X

SOME PROPERTIES OF THE CURRENT ALGEBRAIC STRUCTURE OF ELEMENTARY PARTICLE INTERACTIONS

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by

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PREFACE

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All the material in this dissertation is original except where explicit reference has been made.

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INTRODUCTION

The whole subject of current algebra is extremely vast, and so in this dissertation we are necessarily concerned with only a small part of it. In Chapter I, we discuss the basic ideas of current algebra emphasising the point that the existence of the algebra does not necessarily imply invariance under the corresponding group. We also consider some of the weaker areas of the underlying theory. In the next three chapters, we review one type of application and present further applications of the same type. In fact, we consider sum rules for axial-vector coupling constants for baryons in Chapter II, and for mesons in Chapter III. We also consider the problem of using physical or degenerate masses in various supermultiplets. In Chapter IV, we examine the unpleasant problem of magnetic moments, and come to the conclusion that the best thing we can say about it is that there are no inconsistencies. In Chapter V, worried by the extremely short lifetime of the 3,3-resonance, we consider the role of unstable particles in sum-rules. In the final chapter we examine, in a most rigorous fashion, the definition and domain of the 'charge' operator. Finally, in Appendices A-E we give some results which would not fit elegantly in the text, and which we refer to from time to time.

CHAPTER I

THE BASIC IDEAS OF CURRENT ALGEBRA

In the last few years, we have witnessed some important advances in the theory of elementary particles. Ambitious classification schemes have been suggested for the hadrons, starting in 1961 with SU(3) and culminating in 1964 with nonrelativistic SU(6). Attempts to find higher symmetries, and in particular, relativistic extensions* of SU(6), have so far failed. Indeed, a series of 'impossibility' theorems (of which the most famous is O'Raifeartaigh's theorem) points to the fact that it is impossible to combine an internal symmetry group with the Poincare group in anything more than a trivial way, e.g. by a direct product.

It was soon realized that SU(3) and SU(6) were badly broken symmetries, and that many of the additional assumptions that went into, for example, the mass formula, could only be justified _a posteriori_. To eliminate the shortcomings of symmetry groups, the method of current algebra was proposed. The original idea was that, in nature, there might exist a set of currents which would form a closed algebra. Although this algebra was isomorphic to the Lie algebra of some symmetry group, it was hoped that the mere existence of the algebra would _not_ imply the invariance of nature under the corresponding group. While such ideas were first hinted at in 1961-62, it was not until 1965 that the possible applications of current algebra became clear. Since then, many hundreds of papers have been

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* For example, the U(6,6) theory of Salam et al.

written on the subject. It would therefore be quite impossible, in the short space of one chapter, to trace the whole history and all the applications of current algebra*. Our aim in this chapter is twofold. First, we shall review the hopes and failures of the underlying ideas of current algebra, and secondly, we shall fix the notation for the succeeding chapters.

1. Currents and Charges.

The concept of a current is certainly not new to us. It is, for example, the quantity $J_\mu(x)$ appearing in the interaction Lagrangian of charged particles with the electromagnetic field $L = J_\mu(x)A^\mu(x)$. In the context of the eightfold way, we are familiar with the electric current density

$$J_\mu^{e\ell} = J_\mu^3 + \frac{1}{\sqrt{3}} J_\mu^8$$

and the weak hadron current

$$J_\mu = \cos\theta \left(J_\mu^1 + iJ_\mu^2\right) + \sin\theta \left(J_\mu^4 + iJ_\mu^5\right)$$

Now, if we have any current density $J_\mu(x)$, the charge Q associated with it is the quantity defined by the invariant integral

$$Q(\sigma) = \int_{\sigma(x)} d\sigma_\mu(x') \, J^\mu(x') \tag{1.1}$$

Since we have

$$\frac{\delta Q(\sigma)}{\delta\sigma(x)} = \partial^\mu J_\mu(x) ,$$

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* See, for example, Ref. (8).

local conservation of the current implies that the integral, Eq. (1), is independent of the surface $\sigma(x)$, so that

$$Q = \int d^3x\, J_o(x) \tag{1.2}$$

In fact, we will use this definition, that the charge is the space integral of the current density, whether or not the current is conserved.

The original idea[1,2] was that by turning off various symmetry breaking interactions, we could have conserved vector currents* $\mathcal{V}^{\alpha}_{\mu}(x)$ whose space integrals gave the generators of SU(3), namely

$$V^{\alpha}_{o}(t) = F^{\alpha}(t) = -\, i \int d^3x\ \mathcal{V}^{\alpha}_{o}(x)$$

Since the charges $F^{\alpha}(t)$ satisfy the equal-time commutation relation,

$$\left[F^{\alpha}(t),\, F^{\beta}(t)\right] = i\, f^{\alpha\beta\gamma}\, F^{\gamma}(t)$$

the current densities <u>might</u> satisfy

$$\left[\mathcal{V}^{\alpha}_{o}(\underline{x},t), \mathcal{V}^{\beta}_{o}(\underline{y},t)\right] = -f^{\alpha\beta\gamma}\delta^3(\underline{x}-\underline{y})\mathcal{V}^{\gamma}_{o}(\underline{x},t) \tag{1.3}$$

However, such an equal-time commutation relation is not unique. We could include other singular terms on the r.h.s. of Eq. (3), for example, derivatives of the delta function. For the moment we shall ignore this possibility, although we shall come back to these Schwinger terms in Section 5.

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* Throughout this dissertation, we shall use the notation $\mathcal{S}, \mathcal{V}, \mathcal{T}, \mathcal{A}, \mathcal{P}$ to denote the tensor character (under the Lorentz group) of the current density, and S,V,T,A,P the corresponding charge.

2. U(12) current algebra and its subalgebras.

It was suggested[3,20] that instead of making inspired guesses, we could derive the equal-time commutation relations for current densities from a simple quark model. In such a model, the currents and charges are given by

$$Q_A^\alpha = -i\int d^3x\,\mathcal{G}_A^\alpha(x) = d^3x\ \psi^+(x)\ \gamma_A \tfrac{1}{2}\lambda^\alpha\ \psi(x) \qquad (1.4)$$

$$\text{with}\ \mathcal{G}_A^\alpha(x) = \overline{\psi}(x)\ \Gamma_A\ \tfrac{1}{2}\lambda^\alpha\ \psi(x) \qquad (1.5)$$

where $\alpha = 0,\ldots,8$ is the U(3) internal symmetry index, and $A = 1,\ldots,16$ refers to the appropriate Dirac matrix. Altogether there are 144 currents of the form $\mathcal{G}_A^\alpha(x)$. For future reference we give Eqs. (4 and 5) in tabular form,

Q_A	Γ_A	γ_A
S	$-i$	β
P	γ_5	$i\beta\gamma_5$
V_μ	γ_μ	$(1,\alpha_i)$
A_μ	$\gamma_\mu\gamma_5$	(γ_5,σ_i)
T_{ij}	σ_{ij}	$\varepsilon_{ijk}\beta\sigma_k$
T_{oi}	σ_{oi}	$i\beta\gamma_5\sigma_i$

where $\beta = i\gamma_o$, $\alpha_i = \gamma_i\gamma_o$, $\sigma_i = \gamma_o\gamma_5\gamma_i$, $\sigma_{\mu\nu} = \tfrac{1}{2}\left[\gamma_\mu,\gamma_\nu\right]$

with all the Dirac matrices Hermitian except $\gamma_o = -\gamma_o^+$ and $\sigma_{ij} = -\sigma_{ij}^+$.

If we assume that the quarks obey Fermi statistics*, we can use the anticommutation relations for fermions to show that

$$\left[\mathcal{J}^{\alpha}(\gamma_A, x), \mathcal{J}^{\beta}(\gamma_B, y)\right]_{x_o=y_o}$$

$$\tfrac{1}{2}\,\delta^3(\underline{x}-\underline{y}) \begin{pmatrix} i f^{\alpha\beta\gamma} \mathcal{J}^{\gamma}(\{\gamma_A, \gamma_B\}, x) \\ + d^{\alpha\beta\gamma} \mathcal{J}^{\gamma}([\gamma_A, \gamma_B], x) \end{pmatrix}$$

where $\mathcal{J}^{\alpha}(\gamma_A, x) = \psi^{+}(x)\, \gamma_A \tfrac{1}{2}\lambda^{\alpha}\, \psi(x)$ (1.6)

When we take the equal-time commutators of all the currents in Eq. (6) we find that we obtain a closed algebra, isomorphic to the Lie algebra of the compact group U(12). It is convenient to use the notation $Q_A(\gamma_A)$ for the generators of the group. In the following chapters, we shall be concerned with a few subgroups of U(12). These are

(i) Static U(6), generated by $V_o(1)$ and $A_i(\sigma_i)$

(ii) Collinear U(6) = $U(6)_W$, generated by $V_o(1)$, $A_z(\sigma_z)$, $T_{zx}(\beta\sigma_x)$ and $T_{zy}(\beta\sigma_y)$

(iii) Collinear U(3) $\otimes$ U(3) = Collinear W(3)**, generated by $V_o(1) \pm A_z(\sigma_z)$

(iv) Chiral U(3) $\otimes$ U(3) = Chiral W(3), generated by $V_o(1) \pm A_o(\gamma_5)$

In fact, there are 7 different W(3) subalgebras of U(12) although only the two given above seem to have any physical significance.

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* According to S. Coleman ('discussion' at the 1966 Ettore Majorana Summer School) the possibility of parastatistics has been ruled out by H.J. Borchers who shows that parastatistics ultimately reduces to either Bose or Fermi statistics.

** We also use the notation SW(3) = SU(3) $\otimes$ SU(3).

3. The use of current commutation relations.

It was not until 1965 that a technique was suggested[4,11] for obtaining useful information from equal-time commutation relations. We start with the equal-time commutator of two charges,

$$\left[Q_A, Q_B \right] = i\, C_{ABC}\, Q_C$$

and sandwich this between two one-particle states. A 'complete' set of states* is then inserted in the commutator, giving a sum rule of the form

$$\sum_n \langle \alpha | Q_A | n \rangle \langle n | Q_B | \beta \rangle$$

$$- \sum_n \langle \alpha | Q_B | n \rangle \langle n | Q_A | \beta \rangle$$

$$= i\; C_{ABC} \langle \alpha | Q_C | \beta \rangle \tag{1.7}$$

If the charge Q comes from a conserved current, the matrix elements have a simple kinematical structure. However, if the symmetry is broken, we must introduce an unknown renormalization factor g in the definition of the matrix element. The sum-rule, Eq. (7), enables us to determine g. Because of the importance of this, it is worth spending a moment to see how renormalization and symmetry breaking are connected. Let us take the example of electromagnetic interactions in SU(2). We consider the commutator

$$\left[I^+, I^- \right] = 2I_3 \tag{1.8}$$

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* In practice, we make an approximation to this. See section 4 and also Chapter II.

When the electromagnetic interaction is switched off, the SU(2) symmetry is exact, so that

$$\langle p | I^{+} | n \rangle = \delta^{3}(\underline{p}-\underline{n})$$

Now when the electromagnetic interaction is turned on, a renormalization effect occurs so that

$$\langle p | I^{+} | n \rangle = g\, \delta^{3}(\underline{p}-\underline{n}) \tag{1.9}$$

By taking the equal-time commutator Eq. (8) between proton states, inserting a complete set of states and isolating the neutron contribution, we find that

$$(1-g^{2})\, \delta^{3}(\underline{p}'-\underline{p}) =$$

$$\sum_{\alpha \neq n} \frac{\langle p' | H^{+} | \alpha \rangle \langle \alpha | H^{-} | p \rangle}{(E_{p'}-E_{\alpha})(E_{\alpha}-E_{p})} - (H^{+} \longleftrightarrow H^{-})$$

But, $H^{\pm} = e \int d^{3}x\, J_{\mu}^{\pm}(x)\, A^{\mu}(x)$

so that

$$(1-g^{2})\, \delta^{3}(\underline{p}'-\underline{p}) = O(e^{2}) \tag{1.10}$$

In fact, this is just the Ademollo-Gatto theorem(9), which states that the renormalization is a second order effect in symmetry breaking. In Chapters II and III we shall actually be concerned with the renormalization effects due to weak interactions, however we shall be dealing with sum rules like Eq. (10).

4. Coleman's First Theorem(5).

During the period before Coleman's theorem appeared, current algebra enjoyed a secure position among elementary particle theories. The main advantage of using current algebra instead of a symmetry group for investigating hadron systematics, is that we do not have to assume invariance under an obviously broken symmetry.

Earlier in this chapter we mentioned the idea of saturating a commutator with a complete set of states. Now, from a computational point of view this is quite prohibitive. Instead, we suppose that some dynamical mechanism* is present, which means that, in practice, the charge operators use up approximately all their strength in taking one-particle states into other one-particle states belonging to a reasonably small set of low-lying states. In the jargon of the time, this was described by saying that, there is little or no 'leakage' from the set of low-lying states.

Coleman's Theorem shows that if there is no leakage from the hybrid collinear group, then the Hamiltonian is invariant under this group. Now this is just the sort of situation that current algebra was trying to avoid. Happily, as it turns out, Coleman's Theorem is not relevant tomost applications of current algebra. As we shall see later (Ch. II, Sec. 6) the use of the infinite momentum limit enables us to overcome this difficulty.

5. Schwinger terms.

When we consider the commutator

$$\left[J_o(x),\ J_i(y)\right] = 0 \qquad (1.11)$$

which is based on the naive quark model that we have been using

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* This is analogous to the spurion model of octet dominance.

so far, it is easy to see that a contradiction arises. Eq. (11) implies that $\left[J_0, \partial_0 J_0\right] = 0$ so that

$$\langle 0|\left[J_0, i\partial_0 J_0\right]|0\rangle = \langle 0|\left[J_0, [H, J_0]\right]|0\rangle$$

$$= \sum_\alpha E_\alpha \left|\langle 0| J_0 |\alpha\rangle\right|^2 = 0$$

This situation is quite unphysical, since the operator J_0 will in general have nonvanishing matrix elements between the vacuum and other states of positive energy.

Schwinger[7] has shown that we get this paradoxical situation because the currents, which are bilinear in the quark fields, are not well-defined as products of singular field operators at a coincident point. It is therefore necessary to define the current as a singular limit of the fields. Instead of, for example,

$$\mathcal{A}_i(x) = \psi^+(x)\, \sigma_i\, \psi(x) \tag{1.12}$$

we must define

$$\mathcal{A}_i(x) = \lim_{\underline{\varepsilon} \to 0} \psi^+(x-\underline{\varepsilon})\, \sigma_i\, \psi(x+\underline{\varepsilon}) \tag{1.13}$$

where the limiting procedure is to be performed symmetrically in space. In the naive quark model, i.e. from Eq. (12), we would get the equal-time commutation relation

$$\left[\mathcal{A}_i(x), \mathcal{A}_j(y)\right]_{x_0=y_0} = i\varepsilon_{ijk}\, \mathcal{A}_k(x)\, \delta^3(\underline{x}-\underline{y})$$

On the other hand, if we use the singular limiting procedure we get

$$\left[\mathcal{A}_i(x),\ \mathcal{A}_j(y)\right]_{x_o=y_o} = i\varepsilon_{ijk}\,\mathcal{A}_k(x)\,\delta^3(\underline{x}-\underline{y})$$

$$+\ \frac{1}{3}\,i\varepsilon_{ijk}\ \lim_{\underline{\varepsilon}\to 0}\ \underline{\mathcal{A}}(x)\cdot\underline{\varepsilon}\ \nabla_k\,\delta^3(\underline{x}-\underline{y}) + \ldots\text{(other singular terms)}. \qquad (1.14)$$

If we are working with charges, the Schwinger terms (i.e. anything more singular than the ordinary delta function) vanish on integration, and we have no problems. If, however, we intend to use charge-current or current-current commutation relations, we must examine the sum-rule very carefully when Schwinger terms are present, and try to eliminate them by a suitable symmetrization procedure.

6. Applications of current algebra.

There are basically three types of sum-rules that can be derived from current algebra. These are (i) the Dashen, Gell-Mann, Lee sum-rules, (ii) the superconvergence relations, and (iii) the spectral sum-rules. In this dissertation we shall only be concerned with the first type. Here, we take matrix elements of the equal-time current commutation relations between one-particle states, and obtain sum rules when we saturate the commutator with a complete set of states. By this method, we are able to calculate such things as renormalized coupling constants and magnetic moments.

The second type, the superconvergence relations, are directly concerned with scattering amplitudes, the dispersion relations these amplitudes satisfy, and their convergence properties. The third type, the spectral sum-rules, are the latest advance in current algebra, since they have only appeared within the last few months. So far, these have yielded sum-rules for the spectral functions, on the basis of chiral $SU(3) \otimes SU(3)$ current algebra.

CHAPTER II

BARYON SYSTEMS

In Chapter I we pointed out that the current algebra technique was proposed as an alternative method for obtaining higher symmetry results without actually having to assume invariance under the full symmetry group associated with the algebra. We shall now consider the derivation of the renormalized axial-vector coupling constant in leptonic baryon decays, using the SU(6) current algebra, and various subalgebras with appropriate kinematical restrictions. In sections (1) - (3) we discuss the results of Lee, Ryan, and Gerstein, but omit most of the details of their calculations, as we shall give the explicit derivation of a sum rule, based on chiral $SU(2) \otimes SU(2)$, in section (4). In the remaining sections of this chapter we examine some of the difficulties connected with the sum rules.

1. The SU(6) current algebra.

The method, of obtaining SU(6) results from current commutators, was first proposed by Lee[11]. The space integrals of the time component of the vector current, and the space components of the axial-vector current

$$V_o^\alpha\ (t) = -\ i \int d^3x\, \mathcal{V}_o^\alpha\ (x) \qquad \text{(where } \alpha = 0,\ \ldots, 8$$
$$A_i^\alpha\ (t) = -\ i \int d^3x\, \mathcal{A}_i^\alpha\ (x) \qquad \text{and } i = 1,2,3)$$

are assumed to generate an algebra which is isomorphic to the Lie algebra of the U(6) group, at equal times. In particular, the equal-time commutator of two A_i^α is

$$\left[A_i^\alpha \, , \, A_j^\beta\right] = i \, \delta_{ij} \, f^{\alpha\beta\gamma} \, V_o^\gamma + i \, \varepsilon_{ijk} \, d^{\alpha\beta\gamma} \, A_k^\gamma \tag{2.1}$$

where $\quad d_{\alpha\beta o} = \sqrt{\frac{2}{3}} \; \delta_{\alpha\beta}$

By taking the matrix element of Eq. (1) between $\frac{1}{2}^+$ baryon octet states of zero momentum, we obtain

$$\sum_{C,\gamma} \langle \tfrac{1}{2}^+ \alpha \,|\, A_i^\lambda \,|\, C\gamma \rangle\langle C\gamma \,|\, A_j^\mu \,|\, \tfrac{1}{2}^+ \beta \rangle - \begin{pmatrix} \lambda \longleftrightarrow \mu \\ i \longleftrightarrow j \end{pmatrix}$$

$$= i \, \delta_{ij} \, f^{\lambda\mu\nu} \langle \tfrac{1}{2}^+ \alpha \,|\, V_o^\nu \,|\, \tfrac{1}{2}^+ \beta \rangle + i \varepsilon_{ijk} d^{\lambda\mu\nu} \langle \tfrac{1}{2}^+ \alpha \,|\, A_k^\nu \,|\, \tfrac{1}{2}^+ \beta \rangle \tag{2.2}$$

A complete set of states is inserted in the commutator, and the summation is carried out over C and γ, the SU(3)-dimensionality and magnetic quantum number. The summation also involves an integration over masses, but it is assumed that the integral is highly convergent, and may be replaced by a sum over a few low lying states. We define G_s and G_a as the coupling constants (or form factors at zero momentum transfer) of the decay $\frac{1}{2}^+$ octet $\longrightarrow \frac{1}{2}^+$ octet + e + $\bar{\nu}$, with symmetric and antisymmetric coupling of the two octets. Similarly, we define G^* as the coupling constant for the transition process $\quad \frac{3}{2}^+$ decuplet $\longrightarrow \frac{1}{2}^+$ octet + e + $\bar{\nu}$.

If we assume that only the $\frac{1}{2}^+$ octet contributes to the intermediate state summation in Eq. (2), we obtain a set of six equations

$$G_a^2 - \frac{3}{5} G_s^2 = \sqrt{\frac{5}{3}}\, G_s \qquad G_s = \frac{1}{2}\sqrt{\frac{5}{3}}$$

$$G_a G_s = 0 \qquad G_a^2 + G_s^2 = 3$$

$$G_s^2 = 0 \qquad G_a^2 = \frac{3}{5} G_s^2$$

which are obviously inconsistent. This means that our saturation assumption is an extremely bad approximation. In other words, the axial-vector charge must connect the $\frac{1}{2}^+$ octet to more states than just the $\frac{1}{2}^+$ octet. For this reason, we now suppose that both the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet contribute to the summation. Eq. (2) then yields the unique solution

$$G_a = \frac{2}{\sqrt{3}} \;, \quad G_s = \sqrt{\frac{5}{3}} \quad \text{and} \quad G^* = 2$$

or, in the more usual form

$$G = \sqrt{\frac{1}{3}}\; G_a + \sqrt{\frac{3}{5}}\, G_s = \frac{5}{3}$$

and $$\frac{D}{F} = \frac{3\, G_s}{\sqrt{5}\, G_a} = \frac{3}{2}$$

These are exactly the SU(6) results[12].

In deriving these results, no use was made of SU(6) invariance; in fact, only SU(3) invariance was assumed. However, the fact that the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet saturate the sum rule, to the extent that we reproduce the experimentally incorrect SU(6) results, means that these states form an irreducible representation (namely the $\underline{56}$) of the U(6) algebra, but not necessarily of the group.

2. The SU(4) current algebra.

In the previous calculation by Lee, it was necessary to assume SU(3) invariance. However, Ryan[13,14] has been able to show that one gets essentially the same SU(6) results, if the internal symmetry is reduced from SU(3) to SU(2).

We now consider the commutator algebra SU(4), which is just the non-strange subalgebra of SU(6). In particular, the commutation relation between two axial-vector charges is

$$\left[A_i^\alpha , A_j^\beta\right] = i\delta_{ij}\, \varepsilon^{\alpha\beta\gamma}\, V_o^\gamma + \tfrac{1}{2} i\varepsilon_{ijk}\, \delta^{\alpha\beta}\, A_k^o \qquad (2.3)$$

where the internal symmetry indices α, β now run over 1,2,3. By analogy with the SU(6) sum rule which was saturated by the I.R. 56, we would expect to saturate the SU(4) sum rule with the corresponding I.R., namely the 20-representation of SU(4). This representation just contains the nucleon N, and the 3,3-resonance $N^*(1236)$. The coupling constants are defined as

$$\langle N | A_i^\alpha | N\rangle \sim G \qquad\qquad \langle N | A_k^o | N\rangle \sim g$$

$$\langle N^* | A_i^\alpha | N\rangle \sim G_{14} \qquad\qquad \langle N^* | A_k^o | N^*\rangle \sim g^*$$

$$\langle N^* | A_i^\alpha | N^*\rangle \sim G_{44}$$

In the usual manner, we take the matrix element of Eq. (3) between zero momentum states. However, to obtain a sufficient number of equations, to determine the five coupling constants, we must take the matrix element first between N and N, then N^* and N, and finally between N^* and N^*. This produces six equations

$$G^2 - \frac{8}{9} G_{14}^2 = 1 \qquad G^2 - \frac{8}{9} G_{14}^2 = g$$

$$G = 5\, G_{44} \qquad \frac{1}{9} G_{14}^2 = 2\, G_{44}^2$$

$$\frac{2}{9} G_{14}^2 + 5\, G_{44}^2 = 1 \qquad \frac{2}{9} G_{14}^2 + 5\, G_{44}^2 = g^*$$

which have the unique solution

$$G = \pm \frac{5}{3} \qquad G_{44} = \pm \frac{1}{3}$$

$$G_{14}^2 = 2 \qquad g = g^* = 1$$

Thus the SU(4) current algebra is able to reproduce the SU(6) result $G = 5/3$ except for an ambiguity in the sign, which we discuss in section (7).

Two points are worth commenting on at this stage:

Using the SU(6) algebra, we only have to take the matrix element of the commutator between $\frac{1}{2}^+$ octet states. Whereas, for the SU(4) algebra, we have to go to much greater lengths, and take the matrix element between three different combinations of states, to extract essentially the same information. This is due to the fact that a greater number of channels (dictated by the internal symmetry) is available in the SU(6) case. In SU(3), the current octet can couple to the $\frac{1}{2}^+$ octet through 1, 8_a, 8_s, 10, $\overline{10}$ and 27, whereas in SU(2), the current triplet and nucleon doublet can only couple through $1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$.

The second point is that if we try to saturate Eq. (3) with only the nucleon states, we do, in fact, get a solution, namely $G^2 = 1$ and $g = 1$. In the SU(6) case, when we tried to do this, we ended up with an inconsistent set of equations.

At first sight, there appears to be something wrong here. In going from the SU(6)- to the SU(4)-sum rule we are just restricting ourselves to the nonstrange subspace. Consequently, we might expect the $\frac{1}{2}^+$ baryon states to give an inconsistent set of equations in both cases. The reason why this does not happen can be seen if we consider the spin-unitary spin content of the irreducible representations used in saturating the sum rules.

For $\quad SU(4) \longrightarrow SU(2)_I \otimes SU(2)_J$

$$\underline{20} \longrightarrow (\tfrac{1}{2}, \tfrac{1}{2}) \oplus (\tfrac{3}{2}, \tfrac{3}{2})$$

$$\underline{\bar{4}} \longrightarrow (\tfrac{1}{2}, \tfrac{1}{2})$$

and for $\quad SU(6) \longrightarrow SU(3) \otimes SU(2)_J$

$$\underline{56} \longrightarrow (8, \tfrac{1}{2}) \oplus (10, \tfrac{3}{2})\ ^{*}$$

Therefore, saturating the SU(4) sum rule with the nucleon states alone is equivalent to saturating it with the I.R. $\bar{4}$ of SU(4). ($\bar{4}$ is contained in the decomposition of $4 \otimes 4 \otimes 4$, and is therefore a suitable representation for the nucleons). However, for the SU(6) case, the $\frac{1}{2}^+$ baryon octet, which forms the $(8,\frac{1}{2})$ representation of $SU(3) \otimes SU(2)_J$, corresponds to a 16-dimensional representation of SU(6), and this is certainly not an irreducible representation of the algebra. Consequently, the reason why we get a solution of the SU(4)-, but not of the SU(6)-sum rule, is that the $\frac{1}{2}^+$ baryons themselves form an I.R. in SU(4) but not in

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* We are using the hybrid notation of dimensionality for the SU(3) representation, but the (iso)spin eigenvalue for SU(2).

SU(6). Actually, if we assign "unphysical" nucleon states to the $(3,\frac{1}{2})$ representation of $SU(3) \otimes SU(2)_J$ i.e. the 6-dimensional representation of SU(6), we can saturate the SU(6)-sum rule with these states alone. In this case we get the expected result that $G = 1$ and $g = 1$.

3. The Chiral SU(3) ⊗ SU(3) algebra.

Gerstein's sum rule[15] obtained from the chiral SW(3) algebra, gives the usual result that $G = \pm 5/3$. However, the actual calculation appears to be very different from the SU(6) and SU(4) cases. First, we must use states of infinite momentum in the z-direction and secondly, we assign particles, of definite helicity, to representations of the collinear SW(3) algebra* rather than the chiral algebra. This is because of the equivalence of SU(6) matrix elements at rest, and collinear or chiral SW(3) matrix elements at infinite momentum, as we shall see in section (5).

The chiral SW(3) algebra is generated, at equal-times, by $V_o^\alpha(t) \pm A_o^\alpha(t)$ with $\alpha = 1,\ldots,8$. As usual, we consider the commutator of the axial-vector charges

$$\left[A_o^\alpha \, , \, A_o^\beta \right] = i \, f^{\alpha\beta\gamma} \, V_o^\gamma \tag{2.4}$$

but take its matrix element between $\frac{1}{2}^+$ octet states, moving in the z-direction. The time component of the axial-vector charge $A_\mu^\alpha(t)$ is a helicity conserving operator, and so, if we take the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet as our "complete" set of states, only their helicity $\frac{1}{2}$ components will give a nonvanishing contribution

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* The collinear group we are referring to here is generated by $V_o \pm A_z$, which explains why we have chosen the z-direction for the motion of the particles.

to the matrix element. What we are really doing, is assigning the $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet to the representation*

$$(10,1)_{\frac{3}{2}} \oplus (6,3)_{\frac{1}{2}} \oplus (3,6)_{-\frac{1}{2}} \oplus (1,10)_{-\frac{3}{2}} \qquad \text{of the}$$

collinear SW(3) algebra, and using just the $(6,3) \oplus (3,6)$ representation to saturate the sum rule.

We obtain the solution for G by taking the matrix element of Eq. (4) between $\frac{1}{2}^+$ octet states, and letting their momentum tend to infinity in the z-direction. In addition, as a consistency check, we can take the matrix element between $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet states, and between $\frac{3}{2}^+$ decuplet states. It turns out that altogether we have a set of seven completely consistent equations with the unique solution $G = \pm 5/3$. In carrying out these additional calculations, we require a knowledge of the elements of the SU(3) crossing matrix $(\mu\xi|\beta_{II}(8,\overline{10},8,8)|\mu'\xi')$. These are given in Table I of Appendix B, as they have not been tabulated elsewhere.

4. <u>The chiral SU(2) ⊗ SU(2) algebra</u>.

The use of a chiral algebra, in deriving sum rules, has one great advantage. As we are forced to take the infinite momentum limit, we can remove the mass degeneracy between the baryon octet and the baryon decuplet, which is normally assumed. We shall now present a sum rule[16] based on chiral SW(2), explicitly putting in the nondegenerate masses m and m^* for the nucleon N and the 3,3-resonance N^*.

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* The subscripts $\pm \frac{1}{2}$ and $\pm \frac{3}{2}$ refer to the eigenvalue of $A_z^o = J_z$ i.e. the helicity.

In a quark model, the space integrals, of the time components of the vector and axial-vector current densities, define the operators

$$V_o^\alpha(t) = \int d^3x\ \psi^+(x)\ \frac{\tau^\alpha}{2}\ \psi(x)$$

$$A_o^\alpha(t) = \int d^3x\ \psi^+(x)\ \gamma_5\ \frac{\tau^\alpha}{2}\ \psi(x)$$

(where $\alpha = 1, 2, 3$)

which close under commutation at equal times, to give an algebra isomorphic to the chiral SW(2) algebra. To show that it is, in fact, the chiral SW(2) algebra, rather than some other algebra with six elements, we define the left- and right-handed chirality operators

$$F_\pm^\alpha(t) = \frac{V_o^\alpha(t) \pm A_o^\alpha(t)}{2} = \int d^3x\ \psi^+(x)\left(\frac{1 \pm \gamma_5}{2}\right)\frac{\tau^\alpha}{2}\ \psi(x).$$

We then see that

$$\left[F_+^\alpha(t),\ F_+^\beta(t)\right] = i\varepsilon^{\alpha\beta\gamma}\ F_+^\gamma(t)$$

$$\left[F_+^\alpha(t),\ F_-^\beta(t)\right] = 0$$

$$\left[F_-^\alpha(t),\ F_-^\beta(t)\right] = i\varepsilon^{\alpha\beta\gamma}\ F_-^\gamma(t)$$

In performing the calculation, it is more convenient to work in the nonhermitian spherical basis. The spherical vectors F^μ, with $\mu = \pm 1$ and 0, are given in terms of the usual Cartesian vectors F^α, by $F^\mu = e^\mu_\alpha\ F^\alpha$. The elements of the transformation matrix 'e' are tabulated by Lee(17). In the spherical basis, the charge operators satisfy the equal-time commutation relations

$$\left[A_o^\lambda \,,\, A_o^\mu \right] = - \sqrt{2} \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \end{pmatrix} V_o^\nu \tag{2.5}$$

$$\left[V_o^\lambda \,,\, A_o^\mu \right] = - \sqrt{2} \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \end{pmatrix} A_o^\nu \tag{2.6}$$

$$\left[V_o^\lambda \,,\, V_o^\mu \right] = - \sqrt{2} \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \end{pmatrix} V_o^\nu \;\; * \tag{2.7}$$

We denote the momentum eigenstates of the N and N^* by $\left| \frac{1}{2}\, \alpha\, p \right\rangle$ and $\left| \frac{3}{2}\, \alpha\, p \right\rangle$ where 'α' is the isospin index (in the spherical basis) and 'p' the 4-momentum. The coupling constants are defined by

$$\left\langle \tfrac{1}{2}\alpha p' \right| A_o^\lambda \left| \tfrac{1}{2}\beta p \right\rangle$$
$$= \delta^3(\underline{p}'-\underline{p})\, \frac{m}{p_o}\, \overline{u}(\underline{p}') i\gamma_o\gamma_5 u(\underline{p})\, G_{11} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \beta & \lambda & \alpha \end{pmatrix} \tag{2.8a}$$

$$\left\langle \tfrac{3}{2}\alpha p' \right| A_o^\lambda \left| \tfrac{1}{2}\beta p \right\rangle$$
$$= \delta^3(\underline{p}'-\underline{p}) \sqrt{\frac{mm^*}{p_o p_o^*}}\, \overline{u}_o(\underline{p}') u(\underline{p})\, G_{14} \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ \beta & \lambda & \alpha \end{pmatrix} \tag{2.8b}$$

$$\left\langle \tfrac{3}{2}\alpha p' \right| A_o^\lambda \left| \tfrac{3}{2}\beta p \right\rangle$$
$$= \delta^3(\underline{p}'-\underline{p})\, \frac{m^*}{p_o^*}\, \overline{u}_\mu(\underline{p}') i\gamma_o\gamma_5 u^\mu(\underline{p})\, G_{44} \begin{pmatrix} \frac{3}{2} & 1 & \frac{3}{2} \\ \beta & \lambda & \alpha \end{pmatrix} \tag{2.8c}$$

The scale of the matrix elements is determined by the solution of Eq. (7), namely

$$\left\langle \tfrac{1}{2}\alpha p' \right| V_o^\nu \left| \tfrac{1}{2}\beta p \right\rangle$$
$$= \delta^3(\underline{p}'-\underline{p})\, \frac{\sqrt{3}}{2} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \beta & \nu & \alpha \end{pmatrix} \tag{2.9a}$$

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* We use the notation $\begin{pmatrix} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ for the SU(2) Clebsch-Gordan coefficients. (See Appendix C).

$$\langle \tfrac{3}{2}\alpha p' |\ V_o^\nu | \tfrac{3}{2}\beta p \rangle$$
$$= \delta^3(\underline{p}'-\underline{p})\ \frac{\sqrt{15}}{2} \begin{pmatrix} \frac{3}{2} & 1 & \frac{3}{2} \\ \beta & \nu & \alpha \end{pmatrix} \tag{2.9b}$$

We first take the matrix element of Eq. (5) between N and N and insert N and N^* with momentum k, as the intermediate states. Here we are assigning the helicity $+\frac{1}{2}$ states of the N and N^* to the $(1,\frac{1}{2})_{\frac{1}{2}}$ representation of the collinear SW(2) group. In fact, the $\underline{20}$-representation of SU(4) reduces under the collinear subgroup SW(2) to

$$\underline{20} = (\tfrac{3}{2},0)_{\frac{3}{2}} \oplus (1,\tfrac{1}{2})_{\frac{1}{2}} \oplus (\tfrac{1}{2},1)_{-\frac{1}{2}} \oplus (0,\tfrac{3}{2})_{-\frac{3}{2}}$$

$$\therefore \sum_\gamma \int d^3k \langle \tfrac{1}{2}\alpha p' |\ A_o^\lambda | \tfrac{1}{2}\gamma k \rangle\langle \tfrac{1}{2}\gamma k | A_o^\mu | \tfrac{1}{2}\beta p \rangle$$
$$+ \sum_\gamma \int d^3k \langle \tfrac{1}{2}\alpha p' |\ A_o^\lambda | \tfrac{3}{2}\gamma k \rangle\langle \tfrac{3}{2}\gamma k | A_o^\mu | \tfrac{1}{2}\beta p \rangle$$
$$- (\lambda \longleftrightarrow \mu)$$
$$= -\sqrt{2} \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \end{pmatrix} \langle \tfrac{1}{2}\alpha p' |\ V_o^\nu | \tfrac{1}{2}\beta p \rangle$$

Using the definitions (8) and (9), and integrating over d^3k, we obtain

$$\frac{m^2}{p_o^2}\, \overline{u}(\underline{p}) i\gamma_o\gamma_5 u^\sigma(\underline{p})\overline{u}^\sigma(\underline{p}) i\gamma_o\gamma_5 u(\underline{p})\ G_{11}^2\ (-1)^\lambda \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \alpha-\lambda & \gamma & \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \beta & \mu & \gamma \end{pmatrix}$$
$$+ \frac{mm^*}{p_o p_o^*}\, \overline{u}(\underline{p}) u_o^\sigma(\underline{p})\overline{u}_o^\sigma(\underline{p}) u(\underline{p})\ G_{14}^2\ (-1)^\lambda \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ \alpha-\lambda & \gamma & \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ \beta & \mu & \gamma \end{pmatrix}$$
$$- (\lambda \longleftrightarrow \mu)$$
$$= -\sqrt{\tfrac{3}{2}} \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \beta & \nu & \alpha \end{pmatrix}\ \overline{u}(\underline{p})u(\underline{p}) \tag{2.10}$$

We now perform two manipulations:

(i) We multiply both sides of Eq. (10) by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & J \\ \alpha & -\beta & m' \end{pmatrix} \begin{pmatrix} 1 & 1 & J \\ \lambda & \mu & m' \end{pmatrix} \quad (-1)^{\frac{1}{2}-\beta}$$

and sum over α, β, λ, μ and m'. Using the orthogonality and symmetry properties of the Clebsch-Gordan coefficients, we can write the product of four C-G coefficients as an SU(2) crossing matrix $(J'|\beta_{II}(J_1, J_2, J_3, J_4)|J)$. Details of this are given in Appendix C.

(ii) We sum over the helicities of the intermediate states i.e. insert projection operators*. In order to obtain a covariant equation we then take the limit $p_z = \infty$.

Eq. (10) then becomes

$$\bar{u}(\underline{p})u(\underline{p})\ G_{11}^2\ (1-(-1)^J)\ 2\ (\tfrac{1}{2}|\beta_{II}(\tfrac{1}{2}, 1, \tfrac{1}{2}, 1)|J)$$

$$+ \frac{2}{3}\left(\frac{m+m^*}{2m^*}\right)^2 \bar{u}(\underline{p})u(\underline{p})\ G_{14}^2\ (1-(-1)^J)\ 4\ (\tfrac{3}{2}|\beta_{II}(\tfrac{1}{2}, 1, \tfrac{1}{2}, 1)|J)$$

$$= -3\ \delta_{J1}\ \bar{u}(\underline{p})u(\underline{p})$$

which yields, on evaluating the crossing matrices**

$$\frac{4}{3}\ G_{11}^2 - \frac{8}{9}\left(\frac{m+m^*}{2m^*}\right)^2 G_{14}^2 = 1 \tag{2.11}$$

Similarly, taking the matrix element of Eq. (5) between N and N^* gives

$$G_{11} = \frac{\sqrt{5}}{3}\ G_{44} \tag{2.12}$$

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* See Appendix A.

** The values of the crossing matrices are given in Tables I-III of Appendix C.

In performing the calculation, there is one point that may cause confusion. In defining the coupling constants, we made use of the Wigner-Eckart theorem. Consequently, G_{14} defined by Eq. (8b) is proportional to the reduced matrix element $(\frac{1}{2}\| A(1) \| \frac{3}{2})$. If the order of the N and N^* states is reversed, we must note that

$$(\tfrac{3}{2}\| A(1) \| \tfrac{1}{2}) = - \sqrt{2}\,(\tfrac{1}{2}\| A(1) \| \tfrac{3}{2}) \,.$$

Finally, when we take the matrix element of Eq. (5) between N^* and N^* we obtain

$$\frac{20}{3}\left(\frac{m+m^*}{2m^*}\right)^2 G_{14}^2 + \frac{4}{9} G_{44}^2 = 15 \tag{2.13}$$

$$G_{44}^2 = \frac{15}{4} \tag{2.14}$$

In taking this final matrix element we have to make use of one unusual property of the Rarita-Schwinger wave function*, namely that

$$\begin{aligned} \bar{u}_o^{\sigma}(\underline{p})u_o^{\sigma'}(\underline{p}) &= 0 \qquad \text{for } \sigma,\sigma' = \pm \tfrac{3}{2} \\ &= \frac{2}{3}\left(\frac{p_o^2}{m^2} - 1\right)\delta_{\sigma\sigma'} \quad \text{for } \sigma,\sigma' = \pm \tfrac{1}{2} \end{aligned}$$

Eqs. (11-14) then give the unique solution

$$G_{11}^2 = \frac{25}{12} \;, \quad G_{44}^2 = \frac{15}{4}$$

and $G_{14}^2 = 2\left(\dfrac{2m^*}{m+m^*}\right)^2$

Using the scale factor $G_V = \frac{\sqrt{3}}{2}$ from Eq. (9a), we get the more familiar result

$$G = G_{11}\Big/ G_V = \pm \frac{5}{3}$$

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* See Eq. (A.14)

and $G^* = \frac{2}{\sqrt{3}} G_{14} = \pm\sqrt{\frac{8}{3}}\left(\frac{2m^*}{m+m^*}\right)$

Allowing for the physical masses of N and N^* does nothing to G, and in fact makes G^* slightly worse. An estimate of the value of G^* based on experimental πN cross sections and the P.C.A.C. hypothesis has been made by Adler[18] and Weisberger[19] giving $G^* = 1{\cdot}1 \pm 0{\cdot}1$. On the other hand, this sum rule gives $G^* = 1{\cdot}6$ if $m = m^*$ and $G^* = 1{\cdot}8$ if $m = 938$ and $m^* = 1236$.

5. The relation between the various algebras.

It is hardly surprising that SU(6)-current algebra reproduces SU(6) results. On the other hand, it seems quite remarkable that the chiral SW(2) algebra can give the same results, particularly as it is not a subgroup of SU(6). It is also interesting to note that the states we saturate the SW(2)-sum rule with, almost certainly do not form an irreducible representation of the chiral group. It seems doubtful anyway, whether nature exhibits any trace of invariance under the chiral symmetry (which is only exact in the case of vanishing mass!) It has been suggested[20] that the most likely assignment for the baryon octet, in the chiral group, would be $(3,\bar{3}) \oplus (\bar{3},3)$, but even this is not very appealing as it introduces the problems of parity doublets and the ninth baryon. Inany case, this representation is quite disjoint* from $(10,1) \oplus (1,10)$ to which the baryon decuplet might belong.

However, when we study the matrix elements in more detail,

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* This would mean the vanishing of the transition matrix element $\langle N \,|\, A_o \,|\, N^* \rangle$.

we see a natural link between all the algebras that have been used. Using Eq. (A.16) we find for the nucleons, that

$$\left\langle \tfrac{1}{2}p_z \middle| A_o \middle| \tfrac{1}{2}p_z \right\rangle = \frac{|p_z|}{p_o}\, \bar{u}(p_z)\ \sigma_z\ u(p_z)\ F_1(q^2)$$

and

$$\left\langle \tfrac{1}{2}p_z \middle| A_z \middle| \tfrac{1}{2}p_z \right\rangle = \bar{u}(p_z)\ \sigma_z\ u(p_z)\ F_1(q^2)$$

so that

$$\left\langle \tfrac{1}{2}p_z \middle| A_o \middle| \tfrac{1}{2}p_z \right\rangle = \frac{|p_z|}{p_o} \left\langle \tfrac{1}{2}p_z \middle| A_z \middle| \tfrac{1}{2}p_z \right\rangle \tag{2.15}$$

i.e. the matrix elements of the chiral and collinear algebras become equal in the limit of infinite momentum. A similar argument holds for the N^*. When we consider the transition matrix element between N and N^*, a difficulty arises. We can express the matrix element in the usual covariant form

$$\left\langle \tfrac{3}{2}p' \middle| A_\lambda \middle| \tfrac{1}{2}p \right\rangle =$$

$$\bar{u}^\mu(\underline{p}')\left\{ F_1(q^2)g_{\mu\lambda} + iF_2(q^2)q_\mu\gamma_\lambda + F_3(q^2)q_\mu q_\lambda + F_4(q^2)\sigma_{\lambda\nu}q^\nu q_\mu \right\} u(\underline{p})$$

and with the help of Eqs. (A.15-20) this gives

$$\left\langle \tfrac{3}{2}p_z \middle| A_o \middle| \tfrac{1}{2}p_z \right\rangle = \bar{u}_o(p_z)u(p_z)\left\{ F_1(q^2)+(m^*-m)F_2(q^2)\right\}$$

and

$$\left\langle \tfrac{3}{2}p_z \middle| A_z \middle| \tfrac{1}{2}p_z \right\rangle = \bar{u}_o(p_z)u(p_z)\left\{ \frac{p_o}{|p_z|}\, F_1(q^2)+(m^*-m)F_2(q^2) \right\}$$

In the limit $p_z \to \infty$, the transition matrix elements of the chiral and collinear algebras are indeed equal. However, we notice that for the physical situation of $m^* \neq m$, the matrix element does not have a unique form, but depends on two form factors. In deriving the SW(2)-sum rule, we make the assumption that due to an angular-momentum barrier, all derivative couplings are absent. This means that only the term $F_1(q^2)$ will be present. The fact that this procedure makes the value of the transition coupling constant $G^* \sim F_1(0)$ slightly worse, could mean that the derivative coupling is present. However, it is most unlikely that any experiment will ever clarify this point.

Finally, using the boost matrix, Eq. (A.6) we can show that

$$\langle p_z | A_o | p_z \rangle = \frac{|p_z|}{p_o} \langle \underline{0} | A_z | \underline{0} \rangle \tag{2.16}$$

i.e. the matrix element of the chiral algebra at infinite momentum is equivalent to the matrix element of the collinear subgroup of SU(6) at zero momentum. Eqs. (15-16) thus show the complete equivalence of the matrix elements of SU(6), chiral SW(2) and collinear SW(2) in the appropriate momentum limits.

6. The intermediate states.

The axial-vector current is not conserved, and can therefore connect the vacuum to particle-antiparticle pairs*. In addition to the one-particle intermediate states (Fig. 1) in the matrix element $\langle N|A'|N \rangle \langle N|A|N \rangle$, we should expect three-particle intermediate states (Fig. 2) i.e. $\langle N|A'|N(N\bar{N})\rangle\langle N(N\bar{N})|A|N\rangle \neq 0$

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* At zero momentum there is no such difficulty unless we allow massless particles.

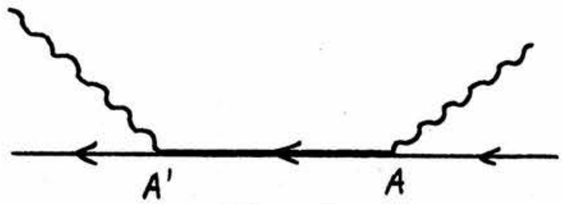

Fig. 1

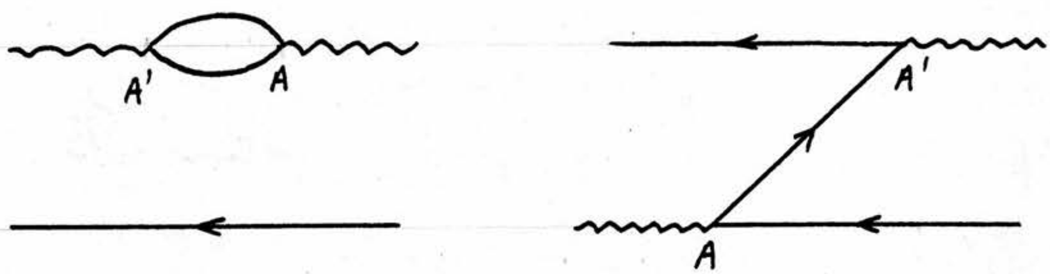

Fig. 2a　　　　　　　　Fig. 2b

In the infinite momentum limit Figs. 2a and 2b correspond to intermediate states of infinite mass. This can easily be seen, if we consider the invariant mass of the intermediate state

$$M^2 = (E + E_i)^2 - \underline{p}^2 = m^2 + E_i^2 + 2EE_i \rightarrow \infty \text{ as } \underline{p} \rightarrow \infty$$

where E_i = energy of the pair

and E, m = energy and mass of the one-particle state .

The fact that such particles have never been observed is perhaps a good enough reason for neglecting these diagrams. Gell-Mann[21] has pointed out that neglecting these infinite mass states is equivalent to assuming unsubtracted dispersion relations. Actually, as long as we are dealing with a commutator, completely disconnected diagrams (Fig. 2a) will always vanish, since they are symmetric in A and A'. Finally, we note that the Z-diagrams (Fig. 2b) will vanish since the particle-antiparticle pair which has negative parity (due to momentum conservation) cannot be connected to the vacuum through the axial-vector current, which has positive parity.

7. G-conjugation.

We noticed, in deriving $G = 5/3$ from the various current algebras, that only SU(6) was able to determine the sign. In the context of SU(4), Ryan[13,14] has shown that the ambiguity in the sign arises from the inability of the SU(4) operators to 'see' any difference between a particle and its conjugate. In other words, we will get two solutions, since the N and N^* may be assigned to either the $\underline{20}$- or $\overline{\underline{20}}$- representation of SU(4).

To see what happens in the case of SW(3), we must examine the properties of the G-conjugation operator. For an arbitrary internal symmetry group SU(n) we can define[22] the G-conjugation operator as $G = C.R$ where C is the "generalized charge"-conjugation operator and R is a generalized rotation in the n-dimensional unitary spin space. Specializing this to the SU(3) case, the action of C and R on the vector and axial-vector currents is given by

$$C\, V^{\alpha}\, C^{-1} = -\, e^{(\alpha)}\, V^{\alpha} \qquad C\, V^{0}\, C^{-1} = -\, V^{0} \quad \text{(no summation)}$$

$$C\, A^{\alpha}\, C^{-1} = e^{(\alpha)}\, A^{\alpha} \qquad C\, A^{0}\, C^{-1} = A^{0} \quad \text{where } \alpha = 1,\ldots,8$$

and

$$R\, Q^{\alpha}\, R^{-1} = -\, e^{(\alpha)}\, Q^{\alpha} \qquad R\, Q^{0}\, R^{-1} = Q^{0} \quad \text{where } Q = V \text{ or } A$$

According to the usual[23] convention

$$e^{(\alpha)} = +1, \text{ for } \alpha = 1, 3, 4, 6, 8$$
$$= -1, \text{ for } \alpha = 2, 5, 7$$

Consequently, under G-conjugation

$$G\, A^{\alpha}\, G^{-1} = -\, A^{\alpha}\,, \qquad G\, A^{o}\, G^{-1} = A^{o}$$

$$G\, V^{\alpha}\, G^{-1} = V^{\alpha}\,, \qquad G\, V^{o}\, G^{-1} = -\, V^{o}$$

Applying these results to the various groups, we find that G-conjugation is an inner automorphism of SU(2), SW(2) and SU(4), an outer automorphism of SU(3) and SW(3), but that it is not an automorphism of SU(6). This means that only SU(6) will be able to unambiguously fix the sign of the coupling constant.

8. Concluding Remarks.

The main thing we have seen in this chapter is that a variety of current algebras lead to the same results of SU(6). However, one feature is not entirely satisfactory. In saturating the sum rules, we assume that the only states with quantum numbers $(I, J^{P}) = (\frac{1}{2}, \frac{1}{2}^{+})$ are the nucleons. *A priori* we would not expect this, and in any case, the intermediate states need not be on the mass shell. If we make allowance for this, we find that instead of the usual result $G^2 = 25/9$, we now have

$$G^2 + \sum_{m} F^2 = \frac{25}{9} \tag{2.17}$$

where $\sum F^2$ is the contribution, from all energies, of $(\frac{1}{2}, \frac{1}{2}^{+})$ states, other than the nucleon. Of course, the current algebra sum rules cannot say anything about the individual values of F and G without making additional assumptions. Since, from experiment, $G^2 = 1{\cdot}38$, we would expect that $\sum F^2 = 1{\cdot}4$, which is quite a sizeable contribution. For the sake of clarity, we have oversimplified things here, by only considering a continuum of $(\frac{1}{2}, \frac{1}{2}^{+})$ states. We would, of course, also expect to have contributions from other $(\frac{3}{2}, \frac{3}{2}^+)$ states.

Adler[18] and Weisberger[19] have, in fact, carried out such a program. They consider the matrix element of the commutator $[A_0^+, A_0^-]$ between proton states. By isolating the neutron contribution to the intermediate state, they are left with an integral over the 4-momentum of all the other states with quantum numbers $(\frac{1}{2},\frac{1}{2}^+)$ and $(\frac{3}{2}, \frac{3}{2}^+)$. It is then possible to relate the quantities in the continuum integral to πN total cross sections, by using the P.C.A.C. hypothesis. Adler uses a model which considers the scattering of zero mass pions off protons, and shows that the off-mass-shell corrections are small. This gives him the result $G = 1{\cdot}24 \pm 0{\cdot}03$. On the other hand, Weisberger uses a dispersion theoretic method to obtain the result $G = 1{\cdot}16$. One interesting point that arises from these calculations is that with the N^* alone in the integral, the value of G is $\sim 1{\cdot}4$. It is the contribution of other $(\frac{1}{2},\frac{1}{2}^+)$ states that depresses the value to $\sim 1{\cdot}2$, which qualitatively we would have expected from Eq.(17).

We have not made any mention, so far, of magnetic moments. As several important problems occur here, we will devote Chapter IV to a discussion of magnetic moment sum rules.

CHAPTER III

MESON SYSTEMS

Following the success of current algebra predictions for baryons, Fayyazuddin, Riazuddin and Razmi[32] obtained analogous results for the vector and pseudoscalar mesons using the SU(6) algebra of currents. We now wish to show that similar results can be obtained from the SU(4), chiral SW(3), and chiral SW(2) current algebras. As the method of calculation is very similar to the example we gave previously (Ch.II, Sec.4), we shall not repeat it in detail here. Values of the SU(2) crossing matrices, used in these calculations, are however given in Tables IV and V of Appendix C. We shall limit our discussion to the essential points of difference. These are (i) the assignment of mesons to supermultiplets, (ii) the vanishing of matrix elements, due to charge conjugation invariance and helicity conservation, (iii) the role of positive parity mesons, (iv) the form of the interaction, and (v) nondegenerate masses and form factors at zero and infinite momentum.

Once again, we wish to emphasize an important distinction between SU(4) matrix elements at rest, and chiral SW(3) or SW(2) matrix elements at infinite momentum. When we expand the matrix element, in the usual covariant way in terms of form factors $F_i(q^2)$*, we have in mind that we ultimately want to determine the SU(6) coupling constants, i.e. $F_1(0)$. If the matrix element is taken between states of physically different mass, we are in

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* q is the momentum transfer.

trouble <u>unless</u> we either go to infinite momentum, or assume degenerate masses when we are at finite (or zero) momentum, as otherwise, $q^2 \neq 0$.

1. <u>Meson sum rule from SU(4) current algebra.</u>

The pseudoscalar (P) and vector (V) mesons belong to the $\underline{1} \oplus \underline{15}$ representation of SU(4), and this reduces under $SU(2)_I \otimes SU(2)_J$ to $(0,0) \oplus (1,0) \oplus (0,1) \oplus (1,1)$. In fact, due to the charge conjugation properties of the matrix elements (assuming invariance under the internal symmetry group $SU(2)_I$), the pseudoscalar singlet, belonging to the $\underline{1}$-representation, is not coupled to any member of the $\underline{15}$-representation. We shall, therefore, try to saturate the sum rule with a vector meson isotriplet (V^3) and an isosinglet (V^0), and a pseudoscalar meson isotriplet (P^3).

We define the coupling constants, for the meson states at rest, as

$$\langle V^0 \mid A_i^\lambda \mid V^3\beta \rangle = i\sqrt{3}\, f_{AO}(\underline{\varepsilon}^+ \times \underline{\varepsilon})_i \begin{pmatrix} 1 & 1 & 0 \\ \beta & \lambda & 0 \end{pmatrix}$$

$$\langle P^3\alpha \mid A_i^\lambda \mid V^3\beta \rangle = \frac{m}{\sqrt{2}}\; h_A\; \varepsilon_i \begin{pmatrix} 1 & 1 & 1 \\ \beta & \lambda & \alpha \end{pmatrix}$$

where 'm' is the average mass of the pseudoscalar and vector mesons, belonging to the $\underline{15}$-representation.

The overall scale is determined by

$$\langle V^3\alpha \mid V_0^\nu \mid V^3\beta \rangle = 2\, \underline{\varepsilon}^+ \cdot \underline{\varepsilon} \begin{pmatrix} 1 & 1 & 1 \\ \beta & \nu & \alpha \end{pmatrix}$$

Other matrix elements for example $\langle V^3\alpha \mid A_i^\lambda \mid V^3\beta \rangle$, vanish in the limit of exact SU(2) symmetry because of charge conjugation invariance.

Taking the matrix element of the SU(4) commutator, Eq. (2.3), between V^3 states at rest, we ultimately get two equations which have the unique solution

$$f_{AO}^2 = 1 \quad \text{and} \quad h_A^2 = \left(\frac{2}{m}\right)^2$$

To check the consistency of these results, we can also take the matrix element between P^3 states at rest, and this finally gives the same value for h_A.

The results we have derived are just those obtained by Razmi et al[32] from the SU(6) algebra of currents. Although the SU(6)-sum rule gives the value $f_A = 1$, for the axial vector coupling constant in the decay $V \to V + e + \bar{\nu}$, the SU(4)-sum rule gives $f_A = 0$. In fact, there is no inconsistency here, since the vector mesons, in the decay $\rho^0 \to \rho^+ + e + \bar{\nu}$, are only coupled to the vector component of the lepton current. The reason why f_A is nonvanishing in the SU(6) case is due to the occurrence of decays such as $\rho^0 \to K^{*+} + e + \bar{\nu}$ and $K^{*+} \to K^{*+} + e + \bar{\nu}$.

2. Meson sum rule from chiral SW(3) current algebra.

We have seen previously how, in deriving chiral algebra sum rules, we assign particles of definite helicity, to representations of the collinear subgroup of SU(6).

Under the reduction

$$SU(6) \longrightarrow SU(3) \otimes SU(3)_{coll}$$

we find for the mesons that

$$35 \to (3,\bar{3})_{+1} \oplus (8,1)_0 \oplus (1,1)_0 \oplus (1,8)_0 \oplus (\bar{3},3)_{-1}$$

This means that the helicity 0 components of the vector meson octet (V^8), plus singlet (V^0), and the pseudoscalar meson octet (P^8) belong to the representation $(8,1) \oplus (1,1) \oplus (1,8)$. Also, the helicity ± 1 components of the vector mesons belong to the representation $(3,\overline{3}) \oplus (\overline{3},3)$.

Making use of the charge-conjugation properties* of the matrix elements, and also the fact that matrix elements, taken between certain helicity eigenstates, vanish (as we shall see in section 4), we can define the coupling constants for the nonvanishing matrix elements as**

$$\langle V^8\, \alpha\, p' \,|\, A_0^\gamma \,|\, V^8 \beta\, p \rangle_{\lambda=1} =$$

$$\frac{\delta^3(\underline{p}'-\underline{p}}{\sqrt{4p_0 p_0'}}\; 2i\sqrt{\frac{5}{3}}\, f_A \begin{pmatrix} 8 & 8 & 8_s \\ \beta & \gamma & \alpha \end{pmatrix} \underline{\varepsilon}^+(\underline{p}') \times \underline{\varepsilon}(\underline{p}) \cdot \underline{p} \qquad (3.1a)$$

$$\langle V^0\, p' \,|\, A_0^\gamma \,|\, V^8\, \beta\, p \rangle_{\lambda=1} =$$

$$\frac{\delta^3(\underline{p}'-\underline{p})}{\sqrt{4p_0 p_0'}}\; \frac{8i}{\sqrt{3}}\; f_{A0} \begin{pmatrix} 8 & 8 & 1 \\ \beta & \gamma & 0 \end{pmatrix} \underline{\varepsilon}^+(\underline{p}') \times \underline{\varepsilon}(\underline{p}) \cdot \underline{p} \qquad (3.1b)$$

$$\langle P^8\, \alpha\, p' \,|\, A_0^\gamma \,|\, V^8\, \beta\, p \rangle_{\lambda=0}$$

$$\frac{\delta^3(\underline{p}'-\underline{p})}{\sqrt{4p_0 p_0'}}\, \sqrt{3}\; mm'\;\; h_A \begin{pmatrix} 8 & 8 & 8_a \\ \beta & \gamma & \alpha \end{pmatrix}\; \varepsilon_0(\underline{p}) \qquad (3.1c)$$

where m = mean mass of the vector meson nonet = 855 MeV.
and m' = mean mass of the pseudoscalar meson octet = 368 MeV.

As usual, by solving the vector commutator equation, we find that the overall scale is given by

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* See Appendix D.

** The index λ is the helicity of the states. Both states must have the same helicity as A_0 in a helicity conserving operator.

$$\left\langle V^8\ \alpha\ p' \,\middle|\, V_o^{\gamma} \,\middle|\, V^8\ \beta\ p \right\rangle_{\lambda=0,1} =$$

$$\frac{\delta^3(\underline{p}'-\underline{p})}{\sqrt{4p_o p_o'}} \sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ \beta & \gamma & \alpha \end{pmatrix} 2p_o \quad \varepsilon^+_\mu(\underline{p})\ \varepsilon^\mu(\underline{p}) \tag{3.1d}$$

First, we take the matrix element of Eq. (2.4) between V^8 states (with helicity = 1), and this gives two equations

$$5f_A^2 + 4f_{AO}^2 = 9,$$

$$f_A^2 = f_{AO}^2 \ .$$

Secondly, we take the matrix element between P^8 states, and obtain one more equation

$$h_A^2 = \frac{4}{mm'} \ .$$

Thus, the chiral SW(3) sum rule gives the solution $f_A = \pm 1$, $f_{AO}^2 = 1$ and $h_A^2 = 4/mm'$. There are two points to notice about this:

Once again, due to the outer automorphism (G-conjugation), the chiral SW(3) algebra is unable to fix the sign of f_A, whereas the SU(6) algebra gives it to be positive.

The second point is that when we put in non-degenerate masses m and m' for the vector and pseudoscalar mesons, only the value of h_A is altered. As we shall see below, this makes an 'unbelievable' improvement on the value of h_A compared with experiment. Unfortunately, due to the fact that the vector mesons decay by strong interactions, we cannot compare the coupling constants directly with experimental values. However, we can use the following round-about method to compare the theoretical predictions with experimentally measurable quantities. The Goldberger-Treiman relations for the V and P mesons are

$$F_\pi = - \frac{h_A \; mm'}{\sqrt{6}\; g_{\rho\pi\pi}} \tag{3.2a}$$

$$F_\pi = \frac{\sqrt{8}\; f_A}{3\; g_{\rho\omega_8\pi}} \tag{3.2b}$$

$$F_\pi = \frac{\sqrt{8}\; f_{AO}}{3\; g_{\rho\omega_1\pi}} \tag{3.2c}$$

The physical ω and ϕ mesons are linear combinations[33] of ω_1 and ω_8, i.e.

$$\begin{pmatrix} \omega \\ \phi \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_8 \end{pmatrix} \tag{3.3}$$

where $\cos\theta \approx \sqrt{\frac{2}{3}}$

Using Eqs. (2,3) we obtain

$$g_{\rho\phi\pi} = 0 \tag{3.4a}$$

$$\frac{g^2_{\rho\omega\pi}}{g^2_{\rho\pi\pi}} = \frac{4}{mm'} \tag{3.4b}$$

Finally, from the Gell-Mann, Sharp, Wagner model[34] for the $\omega \to 3\pi$ decay, the known $\rho \to 2\pi$ decay rate, and Eq.(4b), we find the partial width $\Gamma(\omega \to 3\pi) = 10.6 \pm 5.9$ MeV*. Had we used degenerate masses, or the SU(6)-sum rule, we would have obtained $\Gamma(\omega \to 3\pi) = 6.2 \pm 3.5$ MeV*. These results are to be

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* It is rather difficult to make a realistic estimate of the error here. The Goldberger-Treiman relations hold to about 10°/o, and the Gell-Mann, Sharp, Wagner model to about 25°/o. If there is a conspiracy of errors, we can, at worst, expect a discrepancy of 56°/o in the theoretical partial width. However it is more likely that the errors will cancel out to some extent.

compared with the experimental value[35] of $10\cdot7 \pm 1\cdot5$ MeV. Because of the possibility of large theoretical errors, we should, perhaps, not become too excited with the 'accurate' prediction of the SW(3)-sum rule. If, however, we believe in this result, it suggests that our implicit assumption,* of minimal coupling for the mesons, is quite near the truth.

3. Meson sum rule from chiral SW(2) current algebra.

The mesons belonging to the 15-representation of SU(4) are assigned to representations of the collinear group SW(2), as follows

$$V^3_{\pm1} \oplus V^0_{\pm1} \sim (\tfrac{1}{2},\tfrac{1}{2})_{+1} \oplus (\tfrac{1}{2},\tfrac{1}{2})_{-1}$$

$$V^0_0 \sim (0,0)_0$$

$$V^3_0 \oplus P^3_0 \sim (1,0)_0 \oplus (0,1)_0$$

Here, the only nonvanishing matrix elements of the axial-vector charge operator are

$$\langle V^0 | A_0 | V^3 \rangle_{\lambda=1} \sim f_{AO}$$

$$\langle P^3 | A_0 | V^3 \rangle_{\lambda=0} \sim h_A$$

The definitions here are essentially the same as Eq. (1), except that the numerical factors are different, to allow for the ratio of SU(2) to SU(3) Clebsch-Gordan coefficients.

Taking the matrix element of Eq. (2.4) between either P^3 or V^3 states (with $\lambda = 0$) gives the same result, namely

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* This point will be discussed further in section 4.

$$h_A^2 = \frac{4}{mm'}$$

and taking the matrix element between V^3 states (with $\lambda = 1$) gives

$$f_{AO}^2 = 1$$

To compare these results with experiment, we again make use of the Goldberger-Treiman relations to obtain

$$\frac{g^2_{\rho V^0 \pi}}{4\pi} = \frac{g^2_{\rho\pi\pi}}{4\pi} \cdot \frac{4}{3mm'} \tag{3.5}$$

where in the SW(2) case we have

m = mean mass of vector meson quartet
= 777 MeV.*

m' = mean mass of pseudoscalar meson triplet
= 138 MeV.

At the SU(2) level, there is no way of knowing whether the vector singlet is ω or ϕ or a linear combination of them. For simplicity we assume that ω is the singlet. Then, from the Gell-Mann, Sharp, Wagner model, and Eq. (5) we obtain $\Gamma(\omega \rightarrow 3\pi) = 10{\cdot}4 \pm 5{\cdot}8$ MeV which is in excellent agreement with the experimental value $\Gamma = 10{\cdot}7 \pm 1{\cdot}5$ MeV. If, however, we use degenerate masses for the vector and pseudoscalar mesons, we get the rather poor result that $\Gamma = 2{\cdot}7 \pm 1{\cdot}5$ MeV. As we said in the previous section, this strongly suggests that the mesons are, in fact, minimally coupled.

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* We are assuming that the singlet vector meson is the $\omega(783)$.

4. <u>The form of the meson vertex.</u>

In general, we can express the matrix elements of the axial vector charge as

$$\langle Vp'm \mid A^{\lambda} \mid Vpm \rangle =$$

$$\varepsilon^{+}_{\mu}(\underline{p}') \Big\{ F_1(q^2)\ \varepsilon^{\mu\nu\lambda\eta}\ P_{\eta} + F_2(q^2)\ \varepsilon^{\mu\nu\lambda\eta}\ q_{\eta}$$

$$+ F_3(q^2)\ \varepsilon^{\alpha\beta\lambda\nu}\ q_{\alpha}\ P_{\beta}\ q^{\mu} \Big\} \varepsilon_{\nu}(\underline{p})\ \delta^3\ (\underline{p}'-\underline{p}) \qquad (3.6)$$

where $P = p + p'$ and $q = p - p'$

$$\langle Pp'm' \mid A^{\lambda} \mid Vpm \rangle =$$

$$\Big\{ H_1(q^2)\ g^{\lambda\mu} + H_2(q^2)\ q^{\lambda} q^{\mu}$$

$$+ H_3(q^2)\ P^{\lambda}\ q^{\mu} \Big\} \varepsilon_{\mu}(\underline{p})\ \delta^3(\underline{p}'-\underline{p}) \qquad (3.7)$$

In deriving the sum rules we have made the assumption that f_A and f_{AO} are proportional to $F_1(0)$, and that h_A is proportional to $H_1(0)$. This is certainly true for f_A and f_{AO}, if the vector mesons all have the same mass. However, in the case of h_A, as we shall see below, this assumption is equivalent to saying that the term in H_3 is absent due to an angular momentum barrier.

From the covariant expansions, Eqs. (6,7) we find that

$$\langle Vp'm \mid A_o \mid Vpm \rangle = F_1(0)\ \underline{\varepsilon}^{+}(\underline{p}) \times \underline{\varepsilon}(\underline{p}) \cdot \underline{P} \qquad (3.8)$$

$$\langle Pp'm' \mid A_o \mid Vpm \rangle \rightarrow \Big\{ H_1(0) + H_3(0)(m^2 - m'^2) \Big\}\ \varepsilon_o(\underline{p}) \quad (3.9)$$

as $p_z \rightarrow \infty$

$$\langle Vp'm \mid A_z \mid Vpm \rangle = F_1(0)\ \varepsilon^{\mu\nu z\eta}\ \varepsilon^{+}_{\mu}(\underline{p})\ \varepsilon_{\nu}(\underline{p})\ P_{\eta} \tag{3.10}$$

$$\langle Pp'm' \mid A_z \mid Vpm \rangle \longrightarrow \left\{ H_1(0)\ \frac{p_o}{|p_z|} + H_3(0)(m^2-m'^2) \right\} \varepsilon_o(\underline{p}) \tag{3.11}$$

as $p_z \longrightarrow \infty$

There are three points to notice here, when we make use of the explicit form of the polarization vectors, Eqs. (A.9-11).

(i) Both matrix elements

$$\langle V \mid A_z \mid V \rangle_{\lambda=0} \quad \text{and} \quad \langle V \mid A_o \mid V \rangle_{\lambda=0}$$

vanish, since $\varepsilon_1^o(p_z) = 0 = \varepsilon_2^o(p_z)$.

(ii) As we go to the limit $p_z = \infty$,

$$\langle Vp_z \mid A_z \mid Vp_z \rangle_{\lambda=1} \rightarrow \langle Vp_z \mid A_o \mid Vp_z \rangle_{\lambda=1}$$

and

$$\langle Pp_z \mid A_z \mid Vp_z \rangle \longrightarrow \langle Pp_z \mid A_o \mid Vp_z \rangle$$

so that the matrix elements of the chiral and collinear algebras are the same, at infinite momentum.

(iii) From Eqs. (9 & 11) we see that, to obtain a unique solution from the sum rule when $m \neq m'$, we must eliminate the term in H_3. Otherwise, we will be solving <u>one</u> equation for H_1^2, $H_1\ H_3$ and H_3^2 so that no unique solution will be forthcoming. The only justification for leaving out the term in H_3, and presumably also H_2, is that an angular momentum barrier effectively damps the terms with derivative coupling. In other words, we are assuming a minimal coupling for the PVA vertex.

5. The role of positive parity mesons.

One of the main features, of current algebra, is the ability of the generators, of the algebra, to map states of one representation (of the corresponding group), onto states of some other representation. Of course, the generators of a group can only take states of one representation into states of the same representation. It is, therefore, worth considering what contribution, if any, the positive parity mesons make to the sum rule. There are basically two ways in which we can incorporate the positive parity mesons in our current algebra scheme. These are (i) the orbital excitation model, and (ii) the SU(6) model.

(i) The $SU(6) \otimes O(3)$ model:

In this model, the mesons are considered as bound states of a quark and an antiquark with orbital angular momentum ℓ. The 0^- and 1^- mesons are assigned to the multiplet $35^-(\ell = 0)$, and the 0^+ and 1^+ mesons to the $35^+(\ell = 1)$. The O(3) group, here, is generated by $\underline{L}$, where

$$L_z = -i \int d^3x \, q^+(x) \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) q(x) \tag{3.12}$$

From Eq. (12) it can easily be seen that the axial-vector operators, of our various current algebras, have the selection rule $\Delta \ell = 0$. This means that the positive and negative parity mesons are decoupled in the sum rule, so that we will get two completely independant sets of equations. One set just gives the usual results for the 0^- and 1^- mesons. The ℓ-excitation model therefore forbids the transition of positive to negative parity mesons, i.e.

$M^+ \not\rightarrow M^- + e + \bar{\nu}$. It also leaves unchanged, the value of the axial-vector coupling constants for the 0^- and 1^- mesons.

(ii) The SU(6) quark model:

The smallest number of quarks required to form a positive parity multiplet is four, i.e. $q\bar{q}q\bar{q}$, which contains 1^+, 35^+, 189^+, 280^+, $\overline{280}^+$ and 405^+. The 'known' positive parity mesons are few(35), and at present it is impossible to say, with any conviction, which representations they belong to. However, we may have a scalar octet consisting of $\eta_V(1050)$, $\pi_V(1003)$ and $K_A(1800)$ and also an axial-vector octet consisting of D(1285), A1(1080) and $K_A(1320)$.

Assuming, for the moment that the 0^+ and 1^+ mesons belong to the 35^+-representation of SU(6), we find the following nonvanishing matrix elements for the chiral SW(3) algebra. (We give this in tabular form, indicating the type of coupling, the helicity, and the coupling constant).

Matrix Element	Type of Coupling	λ	Coupling Constant
$\langle V^8 \| A_0 \| V^8 \rangle$	D	1	f_A
$\langle V^0 \| A_0 \| V^8 \rangle$	1	1	f_{A0}
$\langle P^8 \| A_0 \| V^8 \rangle$	F	0	h_A
$\langle A^8 \| A_0 \| V^8 \rangle$	F	0,1	g_1
$\langle S^8 \| A_0 \| P^8 \rangle$	D	0	g_2
$\langle A^8 \| A_0 \| A^8 \rangle$	D	1	g_3
$\langle A^0 \| A_0 \| A^8 \rangle$	1	1	g_{30}
$\langle S^8 \| A_0 \| A^8 \rangle$	D	0	g_4

The other matrix elements vanish for the usual reasons of charge-conjugation invariance, spin and parity conservation, and helicity conservation.

Taking the matrix element of the chiral SW(3) commutator Eq. (2.4) between all possible combinations of states, we get eight different (although not all independant) equations. Of course, we also get many trivial equations like 0 = 0. In deriving these relations, we have assumed, for simplicity, a degenerate mass 'm' for all the mesons. Rearranging the equations, we obtain

$$\frac{20}{3} f_A^2 + \frac{16}{3} f_{AO}^2 = 3m^2 h_A^2$$

$$f_A^2 = f_{AO}^2$$

$$g_1^2 = g_2^2 = 3 - \frac{3m^2}{4} h_A^2$$

$$g_3^2 = \frac{5m^2}{12} h_A^2$$

$$g_{30}^2 = \frac{4m^2}{3} h_A^2$$

$$g_4^2 = \frac{3m^2}{4} h_A^2$$

Although these equations are compatible with the usual solution $f_A^2 = 1 = f_{AO}^2$ and $h_A^2 = 4/m^2$, unfortunately this solution is not unique. However, we have been able to determine the g's in terms of one parameter h_A (which we are probably justified in taking as $4/m^2$, as this gives good agreement with experiment). In the future, when more experimental data becomes available on positive parity mesons, it should be possible to test these results by using the analogous of the Goldberger-Treiman relations and the Gell-Mann, Sharp, Wagner model.

Another likely assignment of the positive parity mesons, is to the 189^+ multiplet, since this also contains 2^+ mesons. Under the reduction $SU(6) \rightarrow SU(3) \otimes SU(2)_J$ we find that

$\underline{189} \rightarrow (1,1) \oplus (8,1) \oplus (27,1) \oplus 2(8,3) \oplus (10,3) \oplus (\overline{10},3) \oplus (1,5) \oplus (8,5)$. In other words, the "189" contains the following states:

$$S^0,\ S^8,\ S^{27}$$
$$A^8,\ A_1^8,\ A^{10},\ A^{\overline{10}}$$
$$T^0,\ T^8$$

where S, A and T are the 0^+, 1^+ and 2^+ mesons, and the superscript refers to the $SU(3)$ multiplicity. With so many unobserved particles here, and in fact, many unknown quantum numbers in the case of the observed particles, it seems futile to try to obtain a sum rule for the coupling constants. A priori, there is no reason why, for example, the two 1^+ octets should not have opposite charge conjugation properties. This, of course, increases the number of nonvanishing matrix elements. In fact, we tried to obtain a solution for eighteen nonvanishing coupling constants (with certain assumptions about the charge-conjugation), but we simply ended up with a vast number of inconsistent equations! Possibly, by a method of trial and error, one might hit on the 'right' charge-conjugation properties of the various multiplets, to give a unique solution.

There is one important conclusion to be drawn from the difference between the $SU(6) \otimes O(3)$ model, and the pure $SU(6)$ model. As the current algebra predictions, based on particle assignments, are quite different for the two models, we should be able to reject one model in favour of the other. Intuitively, it seems most unlikely that there should be an absence of coupling of positive to negative parity mesons by the axial-vector component of the lepton current. If, by using indirect experimental evidence, this turns out to be the case, we can probably reject the ℓ-excitation model. However, the physical particles may actually be mixtures of multiquark bound states, and orbitally excited quark-antiquark states.

CHAPTER IV

MAGNETIC MOMENTS

The problem of magnetic moments is one of the less respectable areas of current algebra. When we consider the commutator of two first-order moment operators, e.g. $M_o^\alpha = -i\int d^3x\, x\mathcal{V}_o^\alpha(x)$, we generate second-order moments. It soon becomes clear that the algebra of moments is not closed. We simply generate higher and higher order moments until we have an infinite parameter algebra. Now this in itself is not worrying. What should worry us, however, is the realization that without any additional assumptions, the matrix elements of all the moment operators vanish. We shall see in the following sections how this problem can be overcome.

1. SU(6) and the Pauli Interaction.

Invoking the well established* principle of minimal electromagnetic coupling, the quark-photon interaction Lagrangian is given by

$$L_{int}(x) = e\, J_\mu^{e\ell}(x)\, A^\mu(x) \qquad (4.1)$$

where $J_\mu^{e\ell}$ is the electromagnetic current of the quarks, and A_μ the electromagnetic field. When the electromagnetic field is just a constant magnetic field $\underline{H}$, then we have $\underline{A} = \frac{1}{2}(\underline{H} \times \underline{x})$, so that the interaction energy becomes

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* At least for electrons.

$$H^{em}_{int} = \tfrac{1}{2} e \int d^3x \, \underline{H} \cdot \underline{x} \times \underline{J}^{e\ell}(x)$$

$$= e \, \underline{\mathcal{M}} \cdot \underline{H} \qquad (4.2)$$

where $\underline{\mathcal{M}}$ is the magnetic moment operator. Explicitly using the quark model, we have

$$\mathcal{M}^{\alpha}_{i} = \tfrac{1}{2} \, \varepsilon_{ijk} \int d^3x \, x_j \, \psi^{+}(x) \, \gamma_5 \sigma_k \, \tfrac{1}{2} \lambda^{\alpha} \, \psi(x) \qquad (4.3)$$

Now this leads to the following commutation relations:

$$\left[\mathcal{M}^{\alpha}_{i} , \mathcal{M}^{\beta}_{j}\right] = i \, f^{\alpha\beta\gamma} Q^{\gamma}_{ij} + \tfrac{2}{3} \, i \, f^{\alpha\beta\gamma} R^{\gamma} + d^{\alpha\beta\gamma} \, i \, \varepsilon_{ijk} \, N^{\gamma}_{k} \qquad (4.4)$$

where Q, R and N are associated with second-order moments by

$$Q^{\alpha}_{ij} = \int d^3x \, (\tfrac{1}{3} \, \delta_{ij} \, x^2 - x_i x_j) \mathcal{V}^{\alpha}_{o}(x)$$

$$R^{\alpha} = \int d^3x \, x^2 \, \mathcal{V}^{\alpha}_{o}(x)$$

$$N^{\alpha}_{i} = \int d^3x \, x_i \, \underline{x} \cdot \underline{\mathcal{A}}^{\alpha}(x)$$

Indeed, Q is the electric quadrupole moment operator, and R the mean square charge radius operator.

$$\left[A^{\alpha}_{i}, \mathcal{M}^{\beta}_{j}\right] = i\varepsilon_{ijk} d^{\alpha\beta\gamma} \mathcal{M}^{\gamma}_{k} + \tfrac{1}{2} i \, \varepsilon_{ijk} f^{\alpha\beta\gamma} \int d^3x \, x_k \mathcal{A}^{\gamma}_{o}(x)$$

$$- \tfrac{1}{2} \, i d^{\alpha\beta\gamma} \int d^3x (\delta_{ij} \underline{x} \cdot \underline{\mathcal{V}}^{\gamma}(x) - x_j \mathcal{V}^{\gamma}_{i}(x)) \qquad (4.5)$$

When we take the matrix elements of either Eq.(4) or Eq.(5) between static baryon states belonging to the 56-representation of SU(6), all the matrix elements vanish. This is a most alarming situation. However, Lee(11) and Ryan(14) have shown that there is a simple explanation for it. If we consider how the various operators in Eqs. (4 and 5) transform under the orbital angular momentum group $O(3)_{\ell}$, we see that they are all $\ell = 1$ operators, except for A_i^{α} which has $\ell = 0$. Now *if* the baryons belong to the $\underline{56}(\ell = 0)$ representation of $SU(6) \otimes O(3)$ then clearly the matrix elements of all $\ell = 1$ operators between these $\ell = 0$ states must vanish. *A priori*, however, there is no reason why the baryons (i.e. qqq states) should have $\ell = 0$, so let us consider the problem of angular momentum in more detail. The total angular momentum operator in the rest frame, $\underline{J}$, is given by

$$\underline{J} = \underline{L} + \underline{S} \tag{4.6}$$

where $\underline{L}$ is the orbital angular momentum, and $\underline{S}$ is the total spin of the quarks. Whereas $\underline{J}$ is the spin of the physical particle, and is therefore independent of any quark model, the operators $\underline{L}$ and $\underline{S}$ refer to the motions of the quarks within the particle. To clarify our notation, we see that in a pure quark model

$$L_z = -i\int d^3x\, \psi^+(x) \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \psi(x) \tag{4.7}$$

$$S_z = \int d^3x\, \psi^+(x)\, \tfrac{1}{2}\, \sigma_z\, \psi(x) \tag{4.8}$$

so that $\underline{S} = \underline{A}^o$ is just one of the generators of the SU(6) algebra. If we are not in the rest frame, we will choose the z-direction to define the various "helicities". We shall denote the eigenvalues of J_z, L_z and S_z by λ, ℓ and s respectively. By definition, the matrix element of $\underline{J}$ between nucleon states at rest is

$$\langle N | J_i | N \rangle = u^+ \tfrac{1}{2} \sigma_i u \tag{4.9}$$

and previously we defined

$$\langle N \mid A_i^o \mid N \rangle = g\, u^+ \tfrac{1}{2} \sigma_i\, u \tag{4.10}$$

In Chapter II, we obtained the solution $g = 1$. Thus, Eqs. (6,8,9 and 10) imply that

$$\begin{aligned} \langle N \mid L_i \mid N \rangle &= \langle N \mid J_i \mid N \rangle - \langle N \mid S_i \mid N \rangle \\ &= 0 \end{aligned}$$

A similar argument holds for the $J^P = \frac{3}{2}^+$ states. Thus, our method of demanding that the sum rule should be saturated by the 56-representation of SU(6) requires, for consistency, that all these states should have $\ell = 0$.

Having seen why the matrix elements of the moment operators vanish, we must now consider how the magnetic moment operator can be modified to give non-trivial results. Ryan(13) has suggested that if the quark-photon interaction is not, in fact, minimal, but also includes a Pauli type of coupling*, then we would have

$$L_{int} = e\, J_\mu^{e\ell}(x)\, A^\mu(x) + \tfrac{1}{2}\, e\, \mu_Q\, T_{\mu\nu}^{e\ell}(x)\, F^{\mu\nu}(x) \tag{4.11}$$

where $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ and $T_{\mu\nu}^{e\ell}$ in the electromagnetic tensor current. If the electromagnetic field is a constant magnetic field $\underline{H}$, then $\underline{A} = \frac{1}{2}(\underline{H} \times \underline{x})$ and $F_{12} = H_3$, $F_{23} = H_1$, $F_{31} = H_2$. The interaction energy now becomes

$$\begin{aligned} H_{int}^{em} &= \tfrac{1}{2}\, e \int d^3x\, \underline{H}\cdot\underline{x} \times \underline{J}^{e\ell}(x) + e\mu_Q \int d^3x\, \underline{H}\cdot\underline{T}^{e\ell}(x) \\ &= e\, \underline{\mathcal{M}}\cdot\underline{H} \end{aligned} \tag{4.12}$$

where $\underline{T}$ is defined by $T_{ij} = \varepsilon_{ijk}\, T_k$

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* This would be present if the quarks possessed an intrinsic anomalous magnetic moment μ_Q.

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The modifield magnetic moment operator is therefore

$$\mathcal{M}_i^{\alpha} = \tfrac{1}{2}\, \varepsilon_{ijk} \int d^3x\, x_j\, \mathcal{V}_k^{\alpha}(x) + \mu_a \int d^3x\, \mathcal{T}_i^{\alpha}(x) \tag{4.13}$$

We now find that its commutator with A_i^{α} is

$$\begin{aligned} \left[A_i^{\alpha}, \mathcal{M}_j^{\beta}\right] &= i\, \varepsilon_{ijk}\, d^{\alpha\beta\gamma} \mathcal{M}_k^{\gamma} + \mu_a i\, \delta_{ij}\, f^{\alpha\beta\gamma}\, S^{\gamma} \\ &+ \tfrac{1}{2}\, i\, \varepsilon_{ijk}\, f^{\alpha\beta\gamma} \int d^3x\, x_k \mathcal{A}_o^{\alpha}(x) \\ &- \tfrac{1}{2}\, i\, d^{\alpha\beta\gamma} \int d^3x (\delta_{ij}\, \underline{x} \cdot \underline{\mathcal{V}}^{\gamma}(x) - x_j \mathcal{V}_i^{\gamma}(x)) \end{aligned} \tag{4.14}$$

where $S^{\alpha} = -i \int d^3x\, \psi^+(x)\, \gamma_o\, \tfrac{1}{2}\, \lambda^{\alpha}\, \psi(x)$

We actually only need to retain the first two terms of this commutator expression as the other terms are all $\ell = 1$ operators.

We take the matrix element of Eq. (14) between nucleon states, and after the usual procedure, we obtain the unique solution $\mu(p)/\mu(n) = -3/2$, which is just the SU(6) result(12). It is clear from the expression for $\underline{\mathcal{M}}$ in the interaction energy, Eq. (12), that we are dealing with the <u>total</u> magnetic moment. We remember that the SU(6) symmetry prediction required the additional assumption that the magnetic moments were 'total', rather than 'anomalous'.

If we use the SU(4), rather than the SU(6) current algebra, we get (as we have come to expect) two solutions, which depend on the sign of $G = G_A/G_V$, namely,

$$\mu(p)/\mu(n) = \frac{(1 + 3G)}{(1 - 3G)} \begin{array}{l} = -\frac{3}{2} \text{ if } G = +\frac{5}{3} \\ = -\frac{2}{3} \text{ if } G = -\frac{5}{3} \end{array}$$

Thus we have seen that by endowing the quarks with an anomalous magnetic moment, we can obtain excellent results. If the quarks are merely mathematical fictions, it may not matter what additional properties we impose on them. However, if the quarks turn out to be real, we may have a different story, although it is possible that by the time their magnetic moments are measured, current algebra will have become another historical curiosity.

2. Chiral SU(3) ⊗ SU(3) and Configuration Mixing.

The SU(6) current algebra gives rise to a total magnetic moment operator of the form $\underline{\mathfrak{M}} = \int d^3x\, \underline{x} \times \underline{\mathcal{V}}(x) + \ldots$, Eq. (13). The chiral SW(3) algebra does not possess the operator $\int d^3x\, \underline{\mathcal{V}}(x)$ and so we may wonder how we are going to be able to determine magnetic moments here. However, when we consider the electric dipole moment operator[39],

$$\underline{D}^{\alpha} = -i \int d^3x\, \underline{x}\, \mathcal{V}_{o}^{\alpha}(x) \tag{4.15}$$

and define

$$D_{\pm}^{\alpha} = -i \int d^3x \left(\frac{x \pm iy}{\sqrt{2}} \right) \ {}_{o}^{\alpha}(x) \tag{4.16}$$

and

$$D_{\pm}^{Q} = D_{\pm}^{3} + \frac{1}{\sqrt{3}}\, D_{\pm}^{8} \tag{4.17}$$

its expectation value at infinite momentum (or zero momentum transfer) gives the <u>anomalous</u> magnetic moment. Making the usual expansion in terms of invariants, with the notation of Drell and Zachariasen[40], it is easy to see that

$$\left\langle B_{1/2}\, ,\ p_z = \infty \left| D_{+}^{Q} \right| B_{-1/2}\, ,\ p_z = \infty \right\rangle = \sqrt{2}\, F_2(0) = \sqrt{2}\, \mu'(B) \tag{4.18}$$

where $\mu'(B)$ is the anomalous magnetic moment of the baryon B_{λ}, with helicity λ.

We must now examine the transformation properties of the electric dipole moment operator, in the infinite momentum frame, with the view to seeing what matrix elements (if any) vanish. First, from Eq. (15) we see that $\underline{D}^{\alpha}$ transforms like $(1,8) \oplus (8,1)$ under the chiral SW(3) algebra. Secondly, we want to see what angular momentum properties $D_{\pm}$ has. The angular momentum operator for the physical particles in the infinite momentum* frame, $\underline{J}'$, is obtained by performing a pure Lorentz transformation on $\underline{J}$, the spin operator in the rest frame.

$$\underline{J}' = \exp(iK_z\xi)\, \underline{J}\, \exp(-iK_z\xi) \tag{4.19}$$

so that

$$J'_x = J_x \cosh \xi - K_y \sinh \xi$$

$$J'_y = J_y \cosh \xi + K_x \sinh \xi$$

$$J'_z = J_z$$

where $\sinh \xi = \lim\limits_{p_z \to \infty} \dfrac{|\underline{p}|}{m}$,

and $\cosh \xi = \lim\limits_{p_z \to \infty} \dfrac{p_o}{m}$,

and where J and K are the infinitessimal generators of rotation and translation. We then see, from Eqs. (16 and 19) that

$$\left[J'_z\, ,\, D^{\alpha}_{\pm}\right] = \pm D^{\alpha}_{\pm} \tag{4.20}$$

and $$\left[L_z\, ,\, D^{\alpha}_{\pm}\right] = \pm D^{\alpha}_{\pm} \tag{4.21}$$

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* Without any loss of generality we choose $p_z = \infty$.

so that $D^{\alpha}_{\pm}$ has the selection rules $\Delta\lambda = \pm 1$ and $\Delta\ell = \pm 1$. Once again, it is clear that matrix elements of $D_{\pm}$, between $\ell = 0$ states, must vanish.

To get round this difficulty, several authors have independently suggested that we should allow a certain admixture of $\ell \neq 0$ states. In other words, the physical baryon states must transform <u>reducibly</u> under chiral SW(3). We shall now briefly consider the merits (or otherwise) of the various configuration mixing schemes. We use the notation $|N_{\lambda}\rangle = |(a,b)_s, \ell\rangle \oplus \ldots$ to mean that a nucleon, with helicity λ belongs to the (a,b)-representation of collinear SW(3) with spin and orbital angular momentum s and ℓ.

Gerstein and Lee[41] have proposed that

$$|N_{1/2}\rangle = \cos\theta\,|(6,3)_{1/2}, 0\rangle \oplus \sin\theta\left\{\cos\alpha\,|(3,\bar{3})_{-1/2}, 1\rangle \oplus \sin\alpha\,|(8,1)_{\sigma}, \tfrac{1}{2}-\sigma\rangle\right\} \quad (4.22a)$$

where σ is arbitrary, and

$$N^{*}_{1/2} = |(6,3)_{1/2}, 0\rangle \quad (4.22b)$$

On the one hand, this scheme has the disadvantage of introducing two free parameters, θ and α. The third parameter*, σ, is actually related to α, since the coefficients $\cos\alpha$ and $\sin\alpha$ are nothing but Clebsch-Gordan coefficients. On the other hand, it has the advantage of not requiring the reducible representation of collinear SW(3) to form a <u>complete</u> representation of $SU(6)_W$,

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* This reminds one of the remark attributed by Lipkin[42] to a famous physicist of an earlier generation. "Give me three parameters and I can fit an elephant - with four I can make him wiggle his trunk".

the hybrid collinear group*.

Harari[43] has proposed that

$$|N_{1/2}\rangle = \cos\theta\, |(6,3)_{1/2}, 0\rangle + \sin\theta \left\{ \sqrt{\tfrac{1}{3}}\, |(\overline{3},3)_{1/2}, 0\rangle - \sqrt{\tfrac{2}{3}}\, |(3,\overline{3})_{-1/2}, 1\rangle \right\} \qquad (4.23a)$$

and

$$|N^{*}_{1/2}\rangle = |(6,3)_{1/2}, 0\rangle \qquad (4.23b)$$

This scheme has the same advantage as the previous one, that we do not have to assume that W-spin is a well conserved quantity. There are no obvious disadvantages in this case.

The final mixing scheme, proposed by Gatto et al.[44] seems to be very seriously wrong. This scheme is based on the assumption that the N and N^{*} states (which form the $\underline{56}$-representation of SU(6) at rest) will completely occupy the $\underline{56}$-representation of $SU(6)_W$, when they are moving in the z-direction. With this scheme, good results are obtained for the axial-vector coupling constants, but disastrous results appear for the magnetic moments, for example $\mu'(n) = 0$. We will consequently ignore their configuration mixing scheme in the ensuing discussion.

We now consider the commutator of the electric dipole moment operator with the axial-vector charge,

$$\left[A_0^{\alpha}, D_{\pm}^{\beta}\right] = -i\int d^3x \left(\frac{x \pm iy}{\sqrt{2}}\right) if^{\alpha\beta\gamma}\, \mathcal{A}_0^{\gamma}(x) \qquad (4.24)$$

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* This is the group generated by $V_0(1)$, $A_z(\sigma_z)$, $T_{zx}(\beta\sigma_x)$, $T_{zy}(\beta\sigma_y)$, thus commuting with the Lorentz transformation in the z-direction. It is to be compared with the static SU(6) group, generated by $V_0(1)$ and $\underline{A}(\underline{\sigma})$.

From this commutator, we obtain

$$\left\langle B_{1/2},\ p_z = \infty \,\middle|\, \left[A_o^{\alpha},\ D_+^{\beta}\right] \,\middle|\, B_{-1/2},\ p_z = \infty \right\rangle = 0 \tag{4.25}$$

since the matrix element of the operator on the r.h.s. of Eq. (24) must vanish. By saturating the commutator in Eq. (25) with a discrete set of states defined by either Eq. (22) or Eq. (23) we obtain sum rules for the anomalous magnetic moments, transition moments, and the axial-vector coupling constant. Since we have given the details of similar calculations, ad nauseam, in Chapters II and III, we will not repeat the procedure here, but only quote the results. Using the transformation properties of D_+^{α} to eliminate some of the matrix elements, we finally obtain (for both the Gerstein-Lee and Harari models),

$$\mu^* \cos \theta = \sqrt{2}\, \mu'(p) \tag{4.26}$$

and

$$\mu'(p) = -\mu'(n) \tag{4.27}$$

where μ^* is the transition moment between the nucleon and the 3,3-resonance. Before we say anything about these results, let us note that we can also take the matrix element of the commutator $\left[A_o^{\alpha}\ ,\ A_o^{\beta}\right]$ between baryon states. Using the same configuration mixing models, we will obtain

$$G = \frac{1}{3}\left(4 \cos^2 \theta + 1\right)$$

Now, if we require that G takes on its experimental value of 1·18, instead of the usual SU(6) or current algebra value of 5/3, we will find that $\theta = 37^{o}$.

With $\theta = 37^{o}$, and the experimental value $\mu'(n) = -1{\cdot}91$, Eqs. (26 and 27) imply that $\mu'(p) = 1{\cdot}91$ and $\mu^* = 3.40$, to be compared with the experimental values $\mu'(p) = 1{\cdot}79$ and $\mu^* = 3{\cdot}36 \pm 0.05$*.

If we had used the chiral SW(2) algebra, instead of SW(3), we would only have obtained Eq. (26). The other relation, Eq. (27), depends on the D/F ratio, so that we need the full SU(3) internal symmetry. This can be seen by looking at the SU(3) Clebsch-Gordan series; both D- and F-type coupling contribute to $\mu'(p)$, whereas only D-type coupling contributes to $\mu'(n)$.

It is an interesting feature of these results that they are quite stable to various representation admixtures. In the Gerstein-Lee model, due to the arbitrariness of σ, we have effectively an infinite number of configuration mixing schemes. However, if we are prepared to accept the quark model in which all baryons consist of qqq states, then $\sigma = 3/2$ is the only possible value.

To conclude this section, let us consider the validity of configuration mixing. It has been shown[46], that the assignment of hadrons to a mixture of representations implies the existence of a one-particle subspace containing more than the observable particles. This is in complete analogy to the problem of configuration mixing in the shell-model of the nucleus[47]. Here, an approximate Hamiltonian is introduced, and by diagonalizing a submatrix, good results are obtained for the low lying states. However, the higher eigenstates, predicted by this approximate Hamiltonian, do not, in general, correspond to any physical levels, i.e. we have a subspace containing unphysical, as well as physics states. We have seen a

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* This has been estimated by Dalitz and Sutherland[45].

clear example of this situation arising in current algebra. The Gatto-model predicts the correct values for the zero order moments, but the first order moments it predicts, do not correspond to the physical ones. One further analogy between current algebra and the shell-model, is that the experimental results can be explained by many different choices of possible configurations. In the context of the shell-model, Flowers[48] came to the conclusion that the task of determining the amount of the respective admixtures was "prohibitively tedious". It certainly looks as if the same remarks apply to current algebra.

3. Final Remarks.

We have seen that current algebra has been able to 'save face' by introducing Pauli moments for SU(6), and configuration mixing for chiral SW(3). However, these procedures seem quite arbitrary. It may turn out, in the course of time, that there is some truth in these ad hoc remedies, but it is unlikely that we will be able to offer any appraisal of this situation until a complete dynamical theory of the quark structure of elementary particles is achieved. At present, therefore, the best that can be said is that current algebra is not inconsistent with magnetic moment results.

CHAPTER V

UNSTABLE STATES

In Chapters II-IV the sum-rules we derived were all based on the assumption that the particles are completely stable as far as strong and electromagnetic interactions are concerned. Now this is obviously far from the truth. For example, although the proton is completely stable and the neutron almost stable ($\tau = 1{\cdot}01 \pm 0{\cdot}03.10^{-3}$ secs.), the $N^*(1236)$ has the extremely short life-time of $\tau = 0{\cdot}548 \pm 0{\cdot}009.10^{-23}$ secs. The probability, that leptonic decays of the N^* will never be observed, is quite irrelevant from the theoretical point of view. However, it would seem most remarkable if no difference is made to the sum-rules when we allow for the extreme instability of the N^*. When we derived the baryon sum rules we had to deal with quantities like $\int d^3x\ d^3y \langle p|J_A(x)|k\rangle\langle k|J_B(y)|p'\rangle$

In section 1. we shall see that when the intermediate state, of momentum k, is completely stable, we get momentum conserving delta functions, $\delta^3(\underline{p}-\underline{p}')\ \delta^3(\underline{k}-\underline{p})$. However, when the intermediate state is no longer stable, we get a smeared-out delta function in k, in fact, $\delta^3(\underline{p}-\underline{p}')\ f_\varepsilon(\underline{k}-\underline{p})$. The astonishing conclusion we are then forced to come to is that there is absolutely no difference between stable and unstable states, as far as the baryon sum rules are concerned. In section 2. we adopt a phenomenological approach. With some slight justification, we ignore the results of section 1, and show that we can associate an instability factor $\mathcal{J}$, where $0 \leq \mathcal{J} \leq 1$, with unstable intermediate states. This is quite compatible with the algebra, and gives excellent agreement with experiment.

1. Theoretical Approach to Unstable Intermediate States.

The essential, physical content of the quantity $\langle p|J_A(x)|k\rangle\langle k|J_B(y)|p'\rangle$ is the following: a local operator $J(y)$ is applied to the initial momentum eigenstate $|p'\rangle$ taking it into another momentum eigenstate $|k\rangle$. Then, a second local operator $J(x)$, at, in general, a different space-time point $x \neq y$, is applied to $|k\rangle$ taking it into the final state $|p\rangle$. The question immediately arises - what happens if the intermediate state is unstable, so that, having been created at the point 'y', it decays before it can reach the point 'x' ? There is one point of difficulty here. As we are dealing with momentum eigenstates, the localization of such states is not compatible with their having exactly determined momenta (due to the Uncertainty Principle). Strictly speaking, we shall have to compromise and deal with wave packets of finite extension. However, as we can make the distance between x and y greater than the extension of the wave packet, we can effectively ignore this complication.

At an early stage in the derivation of the baryon sum-rule we considered a quantity of the form

$$\begin{aligned} I &= \int d^3x\, d^3y\, \langle p|\left[J_A(x),\, J_B(y)\right]|p'\rangle \\ &= \int d^3z\, \delta^3(\underline{p}-\underline{p}')\, e^{i(p_o-p_o')y_o}\, \langle p|\left[J_A(z),\, J_B(0)\right]|p'\rangle \quad (5.1) \end{aligned}$$

where $z = x - y$. The equal-time limit will ultimately be given by putting $z_o = 0$. Inserting intermediate states of momentum k, and integrating over their 3-momentum, Eq. (1) becomes

$$I = \int d^3k\, d^3z \;\; \delta^3(\underline{p}-\underline{p}')\, e^{i(p_o-p_o')y_o} \times$$

$$\times \Big\{ e^{i(k-p)z} \langle p|J_A(0)|\,k\rangle\langle\, k|J_B(0)\,|p'\rangle$$

$$-e^{i(p'-k)z} \langle p\,|J_B(0)|\,k\rangle\langle\, k|\,J_A(0)|\,p'\rangle \Big\} \qquad (5.2)$$

In the conventional manner, when the intermediate state is stable, the d^3z integration is over an infinite volume. This just gives a delta function, $\delta^3(\underline{k}-\underline{p})$, so that when we finally integrate over d^3k, the only contribution from the integrand occurs at $\underline{k} = \underline{p}$.

When the intermediate state is unstable, we have to deal with Eq. (2) in a different manner.

Define τ_o = mean-life (in the rest frame of the particle).

$$\tau = \frac{\tau_o\, \omega_k}{m} = \text{mean-life of particle moving with momentum k.}$$

$$\omega_k = +\sqrt{\underline{k}^2+m^2} = \text{energy.}$$

$$\underline{v} = \underline{k}/\omega_k = \text{velocity.}$$

Thus, the maximum position the unstable particle can get to before it decays, is given by $\underline{z} = \underline{v}\tau$. In other words, the variable 'z' in Eq. (2) is restricted by the inequality

$$|\underline{z}| \leq \frac{\underline{k}\,\tau_o}{m}$$

We are therefore carrying out the d^3z integration over a finite volume, in fact a sphere of radius $\varepsilon = \frac{|\underline{k}|\tau_o}{m}$

We shall digress, for a moment, to consider the properties of the function $f_\varepsilon(k)$ defined by

$$f_\varepsilon(k) = \frac{1}{2\pi} \int_{-\varepsilon}^{+\varepsilon} dx \quad e^{ikx} \tag{5.3}$$

Obviously, in the limit $\varepsilon \to \infty$, this is just the Dirac delta function. However, for finite ε we see, on integration, that

$$f_\varepsilon(k) = \frac{\sin \varepsilon k}{\pi k} \tag{5.4}$$

which looks like

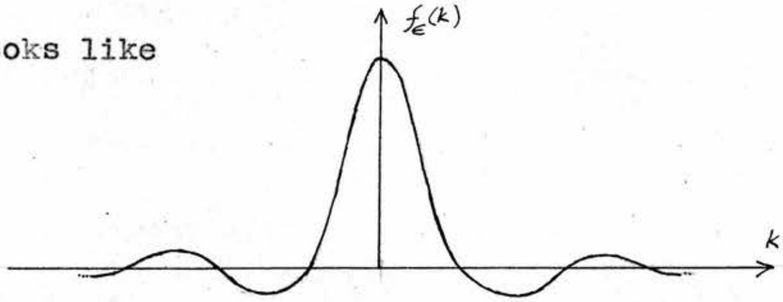


FIG. 1

It is not difficult to find the following properties of $f_\varepsilon(k)$:

(i) Height of main peak = $f_\varepsilon(0) = \frac{\varepsilon}{\pi}$

(ii) Width of main peak (at $f_\varepsilon(k) = 0$) = $\frac{\pi}{\varepsilon}$

(iii) Height of 1^{st} subsidiary peak = $-\frac{2\varepsilon}{3\pi^2}$

(iv) $\int_{-\infty}^{+\infty} dk \; f_\varepsilon(k) = 1$

Unfortunately $f_\varepsilon(k)$ cannot be integrated in closed form for finite limits of integration.

We now return to Eq. (2). For simplicity, we shall work in a two-dimensional space-time. It is trivial to extend this to the four-dimensional space, but the resulting expressions are not so 'transparent', as we have to use spherical coordinates. Integrating Eq. (2) over the finite range $z \leq \frac{k\tau_0}{m}$ we obtain

$$I = \int_{-\infty}^{+\infty} dk\, \delta(p-p')\, e^{i(p_0-p_0')y_0}\, f_\varepsilon(k-p)$$

$$\cdot \left\{ e^{i(p_0-k_0)z_0} \langle p|J_A(0)|k\rangle\langle k|J_B(0)|p'\rangle - e^{i(k_0-p_0')z_0} \langle p|J_B(0)|k\rangle\langle k|J_A(0)|p'\rangle \right\}$$

where $\varepsilon = \frac{|k|\tau_0}{m}$

When we take the equal-time limit, the exponentials in z_0 go out. The exponential in y_0 occurs in each term of the commutator equation and thus cancels out. Essentially what we are left with is

$$I = \int_{-\infty}^{+\infty} dk\, F(p,p',k)\, f_\varepsilon(k-p)$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dk}{(k-p)}\, F(p,p',k)\, \sin \frac{(k-p)|k|\tau_0}{m} \qquad (5.5)$$

When we specify the particular sum rule we are considering, $F(p,p',k)$ is a well known function, i.e., we are just expressing the matrix elements in terms of Dirac spinors etc., whose explicit momentum dependence is given in Appendix A. For example, in our chiral SW(2) sum-rule, $F(p,p',k)$ is of the form

$$F(p,p',k) =$$

$$\overline{u}(p)(a_1+a_2k_0+a_3k_0^2+a_4kk_0+a_5k^2k_0+a_6kk_0^2+a_7k_0^3)u(p') \qquad (5.6)$$

where the a's are combinations of the Dirac matrices, masses, and numerical factors. If we insert this expression in Eq. (5), and carry out a suitable contour integration, we get the unexpected result that

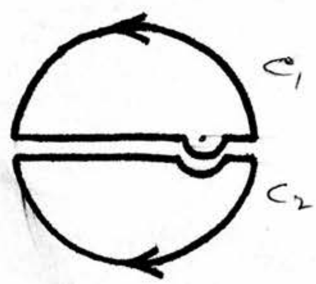
$$I = F(p,p',k=p) \tag{5.7}$$

Because of the importance of this result, it is worth demonstrating briefly the method of integration, and the convergence properties of the integrals. Taking a general k-dependent term, k^n, from the expansion Eq. (6), we must evaluate the integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dk}{(k-p)}\, k^n \sin((k-p)\,|k|\,\lambda) \qquad \text{where } \lambda = \frac{\tau_o}{m}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dk}{(k-p)}\, k^n \left(e^{i(k-p)\,|k|\,\lambda} - e^{-i(k-p)\,|k|\,\lambda}\right)$$

This is easily carried out if we choose the contours C_1 and C_2 respectively, for the two terms of this integral, as shown:



The contribution from the upper (or lower) semicircle vanishes, when the radius tends to infinity, i.e. for the upper half plane

$$\lim_{R\to\infty}\int_{0+\varepsilon}^{\pi-\varepsilon} \frac{i d\theta\, R^{n+1}\, e^{i(n+1)\theta}\, e^{i(Re^{i\theta}-p)R\lambda}}{Re^{i\theta} - p} =$$

$$= \lim_{R \to \infty} \int_{0+\varepsilon}^{\pi-\varepsilon} i \, d\theta \, R^n \, e^{in\theta} \, e^{iR^2 e^{i\theta}\lambda} = 0$$

since $R^n . e^{iR^2 e^{i\theta}\lambda} = R^n . e^{-R^2 \sin\theta\lambda} . e^{iR^2 \cos\theta\lambda}$

$\longrightarrow 0$ as $R \to \infty$ (where $\sin\theta > 0$).

The result we have obtained, Eq. (7), is interesting in two ways. First, from the physical point of view, we see that the matrix element is not affected in any way when the intermediate state becomes unstable. Secondly, from a mathematical point of view, the pseudo-delta function, $f_\varepsilon(k)$, has some of the properties of the Dirac delta function, when we consider it under an infinite integral*. It might, therefore, be useful to consider the application of this function to other problems in physics. For example, we might consider a field theory in which the canonical commutation relations were given in terms of pseudo-delta functions. Such considerations are, however, outside the scope of this dissertation.

2. Phenomenological Approach to the Problem of Unstable States.

Perhaps we should warn against taking the results of this section too seriously, as we saw, in the previous section, that the lifetime of the intermediate state plays no role in the sum-rule. However, we have seen, in other applications** of current algebra, how the "correct" experimental results can be obtained by disregarding certain contradictory features of the theory. In fact, it generally turns out that after a careful

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* See Appendix E.

** For example, the configuration mixing, and "no-leakage" assumptions.

reformulation and reinterpretation of the theory, there is no discrepancy between the current algebra technique, and the underlying theory. With this reservation in mind, we now consider an ad hoc modification to the chiral SW(2) sum-rule.

Intuitively, we might expect that whenever we have a matrix element $\langle p|J|k\rangle\langle k|J|p'\rangle$ with the intermediate state unstable, we should make the replacement $|k\rangle\langle k| \rightarrow \mathcal{F}|k\rangle\langle k|$. We call $\mathcal{F}$ the instability factor, which has the property that $\mathcal{F} = 1$ for completely stable states, and $\mathcal{F} = 0$ for completely unstable states (i.e. vanishing lifetime). The result of section 1 suggests that $\mathcal{F} = 1$ everywhere except where the particle's lifetime is zero, when $\mathcal{F} = 0$. However, we will assume that $\mathcal{F}$ is a smooth function of the lifetime, so that in general $0 \leq \mathcal{F} \leq 1$.

We repeat the derivation of the chiral SW(2) sum rule (Ch. II, Sec. 4) for helicity ½ states, making the replacements

$$|N^*\rangle\langle N^*| \longrightarrow \mathcal{F}\,|N^*\rangle\langle N^*|$$

$$|N\rangle\langle N| \longrightarrow f\,|N\rangle\langle N|$$

This gives us three equations

$$\frac{4}{3}\, f\, G_{11}^2 - \frac{8}{9}\, \mathcal{F}\, G_{14}^2\, M^2 = 1$$

$$f\, G_{11} = \frac{\sqrt{5}}{3}\, \mathcal{F}\, G_{44}$$

$$\frac{20}{3}\, f\, G_{14}^2\, M^2 + \frac{4}{9}\, \mathcal{F}\, G_{44}^2 = 15$$

where $M = \dfrac{m^*+m}{2m^*}$

which have the unique solution

$$G_{11}^2 = \left(\frac{1}{f} + 2M^2 \, \frac{\mathcal{F}}{f^2}\right) \cdot \frac{75}{100+8M^2} \tag{5.8}$$

$$G_{14}^2 = \frac{9}{4f} - \left(\frac{9}{\mathcal{F}} + \frac{18M^2}{f}\right) \cdot \frac{1}{100+8M^2} \tag{5.9}$$

$$G_{44}^2 = \left(\frac{f}{\mathcal{F}^2} + \frac{2M^2}{\mathcal{F}}\right) \cdot \frac{135}{100+8M^2} \tag{5.10}$$

Thus, the idea of damping the effect of the unstable intermediate state is, at least, compatible with the algebra. Since the nucleons are stable (compared with the N^*) we shall assume that $f = 1$. Then, as usual, $G = G_A/G_V$ is given by

$$G^2 = (1+2\mathcal{F}M^2) \cdot \frac{25}{25+2M^2} \tag{5.11}$$

Using the experimental values of G and M as input data, we find that $\mathcal{F} = 0{\cdot}25 \pm 0{\cdot}03$. We can then predict the value of the transition coupling constant G^* to be $1{\cdot}5$. This is to be compared with the experimental value of $1{\cdot}1 \pm 0{\cdot}1$ and the original current algebra prediction of $1{\cdot}6$.

There is one other way in which the role of the instability factor can be tested. Matsuda(36) has obtained a sum rule for the isovector charge radius of the nucleon. If we repeat his calculation, putting in the instability factor, we obtain

$$\frac{1}{3}\langle r^2\rangle = {\frac{\mu'(p)-\mu'(n)}{2m}}^2 - \frac{\mathcal{F}}{\cos\theta}\,{\frac{\mu'(n)}{2m}}^2 \tag{5.12}$$

where $\mu'(p)$ and $\mu'(n)$ are the anomalous magnetic moments of the proton and neutron, and θ is the configuration mixing angle*. This expression is just the same as Matsuda's result

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* See Chapter IV.

when $\mathcal{J} = 1$. Putting in the experimental values $\mu'(p) = 1{\cdot}79$, $\mu'(n) = -1{\cdot}91$, $\theta = 37^{\circ}$ and $\mathcal{J} = 0{\cdot}25$ we find that

$$m_{\pi}^{2} \langle r^{2} \rangle = 0{\cdot}21$$

This is much closer to the experimental value[37] of $0{\cdot}27$ than Matsuda's original value of $0{\cdot}14$.

Having seen the usefulness of the instability factor from a purely phenomenological point of view, it would be interesting to see what functional form it might have. From the well known exponential decay law, we might expect that $\mathcal{J} = e^{-t/\tau}$. However, to be more in keeping with the result of section 1, it is possible that the instability factor has the form

$$\mathcal{J} = (e^{(t-\tau)/\alpha} + 1)^{-2} \tag{5.13}$$

The constant 'α' is the fundamental unit of time. Wheeler[38] has suggested that space has a granular structure, with a "smallest possible distance" of $\sim 10^{-33}$cm. If we accept that velocity is bounded at $c \approx 3.10^{10}$cm/sec., then there is a corresponding "least time", $\alpha \approx \frac{1}{3}\, 10^{-43}$sec. The parameter '$t$' has yet to be explained, but for the moment we will assume that $t \sim 10^{-23}$sec., i.e. the order of magnitude of the lifetime of strongly decaying particles. With these values of α and t, the graph of the instability factor, Eq. (13), is shown in Fig. 2

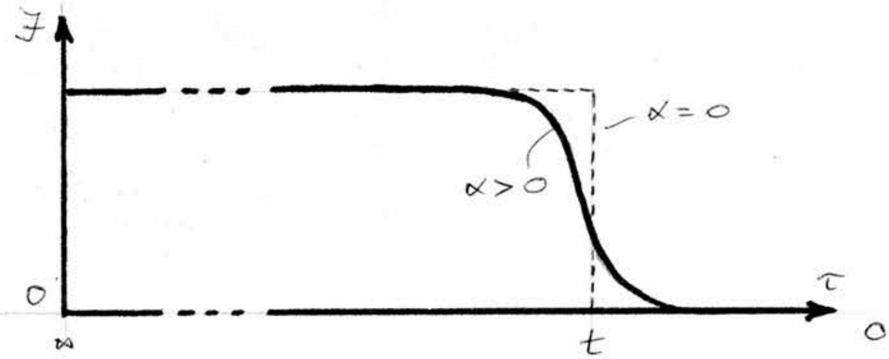

Fig. 2.

It is, interesting to note that by letting $t \to 0$ and $\alpha \to 0$, we can, in fact, reproduce the result of section 1 with this function.

The question still remains - what is the meaning of the parameter t ? From Eqs. (11 and 13) we can compute the values of $\mathcal{F}$ and G for various values of t, using the experimental lifetime of the N^*, $\tau = 0.548 \pm .009.10^{-23}$ sec. The results are given in the following table.

$t-\tau$ $\times 10^{-43}$ sec	$\mathcal{F}$	G
-0.100	0.329	1.240
-0.090	0.321	1.233
-0.080	0.313	1.227
-0.070	0.305	1.221
-0.060	0.296	1.215
-0.050	0.288	1.209
-0.040	0.280	1.202
-0.030	0.273	1.196
-0.020	0.265	1.190
-0.010	0.257	1.184
0.000	0.250	1.179
0.010	0.242	1.173
0.020	0.235	1.167
0.030	0.228	1.161
0.040	0.221	1.155
0.050	0.214	1.150
0.060	0.207	1.144
0.070	0.200	1.139
0.080	0.193	1.134
0.090	0.187	1.128
0.100	0.181	1.123

The strange feature of these results is that the experimental value of G occurs at the point $t = \tau$, within experimental errors. This is surely more than just a remarkable coincidence. We might hope to show that t has some kinematical significance, namely the time taken by a light signal to cross a <u>finite</u>

interaction volume. However, the result of section 1 seems to rule out this possibility. On the other hand there may be some unknown dynamical mechanism at work, which is characterized by the parameter t, and which somehow constrains t to be vanishingly close to the lifetime τ .

3. Conclusion.

In this chapter we have presented two apparently contradictory results. The first, based on a fairly rigorous argument, shows that the instability of particles makes no difference to the sum-rule. The second, based on an *ad hoc* procedure, shows at least, that a damping factor can be introduced consistently with the algebra. It may even hint at some underlying dynamical mechanism, if we are prepared to accept the particular function that was chosen for the instability factor. The dilemma however remains - how do we reconcile the two results? To this, there seems to be no obvious answer.

CHAPTER VI

THE DOMAIN OF THE CHARGE OPERATOR

In all the previous chapters, we have naively assumed that the charge operator

$$Q_\mu = \int d^3x \; j_\mu(x) \qquad (6.1)$$

is a perfectly meaningful quantity, whether or not the current is locally conserved. Now, it turns out that when we have a locally conserved current, there is not too much trouble in giving meaning to the formal definition, Eq. (1). However, when we try to work with nonconserved currents, great difficulty arises, in fact it is impossible to accomodate the charge (associated with a nonconserved current) in Hilbert space. With certain reservations, this does not, however, destroy the whole idea of current algebra.

We shall now consider some well known theorems on current algebra, and then proceed to prove certain additional results. The 'proofs' of some of the theorems often leave much to be desired, and so we shall repeat them in a fair amount of detail in order to point out the possible loopholes.

To make the proofs mathematically readable, we have kept verbal explanation to a minimum. However, the discussion of each proof follows immediately after the proof in a series of notes which are referred to by circled numbers, e.g. 1, in the text of each proof.

Theorem 1. Coleman's Second Theorem(6).

"The invariance of the vacuum is the invariance of the world". Stated symbolically this is,

$$\text{if } \quad Q(t) = \int d^3x \; j_o(x)$$

$$\text{then } \quad Q|0\rangle = 0 \Rightarrow \partial^\mu j_\mu(x) = 0$$

$$\text{and } \quad i\dot{Q} = [Q,H] = 0$$

Proof:

If $\int d^3x \; j_o(x)|0\rangle = 0$ ①② , then for an arbitrary state $|n\rangle$ of zero 3-momentum,

$$\langle n| \int d^3x \; j_o(x)|0\rangle = 0$$

Performing the Lorentz transformation $U(\underline{x})$, and using the translational invariance of the vacuum,

$$\int d^3x \langle n|U(\underline{x}) j_o(t) U^{-1}(\underline{x})|0\rangle = 0 \text{ or } \int d^3x \langle n|j_o(t)|0\rangle = 0.$$

Hence, $\langle n|j_o(x)|0\rangle = 0$ which is equivalent to

$$\langle n|\partial^\mu \; j_\mu(x)|0\rangle = 0 \tag{6.2}$$

It is one of the fundamental axioms of quantum mechanics that Eq. (2) should be true in any Lorentz frame. By applying a Lorentz transformation to $|n\rangle$, we can obtain a complete set of states on the left, so that

$$\sum |n\rangle\langle n|\partial^\mu j_\mu(x)|0\rangle = 0 \Rightarrow \partial^\mu j_\mu(x)|0\rangle = 0$$

But, any local operator that annihilates the vacuum, is itself identically zero③. Therefore,

$$\partial^{\mu} j_{\mu}(x) = 0$$

which means that (4)

$$i\dot{Q} = [Q, H] = 0.$$

Notes:

(1) It is assumed that the vacuum is unique, i.e. we have no massless particles in the theory.

(2) The existence of this integral has only been proved for a dense set of quasilocal states on the left, see Theorem 5. The states $|n\rangle$ are momentum eigenstates rather than quasilocal states. However, Dell'Antonio(49) has shown that the proof can in fact be carried out in this case.

(3) This well known result follows from the Federbush-Johnson argument(50) or from 'Theorem 4-3' of Streater and Wightman(55). Recently, however, a counter-example has been given(51) where $\psi(x)|0\rangle = 0$ does not imply that $\psi(x) = 0$. We are inclined to believe that this situation is pathological, since the $\psi(x)$ are infinite-component fields which lead to a theory without crossing symmetry.

(4) Strictly speaking, all we can conclude from the local conservation law

$$\partial^{\mu} j_{\mu}(x) = 0$$

is that

$$\partial_0 \int d^3x \; j_0(x) = \int_S \underline{j}(x) . d\underline{\sigma}$$

By making the additional assumption that the surface term falls off rapidly at infinity, we get the global conservation law. This assumption implies that we have a detailed knowledge of the interaction.

To avoid writing syllogistic tongue-twisters let us say that roughly speaking, the outcome of this theorem is that if the vacuum is invariant under the group generated by $\int d^3x j_o(x)$, then the Hamiltonian itself is an invariant of the group. The question as to whether the vacuum is in the domain of Q will be taken up later.

Theorem 2. The Fabri-Picasso Theorem(52).

If $Q = \int d^3x\ j_o(x)$ exists as a weak limit* and $\partial^\mu j_\mu(x) \neq 0$, then the vacuum cannot be in the domain of Q .

Proof:

$$Q|0\rangle =: |Q\rangle$$

Since Q is translationally invariant (in $\underline{x}$), so is $|Q\rangle$

$$\text{Now, } \langle Q|Q\rangle = \int d^3x \langle Q|j_o(x)|0\rangle$$
$$= \int d^3x \langle Q|j_o(t)|0\rangle$$

which is either zero or infinity.

If $\langle Q|Q\rangle = \infty$, then $|Q\rangle \notin \mathcal{H}$ (1)

If $\langle Q|Q\rangle = 0$, then $|Q\rangle = 0 = Q|0\rangle$, but from Theorem 1 this implies that $\partial^\mu\ j_\mu(x) = 0$.

So, if $\partial^\mu\ j_\mu(x) \neq 0$, Q cannot be defined on the vacuum.

Notes:

(1) The result of this theorem is true if we restrict the states to Hilbert space. In principle, there is no reason why we

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* See Theorem 5.

should not enlarge this space to include vectors of infinite norm. Katz[53] has, in fact, shown that this can be accomplished by using a rigged Hilbert space.

Theorem 3. The Fabri-Picasso-Strocchi Theorem[54].

If $Q = \int d^3x\, j_o(x)$ exists as a weak limit and $\partial^\mu j_\mu(x) \neq 0$, then Q is not a self-adjoint operator in $\mathcal{H}$.

Proof: (1)

If Q is self-adjoint* then $e^{i\lambda Q}$ exists (2). The state (3) $e^{i\lambda Q}|0\rangle$ is translationally invariant, so that

$e^{i\lambda Q}|0\rangle = e^{i\lambda q}|0\rangle$, where q is a real number.

Then, by Stone's Theorem, $Q|0\rangle = q|0\rangle$.

Now, $\langle 0|j_\mu(x)|0\rangle = 0$, so that in particular,

$\int d^3x \langle 0|j_o(x)|0\rangle = 0$, i.e. $\langle 0|Q|0\rangle = 0$

Hence, $q = 0$. But from Theorem 1,

$Q|0\rangle = 0 = \int d^3x\, j_o(x)|0\rangle$ implies that $\partial^\mu j_\mu(x) = 0$, which contradicts the original assumption.

Hence $Q^+ \neq Q$.

Notes:

(1) This 'proof', given by Fabri, Picasso and Strocchi is entirely fallacious if we accept the result of Theorem 2, that Q is not defined on the vacuum. We shall later prove this result by a different method, (see Theorem 7), so there seems to be no doubt of its validity.

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* We use this in the technical sense to distinguish it from a hermitian operator.

(2) This follows from the spectral theorem.

(3) If Q is not defined on the vacuum then neither is $e^{i\lambda Q}$. Since we are dealing with a continuous group of transformations, the derivatives of $e^{i\lambda Q}$ should also exist, so that, for example, $e^{i\lambda Q} iQ|0\rangle$ should be defined.

We shall now, briefly consider two very important theorems by Schroer and Stichel which tell us how we are to understand the formal definition of the charge operator, Eq. (1). The proofs of these theorems are based on the usual* axioms of quantum field theory, and are quite rigorous, unlike the 'proofs' of the previous theorems. Consequently, we shall merely quote the results. First, however, some explanation is required in order to understand these theorems.

The problem we are mainly concerned with in this chapter is the definition of the charge operator Q in terms of the local current operator $j_\mu(x)$. Between quasilocal states, the matrix element of $j_\mu(x)$

$$\langle \phi | j_\mu(x) | \psi \rangle$$

is a smooth, rapidly decreasing function in $\underline{x}$. The quasilocal states are defined by

$$|\phi\rangle = \int d^3x_1 \ldots d^3x_m f(\underline{x}_1, \ldots \underline{x}_m) A(\underline{x}_1) \ldots A(\underline{x}_m) |0\rangle \qquad (6.3)$$

where $A(\underline{x})$ is the local field, and $f \in \mathcal{S}$, the space of infinitely differentiable functions which, together with their derivatives, approach zero at infinity faster than any power of

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* For the ideas of axiomatic quantum field theory, functional analysis and Hilbert spaces we have relied heavily upon Refs. (55) - (59). In general, we shall not make any further reference to these 'standard' works.

the Euclidean distance. Under these conditions, the integral

$$\int d^3x \langle \emptyset | j_\mu(x) | \psi \rangle$$

always exists. This is true whether or not $j_\mu(x)$ is locally conserved.

When $\partial^\mu j_\mu(x) = 0$, it is possible to define the charge Q as

$$Q = \lim_{R \to \infty} j_o(f_R, f_T) \qquad (6.4)$$

i.e. an operator limit exists for the sequence of unbounded operators $j_o(f_R, f_T)$. The test functions f_R and f_T are space- and time-smearing functions

$$f_R \in \mathcal{D}(\mathbb{R}^3)$$

$$f_T \in \mathcal{D}(\mathbb{R})$$

in fact,

$$\begin{aligned} f_R(\underline{x}) &= 1 \quad \text{if} \quad |\underline{x}| < R \\ &= 0 \quad \text{if} \quad |\underline{x}| \geqslant R \end{aligned}$$

$$f_T(t) \geqslant 0, \ \text{supp}\, f_T(t) \subset [-T, T], \int dt\, f_T(t) = 1$$

The content of the following two theorems, is just giving a meaning to the operator limit in Eq. (4).

<u>Theorem 4.</u> <u>Schroer and Stichel(60).</u>

$Q = \lim_{R \to \infty} j_o(f_R, f_T)$ does not exist as a <u>strong</u> limit, i.e.

$$\langle 0 | j_o(f_R, f_T)\, j_o(f_R, f_T) | 0 \rangle \longrightarrow CR^2$$

as $R \to \infty$, where $C \neq 0$, unless $j_\mu(x) = 0$.

Theorem 5. Schroer and Stichel(60).

$Q = \lim_{R \to \infty} j_o(f_R, f_T)$ exists in a weak sense on a dense set of quasilocal states, i.e.

$\langle \emptyset | j_o(f_R, f_T) | 0 \rangle \to 0$ as $R \to \infty$ where $|\emptyset\rangle$ is defined by

$$|\emptyset\rangle = \int d^3x \; h(x) \; B(x) | 0 \rangle \tag{6.5}$$

$$h(x) \in \mathcal{D}_L (\mathbb{R}^3) \tag{6.6}$$

$$\mathcal{D}_L (\mathbb{R}^3) = \left\{ h(x);\; h(x) \in \mathcal{D} \;;\; \lim_{r \to \infty} r^2 h(x) = 0 \right\}^{*} \tag{6.7}$$

$$B(x) = U(x) \; B \; U^{-1}(x) \tag{6.8}$$

B = quasilocal operator, defined as in Eq. (3).

The implications of these two theorems is the following. As long as the current is locally conserved, the charge operator can be defined as a weak limit, for a dense set of quasilocal states. In fact, the linear form $\lim_{R \to \infty} \langle \emptyset | j_o(f_R, f_T) | 0 \rangle$ is zero, and therefore bounded. However, if $\partial^\mu j_\mu(x) \neq 0$, this linear form is unbounded in $|\emptyset\rangle$ and so we cannot define the charge associated with a nonconserved current, within the realms of Hilbert space.

So far, all these theorems have pointed to the conclusion that $Q|0\rangle$, and therefore $Q|s\rangle$** are not normalizable states,

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* $\mathcal{D}$ is the space of infinitely differentiable functions with compact support.

** $|s\rangle$ are all normalizable states obtained by applying creation operators to the vacuum and smearing with suitable integrable functions.

when the current is not conserved. In the usual context, this means that Q is not a well defined operator. We are, of course, familiar with non-normalizable states in quantum mechanics, e.g. plane waves. However, it is generally asserted that the 'physical' states of our system must be normalizable. For example, by introducing square-integrable smearing functions our plane waves become normalizable wave packets.

There seems to be no reason why we should not use states of infinite norm. It is merely our prejudice and unfamiliarity with working outside Hilbert space that prevents this*. Indeed, Katz[53] has shown that in a rigged Hilbert space, the charge operator can be perfectly well defined even if the current is not locally conserved. Inspired by Katz's work, we have been able to prove two further theorems. Consequently, and also because the idea of a rigged Hilbert space requires some explanation, we shall first give a concise survey of the essential results of his paper.

We define,

$\mathcal{H}$ = a separable Hilbert space

$\mathcal{A}$ = the algebra of observables, consisting of self-adjoint, not necessarily bounded operators

From the Hellinger-Toeplitz theorem** we see that the domain of $\mathcal{A}$, $D(\mathcal{A})$ satisfies

$$D(\mathcal{A}) =: \mathcal{F} \subset \mathcal{H} , \tag{6.9}$$

where $\mathcal{F}$ is assumed to be dense in $\mathcal{H}$, and $\mathcal{F} = \mathcal{H}$ iff

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* I am indebted to Dr. David Judge for this remark.

** See, for example, Ref. (58).

A is bounded $\forall A \in \mathfrak{A}$. In fact,

$$\mathfrak{A}\mathcal{F} \subset \mathcal{F} \subset \mathcal{H}$$

There are two things we must immediately consider, (i) the topology on $\mathcal{F}$, and (ii) the dual space of $\mathcal{F}$.

(i) The action of unbounded operators $A \in \mathfrak{A}$ on $\mathcal{F}$ is not continuous in the topology of $\mathcal{H}$, i.e. if $d_i \in \mathcal{F}$, then $d_i \to 0$ in $\mathcal{H} \not\Rightarrow Ad_i \to 0$ in $\mathcal{H}$. It is therefore necessary to define a <u>finer</u> topology on $\mathcal{F}$ so that all operators $A \in \mathfrak{A}$ are continuous. Convergence in this finer topology means that $d_i \longrightarrow 0$ in $\mathcal{F} \Rightarrow Ad_i \longrightarrow 0$ in $\mathcal{H}$, $\forall A \in \mathfrak{A}$.

(ii) We denote the dual space of $\mathcal{F}$ by $\mathcal{F}'$, and the elements of $\mathcal{F}'$ by $d' \in \mathcal{F}'$. Then, for each $d_i \in \mathcal{F}$, there exists a continuous linear form (or functional) on $\mathcal{F}$, namely

$$\ell(d_i) = (d', d_i)$$

which is continuous in the sense that $d_i \longrightarrow 0$ in $\mathcal{F} \Rightarrow (d', d_i) \to 0$ in $\mathcal{F}'$.

We now have the relation that

$$\mathcal{F} \subset \mathcal{H} \subset \mathcal{F}' \tag{6.10}$$

This triplet of spaces is called a rigged Hilbert space.

The next step is the most vital part of the whole argument. So far $A \in \mathfrak{A}$ has only been defined on $\mathcal{F}$, a space generally smaller than the whole Hilbert space $\mathcal{H}$. Because of the relation between $\mathcal{F}'$ and $\mathcal{F}$, we can extend the definition of <u>all</u> $A \in \mathfrak{A}$ to $\mathcal{F}'$ by defining,

$$(Ad', d) = (d', Ad)$$

The continuity of A on $\mathcal{F}$ of course means that Ad' is continuous in $\mathcal{F}'$.

Having defined our space, the states are now defined as

$$|d\rangle = \int d^3x\, f(\underline{x})\, B(\underline{x})|0\rangle \tag{6.11}$$

where $f(\underline{x}) \in \mathcal{S}(\mathbb{R}^3)$, (6.12)

and $$B(\underline{x}) = \int d^3x_1 \ldots d^3x_m\, g(\underline{x}_1-\underline{x}, \ldots \underline{x}_m-\underline{x})\, A(\underline{x})\, A(\underline{x}_1)\ldots A(\underline{x}_m) \tag{6.13}$$

where $A(\underline{x})$ are the local fields, and $g(\underline{x}_1, \ldots \underline{x}_m) \in \mathcal{S}(\mathbb{R}^{3m})$

Now $|d\rangle \rightarrow 0$ in $\mathcal{F}$ iff $f(x)$ <u>or</u> $g(x) \longrightarrow 0$ in $\mathcal{S}$.

The scalar product of two of these states is,

$$\langle c|d\rangle = \int d^3x\, d^3y\, f_1(\underline{x})\, f_2(\underline{y}) \langle 0|B_1(\underline{x})\, B_2(\underline{y})|0\rangle.$$

Since $\langle 0|B_1(\underline{x})\, B_2(\underline{y})|0\rangle$ is a rapidly decreasing distribution in $(\underline{x}-\underline{y})$, the scalar product is still well defined when $f_1(\underline{x}) \in \mathcal{S}'$ instead of $\mathcal{S}$. ($\mathcal{S}'$, which is the dual space of $\mathcal{S}$, is the space of tempered distributions)*. Through the scalar product, we can therefore define the states of the dual space as,

$$|d'\rangle = \int d^3x\, f'(\underline{x})\, B(\underline{x})|0\rangle$$

where $$f'(x) \in \mathcal{S}'(\mathbb{R}^3)$$

and $B(\underline{x})$ is still defined by Eq. (13)

Now, $|d'\rangle \rightarrow 0$ in $\mathcal{F}'$ iff $f'(\underline{x}) \rightarrow 0$ in $\mathcal{S}'$ <u>or</u> $g(\underline{x}) \rightarrow 0$ in $\mathcal{S}$.

Since the constant 1 belongs to $\mathcal{S}'$, the state

$$|Q\rangle = \int d^3x\, B(\underline{x})|0\rangle$$

is defined in $\mathcal{F}'$. The state $|Q\rangle$ so defined is just the

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* The distribution $\langle 0|B_1(\underline{x})\, B_2(\underline{y})|0\rangle$ decreases exponentially at infinity, whereas $f_1(x) \in \mathcal{S}'$ has, at most, polynomial growth of finite order.

analogue of the state of Fabri and Picasso, (Theorem 2). Thus, in a rigged Hilbert space, the charge operator for a nonconserved current can be well defined. This completes our discussion of Katz's results. We now present two further theorems.

Theorem 6.

The vacuum does not belong to $\mathcal{F}$.

Proof:

The space $\mathcal{F}$ is defined in Eq. (10). If the vacuum $|0\rangle \in \mathcal{F}$, and $|d'\rangle \in \mathcal{F}'$ is given by

$$|d'\rangle = \int d^3x\ f'(\underline{x})\ B(\underline{x})|0\rangle$$

where $f'(\underline{x}) \in \mathcal{S}'(\mathbb{R}^3)$ and $B(\underline{x})$ is the quasilocal operator defined by Eq. (13),

then, from the property of the rigged Hilbert space,

$$\langle d'|0\rangle = \int d^3x\ f'(\underline{x})\langle 0|B(\underline{x})|0\rangle$$

should exist $\forall\ |d'\rangle \in \mathcal{F}'$. However the vacuum expectation value $\langle 0|B(\underline{x})|0\rangle$ is a constant, (by translational invariance), and $f' \in \mathcal{S}'$, so that the integral generally does not exist. Hence, $|0\rangle \notin \mathcal{F}$.

Theorem 7.

The vacuum does not belong to $\mathcal{X}$.

Proof:

The states $|b\rangle \in \mathcal{X}$ are defined as in Eq. (5) by

$$|b\rangle = \int d^3x\ h(\underline{x})\ B(\underline{x})|0\rangle$$

where $h(\underline{x}) \in \mathcal{D}_L\ (\mathbb{R}^3)$

and $B(\underline{x})$ is again defined by Eq. (13).

The states $|b'\rangle$ of the dual space χ' are

$$|b'\rangle = \int d^3x\, h'(\underline{x})\, B(\underline{x})|0\rangle$$

where $h'(\underline{x}) \in \mathcal{D}_L'(\mathbb{R}^3)$

If $|0\rangle \in \chi$, and $|b'\rangle \in \chi'$ then (as in Theorem 6), the scalar product

$$\langle b'|0\rangle = \int d^3x\, h'(\underline{x}) \langle 0|B(\underline{x})|0\rangle$$

should exist $\forall\, |b'\rangle \in \chi'$.

Once again, the vacuum expectation value is a constant, and since $h' \in \mathcal{D}'$, the integral does not generally exist. Hence, $|0\rangle \notin \chi$.

In Theorem 5, the charge operator Q, associated with a locally conserved current, was defined on a dense set of quasi-local states which span the space χ. Now, we have seen from Theorem 7 that the space χ does not include the vacuum. This means that for a conserved current, the domain of Q is $\Omega \cup \chi$, where Ω is the vacuum, and for a nonconserved current, the domain is just χ. This is also the result of Theorem 2. However, when we enlarge the space, the domain of Q is $\Omega \cup \mathcal{F}$, whether or not the current is conserved.

In conclusion, let us recapitulate what has been said in this chapter. Theorem 1 shows that if the vacuum transforms like a singlet under the charge-algebra, then the current must be locally conserved. However, local conservation does not necessarily imply global conservation. Theorems 2, 4, 5 and 7 are concerned with the definition and the domain of the charge operator, and show that with states restricted to Hilbert space, the charge can only be defined when the current is conserved. Katz's work and Theorem 6 show the possibility of defining the charge whether or not the current is conserved, provided we work in a rigged Hilbert space. It therefore seems that there is nothing wrong with the usual manipulations of current algebra if the states we use are taken from the dense set of quasilocal states. The one-particle states are certainly of this form[60].

APPENDIX A

HIGHER SPIN WAVE FUNCTIONS

In this section we indicate the notation and list some of the basic properties of the Dirac, Proca and Rarita-Schwinger wave functions that have been used in the text. We also give a method for constructing helicity eigenstates for arbitrary spin.

The metric used throughout this dissertation is $g_{\mu\nu} = (-1,+1,+1,+1)_{diag}$, and the Dirac matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$

1. Integral spin - n.

We write the boson wave function as $\phi^s_{\mu_1 \ldots \mu_n}(x)$ or $\varepsilon^s_{\mu_1 \ldots \mu_n}(\underline{p})$. The upper index denotes the spin-component, and is suppressed when no ambiguity can arise. The lower indices are the Lorentz indices $\mu = 0,1,2,3$.

$$(\Box - m^2)\, \phi_{\mu_1 \ldots \mu_n}(x) = 0$$

with the subsidiary conditions

$$\partial^\mu \phi_{\mu\nu\ldots} = 0 \qquad \phi_{\mu\mu\nu\ldots} = 0$$

$$\phi_{\ldots\mu_i\ldots\mu_j\ldots} = \phi_{\ldots\mu_j\ldots\mu_i\ldots}$$

Alternatively, $(p^2 + m^2)\, \varepsilon_{\mu_1 \ldots \mu_m}(\underline{p}) = 0$

with $p^\mu \varepsilon_{\mu\nu\ldots} = 0$ etc.

Normalization:

$$g^{\mu\nu}\ldots g^{\eta\xi}\ \overset{+}{\varepsilon}{}^{r}_{\mu\ldots\eta}\ \varepsilon^{s}_{\nu\ldots\xi} = \delta_{rs} \qquad (A.1)$$

Projection operator:

$$\sum \varepsilon_{\mu\nu\ldots}\ \overset{+}{\varepsilon}{}^{\eta\xi\ldots} = \Theta^{\eta\xi\ldots}_{\mu\nu\ldots}$$

where Θ is the spin projection operator defined by Fronsdal[24].

In particular, the spin-1 projection operator is

$$\sum \varepsilon_{\mu}(\underline{p})\ \overset{+}{\varepsilon}_{\nu}(\underline{p}) = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \qquad (A.2)$$

2. Half-integral spin - $(n + \frac{1}{2})$.

The fermion wave functions are written as $\psi^{s}_{\alpha\mu_1\ldots\mu_n}(x)$ or $u^{s}_{\alpha\mu_1\ldots\mu_n}(\underline{p})$. The upper index, again denoting the spin-component, is suppressed whenever possible. The lower index α is the Dirac spinor index and μ_n are the Lorentz indices.

$$(\gamma^{\mu}\partial_{\mu} - m)\ \psi_{\alpha\mu_1\ldots\mu_n}(x) = 0$$

with the subsidiary conditions

$$\gamma^{\mu}\psi_{\alpha\mu\nu\ldots} = 0 \quad \text{i.e.} \quad \partial^{\mu}\psi_{\alpha\mu\nu\ldots} = 0$$

$$\psi_{\alpha\ldots\mu_i\ldots\mu_j\ldots} = \psi_{\alpha\ldots\mu_j\ldots\mu_i\ldots} \qquad \psi_{\alpha\mu\mu\nu\ldots} = 0$$

Alternatively $(i\not{p} - m)\ u_{\alpha\mu_1\ldots\mu_n}(\underline{p}) = 0$, where $\not{p} = \gamma^{\mu}p_{\mu}$

with $p^{\mu}u_{\alpha\mu\nu\ldots} = 0$ etc.

Normalization:

$$g^{\mu\nu}\ldots g^{\eta\xi}\ \bar{u}^{r}_{\alpha\mu\ldots\eta}\ u^{s}_{\alpha\nu\ldots\xi} = \delta_{rs} \qquad (A.3)$$

Projection operator:

$$\sum u_{\alpha\mu\nu\ldots} \; \overline{u}_{\beta}^{\eta\xi\ldots} = \theta^{\eta\xi\ldots}_{\mu\nu\ldots} \qquad \left(\frac{i\not{p}+m}{2m}\right)_{\alpha\beta}$$

We give below the explicit form of the projection operators used in the text.

$$\text{Spin} - \tfrac{1}{2}: \quad \sum u(\underline{p})\overline{u}(\underline{p}) = \frac{i\not{p}+m}{2m} \tag{A.4}$$

$$\text{Spin} - \tfrac{3}{2}: \quad \sum u_{\mu}(\underline{p})\overline{u}_{\nu}(\underline{p}) = D_{\mu\nu}(\underline{p}) \; \frac{i\not{p}+m}{2m} \tag{A.5}$$

with

$$D_{\mu\nu}(\underline{p}) = \frac{1}{3}\left\{3g_{\mu\nu} + 4\,\frac{p_{\mu}p_{\nu}}{m^2} - \gamma_{\mu}\gamma_{\nu} - \frac{(\gamma_{\mu}\gamma\cdot p p_{\nu} + p_{\mu}\gamma\cdot p\gamma_{\nu})}{m^2}\right\}$$

3. Construction of helicity eigenstates for any spin.

Let $u^{s}_{\alpha}(0)$ and $\varepsilon^{s}_{\mu}(0)$ describe the spin-½ and spin-1 rest states, when spin is quantized in the z-direction.

Then $u^{+1/2}_{\alpha}(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $u^{-1/2}_{\alpha}(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

and $\varepsilon^{+1}_{\mu}(0) = (0,\underline{a})$, $\varepsilon^{o}_{\mu}(0) = (0,\underline{a}_3)$, $\varepsilon^{-1}_{\mu}(0) = (0,\underline{a}^*)$

where we have chosen the right-handed orthonormal triad $\underline{a}_1 = (1,0,0)$, $\underline{a}_2 = (0,1,0)$, $\underline{a}_3 = (0,0,1)$ and defined $\underline{a} = \frac{\underline{a}_1 + i\underline{a}_2}{\sqrt{2}}$

By applying the usual boost and rotation matrices(25) to the rest states, we obtain helicity eigenstates for arbitrary momentum,

$$u^{s}_{\alpha}(\underline{p}) = \sum L_{\alpha\beta}(\underline{p}) \; D^{1/2}_{s\sigma}(\emptyset,\theta,-\emptyset) \; u^{\sigma}_{\beta}(0)$$

$$\varepsilon^{s}_{\mu}(\underline{p}) = \sum L^{\nu}_{\mu}\,(\underline{p}) \; D^{1}_{s\sigma}(\emptyset,\theta,-\emptyset) \; \varepsilon^{\sigma}_{\nu}(0)$$

where $p_{\mu} = (p_o,\ p\sin\theta\cos\emptyset,\ p\sin\theta\sin\emptyset,\ p\cos\theta)$

For the spinor representation the boost matrix is

$$L(\underline{p}) = \frac{m + \not{p}\gamma_o}{\sqrt{2m(m+p_o)}} \tag{A.6}$$

so that $u(\underline{p}) = L(\underline{p})\, u(0)$

Written out explicitly the helicity eigenstates are

$$u_\alpha^{+\frac{1}{2}}(\underline{p}) = N(\underline{p}) \begin{pmatrix} & \begin{pmatrix} \cos \frac{1}{2}\theta \\ e^{i\phi}\sin\frac{1}{2}\theta \end{pmatrix} \\ \frac{\underline{p}}{p_o+m} & \begin{pmatrix} \cos \frac{1}{2}\theta \\ e^{i\phi}\sin\frac{1}{2}\theta \end{pmatrix} \end{pmatrix} \tag{A.7}$$

$$u_\alpha^{-\frac{1}{2}}(\underline{p}) = N(\underline{p}) \begin{pmatrix} & \begin{pmatrix} -e^{-i\phi}\sin\frac{1}{2}\theta \\ \cos \frac{1}{2}\theta \end{pmatrix} \\ \frac{-\underline{p}}{p_o+m} & \begin{pmatrix} -e^{-i\phi}\sin\frac{1}{2}\theta \\ \cos \frac{1}{2}\theta \end{pmatrix} \end{pmatrix} \tag{A.8}$$

with the normalization factor $N(\underline{p}) = \left(\frac{p_o+m}{2m}\right)$

$$\varepsilon_\mu^{+1}(\underline{p}) = \frac{1}{2}\, e^{i\phi} \begin{pmatrix} 0 \\ \cos\theta\cos\phi - i\sin\phi \\ \cos\theta\sin\phi + i\cos\phi \\ -\sin\theta \end{pmatrix} \tag{A.9}$$

$$\varepsilon_\mu^{o}(\underline{p}) = \frac{p_o}{m} \begin{pmatrix} p/p_o \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \tag{A.10}$$

$$\varepsilon_\mu^{-1}(\underline{p}) = \frac{1}{2}\, e^{-i\phi} \begin{pmatrix} 0 \\ \cos\theta\cos\phi + i\sin\phi \\ \cos\theta\sin\phi - i\cos\phi \\ -\sin\theta \end{pmatrix} \tag{A.11}$$

The construction of helicity eigenstates for spin - $\frac{3}{2}$ is readily accomplished by coupling the spin - $\frac{1}{2}$ and spin - 1 helicity eigenstates with the appropriate Clebsch-Gordan coefficients. By successively coupling in spin - 1 we can build up helicity eigenstates for any desired half-integral spin. (Actually, a similar procedure of coupling spin - 1 with itself will produce arbitrary integral spin helicity eigenstates which satisfy the Proca equation).

In general, for spin - $(n+\frac{1}{2})$, the helicity eigenstates are

$$u^{s}_{\alpha\mu\mu_2\ldots\mu_n}(\underline{p}) = \sum_{\sigma,\lambda} \begin{pmatrix} n-\frac{1}{2} & 1 & n+\frac{1}{2} \\ \sigma & \lambda & s \end{pmatrix} u^{\sigma}_{\alpha\mu\mu_2\ldots\mu_{n-1}}(\underline{p})\ \varepsilon^{\lambda}_{\mu_n}(\underline{p}) \qquad (A.12)$$

where n = integer.

This type of coupling of the helicity eigenstates with Clebsch-Gordan coefficients ensures that the resulting state is also a helicity eigenstate. Furthermore, it is straightforward to show that this resulting state satisfies the Rarita-Schwinger equation and the subsidiary conditions.

For the particular case of spin - $\frac{3}{2}$ we obtain

$$\begin{aligned}
u^{+\frac{3}{2}}_{\alpha\mu}(\underline{p}) &= u^{+\frac{1}{2}}_{\alpha}(\underline{p})\ \varepsilon^{+1}_{\mu}(\underline{p}) \\
u^{+\frac{1}{2}}_{\alpha\mu}(\underline{p}) &= \sqrt{\tfrac{1}{3}}\ u^{-\frac{1}{2}}_{\alpha}(\underline{p})\ \varepsilon^{+1}_{\mu}(\underline{p}) + \sqrt{\tfrac{2}{3}}\ u^{+\frac{1}{2}}_{\alpha}(\underline{p})\ \varepsilon^{o}_{\mu}(\underline{p}) \\
u^{-\frac{1}{2}}_{\alpha\mu}(\underline{p}) &= \sqrt{\tfrac{1}{3}}\ u^{+\frac{1}{2}}_{\alpha}(\underline{p})\ \varepsilon^{-1}_{\mu}(\underline{p}) + \sqrt{\tfrac{2}{3}}\ u^{-\frac{1}{2}}_{\alpha}(\underline{p})\ \varepsilon^{o}_{\mu}(\underline{p}) \\
u^{-\frac{3}{2}}_{\alpha\mu}(\underline{p}) &= u^{-\frac{1}{2}}_{\alpha}(\underline{p})\ \varepsilon^{-1}_{\mu}(\underline{p})
\end{aligned} \qquad (A.13)$$

From this explicit representation it is easy to show that

$$\begin{aligned}
g^{\mu\nu}\ \bar{u}^{r}_{\alpha\mu}\ u^{s}_{\alpha\nu} &= \delta_{rs} \quad \text{for} \quad r,s = \pm\tfrac{1}{2}\,,\ \pm\tfrac{3}{2} \\
\bar{u}^{r}_{\alpha o}\ u^{s}_{\alpha o} &= \tfrac{2}{3}\left(\frac{p_o^2}{m^2} - 1\right)\delta_{rs} \quad \text{for} \quad r,s = \pm\tfrac{1}{2} \\
&= 0 \quad \text{for } r,s = \pm\tfrac{3}{2}
\end{aligned} \qquad (A.14)$$

We also give a few useful identities relating the Dirac and Rarita-Schwinger wave functions. Omitting the spinor indices, and taking $\underline{p}' = \underline{p}$ but $m' \neq m$ we obtain

$$\bar{u}_{\mu}(p')\ \gamma_5\ u(p) = \frac{p'_o - p_o}{m' + m}\quad \bar{u}_{\mu}(p')\ i\gamma_5\gamma_o\ u(p) \qquad (A.15)$$

$$\bar{u}(p)\ \gamma_A\ u(p) = \frac{1}{2m}\ \bar{u}(p)\ \left\{ i\not{p},\ \gamma_A \right\} u(p) \tag{A.16}$$

where γ_A represents any of the sixteen Dirac matrices.

$$\bar{u}(p)\gamma_\lambda\ u(p) = \frac{ip_\lambda}{m}\ \bar{u}(p)\ u(p) \tag{A.17}$$

$$\bar{u}_\mu(p')\gamma_\lambda\ u(p) = \frac{2ip'_\lambda}{(m'+m)}\ \bar{u}_\mu(p')u(p) + \frac{p_o'-p_o}{m'+m}\ \bar{u}_\mu(p')i\gamma_\lambda\gamma_o u(p) \tag{A.18}$$

$$u_o(p_z) = \sqrt{\frac{2}{3}}\ \frac{|p_z|}{m'}\ u(p_z) \tag{A.19}$$

$$u_z(p_z) = \sqrt{\frac{2}{3}}\ \frac{p'_o}{m'}\ u(p_z) \tag{A.20}$$

APPENDIX B

SU(3) CLEBSCH-GORDAN COEFFICIENTS

The SU(3) C-G coefficients have been tabulated by McNamee and Chilton[26]. In this appendix we give their standard properties, in the notation of de Swart[27]. In the third section we define the crossing matrix and calculate some additional crossing matrices that are needed in the text.

1. Orthogonality.

$$\sum_{\nu_1\nu_2} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 & \mu'_{\gamma'} \\ \nu_1 & \nu_2 & \nu' \end{pmatrix} = \delta_{\mu\mu'}\, \delta_{\gamma\gamma'}\, \delta_{\nu\nu'}$$

$$\sum_{\mu_\gamma \nu} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu'_1 & \nu'_2 & \nu \end{pmatrix} = \delta_{\nu_1\nu'_1}\, \delta_{\nu_2\nu'_2}$$

2. Symmetry.

$$\begin{aligned} \begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} &= \xi_1(\mu_1,\mu_2,\mu_{3\gamma}) \begin{pmatrix} \mu_2 & \mu_1 & \mu_{3\gamma} \\ \nu_2 & \nu_1 & \nu_3 \end{pmatrix} \\ &= \xi_2(\mu_1,\mu_2,\mu_{3\gamma})\, (-1)^{\bar{\nu}_1} \left(\frac{\{\mu_3\}}{\{\mu_2\}}\right)^{1/2} \begin{pmatrix} \mu_1 & \bar{\mu}_3 & \bar{\mu}_{2\gamma} \\ \nu_1 & -\nu_3 & -\nu_2 \end{pmatrix} \\ &= \xi_3(\mu_1,\mu_2,\mu_{3\gamma}) \begin{pmatrix} \bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu}_{3\gamma} \\ -\nu_1 & -\nu_2 & -\nu_3 \end{pmatrix} \end{aligned}$$

The phase factors ξ_1, ξ_2, and ξ_3 are tabulated by de Swart[27].

3. Crossing Matrices.

$$(\mu\ \xi\ \eta\ |\ \beta_{II}(\mu_1,\mu_2,\mu_3,\mu_4)\ |\ \mu'\ \xi'\ \eta')$$

$$= \frac{1}{\{\mu\}} \sum_{\substack{\nu_1\nu_2\nu_3\\ \nu_4\nu\nu'}} (-1)^{\bar{\nu}_2-\bar{\nu}_3} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\eta \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \mu_3 & \mu_4 & \mu_\xi \\ \nu_3 & \nu_4 & \nu \end{pmatrix} \begin{pmatrix} \mu_1 & \bar{\mu}_3 & \mu'_{\eta'} \\ \nu_1 & -\nu_3 & \nu' \end{pmatrix} \begin{pmatrix} \bar{\mu}_2 & \mu_4 & \mu'_{\xi'} \\ -\nu_2 & \nu_4 & \nu' \end{pmatrix}$$

The elements of the crossing matrices $(\mu\xi\eta\ |\ \beta_{II}(8,8,8,8)\ |\ \mu'\xi'\eta')$ and $(\mu\ |\ \beta_{II}(8,10,8,10)\ |\ \mu'\xi')$ have been tabulated by de Swart[28]. Unfortunately, there are no tables for the crossing matrix $(\mu\xi\ |\ \beta_{II}(8,\overline{10},8,8)\ |\ \mu'\xi')$ and so these had to be calculated by hand, by summing the SU(3) Clebsch-Gordan coefficients. It is important to note that a careful choice of the magnetic quantum numbers does much to simplify an otherwise unpleasant computation.

Table I - The elements $(\mu\xi|\beta_{II}(8,\overline{10},8,8)|\mu'\xi')$ of the crossing matrix.

		$\mu'\xi'$		
		10	8_a	8_s
$\mu\ \xi$	$\overline{10}$	$\frac{1}{2}$	$\sqrt{\frac{2}{5}}$	$-\frac{\sqrt{2}}{5}$
	8_a	$\sqrt{\frac{5}{8}}$	0	$\sqrt{\frac{1}{5}}$

APPENDIX C

SU(2) CLEBSCH-GORDAN COEFFICIENTS

In this appendix we give a summary of the usual properties of the SU(2) Clebsch-Gordan coefficients. (See for example Rose(29) and Edmonds(30)). As this dissertation is primarily concerned with internal symmetry, we shall standardize the Clebsch-Gordan notation and write the SU(2) and SU(3) C-G coefficients in the same way, namely $\begin{pmatrix} J_1 & J_2 & J \\ m_1 & m_2 & m \end{pmatrix}$ and $\begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix}$

In SU(2) this notation is generally used for the 3-j symbols. However, we do not use 3-j symbols anywhere in this dissertation.

In the final section we define the SU(2) crossing matrix as the analogue of de Swart's SU(3) crossing matrix, and calculate certain elements which are used in our various sum rules.

1. Orthogonality.

$$\sum_{m_1 m_2} \begin{pmatrix} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \delta_{J_3 J_3'} \quad \delta_{m_3 m_3'}$$

$$\sum_{J_3 m_3} \begin{pmatrix} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \quad \delta_{m_2 m_2'}$$

Note that $\begin{pmatrix} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0$ unless $m_1 + m_2 = m_3$

2. Symmetry.

$$\begin{pmatrix} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{J_1+J_2-J_3} \begin{pmatrix} J_2 & J_1 & J_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$$

$$= (-1)^{J_1-m_1} \left(\frac{2J_3+1}{2J_2+1}\right)^{1/2} \begin{pmatrix} J_1 & J_3 & J_2 \\ m_1 & -m_3 & -m_2 \end{pmatrix}$$

$$= (-1)^{J_1+J_2-J_3} \begin{pmatrix} J_1 & J_2 & J_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}$$

3. Crossing Matrices.

We define the SU(2) crossing matrix as

$$(J|\beta_{II}(J_1, J_2, J_3, J_4)|J')$$

$$= \frac{1}{(2J+1)} \sum_{\substack{m_1 m_2 m_3 \\ m_4 m m'}} (-1)^{(J_2+m_2)-(J_3+m_3)} \begin{pmatrix} J_1 J_2 J \\ m_1 m_2 m \end{pmatrix} \begin{pmatrix} J_3 J_4 J \\ m_3 m_4 m \end{pmatrix} \begin{pmatrix} J_1 & J_3 J' \\ m_1 & -m_3 m' \end{pmatrix} \begin{pmatrix} J_2 J_4 J' \\ -m_2 m_4 m' \end{pmatrix}$$

In order to facilitate computation, we use the orthogonality and symmetry properties of the Clebsch-Gordan coefficients to relate the crossing matrix to either the 6-j symbols or the Racah W-coefficients. We thus obtain

$$(J|\beta_{II}(J_1,J_2,J_3,J_4)|J')$$

$$= (2J'+1)(-1)^{J+J'+J_2-J_3} \begin{Bmatrix} J_1 & J_2 & J \\ J_4 & J_3 & J' \end{Bmatrix}$$

$$= (2J'+1)(-1)^{J+J'+J_1+J_4+2J_2}\ W(J_1,J_2,J_3,J_4;J,J')$$

Formulae for evaluating the 6-j symbols have been given by Edmonds[30] and the Racah W-coefficients have been computer-tabulated* by

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* On page 189 of these tables the entry for $W(\frac{3}{2}, 1, \frac{3}{2}, 1;\ \frac{1}{2}, 1)$ should read + 26352313 instead of -26352313.

Nikoforov et al[31].

We give below some of the crossing matrix elements.

Table I - Values of $(J|\beta_{II}(\frac{1}{2}, 1, \frac{1}{2}, 1)|J')$

		J'	
		0	1
J	$\frac{1}{2}$	$-\sqrt{\frac{1}{6}}$	-1
	$\frac{3}{2}$	$-\sqrt{\frac{1}{6}}$	$\frac{1}{2}$

Table II - Values of $(J|\beta_{II}(\frac{1}{2}, 1, \frac{3}{2}, 1)|J')$

		J'	
		1	2
J	$\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{\frac{25}{12}}$
	$\frac{3}{2}$	$\frac{\sqrt{10}}{4}$	$-\sqrt{\frac{5}{24}}$

Table III - Values of $(J|\beta_{II}(\frac{3}{2}, 1, \frac{3}{2}, 1)|J')$

		J'		
		0	1	2
J	$\frac{1}{2}$	$-\frac{\sqrt{3}}{6}$	$-\frac{\sqrt{10}}{4}$	$-\sqrt{\frac{25}{24}}$
	$\frac{3}{2}$	$-\frac{\sqrt{3}}{6}$	$-\frac{1}{\sqrt{10}}$	$\sqrt{\frac{2}{3}}$

Table IV - Values of $(J|\beta_{II}(1, 1, 1, 1)|J')$

		J'		
		0	1	2
J	0	$\frac{1}{3}$	1	$\frac{5}{3}$
	1	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{5}{6}$
	2	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$

Table V

$(1|\beta_{II}(2, 1, 2, 1)|1) = \sqrt{\frac{9}{20}}$
$(1|\beta_{II}(2, 1, 0, 1)|2) = \sqrt{\frac{5}{3}}$

APPENDIX D

CHARGE-CONJUGATION

As Gell-Mann[23] has shown, every octet that goes into itself under charge-conjugation has a characteristic number $\mathcal{C} = \pm 1$. This is the charge-conjugation quantum number of its 1, 3, 4, 6 and 8 components (in the hermitian Cartesian basis). The charge-conjugation quantum number of its 2, 5 and 7 components is then $-\mathcal{C}$. Thus, a self-conjugate meson octet transforms under charge-conjugation like

$$M_i \longrightarrow \mathcal{C} e^{(i)} M_i \qquad \text{(no summation)}$$

$$\text{where} \quad e^{(i)} = +1, \text{ for } i = 1, 3, 4, 6, 8$$
$$= -1, \text{ otherwise.}$$

For the observed particles we have $\mathcal{C}_V = -1$, and $\mathcal{C}_P = \mathcal{C}_S = \mathcal{C}_A = +1$.

Since a matrix element is an invariant, we shall find that invariance under charge-conjugation imposes certain restrictions on the type of coupling allowed. It is easier to see this if we take a specific example. Consider the matrix element of an axial-vector current octet between pseudoscalar and vector meson octet states. In principle, we can have both F- and D-type coupling, i.e.

$$\langle V \mid A \mid P \rangle \sim V_i \, A_j \, P_k \, (f_{ijk} + d_{ijk})$$

However, under charge-conjugation we find that*

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* We are assuming that the vector and axial-vector currents are purely first class. This means that $\mathcal{C}_V = -1$ and $\mathcal{C}_A = +1$.

$$\langle V|\, A\,|\, P\rangle \longrightarrow -\, V_i\; A_j\; P_k\; e^{(i)}\; e^{(j)}\; e^{(k)}\; (f_{ijk} + d_{ijk})$$

$$= V_i\; A_j\; P_k\; (f_{ijk} - d_{ijk})$$

Consequently, the term in "d" must be absent.

In general, we can show that the matrix element $\langle A^{N\gamma}|\, J^8 \,|\, B^8\rangle$ will vanish unless it satisfies the condition

$$\mathcal{C}_A = \mathcal{C}_J \;\; \mathcal{C}_B\, \xi_3\; (8,\, 8,\, N\gamma)$$

where $\xi_3 = -\,1$, for $N\gamma = 8_a$

$= +\,1$, for $N\gamma = 1,\, 8_s,\, 27$.

Similarly, for SU(2), the matrix element of the isotriplet current $\langle A|\, J^3 |\, B\rangle$ vanishes unless

$$\mathcal{C}_A = \mathcal{C}_J \;\; \mathcal{C}_B\; (-1)^{1+I_B-I_A}$$

where I is the total isospin.

APPENDIX E

PROPERTIES OF THE PSEUDO DELTA FUNCTION

The pseudo delta function is defined by

$$f_\varepsilon(k) = \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} dx\, e^{ikx} = \frac{\sin \varepsilon k}{\pi k} \tag{E.1}$$

In the limit $\varepsilon \longrightarrow \infty$, $f_\varepsilon(k) \longrightarrow \delta(k)$, the Dirac delta function.

By integrating round semicircular contours in the upper and lower half planes we find that

$$\int_{-\infty}^{\infty} dk\ f_\varepsilon(k) = 1 \tag{E.2}$$

By using similar contours, we can show that

$$\int_{-\infty}^{\infty} dk\ k^n\ f_\varepsilon(k-p) = p^n \tag{E.3}$$

where n = positive integer.

It is important to note that this is <u>only</u> true because the limits of integration are infinite. In this respect, $f_\varepsilon(k)$ does differ from the Dirac delta function which satisfies the relation

$$\int_{p-\lambda}^{p+\lambda} dk\ k^n\ \delta(k-p) = p^n \tag{E.4}$$

where $\lambda > 0$ is a small quantity.

Similarly, we can show that

$$f_\varepsilon(-k) = f_\varepsilon(k) \tag{E.5}$$

and

$$k\, f_\varepsilon(k) = 0 \tag{E.6}$$

One further difference between $f_\varepsilon(k)$ and $\delta(k)$ is the following

$$f_\varepsilon(\alpha k) = \frac{1}{\alpha}\, f_{\alpha\varepsilon}(k) \quad \text{where} \quad \alpha > 0 \tag{E.7}$$

whereas

$$\delta(\alpha k) = \frac{1}{\alpha}\, \delta(k) \quad \text{where} \quad \alpha > 0 \tag{E.8}$$

Of course Eq. (7) just reduces to Eq. (8) when $\varepsilon \to \infty$.

Other properties, including the graph, of the pseudo delta function have already been given in Chapter V.

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