

METHODS FOR FINDING RIEMANN-GREEN
FUNCTIONS

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Introduction.

Riemann, in the last century, derived a method of solution for general linear hyperbolic partial differential equations in two independent variables based upon a certain function known as the Riemann-Green function. The aim of this thesis is to bring together some of the methods used to find these functions. This survey is not in any way meant to be exhaustive. Its purpose is to show a unity and cohesion to the development of the work done in this area over the last thirty years, including in particular, the contributions of Cohn, Copson, Mackie, Papadakis and Wood.

The material included can basically be divided into three categories. First, the connection is shown between the Riemann function and the Green's function based upon the work of Mackie. Then, transform techniques for general separable equations are discussed in connection with the work of Copson. Here a certain integral formula is derived. Then, an 'addition formula' of Papadakis and Wood is established which simplifies in the evaluation of Copson's integral formula. The final part of this paper discusses the work of Cohn which is based upon the development of the Riemann function for self-adjoint equations as a Neumann series. Here the Riemann function is required to be functionally dependent upon a single term in the series. An investigation is then made of all possible cases in which the partial differential equation satisfied by the Riemann function can be reduced to an ordinary differential equation. The result obtained is that no new functions can in fact be found and that Cohn's method is exhaustive. Wood has also obtained this result while considering a more general equation.

The Riemann functions discussed in this paper can then be divided into two categories, those which can be expanded into a power series of one variable, as in Cohn's work, and those whose expansions involve multiple power series, such as Papadakis and Wood.

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Declaration.

I declare that this thesis was composed by the author at the University of Edinburgh for the Degree of M.Phil.

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Chapter 1.

Section (1.1)

Linear second order partial differential equations in two independent variables are classified into three main types: elliptic, parabolic and hyperbolic. The concept of characteristics forms a basis of distinction between these three types. The well-posedness of boundary value problems is intimately connected with the dependence of the solution on the form of the boundary conditions. This concept is most dramatic in the distinction between the well-posed problems for hyperbolic and elliptic equations. In general, the meaning of well-posedness is that the solution exists, is unique, and varies continuously with the boundary data. This principle is known as Hadamard's criterion. This paper will deal exclusively with hyperbolic equations where the characteristics are real and distinct. The most general linear hyperbolic equation in two independent variables is

$$L(u) \equiv \frac{\partial^2 u}{\partial x \partial y} + a(x,y) \frac{\partial u}{\partial x} + b(x,y) \frac{\partial u}{\partial y} + c(x,y)u = F(x,y) , \quad (1.1.1)$$

where the characteristics are parallel to the coordinate axes.

The two most common types of well-posed problems associated with hyperbolic equations are the Cauchy problem and the Characteristic boundary value problem. In the former u and its normal derivative are specified on a curve which is nowhere tangent to a characteristic line, while in the latter u is specified on two intersecting characteristic lines.

Figure/

Figure 1.

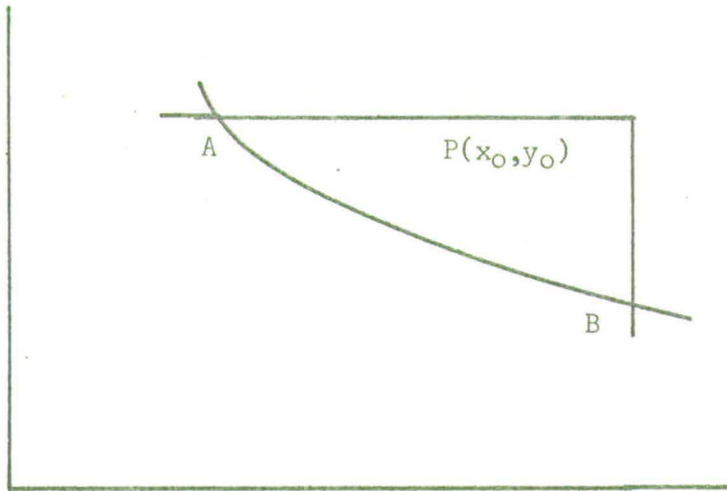
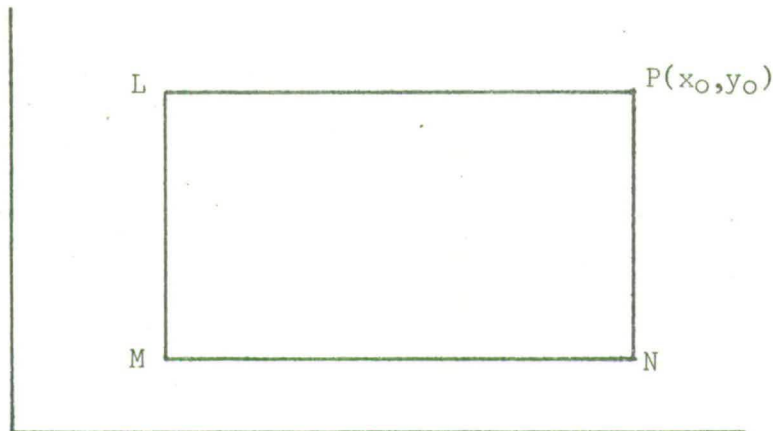


Figure 2.



Riemann developed a general method for the solution of this type of equation utilizing the particular nature of the characteristics. The motivation for this method can be seen from Gauss' theorem

$$\iint_D \operatorname{div} \vec{F} \, dA = \oint_C \vec{F} \cdot \vec{n} \, ds \quad (1.1.2)$$

Considering the Cauchy problem where the solution is sought at an arbitrary point $P(x_0, y_0)$, the domain of integration D is constructed by joining the characteristic lines passing through P and intersecting the data curve at two points A and B . This area is the range of influence for the point P , that is the solution/

solution at P will only depend upon the boundary data along the curve between the points A and B . A divergence can be formed using a particular linear combination of differential operators L and its adjoint \tilde{L} , defined as

$$\tilde{L}(v) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial}{\partial x} (a(x,y)v) - \frac{\partial}{\partial y} (b(x,y)v) + c(x,y)v, \quad (1.1.3)$$

in the following manner

$$\begin{aligned} vL(u) - u\tilde{L}(v) &= \frac{\partial}{\partial x} \left(\frac{1}{2} v \frac{\partial u}{\partial y} - \frac{1}{2} u \frac{\partial v}{\partial y} + a uv \right) \\ &\quad + \frac{\partial}{\partial y} \left(\frac{1}{2} v \frac{\partial u}{\partial x} - \frac{1}{2} u \frac{\partial v}{\partial x} + b uv \right). \end{aligned}$$

When Gauss' theorem is applied and the domain chosen in this particular manner, the equation above assumes the form

$$\begin{aligned} \iint_D (vL(u) - u\tilde{L}(v)) dA &= \oint_C \left(\frac{1}{2} v \frac{\partial u}{\partial y} - \frac{1}{2} u \frac{\partial v}{\partial y} + a uv \right) dy \\ &\quad - \left(\frac{1}{2} v \frac{\partial u}{\partial x} - \frac{1}{2} u \frac{\partial v}{\partial x} + b uv \right) dx, \end{aligned} \quad (1.1.4)$$

where now the line integral can be simplified by choosing v in a special manner. First, however, the line integral along, for example AP can be written as

$$\begin{aligned} & - \int_A^P \left(\frac{1}{2} v \frac{\partial u}{\partial x} - \frac{1}{2} u \frac{\partial v}{\partial x} + b uv \right) dx \\ &= - \int_A^P \left(\frac{1}{2} v \frac{\partial u}{\partial x} + \frac{1}{2} u \frac{\partial v}{\partial x} + (bv - \frac{\partial v}{\partial x})u \right) dx \\ &= - \int_A^P \left(\frac{1}{2} \frac{\partial}{\partial x} (uv) + (bv - \frac{\partial v}{\partial x})u \right) dx \\ &= -\frac{1}{2}(uv)|_P + \frac{1}{2}(uv)|_A - \int_A^P (bv - \frac{\partial v}{\partial x})u dx. \end{aligned}$$

Also the line integral along PB can be similarly rewritten, hence

$$\begin{aligned} \iint_D (vL(u) - u\tilde{L}(v)) dA &= \frac{1}{2} (uv)|_A + \frac{1}{2} (uv)|_B - (uv)|_P \\ &\quad - \int_A^P (bv - \frac{\partial v}{\partial x})u dx - \int_B^P (av - \frac{\partial v}{\partial y})u dy \\ &\quad + \int_A^B \left(\frac{1}{2} v \frac{\partial u}{\partial y} - \frac{1}{2} u \frac{\partial v}{\partial y} + a uv \right) dy - \left(\frac{1}{2} v \frac{\partial u}{\partial x} - \frac{1}{2} u \frac{\partial v}{\partial x} + b uv \right) dx. \end{aligned}$$

Then/

Then v is defined in the following manner

$$\begin{aligned}
\tilde{L}(v) &= 0 \\
\left. \begin{aligned}
\frac{\partial v}{\partial y} - av &= 0 && \text{on } x = x_0 \\
\frac{\partial v}{\partial x} - bv &= 0 && \text{on } y = y_0 \\
v &= 1 && \text{at } x = x_0, y = y_0.
\end{aligned} \right\} \tag{1.1.5}
\end{aligned}$$

The function v , so defined, is known as the Riemann or Riemann-Green function, usually denoted by $R(x,y; x_0,y_0)$. As with the Green's function, the Riemann function is a function of four variables, x, y, x_0 and y_0 , defined with respect to a parameter point (x_0, y_0) . However the Riemann function is defined, as a function of x and y , only in a neighbourhood of this parameter point, since the partial differential equation and boundary conditions which uniquely determine it are defined specifically with reference to such a point where the solution of the particular boundary value problem is sought.

Once the Riemann function has been found the evaluation of u at P can be made explicitly in terms of known quantities

$$\begin{aligned}
u(x_0,y_0) &= \frac{1}{2}(uR)_A + \frac{1}{2}(uR)_B \\
&+ \int_A^B \left(\frac{1}{2} R \frac{\partial u}{\partial y} - \frac{1}{2} u \frac{\partial R}{\partial y} + auR \right) dy \\
&- \left(\frac{1}{2} R \frac{\partial u}{\partial x} - \frac{1}{2} u \frac{\partial R}{\partial x} + buR \right) dx \\
&- \iint_{P_0AB} R(x,y; x_0,y_0) F(x,y) dx dy. \tag{1.1.6}
\end{aligned}$$

Section (1.2)

The characteristic boundary-value problem mentioned earlier can be solved in an analogous manner. The geometry of the problem is now altered so that the rectangular domain D' is formed by joining the characteristics passing through the point (x_0, y_0) and intersecting those at two points L and N on which data has been specified (see Figure 2).

Equation (1.1.4) can be used, v is the Riemann function defined by (1.1.5) and the corresponding range of influence D' is substituted for D to give

$$\iint_{D'} (R L(u) - u \tilde{L}(R)) dA = \oint \left(\frac{1}{2} R \frac{\partial u}{\partial y} - \frac{1}{2} u \frac{\partial R}{\partial y} + a u R \right) dy - \left(\frac{1}{2} R \frac{\partial u}{\partial x} - \frac{1}{2} u \frac{\partial R}{\partial x} + b u R \right) dx .$$

After breaking the integral up, inserting the appropriate limits of integration, and some other small modifications, the equation can be written as

$$\begin{aligned} \iint_{D'} (R L(u) - u \tilde{L}(R)) dA &= \left\{ \int_L^P - \int_M^N \right\} \left(\frac{1}{2} \frac{\partial}{\partial x} (uR) + \left(b R - \frac{\partial R}{\partial x} \right) u \right) dx \\ &+ \left\{ \int_N^P - \int_M^L \right\} \left(\frac{1}{2} \frac{\partial}{\partial y} (uR) + \left(a R - \frac{\partial R}{\partial y} \right) u \right) dy . \end{aligned} \quad (1.2.1)$$

Now using the definition of the Riemann function (1.1.5) and performing some of the integrations the above equation becomes

$$\begin{aligned} \iint_{D'} R(x, y; x_0, y_0) F(x, y) dx dy &= \frac{1}{2} (uR)_P - \frac{1}{2} (uR)_L \\ &- \frac{1}{2} (uR)_N + \frac{1}{2} (uR)_M - \int_M^N \left(b R - \frac{\partial R}{\partial x} \right) u dx \\ &+ \frac{1}{2} (uR)_P - \frac{1}{2} (uR)_N - \frac{1}{2} (uR)_L + \frac{1}{2} (uR)_M \\ &- \int_M^L \left(a R - \frac{\partial R}{\partial y} \right) u dy . \end{aligned}$$

The/

The solution $u(x_0, y_0)$ can be written explicitly in terms of known quantities as

$$\begin{aligned}
 u(x_0, y_0) &= (uR)_L + (uR)_N - (uR)_M \\
 &+ \int_M^N (bR - \frac{\partial R}{\partial x})u \, dx \\
 &+ \int_M^L (aR - \frac{\partial R}{\partial y})u \, dy \\
 &+ \iint_{D'} R(x, y; x_0, y_0) F(x, y) dx dy .
 \end{aligned} \tag{1.2.5}$$

Section (1.3)

A natural question to ask at this stage is whether such a function exists, and if so, is it unique. The easiest method to establish this result will be with the aid of the contraction mapping theorem. This can be stated in two parts, first defining a contraction operator T as, given a metric space X with an operator T defined on X , then T is a contraction operator if there exists a positive constant $\alpha < 1$, such that for any x, x' elements of X then

$$e(Tx, Tx') \leq \alpha e(x, x')$$

where e is a metric associated with the metric space X . Once it has been established that T is a contraction operator, the following theorem furnishes the existence and uniqueness of the Riemann function. If the operator T maps a complete metric space X into itself, then there exists a unique fixed point and this point can be obtained by the method of successive approximations for any point x_0 , an element of X .

The metric space X will be the space of twice differentiable functions and the distance function e will be chosen as follows. But first the operator T must be defined. As has been previously shown, the Riemann function for the operator L is the solution of the adjoint equation $\tilde{L}(R) = 0$. This differential equation with the appropriate boundary conditions can be converted into an integral equation by integrating $\tilde{L}(R) = 0$ over a rectangular domain, as

$$\begin{aligned}
 & \iint_{x_0 y_0}^{x y} \tilde{L}(R) \, dx_1 \, dy_1 \\
 &= \iint_{x_0 y_0}^{x y} (R_{xy} - (aR)_x - (bR)_y + cR) \, dx_1 \, dy_1 \\
 &= R(x,y; x_0, y_0) - 1 - \int_{x_0}^x a(x_1, y) R(x_1, y; x_0, y_0) \, dx_1 \\
 &\quad - \int_{y_0}^y b(x, y_1) R(x, y_1; x_0, y_0) \, dy_1 + \iint_{x_0 y_0}^{x y} c(x_1, y_1) R(x_1, y_1; x_0, y_0) \, dx_1 \, dy_1 = 0 .
 \end{aligned}$$

Thus the operator T will be defined as

$$\begin{aligned}
 T(R) = & 1 + \int_{x_0}^x a(x_1, y_1) R(x_1, y_1; x_0, y_0) \, dx_1 \\
 & + \int_{y_0}^y b(x, y_1) R(x, y_1; x_0, y_0) \, dy_1 - \iint_{x_0 y_0}^{x y} c(x_1, y_1) R(x_1, y_1; x_0, y_0) \, dx_1 \, dy_1 ,
 \end{aligned}$$

defined on the space of twice differential functions X.

Considering R, R* as elements of X, then T is a contraction operator if

$$\| T(R) - T(R^*) \| \leq \alpha \| R - R^* \| ,$$

where $\alpha < 1$ and the metric will be defined as

$$\| \quad \| = \max_{(x,y) \in D} | \quad | , \text{ where } D$$

is the domain of integration. Also the coefficients of the equation will be bounded in the following manner. Let

$$M = \max \{ \|a\| , \|b\| , \|c\| \};$$

then

$$\begin{aligned}
 & \| T(R) - T(R^*) \| \\
 &= \left\| \int_{y_0}^y a(R - R^*) \, dy_1 + \int_{x_0}^x b(R - R^*) \, dx_1 \right. \\
 &\quad \left. - \iint_{x_0 y_0}^{x y} c(R - R^*) \, dx_1 \, dy_1 \right\| \\
 &\leq M \left(\int_{y_0}^y \|R - R^*\| \, dy_1 + \int_{x_0}^x \|R - R^*\| \, dx_1 \right. \\
 &\quad \left. + \iint_{x_0 y_0}^{x y} \|R - R^*\| \, dx_1 \, dy_1 \right)
 \end{aligned}$$

$$\text{If/} \quad \leq M (|x-x_0| + |y-y_0| + |(x-x_0)(y-y_0)|) \|R - R^*\| .$$

If the point (x,y) is restricted to lie in a domain D such that

$$M(|x-x_0| + |y-y_0| + |(x-x_0)(y-y_0)|) < \alpha < 1 ,$$

then the operator T is a contraction operator, therefore has a unique fixed point R such that $T(R) = R$ which can be found by the method of successive approximations. Thus the existence of the Riemann function has been established in a neighbourhood of the point (x_0, y_0) .

Section (1.4)

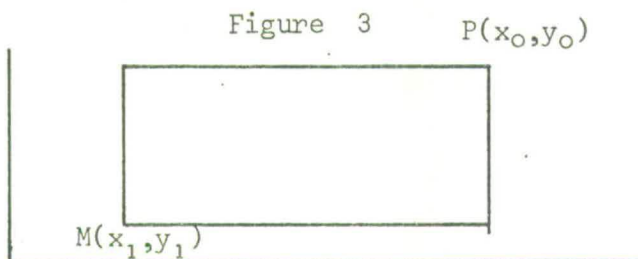
A useful result will be to establish the explicit relationship that exists between the Riemann function of the differential operator L and the Riemann function of the adjoint operator \tilde{L} . For this purpose consider the following pair of characteristic boundary value problems.

$$\begin{aligned}
 \tilde{L}(R) &= 0 \\
 \frac{\partial R}{\partial y} - aR &= 0 \quad \text{on} \quad PN \quad (x = x_0) \\
 \frac{\partial R}{\partial x} - bR &= 0 \quad \text{on} \quad PL \quad (y = y_0) \\
 R &= 1 \quad \text{at} \quad P \quad \begin{matrix} x = x_0 \\ y = y_0 \end{matrix},
 \end{aligned} \tag{1.4.1}$$

defining the Riemann function for the operator L , and choosing an arbitrary point $M(x_1, y_1)$

$$\begin{aligned}
 L(\tilde{R}) &= 0 \\
 \frac{\partial \tilde{R}}{\partial y} + a\tilde{R} &= 0 \quad \text{on} \quad ML \quad (x = x_1) \\
 \frac{\partial \tilde{R}}{\partial x} + b\tilde{R} &= 0 \quad \text{on} \quad MN \quad (y = y_1) \\
 \tilde{R} &= 1 \quad \text{at} \quad M \quad \begin{pmatrix} x = x_1 \\ y = y_1 \end{pmatrix},
 \end{aligned} \tag{1.4.2}$$

defining the Riemann function for the operator \tilde{L} .



The /

The easiest method to resolve these two problems simultaneously will be to return to the solution of the characteristic boundary value problem of Section (1.2). Equation (1.2.3) can be used where the unknown u will be the adjoint Riemann function \tilde{R} . The double integral will vanish and the remaining terms will be

$$\begin{aligned} \tilde{R}(x,y;x_1,y_1) \Big|_P \begin{pmatrix} x = x_0 \\ y = y_0 \end{pmatrix} &= \tilde{R} R \Big|_L + \tilde{R} R \Big|_N \\ &- \tilde{R} R \Big|_M + \int_M^N (bR - \frac{\partial R}{\partial x}) \tilde{R} dx \\ &+ \int_M^L (aR - \frac{\partial R}{\partial y}) \tilde{R} dy . \end{aligned}$$

The first two terms on the right hand side can be eliminated by integrating $-\frac{\partial R}{\partial x} \tilde{R}$ and $-\frac{\partial R}{\partial y} \tilde{R}$ by parts, also then the boundary conditions for \tilde{R} can be applied, that is

$$\begin{aligned} \tilde{R}(x,y;x_1,y_1) \Big|_P \begin{pmatrix} x = x_0 \\ y = y_0 \end{pmatrix} &= \tilde{R}(x,y;x_1,y_1) R(x,y;x_0,y_0) \Big|_M \begin{pmatrix} x=x_1 \\ y=y_1 \end{pmatrix} \\ &+ \int_M^N (b\tilde{R} + \frac{\partial \tilde{R}}{\partial x}) R dx + \int_M^L (a\tilde{R} + \frac{\partial \tilde{R}}{\partial y}) R dy . \end{aligned}$$

Now, the two line integrals will vanish by (1.4.2) and since $\tilde{R} \Big|_M = 1$, also following immediately from (1.4.2), then

$$\tilde{R}(x_0,y_0;x_1,y_1) = R(x_1,y_1;x_0,y_0) .$$

Since /

Since the point $M(x_1, y_1)$ was arbitrary

$$\tilde{R}(x_0, y_0; x, y) = R(x, y; x_0, y_0) ,$$

hence by simply interchanging the roles of x and x_0 , y and y_0 the Riemann functions of the differential operators L and \tilde{L} are also interchanged.

Two results which are useful when transforming the partial differential equation can be conveniently quoted here. They are applicable for making transformations of the independent and dependent variables. The first applies to transformations of the independent variables which are defined by $x = f(r)$ and $y = g(s)$. If the Riemann function of the original equation is $R(x, y; x_0, y_0)$, then the Riemann function of the transformed equation is $R(f(r), g(s); f(r_0), g(s_0))$. Also, when transformations of the dependent variable u of the partial differential equation are of the form $u = \phi(x, y)V$, and if $R(x, y; x_0, y_0)$ is the Riemann function of the original equation, then

$$W(x, y; x_0, y_0) = \frac{\phi(x, y)}{\phi(x_0, y_0)} R(x, y; x_0, y_0)$$

is the Riemann function of the transformed partial differential equation. In each case these results can easily be verified by straight substitution and so will not be proved here.

Section (1.5)

The Riemann and Green's functions were earlier mentioned when the Riemann function was first introduced. At that time a comparison was briefly mentioned; that comparison will now be made more explicit. At first sight the fundamental difference in the definitions of these two quantities would lead one to believe that no close connection existed. However, Mackie (1965) has shown an intimate connection between them. As he says, the Riemann function is totally regular, since the partial differential equation and the boundary conditions that define it are regular, that is assuming the coefficients are regular, while the Green's function has a built-in singularity arising from the delta function. These two apparently contradictory facts can be reconciled when the explicit relationship has been found.

The Riemann function for a particular differential operator as defined from Section (1.1) is the solution of a homogeneous adjoint equation with inhomogeneous boundary conditions dependent upon the coefficients of the particular equation. Thus, it is entirely meaningful to speak of the Riemann function for a differential operator. The Green's function, on the other hand, is the solution of an inhomogeneous differential equation, namely the right-hand side being a delta function, with homogeneous boundary data. These conditions can be written as

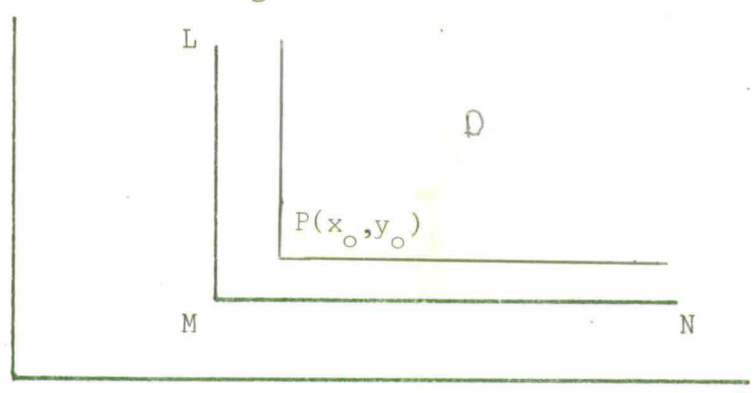
L /

$$L(G) = \delta(x - x_0) \delta(y - y_0)$$

$$G = 0 \text{ on } MN \text{ and } ML,$$

referring to Figure 4, whose solution $G(x,y ; x_0,y_0)$ will be the appropriate Green's function.

Figure 4.



This problem can be regarded as a characteristic boundary value problem solvable by Riemann's method. Using the results of Section (1.2), the solution can be written as

$$G(x,y;x_0,y_0) = \iint_{D'} \delta(x_1-x_0) \delta(y_1-y_0) R(x_1,y_1;x,y) dx_1 dy_1$$

$$= \begin{cases} 0 & \text{if } (x,y) \text{ not in } D \\ R(x_0,y_0;x,y) & \text{if } (x,y) \text{ in } D. \end{cases}$$

Thus when the point (x,y) lies outside the domain D the Green's function is identically zero, as expected, since here the partial differential equation and boundary data are homogeneous. While for $x \geq x_0$ and $y \geq y_0$ the delta function has triggered off a nonzero solution propagating into its range of influence, i.e. the domain D . It is here that the adjoint Riemann function and the Green's function coincide.

As mentioned above a similar discussion, though in noncanonical variables, is described by Mackie. Here the equation /

equation is

$$\frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial s^2} + 2g \frac{\partial u}{\partial r} - 2f \frac{\partial u}{\partial s} + c u = 0 .$$

A Cauchy problem is posed and an analogous analysis shows that

$$G(r,s;r_0,s_0) = \begin{array}{ll} \pm \frac{1}{2} R(r_0,s_0;r,s) & \text{if } (r,s) \text{ in } D \\ 0 & \text{if } (r,s) \text{ not in } D, \end{array}$$

the positive sign taken if the data curve had slope whose modulus is greater than one, and negative if the slope is less than one in modulus.

Chapter 2.

Section (2.1).

The methods that have been developed for finding Riemann functions are diverse, but most are only able to duplicate the results of other techniques. Therefore only four methods will be demonstrated. First, Mackie (1965) has shown that for a given operator a close connection exists between the Riemann function and the Green's function following the earlier results of Section (1.5). Thus, if the Green's function for a particular problem can be found the Riemann function is also known. For many examples this can be accomplished by means of transform techniques since Green's functions are commonly associated with these methods. At the end of this section an example will be given.

Next, Copson (1958) has made the most extensive survey of techniques. In this paper he derives a method applicable to general separable equations based upon an example by Riemann in his original paper. Here an integral formula is derived. This will be the subject of Section (2.2). The ease with which this formula can be applied depends, of course, upon the complexity of the form of the function itself. The most elementary Riemann functions are those that can be expanded in a power series in one variable. That is, by a suitable choice of a particular combination of x , y , x_0 and y_0 , the Riemann function can be written as a power series in this particular combination.

A/

A paper that discusses just this subject, but from a different point of view is Daggit (1970). His paper will not be dealt with for just this reason - that his methods are so different from any of the others. He makes use of infinitesimal transformations of the form

$$x \rightarrow x + \alpha f(x) \qquad y \rightarrow y + \alpha g(y)$$

where α is an infinitesimal and $f(x)$ and $g(y)$ are functions of a single argument. Transformations are found for which the Riemann function remains invariant. When a sufficient number of such transformations exist the form of the particular combination of x , y , x_0 and y_0 can be predicted and the problem has been reduced to finding a function of one variable instead of four.

However, the point of view of this paper will follow is that of Cohn (1947). Chapter 3 will deal with his work. His method seems to be a more direct approach to the problem. The partial differential equation and the boundary conditions are combined into an equivalent integral equation. A series solution is then constructed by an iterative scheme. Necessary conditions are then applied so that the Riemann function is functionally dependent on a single term in the series.

Also, recently Papadakis and Wood (1975) have derived a method following from an example of Copson's. By this method the Riemann functions of two simpler equations are combined to give the Riemann function of a more general equation. The formula derived has been labelled an 'addition formula' for this reason. Also, by means of this formula the Riemann functions so found/

found are of a more general form. That is, the power series expansions of such functions would involve multiple power series. This will be the subject of Section (2.3).

The most direct method to find Riemann functions as has been stated is to use the connection between the Green's function and the Riemann function. An example for this purpose will be the following equation

$$L(u) = u_{xy} + \frac{x}{k} u_x + u = 0 . \quad (2.1.1)$$

The Green's function will satisfy

$$\begin{aligned} \frac{\partial^2 G}{\partial x \partial y} + \frac{x}{k} \frac{\partial G}{\partial x} + G &= \delta(x-x_0) \delta(y-y_0) \\ G &= 0 \quad \text{on} \quad \left[\begin{array}{l} x = 0 \\ y = 0 \end{array} \right] . \end{aligned} \quad (2.1.2)$$

This equation can be solved by means of Laplace Transforms in y , which are defined as

$$\bar{G}(x, p; x_0, y_0) = \int_0^{\infty} G(x, y; x_0, y_0) e^{-py} dy ;$$

then equation (2.1.2) becomes

$$p \frac{\partial \bar{G}}{\partial x} + \frac{x}{k} \frac{\partial \bar{G}}{\partial x} + \bar{G} = \delta(x-x_0) e^{-py_0} ,$$

$$\text{or} \quad \frac{\partial \bar{G}}{\partial x} + \left(\frac{k}{pk+x} \right) \bar{G} = \left(\frac{k}{pk+x} \right) \delta(x-x_0) e^{-py_0} ,$$

$$\text{since} \quad e^{-py} \left. \frac{\partial G}{\partial x} \right|_{y=0}^{\infty} = 0 .$$

The first order equation can be solved as

$$\bar{G}(x, p; x_0, y_0) = \frac{k(pk+x_0)^{k-1}}{(pk+x)^k} e^{-py_0} H(x-x_0) *$$

subject to the boundary condition $G = 0$ when $x = 0$.

The Green's function is found by the inversion formula for the Laplace transform as

* $H(x-x_0)$ is the Heaviside step-function.

$$\begin{aligned}
 G(x,y; x_0,y_0) &= \frac{kH(x-x_0)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(pk+x_0)^{k-1}}{(pk+x_0)^k} e^{p(y-y_0)} dp \\
 &= \frac{H(x-x_0)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(p+x_0)^{k-1}}{(p+x)^k} e^{\frac{p(y-y_0)}{k}} dp .
 \end{aligned}$$

This integral can be evaluated by means of the following formula

$$\begin{aligned}
 &\frac{\Gamma(m+1)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(p-a)^n}{(p-b)^{m+1}} e^{pt} dp \quad \text{Re}(m) > n-1 \\
 &= n! \quad t^{m-n} e^{bt} L_n^{m-n}((b-a)t) .
 \end{aligned}$$

The Green's function is

$$G(x,y; x_0,y_0) = e^{\frac{-x_0(y-y_0)}{k}} L_{-k}^0 \left(-\frac{(x-x_0)(y-y_0)}{k} \right), \quad (2.1.3)$$

when $x > x_0$ and $y > y_0$.

However once the Green's function is known, the Riemann function can be found by using the relations

$$R(x,y; x_0,y_0) = \tilde{R}(x_0,y_0; x,y) = G(x_0,y_0; x,y)$$

from Sections (1.4) and (1.5). Thus the Riemann function for equation (2.2.1) is

$$R(x,y; x_0,y_0) = e^{\frac{x(y-y_0)}{k}} L_{-k} \left(-\frac{(x-x_0)(y-y_0)}{k} \right). \quad (2.1.4)$$

This example is very similar to an example given by Daggit in his paper (1970), although as stated, his method of derivation is totally different.

* L_S^r is the Laguerre polynomial.

Section (2.2)

Copson (1958), in what is probably the most complete summary of techniques for finding Riemann functions, discusses and elaborates a method used by Riemann in his original work on this subject. The method, based on transform techniques, is only applicable to equations with separable variables. The equation that Copson considers is

$$\frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial s^2} + 2a(r) \frac{\partial u}{\partial r} - 2b(s) \frac{\partial u}{\partial s} + (c_1(r) - c_2(s)) u = 0, \quad (2.2.1)$$

whose characteristics are the lines $(r \pm s) = \text{constant}$.

Copson describes the method as follows, "Riemann's original method was based on the fact that the Riemann-Green function does not depend in any way on the curve carrying the Cauchy data. If it is possible to solve by some other means the Problem of Cauchy for a special curve C depending on one variable parameter, a comparison of the two solutions should give the Riemann-Green function."

The method is best elaborated by following Copson's derivation. Two analogous problems are considered; firstly, referring to Figure 5, given data on $s = s_1$ of the form

$$u = 0 \quad \frac{\partial u}{\partial s} = F(r) \quad \text{on } s = s_1,$$

then the solution can be written as

$$u(r_0, s_0) = \frac{1}{2} \int_{r_0 - s_0 + s_1}^{r_0 + s_0 - s_1} R(r, s_1; r_0, s_0) F(r) dr, \quad (2.2.2)$$

and secondly, referring to Figure 6, given data on $r = r_1$ of the form

$$u = 0 \quad \frac{\partial u}{\partial r} = G(s) \quad \text{on } r = r_1,$$

then/

then the solution can be written as

$$u(r_0, s_0) = \frac{1}{2} \int_{s_0 - r_0 + r_1}^{s_0 + r_0 - r_1} R(r_1, s; r_0, s_0) G(s) ds . \tag{2.2.3}$$

Figure 5.

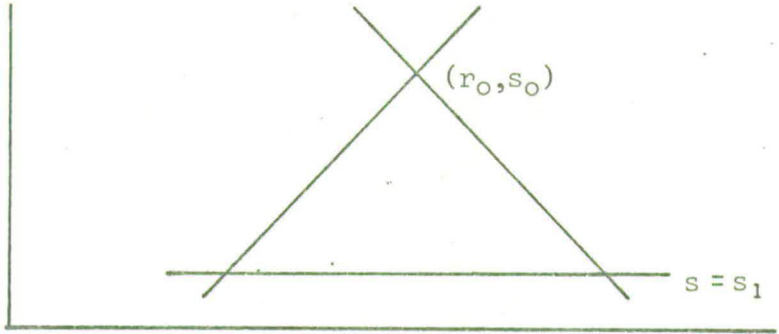
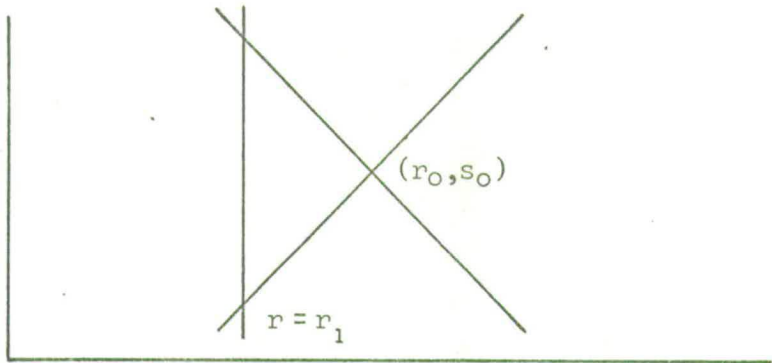


Figure 6.



The comparison solutions to (2.2.2) and (2.2.3) can be formed as follows. By assuming a separation of variables solution, (2.2.1) can be replaced by the equivalent pair of ordinary differential equations

$$\phi'' + a(r) \phi' + [c_1(r) + \lambda^2] \phi = 0 \tag{2.2.4}$$

$$\psi'' + b(s) \psi' + [c_2(s) + \lambda^2] \psi = 0 . \tag{2.2.5}$$

In both equations $-\lambda^2$ is the separation constant. Two solutions can be formed, first by letting $\phi_1(\lambda, r)$ and $\phi_2(\lambda, r)$ be linearly independent solutions of (2.2.4), and $\psi_1(\lambda, s)$ and $\psi_2(\lambda, s)$ be linearly independent solutions of (2.2.5), then

u/

$$u(r,s) = \int (f_1(\lambda)\psi_1(\lambda,s) + f_2(\lambda)\psi_2(\lambda,s))\phi_1(\lambda,r)d\lambda \quad (2.2.6)$$

$$\text{and } u(r,s) = \int (f_3(\lambda)\phi_1(\lambda,r) + f_4(\lambda)\phi_2(\lambda,r))\psi_1(\lambda,s)d\lambda \quad (2.2.7)$$

In both equations the $f(\lambda)$ are functions to be determined from the initial data. This can be accomplished by first considering (2.2.2) with its data, then

$$u(r,s_1) = \int (f_1(\lambda)\psi_1(\lambda,s_1) + f_2(\lambda)\psi_2(\lambda,s_1))\phi_1(\lambda,r)d\lambda = 0$$

$$\left. \frac{\partial u}{\partial s} \right|_{s=s_1} = \int (f_1(\lambda)\psi_1'(\lambda,s_1) + f_2(\lambda)\psi_2'(\lambda,s_1))\phi_1(\lambda,r)d\lambda = F(r) .$$

These conditions can be satisfied if

$$f_1(\lambda)\psi_1(\lambda,s_1) + f_2(\lambda)\psi_2(\lambda,s_1) = 0$$

$$f_1(\lambda)\psi_1'(\lambda,s_1) + f_2(\lambda)\psi_2'(\lambda,s_1) = h(\lambda)$$

where $h(\lambda)$ is determined from the pair of integral equations

$$\begin{aligned} F(r) &= \int \phi_1(\lambda,r) h(\lambda)d\lambda \\ h(\lambda) &= \int \tilde{\phi}_1(\lambda,r) F(r)dr . \end{aligned} \quad (2.2.8)$$

Thus $f_1(\lambda)$ and $f_2(\lambda)$ are determined from the inhomogeneous pair of equations as

$$f_1(\lambda) = - \frac{h(\lambda)\psi_2(\lambda,s_1)}{(\psi_1\psi_2' - \psi_2\psi_1')_{s=s_1}}$$

and

$$f_2(\lambda) = \frac{h(\lambda)\psi_1(\lambda,s_1)}{(\psi_1\psi_2' - \psi_2\psi_1')_{s=s_1}}$$

When these constants are substituted into (2.2.6)

$$u(r,s) = \int \frac{\psi_1(\lambda,s_1)\psi_2(\lambda,s) - \psi_2(\lambda,s_1)\psi_1(\lambda,s)}{(\psi_1\psi_2' - \psi_2\psi_1')_{s=s_1}} h(\lambda)\phi_1(\lambda,r)d\lambda$$

Since both solutions (2.2.2) and (2.2.3) are evaluated at the point (r_0, s_0) then $u(r_0, s_0)$ in this equation is found to be

u/

$$u(r_0, s_0) = \int \frac{[\psi_1(\lambda, s_1)\psi_2(\lambda, s_0) - \psi_2(\lambda, s_1)\psi_1(\lambda, s_0)]\phi_1(\lambda, r_0)h(\lambda)d\lambda}{(\psi_1\psi_2' - \psi_2\psi_1')_{s=s_1}}$$

With the help of the pair of integral equations $u(r_0, s_0)$ is finally found to be

$$u(r_0, s_0) = \int \int \frac{[\psi_1(\lambda, s_1)\psi_2(\lambda, s_0) - \psi_2(\lambda, s_1)\psi_1(\lambda, s_0)]}{(\psi_1\psi_2' - \psi_2\psi_1')_{s=s_1}} \cdot \phi_1(\lambda, r_0)\tilde{\phi}_1(\lambda, r)F(r)dr, \quad (2.2.9)$$

and similarly the analogous comparison solution for (2.2.3) is seen to be

$$u(r_0, s_0) = \int \int \frac{[\phi_1(\lambda, r_1)\phi_2(\lambda, r_0) - \phi_2(\lambda, r_1)\phi_1(\lambda, r_0)]}{(\phi_1\phi_2' - \phi_2\phi_1')_{r=r_1}} \cdot \psi_1(\lambda, s_0)\bar{\psi}_1(\lambda, s_1)G(s)ds \quad (2.2.10)$$

Assuming that (2.2.8) and the corresponding pair of integral equations for (2.2.1) are valid for all x and y , then the two solutions (2.2.9) and (2.2.10) can be equated with (2.2.2) and (2.2.3) respectively. The pair of solutions obtained by Riemann's Method is however only valid within the confines of the limits of integration. Thus corresponding solutions can only be valid within the common domain. Specifically (2.2.2) and (2.2.9) can be equated within the regions

$r /$

$$r_0 + s_0 - s_1 > r > r_0 - s_0 + s_1 \quad (\text{region 1.})$$

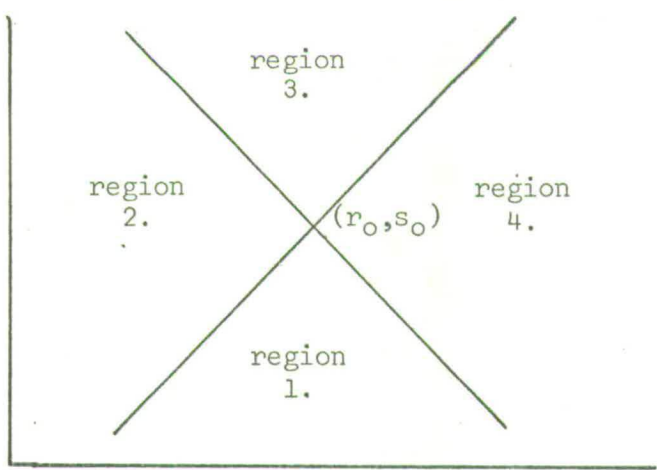
and $r_0 - s_0 + s_1 > r > r_0 + s_0 - s_1 \quad (\text{region 3.})$,

while (2.2.3) and (2.2.10) can be equated within the regions

$$r_0 + s_0 - r_1 > s > -r_0 + s_0 + r_1 \quad (\text{region 2.})$$

and $-r_0 + s_0 + r_1 > s > r_0 + s_0 - r_1 \quad (\text{region 4.})$

Figure 7.



Outside of their respective common domains the values of (2.2.9) and (2.2.10) are zero. This does not mean that the Riemann function is zero, but that the domain of validity has been exceeded and the other formula is now applicable.

As stated, knowing an alternate method of solution should give the Riemann function by a comparison of equations (2.2.2) and (2.2.9), as well as (2.2.3) and (2.2.10). By first noting that $F(r)$ and $G(s)$ are arbitrary functions, and r_1 and s_1 are arbitrary values of r and s , it can be seen that

$R(r,s;r_0,s_0)$

$$= \pm 2 \int \frac{[\psi_1(\lambda,s)\psi_2(\lambda,s_0) - \psi_2(\lambda,s)\psi_1(\lambda,s_0)]}{w(\lambda,s)} \phi_1(\lambda,r_0) \bar{\phi}_1(\lambda,r) d\lambda, \quad (2.2.11)$$

when

$$(r_0 - r) < |s_0 - s| \quad (\text{region 1.})$$

and

$$(s-s_0) > |r-r_0| \quad (\text{region 3.}) ,$$

while

$R(r,s;r_0,s_0)$

$$= \pm 2 \int \frac{[\phi_1(\lambda,r)\phi_2(\lambda,r_0) - \phi_2(\lambda,r)\phi_1(\lambda,r_0)]}{w(\lambda,r)} \psi_1(\lambda,s_0) \bar{\psi}_1(\lambda,s) d\lambda, \quad (2.2.12)$$

when

$$(s_0-s) < |r_0 - r| \quad (\text{region 2.})$$

$$\text{and } (r-r_0) > |s-s_0| \quad (\text{region 4.})$$

A sign change occurs from positive to negative comes about if the point lies within the regions (3) and (4) of Figure (7), since within these areas the solution by Riemann method involves a sign change in formulae (2.2.2) and (2.2.3).

Two examples will be quoted. The first will be the equation

$$\frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial s^2} + u = 0, \quad (2.2.13)$$

and the second equation which first interested Riemann, namely,

$$\frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial s^2} + \frac{2\alpha}{r} \frac{\partial u}{\partial r} = 0. \quad (2.2.14)$$

In the first example the pair of ordinary differential equations is

$$\phi'' + (\lambda^2 + 1)\phi = 0$$

$$\text{and } \psi'' + \lambda^2\psi = 0,$$

whose/

whose solutions are

$$\phi_1(\lambda, r) = \exp(-ir \sqrt{\lambda^2+1}), \quad \phi_2(\lambda, r) = \exp(ir \sqrt{\lambda^2+1})$$

and $\psi_1(\lambda, s) = \exp(-i\lambda s)$, $\psi_2(\lambda, s) = \exp(i\lambda s)$.

The Wronskians are easily found to be

$$W_1(\lambda, r) = 2i \sqrt{\lambda^2+1} \quad \text{and} \quad W_2(\lambda, s) = 2i\lambda.$$

If use is made of formula (2.2.11) the pair of integral equations is

$$F(r) = \int_{-\infty}^{\infty} e^{-ir \sqrt{\lambda^2+1}} h(\lambda) d\lambda$$

$$h(\lambda) = \int_{-\infty}^{\infty} \tilde{\phi}_1(\lambda, r) F(r) dr,$$

then

$$\tilde{\phi}_1(\lambda, r) = \frac{\lambda}{2\pi} \frac{\exp(ir \sqrt{\lambda^2+1})}{\sqrt{\lambda^2+1}}$$

The Riemann function is found by substituting into (2.2.11),

thus

$$R(r, s; r_0, s_0) = \pm \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{[e^{i\lambda(s_0-s)} - e^{-i\lambda(s_0-s)}]}{2i \sqrt{\lambda^2+1}} e^{-i\sqrt{\lambda^2+1}(r_0-r)} d\lambda$$

$$= \pm \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \lambda(s_0-s) e^{-i\sqrt{\lambda^2+1}(r_0-r)}}{\sqrt{\lambda^2+1}} d\lambda$$

$$= \pm \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \lambda(s_0-s) \cos \sqrt{\lambda^2+1}(r_0-r)}{\sqrt{\lambda^2+1}} d\lambda$$

$$= \pm \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda(s_0-s) \cos \sqrt{\lambda^2+1}(r_0-r)}{\sqrt{\lambda^2+1}} d\lambda.$$

The evaluation of this integral is carried out by means of the following formula

$$\int_0^{\infty} (x^2-a^2)^{-\frac{1}{2}} \cos b(x^2-a^2)^{\frac{1}{2}} \sin xy \, dx$$

$$= \begin{cases} \frac{0}{2} J_0(a(y^2-b^2)^{\frac{1}{2}}) & 0 < y < b, \\ \frac{\pi}{2} & b < y < \infty. \end{cases}$$

Thus/

Thus in regions (1) and (3) where

$$(r_0 - r) < |s_0 - s|$$

$$\text{or } (s - s_0) > |r - r_0|$$

the Riemann function is

$$R(r, s; r_0, s_0) = I_0 \left[(s_0 - s)^2 - (r_0 - r)^2 \right]^{\frac{1}{2}} .$$

The sign alternation is removed by considering the argument of the sine in regions (1) and (3).

However when formula (2.2.12) is used, then the pair of integral equations is

$$G(s) = \int_{-\infty}^{\infty} e^{-i\lambda s} h(\lambda) d\lambda$$

$$h(\lambda) = \int_{-\infty}^{\infty} \tilde{\psi}_1(\lambda, s) G(s) ds ,$$

which implies that

$$\tilde{\psi}_1(\lambda, s) = \frac{1}{2\pi} e^{i\lambda s} .$$

Now substituting into (2.2.12) the formula becomes

$$R(r, s; r_0, s_0)$$

$$= \pm \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{[e^{i\sqrt{\lambda^2+1}(r_0-r)} - e^{-i\sqrt{\lambda^2+1}(r_0-r)}]}{2i\sqrt{\lambda^2+1}} e^{i\lambda(s_0-s)} d\lambda$$

$$= \pm \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos\lambda(s_0-s) \sin \sqrt{\lambda^2+1}(r_0-r)}{\sqrt{\lambda^2+1}} d\lambda$$

$$= \pm \frac{2}{\pi} \int_0^{\infty} \frac{\cos\lambda(s_0-s) \sin \sqrt{\lambda^2+1}(r_0-r)}{\sqrt{\lambda^2+1}} d\lambda .$$

The evaluation of this integral is carried out by means of a similar formula to the above one, namely

$$\int_0^{\infty} \frac{\sin b(a^2+x^2)^{\frac{1}{2}} \cos xy}{(a^2+x^2)^{\frac{1}{2}}} dx$$

$$= \begin{cases} \frac{\pi}{2} J_0(a(b-y)^{\frac{1}{2}}) & 0 < y < b \\ 0 & b < y < \infty \end{cases}$$

In regions (2) and (4) where

$$(s_0 - s) < |r_0 - r|$$

$$\text{or } (r - r_0) > |s - s_0|$$

the/

The Riemann function is

$$R(r,s;r_0,s_0) = J_0 \left[((r_0-r)^2 - (s_0-s)^2)^{\frac{1}{2}} \right]$$

Again the sign alternation is removed by considering the argument of the sine in regions (2) and (4).

In the second example

$$\frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial s^2} + \frac{2\alpha}{r} \frac{\partial u}{\partial r} = 0,$$

the pair of ordinary differential equations is

$$\phi'' + \frac{2\alpha}{r} \phi' + \lambda^2 \phi = 0$$

$$\text{and } \psi'' + \lambda^2 \psi = 0.$$

These equations have solutions

$$\phi_1(\lambda,r) = r^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(\lambda r)$$

$$\phi_2(\lambda,r) = r^{\frac{1}{2}-\alpha} J_{\frac{1}{2}-\alpha}(\lambda r)$$

$$\text{and } \psi_1(\lambda,s) = e^{-i\lambda s} \quad \psi_2(\lambda,s) = e^{i\lambda s},$$

with Wronskians

$$W_1(\lambda,r) = \frac{2 \cos \pi \alpha}{\pi r^{2\alpha}} \quad \text{and} \quad W_2(\lambda,s) = 2i\lambda.$$

As Copson notes, a change occurs if $(\alpha-\frac{1}{2})$ is an integer, in which case the first Wronskian is $W(\lambda,r) = \frac{2}{\pi r^{2\alpha}}$.

Using equation (2.2.11) the pair of integral equations is

$$F(r) = \int_0^{\infty} r^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(\lambda r) h(\lambda) d\lambda$$

$$\text{and } h(\lambda) = \int_0^{\infty} \bar{\phi}_1(\lambda,r) F(r) dr,$$

which represents a Hankel transform in r and hence the transform pair is

$$\bar{\phi}_1(\lambda,r) = \lambda r^{\alpha+\frac{1}{2}} J_{\alpha-\frac{1}{2}}(\lambda r), \quad \text{if } \alpha > 0.$$

The Riemann function is then found by evaluating the integral

R/

$$\begin{aligned}
 R(r,s;r_0,s_0) &= \pm 2 \int_0^\infty \frac{(e^{i\lambda(s_0-s)} - e^{-i\lambda(s_0-s)})}{2i\lambda} \lambda r_0^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(\lambda r_0) r^{\alpha+\frac{1}{2}} J_{\alpha-\frac{1}{2}}(\lambda r) d\lambda \\
 &= \pm 2 \frac{r^{\alpha+\frac{1}{2}}}{r_0^{\alpha-\frac{1}{2}}} \int_0^\infty \sin \lambda(s_0-s) J_{\alpha-\frac{1}{2}}(\lambda r_0) J_{\alpha-\frac{1}{2}}(\lambda r) d\lambda .
 \end{aligned}$$

The evaluation of this integral is carried out by means of the formula

$$\begin{aligned}
 &\int_0^\infty J_r(ax) J_r(bx) \sin xy \, dx \\
 &= \begin{cases} 0 & 0 < y < (b-a) \\ \frac{1}{2(ab)^{\frac{1}{2}}} P_{r-\frac{1}{2}}\left(\frac{b^2+a^2-y^2}{2ab}\right) & (b-a) < y < (b+a) \\ -\frac{\cos r\pi}{\pi(ab)^{\frac{1}{2}}} Q_{r-\frac{1}{2}}\left(\frac{y^2-a^2-b^2}{2ab}\right) & (b+a) < y < \infty. \end{cases}
 \end{aligned}$$

Thus when $(r_0-r) < |s_0-s|$ or $(s-s_0) > |r-r_0|$

$$R(r,s; r_0,s_0) = \left(\frac{r}{r_0}\right)^\alpha P_{\alpha-1}\left(\frac{r^2 + r_0^2 - (s_0-s)^2}{2ab}\right)$$

As before the sign alternation is resolved by considering the argument of the sine in regions (1) and (3).

In regions (2) and (4) equation (2.2.12) is used for which the pair of integral equations is

$$\begin{aligned}
 G(s) &= \int_{-\infty}^\infty e^{i\lambda s} h(\lambda) d\lambda \\
 h(\lambda) &= \int_{-\infty}^\infty \tilde{\psi}_1(\lambda,s) G(s) ds
 \end{aligned}$$

hence

$$\psi_1(\lambda,s) = \frac{1}{2\pi} e^{i\lambda s} .$$

The Riemann function is thus found by evaluating the integral

$$\begin{aligned}
 R(r,s; r_0,s_0) &= \pm \frac{\pi r^{2\alpha}}{\cos \pi \alpha} \int_{-\infty}^\infty \left(r^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(\lambda r) r_0^{\frac{1}{2}-\alpha} J_{\frac{1}{2}-\alpha}(\lambda r_0) \right. \\
 &\quad \left. - r^{\frac{1}{2}-\alpha} J_{\frac{1}{2}-\alpha}(\lambda r) r_0^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(\lambda r_0) \right) \frac{e^{i\lambda(s_0-s)}}{2\pi} d\lambda \\
 &= /
 \end{aligned}$$

$$= \pm \frac{1}{\cos \pi \alpha} \frac{r^{\alpha+\frac{1}{2}}}{r_0^{\alpha-\frac{1}{2}}} \int_0^\infty (J_{\alpha-\frac{1}{2}}(\lambda r) J_{\frac{1}{2}-\alpha}(\lambda r_0) - J_{\frac{1}{2}-\alpha}(\lambda r) J_{\alpha-\frac{1}{2}}(\lambda r_0)) \cos \lambda (s_0 - s) d\lambda.$$

The evaluation of this integral is carried out with the aid of

the formula

$$\int_0^\infty \cos(c\lambda) J_r(b\lambda) J_{-r}(a\lambda) d\lambda$$

$$= \begin{cases} \frac{\cos r\pi}{\pi(a-b)^{\frac{1}{2}}} Q_{r-\frac{1}{2}}\left(\frac{a^2+b^2-c^2}{2ab}\right) & 0 < c < a-b \\ \frac{1}{2(ab)^{\frac{1}{2}}} P_{r-\frac{1}{2}}\left(\frac{c^2-a^2-b^2}{2ab}\right) & a-b < c < a+b \\ 0 & c > a+b \end{cases}$$

The Riemann function in region (2) or (4)

$$R(r, s; r_0, s_0)$$

$$= \frac{1}{\cos \pi \alpha} \frac{r^{\alpha+\frac{1}{2}}}{r_0^{\alpha-\frac{1}{2}}} \left\{ \frac{\cos(\alpha-\frac{1}{2})\pi}{\pi(r_0 r)^{\frac{1}{2}}} \left(Q_{\alpha-1} \left(\frac{r_0^2+r^2-(s_0-s)^2}{2rr_0} \right) - Q_{-\alpha} \left(\frac{r_0^2+r^2-(s_0-s)^2}{2rr_0} \right) \right) \right\}$$

$$= \left(\frac{r}{r_0} \right)^\alpha \frac{\tan \pi \alpha}{\pi} \left(Q_{\alpha-1} \left(\frac{r_0^2+r^2-(s_0-s)^2}{2rr_0} \right) - Q_{-\alpha} \left(\frac{r_0^2+r^2-(s_0-s)^2}{2rr_0} \right) \right)$$

$$= \left(\frac{r}{r_0} \right)^\alpha P_{\alpha-1} \left(\frac{r_0^2+r^2-(s_0-s)^2}{2rr_0} \right)$$

Section (2.3)

As has been shown in the last section, the ability to find the Riemann function depends upon being able to solve a pair of ordinary differential equations and then to find their respective transform pairs. Then, substitution in (2.2.11) or (2.2.12) gives the Riemann function in the appropriate domain. For the examples given in Section (2.2) the evaluation of these integrals was relatively easy, since the Riemann function could be expressed in terms of well-known functions. However, in general, this will not be the case. The particular integral to be evaluated is typically of the form

$$R(r,s;r_0,s_0) = 2 \int \phi_1(\lambda,r_0)\bar{\phi}_1(\lambda,r) \left(\frac{\psi_1(\lambda,s)\psi(\lambda,s_0) - \psi_2(\lambda,s)\psi_2(\lambda,s_0)}{w(\lambda,s)} \right) d\lambda$$

when $(r_0-r) < |s_0-s|$. (2.3.1)

In order to simplify the evaluation of integrals of this form, the following device due to Wood and Papadakis (1975) can be used. Consider the two simpler equations

$$u_{rr} - u_{ss} + 2a(r)u_r + c_1(r)u = 0 \quad (2.3.2)$$

and

$$u_{rr} - u_{ss} + 2b(s)u_s + c_1(s)u = 0 . \quad (2.3.3)$$

The method is based upon relating the Riemann functions for the two simpler equations (2.3.2) and (2.3.3) to that of a more general equation

$$u_{rr} - u_{ss} + 2a(r)u_r - 2b(s)u_s + [c_1(r) - c_2(s)]u = 0 . \quad (2.3.4)$$

If/

If Copson's method is applied to equation (2.3.2) the pair of ordinary differential equations is

$$\phi'' + 2a(r)\phi' + [c_1(\lambda) + \lambda^2]\phi = 0$$

and $\psi'' + \lambda^2\psi = 0$,

with linearly independent solutions $\phi_1(\lambda, r)$ and $\phi_2(\lambda, r)$ and $\psi_1 = e^{-i\lambda s}$, $\psi_2 = e^{i\lambda s}$ respectively, where ϕ_1 and ϕ_2 are identical to the ϕ_1 and ϕ_2 of equation (2.3.1) if $R(r, s; r_0, s_0)$ is the Riemann function of (2.3.4). The Riemann function of (2.3.2) is therefore found to be

$$\begin{aligned} R^1(r, s; r_0, s_0) &= 2 \int_{-\infty}^{\infty} \phi_1(\lambda, r_0) \overline{\phi_1}(\lambda, r) \left\{ \frac{e^{+i\lambda(s_0-s)} - i\lambda(s_0-s)}{2i\lambda} \right\} d\lambda \\ &= 2 \int_0^{\infty} 2\phi_1(\lambda, r_0) \overline{\phi_1}(\lambda, r) \frac{\sin\lambda(s_0-s)}{\lambda} d\lambda \end{aligned}$$

when $(r_0 - r) < |s_0 - s|$. (2.3.5)

Also, if Copson's method is applied to equation (2.3.3), the pair of ordinary differential equations is

$$\phi'' + \lambda^2\phi = 0$$

and $\psi'' + 2b(s)\psi' + [c_2(s) + \lambda^2]\psi = 0$,

whose linearly independent solutions are $\phi_1 = e^{-i\lambda r}$, $\phi_2 = e^{i\lambda r}$ and $\psi_1(\lambda, s)$, $\psi_2(\lambda, s)$ respectively. Again note that ψ_1 and ψ_2 are identical to those of formula (2.3.1). The Riemann function is found to be

$$\begin{aligned} R^2(r, s; r_0, s_0) &= 2 \int_{-\infty}^{\infty} \frac{e^{-i\lambda(r_0-r)}}{2\pi} \left\{ \frac{\psi_1(\lambda, s)\psi_2(\lambda, s_0) - \psi_2(\lambda, s)\psi_1(\lambda, s_0)}{w(\lambda, s)} \right\} d\lambda \\ &= 2 \int_0^{\infty} \frac{\cos\lambda(r_0-r)}{\pi} \left\{ \frac{\psi_1(\lambda, s)\psi_2(\lambda, s_0) - \psi_2(\lambda, s)\psi_1(\lambda, s_0)}{w(\lambda, s)} \right\} d\lambda \end{aligned}$$

when $(r_0 - r) < |s_0 - s|$. (2.3.6)

The/

The Riemann function of equation (2.3.4), following the notation of (2.3.1), can be written as

$$R(r,s; r_0, s_0) = 2 \int m(\lambda) n(\lambda) d\lambda$$

$$\text{for } (r_0 - r) < |s_0 - s|, \quad (2.3.7)$$

where

$$m(\lambda) = \phi_1(\lambda, r_0) \bar{\phi}_1(\lambda, r)$$

$$\text{and } n(\lambda) = \frac{(\psi_1(\lambda, s) \psi_2(\lambda, s_0) - \psi_2(\lambda, s) \psi_1(\lambda, s_0))}{w(\lambda, s)}$$

To aid in the evaluation of this integral, the functions $M(u)$ and $N(u)$ will be defined as

$$M(u) = \begin{cases} 0 & 0 < |u| < (r_0 - r) \\ R^1(r, u; r_0, 0) = 2 \int_0^\infty \frac{\sin \lambda u}{\lambda} m(\lambda) d\lambda & (r_0 - r) < |u| \end{cases} \quad (2.3.8)$$

Similarly $N(u)$ will be defined as

$$N(u) = \begin{cases} R^2(u, s; 0, s_0) = 2 \int_0^\infty \frac{\cos \lambda u}{\pi} n(\lambda) d\lambda & u < |s_0 - s| \\ 0 & u > |s_0 - s| \end{cases} \quad (2.3.9)$$

Using a property of Fourier integrals, the evaluation of (2.3.7) can be made as

$$R(r,s; r_0, s_0) = 2 \int_0^\infty m(\lambda) n(\lambda) d\lambda = \int_0^\infty N(u) d(M(u)) \quad (2.3.10)$$

where $M(u)$ and $N(u)$ are defined as above. This is the link between the Riemann functions of equations (2.3.2) and (2.3.3) and the more general equation (2.3.4). This property can be shown as follows:

$$\begin{aligned} \int_0^\infty N(u) d(M(u)) &= \int_0^\infty N(u) d\left(2 \int_0^\infty \frac{\sin \lambda u}{\lambda} m(\lambda) d\lambda \right) \\ &= 2 \int_0^\infty \frac{m(\lambda)}{\lambda} d\lambda \int_0^\infty N(u) d(\sin \lambda u) \\ &= 2 \int_0^\infty m(\lambda) d\lambda \int_0^\infty N(u) \cos \lambda u du \\ &= 2 \int_0^\infty m(\lambda) n(\lambda) d\lambda, \end{aligned}$$

since/

since if

$$N(u) = \frac{2}{\pi} \int_0^{\infty} n(\lambda) \cos \lambda u \, d\lambda,$$

$$\text{then } n(\lambda) = \int_0^{\infty} N(u) \cos \lambda u \, du.$$

Therefore, using (2.3.10) and evaluating this integral in the appropriate domains by means of (2.3.8) and (2.3.9)

$$\begin{aligned} R(r,s; r_0, s_0) &= \int_0^{\infty} N(u) \, d(M(u)) \\ &= \int_0^{|s_0-s|} N(u) \, d(M(u)) \\ &= \left(N(u)M(u) \right) \Big|_{|r_0-r|-0}^{|r_0-r|+0} + \int_{|r_0-r|+0}^{|s_0-s|} N(u) \, d(M(u)) \\ &= R^1(r, r_0-r; r_0, 0) R^2(r_0-r, s; 0, s_0) \end{aligned}$$

$$\begin{aligned} &+ \int_{|r_0-r|+0}^{|s_0-s|} R^2(u, s; 0, s_0) \, d(R^1(r, u; r_0, 0)) \\ &= \exp \int_{r_0}^r a(t) \, dt \, R^2(r_0-r, s; 0, s_0) \end{aligned}$$

$$+ \int_{|r_0-r|+0}^{|s_0-s|} R^2(u, s; 0, s_0) \, d(R^1(r, u; r_0, 0)).$$

The result obtained is that, if the Riemann functions of the equations (2.3.2) and (2.3.3) are known, the Riemann function for the more general equation (2.3.4) can be written as

$$\begin{aligned} R(r,s; r_0, s_0) &= R^2(r_0-r, s; 0, s_0) \exp \int_{r_0}^r a(t) \, dt \\ &+ \int_{r_0-r}^{s_0-s} R^2(u, s; 0, s_0) \, d(R^1(r, u; r_0, 0)). \end{aligned}$$

This result holds generally wherever R^1 and R^2 have meaning.

A similar result can be derived using the other integral formula (2.2.12). Then, the Riemann function can be expressed as

$$\begin{aligned} R(r,s; r_0, s_0) &= R^1(r, s_0-s; r_0, 0) \exp \int_{s_0}^s b(t) \, dt \\ &+ \int_{s_0-s}^{r_0-r} R^1(r, u; r_0, 0) \, d(R^2(u, s; 0, s_0)) \end{aligned}$$

where/

where R^1 and R^2 are again the Riemann functions of the simpler equations (2.3.2) and (2.3.3) respectively.

Chapter 3.

Section (3.1)

An important method different from any of the preceding is that of H. Cohn (1947, 1970, 1971). It is applicable to the self-adjoint equation for which the Riemann function must satisfy the following system

$$\left. \begin{aligned} \frac{\partial^2 R}{\partial x \partial y} + H(x,y)R &= 0 \\ R &= 1 \quad \text{on} \quad x = x_0 \text{ or } y = y_0 . \end{aligned} \right\} \quad (3.1.1)$$

The partial differential equation and boundary conditions can be combined into the integral equation

$$R(x,y; x_0, y_0) = 1 - \int_{x_0}^x \int_{y_0}^y H(x_1, y_1) R(x_1, y_1; x_0, y_0) dx_1 dy_1 .$$

A series solution is constructed by means of an iterative scheme, namely the entire right-hand side is substituted for the value of R under the double integral, as

$$\begin{aligned} R(x,y; x_0, y_0) &= 1 - \int_{x_0}^x \int_{y_0}^y H(x_1, y_1) dx_1 dy_1 \\ &+ \int_{x_0}^x \int_{y_0}^y H(x_1, y_1) \int_{x_0}^x \int_{y_0}^y H(x_2, y_2) R(x_2, y_2; x_0, y_0) dx_2 dy_2 dx_1 dy_1 . \end{aligned}$$

If this process were repeated indefinitely, the Riemann function could be expressed as an infinite series of the form

$$R(x,y; x_0, y_0) = u_0 - u_1 + u_2 - \dots , \quad (3.1.2)$$

where the u_i are defined by the recurrence formula

$$\left. \begin{aligned} u_{n+1} &= \int_{x_0}^x \int_{y_0}^y H(x_1, y_1) u_n dx_1 dy_1 \\ \text{with } u_0 &\equiv 1 \end{aligned} \right\} \quad (3.1.3)$$

Cohn's/

Cohn's method depends upon finding those Riemann functions which are functionally dependent upon the first nonconstant term in the above series, namely u_1 defined by

$$u_1 = \int_{x_0}^x \int_{y_0}^y H(x_1, y_1) dx_1 dy_1 . \quad (3.1.4)$$

In other words, Cohn uses u_1 as a single variable and seeks Riemann functions of the form $R = R(u_1)$. When this form is

used $\frac{\partial^2 R}{\partial x \partial y}$ is found to be

$$\begin{aligned} \frac{\partial R}{\partial x} &= \frac{dR}{du_1} \frac{\partial u_1}{\partial x} \\ \frac{\partial^2 R}{\partial x \partial y} &= \frac{d^2 R}{du_1^2} \frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial y} + \frac{dR}{du_1} \frac{\partial^2 u_1}{\partial x \partial y} \\ &= \frac{d^2 R}{du_1^2} \frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial y} + \frac{dR}{du_1} H(x, y) , \end{aligned}$$

because of (3.1.4), which when substituted into (3.1.1) gives

$$\frac{1}{H} \frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial y} \frac{d^2 R}{du_1^2} + \frac{dR}{du_1} + R = 0 . \quad (3.1.5)$$

Since $u_1 = 0$ when $x = x_0$ or $y = y_0$ the boundary conditions from (3.1.1) can be written as $R(0) = 1$.

It is from (3.1.5) that Cohn is able to derive his results. For convenience, however, the subscript on u_1 will be dropped and further subscripts will denote partial derivatives. The problem can be written as

$$\left. \begin{aligned} H^{-1} u_x u_y \frac{d^2 R}{du^2} + \frac{dR}{du} + R &= 0 \\ R(0) &= 1 . \end{aligned} \right\} \quad (3.1.6)$$

If this equation is to be an ordinary differential equation then the leading coefficient must be functionally dependent upon u , which implies that the Jacobian relation

$$\frac{\partial}{\partial}$$

$$\frac{\partial(H^{-1} u_x u_y, u)}{\partial(x, y)} = 0 \quad (3.1.7)$$

must be satisfied. This Jacobian when expanded can be written as

$$\frac{u_{xx} H - u_x H_x}{u_x^2} - \frac{u_{yy} H - u_y H_y}{u_y^2} = 0 ,$$

$$\frac{\partial}{\partial x} \left(\frac{H}{u_x} \right) - \frac{\partial}{\partial y} \left(\frac{H}{u_y} \right) = 0 ,$$

and since $u_{xy} = H$, then

$$\frac{\partial^2}{\partial x \partial y} (\log u_x) - \frac{\partial^2}{\partial x \partial y} (\log u_y) = 0 ,$$

and finally

$$\frac{\partial^2}{\partial x \partial y} \left(\log \frac{u_x}{u_y} \right) = 0 . \quad (3.1.8)$$

When the above Jacobian is written in this form the general solution is easily seen to be

$$u = H_1 (F(x) + G(y)) ,$$

where H_1 , F and G are arbitrary functions of their arguments.

It is convenient to make a change of variable by defining

$$x_1 = F(x) , \quad y_1 = G(y) \quad \text{and} \quad H = H_1'' .$$

The partial derivatives can then be formed as

$$\frac{\partial^2 R}{\partial x \partial y} = \frac{\partial^2 R}{\partial F \partial G} F'(x) G'(y) ,$$

$$\begin{aligned} \text{and } H(x, y) &= \frac{\partial^2 u}{\partial x \partial y} = H_1'' (F(x) + G(y)) F'(x) G'(y) \\ &= H (F(x) + G(y)) F'(x) G'(y) . \end{aligned}$$

This allows (3.1.1) to be written in the form

$$\frac{\partial^2 R}{\partial x \partial y_1} + H(x_1 + y_1) R = 0 . \quad (3.1.9)$$

In/

In the original paper, Cohn (1947) first considered equations of the type (3.1.8), that is where $H(x,y)$ has the particular form $H(x+y)$. The main result of the second paper (1970) is to recognise that no new results are obtained by considering general $H(x,y)$, since the Jacobian can be written in the compact form (3.1.8), which can then easily be solved as just shown.

The main task, however, of Cohn's approach is to find those functions H for which (3.1.8) is satisfied. For this purpose Cohn expands H as a Taylor series about the point $r = x + y$, then substitutes into the Jacobian relation. By doing so, an infinite series is derived which vanishes identically, and whose coefficients are ordinary differential equations for H from which the functions H can be determined. For this purpose it is somewhat easier to write the Jacobian as

$$u_y^2 (u_{xx} H - u_x H_x) - u_x^2 (u_{yy} H - u_y H_y) = 0.$$

Then from (3.1.4) and (3.1.9) u is seen to be

$$u = \int_{x_0}^x \int_{y_0}^y H(x_1 + y_1) dx_1 dy_1,$$

from which u_x and u_y are determined as

$$u_x = \int_{y_0}^y H(x + y_1) dy_1$$

$$\text{and } u_y = \int_{x_0}^x H(x_1 + y) dx_1.$$

When H is expanded into a double Taylor series about the point $r = x + y$ then

$$u_x = - \left[(y_0 - y) H(r) + \frac{(y_0 - y)^2}{2!} H'(r) + \frac{(y_0 - y)^3}{3!} H''(r) + \dots \right]$$

and

$$u_y = - \left[(x_0 - x) H(r) + \frac{(x_0 - x)^2}{2!} H'(r) + \frac{(x_0 - x)^3}{3!} H''(r) + \dots \right]$$

Since/

Since the above identity requires u_y^2 and u_x^2 , these are found to be

$$u_x^2 = (y_0 - y)^2 H^2(r) + (y_0 - y)^3 H'(r)H(r) + (y_0 - y)^4 \left[\frac{2H''(r)H(r)}{3!} + \left(\frac{H'(r)}{2!} \right)^2 \right] \\ + (y_0 - y)^5 \left[\frac{2H'''(r)H(r)}{4!} + \frac{2H''(r)H'(r)}{3! \cdot 2!} \right] + \dots$$

and

$$u_y^2 = (x_0 - x)^2 H^2(r) + (x_0 - x)^3 H'(r)H(r) + (x_0 - x)^4 \left[\frac{2H''(r)H(r)}{3!} + \left(\frac{H'(r)}{2!} \right)^2 \right] \\ + (x_0 - x)^5 \left[\frac{2H'''(r)H(r)}{4!} + \frac{2H''(r)H'(r)}{3! \cdot 2!} \right] + \dots$$

Finally after the other terms have been calculated the identity becomes

$$\left\{ (x_0 - x)^2 H^2(r) + (x_0 - x)^3 H'(r)H(r) + (x_0 - x)^4 \left(\frac{H''(r)H(r)}{3} + \frac{(H'(r))^2}{4} \right) + \dots \right\} \\ \left\{ \frac{(y_0 - y)^2}{2} (H''(r)H(r) - (H'(r))^2) + \frac{(y_0 - y)^3}{6} (H'''(r)H(r) - H''(r)H'(r)) + \dots \right\} \\ - \left\{ (y_0 - y)^2 H^2(r) + (y_0 - y)^3 H'(r)H(r) + (y_0 - y)^4 \left(\frac{H''(r)H(r)}{3} + \frac{(H'(r))^2}{4} \right) + \dots \right\} \\ \left\{ \frac{(x_0 - x)^2}{2} (H''(r)H(r) - (H'(r))^2) + \frac{(x_0 - x)^3}{6} (H'''(r)H(r) - H''(r)H'(r)) + \dots \right\} \\ \equiv 0 \quad (3.1.10)$$

The first few terms of the left-hand side are

$$\left((x_0 - x)^2 (y_0 - y)^3 - (x_0 - x)^3 (y_0 - y)^2 \right) \\ \left\{ \frac{H^2(r)}{6} (H'''(r)H(r) - H''(r)H'(r)) \right. \\ \left. - \frac{H'(r)H(r)}{2} (H''(r)H(r) - (H'(r))^2) \right\} \\ + \left((x_0 - x)^3 (y_0 - y)^4 - (x_0 - x)^4 (y_0 - y)^3 \right) \\ \left\{ \frac{H'(r)H(r)}{24} (H''''(r)H(r) - H'''(r)H'(r)) \right. \\ \left. - \frac{1}{6} \left(\frac{H''(r)H(r)}{3} + \frac{(H'(r))^2}{4} \right) (H'''(r)H(r) - H''(r)H'(r)) \right\},$$

whose/

whose first term reduces to

$$\frac{H'''(r)}{H(r)} - 4 \frac{H''(r)}{H^2(r)} + 3 \left(\frac{H'(r)}{H(r)} \right)^3 = 0. \quad (3.1.11)$$

This equation can be simplified by means of the substitution $s = \log H$, then

$$s' = \frac{H'}{H},$$

from which it immediately follows that

$$s'' = \frac{H''H - (H')^2}{H^2}$$

and

$$s''' = \frac{H'''}{H} - 3 \frac{H''H'}{H^2} + 2 \left(\frac{H'}{H} \right)^2.$$

Hence (3.1.11) can be written as

$$s''' - s''s' = 0.$$

This equation can be integrated once as

$$s'' - \frac{1}{2}(s')^2 = -2a^2, \quad (a = \text{constant})$$

and then rearranged as

$$\frac{\frac{1}{2a} s''}{1 - \left(\frac{s'}{2a} \right)^2} = -a,$$

which can then be integrated as

$$\text{arc coth} \left(\frac{s'}{2a} \right) = a(-r+b). \quad (b = \text{constant})$$

Again rearranging these results as

$$s' = 2a \frac{\cosh a(-r+b)}{\sinh a(-r+b)},$$

and finally, once again integrating

$$s = -2 \log \sinh a(-r+b) + c, \quad (c = \text{constant}).$$

The function H can be written as

$$H(x+y) = \frac{-\lambda(\lambda+1)\mu^2}{\sinh^2 \mu(x+y+v)}, \quad (3.1.12)$$

where/

where λ, μ and v are arbitrary constants. It is for this family of functions that the Riemann function is functionally dependent upon u . Using (3.1.4) u is calculated as

$$u = -\lambda(\lambda+1) \log(1+V)$$

where

$$V = \frac{\sinh \mu(x-x_0) \sinh \mu(y-y_0)}{\sinh \mu(x+y+v) \sinh \mu(x_0+y_0+v)} \quad (3.1.13)$$

In terms of V equation (3.1.6)

$$V(1+V) \frac{d^2R}{dV^2} + (1+2V) \frac{dR}{dV} - \lambda(\lambda+1) R = 0 .$$

By letting $x = -V$ the equation can be written as

$$x(1-x) \frac{d^2R}{dx^2} + (1-2x) \frac{dR}{dx} + \lambda(\lambda+1) R = 0 , \quad (3.1.14)$$

whose solutions can be recognised as a hypergeometric functions.

The boundary condition $R(0) = 1$ eliminates one of these, thus the Riemann function is

$$R(x,y; x_0,y_0) = F(\lambda+1, -\lambda; 1; -V) . \quad (3.1.15)$$

The three arbitrary constants give various limiting forms.

These will be shown in the following table.

	H(x+y)	U	V	R
All limiting cases are derived from original form, then written in each case with new constants	$\frac{-\lambda(\lambda+1) \mu^2}{\sinh^2 \mu(x+y+v)}$	$-\lambda(\lambda+1) \log(1+V)$	$\frac{\sinh \mu(x-x_0) \sinh \mu(y-y_0)}{\sinh \mu(x+y+v) \sinh \mu(x_0+y_0+v)}$	$F(\lambda+1, -\lambda; 1; -V)$
let $\mu \rightarrow 0$	$\frac{-\lambda(\lambda+1)}{(r+v)^2}$	$-\lambda(\lambda+1) \log(1+V)$	$\frac{(x-x_0)(y-y_0)}{(x+y+v)(x_0+y_0+v)}$	$F(\lambda+1, -\lambda; 1; -V)$
let $\lambda = \lambda e^v$ $v \rightarrow \infty$	$-\lambda e^{\mu r}$	$\frac{-\lambda}{\mu^2} (e^{\mu x} - e^{\mu x_0})(e^{\mu y} - e^{\mu y_0})$	U	$J_0(2V^{\frac{1}{2}})$
let $\lambda = \lambda e^v$ $v \rightarrow \infty$ $\mu \rightarrow 0$	$-\lambda$	$\lambda(x-x_0)(y-y_0)$	U	$J_0(2V^{\frac{1}{2}})$

Section (3.2)

In order to motivate Cohn's method more clearly a more general approach can be taken to the self-adjoint equation. Instead of looking for Riemann functions which are functionally dependent upon the first nonconstant term in the infinite series, suppose that the Riemann function will only be required to be functionally dependent upon an arbitrary function of a single variable. This new variable z , itself a function of four variables, x, y, x_0 and y_0 , will be such that the partial differential equation degenerates into an ordinary differential equation when this new independent variable is substituted into the system defining the Riemann function. The boundary conditions must be such that when $x = x_0$ or $y = y_0$, z must be independent of both x and y , that is z must reduce to a constant. This condition being in agreement with the fact that z must behave as a single variable. It will be helpful to keep in mind the example $z = (x-x_0)(y-y_0)$. This is the appropriate form of z for the equation $u_{xy} + u = 0$, as can be seen from the table from the previous section. In this way no appeal need be made to the Neumann series and there is the prospect of obtaining new Riemann functions.

The ordinary differential equation is derived in the following manner. If the Riemann function is to be a function of a single variable z , then let $R = R(z)$, from which it follows that

$$\frac{\partial^2 R}{\partial x \partial y} = \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \frac{d^2 R}{dz^2} + \frac{\partial^2 z}{\partial x \partial y} \frac{dR}{dz} .$$

Now substituting into the self-adjoint equation and dividing

through by $\frac{\partial^2 z}{\partial x \partial y}$, the equation becomes

$z/$

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$$\frac{\partial^2 R}{\partial x \partial y} = \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \frac{d^2 R}{dz^2} + \frac{\partial^2 z}{\partial x \partial y} \frac{dR}{dz} .$$

Now substituting into the self-adjoint equation and dividing

through by $\frac{\partial^2 z}{\partial x \partial y}$, the equation becomes

$z/$

$$\frac{z}{z} \frac{z}{x} \frac{y}{y} R''(z) + R'(z) + \frac{H}{z} \frac{R(z)}{xy} = 0 \quad (3.2.1)$$

The first case to consider is where $R \equiv z$, that is z has been chosen to be the Riemann function itself. Then (3.2.1) reduces to

$$z_{xy} + H z = 0,$$

and hence the original problem, as could have been anticipated, is retrieved except in a new guise. This case will be ignored. Also Cohn's case can be seen if z is chosen to be the U of the Neumann series. By doing so (3.2.1) becomes

$$\frac{z}{H} \frac{z}{x} \frac{y}{y} R''(z) + R'(z) + R(z) = 0 \quad (3.2.2)$$

which is just equation (3.1.6) of the previous section.

Also this case will be set aside temporarily.

It is more convenient for what follows to rearrange (3.2.1) and write it in the form

$$R''(z) + \frac{z}{z} \frac{xy}{x} \frac{y}{y} R'(z) + \frac{H}{z} \frac{R(z)}{z} \frac{y}{xy} = 0 \quad (3.2.3)$$

Now if this equation is to reduce to a genuine ordinary differential equation then the coefficients of $R'(z)$ and $R(z)$ must both be functions of z separately. This will then give a pair of Jacobian relations, which can be solved in a remarkably simple fashion; specifically it will be required that

$$\frac{\partial \left(\frac{z}{z} \frac{xy}{x} \frac{y}{y}, z \right)}{\partial(x,y)} = 0 \quad (3.2.4)$$

and

$$\frac{\partial \left(\frac{H}{z} \frac{R(z)}{z} \frac{y}{xy}, z \right)}{\partial(x,y)} = 0 \quad (3.2.5)$$

Whereas Cohn, by choosing his new independent variable as the first nonconstant term in the Neumann series, derived only/

only one Jacobian, now there are two which must be satisfied simultaneously. The above Jacobians can be written as

$$\begin{aligned} z_y & \frac{(z_{xx}z_y + z_xz_{xy})z_{xy} - z_xz_y(z_{xy})_x}{(z_xz_y)^2} \\ & = z_x \frac{(z_{xy}z_y + z_xz_{yy})z_{xy} - z_xz_y(z_{xy})_y}{(z_xz_y)^2} \end{aligned}$$

$$\text{or } \frac{z_{xx}z_{xy} - z_x(z_{xy})_x}{(z_x)^2} = \frac{z_{yy}z_{xy} - z_y(z_{xy})_y}{(z_y)^2}$$

$$\text{or } \left(\frac{z_{xy}}{z_x} \right)_x = \left(\frac{z_{xy}}{z_y} \right)_y$$

or finally

$$(\log z_x)_{xy} - (\log z_y)_{xy} = (\log \frac{z_x}{z_y})_{xy} = 0. \quad (3.2.6)$$

Similarly (3.2.5) can be written as

$$\begin{aligned} z_y & \frac{H z_x z_y - H(z_{xx}z_y + z_xz_{xy})}{(z_xz_y)^2} \\ & = z_x \frac{H z_x z_y - H(z_{xy}z_y + z_xz_{yy})}{(z_xz_y)^2} \end{aligned}$$

or

$$\frac{z_x H_x - H z_{xx}}{(z_x)^2} = \frac{z_y H_y - H z_{yy}}{(z_y)^2}$$

$$\text{or finally } \left(\frac{H}{z_x} \right)_x = \left(\frac{H}{z_y} \right)_y. \quad (3.2.7)$$

Keeping in mind the example $z = (x-x_0)(y-y_0)$, then (3.2.6) is obviously satisfied while (3.2.7) becomes

$$(x-x_0) H_x - (y-y_0) H_y = 0,$$

from which it can be seen that $H = \text{constant}$ satisfies this equation and is indeed the only solution for which H is a function of x and y only.

These/

These two equations, (3.2.6) and (3.2.7), must be satisfied simultaneously. The first equation (3.2.6) allows a general solution to be written in the form

$$z = w(F(x) + G(y)) , \quad (3.2.8)$$

where w , F , and G are arbitrary functions of their arguments. As stated earlier, z must be independent of both x and y when $x = x_0$ and $y = y_0$; these conditions can only be satisfied if as $x \rightarrow x_0$ or $y \rightarrow y_0$, then $F(x) \rightarrow \infty$ or $G(y) \rightarrow \infty$ respectively. These conditions will be vital in what follows. It should be noted that the example $z = (x-x_0)(y-y_0)$ does not immediately fit into this category, however by an appropriate choice of the arbitrary functions F and G , for example $F = G = \log$ then $z = \log(x-x_0) + \log(y-y_0)$, which now comes into the form (3.2.8).

If (3.2.8) is substituted into (3.2.7), the second Jacobian relation then assumes the form

$$\left(\frac{H}{F'(x)} \right)_x - \left(\frac{H}{G'(y)} \right)_y = 0 , \quad (3.2.9)$$

whose general solution is

$$H(x,y) = K' [F(x) + G(y)] F'(x) G'(y) . \quad (3.2.10)$$

Here K' , F , and G are again arbitrary functions of their arguments with K' denoting the derivative of an arbitrary function K for later convenience.

In the light of Cohn's work, the function u will become

$$u = \int_{x_0}^x \int_{y_0}^y K' [F(x_1) + G(y_1)] F'(x_1) G'(y_1) dx_1 dy_1 . \quad (3.2.11)$$

The/

The main result of this section will be that

$$\frac{\partial(u,z)}{\partial(x,y)} = 0 .$$

This can be seen if the partial derivatives forming the Jacobian are formed from (3.2.8) and (3.2.11)

$$z_x = w' (F(x) + G(y)) F'(x) \quad (3.2.12)$$

$$z_y = w' (F(x) + G(y)) G'(y) , \quad (3.2.13)$$

$$\begin{aligned} \text{while } u_x &= \int_{y_0}^y K' (F(x) + G(y_1)) F'(x) G'(y_1) dy_1 \\ &= F'(x) \int_{y_0}^y K' (F(x) + G(y_1)) G'(y_1) dy_1 \\ &= F'(x) (K (F(x) + G(y)) - K (F(x) + G(y_0))) \\ &= F'(x) (K (F(x) + G(y)) - K(\infty)) \end{aligned}$$

$$\begin{aligned} \text{and } u_y &= \int_{x_0}^x K' (F(x_1) + G(y)) F'(x_1) G'(y) dx_1 \quad (3.2.14) \\ &= G'(y) \int_{x_0}^x K' (F(x_1) + G(y)) F'(x_1) dx_1 \\ &= G'(y) (K (F(x) + G(y)) - K (F(x_0) + G(y))) \\ &= G'(y) (K (F(x) + G(y)) - K(\infty)) . \quad (3.2.15) \end{aligned}$$

The fact that $K (F(x_0) + G(y))$ and $K (F(x) + G(y_0))$ become $K(\infty)$ follows immediately from the previous arguments about $F(x)$ and $G(y)$.

Once these partial derivatives have been calculated it is easily seen that

$$\frac{\partial(u,z)}{\partial(x,y)} = z_x u_y - z_y u_x = 0 ,$$

which is just what was to be shown.

Thus the apparent generalization leads directly back to Cohn's very restriction, that the only Riemann functions of a single variable are those functionally dependent upon u . The significance of this result is that it demonstrates that the only Riemann functions for self-adjoint equations whose power series expansions involve a single variable are those of Cohn.

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