution with integrand $f(s, t, \mu s^{m_0} t^{n_0})$; and this proves

$$I_4(\delta) \rightarrow 0;$$

but the remaining terms $I_2(\delta), I_3(\delta)$ can be dealt with in a completely analogous manner, and therefore they vanish as well in the limit, thus proving the claim. The proof is then concluded.

Chapter 5

Other surfaces

In this chapter we explore the scope of the method developed so far. In particular, we show how the method can be successfully applied to certain surfaces of the form $(\varphi(s, t), t, \psi(s, t))$ that are beyond the reach of the Stein-Street theory as developed in [Str12],[SS13],[SS12]. While we cannot provide a general theory for such surfaces yet, we study some examples that, we think, already capture some relevant aspects of the problem.

5.1 Surfaces of the form $(s^k, t, P(s, t))$

We wish in this section to provide a sufficiently broad class of surfaces in \mathbb{H}^1 of the form $(s^k, t, P(s, t))$ for which the operator

$$H_{k,P}f(x) := \text{p.v.} \int \int_{|s| \leq 1, |t| \leq 1} f(x \cdot (s^k, t, P(s, t))^{-1}) \frac{ds}{s} \frac{dt}{t}$$

is $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded; here $k \in \mathbb{N} \setminus \{0\}$ and P is a polynomial in $\mathbb{R}[s, t]$. Observe that when k is odd a change of variable can reduce this to the (local) double Hilbert transform along the surface $(s, t, P(s^{1/k}, t))$. Thus we could think of this case as that of (a subclass of) polynomials with fractional exponents. Moreover, this case is obviously not covered by Theorem 3.1, as these fractional polynomials are not in general analytic in any neighbourhood of the origin.

We do not concern ourselves with establishing uniform bounds in the coefficients here; as such, one could extend the results below to a suitable class of real analytic functions without much effort. We will however take advantage of the fact that P is a polynomial in order to slightly simplify the proof.

Remark 5.1. Notice for $k > 1$ the boundedness of the operator cannot be addressed by the Stein-Street theory. Indeed, the pure powers¹ in this example are $s^k X$ and $t Y$, and therefore they do not control any monomial of the form $s^m t^n$ for $m < k$.

We state the main result of this section. The conclusion is somewhat interesting, in that the boundedness of the operator ends up depending on the Newton diagram of a certain part of P quite like in the Euclidean case; contrast this with the fact that

¹See Chapter I, pg. 11, for the definition of pure powers.

when $k = 1$ the Euclidean and Heisenberg case are quite different, since in the latter boundedness holds for all real-analytic functions φ , as we have seen in Chapter 3.

Theorem 5.1. *Let $k \in \mathbb{N} \setminus \{0\}$ be given and let $P(s, t) \in \mathbb{R}[s, t]$ be given by*

$$P(s, t) = \sum_{m, n} c_{mn} s^m t^n,$$

and suppose that st divides $P(s, t)$ (that is, $c_{m0} = c_{0n} = 0$). Let

$$Q(s, t) := \sum_{\substack{m, n: m < k, \\ n \geq 1}} c_{mn} s^m t^n;$$

that is, Q is the polynomial whose monomials are all the monomials of P of degree in s less than k .

Then the operator $H_{k, P}$ is $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ bounded if and only if all the corners (m, n) of $\mathcal{N}(Q)$, the Newton diagram of Q , are such that at least one among m, n is even.

The rest of the section is devoted to the proof of this theorem.

5.1.1 Proof of Theorem 5.1

For every m we denote by ℓ_m the smallest $n > 0$ such that $c_{mn} \neq 0$, if such an n exists. After the usual reductions, we will proceed to remove terms from the phase of the oscillatory integral that arises once one takes the Fourier transform in \mathbb{H}^1 .

In the following ω will denote as before monotone a increasing function such that

$$\int_0^1 \omega(r) \frac{dr}{r} < \infty.$$

By the same arguments as in Chapter 3, §3.1.1, we can reduce the problem of showing $L^2(\mathbb{H}^1) \rightarrow L^2(\mathbb{H}^1)$ boundedness of $H_{k, P}$ to that of showing the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ boundedness uniformly in λ, n of the operator

$$T_{k, P}^{\lambda, n} \phi(y) := \chi(2^n y) \sum_{\substack{j \geq n + C_0, \\ j \geq 0}} \text{p.v.} \int_{|y-t| \sim 2^{-j}} \left(\int_{|s| \leq 1} e^{i\lambda((y+t)s^k + P(s, y-t))} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt.$$

We will write $T_{k, P}$ in place of $T_{k, P}^{\lambda, n}$ to avoid cluttering notation.

Notice that by the same argument as in §4.4.2 we can safely replace $(y+t)s^k$ in the phase by $2ys^k$ (the error is then pointwise dominated by a multiple of $M\phi(y)$, which is acceptable).

The analysis will be in terms of certain quantities, the form of which the reader should now be easily able to guess: for λ, y fixed, define for every m s.t. ℓ_m exists

$$\mathcal{A}_m(j) := \frac{\lambda 2^{-j\ell_m}}{(\lambda|y|)^{m/k}}.$$

5.1.1.1

We show first of all that we can remove all monomials $s^m t^n$ for $m \geq k$ and $n \geq 1$. Fix such a monomial, and let the phase Φ be

$$\Phi(s) = 2ys^k + P(s, y - t);$$

then we claim for $|y - t| \sim 2^{-j}$ we can bound

$$\left| \int_{|s| \leq 1} e^{i\lambda\Phi(s)} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda(\Phi(s) - c_{mn}s^m(y-t)^n)} \frac{ds}{s} \right| \lesssim_P \omega(\mathcal{A}_m(j) \wedge \mathcal{A}_m(j)^{-1})$$

(the implicit constant depending on the coefficients of P). Indeed, suppose first that $\mathcal{A}_m(j) > 1$; then we can bound (since $n \geq \ell_m$)

$$\left| \int_{|s| \leq 1} e^{i\lambda\Phi(s)} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda(\Phi(s) - c_{mn}s^m(y-t)^n)} \frac{ds}{s} \right| \lesssim_P \lambda 2^{-j\ell_m},$$

and since $\ell_m \geq 1$ and $j \geq n + C_0$, one has

$$\lambda 2^{-j\ell_m} \mathcal{A}_m(j)^{k/(m-k)} = \left(\frac{2^{-j\ell_m}}{|y|} \right)^{m/(m-k)} < 1,$$

or in other words the difference above is bounded by

$$\mathcal{A}_m(j)^{-k/(m-k)} \lesssim \omega(\mathcal{A}_m(j)^{-1}).$$

Notice m cannot be equal to k in this case, since $\mathcal{A}_k(j)$ (if defined) is automatically less than 1.

Suppose then that $\mathcal{A}_m(j) \leq 1$. Fix $\theta \in [0, 1]$ to be chosen and split the difference to be estimated as

$$\begin{aligned} & \left| \int_{|s| \leq 1} e^{i\lambda\Phi(s)} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda(\Phi(s) - c_{mn}s^m(y-t)^n)} \frac{ds}{s} \right| \\ & \leq \left| \int_{|s| \leq \theta} e^{i\lambda\Phi(s)} \frac{ds}{s} - \int_{|s| \leq \theta} e^{i\lambda(\Phi(s) - c_{mn}s^m(y-t)^n)} \frac{ds}{s} \right| \\ & \quad + \left| \int_{\theta < |s| \leq 1} e^{i\lambda\Phi(s)} \frac{ds}{s} \right| \\ & \quad + \left| \int_{\theta < |s| \leq 1} e^{i\lambda(\Phi(s) - c_{mn}s^m(y-t)^n)} \frac{ds}{s} \right| \\ & =: I + II + III. \end{aligned}$$

Then we can bound

$$I \lesssim_P \lambda 2^{-j\ell_m} \theta^m.$$

The phase Φ is a polynomial in s and the coefficient of the term s^k is given by

$$2y + \sum_{\ell \geq 1} c_{k\ell} (y - t)^\ell,$$

and by choosing the constant C_0 sufficiently large depending only on the coefficients of P we can assume that

$$\sum_{\ell \geq 1} c_{k\ell} 2^{-j\ell} \ll |y|,$$

and therefore the coefficient of s^k has size $|y|$. Therefore, by Proposition 4.2 we can bound

$$II + III \lesssim_P (\lambda|y|\theta^k)^{-\delta}$$

for some $\delta > 0$. By choosing

$$\theta := \frac{1}{(\lambda|y|)^{1/k}} \frac{1}{\mathcal{A}_m(j)^\sigma} \wedge 1$$

and $0 < \sigma < 1/m$ we see that

$$I + II + III \lesssim_P \omega(\mathcal{A}_m(j)),$$

and the claim is proved.

By repeating the procedure above for all monomials $s^m t^n$ for $m \geq k, n \geq 1$ we obtain

$$\left| \int_{|s| \leq 1} e^{i\lambda\Phi(s)} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda(2ys^k + Q(s, y-t))} \frac{ds}{s} \right| \lesssim_P \sum_{m \geq k} \omega(\mathcal{A}_m(j) \wedge \mathcal{A}_m(j)^{-1});$$

by the same arguments as given in the previous chapters then, since the quantity on the right hand side is summable in j with sum $O_P(1)$, one can control the error by the Hardy-Littlewood maximal function:

$$|T_{k,P}\phi(y) - T_{k,Q}\phi(y)| \lesssim_P M\phi(y).$$

The boundedness of $T_{k,P}$ is therefore equivalent to that of $T_{k,Q}$.

5.1.1.2

Now we consider those $\mathcal{A}_m(j)$ for $m < k$. For $|y - t| \sim 2^{-j}$, we claim that if there is $m < k$ such that $\mathcal{A}_m(j) > 1$, then we can remove the term $2ys^k$ from the phase. Indeed, suppose $m < k$ is such that $\mathcal{A}_m(j) > 1$; we split, for $\theta \in [0, 1]$ to be chosen,

$$\begin{aligned} & \left| \int_{|s| \leq 1} e^{i\lambda(2ys^k + Q(s, y-t))} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda Q(s, y-t)} \frac{ds}{s} \right| \\ & \leq \left| \int_{|s| \leq \theta} e^{i\lambda(2ys^k + Q(s, y-t))} \frac{ds}{s} - \int_{|s| \leq \theta} e^{i\lambda Q(s, y-t)} \frac{ds}{s} \right| \\ & \quad + \left| \int_{\theta < |s| \leq 1} e^{i\lambda(2ys^k + Q(s, y-t))} \frac{ds}{s} \right| \\ & \quad + \left| \int_{\theta < |s| \leq 1} e^{i\lambda Q(s, y-t)} \frac{ds}{s} \right| \\ & =: I + II + III. \end{aligned}$$

Then we can immediately bound

$$I \lesssim \lambda |y| \theta^k.$$

On the other hand, the coefficient of s^m in the phase is

$$\sum_{\ell \geq \ell_m} c_{m\ell} t^\ell,$$

and observe that for $j \geq C_P$, with C_P a constant depending only on P , we have

$$\sum_{\ell \geq \ell_m} |c_{m\ell}| 2^{-j\ell} \sim |c_{m\ell_m}| 2^{-j\ell_m};$$

thus by discarding at most $O_P(1)$ values of j (which is acceptable, since $\int_{|s| \leq 1} \exp(i\lambda(2ys^k + Q(s, y-t))) ds/s$ is $O_P(1)$ by Lemma 3.3) we can assume that the coefficient of s^m is of size $\sim_P 2^{-j\ell_m}$, and therefore by Proposition 4.2 one can bound

$$II + III \lesssim_P (\lambda 2^{-j\ell_m} \theta^m)^{-\delta}$$

for some $\delta > 0$ (in general different from the δ encountered above). By choosing θ of the same form as before with $\sigma > 0$ sufficiently small we have

$$I + II + III \lesssim_P \omega(\mathcal{A}_m(j)^{-1}),$$

and the claim is proved.

5.1.1.3

Let

$$\mathcal{J} := \{j \geq \max\{n + C_0, C_P\} : \exists m < k \text{ s.t. } \mathcal{A}_m(j) > 1\}.$$

Observe first that if $j \geq \max\{n + C_0, C_P\}$ but $j \notin \mathcal{J}$ then for all $m < k$ it is $\mathcal{A}_m(j) \leq 1$ and we claim that

$$\left| \int_{|s| \leq 1} e^{i\lambda(2ys^k + Q(s, y-t))} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda 2ys^k} \frac{ds}{s} \right| \lesssim_P \omega(\mathcal{A}_m(j)).$$

Indeed, the proof is the same as the one given in §5.1.1.1 (notice we never used the fact that $m \geq k$ in the proof for $\mathcal{A}_m(j) < 1$). Thus the operator

$$\phi \mapsto \sum_{\substack{j \geq n + C_0, \\ j \geq C_P, \\ j \notin \mathcal{J}}} \int_{|y-t| \sim 2^{-j}} \left(\int_{|s| \leq 1} e^{i\lambda(2ys^k + Q(s, y-t))} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt$$

is uniformly bounded in λ, n : indeed, up to an error which is pointwise bounded by $O_P(M\phi(y))$, the operator is the same as

$$\phi \mapsto \left(\int_{|s| \leq 1} e^{i\lambda 2ys^k} \frac{ds}{s} \right) \sum_{\substack{j \geq n + C_0, \\ j \geq C_P, j \notin \mathcal{J}}} \int_{|y-t| \sim 2^{-j}} \frac{\phi(t)}{y-t} dt,$$

which is pointwise bounded by $O_P(H^* \phi(y))$.

It follows that we have reduced the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ boundedness of $T_{k,P}$ to that of the *translation invariant* operator

$$\phi \mapsto \sum_{j \in \mathcal{J}} \int_{|y-t| \sim 2^{-j}} \left(\int_{|s| \leq 1} e^{i\lambda Q(s,y-t)} \frac{ds}{s} \right) \frac{\phi(t)}{y-t} dt;$$

therefore it suffices to study its multiplier, which can be written as

$$M_{\lambda,y}(\xi) := \sum_{j \in \mathcal{J}} \sum_{i \geq 0} \text{p.v.} \int \int_{|s|,|t| \sim 1} e^{i(\xi 2^{-j} t + \lambda Q(2^{-i} s, 2^{-j} t))} \frac{ds}{s} \frac{dt}{t} =: \sum_{(i,j) \in \mathbb{N} \times J} I_{ij}.$$

We claim that this is bounded if and only if the Newton diagram $\mathcal{N}(Q)$ has the property that every corner (m, n) has at least one even coordinate².

For the positive direction, assume $\mathcal{N}(Q)$ has such a property. The bulk of the work has already been carried out in §4.4.3.2, for the proof of Lemma 4.9, and thus we will limit ourselves here to illustrate those arguments that are new. Recall we defined for each (m_0, n_0) s.t. $c_{m_0 n_0} \neq 0$

$$\Lambda_{m_0 n_0} := \{(i, j) \in \mathbb{N} \times J \text{ s.t. } \forall m, n \text{ s.t. } c_{mn} \neq 0, |c_{m_0 n_0}| 2^{-im_0 - jn_0} \geq |c_{mn}| 2^{-im - jn}\};$$

we claim that in order to establish the boundedness of the multiplier $M_{\lambda,y}$ it suffices to consider those (m_0, n_0) that are corners of the Newton diagram $\mathcal{N}(Q)$. Indeed, suppose (m, n) is not a corner (thus it is either internal to the diagram or to an edge of the diagram) and $c_{mn} \neq 0$. Then either one of the following cases must be verified:

- i) there exist two corners $(m_0, n_0), (m_1, n_1)$ of $\mathcal{N}(Q)$ such that $(m, n) = \alpha(m_0, n_0) + \beta(m_1, n_1)$ for some $\alpha, \beta > 0$ such that $\alpha + \beta \geq 1$ (thus (m, n) lies on the edge if $\alpha + \beta = 1$);
- ii) there exists a corner (m_0, n_0) such that for some $\alpha > 1$ it is $(m, n) = \alpha(m_0, n_0)$ (thus (m, n) lies on Γ_{m_0, n_0});
- iii) there exists a corner (m_0, n_0) with m_0 minimal (amongst the monomials of Q) such that $n > \frac{mn_0}{m_0}$ or with n_0 minimal such that $n < \frac{mn_0}{m_0}$.

We discuss case *i*), the other cases being similar. We exploit logarithmic convexity. We have that

$$2^{-im - jn} = (2^{-im_0 - jn_0})^\alpha (2^{-im_1 - jn_1})^\beta;$$

if $(i, j) \in \Lambda_{mn}$ then

$$|c_{mn}| 2^{-im - jn} \geq (|c_{m_0 n_0}| 2^{-im_0 - jn_0})^{\alpha/(\alpha+\beta)} (|c_{m_1 n_1}| 2^{-im_1 - jn_1})^{\beta/(\alpha+\beta)}.$$

But since $\alpha + \beta \geq 1$, this can only happen for a finite number of (i, j) , a number which is bounded by $O_P(1)$. In other words, $|\Lambda_{mn}| = O_P(1)$, and since clearly $I_{ij} = O(1)$ we have that those exponent pairs (m, n) that are not corners can only contribute $O_P(1)$ to $\sum_{(i,j) \in \mathbb{N} \times J} |I_{ij}|$.

²That this is so already follows from [CWW00]. Here we reprove this statement with the means developed for Lemma 4.9.

Take then (m_0, n_0) corner of $\mathcal{N}(Q)$. By the same argument as in §4.4.3.2 we reduce to estimate $\sum_{(i,j) \in \Lambda_{m_0 n_0}} I'_{ij}$, where

$$I'_{ij} := \text{p.v.} \int \int_{|s|, |t| \sim 1} e^{i(\xi 2^{-j} t + \lambda \sum_{(m,n) \in \Gamma_{m_0 n_0}} c_{mn} 2^{-im-jn} s^m t^n)} \frac{ds}{s} \frac{dt}{t}.$$

Let p, q be those integers such that $\frac{m_0}{n_0} = \frac{p}{q}$ and $\gcd(p, q) = 1$. If either one of p, q is even, then either all the $(m, n) \in \Gamma_{m_0 n_0}$ have first coordinate even, or they all have the second coordinate even; in this case, $I'_{ij} = 0$ by the oddness of the kernel, and there is nothing else to prove.

Therefore assume that p, q are both odd, and so that m_0, n_0 are both even. Let $\gamma > 1$ be such that $\gamma(m_0, n_0)$ is the smallest element (with respect to the obvious ordering) of $\Gamma_{m_0 n_0}$ that is not (m_0, n_0) . Then by evenness of $\xi t + \lambda c_{m_0 n_0} s^{m_0} t^{n_0}$ we have

$$\begin{aligned} I'_{ij} &= \text{p.v.} \int \int_{|s|, |t| \sim 1} e^{i(\xi 2^{-j} t + \lambda \sum_{(m,n) \in \Gamma_{m_0 n_0}} c_{mn} 2^{-im-jn} s^m t^n)} \\ &\quad - e^{i(\xi 2^{-j} t + \lambda c_{m_0 n_0} 2^{-im_0-jn_0} s^{m_0} t^{n_0})} \frac{ds}{s} \frac{dt}{t} \\ &= O_P(\lambda(2^{-im_0-jn_0})^\gamma). \end{aligned} \tag{5.1.1}$$

Moreover, by discarding at most some other $O_P(1)$ pairs (i, j) , we have

$$|\partial_s^{m_0} \partial_t^{n_0} (\xi 2^{-j} t + \lambda \sum_{(m,n) \in \Gamma_{m_0 n_0}} c_{mn} 2^{-im-jn} s^m t^n)| \gtrsim_P |c_{m_0 n_0}| 2^{-im_0-jn_0},$$

and therefore by Proposition 2.3 we can bound

$$|I'_{ij}| \lesssim_P (\lambda 2^{-im_0-jn_0})^{-\delta} \tag{5.1.2}$$

for some positive $\delta > 0$. But then (5.1.1) and (5.1.2) together imply that $\sum_{(i,j) \in \Lambda_{m_0 n_0}} |I'_{ij}| = O_P(1)$: indeed, if $\lambda \leq 1$ it suffices to sum using only estimate (5.1.1), and if $\lambda > 1$ we have

$$\begin{aligned} \sum_{(i,j) \in \Lambda_{m_0 n_0}} |I'_{ij}| &\lesssim_P \sum_{k \geq 0} \sum_{\substack{(i,j) \in \Lambda_{m_0 n_0}, \\ im_0 + jn_0 = k}} \min\{\lambda 2^{-\gamma k}, (\lambda 2^{-k})^{-\delta}\} \\ &\lesssim \sum_{0 \leq k \leq \frac{1+\delta}{\gamma+\delta} \log_2 \lambda} k (\lambda 2^{-k})^{-\delta} + \sum_{k \geq \frac{1+\delta}{\gamma+\delta} \log_2 \lambda} k \lambda 2^{-\gamma k} \\ &\lesssim \lambda^{-\varepsilon} \end{aligned}$$

for some $\varepsilon > 0$, since $\gamma > 1$. This concludes the proof of the *if* part of the claim.

As for the *only if* part of the statement, we observe that it cannot follow from the same argument in the proof of Proposition 4.6, Chapter 4, §4.5 (except in the special case where st is a monomial in Q), because there are no three parameter automorphic dilations in \mathbb{H}^1 . Thus we have to argue differently: we notice that by the first part of the proof, boundedness of the operator is equivalently reduced to L^∞ boundedness uniform in λ, y, ξ of the multiplier $M_{\lambda, y}(\xi)$ above. It suffices to take $M_{\lambda, y}(0)$ and

observe that this reduces to the integral

$$\int_{|t| \in \mathcal{J}} \int_{|s| \leq 1} e^{i\lambda st} \frac{ds}{s} \frac{dt}{t}$$

where $\mathcal{J} = \{t \text{ s.t. } \exists j \in J, |t| \sim 2^{-j}\}$; the cardinality of J can be made arbitrarily large with the appropriate choice of λ, y . A precise calculation proves that this integral has size at least $\sim \log \lambda$, and thus the operator cannot be bounded uniformly in λ . Such a calculation is carried out explicitly in [Pat08]; we omit the details.

5.2 A further example

In this section we explore another example of surface of the form $(\varphi(s, t), t, \psi(s, t))$ slightly more general from the previous one, to gain a feel for the technical difficulties that stand in the way of extending the L^2 theory developed for $(s, t, \varphi(s, t))$. We are not concerned with presenting statements that are as general as possible but rather with illustrating particular issues and behaviours, so we will make simplifying assumptions where useful.

We choose an example to illustrate how the boundedness of the operator can depend on the properties of φ in $(\varphi(s, t), t, \psi(s, t))$. Consider the surface

$$s, t \mapsto (s^{k_0} + s^h t^\ell, t, P(s, t)),$$

where we assume

- i) $0 < h < k_0$,
- ii) $st \mid P(s, t)$ (for simplicity).

For this surface and the associated analytic vector fields, one clearly cannot control the mixed powers by the pure powers, and thus Stein-Street theory does not apply to it.

Define polynomials $Q_k \in \mathbb{R}[t]$ by

$$P(s, t) =: \sum_{k \geq 0} s^k Q_k(t),$$

and let

$$P_0(s, t) := \sum_{0 \leq k < k_0} s^k Q_k(t),$$

and $P_1 := P - P_0$. Let ℓ_k be the largest integer such that $t^{\ell_k} \mid Q_k(t)$, which by our assumption is necessarily bigger or equal than 1.

The boundedness of operator

$$f \mapsto H_{k_0, h, \ell, P} \phi(y) := \text{p.v.} \int \int_{|s|, |t| \leq 1} f(x \cdot (s^{k_0} + s^h t^\ell, t, P(s, t))^{-1}) \frac{ds}{s} \frac{dt}{t}$$

is reduced as usual to the boundedness uniformly in λ, n of

$$\phi \mapsto T_{k_0, h, \ell, P} \phi(y) := \chi(2^n y) \sum_{\substack{j \geq n + C_0, \\ j \geq C_P}} \text{p.v.} \int_{|y-t| \sim 2^{-j}} m_{k_0, h, \ell, P}(\lambda; y, t) \frac{\phi(t)}{y-t} dt,$$

where C_P is sufficiently large that we can assume $|Q_k(2^{-j})| \sim_P 2^{-j\ell_k}$, and where

$$m_{k_0, h, \ell, P}(\lambda; y, t) := \text{p.v.} \int_{|s| \leq 1} e^{i\lambda((y+t)(s^{k_0} + s^h(y-t)^\ell) + P(s, y-t))} \frac{ds}{s};$$

moreover, by the usual argument we can replace $y+t$ by $2y$, so that the phase in $m_{k_0, h, \ell, P}$ becomes

$$\Phi(s) := 2ys^{k_0} + 2ys^h(y-t)^\ell + P_0(s, y-t) + P_1(s, y-t).$$

If we define the quantities

$$\mathcal{A}_k(j) := \frac{\lambda 2^{-j\ell_k}}{(\lambda|y|)^{k/k_0}}$$

then we can see that the same argument as in §5.1.1.1 works here as well: this is because $k_0 > h$, so that one has

$$\left(\frac{d}{ds} \right)^{k_0} \Phi(s) = 2yk_0! + \partial_s^{k_0} P_1(s, y-t), \quad (5.2.1)$$

that is, the term $2ys^h(y-t)^\ell$ is not relevant for sufficiently high derivatives. Thus we can remove term $P_1(s, y-t)$ from the phase at the price of an error bounded pointwise by $\omega(\mathcal{A}_k(j) \wedge \mathcal{A}_k^{-1}) M\phi(y)$, which is summable in j , and then the $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ boundedness of the operator $T_{k_0, h, \ell, P}$ is equivalent to that of operator T_{k_0, h, ℓ, P_0} .

Now we need to introduce another quantity to measure the relative strength of the terms $2ys^{k_0}$ and $2ys^h(y-t)^\ell$; we define

$$\mathcal{B}(j) := \frac{\lambda|y|2^{-j\ell}}{(\lambda|y|)^{h/k_0}} = 2^{-j\ell} (\lambda|y|)^{(k_0-h)/k_0}.$$

For those j such that $\mathcal{B}(j) \leq 1$, we show we can remove the term $2ys^h(y-t)^\ell$ from the phase of m_{k_0, h, ℓ, P_0} at the usual price of a summable error (in j). Indeed,

$$\left| \int_{|s| \leq \theta} e^{i\lambda(2ys^{k_0} + 2ys^h(y-t)^\ell + P_0(s, y-t))} - e^{i\lambda(2ys^{k_0} + P_0(s, y-t))} \frac{ds}{s} \right| \lesssim \lambda|y|2^{-j\ell}\theta^h,$$

and the k_0 -th derivative of the phase is simply $2yk_0!$ now, thus by Proposition 4.2

$$\left| \int_{\theta < |s| \leq 1} e^{i\lambda(2ys^{k_0} + 2ys^h(y-t)^\ell + P_0(s, y-t))} \frac{ds}{s} \right| \lesssim (\lambda|y|\theta^{k_0})^{-\delta}$$

for some $\delta > 0$; the same estimate holds for the integral with phase $2ys^{k_0} + P_0(s, y-t)$, clearly. Thus if we choose

$$\theta := \frac{1}{(\lambda|y|)^{1/k_0}} \frac{1}{\mathcal{B}(j)^\sigma}$$

with $\sigma > 0$ sufficiently small, we have

$$\left| \int_{|s| \leq 1} e^{i\lambda(2ys^{k_0} + 2ys^h(y-t)^\ell + P_0(s, y-t))} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda(2ys^{k_0} + P_0(s, y-t))} \frac{ds}{s} \right| \lesssim \omega(\mathcal{B}(j)).$$

Thus for those j such that $\mathcal{B}(j) \leq 1$ we have reduced to the operator

$$\phi \mapsto \chi(2^n y) \sum_{\substack{j \geq n + C_0, \\ j \geq C_P, \mathcal{B}(j) \leq 1}} \text{p.v.} \int_{|y-t| \sim 2^{-j}} m_{k_0, 0, 0, P_0}(\lambda; y, t) \frac{\phi(t)}{y-t} dt;$$

but this operator is of the form treated in §5.1, and thus we know by Theorem 5.1 that an equivalent condition for this to be bounded is that for every corner (m, n) of $\mathcal{N}(P_0)$ at least one amongst m, n is even.

Assume then that $\mathcal{B}(j) > 1$, which can happen only for finitely many j , but their cardinality is not uniformly bounded in λ, n . In this case we can remove the term $2ys^{k_0}$ from the phase instead. Indeed,

$$\left| \int_{|s| \leq \theta} e^{i\lambda(2ys^{k_0} + 2ys^h(y-t)^\ell + P_0(s, y-t))} - e^{i\lambda(2ys^h(y-t)^\ell + P_0(s, y-t))} \frac{ds}{s} \right| \lesssim \lambda|y|\theta^{k_0}.$$

Then notice that the coefficient of degree h in s of the phase is given by

$$2y(y-t)^\ell + Q_h(y-t),$$

and thus if $|y|2^{-j\ell} \ll$ or $\gg 2^{-j\ell h}$ we have by Proposition 4.2

$$\left| \int_{\theta < |s| \leq 1} e^{i\lambda(2ys^{k_0} + 2ys^h(y-t)^\ell + P_0(s, y-t))} \frac{ds}{s} \right| \lesssim (\lambda|y|2^{-j\theta^h})^{-\delta'};$$

on the other hand, the above condition can fail for at most $O(1)$ indices j , which can always be discarded. The same holds of the other integral as well. Thus we have by the same choice of θ as before that for all j such that $\mathcal{B}(j) > 1$ (except at most $O(1)$)

$$\left| \int_{|s| \leq 1} e^{i\lambda(2ys^{k_0} + 2ys^h(y-t)^\ell + P_0(s, y-t))} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda(2ys^h(y-t)^\ell + P_0(s, y-t))} \frac{ds}{s} \right| \lesssim \omega(\mathcal{B}(j)^{-1}),$$

which is summable in j .

It remains to establish whether the operator T_{0, h, ℓ, P_0} is bounded, so introduce quantities

$$\mathcal{C}_k(j) := \frac{\lambda 2^{-j\ell k}}{(\lambda|y|2^{-j\ell})^{k/h}}$$

and suppose for simplicity that $\forall k < k_0, \ell_k/k \neq \ell/h$. If there exists $k < k_0$ such that $\mathcal{C}_k(j) > 1$, then we claim we can remove the term $2ys^h(y-t)^\ell$ from the phase. Indeed, by the same argument we have used over and over, for all j such that $\mathcal{B}(j) > 1$, $\mathcal{C}_k(j)$ except at most $O(1)$ one can bound

$$\left| \int_{|s| \leq 1} e^{i\lambda(2ys^h(y-t)^\ell + P_0(s, y-t))} \frac{ds}{s} - \int_{|s| \leq 1} e^{i\lambda P_0(s, y-t)} \frac{ds}{s} \right| \lesssim \omega(\mathcal{C}_k(j)^{-1}),$$

which is a summable error; thus one is reduced to operator $T_{0,0,0,P_0}$, which is translation invariant, and it can be verified it is bounded.

If however $\forall k < k_0$ it is $\mathcal{C}_k(j) \leq 1$ then we claim one can remove the term $P_0(s, y - t)$ instead, and the error is summable in j again. This is the same argument we have given before, and verifying the details is left as an easy exercise to the reader. Thus in this case we have reduced to operator $T_{0,h,\ell,0}$. Now, if h is even we see that $m_{0,h,\ell,0} = 0$ identically, and therefore we have nothing to bound. If h is instead odd, then by the same calculations as in Chapter 1 for $\int \int e^{i\lambda st} / st \, ds \, dt$ we see that up to an error summable in j we can replace $m_{0,h,\ell,0}$ by $\text{sgn}(y(y-t)^\ell) = \text{sgn}(y)\text{sgn}((y-t)^\ell)$. While the factor $\text{sgn}(y)$ is harmless, as we have seen before (in Example 4.1, Chapter 4) the operator will be bounded uniformly if ℓ is even, and will not necessarily be bounded uniformly if ℓ is odd (one can indeed concoct a simple example of a polynomial P such that this is the case). Thus we see that to have uniform boundedness for $T_{0,h,\ell,0}$ we need to have that at least one amongst h, ℓ is even.

This discussion has shown therefore that, as is to be expected, to treat surfaces of the form $(\varphi(s, t), t, \psi(s, t))$ one has to impose conditions on both φ and ψ , in general.

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